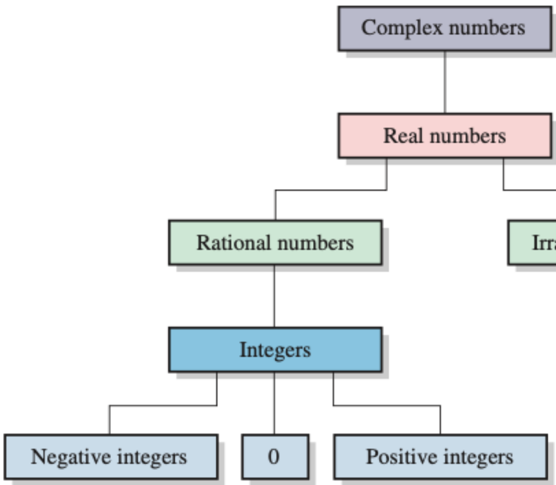
MATH 110: College Algebra

1.1 Real Numbers

1. Numbers



2. The real numbers are not closed to relative to division,
   because division by zero is not defined.
3. The negative of a real number a can be positive.
   e.g. $a=-1$, then $-a=-(-1)=1>0$
4. Trichotomy Law
   If a and b are real numbers, then exactly one of the following is true:
   $a=b$, $a>b$, $a<b$
5. $a<b<c$: b is between a and c
6. P iff Q: P if and only if Q
7. rounding off = 四舍五入; significant figures = 有效数字

1.2 Exponents and Radicals

1. $\sqrt[4]{-16}$ is not a real number

2. $\sqrt{16} \neq \pm 4$, because roots of positive real numbers are positive.

3. $\sqrt[n]{a^n} = a$, if a<0 and n is odd;
   $\sqrt[n]{a^n} = |a|$, if a<0 and n is even.

4. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

5. $\sqrt{a^2 + b^2} \neq a + b$

6. $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

7. $\sqrt{x^2 y} = |x|\sqrt{y}$

8. $\sqrt{x^6} = |x^3|$

9. $\sqrt[5]{-64} = \sqrt[5]{-2^6} = \sqrt[5]{(2)(-2)^5} = -2\sqrt[5]{2}$

10. $(4+x)^{3/2} = \sqrt{(4+x)^3} = (4+x)\sqrt{(4+x)}$

11. We shall assume that all letters — a, b, c, d, x, y and so on — that appear in radicand represent positive real numbers.

12. $a^{1/k} \neq \dfrac{1}{a^k}$

13. $y = x^{-\frac{a}{b}}$

- The Right way: multiplication

  LS: $x^{-\frac{a}{b} * -\frac{b}{a}} = (x^{-\frac{a}{b}})^{-\frac{b}{a}} = x$

  RS: $y^{-\frac{b}{a}}$

  $x = y^{-\frac{b}{a}}$

- The wrong way: addition

  LS: $x^{-\frac{a}{b} + \frac{b-a}{b}} = x^{-\frac{a}{b}} * x^{\frac{b-a}{b}} = x$

  RS: $y * x^{\frac{b-a}{b}}$

- If a is even, then $x = \pm y^{-\frac{b}{a}}$

1.3 Algebraic Expressions

1. R = the set of real numbers;
   Z = the set of integers.

2. If an algebraic expression contains divisions or roots involving a variable x, then it is not a polynomial in x.
   e.g. $\dfrac{1}{x} + 3x$; $\dfrac{x-5}{x^2+2}$; $3x^2 + \sqrt{x} - 2$

3. $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
   $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

4. Factoring is the process of expressing a sum of terms as a product.

5. Every polynomial ax+b of degree 1 is irreducible.

6. To factor a polynomial means to express it as a product of irreducible polynomial.

7. $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$
   $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

8. $4ac + 2bc - 2ad - bd = (4ac + 2bc) + (2ad + bd)$
   $= 2c(2a+b) + d(2a+b) = (2c+d)(2a+b)$

9. $3x^3 + 2x^2 - 12x - 8 = (3x^3 + 2x^2) - (12x + 8) = x^2(3x+2) - 4(3x+2)$
   $= (x^2 - 4)(3x+2) = (x+2)(x-2)(3x+2)$

10. $x^2 + 4x + 5$ is irreducible.

11. $$\frac{9x^2-4}{3x^2-5x+2}\,\frac{9x^4-6x^3+4x^2}{27x^4+8x} = \frac{(3x+2)(3x-2)}{(3x-2)(x-1)}\,\frac{x^2(9x^2-6x+4)}{x(27x^3+8)}$$
    $$= \frac{3x+2}{x-1}\,\frac{x(9x^2-6x+4)}{27x^3+8} = \frac{3x+2}{x-1}\,\frac{x(9x^2-6x+4)}{(3x+2)(9x^2-6x+40)} = \frac{x}{x-1}$$

12. $$\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} = \frac{(\sqrt{x}-\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}{h\sqrt{x(x+h)}(\sqrt{x}+\sqrt{x+h}} = \frac{x-(x+h)}{hx\sqrt{x+h}+h\sqrt{x}(x+h)}$$
    $$= \frac{-h}{hx\sqrt{x+h}+h(x+h)\sqrt{x}} = \frac{-1}{x\sqrt{x+h}+\sqrt{x}(x+h)} = \frac{-1}{\sqrt{x(x+h)}(\sqrt{x}+\sqrt{x+h})}$$

13. $(x^2+9)^4(-\frac{1}{3})(x+6)^{-\frac{4}{3}} + (x+6)^{-\frac{1}{3}}(4)(x^2+9)^3(2x)$

$= [(x+9^3(x+6)^{-\frac{4}{3}}][(-\frac{1}{3})(x^2+9)] + [(x+6)^{-\frac{4}{3}}(x^2+9)^3][8x(x+6)]$

$= [(x^2+9)^3(x+6)^{-\frac{4}{3}}][(-\frac{1}{3}x^2-3) + (8x^2+48x)]$

$= [(x^2+9)^3(x+6)^{-\frac{4}{3}}](\frac{23}{3}x^3+48x-3)$

$= \dfrac{(x^2+9)^3(23x^2+144x-9)}{3(x+6)^{\frac{4}{3}}}$

- Always take the factor whose exponent is the least as the gcd

- Refer to Math 231: Calculus of Functions of One Variable I —
  4.2 What Derivatives Tell Us — Exercises (80)

$(6x^2(1-6x^2)^{-2} + 2x^3(-2(1-6x^2)^{-3})(-12x))(1-3x^2) - 6x(2x^3(1-6x^2)^{-2})$

$= (6x^2(1-6x^2)^{-2} + 48x^4(1-6x^2)^{-3})(1-3x^2) - 12x^4(1-6x^2)^{-2}$

$= (6x^2(1-6x^2)^{-3}(1-6x^2+8x^2))(1-3x^2) - 12x^4(1-6x^2)^{-2}$

$= (1-6x^2)^{-3}(6x^2(1+2x^2)(1-3x^2)) - 12x^4(1-6x^2)^{-2}$

$= (1-6x^2)^{-3}(6x^2(1+2x^2)(1-3x^2) - 12x^4(1-6x^2))$

$= (1-6x^2)^{-3}(36x^6-18x^4+6x^2)$

$= 6x^2(1-6x^2)^{-3}(6x^4-3x^2+1)$

1.4 Equations

1. DO ALWAYS check the solutions when solving an equation, because of extraneous solution.

$$
\begin{aligned}
&3+\sqrt{3x+1}=x\\
&\sqrt{3x+1}=x-3\\
&(\sqrt{3x-1})^2=(x-3)^2\\
&3x-1=x^2-6x+9\\
&x^2-9x+8=0\\
&(x-8)(x-1)=0\\
&x_1=1,\quad x_2=8
\end{aligned}
$$

Check x=1

LS: $3+\sqrt{3+1}=5$

RS: 1

$\because 5\neq 1$

$\therefore x=1$ is not a solution.

Check x=8

LS: $3+\sqrt{3x8+1}=8$

RS: 8

$\because 8=8$

$\therefore x=8$ is a solution.

2. Simple Interest Formula: I=Prt

3. Quadratic Formula: $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

4. If the quadratic formula is excused properly, it is unnecessary to check the solutions because no extraneous solutions are introduced.

5. b^2-4ac = the discriminate.

   - If $b^2-4ac>0$, then $x_1=a,\quad x_2=b\quad(a\neq b)$.
   - If $b^2-4ac=0$, then $x_1=x_2=a$.
   - If $b^2-4ac<0$, then there are no real solutions.

6. $x^{3/2}=x^{1/2}$
   $x^{3/2}-x^{1/2}=0$
   $x^{1/2}(x-1)=0$
   $x_1=0,\quad x_2=1$

   - Avoiding dividing both sides of an equation by an expression that contains variables — always factor instead.

7. $x^{4/3}=16$
   $x=\pm 16^{3/4}=\pm 8$

8. $21x^2 - 13x - 20$

$$x = \frac{-(-13) \pm \sqrt{(-13)^2 - 4 * 21 * (-20)}}{2 * 21}$$

$$x_1 = \frac{4}{3}, \quad x_2 = -\frac{5}{7}$$

$$21x^2 - 13x - 20 = (9x - \frac{4}{3})[x - (-\frac{5}{7})] = 0$$

$$(3x - 4)(7x + 5) = 0$$

$$21x^2 - 13x - 20 = (3x - 4)(7x + 5)$$

9. $2x - 3 = \sqrt{x + 6}$

$$(2x - 3)^2 = (\sqrt{x + 6})^2$$

$$x = \pm 3$$

- Since extraneous solutions may occur, it is absolutely necessary to check all solutions obtained after raising both sides of an equation to an even power.
- Such checks are unnecessary if both sides are raised to an odd power.

10. $x^6 + 7x^3 = 8$

Let $u = x^3$

$$u^2 - 7u - 8 = 0$$

$$(u + 8)(u - 1) = 0$$

$$u_1 = -8, \quad u_2 = 1$$

$$(x_1)^3 = -8, \quad (x_2)^3 = 1$$

$$x_1 = -2, \quad x_2 = 1$$

- Do not forget to to back to x after getting the solutions of u.
- Do not forge to check.

11. $3|x + 1| - 5 = -11$

$$3|x + 1| = -6$$

$$|x + 1| = -2$$

$$\because |x + 1| \geq 0$$

$\therefore$ No Solution

12. $x^{m/n} = a$

- If m is odd, then $x = a^{n/m}$
- if m is even, then$x = \pm a^{n/m}$
- If n is even, extraneous solutions may occur.
  e.g. $x^{3/2} = -8$

  $x = (-8)^{2/3} = 4$

  Check x=4

  LS: $4^{3/2} = 8$

  RS: -8

  $\because 8 \neq -8$

  $\therefore x = 4$ is not a solution
  - Reference: *1.2 Exponents and Radicals — (3)*

$\sqrt[n]{a^n} = a$, if a<0 and n is odd;

$\sqrt[n]{a^n} = |a|$, if a<0 and n is even.

