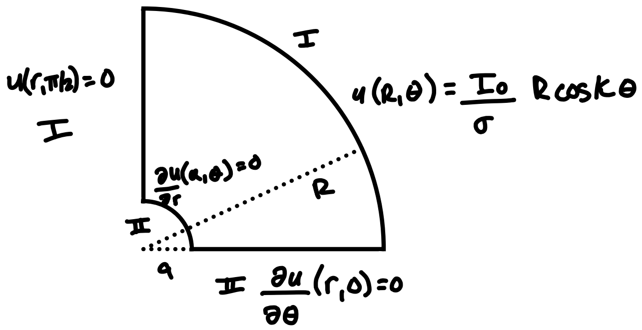
(a)




$$\nabla^2 u = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$u = -\frac{I_o}{\sigma} R \left(\frac{r^k + a^{2k} r^{-k}}{R^k + a^{2k} R^{-k}}\right) \cos k\theta$$

$$\frac{\partial u}{\partial r} = \frac{-I_o R}{\sigma}\left(\frac{k r^{k-1} - a^{2k} k r^{-k-1}}{R^k + a^{2k} R^{-k}}\right) \cos k\theta$$

$$r\frac{\partial u}{\partial r} = \frac{-I_o R}{\sigma}\left(\frac{k r^{k} - a^{2k} k r^{-k}}{R^k + a^{2k} R^{-k}}\right) \cos k\theta$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{-I_o R}{\sigma}\left(\frac{k^2 r^{k-1} + a^{2k} k^2 r^{-k-1}}{R^k + a^{2k} R^{-k}}\right) \cos k\theta$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{-I_o R k^2}{\sigma}\left(\frac{r^{k-2} + a^{2k} r^{-k-2}}{R^k + a^{2k} R^{-k}}\right) \cos k\theta$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{I_o R k^2}{\sigma}\left(\frac{r^{k} + a^{2k} r^{-k}}{R^k + a^{2k} R^{-k}}\right) \cos k\theta$$

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{I_o R k^2}{\sigma}\left(\frac{r^{k-2} + a^{2k} r^{-k-2}}{R^k + a^{2k} R^{-k}}\right) \cos k\theta$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{-I_o R k^2}{\sigma}\left(\frac{r^{k-2} + a^{2k} r^{-k-2}}{R^k + a^{2k} R^{-k}}\right) \cos k\theta \; +$$

$$\frac{I_o R k^2}{\sigma}\left(\frac{r^{k-2} + a^{2k} r^{-k-2}}{R^k + a^{2k} R^{-k}}\right) \cos k\theta = 0$$

∴ it satisfies the diff eq

$$u(r,\pi/2) = -\frac{I_o}{\sigma} R \left(\frac{r^k + a^{2k} r^{-k}}{R^k + a^{2k} R^{-k}}\right) \cos \frac{k\pi}{2}, \quad k \text{ is odd } \therefore u(r,\pi/2) = 0 \checkmark$$

$$u(R,\theta) = -\frac{I_o}{\sigma} R \left(\frac{R^k + a^{2k} R^{-k}}{R^k + a^{2k} R^{-k}}\right) \cos k\theta = -\frac{I_o R}{\sigma} \cos k\theta \checkmark$$

$$\frac{\partial u}{\partial r}(a,\theta) = \frac{-I_o R}{\sigma}\left(\frac{k a^{k-1} - a^{2k} k a^{-k-1}}{R^k + a^{2k} R^{-k}}\right) \cos k\theta = \frac{-I_o R k}{\sigma}\left(\frac{a^{k-1} - a^{2k-k-1}}{R^k + a^{2k} R^{-k}}\right) \cos k\theta = 0 \checkmark$$

$$\frac{\partial u}{\partial \theta} = \frac{I_0}{\sigma} R\, k \left( \frac{r^k + a^{2k} r^{-k}}{R^k + a^{2k} R^{-k}} \right) \sin k\theta$$

$$\frac{\partial u}{\partial \theta}(r,0) = \frac{I_0}{\sigma} R\, k \left( \frac{r^k + a^{2k} r^{-k}}{R^k + a^{2k} R^{-k}} \right) \sin 0 \;=\; 0 \;\checkmark$$

