Example:

$$\nabla\cdot K\nabla u + v\cdot\nabla u + fu = g$$

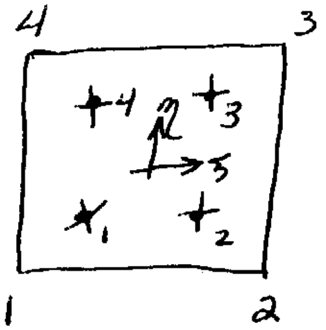
Galerkin:

$$\langle -K\nabla u\cdot\nabla\phi_i\rangle + \langle v\cdot\nabla u\,\phi_i\rangle + \langle fu\,\phi_i\rangle = \langle g,\phi_i\rangle - \oint K\nabla u\cdot\hat{n}\phi_i\,ds$$

$$\sum u_j \underbrace{\left[\langle -K\nabla\phi_j\cdot\nabla\phi_i\rangle + \overbrace{\langle v\cdot\nabla\phi_j\,\phi_i\rangle}^{\text{Non-Symmetric}} + \langle f\phi_j\phi_i\rangle\right]}_{A_{ij}} = \underbrace{\langle g\phi_i\rangle}_{b_j} - \oint K\nabla u\cdot\hat{n}\phi_i\,ds$$

only $\langle\,\rangle$ part will be assembled at the element level

$$A_{ij} = \left\langle -K\frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial x} - K\frac{\partial\phi_j}{\partial y}\frac{\partial\phi_i}{\partial y}\right\rangle + \left\langle V_x\frac{\partial\phi_j}{\partial x}\phi_i + V_y\frac{\partial\phi_j}{\partial y}\phi_i\right\rangle + \langle f\phi_j\phi_i\rangle$$



Bilinear element
2 x 2 Quadrature

$$\phi_1 = (1-\xi)(1-\eta)/4 \qquad \frac{\partial \phi_1}{\partial \xi} = -(1-\eta)/4$$

$$\phi_2 = (1+\xi)(1-\eta)/4 \qquad \frac{\partial \phi_2}{\partial \xi} = (1-\eta)/4$$

$$\phi_3 = (1+\xi)(1+\eta)/4 \qquad \frac{\partial \phi_3}{\partial \xi} = (1+\eta)/4$$

$$\phi_4 = (1-\xi)(1+\eta)/4 \qquad \frac{\partial \phi_4}{\partial \xi} = -(1+\eta)/4$$

$$\frac{\partial \phi_1}{\partial \eta} = -(1-\xi)/4$$

$\vdots$
etc.

Gauss points: 1-D form

$$\xi = \pm .57735\ldots \qquad W = 1$$

Point #	$\xi(m)$	$\eta(m)$	$W(m)$
1	-.57735	-.57735	(1)·(1)
2	.57735	-.57735	(1)·(1)
3	.57735	.57735	(1)·(1)
4	-.57735	.57735	(1)·(1)

at each gauss point need:

$$\left.\begin{matrix} \phi_i \\ \dfrac{\partial \phi_i}{\partial x} \\ \dfrac{\partial \phi_i}{\partial y} \end{matrix}\right\} \text{all } i=1,4$$

$|J|$ for $\langle\ \rangle$

$$\left.\begin{matrix} K \\ V_x \\ V_y \\ f \\ g \end{matrix}\right\} \text{coefficients } K = \sum_1^4 K_k \phi_k \text{ or } \underline{\text{constant}}$$

Subprogram Basis $(\underbrace{\overset{\phi}{P}, \overset{\partial\phi/\partial x}{DPX}, \overset{\partial\phi/\partial y}{DPY}, \overset{|J|}{DJ}}_{\text{Output}}, \underbrace{\overset{\xi}{Z}, \overset{\eta}{E}, \overset{(x,y)\text{ of nodes}}{XL, YL}}_{\text{Input}})$ ↳ 4

Dimension: P, DPX, DPY, $\underbrace{DPZ, DPE}_{\frac{\partial\phi}{\partial\xi} \quad \frac{\partial\phi}{\partial\eta}}$, XL, YL to # nodes/element

$\frac{\partial\phi}{\partial\xi}$ $\frac{\partial\phi}{\partial\eta}$ ⇒ used locally only

* Bases

P(1) = (1−Z)(1−E)/4.
P(2) = (1+Z)(1−E)/4.
P(3) = (1+Z)(1+E)/4.
P(4) = (1−Z)(1+E)/4.
DPZ(1) = −(1−E)/4.
DPZ(2) = (1−E)/4.
⋮
DPE(1) = −(1−Z)/4.
⋮
DPE(4) = (1−Z)/4.

