Two-dimensional Problems

e.g. $\underbrace{\nabla^2 u}_{} + k^2 u = g$ Helmholtz eqn!

$\nabla^2 u = \nabla\cdot(\nabla u)$ 2D: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

WR Statement:

$$\langle \nabla^2 u\, \phi_i \rangle + \langle k^2 u\, \phi_i \rangle = \langle g, \phi_i \rangle$$

"Integrate by parts"... $\nabla\cdot(\nabla u\, \phi_i) = \nabla^2 u\, \phi_i + \nabla u \cdot \nabla \phi_i$
Green's Theorem

$$\Rightarrow \nabla^2 u\, \phi_i = -\nabla u \cdot \nabla \phi_i + \nabla\cdot(\nabla u\, \phi_i)$$

$$\langle -\nabla u \cdot \nabla \phi_i \rangle + \underbrace{\langle \nabla\cdot(\nabla u\, \phi_i) \rangle} + \langle k^2 u\, \phi_i \rangle = \langle g, \phi_i \rangle$$

$$= \oint \underbrace{\hat{n}\cdot\nabla u}_{\frac{\partial u}{\partial n}}\, \phi_i\, ds$$

Galerkin: $$\sum_{j=1}^{N} u_j \Big[\underbrace{\langle -\nabla\phi_j \cdot \nabla\phi_i \rangle + \langle k^2 \phi_j\, \phi_i \rangle}_{a_{ij} \text{ elements in matrix } \mathbf{A}} \Big] = \overbrace{-\oint \nabla u \cdot \hat{n}\, \phi_i\, ds + \langle g, \phi_i \rangle}^{b_i}$$

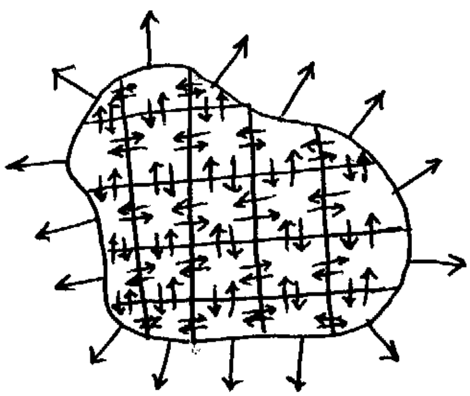
Recall Defn:

$$\nabla \cdot F = \lim_{\Delta V \to 0} \frac{\oint F \cdot \hat{n}\, ds}{\Delta V}$$

"Net outward flux per unit volume as volume shrinks to zero"

$$\therefore \langle \nabla \cdot \underset{\sim}{F} \rangle = \oint \underset{\sim}{F} \cdot \hat{n}\, ds$$

Interior pieces cancel!



priviso: F continuous on V

e.g. Material Heterogeneity : $\nabla \cdot K \nabla u$

FEM WR : $\langle \phi_i \nabla \cdot K \nabla u \rangle = \langle -K \nabla u \cdot \nabla \phi_i \rangle + \oint \underline{\underline{K \nabla u}} \cdot \hat{n}\, ds$

This is the continous quantity not $\frac{\partial u}{\partial n}$

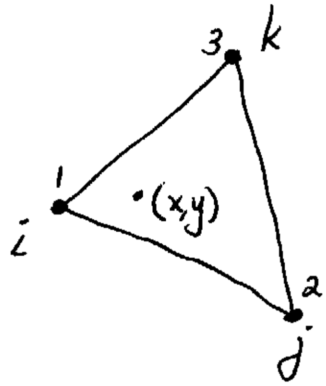
Want to set up our problem so that boundary integral vanishes on interior "element" boundaries so $\alpha_{ij} = -\left\langle \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} \right\rangle - \left\langle \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial y} \right\rangle + \left\langle k^2 \phi_i \phi_j \right\rangle$

$b_i = -\oint \frac{\partial u}{\partial n} \phi_i \, ds \leftarrow$ (vanishes over all interior elements provided $\frac{\partial u}{\partial n}$ continuous)

