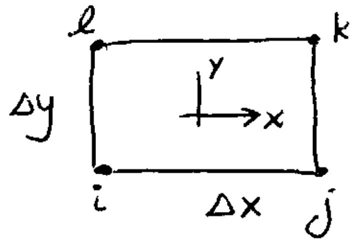
Bilinear Element



$$\phi_i(x,y) = (a_i x + b_i)(c_i y + d_i)$$
$$= A_i + B_i x + C_i y + \underline{\underline{D_i xy}}$$

"bilinear" term

Construct ϕ_i with same rules...

$\phi_i = 1$ at node i

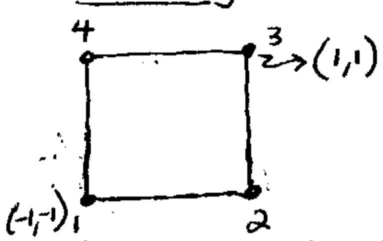
$= 0$ at nodes j, k, l

$$\Rightarrow \begin{bmatrix} 1 & x_i & y_i & (xy)_i \\ 1 & x_j & y_j & (xy)_j \\ 1 & x_k & y_k & (xy)_k \\ 1 & x_l & y_l & (xy)_l \end{bmatrix} \begin{Bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Repeat for ϕ_j, ϕ_k, ϕ_l

Easier... Use a local coordinate system (ξ, η); define element on $(-1,1)$ then map to actual (x,y)-space

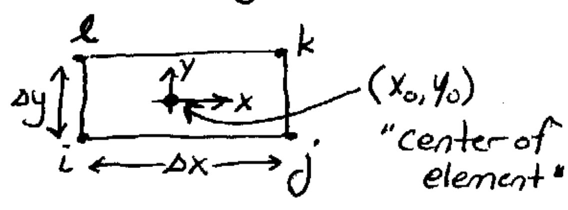
<u>Locally</u>



$$x = x_0 + \frac{\Delta x}{2}\xi$$
$$y = y_0 + \frac{\Delta y}{2}\eta$$

$$\Longleftrightarrow$$

<u>Globally</u>



$$\left.\begin{aligned} \phi_1(\xi,\eta) &= (1-\xi)(1-\eta)/4 \\ \phi_2(\xi,\eta) &= (1+\xi)(1-\eta)/4 \\ \phi_3(\xi,\eta) &= (1+\xi)(1+\eta)/4 \\ \phi_4(\xi,\eta) &= (1-\xi)(1+\eta)/4 \end{aligned}\right\} \quad \phi_i = \frac{1}{4}(1+\xi_i \xi)(1+\eta_i \eta)$$

for $i = 1, 2, 3, 4$

Mapping is simple: $x(\xi)$ only; $y(\eta)$ only; easily inverted; i.e. $\xi(x) = \dfrac{2(x-x_0)}{\Delta x}$

$$\eta(y) = \frac{2(y-y_0)}{\Delta y}$$

element "stretches"; but does not "twist"

But we need terms like $\dfrac{\partial\phi_i}{\partial x}$; $\dfrac{\partial\phi_i}{\partial y}$

$$\frac{\partial\phi_i}{\partial x}(\xi,\eta) = \frac{\partial\phi_i}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\phi_i}{\partial\eta}\cancelto{0}{\frac{\partial\eta}{\partial x}} = \frac{\partial\phi_i}{\partial\xi}\frac{\partial\xi}{\partial x} = \frac{\xi_i(1+\eta_i\eta)}{4}\left(\frac{2}{\Delta x}\right)$$

$$\frac{\partial\phi_i}{\partial y}(\xi,\eta) = \frac{\partial\phi_i}{\partial\xi}\cancelto{0}{\frac{\partial\xi}{\partial y}} + \frac{\partial\phi_i}{\partial\eta}\frac{\partial\eta}{\partial y} = \frac{\partial\phi_i}{\partial\eta}\frac{\partial\eta}{\partial y} = \underbrace{\frac{\eta_i(1+\xi_i\xi)}{4}\left(\frac{2}{\Delta y}\right)}_{\text{No longer constant in an element !!}}$$

Integration is needed in the end... must occur over the physical space ... i.e. $dx\,dy$

But $dx = \dfrac{\Delta x}{2}d\xi$; $dy = \dfrac{\Delta y}{2}d\eta \Rightarrow dx\,dy = \dfrac{\Delta x \Delta y}{4}d\xi\, d\eta$

So $$\int_e (\ )\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} (\ )\underbrace{\frac{A}{4}}_{\text{"Jacobian"}}d\xi\,d\eta$$

"Jacobian" of the transformation tells how $dA(x,y)$ is related to $dA(\xi,\eta)$

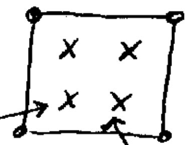
Limits now natural for Gaussian Quadrature

