# FD Solutions to PDEs: Review of Key Points

## I. Elliptic Eqns

- BCs around enclosed domain (Type I or II or III)
- Shadow node techniques for Type II + III ... interior PDE molecule altered
- Type I boundary ... do not use PDE molecule in sol'n ... but this molecule contains flux info after sol'n computed
- FD conservation requires use of all molecules
  Single FD molecule equivalent to Conservation over  "single box" ... sum of all boxes
  Global Conservation
- FD molecules plus BCs lead to coupled algebraic system
- 2D problems pentadiagonal (w/o $\frac{\partial^2 u}{\partial x \partial y}$ term)
  Direct methods "Unnatural"
- Iterative Matrix Sol'n Methods "natural" - take advantage of $(i,j)$ indexing
- All can be cast in form

$$\{u\}^{k+1} = [G]\{u\}^{k} + \{c\}$$ ; guaranteed convergence if $\rho(G) < 1$

- For general Elliptic PDE: $a\frac{\partial^2 u}{\partial x^2} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g$

  Iterative Sol'n guaranteed to converge

  if $a>0$; $c>0$; $f<0$

  or

  if $a>0$; $c>0$; $f=0$ w/ some Type I BCs

  Some restrictions on $h$ relative to size of $d + e$

  (Need some rows strictly diagonally dominant)

- Jacobi, Gauss-Siedel, SOR ... typically sweep through mesh rowwise or columnwise

- ADI ... sweep through mesh in alternating directions ... always tridiagonal ... 2 step process - sweep along rows followed by sweep along columns constitutes a single iteration. Solves system of Eqn's $[A]\{u\} = \{b\}$ w/o explicitly constructing $[A]$.

## II. Parabolic Eqns

- BCs in space same as Elliptic; ICs in time "open boundary" in Time

- Convergence: $U_i^\ell \to u(x_i, t_\ell)$ as $h, k \to 0$
- Consistency: $L_i \to L$ as $h, k \to 0$
- Stability: $|U_i^\ell| < M$ for all $i, \ell$

- Stability + Consistency = Convergence (Lax)

- Spatial discretization same as Elliptic

- Time-stepping ... similar to iteration in matrix soln except each new $u$ is an approximate answer at a given time instant

- 2 time-levels involved due to first derivative in time

- Explicit schemes ... pointwise propagation, stability constraints; $\frac{D\Delta t}{h^2} \equiv r$ key factor

- Implicit schemes ... intrinsically more stable; need to solve system of equations to advance soln one time-level; 1D → tridiagonal
  2D → pentadiagonal or
  iterative matrix methods as in Elliptic; eqn's Diagonally Dominant!

- ADI as time-stepper: Alternate time-level evaluation of $x$ & $y$ derivatives; fully implicit in derivatives tangential to direction of sweep while fully explicit in derivatives normal to direction of sweep ... tridiagonal systems only ... 2 step process: one rowwise followed by one columnwise sweep advances soln one time-level.

- Fourier Analysis ... leads to $\underline{\underline{2}}$ types of information $\Rightarrow$ Stability + Accuracy
- relate all space-time points in molecule to $u_i^l$
- obtain numerical amplification as function of $\sigma h$ (i.e. $\gamma_0 = F(\sigma h)$)
- Stability: $|\gamma_0| \leq 1$ Stable

  $-1 \leq \gamma_0 \leq 0$ bounded oscillations in time ... i.e. $2\Delta t$ wrinkles in soln at fixed space pt as function of $t$ "highest temporal frequency"

- Consider all possible $\sigma h$ supportable on a mesh $\Rightarrow$ $0 < \sigma h \leq \pi$

  why: ICs can have broad spectral content (i.e. discontinuities or sharp transitions in slope)

  Potential for rounding errors

- Shortest wavelengths usually worst offenders
- Accuracy ... compare numerical propagation w/ nature of the analytic propagation

  Consider all possible $\sigma h$ $\Rightarrow$ Short waves most misbehaved

  Can examine difference at single time-step, but also typical to consider propagation over characteristic Time $\Rightarrow$ One analytic Time constant $\Rightarrow$ "Propagation Factor"

# III Hyperbolic PDEs

- Spatial part same as elliptic; temporal part has additional <u>second</u> derivative in time

- BCs same as elliptic; boundaries "open" in time, but need <u>2</u> ICs

- Equivalent (analytically) "Primitive Pair" equations are typical ⇒ arise from "Conservation Laws"

- Time-stepping ... 3 levels; need <u>2</u> to get started (shadow node in time <u>or</u> system at "rest")

  - Explicit ... pointwise propagation, stability constraints; Courant condition is the key $c\frac{\Delta t}{\Delta x} < 1$ or $R \equiv \frac{c^2 \Delta t^2}{h^2} < 1$

  - Implicit possible ... matrix equation sol'n needed at each time-step as before; more stable

  - Fourier Analysis ... yields stability criterion typically in terms of $R$ and $\tau\Delta t$

  - Accuracy issues ... Compare Numerical to Analytic damping <u>and</u> Numerical to Analytic wave speeds

  - $\gamma_0$ has <u>2</u> roots (Quadratic due to 3 time levels) <u>One</u> typically mimics (or tries to) analytic sol'n <u>Other</u> does not ⇒ "parasitic"

- key is often the nature of parasitic root; want it to damp out

- Short wavelengths typically most ill-behaved; can consider propagation over single time-step <u>or</u> over characteristic time: Time to propagate one analytic wavelength

- Primitive Pair schemes plagued by undamped parasitic root; leads to $2\Delta x$ spectral content in sol'n due to abrupt ICs <u>or</u> rounding errors over time.