$+ \langle g, \phi_i \rangle$

2-D linear element : Linear triangle

$\phi_i = a_i + b_i x + c_i y \Rightarrow$ - 3 parameters, 3 nodes



3 conditions $\begin{cases} - \phi_i = 1 \text{ at } (x,y) = (x,y)_i \\ \quad = 0 \text{ at } (x,y) = (x,y)_{j,k} \end{cases}$

- Counterclockwise convention
- 1, 2, 3 local numbering

Solve for a_i's, b_i's, c_i's in an element...

$$\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{Bmatrix} a_i \\ b_i \\ c_i \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$a_i = (x_j y_k - x_k y_j) / Det \quad \cdots$$

Turns out $Det\left\{\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}\right\} = 2A$ ← Area of Triangle

$$A = \frac{1}{2}\sum_{i=1}^{3} x_i \Delta y_i$$

$$\Delta y_i \equiv y_j - y_k$$

So $a_i = (x_j y_k - x_k y_j)/2A$

$$b_i = (y_j - y_k)/2A = \Delta y_i/2A = \frac{\partial \phi_i}{\partial x}$$

$$c_i = -(x_j - x_k)/2A = -\Delta x_i/2A = \frac{\partial \phi_i}{\partial y}$$

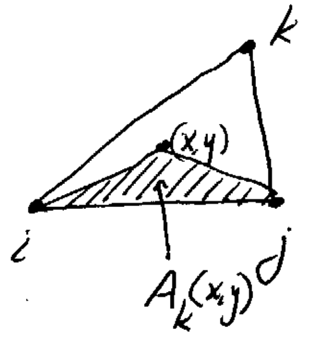
$$\Delta x_i \equiv x_j - x_k$$

Cyclic permutation of i, j, k (i.e. 1, 2, 3)

i.e. $b_j = \dfrac{\Delta y_j}{2A} = \dfrac{\partial \phi_j}{\partial x} = \dfrac{y_k - y_i}{2A}$

etc...

Common to talk about "area coordinates"



$$\phi_k(x,y) = \frac{A_k(x,y)}{A}$$

$$2A_k(x,y) = \begin{vmatrix} 1 & x & y \\ 1 & x_i & y_i \\ 1 & x_j & y_j \end{vmatrix}$$

- A_k linear in x, y
- $A_k = A$ at node k
  $= 0$ at nodes i, j

- $\phi_k = 0$ $\begin{cases} \text{along segment } i \to j \\ \text{varies linearly from 1 to zero along } k \to i,\ k \to j \text{ segments} \end{cases}$

Return to Helmholtz equation example ....

$$\nabla^2 u + k^2 u = g$$

Galerkin:

$$\sum_j u_j \left[\langle -\nabla\phi_j \cdot \nabla\phi_i \rangle + \langle k^2 \phi_j \phi_i \rangle \right] = -\oint \nabla u \cdot \hat{n}\, \phi_i\, ds + \langle g, \phi_i \rangle$$

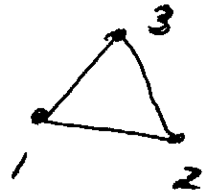
$$\alpha_{ij} = -\left\langle \frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial x}\right\rangle - \left\langle \frac{\partial\phi_j}{\partial y}\frac{\partial\phi_i}{\partial y}\right\rangle + \left\langle k^2\phi_i\phi_j\right\rangle$$

but we know $[A] = \sum_e [A]^e$

$$\alpha_{ij}^e = -\left(\frac{\Delta y_j}{2A_e}\frac{\Delta y_i}{2A_e}\right)A_e - \left(\frac{\Delta x_j}{2A_e}\frac{\Delta x_i}{2A_e}\right)A_e + \frac{k^2 A_e}{12} \qquad i \neq j$$

$$= \quad '' \qquad\qquad '' \qquad + \frac{k^2 A_e}{6} \qquad i = j$$

Locally:



