MATH 231: Calculus of Functions of One Variable 1

2.2 Definitions of Limits

1. The value of $\lim_{x \to a} f(x)$ (if it exists) depends on the values of f near a,

   but it does not depends on the value of *f(a)*.

   In some cases, the limit $\lim_{x \to a} f(x)$ equals *f(a)*.

   In other instances, $\lim_{x \to a} f(x)$ and *f(a)* differ, or *f(a)* may not even be defined.

2. So do one-side limits.

3. Finding limits from a table

   Create a table of values of $f(x) = \dfrac{\sqrt{x} - 1}{x - 1}$ corresponding to values of x near 1.

   Then make a conjecture about the value of $\lim_{x \to 1} f(x)$.

	$\rightarrow 1 \leftarrow$					
x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x) = \dfrac{\sqrt{x} - 1}{x - 1}$	0.5131670	0.5012563	0.5001251	0.4998751	0.4987562	0.4880885

   $\lim_{x \to 1} f(x) = 0.5$

4. Finding limits from a table is less reliable than from a graph.

5. Two-Sided Limits

   $\lim_{x \to a} f(x) = L$, iff $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$.

6. If either $\lim_{x \to a^+} f(x) \neq L$ or $\lim_{x \to a^-} f(x) \neq L$ (or both), then $\lim_{x \to a} f(x) \neq L$.

7. If either $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ does not exist, then $\lim_{x \to a} f(x)$ does not exist.

8. Examine $\lim_{x \to 0} cos(1/x)$

- Examining limits numerically

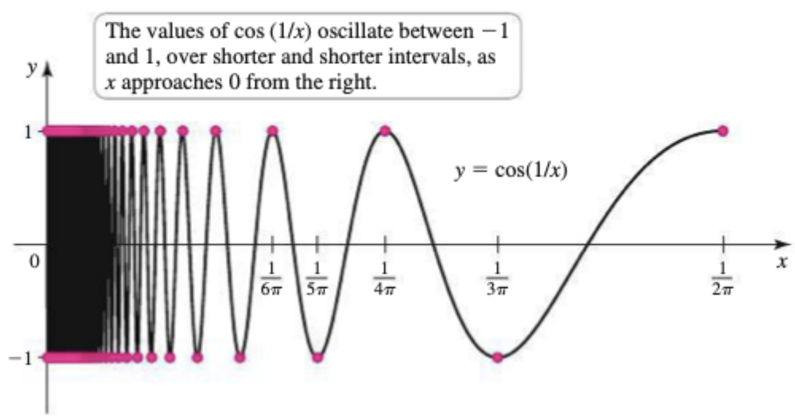
x	cos (1/x)
0.001	0.56238
0.0001	-0.95216
0.00001	-0.99936
0.000001	0.93675
0.0000001	-0.90727
0.00000001	-0.36338

It is tempting to conclude that
$\lim_{x \to 0+} cos(1/c) = -1.$
But this conclusion is not confirmed when we evaluate cos(1/x) for values of x closer to 0.

$\lim_{x \to 0} cos(1/x) \neq -1$

- Examining limits graphically

x=1/(nπ) → $cos\dfrac{1}{x} = cos n\pi = \begin{cases} 1 & if\, n\, is\, even \\ -1 & if\, n\, is\, odd \end{cases}$



$\lim_{x \to 0} cos(1/x)$ does not exist.

9. $\lim_{x \to 0} \sqrt{x} \neq 0$

because its left-side limit does not exist since the domain is x>0.

10. $\lim_{x\to0} sin(\frac{1}{x})$ & $\lim_{x\to\pm\infty} sin(\frac{1}{x})$

- $\lim_{x\to0} sin(\frac{1}{x})$ does not exist.
    - Similarly, all trigonometric functions as such do not exist.

      E.g. $\lim_{x\to0} cos(\frac{1}{x})$, $\lim_{x\to0} tan(\frac{1}{x})$ do not exist.
- $\lim_{x\to\pm\infty} sin(\frac{1}{x}) = sin(\frac{1}{\pm\infty}) = sin0 = 0$
    - Similarly, all trigonometric functions as such exist.

      E.g. $\lim_{x\to\pm\infty} cos(\frac{1}{x}) = cos(\frac{1}{\pm\infty}) = cos0 = 1$

      $\lim_{x\to\pm\infty} tan(\frac{1}{x}) = tan(\frac{1}{\pm\infty}) = tan0 = 0$

2.3 Techniques for Computing Limits

1. Limits of Linear Functions

$$\lim_{x \to a} f(x) = f(a) = ma + b$$

2. Limit Laws

- $\lim_{x \to a}(f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
- $\lim_{x \to a}(cf(x)) = c \lim_{x \to a} f(x)$
- $\lim_{x \to a}(f(x)g(x)) = (\lim_{x \to a} f(x))(\lim_{x \to a} g(x))$
- $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \quad (\lim_{x \to a} g(x) \neq 0)$
- $\lim_{x \to a}(f(x))^n = (\lim_{x \to a} f(x))^n$
- $\lim_{x \to a}(f(x))^{n/m} = (\lim_{x \to a} f(x))^{n/m}, \quad f(x) \geq 0,$
  for x near a, if m is even and n/m is reduced to lowest terms.
  - Refer to *Math 110: College Algebra — 1.4 Equations — (12)*

$x^{m/n} = a$

- If m is odd, then $x = a^{n/m}$
- if m is even, then$x = \pm a^{n/m}$
- If n is even, extraneous solutions may occur.
  e.g. $x^{3/2} = -8$
  $x = (-8)^{2/3} = 4$
  Check x=4
  LS: $4^{3/2} = 8$
  RS: -8
  $\because 8 \neq -8$
  $\therefore x = 4$ is not a solution

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3. Limits of Polynomial and Rational Functions

- Polynomial functions: $\lim_{x \to a} p(x) = p(a)$

- Rational functions: $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(x)}{q(x)}, \quad q(x) \neq 0$

4. Limit laws also apply for one-sided limits.

5. $\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} = \frac{x - 1}{(x - 1)(\sqrt{x} + 1)} = \frac{1}{\sqrt{x} + 1} = \frac{1}{2}$

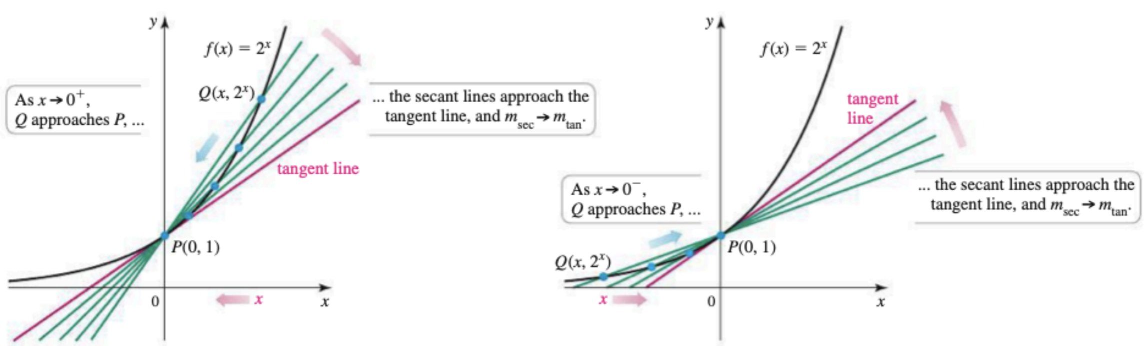
6. $\lim_{x \to 1} \frac{x^2 - 6x + 8}{x^2 - 4} = \frac{(x - 2)(x - 4)}{(x - 2)(x + 2)} = \frac{x - 4}{x + 2} = \frac{2 - 4}{2 + 2} = -\frac{1}{2}$

7. Slope of a tangent line

Estimate the slope of the line tangent to the graph of $f(x) = 2^x$ at the point P(0,1).

- $Q(x, 2^x)$: PQ $\approx$ the tangent line

- $m = \frac{2^x - 1}{x}$: $m_{PQ} \approx$ the slope of the tangent line at P, if Q sufficiently close to P

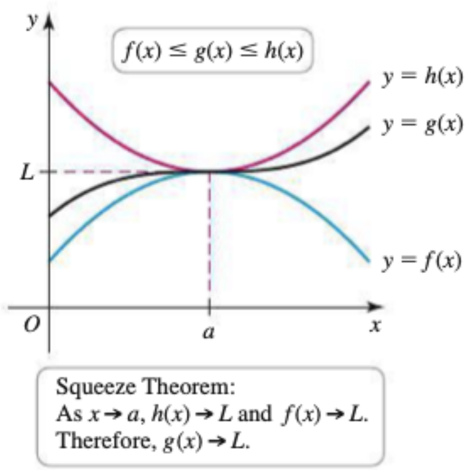


x	1.0	0.1	0.01	0.001	0.0001	0.00001
$m = \frac{2^x - 1}{x}$	1.00000	0.7177	0.6956	0.6934	0.6932	0.6931

$\lim_{x \to 0} \frac{2^x - 1}{x} \approx 0.693 \rightarrow$ the line tangent to f(x) at P is approximately 0.693

8. The Squeeze Theorem

$f(x) \le g(x) \le h(x)$: If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = L$, then $\lim\limits_{x \to a} g(x) = L$.



- Refer to *Math 130: Precalculus Mathematics — 5.6 Additional Trigonometric Graphs*

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(9) Sketching the graph of a damped sine wave: $f(x) = 2^{-x} sin x$

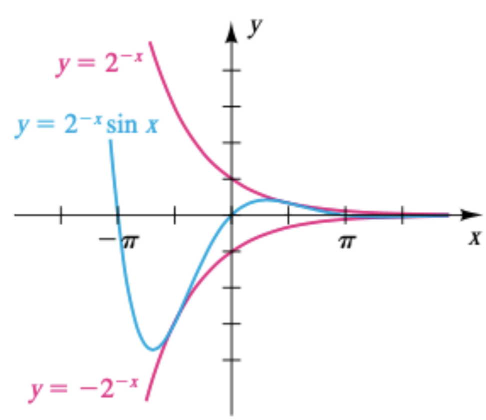
$-1 \le sin x \le 1$

$2^{-x} > 0 \quad \rightarrow \quad -2^{-x} \le 2^{-x} sin x \le 2^{-x}$

- When $sin x = 1 \quad \rightarrow \quad f(x) = 2^{-x} \quad \rightarrow \quad y = f(x)$ coincides with $y = 2^{-x}$
  $\rightarrow \quad x = \dfrac{\pi}{2} + 2n\pi$

- When $sin x = -1 \;\rightarrow\; f(x) = -2^{-x} \;\rightarrow\; y = f(x)$ coincides with $y = -2^{-x}$
  $\rightarrow \quad x = \dfrac{3\pi}{2} + 2n\pi$

- $2^{-x} > 0 \quad \rightarrow$ When $sin x = 0 \quad \rightarrow \quad f(x) = 0 \quad \rightarrow$ x-intercepts
  $\rightarrow \quad x = n\pi$



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9. Use the Squeeze Theorem to verify that $\lim_{x\to 0} x^2 sin(1/x) = 0.$

$-1 \leq sin(1/x) \leq 1 \quad \rightarrow \quad -x^2 \leq x^2 sin(1/x) \leq x^2$

$\because \lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$

$\therefore \lim_{x\to 0} x^2 sin(1/x) = 0$

10. Creating functions satisfying given limit conditions

Find functions f satisfying $\lim_{x\to a}(\frac{f(x)}{x-b}) = c$

$f(x) = (x-b)[x+(c-a)]$

E.g. Find functions f satisfying $\lim_{x\to 1}(\frac{f(x)}{x-1}) = 2$

$$f(x) = (x-1)[x+(2-1)] = (x-1)(x+1) = x^2 - 1$$

11. $\lim_{x\to 0} \frac{sin x}{x} = 0$

12. $\lim_{x\to 0} x^2 ln x^2 = 0$

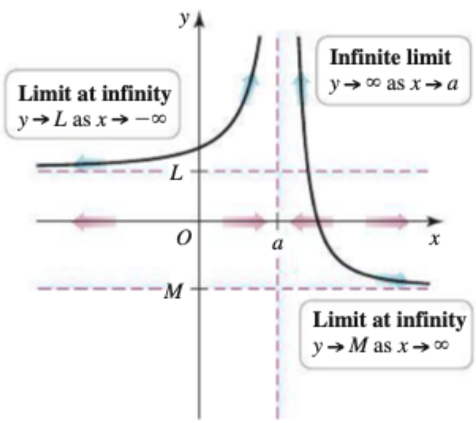
2.4 Infinite Limits

1. The contrast in between

 - Infinite Limits: $\lim_{x \to a} f(x) = \pm\infty$

 - Limits at infinity: $\lim_{x \to \pm\infty} f(x) = L$

 - E.g.



2. Definition of Infinite Limits: $\lim_{x \to a} f(x) = \pm\infty$

3. Definition of Vertical Asymptote
 If $\lim_{x \to a} f(x) = \pm\infty$, $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$,
 x=a is a vertical asymptote of f.

4. Infinite Limits for Two-Sided Limits
 $\lim_{x \to a} f(x) = \pm\infty$ (exists), iff $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = \pm\infty$.

 - Refer to *2.2 Definition of Limits — (6) & (7)*

If either $\lim_{x \to a^+} f(x) \neq L$ or $\lim_{x \to a^-} f(x) \neq L$ (or both), then $\lim_{x \to a} f(x) \neq L$.

If either $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ does not exist, then $\lim_{x \to a} f(x)$ does not exist.

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5. $\lim_{x\to -1^-} \dfrac{x^2-4x+3}{x^2-1} = \lim_{x\to -1^-} \dfrac{(x-1)(x-3)}{(x-1)(x+1)} = \lim_{x\to -1^-} \dfrac{x-3}{x+1} = \dfrac{\lim_{x\to -1^-} x-3}{\lim_{x\to -1^-} x+1}$

$$= \frac{-1-3}{-1^- + 1} = \frac{-4}{0^-} = \infty$$

6. If the numerator and denominator have no common factor,
   then f must have a vertical asymptote x=a.
   - Refer to *Math 110: College Algebra — 3.5 Rational Functions — (3)*
7. Limits of trigonometric functions

   $\lim_{\theta\to 0^+} cot\theta = \lim_{\theta\to 0^+} \dfrac{cos\theta}{sin\theta} = \dfrac{\lim_{\theta\to 0^+} cos\theta}{\lim_{\theta\to 0^+} sin\theta} = \dfrac{1}{0} = \infty$

8. $\lim_{\theta\to \pi/2^+} tanx = \lim_{\theta\to (\pi/2)^+} tanx = \lim_{\theta\to (\frac{\pi}{2})^+} tanx$

9. $q(s) = \dfrac{\pi}{s - sins}$

   s=0 is the only vertical asymptote of f
   because s=0 is the only solution to s-sins=0

10. $g(\theta) = tan\dfrac{\theta\pi}{10}$

    $\dfrac{\theta\pi}{10} = \dfrac{\pi}{2} + n\pi,$ θ=5+10n (n is an integer)

    θ=5+10n (n is an integer) is the vertical asymptotes of g

- Refer to *Math 130: Precalculus Mathematics —*
  5.6 Additional Trigonometric Graphs — (11)

Find the lowest point on an upper brach of the graph $f(x) = 3csc(\frac{1}{2}x - \frac{\pi}{2})$

- $\dfrac{1}{2}x - \dfrac{\pi}{2} = \dfrac{\pi}{2} + 2k\pi \quad \rightarrow \quad x = 2\pi + 4k\pi \ \ (k \in Z)$

  $\therefore (2\pi + 4\pi n,\ 3)$

2.5 Limits at Infinity

1. $\lim_{x\to\infty} tan^{-1}x = \frac{\pi}{2} \quad \leftrightarrow \quad \lim_{x\to(\frac{\pi}{2})^-} tanx = \infty$

   $\lim_{x\to-\infty} tan^{-1}x = -\frac{\pi}{2} \quad \leftrightarrow \quad \lim_{x\to(\frac{\pi}{2})^+} tanx = -\infty$

2. Definitions of Limits at Infinity and Horizontal Asymptotes
   - $\lim_{x\to\pm\infty} f(x) = L$
   - $y = L \rightarrow$ horizontal asymptote

3. Definition of Infinity Limits at Infinity: $\lim_{x\to\pm\infty} f(x) = \pm\infty$

4. $\lim_{x\to\infty}(5 + \frac{sinx}{\sqrt{x}}) = 5 + \lim_{x\to\infty}\frac{sinx}{\sqrt{x}}$

   $-1 \le sinx \le 1 \quad \rightarrow \quad -\frac{1}{\sqrt{x}} \le \frac{sinx}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$

   $\rightarrow \quad \lim_{x\to\infty}(-\frac{1}{\sqrt{x}}) \le \lim_{x\to\infty}\frac{sinx}{\sqrt{x}} \le \lim_{x\to\infty}\frac{1}{\sqrt{x}}$

   $\lim_{x\to\infty}(-\frac{1}{\sqrt{x}}) = -\frac{1}{\lim_{x\to\infty}\sqrt{x}} = \frac{1}{\infty} = 0, \quad \lim_{x\to\infty}\frac{1}{\sqrt{x}} = 0$

   $\therefore \lim_{x\to\infty}\frac{sinx}{\sqrt{x}} = 0$ (The Squeeze Theorem)

   $\lim_{x\to\infty}(5 + \frac{sinx}{\sqrt{x}}) = 5$

- By comparison — refere to *2.4 Infinite Limits — (5)*

$$\lim_{x\to1^-}\frac{x^2-4x+3}{x^2-1} = \lim_{x\to1^-}\frac{(x-1)(x-3)}{(x-1)(x+1)} = \lim_{x\to1^-}\frac{x-3}{x+1} = \frac{\lim_{x\to1^-} x-3}{\lim_{x\to1^-} x-1} = \frac{-4}{0^-} = \infty$$

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5. Limits at Infinity of Powers and Polynomials

- If n is even: $\lim_{x\to\pm\infty} x^n = \infty$

- If n is odd: $\lim_{x\to\infty} x^n = \infty$ & $\lim_{x\to-\infty} x^n = -\infty$

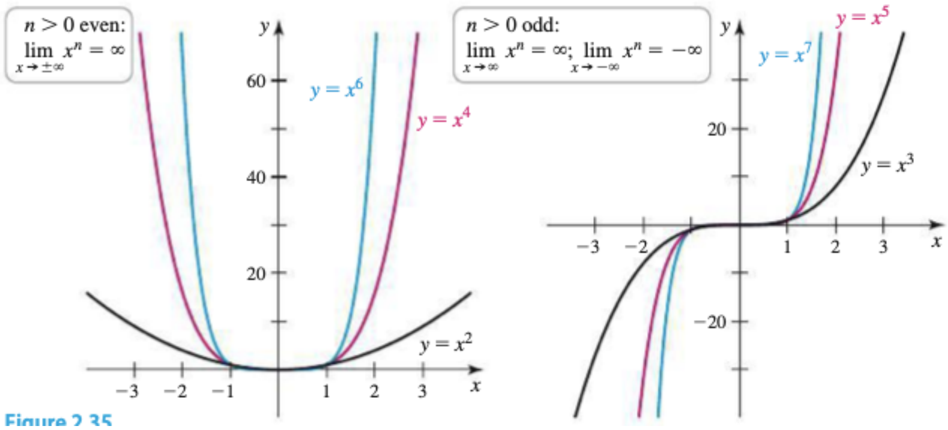

Figure 2.35

- $\lim_{x\to\pm\infty} \frac{1}{x^n} = \lim_{x\to\pm\infty} x^{-n} = 0$

- $\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_n x^n = \pm\infty$

  - $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \quad (a_n \neq 0)$
    $= x^n(a_n + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \ldots + \frac{a_0}{x^n})$
    $\lim_{x\to\pm\infty} \frac{a_{n-1}}{x} = \frac{\lim_{x\to\pm\infty} a_{n-1} = a_{n-1}}{\lim_{x\to\pm\infty} x = \pm\infty} = 0 \to \lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_n x^n = \pm\infty$

  - The sign of the result depends on both the degree of the polynomial and the sign of the leading coefficient a_n

  - E.g. $q(x) = -2x^3 + 3x^2 - 12$

    - $\lim_{x\to\infty} (-2x^3 + 3x^2 - 12) = \lim_{x\to\infty} (-2x^3) = -\infty$

    - $\lim_{x\to-\infty} (-2x^3 + 3x^2 - 12) = \lim_{x\to-\infty} (-2x^3) = \infty$

6. End Behavior and Asymptotes of Rational Functions

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_m x^m + a_{m-1}x^{m-a} + \ldots + a_2x^2 + a_1x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \ldots + b_2x^2 + b_1x + b_0} \quad (a_m \neq 0, b_n \neq 0)$$

- Degree of numerator less than degree of Denominator

  $m < n \quad \rightarrow \quad \lim_{x\to\pm\infty} f(x) = 0 \quad \rightarrow \quad y = 0 \rightarrow$ horizontal asymptote

- Degree of numerator equals degree of denominator

  $m = n \quad \rightarrow \quad \lim_{x\to\pm\infty} f(x) = a_m/b_n \quad \rightarrow \quad y = \dfrac{a_m}{b_n} \rightarrow$ horizontal asymptote

- Degree of numerator greater than degree of denominator

  $m > n \quad \rightarrow \quad \lim_{x\to\pm\infty} f(x) = \pm\infty \quad \rightarrow$ no horizontal asymptote

- Slant asymptote

  $$m = n + 1 \quad \rightarrow \quad \lim_{x\to\pm\infty} f(x) = \pm\infty \quad \rightarrow \quad \begin{cases} 1, & \text{no horizontal asymptote} \\ 2, & \text{a slant asymptote} \end{cases}$$

- Vertical asymptote

  f is in reduced form (p and q share no common factors)

  vertical asymptotes → the zeros of q

7. If $r(x) = 2x^3 - 3x^2 + x$, find a polynomial $p(x)$ of degree ≥ 2 with

- $\lim_{x \to \infty} \dfrac{r(x)}{p(x)} = 5$

  $m = n = 3$

  $5 = \dfrac{a_m}{b_n} = \dfrac{2}{b_n} \quad \to \quad b_n = \dfrac{2}{5}$

  $p(x) = \dfrac{2}{5}x^3$

- $\lim_{x \to \infty} \dfrac{r(x)}{p(x)} = -\infty$

  $n < m = 3,\ n \geq 2 \quad \to \quad n = 2$

  $b_n < 0$

  $p(x) = -x^2$

- $\lim_{x \to \infty} \dfrac{r(x)}{p(x)} = 0$

  $n > m = 3$

  $p(x) = x^4$

- Compared with 2.3 Techniques for Computing Limits — (10)

Creating functions satisfying given limit conditions

Find functions f satisfying $\lim_{x \to a}(\dfrac{f(x)}{x - b}) = c$

$f(x) = (x - b)[x + (c - a)]$

E.g. Find functions f satisfying $\lim_{x \to 1}(\dfrac{f(x)}{x - 1}) = 2$

$f(x) = (x - 1)[x + (2 - 1)] = (x - 1)(x + 1) = x^2 - 1$

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8. The reference of *(6) to Math 110*

- Degree of numerator greater than degree of denominator

$$m > n \quad \rightarrow \quad \lim_{x \to \pm\infty} f(x) = \pm\,\infty \quad \rightarrow \quad \text{no horizontal asymptote}$$

 • Refer to *Math 110: College Algebra — 3.5 Rational Functions — (5)*

Theorem on Horizontal Asymptotes

Let $f(x) = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_k x^k + b_{k-1}x^{k-1} + \ldots + b_1 x + b_0}$, where $a_n \neq 0$ and $b_k \neq 0$.

