MATH 232: Calculus of Functions of One Variable 2

6.1 Velocity and Net Change

1. If the velocity changes sign, then the displacement and the distance traveled are not generally equal.
2. The distance traveled is always nonnegative.
3. Position, Velocity, Displacement and Distance
   - Position: $s(t)$
   - Velocity: $v(t) = s'(t)$
   - Speed: $|v(t)| = |s'(t)|$
   - Displacement: $s(b) - s(a) = \int_a^b v(t)dt$
   - Distance: $\int_a^b |v(t)|dt$
4. Position from Velocity
   - $s(t) = s(0) + \int_0^t v(x)dx$
   - Note that t is the independent variable of the position function.
     Therefore, another (dummy) variable, in this case x, must be used as the variable of integration.
5. Velocity from Acceleration: $v(t) = v(0) + \int_0^t a(x)dx$

6. Net Change and Future Value

- Net Change: $Q(b) - Q(a) = \int_a^b Q'(t)dt$

- Future Value: $Q(t) = Q(0) + \int_0^t Q'(x)dx$

• The units in the integral are consistent.

7. Velocity-Displacement Problems & General Problems

Velocity-Displacement Problems	General Problems
Position: $s(t)$	Quality: $Q(t)$ (such as volume or population)
Velocity: $s'(t) = v(t)$	Rate of Change: $Q'(t)$
Displacement: $s(b) - s(a) = \int_a^b v(t)dt$	Net Change: $Q(b) - Q(a) = \int_a^b Q'(t)dt$
Future Position: $s(t) = s(0) + \int_0^t v(x)dx$	Future Value of Q: $Q(t) = Q(0) + \int_0^t Q'(x)dx$

6.2 Regions Between Curves

1. Area of a Region Between Two Curves

- $A = \int_a^b (f(x) - g(x))dx$

  - It is helpful to interpret the area formula $f(x) - g(x)$ is the length of a rectangle and dx represents its width. We sum (integrate) the areas of the rectangles $(f(x) - g(x))dx$ to obtain the area of the region.
  - The area formula is valid even if one or both curves lie below the x-axis, as long as $f(x) \geq g(x)$ on $[a, b]$.

2. Area of a Region Between Two Curves with Respect to y

$$A = \int_c^d (f(y) - g(y))dy$$

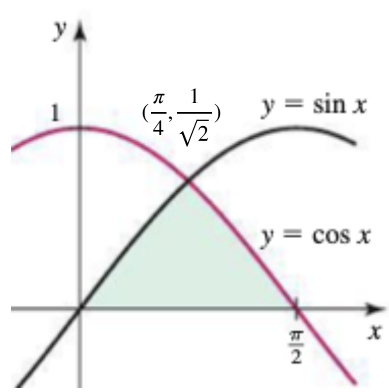
- The positive direction is with respect to x-axis, which means right.

3. Slicing the region vertically and integrating with respect to x requires two integrals. Slicing the region horizontally requires a single integral with respect to y.
   The second approach appears to involve less work.

4. Find the area.

- $\int_0^{\frac{\pi}{4}} sinxdx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cosxdx$

- It is tempting to integrate with respect to y,
  but $\int_0^{\frac{1}{\sqrt{2}}} (cos^{-1}y - sin^{-1}y)dy$ is unresolvable now.



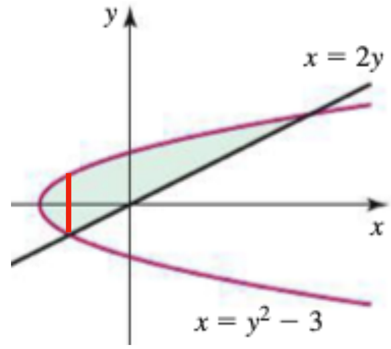
$\int_0^{\frac{1}{\sqrt{2}}} (cos^{-1}y - sin^{-1}y)dy \neq (\frac{-1}{\sqrt{1-y^2}} - \frac{1}{\sqrt{1-y^2}})|_0^{\frac{1}{\sqrt{2}}} = \frac{-2}{\sqrt{1-y^2}}|_0^{\frac{1}{\sqrt{2}}}$

Because $\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ instead of $\frac{d}{dx}(\frac{1}{\sqrt{1-x^2}}) = sin^{-1}x$

5. Watch out the specific condition “in the first quadrant”.

6. Find the area

- Integrate with respect to x



$x = 2y \ \rightarrow \ y = \frac{1}{2}x$

$x = y^2 - 3 \ \rightarrow \ y = \pm\sqrt{x+3}$

$\frac{1}{2}x = \pm\sqrt{x+3} \ \rightarrow \ x = -2, 6$

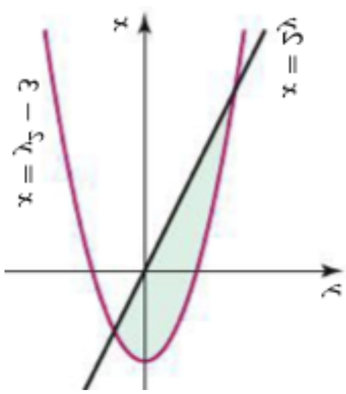
$\pm\sqrt{x+3} = 0 \ \rightarrow \ x = -3$

$$\int_{-3}^{-2}(\sqrt{x+3} - (-\sqrt{x+3}))dx + \int_{-2}^{6}(\sqrt{x+3} - \frac{1}{2}x)dx$$

- Integrate with respect to y

$2y = y^2 - 3 \ \rightarrow \ y = -1, 2$

$$\int_{-1}^{2}(2y - (y^2 - 3))dy$$



6.3 Volume by Slicing

1. General Slicing Method

- $V = \int_a^b A(x)dx$

• $A(x) = (y - axis)^2 = (y - axis) * (z - axis)$

• dx represents its thickness.

  - $dx = \Delta x \quad (dy = \Delta y)$

Reference: Math 231: Calculus of Function of One Variable 1
— 4.5 Linear Approximation and Differentials

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2. Disk Method about the x-Axis

- $V = \int_a^b \pi f^2(x)dx$

• By rotating $y - axis$ to substitute for $y - z\ plane$

3. Washer Method about the x-Axis

- $v = \int_a^b \pi(f^2(x) - g^2(x))dx$

• The washer method is really two applications of the disk method.
We compute the volume of the entire solid without the hole (by the dis method) and then subtract the volume of the hole (also competed by the disk method).

• The integrand is $f^2(x) - g^2(x)$ instead of $(f(x) - g(x))^2$

4. Disk and Washer Methods about the y-Axis

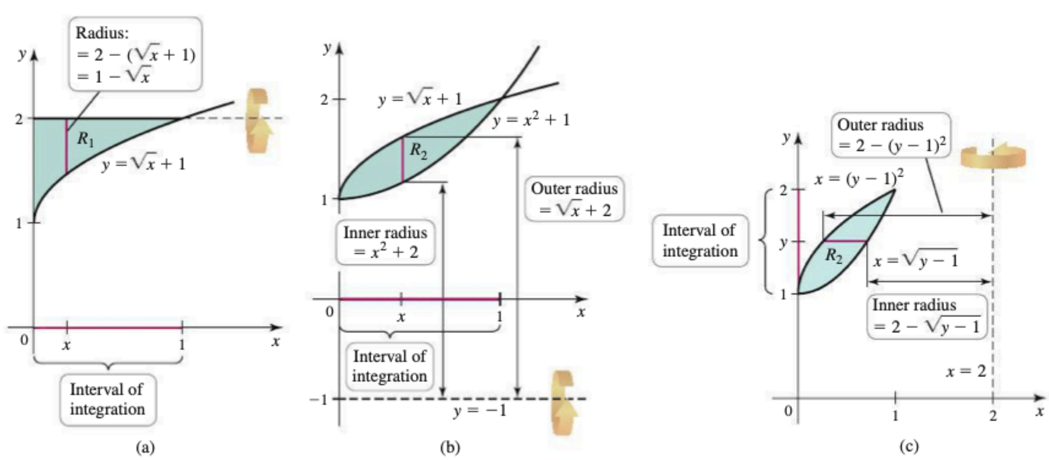
- Disk Method: $V = \int_c^d \pi p^2(y)dy$

- Washer Method: $V = \int_c^d \pi(p^2(y) - q^2(y))dy$

5. Revolving about other lines

- $A(x) = \pi(R^2 - r^2) = \pi(Outside^2 - Inside^2)$
- $R \,\&\, r = Positive - Negative$

- E.g. Let $f(x) = \sqrt{x} + 1$ and $g(x) = x^2 + 1$.
  - Find the volume of the solid generated when the region R_1 bounded by the graph of f and the line $y = 2$ on the interval $[0,\ 1]$ is revolved about the line $y = 2$.
  $$\int_0^1 A(x)dx = \int_0^1 \pi(2 - (\sqrt{x} + 1))^2 dx = \frac{\pi}{6}$$
  - Find the volume of the solid generated when the region R_2 bounded by the graphs of f and g on the interval $[0,\ 1]$ is revolved about the line $y = -1$
  $$\int_0^1 A(x)dx = \int_0^1 \pi\{[(\sqrt{x} + 1) - (-1)]^2 - [(x^2 + 1) - (-1)]^2\}dx = \frac{49\pi}{30}$$
  - Find the volume of the solid generated when the region R_2 bounded by the graphs of f and g on the interval $[0,\ 1]$ is revolved about the line $x = 2$
  $$\begin{cases} y = \sqrt{x} + 1 & \rightarrow \quad x = (y - 1)^2 \\ y = x^2 + 1 & \rightarrow \quad x = \sqrt{y - 1} \end{cases}$$
  $$\int_1^2 A(y)dy = \int_1^2 \pi\{[(2 - (y - 1)^2]^2 - (2 - \sqrt{y - 1})^2\}dy = \frac{31\pi}{30}$$



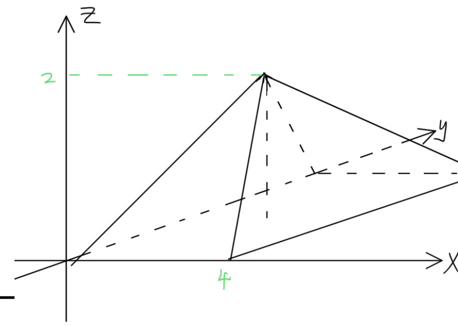
6. The pyramid with a square base 4m on a side and a height of 2m (Use calculus).

In the xz-plane:

$$z = \frac{1}{2}x \ \rightarrow \ x = 2z$$

$$A(z) = x^2 = (2z)^2 = 4z^2$$

$$\int_0^2 A(z)dz = \int_0^2 4z^2 dz = \frac{4}{3}z^3 |_0^2 = \frac{32}{3}$$



- Refer to (1): dx represents its thickness.

- In this case, the thickness of the pyramid is its height, which means z-axis, therefore dz
- The domain of $\int$ should be with respect to dx (in this case dz), therefore $\int_0^2$

7. The degree angle of the thickness does not affect the volume.

6.5 Length of Curves

1. Arc Length for $y = f(x)$
   - $L = \int_a^b \sqrt{1 + (f'(x))^2} dx$
   - $1 + (f'(x))^2$ is positive, so the square root in the integrand is defined whenever f' exists.

     To ensure that $\sqrt{1 + (f'(x))^2}$ is integrable on $[a, b]$, we require that f' be continuous.
2. Arc Length for $x = f(y)$

   $L = \int_c^d \sqrt{1 + (g'(y))^2} dy$
3. Δx is the same for each subinterval, but Δy_k depends on the subinterval.
4. A typical question
   - $f'(x) = ax^m - bx^{-m}$ where $ab = \dfrac{1}{4}$
   - $(f'(x))^2 = (ax^m - bx^{-m})^2$
     $= a^2x^{2m} + b^2x^{-2m} - 2ax^m bx^{-m}$
     $= a^2x^{2m} + b^2x^{-2m} - 2ab$
     $= a^2x^{2m} + b^2x^{-2m} - \dfrac{1}{2}$
   - $1 + (f'(x))^2$
     $= 1 + (a^2x^{2m} + b^2x^{-2m} - \dfrac{1}{2})$
     $= a^2x^{2m} + b^2x^{-2m} + \dfrac{1}{2}$
     $= a^2x^{2m} + b^2x^{-2m} + 2ab$
     $= a^2x^{2m} + b^2x^{-2m} + 2ax^m bx^{-m}$
     $= (ax^m + bx^{-m})^2$
   - $L = \int \sqrt{1 + (f'(x))^2} dx$
     $= \int \sqrt{(ax^m + bx^{-m})} dx$
     $= \int (ax^m + bx^{-m}) dx$

   • E.g. See below → (5)

5. Arc length calculations with respect to y: $y = ln(x - \sqrt{x^2 - 1})$, for $1 \leq x \leq \sqrt{2}$

$$y = ln(x - \sqrt{x^2 - 1})$$
$$e^y = x - \sqrt{x^2 - 1}$$
$$x - e^y = \sqrt{x^2 - 1}$$
$$x^2 - 2xe^y + e^{2y} = x^2 - 1$$
$$2xe^y = e^{2y} + 1$$
$$x = \frac{e^{2y} + 1}{2e^y} = \frac{e^y + e^{-y}}{2}$$
$$x' = \frac{e^y - e^{-y}}{2}$$
$$1 \leq x \leq \sqrt{2}$$
$$ln(1 - \sqrt{1^2 - 1}) \leq y \leq ln(\sqrt{2} - \sqrt{(\sqrt{2})^2 - 1})$$
$$0 \leq y \leq ln(\sqrt{2} - 1)$$
$$L = \int_{ln(\sqrt{2}-1)}^{0} \sqrt{1 + (x')^2} dy$$
$$= \int_{ln(\sqrt{2}-1)}^{0} \sqrt{1 + (\frac{e^y - e^{-y}}{2})^2} dy$$
$$= \int_{ln(\sqrt{2}-1)}^{0} \sqrt{1 + \frac{1}{4}(e^{2y} - 2 + e^{-2y})} dy$$
$$= \int_{ln(\sqrt{2}-1)}^{0} \sqrt{\frac{1}{4}(e^{2y} + 2 + e^{-2y}} dy$$
$$= \int_{ln(\sqrt{2}-1)}^{0} \frac{e^y + e^{-y}}{2} dy$$
$$= \frac{1}{2}(e^y - e^{-y})\,\Big|_{ln(\sqrt{2}-1)}^{0}$$
$$= \frac{1}{2}((e^0 - e^{-0})(e^{ln(\sqrt{2}-1)} - e^{-ln(\sqrt{2}-1)}))$$
$$= \frac{1}{2}((\frac{1}{\sqrt{2} - 1}) - (\sqrt{2} - 1))$$
$$= \frac{1}{2}(\frac{1 - (\sqrt{2} - 1)^2}{\sqrt{2} - 1})$$
$$= \frac{1}{2}(\frac{1 - (2 + 1 - 2\sqrt{2})}{\sqrt{2} - 1})$$
$$= \frac{1}{2}(\frac{2\sqrt{2} - 2}{\sqrt{2} - 1}) = 1$$

6.6 Surface Area

1. Area of a Surface of Revolution
$$S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$
   - $2\pi f(x)$ = the loop with respect to x-axis
2. A surface area problem is "between" a volume problem (which is three-dimensional) and an arch length problem (which is one-dimensional).
3. If the curve $y = f(x)$ on the interval $[a,\ b]$ is resolved about the y-axis, the area of the surface generated
   - is $\int_{f(a)}^{f(b)} 2\pi f^{-1}(y)\sqrt{1+f^{-1'}(y)^2}\,dy$
   - Instead of $\int_{f(a)}^{f(b)} 2\pi f(y)\sqrt{1+f'(y)^2}\,dy$.
   - Because $f(y) \neq f^{-1}(y)$: if $f(x) = x^2$, $f(y) = y^2$ and $f^{-1}(y) = \sqrt{y}$
4. If f is not one-to-one on the interval $[a,\ b]$, then the area of the surface generated when the graph of f on $[a,\ b]$ is resolved about the x-axis not defined.
   False.