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13. $\dfrac{-3}{x+4} + \dfrac{7}{x-4} = \dfrac{-5x+4}{x^2-16}$

$\dfrac{-3(x-4)}{(x+4)(x-4)} + \dfrac{7(x+4)}{(x+4)(x-4)} = \dfrac{-5x+4}{x^2-16}$

$\dfrac{-3x+12}{x^2-16} + \dfrac{7x+28}{x^2-16} = \dfrac{-5x+4}{x^2-16}$

$4x + 4 - = -5x + 4$

$9x = -36$

$x = -4$

Check $x = -4$

$x + 4 \neq 0, \quad x \neq -4$

$x - 4 \neq 0, \quad x \neq 4$

$x^2 - 16 \neq 0, \quad x \neq \pm 4$

$\therefore x = -4$ is not a solution

$\therefore$ No Solution

14. $x^{3/4} = -8$

$x = (-8)^{4/3} = 16$

Check $x = 16$

LS: $16^{3/4} = 8$

RS: -8

$\because 8 \neq -8$

$\therefore x = 16$ is not a solution

$\therefore$ No Solution

15. $x = 3 + \sqrt{5x - 9}$

$x - 3 = \sqrt{5x - 9}$

$(x - 3)^2 = (\sqrt{5x - 9})^2$

$x^2 - 6x + 9 = 5x - 9$

$x^2 - 11x + 18 = 0$

$(x - 2)(x - 9) = 0$

$x_1 = 2, \quad x_2 = 9$

Check $x = 2$

LS: 2

RS: $3 + \sqrt{5 * 2 - 9} = 4$

$\because 2 \neq 4$

$\therefore x = 2$ is not a solution

Check $x = 9$

LS: 9

RS: $3 + \sqrt{5 * 9 - 9} = 9$

$\because 9 = 9$

$\therefore x = 9$ is a solution

1.6 Inequalities

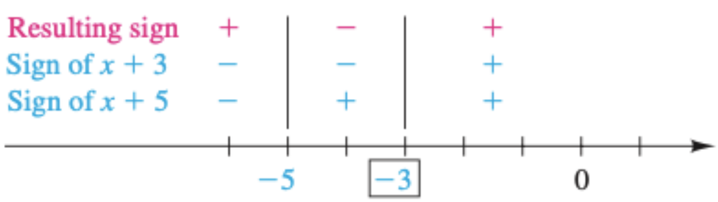
1. Multiplying or dividing both sides of an inequalities by a negative real number reverses the inequality sign.
2. If 0<a<b, then $\frac{1}{a} > \frac{1}{b}$
3. (x+1)(2x-3)<0 ≠ x+1<0 or 2x-3<0
4. $\frac{x+1}{x+3} \le 2$

   $\frac{x+1}{x+3} - 2 \le 0$

   $\frac{x+1}{x+3} - \frac{2(x+3)}{x+3} \le 0$

   $\frac{-x-5}{x+3} \le 0$

   $\frac{x+5}{x+3} \ge 0$



   $x+3 \neq 0, \quad x \neq -3$

   $\therefore (-\infty, -5] \cup (-3, \infty)$

5. 5>x≥-2 ≠ (5,-2] = [-2,5)
6. $\frac{x^2(x+2)}{(x+2)(x+1)} \le 0$

   $(-\infty, -2) \cup (-2, -1) \cup \{0\}$

7. $$\frac{x-2}{x^2-3x-10} \geq 0$$

$$\frac{x-2}{(x-5)(x+2)} \geq 0$$

$(-2,2] \cup (5,\infty)$

8. $$\frac{x+6}{x^2-7x+12} \leq 0$$

$$\frac{x+6}{(x-3)(x-4)} \leq 0$$

$x-3 \neq 0, \quad x \neq 3$

$x-4 \neq 0, \quad x \neq 4$

$(-\infty,-6] \cup (3,4)$

2.1 Rectangular Coordinate System

1. Distance Formula
   $d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

2. Midpoint Formula
   $M_{P_1P_2} = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$

2.2 Graphs of Equations

1. Standard Equation of a Circle with Centre (h, k) and Radius r
   $r^2 = (x - h)^2 + (y - k)^2$

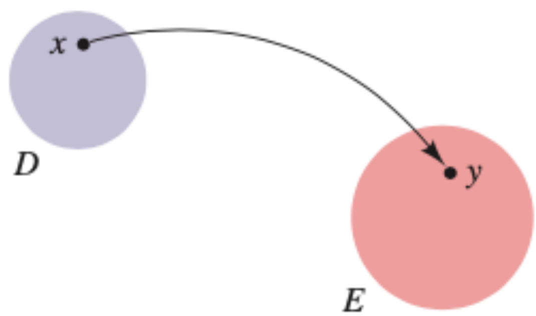
2. If r>0, then a circle with centre=(h, k) and radius=r
   if r=0, then the point (h, k)
   if r<0, then no graph.

2.3 Lines

1. Definition of Slope of a Line: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
2. If l is parallel to the y-axis, then the slope of l is not defined, because the dominator is zero.
3. In finding the slope of a line, it is immaterial which point we label as P_1 and which as P_2.
4. If the slope of a line is a/b and b is positive, then for every change of b units in the horizontal direction, the line rises or falls |a| units, depending on whether a is positive or negative, respectively.
5. Point-Slop Form for the Equation of a Line
   $y - y_1 = m(x - x_1)$
6. Slope-Intercept Form for the Equation of a Line
   $y = mx + b$
7. General Form for the Equation of a Line
   $ax + by = c$
8. Theorem on Slopes of Parallel Lines
   Two non vertical lines are parallel if and only if they have the same slope.
9. If the slopes pf two non vertical lines are not the same, then the lines are not parallel and intersect at exactly one point.
10. Theorem on Slopes of Perpendicular Lines
    $m_1 m_2 = -1$

2.4 Definition of Function

1. Definition of Function
   A function f from a set D to a set E is a correspondence that assigns to each element x of D exactly one element y of E.



2. To each x in D there corresponds exactly one y in E.
   However, the same element of E may correspond to different of D.
3. $f(a+b) \neq f(a) + f(b)$
4. The symbols used for the function and variable are immaterial.
5. Implied Domain of f
   If a function is defined by means of an expression and the domain D is not stated, then we will consider D to be the totality of real numbers x such that $f(x)$ is real.
6. The terminology f is undefined at x means that x is not the domain of f.
7. Vertical Line Test
   The graph of a set of points in a coordinate plane is the graph of a function if every vertical line intersects the graph in at most one point.
8. The graph of a function cannot be a figure such as a circle.
9. Identity Function: $f(x) = x$
10. Linear Function: $f(x) = ax + b$
11. The graph of a linear function is a line. The domain is R.

12. $y = f(x)$
    x = independent variable; y = dependent variable.

13. Alternative Definition of Function: W = {(x, $f(x)$): x is in D}
    A function with domain D is set W of ordered pairs such that, for each x in D, there is exactly one ordered pair (x,y) in W having x in the first position.

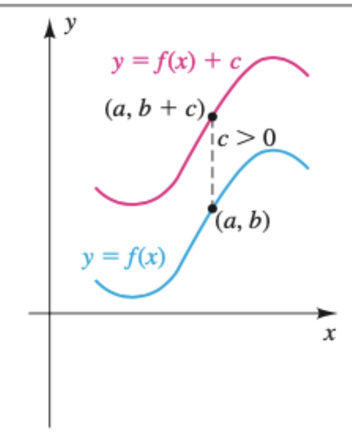
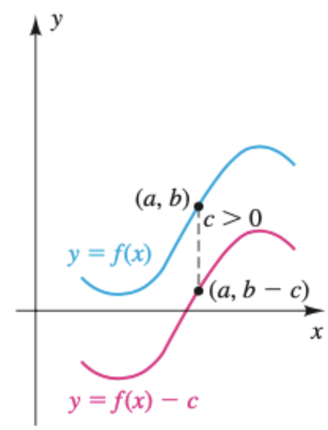
14. If W = {(x, y): $x^2 = y^2$}, then W is not a function,
    since for a given x there may be more than one pair in W with x in the first position.

15. W = {(x, y): |y| = |x|} is not a function.

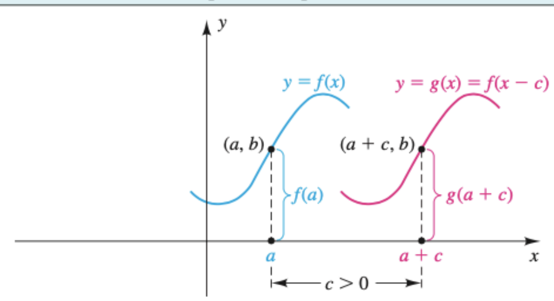
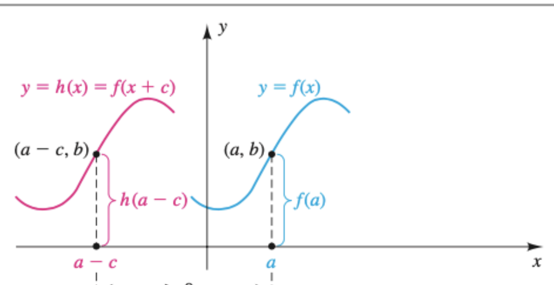
2.5 Graphs of Functions

1. Even Function: $f(-x) = f(x)$
   Odd Function: $f(-x) = -f(x)$

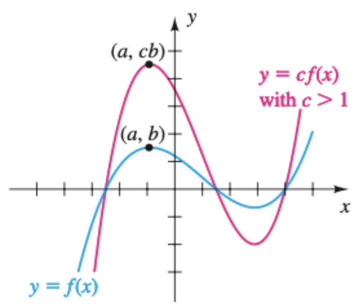
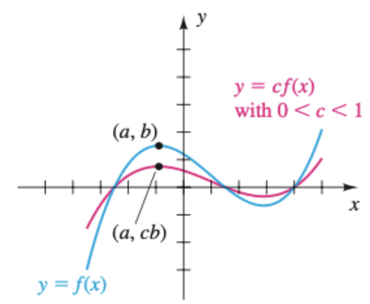
2. Vertically shifting the Graph of $y = f(x)$

Equation	$y = f(x) + c$ with $c > 0$	$y = f(x) - c$ with $c > 0$
Effect on graph	The graph of f is shifted vertically upward a distance c.	The graph of f is shifted vertically downward a distance c.
Graphical interpretation		

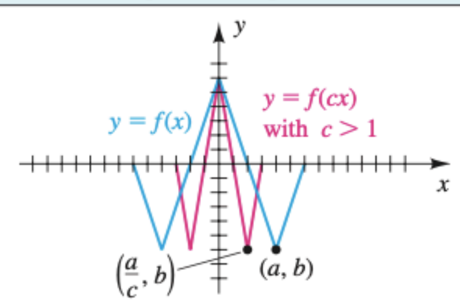
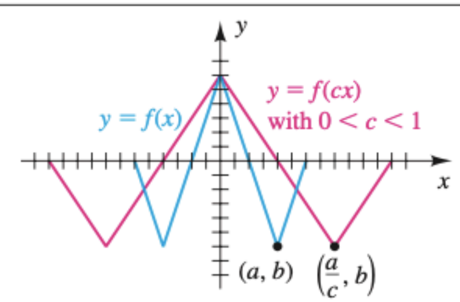
3. Horizontally shifting the Graph of $y = f(x)$: reverse order

Equation	**Effect on graph**	**Graphical interpretation**
$y = g(x)$ $= f(x - c)$ with $c > 0$	The graph of f is shifted horizontally to the *right* a distance c.	
$y = h(x)$ $= f(x + c)$ with $c > 0$	The graph of f is shifted horizontally to the *left* a distance c.	

