MATH 130: Precalculus Mathematics

2.4 Definition of Function

See MATH 110

2.5 Graphs of Function

See MATH 110

3.5 Rational Functions

See MATH 110

5.1 Angles

1. Definitions of special types of angles
   - Complementary Angles: $\alpha + \beta = 90^\circ$
   - Supplementary Angles: $\alpha + \beta = 180^\circ$
2. Smaller measurements of Angles
   - Minutes ('): 1°=60'
   - Seconds ("): 1'=60"
3. Relationships Between Degrees and Radians
   - 180° = π radians
   - $1^\circ = \dfrac{\pi}{180}$ radian ≈ 0.0175 radian
   - 1 radian = $(\dfrac{180^\circ}{\pi})$ ≈ 57.2958°
4. Changing Angular Measures
   - Degrees to radians = $\dfrac{\pi}{180^\circ}$
   - Radians to degrees = $\dfrac{180^\circ}{\pi}$
5. The corresponding radian and degree measure of special angles

Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°

6. Formula for the Length of a Circular Arc: $s = r\theta$ ($\theta \rightarrow$ radian measure)
7. Formula for the Area of a Circular Sector: $A = \dfrac{1}{2}r^2\theta$ ($\theta \rightarrow$ radian measure)
8. A Relationship for Linear Speed and Angular Speed: $\dfrac{s}{t} = r\dfrac{\theta}{t}$

5.2 Trigonometric Functions of Angles

1. Less Common Trigonometric Functions

- $vers\theta = 1 - cos\theta$
- $exsec\theta = sec\theta - 1$
- $covers\theta = 1 - sin\theta$
- $hav\theta = \frac{1}{2}vers\theta$

2. Reciprocal Identities

- $sin\theta = \frac{1}{csc\theta}$
- $csc\theta = \frac{1}{sin\theta}$
- $cos\theta = \frac{1}{sce\theta}$
- $sec\theta = \frac{1}{cos\theta}$
- $tan\theta = \frac{1}{cot\theta}$
- $cot\theta = \frac{1}{tan\theta}$

3. Special Values of the Trigonometric Functions

θ (radians)	θ (degrees)	$sin\theta$	$cos\theta$	$tan\theta$	$cot\theta$	$sec\theta$	$csc\theta$
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

4. The Fundamental Identities

- The tangent and cotangent identities:
  $tan\theta = \frac{sin\theta}{cos\theta}$; $cot\theta = \frac{cos\theta}{sin\theta}$
- The Pythagorean identities:
  - $sin^2\theta + cos^2\theta = 1$
  - $1 + tan^2\theta = sec^2\theta$
  - $1 + cot^2\theta = csc^2\theta$

5. Definition of the Trigonometric Functions of Any Angle

- $sin\theta = \frac{y}{r}$
- $cos\theta = \frac{x}{r}$
- $tan\theta = \frac{y}{x}$ (if x≠0)
- $csc\theta = \frac{r}{y}$ (if y≠0)
- $sec\theta = \frac{r}{x}$ (if x≠0)
- $cot\theta = \frac{x}{y}$ (if y≠0)

6. The Domains of Trigonometric Functions

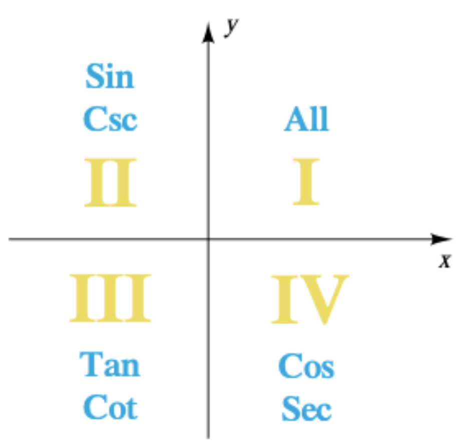
Functions		Domain
Sine	Cosine	Every angle θ
Tangent	Secant	Every angle θ except $\theta = \frac{\pi}{2} + \pi n = 90^\circ + 180^\circ \bullet n$
Cotangent	Cosecant	Every angle θ except $\theta = \pi n = 180^\circ \bullet n$

7. $|sin\theta| \leq 1 \quad |cos\theta| \leq 1 \quad |csc\theta| \geq 1 \quad |sec\theta| \geq 1$

8. Signs of the Trigonometric Functions

Quadrant containing θ	Positive Functions	Negative Functions
I	All	None
II	sin, csc	cos, csc, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

9. Quadrants & Positive Trigonometric Functions



10. $cot\theta = \frac{7}{24}$

$r = \sqrt{7^2 + 24^2} = 25$

$sin\theta = \frac{24}{25} \quad cos\theta = \frac{7}{25} \quad tan\theta = \frac{24}{7}$

11. $\frac{sin^3\theta + cos^3\theta}{sin\theta + cos\theta} = \frac{(sin\theta + cos\theta)(sin^2\theta - sin\theta cos\theta + cos^2\theta)}{sin\theta + cos\theta}$

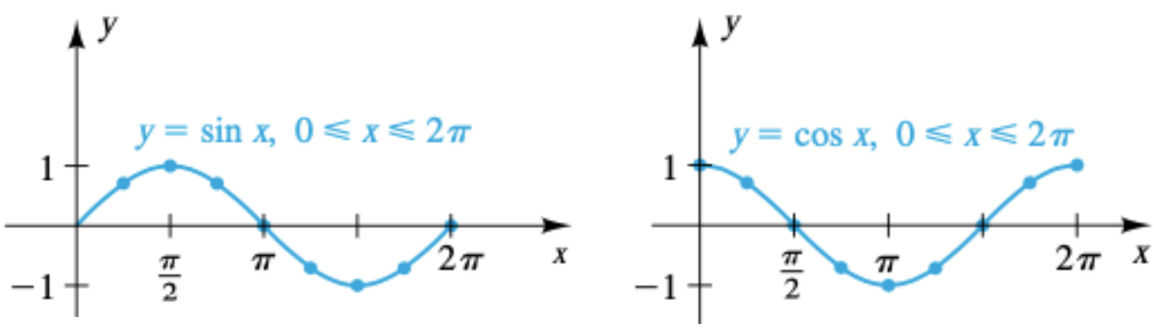
$= sin^2\theta - sin\theta cos\theta + cos^2\theta = 1 - sin\theta * cos\theta$

- Refer to *Math 110: College Algebra — 1.3 Algebra Expressions — (7)*

- $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
- $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

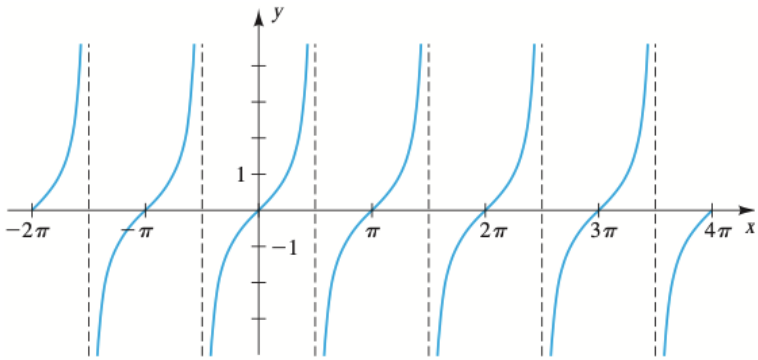
5.3 Trigonometric Functions of Real Numbers

1. Definition of the Trigonometric Functions in Terms of a Unit Circle
   - $sin\ t = y$
   - $cos\ t = x$
   - $tan\ t = \frac{y}{x}$ (if x≠0)
   - $csc\ t = \frac{1}{y}$ (if y≠0)
   - $sec\ t = \frac{1}{x}$ (if x≠0)
   - $cot\ t = \frac{x}{y}$ (if y ≠0)
2. Theorem on Repeated Function Values for sin and cos
   - $sin(t + 2\pi n) = sin\ t$
   - $cos(t + 2\pi n) = cos\ t$
3. Definition of Periodic Function: $f(t + k) = f(t)$
4. The graphs of y=sinx and y=cosx

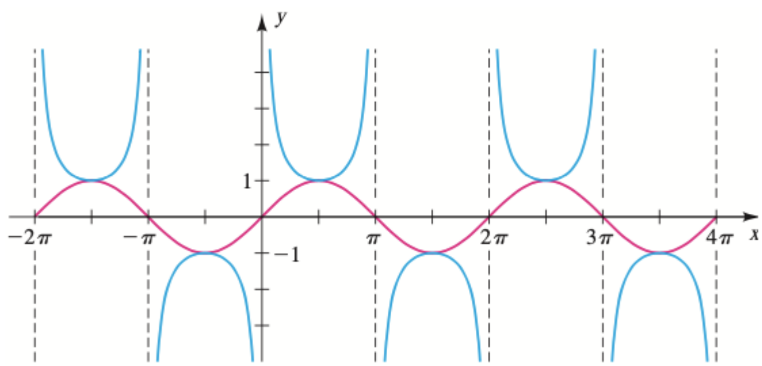


5. Formulas for Negatives
   - $sin(-t) = -sin\ t$
   - $cos(-t) = cos\ t$
   - $tant(-t) = -tan\ t$
   - $csc(-t) = -csc\ t$
   - $sec(-t) = sec\ t$
   - $cot(-t) = -cot\ t$
6. Theorem on Even and Odd Trigonometric Functions
   - cosx & secx → even
   - sinx & tanx & cotx & cscx → odd

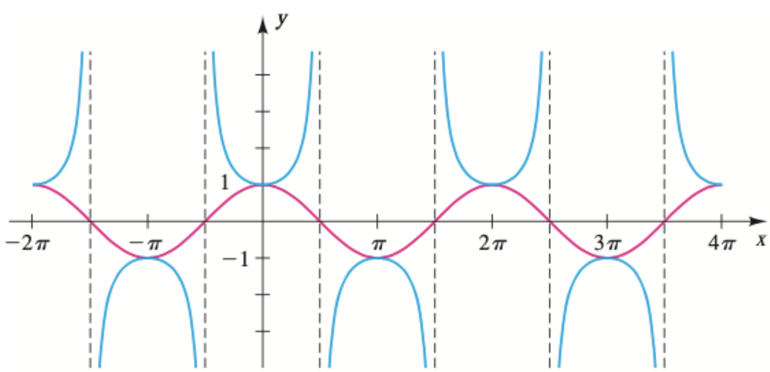
7. The graph of y=tanx



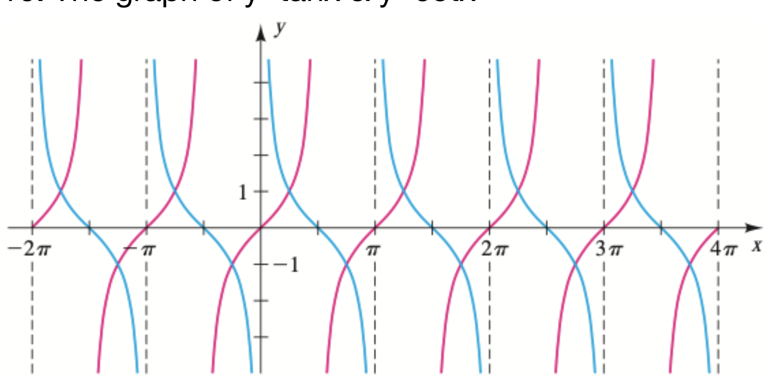
8. The graph of y=sinx & y=cscx



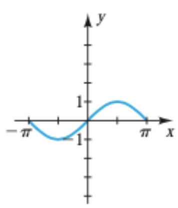
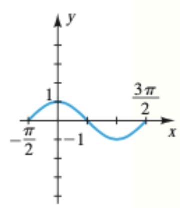
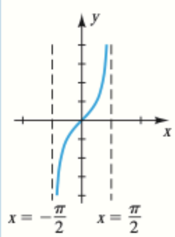
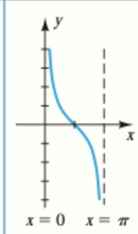
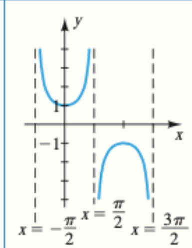
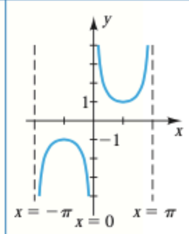
9. The graph of y=cosx & y=secx



10. The graph of y=tanx & y=cotx



11. Summary of Features of the Trigonometric Functions and Their Graphs

Feature	$y = \sin x$	$y = \cos x$	$y = \tan x$	$y = \cot x$	$y = \sec x$	$y = \csc x$
Graph (one period)						
Domain	$\mathbb{R}$	$\mathbb{R}$	$x \neq \frac{\pi}{2} + \pi n$	$x \neq \pi n$	$x \neq \frac{\pi}{2} + \pi n$	$x \neq \pi n$
Vertical asymptotes	none	none	$x = \frac{\pi}{2} + \pi n$	$x = \pi n$	$x = \frac{\pi}{2} + \pi n$	$x = \pi n$
Range	$[-1, 1]$	$[-1, 1]$	$\mathbb{R}$	$\mathbb{R}$	$(-\infty, -1] \cup [1, \infty)$	$(-\infty, -1] \cup [1, \infty)$
x-intercepts	πn	$\frac{\pi}{2} + \pi n$	πn	$\frac{\pi}{2} + \pi n$	none	none
y-intercept	0	1	0	none	1	none
Period	2π	2π	π	π	2π	2π
Even or odd	odd	even	odd	odd	even	odd
Symmetry	origin	y-axis	origin	origin	y-axis	origin

12. $f(x) = \dfrac{sin x}{x}$

$$\lim_{x \to 0} \frac{sin x}{x} = 1$$

if $x \approx 0$, then $\dfrac{sin x}{x} \approx 1 \quad \rightarrow \quad sin x \approx x$

$$sin(0.03) \approx 0.0299955 \approx 0.03$$
$$sin(0.02) \approx 0.0199987 \approx 0.03$$
$$sin(0.01) \approx 0.0099998 \approx 0.01$$

13. $csc(\frac{\pi}{2}) = sec(\frac{\pi}{2}) = \sqrt{2}$

5.4 Values of the Trigonometric Functions

1. Definition of Reference Angle: $\theta_R \in (0^\circ, 90^\circ)$ or $\theta_R \in (0, \pi/2)$

2. Theorem on Reference Angles

   If θ is a non-quadrantal angle in standard position, then to find the value of a trigonometric function at θ, find its value for the reference angle θ_R and prefix the appropriate sign.