6.7 Physical Applications

1. When the density of an object varies, this formula no longer holds, and we must appeal to calculus.
2. Mass of One-Dimensional Object
   - $m = \int_a^b \rho(x)dx$
   - $M = \rho * V = \rho(x) * dx \quad (dx =$ in this case, length)
   - Linear Density
     - For one-dimensional objects,
       we use linear density with units of mass per length (e.g. g/cm).
     - We will return to mass calculations for two- and three-dimensional objects (plates and solids) later.
   - The units in the integral are consistent.
     ρ has units of mass per length and dx has units of length,
     so $\rho(x)dx$ has units of mass.
3. Work
   - $W = \int_a^b F(x)dx$
   - $W = F * d = F(x) * dx$

4. Lifting Problems

- $W = \int_a^b \rho g A(y) D(y) dy$

- $W = m * g * y = (\rho V) * g * y = (\rho * (A * h)) * g * y$
  $= (\rho(x) * (A(y) * dy)) * g * D(y)$

  - y = the distance to lift (in the future) ≠ height: $D(y) = h - y$
  - $A(y)$ = cross section
    - Geometrically, it is temptable to think that the cross section $= A(x)$
    - But, the cross section $= 2 * [half\ A(y)] * length$

Refer to 6.3 Volume by Slicing — (1)

- E.g. Pumping gasoline
  A cylindrical tank with a length of 10m and a radius of 5m is on its side and half-full of gasoline. How much work is required to empty the tank through an outlet pipe at the top of the tank? (The density of gasoline is $\rho \approx 737 kg/m^3$.)

  $x^2 + y^2 = 5^2 \ \rightarrow \ x = \sqrt{25 - y^2}$

  $A(y) = 2 * \sqrt{25 - y^2} * 10 = 20\sqrt{25 - y^2}$

  $W = \int_{-5}^{0} \rho g A(y) D(y) dy = \int_{-5}^{0} 737 * 9.8 * 20\sqrt{25 - y^2} * (5 - y) dy$

  $= 737 * 9.8 * 20 * (5 \int_{-5}^{0} \sqrt{25 - y^2} dy - \int_{-5}^{0} y\sqrt{25 - y^2} dy)$

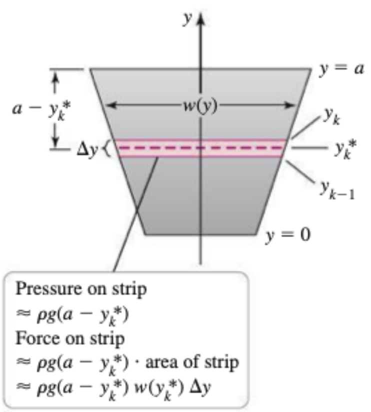
  $\int_{-5}^{0} \sqrt{25 - y^2} dy$ = a quarter of a circle $= \frac{1}{4}\pi * 5^2 = \frac{25\pi}{4}$

  $\int_{-5}^{0} y\sqrt{25 - y^2} dy = -\frac{1}{2}\int_{0}^{25} u^{\frac{1}{2}} du = -\frac{1}{2} * \frac{2}{3} u^{\frac{3}{2}} |_0^{25} = -\frac{1}{3} u^{\frac{3}{2}} |_0^{25} = -\frac{125}{3}$

  $W = 737 * 9.8 * 20 * (5 * \frac{25\pi}{4} - (-\frac{125}{3})) = 20.2\ million\ J$

5. Force and Pressure

- $F = \int_0^a \rho g(b-y)w(y)dy$
- $F = p * A = (\rho * g * h) * A = (\rho * g * (b-y)) * (w(y) * dy)$
- Hydrostatic Pressure: $p = \rho * g * h$
- $y =$ depth $\neq$ height: $y = b - h$
    - Because $A =$ area of strip and $p =$ pressure of strip



Compare with (4)

- $w(y) =$ with respect to y
    - Because the direction of gravity is with respect to y-axis
    - After getting $w(x)$, switch $w(x)$ to $w(y)$

Compare with (4)

7.1 Basic Approaches

1. A variant of substitution

- When you can NEITHER find the denominator's derivative from the numerator NOR covert it into trigonometric integrals.
- $\int \frac{dx}{\pm\sqrt{x} \pm a}$

$u = \pm\sqrt{x} \pm a \ \rightarrow \pm\sqrt{x} = u \pm a$

$du = \pm \frac{dx}{2\sqrt{x}} \ \rightarrow \ dx = \pm 2\sqrt{x}du = 2(u-a)du$

$= 2\int \frac{u-a}{u}du = 2(\int du - a\int \frac{du}{u}) = 2(u - aln|u|) + C$

- E.g. $\int_4^9 \frac{dx}{1-\sqrt{x}}$

$u = 1-\sqrt{x} \ \rightarrow -\sqrt{x} = u-1$

$du = -\frac{dx}{2\sqrt{x}} \ \rightarrow \ dx = -2\sqrt{x}du = 2(u-1)du$

$u(9) = 1-\sqrt{9} = -2 \quad u(4) = 1-\sqrt{4} = -1$

$\int_{-1}^{-2} \frac{2(u-1)du}{u} = 2\int_{-2}^{-1} \frac{1-u}{u}du = 2(\int_{-2}^{-1} \frac{du}{u} - \int_{-2}^{-1} du)$

$= 2(ln|u| \ |_{-2}^{-1} - u|_{-2}^{-1}) = 2((ln|-1| - ln|-2|) - ((-1)-(-2))$

$= 2(-ln2 - 1) = -2ln2 - 2$

Compared with Math 231: Calculus of Functions of One Variables 1

5.5 Substitution Rule — (7)

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2. A variant of trigonometric integrals

- $\int \frac{\pm x \pm a}{x^2 \pm 2bx + c^2}dx$

$= \int \frac{\pm x \pm a}{(x \pm b)^2 + (c^2 - b^2)}dx$

$u = x \pm b \;\; \rightarrow \;\; x = u \pm b$

$\int \frac{\pm(u \pm b) \pm a}{u^2 + (c^2 - b^2)}du$

$= \int \frac{\pm u \pm (b \pm a)}{u^2 + (c^2 - b^2)}du$

$= \int \frac{\pm u}{u^2 + (c^2 - b^2)}du \pm \int \frac{b \pm a}{u^2 + (c^2 - b^2)}du$

$v_1 = u^2 + (c^2 - b^2) \;\; \rightarrow \;\; dv_1 = 2udu$

$\int \frac{\pm u}{u^2 + (c^2 - b^2)}du$

$= \pm \frac{1}{2}\int \frac{2udu}{u^2 + (c^2 - b^2)} = \pm \frac{1}{2}\int \frac{dv_1}{v_1}$

$= \pm \frac{1}{2}ln|v_1| + C$

$= \pm \frac{1}{2}ln|(x \pm b)^2 + (c^2 - b^2)| + C$

$\int \frac{b \pm a}{u^2 + (c^2 - b^2)}du$

$= \frac{b \pm a}{\sqrt{c^2 - b^2}}\int \frac{\frac{du}{\sqrt{c^2 - b^2}}}{(\frac{u}{\sqrt{c^2 - b^2}})^2 + 1}$

$v_2 = \frac{u}{\sqrt{c^2 - b^2}} \;\; \rightarrow \;\; dv_2 = \frac{du}{\sqrt{c^2 - b^2}}$

$= \frac{b \pm a}{\sqrt{c^2 - b^2}}\int \frac{dv_2}{(v_2)^2 + 1}$

$= \frac{b \pm a}{\sqrt{c^2 - b^2}}tan^{-1}v_2 + C$

$= \frac{b \pm a}{\sqrt{c^2 - b^2}}tan^{-1}(\frac{x \pm b}{c^2 - b^2}) + C$

$\int \frac{\pm x \pm a}{x^2 \pm 2bx + c^2}dx = \pm \frac{1}{2}ln|(x \pm b)^2 + (c^2 - b^2)| \pm (b \pm a)tan^{-1}(\frac{x \pm b}{c^2 - b^2}) + C$

- E.g. $\int \frac{x - 2}{x^2 + 6x + 13}dx$

$= \int \frac{x - 2}{(x + 3)^2 + 4}dx$

$u = x + 3 \;\; \rightarrow \;\; x = u - 3$

$\int \frac{(u - 3) - 2}{u^2 + 4}du = \int \frac{u - 5}{u^2 + 4}du$

$= \int \frac{u}{u^2 + 4}du + 5\int \frac{du}{u^2 + 4}du$

$v_1 = u^2 + 4 \;\; \rightarrow \;\; dv_1 = 2udu$

$\int \frac{u}{u^2 + 4}du = \frac{1}{2}\int \frac{2udu}{u^2 + 4} = \frac{1}{2}\int \frac{dv_1}{v_1}$

$= \frac{1}{2}ln|v_1| + C = \frac{1}{2}ln((x + 3)^2 + 4) + C$

$\int \frac{du}{u^2 + 4}du = \frac{1}{2}\int \frac{\frac{du}{2}}{(\frac{u}{2})^2 + 1}$

$v_2 = \frac{u}{2} \;\; \rightarrow \;\; dv_2 = \frac{du}{2}$

$= \frac{1}{2}\int \frac{dv_2}{(v_2)^2 + 1} = \frac{1}{2}tan^{-1}v_2 + C$

$= \frac{1}{2}tan^{-1}(\frac{x + 3}{2}) + C$

$\int \frac{x - 2}{x^2 + 6x + 13}dx$

$= \frac{1}{2}ln((x + 3)^2 + 4) + \frac{5}{2}tan^{-1}(\frac{x + 3}{2}) + C$

3. A variant of the application of half-angle formulas

- $$\begin{cases} sin x = \sqrt{\frac{1 - cos 2x}{2}} \\ cos x = \sqrt{\frac{1 + cos 3x}{2}} \end{cases}$$

- $$\int \sqrt{1 + cos(2ax)} dx$$
$$= \sqrt{2} \int \sqrt{\frac{1 + cos(2ax)}{2}} dx$$
$$= \sqrt{2} \int \sqrt{cos^2(ax)} dx$$
$$= \sqrt{2} \int cos(ax) dx$$
$$= \frac{\sqrt{2}}{a} \int cos(ax)\ a dx$$
$$= \frac{\sqrt{2}}{a} \int cos(ax) d(ax)$$
$$= \frac{\sqrt{2}}{a} sin(ax) + C$$

- $$\int \sqrt{1 - cos(2ax)} dx$$
$$= \sqrt{2} \int \sqrt{\frac{1 - cos(2ax)}{2}} dx$$
$$= \sqrt{2} \int \sqrt{sin^2(ax)} dx$$
$$= \sqrt{2} \int sin(ax) dx$$
$$= \frac{\sqrt{2}}{a} \int sin(ax)\ a dx$$
$$= \frac{\sqrt{2}}{a} \int sin(ax) d(ax)$$
$$= -\frac{\sqrt{2}}{a} cos(ax) + C$$

4. A variant of the application of Pythagorean identities
   A subdivision of "Multiply by 1"

- $\begin{cases} 1 + tan^2x = sec^2x \\ 1 + cot^2x = csc^2x \end{cases}$

• $\int \frac{dx}{secx - 1}$

$$= \int \frac{secx + 1}{(secx + 1)(secx - 1)}dx$$

$$= \int \frac{secx + 1}{sec^2x - 1}dx$$

$$= \int \frac{secx + 1}{tan^2x}dx$$

$$= \int \frac{secx}{tan^2x}dx + \int \frac{1}{tan^2x}dx$$

$$= \int cotx * cscx\ dx + \int cot^2xdx$$

$$= \int cotx * cscx + \int (csc^2x - 1)dx$$

$$= - cscx - cotx - x + C$$

• $\int \frac{dx}{1 - cscx}$

$$= \int \frac{1 + cscx}{(1 + cscx)(1 - cscx)}dx$$

$$= \int \frac{cscx + 1}{1 - csc^2x}dx$$

$$= - \int \frac{1 + cscx}{cot^2x}dx$$

$$= - (\int \frac{1}{cot^2x} + \int \frac{cscx}{cot^2x}dx)$$

$$= - (\int tan^2x + \int tanx * secx\ dx)$$

$$= - (\int (sec^2x - 1)dx + \int tanxsecxdx)$$

$$= - (tanx - x + secx) + C$$

$$= - secx - tanx + x + C$$

5. $\int \frac{dx}{x^{\frac{1}{2}} + x^{\frac{3}{2}}} = \int \frac{dx}{x^{\frac{1}{2}}(1+x)} = \int \frac{dx}{\sqrt{x}(1+(\sqrt{x})^2)}$

$u = \sqrt{x} \;\; \rightarrow \;\; du = \frac{dx}{2\sqrt{x}}$

$= 2\int \frac{\frac{dx}{2\sqrt{x}}}{1+(\sqrt{x})^2} = 2\int \frac{du}{1+u^2}$

$= 2tan^{-1}u + C = 2tan^{-1}\sqrt{x} + C$

- It is tempting to split the denominator, but only the numerator can be split

  $\int \frac{dx}{x^{\frac{1}{2}} + x^{\frac{3}{2}}} \neq \int (x^{-\frac{1}{2}} + x^{-\frac{3}{2}})dx$