- If n<k, then the x-axis (the line y=0) is the horizontal asymptote for the graph of f
- If n=k, then the line $y = a_n / b_k$ (the ratio of leading coefficients) is the horizontal asymptote for the graph of f
- If n>k, the graph of f has no horizontal asymptote.
  Instead, either $f(x) \rightarrow \pm\,\infty$ as $x \rightarrow \pm\,\infty$.

So

- Slant asymptote

$$m = n + 1 \quad \rightarrow \quad \lim_{x \to \pm\infty} f(x) = \pm\,\infty \quad \rightarrow \quad \begin{cases} 1, & \text{no horizontal asymptote} \\ 2, & \text{a slant asymptote} \end{cases}$$

 • Refer to *Math 110: College Algebra — 3.5 Rational Functions*
 — (10) Asymptotes

- $f(x) = \dfrac{g(x)}{h(x)} = q(x) + \dfrac{r(x)}{h(x)}$: The Oblique Asymptote $\rightarrow y = q(x)$
- If $r(x)$ has zero(s): $f(x)$ will intersect the asymptote $q(x)$ at the zeros of $r(x)$

$$r(x) = 0 \quad \rightarrow \quad \frac{r(x)}{h(x)} = 0 \quad \rightarrow \quad f(x) = q(x)$$

- An Oblique Asymptotes: if $r(x) = ax + b$
- A Curvilinear Asymptote: if $r(x) = x^a + b$

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9. The graph of a function can have at most two horizontal asymptotes.
    - Compared with (10) — (I) — as below
10. Rational Function & Horisontal Asymptote
    - A rational function can have at most one horizontal asymptote.
    - Whenever there is a horizontal asymptote, $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x)$
    - It does not apply for other functions.
11. End Behavior
    - Dividing both the numerator and denominator by x^n, where n is the greatest degree of the denominator.
    - This strategy forces the terms corresponding to lower powers of x to approach 0 in the limit.
12. End Behavior of Transcendental Functions
    - $\lim_{x\to\infty} e^x = \infty \quad \& \quad \lim_{x\to-\infty} e^x = 0$
        - $\lim_{x\to-\infty} e^x = \lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} \frac{1}{e^x}$
    - $\lim_{x\to\infty} e^{-x} = 0 \quad \& \quad \lim_{x\to-\infty} e^x = \infty$
    - $\lim_{x\to0^+} lnx = -\infty \quad \& \quad \lim_{x\to\infty} lnx = \infty$
        - e^x is the inverse function of lnx

          $\rightarrow$ the domain of e^x = $(-\infty, \infty)$ = the range of lnx

          $\rightarrow \lim_{x\to\infty} lnx = \infty$
13. Hyperbolic sine and cosine functions
    - $sinhx = \dfrac{e^x - e^{-x}}{2}$
    - $coshx = \dfrac{e^x + e^{-x}}{2}$

14. End Behavior of Algebraic Functions

$$f(x) = \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$$

- The highest power of the denominator is $\sqrt{x^6} = x^3$
- For x>0, dividing the numerator and denominator by x^3

$$f(x) = \frac{10 - \frac{3}{x} + \frac{8}{x^3}}{\sqrt{25 + \frac{1}{x^2} + \frac{2}{x^6}}}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{10 - \frac{3}{x} + \frac{8}{x^3}}{\sqrt{25 + \frac{1}{x^2} + \frac{2}{x^6}}} = \frac{10 - \frac{3}{\infty} + \frac{8}{\infty}}{\sqrt{25 + \frac{1}{\infty} + \frac{2}{\infty}}} = \frac{10 - 0 + 0}{\sqrt{25 + 0 + 0}}$$

$$= \frac{10}{\sqrt{25}} = \frac{10}{5} = 2$$

- For $x > 0 \quad \to \quad x^3 < 0,$
  then dividing the numerator and denominator by $-x^3$(>0)

  - Since the domain of $y = \sqrt{x}$ is $x \geq 0$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{-10 - \frac{3}{-x} + \frac{8}{-x^3}}{\sqrt{25 + \frac{1}{x^2} + \frac{2}{x^6}}} = \lim_{x\to-\infty} \frac{-10 + \frac{3}{x} - \frac{8}{x^3}}{\sqrt{25 + \frac{1}{x^2} + \frac{2}{x^6}}} = \frac{-10 + \frac{3}{\infty} - \frac{8}{\infty}}{\sqrt{25 + \frac{1}{\infty} + \frac{2}{\infty}}}$$

$$= \frac{-10 - 0 + 0}{\sqrt{25 + 0 + 0}} = \frac{-10}{\sqrt{25}} = \frac{-10}{5} = -2$$

15. $sec^{-1}x \neq (\frac{1}{cosx})^{-1} \neq cosx$

16. $f(x) = \dfrac{x^2 - 3}{x + 6}$

Long Division of Polynomials

		x	-6
$x+6$	x^2	$+0x$	-3
	x^2	$+6x$	
		$-6x$	-3
		$-6x$	-36
			33

Synthetic Division of Polynomials

-6	1	0	-3
		-6	36
	1	-6	33

$$f(x) = x - 6 + \frac{33}{x+6}$$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} x - 6 + \frac{33}{x+6} = \lim_{x\to\pm\infty} (x-6) + \lim_{x\to\pm\infty} \frac{33}{x+6}$$

$$= \lim_{x\to\pm\infty} (x-6) + \frac{33}{\pm\infty} \lim_{x\to\pm\infty} (x-6) + 0 = \lim_{x\to\pm\infty} x - 6$$

y=x-6 is the slant asymptote of f

- Refer to *Math 110: College Algebra — 3.3 Zeros of Polynomials — (14)*

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17. $f(x) = \dfrac{4x^3 + 4x^2 + 7x + 4}{1 + x^2}$

			$4x$	$+4$
$1+x^2$	$4x^3$	$+4x^2$	$+7x$	$+4$
	$4x^3$		$4x$	
		$+4x^2$	$+3x$	$+4$
		$+4x^2$		$+4$
			$3x$	

$$f(x) = 4x + 4 + \frac{3x}{1+x^2} = 4x + 4 + \frac{\frac{3}{x}}{\frac{1}{x^2} + 1}$$

y=4x+4 is the asymptote of f

18. $f(x) = \dfrac{x^2 - 2x + 5}{3x - 2}$

- Long Division of Polynomials

		$\frac{1}{3}x$	$-\frac{4}{9}$
$3x - 2$	x^2	$-2x$	$+5$
	x^2	$-\frac{2}{3}x$	
		$-\frac{4}{3}x$	$+5$
		$-\frac{4}{3}x$	$\frac{8}{9}$
			$\frac{37}{9}$

$$f(x) = \frac{1}{3}x - \frac{4}{9} + \frac{37}{27x - 18}$$

$y = \dfrac{1}{3}x - \dfrac{4}{3}$ is the slant asymptote of f

- Synthetic Division of Polynomials

$$f(x) = \frac{x^2 - 2x + 5}{3x - 2} = \frac{1}{3} * \frac{x^2 - 2x + 5}{x - \frac{2}{3}}$$

$$\begin{array}{c|ccc} \frac{2}{3} & 1 & -2 & 5 \\ & & \frac{2}{3} & -\frac{8}{9} \\ \hline & 1 & -\frac{4}{3} & \frac{37}{9} \end{array}$$

$$f(x) = \frac{1}{3} * (x - \frac{4}{3} + \frac{\frac{37}{9}}{x - \frac{2}{3}}) = \frac{1}{3}x - \frac{4}{9} + \frac{37}{27x - 18}$$

- Refer to *Math 110: College Algebra — 3.2 Properties of Division —(11)*

The power of x must be 1, but the coefficient may not: $ax - c = a(x - \dfrac{c}{a})$

---

19. $f(x) = \dfrac{\sqrt{x^2 + 2x + 6} - 3}{x - 1} = \dfrac{(\sqrt{x^2 + 2x + 6} - 3)(\sqrt{x^2 + 2x + 6} + 3)}{(x - 1)(\sqrt{x^2 + 2x + 6} + 3)}$

$$= \frac{x^2 + 2x + 6 - 9}{(x - 1)(\sqrt{(x^2 + 2x + 1) + 5} + 3)} = \frac{x^2 + 2x - 3}{(x - 1)(\sqrt{(x + 1)^2 + 5} + 3)}$$

$$= \frac{(x + 3)(x - 1)}{(x - 1)(\sqrt{(x + 1)^2 + 5} + 3)} = \frac{x + 3}{(\sqrt{(x + 1)^2 + 5} + 3)} = \frac{x + 3}{(\sqrt{(x + 1)^2 + 5} + 3)}$$

$\therefore f(x)$ has no vertical asymptote.

20. $f(x) = 4x(3x - \sqrt{9x^2+1}) = \dfrac{4x(3x - \sqrt{9x^2+1})(3x + \sqrt{9x^2+1})}{3x + \sqrt{9x^2+1}}$

$= \dfrac{4x[9x^2 - (9x^2 - 1)]}{3x + \sqrt{9x^2+1}} = \dfrac{-4x}{3x + \sqrt{9x^2+1}}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{-4}{3 + \sqrt{9 + \frac{1}{x^2}}} = \frac{-4}{3 + \sqrt{9 + \frac{1}{\infty}}} = \frac{-4}{3 + \sqrt{9+0}} = -\frac{2}{3}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{4}{-3 + \sqrt{9 + \frac{1}{x^2}}} = \frac{4}{-3 + \sqrt{9+0}} = \frac{4}{-3+3} = \frac{4}{0} = \infty$$

21. $f(x) = \dfrac{|1 - x^2|}{x(x+1)}$

$1 - x^2 \geq 0 \quad \rightarrow \quad [-1,1]$

$1 - x^2 < 0 \quad \rightarrow \quad (-\infty, -1) \cup (1,\infty)$

$x(x+1) \neq 0 \quad \rightarrow \quad x \neq -1, 0$

$$f(x) = \begin{cases} \frac{1-x^2}{x(x+1)}, & (-1,1] \\ \frac{x^2-1}{x(x+1)}, & (-\infty,-1) \cup (1,\infty) \end{cases}$$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2-1}{x(x+1)} = \lim_{x\to\pm\infty} \frac{(x+1)(x-1)}{x(x+1)} = \lim_{x\to\pm\infty} \frac{x-1}{x}$$

$$= \lim_{x\to\pm\infty} (1 + \frac{1}{x}) = 1 + \frac{1}{\pm\infty} = 1 + 0 = 1$$

$$\lim_{x\to 0^\pm} f(x) = \lim_{x\to 0^\pm} \frac{1-x^2}{x(x+1)} = \lim_{x\to 0^\pm} \frac{(1+x)(1-x)}{x(x+1)} = \lim_{x\to 0^\pm} \frac{1-x}{x} = \frac{1\pm 0}{\pm 0} = \pm\infty$$

22. $f(x)=\dfrac{3x^4+3x^3-36x^2}{x^4-25x^2+144}=\dfrac{3x^2(x^2+x-12)}{(x^2-9)(x^2-16)}=\dfrac{3x^2(x-4)(x+3)}{(x+3)(x-3)(x+4)(x-4)}$

$=\dfrac{3x^2}{(x-3)(x+4)}$

- x=-4, 3 are the vertical asymptotes of f.
- x=-3, 4 are the unidentified points of f.

23. $f(x)=\sqrt{|x|}-\sqrt{|x-1|}$

Intervals	$(-\infty,0)$	$(0,1)$	$(1,\infty)$
$\sqrt{\lvert x\rvert}$	$\sqrt{-x}$	$\sqrt{x}$	$\sqrt{x}$
$\sqrt{\lvert x-1\rvert}$	$\sqrt{1-x}$	$\sqrt{1-x}$	$\sqrt{x-1}$

$$f(x)=\begin{cases}\sqrt{-x}-\sqrt{1-x}, & (-\infty,0)\\ \sqrt{x}-\sqrt{1-x}, & (0,1)\\ \sqrt{x}-\sqrt{x-1}, & (1,\infty)\end{cases}$$

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\sqrt{x}-\sqrt{x-1}=\lim_{x\to\infty}\frac{(\sqrt{x}-\sqrt{x-1})(\sqrt{x}+\sqrt{x+1})}{\sqrt{x}+\sqrt{x-1}}$$

$$=\lim_{x\to\infty}\frac{x-(x-1)}{\sqrt{x}+\sqrt{x-1}}=\lim_{x\to\infty}\frac{1}{\sqrt{x}+\sqrt{x-1}}=\frac{1}{\sqrt{\infty}+\sqrt{\infty-1}}=0$$

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\sqrt{-x}-\sqrt{1-x}=\lim_{x\to-\infty}\frac{(\sqrt{-x}-\sqrt{1-x})(\sqrt{-x}+\sqrt{1-x})}{\sqrt{-x}+\sqrt{1-x}}$$

$$=\lim_{x\to-\infty}\frac{-x-(1-x)}{\sqrt{-x}+\sqrt{1-x}}=\lim_{x\to-\infty}\frac{-1}{\sqrt{-x}+\sqrt{1-x}}=\frac{-1}{\sqrt{-(-\infty)}+\sqrt{1-(-\infty)}}$$

$$=\frac{-1}{\sqrt{\infty}+\sqrt{1+\infty}}=0$$

f has no vertical asymptote.

24. $f(x)=\dfrac{x-1}{x^{2/3}-1}=\dfrac{x^{1/3}-\frac{1}{x^{2/3}}}{1-\frac{1}{x^{2/3}}}$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x^{1/3}-\frac{1}{x^{2/3}}}{1-\frac{1}{x^{2/3}}}=\frac{\infty-\frac{1}{\infty}}{1-\frac{1}{\infty}}=\frac{\infty-0}{1-0}=\frac{\infty}{1}=\infty$$

$$\lim_{x\to-\infty} f(x)=-\infty$$

25. Limits of exponentials

- $\displaystyle\lim_{x\to\infty} f(x)=\frac{Ae^{ax}\pm Be^{bx}}{Ce^{cx}\pm De^{dx}} \quad\to\quad \frac{g(x)}{m\pm\frac{l}{e^{nx}}}$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{g(x)}{m\pm\frac{l}{e^{nx}}}=\lim_{x\to\infty}\frac{\lim_{x\to\infty} g(x)}{m\pm\frac{l}{\infty}}=\lim_{x\to\infty}\frac{\lim_{x\to\infty} g(x)}{m\pm 0}=\frac{k}{m}$$

- $\displaystyle\lim_{x\to-\infty} f(x)=\frac{Ae^{ax}\pm Be^{bx}}{Ce^{cx}\pm De^{dx}} \quad\to\quad \frac{g(x)}{m\pm e^{nx}}$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{g(x)}{m\pm e^{nx}}=\lim_{x\to\infty}\frac{\lim_{x\to\infty} g(x)}{m\pm 0}=\frac{k}{m}$$

- E.g. $f(x)=\dfrac{2e^{x}+3e^{2x}}{e^{2x}+e^{3x}}$

$$\lim_{x\to\infty} f(x)=\frac{\frac{2}{e^{2x}}+\frac{3}{e^{x}}}{\frac{1}{e^{x}}+1}=\frac{\frac{2}{\infty}+\frac{3}{\infty}}{\frac{1}{\infty}+1}=\frac{0+0}{0+1}=0$$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{\frac{2}{e^{x}}+3}{1+e^{x}}=\frac{\frac{2}{0}+3}{1+0}=\frac{\infty+3}{1+0}=\infty$$

2.6 Continuity

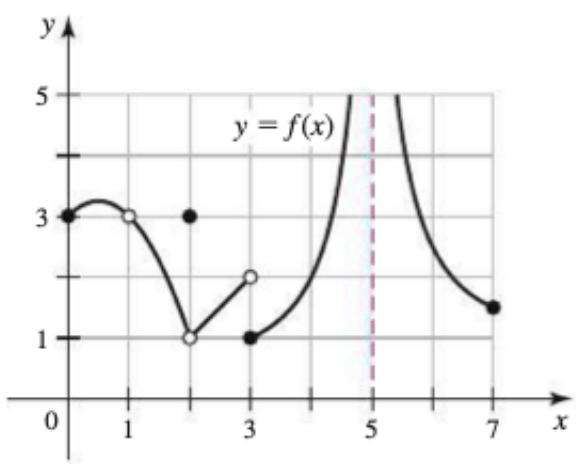
1. Continuity Checklist

- $f(a)$ is defined (a is in the domain of *f*).
- $\lim_{x \to a} f(x)$ exists.
- $\lim_{x \to a} f(x) = f(a)$ (the value of *f* equals the limit of *f* at a).

2. If *f* is continuous at a, then $\lim_{x \to a} f(x) = f(a)$,

   and directly substitution may be used to evaluate $\lim_{x \to a} f(x)$.

3. Classifying discontinuities



- $f(1)$ & $f(2)$ = Removable Discontinuity

  $f(1)$ is undefined; $\lim_{x \to 2} f(x) \neq f(2)$

- $f(3)$ = Jump Discontinuity

  $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

- $f(5)$ = Infinity Discontinuity

  $f(5)$ is undefined

- $f(0)$ & $f(7)$ = Continuity

  The limits of the domain is continuous.

4. Continuity Rules

   If *f* and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and n>0 is an integer.

- $f + g$
- $f - g$
- cf
- fg
- f/g, provided $g(a) \neq 0$
- $(f(x))^n$

5. Polynomial and Rational Functions

- A polynomial function is continuous for all x.
- A rational function (a function of the form $\frac{p}{q}$, where p and q are polynomials) is continuous for all x for which $q(x) \neq 0$.

6. Continuity of Composition Functions at a Point

- If g is continuous at a and f is continuous at $g(a)$, then the composite function fog is continuous at a.
- $\lim_{x \to a} f(g(x)) = f(g(a))$

• E.g. $g(2) = 3, f(2)$ is undefined, $f(3) = 4$

$$\lim_{x \to 2} f(g(x)) = f(g(2)) = f(3) = 4$$

7. Limits of Composite Functions: $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$

- If g is continuous at a and f is continuous at $g(a)$
  - if f and g is continuous at a, the composite function is not necessarily continuous at a

    E.g. $f(x) = \dfrac{1}{x-1} \quad g(x) = \dfrac{1}{x+1}$
- If $\lim_{x \to a} g(x) = L$ and f is continuous at L

8. Indications of Limits of Composite Functions

- The order of a limit and a function evaluation is interchangeable.
- The inner function needn't be continuous at the point of interest, but is must have a limit at that point.
- $\lim_{x \to a}$ can be replaced with $\lim_{x \to a^+}$ or $\lim_{x \to a^-}$

9. Continuity of Inverse Function: f is continuous $\to f^{-1}$ is continuous.

10. Definition of Continuity at Endpoints

- Left-Continuity: $\lim_{x \to a^-} f(x) = f(a)$

- Right-Continuity: $\lim_{x \to a^+} f(x) = f(a)$

11. Continuity of Functions with Roots
Assume that m and n are positive integers with no common factors.

- If m is an odd integer,
then $(f(x))^{n/m}$ is continuous at all points at which f is continuous.

- If m is an even integer,
then $(f(x))^{n/m}$ is continuous at all points at which f is continuous and $f(a) > 0$.

  - Since the domain of $y = \sqrt[m]{x}$ (m is an even integer) is $x \geq 0$

12. At points where $f(a) = 0$, the behavior of $(f(x))^{n/m}$ varies.
Often we find that $(f(x))^{n/m}$ is left- or right continuous at that point,
or it may be continuous from both sides.

- One-side continuity: m is an even integer

- Both-side continuity: m is an odd integer

13. Finding an interest rate: $A(r) = A(1 + \frac{r}{12})^t$

Use the intermediate Value Theorem to show there is a value of r in (a, b) —
that is, an interest rate between a% and b% — for which $A(r) = k$

14. Continuity of Transcendental Functions

The following functions are continuous at all points of their domains.

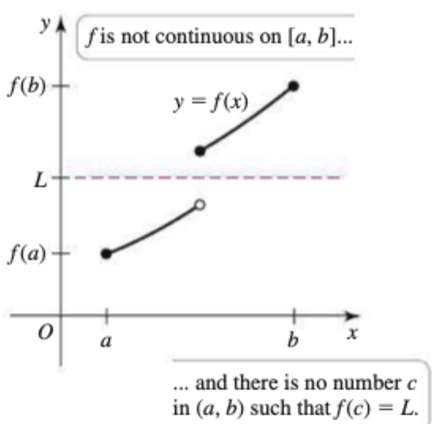
Trigonometric		Inverse Trigonometric		Exponential	
$sin x$	$csc x$	$sin^{-1}x$	$csc^{-1}x$	b^x	e^x
$cos x$	$sec x$	$cos^{-1}x$	$sec^{-1}x$	Logarithmic	
$tan x$	$cot x$	$tan^{-1}x$	$cot^{-1}x$	$log_b x$	$ln x$

- In order for $f(x) = a^x$ to be continuous for all real numbers, it must also be defined when x is an irrational number. Providing a working definition for an expression such as $a^{\sqrt{2}}$ requires mathematical results that haven't appeared yet. Thus, we assume without proof that the domain $f(x) = a^x$ is the set of all real numbers and that f is continuous at all points of its domains.

15. The Intermediate Value Theorem

Suppose f is continuous on the interval [a, b] and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.

- The importance of continuity



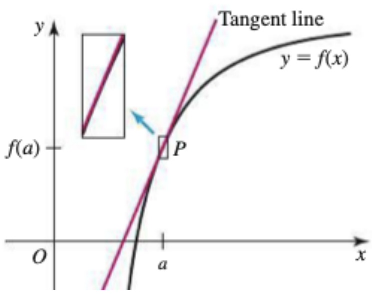
16. Continuity of the absolute value function: $y = |x|$ is continuous in R.

17. Sketch the graph of a function that is not continuous at 1, but has a limit at 1.

$$y = \frac{sin(x-1)}{x-1}$$

3.1 Introducing the Derivative

1. A smooth curve has the property of local linearity, which means that if we look at a point on the curve locally (by zooming in), then the curve appears linear.



2. Definition: Rate of Change and the Slope of the Tangent Line

- The average rate of change: $M_{sec} = \dfrac{f(x) - f(a)}{x - a}$
- The instantaneous rate of change: $m_{tan} = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$
- The slope of the tangent line: $y - f(a) = m_{tan}(x - a)$

• Refer to *Midterms & Final Exams — (1)*

- Secant line = average rate of change = slope
- Tangent line = instantaneous rate of change = derivative
- $m_{tan} = \lim\limits_{x \to a} m_{sec}$: the tangent line = the limit of the secant line

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3. If x and y have physical units, then the average and instantaneous rates of change have units of $\dfrac{units\ of\ y}{units\ of\ x}$.

