Classical Point Iteration Methods
(For Solving System $Au = b$)

- Start w/ u_{ij}^{0} ... "Initial Guess" for each $(i,j) \Rightarrow \underset{\sim}{u}^{0}$
  Update sol'n repeatedly point-by-point

- "Infinite" Algorithm ... need a stopping rule

  Typical: $\| \underset{\sim}{u}^{\ell+1} - \underset{\sim}{u}^{\ell} \| < \epsilon$ Absolute

  Better: $\dfrac{\| \underset{\sim}{u}^{\ell+1} - \underset{\sim}{u}^{\ell} \|}{\| \underset{\sim}{u}^{\ell+1} \|} < \epsilon$ relative

Relative criterion accounts for Sol'n size!

- Jacobi

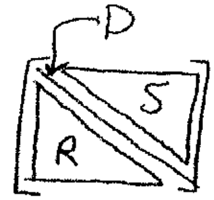
$$u_{ij}^{\ell+1} = \frac{1}{\beta_0}\left[\beta_1 u_{i+1,j}^{\ell} + \beta_2 u_{i-1,j}^{\ell} + \beta_3 u_{i,j+1}^{\ell} + \beta_4 u_{i,j-1}^{\ell} - h^2 g_{ij}\right]$$

$\ell \equiv$ iteration # ; u_{ij}^{0} "initial guess"

- Simple
- 2 arrays: $u^{\ell}, u^{\ell+1}$ of length Total # Nodes
- get one new value at a time
- order of calculation does not matter

Common to analyze from matrix perspective (but don't build these in practice)

$$[A] = \underbrace{[R]}_{\text{below}} + \underbrace{[D]}_{\text{on diagonal}} + \underbrace{[S]}_{\text{above}} \Rightarrow$$

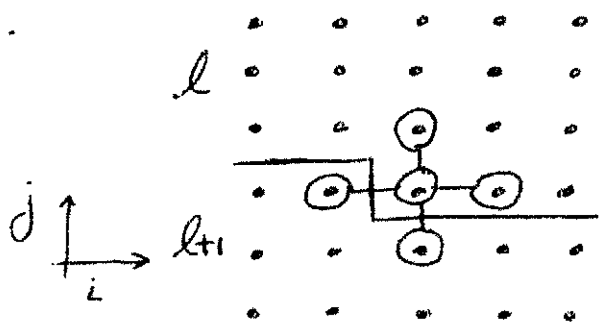


then $[D]\{U\}^{l+1} = -[R]\{U\}^{l} - [S]\{U\}^{l} + \{b\}$

$$\{U\}^{l+1} = \underbrace{-[D]^{-1}[R+S]}_{G_J \,\equiv\, \text{Jacobi Iteration matrix}}\{U\}^{l} + [D^{-1}]\{b\}$$

- Gauss-Seidel
  - use latest info
  - order does make a difference

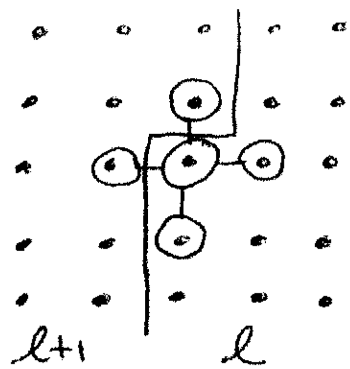
e.g.



proceed row-wise

$$u_{ij}^{l+1} = \frac{1}{\beta_0}\left[\beta_1 u^{l}_{i+1,j} + \beta_2 u^{l+1}_{i-1,j} + \beta_3 u^{l}_{i,j+1} + \beta_4 u^{l+1}_{i,j-1} - h^2 g_0\right]$$

proceed column-wise
(downward)



$$u_{ij}^{\ell+1} = \frac{1}{\beta_0}\left[\beta_1 u^{\ell}_{i+1,j} + \beta_2 u^{\ell+1}_{i-1,j} + \beta_3 u^{\ell+1}_{i,j+1} + \beta_4 u^{\ell}_{i,j-1} - g_{ij} h^2\right]$$

In matrix form ...

$$[R+D]\{u\}^{\ell+1} = -[S]\{u\}^{\ell} + \{b\}$$

$$\{u^{\ell+1}\} = \underbrace{-[R+D]^{-1}[S]}_{\text{Gauss-Seidel Iteration Matrix} \equiv G_{GS}}\{u\}^{\ell} + [R+D]^{-1}\{b\}$$