4. Vertically Stretching or Compressing the Graph of $y = f(x)$

Equation	$y = cf(x)$ with $c > 1$	$y = cf(x)$ with $0 < c < 1$
Effect on graph	The graph of f is stretched vertically by a factor c.	The graph of f is compressed vertically by a factor $1/c$.
Graphical interpretation		

5. Horizontally Compressing or Stretching the Graph of $y = f(x)$: reverse order

Equation	**Effect on graph**	**Graphical interpretation**
$y = f(cx)$ with $c > 1$	The graph of f is compressed horizontally by a factor c.	
$y = f(cx)$ with $0 < c < 1$	The graph of f is stretched horizontally by a factor $1/c$.	

6. x反；y正。

7. Piecewise-Defined Function
   Functions are described by more than one expression.

8. When finishing sketching the graph of a piecewise-defined function, check that it passes the vertical line test.
9. It is a common misconception to think that if you move up to a higher tax bracket, all your income is taxed at the higher rate.
10. Greatest Integer Function: $f(x) = [x]$
11. $f(x) = a$ (a is a constant) is an even function.
12. $f(x) = \sqrt[3]{x^3 - x}$ is an odd function.

$$f(-x) = \sqrt[3]{(-x)^3 - (-x)} = \sqrt[3]{-x^3 + x} = \sqrt[3]{-(x^3 - x)} = -\sqrt[3]{x^3 - x} = -f(x)$$

2.6 Quadratic Functions

1. Quadratic Function: $f(x) = ax^2 + bx + c \quad (a \neq 0)$

2. Standard Equation of a Parabola with Vertical Axis
   $y = a(x-h)^2 + k \quad (a \neq 0)$

3. Vertex of a Parabola: $(-\dfrac{b}{2a}, f(-\dfrac{b}{2a}))$

4. Relationship Between Quadratic Function Forms and Their Vertex and x-intercepts

Form	Vertex (h, k)	x-intercepts (if there are any)
(1) $y = f(x) = a(x-h)^2 + k$	h and k as in the form	$x = h \pm \sqrt{-k/a}$ (see (*) on next page)
(2) $y = f(x) = a(x-x_1)(x-x_2)$	$h = \dfrac{x_1 + x_2}{2}$, $k = f(h)$	$x = x_1, x_2$
(3) $y = f(x) = ax^2 + bx + c$	$h = -\dfrac{b}{2a}$, $k = f(h)$	$x = -\dfrac{b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$ (see (*) on next page)

5. If $b^2 - 4ac < 0$, then there are no x-intercepts.

2.7 Operations on Functions

1. $(f+g)(x) = f(x) + g(x)$
   $(f-g)(x) = f(x) - g(x)$
   $(fg)(x) = f(x)g(x)$
   $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

2. Polynomial Function
   $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$

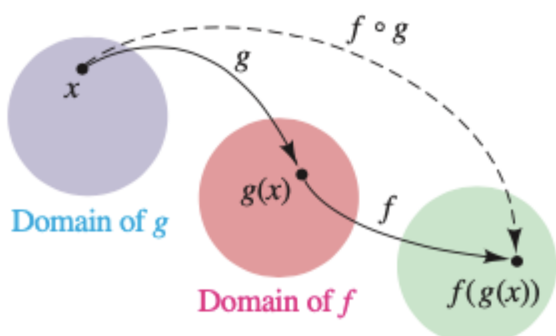
3. Quadratic functions are polynomial functions.

4. A algebraic function is a function that can be expressed in terms of finite sums, differences, products, quotients, or roots of polynomial functions.

5. Functions that are not algebraic are transcendental,
   such as the exponential and logarithmic functions.

6. Composite Function: $(fog)(x) = f(g(x))$

7. The domain of fog is the set of all x in the domain of g such that $g(x)$ is in the domain of f.



8. A number x is in the domain $(fog)(x)$ id and only if both $g(x)$ and $f(g(x))$ are defined.

9. $fog \neq gof$

10. Find composite function values from tables

x	1	2	3	4		x	1	2	3	4
$f(x)$	3	4	2	1		$g(x)$	4	1	3	2

Find $(fog)(2)$, $(gof)(2)$, $(fof)(2)$, and $(gog)(2)$

$(fog)(2) = f(g(2)) = f(1) = 3$

$(gof)(2) = g(f(2)) = g(4) = 2$

$(fof)(2) = f(f(2)) = f(4) = 1$

$(gog)(2) = g(g(2)) = g(1) = 4$

11. The composite function form is never unique.
There are unlimited number of composite function forms.

12. $f(x) = \dfrac{1}{x-7}, \quad g(x) = \sqrt{x-4}$

$x - 7 \neq 0, \quad x \neq 7$

$x - 4 \neq 0, \quad x \neq 4$

$(gof)(x) = \sqrt{\dfrac{1}{x-7} - 4}$

$\dfrac{1}{x-7} - 4 \geq 0, \quad \dfrac{29-4x}{x-7} \geq 0$

Interval	$(-\infty,7)$	$(7,\frac{29}{4})$	$(\frac{29}{4},\infty)$
Sign of -4x+29	+	+	-
Sign of x-7	-	+	+
Resulting sign	-	+	-

The domain is $(7,\frac{29}{4}]$

13. By comparison with the two examples below, we can conclude:

Domain ≠ { the composite function } U { $f(x)$ } U { $g(x)$ }

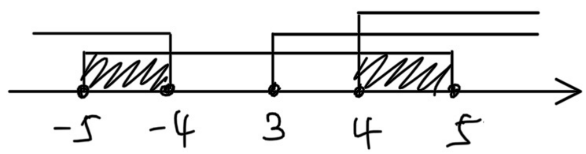
$f(x) = \sqrt{3-x}, \quad g(x) = \sqrt{x^2-16}$

$3 - x \geq 0, \quad (-\infty, 3]$

$x^2 - 16 \geq 0, \quad (-\infty, -4] \cup [4, \infty)$

- $(fog)(x) = \sqrt{3 - \sqrt{x^2-16}}$

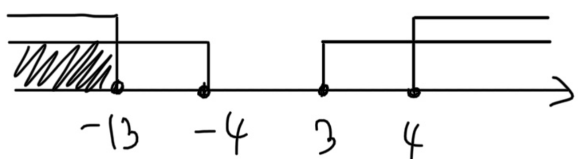
  $3 - \sqrt{x^2-16} \geq 0, \quad (-\infty, -13]$



The domain is $[-5, -4] \cup [4, 5]$

- $(gof)(x) = \sqrt{(\sqrt{3-x})^2 - 16} = \sqrt{-x-13}$

  $-x - 13 \geq 0, \quad (-\infty, -13]$



The domain is $(-\infty, -13]$

$f(x) = \dfrac{x-1}{x-2}, \quad g(x) = \dfrac{x-3}{x-4}$

$x - 2 \neq 0, \quad x \neq 2$

$x - 4 \neq 0, \quad x \neq 4$

- $(fog)(x) = \dfrac{\dfrac{x-3}{x-4} - 1}{\dfrac{x-3}{x-4} - 3} = \dfrac{1}{5-x}$

  $5 - x \neq 0, \quad x \neq 5$

  The domain is R - {4, 5}

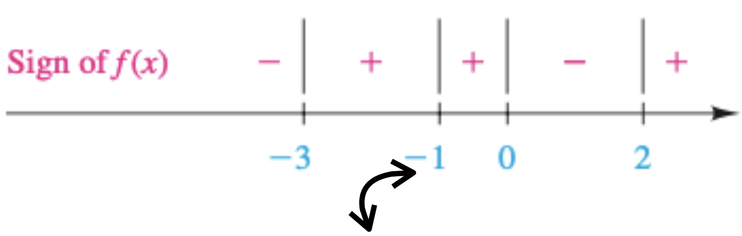
- $(gof)(x) = \dfrac{\dfrac{x-1}{x-2} - 3}{\dfrac{x-1}{x-2} - 4} = \dfrac{5-2x}{7-3x}$

  $7 - 3x \neq 0, \quad x \neq \dfrac{7}{3}$

  The domain is R - {2, $\dfrac{7}{3}$}

3.1 Polynomial Functions of Degree Greater Than 2

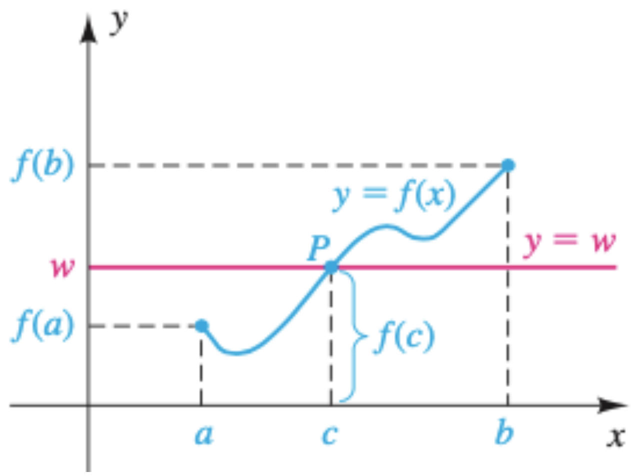
1. All polynomial functions are continuous functions — that is, their graphs can be draw without any breaks.
2. $f(x) = ax^2$
   - If n is an odd positive integer, then f is an odd function and the graph is symmetric with respect to the origin.
     - For a>0, as either n or a increase, the graph rises more rapidly for x>1.
   - If n is an even positive integer, then f is an even function and the graph is symmetric with respect to the y-axis.
     - As the exponent increases, the graph becomes falter at the origin.
     - The graph intersects the x-axis at the origin, but it does not cross the x-axis.
3. The graphs of polynomial functions always have a smooth appearance with a number of high points and low point.
4. An n-degree polynomial has at most n-1 turning points.
5. At an extremum, f changes from an increasing function to a decreasing function, or vice versa.
6. Sketch the graph of a polynomial knowing its sign.