- E.g. $\begin{cases} y \rightarrow meters \\ x \rightarrow seconds \end{cases}$ $\rightarrow$ the rate of change $\rightarrow \dfrac{meters}{seconds}$

4. Alternative Definition: Rate of Change and the Slope of the Tangent Line
   - The average rate of change: $M_{sec} = \dfrac{f(a+h) - f(a)}{h}$
   - The instantaneous rate of change: $m_{tan} = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$
5. Definition: The Derivative Function

   $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$
   - The domain of f' is no larger than the domain of f.
   - If the limit in the definition of f' fails to exist at some points, then the domain of f' is a subset of the domain of f.
   - E.g. $f(x) = \dfrac{x^2 - 5x + 6}{x - 2}$ is not differentiable at $x = 2$.
6. The derivative of f evaluated at a

   $f'(a) \quad y'(a) \quad \dfrac{df}{dx}\Big|_{x=a} \quad \dfrac{dy}{dx}\Big|_{x=a}$
7. For linear functions, the slope of any secant line always equals to the slope of any tangent line.
8. The slope of the secant line passing through the points P and Q is less than the slope of the tangent line at P.

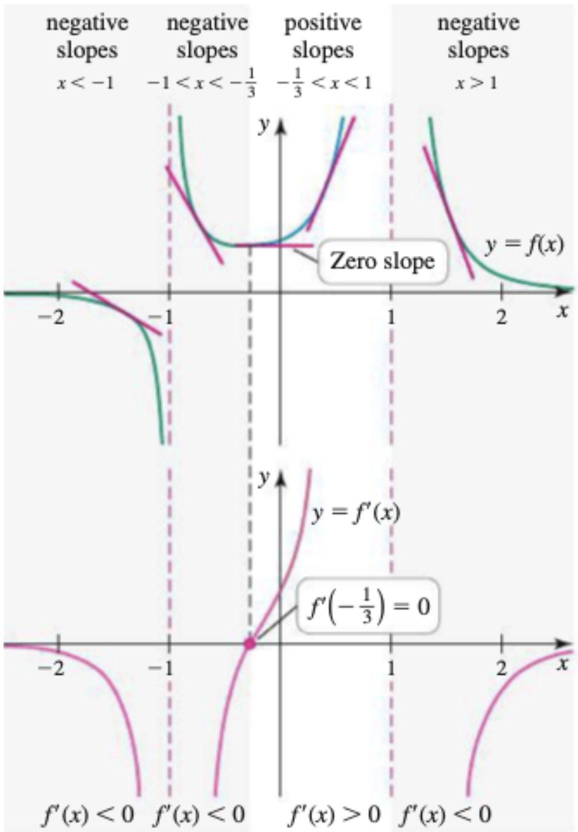
   False.

   E.g. $y = x^2 \quad P(0, 0) \quad Q(1, 1)$

   $m_{sec} = 1 \ > \ m_{tan} = 0$
9. f' = f prime

3.2 Working with Derivatives

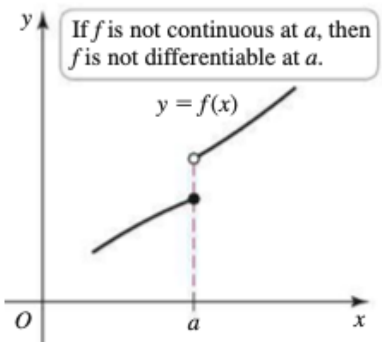
1. f and f' have the same vertical asymptote.



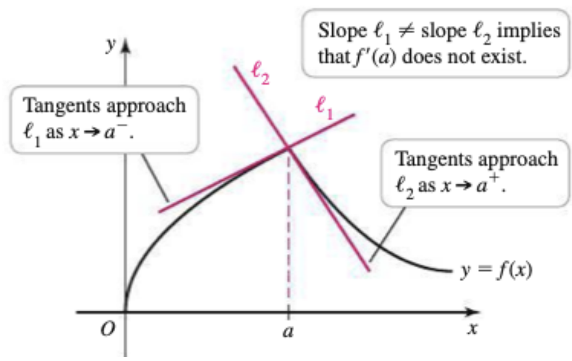
2. Differentiable Implies Continuous
   - If f is differentiable at a, then f' is continuous at a.
   - E.g. $f(x) = |x|$: it is continuous everywhere but not differentiable at 0.
3. Continuity & Differentiability
   - Continuity: $\lim_{x \to a}(f(x) - f(a)) = 0$
   - Differentiability requires more: $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ must exist.

4. When is a Function Not Differentiable at a Point?
(At least one of the following conditions holds)

- f is not continuous at a: e.g. (1)
- f has a corner at a: e.g. (2)
- f has a vertical tangent at a: e.g. (3)&(4)



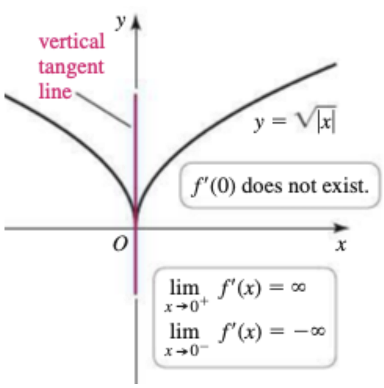
(1)



(2)

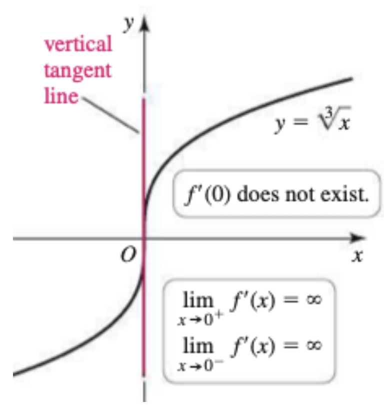
The slope of l_1 and l_2 are not equal. Because of the abrupt change in the slope of the curve at a,

f is not differentiable at a: the limit that defines f' does not exist at a.



(3)

- $f'(a)$ is undefined because the slope of a vertical line is undefined.
- Cusp: a vertical line may occur at a sharp point on the curve.



(4)

A vertical tangent line may occur without a cusp.

5. If f is continuous at a point, f is not necessarily differentiable at that point.

6. Given the function f and the point Q, find all points P on the graph of f such that the line tangent to f at P passes through P.

$$f(x) = \frac{1}{x};\ \ Q(-2,4)$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

$$P(x, \frac{1}{x})$$

$$\frac{4 - \frac{1}{x}}{-2 - x} = -\frac{1}{x^2} \quad \to \quad 2x^2 - x - 1 = 0 \quad \to \quad x = 1,\ -\frac{1}{2}$$

$$P(1,\ 1),\quad (-\frac{1}{2},\ -2)$$

3.3 Rules of Differentiation

1. Constant Rule
   $f(x) = c \quad \rightarrow \quad f'(x) = 0$
2. Power Rule
   $f(x) = x^n \quad \rightarrow \quad f'(x) = n x^{n-1} \ (n > 0)$
   - This formula agrees with familiar factoring formulas for differences of perfect squares and cubes
     - $x^2 - a^2 = (x - a)(x + a)$
     - $x^3 - a^3 = (x - a)(x^2 + a x + a^2)$
3. The Constant Rule is consistent with the Power Rule with $n = 0$
4. Constant Multiple Rule
   $g(x) = cf(x) \quad \rightarrow \quad g'(x) = cf'(x)$
5. Sum Rule
   $h(x) = g(x) + f(x) \quad \rightarrow \quad h'(x) = g'(x) + f'(x)$
6. Generalized Sum Rule
   $g(x) = f_1(x) + f_2(x) + \ldots + f_n(x)$
   $g'(x) = f_1'(x) + f_2'(x) + \ldots + f_n'(x)$
7. Difference Rule
   $h(x) = g(x) - f(x) \quad \rightarrow \quad h'(x) = g'(x) - f'(x)$
8. The Derivative of e^x
   $f(x) = e^x \quad \rightarrow \quad f'(x) = e^x$
9. The Power Rule cannot be applied to exponential functions
   - $f(x) = e^x \quad \rightarrow \quad f'(x) \neq x e^{x-1}$
   - $f(x) = e^n \quad \rightarrow \quad f'(x) \neq e^n \quad \rightarrow \quad f'(x) = 0$

10. Higher-Order Derivatives

- The second derivative of f is $f''(x)$
- The nth derivative of f is $f^{(n)}(x)$

11. In general, the nth derivative of an nth-degree polynomial is a constant, which implies that derivatives of order $k > n$ are 0.

12. Suppose $f(3) = 1$ and $f'(3) = 4$. Let $g(x) = x^2 + f(x)$.
    Find an equation of the line tangent to $y = g(x)$ at $x = 3$.
    $g'(x) = 2x + f'(x)$
    $g'(3) = 6 + f'(3) = 6 + 4 = 10$
    $g(3) = 3^2 + f(3) = 9 + 1 = 10 \quad \rightarrow \quad (3,\ 10)$
    $10(x - 3) = y - 10 \quad \rightarrow \quad y = 10x - 20$

3.4 The Product and Quotient Rules

1. Product Rule
   $h(x) = f(x) * g(x) \quad \rightarrow \quad h'(x) = f'(x)g(x) + g'(x)f(x)$

2. Quotient Rule
   $h(x) = \dfrac{f(x)}{g(x)} \quad \rightarrow \quad h'(x) = \dfrac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$

3. Extended Power Rule
   $f(x) = x^n \quad \rightarrow \quad f'(x) = nx^{n-1}$ (n is any integer including negative ones)

4. The derivative of e^{kx}
   $f(x) = e^{kx} \quad \rightarrow \quad f'(x) = ke^{kx}$

5. $g(t) = 2te^{\frac{t}{2}}$
   $g'(t) = 2e^{\frac{t}{2}} + 2t * \dfrac{1}{2}e^{\frac{t}{2}} = 2e^{\frac{t}{2}} + te^{\frac{t}{2}} = e^{\frac{t}{2}}(2+t)$

6. $h(x) = \dfrac{(x-1)(2x^2-1)}{(x^3-1)}$

$$h(x) = \frac{(x-1)(2x^2-1)}{(x-1)(x^2+x+1)} = \frac{(2x^2-1)}{(x^2+x+1)}$$

$$h'(x) = \frac{(2x^2-1)'(x^2+x+1) - (x^2+x+1)'(2x^2-1)}{(x^2+x+1)^2}$$

$$= \frac{4x(x^2+x+1) - (2x+1)(2x^2-1)}{(x^2+x+1)^2} = \frac{2x^2+6x+1}{(x^2+x+1)^2}$$

- Refer to Math 110: College Algebra — 1.3 Algebra Expressions — (7)

  - $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$
  - $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

---

7. $\dfrac{d^n}{dx^n}(e^{3x}) = 3^n e^{3x}$, for any integer $n \geq 1 \quad \rightarrow$ True

8. Quotient Rule & $\frac{d}{dx}(\frac{1}{x})$

- Both the numerator and denominator have x $\rightarrow$ Quotient Rule
- Only the denominator has x $\rightarrow$ $\frac{d}{dx}(\frac{1}{x}) = \frac{d}{dx}(x^{-1}) = -\frac{1}{x^2}$

3.5 Derivatives of Trigonometric Functions

1. Two Special Limits

- $\lim_{x \to 0} \frac{sinx}{x} = 1$
- $\lim_{x \to 0} \frac{cosx - 1}{x} = 0$

2. Derivatives of Sine and Cosine

- $f(x) = sinx \quad \to \quad f'(x) = cosx$
- $f(x) = cosx \quad \to \quad f'(x) = - sinx$

3. Because $f(x) = sinx$ is a periodic function, we expect its derivative to be periodic.

4. Derivatives of the Trigonometric Functions

- $f(x) = tanx \quad \to \quad f'(x) = sec^2x$
- $f(x) = cotx \quad \to \quad f'(x) = - csc^2x$
- $f(x) = secx \quad \to \quad f'(x) = tanx * secx$
- $f(x) = cscx \quad \to \quad f'(x) = - cotx * cscx$

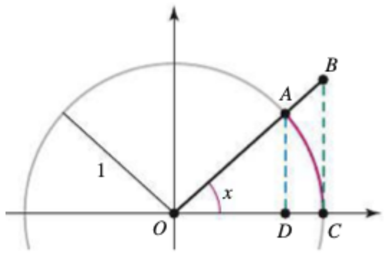
5. How to remember: move something over "="

- $\frac{d}{dx}(sinx) = cosx \quad \leftrightarrow \quad \frac{d}{dx}(cosx) = - sinx$
- $\frac{d}{dx}(tanx) = sec^2x \quad \leftrightarrow \quad \frac{d}{dx}(secx) = tanx * secx$
- $\frac{d}{dx}(cotx) = - cscx \quad \leftrightarrow \quad \frac{d}{dx}(cscx) = - cot * cscx$

6. Higher-Order Trigonometric Derivatives

- $f(x) = sinx \quad \to \quad f^{2n}(x) = (-1)^n sinx$
- $f(x) = cosx \quad \to \quad f^{2n}(x) = (-1)^n cosx$

7. Proof of $\lim_{x \to 0} \frac{sinx}{x} = 1$



area of $\triangle\ OAD$ < area of sector OAC < area of $\triangle\ OBC$

$OA = OC = 1$

$sinx = \frac{AD}{OA} = AD$

$cosx = \frac{OD}{OA} = OD$

$tanx = \frac{BC}{OC} = BC$

$S_{\triangle OAD} = \frac{1}{2} * OD * AD = \frac{1}{2} cosx * sinx$

$S_{OAC} = \frac{1}{2} * 1^2 * x = \frac{x}{2}$

$S_{\triangle OBC} = \frac{1}{2} * OC * BC = \frac{1}{2} tanx$

$\frac{1}{2} cosx * sinx \ < \ \frac{x}{2} \ < \ \frac{1}{2} tanx = \frac{1}{2} * \frac{sinx}{cosx}$

$cosx \ < \ \frac{sinx}{x} \ < \ \frac{1}{cosx}$

$\lim_{x \to 0} cosx \ < \ \lim_{x \to 0} \frac{sinx}{x} \ < \ \lim_{x \to 0} \frac{1}{cosx}$

$\lim_{x \to 0} cosx = 1 \quad \lim_{x \to 0} \frac{1}{cosx} = 1$

The Squeeze Theorem $\rightarrow$ $\lim_{x \to 0} \frac{sinx}{x}$

8. Proof of $\lim_{x\to 0}\frac{cosx-1}{x}=0$

- $\lim_{x\to 0}\frac{cosx-1}{x}=\lim_{x\to 0}\frac{(cosx-1)(cosx+1)}{x(cosx+1)}=\lim_{x\to 0}\frac{cos^2x-1}{x(cosx+1)}$
  $=\lim_{x\to 0}\frac{-sin^2x}{x(cosx+1)}=-\lim_{x\to 0}\frac{sinx}{x}*\lim_{x\to 0}\frac{sinx}{cosx+1}$
  $=-1*\frac{0}{1+1}=0$

- $sin^2x=\frac{1-cos2x}{2}\ \rightarrow\ cosx-1=-2sin^2(\frac{x}{2})$
  $\lim_{x\to 0}\frac{cosx-1}{x}=\lim_{x\to 0}\frac{-2sin^2(\frac{x}{2})}{x}=-\lim_{x\to 0}\frac{sin^2(\frac{x}{2})}{x/2}$
  $-\lim_{x\to 0}sin(\frac{x}{2})*\lim_{x\to 0}\frac{sin(\frac{x}{2})}{x/2}=-0*1=0$

9. Variants of $\lim_{x\to 0}\frac{sinx}{x}=1$

- $\lim_{x\to 0}\frac{sinax}{b}=\lim_{x\to 0}\frac{sinax}{ax}*\frac{a}{b}=\frac{a}{b}\lim_{t\to 0}\frac{sint}{t}=\frac{a}{b}$
  - E.g. $\lim_{x\to 0}\frac{sin5x}{3x}=\frac{5}{3}$
- $\lim_{x\to 0}\frac{sinax}{sinbx}=\lim_{x\to 0}\frac{(sinax)/ax}{(sinbx)/bx}*\frac{a}{b}=\frac{a}{b}\frac{\lim_{u\to 0}\frac{sinu}{u}}{\lim_{v\to 0}\frac{sinv}{v}}=\frac{a}{b}$
  - E.g. $\lim_{x\to 0}\frac{sin7x}{sin3x}=\frac{7}{3}$
- $\lim_{x\to 0}\frac{sin(x+3)}{x^2+8x+15}=\lim_{x\to 0}\frac{sin(x+3)}{(x+3)(x+5)}=\lim_{x\to 0}\frac{1}{x+5}*\lim_{t\to 0}\frac{sint}{t}=\frac{1}{5}$

10. $y = \dfrac{tanx}{1 + secx}$

$$y' = \frac{sec^2x(1 + secx) - tanx * secx * tanx}{(1 + secx)^2} = \frac{sex^2x + sec^3x - tan^2x * secx}{(1 + secx)^2}$$

$$= \frac{sec^2x + sec^3x - \frac{sin^2}{cos^3x}}{(1 + secx)^2} = \frac{sec^2x + sec^3x - \frac{1 - cos^2x}{cos^3x}}{(1 + secx)^2}$$

$$= \frac{sec^2x + sec^3x - sec^3x + secx}{(1 + secx)^2} = \frac{secx(secx + 1)}{(1 + secx)^2} = \frac{secx}{1 + secx}$$

11. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{cosx}{x - \frac{\pi}{2}}$

$$t = x - \frac{\pi}{2} \ \rightarrow \ x = t + \frac{\pi}{2}$$

$$\lim_{t \to 0} \frac{cos(t + \frac{\pi}{2})}{t} = \lim_{t \to 0} \frac{sint}{t} = 1$$

12. $y = \dfrac{sinx}{sinx - cosx}$

$$y' = \frac{cosx(sinx - cosx) - sinx(cosx + sinx)}{(sinx - cosx)^2}$$

$$= \frac{sinxcosx - cos^2x - sinxcosx - sin^2x}{sin^2x - 2sinxcosx + cos^2x}$$

$$= \frac{-(sin^2x + cos^2x)}{(sin^2x + cos^2x) - 2sinxcosx}$$

$$= \frac{-1}{1 - sin2x} = \frac{1}{sin2x - 1}$$

13. $tan x \leftrightarrow sec x$ & $cot x \leftrightarrow csc x$

- Derivatives of the Trigonometric Functions
  - $f(x) = tan x \quad \rightarrow \quad f'(x) = sec^2 x$
  - $f(x) = cot x \quad \rightarrow \quad f'(x) = -csc^2 x$
- The Pythagorean identities
  - $1 + tan^2\theta = sec^2\theta$
  - $1 + cot^2\theta = csc^2\theta$

- Refer to *Math 130: Precalculus mathematics —*
  5.2 Trigonometric Functions of Angles — (4)-2

3.6 Derivatives as Rates of Change

1. Average and Instantaneous Velocity
   - $v_{av} = \dfrac{f(a+\Delta t) - f(a)}{\Delta t}$
   - $v(a) = \lim_{\Delta t \to 0} \dfrac{f(a+\Delta t) - f(a)}{\Delta t} = f'(a)$
2. Velocity, Speed, and Acceleration: $s = f(t)$
   - Velocity: $v = \dfrac{ds}{dt} = f'(t)$
   - Speed: $|v| = |f'(t)|$
   - Acceleration: $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} = f''(t)$
3. Average and Marginal Cost
   - Average cost: $\bar{C}(x) = C(x)/x$
   - Marginal cost: $C'(x)$

4. Elasticity

- The price elasticity of the demand

$$E(p) = \lim_{\Delta p \to 0} \frac{\Delta D/D}{\Delta p/p} = \frac{dD}{dp} * \frac{p}{D}$$

- E.g. Elasticity in pork prices

The demand for processed pork in Canada is described by the function

$D(p) = 286 - 20p$

- Computing and graph the price elasticity of the demand.

$$E(p) = \frac{dD}{dp} * \frac{p}{D} = \frac{d}{dp}(286 - 20p)(\frac{p}{286 - 20p}) = \frac{-10p}{143 - 10p}$$

The elasticity is undefined at $p = 14.3$, which is the price at which the demand reaches zero. (According to the model, no pork can be sold at prices above \$14.30.) Therefore, the domain of the elasticity function is $[0,\ 14.3)$

- When $-\infty < E < -1$, the demand is said to be elastic.

When $-1 < E < 0$, the demand is said to be inelastic.

Interpret these terms.

increasing the price $\rightarrow\ \dfrac{dD}{d} > 0$

the demand is, in turn, decreased $\rightarrow\ \dfrac{dp}{p} < 0$

$$-\infty < E = \frac{dD/d}{dp/d} < -1 \quad \rightarrow \quad -\frac{dD}{d} > \frac{dp}{p}$$

The increase rate of the price > the decrease rate of the demand → more profits

$$-1 < E = \frac{dD/d}{dp/d} < 0 \quad \rightarrow \quad -\frac{dD}{d} < \frac{dp}{p}$$

The increase rate of the price < the decrease rate of the demand → less profits

- For what prices is the demand for pork elastic? Inelastic?

$E(p) < -1\ \rightarrow\ p > 7.15\ \rightarrow\ p \in (7.15,\ 14.3)$ is elastic

$E(p) > -1\ \rightarrow\ p < 7.15\ \rightarrow\ p \in (0,\ 7.15)$ is inelastic

3.7 The Chain Rule

1. The Chain Rule

- Version 1: $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$
- Version 2: $\frac{d}{dx}(f(g(x))) = f'(g(x)) * g'(x)$

2. Replace $g(x)$ with u to differentiate $f(x)$ and then replace back u with $g(x)$.

3. Chain Rule for Powers

   $h(x) = f(g(x)^n) \quad \rightarrow \quad h'(x) = ng(x)^{n-1} * g'(x)$

4. The composition of three or more functions

$$y = \sqrt{x + \sqrt{x + \sqrt{x}}}$$

$$y' = \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} * \frac{d}{dx}(x + \sqrt{x + \sqrt{x}})$$

$$= \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} * (1 + \frac{1}{2\sqrt{x + \sqrt{x}}} * \frac{d}{dx}(x + \sqrt{x}))$$

$$= \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} * (1 + \frac{1}{2\sqrt{x + \sqrt{x}}} * (1 + \frac{1}{2\sqrt{x}}))$$

5. $y = (\frac{3x}{4x + 2})^5$

$$y' = 5(\frac{3x}{4x+2})^4 * \frac{d}{dx}(\frac{3x}{4x+2}) = 5(\frac{3x}{4x+2})^4 * \frac{3(4x+2) - 4 * 3x}{(4x+2)^2}$$

$$= \frac{405x^4}{(4x+2)^4} * \frac{6}{(4x+2)^2} = \frac{2430x^4}{(4x+2)^6}$$

6. The derivative of a product is not the product of the derivatives, but the derivative of a composition is a product of derivatives.

7. $\frac{d}{dx}P(Q(x)) \neq P'(x) * Q'(x) = P'(Q(x)) * Q'(x)$

- E.g.

  $h(x) = f(g(x)) \quad g(3) = 1 \quad f'(1) = 5 \quad g'(3) = 20$

  $h'(3) = f'(g(3)) * g'(3) = f'(1) * g'(3) = 100$

8. The application of the chain rule

EXAMPLE 4 **Applying the Chain Rule** A trail runner programs her GPS unit to record her altitude a (in feet) every 10 minutes during a training run in the mountains; the resulting data are shown in Table 3.4. Meanwhile, at a nearby weather station, a weather probe records the atmospheric pressure p (in hectopascals, or hPa) at various altitudes, shown in Table 3.5.

