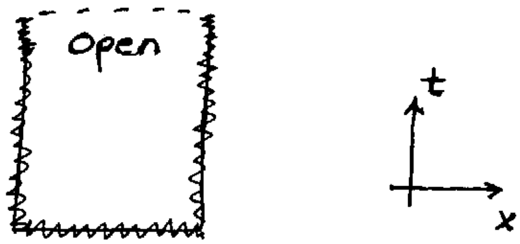
Parabolic Equations

Prototype Form: $\dfrac{\partial u}{\partial t} = L(u)$

Elliptic Operator
(spatial derivatives)

Characteristics (2 identical real curves) dictate some "open" boundaries

e.g.



- Space ... handle same as Elliptic Case

$$u(x,t) \Longrightarrow u_i(t)$$

$$L(u) \Longrightarrow L_i(u_i) \quad \text{"FD molecule"}$$

$$\frac{du_i}{dt} = \underbrace{L_i(u_i)}_{F_i(t,\,u_i)}$$

"Lumped System"
discrete in space
continuous in time

Can consider as coupled set of ODEs involving variables u_i for $i=1,2\ldots N$

All of the techniques for ODEs from EN65 69 are available

- Time ... can take several views:

1.) FD in time
$$\frac{U_i(t+\Delta t) - U_i(t)}{\Delta t} = L_i(U_i)^*$$

U_i evaluated at some time point on $[t, t+\Delta t]$

2.) Integrate in time
$$\int_t^{t+\Delta t} \frac{dU_i}{dt}\,dt = \int_t^{t+\Delta t} L_i(U_i)\,dt$$

$$U_i(t+\Delta t) - U_i(t) = \Delta t\, \overline{L_i(U_i)}$$

Some type of averaged value on $[t, t+\Delta t]$

These are alternate views of the same thing.
Speak of "Discrete System" when fully discretized
(i.e. discrete in space
discrete in time)

- Numerically then sol'n will propagate through time... begin with IC's, constrained by BCs

Can accomplish this point by point or line by line

Classic equation to study: $\frac{\partial U}{\partial t} = \nabla \cdot (D \nabla U)$

Possible Discrete Systems $\begin{pmatrix} \text{discrete in space} \\ \text{discrete in time} \end{pmatrix}$

Example: $\dfrac{\partial u}{\partial t} = D \dfrac{\partial^2 u}{\partial x^2}$

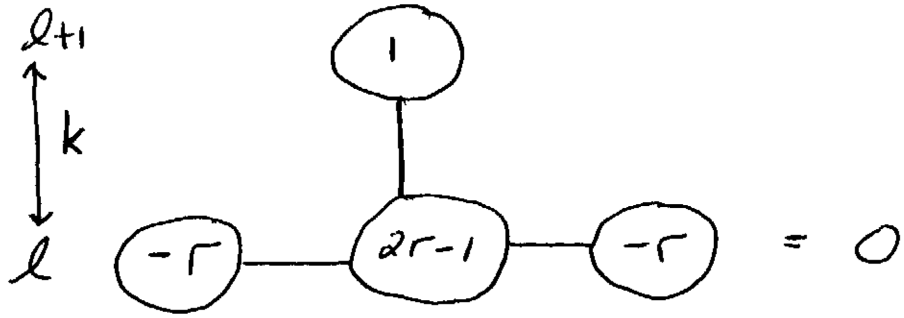
Simplest treatment: $\dfrac{\partial^2}{\partial x^2} \Rightarrow \dfrac{\delta_x^2}{h^2} \Rightarrow \underbrace{\dfrac{\partial u_i}{\partial t} = \dfrac{D\delta_x^2 u_i}{h^2}}_{\text{Lumped System}}$

$$\frac{\partial}{\partial t} \Rightarrow \frac{\Delta}{k}$$

FD molecule: $\dfrac{\Delta u_i^l}{k} = \dfrac{D\delta_x^2 u_i^l}{h^2}$

$$u_i^{l+1} - u_i^l - \frac{Dk}{h^2}\left(u_{i-1}^l - 2u_i^l - u_{i+1}^l\right) = 0$$

$$r \equiv \frac{Dk}{h^2}$$



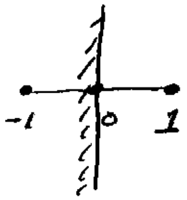
- "Explicit"
- pointwise propagation
- $O(k + h^2)$
- Forward Difference in t (or Euler integration)
- Conditional stability