6. $\int 3\sqrt{1+sin2x}dx$

$= 3\int \frac{\sqrt{(1-sin2x)(1+sin2x)}}{\sqrt{1-sin2x}}dx$

$= 3\int \frac{\sqrt{1-sin^2 2x}}{\sqrt{1-sin2x}}dx$

$= 3\int \frac{\sqrt{cos^2 2x}}{\sqrt{1-sin2x}}dx$

$= 3\int \frac{cos2x}{\sqrt{1-sin2x}}dx$

$u = 1 - sin2x \;\; \rightarrow \;\; du = -2cos2xdx$

$= -\frac{3}{2}\int \frac{-2cos2xdx}{\sqrt{1-sin2x}}dx$

$= -\frac{3}{2}\int \frac{du}{u^{\frac{1}{2}}} = -\frac{3}{2}(2u^{\frac{1}{2}}) + C$

$= -3\sqrt{u} + C$

7. Different substitutions

- $\int tanx sec^2x \; dx$

  - $u = tanx$
    $du = sec^2x dx$
    $\int udu = \frac{1}{2}u^2 + C$
    $= \frac{1}{2}tan^2x + C$

  - $u = secx$
    $du = tanxsecx \; dx$
    $\int udu = \frac{1}{2}u^2 + C$
    $= \frac{1}{2}sec^2x + C$

- $tan^2x + 1 = sec^2x$
  $\frac{1}{2}sec^2x + C = \frac{1}{2}(tan^2x + 1) + C = \frac{1}{2}tan^2x + \frac{1}{2} + C = \frac{1}{2}tan^2x + C$

- $\int cotxcsc^2x \; dx \;\; : \quad u = cotx \;\&\; u = cscx$

7.2 Integration by Parts

1. Integral by parts

$$\int u dv = uv - \int v du$$

- The integral by parts calculation maul be done without including the constant of integration — as long as it is included in the final result.
- How to choose u? — LIPET
  Logarithmic, Inverse trigonometric, Polynomial, Exponential, Trigonometric

2. Reduction Formula

$$\int x^n e^x dx = x^n e^x - n\int x^{n-1} e^x dx$$

3. Integration by Parts for Definite Integrals

$$\int_a^b u(x)v'(x)dx = u(x) * v(x)\,|_a^b - \int_a^b v(x)u'(x)dx$$

4. Integral of lnx

$$\int lnx dx = xlnx - x + C$$

5. $$\int (sec^2x) * ln(tanx + 2)dx = \int lnw\ dw$$

$$u = lnw \ \rightarrow \ du = \frac{dw}{w}$$

$$dv = dw \ \rightarrow \ v = w$$

$$\int lnw\ dw = wlnw - \int w * \frac{dw}{w} = wlnw - \int dw = wlnw - w + C$$

$$\int (sec^2x) * ln(tanx + 2)dx = (tanx + 2)ln(tanx + 2) - (tanx + 2) + C$$

6. Repeated integration by parts

- $$\int e^{\pm ax} sin(bx) dx$$

  - $u = e^{\pm ax} \ \rightarrow \ du = \pm a e^{\pm ax} dx$

    $dv = sin(bx)dx \ \rightarrow \ v = -\frac{1}{b}cos(bx)$

    $\int u dv = uv - \int v du$

    $\int e^{\pm ax} sin(bx) dx = e^{\pm ax}(-\frac{1}{b}cos(bx)) - \int -\frac{1}{b}cos(bx)(\pm a e^{\pm ax})dx$

    $\int e^{\pm ax} sin(bx) dx = -\frac{1}{b}e^{\pm ax}cos(bx) \pm \frac{a}{b}\int e^{\pm ax}cos(bx)dx$

  - $\int e^{\pm ax} cos(bx) dx$

    $u = e^{\pm ax} \ \rightarrow \ du = \pm a e^{\pm ax} dx$

    $dv = cos(bx)dx \ \rightarrow \ v = \frac{1}{b}sin(bx)$

    $\int u dv = uv - \int v du$

    $\int e^{\pm ax} cos(bx) dx = e^{\pm ax} * \frac{1}{b}sin(bx) - \int \frac{1}{b}sin(bx)(\pm a e^{\pm ax})dx$

    $\int e^{\pm ax} cos(bx) dx = \frac{1}{b}e^{\pm ax}sin(bx) \pm \frac{a}{b}\int e^{\pm ax}sin(bx)dx$

  - $\int e^{\pm ax} sin(bx) dx = -\frac{1}{b}e^{\pm ax}cos(bx) \pm \frac{a}{b}(\frac{1}{b}e^{\pm ax}sin(bx) \pm \frac{a}{b}\int e^{\pm ax}sin(bx)dx)$

    $\int e^{\pm ax} sin(bx) dx = -\frac{1}{b}e^{\pm ax}cos(bx) \pm \frac{a}{b^2}e^{\pm ax}sin(bx) - \frac{a^2}{b^2}\int e^{\pm ax}sin(bx)dx$

    $\frac{a^2 \pm b^2}{b^2}\int e^{\pm ax} sin(bx) dx = -\frac{1}{b^2}e^{\pm ax}(bcos(bx) \pm asin(bx))$

    $\int e^{\pm ax} sin(bx) dx = -\frac{e^{\pm ax}}{a^2 + b^2}(bcos(bx) \pm asin(bx))$

  - $$\begin{cases} \int e^{ax} sin(bx)dx = \frac{e^{ax}}{a^2+b^2}(asin(bx) - bcos(bx)) \\ \int e^{-ax} sin(bx)dx = \frac{-e^{-ax}}{a^2+b^2}(asin(bx) + bcos(bx)) \end{cases}$$

- $\int e^{\pm ax} cos(bx)dx$

- $u = e^{\pm ax} \;\rightarrow\; du = \pm a e^{\pm ax} dx$

  $dv = cos(bx)dx \;\rightarrow\; v = \frac{1}{b} sin(bx)$

  $\int u dv = uv - \int v du$

  $\int e^{\pm ax} cos(bx)dx = e^{\pm ax} * \frac{1}{b} sin(bx) - \int \frac{1}{b} sin(bx)(\pm a e^{\pm ax})dx$

  $\int e^{\pm ax} cos(bx)dx = \frac{1}{b} e^{\pm ax} sin(bx) \pm \frac{a}{b} \int e^{\pm ax} sin(bx)dx$

- $\int e^{\pm ax} sin(bx)dx$

  $u = e^{\pm ax} \;\rightarrow\; du = \pm a e^{\pm ax} dx$

  $dv = sin(bx)dx \;\rightarrow\; v = -\frac{1}{b} cos(bx)$

  $\int u dv = uv - \int v du$

  $\int e^{\pm ax} sin(bx)dx = e^{\pm ax}(-\frac{1}{b} cos(bx)) - \int -\frac{1}{b} cos(bx)(\pm a e^{\pm ax})dx$

  $\int e^{\pm ax} sin(bx)dx = -\frac{1}{b} e^{\pm ax} cos(bx) \pm \frac{a}{b} \int e^{\pm ax} cos(bx)dx$

- $\int e^{\pm ax} cos(bx)dx = \frac{1}{b} e^{\pm ax} sin(bx) \pm \frac{a}{b}(-\frac{1}{b} e^{\pm ax} cos(bx) \pm \frac{a}{b} \int e^{\pm ax} cos(bx)dx)$

  $= \frac{1}{b} e^{\pm ax} sin(bx) \pm \frac{a}{b^2} e^{\pm ax} cos(bx) - \frac{a^2}{b^2} \int e^{\pm ax} cos(bx)dx$

  $\frac{a^2 + b^2}{b^2} \int e^{\pm ax} cos(bx)dx = \frac{e^{\pm ax}}{b^2}(bsin(bx) \pm acos(bx))$

  $\int e^{\pm ax} cos(bx)dx = \frac{e^{\pm ax}}{a^2 + b^2}(bsin(bx) \pm acos(bx))$

- $\begin{cases} \int e^{ax} cos(bx)dx = \frac{e^{ax}}{a^2 + b^2}(bsin(bx) + acos(bx)) \\ \int e^{-ax} cos(bx)dx = \frac{e^{-ax}}{a^2 + b^2}(bsin(bx) - acos(bx)) \end{cases}$

7.3 Trigonometric Integrals

1. $\int sin^m x dx \quad \& \quad \int cos^m dx$

- m is odd → Pythagorean Identities
  - $cos^2x + sin^2x = 1$
  - $1 + tan^2x = sec^2x$
  - $1 + cot^2x = csc^2x$
- m is even → Half-Angle Formulas
  - $sin^2x = \dfrac{1 - cos2x}{2}$
  - $cos^2x = \dfrac{1 + cos2x}{2}$
- Easy way: see below → Reduction Formulas

2. Reduction Formulas

- $\int sin^n x\ dx = -\dfrac{sin^{n-1}x * cosx}{n} + \dfrac{n-1}{n}\int sin^{n-2}x\ dx$
- $\int cos^n x\ dx = \dfrac{cos^{n-1} * sinx}{n} + \dfrac{n-1}{n}\int cos^{n-2}x\ dx$
- $\int tan^n x\ dx = \dfrac{tan^{n-1}x}{n-1} - \int tan^{n-2}x\ dx \quad n \neq 1$
- $\int sec^n x\ dx = \dfrac{sec^{n-2}x * tanx}{n-1} + \dfrac{n-2}{n-1}\int sec^{n-2}x\ dx \quad n \neq 1$

- For odd powers of $tanx$ and $secx$, the use of reduction formula will eventually lead to $\int tan\ dx$ and $\int sex\ dx$.
  - Refer to (4) Integrals of $tanx,\ cotx,\ secx,\ cscx$

3. $\int sin^m x * cos^n x \ dx$

- m is odd and positive $\rightarrow$ $m = 2a + 1$

$$\int sin^m x * cos^n x \ dx = \int cos^n x(1 - cos^2 x)^a * sinx \ dx$$

$$= - \int cos^n x(1 - cos^2 x)^a(-sinx)dx$$

$$= - \int u^n(1 - u^2)^a \ du \quad (u = cosx)$$

- When $a = 2$: $- \int u^n(u^2 - 1)^2 \ du$

$$= - \int u^n(u^4 - 2u^2 + 1)du \quad \neq - \int u^n(u^2 - 2u + 1)du$$

- n is odd and positive $\rightarrow$ $n = 2a + 1$

$$\int sin^m x * cos^n x \ dx = \int sin^m x(1 - sin^2 x)^a * cosxdx$$

$$= \int u^m(1 - u^2)^a du \quad (u = sinx)$$

- When $a = 2$: $\int u^m(u^2 - 1)^2 \ du$

$$= \int u^m(u^4 - 2u^2 + 1)du \quad \neq \int u^m(u^2 - 2u + 1)du$$

- Both m and n are even and positive $\rightarrow$ half-angle formulas

$$\int sin^m x * cos^n x \ dx = \int (\frac{1 - cos2x}{2})^{\frac{m}{2}}(\frac{1 + cos2x}{2})^{\frac{n}{2}} \ dx$$

$$= \frac{1}{2} * \frac{1}{2^{\frac{m+n}{2}}} \int (1 - cos2x)^{\frac{m}{2}}(1 + cos2x)^{\frac{n}{2}} \ 2dx$$

$$= \frac{1}{2^{\frac{m+n}{2}+1}} \int (1 - cosu)^{\frac{m}{2}}(1 + cosu)^{\frac{n}{2}} \ du \quad (u = 2x)$$

4. Integrals of $tanx$, $cotx$, $secx$, $cscx$

- $\int tanx\ dx = -ln|cosx| + C = ln|secx| + C$
- $\int cotx\ dx = ln|sinx| + C$
- $\int secx\ dx = ln|secx + tanx| + C$
- $\int cscx\ dx = -ln|cscx + cotx| + C$

5. $\int sec^{-2}x * tan^3x\ dx$

$$= \int sec^{-2}x(sec^2x - 1)tanx\ dx = \int (tanx - \frac{tanx}{sec^2x})dx$$
$$= -\int \frac{-sinx}{cosx}dx - \int \frac{tanx * secx}{sec^3x}dx = -\int \frac{d(cosx)}{cosx} - \int \frac{d(secx)}{sec^3x}$$
$$= -ln|cosx| - \frac{1}{2}sec^{-2}x + C = -ln|cosx| - \frac{1}{2}cos^2x + C$$

6. $\int_{\frac{\pi}{3}}^{\frac{\pi}{3}} \sqrt{sec^2x - 1}\ dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{3}} \sqrt{tan^2x}\ dx$

- Right solution

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{3}} |tanx|\ dx = 2\int_0^{\frac{\pi}{3}} tanx\ dx$$
$$= 2ln|secx|\ |_0^{\frac{\pi}{3}} = 2ln|\frac{cos0}{cos\frac{\pi}{3}}|$$
$$= 2ln2$$

- Wrong solution

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{3}} tanx\ dx = ln|secx|\ |_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$
$$= ln|\frac{cos(-\frac{\pi}{3})}{cos(\frac{\pi}{3})} = ln|\frac{cos(\frac{\pi}{3})}{cos(\frac{\pi}{3})}|$$
$$= ln1 = 0$$

Refer to 7.4 Trigonometric Substitutions — (3)

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7. $\int tan^m x * sec^n x\ dx$ & $\int cot^m x * csc^n x\ dx$

- n is even and positive $\rightarrow$ $n = 2a$
  - $\int tan^m x * sec^{2a+2} x\ dx = \int tan^m x (tan^2 + 1)^a * sec^2 x dx$
    $= \int u^m (u^2 + 1)^a\ du$
  - $\int cot^m x * csc^{2a+2} x\ dx = -\int cot^m x (cot^2 x + 1)^a * (-cscx) dx$
    $= -\int u^m (u^2 + 1)^a du$
- m is odd and positive $\rightarrow$ $m = 2a + 1$
  - $\int tan^{2a+1} x * sec^n x\ dx = \int sec^n x (sec^2 x - 1)^a (tanx * secx) dx$
    $\int u^n (u^2 - 1)^a\ du$
  - $\int cot^{2a+1} x * csc^n x\ dx = -\int csc^n x (csc^2 x - 1)^a (-cscx * cotx) dx$
    $= -\int u^n (u^2 - 1)^a\ du$
- m is even and n is odd, both of them are positive
  $\rightarrow$ $m = 2a,\ n = 2b + 1$
  - $\int tan^{2a} * sec^{2b+1} x\ dx = \int (sec^2 x - 1)^a sec^{2b+1} x\ dx$
  - $\int sec^n x\ dx = \dfrac{sec^{n-2} x * tanx}{n - 1} + \dfrac{n - 1}{n - 2} \int sec^{n-2} x\ dx \quad n \neq 1$

7.4 Trigonometric Substitutions

1. It turns out that integrals containing the terms $a^2 \pm x^2$ or $x^2 - a^2$ can be simplified using somewhat substitutions involving trigonometric functions.