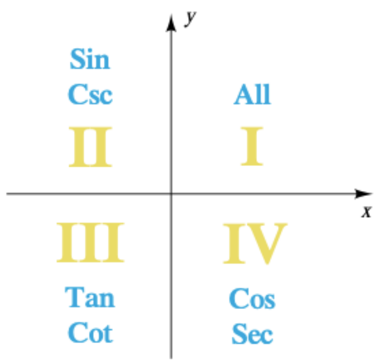
- Prefix the appropriate sign
  - Refer to *5.2 Trigonometric Functions of Angles — (8) or (9)*

---

(8) Signs of the Trigonometric Functions

Quadrant containing θ	Positive Functions	Negative Functions
I	All	None
II	sin, csc	cos, csc, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

(9) Quadrants & Positive Trigonometric Functions



---

3. $sin\theta_R \quad \& \quad cos\theta_R \quad \& \quad tan\theta_R$

- $|sin\theta| = sin\theta_R$
- $|cos\theta| = cos\theta_R$
- $|tan\theta| = tan\theta_R$

4. If θ is an acute angle and $sin\theta = 0.06635$, approximate θ.

- Calculator → sin^{-1} / cos^{-1} / tan^{-1}
- When finding an angle, we will usually round off degree measure to the nearest 0.001° and radian measure to four decimal places.

5. Finding Acute Angle Solutions of Equations with a Calculator

Equation	Calculator solution (degree and radian)
$sin\theta = k$	$\theta = sin^{-1}(k)$
$cos\theta = k$	$\theta = cos^{-1}(k)$
$\tan\theta = k$	$\theta = tan^{-1}(k)$
$csc\theta = k$	$\theta = sin^{-1}(\frac{1}{k})$
$sec\theta = k$	$\theta = cos^{-1}(\frac{1}{k})$
$cot\theta = k$	$\theta = tan^{-1}(\frac{1}{k})$

6. Find the reference angle θ_R if θ has the given measure.

- $\theta = 260^{\circ}$: 260° < 270° → θ_R = 260° - 180°=80°
  - Compared with $\frac{3\pi}{2}$ = 270°
- $\theta = 5.5$: 5.5 > 4.71($\frac{3\pi}{2}$) → θ_R = 6.28(2π) - 5.5=0.78
  - Compared with $\frac{3\pi}{2} \approx$ 4.71

7. Watch out negative angles

E.g. $sin(-\frac{5\pi}{4}) = sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

5.5 Trigonometric Graphs

1. Theorem on Amplitudes and Periods

   $y = a\sin\ bx \quad \& \quad y = a\cos\ bx$

   - the amplitude = $|a|$
   - the period = $\dfrac{2\pi}{|b|}$

2. Sketching the graphs

   - If |b|>1 → being compressed horizontally by a factor b
   - If 0<|b|<1 → being stretched horizontally by a factor b
   - If b<0 → $sin(-x) = -sinx$ & $cos(-x) = cosx$

3. Theorem on Amplitudes, Periods, and Phase Shifts

   $y = a\sin(bx + c) \quad \& \quad y = a\cos(bx + c)$

   - the phase shift = $-\dfrac{c}{b}$
     - negative → left
     - Positive → right
     - Compare with $y = f(x \pm c)$: 左加右减
       - Refer to *Math 110: College Algebra — 2.5 Graphs of Functions — (3)*
   - $0 \leq bx + c \leq 2\pi$

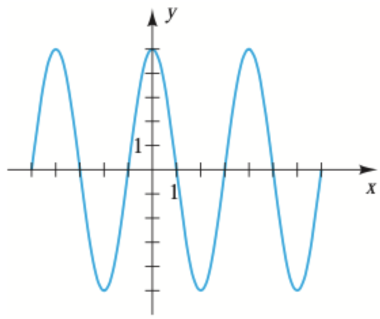
4. Period → Phase Shift: 先伸缩，再平移

   - Stretch or compress the graph at first
   - Then move the graph
   - $-\dfrac{c}{b}$ is the phase shift of $y = a\sin bx$ instead of $y = a\sin x$

5. Move must be from the y-axis. (Determine c by displacement.)

6. Finding an equation for a sine wave

$y = asin(bx + c)$, a>0, b>0, c>o



- $|a| = 5 \ + \ a > 0 \quad \rightarrow \quad a = 5$
- $\dfrac{2\pi}{|b|} = 4 \ + \ b > 0 \quad \rightarrow \quad b = \dfrac{\pi}{2}$
- $-\dfrac{c}{b} = -1 + c > 0 \quad \rightarrow \quad c = \dfrac{\pi}{2}$
    - c>0 → the phase shift must be negative
- $y = 5sin(\dfrac{\pi}{2}x + \dfrac{\pi}{2})$

- An alternative solution

  $y = asin(bx + c) = asin[b(x + \dfrac{c}{b})]$

  x-intercepts at x=−1 → a horizontal shift of $y = 5sin(\dfrac{\pi}{2}x)$ to the left by 1 unit

  $y = 5sin[\dfrac{\pi}{2}(x + 1)] = 5sin(\dfrac{\pi}{2}x + \dfrac{\pi}{2})$

  - 此方法适用于左加右减

7. Find an amplitude, a period, and a phase shift
   $y = 2cos(3x - \pi)$

- the amplitude is $|a| = 2$
- the period is $\frac{2\pi}{|b|} = \frac{2\pi}{3}$
- the phase shift is $-\frac{c}{b} = -\frac{-\pi}{3} = \frac{\pi}{3}$

  - A cycle between maximums

    $0 \le 3x - \pi \le 2\pi$

    $\pi \le 3x \le 3\pi$

    $\frac{\pi}{3} \le x \le \pi$

  - A cycle between x-intercepts

    $-\frac{\pi}{2} \le 3x - \pi \le \frac{3\pi}{2}$

    $\frac{\pi}{2} \le 3x \le \frac{5\pi}{2}$

    $\frac{\pi}{6} \le x \le \frac{5\pi}{5}$

5.6 Additional Trigonometric Graphs

1. Since the tangent, cotangent, secant, and cosecant functions have no largest values, the notion of amplitude has no meaning. Moreover, we do not refer cycles.

2. Theorem on the Graph of $y = atan(bx + c)$

   - the period = $\dfrac{\pi}{|b|}$

   - the phase shift = $-\dfrac{c}{b}$ (左减右加)

   - $-\dfrac{\pi}{2} < bx + c < \dfrac{\pi}{2}$

3. Normal Phase Shift & The Phase Shift of Trigonometry

   - Normal Phase Shift: 左加右减

   - The Phase Shift of Trigonometry: 左减右加

4. Find a period, and a phase shift
   $y = \dfrac{1}{3}tan(x + \dfrac{\pi}{4})$

   - the period is $\dfrac{\pi}{|b|} = \pi$

   - the phase shift is $-\dfrac{c}{b} = -\dfrac{\pi}{4}$

   - $-\dfrac{\pi}{2} < x + \dfrac{\pi}{4} < \dfrac{\pi}{2}$
     $-\dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$

   - Or, shifting the graph of $y = \dfrac{1}{3}tan\,x$ to the left $\dfrac{\pi}{4}$

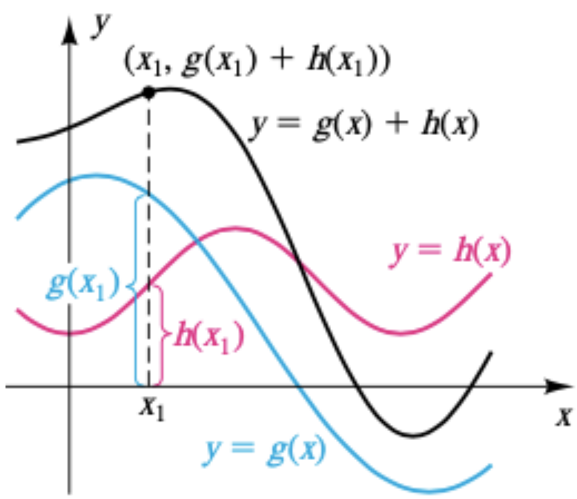
5. $y = acot(bx + c) \quad \rightarrow \quad 0 < bx + c < \pi$

6. Theorem on the Graph of $y = acsc(bx + c)$ & $y = asec(bx + c)$

- Similar to the methods for the tangent and cotangent functions
  - $y = acsc(bx + c) \quad \rightarrow \quad 0 < bx + c < 2\pi$
  - $y = asec(bx + c) \quad \rightarrow \quad -\dfrac{\pi}{2} < bx + c < \dfrac{3\pi}{2}$
- Taking reciprocals of the sine and cosine functions
  - The zeros of the sine/cosine function
    = the asymptotes of the cosecant/secant function

7. Additional of y-coordinates:

- $f(x) = g(x) + h(x)$



- $f(x)$ intersects $\begin{cases} g(x) \text{ when } h(x) = 0 \\ h(x) \text{ when } g(x) = 0 \end{cases}$

8. Damped Sine Waves & Damped Cosine Waves

- $y = f(x)\ sin(ax + b)$
- $y = f(x)\ cos(ax + b)$
- $f(x)$ is the damping factor

9. Sketching the graph of a damped sine wave

$f(x) = 2^{-x} sin x$

$-1 \leq sin x \leq 1$

$2^{-x} > 0$

$-2^{-x} \leq 2^{-x} sin x \leq 2^{-x}$

- When $sin x = 1 \quad \rightarrow \quad f(x) = 2^{-x} \quad \rightarrow \quad y = f(x)$ coincides with $y = 2^{-x}$

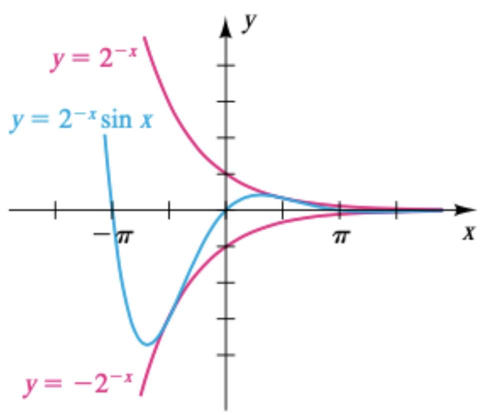
  $\rightarrow \quad x = \dfrac{\pi}{2} + 2n\pi$

- When $sin x = -1 \quad \rightarrow \quad f(x) = -2^{-x} \quad \rightarrow \quad y = f(x)$ coincides with $y = -2^{-x}$

  $\rightarrow \quad x = \dfrac{3\pi}{2} + 2n\pi$

- $2^{-x} > 0 \quad \rightarrow$ When $sin x = 0 \quad \rightarrow \quad f(x) = 0 \quad \rightarrow$ x-intercepts

  $\rightarrow \quad x = n\pi$



10. Find a special point $P(x_P, y_P)$ of the graph of f

$f(x) = a sin(bc + c)$

Let $Q(x_Q, y_Q)$ is the special point of $y = sin x$

$$bx + c = x_Q \quad \rightarrow \quad x = \frac{x_Q - c}{b}$$

the period is $\dfrac{2\pi}{|b|} \quad \rightarrow \quad x_P = \dfrac{x_Q - c}{b} + \dfrac{2\pi}{|b|}n$