Recall: $\displaystyle\int_{-1}^{1} f(x)\,dx = \sum_{k=1}^{NG} f(x_k)W_k \Rightarrow$ exact for polynomial of degree $2NG-1$

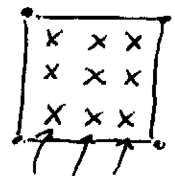
2-D extension:

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi\, d\eta = \sum_{l=1}^{NG}\sum_{k=1}^{NG} f(\xi_k,\eta_l)\, W_k W_l$$

e.g. 2X2



3X3



2 pts from 1D set of gauss pts + weights but in each dimension

3 pts from 1D set in each dimension

From $\sum_{l=1}^{2}\sum_{k=1}^{2}$: $(\xi_1,\eta_1)=(-.577,-.577)$; $W_1=1$

$(\xi_2,\eta_1)=(.577,-.577)$; $W_1=W_2=1$

$\vdots$

etc

these are the locations where we must evaluate the integrand written as a function of the local coordinates

e.g. $$\left\langle \frac{\partial\phi_i}{\partial x}\frac{\partial\phi_j}{\partial x}\right\rangle^e = \int_{-1}^{1}\int_{-1}^{1} \frac{\partial\phi_i}{\partial\xi}\frac{\partial\phi_j}{\partial\xi}\left(\frac{2}{\Delta x}\right)^2 \frac{A}{4}\, d\xi\, d\eta$$

$$= \sum_{k=1}^{NG}\sum_{l=1}^{NG} \frac{\partial\phi_i}{\partial\xi}(\xi_k,\eta_l)\frac{\partial\phi_j}{\partial\xi}(\xi_k,\eta_l)\left(\frac{2}{\Delta x}\right)^2\frac{A}{4}W_k W_l$$

$$\left\langle \phi_j \phi_i\right\rangle^e = \sum_{k=1}^{NG}\sum_{l=1}^{NG} \phi_j(\xi_k,\eta_l)\,\phi_i(\xi_k,\eta_l)\frac{A}{4}W_k W_l$$

$$\left\langle \frac{\partial\phi_j}{\partial y}\phi_i\right\rangle^e = \sum_{k=1}^{NG}\sum_{l=1}^{NG} \frac{\partial\phi_j}{\partial\eta}(\xi_k,\eta_l)\left(\frac{2}{\Delta y}\right)\phi_i(\xi_k,\eta_l)\frac{A}{4}W_k W_l$$

Need to do this for each (i,j) combination ... Sum over Gauss pts only performs the integration; (i,j) locally set the position in the element matrix; (i,j) globally set the position for the contribution to the global matrix

Now simplify Notation ... collapse double sum to single

e.g. $\sum_k \sum_\ell \longrightarrow \sum_{m=1}^{M=NG^2}$; $W_k W_\ell = W_m$

let f_{ij} symbolize a general integrand, then

$$\langle f_{ij} \rangle^e = \sum_{m=1}^{M} f_{ijm} \left(\frac{A}{4}\right) W_m$$

$$\text{So} \quad A_{ij}^e = \langle f_{ij} \rangle^e = \sum_{m=1}^{M} \underbrace{f_{ijm}}_{\equiv [f]_m^e} \frac{A}{4} W_m \quad \text{for } i,j = 1 \text{ to \# nodes per element}$$

"gauss pt matrix"

i.e.

$$[f]_m^e = \begin{bmatrix} f_{11m} & f_{12m} & f_{13m} & f_{14m} \\ f_{21m} & f_{22m} & f_{23m} & f_{24m} \\ f_{31m} & f_{32m} & f_{33m} & f_{34m} \\ f_{41m} & f_{42m} & f_{43m} & f_{44m} \end{bmatrix}^e$$

only need to be able to evaluate f_{ij} ... i.e. the integrand at a gauss point i.e. $A_{ij}^e = \frac{A}{4} \sum_{m=1}^{M} [f]_m^e W_m$

Conclude that the element matrix is constructed as a weighted sum of M $\underbrace{\text{gauss point matrices}}$

purely the integrand evaluated at a gauss pt

easy once we have $\phi_i, \frac{\partial \phi_i}{\partial x}, \frac{\partial \phi_i}{\partial y}$ etc as functions of (ξ, η)

requires $x(\xi), y(\eta)$

Global Assembly ...

$$[A] = \sum_e [A]^e = \sum_e \sum_{m=1}^{M} \frac{A^e}{4} [f]_m^e \omega_m$$

have added another loop to the procedure

i.e. Loop over elements

Evaluate element matrix

Loop over gauss points

Evaluate gauss pt matrix

sum (weighted) into element matrix

Assemble in global matrix

Apply BCs

Solve

Analyze solution