2. Trigonometric substitutions

The Integral Contains …	Corresponding Substitution	Useful Identity
$a^2 - x^2$	$x = asin\theta \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \quad for\ \lvert x \rvert \leq a$	$a^2 - a^2sin^2\theta = a^2cos^2\theta$
$a^2 + x^2$	$x = atan\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$a^2 + a^2tan^2\theta = s^2sec^2\theta$
$x^2 - a^2$	$x = asec\theta \begin{cases} 0 \leq \theta < \frac{\pi}{2} & for\ x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & for\ x \leq -a \end{cases}$	$a^2sec^2\theta - a^2 = a^2tan^2\theta$

3. Technicalities of the domain of $x = atan\theta$ & $x = asec\theta$

- $\sqrt{a^2 + x^2} = \sqrt{a^2 + (atan\theta)^2} = \sqrt{a^2(1 + tan^2\theta)} = a\lvert sec\theta \rvert$

  When $-\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad sec\theta > 0$

- $\sqrt{x^2 - a^2} = \sqrt{a^2(sec^2\theta - 1)} = \lvert atan\theta \rvert = \begin{cases} atan\theta & if\ 0 \leq \theta < \frac{\pi}{2} \\ -atan\theta & if\ \frac{\pi}{2} < \theta \leq \pi \end{cases}$

Compared with 7.3 Trigonometric Integrals — (6)

- When evaluating a definite integral, you should check the limits of integration to see which of these two cases applies.

4. When the coefficient of x is not 1.

- $1 - \frac{x^2}{a^2} \quad \rightarrow \quad \frac{x}{a} = sin\theta \ \rightarrow\ x = asin\theta$

- $a^2 - b^2x^2 \quad \rightarrow \quad bx = asin\theta \ \rightarrow\ x = \frac{a}{b}sin\theta$

5. Do not further transform x^2

- E.g. $\int \frac{dx}{x^2\sqrt{x^2+9}}$

$x = 3tan\theta \;\; \rightarrow \;\; dx = 3sec^2\theta d\theta$

$\sqrt{x^2+9} = \sqrt{(3tan\theta)^2+9} = 3sec\theta$

$x^2 = 9tan^2\theta \quad \cancel{= 9(sec^2\theta - 1)}$

$$\int \frac{3sec^2\theta d\theta}{(3tan\theta)^2 * 3sec\theta} = \frac{1}{9}\int \frac{sec\theta}{tan^2\theta}d\theta = \frac{1}{9}\int csc\theta * cot\theta \ d\theta$$

$$= -\frac{1}{9}csc\theta + C = -\frac{1}{9}\frac{sec\theta}{tan\theta} + C = -\frac{1}{9} * \frac{\frac{1}{3}\sqrt{x^2+9}}{\frac{1}{3}x} + C$$

$$= -\frac{\sqrt{x^2+9}}{9x} + C$$

6. Watch out the power of $a^2 \pm x^2 \; \& \; x^2 - a^2$
Do not forget to square/cube what you got after transformation

- E.g. $\int \frac{x^2}{(25+x^2)^2}dx$

$x = 5tan\theta \;\; \rightarrow \;\; dx = 5sec^2\theta d\theta$

$25 + x^2 = 25 + (5tan\theta)^2 = 25sec^2\theta$

$$\int \frac{(5tan\theta)^2}{(25sec^2\theta)^2} * 5sec^2\theta d\theta = \frac{1}{5}\int \frac{tan^2\theta}{sec^2\theta}d\theta = \frac{1}{5}\int \sin^2\theta d\theta$$

$$= \frac{1}{5} * \frac{1}{2}(\theta - sin\theta cos\theta) + C = \frac{1}{10}(\theta - \frac{tan\theta}{sec^2\theta}) + C$$

$$= \frac{1}{10}(tan^{-1}(\frac{x}{5}) - \frac{\frac{1}{5}x}{\frac{1}{25}(25+x^2)}) + C$$

$$= \frac{1}{10}(tan^{-1}(\frac{x}{5}) - \frac{5x}{25+x^2}) + C$$

7.5 Partial Fractions

1. Partial fraction decomposition

- Rational functions $\dfrac{3x}{x^2+2x-8} \xrightarrow{\text{method of partial fractions}}$ Partial fraction decomposition $\dfrac{1}{x-1}+\dfrac{2}{x+4}$

- Difficult to integrate $\displaystyle\int \frac{3x}{x^2+2x-8}dx$ Easy to integrate $\displaystyle\int (\frac{1}{x-2}+\frac{2}{x+4})dx$

2. Partial Fractions with Simple Linear Factors

- $f(x)=\dfrac{p(x)}{q(x)}=\dfrac{ax^m \pm \dots}{bx^n \pm \dots} \quad (m<n)$

- $f(x)=\dfrac{ax^m \pm \dots}{(x-r_1)(x-r_2)\dots(x-r_n)}$

- $\dfrac{ax^m \pm \dots}{(x-r_1)(x-r_2)\dots(x-r_n)}=\dfrac{A}{x-r_1}+\dfrac{B}{x-r_2}+\dots+\dfrac{Z}{x-r_n}$

- $ax^m \pm \dots = A(x-r_1)(x-r_2)\dots(x-r_n)+B(x-r_2)\dots(r-x_n)+\dots$

• A Short Cut: let $x=r_1,\ r_2,\ \dots,\ r_n$

3. Partial Fractions for Repeated Linear Factors

$$\frac{ax^m \pm \dots}{(x-r)^n}=\frac{A}{x-r}+\frac{B}{(x-r)^2}+\dots+\frac{Z}{(x-r)^n}$$

4. Partial Fractions with Simple Irreducible Quadratic Factors

$$\frac{ax^m \pm \dots}{(x-r)(ax^2 \pm bx \pm c)}=\frac{A}{x-r}+\frac{Bx+C}{ax^2 \pm bx \pm c}$$

5. Long Division

$$f(x)=\frac{p(x)}{q(x)}=\frac{ax^m \pm \dots}{bx^n \pm \dots}$$

- When $m>n$ $\rightarrow$ Long division

7.8 Improper Integrals

1. Improper Integrals
   - The interval of integration is infinite, or
   - The integrand is unbounded on the interval of integration
2. Improper Integrals over Infinite Intervals
   - f is continuous on $[a, \infty)$: $\displaystyle\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$
   - f is continues on $(-\infty, b]$: $\displaystyle\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$
   - f is continuous in $(-\infty, \infty)$

     $$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_a^c f(x)dx + \lim_{b\to\infty}\int_c^b f(x)dx$$
3. Improper Integrals with an Unbounded Integrand
   - f is continuous on $(a, b]$ with $\lim_{x\to a^+} f(x) = \pm\infty$

     $$\int_a^b f(x)dx = \lim_{c\to a^+}\int_c^b f(x)dx$$
   - f is continuous on $[a, b)$ with $\lim_{x\to b^-} f(x) = \pm\infty$

     $$\int_a^b f(x)dx = \lim_{c\to b^-}\int_a^c f(x)dx$$
   - f is continuous on $[a, b]$ except at the interior point p where f is unbounded

     $$\int_a^b f(x)dx = \lim_{c\to p^-}\int_a^c f(x)dx + \lim_{d\to p^+}\int_d^b f(x)dx$$
   - If the limits exist, then the improper integrals converge;
     otherwise, they diverge.

4. The family $f(x) = \dfrac{1}{x^p}$

$$\int_1^{\infty} \frac{1}{x^p}dx = \frac{1}{p-1} : \quad p > 1 \ \rightarrow \text{divergent}$$

5. $\displaystyle\int_{e^2}^{\infty} \frac{dx}{xln^p x} \quad p > 1$

$$\int \frac{dx}{xln^p x} = \int ln^{-p}x \ d(lnx) = \frac{ln^{-p+1}x}{-p+1} + C = \frac{1}{1-p} * \frac{1}{ln^{p-1}x} + C$$

$$\int_{e^2}^{\infty} \frac{dx}{xln^p x} = \lim_{b\to\infty} \int_{e^2}^{b} \frac{dx}{xln^p x} = \lim_{b\to\infty} \frac{1}{1-p} * \frac{1}{ln^{p-1}x} |_{e^2}^{b}$$

$$= \lim_{b\to\infty} \frac{1}{1-p}(\frac{1}{ln^{p-1}b} - \frac{1}{ln^{p-1}(e^2)}) = \frac{1}{1-p}(\frac{1}{ln^{p-1}\infty} - \frac{1}{2^{p-1} * ln^{p-1}e})$$

$$= \frac{1}{1-p}(0 - \frac{1}{2^{p-1}}) = -\frac{2^{1-p}}{1-p}$$

6. $\displaystyle\int_{-1}^{1} lny^2 dy$

$$u = lny^2 \quad du = \frac{2y}{y^2}dy = \frac{2dy}{y}$$

$$dv = dy \quad v = y$$

$$\int lny^2 dy = ylny^2 - \int y * \frac{2dy}{y} = ylny^2 - \int 2dy = 2ylny - 2y + C$$

$$\int_{-1}^{1} lny^2 dy = 2\int_{0}^{1} lny^2 dy = \lim_{a\to 0^+} 2\int_{a}^{1} lny^2 dy = 4(ylny - y)|_{a}^{1}$$

$$= \lim_{a\to 0^+} 4(-1 - (alna - a)) = -4 - 4(\lim_{a\to 0^+} alna - 0) = -4 - 4\lim_{a\to 0^+} alna$$

$$\lim_{a\to 0^+} alna = \lim_{a\to 0^+} \frac{lna}{1/a} = \lim_{a\to 0^+} \frac{1/a}{-1/a^2} = \lim_{a\to 0^+} -a = 0$$

$$\int_{-1}^{1} lny^2 dy = -4$$

7. $\displaystyle\int_{-2}^{6} \frac{dx}{\sqrt{|x-2|}}$

$$= \lim_{c\to 2^-}\int_{-2}^{c}\frac{dx}{\sqrt{2-x}} + \lim_{d\to 2^+}\int_{d}^{6}\frac{dx}{\sqrt{x-2}}$$

$$= \lim_{c\to 2^-} -\int_{-2}^{c}(2-x)^{-\frac{1}{2}}d(2-x) + \lim_{d\to 2^+}\int_{d}^{6}(x-2)^{-\frac{1}{2}}d(x-2)$$

$$= \lim_{d\to 2^+} 2(x-2)^{\frac{1}{2}}|_{d}^{6} - \lim_{c\to 2^-} 2(2-x)^{-\frac{1}{2}}|_{-2}^{c}$$

$$= \lim_{d\to 2^+} 2((6-2)^{\frac{1}{2}} - (d-2)^{\frac{1}{2}}) - \lim_{c\to 2^-} 2((2-c)^{\frac{1}{2}} - (2+2)^{\frac{1}{2}})$$

$$= 2(2 - \lim_{d\to 2^+}(d-2)^{\frac{1}{2}}) - 2(\lim_{c\to 2^-}(2-c)^{\frac{1}{2}} - 2)$$

$$= 2(2-0) - 2(0-2) = 8$$

A Comparison of different ways of integration

1. Rational functions $\int \frac{xxx}{xxx} dx$

- Numerator does not include x $\quad \rightarrow \quad u =$ denominator
- The power of numerator < that of denominator
    - Denominator $= (ax + b)^2 + c$ $\quad \rightarrow$ Trigonometric integration
    - Denominator $= (xxx)^m (xxx)^n \ldots$ $\quad \rightarrow$ Partial fractions

2. Integration by parts

- $e^x \quad x^n \quad lnx$ trigonometric functions
- Include two or more of factors above or their variants

3. Trigonometric Integration

- $sin^m x \quad cos^m x \quad tan^m x \quad sec^m x$ $\quad \rightarrow$ Reduction formulas
- $\int tanx dx \quad \int cotx dx \quad \int secx dx \quad \int cscx dx$
- $sin^n x * cos^m x \quad tan^m x * sec^n x \quad cot^m x * csc^n x$

4. Trigonometric Substitutions: $a^2 \pm x^2 \quad x^2 - a^2$
5. Half-Angle Formulas: replaced by reduction formulas
6. Pythagorean Identities

- $1 + tan^2 x = sec^2 x$
- $1 + cot^2 x = csc^2 x$

8.1 An Overview

1. Sequences
   $\{a_1, a_2, a_3, \ldots, a_n \ldots\} \quad \{a_n\}_{n=1}^{\infty} \quad \{a_n\}$
2. Sequence
   - Recurrence relation: $a_{n+1} = f(a_n) \quad (n = 1,2,3, \ldots)$
   - Explicit formula: $a_n = f(n) \quad (n = 1,2,3 \ldots)$
3. The switch $(-1)^n$ is used frequently to alternative the signs of the terms of sequence and series.
4. Limit of a Sequence
   $\lim_{n\to\infty} a_n = L$: - exists → convergence

   - does not exist → divergence
5. Infinite Series
   - $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L$
   - If the sequence of partial sums diverges, the infinite series also diverges.
6. A bouncing ball
   - $h_n = h_0 * r^n \quad (n = 0,1,2, \ldots)$
   - $\lim_{n\to\infty} h_n = 0$

7. An analogy between sequences/series and functions

	Sequences/series	Functions
Independent variable	n	x
Dependent variable	a_n	$f(x)$
Domain	Integers e.g., $n = 1,2,3, \ldots$	Real numbers e.g., $\{x : x \geq 1\}$
Accumulation	Sums	Integrals
Accumulation over a finite interval	$\sum_{k=1}^{n} a_k$	$\int_1^n f(x)dx$
Accumulation over an infinite interval	$\sum_{k=1}^{\infty} a_k$	$\int_1^\infty f(x)dx$

8. $\{1, -2, 3, -4, 5, \ldots\}$
   $a_1 = 1 \quad a_{n+1} = -(|a_n| + 1) \quad a_n = (-1)^{n-1} * n$

9. If a sequence of positive numbers converges, then the terms of the sequence must decrease in size.
   False: $a_n = 1 - \dfrac{1}{2^n}$

8.2 Sequences

1. Limits of sequences are no different from limits at an infinite of functions except that the variable n assumes only integer values as $n \to \infty$.

2. Limits of Sequences from Limits of Functions

   - If $\lim_{x \to \infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L.
   - The converse is not true.
     E.g. $a_n = cos 2\pi n \quad \to \quad \lim_{n \to \infty} a_n = 1$ & $\lim_{x \to \infty} cos 2\pi n$ does not exist

3. Limit Laws for Sequences

   - $\lim_{n \to \infty} a_n = A \quad \& \quad \lim_{n \to \infty} b_n = B$
   - $\lim_{n \to \infty} (a_n \pm b_n) = A \pm B$
   - $\lim_{n \to \infty} ca_n = cA$
   - $\lim_{n \to \infty} a_n b_n = AB$
   - $\lim_{n \to \infty} \dfrac{a_n}{b_n} = \dfrac{A}{B} \quad (B \neq 0)$

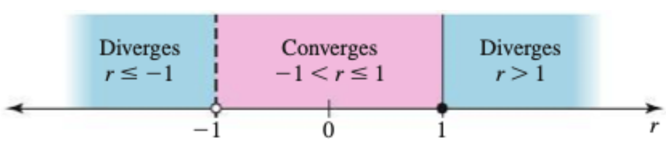
4. The limit of a sequence $\{a_n\}$ is determined by the terms in the tail of the sequence — the terms with large values of n. If the sequences $\{a_n\}$ and $\{b_n\}$ differ in their first 100 terms but have identical terms for $n > 100$, then they have the same limit. For this reason, the initial index of a sequence (for example, $n = 0$ or $n = 1$) is often not specified.