Table 3.4

t (minutes)	0	10	20	30	40	50	60	70	80
$a(t)$ (altitude)	10,000	10,220	10,510	10,980	11,660	12,330	12,710	13,330	13,440

Table 3.5

a (altitude)	5485	7795	10,260	11,330	12,330	13,330	14,330	15,830	16,230
$p(a)$ (pressure)	1000	925	840	821	793	765	738	700	690

Use the Chain Rule to estimate the rate of change in pressure per unit time experienced by the trail runner when she is 50 minutes into her run.

$$\frac{da}{dt}\Big|_{t=50} = \frac{a(50+10) - a(50)}{10} = \frac{12710 - 12330}{10} = 38\frac{ft}{min}$$

$$\frac{dp}{da}\Big|_{a=12330} = \frac{p(12330+1000) - p(12330)}{1000} = \frac{738 - 765}{1000} = -0.028\frac{hPa}{ft}$$

$$\frac{dp}{dt} = \frac{dp}{da} * \frac{da}{dt} = -0.028\frac{hPa}{ft} * 38\frac{ft}{tmin} = -1.06\frac{hPa}{min}$$

9. Derives of even and odd functions

- $f(x)$ is an even function $f(-x) = f(x)$

  $$f'(-x) = f'(-x) * \frac{d}{dx}(-x) = -f'(-x)$$

  $f'(x)$ is odd

- $f(x)$ is an odd function $f(-x) = -f(x)$

  $$f'(-x) = f'(-x) * \frac{d}{dx}(-x) = -f'(-x) = -f'(x)$$

  $f'(x)$ is even

10. $y = sin^2x$ is a composite function

$$y' = 2sinx * cosx = sin2x$$

$$y' \neq 2sinx$$

3.8 Implicit Differentiation

1. Power Rule for Rational Exponents

$$f(x) = x^{\frac{p}{q}} \quad \rightarrow \quad f'(x) = \frac{p}{q} x^{\frac{p}{q}-1}$$

- Compared with the derivative of $f(y) = x$

Refer to Math 232: Calculus of Functions of One Variable 2

— 6.5 Length of Curves — (4)

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2. $x^2 + y^2 = 1$

- $\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) * \frac{d}{dx}(y) = \frac{d}{dx}(1)$

  $2x + 2y * y' = 0$

  $y' = -\frac{x}{y}$

- $\frac{d}{dx}(y') = \frac{d}{dx}(-\frac{x}{y})$

  $y'' = -\frac{y - xy'}{y^2} = -\frac{y - x(-\frac{x}{y})}{y^2} = -\frac{x^2 + y^2}{y^3} = -\frac{1}{y^3}$

3. If the implicit differentiation of the function is too complicated to simplify, just directly substitute numerical values for variables x and y.

- E.g. $(x^2 + y^2 - 2x)^2 = 2(x^2 + y^2)$; $(2,\ 2)$
  $2(x^2 + y^2 - 2x)(2x + 2y * y' - 2) = 2(2x + 2y * y')$

  - $4(x^2 + y^2 - 2x)(x + y * y' - 1) = 4(x + y * y')$
    $(x^2 + y^2 - 2x)(x + y * y' - 1) = x + y * y'$
    $x^3 + x^2y * y' - x^2 + xy^2 + y^3 * y' - y^2 - 2x^2 - 2xy * y' + 2x = x + y * y'$
    $$y' = \frac{x^3 - 3x^2 + x - y^2 + xy^2}{y(1 - x^2 + y^2 - 2x)}$$
    $$y'|_{(2,\ 2)} = \frac{8 - 12 + 2 - 4 + 8}{2(1 - 4 + 4 - 4)} = -\frac{1}{3}$$

  - $2(4 + 4 - 4)(4 + 4y' - 2) = 2(4 + 4y')$
    $$y' = -\frac{1}{3}$$

4. $sinx + x^2y = 10$
   $cosx + 2xy + x^2y' = 0$
   $$y' = -\frac{cosx + 2xy}{x^2}$$
   $$y'' = -\frac{(-sinx + 2y + 2xy')x^2 - 2x(cosx + 2xy)}{x^4}$$
   $$= \frac{2x(cosx + 2xy) - (2y + 2xy' - sinx)x^2}{x^4}$$
   $$= \frac{2xcosx + 4x^2y - 2x^2y - 2x^3y' + x^2sinx}{x^4}$$
   $$= \frac{2xcosx + 2x^2y - 2x^3(-\frac{cosx + 2xy}{x^2}) + x^2sinx}{x^4}$$
   $$= \frac{2xcosx + 2x^2y + 2xcosx + 4x^2y + x^2sinx}{x^4}$$
   $$= \frac{4xcosx + 6x^2y + x^2sinx}{x^4}$$
   $$= \frac{4cosx + 6xy + xsinx}{x^3}$$

5. $y = \sqrt[3]{x^2 - x + 1} \quad \rightarrow \quad y' = \dfrac{2x - 1}{3(x^2 - x + 1)^{\frac{2}{3}}}$

6. $(xy + 1)^3 = x - y^2 + 8 \quad \rightarrow \quad y' = \dfrac{1 - 3y(xy + 1)^2}{3x(xy + 1)^2 + 2y}$

7. $(x^2 + y^2)(x^2 + y^2 + x) = 8xy^2 \quad \rightarrow \quad y' = \dfrac{7y^2 - 3x^2 - 4xy^2 - 4x^3}{2y(2x^2 + 2y^2 - 7x)}$

8. $\sqrt{3x^7 + y^2} = sin^2y + 100xy$

$$y' = \frac{200y\sqrt{3x^7 + y^2} - 21x^6}{2y - 4siny * cosy\sqrt{3x^7 + y^2} - 200x\sqrt{3x^7 + y^2}}$$

9. $\sqrt{y} + xy = 1 \quad \rightarrow \quad y' - \dfrac{2y^{\frac{3}{2}}}{2x\sqrt{y} + 1} \quad \rightarrow \quad y'' = \dfrac{10y^2 + 16xy^2\sqrt{y}}{(2x\sqrt{y} + 1)^3}$

3.9 Derivatives of Logarithmic and Exponential Functions

1. Inverse Properties for e^x and $ln\,x$
   - $e^{ln\,x} = x\ \ (x > 0);\quad ln(e^x) = x$
   - $y = ln\,x$ iff $x = e^x$
   - $b^x = e^{lnb^x} = e^{xlnb}\ (b > 0)$
2. Because e^x is differentiable on its domain, its inverse $ln\,x$ is also differentiable on its domain.
3. The graph of $ln\,x$ is smooth with no jumps or cusps.
4. Derivative of $ln\,x$
   - $f(x) = ln\,x\ \ (x > 0) \quad \rightarrow \quad f'(x) = \dfrac{1}{x}$
   - $f(x) = ln\,|\,x\,|\ \ (x \neq 0) \quad \rightarrow \quad f'(x) = \dfrac{1}{x}$
   - $f(x) = ln\,|\,g(x)\,|\ \ (g(x) \neq 0) \quad \rightarrow \quad f'(x) = \dfrac{g'(x)}{g(x)}$
5. Derivative of b^x
   - $f(x) = b^x\ (b > 0,\ b \neq 1) \quad \rightarrow \quad f'(x) = b^x lnb$
   - When $b > 1\ \ \rightarrow\ \ lnb > 0$, the graph $y = b^x$ has tangent lines with positive slopes for all x.

     When $0 < b < 1\ \ \rightarrow\ \ lnb < 0$, the graph $y = b^x$ has tangent lines with negative slopes for all x.
6. General Power Rule
   - $f(x) = x^p \quad \rightarrow \quad f'(x) = px^{p-1}$
   - $g(x) = f^p(x) \quad \rightarrow \quad p * f^{p-1}(x) * f'(x)$
7. $f(x) = ln(kx)\ (x > 0,\ k > 0) \quad \rightarrow \quad f'(x) = \dfrac{k}{kx} = \dfrac{1}{x}$

8. Derivative of $log_b x$ $(b > 0,\ b \neq 1)$

- $f(x) = log_b x\ (x > 0) \quad \rightarrow \quad f'(x) = \dfrac{1}{xlnb}$
- $f(x) = log_b|x|\ (x \neq 0) \quad \rightarrow \quad f'(x) = \dfrac{1}{xlnb}\ (x \neq 0)$

9. Two ways to solve $f(x) = h(x)^{g(x)}$

- Exponential Differentiation

$$f(x) = h(x)^{g(x)} = e^{g(x)*ln(h(x))}$$

$$f'(x) = e^{g(x)*ln(h(x))}(g'(x)*ln(h(x)) + \frac{g(x)*h'(x)}{h(x)})$$

$$= h(x)^{g(x)}(g'(x)*ln(h(x)) + \frac{g(x)*h'(x)}{h(x)})$$

- Logarithmic Differentiation

$$f(x) = h(x)^{g(x)}$$

$$ln(f(x)) = ln(h(x)^{g(x)}) = g(x)*ln(h(x))$$

$$\frac{f'(x)}{f(x)} = g'(x)*ln(h(x)) + \frac{g(x)*h'(x)}{h(x)}$$

$$f'(x) = f(x)(g'(x)*ln(h(x)) + \frac{g(x)*h'(x)}{h(x)})$$

$$= h(x)^{g(x)}(g'(x)*ln(h(x)) + \frac{g(x)*h'(x)}{h(x)})$$

- E.g. $f(x) = x^x\ (x > 0)$

- Exponential Differentiation

$$f(x) = x^x = e^{xlnx}$$

$$f'(x) = e^{xlnx}(lnx + x*\frac{1}{x})$$

$$= x^x(1 + lnx)$$

- Logarithmic Differentiation

$$f(x) = x^x$$

$$ln(f(x)) = ln(x^x) = xlnx$$

$$\frac{1}{f(x)}*f'(x) = lnx + 1$$

$$f'(x) = f(x)(1 + lnx) = x^x(1 + lnx)$$

10. When both the base and the exponent contain x $\rightarrow$ logarithmic differnetiation

11. $f(g($trigonometry$))$ is a third composite function

- E.g. $f(x) = ln(cos^2x)$

$$f'(x) = \frac{1}{cos^2x} * 2cosx * (-sinx) = -2tanx$$

12. $y = ln10^x = xln10 \quad \rightarrow \quad y' = ln10$

13. $g(y) = e^y * y^e$

$$g'(y) = y^e * e^y lne + e^y * ey^{e-1} = y^e * e^y + e^{y+1} * y^{e-1}$$

14. $y = log_2(log_2x)$

$$y' = \frac{\frac{1}{xln2}}{log_2x * ln2} = \frac{1}{x * (ln2)^2 log_2x} = \frac{1}{x * (ln2)^2 * \frac{lnx}{ln2}} = \frac{1}{x * ln2 * lnx}$$

15. The graph of $y = x^{2x}$ has two horizontal tangent lines. Find equations for both of them.

$$y = x^{2x}$$

$$lny = ln(x^{2x}) = xln(x^2) \ (x \neq 0)$$

$$\frac{y'}{y} = ln(x^2) + \frac{x * 2x}{x^2} = ln(x^2) + 2$$

$$y' = x^{2x}(ln(x^2) + 2)$$

$$y' = 0 \quad \rightarrow \quad ln(x^2) + 2 = 0 \quad \rightarrow \quad x = \pm\frac{1}{e}$$

$$y_1 = (\frac{1}{e})^{\frac{2}{e}} = \frac{1}{e^{\frac{2}{e}}} y_2 = (-\frac{1}{e})^{-\frac{2}{e}} = e^{\frac{2}{e}}$$

16. $\frac{d^n}{dx^n}(2^x) = 2^x(ln2)^n$

17. $b^{\frac{1}{lnb}} = e^{\frac{1}{lnb}lnb} = e$

18. Power Rule (The applicability gets bigger)

- 3.3: the exponent can be a positive integer ($n > 0$)

Power Rule

$f(x) = x^n \quad \rightarrow \quad f'(x) = nx^{n-1}$ $(n > 0)$

- 3.4: the exponent can be a positive integer ($n < 0$)

Extended Power Rule

$f(x) = x^n \quad \rightarrow \quad f'(x) = nx^{n-1}$ (n is any integer including negative ones)

- 3.7: power rule applies for "y" (not only for "x")

Chain Rule for Powers

$h(x) = f(g(x)^n) \quad \rightarrow \quad h'(x) = ng(x)^{n-1} * g'(x)$

- 3.8: the exponent can be e.g. $\frac{a}{b}$, a^b

Power Rule for Rational Exponents

$f(x) = x^{\frac{p}{q}} \quad \rightarrow \quad f'(x) = \frac{p}{q}x^{\frac{p}{q}-1}$

- 3.9: power rule applies for <u>all</u>

General Power Rule

- $f(x) = x^p \quad \rightarrow \quad f'(x) = px^{p-1}$
- $g(x) = f^p(x) \quad \rightarrow \quad p * f^{p-1}(x) * f'(x)$

3.10 Derivatives of Inverse Trigonometric Functions

1. Derivatives of Inverse Trigonometric Functions

- $f(x) = sin^{-1}x \quad \rightarrow \quad f'(x) = \dfrac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1)$

- $f(x) = cos^{-1}x \quad \rightarrow \quad f'(x) = -\dfrac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1)$

- $f(x) = tan^{-1}x \quad \rightarrow \quad f'(x) = \dfrac{1}{1+x^2}$

- $f(x) = cot^{-1}x \quad \rightarrow \quad f'(x) = -\dfrac{1}{1+x^2}$

- $f(x) = sec^{-1}x \quad \rightarrow \quad f'(x) = \dfrac{1}{|x|\sqrt{x^2-1}} = \begin{cases} \frac{1}{x\sqrt{x^2-1}} & (x > 1) \\ -\frac{1}{x\sqrt{x^2-1}} & (x < -1) \end{cases}$

- $f(x) = csc^{-1}x \quad \rightarrow \quad f'(x) = -\dfrac{1}{|x|\sqrt{x^2-1}} = \begin{cases} \frac{1}{x\sqrt{x^2-1}} & (x < -1) \\ -\frac{1}{x\sqrt{x^2-1}} & (x > 1) \end{cases}$

2. Derivative of the Inverse Function

- $(f^{-1})'(y_0) = \dfrac{1}{f'(x_0)} \quad (y_0 = f(x_0))$

• $(f^{-1})'(y_0)$ is the reciprocal of $f'(x_0)$

3. We can evaluate the derivative of the inverse function without finding the inverse function itself.

- When the specific number is given, no needs to find $f^{-1}(x)$.

  - E.g. $f(x) = x^2 + 1;\ (5,\ 2)$

    $f'(x) = 2x \quad \rightarrow \quad (f^{-1})'(x) = \dfrac{1}{f'(y)}$

    $(f^{-1})'(5) = \dfrac{1}{f'(2)} = \dfrac{1}{4}$

- When there is no given specific number, we need to find $f^{-1}(x)$ at first.

  - E.g. $f(x) = x^2 + 1$

    $f^{-1}(x) = \sqrt{x-1}$

    $(f^{-1})'(x) = \dfrac{1}{2\sqrt{x-1}}$

    (Verification: $(f^{-1})'(5) = \dfrac{1}{2\sqrt{5-1}} = \dfrac{1}{4}$)

- E.g. If $f(x) = \dfrac{1}{x}$, then $(f^{-1})'(x) = -\dfrac{1}{x^2}$ $\quad\rightarrow$True

4. $sin(sin^{-1}x) = x \quad sin(csc^{-1}x) = \dfrac{1}{x} \quad csc(sin^{-1}x) = \dfrac{1}{x}$

- E.g.1: $f(w) = cos(sin^{-1}2w)$

  $$f'(w) = -sin(sin^{-1}2w) * \frac{1}{\sqrt{1-(2w)^2}} * 2 = -2w * \frac{2}{\sqrt{1-4w^2}} = -\frac{4w}{\sqrt{1-4w^2}}$$

- E.g. 2: $f(w) = sin(sec^{-1}2w)$

  $$f'(w) = cos(sec^{-1}2w) * \frac{2}{|2w|\sqrt{(2w)^2-1}} = \frac{1}{2w} * \frac{1}{|w|\sqrt{4w^2-1}}$$

  $$= \frac{1}{2w * |w|\sqrt{4w^2-1}} \neq \frac{1}{2w^2\sqrt{4w^2-1}}$$

5. Derivatives of inverse function from a table

x	-4	-2	0	2	4
$f(x)$	0	1	2	3	4
$f'(x)$	5	4	3	2	1

- $(f^{-1})'(0) = \dfrac{1}{f'(-4)} = \dfrac{1}{5}$

- $(f^{-1})'(f(0)) = (f^{-1})'(2) = \dfrac{1}{f'(0)} = \dfrac{1}{3}$

6. $f(x) = sin^{-1}(\dfrac{x}{4})$

$$f'(x) = \frac{\frac{1}{4}}{\sqrt{1-(\frac{x}{4})^2}} = \frac{\frac{1}{4}}{\sqrt{\frac{16-x^2}{16}}}\frac{\frac{1}{4}}{\frac{1}{4}\sqrt{16-x^2}} = \frac{1}{\sqrt{16-x^2}}$$

7. $f(x) = csc^{-1}t \quad \rightarrow \quad f'(x) = \dfrac{t'}{|t|\sqrt{t^2-1}} \neq \dfrac{t'}{|x|\sqrt{t^2-1}}$

   So does $f(x) = sec^{-1}t$.

- E.g. $f(x) = sec^{-1}(e^x)$

$$f'(x) = \frac{e^x}{|e^x|\sqrt{(e^x)^2-1}} = \frac{1}{\sqrt{e^{2x}-1}}$$

8. Summary of the Derivatives of Trigonometric Functions

Function	Derivative
$f(x) = sin x$	$f'(x) = cos x$
$f(x) = cos x$	$f'(x) = - sin x$
$f(x) = tan x$	$f'(x) = sec^2 x$
$f(x) = csc x$	$f'(x) = - csc x * cot x$
$f(x) = sec x$	$f'(x) = sec x * tan x$
$f(x) = cot x$	$f'(x) = - csc^2 x$
$f(x) = sin^{-1} x$	$f'(x) = \frac{1}{\sqrt{1 - x^2}} \quad (\lvert x \rvert < 1)$
$f(x) = cos^{-1} x$	$f'(x) = - \frac{1}{\sqrt{1 - x^2}} \quad (\lvert x \rvert < 1)$
$f(x) = tan^{-1} x$	$f'(x) = \frac{1}{1 + x^2}$
$f(x) = cot^{-1} x$	$f'(x) = - \frac{1}{1 + x^2}$
$f(x) = sec^{-1} x$	$f'(x) = \frac{1}{\lvert x \rvert \sqrt{x^2 - 1}} \quad (\lvert x \rvert > 1)$
$f(x) = csc^{-1} x$	$f'(x) = - \frac{1}{\lvert x \rvert \sqrt{x^2 - 1}} \quad (\lvert x \rvert > 1)$

3.11 Related Rates

1. The essential feature of these problems is that two or more variables, which are related in a know way, are themselves changing in time.

2. A spherical snowball is placed in the sun. The sun melts the snowball so that its radius decreases 0.25 in/hr. Find the rate of change in volume w.r.t time at the instant the radius is 4 inches.

$$\frac{dr}{dt} = -0.25 \quad \rightarrow \quad \frac{dV}{dt}\Big|_{r=4} = ?$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi * 3r^2 * \frac{dr}{dt}$$

$$= \frac{4}{3}\pi * 3r^2 * (-0.25) = -\pi r^2$$

$$\frac{dV}{dt}\Big|_{r=4} = -\pi * 4^2 = -16\pi$$

- Decrease → negative

Logic Chain

- $\frac{dr}{dt} = -0.25 \quad \rightarrow \quad \frac{dV}{dt}\Big|_{r=4} = ?$
- Volume/Surface Area Formula
- Differentiate both sides of the equation
- Do not forget the derivative of the independent variable $\frac{dr}{dt}$
- Watch out the sign of the derivative

3. A two-piece extension ladder leaning against a wall is collapsing at a rate of 2 ft/sec. while the foot of the ladder remains a constant 5 ft from the wall. How fast is the ladder moving down the wall when the ladder is 13 ft. long?

$$\frac{dy}{dx} = -2 \quad \rightarrow \quad \frac{dx}{dt}\Big|_{y=13} = ?$$

$$x^2 + 5^2 = y^2$$

$$2x * \frac{dx}{dt} = 2y * \frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{y}{x} * \frac{dy}{dt} = -2\frac{dy}{dt}$$

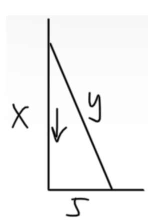
$$x\Big|_{y=13} = \sqrt{13^2 - 5^2} = 12$$

$$\frac{dx}{dt}\Big|_{y=13} = -2 * \frac{13}{12} = -\frac{13}{6}$$

- Collapse → negative

Logic Chain

- $\frac{dy}{dx} = -2 \quad \rightarrow \quad \frac{dx}{dt}\Big|_{y=13} = ?$
- The Pythagorean Theorem
- Differentiate both sides of the equation
- Watch out the sign of the derivative



4. A beacon 0.5 miles (perpendicular distance) from a straight shore revolves at 2 revolutions per minute (4π radians per minute). At what speed is the beam moving along the shore when it hits the shore 1 mile from the lighthouse?

$$\frac{d\theta}{dt} = 4\pi \quad \rightarrow \quad \frac{dx}{dt}|_{y=1} = ?$$

$$tan\theta = \frac{x}{0.5} = 2x$$

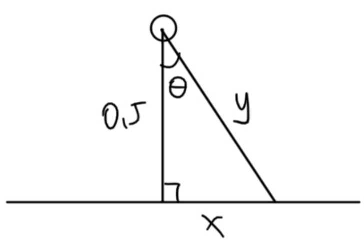
$$sec^2\theta * \frac{d\theta}{dt} = 2\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{1}{2cos^2\theta} * \frac{d\theta}{dt}$$

$$= \frac{1}{2cos^2\theta} * 4\pi = \frac{2\pi}{cos^2\theta}$$

$$cos\theta|_{y=1} = \frac{0.5}{1} = 0.5$$

$$\frac{dx}{dt}|_{y=1} = \frac{2\pi}{0.5^2} = 8\pi$$

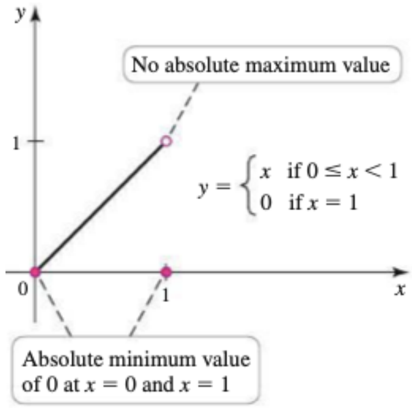


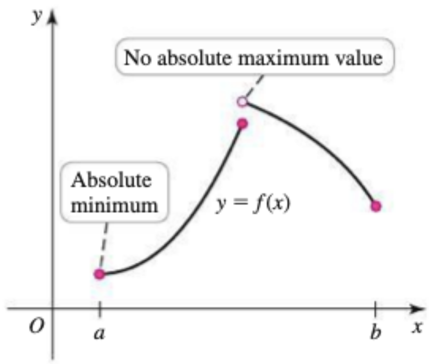
Logic Chain

- $\frac{d\theta}{dt} = 4\pi \quad \rightarrow \quad \frac{dx}{dt}|_{y=1} = ?$
- Trigonometric Equations
- Differentiate both sides of the equation
- Do not forget the derivative of the angle $\frac{d\theta}{dt}$

4.1 Maxima and Minima

1. Defining a function on a closed interval is not enough to guarantee the existence of absolute extremum.

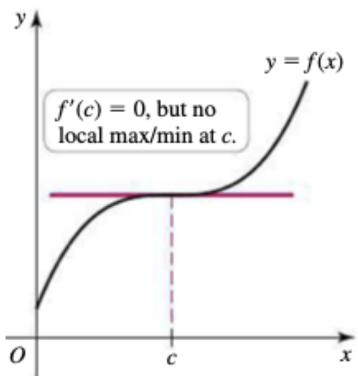


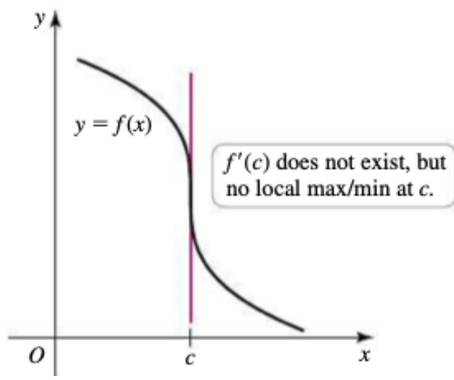


2. Extreme Value Theorem (Absolute Maximum and Minimum Values)
   - Continuous
   - Closed and bounded
3. Local Maximum and Minimum Values
   - Local max: its value is greatest among values at nearby points.
   - Local min: its value is least among values at nearby points.
4. Critical Point
   - $f'(c) = 0$
   - $f'(c)$ does not exist.
5. Constant functions have an absolute maximum and minimum, but no local maximum and minimum.