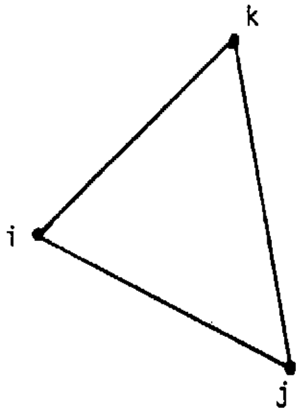
(requires CC order only)

$$[A]^e = \begin{bmatrix} -\frac{\Delta y_1^2}{4A_e} - \frac{\Delta x_1^2}{4A_e} + \frac{k_e^2 A_e}{6} & -\frac{\Delta y_1 \Delta y_2}{4A_e} - \frac{\Delta x_1 \Delta x_2}{4A_e} + \frac{k_e^2 A_e}{12} & -\frac{\Delta y_1 \Delta y_3}{4A_e} - \frac{\Delta x_1 \Delta x_3}{4A_e} + k_e^2 A_e/12 \\ -\frac{\Delta y_2 \Delta y_1}{4A_e} - \frac{\Delta x_2 \Delta x_1}{4A_e} + \frac{k_e^2 A_e}{12} & -\frac{\Delta y_2^2}{4A_e} - \frac{\Delta x_2^2}{4A_e} + \frac{k_e^2 A_e}{6} & -\frac{\Delta y_2 \Delta y_3}{4A_e} - \frac{\Delta x_2 \Delta x_3}{4A_e} + k_e^2 A_e/12 \\ -\frac{\Delta y_3 \Delta y_1}{4A_e} - \frac{\Delta x_3 \Delta x_1}{4A_e} + \frac{k_e^2 A_e}{12} & -\frac{\Delta y_3 \Delta y_2}{4A_e} - \frac{\Delta x_3 \Delta x_2}{4A_e} + \frac{k_e^2 A_e}{12} & -\frac{\Delta y_3^2}{4A_e} - \frac{\Delta x_3^2}{4A_e} + \frac{k_e^2 A_e}{6} \end{bmatrix}$$

$\Delta y_1 = y_2 - y_3, \quad \Delta y_2 = y_3 - y_1, \quad \Delta y_3 = y_1 - y_2$

$\Delta x_1 = x_2 - x_3, \quad \Delta x_2 = x_3 - x_1, \quad \Delta x_3 = x_1 - x_2$

INTEGRATION FORMULAS FOR LINEAR TRIANGLES



$$\phi_i = a_i + b_i x + c_i y$$

$$a_i = (x_j y_k - x_k y_j)/2A$$

$$b_i = \Delta y_i / 2A$$

$$c_i = -\Delta x_i / 2A$$

$$\Delta x_i = x_j - x_k$$

$$\Delta y_i = y_j - y_k$$

$$A = \tfrac{1}{2} \sum_{i=1}^{3} x_i \Delta y_i$$

$$= -\tfrac{1}{2} \sum_{i=1}^{3} y_i \Delta x_i$$

$$\langle 1 \rangle = A$$

$$\langle \phi_i \rangle = A/3$$

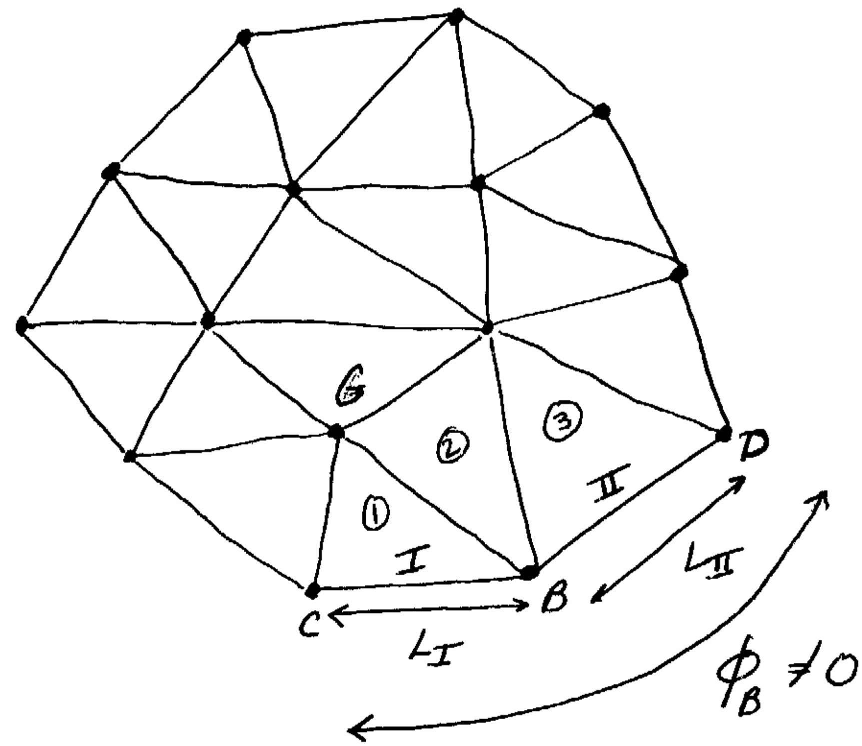
$$\langle \phi_i \phi_j \rangle = A/12 \qquad (i \neq j)$$

