Interpretation of Mueller matrices based on polar decomposition

Shih-Yau Lu and Russell A. Chipman

Department of Physics, University of Alabama in Huntsville, Huntsville, Alabama 35899

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We present an algorithm that decomposes a Mueller matrix into a sequence of three matrix factors: a diattenuator, followed by a retarder, then followed by a depolarizer. Those factors are unique except for singular Mueller matrices. Based on this decomposition, the diattenuation and the retardance of a Mueller matrix can be defined and computed. Thus this algorithm is useful for performing data reduction upon experimentally determined Mueller matrices.

1. INTRODUCTION

The objective of this paper is to decompose an arbitrary Mueller matrix to determine its diattenuation, retardance, and depolarization. For the diattenuation the three diattenuation components x/y, 45/135, and right/left provide a complete description. Similarly the x/y, 45/135, and right/left components of retardances provide a full specification of retardance.

The decomposition of Mueller matrices has been addressed by many authors.[1–4] Cloude[1] showed that any (physical) Mueller matrix can be expressed as a sum of four nondepolarizing Mueller matrices. This decomposition into a sum was also discussed by Anderson and Barakat.[5] Gil and Bernabeu[2] applied the polar decomposition to nondepolarizing Mueller matrices. Xing[3] also discussed the polar decomposition for nondepolarizing Mueller matrices. In addition, Xing[3] pointed out that any Mueller matrix can be expressed as a product of three factors: one nondepolarizing Mueller matrix, followed by a diagonal Mueller matrix, then followed by another nondepolarizing Mueller matrix. This decomposition was further elaborated by Sridhar and Simon.[4]

The theories presented by Cloude and by Sridhar and Simon are useful in discussing the physical realizability of a Mueller matrix. But these methods have not been applied to determine the diattenuation and the retardance of a Mueller matrix. The polar decomposition has been proven to be a feasible approach for this purpose.[2,6] This paper extends the methods of the two previous works in Refs. 2 and 6 to the derivation of the diattenuation and the retardance for a Mueller matrix using the polar decomposition. We show that any Mueller matrix can be decomposed into three factors: a diattenuator, followed by a retarder, then followed by a purely depolarizing factor. Those factors are uniquely defined except for singular Mueller matrices such as polarizers and analyzers. These algorithms also provide a method to perform data reduction upon a Mueller matrix.

2. DIATTENUATION AND RETARDANCE: A REVIEW

Nondepolarizing Mueller matrices describe nondepolarizing polarization elements that convert completely polarized light into completely polarized light. Those Mueller matrices have equivalent Jones matrices. The relationship between nondepolarizing Mueller matrices and Jones matrices has been well presented in the literature; see, for example, Ref. 5. On the other hand, depolarizing Mueller matrices describe depolarizing elements that may convert completely polarized light into partially polarized light. Those Mueller matrices have no equivalent Jones matrices.

We begin by reviewing the diattenuation and the retardance of nondepolarizing elements. For a more complete treatment refer to Refs. 6 and 7. The Jones matrix representation of a polarization element is used in this section, in addition to the Mueller matrix representation, because of the simplicity of the polarization element forms when expressed as Jones matrices.

Physically, a polarization element alters the polarization state of light by changing the amplitudes and/or the phases of the components of its electric-field vector. Two types of nondepolarizing elements, called diattenuators and retarders, are basic. The diattenuator changes only the amplitudes of the components of the electric-field vector. A diattenuator is described by a Hermitian Jones matrix. The retarder changes only the phases of components of the electric-field vector. A retarder has a unitary Jones matrix. Polarizers and wave plates are examples of diattenuators and retarders, respectively.

A diattenuator has an intensity transmission that depends on the incident polarization state. The diattenuation of a diattenuator is defined as[7]

$$D \equiv \frac{|T_q - T_r|}{T_q + T_r} = \frac{||\xi_q|^2 - |\xi_r|^2|}{|\xi_q|^2 + |\xi_r|^2}, \qquad 0 \le D \le 1. \tag{1}$$

Here ξ_q and ξ_r are the eigenvalues of the diattenuator Jones matrix, and $T_q = |\xi_q|^2$ and $T_r = |\xi_r|^2$ are trans-

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mittances for (orthogonal) eigenpolarizations. It turns out that T_q and T_r are also the maximum and minimum transmittances. The diattenuation characterizes the dependence of transmission upon the incident polarization state. Since different diattenuators can have the same diattenuation, additional degrees of freedom are needed for a complete description.

First, the eigenpolarizations of a diattenuator describe its principal axes. The axis (or direction) of diattenuation for a diattenuator is defined to be along the eigenpolarization with larger transmittance. Let this diattenuation axis be along the eigenpolarization described by the Stokes vector $(1, d_1, d_2, d_3)^T = (1, \hat{D}^T)^T$, with $\sqrt{d_1{}^2 + d_2{}^2 + d_3{}^2} = |\hat{D}| = 1$. Define a diattenuation vector $\overrightarrow{D}$ as

$$\overrightarrow{D} \equiv D\hat{D} = \begin{pmatrix} Dd_1 \\ Dd_2 \\ Dd_3 \end{pmatrix} \equiv \begin{pmatrix} D_H \\ D_{45} \\ D_C \end{pmatrix}. \quad (2)$$

The three components of $\overrightarrow{D}$ are called the horizontal, 45°-linear, and circular diattenuations of this diattenuation, respectively. The linear diattenuation is defined as

$$D_L \equiv \sqrt{D_H{}^2 + D_{45}{}^2}. \quad (3)$$

Note that

$$D = \sqrt{D_H{}^2 + D_{45}{}^2 + D_C{}^2} = \sqrt{D_L{}^2 + D_C{}^2} = |\overrightarrow{D}|. \quad (4)$$