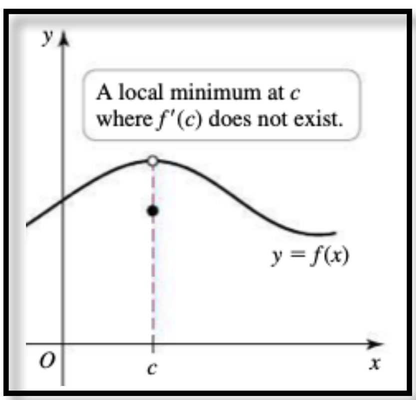
6. Local Extreme Value Theorem

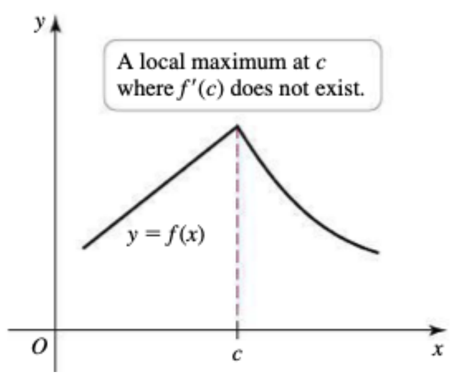
- $f(c)$ = local max/min + $f'(c)$ exists $\rightarrow f'(c) = 0$
- A necessary but not sufficient condition
  - $f'(c) = 0$ but no a local max/min
  - $f'(c)$ fails to exist, with no local max/min





  - Local extremum can also occur at points c where $f'(c)$ does not exist.





- Compared with 3.2 Working with Derivatives — (4)

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7. Local absolute extreme values on a closed interval

   Assume the function f is continuous on the closed interval $[a, b]$

   - The critical points: $f'(c) = 0$ or $f'(c)$ does not exist. (e.g. See below (8).)
   - The endpoints: $f(a)$ and $f(b)$
   - Choose the largest and smallest values.

8. If the interval of interest is an open interval, then absolute extreme values — if they exist — occur it interior points.

9. $f(x) = x^{\frac{2}{3}}(x-5)$ on $[-5, 5]$

   $$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}(x-5) + x^{\frac{2}{3}} = \frac{5}{3}x^{-\frac{1}{3}}(x-2)$$

   $$f'(x) = 0 \;\rightarrow\; x = 2; \quad x \neq 0$$

Sign	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
$\frac{5}{3}x^{-\frac{1}{3}}$	$-$	$+$	$+$
$x-2$	$-$	$-$	$+$
$f'(x)$	$+$	$-$	$+$

   Local maximum: $x = 0$

   Local minimum: $x = 2$

10. $f(x) = \dfrac{4x^5}{5} - 3x^3 + 5$ on $[-2, 2]$

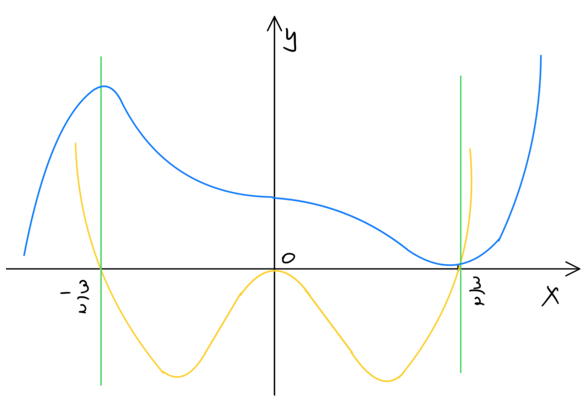
   $$f'(x) = 4x^4 - 9x^2 = x^2(4x^2 - 9)$$

   $$f'(x) = 0 \;\rightarrow\; x = 0, \; \pm\frac{3}{2}$$

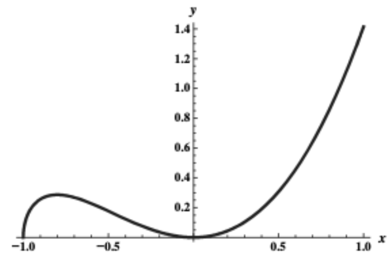
   $x^2 = 0 \;\rightarrow\; x = 0$ is a fake zero point (the sign does not change)

   Local Maximum: $x = \dfrac{3}{2}$

   Local Minimum: $x = -\dfrac{3}{2}$



11. $f(x) = x^2\sqrt{x+1}$ on $[-1,\ 1]$



$f(x) = x^2(x+1)^{\frac{1}{2}}$

$f'(x) = 2x(x+1)^{\frac{1}{2}} + \frac{1}{2}x^2(x+1)^{-\frac{1}{2}}$

$= x(x+1)^{-\frac{1}{2}}[2(x+1) + \frac{1}{2}x]$

$= x(x+1)^{\frac{1}{2}}(\frac{5}{2}x + 2)$

$f'(x) = 0 \ \rightarrow \ x = 0, -\frac{4}{5}$

Local Maximum: $x = -\frac{4}{5}$

Local Minimum: $x = 0$

- Refer to *Math 110: College Algebra — 1.3 Algebra Expressions — (13)*

$(x^2+9)^4(-\frac{1}{3})(x+6)^{-\frac{4}{3}} + (x+6)^{-\frac{1}{3}}(4)(x^2+9)^3(2x)$

$= [(x+9^3(x+6)^{-\frac{4}{3}}][(-\frac{1}{3})(x^2+9)] + [(x+6)^{-\frac{4}{3}}(x^2+9)^3][8x(x+6)]$

$= [(x^2+9)^3(x+6)^{-\frac{4}{3}}][(-\frac{1}{3}x^2 - 3) + (8x^2 + 48x)]$

$= [(x^2+9)^3(x+6)^{-\frac{4}{3}}](\frac{23}{3}x^3 + 48x - 3)$

$= \dfrac{(x^2+9)^3(23x^2+144x-9)}{3(x+6)^{\frac{4}{3}}}$

- Always take the factor whose exponent is the least as the gcd

---

12. $f(x) = sinx * cosx$ on $[0,\ 2\pi]$

- $f'(x) = cos^2x - sin^2x$

$f'(x) = 0 \ \rightarrow \ sinx = cosx$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

- $f(x) = \frac{1}{2}sin(2x) \quad 2x \in [0,\ 4\pi]$

$f'(x) = \frac{1}{2}cos(2x) * 2 = cos(2x)$

$f'(x) = 0 \ \rightarrow \ 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

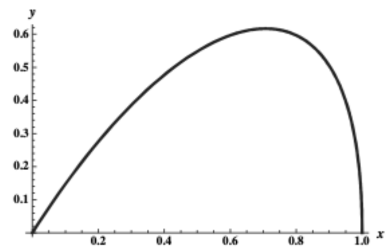
13. $sin\theta = cos\theta \ \&\ sin^{-1}x = cos^{-1}x \quad \rightarrow \quad \theta = \frac{\pi}{4} + \frac{\pi}{2}k \ (k \in Z)$

- E.g. $f(x) = (sin^{-1}x)(cos^{-1}x)$ on $[0,\ 1]$

$$f'(x) = \frac{cos^{-1}x}{\sqrt{1-x^2}} - \frac{sin^{-1}x}{\sqrt{1-x^2}}$$

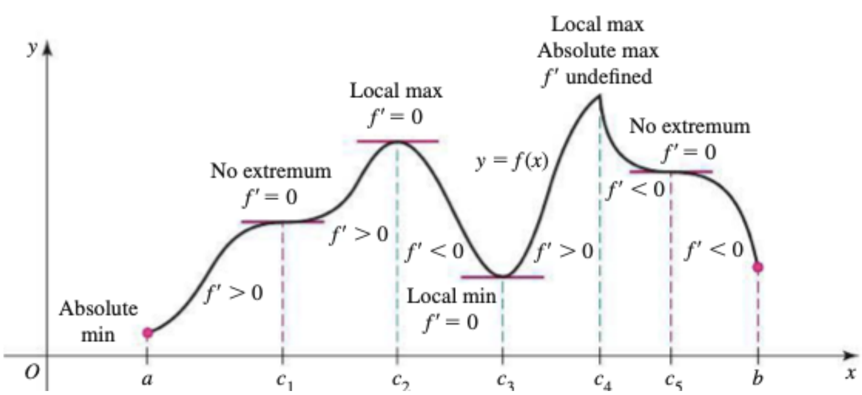
$$f'(x) = 0 \ \rightarrow \ sin^{-1}x = cos^{-1}x \ \rightarrow \ x = \frac{1}{\sqrt{2}}$$

Local Maximum: $x = \frac{1}{\sqrt{2}}$



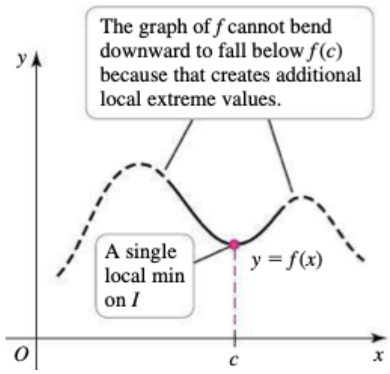
4.2 What Derivatives Tell Us

1. Increasing and decreasing functions
   - Monotonic Function: either increasing or decreasing.
   - Non-decreasing: $f(x_2) \geq f(x_1) \quad x_2 > x_1$
   - Non-increasing: $f(x_2) \leq f(x_1) \quad x_2 > x_1$
2. Test for Intervals of Increase and Decrease
   - Increasing $\rightarrow$ $f'(x) > 0$
   - Decreasing $\rightarrow$ $f'(x) < 0$
   - But not vice versa
     - $f'(x) > 0 \neq$ increasing
     - $f'(x) < 0 \neq$ decreasing
     - E.g. $f(x) = x^3$: increasing but $f'(x) \geq 0$ $(f'(0) = 0)$
3. Critical points do not always correspond to local extreme values.

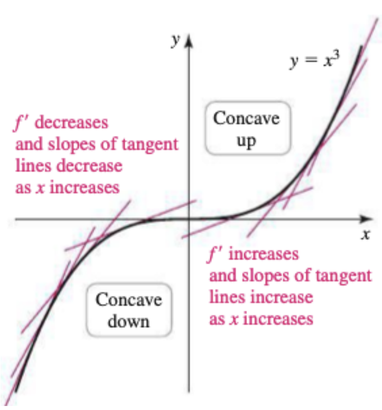


4. First Derivative Test
   - $f'(x) > 0 \rightarrow f'(x) < 0 \quad =$ local max
   - $f'(x) < 0 \rightarrow f'(x) > 0 \quad =$ local min
   - $f'(x)$ no sign change $\rightarrow$ no local extremum

5. One Local Extremum Implies Absolute Extremum
   Suppose f is continuous on an interval I that contains exactly one local extremum.
   - Local max = absolute max
   - Local min = absolute min



6. Concavity and Inflection Point
   - f' is increasing → concave up
   - f' is decreasing → concave down
   - Inflection Point: f changes concavity (from up to down, or vice versa).
7. Concavity & tangent line
   - If a function is concave up at a point, then its graph near that point lies above the tangent line at that point.
   - If a function is concave down at a point, then its graph near that pint lies below the tangent line at that point.

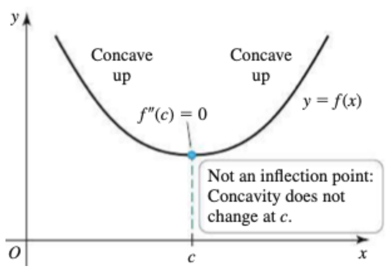


8. The second derivative measures concavity.

9. Test for Concavity

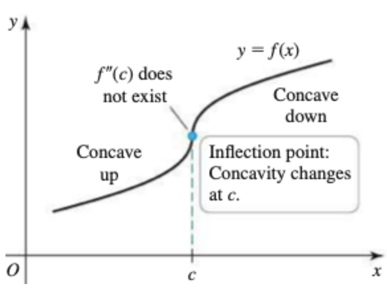
- $f''(x) > 0 \;\rightarrow$ concave up
- $f''(x) < 0 \;\rightarrow$ concave down
- $f''(x)$ changes sign $\rightarrow$ an inflection point

10. $f''(c) = 0$ does not necessarily implies an inflection point.

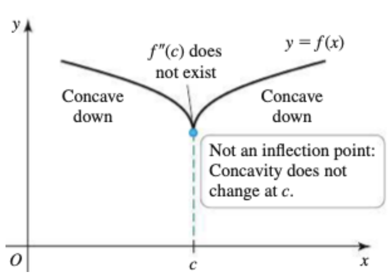


- There is a fake zero point of $f''(x)$.
- It may probably because the power of x is even.
- E.g. $x^2 = 0$, or $(ax \pm b)^2 = 0$

11. An inflection point may also occur if f'' does not exist. <u>(A vertical inflection point)</u>



12. That f'' does not exist does not necessarily implies an inflection point.



13. $f(x) = ln|x| \;\rightarrow\; f'(x) = \dfrac{1}{x} \neq \dfrac{1}{|x|}$

14. Second Derivative Test for Local Extrema

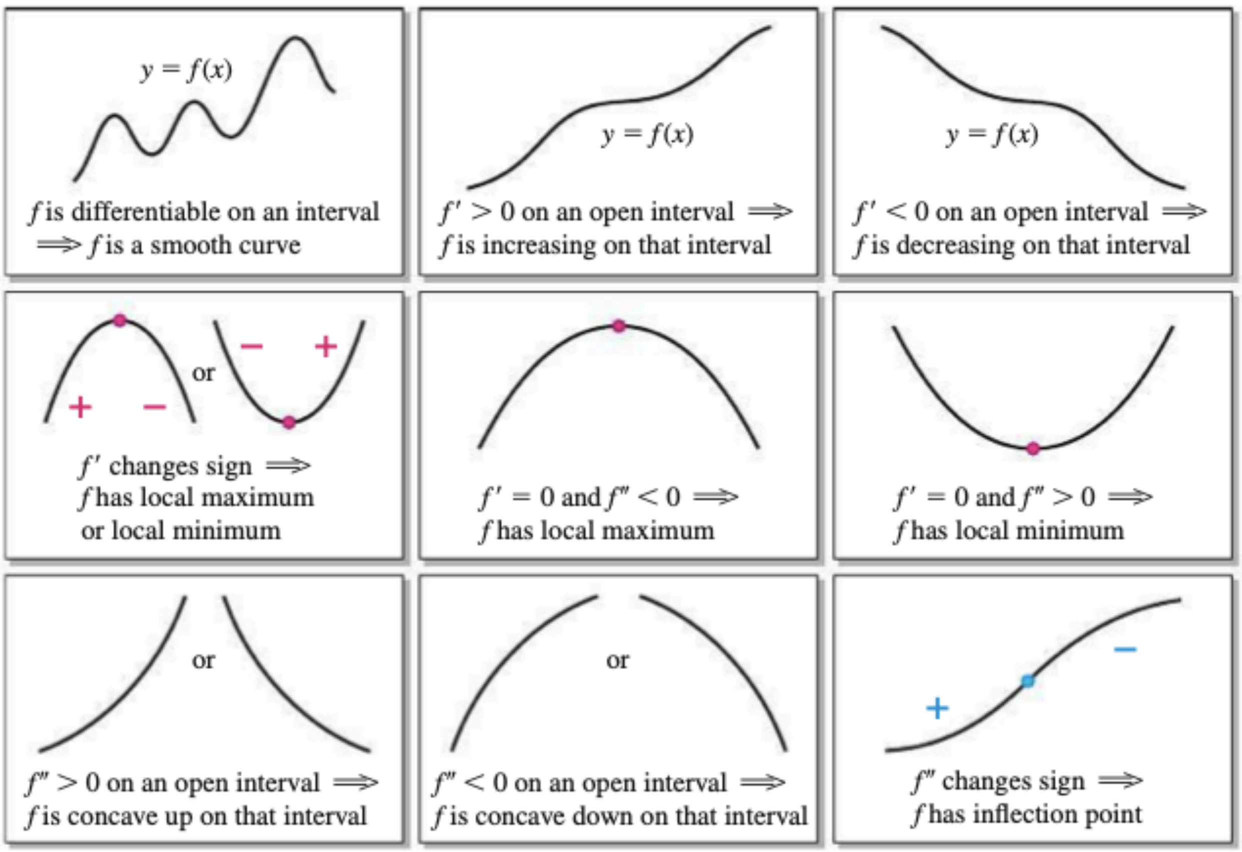
When $f'(c) = 0$

- $f''(c) > 0 \rightarrow$ local min
- $f''(c) < 0 \rightarrow$ local max

← Reverse Order

- $f''(c) = 0 \rightarrow$ the test is inconclusive: local max/min or no local extremum

15. Recap of Derivative Properties



16. Two functions that differ by an additive constant both increase and decrease on the same intervals.

- True: In fact, if two functions differ by a constant, then all of their derivatives are the same.
- E.g. $f_1(x) = x + 1 \quad f_2(x) = x + 2 \quad \rightarrow \quad f_1'(x) = f_2'(x) = 1$

17. A continuous function with two local maxima must have a local minimum in between, and vice versa.

18. Critical Point & Inflection Point

- Definition
  - Critical Point $\to f'(x)$: $f'(x) = 0$ or $f'(x)$ does not exist.
  - Inflection Point $\to f''(x)$: f changes concavity.
- $f'(x)$ & $f''(x)$: $= 0$ or undefined
  - $= 0 \to$ a horizontal tangent line
  - Undefined: a vertical line, discontinuity, a corner, a cusp

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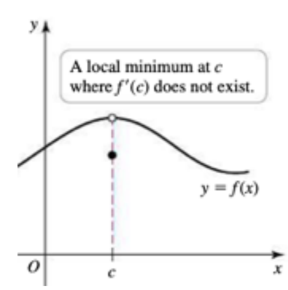
- Compared with 3.2 *Working with Derivatives — (4)*

- Necessarily & Not Necessarily
  - Critical Point $\leftrightarrow$ $f'(x) = 0$ or $f'(x)$ does not exist.
  - Inflection Point $\not\leftrightarrow$ $f''(x) = 0$ or $f''(x)$ does not exist.
    - An inflection point $\leftrightarrow$ $f''(x) = 0$: maybe
    - An inflection point $\nrightarrow$ $f''(x) = 0$
      - E.g. $f''(x)$ does not exist: a vertical inflection point, a corner
    - $f''(x) = 0$ $\nrightarrow$ an inflection point: e.g. a fake zero

---

- An inflection point $\leftrightarrow$ $f''(x)$ does not exist: e.g. a vertical inflection point
- An inflection point $\nrightarrow$ $f''(x)$ does not exist: e.g. $f''(x) = 0$
- $f''(x)$ does not exist $\nrightarrow$ an inflection point: e.g. a cusp, discontinuity

- Continuity
  - Critical points may be discontinuity: e.g. $\to$
  - Inflection points must be continuity



19. $f(x) = \dfrac{x^4}{4} - \dfrac{5x^3}{3} - 4x^2 + 48x$

$f'(x) = x^3 - 5x^2 - 8x + 48$

c_1	± 1	± 2	± 3	± 4	± 6	± 8
	± 12	± 16	± 24	± 48		
d_1	± 1					
$\dfrac{c_1}{d_1} = c_1$	± 1	± 2	± 3	± 4	± 6	± 8
	± 12	± 16	± 24	± 48		

$f'(\pm 1) \neq 0 \quad f'(\pm 2) \neq 0 \quad f'(3) \neq 0 \quad f'(-3) = 0$

-3	1	-5	-8	48
		-3	24	-48
	1	-8	16	0

$f'(x) = (x+3)(x^2 - 8x + 16) = (x+3)(x-4)^2$

$f'(x) = 0 \;\; \rightarrow \;\; x = -3,\ 4$

$f''(x) = 3x^2 - 10x - 8$

$f''(-3) > 0 \;\; \rightarrow$ local minimum

$x = 4$ is a fake critical point.

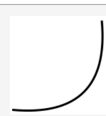
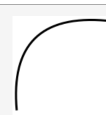
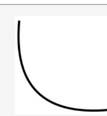
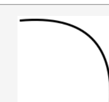
$x = -3$ is a local minimum

- Refer to *Math 110: College Algebre —*
  3.4 Complex and Rational Zeros of Polynomials

4.3 Graphing Functions

1. Graphing Guidelines for $y = f(x)$

- Identify the domain or interval of interest
- Exploit symmetry
    - Odd: $f(-x) = -f(x)$
    - Even: $f(-x) = f(x)$
    - Neither
- Find the first and second derivatives
- Find critical points and possible inflection points
    - $f'(x)$ & $f''(x)$: $= 0$ or undefined
- Find intervals on which the function is increasing/decreasing and concave up/down
    - $f'(x)$: increasing/decreasing
    - $f''(x)$: concave up/down
    - E.g.

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
$f'(x)$	$+$	$+$	$-$	$-$
$f''(x)$	$+$	$-$	$+$	$-$
$f(x)$	Increasing	Increasing	Decreasing	Decreasing
	Concave up	Concave down	Concave up	Concave down
Graph of f				

- The sign of f' and f'' may or may not change at an asymptote.