5. Geometric sequences

- $$\lim_{n\to\infty} r^n = \begin{cases} 0 \quad if \ |r| < 1 \\ 1 \quad if \ r = 1 \end{cases} \Bigg\} \rightarrow \ convergent$$
  $$\quad \textit{does not exist} \quad if \ |r| > 1 \ \rightarrow divergent$$

  - However, if $r \leq -1$ or $r > 1$, $\lim_{x\to\infty} r^n$ exists.
    - $r \leq -1: \quad \lim_{x\to\infty} r^n = -\infty$
    - $r > 1: \quad \lim_{x\to\infty} r^n = \infty$

- If $r > 0$, then $\{r^n\}$ is a monotonic sequence.
  If $r < 0$, then $\{r^n\}$ oscillates.



6. The terms of a sequence may remain bounded, but wander chaotically forever without a pattern. In this case, the sequence also diverges.

7. Squeeze Theorem for Sequence
   $a_n \leq b_n \leq c_n \ + \ \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \quad \rightarrow \quad \lim_{n\to\infty} b_n = L$

8. Some optional terminology

- $M =$ upper bound; $N =$ lower bound

- The number M^* is the least upper bound of a sequence (or a set) if it is the smallest of all the upper bounds. It is a fundamental property of the real number that if a sequence (or a nonempty set) is bounded above, then it has a least upper bound. It can be shown that an increasing sequence that is bounded above converges to its least upper bound.
  Similarly, a decreasing sequence that is bounded below converges to its greatest lower bound.

9. Bounded Monotonic Sequences
   A bounded monotonic sequence converges.

10. Growth Rates of Sequences

- $\lim_{n \to \infty} \frac{b_n}{a_n} = 0 \;\; \rightarrow \;\; \{a_n\} >> \{b_n\}$

- $\lim_{n \to \infty} \frac{b_n}{a_n} = \infty \;\; \rightarrow \;\; \{a_n\} << \{b_n\}$

• $\{ln^q n\} << \{n^p\} << \{n^p ln^r n\} << \{n^{p+s}\} << \{b^n\} << \{n!\} << \{n^n\}$
  $p, q, r, s > 0 \quad b > 1$
  - The rankings do not change if a sequence is multiplied by a positive constant.

Compared with Math 231: Calculus of Functions of One Variables 1

4.7 L'Høpital's Rule — (3) & (4)

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11. Limits of a Sequence

- The sequence $\{a_n\}$ converges to L provided the terms of a_n can be made arbitrarily close to L by taking n sufficiently large.

- More precisely, $\{a_n\}$ has the unique limit L if given any $\epsilon > 0$, it is possible to find a positive integer N (depending only on ϵ) such that
  $|a_n - L| < \epsilon \quad$ whenever $n > N$
  • Way to prove $\lim_{n \to \infty} a_n = L$: See below (14) →

- A sequence that does not converge is said to diverge.

- Given any value of $\epsilon > 0$ (no matter how small), you must find a value of N such that all terms beyond a_N are within ϵ of L.

12. If $\lim_{n \to \infty} a_n = 0$ and $\lim_{n \to \infty} b_n = \infty$, then $\lim_{n \to \infty} a_n * b_n = 0$
   False: L'Hopital's Rule.

13. If $\{a_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$ and $\{b_n\} = \{1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \ldots\}$,

then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

True: $\{a_n\}$ converges to 0. The nonzero terms of $\{b_n\}$ converges to 0.

14. Formal proofs of limits: e.g. $\lim_{n\to\infty} \frac{1}{n} = 0$

$$\lim_{n\to\infty} \frac{1}{n} = 0 \;\rightarrow\; a_n = \frac{1}{n} \;\&\; L = 0$$

$$|a_n - L| = |\frac{1}{n} - 0| = |\frac{1}{n}| = \frac{1}{n} < \epsilon$$

$$n > \frac{1}{\epsilon}$$

Given any $\epsilon > 0$, it is possible to find

a positive integer N where $N > \frac{1}{\epsilon}$

such that $|\frac{1}{n} - 0| < \epsilon$ whenever $n > N$.

Therefore, $\{\frac{1}{n}\}$ has the unique limit 0.

Logic Chain

- $|a_n - L| < \epsilon$
- Definition of Limit of a Sequence

15. Find the limit of sequence: $a_n = \dfrac{4^n + 5n!}{n! + 2^n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4^n + 5n!}{n! + 2^n} = \lim_{n\to\infty} \frac{\frac{4^n}{n!} + 5}{1 + \frac{2^n}{n!}} = \frac{0+5}{1+0} = 5$$

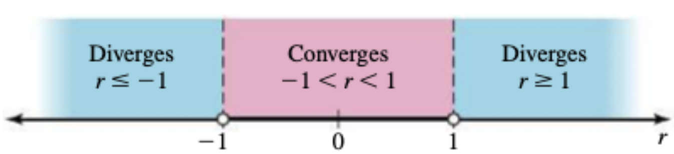
8.3 Infinite Series

1. Geometric Sums

- $S_n = \sum_{k=m}^{n} ar^k = a * \frac{1 - r^{n+1-m}}{1 - r}$
    - $S_n = \sum_{k=0}^{n} ar^k = a * \frac{1 - r^{n+1}}{1 - r}$
    - $S_n = \sum_{k=1}^{n} ar^k = a * \frac{1 - r^n}{1 - r}$

2. Geometric Series

- $|r| < 1 \rightarrow \sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r} \quad (\sum_{k=0}^{\infty} ar^k = \frac{ar^0}{1 - r} = \frac{a}{1 - r})$
    - $\sum_{k=m}^{\infty} ar^k = \frac{ar^m}{1 - r} \quad (= \frac{a_m}{1 - r})$
    - $\sum_{k=1}^{\infty} ar^k = \frac{ar}{1 - r}$
- $|r| \geq 1 \rightarrow$ divergent



3. $\sum_{k=0}^{\infty} 2^{2k} = \sum_{k=0}^{\infty} 4^k = \frac{1}{1 - 4} = -\frac{1}{3}$

4. Decimal expansions as geometric series

- $0.\overline{abc} = \sum_{k=0}^{\infty} 0.abc * (decimal\ places)^k = \dfrac{0.abc}{1-(decimal\ places)}$

  Or $0.\overline{abc} = \sum_{k=1}^{\infty} abc * (decimal\ places)^k = \dfrac{0.abc}{1-(decimal\ places)}$

- $x\,.\,\overline{abc} = x + \sum_{k=0}^{\infty} 0.abc(decimal\ places)^k$

  Or $x\,.\,\overline{abc} = x + \sum_{k=1}^{\infty} abc(decimal\ places)^k$

5. True or False

- Viewed as a function of r,
  the series $1 + r^2 + r^3 + \dots$
  takes on all values
  in the interval $(\frac{1}{2},\ \infty)$

  $S_n = \sum_{k=0}^{\infty} r^k = \dfrac{1}{1-r}$

  $r = 1 - \dfrac{1}{S_n}$

  $|r| = |1 - \dfrac{1}{S_n}| < 1$

  $-1 < 1 - \dfrac{1}{S_n} < 1$

  $0 < \dfrac{1}{s_n} < 2$

  $S_n > \dfrac{1}{2}$

  TRUE

- Viewed as a function of r,
  the series $\sum_{k=1}^{\infty} r^k$ takes on all values
  in the interval $(-\frac{1}{2}, \infty)$

  $S_n = \sum_{k=1}^{\infty} r^k = \dfrac{r}{1-r} \quad \rightarrow \quad r = 1 - \dfrac{1}{s_n + 1}$

  $|r| = |1 - \dfrac{1}{S_n - 1}| < 1$

  $-1 < 1 - \dfrac{1}{S_n - 1} < 1$

  $0 < \dfrac{1}{S_n + 1} < 2$

  $S_n > -\dfrac{1}{2}$

  TRUE

6. Telescopic Series

- $$\sum_{k=m}^{\infty} \frac{a}{(bk+c)(bk+d)} \quad (c > d)$$
$$= \sum_{k=m}^{\infty} \frac{a}{c-d}\left(\frac{1}{bk+d} - \frac{1}{bk+c}\right)$$
$$= \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a}{c-d}\left(\frac{1}{bm+d} - \frac{1}{bn+c}\right)$$
$$= \frac{a}{c-d} \lim_{n\to\infty} \frac{bn+(c-bm+d)}{(bm+d)(bn+c)}$$
$$= \frac{a}{c-d} * \frac{b}{b(bm+d)} = \frac{a}{(c-d)(bm+d)}$$

- $c > d :$ $bm+d$ (always the smaller term)

• E.g. $\sum_{k=1}^{\infty} ln\frac{k+1}{k} = \sum_{k=1}^{\infty} (ln(k+1) - lnk)$
$$= \lim_{n\to\infty} (ln(n+1) - ln1) = \lim_{n\to\infty} ln(n+1) = \lim_{n\to\infty} ln\infty = \infty$$

- How many terms are left $= \frac{c-d}{b}$

• E.g.1: $\sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+4)}$
$$= \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{1}{3k+1} - \frac{1}{3k+4}\right)$$
$$\frac{c-d}{b} = \frac{4-1}{3} = 1$$
$$= \frac{1}{3}\lim_{n\to\infty}\left(1 - \frac{1}{3n+4}\right)$$
$$= \lim_{n\to\infty} \frac{n+1}{3n+4} = \frac{1}{3}$$

• E.g.2: $\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k+3}}\right)$
$$\frac{c-d}{b} = \frac{3-1}{1} = 2$$
$$= \lim_{n\to\infty}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{n-1+3}} - \frac{1}{\sqrt{n+3}}\right)$$
$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{\infty+2}} - \frac{1}{\sqrt{\infty+3}}\right)$$
$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - 0 - 0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}$$

- A general way to figure out how many terms are left

$$\sum_{k=m}^{\infty}(A-B)=A_m-B_m+A_{m+1}-B_{m+1}+\ldots+A_\infty-B_\infty$$

Among $A_n \quad -B_n \quad A_{n+1} \quad -B_{n+1}$, the sum of which two $=0$

- E.g. $\sum_{k=1}^{\infty}(sin(\frac{(k+1)\pi}{2k+1}-sin(\frac{k\pi}{2k-1}))$

$$A_n=sin(\frac{(n+1)\pi}{2n+1}) \qquad -B_n=-sin(\frac{n\pi}{2n-1})$$

$$A_{n+1}=sin(\frac{(n+2)\pi}{2(n+1)-1})=sin(\frac{(n+2)\pi}{2n+1})$$

$$-B_{n+1}=-sin(\frac{(n+1)\pi}{2(n+1)-1})=-sin(\frac{(n+1)\pi}{2n+1})$$

$$A_n-B_{n+1}=0 \quad \rightarrow \quad A_n-B_n+A_{n+1}-B_{n+1}=-B_n+A_{n+1}$$

$$S_n=-B_m+A_n=-sin(\frac{m\pi}{2m-1})+sin(\frac{(n+1)\pi}{2n+1})$$

$$=-sin(\frac{\pi}{2-1})+sin(\frac{(n+1)\pi}{2n+1})=-sin\pi+sin(\frac{(n+1)\pi}{2n+1})$$

$$=sin(\frac{(n+1)\pi}{2n+1})$$

$$\sum_{k=1}^{\infty}(sin(\frac{(k+1)\pi}{2k+1}-sin(\frac{k\pi}{2k-1}))=\lim_{n\to\infty}sin(\frac{(n+1)\pi}{2n+1})$$

$$=sin(\lim_{n\to\infty}\frac{(n+1)\pi}{2n+1})=sin(\frac{\pi}{2})=1$$

7. Geometric Series & Telescopic Series

$$\sum_{k=1}^{\infty}(\frac{1}{2^k}-\frac{1}{2^{k+1}})$$

- Telescopic Series

$$\sum_{k=1}^{\infty}(\frac{1}{2^k}-\frac{1}{2^{k+1}})$$

$$=\lim_{n\to\infty}(\frac{1}{2}-\frac{1}{2^{n+1}})$$

$$=\frac{1}{2}$$

- Geometric Series

$$\frac{1}{2^k}-\frac{1}{2^{k+1}}=\frac{1}{2^{k+1}}=\frac{1}{2}*(\frac{1}{2})^k$$

$$\sum_{k=1}^{\infty}(\frac{1}{2^k}-\frac{1}{2^{k+1}})=\sum_{k=1}^{\infty}\frac{1}{2}*(\frac{1}{2})^k$$

$$=\frac{\frac{1}{2}*\frac{1}{2}}{1-\frac{1}{2}}=\frac{1}{2}$$

8.4 The Divergence and Integral Tests

1. Divergence Test

- $\sum a_k$ converges $\rightarrow$ $\lim_{k\to\infty} a_k = 0$
    - $\lim_{k\to\infty} a_k = 0 \nrightarrow \sum a_k$ converges
- $\lim_{k\to\infty} \neq 0 \rightarrow \sum a_k$ diverges

2. Harmonic Series

   The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$ diverges,

   even though the terms of the series approach zero.

3. Integral Test

- Suppose f is a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1,2,3, \ldots.$

  Then $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)dx$ either both converge or both diverge.

    - $\sum_{k=m}^{\infty} a_k$ & $\int_m^{\infty} f(x)dx \quad \rightarrow \quad x \geq m$

- In the case of convergence, the value of the integral is not equal to the value of the series.

- The outcome of the test is not affected by
    - Adding or subtracting a few terms in the series
    - Changing the lower limit of integration to another finite point

4. The fact that infinite series are sums and that integrals are limits of sums suggests a connection between series and integrals.

5. Steps of integral tests

- Check if the function is continuous and positive when $x \geq m$
- Check if the function is decreasing: $f'(x) < 0$
- $\displaystyle\sum_{k=m}^{\infty} a_k \ \rightarrow \int_m^{\infty} f(x)dx = \lim_{b\to\infty} \int_m^b f(x)dx$

6. Convergence of the p-Series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$

- $p > 1 \ \rightarrow$ convergence
- $p \leq 1 \ \rightarrow$ divergence

7. Estimating Series with Positive Terms

Let f be continuous, positive, decreasing function, for $x \geq 1$,

Let $a_k = f(k)$, for $k = 1,2,3, \ \ldots.$

Let $S = \displaystyle\sum_{k=1}^{\infty} a_k$ be a convergent series

Let $S_n = \displaystyle\sum_{k=1}^{n} a_k$ be the sum of the first n terms of the series.

