Instructions

- The homework is due on **Friday 4/15 at 5pm ET**.
- There are 4 problems. The last problem is on Git, and it is a programming assignment.
- No extension will be provided, unless for serious documented reasons.
- **Start early!**
- Study the material taught in class, and feel free to do so in small groups, but the solutions should be a product of your own work.
- This is not a multiple choice homework; reasoning, and mathematical proofs are required before giving your final answer.
- Unless otherwise told, the points are distributed evenly between the different sub-problems. E.g., each sub-problem in Problem 1 is worth 2.5 points.

1 PCA [10 points]

In this exercise, you may use Python, Julia, Wolfram Alpha, or your favorite software to do calculations. Explain briefly your steps, and show your answers. Consider the data shown in Table 1 as a $\mathbb{R}^{k\times n}$ matrix where $k=2$ and $n=5$.

(a) Find the sample mean vector $\mu \in \mathbb{R}^2$, and subtract it from the observation vectors.

(b) Let B be the resulting $\mathbb{R}^{2\times 5}$ matrix from step (a). Compute the sample covariance matrix $S = \frac{1}{n-1}BB^T$.

(c) What are the two eigenvalues$\lambda_1 > \lambda_2$ and the respective eigenvectors of S? Compute the variance of the data captured by the top PC as $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

(d) Plot the data points, and visualize the top PC. What do you observe?

Weight (lb)	120	125	125	135	145
Height (in.)	61	60	64	68	72

Table 1: Measurements of weight and height for 5 people.

2 Proofs [20 points]

(a) Prove that the determinant of an orthogonal matrix is equal to either +1 or -1.
Reminders: For square matrices X, Y the determinant satisfies properties (i)$det(X) = det(X^T)$, (ii)$det(XY) = det(X)det(Y)$.

(b) Prove that for any square matrix the absolute value of its determinant $|det(A)|$ is equal to the product of its singular values, i.e., $|det(A)| = \prod_{i=1}^{n} \sigma_i$.

Hint: Use the SVD of A, the fact that $det(XY) = det(X)det(Y)$, and (a).

3 SVD [20 points]

Let $A = U\Sigma V^T$ be the singular value decomposition of a matrix $A \in \mathbb{R}^{n\times m}$.

(a) (6pts) Show that $A^T u_j = \sigma_j v_j, 1 \leq j \leq rank(A)$. Here u_j, v_j are the left, and right singular vectors of A respectively for $1 \leq j \leq rank(A)$.

(b) (6pts) Suppose $n = m$, i.e., A is square. Furthermore, suppose A is invertible. What is the SVD of A^{-1}?

(c) (8pts) Let $A^{n\times m}$ be a real matrix. Show that if $P \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, PA has the same singular values as A.

4 Eigenfaces [50 points]

For the coding assignment, see the Jupyter notebook in our Git repo.