Transient Problems on Finite Elements

e.g. Diffusion equation: $c\frac{\partial u}{\partial t} - \nabla\cdot K\nabla u = g$

$$u(x,y,t) \approx \hat{u}(x,y,t) = \sum_{j=1}^{N} u_j(t)\,\phi_j(x,y)$$

- ϕ_j are time-invariant
- $u_j(t)$ are nodal "histories"
- Derivatives: $\frac{\partial \hat{u}}{\partial t} = \sum_{j=1}^{N} \frac{du_j}{dt}\phi_j$; $\underbrace{\frac{d\hat{u}}{dx} = \sum_{j=1}^{N} u_j \frac{\partial \phi_j}{\partial x}}_{\text{same as steady-state case}}$

Galerkin:

$$\left\langle c\frac{d\hat{u}}{dt}, \phi_i\right\rangle + \left\langle K\nabla\hat{u}\cdot\nabla\phi_i\right\rangle = \langle g,\phi_i\rangle + \oint K\nabla\hat{u}\cdot\hat{n}\,\phi_i\,ds$$

$$\left\langle c\sum_{j=1}^{N}\frac{du_j}{dt}\phi_j\phi_i\right\rangle + \left\langle K\sum_j u_j\nabla\phi_j\cdot\nabla\phi_i\right\rangle = \langle g\phi_i\rangle + \oint(\,)\phi_i\,ds$$

$$\sum_j \frac{du_j}{dt}\langle c\,\phi_j\phi_i\rangle + \sum_j u_j\langle K\nabla\phi_j\cdot\nabla\phi_i\rangle = \langle g\phi_i\rangle + \oint(\,)\phi_i\,ds$$

$$[M]\left\{\frac{du_j}{dt}\right\} + [K]\{u\} = \{r\}$$

$$[M] \Rightarrow m_{ij} = \langle c\phi_j \cdot \phi_i \rangle \quad \text{"Mass" Matrix}$$

$$[K] \Rightarrow k_{ij} = \langle K \nabla\phi_j \cdot \nabla\phi_i \rangle \quad \text{"Stiffness" Matrix}$$

Lumped System $\Rightarrow$ ODE's in t

Formally Sol'n is

$$\left\{\frac{du}{dt}\right\} = -[M^{-1}K]\{u\} + [M^{-1}]\{r\}$$

$$u = \{u(0)\} - [M^{-1}K]\left\{\int u\,dt\right\} + [M^{-1}]\left\{\int r\,dt\right\}$$

IC's $\qquad\qquad\qquad\qquad$ BC's

M^{-1} seldom calculated : M banded; M^{-1} <u>full</u>

<u>Discrete System</u>: <u>2 levels in t</u>

Two views which lead to the same result

a.) $\displaystyle\int_t^{t+\Delta t} (\text{Lumped System ODE})\,dt$

$$\int \frac{du}{dt}dt \rightarrow u^{k+1} - u^k$$

$$\int u\,dt \rightarrow \Delta t\left[\theta u^{k+1} + (1-\theta)u^k\right]$$

<u>or</u>

b.) FD in time-domain:

$$\frac{du}{dt} \Rightarrow \frac{u^{k+1} - u^k}{\Delta t}$$

$$u \Rightarrow \theta u^{k+1} + (1-\theta)u^k$$

Same Result:

$$[M]\left(\{u\}^{k+1} - \{u\}^k\right) + \left[\Delta t K\right]\left(\theta\{u\}^{k+1} + (1-\theta)\{u\}^k\right)$$

$$= \left\{\int r\,dt\right\} \equiv \{R\}^{k+\theta}$$

$$\underbrace{\left[M + \theta\Delta t K\right]}_{[A]}\{u\}^{k+1} = \underbrace{\left[M - \Delta t(1-\theta)K\right]}_{[B]}\{u\}^k + \{R\}^{k+\theta}$$

$$[A]\{u\}^{k+1} = [B]\{u\}^{k} + \{R\}^{k+\theta}$$

Formally ....

$$\{u\}^{k+1} = [A^{-1}B]\{u\}^{k} + [A^{-1}]\{R\}^{k+\theta}$$

- Like FD, build solution from IC's step-by-step
- $[A] = [M + \theta \Delta t K]$ is sparse ; A^{-1} full
- don't do it this way ⇒ instead use LR decomposition (e.g. LU ⇒ Lower/Upper)

$$[A] = [L][R] \Rightarrow [L][R]\{u\}^{k+1} = \{C\} \equiv [B]\{u\}^{k} + \{R\}^{k+1/2}$$

$$[R]\{u\}^{k+1} = [L^{-1}]\{C\}$$

or $y \equiv [R]\{u\}^{k+1}$

$[L]\{y\} = \{c\}$ ← solve this then

$[R]\{u\}^{k+1} = \{y\}$ ← this

Do this only one for all time

Step 1: Find R, L^{-1} ; Store in $[A]$ (decompose)