$f(x) = (x+3)(x+1)^2(x)(x-2)$

7. Intermediate Value Theorem for Polynomial Functions
   If f is a polynomial function and $f(a) \neq f(b)$ for a<b, then f takes on even value between $f(a)$ and $f(b)$ in the interval [a, b].
8. If w is any number between $f(a)$ and $f(b)$,
   there is at least one number c between a and b such that $f(c) = w$.

9. For any number w between $f(a)$ and $f(b)$, the horizontal line y=w intersects the graph in at least one point P. The x -coordinate c of P is a number such that $f(c) = w$.



10. If $f(a)$ and $f(b)$ have opposite signs (one positive and one negative), there is at least one number c between a and b such that $f(c) = 0$; that is, f has a zero at c.
11. By using successive approximations, we can approximate each zero at any degree of accuracy by locating it in smaller and smaller intervals.
12. If c and d are successive at real zeros of $f(x)$ — that is, there are no other zeros between c and d — then $f(x)$ does not change sign on the interval (c, d). Thus if we choose any number k such that c<k<d and if $f(k)$ is positive, then $f(x)$ is positive throughout (c, d). Similarly, if $f(k)$ is negative, then $f(x)$ is negative throughout (c,d).
13. The sign of $f(x)$ does not change at x=0. Thus (c, d) instead of [c, d].

~~3.2 Properties of Division~~

1. Long Division of Polynomials

- $f(x) = x^5 - 3x^3 - x^2 + 1;\ p(x) = x^2 - 3$

$$\begin{array}{r|rrrrr} & & & x^3 & & -1 \\ \cline{2-6} x^2-3 & x^5 & +0x^4 & -3x^3 & -x^2 & +1 \\ & x^5 & & -3x^3 & & \\ \cline{2-6} & & & & -x^2 & +1 \\ & & & & -x^2 & +3 \\ \cline{5-6} & & & & & -2 \end{array}$$

$$f(x) = (x^2 - 3)(x^3 - 1) - 2$$

- Watch out the position of "x^3" and "-1":
  Especially, x^3 should be in the column of "$-3x^3$" instead of "$+0x^4$".

- $f(x) = 3x^3 - 5x^2 - 4x - 8;\ p(x) = 2x^2 + 1$

$$\begin{array}{r|rrrr} & & & \frac{3}{2}x & \\ \cline{2-5} 2x^2+1 & 3x^3 & +0x^2 & +2x & -4 \\ & 3x^3 & & +\frac{3}{2}x & \\ \cline{2-4} & & & \frac{1}{2}x & -4 \end{array}$$

$$f(x) = \frac{3}{2}x(2x^2 + 1) + (\frac{1}{2}x - 4)$$

- Watch out "$+0x^2$":
  Even if the coefficient of x^2 is 0, we should not omit it.

2. The long division process ends when we arrive at a polynomial (the remainder) that either is 0 or has smaller degree than the divisor.

3. Division Algorithm for Polynomial
   - $f(x) = p(x) * q(x) + r(x)$
   - $q(x) =$ quotient; $r(x) =$ remainder

4. Remainder Theorem
   If a polynomial $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

5. Factor Theorem
   A polynomial $f(x)$ has a factor $x - c$ iff $f(c) = 0$

6. Synthetic Division of Polynomials by $x - c$
   $x^3 - 8x - 5; \quad x + 3$

−3	1	0	−8	−5
		−3	9	−3
	1	−3	1	-8

   $x^3 - 8x - 5 = (x + 3)(x^2 - 3x + 1) - 8$

- “−3”: x+3=x−(−3)
- “0”: Even if the coefficient of x^2 is 0, we should not omit it.
- “1”: Do not forget the leading coefficient.
- “-8”: the last one is the remainder

7. Show that $x - c$ is not a factor of $f(x)$ for any real number c
   $f(x) = -x^4 - 3x^2 - 2$
   $f(c) = -c^4 - 3c^2 - 2 \leq -2$
   $f(c)$ has no zeros $\rightarrow$ $f(c) \neq 0$

8. Find all values of k such that $f(x)$ is divided by the given linear polynomial.
   $f(x) = kx^3 + x^2 + k^2x + 3k^2 + 11; \;\; x + 2$
   $f(-2) = k^2 - 8k + 15 = (k - 3)(k - 5) = 0$
   $k = 3$ or $k = 5$

9. Equivalent statements for $f(a) = b$

- The point $(a,\ b)$ is on the graph of f.
- The value of f at $x = a$ equals b: $f(a) = b$.
- If $f(x)$ is divided by $x - a$, then the remainder is b.

10. Additional equivalent statements for $f(a) = 0$

- The number a is a zero of the function f.
- The point $(a,\ 0)$ is on the graph of f: a is an x-intercept.
- The number a is a solution of the equation $f(x) = 0$
- The binomial $x - a$ is a factor of the polynomial $f(x)$.

11. Long Division & Synthetic Division

- Synthetic division is applicable only when the divisor is of the form $x - c$.
  - The power of x must be 1, but the coefficient may not: $ax - c = a(x - \frac{c}{a})$
- E.g. $f(x) = 2x^3 - 3x^2 + 4x - 5;\ \ x - 2$

Long Division

$$\begin{array}{r|rrrr} & 2x^2 & +x & +6 & \\ \hline x-2 & 2x^3 & -3x^2 & +4x & -5 \\ & 2x^3 & -4x^2 & & \\ \hline & & x^2 & +4x & \\ & & x^2 & -2x & \\ \hline & & & 6x & -5 \\ & & & 6x & -12 \\ \hline & & & & 7 \end{array}$$

Synthetic Division

2	2	−3	4	−5
		4	2	12
	2	1	6	7

3.3 Zeros of Polynomials

1. A complex number $c = a + bi$ is a zero of a polynomial $f(x)$ is if and only if $x - c$ is a factor of $f(x)$.
   E.g. Please see (9).

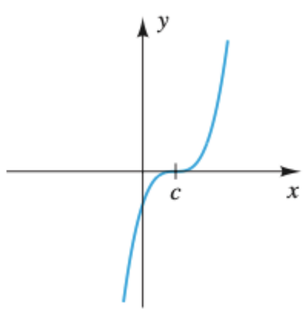
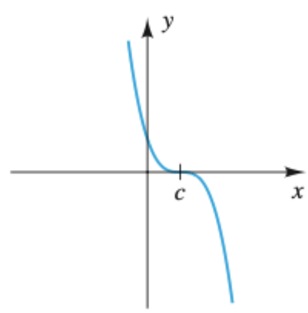
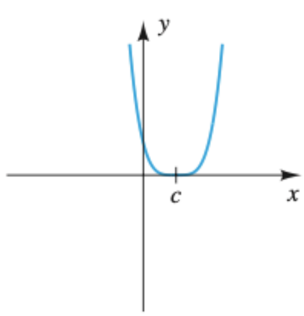
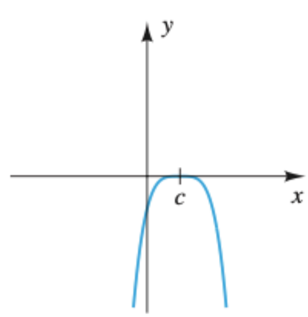
2. Fundamental Theorem of Algebra
   If a polynomial $f(x)$ has positive degree and complex coefficients, then $f(x)$ has at least one complex zero.

3. As a special case of the fundamental theorem of algebra, if all the coefficients of $f(x)$ are real, then $f(x)$ has at least one complex zero. If $a + bi$ is a complex zero, it may happen that b=0, in which case the number a is a real zero.

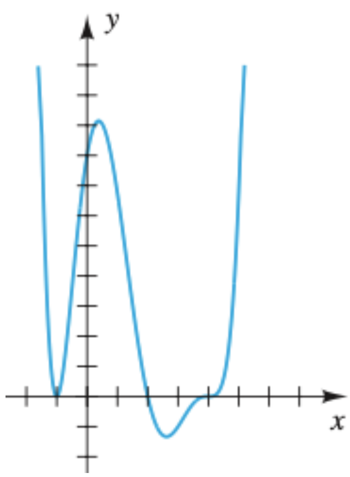
4. $f(x) = a(x - c_1)(x - c_2)\ldots(x - c_n)$

5. Theorem on the Maximum Number of Zeros of a Polynomial
   A polynomial of degree n>0 has at most n different complex zeros.

6. Finding a polynomial with prescribed zeros

Factor of $f(x)$	**General shape of the graph of f near $(c, 0)$**	
$(x - c)^m$, with m odd and $m \neq 1$		
$(x - c)^m$, with m even		

7. $f(x) = \frac{1}{16}(x-2)(x-4)^3(x+1)^2$



The graph is “flatter” at 4 than at 2.

8. Theorem on the Exact Number of Zeros of a Polynomial

   If $f(x)$ is a polynomial of degree n>0 and if a zero of multiplicity m is counted m times, then $f(x)$ has precisely n zeros.

9. Finding the zeros of a polynomial

   $f(x) = x^5 - 4x^4 + 13x^3 = x^3(x^2 - 4x + 13)$

   $$\frac{-(-4) \pm \sqrt{(-4)^2 - 4*1*13}}{2*1} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

   $f(x) = x*x*x(x-2-3i)(x-2+3i)$

   The five zeros of $f(x)$ are 0, 0, 0, $2+3i$, $2-3i$.

10. There is a variation of sign in $f(x)$ if two consecutive coefficients have opposite signs.

    E.g. Variations of Sign in $f(x) = 2x^5 - 7x^4 + 3x^2 + 6x - 5$

$$f(x) = 2x^5 \overbrace{\qquad}^{+\text{ to }-} - 7x^4 \overbrace{\qquad}^{-\text{ to }+} + 3x^2 \overbrace{\qquad\qquad}^{\text{no variation}} + 6x \overbrace{\qquad}^{+\text{ to }-} - 5$$

- There is no x^3 (x to the 3rd power) in this polynomial → omit it

11. Descartes' Rule of Signs

Let $f(x)$ be a polynomial with real coefficients and a nonzero constant term.