A diattenuator described by a diattenuation vector $\overrightarrow{D}$ has a diattenuation $|\overrightarrow{D}|$ and its axis along $(1, \hat{D}^T)^T$. Hence the diattenuation vector describes both the magnitude and the axis of diattenuation, providing a complete description of the diattenuation properties of a diattenuator. The diattenuation vector also characterizes the diattenuator Jones matrix up to a complex factor. The diattenuator Jones matrix with a diattenuation vector $\overrightarrow{D}$ is given by

$$\begin{aligned} \mathbf{J}_D \propto \exp\left(\frac{\alpha}{2}\hat{D}\cdot\vec{\boldsymbol{\sigma}}\right) &= \boldsymbol{\sigma}_0 \cosh\left(\frac{\alpha}{2}\right) \\ &\quad + (\hat{D}\cdot\vec{\boldsymbol{\sigma}})\sinh\left(\frac{\alpha}{2}\right) \\ &\propto \boldsymbol{\sigma}_0 + \frac{\overrightarrow{D}\cdot\vec{\boldsymbol{\sigma}}}{1+\sqrt{1-D^2}}, \end{aligned} \quad (5)$$

where α relates to the diattenuation through $D = \tanh \alpha$. Here $\vec{\boldsymbol{\sigma}} \equiv (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3)^T$. Those $\boldsymbol{\sigma}$ matrices are the Pauli spin matrices, and $\boldsymbol{\sigma}_0$ is the 2×2 identity matrix:

$$\boldsymbol{\sigma}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{\sigma}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$\boldsymbol{\sigma}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \boldsymbol{\sigma}_3 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \quad (6)$$

A retarder causes different phase changes for its (orthogonal) eigenpolarizations. A retarder has a constant intensity transmittance independent of the incident polarization state. The retardance for a retarder is defined as[7]

$$R \equiv |\delta_q - \delta_r|, \qquad 0 \le R \le \pi, \quad (7)$$

in which δ_q and δ_r are the phase changes for the eigenpolarizations. The fast axis (or direction) of retardance is defined to be along the eigenpolarization that emerges first from the retarder, i.e., the eigenpolarization with the leading phase. Let this retardance axis be characterized by the Stokes vector $(1, a_1, a_2, a_3)^T = (1, \hat{R}^T)^T$, with $\sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2} = |\hat{R}| = 1$. The retardance and the associated fast axis are described by a retardance vector $\overrightarrow{R}$ defined as

$$\overrightarrow{R} \equiv R\hat{R} = \begin{pmatrix} Ra_1 \\ Ra_2 \\ Ra_3 \end{pmatrix} \equiv \begin{pmatrix} R_H \\ R_{45} \\ R_C \end{pmatrix}. \quad (8)$$

The components of $\overrightarrow{R}$ give the horizontal, 45°-linear, and circular retardance components. The linear retardance is defined as

$$R_L \equiv \sqrt{R_H{}^2 + R_{45}{}^2}. \quad (9)$$

Note also that

$$R = \sqrt{R_H{}^2 + R_{45}{}^2 + R_C{}^2} = \sqrt{R_L{}^2 + R_C{}^2} = |\overrightarrow{R}|. \quad (10)$$

A retarder described by a retardance vector $\overrightarrow{R}$ has a retardance $|\overrightarrow{R}|$ and its fast axis along $(1, \hat{R}^T)^T$. Therefore the retardance vector provides a complete description for the retarding properties of a retarder and completely characterizes a retarder Jones matrix up to a complex factor. The Jones matrix of a retarder with retardance vector $\overrightarrow{R}$ is

$$\begin{aligned} \mathbf{J} \propto \exp\left(\frac{i}{2}\overrightarrow{R}\cdot\vec{\boldsymbol{\sigma}}\right) &= \boldsymbol{\sigma}_0 \cos\left(\frac{R}{2}\right) \\ &\quad + i(\hat{R}\cdot\vec{\boldsymbol{\sigma}})\sin\left(\frac{R}{2}\right). \end{aligned} \quad (11)$$

The diattenuation and retardance vectors are vectors in a polarization space that corresponds to the three elements of a normalized Stokes vector. The retardance and diattenuation vectors can be represented on the Pioncaré sphere.

A diattenuator (or retarder) is said to be linear if it has no circular diattenuation (or retardance). Likewise, a diattenuator (or retarder) is said to be circular if it has no linear diattenuation (or retardance). Diattenuators and retarders belong to a group of polarization elements called homogeneous polarization elements. A homogeneous element has two orthogonal eigenpolarizations, and its Jones matrix, and Mueller matrix as well, is also said to be homogeneous. The concepts of the diattenuation and the retardance can be easily extended to include all homogeneous elements. First, note that a diattenuator is a homogeneous element with zero retardance and that a retarder is a homogeneous polarization element with no diattenuation. Generally, a homogeneous element can have both diattenuation and retardance. Let $\xi_q = |\xi_q|\exp(i\delta_q)$ and $\xi_r = |\xi_r|\exp(i\delta_r)$ be the eigenvalues of the Jones matrix of a homogeneous element. Those eigenvalues describe the amplitude and phase changes of the orthogonal eigenpolarizations. Thus the diattenuation and the retardance of this homogeneous element can still be obtained by Eqs. (1) and (7).[6,7] In addition, the diattenuation and retardance vectors are along the directions of the eigenpolarizations. It is easy to see that the diattenuation and retardance vectors are parallel for homogeneous elements.

The diattenuation and the retardance of an inhomogeneous element are more complex, since it has nonorthogonal eigenpolarization. Although the eigenvalues are also well defined for inhomogeneous elements, Eqs. (1) and (7) do not yield correct retardances and diattenuation. Instead we define the diattenuation and the retardance through the polar decomposition of the Jones matrix, $\mathbf{J}$,[8,9]

$$\mathbf{J} = \mathbf{J}_R \mathbf{J}_D\,, \tag{12}$$

where $\mathbf{J}_R$ is a unitary matrix denoting a retarder and $\mathbf{J}_D$ is a nonnegative definite Hermitian matrix denoting a diattenuator. Therefore any nondepolarizing polarization element described by a Jones matrix can be represented as a cascade of a retarder and a diattenuator.[2,6] A similar decomposition with a retarder followed by a diattenuator can also be constructed and is equivalent to Eq. (12) but will not be elaborated here.[6] The diattenuation and retardance vectors for $\mathbf{J}$ can be defined as[6]

$$\vec{D}(\mathbf{J}) \equiv \vec{D}(\mathbf{J}_D), \qquad \vec{R}(\mathbf{J}) \equiv \vec{R}(\mathbf{J}_R)\,. \tag{13}$$

For homogeneous elements these definitions reduce to Eqs. (1) and (7). The diattenuation axis of an element is along the polarization state with the maximum transmittance, which is not generally one of the eigenpolarizations.[6] In addition, the diattenuation and retardance vectors of an inhomogeneous element are not parallel.

3. RETARDER AND DIATTENUATOR MUELLER MATRICES

This section summarizes the properties of retarder and diattenuator Mueller matrices, which are the building blocks for nondepolarizing elements.