- BCs handled same as elliptic problem

  Type I : stop molecule "inside" the boundary

  Type II, III : use shadow node strategy

e.g. at $i=0$, $-D\frac{\partial u}{\partial x} = q_0$

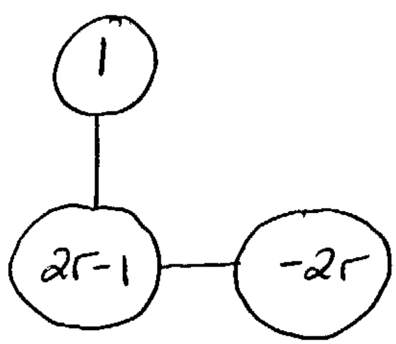


"shadow" $\longrightarrow$ $\frac{u_{-1} - u_1}{2h} = \frac{q_0}{D} \Rightarrow u_{-1} = \frac{2hq_0}{D} + u_1$

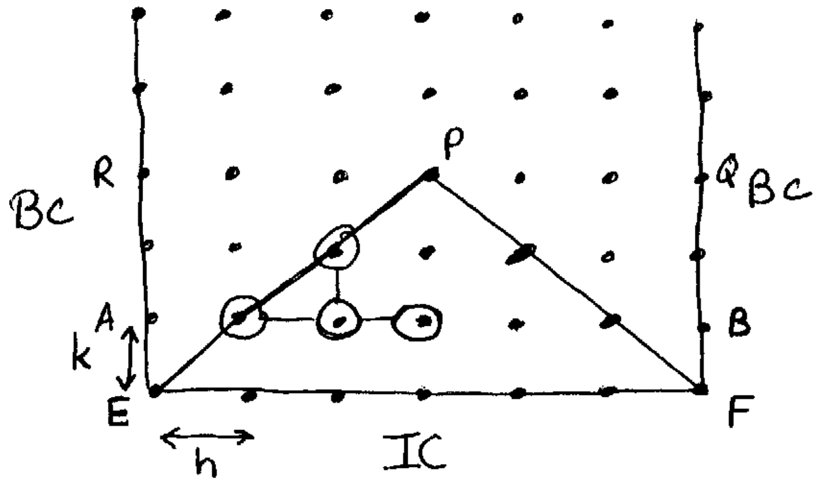
PDE template : $\frac{du_0}{dt} = \frac{D}{h^2}\left(u_{-1} - 2u_0 + u_1\right)$

Combine : $\frac{du_0}{dt} = \frac{D}{h^2}\left(2u_1 - 2u_0 + \frac{2hq_0}{D}\right)$

Molecule at $i=0$:



$= \frac{2rhq_0}{D}$



- Sol'n runs ahead of itself... knowledge of BCs not required (along RE and QF) ... but we know for parabolic system, continuous sol'n at P not determined until BC info along ER + QF prescribed (i.e. has single-valued characteristics, $t = \text{constant}$, either EF or RQ)

- Behavior for explicit discrete system not appropriate for parabolic eqn... expect critical value of k which must not be exceeded ... i.e. sol'n can't get too far ahead of itself