$y_P = ay_Q$

$$\therefore P\ (\frac{x_Q - c}{b} + \frac{2\pi}{|b|}n,\ ay_Q)$$

11. Find the lowest point on an upper brach of the graph $f(x) = 3csc(\frac{1}{2}x - \frac{\pi}{2})$

- $\frac{1}{2}x - \frac{\pi}{2} = \frac{\pi}{2} + 2k\pi \quad \rightarrow \quad x = 2\pi + 4k\pi \ (k \in Z)$

  $\therefore (2\pi + 4\pi n,\ 3)$

- An alternative method

  the period = $\frac{2\pi}{\frac{1}{2}} = 4\pi$

  the lowest point on an upper branch of $y = 3csc(\frac{1}{2}x)$ is $(\pi,3)$

  the phase shift is $-\frac{c}{b} = -\frac{-\frac{\pi}{2}}{\frac{1}{2}} = \pi$

  a lowest point on an upper branch of $y = 3csc(\frac{1}{2}x - \frac{\pi}{2})$ is $(2\pi,3)$

  the lowest point on an upper branch of $y = 3csc(\frac{1}{2}x - \frac{\pi}{2})$ is

- Instead of

  $y = 3cscx \quad \rightarrow \quad (\frac{\pi}{2},3)$

  the phase shift is $\pi \quad \rightarrow \quad (\frac{\pi}{2} + \pi,\ 3) = (\frac{3\pi}{2},3)$

  the period is $4\pi \quad \rightarrow \quad (\frac{3\pi}{2} + 4\pi n,\ 3)$ X

- Refer to *5.5 Trigonometric Graphs — (4)*

Period → Phase Shift: 先伸缩，再位移

- Stretch or compress the graph at first
- Then move the graph

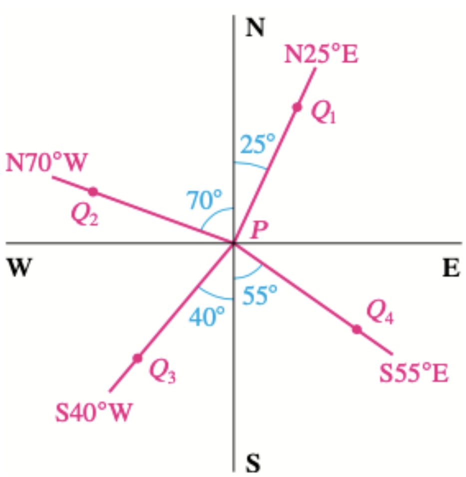
- $-\frac{c}{b}$ is the phase shift of $y = asinbx$ instead of $y = asinx$

5.7 Applied Problems

1. When working with triangles, we usually round off.
   - This is because in most applications the lengths of sides of triangles and measures of angles are found by mechanical devices and hence are only approximations to the exact values.
   - Consequently, it is assumed to have been rounded off to the nearest tenth.
   - We cannot expect more accuracy in the calculated values for the remaining sides.
2. Round-Off

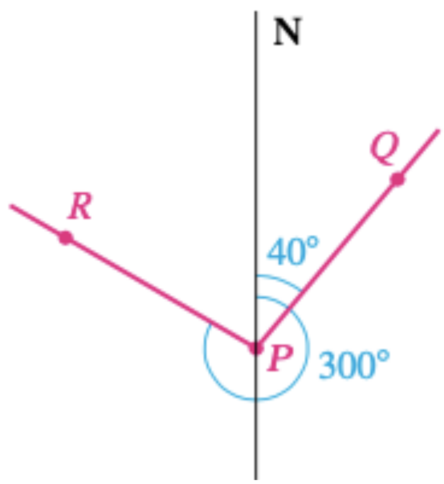
Number of significant figures for sides	Round off degree measure of angles to the nearest
2	1°
3	0.1° or 10’
4	0.01° or 1’

3. Navigation
   - Acute Angle
   - N or S → left
   - W or E → right
   - E.g. N25°E



Air Navigation

- From the north in a clockwise direction
- E.g. PR=300°

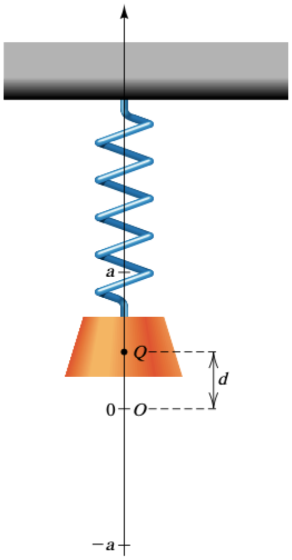


4. Definition of Simple Harmonic Motion

   $d = acos\omega t$ or $d = asin\omega t$ (ω>0)

5. Describing harmonic motion

   Supposes that the oscillation of the weight is given by $d = 10cos(\frac{\pi}{6}t)$.



$d_0 = 10cos0 = 10$

The initial position of Q is 10 cm above the origin O.

6. Angle of elevation & Angle of depression
7. Tree problems: always assume that the tree is perpendicular to the ground.
8. In most cases, draw a right angle or an xy-plane at first.

6.2 Trigonometric Equations

1. We first solve the trigonometric equation for sins, cosx and so on, and then find values of x that satisfy the equation.

2. If degree measure is not specified, then solutions of a trigonometric equation should be expressed in maidan measure (or as real numbers).

3. $sin\theta tan\theta = sin\theta$
   $sin\theta tan\theta - sin\theta = 0$
   $sin\theta(tan\theta - 1) = 0$

   - $sin\theta = 0 \quad \rightarrow \quad \theta = \pi n$

   - $tan\theta - 1 = 0$
     $tan\theta = 1$
     $\theta = \dfrac{\pi}{4} + \pi n$

   $\therefore \theta = \dfrac{\pi}{4} + \pi n \ \& \ \pi n$

   - It is incorrect to begin by dividing both sides by $sin\theta$,
     since we will lose the solutions of $sin\theta = 0$

4. $cosx = -\dfrac{\pi}{3}$

   $cosx < 1 \quad \rightarrow$ no solution

5. $csc\theta sin\theta = 1$
   $\dfrac{1}{sin\theta} sin\theta = 1$
   $\theta \neq \pi n$
   $\theta \in$ R − {πn} (n is any integer)

- Watch out the domain, when the equation involves cscx, secx, tanx, cotx

6. $sinx - cosx = 0$
   $sinx = cosx$
   $x = \dfrac{\pi}{4} + \pi n$

7. $cos(4x - \frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

- $4x - \frac{\pi}{4} = \frac{\pi}{4} + 2\pi n$

  $4x = \frac{\pi}{2} + 2\pi n$

  $x = \frac{\pi}{8} + \frac{\pi}{2}n$

- $4x - \frac{\pi}{4} = \frac{7\pi}{4} + 2\pi n$

  $4x = 2\pi + 2\pi n = 2\pi n$

  $x = \frac{\pi}{2}n$

$\therefore \theta = \frac{\pi}{8} + \frac{\pi}{2}n \ \& \ \frac{\pi}{2}n$

8. $cosx + 1 = 2sin^2x$

   $cosx + 1 = 2 - 2cos^2x$

   $2cos^2x + cosx - 1 = 0$

   $(2cosx - 1)(cosx + 1) = 0$

- $2cosx - 1 = 0$

  $cosx = \frac{1}{2}$

  $x = \frac{\pi}{6} + 2\pi n \ \& \ \frac{5\pi}{6} + 2\pi n$

- $cosx + 1 = 0$

  $cosx = -1$

  $x = \frac{3\pi}{2} + 2\pi n$

$\therefore x = \frac{\pi}{6} + 2\pi n \ \& \ \frac{5\pi}{6} = 2\pi n \ \& \ \frac{3\pi}{2} + 2\pi n$

9. $(cos\theta - 1)sin\theta = 0$

- $cos\theta - 1 = 0$
  $cos\theta = 1$
  $\theta = 2\pi n$

- $sin\theta = 0$
  $\theta = \pi n$

$\therefore \theta = 2\pi n \ \& \ \pi n$

$= \pi n$

10. $sin^2\theta + sin\theta - 6 = 0$
    $(sin\theta + 3)(sin\theta - 2) = 0$

- $sin\theta + 3 = 0$
  $sin\theta = -3$
  $sin\theta < 1$ $\rightarrow$ no solution

- $sin\theta - 2 = 0$
  $sin\theta = 2$
  $sin\theta < 1$ $\rightarrow$ no solution

11. $2tant - sec^2t = 0$
    $2tant - (1 + tan^2t) = 0$
    $tan^2t - 2tant + 1 = 0$
    $(tant - 1)^2 = 0$
    $tant - 1 = 0$
    $tnat = 1$
    $t = \dfrac{\pi}{4} + \pi n$

- Refer to *5.2 Trigonometric Functions of Angles — (4)*

- $1 + tan^2\theta = sec^2\theta$
- $1 + cot^2\theta = csc^2\theta$

---

12. $1 - sint = \sqrt{3}cost$

$(1 - sint)^2 = (\sqrt{3}cost)^2$

$1 - 2sint + sin^2t = 3cos^2t$

$1 - 2sint + sin^2t = 3 - 3sin^2t$

$4sin^2t - 2sint - 2 = 0$

$2sin^2t - sint - 1 = 0$

$(2sint + 1)(sint - 1) = 0$

- $2sint + 1 = 0$

  $sint = -\frac{1}{2}$

  $t = \frac{7\pi}{6} + 2\pi n \ \& \ \frac{11\pi}{6} + 2\pi n$

- $sint - 1 = 0$

  $sint = 1$

  $t = \frac{\pi}{2} + 2\pi n$

- Do not forget to Check solutions

- Check $t = \frac{7\pi}{6} + 2\pi n$

  LS: $1 - sin(\frac{7\pi}{6}) = 1 - (-\frac{1}{2}) = \frac{3}{2}$

  RS: $\sqrt{3}cos(\frac{7\pi}{6}) = \sqrt{3} * (-\frac{\sqrt{3}}{2}) = -\frac{3}{2}$

  LS ≠ RS

  $\therefore t = \frac{7\pi}{6} + 2\pi n$ is not a solution

- Check $t = \frac{11\pi}{6} + 2\pi n$

  LS: $1 - sin(\frac{11\pi}{6}) = 1 - (-\frac{1}{2}) = \frac{3}{2}$

  RS: $\sqrt{3}cos(\frac{11\pi}{6}) = \sqrt{3} * \frac{\sqrt{3}}{2} = \frac{3}{2}$

  LS = RS

  $\therefore t = \frac{11\pi}{6} + 2\pi n$ is a solution

- Check $t = \frac{\pi}{2} + 2\pi n$

  LS: $1 - sin(\frac{\pi}{2}) = 1 - 1 = 0$

  RS: $\sqrt{3}cos(\frac{\pi}{2}) = \sqrt{3} * 0 = 0$

  LS = RS

  $\therefore t = \frac{\pi}{2} + 2\pi n$ is a solution

- $t = \frac{11\pi}{6} + 2\pi n \ \& \ \frac{\pi}{2} + 2\pi n$

13. $sin^2t - 2sint + 1 = 0$

$$sint = \frac{-(-2) \pm \sqrt{(-2)^2 - 4}}{2} = 2 \pm \sqrt{3}$$

$-1 < sint < 1 \quad \rightarrow \quad sint = 2 - \sqrt{3}$

14. $cos\theta - sin\theta = 1$

- $cos\theta = 1 + sin\theta$
  $cos^2\theta = (1 - sin\theta)^2$
  $1 - sin^2\theta = 1 - 2sin\theta + sin^2\theta$
  $2sin^2\theta - 2sin\theta = 0$
  $sin^2\theta - sin\theta = 0$
  $sin\theta(sin\theta - 1) = 0$

  - $sin\theta = 0$
    $\theta = 2\pi n\ \ \&\ \ \pi + 2\pi n$

  - $sin\theta - 1 = 0$
    $sin\theta = 1$
    $\theta = \dfrac{\pi}{2} + 2\pi n$

- An alternative method
  $(cos\theta - sin\theta)^2 = 1^2$
  $cos^2\theta - 2sin\theta cos\theta + sin^2\theta = 1$
  $1 - 2sin\theta cos\theta = 1$
  $-2sin\theta cos\theta = 0$
  $sin\theta cos\theta = 0$

  - $sin\theta = 0$
    $\theta = 2\pi n\ \ \&\ \ \pi + 2\pi n$

  - $cos\theta = 0$
    $\theta = \dfrac{\pi}{2} + 2\pi n\ \ \&\ \ \dfrac{3\pi}{2} + 2\pi n$

- Do not forget to check solutions

- Check $\theta = 2\pi n$
  LS: $cos0 - sin0 = 1 - 0 = 1$
  RS: 1
  LS = RS
  $\therefore \theta = 2\pi n$ is a solution

- Check $\theta = \pi + 2\pi n$
  LS: $cos\pi - sin\pi = -1 - 0 = -1$
  RS: 1
  LS ≠ RS
  $\therefore \theta = \pi + 2\pi n$ is not a solution

