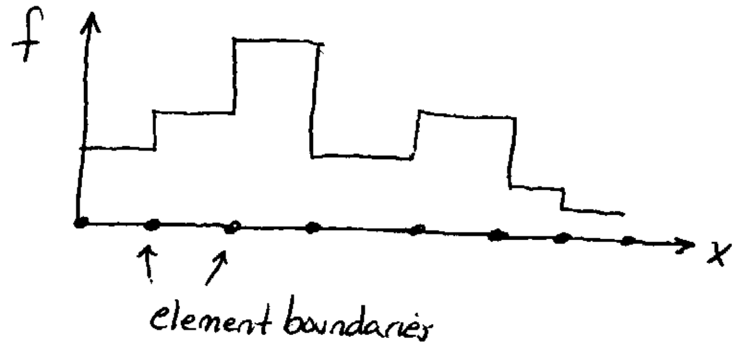
Variable coefficients

Above assumed f, g constant $\Rightarrow \langle f \phi_i \phi_j \rangle = f \langle \phi_i \phi_j \rangle$

Strategy 1: assume f, g constant on an element
different in different elements



f "piecewise constant"

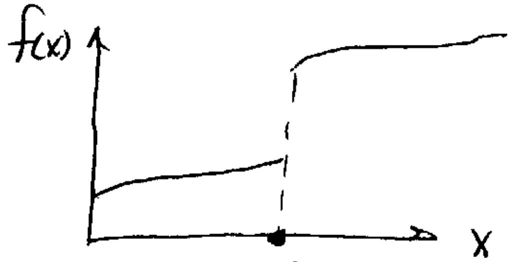
$$\langle f \phi_i \phi_j \rangle^e = f^e \langle \phi_i \phi_j \rangle^e$$

- No difference in element matrix, RHS from above
- Crude when $f(x)$ smooth
- Common to choose "average" value of f ... e.g. at center of element

$$\left(\text{exact: } f_{ij}^e \equiv \frac{\langle f \phi_i \phi_j \rangle^e}{\langle \phi_i \phi_j \rangle^e} \right)$$ MVT for integrals

- popular, easy ... $\langle \ \rangle^e$ simple

- Best when have jumps in $f(x)$



$f(x)$ dictates node placement

Make it an element boundary

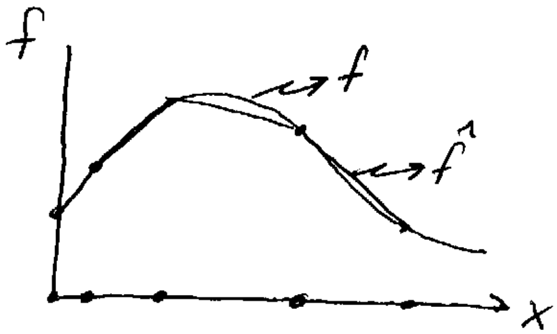
"element-based" coefficients

Strategy #2: $f(x) \approx \hat{f}(x) = \sum f_i \phi_i(x)$ "Functional coefficients"

i.e. interpolate f among nodal values with the basis functions

"Node-based" coefficients

Linear:



Nodes/element

$$\langle f \phi_i \phi_j \rangle^e = \sum_{1}^{N} f_k \langle \phi_k \phi_i \phi_j \rangle^e$$

e.g. Linear Elements ... $\langle f\phi_i\phi_j\rangle^e = \sum_{k=1}^{2} f_k \langle\phi_k\phi_i\phi_j\rangle^e$

$$= \underbrace{f_1\langle\phi_1\phi_i\phi_j\rangle^e + f_2\langle\phi_2\phi_i\phi_j\rangle^e}$$

Need to do these integrations for <u>all</u> (i,j) combinations in the element

i.e. $i=1, j=1$: $\langle f\phi_i\phi_j\rangle^e = f_1\langle\phi_1^3\rangle^e + f_2\langle\phi_2\phi_1^2\rangle^e$

$$= \frac{f_1 h}{4} + \frac{f_2 h}{12} \Rightarrow \text{favors } f_1$$

If $f_1 = f_2 \Rightarrow \langle f\phi_1\phi_1\rangle^e = \frac{fh}{3}$; same as $f^e\langle\phi_1\phi_1\rangle^e$

$i=1, j=2$: $\langle f\phi_i\phi_j\rangle^e = f_1\langle\phi_1^2\phi_2\rangle^e + f_2\langle\phi_1\phi_2^2\rangle^e$

$$= \frac{f_1 h}{12} + \frac{f_2 h}{12} \Rightarrow \text{"Neutral"}$$

If $f_1 = f_2 \Rightarrow \langle f\phi_1\phi_2\rangle^e = \frac{fh}{6}$; same as $f^e\langle\phi_1\phi_2\rangle^e$

$i=2, j=1$: Same as $i=1, j=2$ due to symmetry

$i=2, j=2$: $\langle f\phi_i\phi_j\rangle^e = f_1\langle\phi_1\phi_2^2\rangle^e + f_2\langle\phi_2^3\rangle^e = \frac{f_1 h}{12} + \frac{f_2 h}{4}\Big\}$ favors f_2

Same as $f^e\langle\phi_2\phi_2\rangle^e$ if $f_1 = f_2$

So element matrix for:

$$\langle f\phi_i\phi_j\rangle^e = \begin{bmatrix} \frac{f_1 h}{4} + \frac{f_2 h}{12} & \frac{f_1 h}{12} + \frac{f_2 h}{12} \\ \frac{f_1 h}{12} + \frac{f_2 h}{12} & \frac{f_1 h}{12} + \frac{f_2 h}{4} \end{bmatrix}$$

where, $f = \sum_{k=1}^{2} f_k\phi_k$

Strategy 3:

a.) Interpolate with more detail than ϕ_i

e.g. $f(x)$ sampled at several points in an element $\quad \sum f_k \psi_k(x) \approx f(x)$

b.) Analytic expression available for $f(x)$

$$\langle f \phi_i \phi_j \rangle^e = \sum f_k \langle \psi_k \phi_i \phi_j \rangle^e$$

$$\langle f \phi_i \phi_j \rangle^e = \langle f(x) \phi_i \phi_j \rangle^e$$

In either case: $\langle \; \rangle^e$ becomes tiresome (on paper)

$\Rightarrow$ evaluate $\langle \; \rangle^e$ numerically

e.g. $\langle (\,) \rangle = \int (\,)\, dx \approx \sum_{\ell=1}^{n} (\,)_\ell w_\ell$

"Numerical Quadrature"

"Readily Automated"

Rules for level of accuracy

Popular with FE (more later!)