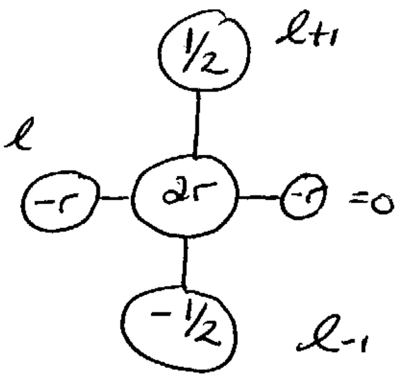
- All pointwise propagation schemes have this property

- "Natural" parabolic FD approximation would be incapable of producing any values along AB if BCs at A and B were not specified ... need values at each mesh point along AB given as functions of values along EF and BCs at A, B $\Rightarrow$ implicit

- Other time discretization possibilities...

1.) "Leapfrog"

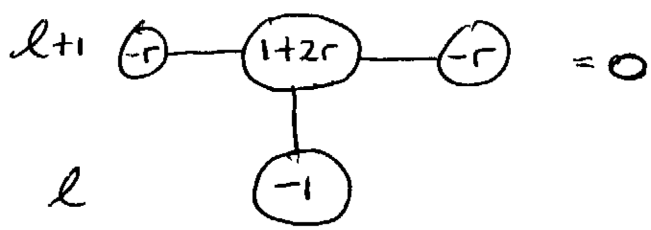
$$\frac{u_i^{\ell+1} - u_i^{\ell-1}}{2k} = \frac{D\delta_x^2 u_i^\ell}{h^2} \Rightarrow \frac{u_i^{\ell+1} - u_i^{\ell-1}}{2} - r\delta_x^2 u_i^\ell$$



Centered in t (equivalent to Midpoint Rule $\int dt$)

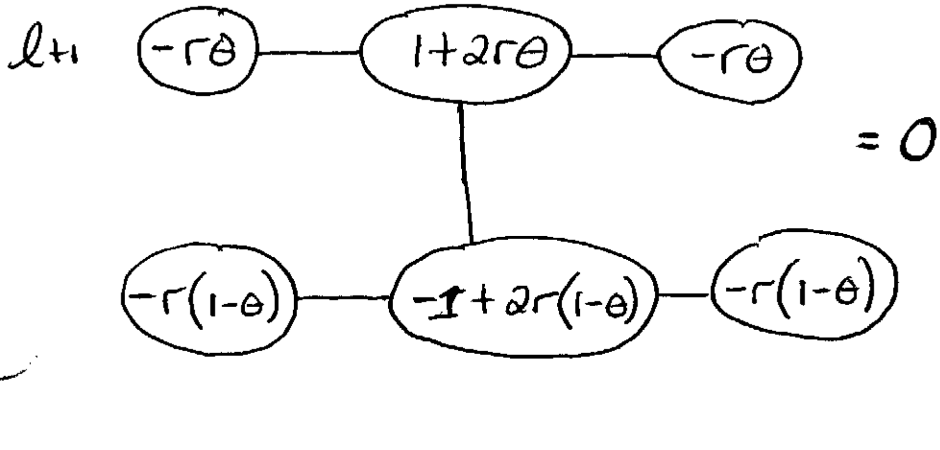
$O(h^2 + k^2)$, but unconditionally unstable! Perfect example of intuition gone astray

2.) Backward difference: $u_i^{\ell+1} - u_i^\ell = r\delta_x^2 u_i^{\ell+1}$ (Implicit)



Matrix sol'n

3.) Generalization: $u_i^{\ell+1} - u_i^\ell = \underbrace{r\theta\,\delta_x^2 u_i^{\ell+1} + r(1-\theta)\,\delta_x^2 u_i^\ell}_{\text{averaging of } u_i^\ell + u_i^{\ell+1} \text{ weighted by } \theta}$



$\theta = 1$ fully implicit
$\theta = 0$ explicit
$\theta = .5$ "Crank-Nicolson" (molecule centered at $\ell + 1/2$) $O(k^2 + h^2)$
$\theta \geq .5$ Unconditional stability
$< .5$ Conditional Stability