- Identify extreme values
  - The First and Second Derivative Test
  - Local Maximums/Minimums
  - Absolute Maximums/Minimums
- Locate all asymptotes and determine end behavior
  - Vertical Asymptotes: zeros of denominators
  - Horizontal Asymptotes: $\lim_{x\to\pm\infty} f(x)$
  - There may be a slant asymptote, if m=n+1.

- Refer to *2.5 Limits at Infinity — (6)*

- Find the intercepts
  - Y-intercepts: $= f(0)$
  - X-intercept: $f(x) = 0$
- Choose an appropriate graphing window and plot a graph
  - Derivative information sometimes may not be sufficient to determine the y-coordinates of points on the curve.

2. The family of functions $f(x) = ce^{-ax^2}$ are central to the study of statistics. They have bell-shaped graphs and describe Gaussian or normal distributions.

3. If the zeros of the denominator of f are -3 and 4, then f has vertical asymptotes at these points.

   False: e.g. $f(x) = \dfrac{x+3}{(x+3)(x-4)}$

4. $f(x) = x - 3x^{\frac{2}{3}}$

- The domain of f is $(-\infty,\ \infty)$

- $f(-x) = -x - 3(-x)^{\frac{2}{3}} = -x - 3x^{\frac{2}{3}}$

  $\neq f(x) \neq -f(x)$

  f is neither odd nor even

- $f'(x) = 1 - 2x^{-\frac{1}{3}}$

  $f''(x) = \dfrac{2}{3}x^{-\frac{4}{3}}$

- $f'(x) = 0$

  $1 - 2x^{-\frac{1}{3}} = 0,\ x = 8$

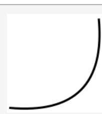
  $f(8) = 8 - 3 * 8^{\frac{2}{3}} = -4$

  $f''(8) > 0$

  $(8,\ -4)$ is a local minimum

  $x \neq 0$

  $f''(x) > 0$

	$(-\infty,\ 0)$	$(0,\ 8)$	$(8,\ \infty)$
$f'(x)$	+	$-$	+
$f''(x)$	+	+	+
$f(x)$			

- $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} (x - 3x^{\frac{2}{3}})$

  $= \lim\limits_{x\to\infty} x = \infty$

  $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} (x - 3x^{\frac{2}{3}})$

  $= \lim\limits_{x\to-\infty} x = -\infty$

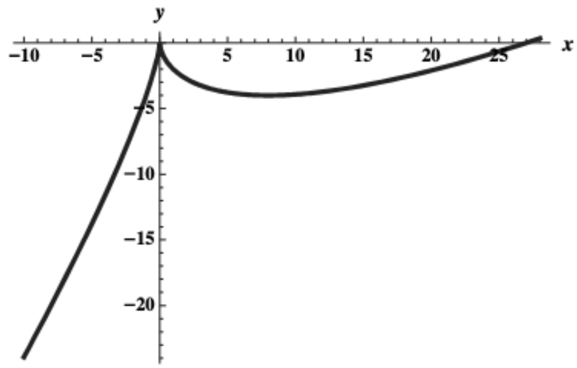
- $f(0) = 0$

  $f(x) = x(1 - 3x^{-\frac{1}{3}}) = 0$

  $x = 0$

  $1 - 3x^{-\frac{1}{3}} = 0,\ x = 3\sqrt{3}$

  $x = 0,\ 3\sqrt{3}$



5. $f(x) = x - 3x^{\frac{1}{3}}$

- The domain of f is $(-\infty,\ \infty)$
- $f(-x) = -x - 3 * (-x)^{\frac{1}{3}} = -x + 3x^{\frac{1}{3}}$

  $= -(x + 3x^{\frac{1}{3}}) = -f(x)$

  f is an odd function
- $f'(x) = 1 - x^{-\frac{2}{3}}$

  $f''(x) = \dfrac{2}{3}x^{-\frac{5}{3}}$
- $f'(x) = 0$

  $1 - x^{-\frac{2}{3}} = 0,\ x = \pm 1$

  $f(1) = 1 - 3 = -2$

  $f(-1) = -1 + 3 = 2$

  $f''(1) > 0$

  $(1,\ -2)$ is a local minimum

  $f''(-1) = < 0$

  $(-1,\ 2)$ is a local maximum

  $x \neq 0$

  $f''(x) \neq 0$

- $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} (x - 3x^{\frac{1}{3}})$

  $\lim\limits_{x\to\infty} x = \infty$

  $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} (x - 3x^{\frac{1}{3}})$

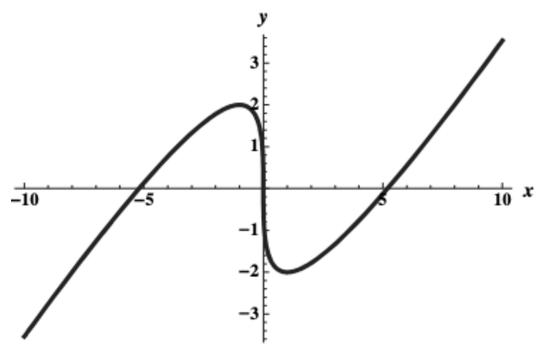
  $\lim\limits_{x\to-\infty} x = -\infty$
- $f(0) = 0$

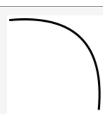
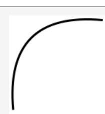
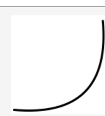
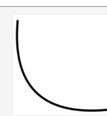
  $f(x) = x(1 - 3x^{-\frac{2}{3}}) = 0$

  $x = 0$

  $1 - 3x^{-\frac{2}{3}} = 0,\ x = 3\sqrt{3}$

  $x = 0,\ 3\sqrt{3}$



	$(-\infty,\ -1)$	$(-1,\ 0)$	$(0,\ 1)$	$(1,\ \infty)$
$f'(x)$	$-$	$+$	$+$	$-$
$f''(x)$	$-$	$-$	$+$	$+$
$f(x)$				

4.4 Optimization Problems

1. Find the base radius r and the height h of the right circular cone of maximum volume that will fit inside a sphere of radius 3 units.

$(h-3)^2 + r^2 = 3^2 = 9$

$r^2 = 9 - (h-3)^2$

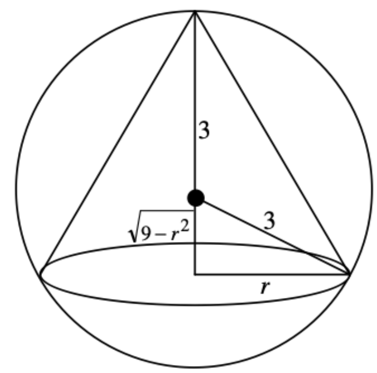
$V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi(9-(h-3)^2)h$

$= -\dfrac{1}{3}\pi h^3 + 2\pi h^2$

$\dfrac{dV}{dh} = -\pi h^2 + 4\pi h$

$\dfrac{dV}{dh} = 0 \;\rightarrow\; h = 4$

$r = \sqrt{9-(h-3)^2} = 2\sqrt{2}$



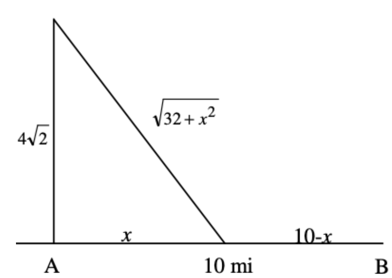
2. A motorist is in the desert in a Jeep. The closest point on a straight road is town A and it is $4\sqrt{2}$ miles from the motorist. He wishes to reach town B, 10 miles from town A on the road, in the shortest time. If he can drive 15 mi/hr on the desert and 45 mi/hr on the road, where should he intersect the road?

$T = \dfrac{\sqrt{(4\sqrt{2})^2 + x^2}}{15} + \dfrac{10-x}{45}$

$= \dfrac{\sqrt{32+x^2}}{15} + \dfrac{10-x}{45}$

$\dfrac{dT}{dx} = \dfrac{\frac{1}{2}(32+x^2)^{-\frac{1}{2}}(2x)}{15} - \dfrac{1}{45}$

$\dfrac{dT}{dx} = 0 \;\rightarrow\; x = \pm 2 \rightarrow\; x = 2$



4.5 Linear Approximation and Differentials

1. Local linearity: smooth curves appear straighter on smaller scales.
2. Linear Approximation to f at a
   $L(x) = f(a) + f'(a)(x - a)$
3. While linear approximation does a decent job of estimating function values when x is near a, we can generally do better with higher-degree polynomials.
4. Linear approximation also allows us to discover simple approximation to complicated functions.
5. Linear Approximation & Concavity
   - concave up → underestimate
   - concave down → overestimate
6. Curvature: the degree of concavity.
7. Absolute errors in linear approximation are larger when $|f''(a)|$ is large.
8. Relationship Between Δx and Δy
   $\Delta y = f'(a)\Delta x$
9. A change in y (the function value) can be approximated by the corresponding change in x magnified or diminished by a factor of $f'(a)$. This interpretation states the familiar fact that $f'(a)$ is the rate of change of y with respect to x.
10. Uses of Linear Approximation
    - To approximate f near $x = a$: $f(x) \approx L(x) = f(a) + f'(a)(x - a)$
    - To approximate the change Δy in the dependent variable when x changes from a to $a + \Delta x$: $\Delta y \approx f'(a)\Delta x$
11. Differentials
    $\Delta y = f(x + dx) - f(x) \approx dy = f'(x)dx$
12. The notation for differentials is consistent with the notation for the derivative
    $\dfrac{dy}{dx} = \dfrac{f'(x)dx}{dx} = f'(x)$

13. If $\Delta x = dx$ is small, the change in $f(\Delta y)$ is well approximated by the change in the linear approximation (dy).

    The approximation $\Delta y \approx dy$ improves as dx approaches 0.

14. Estimate with linear approximation: $sin 2.5^\circ$

    $f(x) = sinx \rightarrow f'(x) = cosx$

    $a = 0 = 0^\circ \rightarrow L(x) = f(0) + f'(0)(x - 0) = sin0 + cos0 * x = x$

    $\frac{2.5^\circ}{180^\circ} = \frac{\theta}{\pi} \rightarrow \theta \approx 0.04363(rad) \rightarrow f(0.04363) \approx L(0.04363) = 0.04363$

    - Before using $L(x)$ to approximate $sin 2.5^\circ$, DO convert to radian measure. (The derivative formulas for trigonometric functions require angles in radian.)

15. The small-angle approximation to the sine function: $sinx \approx x$

16. Estimate with linear approximation: $\sqrt{\frac{5}{29}}$

    $f(x) = \sqrt{x} \rightarrow f'(x) = \frac{1}{2\sqrt{x}}$

    $\frac{5}{29} \approx 0.17 \rightarrow a = 0.16$

    $L(x) = f(0.16) + f'(0.16)(x - 0.16)$

    $= 0.4 + 0.4(x - 0.16)$

    $f(\frac{5}{29}) \approx L(\frac{5}{29}) = 0.4 + 0.4(\frac{5}{29} - 0.16) \approx 0.416$

17. Approximate the change in the volume of a right circular cylinder of fixed radius $r = 20cm$ when its height decreases from $h = 12cm$ to $h = 11.9cm$.

    $V(h) = \pi r^2 h \rightarrow V'(h) = \pi r^2 = \pi * 20^2 = 400\pi$

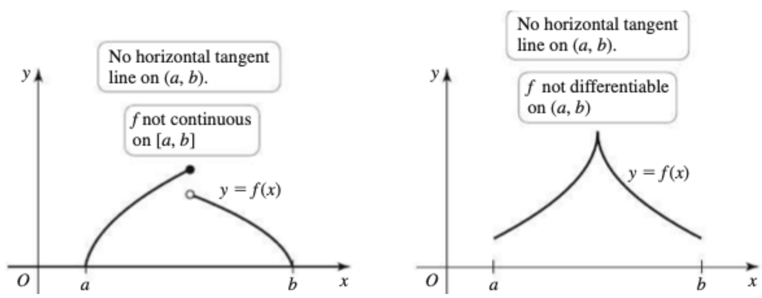
    $a = 12 \rightarrow \Delta y = f'(12)\Delta x = 400\pi * (11.9 - 12) = 400\pi * (-0.1) = -40\pi$

18. Linear approximation at $x = 0$ provides a good approximation to $f(x) = |x|$.

    False. Because $f(x)$ is not differentiable at $x = 0$.

4.6 Mean Value Theorem

1. Rolle's Theorem
   Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. There is at least one point c in (a, b) such that $f'(c) = 0$
2. Why does Rolle's Theorem require continuity?
   - Not continuous → may no horizontal tangent line
   - Not differentiable → may no horizontal tangent line



3. Mean Value Theorem
   If f is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that
   $$\frac{f(b)-f(a)}{b-a} = f'(c)$$
4. Rolle's Theorem & Mean Value Theorem
   - Continuous
   - Differentiable
   - See (9) below →
5. The proofs of Rolle's Theorem and the Mean Value Theorem are nonconstructive: The theorems claim that a certain point exists, but their proofs do not say how to find it.
6. Zero Derivative Implies Constant Function
   If f is differentiable and $f'(x) = 0$ at all points of an interval I, then f is a constant function on I.

7. Functions with Equal Derivative Differ by a Constant
   If two functions have the property that $f'(x) = g'(x)$, for all x of an interval I, then $f(x) - g(x) = C$ on I, where C is a constant; that is, f and g differ by a constant.

8. Intervals of Increase and Decrease
   Suppose f is constant on an interval I and differentiable at all interior point of I.

- If $f'(x) > 0$ at all interior points of I, then f is increasing on I.
- If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.

9. Theorems not apply

- Not differentiable
  - $f(x) =$ Absolute value $\rightarrow$ when $|u| = 0$
    E.g. $y = |x|$ is not differentiable at $x = 0$
  - $f'(x) =$ rational function $\rightarrow$ when $denominator = 0$
    E.g. $y = \sqrt{x}$ is not differentiable at $x = 0$, because $y' = \dfrac{1}{2\sqrt{x}}$
- Not continuous
  - $f(x) =$ Rational functions $\rightarrow$ when $denominator = 0$
    E.g. $y = \dfrac{1}{x}$ is not continuous at $x = 0$

10. Rolle's Theorem: $f(x) = x(x-1)^2;\ \ [0,\ 1]$

$f(0) = f(1) = 0$

$f'(x) = 3x^3 - 4x + 1 = (3x-1)(x-1) = 0$

$x = \dfrac{1}{3} \qquad x = 1$ dropped, because $(0,\ 1)$

- Refer to *(1)*

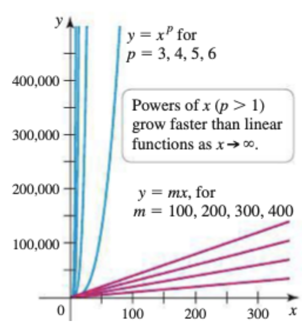
Rolle's Theorem

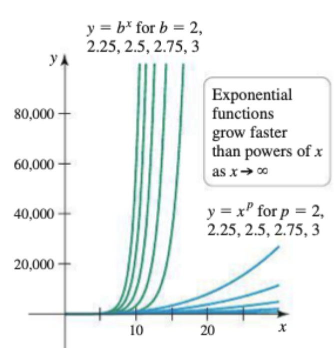
Let f be continuous on a closed interval $[a,\ b]$ and differentiable on $(a,\ b)$ with $f(a) = f(b)$. There is at least one point c in $(a,\ b)$ such that $f'(c) = 0$

4.7 L'Høpital's Rule

1. L'Hopital's Rule
   - In general, a limit with the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$ can have any value — which is why these limits must be handled carefully.
   - Indeterminate Forms: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty} / \frac{0}{0}$
   - $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$
   - It also applies for $x \to \pm\infty$; $x \to a^{\pm}$; $x \to 0^{\pm}$
2. All the limits in *2.4 Infinity Limits and 2.5 Limits at Infinity* are polynomials and rational functions, which do not involve limits of such forms as $0 * \infty$; $1^{\infty}/0^{0}/\infty^{0}$.
3. Growth Rates of Functions (as $x \to \infty$)

   Suppose f and g are functions with $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty$
   - f grows faster than g: $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$ or $\lim_{x \to \infty} \frac{g(x)}{f(x)} = 0$
   - f and g have comparable growth rates: $\lim_{x \to \infty} \frac{f(x)}{g(x)} = M$
4. Raking Growth Rates as $x \to \infty$
   - $ln^q x << x^p << x^p ln^r x << x^{p+s} << b^x << x^x$
   - Graphs





5. L'Hopital's Rule: $\frac{0}{0}/\frac{\infty}{\infty}$ Form & Infinite Limits and Limits at Infinity

- L'Hopital's Rule: $\frac{0}{0}/\frac{\infty}{\infty}$ Form & Infinity Limits

E.g. $\lim_{x\to 1}\frac{x^2-4x+3}{x^2-1}$

- Infinity Limits

$$\lim_{x\to 1}\frac{x^2-4x+3}{x^2-1}=\lim_{x\to 1}\frac{(x-3)(x-1)}{(x+1)(x-1)}$$

$$\lim_{x\to 1}\frac{x-3}{x+1}=\frac{1-3}{1+1}=\frac{-2}{2}=-1$$

- L'Hopital's Rule: $\frac{0}{0}/\frac{\infty}{\infty}$ Form

$$\lim_{x\to 1}\frac{x^2-4x+3}{x^2-1}=\frac{1^2-4+3}{1^2-1}=\frac{0}{0}$$

$$\lim_{x\to 1}\frac{x^2-4x+3}{x^2-1}=\lim_{x\to 1}\frac{2x-4}{2x}$$

$$=\frac{2*1-4}{2*1}=\frac{-2}{2}=-1$$

- L'Hopital's Rule: $\frac{0}{0}/\frac{\infty}{\infty}$ Form & Limits at Infinity

E.g. $\lim_{x\to\infty}\frac{x^2-4x+3}{x^2-1}$

- Limits at Infinity

$$\lim_{x\to\infty}\frac{x^2-4x+3}{x^2-1}$$

$$\begin{cases} m=2 \\ n=2 \end{cases}\rightarrow\ m=n$$

$$\lim_{x\to\infty}\frac{x^2-4x+3}{x^2-1}=\frac{a_m}{a_n}=\frac{1}{1}=1$$

- L'Hopital's Rule: $\frac{0}{0}/\frac{\infty}{\infty}$ Form

$$\lim_{x\to\infty}\frac{x^2-4x+3}{x^2-1}=\lim_{x\to\infty}\frac{2x-4}{2x}$$

$$=\lim_{x\to\infty}\frac{2}{2}=\lim_{x\to\infty}1=1$$

- Limits at Infinity

- $m<n\ \rightarrow\ \lim_{x\to\pm\infty}f(x)=0$
- $m=n\ \rightarrow\ \lim_{x\to\pm\infty}f(x)=\frac{a_m}{a_n}$
- $m>n\ \rightarrow\ \lim_{x\to\pm\infty}f(x)=\pm\infty$
- Details see below →

- Reference: *2.5 Limits at Infinity — (6)*

End Behavior and Asymptotes of Rational Functions

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_m x^m + a_{m-1}x^{m-a} + \ldots + a_2 x^2 + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \ldots + b_2 x^2 + b_1 x + b_0} \quad (a_m \neq 0, b_n \neq 0)$$

- Degree of numerator less than degree of Denominator

  $m < n \quad \rightarrow \quad \lim_{x \to \pm\infty} f(x) = 0 \quad \rightarrow \quad y = 0 \rightarrow$ horizontal asymptote

- Degree of numerator equals degree of denominator

  $m = n \quad \rightarrow \quad \lim_{x \to \pm\infty} f(x) = a_m / b_n \quad \rightarrow \quad y = \dfrac{a_m}{b_n} \rightarrow$ horizontal asymptote

- Degree of numerator greater than degree of denominator

  $m > n \quad \rightarrow \quad \lim_{x \to \pm\infty} f(x) = \pm\infty \quad \rightarrow$ no horizontal asymptote

- Slant asymptote

  $$m = n + 1 \quad \rightarrow \quad \lim_{x \to \pm\infty} f(x) = \pm\infty \quad \rightarrow \quad \begin{cases} 1, \text{ no horizontal asymptote} \\ 2, \text{ a slant asymptote} \end{cases}$$

- Vertical asymptote

  f is in reduced form (p and q share no common factors)

  vertical asymptotes $\rightarrow$ the zeros of q

---

6. Repeated L'Hopital's Rule

- It may happen that this second limit is another indeterminate form to which l'Hopital's Rule may again be applied.

- E.g. $\lim_{x \to \frac{\pi}{2}^-} \frac{tanx}{3/(2x-\pi)}$

• $\lim_{x \to \frac{\pi}{2}^-} \frac{tanx}{3/(2x-\pi)} = \lim_{x \to \frac{\pi}{2}^-} \frac{tan\frac{\pi}{2}}{3/(2*\frac{\pi}{2}-\pi)} = \frac{\infty}{3/0} = \frac{\infty}{\infty}$

• $\lim_{x \to \frac{\pi}{2}^-} \frac{tanx}{3/(2x-\pi)} = \lim_{x \to \frac{\pi}{2}^-} \frac{sec^2x}{\frac{-3*2}{(2x-\pi)^2}} = \lim_{x \to \frac{\pi}{2}^-} (-\frac{1}{6} * \frac{(2x-\pi)^2}{cos^2x})$

$= -\frac{1}{6} * \frac{(2*\frac{\pi}{2}-\pi)^2}{cos^2\frac{\pi}{2}} = -\frac{1}{6} * \frac{0}{0} = \frac{0}{0}$

• $\lim_{x \to \frac{\pi}{2}^-} (-\frac{1}{6} * \frac{(2x-\pi)^2}{cos^2x}) = \lim_{x \to \frac{\pi}{2}^-} (-\frac{1}{6} * \frac{2(2x-\pi)*2}{2cosx*(-sinx)}) = \lim_{x \to \frac{\pi}{2}^-} (\frac{2}{3} * \frac{2x-\pi}{sin2x})$

$= \frac{2}{3} * \frac{2*\frac{\pi}{2}-\pi}{sin(2*\frac{\pi}{2})} = \frac{2}{3}\frac{\pi-\pi}{sin\pi} = \frac{2}{3} * \frac{0}{0} = \frac{0}{0}$

• $= \lim_{x \to \frac{\pi}{2}^-} (\frac{2}{3} * \frac{2x-\pi}{sin2x}) = \lim_{x \to \frac{\pi}{2}^-} (\frac{2}{3} * \frac{2}{2cos2x})$

$= \frac{2}{3} * \frac{2}{2cos(2*\frac{\pi}{2})} = \frac{2}{3} * \frac{2}{2cos\pi} = \frac{2}{3} * \frac{2}{2*(-1)} = -\frac{2}{3}$

7. $0 * \infty$ Form

- A limit of the form $0 * \infty$, in which the two functions compete with each other, may have any value or may not exist.

- It is risky to jump to conclusions about such limits.

  - E.g. $f(x) = x \;\; g(x) = \dfrac{1}{x^2}$ $\qquad \lim_{x\to 0} f(x) = 0 \;\; \lim_{x\to 0} g(x) = \dfrac{1}{0} = \infty$

    $$\lim_{x\to 0}(f(x) * g(x)) = \lim_{x\to 0}(x * \frac{1}{x^2}) = \lim_{x\to 0} \frac{1}{x} = \frac{1}{0} = \infty$$

- L'Hopital's Rule cannot be directly applied to limits of $0 * \infty$ form.

