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1) $h(x) = f \cdot g = \int f(\chi)\, g(x-\chi)\, dx$

in discrete limit $= \sum f(\chi) g(x-\chi)$

$h(x) = \sum_{\chi} \frac{1}{N} \sum F(k) e^{2\pi i k\chi/N} \frac{1}{N} \sum G(k') e^{2\pi i k'(x-\chi)}$

$= \frac{1}{N^2} \sum_{k} \sum_{k'} F(k) G(k') e^{\frac{2\pi i k' x}{N}} \sum_{\chi} e^{2\pi i (k-k')\chi}$

$= \frac{1}{N} \sum_{k} F(k) G(k) e^{2\pi i k x/N}$ inverse fourier transform

$h(x) = IFT(F \times G)$

if $h = f \otimes g$, $H = FG$

let f be the gaussian, g be a delta function centred around n

2) $f \star g = \int f(x) \cdot g(x+y)\, dx$

$= ift(dft(f) * conj(dft(g)))$

3) The function shifts as expected, as can be seen it will be centred when told to shift by half the array length. When shifted by full length it returns back to its original spot

4. The array is 1182 length vs the inputs of 1000 for f and 1001 for g

5) Show $\sum_{x=0}^{N-1} e^{\frac{-2\pi i k x}{N}} = \frac{1-e^{-2\pi i k}}{1-e^{-2\pi i k/N}}$

$$\sum_{x=0}^{N-1} e^{-\frac{2\pi i k}{N}x} = \sum_{x=0}^{N-1} \alpha^x \quad \text{where } \alpha = e^{-\frac{2\pi i k}{N}}$$

Sum of geometric series

$$S = \sum_{x=0}^{N-1} \alpha^x = 1 + \alpha + \alpha^2 + \dots + \alpha^{N-1}$$

$$\alpha S = \alpha + \alpha^2 + \alpha^3 + \dots + \alpha^N$$

$$S - \alpha S = 1 - \alpha^N$$

$$S(1-\alpha) = 1 - \alpha^N$$

$$S = \frac{1-\alpha^N}{1-\alpha} = \boxed{\frac{1-e^{-2\pi i k}}{1-e^{-2\pi i k/N}}}$$

b) $S = \sum e^{-\frac{2\pi i k}{N}x}$

$$S = e^{-2\pi i k/N(0)} + e^{-2\pi i k/N(1)} + e^{-\frac{2\pi i k}{N}(2)} + \dots + e^{-\frac{2\pi i k}{N}(N-1)}$$

$$\lim_{k\to 0} e^0 = 1 \Rightarrow \sum_{x=0}^{N-1} 1 = N$$

$n \in \mathbb{N}$

Since k not an integer multiple of N $-\frac{2\pi k}{N} \neq n2\pi$

$$\frac{1-e^{-2\pi i k}}{1-e^{-2\pi i k/N}} = \frac{1-(\overset{\to 1}{\cos(2\pi k)} + i\overset{\to 0}{\sin(2\pi k)})}{\sim} = 0$$

$k \in \mathbb{N}$

Note $e^{-2\pi i k/N}$ will not equal 1 for any k so the denominator will not equal zero.

c) $f(k) = \sum_{x=0}^{N-1} e^{-2\pi i k x/N} = \frac{1-e^{-2\pi i k}}{1-e^{-2\pi i k/N}}$ where $f(x) = 1$

for non-integer sin wave :

$f(x) = \sin(2\pi k x/N)$ for $k \notin \mathbb{N}$ $\quad \sin(2\pi(2.5)x/N)$

$$F(k) = \sum_{x}^{N-1} f(x) e^{-2\pi i k' x/N}$$

$k = \frac{2\pi}{\lambda}$ $\quad \lambda = \frac{2\pi}{2}$

$$= \sum_{x}^{N-1} \sin(2\pi k x/N) e^{-2\pi i k' x/N}$$

$$= \sum_{x}^{N-1} \left(\frac{e^{2\pi i k x/N} - e^{-2\pi i k x/N}}{2i} \right) e^{-2\pi i k' x/N}$$

$$= \frac{1}{2i} \sum_{x} e^{2\pi i (k-k')x/2} - e^{-(2\pi i (k+k')x/N}$$

$$= \frac{1}{2i} \sum_{x} e^{-2\pi i (k'-k)x/2} - e^{-(2\pi i (k'+k)x/N}$$

$$= \frac{1}{2i} \left[\left(\frac{1-e^{-2\pi i (k'-k)}}{1-e^{-2\pi i (k'-k)/N}} \right) - \left(\frac{1-e^{-2\pi i (k'+k)x}}{1-e^{-2\pi i ((k'+k)x/N}} \right) \right.$$

d) unfortunatly, potentinally due to a bug, it does not improve dramaticaly

e) $\text{yft} = \left[\frac{N}{2} - \frac{N}{4} - \frac{N}{4}\right] \text{yft}$

window $= \frac{1}{2} + \frac{1}{2}\cos(2\pi x/N)$

As can be seen in the figure "window_vs_smooth.png" the window scales as $\left(\frac{N}{2} - \frac{N}{4} - \frac{N}{4}\right)$.

6.

a) random walk:

$$\langle (f(x) - f(x+dx))^2 \rangle = n\,dx$$

$$n\,dx = \langle f(x)^2 - 2\underbrace{f(x)f(x+dx)}_{\text{correlation}} + f(x+dx)^2 \rangle$$

say $f(x)^2 = f(x+dx)^2 = N$

$$n\,dx = 2N - 2\langle f(x) f(x+dx) \rangle$$

$$\therefore \langle f(x) f(x+dx) \rangle = N - \frac{n\,dx}{2} = g(dx)$$

$$G(k) = \int e^{2\pi i k x/N} g(x)\, dx$$

$$= \int e^{2\pi i k x/N} \left(N - \frac{nx}{2}\right) dx$$

$$= \int N e^{2\pi i k x/N} dx - \frac{n}{2} \int x e^{2\pi i k x/N} dx \qquad u = x \quad dv = e^{2\pi i k x/N}$$

$$= -i \frac{N^2}{2\pi k} e^{2\pi i k x/N} - \frac{n}{2}\left[-\frac{iNx}{2\pi k} e^{2\pi i k x/N} - \int -\frac{iN}{2\pi k} e^{2\pi i k x/N} dx \right]$$

$$u = \frac{2\pi i k x}{N}, \quad du = \frac{2\pi i k}{N}$$

$$= -\frac{iN^2}{2\pi k} e^{2\pi i k x/N} - \frac{1}{2}\left[-\frac{iNx}{2\pi k} e^{2\pi i k x/N} + \frac{iN}{2\pi k}\left(\frac{-iN}{2\pi k} e^{2\pi i k x/N}\right)\right]$$

$$= -\frac{iN^2}{2\pi k} e^{2\pi i k x/N} - \frac{n}{2}\left[-\frac{iNx}{2\pi k} e^{2\pi i k x/N} + \frac{N^2}{(2\pi k)^2} e^{2\pi i k x/N}\right]$$

$$= -\frac{iN^2}{2\pi k} e^{2\pi i k x/N} + \frac{i n N x e^{2\pi i k x/N}}{4\pi k} - \frac{n N^2 e^{2\pi i k x/N}}{8\pi^2 k^2}$$

$$g(dx) = \frac{(N n 2\pi i k x - N^2 n - N^2 4\pi i k) e^{2\pi i k x/N}}{8\pi^2 k^2} \sim \frac{1}{k^2}$$