Method of Weighted Residuals

- Consider $\nabla^2 u + fu = g \Rightarrow \underbrace{(\nabla^2 + f)}_{L} u = g$
  - FD approach:
    - approximate L w/ $L_{ij} \Rightarrow \underbrace{(\delta^2 + f)}_{L_{ij}} u_{ij} = g_{ij}$

      i.e. replace "differential" operator w/ "difference" operator $\Rightarrow$ get "exact sol'n" to "approximate operator"

    - Limitations: Cumbersome on irregular mesh
      Curved boundaries difficult to handle
      u only found at (i,j) points.

  - Weighted Residuals:
    - approximate u as $\hat{u} = \sum_{j=1}^{N} c_j \phi_j(x,y)$

      c_j: unknown coefficients

      ϕ_j: known function, "Basis" fctn's, "Trial" fctn's

    - Define "Residual" $\Rightarrow R(u) = Lu - g$

- For exact Sol'n: $R(u) = 0$ everywhere

then $\iint R(u) \underbrace{W(x,y)}_{\text{any function of position}} \, dx\,dy = 0$

"$R(u)$ orthogonal to all $W(x,y)$"

- But for $\hat{u}$: $R(\hat{u}) \neq 0$ ... choose N c_j's such that

$$\langle R(\hat{u}), W_i \rangle = 0 \quad \text{for } i = 1, 2, \ldots N$$

"Inner Product" $\Rightarrow \langle a, b \rangle \equiv \iint a b \, dx\,dy$

- W_i's set of "weighting" functions $\Rightarrow$ finite!
  "Testing" functions
- Use N independent W_i's $\Rightarrow$ generate N equations in N unknown c_j's

$$\langle R(\hat{u}), W_i \rangle = \sum_{j=1}^{N} c_j \langle L(\phi_j) W_i \rangle = \langle g, W_i \rangle$$

for each $W_i(x,y)$ $\quad i = 1, 2, \ldots N$

- Necessary, but not sufficient for $\hat{u} = u$

IF W_i a "complete set" (Only possible if $N \to \infty$)

- Contains all possible functions
- for any function, f:

$$\langle f, W_i \rangle = 0 \text{ for all } i \Rightarrow f = 0 \text{ everywhere}$$

- $\langle R(\hat{u}), W_i \rangle = 0$ for all $i = 1, 2, \ldots \infty \Rightarrow R(\hat{u}) = 0$

then $\hat{u} = u \Rightarrow$ Convergence to exact sol'n

- Weighted Residual Method:
  - W_i not complete in practice (N finite), but make "$R(\hat{u})$ orthogonal to 1ST N members of a complete set"
  - "Approximate sol'n" which exactly satisfies "differential relations" in PDE
  - Is an "Integral" formulation
  - L is typically a differential Operator

Example: $\frac{d^2u}{dx^2} + fu = g$ w/ $u=0$ at $x=0,\ell$

Write: $\hat{u} = \sum_{j=1}^{N} c_j \underbrace{\sin\frac{j\pi x}{\ell}}_{\phi_j}$

$\phi_j \Rightarrow$ eigenfunctions of L

Pick: $W_i = \sin\frac{i\pi x}{\ell} \Rightarrow$ same as ϕ_i

$$\frac{d^2\hat{u}}{dx^2} = -\sum_{j=1}^{N}\left(\frac{j\pi}{\ell}\right)^2 c_j \sin\frac{j\pi x}{\ell}$$

$$R(\hat{u}) = \sum_{j=1}^{N} c_j\left(f-\left(\frac{j\pi}{\ell}\right)^2\right)\sin\frac{j\pi x}{\ell} - g$$

$\langle R(\hat{u}), W_i\rangle = 0$:

$$\left\langle \sum_{j=1}^{N} c_j\left(f-\left(\frac{j\pi}{\ell}\right)^2\right)\sin\frac{j\pi x}{\ell}\sin\frac{i\pi x}{\ell}\right\rangle = \left\langle g\sin\frac{i\pi x}{\ell}\right\rangle$$