- Check $\theta = \dfrac{\pi}{2} + 2\pi n$
  LS: $cos(\dfrac{\pi}{2}) - sin(\dfrac{\pi}{2}) = 0 - 1 = -1$
  RS: 1
  LS ≠ RS
  $\therefore \theta = \dfrac{\pi}{2} + 2\pi n$ is not a solution

- $\theta = 2\pi n$

6.3 The Additional and Subtraction Formulas

1. Sine and cosine is cofunctions of each other.
2. Cofunction formulas
   - $cos(\frac{\pi}{2} - u) = sinu$
   - $sin(\frac{\pi}{2} - u) = cosu$
   - $tan(\frac{\pi}{2} - u) = cotu$
   - $cot(\frac{\pi}{2} - u) = tanu$
   - $sec(\frac{\pi}{2} - u) = cscu$
   - $csc(\frac{\pi}{2} - u) = secu$
3. Addition and Subtraction Formulas for Sine
   - $sin(u + v) = sinu\ cosv\ +\ cosu\ sinv$
   - $sin(u - v) = sinu\ cosv\ -\ cosu\ sinv$
4. Addition and Subtraction Formulas for Cosine
   - $cos(u - v) = cosu\ cosv\ +\ sinu\ sinv$
   - $cos(u + v) = cosu\ cosv\ -\ sinu\ sinv$
5. Addition and Subtraction Formulas for Tangent
   - $tan(u + v) = \frac{tanu + tanv}{1 - tanu\ tanv}$
   - $tan(u - v) = \frac{tanu - tanv}{1 + tanu\ tanv}$
6. $acosBx + bsinBx = Acos(Bx - C)$
   - a>0
   - $A = \sqrt{a^2 + b^2}$
   - $tanC = \frac{b}{a}\ (-\frac{\pi}{2} < C < \frac{\pi}{2})$

7. Combining a sum involving the sine and cosine functions
   Let a and b be real numbers with a>0. Show that for every x,
   $acosBx + bsinBx = Acos(Bx - C)$,
   where $A = \sqrt{a^2 + b^2}$ and $tanC = \frac{b}{a}$ with $-\frac{\pi}{2} < C < \frac{\pi}{2}$.

   $$tanC = \frac{b}{a} \quad \rightarrow \quad b = atanC$$

   $$acosBx + bsinBx = acosBx + atanC * sinBx$$

   $$= acosBx + a * \frac{sinC}{cosC} * sinBx = acosBx * cosC + asinC * sinBx$$

   $$= \frac{a}{cosC}(cosBx * cosC + sinC * sinBx) = \frac{a}{cosC}cos(Bx - C)$$

   $$= asecC * cos(Bx - C)$$

   $$= a\sqrt{1 + tan^2C} * cos(Bx - C) = a\sqrt{1 + \frac{b^2}{a^2}} * cos(Bx - C)$$

   $$= \sqrt{a^2 * \frac{a^2 + b^2}{a^2}} * cos(Bx - C) = \sqrt{a^2 + b^2} * cos(Bx - C)$$

   $$= Acos(Bx - C)$$

8. Applications of $acosBx + bsinBx = Acos(Bx - c)$

   $$f(x) = 2cos3x - 2sin3x$$

   $$a = 2 \quad b = -2 \quad B = 3$$

   $$A = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$$

   $$tanC = \frac{b}{a} = \frac{-2}{2} = -1 \quad \rightarrow \quad C = -\frac{\pi}{4}$$

   $$f(x) = 2\sqrt{2}cos[3x - (-\frac{\pi}{4})] = 2\sqrt{2}cos(3x + \frac{\pi}{4})$$

9. $sin89°41' = cos0°19'$

10. $tan2t + tant = 1 - tant2t * tant$

    $$\frac{tan2t + tant}{1 - tan2t * tant} = 1$$

    $$tan(2t + t) = tan3t = 1$$

11. Trigonometric functions of $\theta \pm \pi$, $\pi - \theta$

- $\theta \pm \pi: \quad QI \rightarrow QIII$
  - $sin(\theta \pm \pi) = -sin\theta$
  - $cos(\theta \pm \pi) = -cos\theta$
  - $tan(\theta \pm \pi) = tan\theta$
  - $csc(\theta \pm \pi) = -csc\theta$
  - $sec(\theta \pm \pi) = -sec\theta$
  - $cot(\theta \pm \pi) = cot\theta$
- $\pi - \theta: \quad QI \rightarrow QII$
  - $sin(\pi - \theta) = sin\theta$
  - $cos(\pi - \theta) = -cos\theta$
  - $tan(\pi - \theta) = tan\theta$
  - $csc(\pi - \theta) = csc\theta$
  - $sec(\pi - \theta) = -sec\theta$
  - $cot(\pi - \theta) = cot\theta$

6.4 Multiple-Angle Formulas

1. Double-Angle Formulas

- $sin2u = 2sinu * cosu$
- $cos2u = cos^2u - sin^2u = 1 - 2sin^2u = 2cos^2u - 1$
- $tan2u = \dfrac{2tanu}{1 - tan^2u}$

2. Half-Angle Identities

- $sin^2u = \dfrac{1 - cos2u}{2}$
- $cos^2u = \dfrac{1 + cos2u}{2}$
- $tan^2u = \dfrac{1 - cos2u}{1 + cos2u}$

3. Half-Angle Formulas

- $sin\dfrac{v}{2} = \pm\sqrt{\dfrac{1 - cosv}{2}}$
- $cos\dfrac{v}{2} = \pm\sqrt{\dfrac{1 + cosv}{2}}$
- $tan\dfrac{v}{2} = -\sqrt{\dfrac{1 - cosv}{1 + cosv}}$
- $tan\dfrac{v}{2} = \dfrac{1 - cosv}{sinv}$
- $tan\dfrac{v}{2} = \dfrac{sinv}{1 + cosv}$

• The sign depends on which quadrant $\dfrac{v}{2}$ is in.

4. An alternative form for the half-angle formula for tan(v/2)

- $tan(\frac{v}{2}) = \pm \frac{1 - cosv}{sinv}$

- $tan(\frac{v}{2}) = \pm \sqrt{\frac{1 - cosv}{1 + coxv}} = \pm \sqrt{\frac{1 - cosv}{1 + cosv} * \frac{1 - cosv}{1 - cosv}} = \pm \sqrt{\frac{(1 - cosv)^2}{1 - cos^2 v}}$

$$= \pm \sqrt{\frac{(1 - cosv)^2}{sin^2 v}} = \pm \frac{1 - cosv}{sinv}$$

5. $tan\theta = 1 \quad -180^\circ < \theta < -90^\circ$

$cos\theta = -\frac{\sqrt{2}}{2}$

$-90^\circ < \frac{\theta}{2} < 45^\circ$

$$sin(\frac{\theta}{2}) = -\sqrt{\frac{1 - cos\theta}{2}} = -\sqrt{\frac{1 - (-\frac{\sqrt{2}}{2})}{2}} = -\frac{1}{2}\sqrt{2 + \sqrt{2}}$$

6. Find the solutions of the equation that is in the interval [0, 2π).

$cos2\theta - tan\theta = 1$

- A way of increasing the angle

$$cos2\theta - \frac{1 - cos2\theta}{1 + cos2\theta} = 1$$

$$cos2\theta(1 + cos2\theta) - (1 - cos2\theta) = 1 + cso2\theta$$

$$cos2\theta + cos^2 2\theta - 1 + cos2\theta = 1 + cos2\theta$$

$$cos^2 2\theta + cos2\theta - 2 = 0$$

$$(cos2\theta + 2)(cos2\theta - 1) = 0$$

- $cos2\theta + 2 = 0$

$cos2\theta = -2$

$cos2\theta \leq 1$

$cos2\theta \neq -2$

$cos2\theta + 2 \neq 0$

- $cos2\theta - 1 = 0$

$cos2\theta = 1$

$0 \leq \theta < 2\pi \quad \rightarrow \quad 0 \leq 2\theta < 4\pi$

$2\theta = 0,\ 2\pi$

$\theta = 0,\ \pi$

$$X \quad \rightarrow \quad tan\theta = \pm\sqrt{\frac{1 - cos2\theta}{1 + cos2\theta}}$$

- A way of decreasing the angle

$$1 - 2sin^2\theta - \frac{sin\theta}{cos\theta} = 1$$

$$2sin^2\theta cos\theta - sin\theta = 0$$

$$sin\theta(2sincos\theta - 1) = 0$$

$$sin\theta(sin2\theta - 1) = 0$$

- $sin\theta = 0$

  $\theta = 0,\ \pi$

- $sin2\theta - 1 = 0$

  $sin2\theta = 1$

  $0 \leq \theta < 2\pi \quad \rightarrow \quad 0 \leq 2\theta < 4\pi$

  $2\theta = \frac{\pi}{2}, \frac{5\pi}{2}$

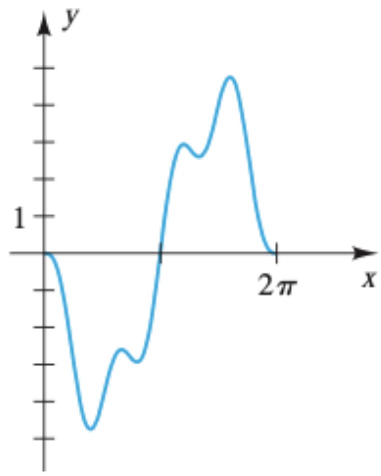
  $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

- $\theta = 0,\ \pi, \frac{\pi}{4}, \frac{5\pi}{4}$

7. $\theta = 2\beta$

$$\theta = 74^\circ \begin{cases} sin\theta = \frac{24}{25} \\ cos\theta = \frac{7}{25} \\ tan\theta = \frac{24}{7} \end{cases} \qquad \beta = 37^\circ \begin{cases} sin\beta = \frac{3}{5} \\ cos\beta = \frac{4}{5} \\ tan\beta = \frac{3}{4} \end{cases}$$

8. A graph of $y = sin4x - 4sinx$ for $0 \leq x \leq 2\pi$ is shown in the figure. Find the x-intercepts



$sin4x - 4sinx = 0$
$2sin2x * cos2x - 4sinx = 0$
$4sinx * cosx(2cos^2x - 1) - 4sinx = 0$
$sinx * cosx(2cos^2x - 1) - sinx = 0$
$sinx * 2cos^3x - sinx * cosx - sinx = 0$
$sinx(2cos^3x - cosx - 1) = 0$

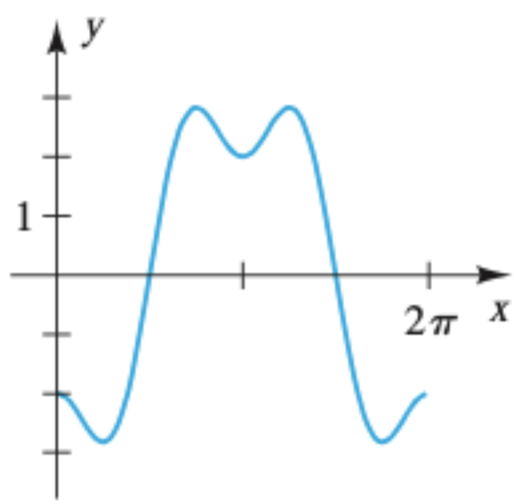
- $sinx = 0$
  $x = 0,\ \pi,\ 2\pi$
- $2cos^3x - cosx - 1 = 0$
  $(cosx - 1)(2cos^2 + 2cosx1 + 1) = 0$
  $2cos^2x + 2cosx + 1 = 2(cosx + \frac{1}{2})^2 + \frac{5}{3} \neq 0$
  $cosx - 1 = 0$
  $cosx = 1$
  $x = 0,\ 2\pi$
- $x = 0,\ \pi,\ 2\pi$

- $2x^3 - x - 1 = (x - 1)(2x^2 + 2x + 1)$

9. A graph of $y = cos3x - 3cosx$ for $0 \leq x \leq 2\pi$ is shown in the figure.
   (1) Find the x-intercepts.
   (2) The x-coordinates of the <u>seven</u> turning points on the graph are solutions of $sin3x - sinx = 0$. Find these x-coordinates>

$cos3x - 3cosx = 0$
$cos(2x + x) - 3cosx = 0$
$cos2x * cosx - sin2x * sinx - 3cosx = 0$
$(2cos^2x - 1)cosx - 2sinx * cosx * sinx - 3cosx = 0$
$2cos^3x - cosx - 2sin^2x * cosx - 3cosx = 0$
$2cos^3x - 2(1 - cos^2x)cosx - 4cosx = 0$
$2cos^3x - 2cosx + 2cos^3x - 4cosx = 0$
$4cos^3x - 6cosx = 0$
$2cosx(2cos^2x - 3) = 0$
$cosx(2cos^2x - 3) = 0$
$2cos^2x - 3 = 0$

- $cosx = 0$
  $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$
- $2cos^2\theta - 3 = 0$
  $cos^2\theta = \dfrac{3}{2}$
  $-1 \leq cos\theta \leq 1$
  $cos^2\theta \leq 1$
  $cos^2\theta \neq \dfrac{3}{2}$
  $2cos^2\theta - 3 \neq 0$