Step 2: a.) get $[L^{-1}]\{c\}$

b.) back-substitute

"Solve" does this

Summarize:

$$[A]\{u\}^{k+1} = \{c\}^k$$

$\{c\}^k$ ← evaluate at beginning of each time step

$[A]$ ← evaluate once at start; then perform LR decomposition

Right-hand-side:

$$\{c\}^k \equiv [B]\{u\}^k + \{R\}^{k+\theta}$$

$\{R\}^{k+\theta}$ → Known: $\int \langle g, \phi_i \rangle dt + \int \left[\oint (\ ) \phi_i ds \right] dt$

$$[B] = [M - (1-\theta)\Delta t K]$$

$$b_{ij} = \langle c \phi_j \phi_i \rangle - (1-\theta)\Delta t \langle K \nabla\phi_i \cdot \nabla\phi_j \rangle$$

Strategy a.) evaluate $[B]$ at start $(t=0)$; Store

- Banded form OK, but doubles storage requirement
- Then multiply $[B]\{u\}^k$ at each time-step ($\{u\}^k$ ← known)

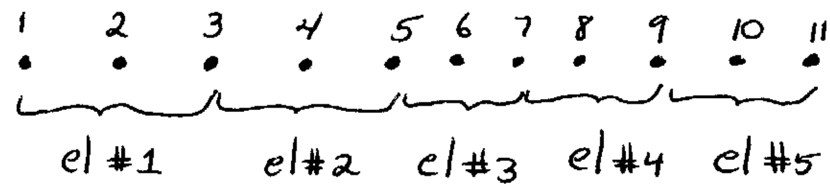
Better... Use "sparse" storage scheme ⇒ i.e. only store NON-zero coefficients; Bandwidth irrelevant since $[B]$ does not have to be decomposed!

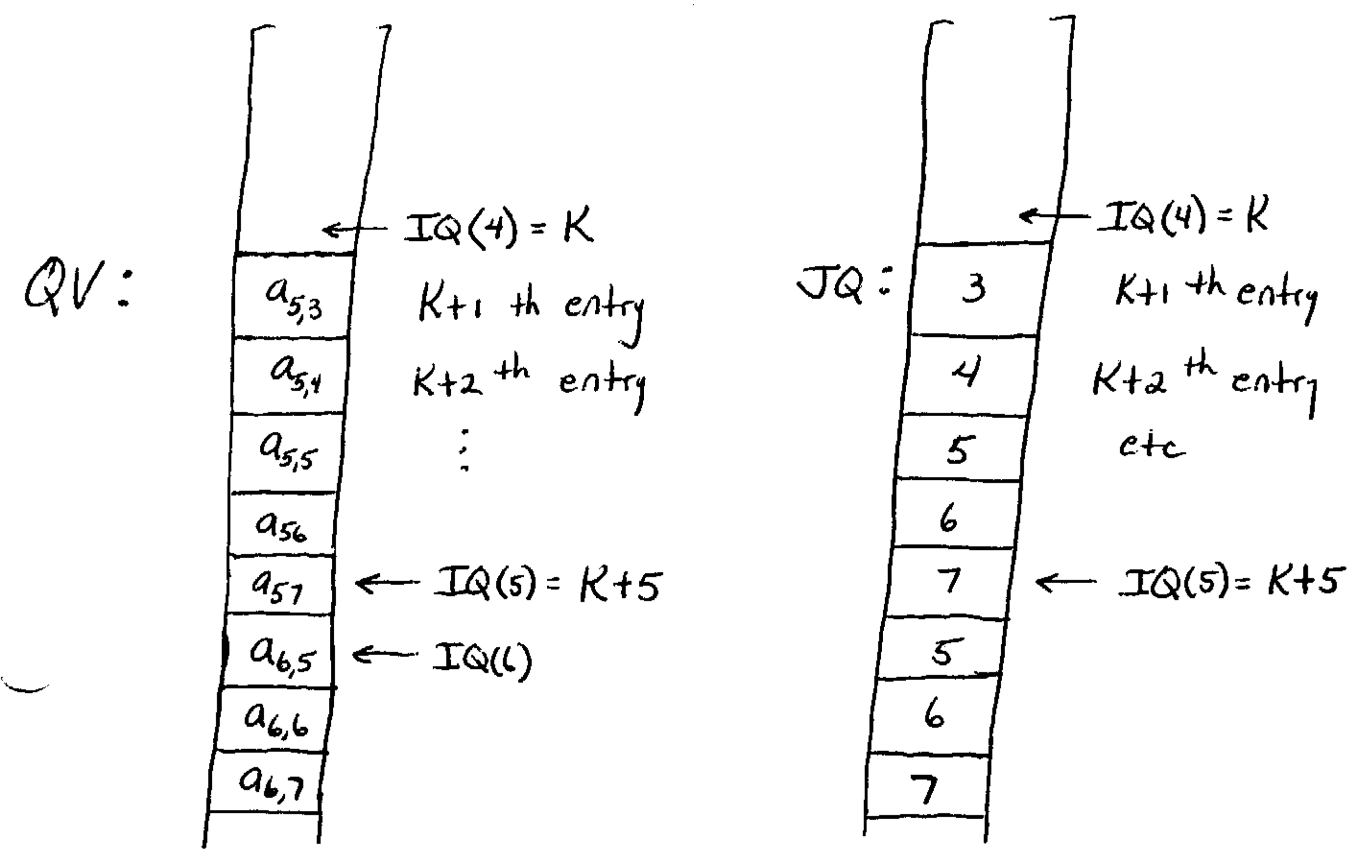
Store [B] as a 1-D vector (QV) which contains only non-zero coefficients.