Common to all elements if Gauss pts are same in all elements

* Jacobian

DXZ = Σ XL(I) * DPZ(I)
DXE = Σ XL(I) * DPE(I)
DYZ = Σ YL(I) * DPZ(I)
DYE = Σ YL(I) * DPE(I)

Loop over I = 1→4

$$\frac{\partial x}{\partial \xi} = \sum x_i \frac{\partial \phi_i}{\partial \xi}$$

$$\frac{\partial y}{\partial \xi} = \sum y_i \frac{\partial \phi_i}{\partial \xi} \quad \text{etc...}$$

DJ = DXZ * DYE − DYZ * DXE

* Derivatives

DPX(I) = [DYE * DPZ(I) − DYZ * DPE(I)]/DJ
DPY(I) = [−DXE * DPZ(I) + DXZ * DPE(I)]/DJ

Do for I = 1→4

* Done !!

Element Matrix Assembly $L \equiv$ el

Load $XL(I)$, $YL(I)$ $I = 1 \to 4$ Local node coordinates

Loop over Gauss Points $m = 1 \to M$

get $Z = \xi(m)$, $E = \eta(m)$ } Gauss point coordinates

Call Basis (P, DPX, DPY, DJ, Z, E, XL, YL)

assemble coefficients

KM = Σ K (IN(L,I)) * P(I) $I = 1 \to 4$

VXM = Σ VX(IN(L,I)) * P(I) $K = \sum K_i \phi_i$ etc

⋮

when need derivative of coeffs } → DHX = Σ H(IN(L,I)) * DPX(I) $\Rightarrow \frac{\partial H}{\partial x} = \sum H_i \frac{\partial \phi_i}{\partial x}$

⋮

Gauss Pt. Matrix

Loop over I = 1, 4   (row)

Loop over J = 1, 4   (column)

element matrix → AE(I,J) = AE(I,J) +

DJ * W(m) * { − KM * [DPX(I) * DPX(J) + DPY(I) * DPY(J)]

PDE → + [VXM * DPX(J) + VYM * DPY(J)] * P(I)

+ FM * P(I) * P(J) }

End loop J

RE(I) = RE(I) + DJ * W(m) * { GM * P(I) }

END Loop I

End Gauss Pt Loop

Direct assembly of $[A]^b$ from Gauss Pt Level

$$
\begin{tikzcd}[column sep=huge]
\text{Gauss Pt Matrix} & \text{Element Matrix} & \text{Global Matrix} \\
AEG(I,J) \arrow[r] \arrow[rr, bend right, "\text{Bypass}"'] & AE(I,J) \arrow[r] & A^b(M,N)
\end{tikzcd}
$$

$$M = IN(L,J)$$

$$N = \underbrace{NH+1}_{NDIAG} + IN(L,J) - IN(L,I)$$

Gauss pt matrix: evaluate and assemble globally

```
Loop over I=1,4   (rows)
   M = IN(L,J)
   Loop over J=1,4   (columns)
      N = NDIAG + IN(L,J) - IN(L,I)
      A(M,N) = A(M,N) + DJ* W(m) * { PDE }
   END J Loop
   R(m) = R(m) + DJ*W(m) * {RHS of PDE}
END I Loop
```

END of Gauss point Loop; end of Element Loop !!

Schematic of FE Structure:

Element Loop $\quad \langle f(x,y)\rangle = \sum_e \langle f(x,y)\rangle^e$

Gauss point loop $\quad \langle f\rangle^e = \sum f_m W(m)\, |J|_m$

depends on quadrature $\rightarrow$ Look up $\xi(m)$, $\eta(m)$

depends on element $\rightarrow$ Call Basis: ϕ, $\nabla\phi$, $|J|$

Assemble coefficients

Row Loop

Column loop

depends on PDE $\rightarrow$ $A(m,N) = A(m,N) + |J| * W(m) * \{PDE\}$

END column loop

$R(m) = R(m) + |J| * W(m) * \{RHS\}$

END Row Loop

END Gauss pt Loop

END element loop

"Simultaneous"

Apply BC's

Solve

Examine Solution!