- The reminder $R_n = S - S_n$ satisfies $R_n < \displaystyle\int_n^{\infty} f(x)dx$.
- The exact value of the series is bounded as follows

$$S_n + \int_{n+1}^{\infty} f(x)dx < \sum_{k=1}^{\infty} a_k < S_n + \int_n^{\infty} f(x)dx$$

Or $\quad L_n < S < U_n$

- $L_n = S_n + \displaystyle\int_{n+1}^{\infty} f(x)dx$
- $U_n = S_n + \displaystyle\int_n^{\infty} f(x)dx$

8. Properties of Convergent Series

- $\sum a_k$ & $\sum ca_k$
    - $\sum a_k$ converges $\rightarrow$ $\sum ca_k$ converges
    - $\sum a_k = A \rightarrow \sum ca_k = cA$
- $\sum a_k$ & $\sum b_k$
    - $\sum a_k$ converges & $\sum b_k$ converges $\rightarrow$ $\sum (a_k \pm b_k)$ converges
    - $\sum a_k = A$ & $\sum b_k = B \rightarrow \sum (a_k \pm b_k) = A \pm B$

Compared with 9.2 Properties of Power Series — (8)

- If M is a positive integer, $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=M}^{\infty} a_k$ either both converge or both diverge.
    - Whether a series converges does not depend on a finite number of terms added to or removed from the series.

Compared with (3):

The outcome of the integral test is not affected by

- Adding or subtracting a few terms in the series
- Changing the lower limit of integration to another finite point

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- However, the value of a convergent series does changes if nonzero terms are added or removed.

- $\sum a_k = \sum b_k \pm \sum c_k$
    - Convergence $\pm$ convergence = convergence
    - Divergence $\pm$ convergence = divergence
    - Divergence $\pm$ divergence $\neq$ divergence

9. Integral Test & Estimating Series with Positive Terms

- "Estimating Series with Positive Terms" is based on "Integral Test".
- "Estimating Series with Positive Terms"
  - Estimate the value of a convergent series with positive values.
  - R_n is the error in approximating a convergent series.

10. Geometric Series > p-Series > Harmonic Series

- Geometric Series & p-Series

	Form	Variable	Convergence	Divergence
Geometric Series	$\sum_{k=0}^{\infty} ar^k$	Power	$\lvert base \rvert < 1$	$\lvert base \rvert \geq 1$
p-Series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	Base	$power > 1$	$power \leq 1$

  - The application of p-Series is more common than Geometric Series

- Harmonic Series is a derivative (衍生品) of p-Series.
  - When p-Series $p = 1$ is harmonic Series

8.5 The Ratio, Root, and Comparison Tests

1. Ratio Test

- Let $\sum a_k$ be an infinite series with positive terms
- Let $r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.
  - $0 \le r < 1$ $\to$ convergent
  - $r > 1$ (including $r = \infty$) $\to$ divergent

  - It comes from Divergence Test

  - $r = 1$ $\to$ inconclusive

2. Root Test

- Let $\sum a_k$ be an infinite series with nonnegative terms
- Let $\rho = \lim_{k\to\infty} \sqrt[k]{a_k}$.
  - $0 \le \rho < 1$ $\to$ convergent
  - $\rho > 1$ (including $\rho = \infty$) $\to$ divergent
  - $\rho = 1$ $\to$ inconclusive

3. Comparison Test

- Let $\sum a_k$ and $\sum b_k$ be series with positive terms
  - $0 < a_k \le b_k$ & $\sum b_k$ converges $\to$ $\sum a_k$ converges
  - $0 < b_k \le a_k$ & $\sum b_k$ diverges $\to$ $\sum a_k$ diverges
- Whether a series converges depends on the behaviour of terms in the tail (large values of the index). So the inequalities $0 < a_k \le b_k$ and $0 < b_k \le a_k$ need not hold for all terms of the series. They must hold for all $k > N$ for some positive integer N.

4. Limit Comparison Test

- Let $\sum a_k$ and $\sum b_k$ be series with positive terms
- Let $\lim_{k\to\infty} \frac{a_k}{b_k} = L$
  - $0 < L < \infty \ \rightarrow \sum a_k$ and $\sum b_k$ either both converge or diverge
  - $L = 0 \ \& \ \sum b_k$ converges $\rightarrow \sum a_k$ converges
  - $L = \infty \ \& \ \sum b_k$ diverges $\rightarrow \sum a_k$ diverges

5. Divergence Test & Root Test

- Root test is a variant of divergence test.
  - Divergence test: $\lim_{n\to\infty} a_k$
  - Root test: $\lim_{n\to\infty} \sqrt[k]{a}_k$
- E.g.
  - $a_k = \dfrac{ak^n \pm \ldots}{bk^n \pm \ldots}$ $\rightarrow$ divergence test
  - $a_k = (\dfrac{ak^n \pm \ldots}{bk^n \pm \ldots})^{k \ \& \ k^2 \ \& \ ak}$ $\rightarrow$ root test
- Divergence & Convergence
  - Divergence test → divergence ONLY
  - Convergence → Root test ONLY
- 有些明显极限=0的 → 排除 Divergence test

6. Comparison Test & Limit Comparison Test

- Limit Comparison Test

  $L = 0 \;\&\; \sum b_k$ converges $\rightarrow \sum a_k$ converges

  $L = \lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = 0 \;\rightarrow\; a_k < b_k :\quad b_k$ converges $\rightarrow\; a_k$ converges

- Comparison Test

  $0 < a_k \le b_k \;\&\; \sum b_k$ converges $\rightarrow \sum a_k$ converges

• Limit Comparison Test: $\lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = L$

  $L = \infty \;\&\; \sum b_k$ diverges $\rightarrow \sum a_k$ diverges

  $L = \lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = \infty \;\rightarrow\; a_k > b_k :\quad b_k$ diverges $\rightarrow\; a_k$ diverges

• Comparison Test

  $0 < b_k \le a_k \;\&\; \sum b_k$ diverges $\rightarrow \sum a_k$ diverges

7. Ratio test and limit comparison test are both according to growth rates of functions

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Compared with Math 231: Calculus of Functions of One Variables 1

4.7 L'Høpital's Rule — (3) & (4)

Growth Rates of Functions (as $x \to \infty$)

Suppose f and g are functions with $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = \infty$

- f grows faster than g: $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \infty$ or $\lim\limits_{x\to\infty} \dfrac{g(x)}{f(x)} = 0$

- f and g have comparable growth rates: $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = M$

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8. Guidelines for choosing a test

- Divergence Test — the first test of trying / being ruled out

  - $a_k = \dfrac{ak^n \pm \dots}{bk^n \pm \dots}$

- Comparison Test — the most usual test

  - Apparently $\lim\limits_{n\to\infty} a_k = 0$
    (those divergence test apparently does not apply for)

  - About p-Series

    $$a_k = \frac{1}{ak^n \pm \dots} = \begin{cases} n > 1: & \frac{1}{ak^n + \dots} < \frac{1}{k^n} \quad \to \ convergence \\ n = 1: & \frac{1}{ak - \dots} > \frac{1}{k} \quad \to \ divergence \end{cases}$$

  - a_k includes lnk : $\quad lnk > 1$

    - Numerator includes lnk $\to$ convergence

    - Denominator includes lnk $\to$ divergence

    - $a_k = \begin{cases} \frac{1}{ak^n * lnk} < \frac{1}{k^n} \ (n > 1) & \to \ convergence \\ \frac{alnk}{bk} > \frac{1}{k} & \to \ divergence \end{cases}$

  - a_k includes trigonometry: $\quad -1 \leq sinx \,\&\, cosx \leq 1$

    - $a_k = \begin{cases} n = 1: & \frac{sin(\dots) \,\&\, cos(\dots)}{k} \geq -\frac{1}{k} \quad \to \quad divergence \quad (idk) \\ n > 1: & \frac{sin(\dots) \,\&\, cos(\dots)}{k^n} \leq \frac{1}{k^n} \quad \to \quad convergence \end{cases}$

- Limit Comparison Test — Comparison Test's secondary substitute
  - In most cases, it can be replaced by Comparison Test
    - $a_k = \dfrac{ak^m \pm \dots}{bk^n \pm \dots} \approx \dfrac{k^m}{k^n} = \dfrac{1}{k^{n-m}} = b_k \quad (m < n)$
    - $a_k = \dfrac{\sqrt[p]{ak^m \pm \dots}}{\sqrt[q]{bk^n \pm \dots}} \approx \dfrac{k^{\frac{m}{p}}}{k^{\frac{n}{q}}} = \dfrac{1}{k^{\frac{n}{q}-\frac{m}{p}}} = b_k \quad (\dfrac{n}{p} > \dfrac{m}{p})$
  - Unless Comparison Test does not work
- Root Test — a variant of Divergence Test
  - Exponent includes k
  - L'Høpital's Rule: $1^\infty \quad 0^0 \quad \infty^0$
- Ratio Test — those including $k!$ must be ratio test
  - $k! \quad k^k \quad a^k$

9. $(a(k+1))! \ \rightarrow \ (ak)!$

e.g. $(3(k+1))! = (3k+3)! = (3k+3)(3k+2)(3k+1)(3k)!$

10. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^{lnk}}$

$a_k = \dfrac{1}{k^{lnk}} > \dfrac{1}{k^2} = b_k$

$b_k = \dfrac{1}{k^2}: \quad r = 2 > 1 \ \rightarrow \ b_k$ converges

$a_k > b_k \ \rightarrow \ a_k$ converges

11. $\displaystyle\sum_{k=1}^{\infty} sin^2\frac{1}{k}$

$a_k = sin^2\dfrac{1}{k} \quad b_k = \dfrac{1}{k^2}$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{sin^2\frac{1}{k}}{\frac{1}{k^2}} = \lim_{k\to\infty} (\frac{sin\frac{1}{k}}{\frac{1}{k}})^2$$

$t = \dfrac{1}{k}: \quad as\ k \rightarrow \infty,\ t \rightarrow 0$

$$\lim_{t\to\infty} (\frac{sint}{t})^2 = (\lim_{t\to\infty} \frac{sint}{t})^2 = 1$$

b_k converges $\rightarrow$ a_k converges

12. $\displaystyle\sum_{k=1}^{\infty} tan\frac{1}{k}$

13. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{lnk}$ diverges

Comparison Test: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges; $\quad \displaystyle\sum_{k=1}^{\infty} \frac{1}{k} < \sum_{k=1}^{\infty} \frac{1}{lnk}$

8.6 Alternating Series

1. For the alternating harmonic series, the odd terms of the sequence of partial sums forma decreasing sequence and the even terms form an increasing sequence.
   As a result, the limit of partial sums lies between any two consecutive terms of the sequence.

2. Alternating Series Test

   - $\sum (-1)^{k+1} a_k$ convergee if

   - Non-increasing in magnitude: $0 \leq a_{k+1} \leq a_k$
   - $\lim_{k \to \infty} a_k = 0$

3. There is a potential confusion.

   - For series of positive terms, $\lim_{k \to \infty} a_k = 0$ does not imply convergence.
   - For series with non-increasing terms, $\lim_{k \to \infty} a_k = 0$ does imply convergence.

4. Depending on the sign of the first term of the series, an alternating series may be written with $(-1)^k$ or $(-1)^{k+1}$

5. Alternating Harmonic Series

   - $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} \ldots$ converges.
   - Even though $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} \ldots$ diverges.

6. Ramainder in Alternating Series

- Let $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ be a convergent alternating series with terms that are non-increasing in magnitude.
- Let $R_n = S - S_n$ be the remainder in approximating the value of that series by the sum of its first n terms.
- Then $|R_n| \leq a_{n+1}$.
- In other words, the magnitude of the remainder is less than or equal to the magnitude of the first neglected term.

7. Absolute and Conditional Convergence

- $\sum |a_k|$ converges $\rightarrow \sum a_k$ converges absolutely.
- $\sum |a_k|$ diverges $\rightarrow \sum a_k$ converges conditionally.

• The distinction between absolute and conditional convergence is relevant only for series of mixed sign, which includes alternating series. If a series of positive terms converges, it converges absolutely; conditional convergence does not apply.

• Absolutely & Conditionally
  - Absolutely: in terms of absolute values
  - Conditionally: without absolute values

8. Absolute Convergence Implies Convergence

- $\sum |a_k|$ converges $\rightarrow \sum a_k$ converges.
- $\sum a_k$ diverges $\rightarrow \sum |a_k|$ diverges.

9. Typical convergent limits

$$ln2 = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} \qquad \frac{1}{e} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!} \qquad \frac{\pi}{4} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$

10. Determine how many terms of the following convergent series must be summed to be sure that the remainder is less than 10^{-3} in magnitude. Although you do not need it, the exact value of the series is given in each case.

- $ln2 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$

$$|R_n| = |S - S_n| = a_{n+1} \leq \frac{1}{n+1} < 10^{-3} \quad \rightarrow \quad n = 1000$$

- When it starts from $k = 0$, then $|R_n| = |S - S_n| = a_n$

• $\frac{1}{e} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$

$$|R_n| = |S - S_n| = a_{n+1} = \frac{1}{k!} \leq 10^{-3} \quad \rightarrow \quad n = 7$$

11. Estimate the value of the following convergent series with an absolute error less than 10^{-3}.

- $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^5}$

$$|R_n| = |S - S_n| = a_{n+1} \leq \frac{1}{(n+1)^5} < 10^{-3} \quad \rightarrow \quad n = 3$$

$$\sum_{k=1}^{3} \frac{(-1)^k}{k^5} \approx -0.973$$

12. $\sum_{k=1}^{\infty} \frac{cos\pi k}{k^2}$: $cos\pi k = (-1)^k$

13. A series that converges conditionally must converge.

14. Steps of Alternating Series Test

- Check if $\lim_{k\to\infty} a_k = 0$ first
    - The Divergence Test applies to all series (including alternating series).

Compared with (14): see below →

- Check if a_k is non-increasing in magnitude
    - $f(k) = a_k \quad \rightarrow \quad f'(k) \le 0$
        - $f(k) = g^k(k) \quad \rightarrow \quad g'(k) \le 0$

          E.g. $(1+\frac{1}{k})^k$: $\quad g(k) = 1 + \frac{1}{k} \quad \rightarrow \quad g'(k) = -\frac{1}{k^2} < 0$
    - $\frac{a_{k+1}}{a_k} \le 1$: when a_k includes $k!$

      E.g. $\sum_{k=1}^{\infty} (-1)^{k+1}\frac{k!}{k^k}$: $\quad \frac{a_{n+1}}{a_n} = \frac{\frac{(k+1)!}{(k+1)^{k+1}}}{\frac{k!}{k^k}} = \frac{\frac{(k+1)k!}{(k+1)(k+1)^k}}{\frac{k!}{k^k}} = (\frac{k}{k+1})^k < 1$

      $a_k = \frac{k!}{k^k}$ is non-increasing in magnitude

Compared with Ratio Test: 8.5 — (1)

If you find this message, thank you for taking my notes seriously.
My depression has been triggered since here, because something happened.
24th March, 2022 — I will always remember this day.
I will continue my work till the end.
May all of my readers do well in your lives!