Convergence of Point Iterative Methods

- Have general form

$$u^{\ell+1} = G u^{\ell} + r$$

Primary result: Spectral Radius $\equiv \rho(G) < 1$
guarantees it

Proof has two key observations:

- $e^{\ell+1} = Ge^{\ell}$ where $e^{\ell+1} = u^{\ell+1} - u$

Recall: $u^{\ell+1} = Gu^{\ell} + r$

$$\begin{array}{l} u^{\ell+1} = Gu^{\ell} + r \\ \underline{u = Gu + r} \\ u^{\ell+1} - u = G(u^{\ell} - u) \end{array}$$

- e^{0} can be expressed in terms of eigenvectors of G:

$$e^{0} = \sum_{i=1}^{n} b_i V_i \qquad V_i = \text{eigenvector of } G$$

then $e^{1} = Ge^{0} = \sum_{i=1}^{n} b_i G V_i = \sum_{i=1}^{n} b_i \lambda_i V_i$

$$\vdots$$

$$e^{\ell} = \sum_{i=1}^{n} b_i \lambda_i^{\ell} V_i$$

If we want $\lim_{\ell \to \infty} e^{\ell} = 0$, then need $|\lambda_i| < 1$

So eigenvalues of G are critical!

<u>Eigenvalues of Iteration Matrices</u>

- Jacobi: $G_J = -D^{-1}[R+S]$

so $\det(\lambda I - G_J) = 0 \Rightarrow \det(\lambda I + D^{-1}(R+S)) = 0$

$$\Rightarrow \det\left(D^{-1}\lambda D + D^{-1}(R+S)\right) = \det D^{-1}(\lambda D + R + S) = 0$$

$$= \det D^{-1} \det(\lambda D + R + S) = 0$$

$$\boxed{\therefore \det(\lambda D + R + S) = 0}$$

- Gauss-Seidel : $G_{GS} = -(D+R)^{-1} S$

Same form as above ... replace: D w/ $D+R$
$R+S$ w/ S

$$\boxed{\therefore \det(\lambda D + \lambda R + S) = 0}$$

- Recall from EN6569 ... Strict diagonal dominance guarantees convergence ... but we don't have this for general elliptic FD molecules

have instead for A :

i) $\alpha_{ii} > 0, \ \alpha_{ij} \leq 0 \quad i \neq j$

ii) $\alpha_{ii} \geq \sum_{\substack{j=1 \\ i \neq j}}^{N} |\alpha_{ij}|$ w/ strict inequality for some "i"
($f_{ij} < 0$ or $f_{ij} = 0$ w/ Type I)

Can still show that this will produce $\rho(G) < 1$

Proof: want to show existence of $|\lambda| \geq 1$ is a contradiction....

e.g. Jacobi: $G = -D^{-1}(R+S)$

so $\underset{i \neq j}{g_{ij}} = -\frac{\alpha_{ij}}{\alpha_{ii}} \Rightarrow g_{ij} \geq 0$ since (i)

If $|\lambda| \geq 1$ exists, then must have eigenvector

$$Gw = \lambda w \Rightarrow (G - \lambda I)w = 0$$

or $\underbrace{\left(I - \frac{1}{\lambda} G\right)}_{F} w = 0$

if w is an eigenvector, $\det F = 0$

But $\sum_{\substack{j=1 \\ i \neq j}}^{N} |f_{ij}| = \sum_{\substack{j=1 \\ i \neq j}}^{N} \left|\frac{g_{ij}}{\lambda}\right| = \underbrace{\left|\frac{1}{\lambda}\right|}_{\leq 1} \underbrace{\sum_{\substack{j=1 \\ i \neq j}}^{N} |g_{ij}|}_{\leq 1} \leq 1 = f_{ii}$

from ii)

So F has the properties

$$\left.\begin{array}{l} f_{ii} > 0, \; f_{ij} \leq 0 \\ f_{ii} \geq \sum_{\substack{j=1 \\ i \neq j}}^{N} |f_{ij}| \end{array}\right\}$$ same properties as A

$\therefore \det F \neq 0$

$\lambda \neq$ eigenvalue

(i.e. a matrix which has these properties is non-singular)

– Determing $\rho(G)$ in practice

- Can compute w/ Power Method
  requires us to actually construct G
- estimate during iteration

$$\rho \approx \frac{\|\delta^{\ell}\|}{\|\delta^{\ell-1}\|} \quad \text{where } \delta^{\ell} = u^{\ell} - u^{\ell-1}$$