$$\langle \phi_i^2 \rangle = A/6$$

$$\langle \phi_i^\ell \phi_j^m \phi_k^n \rangle = 2A \left[\frac{\ell! m! n!}{(\ell+m+n+2)!} \right]$$

$$\left. \begin{aligned} \frac{\partial \phi_i}{\partial x} &= \Delta y_i / 2A \\ \frac{\partial \phi_i}{\partial y} &= -\Delta x_i / 2A \end{aligned} \right\} \text{constant}$$

BC's



on an element: $g^e \langle \phi_i \rangle^e = g^e \frac{A^e}{3}$

on a node: $\sum_{k=1}^{3} g_k \langle \phi_k \phi_i \rangle^e = g_1 \langle \phi_1 \phi_i \rangle^e + g_2 \langle \phi_2 \phi_i \rangle^e + g_3 \langle \phi_3 \phi_i \rangle^e$

Helmholtz eqn:

$$a_{ij} = \langle -\nabla\phi_i \cdot \nabla\phi_j \rangle + \langle k^2 \phi_i \phi_j \rangle$$

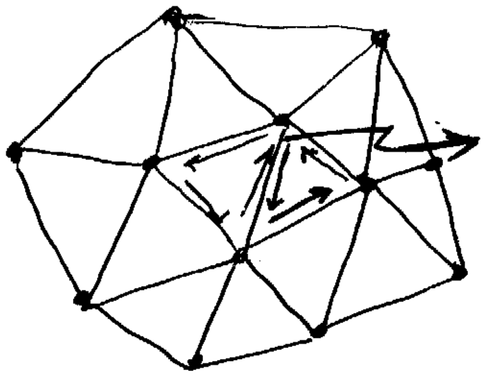
$$b_i = -\oint \underbrace{(\nabla u \cdot \hat{n})}_{\frac{\partial u}{\partial n}} \phi_i \, ds$$

drop the $\langle g, \phi_i \rangle$ term for the moment!

at: interior node (e.g. G) $\Rightarrow$ $\phi_G = 0$ on boundary

boundary node (e.g. B) $\Rightarrow$ $\phi_B = 0$ on most of boundary (only nonzero in elements 1, 3)

Alternately ... consider $\oint \nabla u \cdot \hat{n} \, \phi_i \, ds$ over each element $\Rightarrow$ all vanish provided $\nabla u \cdot n$ continuous (needed for application of divergence theorem) on interior



$\oint \nabla u \cdot \hat{n} \phi_i \, ds$ vanishes across common segments

Now on boundary segment where ϕ_B does not vanish

$$\Rightarrow \quad \frac{\partial u}{\partial n} = \frac{\partial u}{\partial n}_{I} \quad \text{across } \underbrace{\text{"element"}}_{\text{segment}} \, I$$

"boundary element"

$$= \frac{\partial u}{\partial n}_{II} \quad \text{across "element" } II$$

element values of $\frac{\partial u}{\partial n}$ $\Rightarrow$ Type II info.!

$$\text{So} \quad b_B = -\frac{\partial u}{\partial n}_{I} \underbrace{\int_{I} \phi_B \, ds}_{\phi_B \text{ linear from 0 to 1}} - \frac{\partial u}{\partial n}_{II} \underbrace{\int_{II} \phi_B \, ds}_{\text{same here}}$$

$$b_B = -\frac{\partial u}{\partial n}_{I} \frac{L_I}{2} - \frac{\partial u}{\partial n}_{II} \frac{L_{II}}{2}$$

(element based info!)