$\therefore\; u = -\dfrac{I_0}{\sigma} R \left( \dfrac{r^k + a^{2k} r^{-k}}{R^k + a^{2k} R^{-k}} \right) \cos k\theta$ is analytical sol

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b)

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta}$$

$$-\nabla u = \frac{I_0}{\sigma} R k \left( \frac{r^{k-1} - a^{2k} r^{-k-1}}{R^k + a^{2k} R^{-k}} \right) \cos k\theta\, \hat{r} - \frac{I_0}{\sigma} R k \left( \frac{r^{k-1} + a^{2k} r^{-k-1}}{R^k + a^{2k} R^{-k}} \right) \sin k\theta\, \hat{\theta}$$

$$= \frac{I_0}{\sigma} R k \left( \left( \frac{r^{k-1} - a^{2k} r^{-k-1}}{R^k + a^{2k} R^{-k}} \right) \cos k\theta\, \hat{r} - \left( \frac{r^{k-1} + a^{2k} r^{-k-1}}{R^k + a^{2k} R^{-k}} \right) \sin k\theta\, \hat{\theta} \right)$$

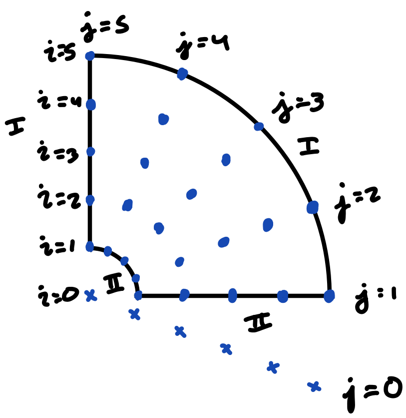
---

$$\hat{r} = \cos\theta\, i + \sin\theta\, j$$
$$\hat{\theta} = -\sin\theta\, i + \cos\theta\, j$$

c)

$$\nabla^2 u = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$0 = \frac{1}{r}\left( \frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2} \right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$




$$\frac{\partial u_i}{\partial r} = \frac{u_{i+1} - u_{i-1}}{2\Delta r}$$

$$\frac{\partial^2 u_i}{\partial r^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta r^2}$$

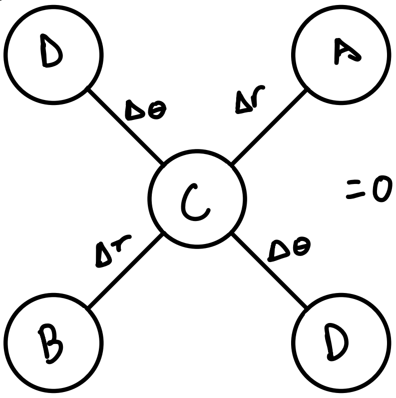
$$\frac{\partial^2 u}{\partial \theta^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta \theta^2}$$

$$\frac{1}{r_i}\left( \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta r} + r_i \left( \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta r^2} \right) \right) + \frac{1}{r_i^2}\left( \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta \theta^2} \right) = 0$$

$$\left( \left(\frac{1}{r_i}\right)\left(\frac{1}{2\Delta r}\right) + \left(\frac{1}{\Delta r^2}\right) \right) u_{i+1,j} + \left( -\left(\frac{1}{r_i}\right)\left(\frac{1}{2\Delta r}\right) + \left(\frac{1}{\Delta r^2}\right) \right) u_{i-1,j} - 2\left( \frac{1}{\Delta r^2} + \frac{1}{r_i^2 \Delta\theta^2} \right) u_{i,j} +$$

$$\left( \frac{1}{r_i^2 \Delta\theta^2} \right) u_{i,j+1} + \left( \frac{1}{r_i^2 \Delta\theta^2} \right) u_{i,j-1}$$

(c)