- The number of positive real zeros of $f(x)$ either is equal to the number of variations of sign in $f(x)$ or is less than that number by an even integer.
- The number of negative real zeros of $f(x)$ either is equal to the number of variations of sign in $f(-x)$ or is less than that number by an even integer.

• E.g. $f(x) = 2x^5 - 7x^4 + 3x^2 + 6x - 5$

Variations of sign in $f(x)$ is 3

$f(-x) = -2x^5 - 7x^4 + 3x^2 - 6x - 5$

Variations of sign in $f(-x)$ is 2

Number of positive real solutions	3	3	1	1
Number of negative real solutions	2	0	2	0
Number of imaginary solutions	0	2	2	4
Total number of solutions	5	5	5	5

12. If the constant term is 0.

$f(x) = x^4 - 3x^3 + 2x^2 - 5x = x(x^3 - 3x^2 + 2x - 5)$

One solution is x=0, then applying Descartes' rule to $x^3 - 3x^2 + 2x - 5$ to determine the remaining three nature of solutions.

13. First Theorem on Bounds for Real Zeros of Polynomials

Suppose that $f(x)$ is a polynomial with real coefficients and a positive leading coefficient and that $f(x)$ is divided synthetically by x-c.

- If c>0 and if all numbers in the third row of the division process are either positive or zero, then c is an upper bound for the real zeros of $f(x)$. → No Negative
- If c<0 and if the numbers in the third row of the division process are alternatively positive and negative (and a 0 in the third row is considered to be either positive or negative), then c is a lower bound for the real zeros of $f(x)$. →Positive + Negative

14. Finding bounds for the solutions of an equation

$f(x) = 2x^3 + 5x^2 - 8x - 7$

- Upper bound → c>0
    - Trail-and-error attempt 1: let c=1 → x-1

$$\begin{array}{r|rrrr} 1 & 2 & 5 & -8 & -7 \\ & & 2 & 7 & -1 \\ \hline & 2 & 7 & -1 & -8 \end{array}$$

The third row contains negative numbers,
then x-1 does not apply.

    - Train-and-error attempt 2: let c=2 → x-2

$$\begin{array}{r|rrrr} 2 & 2 & 5 & -8 & -7 \\ & & 4 & 18 & 20 \\ \hline & 2 & 9 & 10 & 13 \end{array}$$

The third row does not contain negative numbers,
thus the upper bound is x=2.

- Lower bound → c<0
    - Trail-and-error attempt 1: let c=-1 → x-(-1)=x+1
……
    - Trail-and-error attempt 4: let c=-4 → x-(-4)=x+4

$$\begin{array}{r|rrrr} -4 & 2 & 5 & -8 & -7 \\ & & -8 & 12 & -16 \\ \hline & 2 & -3 & 4 & -23 \end{array}$$

The numbers in the third row are alternatively positive and negative,
thus the lower bound is x=-4.

15. Second Theorem on Bounds for Real Zeros of Polynomials

Suppose $f(x) = a_n x^n + a_n - 1x^{n-1} + ... + a_1 x + a_0$ is a polynomial with real coefficients. All of the real zeros of $f(x)$ are in the interval $(M, -M)$, where

$$M = \frac{max(|a_n|, (a_{n-1}|, ..., |a_1|, |a_0|}{|a_n|} + 1$$

16. $f(x) = x^4 - 9x^3 + 22x^2 - 32$ 4 (mult. 2)

4	1	-9	22	0	-32
		4	-20	8	32
	1	-5	2	8	0

4	1	-5	2	8
		4	-4	-8
	1	-1	-2	0

$$f(x) = (x - 4)^2(x^2 - x - 2) = (x - 4)^2(x - 2)(x + 1)$$

- The coefficient of x is 0. Do not omit it.

17. $f(x) = x^4 + 5x^3 + 6x^2 - 4x - 8$ -2 (mult. 3)

-2	1	5	6	-4	-8
		-2	-6	0	8
	1	3	0	-4	0

-2	1	3	0	-4
		-2	-2	4
	1	1	-2	0

-2	1	1	-2
		-2	2
	1	-1	0

$$f(x) = (x + 2)^3(x - 1)$$

- Do not forget the 1.

18. $3x^4 + 2x^3 - 4x + 2 = 0$

$f(x) = 3x^4 + 2x^3 - 4x + 2 \quad P = 2$

$f(-x) = 3x^4 - 2x^3 + 4x + 2 \quad N = 2$

Number of positive real solutions	2	2	0	0
Number of negative real solutions	2	0	2	0
Number of imaginary solutions	0	2	2	4
Total number of solutions	4	4	4	4

- When the coefficient of x^2 is 0, omit it.

19. $f(x) = x^6 - 8x^5 + 20x^4 - 80x^2 + 128x - 64$ 2 (multi. 5)

$= (x-2)^5(x+2)$

20. A comparison of zero coefficient(s)

If there is x(s) with zero coefficient(s).

(If the decreasing degree powers of a polynomial are not continuous.)

E.g. $f(x) = ax^3 + 0x^2 + bx + c = ax^3 + bx + c$

- Finding the zeros of a polynomial (variations of sign) → Omit it!
- Finding bounds for the solutions of an equation → Do not omit it!
- E.g. (16) & (18)

~~3.4 Complex and Rational Zeros of Polynomials~~

1. Theorem on Conjugate Pair Zeros of a Polynomial
   If a polynomial $f(x)$ of degree $n > 1$ has real coefficients and if $z = a + bi$ with $b \neq 0$ is a complex zero of $f(x)$, then the conjugate $\overline{z} = a - bi$ is also a zero of $f(x)$.

2. $[x - (a + bi)][x - (a - bi)] = x^2 - 2ax + a^2 + b^2$

3. Theorem on Expressing a Polynomial as a Product of Linear and Quadratic Factors
   Every polynomial with real coefficients and positive degree n can be expressed as a product of linear and quadratic polynomials with real coefficients such that the quadratic factors are irreducible over R.

4. Theorem on Rational Zeros of Polynomial
   - $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$
   - $\dfrac{c}{d}$ is a rational zero of $f(x)$
   - the numerator c of the zero is a factor of the constant term a_0
   - the denominator d of the zero is a factor of the leading coefficient a_n

5. Possible rational zeros $= \dfrac{factors\ of\ the\ constant\ a_0}{facotors\ of\ the\ leading\ coefficient\ a_n}$

6. The Theorem on rational zeros of a polynomial may be applied to equations with rational coefficients by merely multiplying both sides of the equation by the lcd of all the coefficients to obtain an equation with integral coefficients.

7. Find all solutions of the equation

$3x^4 + 14x^3 + 14x^2 - 8x - 8 = 0$

$f(x) = 3x^4 + 14x^3 + 14x^2 - 8x - 8$

Choices for the numerator c_1	±1	±2	±4	±8
Choices for the denominator d_1	±1	±3		
Choices for $\frac{c_1}{d_1}$	±1	±2	±4	±8
	$\pm\frac{1}{3}$	$\pm\frac{2}{3}$	$\pm\frac{4}{3}$	$\pm\frac{8}{3}$

$f(\pm 1) = 0,\ f(2) \neq 0, \quad f(-2) = 0$

−2	3	14	14	−8	−8
		−6	−16	4	8
	3	8	−2	−4	0

$f(x) = (x+2)(3x^3 + 8x^2 - 2x - 4)$

$g(x) = 3x^3 + 8x^2 - 2x - 4$

Choices for the numerator c_2	±1	±2	±4
Choices for the denominator d_2	±1	±3	
Choices for $\frac{c_2}{d_2}$	±1	±2	±4
	$\pm\frac{1}{3}$	$\pm\frac{2}{3}$	$\pm\frac{4}{3}$

- $\frac{c_2}{d_2} = \pm 1,\ \pm 2$: since they were proved in $f(x)$, thus no need to be done again in $g(x)$
- Also, since $\frac{c_2}{d_2} \neq \pm 8,\ \pm\frac{8}{3}$, nor is $\frac{c_1}{d_1}$.

$g(\pm 4) \neq 0,\ g(\pm\frac{1}{3}) \neq 0,\ g(\frac{2}{3}) \neq 0, \quad g(-\frac{2}{3}) = 0$

$-\frac{2}{3}$	3	8	−2	−4
		−2	−4	4
	3	6	−6	0

$g(x) = (x + \frac{2}{3})(3x^2 + 6x - 6) = 3(x + \frac{2}{3})(x^2 + 2x - 2)$

$h(x) = x^2 + 2x - 2$

$$x = \frac{-2 \pm \sqrt{2^2 - 4 * (-2)}}{2} = \pm\sqrt{3} - 1$$

$h(x) = [x - (\sqrt{3} - 1)][x - (-\sqrt{3} - 1)] = [x - (\sqrt{3} - 1)][x + (\sqrt{3} + 1)]$

$f(x) = (x + 2)g(x) = (x + 2)(x + \frac{2}{3})h(x)$

$= (x + 2)(x + \frac{2}{3})[x - (\sqrt{3} - 1)][x + (\sqrt{3} + 1)]$

Rational solutions: $x = -2,\ -\frac{2}{3}$

Irrational solutions: $x = \pm\sqrt{3} - 1$

3.5 Rational Functions

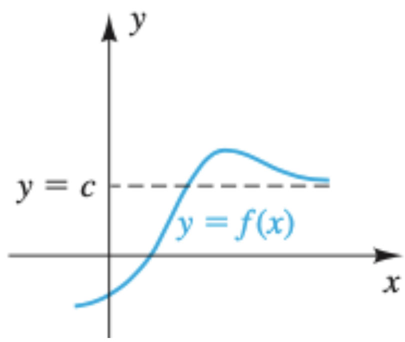
1. $f(x) = \dfrac{x^2 - 4}{x - 2}$

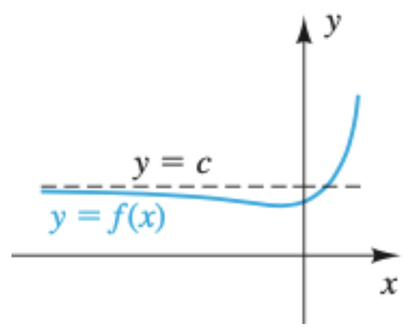
   There is a hole in the graph of $f(x)$ when x=2.
   To find the y-value of the hole, we can substitute 2 for the x in the reduced function.
   $\dfrac{x^2 - 4}{x - 2} = \dfrac{(x+2)(x-2)}{x-2} = x + 2 = g(x)$
   $g(2) = 2 + 2 = 4$

2. Vertical asymptote = 垂直渐进线; horizontal asymptote = 水平渐近线.

3. If the numerator and denominator have no common factor,
   then f must have a vertical asymptote x=a.

4. The graph of f may cross a horizontal asymptote.
   E.g.





5. Theorem on Horizontal Asymptotes

   Let $f(x) = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_k x^k + b_{k-1}x^{k-1} + \ldots + b_1 x + b_0}$, where $a_n \neq 0$ and $b_k \neq 0$.