$sin3x - sinx = 0$
$sin(2x + x) - sinx = 0$
$sin2x * cosx + cos2x * sinx - sinx = 0$
$2sinx * cosx * cosx + (1 - 2sin^2x)sinx - sinx = 0$
$2sinx * cos^2x + (1 - 2sin^2x)sinx - sinx = 0$
$sinx(2cos^2x + 1 - 2sin^2x - 1) = 0$
$2sinx(cos^2x - sin^2x) = 0$
$sinx * cos2x = 0$

- $sinx = 0$
  $x = 0, \pi, 2\pi$

- $cos2x = 0$
  $0 \le x \le 2\pi \quad \rightarrow \quad 0 \le 2x \le 4\pi$
  $2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$
  $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

- $x = 0, \pi, 2\pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

- $f(0) = cos0 - 3cos0 = 1 - 3 = -2$
  $f(\pi) = cos3\pi - 3cos\pi = -1 - 3(-1) = 2$
  $f(2\pi) = cos6\pi - 3cos2\pi = 1 - 3 = -2$
  $f(\frac{\pi}{4}) = cos\frac{3\pi}{4} - 3cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{2} = -2\sqrt{2}$
  $f(\frac{3\pi}{4}) = cos\frac{9\pi}{4} - 3cos\frac{3\pi}{4} = \frac{\sqrt{2}}{2} - 3(-\frac{\sqrt{2}}{2}) = \sqrt{2}$
  $f(\frac{5\pi}{4}) = cos\frac{15\pi}{4} - 3cos\frac{5\pi}{4} = \frac{\sqrt{2}}{2} - 3(-\frac{\sqrt{2}}{2}) = \sqrt{2}$
  $f(\frac{7\pi}{4}) = cos\frac{21\pi}{4} - 3cos\frac{7\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{2} = -2\sqrt{2}$

$(0, -2)\ (\pi, 2)\ (2\pi, -2)\ (\frac{\pi}{4}, -2\sqrt{2})\ (\frac{3\pi}{4}, \sqrt{2})\ (\frac{5\pi}{4}, \sqrt{2})\ (\frac{7\pi}{4}, -2\sqrt{2})$

~~6.5 Product-to-Sum and Sum-to-Product Formulas~~

1. Product-to-Sum Formulas

- $sinu * cosv = \frac{1}{2}[sin(u+v) + sin(u-v)]$
- $cosu * sinv = \frac{1}{2}[sin(u+v) - sin(u-v)]$
- $cosu * cosv = \frac{1}{2}[cos(u+v) + cos(u-v)]$
- $sinu * sinv = \frac{1}{2}[cos(u-v) - cos(u+v)]$

2. Sum-to-Product Formulas

- $sina + sinb = 2sin\frac{a+b}{2} * cos\frac{a-b}{2}$
- $sina - sinb = 2cos\frac{a+b}{2} * sin\frac{a-b}{2}$
- $cosa + cosb = 2cos\frac{a+b}{2} * cos\frac{a-b}{2}$
- $cosa - cosb = -2sin\frac{a+b}{2} * sin\frac{a-b}{2}$

6.6 The Inverse Trigonometric Functions

1. Relationships Between f^{-1} and f

   - $y = f^{-1}(x)$ iff $x = f(y)$,
     where x is in the domain of f^{-1} and y is in the domain of f
   - domain of f^{-1} = range of f
   - range of f^{-1} = domain of f
   - $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}
   - $f^{-1}(f(y)) = y$ for every y in the domain of f
   - The point (a, b) is on the graph of f iff the point (b, a) is on the graph f^{-1}
   - The graph of f^{-1} and f are reflections of each other through the line $y = x$

2. The sine and cosine functions are not one-to-one.

3. Definition of the Inverse Sine Function
   $y = sin^{-1}x$ iff $x = siny$
   for $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

4. $(sinx)^{-1} = \frac{1}{sinx} = cscx \quad \neq sin^{-1}x$

5. $y = sin^{-1}x$

   - y is the inverse sine of x
   - y is the angle whose sine is x

6. It is essential to choose the value y in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$ of sin^{-1}.

   Thus, even though $sin\frac{5\pi}{6} = \frac{1}{2}$,
   the number $y = \frac{5\pi}{6}$ is not the inverse function value $sin^{-1}\frac{1}{2}$

7. Properties of sin^{-1}

- $sin(sin^{-1}x) = sin(arcsin x) = x$ if $-1 \le x \le 1$
- $sin^{-1}(sin y) = arcsin(sin y) = y$ if $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

8. Determine a convenient subset of the domain in order to obtain a one-to-one function.

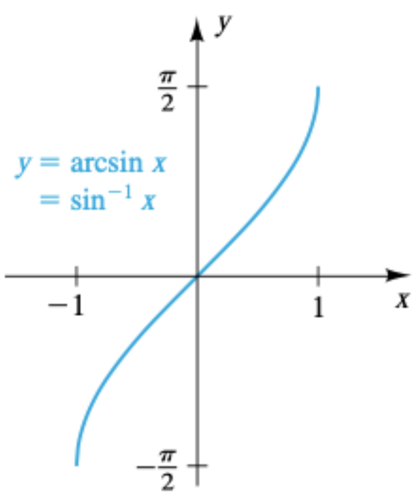
9. Definition of the Inverse Cosine Function
$y = cos^{-1}x$ iff $x = cos y$
for $1 - \le x \le 1$ and $0 \le y \le \pi$

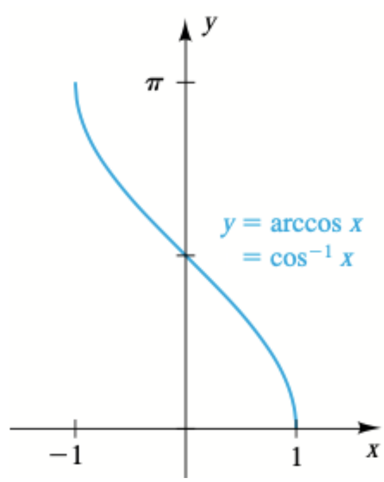
10. The range of cos^{-1} is not the same as the range of sin^{-1} but their domains are equal.

11. It is essential to choose the value y in the range $[0, \pi]$ of cos^{-1}.

12. The value of the inverse cosine function are never negative.

13. $y = sin^{-1}x = arcsin x$ $\qquad$ $y = cos^{-1}x = arccos x$





14. Properties of cos^{-1}

- $cos(cos^{-1}x) = cos(arccos x) = x$ if $-1 \le x \le 1$
- $cos^{-1}(cos y) = arccos(cos y) = y$ if $0 \le y \le \pi$

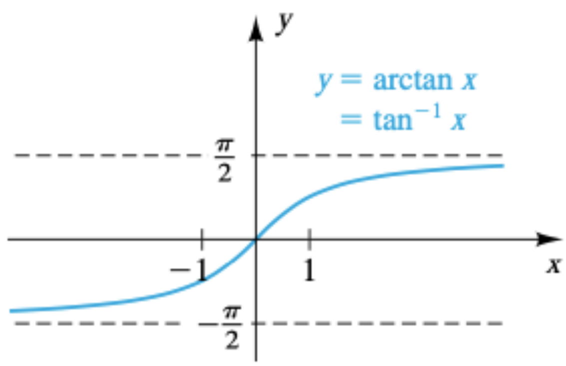
15. Definition of the Inverse Tangent Function

$y = tan^{-1}x = arctanx$ iff $x = tany$

for any real number x and for $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

16. The Domain of the arctangent function is R, and the range is the open interval $(-\dfrac{\pi}{2}, \dfrac{\pi}{2})$.

17. $y = tan^{-1}x = arctanx$



18. The two vertical asymptotes $x = \pm\dfrac{\pi}{2}$ of the tangent function correspond to the two horizontal asymptotes $y = \pm\dfrac{\pi}{2}$ of the arctangent function.

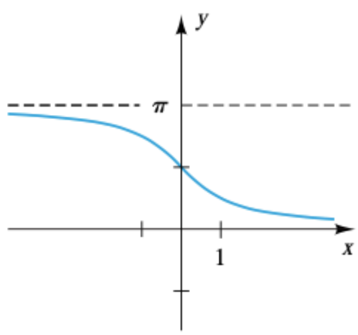
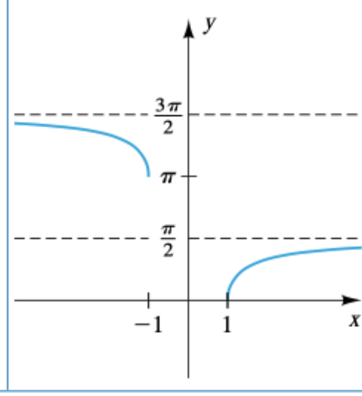
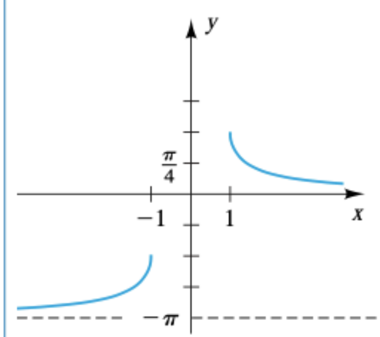
19. Properties of tan^{-1}

- $tan(tan^{-1}x) = tan(arctanx) = x$ for every x
- $tan^{-1}(tany) = arctan(tany) = y$ if $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

20. $tan[sin^{-1}(-1)] = tan(-\dfrac{\pi}{2})$ is not defined.

21. The function sec^{-1} is used calculus; however, cot^{-1} and csc^{-1} are seldom used. Because of their limited used in applications, we will not consider examples or exercises pertaining to these functions.

22. Summary of Features of cot^{-1}, sec^{-1}, $and\ csc^{-1}$

Feature	$y = \cot^{-1} x$	$y = \sec^{-1} x$	$y = \csc^{-1} x$
Domain	$\mathbb{R}$	$\lvert x \rvert \geq 1$	$\lvert x \rvert \geq 1$
Range	$(0, \pi)$	$\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$	$\left(-\pi, -\frac{\pi}{2}\right] \cup \left(0, \frac{\pi}{2}\right]$
Graph			

23. Verifying an identity involving inverse trigonometric functions

$$sin^{-1}x + cos^{-1}x = \frac{\pi}{2}, \text{ for } -1 \leq x \leq 1$$

$$\alpha = sin^{-1}x \quad \rightarrow \quad sin\alpha = x,\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \quad \rightarrow \quad cos\alpha = \sqrt{1 - x^2}$$

$$\beta = cos^{-1}x \quad \rightarrow \quad cos\beta = x,\ 0 \leq x \leq \pi \quad \rightarrow \quad sin\beta = \sqrt{1 - x^2}$$

$$-\frac{\pi}{2} \leq \alpha + \beta \leq \frac{3\pi}{2}$$

$$sin(\alpha + \beta) = sin\alpha * cos\beta + sin\beta * cos\alpha$$

$$= x * x + \sqrt{1 - x^2} * \sqrt{1 - x^2} = x^2 + 1 - x^2 = 1$$

$$\therefore \alpha + \beta = \frac{\pi}{2}$$

24. Writing the expression as an algebraic expression in x for x>0

$$cos(2tan^{-1}x)$$

$$\alpha = tan^{-1}x \quad \rightarrow \quad tan\alpha = x = \frac{a}{b} \quad \rightarrow \quad a = x,\ b = 1 \quad \rightarrow \quad c = \sqrt{x^2 + 1}$$

$$sin\alpha = \frac{x}{\sqrt{x^2 + 1}}, \quad cos\alpha = \frac{1}{\sqrt{x^2 + 1}}$$

$$cos(2tan^{-1}x) = cos2\alpha = cos^2\alpha - sin^2\alpha = \frac{1}{x^2 + 1} - \frac{x^2}{x^2 + 1} = \frac{1 - x^2}{1 + x^2}$$

25. Composite Functions of $sin x$ and $sin^{-1}x$

- $y = sin(sin^{-1}x)$: watch out the domain, may be undefined.
  - E.g. $cos^{-1}\frac{\pi}{2}$ is not defined, since $\frac{\pi}{2} \approx 1.57 > 1$
- $y = sin^{-1}(sin x)$: never be undefined, but need to convert x into the domain.
  - E.g. $sin^{-1}(sin\frac{2\pi}{3}) = \frac{\pi}{3}$; $arctan(tan\pi) = 0$

26. Summary of Properties of the Inverse Trigonometric Functions

Functions	Domain	Range
$y = sin^{-1}x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$y = cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$y = tan^{-1}x$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$y = csc^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi, -\frac{\pi}{2}] \cup [0, \frac{\pi}{2}]$
$y = sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$
$y = cot^{-1}x$	$(-\infty, \infty)$	$(0, \pi)$

