Prof. C.E. Tsourakakis

Instructions

- The midterm will span the material taught from Lecture 1, till Lecture 9 (2/17).
- For the EM algorithm, study the problem I taught you on the mixture of two coins.
- You should always study carefully the lecture material, the mandatory readings.
- I will hold extra time office hours on Thursday morning to answer your questions, from 9.30am to 11am.

Exercises

1. Consider tossing a coin three times. (a) What is the sample space? (b) Let X be the number of heads. What is the CDF of X?

2. Suppose $X \sim U(0, 2\pi)$. What is the CDF of $Y = \sin^2(X)$?

3. Let X be a Cauchy random variable, that is its pdf is $f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}, -\infty < x < \infty$. Prove that $\mathbb{E}[X]$ does not exist.

4. When can we write $\int\sum = \sum\int$, i.e., exchange the summation with an integral? To answer this question fill in the equation below where $f(x,\theta), a(\theta), b(\theta)$ are differentiable with respect to θ

$$\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} f(x,\theta)dx = ??$$

5. Let X be a continuous, nonnegative RV. Show that $\mathbb{E}[X] = \int_0^\infty (1 - F_X(x))dx$ where $F_X(x)$ is the CDF of X.

6. Consider a pair of discrete random variables (X, Y). Design two different joint probability distributions for (X, Y) that have the same marginals.

7. Let X be a continous RV. What is the value of $x^\star = \arg\min_x \mathbb{E}|X - x|$?

8. What is the probability of throwing at least one 6 in k rolls of a fair die?

9. Define the correlation of X, Y to be the number $\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$. Prove that $|\rho_{XY}| \leq 1$. When is the inequality tight?

10. Consider two continuous RVs X, Y with the following joint pdf: $f(x, y) = 1$ for $0 < x < 1, x < y < x + 1$. Compute ρ_{XY}.

11. Let $X \sim Bin(n, p)$. Compute the MGF $M_X(t)$ in two different ways; first, by (i) starting with the definition of mgf, and (ii) by expressing X as a sum of iid Bernoulli variables and using properties of MGFs we saw in class.

12. Let $X_1, \ldots, X_n \sim Bernoulli(p)$. We have a $beta(\alpha, \beta)$ prior on the parameter p. Write down the joint probability distribution of Y, p, where $Y = \sum_{i=1}^{n} X_i$.

13. The bias of an estimator X is the difference between the expected value of X and the parameter θ, i.e., $\text{bias}_\theta(X) = \mathbb{E}(X) - \theta$. The mean squared error (MSE) of an estimator X of a parameter θ is defined by $\mathbb{E}(X-\theta)^2$. Prove that the MSE of X can be written as follows $\mathbb{E}(X-\theta)^2 = \text{Var}(X) + (\text{bias}_\theta(X))^2$.

14. Suppose you have access to a single sample x from a normal $\mathcal{N}(0, \sigma^2)$ distribution, where σ is the only unknown parameter. Give an unbiased estimator of σ^2.

15. (a) Suppose you get n samples from a Poisson distribution with parameter λ. Find λ_{MLE}.

    (b) Continuing the previous question, suppose we impose a prior $\gamma(\alpha, \beta)$ on λ, that is the conjugate prior for Poisson. Find the posterior distribution of λ.

16. Suppose the probability of a rare disease is 10^{-5}. How large should a sample be from a population of a country to ensure that a person with this disease will appear with probability at least 95%? You may assume the population of the country is large enough, and that the sample is uniformly at random.

17. Prove the following on your own (i) WLLN, and (ii) the sampling theorem we saw in class.

18. Study the EM algorithm for the mixture of two coins, and perform on your own the steps we saw in class, step by step.

19. What is the difference between MAP and MLE inference?

20. If in the image denoising lecture, we expect the image to be similar to a chessboard, what prior would you impose on x?

21. Review the Bayes' problems we saw in class, and in the HW.