Need 2 pointers: $IQ(i) \Rightarrow$ Location of the end of row "i" in QV. Row "i" is Stored in $QV(k)$; $k = IQ(i-1)+1, IQ(i)$

$JQ(k)$ is the column idex associated w/ $QV(k)$ (i.e. the variable which multiplies the coefficient in $QV(k)$)

e.g.





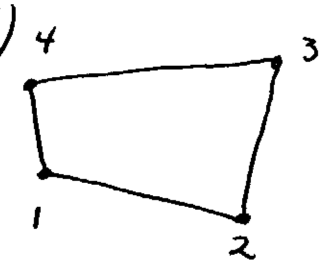
fast efficient, but costs some storage $\Rightarrow$ minimize with sparse storage format

Strategy b: assemble $[B]\{u\}^k$ element-by-element at each time step $\Rightarrow$ eliminates the need to store $[B]$

$$[B]\{u\}^k = \sum_e [B]^e \{u\}^{e^k}$$

Economizes memory requirements, but can be slow due to continuous rebuilding of element matrices (e.g. looping over elements)

e.g. Bilinear element (deformed)



local #	global #
1	10
2	12
3	20
4	25

$[B^e]$ is 4×4 ; $\{u\}^{e^k} = \begin{Bmatrix} u_{10} \\ u_{12} \\ u_{20} \\ u_{25} \end{Bmatrix}^k$

$$[B]^e\{u\}^{ek} = \sum_\ell |J_\ell| W_\ell [B]^e_\ell \{u\}^{ek}$$

everything at the gauss point level

So Right-hand-side for ith row in $[A]\{u\}^{k+1} =$

$$\underbrace{[B]\{u\}^k + \{R\}^{k+1/2}}_{\equiv \{C\}}$$

$$C_i = \langle c\hat{u}, \phi_i \rangle^k - (1-\theta)\Delta t \langle K\nabla\hat{u} \cdot \nabla\phi_i \rangle + R_i$$

(only need to evaluate integrand at Gauss pts)

- $\hat{u}^k$ known; treat like any other known coefficient

$\{R\}^{k+\theta}$ Term:

$$= \int_k^{k+1} \left[\langle g, \phi_i \rangle + \oint K \frac{\partial u}{\partial n} \phi_i \right] dt$$

Forcing: $\displaystyle \int_k^{k+1} \langle g, \phi_i \rangle dt = \Delta t \left[\theta \langle g, \phi_i \rangle^{k+1} + (1-\theta) \langle g, \phi_i \rangle^k \right]$

$$= \Delta t \left[\theta \langle g^{k+1} \phi_i \rangle + (1-\theta) \langle g^k \phi_i \rangle \right]$$

Boundary Integral: $\displaystyle \int_k^{k+1} \oint K \frac{\partial u}{\partial n} \phi_i \, ds =$

$$\Delta t \left[\theta \oint K \frac{\partial u}{\partial n}^{k+1} \phi_i \, ds + (1-\theta) \oint K \frac{\partial u}{\partial n}^k \phi_i \, ds \right]$$

Type II: $K \frac{\partial u}{\partial n}$ known as function of t!

$\therefore \frac{\partial u}{\partial n}^{k+1}, \frac{\partial u}{\partial n}^k$ known ... everything stays on RHS

Type III: $K \frac{\partial u}{\partial n} = au + c$ ...

$$\underbrace{\Delta t \theta \oint (au+c)^{k+1} \phi_i \, ds}_{\text{Unknown, goes back in Left-side Matrix; Expand } u^{k+1} = \sum \phi_j u_j^{k+1}} + \underbrace{(1-\theta) \Delta t \oint (au+c)^k \phi_i \, ds}_{\text{known stays on RHS}}$$