27. $cos[arctan(-\frac{4}{3}) - arcsin\frac{4}{5}]$

$$\alpha = arctan(-\frac{4}{3}) \quad \rightarrow \quad tan\alpha = -\frac{4}{3} \quad \rightarrow \quad sin\alpha = -\frac{4}{5}, \quad cos\alpha = \frac{3}{5}$$

$$\beta = arcsin\frac{4}{5} \quad \rightarrow \quad sin\beta = \frac{4}{5} \quad \rightarrow \quad cos\beta = \frac{3}{5}$$

$$cos[arctan(-\frac{4}{3}) - arcsin\frac{4}{5}] = cos(\alpha - \beta)$$

$$= cos\alpha * cos\beta + sin\alpha * sin\beta = \frac{3}{5} * \frac{3}{5} + (-\frac{3}{5} * \frac{3}{5}) = 0$$

28. $tan(\frac{1}{2}cos^{-1}\frac{1}{x})$

$$\alpha = cos^{-1}\frac{1}{x} \quad \rightarrow \quad cos\alpha = \frac{1}{x}$$

$$sin\alpha = \frac{\sqrt{1-x^2}}{x}, \quad cos\alpha = \frac{1}{x}$$

$$tan(\frac{1}{2}cos^{-1}\frac{1}{x}) = tan\frac{1}{2}\alpha = \frac{1-cos\alpha}{sin\alpha} = \frac{1-\frac{1}{x}}{\frac{\sqrt{1-x^2}}{x}} = \frac{x-1}{\sqrt{x^2-1}}$$

- Refer to *6.4 Multiple-Angle Formulas — (4)*
  An alternative form for the half-angle formula for tan(v/2)
- $tan(\frac{v}{2}) = \pm\frac{1-cosv}{sinv}$
- $tan(\frac{v}{2}) = \pm\sqrt{\frac{1-cosv}{1+coxv}} = \pm\sqrt{\frac{1-cosv}{1+cosv} * \frac{1-cosv}{1-cosv}} = \pm\sqrt{\frac{(1-cosv)^2}{1-cos^2v}}$

$$= \pm\sqrt{\frac{(1-cosv)^2}{sin^2v}} = \pm\frac{1-cosv}{sinv}$$

29. $tan^{-1}(1) = \frac{\pi}{4} \neq \frac{\pi}{2}$

7.1 The Law of Sines

1. The Law of Sines: $\dfrac{sin\alpha}{a} = \dfrac{sin\beta}{b} = \dfrac{sin\gamma}{c}$

   - $\dfrac{sin\alpha}{a} = \dfrac{sin\beta}{b}$
   - $\dfrac{sin\alpha}{a} = \dfrac{sin\gamma}{c}$
   - $\dfrac{sin\beta}{b} = \dfrac{sin\gamma}{c}$

2. Find the remaining parts of an oblique triangle by the law of sine

   - Can
     - SSA: two sides and an angle opposite one of them
     - AAS or ASA: two angles and any side
   - Can't
     - SAS: two sides and the triangle between them
     - SSS: three sides

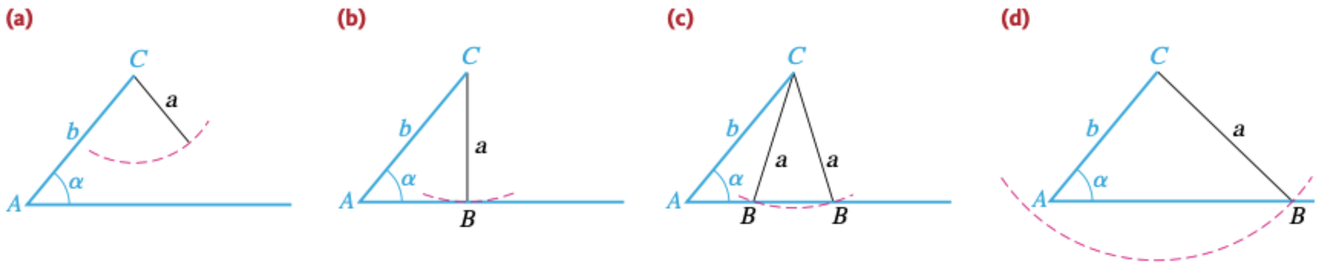
3. If known sides or angles are stated to a certain accuracy, then unknown sides or angles should be calculated to the same accuracy.

4. SSA: $a + b + \alpha$

   - There are always two βs: β_1 and β_2
     - β_1 is the one you get from your calculator by the inverse sine function
     - β_2 is the reference angle for β_1 in the second quadrant
   - Check the sum of these two angles: $\alpha + \beta$
     - $\alpha + \beta > 180^\circ$ $\rightarrow$ no triangle exists
     - $\alpha + \beta < 180^\circ$ $\rightarrow \surd$

5. The four possibilities of SSA

- $sin\beta > 1$ $\rightarrow$ no triangle exists $\rightarrow$ (a)
- $sin\beta = 1$ $\rightarrow$ a right triangle $\rightarrow$ (b)
- $sin\beta < 1$ $\rightarrow$ two possible triangles $\rightarrow$ (c) or (d)



6. $B = 40^{\circ}$ $\quad b = 4$ $\quad c = 3$

$$\frac{sinB}{b} = \frac{sinC}{c} \rightarrow sinC = \frac{c * sinB}{b} = \frac{3sin40^{\circ}}{4}$$

$$C_1 = sin^{-1}(\frac{3sin40^{\circ}}{4}) \quad C_2 = \pi - C_1 = 180^{\circ} - sin^{-1}(\frac{3sin40^{\circ}}{4})$$

$$C_1 = sin^{-1}(\frac{3sinB}{4}) < B$$

$$\pi - C_2 < B \rightarrow C_2 + B > \pi \rightarrow C_2 = DNE$$

$$C = sin^{-1}(\frac{3sinB}{4})$$

$$A = \pi - B - C = 140^{\circ} - sin^{-1}(\frac{3sin40^{\circ}}{4})$$

$$\frac{sinA}{a} = \frac{sinB}{b} \rightarrow a = \frac{b * sinA}{sinB} = \frac{4sin(140^{\circ} - sin^{-1}(\frac{3sinB}{4}))}{sin40^{\circ}}$$

7.2 The Law of Cosine

1. The Law of Cosine
   - $a^2 = b^2 + c^2 - 2bc * cos\alpha$
   - $b^2 = a^2 + c^2 - 2ac * cos\beta$
   - $c^2 = a^2 + b^2 - 2ab * cos\gamma$
2. Find the remaining parts of an oblique triangle by the law of cosines
   - SAS: two sides and the angle between them
   - SSS: three sides
3. $\alpha = 90°$ $\rightarrow$ $cos\alpha = 0$ $\rightarrow$ The Pythagorean theorem: $a^2 = b^2 + c^2$
4. SAS: $a + b + \gamma$
   - the law of cosines $\rightarrow c$
   - the law of sines $\rightarrow \alpha$
     - It is best to find the angle opposite to the shortest side, since that angle is always acute.
     - In this way, we avoid the possibility of obtaining two solutions when solving a trigonometric equation involving that angle.
   - E.g. $\triangle ABC: \ a = 5.0, \ c = 8.0, \ \beta = 77°$

     $b^2 = a^2 + c^2 - 2ac * cos\beta \quad \rightarrow \quad b \approx 8.4$

     $\frac{sin\alpha}{a} = \frac{sin\beta}{b} \quad \rightarrow \quad sin\alpha = 0.5782 \quad \rightarrow \quad \alpha \approx 35°$ (since α is acute)

     $\gamma = 180° - \alpha - \beta = 68°$
5. Heron's Formula
   - $A = \sqrt{s(s-a)(s-b)(s-c)}$
   - $s = \frac{1}{2}(a + b + c)$

6. Area of a Triangle

- $A = \frac{1}{2}bc * sin\alpha$
- $A = \frac{1}{2}ac * sin\beta$
- $A = \frac{1}{2}ab * sin\gamma$

7. SSS: a+b+c

- the law of cosines → any of the three angles (α)
  - We shall always find the largest angle first (the angle opposite the longest side), since this practice will guarantee that the remaining angles are acute.
- Another angle of the triangle (β)
  - Either the law of sines or the law of cosines
  - The law of cosines (better)
    There is no ambiguous case, since we always obtain a unique angle between 0° and 180°.
- The remaining angle of triangle (γ)
  - Either $\alpha + \beta + \gamma = 180^\circ$ or the law of cosines
  - The law of cosines (better)
    A. If either α or β is incorrectly calculated, then γ will be incorrect.
    B. Check by $\alpha + \beta + \gamma = 180^\circ$
- E.g. $\triangle ABC: \ a = 90, \ b = 70, \ c = 40$

  $a^2 = b^2 + c^2 - 2bc * cos\alpha \quad \rightarrow \quad cos\alpha = -\frac{2}{7} \quad \rightarrow \quad \alpha \approx 107^\circ$

  $b^2 = a^2 + c^2 - 2ac * cos\beta \quad \rightarrow \quad cos\beta = \frac{2}{3} \quad \rightarrow \quad \beta \approx 48^\circ$

  $c^2 = a^2 + b^2 - 2ab * cos\gamma \quad \rightarrow \quad \gamma \approx 25^\circ$

  Check: $\alpha + \beta + \gamma = 180^\circ$

10.1 Parabolas

1. Properties of Parabolas

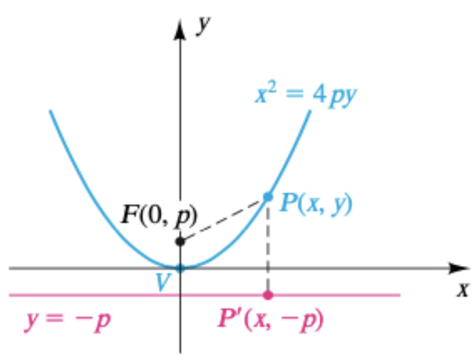
	y-axis	x-axis	y-axis	x-axis
Standard Form	$4py = x^2$	$4px = y^2$	$4p(y-k) = (x-h)^2$	$4p(x-h) = (y-k)^2$
General Form	$y = ax^2$	$x = ay^2$	$y = ax^2 + bx + c$	$x = ay^2 + by + c$
Vertex	$V(0,0)$		$V(h,k)$	
Focus	$F(0,p)$	$F(p,0)$	$F(h, k+p)$	$F(h+p, k)$
Directrix	$l: \ y = -p$	$l: \ x = -p$	$l: \ y = k-p$	$l: \ x = h-p$

- $a = \dfrac{1}{4p}$ or $p = \dfrac{1}{4a}$
- l is a distance $|p|$ from V

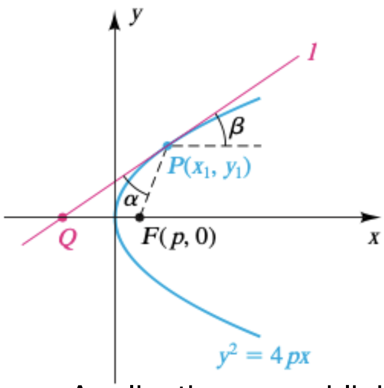
2. $x^2 = 4py \quad \rightarrow \quad (x-h)^2 = 4p(y-k)$
   - Shifting it h units to the right
   - Shifting it k units up

3. 所有复杂操作只发生在同一边（中心轴）。

4. $d(P, F) = d(P, P')$



5. A Reflective Property of Parabolas: $\alpha = \beta$



- Application: searchlight mirror, telescope mirror

10.2 Ellipses

1. Properties of Ellipses

	x-axis	y-axis	x-axis	y-axis
Standard Form	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$
General Form	$Ax^2+By^2+C=0$		$Ax^2+By^2+Cx+Dy+E=0$	
Center	$C(0,\ 0)$		$C(h,\ k)$	
Foci	$F(\pm c,\ 0)$	$F(0,\ \pm c)$	$F(h\pm c,\ k)$	$F(h,\ k\pm c)$
Vertices	$V(\pm a,\ 0)$	$V(0,\ \pm a)$	$V(h\pm a,\ k)$	$V(h,\ k\pm a)$
Minor Axis	$M(0,\ \pm b)$	$M(\pm b,\ 0)$	$M(h,\ k\pm b)$	$M(h\pm b,\ k)$
Eccentricity	$e=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}\quad(0<e<1)$			

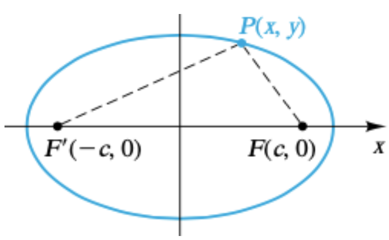
- $c^2=a^2-b^2\quad(a>b>0)$

2. Special Scenarios

   - $c=0\quad\rightarrow\quad b^2=a^2\quad\rightarrow$ a circle
   - $c=a\quad\rightarrow\quad b=0\quad\rightarrow$ a point (a degenerate conic)

3. The major axis is always longer than the minor axis, since $a>b$

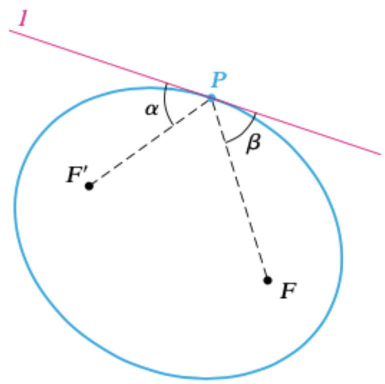
4. $d(P,\ F)+d(P,\ F')=2a$ (constant)



5. Two Scenarios

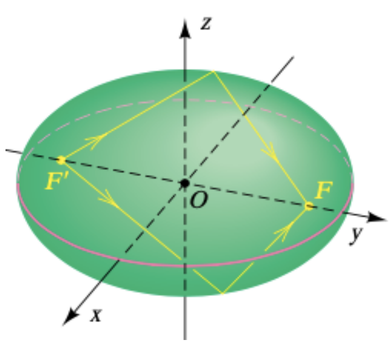
   - If b is close to 0 (c is close to a), then $\sqrt{a^2-b^2}\approx a$, $e\approx 1$, and the ellipses is very flat.
   - If b is close t a (c is close to 0), then $\sqrt{a^2-b^2}\approx 0$, $e\approx 0$, and the ellipses is almost circle.

