# Summary of Von Neumann Approach

- Expand the spatial distribution of ICs as Fourier Series such that

$$u_i^0 = \sum_n A_n e^{j\sigma_n x_i}$$

- Examine how each term in the sum is propagated as $l = 1, 2, \ldots$ (in general $t_l$ to $t_{l+1}$) by the FD molecule

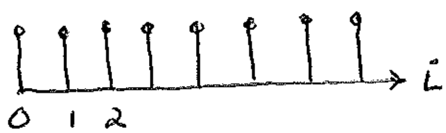
- Sufficient to look a general form of a single term and consider all possible $\sigma$ values: $u_i = e^{j\sigma x_i} = e^{j(\sigma h)i}$ $(j = \sqrt{-1})$

- $0 \le \sigma h \le \pi$ Since the most rapid variation on a mesh is node-to-node oscillation

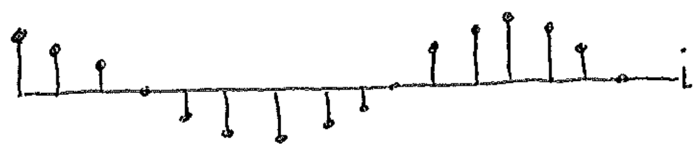
- As $\sigma h$ increases from zero, $e^{j\sigma h i}$ has increasing rate of oscillation which peaks at $\sigma h = \pi$

e.g. $\cos(\sigma h i)$

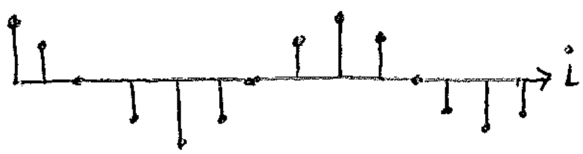
$\cos(0 \cdot i)$



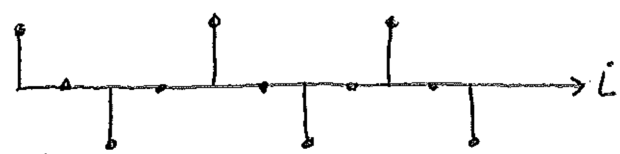

$\sigma h = \pi/6$



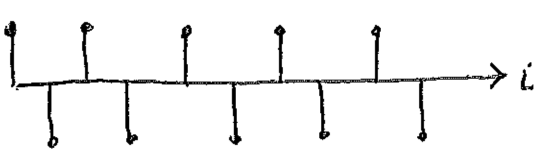

$\sigma h = \pi/4$



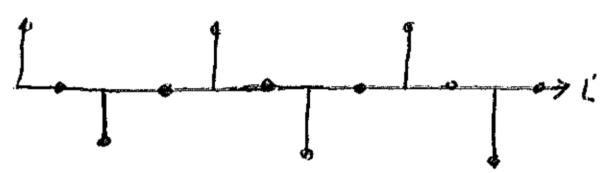

$\sigma h = \pi/2$




$\sigma h = \pi$




$\sigma h = 3\pi/2$




$\sigma h = 7\pi/4$


$\sigma h = 2\pi$

- Define "Amplification factor" of the FD eqns

$$U_i^{l+1} = U_i^l \gamma_o$$

(Analytically we know $\underbrace{\gamma \equiv e^{\alpha \Delta t}}$ )

$$U(x,t) = e^{\alpha t} e^{j\sigma x} \Rightarrow \frac{U(x,t+\Delta t)}{U(x,t)} = e^{\alpha \Delta t}$$

- Relate all (space, time) points in the FD molecule to pt $(i, l)$ using the defining relations

e.g. $U_i^{l+1} = \gamma_o U_i^l$

$$U_{i-1}^l = e^{-j\sigma h} U_i^l \Rightarrow e^{j\sigma(x_i - h)} = e^{-j\sigma h} e^{j\sigma x_i} = e^{-j\sigma h} U_i$$

$$U_{i+1}^{l+1} = e^{j\sigma h} \gamma_o U_i^l$$

etc

⋮

- Provides a relationship between $\gamma_o$ and $\sigma h$

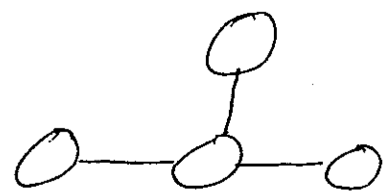
- Stability requires $-1 \le \gamma_o \le 1$ for all possible $\sigma h$ (i.e. $0 \le \sigma h \le \pi$)

- Bounded oscillations develop for $-1 \le \gamma_o < 0$

- Formally method only valid for
  - linear equations w/ constant coefficients
  - Uniform mesh
  - BCs at infinity

- Generally get same results as Matrix method (i.e. BCs effect stability in minor way relative to FD equations themselves)

e.g. Examine Euler Explicit



$$u_i^{l+1} - u_i^{l} = r\delta_x^2 u_i^l$$
$$= r\left(u_{i-1}^l - 2u_i^l + u_{i+1}^l\right)$$

$$\Rightarrow \left(\gamma_0 - 1\right)u_i^l = r\left(e^{-j\sigma h} - 2 + e^{j\sigma h}\right)u_i^l$$

$$\gamma_0 = 1 - 2r\left(1 - \cos\sigma h\right) \Rightarrow \text{Note: } \cos\sigma h = 1 - 2\sin^2\frac{\sigma h}{2}$$
$$\gamma_0 = 1 - 4r\sin^2\frac{\sigma h}{2}$$

$$\Rightarrow \text{For stability} \ldots \; |\gamma_0| < 1 \Rightarrow \underbrace{-1 \leq 1 - 2r\left(1 - \cos\sigma h\right) \leq 1}_{\text{For } \underline{\underline{\text{all}}} \text{ possible } \sigma\text{'s}}$$

But $0 < \sigma h \leq \pi \Rightarrow 0 < 1 - \cos\sigma h \leq 2$