   - If n<k, then the x-axis (the line y=0) is the horizontal asymptote for the graph of f
   - If n=k, then the line $y = a_n / b_k$ (the ratio of leading coefficients) is the horizontal asymptote for the graph of f
   - If n>k, the graph of f has no horizontal asymptote.
     Instead, either $f(x) \to \pm \infty$ as $x \to \pm \infty$.

6. $f(x) = \dfrac{3x-1}{x^2-x-6}$, the horizontal asymptote is y=0

$f(x) = \dfrac{5x^2+1}{3x^2-4}$, the horizontal asymptote is $y = \dfrac{5}{3}$

$f(x) = \dfrac{2x^4-3x^2+5}{x^2+1}$, the graph has no horizontal asymptote.

7. Guidelines for Sketching the Graph of a Rational Function

Assume that $f(x) = \dfrac{g(x)}{h(x)}$, where g(x) and h(x) are polynomials that have not common factor.

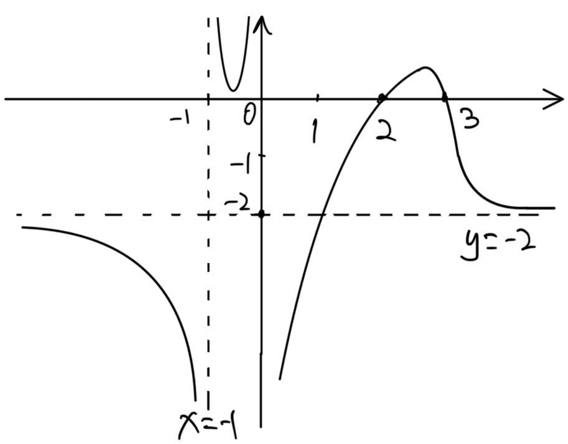
- Find the x-intercepts — that is, the real zeros of numerator g(x) — and plot the corresponding points on the x-axis.
(Please refer to *(3)*.)
- Find the real zeros of the denominator h(x). For each real zero a, sketch the vertical asymptote x=a with dashes.
- Find the y-intercept $f(0)$, if it exist, and plot the point $(0, f(0))$ on the -axis.
- Apply the theorem on horizontal asymptotes. If there is a horizontal asymptote y=c, sketch it with dashes.
- If there is a horizontal asymptote y=c, determine whether is intersects the graph. The x-coordinates of the points of intersection are the solutions of the equation $f(x) = c$. Plot these points, if they exist.
(Whether y=c intersects the graph determine if the graph is 'S' type or 'U' type.)
- Sketch the graph of f in each of the regions in the xy-plane determined by the vertical asymptotes in guideline 2. If necessary, use the sign of specific function values to tell whether the graph is above or below the x-axis or the horizontal asymptote. Use guideline 5 to decide whether the graph approaches the horizontal asymptote from above or below.
(Please refer to 1.6 Inequalities — (4).)

8. $f(x) = \dfrac{-2x^2 + 10x - 12}{x^2 + x}$

$$f(x) = \frac{-2(x^2 - 5x + 6)}{x^2 + x} = \frac{-2(x-2)(x-3)}{x(x+1)}$$

- -2(x-2)(x-3)=0, x=2, 3
- x(x+1)≠0, x≠-1, 0
- f(0) does not exist
- y=-2
- f(x)=-2, x=1 → (1, -2)

Interval	$(-\infty, -1)$	(-1, 0)	(0, 2)	(2, 3)	$(3, +\infty)$
Sign of x-2	-	-	-	+	+
Sign of x-3	-	-	-	-	+
Sign of x	-	-	+	+	+
Sign of x+1	-	+	+	+	+
Sign of -2	-	-	-	-	-
Resulting Sign	-	+	-	+	-



The lowest point of the graph in (-1, 0) cannot be figured out.

Do not forget the hole(s) if *f* has.

9. Find an equation of a rational function f that satisfies the given conditions.
   vertical asymptotes: x=-3, x=1;
   horizontal asymptote: y=0;
   x-intercept: -1;
   $f(0) = 2;$
   hole at x=2

$$f(x) = \frac{a(x+1)(x-2)}{(x+3)(x-1)(x-2)}$$

$$f(0) = \frac{a(-2)}{3(-1)(-2)} = 2, \quad a = -2$$

$$f(x) = \frac{6(x+1)(x-2)}{(x-1)(x+3)(x-2)}$$

10. Asymptotes

- $f(x) = \dfrac{g(x)}{h(x)}$ (The degree of $g(x)$ is one greater than that of $h(x)$.)
- $f(x) = \dfrac{g(x)}{h(x)} = q(x) + \dfrac{r(x)}{h(x)}$
- $\dfrac{r(x)}{h(x)} \rightarrow 0$, as $x \rightarrow \pm\infty$
- The Oblique Asymptote: $y = q(x)$
- If $r(x)$ has zero(s), then $f(x)$ will intersect the asymptote $q(x)$ at the zeros of $r(x)$
  $r(x) = 0 \rightarrow \dfrac{r(x)}{h(x)} = 0 \rightarrow f(x) = q(x)$
- An Oblique Asymptotes: if $r(x) = ax + b$
- A Curvilinear Asymptote: if $r(x) = x^a + b$

• Refer to *Math 231: Calculus of Functions of One Variable I — 2.5 Limits at Infinity*

11. f intersects with its oblique asymptote

$$f(x) = \frac{x^3+1}{x^2-9}$$

$$x^2 - 9 \neq 0 \rightarrow x \neq \pm 3$$

The vertical asymptote: $x = \pm 3$

$$f(x) = 0 \rightarrow x^3 + 1 = 0 \rightarrow x = -1 \rightarrow f(-1) = 0$$

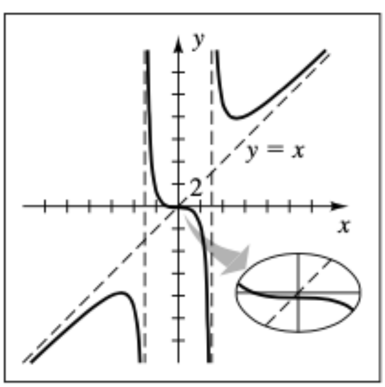
$$\begin{array}{r|llll} & & & x & \\ \hline x^2-9 & x^3 & +0x^2 & +0x & +1 \\ & x^3 & & -9x & \\ \hline & & & 9x & +1 \end{array}$$

$$f(x) = x + \frac{9x+1}{x^2-9}$$

The asymptote: $y = x$

$f(0) = -1$: When x=0, f intersects with $y = x$ at $(0, -1)$

Intervals	$(-\infty, -3)$	$(-3, -1)$	$(-1, 3)$	$(3, \infty)$
Sign of x^3+1	−	−	+	+
Sign $x+3$	−	+	+	+
Sign $x-3$	−	−	−	+
Sign $f(x)$	−	+	−	+



12. f intersects with its curvilinear asymptote

$$f(x) = \frac{x^5 - 3x^3 - x^2 + 1}{x^2 - 3}$$

			x^3			-1
$x^2 - 3$	x^5	$+0x^4$	$-3x^3$	$-x^2$	$+0x$	$+1$
	x^5		$-3x^3$			
				$-x^2$		$+1$
				$-x^2$		$+3$
						-2

$f(x) = x^3 - 1 - \dfrac{2}{x^2 - 3}$ $\rightarrow$ the curvilinear asymptote: $y = x^3 - 1$

4.1 Inverse Functions

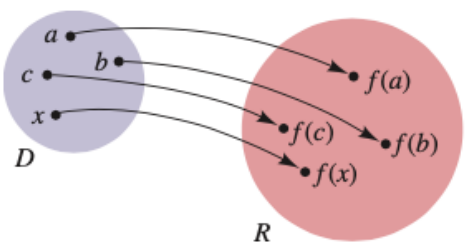
1. Definition of One-to-One Function

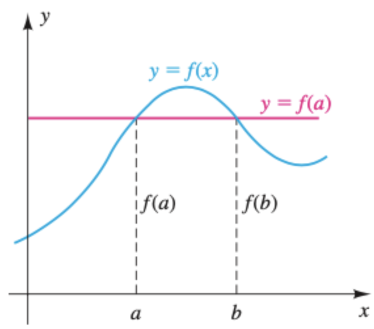
   A function f with domain D and range R is a one-to-one function if either of the following equivalent conditions is satisfied:

   - Whenever a≠b in D, then $f(a) \neq f(b)$ in R.
   - Whenever $f(a) = f(b)$ in R, then a=b in D.

2. One-to-one functions are that different numbers in the domain always gives different values of f.

3. The left is a one-to-one function. This is not a one-to-one function.





4. Horizontal Line Test

   A function f is one-to-one if and only if every horizontal line intersects the graph of f in at most one point.

   - By comparison with 2.4 Definition of Function — (7)

Vertical Line Test

The graph of a set of points in a coordinate plane is the graph of a function if every vertical line intersects the graph in at most one point.

- Vertical Line Test → Function
- Horizontal Line Test → One-to-One Function

5. Theorem: increasing or decreasing functions are one-to-one.

6. Inverse functions are one-to-one.

   (Both $y = f(x)$ and $x = g(y)$ are one-to-one functions.)

7. The inverse function $x = g(y)$ exists iff $y = f(x)$ is a one-to-one function.

8. $f^{-1}(y)$ does not mean $1/[f(y)]$

9. Domain and Range of f and f^{-1}

- domain of f^{-1} = range of f
- range of f^{-1} = domain of f
- E.g. $f(x) = -\sqrt{9 - x^2} \le 0, \quad -3 \le x \le 0$
  $f^{-1}(x) = -\sqrt{9 - x^2}, \quad x \le 0$

10. Guidelines for Finding f^{-1} in Simple Cases

- Verify that f is a one-to-one function throughout its domain.
- Solve the equation $y = f(x)$ for x in terms of y, obtaining an equation of the form $x = f^{-1}(y)$.
- Verify the following two conditions:
  - $f^{-1}(f(x)) = x$ for every x in the domain of f
  - $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}

11. $f(x) = \sqrt{3 - x}$

- $f(x)$ is a one-to-one function
- $y = \sqrt{3 - x}$
  $y^2 = 3 - x$
  $x = 3 - y^2$
  $f(x) = \sqrt{3 - x} \ge 0$
  $f^{-1}(x) = 3 - x^2, \quad x \ge 0$
- $f^{-1}(f(x)) = 3 - (\sqrt{3 - x})^2 = x$
  $f(f^{-1}(x)) = \sqrt{3 - (3 - x^2)} = x$
  Do not forget to verify.