6. Kepler's First Law
   The orbit of each planet in the solar system is an ellipses with the sun at one focus.
7. Most of these orbits are almost circle,
   so their corresponding eccentricities are close to 0.
8. Many comets have ellipses orbits with the sun at a focus.
9. The Astronomical Unit (AU): the average distance from Earth to the sun.
10. A Reflective Property of Ellipses: $\alpha = \beta$



- If a ray of light or sound emanates for one focus, it is reflected to the other focus.
- Applications: optical equipments

11. A Reflective Property of Ellipsoid



- Sounds waves or other impulses that are emitted from the focus F' will be reflected of the ellipsoid into the focus F.
- Applications: whispering galleries, lithotripter (kidney stones)

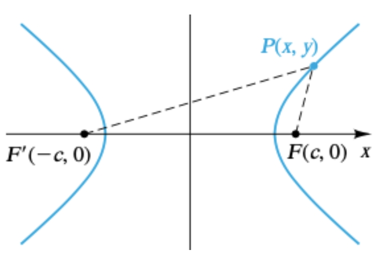
10.3 Hyperbolas

1. Properties of Hyperbolas

	x-axis	y-axis	x-axis	y-axis
Standard Form	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
General Form	$Ax^2 - By^2 + C = 0$		$Ax^2 - By^2 + Cx - Dy + E = 0$	
Center	$C(0, 0)$		$C(h, k)$	
Foci	$F(\pm c, 0)$	$F(0, \pm c)$	$F(h \pm c, k)$	$F(h, k \pm c)$
Vertices	$V(\pm a, 0)$	$V(0, \pm a)$	$V(h \pm a, k)$	$V(h, k \pm a)$
Conjugate Axis	$W(0, \pm b)$	$W(\pm b, 0)$	$W(h, k \pm b)$	$W(h \pm b, k)$
Asymptotes	$y = \pm \frac{b}{a} x$	$y = \pm \frac{a}{b} x$	$y - k = \pm \frac{b}{a}(x - h)$	$y - k = \pm \frac{a}{b}(x - h)$

- $c^2 = a^2 + b^2$

2. $|d(P, F) - d(P, F')| = 2a$ (constant)



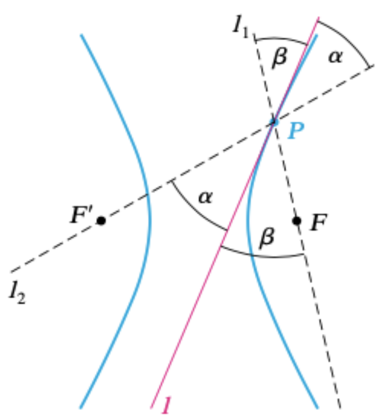
- $d(P, F) - d(P, F') < d(F', F)$
- $d(P, F') - d(P, F) < d(F', F)$

   1. The graph has no y-intercept,

      since the equation $-\frac{y^2}{b^2} = 1$ has the complex solutions $y = \pm bi$.

3. If $x^2 - a^2 < 0 \quad \rightarrow \quad -a < x < a$, then there are no points (x, y) on the graph.
   There are points P(x, y) on the graph if $x \geq a$ or $x \leq -a$.

4. Whether the vertices are on the x-axis or y-axis depends on the sign of the coefficient of x^2-term and y^2-term, instead of if a>b.
   a is always under x, and b is always under y.
   - $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad \rightarrow$ x-axis
   - $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1 \quad \rightarrow$ y-axis
   - E.g. $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$: a<b but the vertices are on the x-axis.
5. Applications of hyperbolas
   - Locating a ship
   - The navigational system LORAN (for Long Range Navigation)
6. A Reflective Property Hyperbolas: $\alpha = \beta$



- If a ray or light is directed along the line l_1 toward F, it will be reflected back at P along the line l_2 toward F'.
- Applications: telescopes of the Cassegrain type

7. Conjugate hyperbola
   $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad \& \quad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$

A Comparison Between Parabolas, Ellipses, and Hyperbolas

1. Properties

Parabolas	y-axis	x-axis	y-axis	x-axis
Standard Form	$4py = x^2$	$4px = y^2$	$4p(y-k) = (x-h)^2$	$4p(x-h) = (y-k)^2$
General Form	$y = ax^2$	$x = ay^2$	$y = ax^2 + bx + c$	$x = ay^2 + by + c$
Vertex	$V(0,\ 0)$		$V(h,\ k)$	
Focus	$F(0,\ p)$	$F(p,\ 0)$	$F(0,\ p+k)$	$F(p+h,\ 0)$
Directrix	$l:\ \ y = -p$	$l:\ \ x = -p$	$l:\ \ y = k-p$	$l:\ \ x = h-p$

Ellipses	x-axis	y-axis	x-axis	y-axis
Standard Form	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$\dfrac{y^2}{a^2} + \dfrac{x^2}{b^2} = 1$	$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$	$\dfrac{(y-k)^2}{a^2} + \dfrac{(x-h)^2}{b^2} = 1$
General Form	$Ax^2 + By^2 + C = 0$		$Ax^2 + By^2 + Cx + Dy + E = 0$	
Center	$C(0,\ 0)$		$C(h,\ k)$	
Foci	$F(\pm c,\ 0)$	$F(0,\ \pm c)$	$F(h \pm c,\ k)$	$F(h,\ k \pm c)$
Vertices	$V(\pm a,\ 0)$	$V(0,\ \pm a)$	$V(h \pm a,\ k)$	$V(h,\ k \pm a)$
Minor Axis	$M(0,\ \pm b)$	$M(\pm b,\ 0)$	$M(h,\ k \pm b)$	$M(h \pm b,\ k)$
Eccentricity	$e = \dfrac{c}{a} = \dfrac{\sqrt{a^2 - b^2}}{a} \quad (0 < e < 1)$			

Hyperbolas	x-axis	y-axis	x-axis	y-axis
Standard Form	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$	$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$	$\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$
General Form	$Ax^2 - By^2 + C = 0$		$Ax^2 - By^2 + Cx - Dy + E = 0$	
Center	$C(0,\ 0)$		$C(h,\ k)$	
Foci	$F(\pm c,\ 0)$	$F(0,\ \pm c)$	$F(h \pm c,\ k)$	$F(h,\ k \pm c)$
Vertices	$V(\pm a,\ 0)$	$V(0,\ \pm a)$	$V(h \pm a,\ k)$	$V(h,\ k \pm a)$
Conjugate Axis	$W(0,\ \pm b)$	$W(\pm b,\ 0)$	$W(h,\ k \pm b)$	$W(h \pm b,\ k)$
Asymptotes	$y = \pm \dfrac{b}{a}x$	$y = \pm \dfrac{a}{b}x$	$y - k = \pm \dfrac{b}{a}(x-h)$	$y - k = \pm \dfrac{a}{b}(x-h)$

2. $Ax^2 + By^2 + Cx + Dy + E = 0$

- $A = B \ (A \neq 0, \ B \neq 0)$ → a circle (or a point)
- $A \neq B \ (A \neq 0, \ B \neq 0)$ + the same sign → an ellipse
- Opposite sign → a hyperbola (or two intersecting straight lines)
- $A = 0$ and/or $B = 0$ → a parabola (or a pair of parallel lines)

3. $\dfrac{x^2}{a^2} \pm \dfrac{y^2}{b^2} = 1$

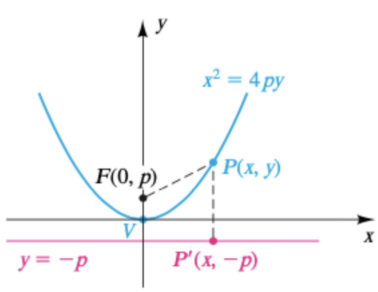
- Negative Sign: "—"
    - Whether a=b or a≠b → a hyperbola
- Positive Sign: "+"
    - a=b (a≠0, b≠0) → a circle
    - a≠b (a≠0, b≠0) → an ellipse

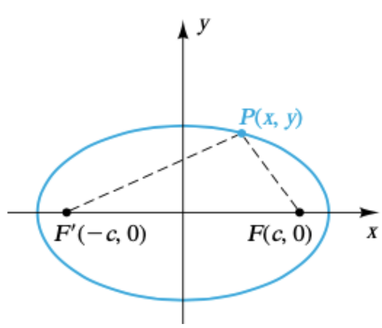
4. Ellipses & Hyperbolas

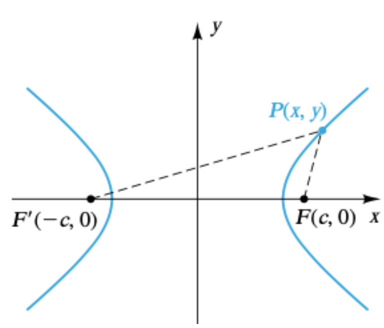
- Ellipses: $c^2 = a^2 - b^2$ & $a > c$
- Hyperbolas: $c^2 = a^2 + b^2$ & $a < c$

5. Definitions in Brief

- Parabolas: $d(P, \ F) = d(P, \ P')$
- Ellipses: $d(P, \ F) + d(P, \ F') = 2a$
- Hyperbolas: $|\, d(P, \ F) - d(P, \ F') \,| = 2a$

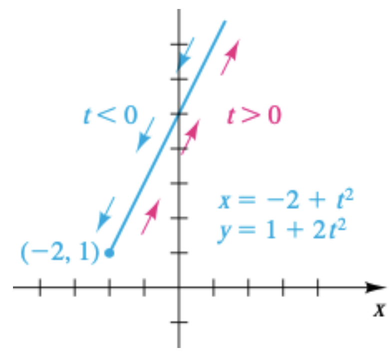






10.4 Plane Curves and Parametric Equations

1. Definition of Parametric Equations
   Let C be the curve consisting of all ordered pairs $(f(t),\ g(t))$, where f and g are defined on an interval I.
   The equation $x = f(t),\ y = g(t)$, for t in I, are parametric equations for C with parameter t.
2. Eliminate the parameter to obtain a familiar equation in x and y for C.
3. $P(x,\ y)$ may oscillate back and forth along C as t increases.
   - E.g.
     $x = -2 + t^2,\ y = 1 + 2t^2;\ \ t \in R$
     $y = 2x + 3$
     $x = -2 + t^2 \geq -2$
     $y = 1 + 2t^2 \geq 1$



     - (-2, 1) corresponds to t=0
     - As t increases in the interval $(-\infty,\ 0]$, P(x, y) moves down the curve toward the point (-2, 1).
     - As t increases in $[0,\ \infty)$, P(x, y) moves up the curve away from (-2, 1).
4. A curve may have different orientations, depending on the parametrization.
5. It is useful to eliminate the parameter before plotting points.
6. If C is a circle or an ellipse
   - The direction may be clockwise or counterclockwise
   - It may start from any points instead of the right x-intercept (r, 0)

7. $x = acost, y = asint;\ t \in R$

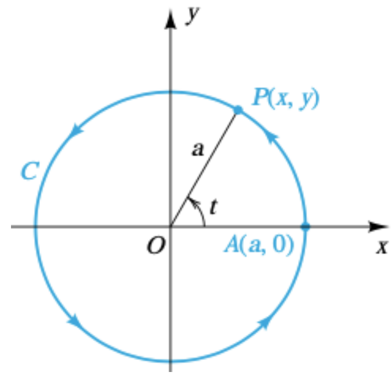
$x = acost \quad \rightarrow \quad cost = \dfrac{x}{a}$

$y = asint \quad \rightarrow \quad sint = \dfrac{y}{a}$

$sin^2t + cos^2t = 1$

$\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} = 1$

$x^2 + y^2 = a^2$



- In this case, we may interpret t geometrically as the radian measure of the angle generated by the line OP.

