Fitting Models to Data

Inverting Data

- Start w/ well-posed FD/FE model

$$[K]\{u\} = \{b\}$$

(i.e. $[K]^{-1}$ exists)

- measure some u_i with uncertainty
- deduce $\{b\}$ through LS fit to data
- estimate complete $\{u\}$ from model for estimated $\{b\}$
- estimate inverse noise in $\{u\}$ and system noise in $\{b\}$
- examine misfit between data and model

To do this, need prior estimates of $Cov(u)$, $Cov(b)$ and expected data-model misfit

Data-Model Misfit

$$\{d\} = [S]\{u\} + \{\delta\}$$

data

sampling matrix (e.g. point, average, derivative, etc. ⋮)

model output

misfit (2 pieces) → measurement error; → modeling error (difference between reality and model)

- In FE case, sampling matrix, $[S]$, is determined from basis functions

e.g. given true u at point $(x,y,z)_k$, then

FE sol'n there is $u(x,y,z)_k = \sum_j u_j \phi_j(x,y,z)_k$

$\therefore \quad S_{kj} = \phi_j(x,y,z)_k$ for row index k, column index j

e.g.  only 3 non-zero basis, i.e. j's $\Rightarrow$ sparse

- Common to "de-mean" everything, i.e.

$$\left.\begin{aligned} \{u\} &= \{\bar{u}\} + \{\tilde{u}\} \\ \{b\} &= \{\bar{b}\} + \{\tilde{b}\} \\ \{d\} &= \{\bar{d}\} + \{\tilde{d}\} \end{aligned}\right\}$$

a "mean" value + perturbation

sometimes called Best prior estimate, BPE

such that $\{\bar{u}\} = [K]^{-1}\{\bar{b}\}$

$\{\bar{d}\} = [S]\{\bar{u}\}$

$\therefore$ perturbations are the unknowns

BPE is known

and then $\{\tilde{u}\} = [K]^{-1}\{\tilde{b}\}$

$\{\tilde{d}\} = [S]\{\tilde{u}\} + \{\delta\}$

Drop "~" over top for convenience, but are only working w/ perturbations now then $\{\delta\} = \{d\} - [S]\underbrace{[K^{-1}]\{b\}}_{\{x\}}$

- want to find $\{b\}$ such that $\{\delta\} \cong 0$

i.e. $\underbrace{[S][K^{-1}]}_{\text{Not square, no inverse, solve in LS sense}}\{b\} \cong \{d\}$

i.e. $[S][K^{-1}]\{b\} = \{d\}$

- Normal Eqn's (minimum variance misfit sol'n)

$$\left[[S][K^{-1}]\right]^T\left[[S][K^{-1}]\right]\{b\} = \left[[S][K^{-1}]\right]^T\{d\}$$

$$\{b\} = \underbrace{\left[[SK^{-1}]^T[SK^{-1}]\right]^{-1}[SK^{-1}]^T}_{[B]}\{d\}$$

- So $\{b\}$ is linear in $\{d\}$ i.e. $\{b\} = [B]\{d\}$

i.e. estimate for $\{b\}$ is linear in the data

- Can get $\{u\}$ from estimated $\{b\}$ as

$$\{u\} = [K^{-1}][B]\{d\}$$

- given data w/ noise, then

$$[Cov(b)] = [B][Cov(d)][B]^T$$

$$[Cov(u)] = [K^{-1}][Cov(b)][K^{-T}]$$

- Different Sol'n strategies lead to different linear estimators $[B]$

e.g. What about SVD?

$$[S][K^{-1}]\{b\} = \{d\}$$

$$\{b\} = \underbrace{[V][\text{diag } 1/w][U]^T}_{[B]_{SVD}}\{d\}$$

where $[S][K^{-1}] = [U][\text{diag } w][V^T]$

or $[B]_{WLS} = \ldots$

or $[B]_{GLS} = \ldots$

Aside: Normal Eqn's for OLS same as SVD (i.e. are minimum residual variance sol'n to $[A]\{X\} = \{b\}$ where $m > n$)

Normal Eqn's for $[A]\{x\}=\{b\}$

$$[A^TA]\{x\}=[A^T]\{b\}$$

SVD:

$$[A]=[U][\text{diag}\,w][V^T]$$

$$[A^T]=[V][\text{diag}\,w][U^T]$$

$$\underbrace{[V][\text{diag}\,w][U^T]}_{[A^T]}\underbrace{[U][\text{diag}\,w][V^T]}_{[A]}\{x\}=[V][\text{diag}\,w][U^T]\{b\}$$

$$[U^T][U]=[I]$$

$$[V][\text{diag}\,w^2][V^T]\{x\}=[V][\text{diag}\,w][U^T]\{b\}$$

multiply through by $[V^T]$; (recall $[V^T][V]=[I]=[V][V^T]$)

$$[\text{diag}\,w^2][V^T]\{x\}=[\text{diag}\,w][U^T]\{b\}$$

$$[V^T]\{x\}=[\text{diag}\,1/w][U^T]\{b\}$$

multiply through by $[V]$

$$\{x\}=\underbrace{[V][\text{diag}\,1/w][U^T]\{b\}}_{\text{Exactly the SVD sol'n to }[A]\{x\}=\{b\}}$$

- Can consider $\{\delta\}$ as having two sources of uncertainty

    1. Model "mismatch" with nature

    i.e. $$\underbrace{\{u\}}_{\text{model Sol'n}} = \underbrace{\{u_{TR}\}}_{\text{"Truth"}} + \underbrace{\{\epsilon_m\}}_{\text{model error}}$$

    2. data uncertainty (measurement noise)

    i.e. $$\underbrace{\{d\}}_{\text{data}} = \underbrace{[S]\{u_{TR}\}}_{\text{Sampled "Truth"}} + \underbrace{\{\epsilon_d\}}_{\text{measurement error (noise)}}$$

then
$$\begin{aligned}\{\delta\} &= \{d\} - [S]\{u\} \\ &= \{d\} - [S]\left(\{u_{TR}\} + \{\epsilon_m\}\right) \\ &= \underbrace{\{\epsilon_d\}}_{\text{error due to measurement}} - \underbrace{[S]\{\epsilon_m\}}_{\text{error due to model imperfections}}\end{aligned}$$

- misfit is superposition of these two error contributions