4.2 Exponential Functions

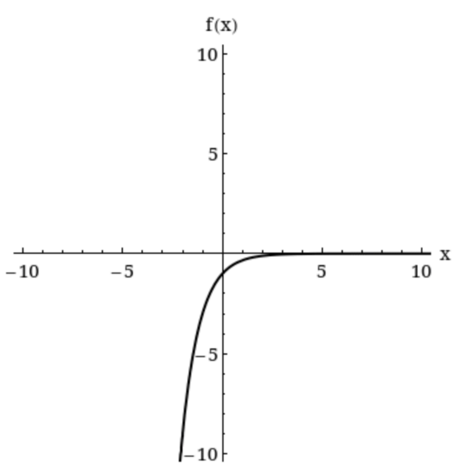
1. Exponent functions: $f(x) = a^x$, 0<a<1 or a>1
2. Exponent functions are a one-to-one.
3. Compound Interest Formula: $A = P(1 + \frac{r}{n})^{nt}$

   P = principal
   r = annual interest rate expressed as a decimal
   n = number of interest periods per year
   t = number of years P is invested
   A = amount after t years

4. Using the compound interest formula
   Suppose that $1000 is invested at an interest rate of 9% compounded monthly. Find the new amount of principal after 5 years, after 10 years, and after 15 years.
   $A = 1000(1 + \frac{0.09}{12})^{12t} = 1000(1.0075)^{12t}$

Number of years	Amount
5	A = $1000(1.0075)^{60} = $1565.68
10	A = $1000(1.0075)^{120} = $2451.36
15	A = $1000(1.0075)^{180} = $3838.04

5. The graph of $f(x) = -a^x$ if $a = \frac{1}{3}$



4.3 The Natural Exponential Function

1. The compound interest formula
   After we reach an interest period of one hour, the number of interest periods per year has no effect on the final amount.
   (The number is too little to affect the final amount.)

2. $(1+\frac{1}{n})^n \to e \approx 2.71828$ as $n \to \infty$

3. Continuously Compounded Interest Formula: $A = Pe^{rt}$

4. $A = P(1+\frac{r}{n})^{nt}$
   let $\frac{1}{k} = \frac{r}{n}$, then $n = kr$
   $A = P(1+\frac{r}{n})^{nt} = P(1+\frac{1}{k})^{krt} = P[(1+\frac{1}{k})^k]^{rt} = Pe^{rt}$

5. By comparison

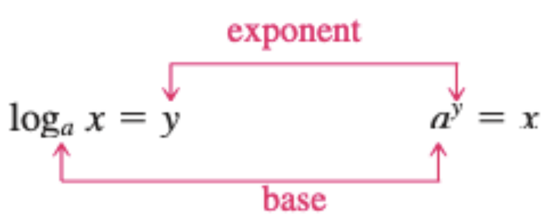
   - Compound Interest Formula: $A = P(1+\frac{r}{n})^{nt}$

   - Continuously Compounded Interest Formula: $A = Pe^{rt}$

4.4 Logarithmic Functions

1. Logarithmic Functions: $f(x) = log_a x$, iff $x = a^y$, x>0
2. Logarithm functions are one-to-one.
3. $log_6(4x - 5) = log_6(2x + 1)$
   4x-5=2x+1
   x=3
   Check x=3
   LS: $log_6(4 * 3 - 5) = log_6 7$
   RS: $log_6(2 * 3 + 1) = log_6 7$
   $\because log_6 7 = log_6 7$
   $\therefore x = 3$ is a solution
   Do not forget to check.
4. A comparison of check

   - Solving logarithmic equations → check;
   - Solving exponential equations → no need to check.
5. Arguments must be positive: check x>0 in $log_a x$.
6. Logarithmic Form & Exponential Form



7. Common Logarithm: $logx = log_{10}x, \quad x > 0$
8. Natural Logarithm: $lnx = log_e x, \quad x > 0$
9. Coverting to Base e Expressions

   - $a^x = e^{xlna}$
   - $x^a = e^{alnx}$

- $a * b^x = a * e^{xlna}$

10. Properties of Logarithms

Logarithms with base a	Common logarithms	Natural logarithms
$log_a 1 = 0$	$log1 = 0$	$ln1 = 0$
$log_a a = 1$	$log10 = 1$	$lne = 1$
$log_a a^x = x$	$log10^x = x$	$lne^x = x$
$a^{log_a x} = x$	$10^{logx} = x$	$e^{lnx} = x$

11. Approximating a double time

- Assume that a population is growing continuously at a rate of 4% per year. Approximate the amount of time it takes for the population to double its size — that is, its doubling time.
  $q = q_0 e^{rt}$
  $2q_0 = q_0 e^{0.004t}$
  $t = 25ln2 \approx 17.3$ yr.

- The fact that q_0 did not have any effect on the answer indicates that the doubling time for a population of 1000 is the same as the doubling time for a population of 1,000,000 or any other reasonable initial population.

- A General Formula for the Doubling Time of A Population: $t = \dfrac{ln2}{r}$

- The Rule of 70 or the Rule of 72

4.5 Properties of Logarithms

1. Laws of Logarithms
   - $log_a(uw) = log_a u + log_a w$
   - $log_a(\frac{u}{w}) = log_a u - log_a w$
   - $log_a(u^c) = clog_a u$
2. So do common logarithms and natural logarithms
3. $log_a(u \pm w) \neq log_a u \pm log_a w$
4. If we apply these laws to equations in which u and w are expressions involving a variable, then extraneous solutions may occur. Answers should therefore be substituted for the variable in u and w to determine whether these expressions are defined (Need to be checked!).
   E.g. $log_2 x + log_2(x+2) = 3$
   $x_1 = 2, \quad x_2 = -4$
   Check $x = 2$
   LS: $log_2 2 + log_2(2+2) = 3$
   RS: 3
   $\because 3 = 3$
   $\therefore x = 2$ is a solution
   Check $x = -4$
   LS: $log_2(-4) + log_2(-4+2)$
   $\because -4+2 = -2 < 0$
   $\therefore x = -4$ is not a solution

4.6 Exponential and Logarithmic Equations

1. Theorem: Change of Base Formula: $log_b u = \dfrac{log_a u}{log_a b}$

2. $\dfrac{log_a u}{log_a b} \neq log_a \dfrac{u}{b}$; $\quad \dfrac{log_a u}{log_a b} \neq log_a(u - b)$

3. Special Change of Base Formulas:

- $log_b u = \dfrac{log_{10} u}{log_{10} b} = \dfrac{log u}{log b}$

- $log_b u = \dfrac{log_e u}{log_e b} = \dfrac{ln u}{ln b}$

4. $2^{-x^2} = 5$
   $log_2 5 = -x^2$
   $x^2 = -log_2 5 < 0$
   There is no solution.

5. Solving an exponential equation
   $5^{2x+1} = 6^{x-2}$
   $log(5^{2x+1}) = log(6^{x-2})$
   $(2x + 1)log5 = (x - 2)log6$
   $log5 + 2log6 = xlog6 - 2xlog5$
   $x = \dfrac{log5 + 2log6}{log6 - 2log5} = \dfrac{log(5 * 6^2)}{log\frac{6}{5^2}} = \dfrac{log180}{log\frac{6}{25}} \approx -3.64$

6. Solving an equation involving logarithms need to be checked.

E.g. $log\sqrt{x} = \sqrt{logx}$

$$\frac{1}{2}logx = \sqrt{logx}$$

$$\frac{1}{4}(logx)^2 = logx$$

$$logx(\frac{1}{4}logx - 1) = 0$$

$$logx = 0, \quad 4$$

$$x = 1, \quad 10000$$

Check x=1

LS: $log\sqrt{1} = 0$

RS: $\sqrt{log1} = 0$

$\because 0 = 0$

$\therefore x = 1$ is a solution

Check x=10000

LS: $log\sqrt{10000} = 2$

RS: $\sqrt{log10000} = 2$

$\because 2 = 2$

$\therefore x = 10000$ is a solution

7. Hyperbolic secant Function = 双曲正割函数

$$f(x) = \frac{2}{e^x + e^{-x}}$$

8. Finding an inverse hyperbolic function

$$y = \frac{2}{e^x + e^{-x}}$$

$$ye^x + \frac{y}{e^x} = 2$$

$$y(e^x)^2 - 2e^x + y = 0$$

$$e^x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4y^2}}{2y} = \frac{1 \pm \sqrt{1 - y^2}}{y}$$

$$x = ln\frac{1 \pm \sqrt{1 - y^2}}{y}$$

$$f^{-1}(x) = ln\frac{1 \pm \sqrt{1 - y^2}}{y}$$

Since the hyperbolic secant it not one-to-one,
it can not have one simple equation for its inverse.

8.1 Systems of Equations

1. Solving a system of equations need to be checked.

2. The method of substitution

3. Solving a system of three equations

$$\begin{cases} x^2 + z^2 = 5 \\ 2x + y = 1 \\ y + z = 1 \end{cases}$$

$$y + z = 1, \quad y = 1 - z$$
$$2x + y = 1$$
$$2x + 1 - z = 1, \quad z = 2x$$
$$x^2 + z^2 = 5$$
$$x^2 + (2x)^2 = 5$$
$$x^2 = 1$$
$$x_1 = 1, \quad x_2 = -1$$
$$2 * 1 + y_1 = 1, \quad y_1 = -1$$
$$2 * (-1) + y_2 = 1, \quad y_2 = 3$$
$$-1 + z_1 = 1, \quad z_1 = 2$$
$$3 + z_2 = 1, \quad z_2 = -2$$
$$(1, -1, 2) \quad or \quad (-1, 3, -2)$$

Check $\begin{cases} x = 1 \\ y = -1 \\ z = 2 \end{cases}$

$$\begin{cases} 1^2 + 2^2 = 5 \\ 2 * 1 + (-1) = 1 \\ -1 + 2 = 1 \end{cases}$$

$\therefore (1, -1, 2)$ is a solution.

Check $\begin{cases} x = -1 \\ y = 3 \\ z + -2 \end{cases}$

$$\begin{cases} (-1)^2 + (-2)^2 = 5 \\ 2 * (-1) + 3 = 1 \\ 3 + (-2) = 1 \end{cases}$$

$\therefore (-1, 3, -2)$ is a solution.