The effect of a retarder upon polarization states is equivalent to a rotation on the Poincaré sphere. Therefore a retarder has a Mueller matrix $\mathbf{M}_R$ given by

$$\mathbf{M}_R = \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & \mathbf{m}_R \end{bmatrix}, \tag{14}$$

$$(\mathbf{m}_R)_{ij} = \delta_{ij}\cos R + a_i a_j (1-\cos R) + \sum_{k=1}^{3} \epsilon_{ijk} a_k \sin R\,, \qquad i, j = 1, 2, 3\,, \tag{15}$$

in which $\vec{0}$ denotes the three-element zero vector, R is the retardance, $(1, a_1, a_2, a_3)^T = (1, \hat{R}^T)^T$ is the normalized Stokes vector for the fast axis, δ_{ij} is the Kronecker delta, ϵ_{ijk} is the Levi–Cività permutation symbol, and $\mathbf{m}_R$ is a 3×3 submatrix of $\mathbf{M}_R$ obtained by striking out the first row and the first column of $\mathbf{M}_R$. Note that $\mathbf{m}_R$ is a three-dimensional rotation matrix. In Eqs. (14) and (15) we assume that the transmittance of this retarder is unity. Thus a retarder Mueller matrix has 3 degrees of freedom given by its retardance vector. The eigenpolarizations of a retarder are along the fast and slow axes, i.e.,

$$\mathbf{M}_R \begin{pmatrix} 1 \\ \pm\hat{R} \end{pmatrix} = \begin{pmatrix} 1 \\ \pm\hat{R} \end{pmatrix}, \qquad \mathbf{m}_R \hat{R} = \hat{R}\,. \tag{16}$$

Conversely, the retardance and the fast axis of a given retarder Mueller matrix can be obtained by

$$R = \cos^{-1}\left[\frac{\mathrm{tr}(\mathbf{M}_R)}{2} - 1\right],$$
$$a_i = \frac{1}{2\sin R} \sum_{j,k=1}^{3} \epsilon_{ijk} (\mathbf{m}_R)_{jk}\,. \tag{17}$$

A diattenuator has a symmetric Mueller matrix. The Mueller matrix $\mathbf{M}_D$ for a diattenuator with a diattenuation vector $\vec{D}$ is given by

$$\mathbf{M}_D = T_u \begin{bmatrix} 1 & \vec{D}^T \\ \vec{D} & \mathbf{m}_D \end{bmatrix}, \tag{18}$$

$$\mathbf{m}_D = \sqrt{1-D^2}\,\mathbf{I} + (1-\sqrt{1-D^2})\hat{D}\hat{D}^T, \tag{19}$$

in which $\mathbf{I}$ is the 3×3 identity matrix, $\hat{D}(=\vec{D}/|\vec{D}|)$ denotes the unit vector along $\vec{D}$, and T_u is the transmittance for unpolarized light. There are other important properties of a diattenuator Mueller matrix:

$$\mathbf{M}_D \begin{pmatrix} 1 \\ \pm\hat{D} \end{pmatrix} = T_u(1 \pm D) \begin{pmatrix} 1 \\ \pm\hat{D} \end{pmatrix}, \tag{20}$$

$$\mathbf{m}_D \vec{D} = \vec{D}, \qquad \mathbf{m}_D \vec{D}_\perp = \sqrt{1-D^2}\,\vec{D}_\perp\,, \tag{21}$$

in which $\vec{D}_\perp$ denotes a vector normal to $\vec{D}$. It is easy to see that a diattenuator Mueller matrix has 4 degrees of freedom: the diattenuation vector and the transmittance for an unpolarized state.

Diattenuators also have interesting geometrical effects that are not well presented in the literature. One notable exception is the work by de Lang.[10] Consider a diattenuator with its Mueller matrix given by Eqs. (18) and (19). Let the incident state have a normalized Stokes vector $\hat{\mathbf{S}} = (1, \vec{s}^T)^T$. The exiting Stokes vector $\mathbf{S}'$ is given by

$$\mathbf{S}' = T_u \begin{pmatrix} 1 + \vec{D}\cdot\vec{s} \\ (\vec{D} + \vec{s}_{D\parallel}) + \sqrt{1-D^2}\,\vec{s}_{D\perp} \end{pmatrix}, \tag{22}$$

in which $\vec{s}_{D\parallel} \equiv (\vec{s}\cdot\hat{D})\hat{D}$ and $\vec{s}_{D\perp} \equiv \vec{s} - \vec{s}_{D\parallel}$ are the projections of $\vec{s}$ along $\vec{D}$ and $\vec{D}_\perp$, respectively. It follows that, in the Poincaré sphere, this exiting state always lies on the same plane containing the incident state and the diattenuation axis. For a completely polarized incident state Eq. (22) leads to

$$\cos\Theta'_{\mathrm{DS}} = \frac{D + \cos\Theta_{\mathrm{DS}}}{1 + D\cos\Theta_{\mathrm{DS}}}\,, \qquad \Theta'_{\mathrm{DS}} \le \Theta_{\mathrm{DS}}\,, \tag{23}$$

in which Θ_{DS} is the angle between the diattenuation axis and the incident state in the Poincaré sphere and Θ'_{DS} is the angle for the exiting state. Completely polarized incident states are moved along the great circles toward the diattenuation axis, which is the state of the maximum transmittance. This motion approaches zero as the incident state approaches either of the eigenpolarizations. Equation (22) also gives the intensity transmittance function $T(\hat{\mathbf{S}})$:

$$T(\hat{\mathbf{S}}) = T_u(1 + D|\vec{s}|\cos\Theta_{\mathrm{DS}}), \tag{24}$$

in which $|\vec{s}|$ is the degree of polarization (DOP) for the incident state.

4. DIATTENUATION VECTOR AND POLARIZANCE VECTOR

Consider an arbitrary Mueller matrix, denoted by $\mathbf{M}$, whose elements are denoted by $m_{\alpha\beta}$ ($\alpha, \beta = 0, \ldots, 3$). Note that the first row of a Mueller matrix completely determines the intensity transmittance. It can be shown that the maximum and minimum transmittances $T_{\max}$ and $T_{\min}$ for $\mathbf{M}$ are given by[11]

$$T_{\substack{\max\\ \min}} = m_{00} \pm \sqrt{m_{01}^2 + m_{02}^2 + m_{03}^2}, \tag{25}$$

in which the plus sign is for $T_{\max}$ and the minus sign is for $T_{\min}$. In addition, their associated incident Stokes vectors are given by[11]

$$\hat{\mathbf{S}}_{\max} = \begin{pmatrix} 1 \\ m_{01} \Big/ \sqrt{m_{01}^2 + m_{02}^2 + m_{03}^2} \\ m_{02} \Big/ \sqrt{m_{01}^2 + m_{02}^2 + m_{03}^2} \\ m_{03} \Big/ \sqrt{m_{01}^2 + m_{02}^2 + m_{03}^2} \end{pmatrix},$$

$$\hat{\mathbf{S}}_{\min} = \begin{pmatrix} 1 \\ -m_{01} \Big/ \sqrt{m_{01}^2 + m_{02}^2 + m_{03}^2} \\ -m_{02} \Big/ \sqrt{m_{01}^2 + m_{02}^2 + m_{03}^2} \\ -m_{03} \Big/ \sqrt{m_{01}^2 + m_{02}^2 + m_{03}^2} \end{pmatrix}. \tag{26}$$