Theoretical basis:

$$\begin{aligned} u^{\ell+1} &= Gu^{\ell} + r \\ u^{\ell} &= Gu^{\ell-1} + r \\ \hline \delta^{\ell+1} &= G\delta^{\ell} \end{aligned}$$

same as before... expand $\delta^{1} = \sum c_i v_i$

– Rate of convergence

- fundamentally governed by $\rho(G)$
- for "large" $\ell \Rightarrow e^{\ell+1} \approx \rho(G) e^{\ell}$

$$\therefore \frac{\|e^{\ell}\|}{\|e^{\ell+1}\|} \approx \frac{1}{\rho(G)}$$

$-\log \rho(G)$ indicates # digits by which each iteration reduces the error

so for error reduction by factor K

$\|e\|^{\ell+m} = \rho^m \|e\|^{\ell}$, $K = \rho^m \Rightarrow m > \frac{\log K}{\log \rho}$

- SOR

$$u_{ij}^{l+1} = \frac{\omega}{\beta_0}\left[\beta_1 u_{i+1,j}^{l} + \beta_2 u_{i-1,j}^{l+1} + \beta_3 u_{i,j+1}^{l} + \beta_4 u_{i,j-1}^{l+1} - h_{ij}\right] + (1-\omega)u_{ij}^{l}$$

ω = relaxation parameter ($\omega = 1$ is Gauss Seidel)

Matrix form:

$$[D]\{u^{l+1}\} = (1-\omega)[D]\{u^{l}\} - \omega[R]\{u^{l+1}\} - \omega[S]\{u^{l}\} + \omega\{b\}$$

$$[D+\omega R]\{u^{l+1}\} = [(1-\omega)D - \omega S]\{u^{l}\} + \omega\{b\}$$

$$\{u\}^{l+1} = \underbrace{[D+\omega R]^{-1}[(1-\omega)D - \omega S]}_{G_{SOR}}\{u\}^{l} + \omega[D+\omega R]^{-1}\{b\}$$

- Have a number of variations on a theme

SSOR $\equiv$ Symmetric SOR
USSOR $\equiv$ Unsymmetric SOR

- SSOR

Single cycle of an iteration consists of <u>two</u> passes through the nodes in the mesh

In first pass $\Rightarrow$ use SOR and sweep in forward direction

In 2nd pass $\Rightarrow$ repeat but in reverse order

e.g. $\{u\}^{\ell+1/2} = [G_{SOR}]\{u\}^{\ell} + C_{SOR}$

$$\{u\}^{\ell+1} = [\bar{G}_{SOR}]\{u^{\ell+1/2}\} + \bar{C}_{SOR}$$

where $\bar{G}_{SOR} = [D+\omega S]^{-1}[(1-\omega)D - \omega R]$

$$\bar{C}_{SOR} = \omega [D+\omega S]^{-1}\{b\}$$

- USSOR

Same as SSOR except use different relaxation parameter on the reverse sweep

i.e. $\{u\}^{\ell+1/2} = [G_{SOR}]\{u\}^{\ell} + C_{SOR}$

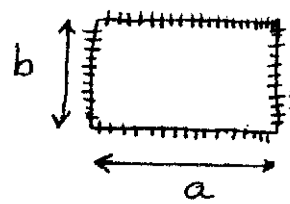
$$\{u\}^{\ell+1} = [\bar{G}_{SOR}]\{u\}^{\ell+1/2} + \bar{C}_{SOR}$$

where $[\bar{G}_{SOR}] = [D+\bar{\omega}S]^{-1}[(1-\bar{\omega})D - \bar{\omega}R]$

$$\bar{C}_{SOR} = \bar{\omega}[D+\bar{\omega}S]^{-1}\{b\}$$

- Consider Jacobi on model problem ...

  Laplace / Poisson w/ Type I BCs



  Want spectrum of G_J ....

- First get spectrum of $A \Rightarrow Au = \lambda u$ subject to $u=0$ on boundaries

We know: $\dfrac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}}{h^2} = \lambda u_{ij}$

But u_{ij} has general sol'n form: $u_{ij} = e^{\pm \tilde{i}(\sigma x_i + \gamma y_j)}$

so: $u_{i+1,j} = e^{\pm \tilde{i}(\sigma(x_i+h) + \gamma y_j)} = e^{\pm \tilde{i}\sigma h} e^{\pm \tilde{i}(\sigma x_i + \gamma y_j)} = e^{\pm \tilde{i}\sigma h} u_{ij}$

$u_{i-1,j} = e^{\pm \tilde{i}(\sigma(x_i-h) + \gamma y_j)} = e^{\mp \tilde{i}\sigma h} u_{ij}$