8. $y = lnx$

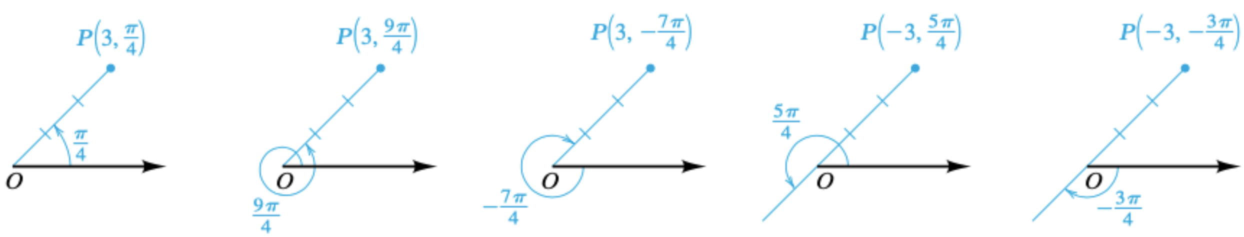
- Find three parametrizations that give the same graph as the given equation.
  - $x = t, y = lnt;\ t > 0$ — Normal Substitution: original functions
  - $x = e^t, y = t;\ t \in R$ — Inverse Functions
  - $x = t^2, y = 2lnt;\ t > 0$ — +, —, x, ÷, exponential functions, logarithmic functions
- Find three parametrizations that give only a portion of the graph of the given equation.

- Trigonometric Functions
- $x = sint, y = lnsint;\ 0 < t \leq \pi$
- $x = cost, y = lncost;\ -\dfrac{\pi}{2} < t \leq \dfrac{\pi}{2}$
- $x = tant, y = lntant;\ 0 < t \leq \dfrac{\pi}{2}$

10.5 Polar Coordinates

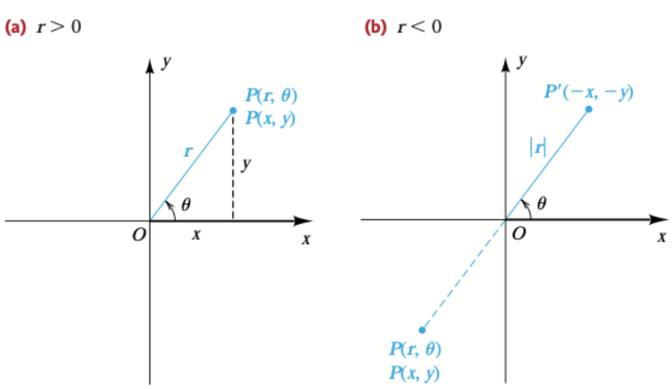
1. The polar coordinates of a point are not unique.

- E.g. $(3, \dfrac{\pi}{4})$
  - $r > 0$: $(3, \dfrac{\pi}{4}) = (3, \dfrac{9\pi}{4}) = (3, -\dfrac{7\pi}{4})$
  - $r < 0$: $(3, \dfrac{\pi}{4}) = (-3, \dfrac{5\pi}{4}) = (-3, -\dfrac{3\pi}{4})$
    - If $r < 0$, instead of measuring $|r|$ units along the terminal side of the angle θ, we measure along the half-line with endpoint O that has direction opposite that of the terminal side.
    - $(r, \theta) = (-r, \theta \pm \pi)$



2. Relationships Between Rectangular and Polar Coordinates

- $P = (x, y) = (r, \theta)$
- $x = r\cos\theta, \quad y = r\sin\theta$
- $r^2 = x^2 + y^2, \quad \tan\theta = \dfrac{y}{x}$ (if $x \neq 0$)



3. The simplest polar equations are $r = a$ and $\theta = a \ \ (a \neq 0)$

- $r = a \ \ \rightarrow \ \ (a, \, \theta)$ is a circle of radius $|a|$ with center at the pole.

- $\theta = a \ \ \rightarrow \ \ (r, \, a)$ is a line thought the origin.

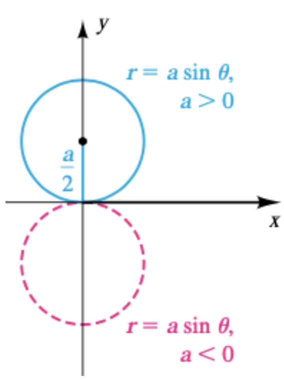
4. Find a polar equation of an arbitrary line.

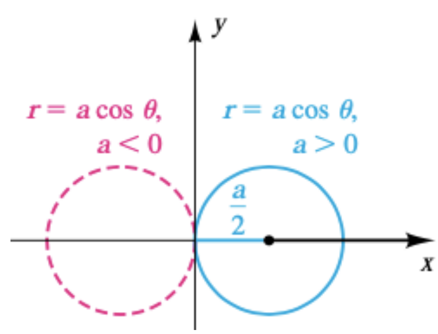
$$ax + by = c$$
$$arcos\theta + brsin\theta = c$$
$$r(acos\theta + bsin\theta) = c$$
$$r = \frac{c}{acos\theta + bsin\theta} \ \ (acos\theta + bsin\theta \neq 0)$$

5. The graphs of $r = asin\theta \quad \& \quad r = acos\theta$

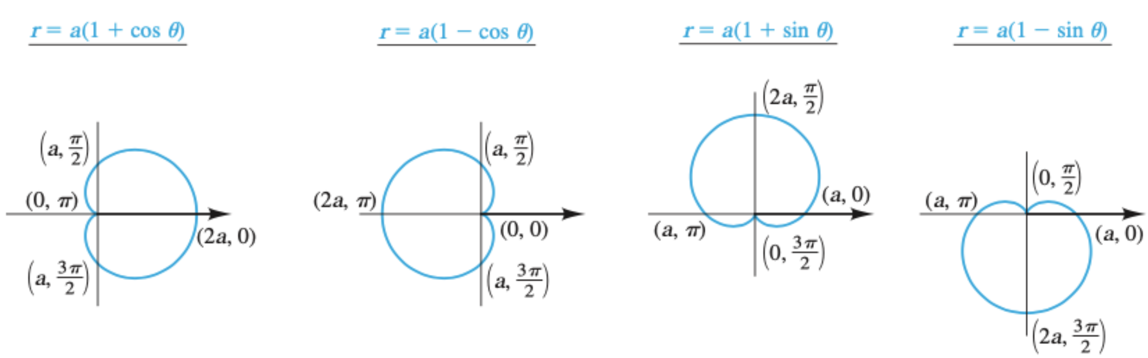
- $r = asin\theta \ (a \neq 0)$ is a circle with center $(0, \, \dfrac{a}{2})$ and radius $\dfrac{|a|}{2}$.

- $r = acos\theta \ (a \neq 0)$ is a circle with center $(\dfrac{a}{2}, \, 0)$ and radius $\dfrac{|a|}{2}$.





6. Cardioids: $r = a(1 \pm sin\theta) \quad \& \quad r = a(1 \pm cos\theta)$



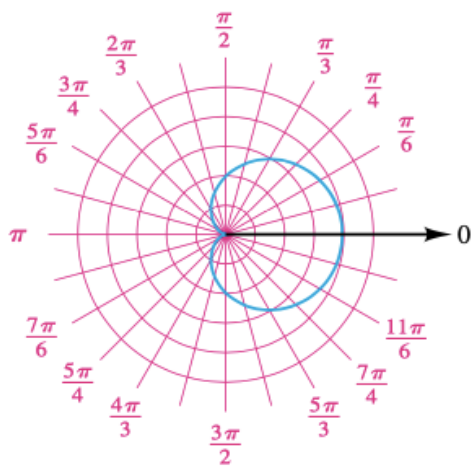
7. Sketching the graph of a polar equation

- As an aid to plotting points in the $r\theta$-plane, we have extended the polar axis in the negative direction and introduced a vertical line through the polar (this line is the graph of the equation $\theta = \frac{\pi}{2}$).

- Additional points obtained by letting θ vary from π to 2π lie on the same circle.

- If we let θ increase through all real numbers, we obtain the same points again and again because of the periodicity of the sine/cosine function.

- Polar coordinate graph paper: it displays lines through O at various angles and concentric circles with centers at the pole.



- The θ-interval [0, 2π] (or [-π, π]) is usually sufficient to graph polar equations.
  - The pole value: r=0

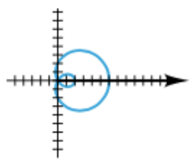
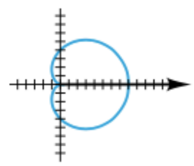
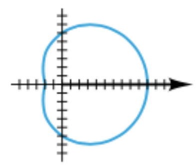
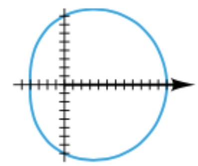
8. Test for Symmetry

- The polar axis: $\theta \to -\theta$
- The vertical line $\theta = \frac{\pi}{2}$:
  - $\theta \to \pi - \theta$
  - $r \to -r$ and $\theta \to -\theta$
- The pole: either $\theta \to \pi + \theta$ or $r \to -r$

9. Limaçons: $r = a + b\cos\theta$ & $r = a + b\sin\theta$

- $|a| = |b|$ → limaçons = cardioids

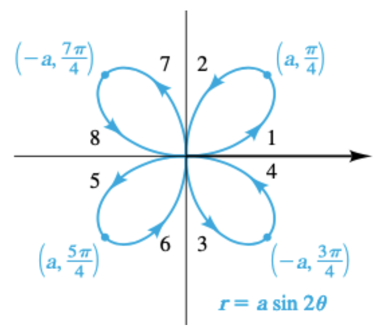
Limaçons $a \pm b\cos\theta$, $a \pm b\sin\theta$ ($a > 0$, $b > 0$)

Name	Limaçon with an inner loop	Cardioid	Limaçon with a dimple	Convex limaçon
Condition	$\frac{a}{b} < 1$	$\frac{a}{b} = 1$	$1 < \frac{a}{b} < 2$	$\frac{a}{b} \geq 2$
Specific graph				
Specific equation	$r = 2 + 4\cos\theta$	$r = 4 + 4\cos\theta$	$r = 6 + 4\cos\theta$	$r = 8 + 4\cos\theta$

10. $r = a\sin n\theta$ & $r = a\cos n\theta$

n>1: n is any positive integer greater than 1; a≠0

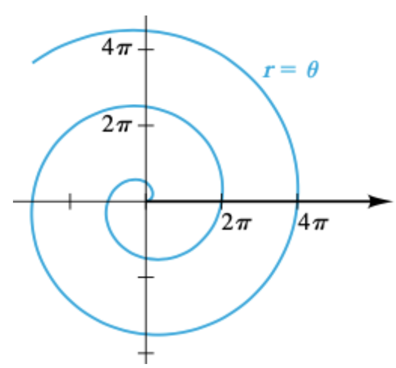
The graph consists of a number of loops through the origin.



- n=even → 2n loops
- n=odd → n loops
  - E.g. refer to *(5)*

    The graphs of $r = a\sin\theta$ & $r = a\cos\theta$

11. Spirals of Archimedes: $r = a\theta$ (a≠0)



- E.g. $r = \theta$, for $\theta > 0$: counterclockwise
  - The graph consists of all points that have polar coordinates of the form (c, c) for every real number c≥0.
  - If $\theta < 0$: clockwise

    as θ decreases through negative values, the resulting spiral winds around (缠绕) the origin and is the symmetric image, with respect to the vertical axis, of the curve of $r = \theta$ ($\theta > 0$).

12. Unlike the graph of an equation in x and y, the graph of a polar equation $r = f(\theta)$ can be symmetric with respect to the polar axis, the line $\theta = \frac{\pi}{2}$, or the pole without satisfying one of the preceding tests for symmetry. This is true because of the many different ways of specifying a point in polar coordinates.

13. The points of intersection of two graphs cannot always be found by solving the polar equations simultaneously. Thus, in searching for points of intersection of polar graphs, it is sometimes necessary to refer to the graphs themselves.

14. $(2, -2), r > 0, 0 \le \theta \le 2\pi$

    $r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$

    $tan\theta = \frac{y}{x} = -1, \; cos\theta = \frac{2}{r}, \; sin\theta = \frac{-2}{r}$

    θ in QIV $\rightarrow \theta = \frac{7\pi}{4} \neq \frac{3\pi}{4}$

15. $x^2 = 5y$

    $(rcos\theta)^2 = 5rsin\theta$

    $r = 5tan\theta sec\theta$

    - $r = 0$ (the pole) is included in the graph.

16. $r - 6sin\theta = 0$

    $r^2 - 6rsin\theta = 0$

    $x^2 + y^2 - 6y = 0$

    $x^2 + (y - 3)^2 = 9$

    Instead of

    $r = 6sin\theta$

    $r^2 = 36sin^2\theta$

17. $\theta = \frac{\pi}{4}$

    $tan\theta = tan\frac{\pi}{4} = 1 = \frac{y}{x}$

    $y = x$