Note that $\hat{\mathbf{S}}_{\max}$ and $\hat{\mathbf{S}}_{\min}$ are completely polarized and orthogonal to each other. However, in the presence of depolarization the exiting states of $\hat{\mathbf{S}}_{\max}$ and $\hat{\mathbf{S}}_{\min}$ may not be completely polarized nor be orthogonal.

Based on the definition derived in Ref. 6, the diattenuation of this Mueller matrix is given by

$$D = \frac{T_{\max} - T_{\min}}{T_{\max} + T_{\min}} = \frac{1}{m_{00}}\sqrt{m_{01}^2 + m_{02}^2 + m_{03}^2}\ . \tag{27}$$

The axis of diattenuation is along the state $\hat{\mathbf{S}}_{\max}$. It follows that the diattenuation vector of $\mathbf{M}$ is given by

$$\vec{D} = \begin{pmatrix} D_H \\ D_{45} \\ D_R \end{pmatrix} = \frac{1}{m_{00}} \begin{pmatrix} m_{01} \\ m_{02} \\ m_{03} \end{pmatrix}. \tag{28}$$

Thus the first row of $\mathbf{M}$ gives its diattenuation vector. The intensity transmittance function of $\mathbf{M}$ is completely described by this diattenuation vector through Eq. (24). With this diattenuation vector the expressions for $\hat{\mathbf{S}}_{\max}$ and $\hat{\mathbf{S}}_{\min}$ can be rewritten as

$$\hat{\mathbf{S}}_{\max} = \begin{pmatrix} 1 \\ \hat{D} \end{pmatrix}, \qquad \hat{\mathbf{S}}_{\min} = \begin{pmatrix} 1 \\ -\hat{D} \end{pmatrix}. \tag{29}$$

Use the symbol T_H to denote the transmittance for horizontally polarized light; let T_V, T_{45}, T_{135}, T_R, and T_L have analogous definitions. It follows that

$$\begin{aligned} \frac{T_H - T_V}{T_H + T_V} &= \frac{m_{01}}{m_{00}} = D_H\,, \\ \frac{T_{45} - T_{135}}{T_{45} + T_{135}} &= \frac{m_{02}}{m_{00}} = D_{45}\,, \\ \frac{T_R - T_L}{T_R + T_L} &= \frac{m_{03}}{m_{00}} = D_C\,. \end{aligned} \tag{30}$$

Equations (30) provide a physical explanation, as well as an operational definition, for the diattenuation vector.

Consider the case of the unpolarized incident state. Its exiting state is determined only by the first column of $\mathbf{M}$. The DOP of this exiting light resulting from unpolarized light is called the polarizance.[6,12] Thus the polarizance of $\mathbf{M}$ is given by

$$P = \frac{1}{m_{00}}\sqrt{m_{10}^2 + m_{20}^2 + m_{30}^2}, \qquad 0 \le P \le 1\,. \tag{31}$$

Similarly, the concept of polarizance can be further generalized into the polarizance vector $\vec{P}$ defined as

$$\vec{P} \equiv \begin{pmatrix} P_H \\ P_{45} \\ P_R \end{pmatrix} = \frac{1}{m_{00}} \begin{pmatrix} m_{10} \\ m_{20} \\ m_{30} \end{pmatrix}, \qquad |\vec{P}| = P\,. \tag{32}$$

Thus the first column of $\mathbf{M}$ gives its polarizance vector. With this polarizance vector the exiting Stokes vector for the unpolarized incident light is given by $m_{00}(1, \vec{P}^T)^T$. This is also the average emerging Stokes vector when integrating the incident Stokes vector over the entire Poincaré sphere. Physically, components of $\vec{P}$ are equal to the horizontal DOP, 45°-linear DOP, and circular DOP produced by unpolarized light. For nondepolarizing Mueller matrices, the exiting states for $\hat{\mathbf{S}}_{\max}$ and $\hat{\mathbf{S}}_{\min}$ can be expressed as $T_{\max}(1, \hat{P}^T)^T$ and $T_{\min}(1, -\hat{P}^T)^T$, respectively.

5. DECOMPOSITION OF NONDEPOLARIZING MUELLER MATRICES

This section considers only nondepolarizing Mueller matrices. Let $\mathbf{M}$ denote a nondepolarizing Mueller matrix. According to the discussion in Refs. 2 and 6, there exists a diattenuator $\mathbf{M}_D$ and retarder $\mathbf{M}_R$ pair such that

$$\mathbf{M} = \mathbf{M}_R\mathbf{M}_D\,, \tag{33}$$

in which $\mathbf{M}_D$ and $\mathbf{M}_R$ describe the diattenuation and retardance properties of this Mueller matrix, respectively. If one applies those general expressions for $\mathbf{M}_D$ and $\mathbf{M}_R$ given in Section 3, it follows immediately that $\mathbf{M}_D$ can be determined from the following parameters:

$$T_u = m_{00}, \qquad \vec{D} = \frac{1}{m_{00}} \begin{pmatrix} m_{01} \\ m_{02} \\ m_{03} \end{pmatrix}. \tag{34}$$

This echoes the results of Section 4. If this Mueller matrix does not describe an analyzer or a polarizer, then $\mathbf{M}_D$ is nonsingular and $\mathbf{M}_R$ can be simply determined by

$$\mathbf{M}_R = \mathbf{M}\mathbf{M}_D{}^{-1}. \tag{35}$$

The retardance vector of $\mathbf{M}$ is then determined from $\mathbf{M}_R$ by Eqs. (17). It should be emphasized that those diattenuation and retardance vectors determine a nondepolarizing Mueller matrix up to a constant factor, and they completely characterize its polarization properties.