15. Determine if an alternating series is absolute or conditional convergence

- Divergence Test → absolute divergence
- Comparison / Ratio Test → absolute convergence
- Absolute and Conditional Convergence
  - $\sum |a_k|$ converges → $\sum a_k$ converges absolutely
  - $\sum |a_k|$ diverges → $\sum a_k$ diverges conditionally
    - Alternating Series Test
      - Converges conditionally
      - Does not converge conditionally
- E.g. $\sum_{k=1}^{\infty} \frac{(-1)^k k}{2k+1}$

$\sum |\frac{(-1)^k k}{2k+1}| = \sum \frac{k}{2k+1}$

Divergence Test: $\lim_{k\to\infty} \frac{k}{2k+1} = \frac{1}{2} \neq 0$ → divergence

→ $\sum_{k=1}^{\infty} \frac{(-1)^k k}{2k+1}$ converges conditionally

Alternating Series Test:

$a_k = \frac{k}{2k+1}$ is non-increasing; $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{2k+1} = \frac{1}{2} \neq 0$

→ $\sum_{k=1}^{\infty} \frac{(-1)^k k}{2k+1}$ does not converge conditionally

A summary of Special Series and Convergence Tests

1. Special Series and Convergence Tests

- Geometric Series: $\sum_{k=0}^{\infty} ar^k \; (a \neq 0)$
  - Convergence: $|r| < 1$
  - Divergence: $|r| \geq 1$
  - $|r| < 1 \rightarrow \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$
- Divergence Test
  - $\lim_{k\to\infty} a_k \neq 0$
  - Cannot be used to prove convergence
- Integral Test
  - Continuous, positive and decreasing
  - Convergence: $\int_1^{\infty} f(x)dx$ converges
  - Divergence: $\int_1^{\infty} f(x)dx$ diverges
  - The value of the integral is not the value of the series.
- p-Series: $\sum_{k=0}^{\infty} \frac{1}{k^p}$
  - Convergence: $p > 1$
  - Divergence: $p \leq 1$
  - Useful for comparison tests
- Ratio Test
  - Convergence: $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$
  - Divergence: $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} > 1$
  - Inconclusive: $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$
- Root Test
  - Convergence: $\lim_{k\to\infty} \sqrt[k]{a_k} < 1$
  - Divergence: $\lim_{k\to\infty} \sqrt[k]{a_k} > 1$
  - Inconclusive: $\lim_{k\to\infty} \sqrt[k]{a_k} = 1$
- Comparison Test
  - Convergence

    $0 < a_k \leq b_k \quad \sum_{k=1}^{\infty} b_k$ converges
  - Divergence

    $0 < b_k \leq a_k \quad \sum_{k=1}^{\infty} b_k$ diverges
- Limit Comparison Test
  - Convergence

    $0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty \quad \sum_{k=1}^{\infty} b_k$ converges
  - Divergence

    $\lim_{k\to\infty} \frac{a_k}{b_k} > 0 \quad \sum_{k=1}^{\infty} b_k$ diverges

- Alternating Seires Test
    - $\sum_{k=1}^{\infty} (-1)^k a_k \quad 0 < a_{k+1} \leq a_k$
    - Convergence: $\lim_{k \to \infty} a_k = 0$
    - Divergence: $\lim_{k \to \infty} a_k \neq 0$
    - $|R_n| \leq a_{k+1}$
- Absolute Convergence
    - Convergence: $\sum_{k=1}^{\infty} |a_k|$ converges
    - Applies to arbitrary series

9.1 Approximating Functions with Polynomials

1. Taylor Polynomials

- $p_n(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \ldots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$

- $p_n(x) = \displaystyle\sum_{k=0}^{n} c_k(x-a)^k$

• $c_k = \dfrac{f^{(k)}(a)}{k!} \quad (k = 0,\ 1,\ 2,\ \ldots,\ n)$

2. Remainder in a Taylor Polynomial

   $R_n(x) = f(x) - p_n(x)$

3. Taylor's Theorem (Remainder Theorem)

- $f(x) = p_n(x) + R_n(x)$

- $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

• The term $f^{(n+1)}(c)$ in Taylor's Theorem comes from a Mean Theorem argument.

• Replacing a with c

  - The highest-degree term of the $(n+1)st$ Taylor polynomial p_{n+1} is $\dfrac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}$

  - $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

4. Estimate of the Remainder

   $|R_n(x)| = |f(x) - p_n(x)| \le M\dfrac{|x-a|^{n+1}}{(n+1)!}$

9.2 Properties of Power Series

1. Power Series: $\sum_{k=0}^{\infty} c_k(x-a)^k$

2. Convergence of Power Series

	Interval of Convergence	Radius of Convergencec
1	$(-\infty,\ \infty)$	$R = \infty$
2	$\lvert x-a \rvert < R$	R
3	At a	$R = 0$

- The interval of convergence is symmetric about the center of the series.
- The radius of convergence R is determined by analyzing
  - r from the Ratio Test
  - ρ from the Root Test
  - It says nothing about convergence at the endpoint.

3. Steps of Convergence of Power Series

- Root/Ratio Test
  - Divergence on $(-\infty,\ \infty)$
    - Sometimes, convergence at $x = a$
  - Convergence on $(-\infty,\ \infty)$
  - Convergence on $(-a,\ a)$ $\rightarrow$ see below
- Divergence Test
  Determine if the two endpoints are convergent or divergent.

4. $\sum (\pm ax)^{mk}$ and its variants

- Root Test

$\rho = \lim_{k\to\infty} \sqrt[k]{|\pm ax|^{mk}}$

$= \lim_{k\to\infty} |ax|^m = |ax|^m$

Convergence: $0 \le \rho < 1$

$0 \le |ax|^m < 1$

$-1 < ax < 1$

$-\frac{1}{a} < x < \frac{1}{a}$

Inconclusive: $\rho = 1$

$|ax|^m = 1$

$x = \pm \frac{1}{a}$

Divergence Test

$\lim_{k\to\infty} (\pm ax)^{mk} = \lim_{k\to\infty} (\pm 1)^{mk} \neq 0$

$\to$ Divergence

The interval of convergence

is $(-\frac{1}{a}, \frac{1}{a})$

Variant: $\sum \frac{x^{mk+b}}{a^{nk+c}}$

- Ratio Test

$r = \lim_{k\to\infty} |\frac{a_{k+1}}{a_k}|$

$= \lim_{k\to\infty} |\frac{x^{(mk+b)+m}/a^{(nk+c)+n}}{x^{mk+b}/a^{nk+c}}|$

$= \lim_{k\to\infty} |\frac{x^m}{a^n}| = |\frac{x}{a^{\frac{n}{m}}}|$

Convergence: $0 \le r < 1$

$0 \le |\frac{x}{a^{\frac{n}{m}}}| < 1, \quad -1 < \frac{x}{a^{\frac{n}{m}}} < 1$

$-a^{\frac{n}{m}} < x < a^{\frac{n}{m}}$

Inconclusive: $r = 1 \quad \to \quad x = \pm a^{\frac{n}{m}}$

Divergence Test

$\lim_{k\to\infty} \frac{x^{mk+b}}{a^{nk+c}} = \lim_{k\to\infty} a^{\frac{n}{m}*b-c} \neq 0$

$\to$ Divergence

The interval of convergence

is $(-a^{\frac{n}{m}}, a^{\frac{n}{m}})$

5. $\sum \frac{(\pm ax \pm b)^k}{k!}$ and its variant

- $r = \lim_{k\to\infty} |\frac{a_{k+1}}{a_k}|$

$= \lim_{k\to\infty} |\frac{(\pm ax \pm b)^{k+1}/(k+1)!}{(\pm ak \pm b)^k/k!}|$

$= \lim_{k\to\infty} |\frac{\pm ax \pm b}{k+1}| = 0$

$\to$ Convergence

The interval of convergence

is $(-\infty, \infty)$

- Variant: $\sum k!(\pm ax \pm b)^k$

$r = \lim_{k\to\infty} |\frac{a_{k+1}}{a_k}|$

$= \lim_{k\to\infty} |\frac{(k+1)! * (\pm ax \pm b)^{k+1}}{k! * (\pm ax \pm b)^k}|$

$= \lim_{k\to\infty} |(k+1)(\pm ax \pm b)|$

When $\pm ax \pm b \neq 0 \quad \to \quad x \neq \pm \frac{b}{a}$

$r = \infty \quad \to$ Divergence

When $x = \pm \frac{b}{a} \to r = 0 \to$ Convergence

$\sum k!(\pm ax \pm b)^k$ converges at $x = \pm \frac{b}{a}$

6. Combining Power Series

- Suppose the power series $\sum c_k x^k$ and $\sum d_k x^k$ converge to $f(x)$ and $g(x)$, respectively, on an interval I.

- Sum and difference
  $\sum (c_k \pm d_k)x^k$ converges to $f(x) \pm g(x)$ on I.

- Multiplication by a power
  - $x^m \sum c_k x^k = \sum c_k x^{k+m} \quad (k + m \geq 0)$
  - This series converges to $x^m f(x)$ for all $x \neq 0$ in I.
    When $x = 0$, the series converges to $\lim\limits_{x \to \infty} x^m f(x)$.

- Composition
  - If $g(x) = bx^m$, $\sum c_k g^k(x)$ converges to $f(g(x))$ in I.
  - $f(x) = g(h(x)) \neq g(h(x))h'(x)$
    E.g. $f(x) = \dfrac{1}{1-2x}$

$$g(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$f(x) = g(2x) = \frac{1}{1-2x} = \sum_{k=0}^{\infty} (2x)^k$$

$$\neq (2x)' \ g(2x) = 2g(2x) = \frac{2}{1-2x} = 2\sum_{k=0}^{\infty} (2x)^k$$

- It also applies to power series centered at points other than $x = 0$.
  - “Sum and difference” applies directly.
  - “Multiplication by a power” and “Composition” apply with slight modifications.

Compared with 8.4 The Divergence and Integral Tests — (8)

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7. Differentiating and Integrating Power Series

- $f(x) = \sum c_k(x-a)^k$
- $f'(x) = \sum k * c_k(x-a)^{k-1}$
- $\int f(x)\ dx = \sum c_k \frac{(x-a)^{k+1}}{k+1} + C$

• The differentiated and integrated power series converge on the same interval of convergence: $|x-a| < R$, maybe differ by the two endpoints.

• The differentiated and integrated power series converge to the derivative and indefinite integral of f, respectively.

• It makes no claim about the convergence of the differentiated or integrated series at the endpoints off the interval of convergence.

8. The issue of k when integrating power series

- $f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$
- $f'(x) = \sum_{k=1}^{\infty} k * c_k(x-a)^{k-1} \quad \left(= \sum_{k=0}^{\infty} (k+1)c_k(x-a)^k \ \right)$

9. Two Prototypes of power series

- Fundamental power series: $f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad$ for $|x| < 1$
- A variant: $f(x) = ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \quad$ for $-1 \le x < 1$
- A less common one: $f(x) = tan^{-1}x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$

10. Combining power series

- $f(x) = \dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $|x| < 1$

  - $f(\pm ax) = \dfrac{1}{1 \pm ax} = \sum_{k=0}^{\infty} (\pm ax)^k$

    $|r| = |\pm ax| = a|x| < 1$

    $\rightarrow \ |x| < \dfrac{1}{a}$

    $f(\pm ax)$ converges for $|x| < \dfrac{1}{a}$,

    and diverges at the endpoints.

  - $\pm ax^m f(x) = \dfrac{\pm ax}{1-x} = \pm a \sum_{k=0}^{\infty} x^{m+k}$

    $|r| = |x| < 1$

    $\pm ax^m f(x)$ converges for $|x| < 1$,

    and diverges at the endpoints.

  - $f(\pm ax^m) = \dfrac{1}{1 \pm ax^m} = \sum_{k=0}^{\infty} (\pm 1)^{mk} a x^{mk}$

    Root Test

    $\rho = \lim_{k\to\infty} \sqrt[k]{|(-1)^{mk} a x^{mk}} = \lim_{k\to\infty} a|x|^m$

    $0 \le a|x^m| < 1 \quad |x| < \dfrac{1}{a}$

    $f(\pm ax^m)$ converges for $|x| < \dfrac{1}{a}$,

    and diverges at the endpoints.

- $f(x) = ln(1-x) = -\sum_{k=1}^{\infty} \dfrac{x^k}{k}$

  for $-1 \le x < 1$

  Ratio Test

  $r = \lim_{k\to\infty} |\dfrac{a_{k+1}}{a_k}| = \lim_{k\to\infty} |\dfrac{x^{k+1}/(k+1)}{x^k/k}|$

  $= |x| \lim_{k\to\infty} |\dfrac{k}{k+1}| = |x|$

  $0 \le |x| < 1 \ \rightarrow \ |x| < 1$

  - $f(\pm ax): \quad a|x| < 1 \ \rightarrow \ |x| < \dfrac{1}{a}$

    $f(\pm ax)$ diverges at $x = \dfrac{1}{a}$

    (harmonic series),

    and converges at $x = -\dfrac{1}{a}$

    (alternating harmonic series).