Likewise: $u_{i,j+1} = e^{\pm \tilde{i}\gamma h} u_{ij}$; $u_{i,j-1} = e^{\mp \tilde{i}\gamma h} u_{ij}$

Plug in: $\dfrac{1}{h^2}\Big(\underbrace{e^{\pm \tilde{i}\sigma h} + e^{\mp \tilde{i}\sigma h}}_{2\cos\sigma h} + \underbrace{e^{\pm \tilde{i}\gamma h} + e^{\mp \tilde{i}\gamma h}}_{2\cos\gamma h} - 4\Big) u_{ij} = \lambda u_{ij}$

$\therefore \lambda = \dfrac{1}{h^2}\left(2\cos\sigma h + 2\cos\gamma h - 4\right)$

Now $G_J = -D^{-1}[R+S] = I + \dfrac{h^2}{4} A$

So $$G_J u = \left(I + \frac{h^2}{4}A\right)u = \lambda_J u$$

$$= u + \frac{h^2}{4}Au = u + \frac{h^2}{4}\lambda u = \lambda_J u$$

$$\therefore \lambda_J = 1 + \frac{2\cos\sigma h + 2\cos\gamma h}{4} - 1$$

$$= \frac{1}{2}\left(\cos\sigma h + \cos\gamma h\right)$$

But we know u is subject to homogeneous BCs all around the boundary $\Rightarrow$ sin modes !!

$$\sigma = \frac{n\pi}{a} \; ; \; \gamma = \frac{m\pi}{b} \qquad n, m = 1, 2, \dots N$$

$\uparrow$ # interior nodes in each directi

So $$\lambda_J^{n,m} = \frac{1}{2}\left(\cos\frac{n\pi h}{a} + \cos\frac{m\pi h}{b}\right)$$

Max occurs for $m = n = 1$

$$\therefore \rho_J = \frac{1}{2}\left(\cos\frac{\pi h}{a} + \cos\frac{\pi h}{b}\right)$$

$$\approx 1 - \frac{1}{4}\left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}\right)h^2$$

$\Rightarrow$ For square $a = b \Rightarrow \rho_J \approx 1 - \dfrac{\pi^2 h^2}{2a^2}$

- Note: $\rho_J \to 1$ as $h \to 0$ .... Jacobi slows as mesh grows

- Rate of Convergence: $R(G_J) = -\log(\rho_J)$

$$= -\log\left(\cos\frac{\pi h}{a}\right) \approx \frac{\pi^2 h^2}{2a^2} + O(h^4)$$

IF $a=\pi$, then $R(G_J) \approx \frac{h^2}{2}$ (for h relative to π)

- Similar analysis for Gauss-Siedel shows

$$\lambda_{GS}^{n,m} = \frac{1}{4}\left(\cos\frac{n\pi h}{a} + \cos\frac{m\pi h}{b}\right)^2$$

so $$\rho_{GS} = \left[\frac{1}{2}\left(\cos\frac{\pi h}{a} + \cos\frac{\pi h}{b}\right)\right]^2$$

$$= \rho_J^2 \Rightarrow a=b : \rho_{GS} = \left(\cos\frac{\pi h}{a}\right)^2 \approx 1 - \left(\frac{\pi h}{a}\right)^2$$

- $R(G_{GS}) = -\log(\rho_{GS}) = -\log(\rho_J^2) = -2\log(\rho_J)$

$$= 2R(G_J) \approx \frac{\pi^2 h^2}{a^2} \Rightarrow h^2 \text{ (relative to } \pi)$$

Gauss-Siedel twice as fast as Jacobi !!

All based on Laplace w/ Type I BCs

So largest parabola given by largest $|\mu|$
i.e. $\rho(G_J)$ ... leads to largest minimum
corresponding λ ... i.e. $\rho(G_{SOR})$

$$\therefore \quad \omega_{opt} = \frac{2}{1+(1-\rho^2(G_J))^{1/2}}$$

and $\rho(G_{SOR}) = \omega_{opt} - 1$

(get this by plugging $\omega = \dfrac{2(1-(1-\mu^2)^{1/2})}{\mu^2}$ into
quadratic for λ ... $\lambda = \dfrac{-b}{2a}$ since $b^2-4ac=0$
to get value of λ at tangent point)

- Turns out for $\omega_{opt} \leq \omega < 2$ all eigenvalues
of G_{SOR} have same modulus $\omega - 1$

i.e. $\rho(G_{SOR}) = \omega - 1$

λ's turn complex for $\omega > \omega_{opt}$