FEM Summary

- Belongs to general class of WR methods
  - Residual "orthogonal" to first N members of complete set
  - Residual "on average" equal to zero
  - Mimics familiar analytic techniques

- FEM utilizes localized low order C^0 polynomials as basis/weighting functions
  - Simple functions ... easy to integrate/differentiate
  - Single choice suitable for many problems
  - members of a complete set.

- Lagrange Bases
  - single function centered at each node ... local variation over an element
  - easy to extend from linear to quadratic to cubic etc ... only C^0
  - element becomes local unit of interpolation and integration ... # nodes/element dictate basis order.
  - FEM on 1D Lagrange (linear) looks just like FD except for Simpson's Rule like spreading of non-derivative terms

- Matrix Assembly
  - done element by element ... $[A] = \sum_e [A]^e$
  - assemble all contributions from a single element once ... i.e. deal with multiple rows and columns in global system of equations
  - use local node numbering scheme with a mapping (incidence list) to global system of equations

  - BCs ... Type I: remove Galerkin equation and enforce BC statement directly
    Type II: place value in boundary integral and integrate ... Type III similar except get contribution multiplying u ... must place back into LHS.
  - Unused Galerkin at Type I boundary is equation for the boundary flux.

- Variable Coefficients
  - several strategies are possible ... can use element-based values ... i.e. constant on an element and different in different elements
  - Node-based coefficients ... interpolate with same bases used to expand the solution
  - express analytically and integrate accordingly.

- Two-dimensional Problems
  - invoke Green's Theorem to allow use of linear bases ... also yields boundary integral convenient for type 2 and type 3 boundary data
  - Linear triangle ... popular, integrations simple and exact ... counterclockwise numbering convention ... element matrix is 3×3
  - Boundary conditions ... through the boundary integral ... these become 1D integrations so 1D linear basis is appropriate
  - Banded Storage mode essential ... rows stay the same, but columns compressed ... half bandwidth is maximum difference between node numbers within an element
  - FEM has similar conservation properties as FD. Proviso: must use all Galerkin equations Hinges of $\sum_{i=1}^{3} \phi_i = 1$ and $\sum_{i=1}^{3} \frac{\partial \phi_i}{\partial x} = 0$
- Bilinear element
  - defined in local coordinate system ... for ease of integration
  - leads to concept of a Gauss point matrix ... i.e. evaluate all possible integrands at a gauss point for a given element ... sum of Gauss point matrices results in element integration

- Need Jacobian of the transformation ... simple in this case; constant $A/4$ ... use Gaussian quadrature which can integrate a polynomial of degree $2N_G-1$ exactly for N_G gauss points

- Deformed Bilinear Element

  - utilizes concept of isoparametric element... i.e. geometry interpolated by a "shape function"... in isoparametric case, shape function is same as basis for the solution... can have sub-and superparametric elements as well.

  - Bilinear case... lines of constant ξ or η are lines of constant slope in x-y space... must get the Jacobian and express derivatives in x-y space as derivatives in ξ-η space

  - Higher order elements easy to generate... always best to work in local coordinate space

  - Can do the same for the linear triangle... procedurally everything the same once the parent element is defined and associated bases in ξ, η space

- Transient Problems

  - bases are time invariant ... nodal values become time-varying.

  - spatial discretization leads to coupled ODEs in time ... can use any methods from ENGS 69

- Simplest … θ weighting for time integration essentially the same as FD … but no explicit method occurs when $\theta = 0$ … parabolic equation leads to a 2 level scheme whereas hyperbolic problem is 3 level in time.

- Can identify the "Mass" and "Stiffness" matrices RHS involves matrix/vector product ….
Several strategies: ① build a matrix … prefer to store in sparse mode; banded OK, but 1D vector best; ② reconstruct each time; involves element assembly loop … do everything at element level

- Explicit Schemes on FEM possible … best approach "Nodal quadrature" … i.e. use a weaker quadrature which diagonalizes the mass matrix; Strategy is to apply "gauss points" at the nodes; must find new weights so that can integrate the area of the element exactly; apply this quadrature rule to <u>all</u> terms in the equation

- Stability / Accuracy … Can use the Von Neumann analysis and amplification/propagation factor as in FD situation. Analysis done on uniform mesh of bilinear elements. Integral lumping leads exactly to FD molecule.

- Point Sources … easy on FEM since everything is already an integral … when possible place at a node … when interior to an element must distribute proportionally amongst nodes in the element.

- Vector Problems on FE
  - have multiple equations and unknowns at each node ... generally, apply FE methods to each scalar component
  - size of system grows. In 2D with x and y component where there had been a single number associated with each (i,j) combination; now have 4 numbers
  - Best to use vector identities to lower derivative orders in the area integrals and produce appropriate boundary integrals
  - Boundary quantities often supplied in a normal/tangential framework ... best to rotate the projections of the vector equations into normal tangential coordinate system; for symmetry also want to rotate variables ... leads to a similarity transformation of pre and post multiplication of the system matrix by a rotation matrix and its transpose ... only do this for weighting and basis functions associated with boundary nodes.
- Can have situations with interface relations which need to be imposed as constraints ... conceptually can view as a doubling of nodes at the interface. Instead, use interface relations to eliminate one unknown in terms of the other.

- Rotations etc, require a suitable nodal normal. take as a weighted average of the 2 normals for each neighboring line segment eminating from a boundary point.
- Bandwidth grows to twice the full bandwidth of the scalar system plus 1.
- examples considered include continuum mechanics and electromagnetics.