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Problem 2

CFL condition: $\frac{v\,dt}{dx} \leq 1$ sol of form $f(t,x) = \xi^t e^{ikx}$

$$\frac{f(t+dt,x) - f(t-dt,x)}{2dt} = -v\,\frac{f(t,x+dx) - f(t,x-dx)}{2dx}$$

$$\frac{\xi^{t+dt}e^{ikx} - \xi^{t-dt}e^{ikx}}{2dt} = -v\,\frac{\xi^t e^{ik(x+dx)} - \xi^t e^{ik(x-dx)}}{2dx}$$

$$\xi^{\cancel{t}+dt}\cancel{e^{ikx}} - \xi^{\cancel{t}-dt}\cancel{e^{ikx}} = -\frac{v\,dt}{dx}\left(\cancel{\xi^t} e^{ik(\cancel{x}+dx)} - \cancel{\xi^t} e^{ik(\cancel{x}-dx)}\right)$$

$$\xi^{dt} - \xi^{-dt} = -\frac{v\,dt}{dx}\left(e^{ikdx} - e^{-ikdx}\right)$$

$$\xi^{2dt} - 1 = -\frac{v\,dt}{dx}\,\xi^{dt}\,2i\sin(kdx)$$

$$\xi^{2dt} + 2i\frac{v\,dt}{dx}\sin(kdx)\,\xi^{dt} - 1 = 0 \qquad (\text{let } dt = 1)$$

$$\xi = -\frac{i\,v\,dt}{dx}\sin(kdx) \pm \sqrt{1 - \left(\frac{v\,dt}{dx}\sin(kdx)\right)^2}$$

if $|\xi| = 1$, energy stable

if $\frac{v\,dt}{dx} > 1$ then $\sqrt{1-\left(\frac{v\,dt}{dx}\sin(kdx)\right)^2}$ complex and $|\xi| > 1$

if $\frac{v\,dt}{dx} \leq 1$ then $\sqrt{1-\left(\frac{v\,dt}{dx}\sin(kdx)\right)^2}$ real

$$|\xi|^2 = \cancel{\left(\frac{v\,dt}{dx}\sin(kdx)\right)^2} + 1 - \cancel{\left(\frac{v\,dt}{dx}\sin(kdx)\right)^2}$$

$$|\xi| = 1$$

$$\boxed{\therefore\ \frac{v\,dt}{dx} \leq 1 \text{ necessary!}}$$

2.

a) $V = \frac{-1}{4\pi\epsilon_0} \rho \ln(r)$ $\quad \nabla^2 V = \rho$ $\quad \rho = V - \bar{V}_{neighbours}$

Convolve ρ with $\frac{1}{r}$ to get V

Solving $Av = b$ $\begin{cases} \neq 0 & \text{surface} \\ 0 & \text{elsewhere} \end{cases}$

Conjugate gradient

way of approximatly solving $Ax = b$

Search directions : p_k

$p_k^T A p_{k'} \propto \delta_{kk'}$ $\quad$ p_k are orthogonal to eachother

say $Ax = b$ $\begin{cases} x = \sum \alpha_k p_k \\ b = \sum \alpha_k A p_k \end{cases}$

$p_k^T A x = p_k^T b$

$= p_k^T A p_k \alpha_k$ $\quad = 0$ unless $k' = k$

$= \alpha_k \delta_{kk'}$

$Ax = b \Rightarrow$ remap to minimize $\frac{x^T A x}{2} - x^T b$

$V_{on\ boundary} = \rho_{boundary} \circledast g(r)$ $\quad$ $g(r)$: green's function

$V_{boundary} = \text{IFFT}(\hat{g}(k) \cdot \text{FFT}(\rho))$

$V_b = \text{IFT} \cdot \hat{g} \cdot \text{FT} \cdot \rho \Rightarrow \text{IFT} \cdot \hat{g} \cdot \text{FT} = A$

$V_b = A \rho_b$

get a and plot