  - $\pm ax^m f(x): \quad |x| < 1$

    $\pm ax^m f(x)$ diverges at $x = 1$,

    and converges at $x = -1$

  - $f(\pm ax^m)$

    $0 \le |r| = |\pm ax^m| = a|x^m| < 1$

    $\rightarrow \ |x| < \dfrac{1}{a}$

    $f(\pm ax^m)$ diverges at $x = \dfrac{1}{a}$,

    and converges at $x = -\dfrac{1}{a}$.

	m is even	m is odd
$-x^m$	$[-1, 1]$	$(-1, 1]$
x^m	$(-1, 1)$	$[-1, 1)$

11. Differentiating and integrating power series

- Differentiating power series

- $f(\pm ax) = \dfrac{1}{1 \pm ax} = \sum_{k=0}^{\infty} (\pm ax)^k$

  $|\pm ax| = a|x| < 1 \;\rightarrow\; |x| < \dfrac{1}{a}$

  $f(\pm ax)$ converges for $|x| < \dfrac{1}{a}$

- $f'(\pm ax) = -\dfrac{\pm a}{(1 \pm ax)^2}$

  $= \pm a \sum_{k=1}^{\infty} k(\pm ax)^{k-1}$

  $g_1(x) = -\dfrac{1}{\pm a} f'(\pm ax)$

  $= \dfrac{1}{(1 \pm ax)^2} = -\sum_{k=1}^{\infty} k(\pm ax)^k$

  $g_1(x)$ converges for $|x| < \dfrac{1}{a}$

  $g_2(x) = \dfrac{1}{2a^2} f''(\pm ax)$

  $g_2(x)$ converges for $|x| < \dfrac{1}{a}$

- $f(\pm x^m) = \dfrac{1}{1 \pm x^m}$

  $= \sum_{k=0}^{\infty} (\pm 1)^k x^{mk}$

  Root Test: converges for $|x| < 1$

  $f'(\pm x^m) = -\dfrac{\pm m x^{m-1}}{(1 \pm x^m)^2}$

  $= \sum_{k=1}^{\infty} (\pm 1)^k * m x^{mk-1}$

  $g(x) = -\dfrac{1}{\pm m} f'(\pm x^m) = \dfrac{x^{m-1}}{(1 \pm x^m)^2}$

  $= \sum_{k=1}^{\infty} (\pm 1)^{k-1} x^{mk-1}$

  $g(x)$ converges for $|x| < 1$

- Integrating power series

- $f(x) = \dfrac{1}{1 - x} = \sum_{k=0}^{\infty} x^k$

  $\int f(\pm ax)dx = \int \dfrac{dx}{1 \pm ax}$

  $= \dfrac{1}{\pm a} ln(1 \pm ax) + C = \sum_{k=0}^{\infty} \dfrac{(\pm ax)^{k+1}}{\pm a(k+1)}$

  $g(x) = \pm a \int f(\pm ax)dx$

  $= ln(1 \pm ax) = \sum_{k=0}^{\infty} \dfrac{(\pm ax)^{k+1}}{k+1}$

  $1 \pm ax \neq 0 \;\rightarrow\; x \neq \pm \dfrac{1}{a}$

  $g(x)$ converges on $[-\dfrac{1}{a}, \dfrac{1}{a})$ or $(-\dfrac{1}{a}, \dfrac{1}{a}]$

- $f(x^m) = \dfrac{x^{m-1}}{1 - x^m} = \sum_{k=0}^{\infty} x^{mk}$

  $\int f(\pm x^m)dx = \int \dfrac{x^{m-1}}{1 \pm x^m} dx$

  $= \dfrac{1}{\pm m} ln(1 \pm x^m) + C$

  $= \sum_{k=0}^{\infty} (\pm 1)^{k+1} \dfrac{1}{m(k+1)} x^{mk+m}$

  $g(x) = \pm m \int f(x^m)dx = ln(1 \pm x^m)$

  $= \sum_{k=0}^{\infty} (\pm 1)^k \dfrac{1}{k+1} x^{m(k+1)}$

  convergence: $|x| < 1$

  inconclusive: $|x| = 1 \;\rightarrow\; x = \pm 1$

12. Endpoints issues

- Combining power series

  - $f(x) = \dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, for $|x| < 1$: divergence at the endpoints

  - $f(x) = ln(1-x) = -\sum_{k=1}^{\infty} \dfrac{x^k}{k}$, for $-1 \leq x < 1$

    divergence at the positive endpoint (harmonic series);
    convergence at the negative endpoint (alternative harmonic series).

- Differentiating and integrating power series

  - Differentiating power series: divergence at the endpoints

  - Integrating power series: it depends

13. Power series satisfy the defining property of all functions:
    For each admissible value of x, a power series has at most one value.

Compared with Math 231: Calculus of Functions of One Variable I
— 5.3 Fundamental Theorem of Calculus — (1)

- Power Series: $f(x) = \dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

- Area Functions: $A(x) = \int_a^x f(t)dt$

14. Functions to power series: $ln(\frac{1+x}{1-x})$

- $g(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

$$\int g(x)dx = \int \frac{1}{1-x}dx = -\int \frac{-dx}{1-x} = -\int \frac{d(1-x)}{1-x}$$

$$= -ln(1-x) + C = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$$

- $g(-x) = \frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$

$$\int g(-x)dx = \int \frac{1}{1+x}dx = \int \frac{d(1+x)}{1+x}$$

$$= ln(1+x) + C = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}$$

- $ln(\frac{1-x}{1+x}) = ln(1-x) - ln(1+x) = \int g(x)dx + \int g(-x)dx$

$$= \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} + \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = 2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$$

- $= \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} + \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = \sum_{k=1}^{\infty} \frac{x^k}{k} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$

$$= (x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} \ldots) + (x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots)$$

$$= 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots) = 2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$$

15. $\sum (kx)^k$

- $\rho = \lim\limits_{k\to\infty} \sqrt[k]{|kx|^k} = |x| \lim\limits_{k\to\infty} |k|$

  When $x \neq 0$

  $\rho = \infty > 1$ $\rightarrow$ divergence

- When $x = 0$

  $\rho = 0$ $\rightarrow$ convergence

Refer to Math 231: Calculus of Functions of One Variable 1
— 4.7 L'Hopital's Rule — (12)

9.3 Taylor Series

1. Taylor / Maclaurin Series for a Function: $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

2. There is only one power series representation for a given function about a given point; however, there may be several ways to find it.

3. Binomial Coefficients

   - $\binom{p}{k} = \frac{p(p-1)(p-2)\ldots(p-k+1)}{k!}$

   - $\binom{p}{0} = 1$

4. Binomial Series

   - The Taylor series for $f(x) = (1+x)^p$ centers at 0 is the binomial series.

   - $f(x) = (1+x)^p = \binom{p}{0} + \binom{p}{1} x + \binom{p}{2} x^2 + \ldots + \binom{p}{p} x^p$

   - $\sum_{k=0}^{\infty} \binom{p}{k} x^k = 1 + \sum_{k=0}^{\infty} \frac{p(p-1)(p-2)\ldots(p-k+1)}{k!} x^k$
     $= 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$

   - The series converges for $|x| < 1$ (and possibly at the endpoints, depending on p).

   - If p is a nonnegative integer, the series terminates and results in a polynomial of degree p.

5. Endpoints issue of the binomial series

   - $p \leq -1 \quad \rightarrow \quad (-1, 1)$

   - $-1 < p < 0 \quad \rightarrow \quad (-1, 1]$

   - $p > 0 \; and \; not \; an \; integer \quad \rightarrow \quad [-1, 1]$

6. Convergence of Taylor Series

- The Tylor Series converges to f iif $\lim_{n\to\infty} R_n(x) = 0$
- $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$
- $a \le c \le x$

7. Because the Taylor series is a form of power series, every Taylor series also has an interval of convergence.

- $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \to \quad alwasy: \lim_{n\to\infty} R_n(x) = 0$
- $\lim_{n\to\infty} |R_n(x)| = |f^{(n+1)}(c)| * \lim_{n\to\infty} \frac{|x-a|^{n+1}}{(n+1)!} = |f^{(n+1)}(c)| * 0 = 0$

8. Remainders

- $f(x) = sinx/cosx$

  $f^{(k)}(x) = \pm sinx / \pm cosx \ \le 1$

  $|R_n(x)| = |\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}|$

  $\le \frac{|x-a|^{n+1}}{(n+1)!}$

  $\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \frac{|x-a|^{n+1}}{(n+1)!} = 0$

- Variant of $f(x) = sinx/cosx$

  $f(x) = sin(ax)/cos(ax)$

  $f^{(k)}(x) = \pm a^k sin(ax)/ \pm a^k cos(ax) \ \le a^k$

  $|R_n(x)| = |\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}|$

  $\le a^k \frac{|x-a|^{n+1}}{(n+1)!}$

  $\lim_{n\to\infty} |R_n(x)| = a^k \lim_{n\to\infty} \frac{|x-a|^{n+1}}{(n+1)!} = 0$

- $f(x) = e^{\pm x}$

  $f^{(k)}(x) = \pm e^{\pm x} \le e^{\pm x}$

  $|R_n(x)| = |\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}| \le e^{\pm x} \frac{|x-a|^{n+1}}{(n+1)!}$

  $\lim_{n\to\infty} |R_n(x)| = e^{\pm x} \lim_{n\to\infty} \frac{|x-a|^{n+1}}{(n+1)!} = 0$

9.4 Working with Taylor Series

1. Limits by Taylor Series

   E.g. $\lim_{x \to \infty} (6x^5 sin\frac{1}{x} - 6x^4 + x^2)$

   $t = \frac{1}{x}$: $x \to \infty$, $t \to 0^+$

   $$\lim_{t \to 0^+} (\frac{6sint}{t^5} - \frac{6}{t^4} + \frac{1}{t^2}) = \lim_{t \to 0^+} \frac{6sint - 6t + t^3}{t^5}$$

   $$= \lim_{t \to 0^+} \frac{6(t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{7!} \dots) - 6t + t^3}{t^5}$$

   $$= \lim_{t \to 0^+} \frac{6t - t^3 + \frac{t^5}{20} - 6(\frac{t^7}{7!} - \dots) - 6t + t^3}{t^5}$$

   $$= \lim_{t \to 0^+} \frac{\frac{t^5}{20} - 6(\frac{t^7}{7!} - \dots)}{t^5} = \lim_{t \to 0^+} (\frac{1}{20} - 6(\frac{t^2}{7!} - \dots))$$

   $$= \frac{1}{20} - 0 = \frac{1}{20}$$

2. Power Series for Derivatives

   E.g. $f(x) = sinx$

   $$f'(x) = \frac{d}{dx}(sinx) = cosx$$

   $$= \frac{d}{dx}(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots)$$

   $$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = cosx$$

3. Differential Equations

$$y'(t) = \pm ay \pm b \quad y(0) = k$$

$$y(t) = \sum_{k=0}^{\infty} c_k t^k = c_0 + c_1 t + c_2 t^2 + \dots \ \rightarrow \ c_k = \frac{y^{(k)}(0)}{k!}$$

$$y(0) = c_0 + c_1 * 0 + c_2 * 0^2 + \dots = c_0 = k$$

$$y'(0) = \pm ay(0) \pm b = \pm ak \pm b \ \rightarrow \ c_1 = \frac{y'(0)}{1!} = \pm ak \pm b$$

$$y''(0) = \pm ay'(0) = \pm a(\pm ak \pm b) \ \rightarrow \ c_2 = \frac{\pm a(\pm ak \pm b)}{2!}$$

$$y'''(0) = \pm ay''(0) = \pm a(\pm ak \pm b) \ \rightarrow \ c_3 = \frac{(\pm a)^2(\pm ak \pm b)}{3!}$$

$$y^{(k)}(0) = (\pm a)^{k-1}(\pm ak \pm b) = \pm \frac{\pm ak \pm b}{a}(\pm a)^k \quad (k \geq 1)$$

$$c_k = \frac{y^{(k)}(0)}{k!} = \frac{\pm \frac{\pm ak \pm b}{a}(\pm a)^k}{k!} = \pm \frac{\pm ak \pm b}{a} * \frac{(\pm a)^k}{k!} \quad (k \geq 1)$$

$$y(t) = c_0 + (c_1 t + c_2 t^2 + \dots) = k \pm \frac{\pm ak \pm b}{a} \sum_{k=1}^{\infty} \frac{(\pm ax)^k}{k!}$$

$$= k \pm \frac{\pm ak \pm b}{a}(e^{\pm at} - 1)$$

4. Approximating a Definite Integral

E.g. Approximating the value of the integral $\int_0^1 e^x dx$

with an error no greater than 10^{-2}.

$$\int_0^1 e^x dx = \int_0^1 (1 + x + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!} + \ldots)dx$$

$$= (1 + \frac{x}{1!} + \ldots + \frac{x^{k-1}}{(k-1)!} + \ldots)|_0^1$$

$$= 1 + \frac{1}{1!} + \ldots + \frac{1^{k-1}}{(k-1)!} + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!}$$

$$a_n = \frac{1^{k-1}}{(k-1)!} \rightarrow a_{n+1} = \frac{1^k}{k!}$$

$$|R_n| = |S - S_n| = |a_{n+1}| = |\frac{1^k}{k!}| = \frac{1}{k!} \leq 10^{-2}$$

$$k! \geq 10^2 \rightarrow k \geq 5$$

$$\int_0^1 e^x dx = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = 2.72$$

Refer to 8.6 Alternating Series — (11)

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5. Evaluating Infinite

E.g. $1 - \frac{1}{3} + \frac{1}{5} - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$

$$f(x) = tan^{-1}x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \rightarrow f(1) = tan^{-1}1 = \frac{\pi}{4}$$

6. Representing Functions as Power Series (Mystery Series)

E.g. $\sum_{k=2}^{\infty} \frac{k(k-1)}{3^k} x^k$

- Obtaining a geometric series in order to find the function represented by the series
- The series is not a geometric series because of $k(k-1)$

$\frac{d}{dx}(x^k) = k * x^{k-1} \quad \frac{d^2}{dx^2}(x^k) = k(k-1)x^{k-2}$

- $\sum_{k=2}^{\infty} \frac{k(k-1)}{3^k} x^k = x^2 \sum_{k=2}^{\infty} \frac{1}{3^k} * k(k-1)x^{k-2}$

$= x^2 \sum_{k=2}^{\infty} \frac{1}{3^k} * \frac{d^2}{dx^2}(x^k) = x^2 * \frac{d^2}{dx^2}(\sum_{k=2}^{\infty} (\frac{x}{3})^k)$

$= x^2 * \frac{d^2}{dx^2}(\frac{x^2}{9} * \frac{1}{1-\frac{x}{3}}) = x^2 * \frac{d^2}{dx^2}(\frac{x^2}{9-3x})$

$= x^2 * \frac{-6}{(x-3)^3} = \frac{-6x^2}{(x-3)^3}$

Prototypes of Power Series

1. $$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad |x| < 1$$
$$= 1 + x + x^2 + \ldots + x^k + \ldots$$

2. $$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k \quad |x| < 1$$
$$= 1 - x + x^2 - \ldots + (-1)^k x^k + \ldots$$

3. $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad |x| < \infty$$
$$= 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!} + \ldots$$

4. $$sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad |x| < \infty$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \ldots$$

5. $$cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \quad |x| < \infty$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^k x^{2k}}{(2k)!} + \ldots$$

6. $$tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \quad |x| \le 1$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + \frac{(-1)^k x^{2k+1}}{2k+1} + \ldots$$

7. $$ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} \quad -1 < x \le 1$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{k+1} x^k}{k} + \ldots$$

8. $$-ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k} \quad -1 \le x < 1$$
$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{x^k}{k} + \ldots$$

9. $$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k \quad |x| < 1$$
$$= 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \ldots$$

10. $$sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \quad |x| < \infty$$
$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2k+1}}{(2k+1)!} + \ldots$$

11. $$cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \quad |x| < \infty$$
$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2k}}{(2k)!} + \ldots$$

- It asserts, without proof, that in several cases, the Taylor series for f converges to f at the endpoints of the interval of convergence.