Rewrite $\mathbf{M}$ as

$$\mathbf{M} = m_{00}\begin{bmatrix} 1 & \vec{D}^T \\ \vec{P} & \mathbf{m} \end{bmatrix}, \tag{36}$$

where $\vec{D}$ and $\vec{P}$ are the diattenuation and polarizance vectors of $\mathbf{M}$, and $\mathbf{m}$ is a 3×3 matrix obtained by striking out the first row and the first column of $\mathbf{M}$ and then dividing by m_{00}. It is easy to see that

$$\mathbf{m} = \mathbf{m}_R\mathbf{m}_D = \sqrt{1 - D^2}\,\mathbf{m}_R + (1 - \sqrt{1 - D^2})\hat{P}\hat{D}^T. \tag{37}$$

It follows that

$$\mathbf{m}_R = \frac{1}{\sqrt{1 - D^2}}[\mathbf{m} - (1 - \sqrt{1 - D^2})\hat{P}\hat{D}^T]. \tag{38}$$

If this Mueller matrix $\mathbf{M}$ describes an analyzer or a polarizer, then $\mathbf{M}_D$ is singular and Eqs. (35) and (38) are not valid. In such a case there are infinite numbers of $\mathbf{M}_R$ that satisfy Eq. (33). However, we choose the $\mathbf{M}_R$ with the minimum retardance.[6] From geometrical considerations it follows that the retardance vector of $\mathbf{M}_R$, and $\mathbf{M}$ as well, for an analyzer or a polarizer is given by

$$\vec{R} = \frac{\vec{P} \times \vec{D}}{|\vec{P} \times \vec{D}|}\cos^{-1}(\vec{P} \cdot \vec{D}). \tag{39}$$

With Eq. (39) $\mathbf{M}_R$ can be easily determined. In fact, the general Mueller matrix for a nondepolarizing analyzer or polarizer can be expressed as

$$\mathbf{M}_{\substack{\text{ND polarizer}\\ \text{ND analyzer}}} = m_{00}\begin{pmatrix} 1 \\ \vec{P} \end{pmatrix}\begin{pmatrix} 1 \\ \vec{D} \end{pmatrix}^T = m_{00}\begin{bmatrix} 1 & \vec{D}^T \\ \vec{P} & \vec{P}\vec{D}^T \end{bmatrix},$$
$$|\vec{D}| = |\vec{P}| = 1. \tag{40}$$

Here ND denotes nondepolarizing. Equation (40) describes a nondepolarizing analyzer for the state $(1, \vec{D}^T)^T$ and a nondepolarizing polarizer for $(1, \vec{P}^T)^T$. When $\vec{D} = \vec{P}$, Eq. (40) describes a (nondepolarizing) homogeneous analyzer and polarizer, i.e., a diattenuator with $D = 1$. Appendix A gives a more complete discussion on analyzer and polarizer Mueller matrices.

Here we examine the relation between the polarizance vector and the diattenuation vector of a nondepolarizing Mueller matrix. It follows from Eq. (33) that

$$\vec{P} = \mathbf{m}_R\vec{D}. \tag{41}$$

Equation (41) implies the equivalence of the magnitude of the diattenuation and the polarizance for all nondepolarizing elements.[6] Thus the following relation must be satisfied by nondepolarizing Mueller matrices:

$$m_{01}{}^2 + m_{02}{}^2 + m_{03}{}^2 = m_{10}{}^2 + m_{20}{}^2 + m_{30}{}^2. \tag{42}$$

In particular, a nondepolarizing Mueller matrix is homogeneous if and only if its polarizance vector and diattenuation vector are identical, i.e., $\vec{D} = \vec{P}$.

6. DEPOLARIZER MUELLER MATRICES

The diagonal Mueller matrix given below represents a pure depolarizer, an element with zero diattenuation or retardance:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{bmatrix}, \qquad |a|, |b|, |c| \le 1. \tag{43}$$

The principal axes of this depolarizer are along the s_1 axes, the s_2 axis, and the s_3 axis. Its principal depolarization factors are $1 - |a|$, $1 - |b|$, and $1 - |c|$, which describe the depolarization capabilities of this polarizer along its principal axes. Define a parameter Δ as

$$\Delta \equiv 1 - \frac{|a| + |b| + |c|}{3}, \qquad 0 \le \Delta \le 1. \tag{44}$$

Δ is the average of the principal depolarization factors. Roughly speaking, this parameter indicates the averaged depolarization capability of this depolarizer, and this parameter can be called the depolarization power of this depolarizer. In general, the principal axes of a depolarizer can be along any three orthogonal axes. Thus a more general expression for a depolarizer is given by

$$\begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & \mathbf{m}_\Delta \end{bmatrix}, \qquad \mathbf{m}_\Delta{}^T = \mathbf{m}_\Delta, \tag{45}$$

in which $\mathbf{m}_\Delta$ is a 3×3 symmetric matrix. The eigenvalues of $\mathbf{m}_\Delta$ give the three principal depolarization factors, and its depolarizing power can be determined by Eq. (44). The eigenvectors of $\mathbf{m}_\Delta$ give the three (orthogonal) principal axes, and those principal axes can be conveniently described by the Euler angles. This matrix can be diagonalized into the form of Eq. (44) by retarder Mueller matrices.

The matrix given by expression (45) has only 6 degrees of freedom, since it does not include the polarizance that a depolarizer may display. The most general expression for a depolarizer with polarizance is

$$\begin{bmatrix} 1 & \vec{0}^T \\ \vec{P}_\Delta & \mathbf{m}_\Delta \end{bmatrix} \equiv \mathbf{M}_\Delta, \qquad \mathbf{m}_\Delta{}^T = \mathbf{m}_\Delta, \tag{46}$$

in which $\vec{P}_\Delta$ denotes the polarizance vector of this depolarizer. This depolarizer has 9 degrees of freedom, and it has zero diattenuation or retardance. Therefore we regard Eq. (46) as the most general form for a pure depolarizer. Here the polarizance P_Δ characterizes the polarizing capability of this depolarizer, while the eigenvalues and the eigenvectors of $\mathbf{m}_\Delta$ characterize its depolarization properties. So a depolarizer may have certain polarizing capability that is due to nonzero polarizance. Some depolarizing elements even behave as polarizers, such as a depolarizer followed by a polarizer.[13] On the other hand, nondepolarizing elements may decrease the DOP for partially polarized incident states.[13]

7. DECOMPOSITION OF DEPOLARIZING MUELLER MATRICES

Let $\mathbf{M}$ be a depolarizing Mueller matrix. The diattenuator Mueller matrix $\mathbf{M}_D$ can still be found by the same method as that in the case of a nondepolarizing element with the use of Eqs. (18) and (19). Thus $\mathbf{M}_D$ is completely determined from the first row of $\mathbf{M}$. In the following, assume that $\mathbf{M}_D$ is not singular. The case in which $\mathbf{M}_D$ is singular is treated in Appendix A.