$=0$

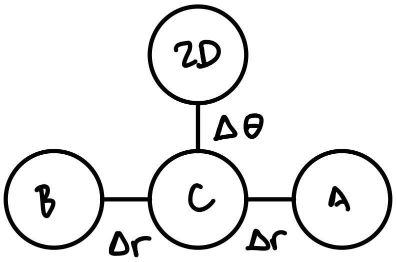
$$A = \left(\left(\frac{1}{r_i}\right)\left(\frac{1}{2\Delta r}\right) + \frac{1}{\Delta r^2}\right)$$

$$B = \left(-\left(\frac{1}{r_i}\right)\left(\frac{1}{2\Delta r}\right) + \frac{1}{\Delta r^2}\right)$$

$$C = -2\left(\frac{1}{\Delta r^2} + \frac{1}{r_i^2 \Delta\theta^2}\right)$$

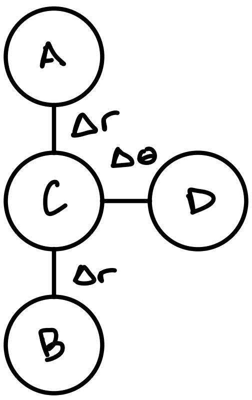
$$D = \left(\frac{1}{r_i^2 \Delta\theta^2}\right)$$

$$A u_{i+1,j} + B u_{i-1,j} + C u_{i,j} + D u_{i,j-1} + D u_{i,j+1}$$

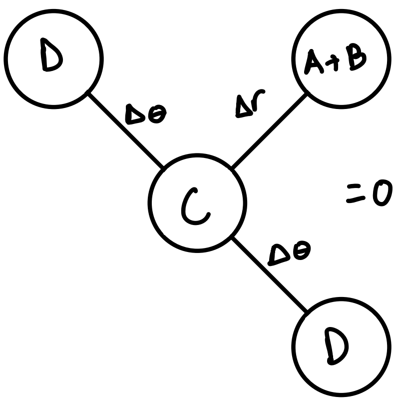



$$\frac{\partial u}{\partial \theta}(r,0) = 0 = \frac{u_{i,2} - u_{i,0}}{2\Delta\theta}$$

$=0$ $\quad u_{i,0} = u_{i,2}$



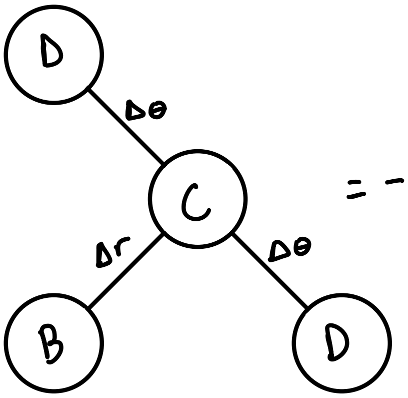

$=0$ $\quad u_{i,s} = 0$



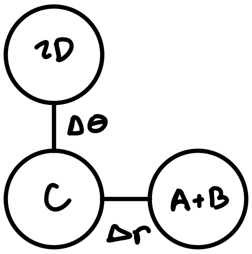

$$\frac{\partial u}{\partial r}(a,\theta) = 0 = \frac{u_{2,j} - u_{0,j}}{2\Delta r}$$

$=0$ $\quad u_{0,j} = u_{2,j}$

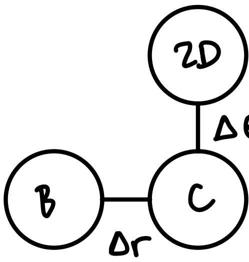
$$u(R,\theta) = f(\theta) = \frac{-I_0}{\sigma} R\cos k\theta$$



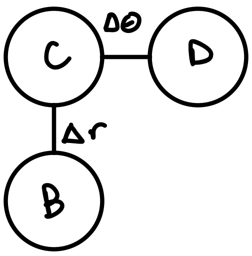

$= -A f(\theta)$



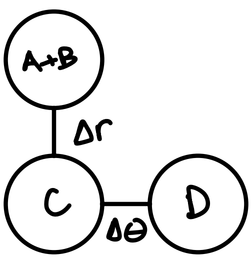

$= 0$




$= -A f(0)$




$= -A f(\pi/2)$




$= 0$