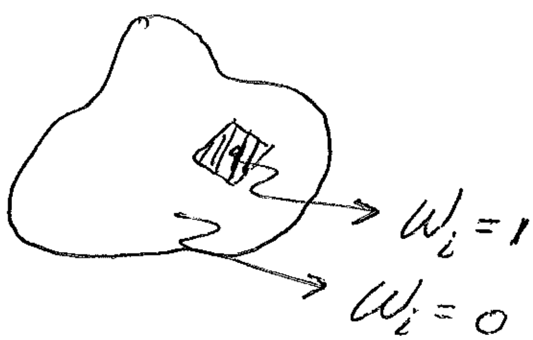
But $\sum_{j=1}^{N} c_j\left(f-\left(\frac{j\pi}{\ell}\right)^2\right)\underbrace{\left\langle\sin\frac{j\pi x}{\ell}\sin\frac{i\pi x}{\ell}\right\rangle}_{=0 \text{ if } i\neq j} = \left\langle g\sin\frac{i\pi x}{\ell}\right\rangle$

so $$c_j = \frac{\left\langle g\sin\frac{j\pi x}{\ell}\right\rangle}{\left\langle\sin^2\frac{j\pi x}{\ell}\right\rangle}\left(\frac{1}{f-\left(\frac{j\pi}{\ell}\right)^2}\right)$$

- Similar to getting coefficients in Fourier Series
- More generally ... can't find eigenfunctions for complex boundaries ∴ can't tailor ϕ_i to the problem so easily! ... want simple functions which can be used for a wide class of problems

- "Famous" Weighted Residual Methods

a.) Subdomain:



Subdivide domain of interest into N small regions each of which has $\underbrace{\text{weighting function}}_{\text{unique}}$ equal to unity

b.) Collocation:

$$W_i = \delta(x - x_i, y - y_i)$$

$$\Rightarrow \langle R(\hat{u}), W_i \rangle = R(\hat{u}, x_i, y_i) = 0$$

Residual exactly zero at a finite # points c.) Least Squares:

Minimize $\langle R^2 \rangle$ by selection of c_j's

$$\Rightarrow \left\langle 2R \frac{\partial R}{\partial c_i} \right\rangle = 0 \qquad i = 1, 2, \ldots N$$

$$R(\hat{u}) = L\hat{u} - g = \sum_j c_j L\phi_j - g$$

$$\frac{\partial R}{\partial c_i} = L\phi_i$$

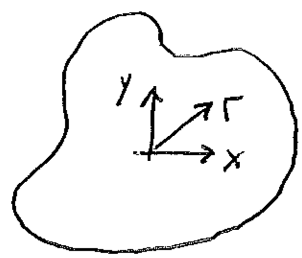
So $W_i = L\phi_i$

d.) Galerkin: $W_i = \phi_i$

Only use one set of functions

e.) Moments: $W_i = r^i$

$\langle R(\hat{u}), r^i \rangle = 0 \quad i = 0, 1, \ldots N-1$



Finite Element Method:

- Is a Weighted Residual Method (Integral)
- L is a differential operator (usually)
- Utilizes piecewise smooth functions as basis + weighting functions
- Are continuous, but only enough to allow integrals to be evaluated

Summarize Weighted Residual Method

- Work directly on PDE ... weight it + integrate over problem domain

$$\iint (Lu - g)\, W \, dx\, dy = 0$$

- Is a "weak" formulation ("weak" form)

$\iint (\ )\, W\, dx\, dy = 0$ does not require $(\ ) = 0$ everywhere

But ...
$(\ ) = 0$ everywhere then $\iint (\ )\, W\, dx\, dy = 0$

- Numerically ...
  a) Expand unknown $\hat{u} = \sum_{j=1}^{N} c_j \phi_j (x,y)$
  - Finite sum
  - We pick → $\phi_j (x,y)$ known function of (x,y)
  - Solve for this → c_j's unknown coefficients to determine

b.) Generate system of Eqn's in unknown C_j's

$$\left\langle (L\hat{u}-g)\, W_i(x,y) \right\rangle = 0 \quad i=1,2,\dots N$$

$$\sum_{j=1}^{N} C_j \left\langle L\phi_j\, W_i(x,y) \right\rangle = \left\langle g\, W_i(x,y) \right\rangle \quad i=1,2,\dots N$$

We choose $\rightarrow$ • $W_i(x,y)$ known functions of (x,y)

• For each i, generate algebraic relation in C_j's

• Creates N equations in N unknowns

- Procedure mimics familiar analytic methods

e.g. Fourier Series representation of a function
PDE sol'n through expansions of orthogonal eigenfunctions

But analytic methods ...

• require special knowledge of how to choose ϕ_j's + W_i's

• different choices are needed for different problems

• usually need an infinite # of them

• Can't find them for many practical problems

- Numerically want...
  - ϕ_j's + w_i's to be simple
  - single choice suitable for many problems
  - can only use finite #, but want convergence as number used increases

- Various numerical methods distinguished by their choice of ϕ_j's + w_i's