FD Soln's to Elliptic PDEs: Summary

- Complex characteristics ($b^2 - ac < 0$); boundary data influences all points instantaneously; need BCs around enclosed domain (Type I or Type II or Type III)

- Type II + III BCs: use molecule on boundary; remove "shadow node" w/ FD representation of boundary condition.

- Type I: do not use PDE molecule on boundary to get soln ... but PDE molecule contains flux info which can be extracted after soln computed

- FD Conservation requires use of all molecules; Single PDE molecule equivalent to conservation over ⊡ "Single box"... Sum of all boxes leads to Global conservation. requires treatment of source as "average value in the box" ... i.e. integration of source over the box divided by box size.

- FD molecules plus BCs lead to coupled algebraic system

- 2D problems pentadiagonal (w/o $\frac{\partial^2 u}{\partial x \partial y}$ term) Direct methods "unnatural"

- Iterative methods "natural"... take advantage of (i,j) indexing ... All can be cast in form

$$\{u\}^{k+1} = [G]\{u\}^k + \{c\}$$; convergence guaranteed if $\rho(G) < 1$

- For general Elliptic PDE:

$$a\frac{\partial^2 u}{\partial x^2} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g$$

Iterative Soln guaranteed to converge

if $a>0$; $c>0$; $f<0$

Some restriction on h relative to size of d + e OR

if $a>0$; $c>0$; $f=0$ w/ some Type I BCs

- Simple Point Iterative Methods ... Jacobi, Gauss-Siedel used easily ... sweep through mesh row or column-wise Iteration Matrices have predictable Spectrums for simple problem (Laplace/Possion w/ Type I BCs)

- For model problem $\rho_{GS} = \rho_J^2$ ... leads to "Rule of Thumb": "Gauss-Siedel twice as fast as Jacobi" ... can examine "Rate of Convergence" $= -\log(\rho) \Rightarrow$ "number of digits error reduced per step" ... found $R_J \approx h^2/2$; $R_{GS} \approx h^2$

- Can speed-up through "acceleration" methods ... Lyusternik, Aitkens, SOR ... Lyusternik, Aitkens extrapolate previous iterates ... SOR offers "iteration parameter"

- Predictable ω_{opt} for system matrices that are "consistently ordered" ... have a special block tridiagonal form ... or matrices which have "Property A" ... these can be put in the appropriate block tridiagonal form through simple row + column interchange (Similarity Transformations)

- The model elliptic problem has Property A ... therefore, has a predictable ω_{opt} ... most any ordering of nodes which has a logical progression leads to a consistently ordered system matrix

- Can get rate of convergence ... $R_{G_{SOR}}^{opt} \simeq 2h$

- Possible to speed-up convergence rate using Block iterative methods ... but cost/iteration increases

- Simplest ... create line iterations from Jacobi, Gauss Siedel, SOR ... solve for new update on <u>all</u> variables on a constant "j" row <u>or</u> constant "i" column simultaneously ... use direct soln method to do it ... for "single" line ⇒ tridiagonal ⇒ fast

- In general get doubling of convergence rate (SOR_{opt} only factor of $\sqrt{2}$)

- Better performance if alternate rows + columns, especially if split the operator ... i.e. X derivatives to one side; y derivatives to other ⇒ ADI; two step procedure; one iteration constitutes two sweeps (one rowwise, one columnwise)

- Model problem has predictable optimal iteration parameter best performance found when iteration parameter sequence is used (i.e. ω^k for k-th iteration)