Define a Mueller matrix $\mathbf{M}'$ based on $\mathbf{M}$ as

$$\mathbf{M}' \equiv \mathbf{M}{\mathbf{M}_D}^{-1}. \tag{47}$$

Note that $\mathbf{M}'$ has no diattenuation. However, $\mathbf{M}'$ is not a pure retarder as in the case of a nondepolarizing element, because $\mathbf{M}'$ contains both retardance and depolarization. $\mathbf{M}'$ is then further decomposed as a retarder followed by a depolarizer:

$$\begin{aligned}\mathbf{M}_\Delta \mathbf{M}_R &= \begin{bmatrix} 1 & \vec{0}^T \\ \vec{P}_\Delta & \mathbf{m}_\Delta \end{bmatrix}\begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & \mathbf{m}_R \end{bmatrix} = \begin{bmatrix} 1 & \vec{0}^T \\ \vec{P}_\Delta & \mathbf{m}_\Delta \mathbf{m}_R \end{bmatrix} \\ &= \begin{bmatrix} 1 & \vec{0}^T \\ \vec{P}_\Delta & \mathbf{m}' \end{bmatrix} = \mathbf{M}'. \end{aligned} \tag{48}$$

Here $\mathbf{m}'$ is a 3×3 submatrix of $\mathbf{M}'$. In this section we are going to demonstrate that the decomposition given by Eq. (48) is valid. Therefore $\mathbf{M}$ can be decomposed as

$$\mathbf{M} = \mathbf{M}_\Delta \mathbf{M}_R \mathbf{M}_D\,, \tag{49}$$

in which $\mathbf{M}_D$ is a diattenuator, $\mathbf{M}_R$ is a retarder, and $\mathbf{M}_\Delta$ is a depolarizer. Those three factors in Eq. (49) provide diattenuation, retardance, and depolarization properties about $\mathbf{M}$. Thus Eq. (49) can be regarded as a generalized polar decomposition for depolarizing Mueller matrices.

Equations (47) and (48) lead to

$$\vec{P}_\Delta = \frac{\vec{P} - \mathbf{m}\vec{D}}{1 - D^2}\,, \tag{50}$$

$$\mathbf{m}' = \mathbf{m}_\Delta \mathbf{m}_R\,. \tag{51}$$

Now the 4×4 matrix decomposition reduces to a 3×3 matrix decomposition given by Eq. (51). Actually, this is equivalent to the polar decomposition of a 3×3 real matrix. Let λ_1, λ_2, and λ_3 be the eigenvalues of $\mathbf{m}'(\mathbf{m}')^T$. From Eq. (51) $\mathbf{m}_\Delta$ has $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$, and $\sqrt{\lambda_3}$ as eigenvalues. It follows that $\mathbf{m}_\Delta$ can be obtained by

$$\begin{aligned}\mathbf{m}_\Delta = &\pm [\mathbf{m}'(\mathbf{m}')^T + (\sqrt{\lambda_1\lambda_2} + \sqrt{\lambda_2\lambda_3} + \sqrt{\lambda_3\lambda_1})\mathbf{I}]^{-1} \\ &\times [(\sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3})\mathbf{m}'(\mathbf{m}')^T + \sqrt{\lambda_1\lambda_2\lambda_3}\,\mathbf{I}].\end{aligned} \tag{52}$$

If the determinant of $\mathbf{m}'$ is negative, then the minus sign is applied. Otherwise, the plus sign is applied. $\mathbf{M}_\Delta$ can be determined through Eqs. (50) and (52). $\mathbf{M}_R$ can be obtained by

$$\mathbf{M}_R = {\mathbf{M}_\Delta}^{-1}\mathbf{M}'. \tag{53}$$

If $\mathbf{m}'$ is singular, then so are $\mathbf{m}_\Delta$ and $\mathbf{M}_\Delta$. Equations (52) and (53) cannot apply. $\mathbf{M}_R$ still exists, though it is not unique. We again choose the one with the minimum retardance. Appendix B gives a detailed account for this case.

The depolarizing Mueller matrix $\mathbf{M}_\Delta$ obtained is always a semidefinite matrix. Its depolarization power, denoted by Δ, is given by

$$\Delta = 1 - \frac{|\mathrm{tr}(\mathbf{m}_\Delta)|}{3} = 1 - \frac{|\mathrm{tr}(\mathbf{M}_\Delta) - 1|}{3}, \qquad 0 \le \Delta \le 1\,. \tag{54}$$

8. DISCUSSION

Since this matrix multiplication is not commutative, the three factors in the generalized polar decomposition given by Eq. (49) are order dependent. This section briefly addresses this problem of order dependence.

With three factors six different arrangements are possible. Thus there are five other variations to the decompositions given by Eq. (49), as follows:

$$\mathbf{M} = \mathbf{M}_{\Delta 2}\mathbf{M}_{D2}\mathbf{M}_{R2}\,, \tag{55}$$

$$\mathbf{M} = \mathbf{M}_{R3}\mathbf{M}_{D3}\mathbf{M}_{\Delta 3}\,, \tag{56}$$

$$\mathbf{M} = \mathbf{M}_{D4}\mathbf{M}_{R4}\mathbf{M}_{\Delta 4}\,, \tag{57}$$

$$\mathbf{M} = \mathbf{M}_{R5}\mathbf{M}_{\Delta 5}\mathbf{M}_{D5}\,, \tag{58}$$

$$\mathbf{M} = \mathbf{M}_{D6}\mathbf{M}_{\Delta 6}\mathbf{M}_{R6}\,. \tag{59}$$

Like Eq. (49), Eq. (55) has its depolarizer factors on the left. Equations (56) and (57) have their depolarizer factors on the right, and Eqs. (58) and (59) have their depolarizer factors in the middle. A comparison of Eqs. (49) and (55) leads to the relationship

$$\mathbf{M}_{\Delta 2} = \mathbf{M}_\Delta, \qquad \mathbf{M}_{R2} = \mathbf{M}_R, \qquad \mathbf{M}_{D2} = \mathbf{M}_R\mathbf{M}_D{\mathbf{M}_R}^T\,. \tag{60}$$

Thus the retarders and the depolarizers are the same in those two decompositions, and the diattenuators are related by a similarity transformation. Likewise, a comparison of Eqs. (49) and (58) leads to

$$\mathbf{M}_{\Delta 5} = {\mathbf{M}_R}^T\mathbf{M}_\Delta\mathbf{M}_R, \qquad \mathbf{M}_{R5} = \mathbf{M}_R, \qquad \mathbf{M}_{D5} = \mathbf{M}_D\,. \tag{61}$$

In this case the retarders and the diattenuators are the same, and the depolarizers are related by a similarity transformation. Thus the same diattenuation, retardance, and depolarization power are computed from those three decompositions given by Eqs. (49), (55), and (58). However, the decompositions given by Eqs. (56), (57), and (59) bear no simple relationship to Eq. (49).