8.2 Systems of Linear Equations in Two Variables

1. Characteristics of a System of Two Linear Equations in Two Variables

Graphs	Number of solutions	Classification
Nonparallel lines	One solution	consistent system
Identical lines	Infinite solutions	Dependent and consistent system
Parallel lines	No solution	Inconsistent system

$$\begin{cases} \frac{2}{x} + \frac{3}{y} = -2 \\ \frac{4}{x} - \frac{5}{y} = 1 \end{cases}$$

let $\begin{cases} u = \frac{1}{x} \\ w = \frac{1}{y} \end{cases}$

$$\begin{cases} 2u + 3w = -2 \\ 4u - 5w = 1 \end{cases}$$

$$\begin{cases} u = -\frac{7}{22} \\ w = -\frac{5}{11} \end{cases}$$

$$\begin{cases} x = -\frac{22}{7} \\ y = -\frac{11}{5} \end{cases}$$

Do not forget to go back to x and y.

Check $\begin{cases} x = -\frac{22}{7} \\ y = -\frac{11}{5} \end{cases}$

$$\begin{cases} \frac{2}{-\frac{22}{7}} + \frac{3}{-\frac{11}{5}} = -2 \\ \frac{4}{-\frac{22}{7}} - \frac{5}{\frac{11}{5}} = 1 \end{cases}$$

$\therefore (-\frac{22}{7}, \frac{11}{5})$ is a solution

$$\begin{cases} 2^{x+1} + 3^y = 11 \\ 2^x - 3^{y+1} = -26 \end{cases}$$

let $\begin{cases} u = 2^x \\ w = 3^y \end{cases}$

$$\begin{cases} 2u + w = 11 \\ u - 3w = -26 \end{cases}$$

$$\begin{cases} u = 1 \\ w = 9 \end{cases}$$

$$\begin{cases} x = 0 \\ y = 2 \end{cases}$$

Do not forget to go back to x and y.

Check $\begin{cases} x = 0 \\ y = 2 \end{cases}$

$$\begin{cases} 2^{0+1} + 3^2 = 11 \\ 2^0 - 3^{2+1} = -26 \end{cases}$$

$\therefore (0,2)$ is a solution

As a back rolls down an inclined plane, its velocity v(t) (in cm/sec) at time t (in seconds) is given by $v(t) = v_0 + at$ fir initial velocity v_0 and acceleration a (in cm/sec^2). If v(2)=23 and v(5)=35, find v_0 and a.

$$\begin{cases} v(2) = v_0 + 2t = 23 \\ v(5) = v_0 + 5t = 35 \end{cases}$$

$$3t = 35 - 23 = 12 \quad \neq 8$$

$$\begin{cases} a = 4 \\ v_0 = 15 \end{cases}$$

Check $\begin{cases} a = 4 \\ v_0 = 15 \end{cases}$

$$\begin{cases} 15 + 2 * 4 = 23 \\ 15 + 5 * 4 = 35 \end{cases}$$

$\therefore \begin{cases} a = 4 \\ v_0 = 15 \end{cases}$ is a solution.

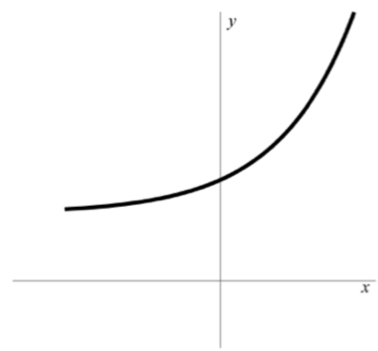
Midterm Exams & Final Exams

1. Suppose $log(m) = 4$, $log(n) = 100$, and $log(w) = -2$, find the exact value of $log(\frac{m * \sqrt{n}}{w^3})$.

$$log(\frac{m * \sqrt{n}}{w^3}) = log(m) + \frac{1}{2}log(n) - 3log(w) \neq \frac{log(m) + \frac{1}{2}log(n)}{3log(w)}$$

$$= 4 + \frac{1}{2} * 100 - [3 * (-2)] = 60$$

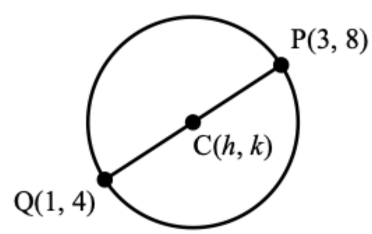
2. Choose the statement that best describes the function with the given graph.
   A. Polynomial function
   B. Quadratic function
   C. Exponential function
   D. Rational function



Rational Function:

A function f is a rational function if $f(x) = \frac{g(x)}{h(x)}$, where g(x) and h(x) are polynomials. The domain of f consists of all real numbers except the zeros of the denominator h(x).

3. Construct an equation for the circle shown below, where point C is the center of the circle.



$$d(P, Q) = \sqrt{(3-1)^2 + (8-4)^2} = 2\sqrt{5}$$

$$r = \frac{1}{2}d(P, Q) = \sqrt{5}$$

$$C = (\frac{3+1}{2}, \frac{8+4}{2}) = (2,6)$$

$$(x-2)^2 + (y-6)^3 = (\sqrt{5})^2 = 5$$

4. Determine the interval(s) where the graph of $f(x) = \dfrac{9-x}{x^2+5x-66}$ is above the x-axis. Given answer using interval notation.

$$f(x) = \frac{9-x}{x^2+5x-66} = \frac{9-x}{(x+11)(x-6)}$$

$$9 - x = 0, \quad x = 9$$

$$(x+11)(x-6) \neq 0, \quad x \neq -11{,}6$$

Intervals	$(-\infty, -11)$	(-11, 6)	(6, 9)	$(9,\infty)$
Sign of 9-x	+	+	+	-
Sign of x+11	-	+	+	+
Sign of x-6	-	-	+	+
Resulting sign	+	-	+	-

$$(-\infty, -11) \cup (5{,}9)$$

- When the coefficient of x is negative, the sign of the factor should be reverse.

5. $log(x-y) = \dfrac{log(x)}{log(y)} \rightarrow FALSE$

E.g.1 If x=10, y=10

LS: $log(10-10) = log(0)$ is undefined

RS: $\dfrac{log(10)}{log(10)} = 1$

E.g.2 If x=10, y=1

LS: $log(10-1) = log(9)$

RS: $\dfrac{log(10)}{log(1)} = \dfrac{1}{0}$ is undefined

- The domains of the left side and right side are different.

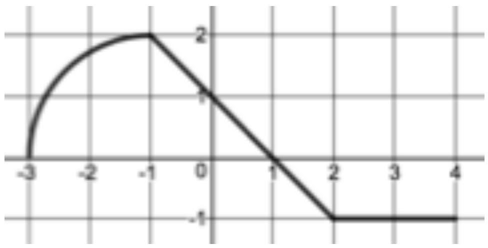
6. A wedding dress was purchased fr \$1200. Suppose that its value decreases by a fixed percent each year, and two years after purchase, the value is \$900. Write an equation to express the value V in terms of the time in year t since purchase. Your equation should only have two variables: V and t.

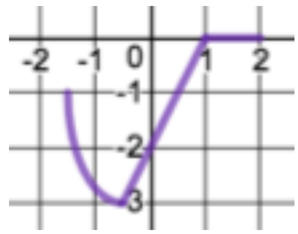
$$V = 1200e^{2r}$$

$$r = ln(\frac{\sqrt{3}}{2})$$

$$V = 1200e^{ln(\frac{\sqrt{3}}{2})t} = 1200(\frac{\sqrt{3}}{2})^t$$

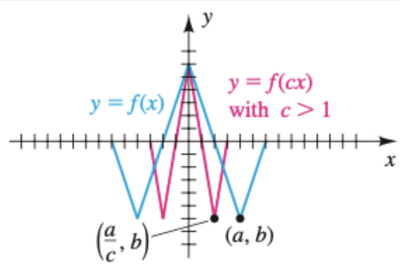
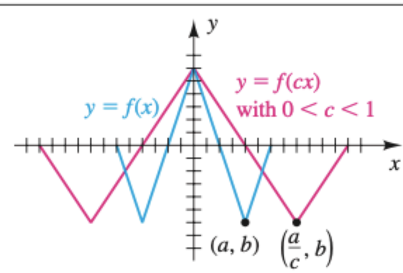
7. The graph of the function $y = f(x)$ is shown below.





Use transformations to draw the graphs of $y = -f(2x) - 1$.

- Refer to *2.5 Graphs of Functions — (5)*

Equation	Effect on graph	Graphical interpretation
$y = f(cx)$ with $c > 1$	The graph of f is compressed horizontally by a factor c.	
$y = f(cx)$ with $0 < c < 1$	The graph of f is stretched horizontally by a factor $1/c$.	

8. The tail behavior of the graph of $y = x^9$ is such that: D

A. Both left and right tails go to positive infinity (upward)

B. Both left and right tails go to negative infinity (downward)

C. the left tail goes to positive infinity (upward) while the right tail goes to negative infinity (downward)

D. the right tail goes to positive infinity (upward) while the left tail goes to negative infinity (downward)

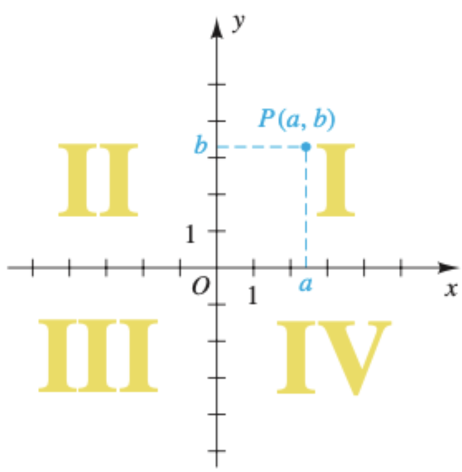
9. The graph of y=|x| passes the vertical test. $\rightarrow TRUE$
   y=|x| is a function, therefore passes the vertical test.

Refer to *2.4 Definition of Function — (7) and 4.1 Inverse Functions — (4)*

- Vertical Line Test → Function
- Horizontal Line Test → One-to-One Function
  - Horizontal Line Test
    A function f is one-to-one if and only if every horizontal line intersects the graph of f in at most one point.
  - Vertical Line Test
    The graph of a set of points in a coordinate plane is the graph of a function if every vertical line intersects the graph in at most one point.

---

10. The point (-3, -7) is in: C
    A. Quadrant 1;   B. Quadrant 2;   C. Quadrant 3;   D. Quadrant 4



11. How much money, invested at an interest rate of 3.4% per year compounded continuously, will amount of $1000 after 4 years? Show all work. You may use your calculator for this problem.

$$1000 = Pe^{0.034*4}$$

$$P = \frac{1000}{e^{0.034*4}} \approx \$872.84$$

题没读懂，英语水平问题。