- A common technique that converts this form to either $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ is to divide by the reciprocal.

  - $0 * \infty = \dfrac{\infty}{1/0} = \dfrac{\infty}{\infty}$

  - $0 * \infty = \dfrac{0}{1/\infty} = \dfrac{0}{0}$

- E.g. $\lim_{x\to 0} x * cscx$

  - $\lim_{x\to 0} x * cscx = 0 * \infty$

  - $\lim_{x\to 0} x * cscx = \lim_{x\to 0} \dfrac{x}{sinx} = \dfrac{0}{sin0} = \dfrac{0}{0}$

  - $\lim_{x\to 0} \dfrac{x}{sinx} = \lim_{x\to 0} \dfrac{1}{cosx} = \dfrac{1}{cos0} = \dfrac{1}{1} = 1$

- According to 3.5 Derivatives of Trigonometric Functions — (1)

$$\lim_{x\to 0} x * cscx = \lim_{x\to 0} \frac{x}{sinx} = \frac{1}{\lim_{x\to 0} \frac{sinx}{x}} = \frac{1}{1} = 1$$

8. $\infty - \infty$ Form

- $\lim\limits_{x \to a}(f(x) - g(x)) = \lim\limits_{x \to a}(\infty - \infty)$ is an indeterminate form.
- It also applies for $x \to \pm\infty;\ \ x \to a^{\pm};\ \ x \to 0^{\pm}$

- Indeterminate forms are deceptive
  - E.g.1: $f(x) = 3x + 5 \ \ g(x) = 3x$

    $\lim\limits_{x \to \infty} f(x) = 3\infty + 5 = \infty \ \ \lim\limits_{x \to \infty} g(x) = \infty$

    $\lim\limits_{x \to \infty} (f(x) - g(x)) = \lim\limits_{x \to \infty} (3x + 5 - 3x) = \lim\limits_{x \to \infty} 5 = 5$
  - E.g.2: $f(x) = 3x \ \ g(x) = 2x \qquad \lim\limits_{x \to \infty} f(x) = \infty \ \ \lim\limits_{x \to \infty} g(x) = \infty$

    $\lim\limits_{x \to \infty} (f(x) - g(x)) = \lim\limits_{x \to \infty} (3x - 2x) = \lim\limits_{x \to \infty} x = \infty$

- L'Hopital's Rule cannot be directly applied to limits of $\infty - \infty$ form.

- A common technique

$$\infty - \infty \ \to \ 0 * \infty = \begin{cases} \dfrac{0}{1/\infty} = \dfrac{0}{0} \\ \dfrac{\infty}{1/0} = \dfrac{\infty}{\infty} \end{cases} \to \begin{cases} \lim_{x \to 0^+} f(x) = \lim_{t \to \infty} f(\frac{1}{t}) \\ \lim_{x \to 0^-} f(x) = \lim_{t \to -\infty} f(\frac{1}{t}) \\ \lim_{x \to \infty} f(x) = \lim_{t \to 0} f(\frac{1}{t}) \end{cases}$$

  - E.g. $\lim\limits_{x \to \infty} (x - \sqrt{x^2 - 3x})$

$$\lim_{x \to \infty} (x - \sqrt{x^2 - 3x}) = \infty - \sqrt{\infty^2 - 3\infty} = \infty - \infty ?$$

$$\lim_{x \to \infty} (x - \sqrt{x^2 - 3x}) = \lim_{x \to \infty} (x - \sqrt{x^2(1 - \frac{3}{x})}) = \lim_{x \to \infty} (x(1 - \sqrt{1 - \frac{3}{x}}))$$

$$= \infty * (1 - \sqrt{1 - \frac{3}{\infty}}) = \infty * (1 - \sqrt{1 - 0}) = \infty * (1 - 1) = \infty * 0$$

$$= \lim_{x \to \infty} (x(1 - \sqrt{1 - \frac{3}{x}})) = \lim_{x \to \infty} \frac{1 - \sqrt{1 - \frac{3}{x}}}{1/x}$$

$$t = \frac{1}{x}: \quad x \to \infty \Rightarrow t = \frac{1}{x} \to \frac{1}{\infty} = 0$$

$$= \lim_{x \to \infty} \frac{1 - \sqrt{1 - \frac{3}{x}}}{1/x} = \lim_{t \to 0} \frac{1 - \sqrt{1 - 3t}}{t} = \lim_{t \to 0} \frac{-3}{-2\sqrt{1 - 3t}} = \frac{3}{2\sqrt{1 - 0}} = \frac{3}{2}$$

9. L'hopital's Rule: $\infty - \infty$ Form & Limits at Infinity

- E.g.1: $\lim_{x \to \infty} (x - \sqrt{x^2+1})$

  - L'hopital's Rule: $\infty - \infty$ Form

    - $\lim_{x \to \infty} (x - \sqrt{x^2+1})$
      $= \infty - \sqrt{\infty^2 + 1} = \infty - \infty$

    - $\lim_{x \to \infty} (x - x\sqrt{1 + \frac{1}{x^2}})$
      $= \lim_{x \to \infty} (x(1 - \sqrt{1 + \frac{1}{x^2}}))$
      $= \infty * (1 - \sqrt{1 + \frac{1}{\infty^2}})$
      $= \infty * (1 - \sqrt{1+0})$
      $= \infty * (1-1) = \infty * 0$

    - $= \lim_{x \to \infty} (x(1 - \sqrt{1 + \frac{1}{x^2}}))$
      $= \lim_{x \to \infty} \dfrac{1 - \sqrt{1 + \frac{1}{x^2}}}{1/x}$

    - $t = \dfrac{1}{x}$: $x \to \infty \Rightarrow t = \dfrac{1}{x} \to \dfrac{1}{\infty} = 0$
      $= \lim_{x \to \infty} \dfrac{1 - \sqrt{1 + \frac{1}{x^2}}}{1/x}$
      $= \lim_{t \to 0} \dfrac{1 - \sqrt{1+t^2}}{t}$
      $= \dfrac{1 - \sqrt{1+0^2}}{0} = \dfrac{1-1}{0} = \dfrac{0}{0}$

    - $= \lim_{t \to 0} \dfrac{1 - \sqrt{1+t^2}}{t} = \lim_{t \to 0} \dfrac{-\frac{2t}{2\sqrt{1+t^2}}}{1} = \lim_{t \to 0} (-\dfrac{t}{\sqrt{1+t^2}})$
      $= -\dfrac{0}{\sqrt{1+0^2}} = -\dfrac{0}{1} = 0$

- E.g. 2: $\lim_{x\to 1^+}(\frac{1}{x-1}-\frac{1}{\sqrt{x-1}})$

- L'hopital's Rule: $\infty - \infty$ Form

- $t = \frac{1}{\sqrt{x-1}}$: $x \to 1^+ \Rightarrow t \to \infty$

- $\lim_{x\to 1^+}(\frac{1}{x-1}-\frac{1}{\sqrt{x-1}})$

$= \lim_{t\to\infty}(t^2-t) = \infty - \infty$

- $\lim_{t\to\infty}(t^2-t) = \lim_{t\to\infty}(2t-1)$

$= 2\infty - 1 = \infty$

- Limits at Infinity

$\lim_{x\to 1^+}(\frac{1}{x-1}-\frac{1}{\sqrt{x-1}})$

$= \lim_{x\to 1^+}\frac{1-\sqrt{x-1}}{x-1}$

$= \frac{1-\sqrt{1^+-1}}{1-1} = \frac{1-0}{0} = \frac{1}{0} = \infty$

- Limits at Infinity

$\lim_{x\to\infty}(x-\sqrt{x^2+1})$

$= \lim_{x\to\infty}\frac{(x-\sqrt{x^2+1})(x+\sqrt{x^2+1})}{x+\sqrt{x^2+1}}$

$= \lim_{x\to\infty}\frac{x^2-(x^2+1)}{x+\sqrt{x^2+1}}$

- Compare with *2.5 Limits at Infinity*:

$\lim_{x\to\infty}f(x) = \lim_{t\to 0}f(\frac{1}{t})$

$\lim_{x\to\infty}\frac{3x^2-7}{x^2+5x} = \lim_{x\to\infty}\frac{3-\frac{7}{x^2}}{1+\frac{5}{x}} = \frac{3-\frac{7}{\infty^2}}{1+\frac{5}{\infty}} = \frac{3-0}{1+0} = 3$

- Compared with 2.5 Limits at Infinity — (25) Limits of exponentials

10. $1^{\infty},\ 0^0,\ \infty^0$ Form

Assume $\lim_{x\to a} f(x)^{g(x)}$ has the indeterminate form $1^{\infty},\ 0^0,\ \infty^0$

- $L = \lim_{x\to a} g(x) * lnf(x)$
- $\lim_{x\to a} f(x)^{g(x)} = e^L$

- It also applies for $x \to \pm\infty;\ \ x \to a^{\pm};\ \ x \to 0^{\pm}$
- L'Hopital's Rule cannot be directly applied to limits of $1^{\infty},\ 0^0,\ \infty^0$ form
- $\lim_{x\to a} f(x)^{g(x)} = 1^{\infty},\ 0^0,\ \infty^0$ is an indeterminate form
- $L = \lim_{x\to a} g(x) * lnf(x)$
  - $1^{\infty}:\quad L = \infty * ln1 = \infty * 0$
  - $0^0:\quad L = 0 * ln0 = 0 * -\infty$
  - $\infty^0:\quad L = 0 * ln\infty = 0 * \infty$
- E.g. $\lim_{x\to 0^+} x^x$
  - $\lim_{x\to 0^+} x^x = 0^0$
  - $L = \lim_{x\to 0} xlnx = \lim_{x\to 0} \frac{lnx}{1/x} = \frac{ln0}{1/0} = \frac{-\infty}{\infty}$
  - $= \lim_{x\to 0} \frac{lnx}{1/x} = \lim_{x\to 0} \frac{1/x}{-\frac{1}{x^2}} = \lim_{x\to 0}(-x) = 0$
  - $\lim_{x\to 0^+} x^x = e^L = e^0 = 1$
- 0^{∞} is NOT an indeterminate form

  E.g. $\lim_{x\to 0^+} x^{1/x} = 0^{1/0} = 0^{\infty} = 0 \neq 0^0$

11. Trigonometric Functions & L'Hopital's Rule

- $\lim_{x\to 0}(\frac{sinx}{x})^{\frac{1}{x^2}}$
  - $\lim_{x\to 0}(\frac{sinx}{x})^{\frac{1}{x^2}} = (\lim_{x\to 0}\frac{sinx}{x})^{(\lim_{x\to 0}\frac{1}{x^2})} = 1^{\frac{1}{0^2}} = 1^{\infty}$
  - $L = \lim_{x\to 0}\frac{ln\frac{sinx}{x}}{x^2} = \frac{ln(\lim_{x\to 0}\frac{sinx}{x})}{\lim_{x\to 0}x^2} = \frac{ln1}{0^2} = \frac{0}{0}$
  - $L = \lim_{x\to 0}\frac{ln\frac{sinx}{x}}{x^2} = \lim_{x\to 0}\frac{\frac{x}{sinx}*\frac{xcosx-sinx}{x^2}}{2x} = \lim_{x\to 0}\frac{xcosx-sinx}{2x^2sinx} = \frac{0cos0-sin0}{2*0^2*sin0} = \frac{0}{0}$
  - $= \lim_{x\to 0}\frac{xcosx-sinx}{2x^2sinx} = \lim_{x\to 0}\frac{cosx-xsinx-cosx}{4xsinx+2x^2cosx} \neq \lim_{x\to 0}\frac{cosx-xsinx-sinx}{4xsinx+2x^2cosx}$
    $= \lim_{x\to 0}\frac{-sinx}{4sinx+2xcosx} = \frac{-sin0}{4sin0+2*0*cos0} = \frac{0}{0}$
  - $= \lim_{x\to 0}\frac{-sinx}{4sinx+2xcosx} = \lim_{x\to 0}\frac{-cosx}{4cosx+2cosx-2xsinx} = \lim_{x\to 0}\frac{-cosx}{6cosx-2xsinx}$
    $= \frac{-cos0}{6cos0-2*0*sin0} = \frac{-1}{6-0} = -\frac{1}{6}$

- $\lim_{x\to\infty}(cot\frac{1}{x}-x)$
  - $u = \frac{1}{x}: \quad x\to\infty \Rightarrow u = \frac{1}{x} \to \frac{1}{\infty} = 0$
  - $\lim_{x\to\infty}(cot\frac{1}{x}-x) = \lim_{u\to 0}(cotu-\frac{1}{u}) = \lim_{u\to 0}(\frac{cosu}{sinu}-\frac{1}{u}) = \lim_{u\to 0}\frac{ucosu-sinu}{usinu}$
    $= \frac{0*cos0-sin0}{0*sin0} = \frac{0-0}{0} = \frac{0}{0}$
  - $= \lim_{u\to 0}\frac{ucosu-sinu}{usinu} = \lim_{u\to 0}\frac{cosu-usinu-cosu}{sinu+ucosu} \neq \lim_{u\to 0}\frac{cosu-usinu-sinu}{sinu+ucosu}$
    $= \lim_{u\to 0}\frac{-usinu}{sinu+ucosu} = \frac{0*sin0}{sin0+0*cos0} = \frac{0}{0+0} = \frac{0}{0}$
  - $\lim_{u\to 0}\frac{-usinu}{sinu+ucosu} = \lim_{u\to 0}\frac{-(sinu+ucosu)}{cosu+cosu-usinu} = \lim_{u\to 0}\frac{-(sinu+ucosu)}{2cosu-usinu}$
    $= \frac{-(sin0+0*cos0)}{2cos0-0*sin0} = \frac{-(0+0)}{2-0} = 0$

12. Pitfalls in using L'Hopital's Rule

- L'Hopital's Rule & Quotient Rule
  - L'Hopital's Rule: $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$
  - Quotient Rule: $h(x) = \frac{f(x)}{g(x)} \quad \rightarrow \quad h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$

- Reference: *3.4 The Product and Quotient Rules — (2)*

- Check out if the limit is an indeterminate form: $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\infty}{\infty}/\frac{0}{0}$
  - E.g. $\lim_{x\to 0}\frac{1 - sinx}{cosx} = \frac{1 - sin0}{cos0} = \frac{1-0}{1} = 1$

    $\neq \lim_{x\to 0}\frac{-cosx}{-sinx} = \lim_{x\to 0}\frac{cox}{sinx} = \lim_{x\to 0} cotx$
- When using l'Hopital's Rule repeatedly
  - Simplify expression as much as possible
  - Evaluate the limit as soon as the new limit is no longer an indeterminate form
- Repeated use of l'Hopital's Rule occasionally leads to unending cycles.
  - E.g. $\lim_{x\to\infty}\frac{\sqrt{ax+b}}{\sqrt{cx+d}}$
- Be sure that the limit produced by l'Hopital's Rule exists.
  - E.g. $\lim_{x\to\infty}\frac{3x + cosx}{x} = \lim_{x\to\infty}(3 + \frac{cosx}{x}) = 3 + \frac{cos\infty}{\infty} = 3 + 0 = 3$

    Incorrect Answer

    $\lim_{x\to\infty}\frac{3x + cosx}{x} = \frac{\infty+\infty}{\infty} = \frac{\infty}{\infty}$

    $\lim_{x\to\infty}\frac{3x + cosx}{x} \neq \lim_{x\to\infty}\frac{3 - sinx}{1} = \lim_{x\to\infty}(3 - sinx)$ does not exist

13. $\lim_{x \to 0^+} (a^x - b^x)^x$, $a > b > 0$

- $\lim_{x \to 0^+} (a^x - b^x)^x = (a^0 - b^0)^0 = (1-1)^0 = 0^0$

- $L = \lim_{x \to 0^+} x ln(a^x - b^x) = \lim_{x \to 0^+} \frac{ln(a^x - b^x)}{1/x} = \lim_{X \to 0^+} \frac{\frac{a^x lna - b^x lnb}{a^x - b^x}}{-\frac{1}{x^2}}$

$= \lim_{x \to 0^+} \frac{-x^2(a^x lna - b^x lnb)}{a^x - b^x}$

- To avoid complicating matters further, we separate the complicated chunk off.

$= \lim_{x \to 0^+} \frac{-x^2(a^x lna - b^x lnb)}{a^x - b^x} = \lim_{x \to 0^+} (a^x lna - b^x lnb) * \lim_{x \to 0^+} \frac{-x^2}{a^x - b^x}$

- $\lim_{x \to 0^+} \frac{-x^2}{a^x - b^x} = \frac{-0^2}{a^0 - b^0} = \frac{0}{1-1} = \frac{0}{0}$

- $\lim_{x \to 0^+} \frac{-x^2}{a^x - b^x} = \lim_{x \to 0^+} \frac{-2x}{a^x lna - b^x lnb}$

- $\lim_{x \to 0^+} (a^x lna - b^x lnb) * \lim_{x \to 0^+} \frac{-x^2}{a^x - b^x}$

$= \lim_{x \to 0^+} ((a^x lna - b^x lnb) * \frac{-x^2}{a^x lna - b^x lnb})$

$= \lim_{x \to 0^+} (-x^2) = 0$

- $\lim_{x \to 0^+} (a^x - b^x)^x = e^L = e^0 = 1$

4.8 Newton's Method

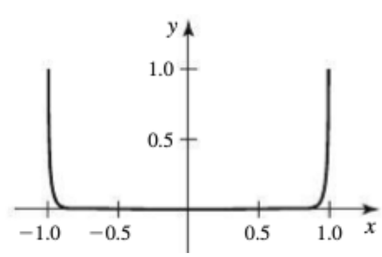
1. Newton's Method for Approximation Roots of $f(x) = 0$

   - Choose an initial approximation x_0 as close to a root as possible.
   - $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$: for $n = 1, 2, \ldots$; provided $f'(x_n) \neq 0$
   - End the calculations when a termination condition is met.

2. Newton's method is an example of a repetitive loop calculation called an iteration.

3. When do you stop?

   - Until either two successive approximations agree to p digits
   - Until some maximum number of iterations is exceeded
     （超过规定次数之后还没算出来就停。）
     (in which case Newton's method has failed to find an approximation of the root with the desired accuracy).
   - Until the residual is zero: $f(x_n) = 0$
     - Because Newton's method generates approximations to a root of f, it follows that as the approximations x_n approach the root $f(x_n)$ should approach zero.
     - The quantity $f(x_n)$ is called a residual, and small residuals usually <u>(but not always)</u> suggest that the approximations have small errors.
     - Small residuals do not always imply small errors
       - E.g. The function shown below has a zero at $x = 0$.
         An approximation such as 0.5 has a small residual but a large error.



4. Pitfalls of Newton's Method

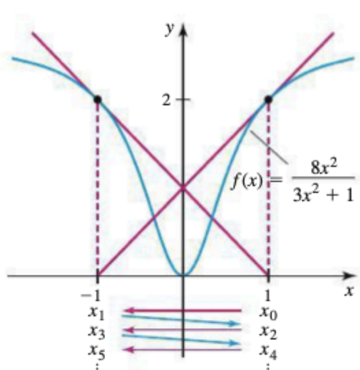
- If at any step $f'(x_n) = 0$, then the method breaks down.
- If $f'(x_n)$ is close to zero at any step, then the method
  - may converge slowly
    - If the approximations x_n approach a zero of f', the rate of convergence is often compromised.
  - or may fail to converge
- Several ways that Newton's method may fail to converge as its usual rate They may cycle, wander, converge slowly, or diverge (often at a rapid rate).
- E.g. See below →

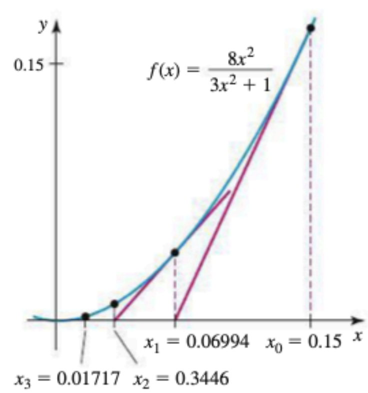
5. Find the root of $f(x) = \dfrac{8x^2}{3x^2+1}$ using Newton's method with initial approximation of $x_0 = 1$, $x_0 = 0.15$ and $x_0 = 1.1$.

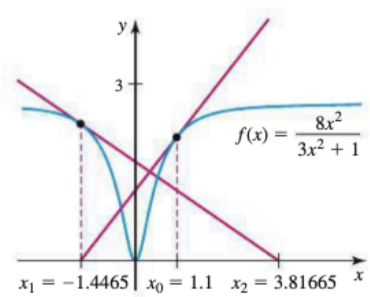
- $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = \dfrac{x_n}{2}(1 - 3x_n^2)$

n	x_n	x_n	x_n
0	1	0.15	1.1
1	−1	0.0699375	−1.44665
2	1	0.0344556	3.81665
3	−1	0.0171665	−81.4865
4	1	0.00857564	8.11572E+05
5	−1	0.00428687	−8.01692E+17

- $x_0 = 1$: cycle
  getting stuck in a cycle that alternates between +1 and −1.
- $x_0 = 0.15$: a slow convergence
  Due to the fact that both f and f' have zeros at the same position ($x = 0$).
- $x_0 = 1.1$: wandering and divergence (at a rapid rate)
  Increasing in magnitude quickly and not converging to a finite value.
  (even though it seems reasonable)







4.9 Antiderivatives

1. Indefinte Integrals

- $\int x^p dx = \frac{x^{p+1}}{p+1} + C \quad (p \neq -1)$
- $\int e^{ax} dx = \frac{1}{a} e^{ax} + C$

- No need for Substitution Rule — Reference: 5.5

- $\int b^x dx = \frac{b^x}{lnb} + C \quad (b > 0,\ b \neq 1)$
- $\int \frac{1}{x} dx = ln|x| + C$

2. Constant Multiple and Sum Rules

- Constant Multiple Rule: $\int cf(x)dx = c\int f(x)dx$
- Sum Rule: $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

- All of arbitrary constants may be combined in one.
- In general, the indefinite integral of a product or quotient is not the product or quotient of indefinite integrals.

3. dx means that x is the independent variable, or the variable of integration. When the integrand is a function of a variable different from x — say, $g(t)$ — then we write $\int g(t)dt$ to represent the antiderivative of g.

4. $\int dx = \int 1dx: \quad \int dx = x + C$

5. Checking results by differentiation is necessary!

6. Indefinite Integrals of Trigonometric Functions

- $\int cosax\ dx = \frac{1}{a}sinax + C$
- $\int sinax\ dx = -\frac{1}{a}cosax + C$
- $\int sec^2ax\ dx = \frac{1}{a}tanax + C$
- $\int csc^2ax\ dx = -\frac{1}{a}cotax + C$
- $\int secax * tanax\ dx = \frac{1}{a}secax + C$
- $\int cscax * cotax\ dx = -\frac{1}{a}cscax + C$
- $\int \frac{1}{\sqrt{a^2 - x^2}}dx = sin^{-1}\frac{x}{a} + C \quad \neq \frac{1}{a}sin^{-1}\frac{x}{a} + C$
- $\int \frac{1}{a^2 + x^2}dx = \frac{1}{a}tan^{-1}\frac{x}{a} + C$
- $\int \frac{1}{x\sqrt{x^2 - a^2}}dx = \frac{1}{a}sec^{-1}|\frac{x}{a}| + C \quad (a > 0)$

7. Initial Value Problems for Velocity and Position

- The position is an antiderivative of velocity.
  $s'(t) = v(t) \quad s(0) = s_0$
- Te velocity is an antiderivative of the acceleration.
  $v'(t) = a(t) \quad v(0) = v_0$

• But there are infinitely many antiderivatives that differ by a constant.