As mentioned in Section 1, the decomposition of the Mueller matrix has been an important topic in the polarization calculus. However, only those decompositions that reveal useful physical properties and are readily mathematically obtained are the most interesting. The decomposition given by Eq. (49) is particularly important. Not only does this decomposition meet the criteria above, but also it clearly separates the depolarizing component ($\mathbf{M}_\Delta$) from the nondepolarizing component ($\mathbf{M}_R\mathbf{M}_D$). For completely polarized incident light the decrease of the DOP is solely caused by this depolarizing component, since light is still completely polarized after passing through the nondepolarizing component. Thus it is useful, for the interpretation of experimental data, to have the depolarizing component following the nondepolarizing component. Besides, this decomposition is a natural generalization of the polar decomposition.

APPENDIX A: ANALYZER AND POLARIZER MUELLER MATRICES

If a polarization element transmits a certain (completely polarized) state, say $\mathbf{S}_a$, and blocks its orthogonal state $\mathbf{S}_{a\perp}$, then this element is called a perfect analyzer, or simply an analyzer, for $\mathbf{S}_a$. Mathematically, an $\mathbf{S}_a$ analyzer must satisfy

$$T(\mathbf{S}_a) = T_{\max} \neq 0, \qquad T(\mathbf{S}_{a\perp}) = T_{\min} = 0\,. \tag{A1}$$

A criterion for analyzers follows immediately: a Mueller matrix describes an analyzer if and only if it has unity diattenuation. Moreover, the Mueller matrix of an $\mathbf{S}_a$ analyzer has its first row parallel to ${\mathbf{S}_a}^T$.

On the other hand, if a polarization element always produces completely polarized light for arbitrary incident light, then this element is called a perfect polarizer, or simply a polarizer. This definition leads to a criterion for polarizers: a Mueller matrix describes a polarizer if and only if its polarizance is unity. This can be understood through the fact that the unpolarized state is equivalent to an incoherent sum of any two orthogonal polarization states. Moreover, if an element generates completely polarized light from one particular partially polarized light, then it always generates completely polarized light and it is a polarizer. It turns out that the exiting state of a polarizer is not only completely polarized but also fixed. That is, the exiting Stokes vector of a polarizer is always parallel to a completely polarized Stokes vector, say $\mathbf{S}_p$. Such a polarizer is called an $\mathbf{S}_p$ polarizer. The Mueller matrix of an $\mathbf{S}_p$ polarizer has its first column parallel to the Stokes vector $\mathbf{S}_p$.

With the definitions above, a polarizer or an analyzer can be either homogeneous or inhomogeneous and either nondepolarizing or depolarizing. For nondepolarizing elements an analyzer is also a polarizer, and vice versa. However, this is not true for depolarizing elements. Nevertheless, the following statement is always true: a Mueller matrix $\mathbf{M}$ describes an analyzer (or polarizer) if and only if $\mathbf{M}^T$ describes a polarizer (or analyzer). Additional analysis leads to the general form for analyzer and polarizer Mueller matrices:

$$m_{00}\begin{bmatrix} 1 & \vec{D}^T \\ \vec{P} & \vec{P}\,\vec{D}^T \end{bmatrix} = \begin{cases} \mathbf{M}_{\text{analyzer}}\,, & |\vec{D}| = 1 \\ \mathbf{M}_{\text{polarizer}}\,, & |\vec{P}| = 1 \end{cases}. \tag{A2}$$

Note that the Mueller matrix for an analyzer or a polarizer is always singular. The exiting state of an analyzer is also fixed, though it is partially polarized for a depolarizing analyzer. Thus a depolarizing analyzer is a perfect analyzer but an imperfect polarizer. On the other hand, a depolarizing polarizer has a diattenuation less than unity. Thus a depolarizing polarizer is a perfect polarizer but an imperfect analyzer. This is summarized in Table 1. Note also that a depolarizing analyzer is equivalent to a nondepolarizing analyzer followed by a pure depolarizer and that a depolarizing polarizer is equivalent to a pure depolarizer followed by a nondepolarizing polarizer.

A depolarizing analyzer has the decomposition given by

$$\begin{aligned} m_{00}\begin{bmatrix} 1 & \vec{D}^T \\ \vec{P} & \vec{P}\,\vec{D}^T \end{bmatrix} &= \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & P\mathbf{I} \end{bmatrix} \times m_{00}\begin{bmatrix} 1 & \vec{D}^T \\ \hat{P} & \hat{P}\vec{D}^T \end{bmatrix} \\ &= \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & P\mathbf{I} \end{bmatrix} \times \mathbf{M}_R \times m_{00}\begin{bmatrix} 1 & \vec{D}^T \\ \vec{D} & \vec{D}\vec{D}^T \end{bmatrix}, \\ &\qquad |\vec{D}| = 1\,, \end{aligned} \tag{A3}$$

in which $\hat{P}(=\vec{P}/P)$ is a unit vector along the polarizance vector $\vec{P}$. In the decomposition given by Eq. (A3) the diattenuator is the analyzer and also the polarizer for the state $(1, \vec{D}^T)^T$. The second factor is a retarder that transfers $(1, \vec{D}^T)^T$ into $(1, \hat{P}^T)^T$, and it is followed by a pure depolarizer. Because the diattenuator is singular, the retarder and the depolarizer are not uniquely determined, as a family of retarders and depolarizers can map the analyzed state into the single exiting state. Nevertheless, we choose the retarder Mueller matrix $\mathbf{M}_R$ to be the unique retarder that rotates $\vec{D}$ into $\hat{P}$ along the shortest path on the Poincaré sphere. Thus the retardance for an analyzer is given by

$$\vec{R} = \frac{\hat{P} \times \vec{D}}{|\hat{P} \times \vec{D}|} \cos^{-1}(\hat{P} \cdot \vec{D})\,. \tag{A4}$$

Note the similarity between Eqs. (39) and (A4). We also choose the depolarizer to be the one given in Eq. (A3). With this choice the depolarization power of this analyzer is $1 - |\vec{P}|$.

