Camryn Mullin 260926298

Problem 1

$\frac{dy}{dx} = \frac{y}{1+x^2}$ $\quad x\ (-20, 20) \quad y(-20) = 1 \quad nstep = 200$

$\frac{dy}{y} = \frac{dx}{1+x^2} \rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{1+x^2}\,dx$

$\ln(y) = \arctan(x) + c \rightarrow y = e^{\arctan(x)+c} = ce^{\arctan(x)}$

$1 = e^{\arctan(-20)}c \rightarrow c = 4.576$

$$\left.\begin{aligned}
k_1 &= hf(x,y)\\
k_2 &= hf(x+\tfrac{h}{2}, y+\tfrac{k_1}{2})\\
k_3 &= hf(x+\tfrac{h}{2}, y+\tfrac{k_2}{2})\\
k_4 &= hf(x+h, y+k_3)\\
y(x+h) &= y(x) + (k_1 + 2k_2 + 2k_3 + k_4)\tfrac{1}{6}
\end{aligned}\right\}\ \text{rk4 method}$$

$$\left.\begin{aligned}
y_1 &= \text{rk4_step}(fun, x, y, h)\\
y_{2a} &= \text{rk4_step}(fun, x, y, h/2)\\
y_{2b} &= \text{rk_step}(fun, x+h/2, y+y_{2a}, h/2)\\
y_2 &= y_{2a} + y_{2b}
\end{aligned}\right\}\ \text{rk4_stepd}$$

$y_1 = y\_true + err$

$y_2 = y\_true + err/4$

$4y_2 = 4y\_true + err$

$4y_2 - y_1 = 3y\_true$ (errors are cancelled), $y\_true = \frac{4y_2 - y_1}{3}$

The rk4_stepd takes 11 function evals per step.

forcing the function evals rk4_step preforms better. If rk4_stepd is allowed more function evals it will preform better by a factor of 10^2

Problem 2

U238 Decay

	Half-Life	Time unit	Emitter
Uranium-238	4,468	billion of years	alpha
Thorium-234	24,10	days	beta -
Protactinium-234	6,70	hours	beta -
Uranium-234	245 500	years	alpha
Thorium-230	75380	years	alpha
Radium-226	1 600	years	alpha
Radon-222	3,8235	days	alpha
Polonium-218	3,10	minutes	alpha
Plomb-214	26,8	minutes	beta -
Bismuth-214	19,9	minutes	beta -
Polonium-214	164,3	microseconds	alpha
Plomb-210	22,3	years	beta
Bismuth-210	5,015	years	beta
Polonium-210	138,376	days	alpha
Plomb-206	Stable		

a)

Use integrate.solve_IVP(. . . method='Radau')

y_a decays into y_b $\Rightarrow$ with rate τ_a

y_b decays into y_c $\Rightarrow$ with rate τ_b

$$\frac{dy_a}{dt} = -\tau_a y_a \quad , \quad \frac{dy_b}{dt} = \tau_a y_a - \tau_b y_b \qquad \frac{dy_c}{dt} = \tau_b y_b$$

$$\begin{pmatrix} -\tau_a & 0 & 0 \\ \tau_a & -\tau_b & 0 \\ 0 & \tau_b & 0 \end{pmatrix} \begin{pmatrix} y_a \\ y_b \\ y_c \end{pmatrix} = y(t)$$

$$y(t) = V e^{-t\lambda} V^{-1} y_0$$

b)

It can be seen that as U238 decays into Pb206 the ratio of Pb206/U238 trends towards infinity. When U238 is about half decayed, the ratio is close to 1.

For Th230 and U234 the ratio starts at zero and then steadily increases as U234 decays into Th230. Eventually it reaches a steady amount for both.

Problem 3

a) $z - z_0 = a((x-x_0)^2 + (y-y_0)^2)$

$$z = a((x-x_0)^2 + (y-y_0)^2) + z_0$$

$$= a(x^2 - 2xx_0 + x_0^2) + a(y^2 - 2yy_0 + y_0^2) + z_0$$

$$= ax^2 - 2ax_0x + ax_0^2 + ay^2 - 2ay_0y + ay_0^2 + z_0$$

$$= a(x^2+y^2) \underbrace{-2ax_0}_{b}x \underbrace{-2ay_0}_{c}y + \underbrace{ax_0^2 + ay_0^2 + z_0}_{d}$$

$$\boxed{\begin{array}{l} z = a(x^2+y^2) - bx - cy + d \\ \text{parameters } a, b, c, d \text{ where } b = -2ax_0,\ c = -2ay_0 \\ \text{and } d = ax_0^2 + ay_0^2 + z_0 \end{array}}$$

b) $b = -2ax_0$, $c = -2ay_0$ $d = ax_0^2 + ay_0^2 + z_0$

$a = 1.6670 \times 10^{-4}$

$b = 4.536 \times 10^{-4} = -2(1.667\times10^{-4})x_0$

$c = -1.944\times10^{-2} = -2(1.667\times10^{-4})y_0$

$d = 1.667\times10^{-4}x_0^2 + 1.667\times10^{-4}y_0^2 + z_0$

c) The noise is a diagonal matrix of the fit values minus the predicted, squared.

$y = \frac{x^2}{4f} \rightarrow f = \frac{x^2}{4y}$ z_{max} = focal length

error: $A(A^TN^{-1}A)^{-1}A^T$

focal length is $f = 1.516\text{m} \pm 7.78\times10^{-5}\text{m}$

$f \sim \frac{1}{4a}$

$\frac{1}{4a} = \frac{x2}{4y} \rightarrow y = ax^2$

$y'(x) = 2ax$

$y''(x) = 2a$

$y'''(x) = 0$

$P(x) = y(x_0) + y'(x - x_0) + y''\frac{(x - x_0)^2}{2}$

$P(x) = ax_0^2 + 2ax(x - x_0) + a(x - x_0)^2$