8. Integrals of Inverse Trigonometric Functions & Other Functions

- Inverse Trigonometric Integrands and Logarithmic Integrands

$$\int \frac{ax}{b+cx^2}dx \quad \& \quad \int \frac{a}{b+cx^2}dx$$

- E.g.1: $\int \frac{x}{1+x^2}dx$

$$= \frac{1}{2}\int \frac{2x}{1+x^2}dx$$

$$= \frac{1}{2}\int \frac{(1+x^2)'}{1+x^2}dx$$

$$= \frac{1}{2}\int \frac{1}{u}du \quad = \frac{1}{2}ln|u| + C$$

- E.g.2: $\int \frac{1}{1+x^2}dx$

$$= tan^{-1}x + C$$

- Integrals of Trigonometric Functions & Power Rule for Integrals

$$\int \frac{ax}{\sqrt{b+cx^2}}dx \quad \& \quad \int \frac{a}{\sqrt{b+cx^2}}dx$$

- E.g.3: $\int \frac{x}{\sqrt{1-x^2}}dx$

$$= -\frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}dx$$

$$= -\frac{1}{2}\int \frac{(1-x^2)'}{\sqrt{1-x^2}}dx$$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{u}}du \quad = -\frac{1}{2}\int u^{-\frac{1}{2}}du$$

$$= -\frac{1}{2}(2u^{\frac{1}{2}}) + C \quad = -u^{\frac{1}{2}} + C$$

- E.g.4: $\int \frac{1}{\sqrt{1-x^2}}dx$

$$= sin^{-1}x + C$$

- Reference: 5.5 Substitution Rule — (1)

Substitution Rule for Indefinite Integrals

$$\int f(g(x))g'(x)dx = \int f(u)du$$

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9. $\int \frac{2}{16z^2+25}dz$

$= \int \frac{2 * \frac{1}{16}}{\frac{1}{16}(16z^2+25)}dz$

$= \frac{1}{8}\int \frac{1}{z^2+(\frac{5}{4})^2}dz$

$= \frac{1}{8}tan^{-1}(\frac{z}{5/4}) * \frac{1}{5/4}$

$= \frac{1}{8}tan^{-1}(\frac{4z}{5}) * \frac{4}{5}$

$= \frac{1}{10}tan^{-1}(\frac{4z}{5})$

- Check

$\frac{d}{dx}(\frac{1}{10}tan^{-1}(\frac{4z}{5}))$

$= \frac{1}{10} * \frac{1}{1+(\frac{4z}{5})^2} * \frac{4}{5}$

$= \frac{2}{25} * \frac{25}{25+16z^2}$

$= \frac{2}{16z^2+25}$

5.1 Approximation Areas under Curves

1. Regular Partition
   - $\Delta x = \dfrac{b-a}{n}$
   - $x_k = a + k\Delta x$
2. Sums of Powers of Integers
   - $\displaystyle\sum_{k=1}^{n} c = cn$
   - $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
   - $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
   - $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$
3. Formulas for $\displaystyle\sum_{k=1}^{n} k^p$ become complicated as p increases.
4. Left, Right, and Midpoint Riemann Sums in Sigma Notation
   - Riemann Sum: $\displaystyle\sum_{k=1}^{n} f(x_k{}^*)\Delta x$
   - $x_k{}^*$ is any point in the kth subinterval $[x_{k-1}, x_k]$.
   - Left Riemann Sum: $x_k{}^* = a + (k-1)\Delta x$
   - Right Riemann Sum: $x_k{}^* = a + k\Delta x$
   - Midpoint Riemann Sum: $x_k{}^* = a + (k - \dfrac{1}{2})\Delta x$
5. Comparing the midpoint Riemann sum with the left and right, Riemann sum suggests that the midpoint sum is a more accurate estimate of the area under the curve.
6. The value of f at the midpoint will always be between the value of f at the endpoints, if f is monotonic increasing or monotonic decreasing.

7. Overestimation & Underestimation
   - Monotonic increasing functions
     - Left Riemann sums = underestimation
     - Right Riemann sums = overestimation
   - Monotonic decreasing functions
     - Left Riemann sums = overestimation
     - Right Riemann sums = underestimation
8. $\sum_{n=0}^{40} (n^2 + 3n - 1)$

$$= \sum_{n=1}^{40} n^2 + 3\sum_{n=0}^{40} n - \sum_{n=0}^{40} 1$$

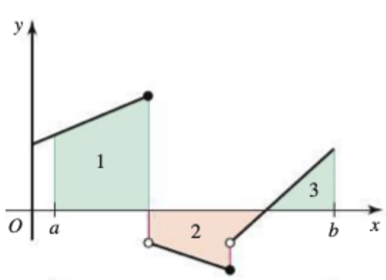
$$= \frac{40(40+1)(2*40+1)}{6} + 3*\frac{(0+40)*41}{2} - 41*1$$

$$= 24559$$

5.2 Difinite Integrals

1. On intervals where $f(x) < 0$, Riemann sums approximate the negative of the area of the region bounded by the curve.
2. Definite Integral
   - $\int_a^b f(x)dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k{}^*)\Delta x_k$
   - $\Delta \to 0$ forces all $\Delta x_k \to 0$, which forces $n \to \infty$.
     Therefore, it suffices to write $\Delta \to 0$ in the limit.
   - For polynomials of degree 4 and higher, the calculations are more challenging, and for rational and transcendental functions, advanced mathematical results are needed.
3. The indefinite integral $\int_a^b f(x)dx$ is a family of functions of x

   (the antiderivatives of f) and that the definite integral
   $\int_a^b f(x)dx$ is a real number (the net area of a region).
4. Slice-and-Sum Method
   The strategy of slicing a region into smaller parts, summing the results from the parts, and taking a limit is used repeatedly in calculus and its applications.
5. Integral Functions
   - If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuity, then f is integrable on $[a, b]$.
   - A bounded piecewise continuous function is integrable.



6. Properties of definite integrals

- $\int_b^a f(x)dx = -\int_a^b f(x)dx$
- $\int_a^a f(x)dx = 0$
- $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
- $\int_a^b f(x)dx = \int_a^p f(x)dx + \int_p^b f(x)dx$
- $\int_a^b |f(x)|dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a, b]$.

7. Evaluating Definite Integrals Using Limits

- Right Riemann Sum: $\int_a^b f(x)dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k{}^*)\Delta x_k = \lim_{n \to \infty} \sum_{k=1}^{n} f(a + k\Delta x)\Delta x$
- An analogous calculation could be done using left Riemann sums or midpoint Riemann sums.

8. $\int_0^{\frac{2\pi}{a}} sinax\ dx = \int_0^{\frac{2\pi}{a}} cosax\ dx = 0$

5.3 Fundamental Theorem of Calculus

1. Area Functions: $A(x) = \int_a^x f(t)dt$

- x is the upper limit of the integral
- x is the independent variable of the area function.

2. Fundamental Theorem of Calculus

- $A'(x) = \dfrac{d}{dx}\int_a^x f(t)dt = f(x)$
  - $A'(x) = \dfrac{d}{dx}\int_a^x f'(t)dt = f'(x) \neq f'(t)$
  - $A'(x) = \dfrac{d}{dx}\int_a^x f'(t)dt = f'(x) \neq$ antiderivative $+C$

    It is differentiation rather than antidifferentiation, no need to $+C$.
- $\int_a^b f(x)dx = F(b) - F(a) = F(x)\,|_a^b$
  - The integral "undoes" the derivative.
  - $A(x) \quad \& \quad F(x)$
    - $A(x) = \int_a^x f(t)dt = F(x) - F(a) = F(t)\,|_a^x$
    - $A(x) =$ Indefinite Integral; $F(x) =$ Definite Integral
    - $F(x)\,|_a^b$ can be simplified
      - $a(F(x))\,|_a^b = aF(x)\,|_a^b$
      - $-(-F(x))\,|_a^b = F(x)\,|_a^b$
      - $-F(x)\,|_a^b = F(x)\,|_b^a$

3. Leibniz's Rule: $\dfrac{d}{dx}\displaystyle\int_a^{g(x)} f(t)dt = f(g(x)) * g'(x)$

- $\displaystyle\int_a^x f(t)dt = A(x) \quad \rightarrow \quad \frac{d}{dx}\int_a^x f(t)dt = \frac{d}{dx}A(x) = A'(x) = f(x)$

- $\dfrac{d}{dx}\displaystyle\int_a^{g(x)} f(t)dt = \frac{d}{dx}A(g(x))$

  $= \dfrac{dA(x)}{dg(x)} * \dfrac{dg(x)}{dx} = \dfrac{d}{dx}A(g(x)) * \dfrac{d}{dx}g(x)$

  $= A'(g(x)) * g'(x) = f(g(x)) * g'(x)$

4. $f(x) \quad \& \quad A(x) = \displaystyle\int_0^x f(t)dt$

- f is a positive decreasing function on $x > 0$

  $\rightarrow A(x) = \displaystyle\int_0^x f(t)dt$ is an increasing function

- f is a negative increasing function on $x > 0$

  $\rightarrow A(x) = \displaystyle\int_0^x f(t)dt$ is a decreasing function

5. Two area functions of the same function differ by a constant.

- E.g. If $A(x) = 3x^2 - x - 3$ is an area function for f, then $B(x) = 3x^2 - x$ is also an area function of f.

  - $A(x) = \displaystyle\int_a^x f(t)dt = F(x) - F(a) = F(t)\,|_a^x$

  - $B(x) = \displaystyle\int_b^x f(t)dt = F(x) - F(b) = F(t)\,|_b^x$

  - $A(x)$ and $B(x)$ differ by different lower limits of the integral of the same function f.

6. $\frac{d}{dx}\int_a^b f(t)dt = 0$

- $\int_a^b f(t)dt$ is a constant, whose derivative is 0.

- There is no variable $x \rightarrow \int_a^b f(t)dt \neq A(x)$

7. Watch out: area & net area

8. $\frac{d}{dx}\int_0^{cosx} (t^4 + 6)dt = (cos^4 x + 6)(-sin x) = -sin x(cos^4 x + 6)$

- By comparison, $\frac{d}{dx}\int_x^1 e^{t^2}dt = e^{t^2}$

9. Definite Integrals & Indefinite Integrals

- Indefinite Integrals: $\int f(x)dx \rightarrow +C$

- Definite Integrals: $\int_a^b f(x)dx \rightarrow$ no need to $+C$

10. Derivatives of integrals

- $\frac{d}{dx}\int_a^x f(t)dt = f(x) \neq f(t)$

- $\frac{d}{dx}\int_x^a f(t)dt = -f(x)$

5.4 Working with Integrals

1. Integrals of Even and Odd Functions

   - If f is even: $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$
   - If f is odd: $\int_{-a}^{a} f(x)dx = 0$

2. Integration of the even-powered terms may be simplified because the lower limit is zero.

   - But not for trigonometric function (E.g. $cos0 = 1$)

3. Average Value of a Function: $\bar{f} = \dfrac{1}{b-a}\int_{a}^{b} f(x)dx$

   - $\bar{f}$ may be negative or zero.

4. Mean Value Theorem for Integrals
   Let f be continuous on the interval $[a, b]$. There exists a point c in (a, b) such that

   $$f(c) = \bar{f} = \frac{1}{b-a}\int_{a}^{b} f(t)dt$$

   - f may also equal its average value at an endpoint of that interval.

5. A general form of the Mean Value Theorem states that if f and g are continuous on $[a, b]$ with $g(x) \geq 0$ on $[a, b]$, then there exists a number c in (a, b) such that

   $$\int_{a}^{b} f(x)g(x)dx = f(c)\int_{a}^{b} g(x)dx$$

6. If f has the property $f(a+x) = -f(a-x)$, for all x, where a is a constant, then $\int_{a-2}^{a+2} f(x)dx = 0$.

   ????????????????????????????????

7. The average value of a linear function on an interval $[a, b]$ is the function value at the midpoint of $[a, b]$.

8. The mean value of $f(x) = \frac{\pi}{4} sinx$ on $[0, \pi]$

   $$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{\pi - 0}\int_0^\pi \frac{\pi}{4} sinxdx = \frac{1}{\pi} * \frac{\pi}{4}(-cosx)\,|_0^\pi = -\frac{1}{4}(cosx)\,|_0^\pi$$

   $$-\frac{1}{4}(cos\pi - cos0) = -\frac{1}{4} * (-1-1) = \frac{1}{2}$$

   $$f(c) = \frac{\pi}{4} sinc = \frac{1}{2} \quad c = sin^{-1}(\frac{\pi}{2})$$

   c=0.7 and 2.5 on [0, π]

9. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} cos2xdx = 2\int_0^{\frac{\pi}{4}} cos2xdx$

   $f(x) = cos2x$ is an even function. The period is π.

5.5 Substitution Rule

1. Substitution Rule for Indefinite Integrals
$$\int f(g(x))g'(x)dx = \int f(u)du$$
   - Do not forget to use $u = g(x)$ to return to the original variable.
2. When the integrands has the form $f(ax+b)$, the substitution $u = ax + b$ is often effective.
3. Substitution Rule for Definite Integrals
$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)gu$$
   - The upper limit $= u(b)$; the lower limit $= u(a)$
4. Variations on the substitution method
   - $u = f(x) \ \rightarrow \ x = g(u) \ \rightarrow \ dx = g'(u)du$
   - E.g. $\int x\sqrt[3]{2x+1}\ dx$

$$u = 2x + 1 \ \rightarrow \ x = \frac{1}{2}(u-1) \ \rightarrow \ dx = \frac{1}{2}du$$
$$\int x\sqrt[3]{2x+1}\ dx = \int \frac{1}{2}(u-1)u^{\frac{1}{3}}\ \frac{1}{2}du$$
$$= \frac{1}{4}\int (u^{\frac{4}{3}} - u^{\frac{1}{3}})du = \frac{1}{4}(\frac{3}{7}u^{\frac{7}{3}} - \frac{3}{4}u^{\frac{4}{3}}) + C$$
$$= \frac{3}{28}u^{\frac{7}{3}} - \frac{3}{16}u^{\frac{4}{3}} + C = \frac{3}{28}(2x+1)^{\frac{7}{3}} - \frac{3}{16}(2x+1)^{\frac{4}{3}} + C$$

5. $\int sin^2xdx$ and $\int cos^2xdx$ $\rightarrow$ Half-Angle Formulas
   - $\int sin^2xdx = \int \frac{1-cos2x}{2}dx = \frac{1}{2}x - \frac{1}{4}sin2x + C$
   - $\int cos^xdx = \int \frac{1+cos2x}{2}dx = \frac{1}{2}x + \frac{1}{4}sin2x + C$

6. Conclusions of (5)

- $\int_a^b sin^2 x dx$

$= \int_a^b \frac{1 - cos2x}{2} dx$

$= \frac{1}{2}(\int_a^b dx - \int_a^b cos2xdx)$

  - $\int_a^b dx = x|_a^b = b - a$
  - $\int_a^b cos2xdx$

    $= \frac{1}{2}\int_{2a}^{2b} cosudu$

    $= \frac{1}{2} sinu|_{2a}^{2b}$

    $= \frac{1}{2}(sin(2b) - sin(2a))$
  - $\int_a^b sin^2 xdx$

    $= \frac{1}{2}((b - a) - \frac{1}{2}(sin(2b) - sin(2a)))$

    $= \frac{1}{2}(b - a) - \frac{1}{4}(sin(2b) - sin(2a))$

- $\int_a^b cos^2 xdx$

$= \int_a^b \frac{1 + cos2x}{2} dx$

$= \frac{1}{2}(\int_a^b dx + \int_a^b cos2xdx)$

  - $\int_a^b dx = x|_a^b = b - a$
  - $\int_a^b cos2xdx$

    $= \frac{1}{2}\int_{2a}^{2b} cosudu$

    $= \frac{1}{2} sinu|_{2a}^{2b}$

    $= \frac{1}{2}(sin(2a) - sin(2b))$
  - $\int_a^b cos^2 xdx$

    $= \frac{1}{2}((b - a) + (sin(2a) - sin(2b)))$

    $= \frac{1}{2}(b - a) + \frac{1}{4}(sin(2a) - sin(2b))$

- $\int_a^b sin^2 xdx = \frac{1}{2}(b - a) - \frac{1}{4}(sin(2b) - sin(2a))$
- $\int_a^b cos^2 xdx = \frac{1}{2}(b - a) + \frac{1}{4}(sin(2a) - sin(2b))$

7. The Prototype & Variant on the Substation Method

- Mutual Exclusion: the two can never be used in the same problem.
- Generally
  - If you can find the basis of $f'(x)$ → Prototype
    The basis of $f'(x)$: In most cases, a nugget of x.
  - If you can't → Variant
- Specifically, rational integrands (fractional integrands)
  - If the numerator includes x → Variant
  - If not → Prototype

- E.g.1: $\int_3^5 \frac{1}{x-2}dx$

$u = x - 2 \ \rightarrow \ du = dx$

$u(5) = 5 - 2 = 3$

$u(3) = 3 - 2 = 1$

$= \int_1^3 \frac{1}{u}du$

$= lnu|_1^3 \quad = ln3$

- E.g.2: $\int_3^5 \frac{x}{x-2}dx$

$u = x - 2 \ \rightarrow \ x = u + 2 \ \rightarrow \ dx = du$

$u(5) = 5 - 2 = 3$

$u(3) = 3 - 2 = 1$

$= \int_1^3 \frac{u+2}{u}du = \int_1^3 (1 + \frac{2}{u})du$

$= (u + 2lnu)|_1^3 \quad = 2 + ln9$

- If not → Prototype → the inverse trigonometric integrands

- Refer to 4.9 Antiderivatives — (8)

- Counterexample: $\int \frac{x}{1+x^2}dx$

$= \frac{1}{2}\int \frac{2x}{1+x^2}dx$

$= \frac{1}{2}\int \frac{(1+x^2)'}{1+x^2}dx$

$= \frac{1}{2}\int \frac{1}{u}du \quad = \frac{1}{2}ln|u| + C$

8. A comparison of the Substitution Rule

- E.g. $\int \frac{1}{x+1} dx$
  - $= \int \frac{(x+1)'}{x+1} dx$
    $= \int \frac{1}{u} du$
    $= ln|u| + C$
    $= ln|x+1| + C$
  - $= \int \frac{(x+1)'}{x+1} dx$
    $= \int \frac{1}{x+1} d(x+1)$
    $= ln|x+1| + C$

9. Do not forget to change the upper and lower limits (even if $g'(x) = 1$)

- E.g. $\int_0^3 \sqrt{x+1} dx$

  $u = x + 1 \quad \rightarrow \quad u' = 1$

  $u(0) = 1 \quad u(3) = 4$

  $\int_0^1 \sqrt{x+1} dx = \int_1^4 u^{\frac{1}{2}} du = \frac{2}{3} u^{\frac{3}{2}}|_1^4 = \frac{2}{3}(\sqrt{4} - \sqrt{1}) = \frac{2}{3}$

10. $\int \frac{2}{x\sqrt{4x^2 - 1}} dx \quad x > \frac{1}{2}$

$u = 2x \ \rightarrow \ du = 2dx$

$= 2\int \frac{2dx}{(2x)\sqrt{(2x)^2 - 1}} = 2\int \frac{du}{u\sqrt{u^2 - 1}}$

$= 2sec^{-1}u + C = 2sec^{-1}(2x) + C$

11. $\int sinx * sec^8x \ dx$

$= \int \frac{sinx}{cos^8x} dx = -\int \frac{-sinx}{cos^8x} dx = -\int \frac{(cosx)'}{cos^8x} dx$

$= -\int u^{-8} du = -(-\frac{u^{-7}}{7}) + C = \frac{u^{-7}}{7} + C$

$= \frac{cos^{-7}x}{7} + C = \frac{1}{7cos^7x} + C$

12. $\int_0^{\frac{\pi}{2}} sin^4\theta d\theta$

$$= \int_0^{\frac{\pi}{2}} (\frac{1 - cos(2\theta}{2})^2 d\theta = \int_0^{\frac{\pi}{2}} (\frac{1 - 2cos(2\theta) + cos^2(2\theta)}{4}) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4} d\theta - \frac{1}{2}\int_0^{\frac{\pi}{2}} cos(2\theta) d\theta + \frac{1}{4}\int_0^{\frac{\pi}{2}} cos^2(2\theta) d\theta$$

- $\int_0^{\frac{\pi}{2}} \frac{1}{4} d\theta = \frac{\theta}{4}|_0^{\frac{\pi}{2}} = \frac{1}{4}(\frac{\pi}{2} - 0) = \frac{\pi}{8}$

- $\frac{1}{2}\int_0^{\frac{\pi}{2}} cos(2\theta) d\theta$

$u_1 = 2\theta \ \rightarrow \ du_1 = 2d\theta$

$u_1(\frac{\pi}{2}) = \pi \quad u_2(0) = 0$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} cos(2\theta) d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} 2cos(2\theta)\ d\theta$$

$$= \frac{1}{4}\int_0^{\pi} cosu_1\ du_1$$

$$= \frac{1}{2} sinu_1|_0^{\pi} = \frac{1}{2}(sin\pi - sin0) = 0$$

- $\frac{1}{4}\int_0^{\frac{\pi}{2}} cos^2(2\theta) d\theta = \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{1 + 2cos(4\theta)}{2} d\theta$

$$= \frac{1}{8}\int_0^{\frac{\pi}{2}} d\theta + \frac{1}{4}\int_0^{\frac{\pi}{2}} cos(4\theta) d\theta$$

- $\frac{1}{8}\int_0^{\frac{\pi}{2}} d\theta = \frac{\theta}{8}|_0^{\frac{\pi}{2}} = \frac{1}{8}(\frac{\pi}{2} - 0) = \frac{\pi}{16}$

- $\frac{1}{4}\int_0^{\frac{\pi}{2}} cos(4\theta) d\theta$

$u_2 = 4\theta \ \rightarrow \ du_2 = 4d\theta$

$u_2(\frac{\pi}{2}) = 2\pi \quad u_2(0) = 0$

$$\frac{1}{4}\int_0^{\frac{\pi}{2}} cos(4\theta) d\theta = \frac{1}{16}\int_0^{\frac{\pi}{2}} 4cos(4\theta) d\theta$$

$$= \frac{1}{16}\int_0^{2\pi} cosu_2 du_2$$

$$= \frac{1}{16} sinu_2|_0^{2\pi} = \frac{1}{16}(sin(2\pi) - sin0) = 0$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{4} d\theta - \frac{1}{2}\int_0^{\frac{\pi}{2}} cos(2\theta) d\theta + \frac{1}{4}\int_0^{\frac{\pi}{2}} cos^2(2\theta) d\theta$$

$$= \frac{\pi}{8} + \frac{\pi}{16} = \frac{3\pi}{16}$$