APPENDIX B: DECOMPOSITION OF SINGULAR MUELLER MATRICES

This appendix discusses Eq. (51) when $\mathbf{m}'$ is singular by applying the singular-value decomposition. The singular-value decomposition of $\mathbf{m}'$ is as follows:

$$\begin{aligned} \mathbf{m}' &= (\hat{\mathbf{v}}_1\,\hat{\mathbf{v}}_2\,\hat{\mathbf{v}}_3)\mathrm{diag}(\sqrt{\lambda_1}\,, \sqrt{\lambda_2}\,, \sqrt{\lambda_3})(\hat{\mathbf{u}}_1\,\hat{\mathbf{u}}_2\,\hat{\mathbf{u}}_3)^T \\ &= \sqrt{\lambda_1}\,\hat{\mathbf{v}}_1{\hat{\mathbf{u}}_1}^T + \sqrt{\lambda_2}\,\hat{\mathbf{v}}_2{\hat{\mathbf{u}}_2}^T + \sqrt{\lambda_3}\,\hat{\mathbf{v}}_3{\hat{\mathbf{u}}_3}^T\,. \end{aligned} \tag{B1}$$

Here the $\hat{\mathbf{v}}$'s are a set of (real) orthonormal vectors, as are the $\hat{\mathbf{u}}$'s. The λ's are the eigenvalues of $\mathbf{m}'(\mathbf{m}')^T$. It follows from Eq. (B1) that

$$\mathbf{m}_\Delta = \pm(\sqrt{\lambda_1}\,\hat{\mathbf{v}}_1{\hat{\mathbf{v}}_1}^T + \sqrt{\lambda_2}\,\hat{\mathbf{v}}_2{\hat{\mathbf{v}}_2}^T + \sqrt{\lambda_3}\,\hat{\mathbf{v}}_3{\hat{\mathbf{v}}_3}^T)\,, \tag{B2}$$

$$\mathbf{m}_R = \pm(\hat{\mathbf{v}}_1{\hat{\mathbf{u}}_1}^T + \hat{\mathbf{v}}_2{\hat{\mathbf{u}}_2}^T + \hat{\mathbf{v}}_3{\hat{\mathbf{u}}_3}^T)\,. \tag{B3}$$

The sign convention for Eqs. (B2) and (B3) is the same as that in Eq. (52). For singular matrices those $\hat{\mathbf{v}}$'s and $\hat{\mathbf{u}}$'s corresponding to zero singular values are not uniquely defined. Nevertheless, we choose the $\mathbf{m}_R$ with the minimum retardance as the retarder because it is a uniquely defined member of the solution set. The determination of $\mathbf{m}_R$ is elaborated in the following.

Table 1. Classification of Analyzers and Polarizers

$D = P = 1$	Nondepolarizing analyzer and polarizer (perfect analyzer and perfect polarizer)
$D = 1\,,\ 0 \leq P < 1$	Depolarizing analyzer (perfect analyzer and imperfect polarizer)
$0 \leq D < 1\,,\ P = 1$	Depolarizing polarizer (perfect polarizer and imperfect analyzer)

Assume that $\mathbf{m}'$ is singular. First, consider the case in which $\lambda_1 = \lambda_2 = \lambda_3 = 0$, i.e., $\mathbf{m}'$ is a zero matrix. Simply choose

$$\mathbf{M}_\Delta = \begin{bmatrix} 1 & \vec{0}^T \\ \vec{P}_\Delta & \mathbf{0} \end{bmatrix}, \qquad \mathbf{M}_R = \mathbf{I}\,. \tag{B4}$$

Next, consider the case in which $\lambda_1 \neq 0$ and $\lambda_2 = \lambda_3 = 0$. $\mathbf{m}'$ can be expressed as

$$\mathbf{m}' = \sqrt{\lambda_1}\,\hat{\mathbf{v}}\hat{\mathbf{u}}^T = (\sqrt{\lambda_1}\,\hat{\mathbf{v}}\hat{\mathbf{v}}^T)\mathbf{m}_R\,. \tag{B5}$$

Hence

$$\mathbf{m}_\Delta = \sqrt{\lambda_1}\,\hat{\mathbf{v}}\hat{\mathbf{v}}^T = \frac{\mathbf{m}'(\mathbf{m}')^T}{\sqrt{\mathrm{tr}[\mathbf{m}'(\mathbf{m}')^T]}}\,. \tag{B6}$$

Clearly, $\mathbf{m}_R$ is a three-dimensional rotation matrix that transfers $\hat{\mathbf{u}}$ into $\hat{\mathbf{v}}$. From geometrical considerations the minimum retardance and its fast axis are given by

$$R = \cos^{-1}(\hat{\mathbf{v}} \cdot \hat{\mathbf{u}}) = \cos^{-1}\left\{\frac{\mathrm{tr}(\mathbf{m}')}{\sqrt{\mathrm{tr}[\mathbf{m}'(\mathbf{m}')^T]}}\right\},$$
$$\hat{R} = \frac{\hat{\mathbf{v}} \times \hat{\mathbf{u}}}{|\hat{\mathbf{v}} \times \hat{\mathbf{u}}|}\,. \tag{B7}$$

Finally, consider the case in which $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, and $\lambda_3 = 0$. Consequently, Eq. (52) reduces to

$$\mathbf{m}_\Delta = (\sqrt{\lambda_1} + \sqrt{\lambda_2})[\mathbf{m}'(\mathbf{m}')^T + \sqrt{\lambda_1\lambda_2}\,\mathbf{I}]^{-1}\mathbf{m}'(\mathbf{m}')^T. \tag{B8}$$

Also, from geometrical considerations, it follows that

$$\mathbf{m}_R = \hat{\mathbf{v}}_1\hat{\mathbf{u}}_1{}^T + \hat{\mathbf{v}}_2\hat{\mathbf{u}}_2{}^T + \frac{\hat{\mathbf{v}}_1 \times \hat{\mathbf{v}}_2}{|\hat{\mathbf{v}}_1 \times \hat{\mathbf{v}}_2|}\,\frac{(\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2)^T}{|\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2|}\,. \tag{B9}$$

Note added in proof: The diagonalization of Mueller matrices discussed in Refs. 3 and 4 is not always possible, as pointed out by van der Mee.[14] We thank one of the reviewers for bringing this publication to our attention.

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