Alternately on Type II boundary:

$$\frac{\partial u}{\partial n} = \sum \frac{\partial u}{\partial n}_i \phi_i \quad ; \quad \sum_i \text{ over boundary nodes only}$$

(nodal-based info on $\frac{\partial u}{\partial n}$)

$$b_i = -\sum_k \frac{\partial u}{\partial n}_k \oint \phi_k \phi_i \, ds$$

$$b_B = -\frac{\partial u}{\partial n}_C \underbrace{\int_I \phi_C \phi_B \, ds}_{\frac{L_I}{6}} - \frac{\partial u}{\partial n}_B \underbrace{\int_{I+II} \phi_B \phi_B \, ds}_{\frac{L_I}{3} + \frac{L_{II}}{3}} - \frac{\partial u}{\partial n}_D \underbrace{\int_{II} \phi_D \phi_B \, ds}_{\frac{L_{II}}{6}}$$

$$= -\frac{1}{6}\left[L_I \left(\frac{\partial u_C}{\partial n} + 2\frac{\partial u_B}{\partial n} \right) + L_{II} \left(2\frac{\partial u_B}{\partial n} + \frac{\partial u_D}{\partial n} \right) \right]$$

Type III BC: $\frac{\partial u}{\partial n} = au + c$

element-based strategy...

$$b_i = -\oint \frac{\partial u}{\partial n} \phi_i \, ds = \underbrace{-\oint a u \phi_i \, ds}_{\text{goes back into } A \text{ matrix}} - \oint c \phi_i \, ds$$

$a = a_I, a_{II} \quad c = c_I, c_{II}$: so

$$b_B = \sum_{\substack{j \\ \uparrow \\ u_j}} -\left[a_I \int_I \phi_j \phi_B \, ds + a_{II} \int_{II} \phi_j \phi_B \, ds \right] - c_I \int_I \phi_B \, ds - c_{II} \int_{II} \phi_B \, ds$$

$$= \underbrace{-\frac{1}{6}\left[L_I a_I (u_C + 2u_B) + L_{II} a_{II} (2u_B + u_D) \right]}_{\text{goes into } A}$$

$$-\frac{1}{2}\left[c_I L_I + c_{II} L_{II} \right]$$

More generally . . .

$$b_B = -u_C \int_I a \phi_C \phi_B \, ds - u_B \int_{I+II} a \phi_B \phi_B \, ds - u_D \int_{II} a \phi_D \phi_B \, ds$$

$$- \int_I c \phi_B \, ds - \int_{II} c \phi_B \, ds$$

This info goes into A matrix in row B

$$A_{B,C} = A_{B,C} + \int_I a \phi_C \phi_B \, ds$$

$$A_{B,B} = A_{B,B} + \int_{I+II} a \phi_B \phi_B \, ds$$

$$A_{B,D} = A_{B,D} + \int_{II} a \phi_D \phi_B \, ds$$

Node-Based...

$a = \sum a_k \phi_k$

$c = \sum c_k \phi_k$ etc.

element-based...

$\int_I a \phi_C \phi_B \, ds = a_I \int \phi_C \phi_B \, ds$

$\int_I c \phi_B \, ds = c_I \int \phi_B \, ds$

etc.

Type I BC: e.g. $u_B = 5$

$$\begin{array}{r} \\ \text{Row } B \rightarrow \\ \\ \end{array} \begin{bmatrix} \ddots & & & & \\ 0\;0\;0\;0\;0 & 1 & 0\;0\;0\;0\;0\;0 \\ & & \ddots & & \end{bmatrix} \begin{Bmatrix} \\ u_B \\ \\ \end{Bmatrix} = \begin{Bmatrix} \\ 5 \\ \\ \end{Bmatrix}$$

- After element assembly
- Destroy the B-row + b_B
- Put "1" on A_{BB}
- put "5" on b_B

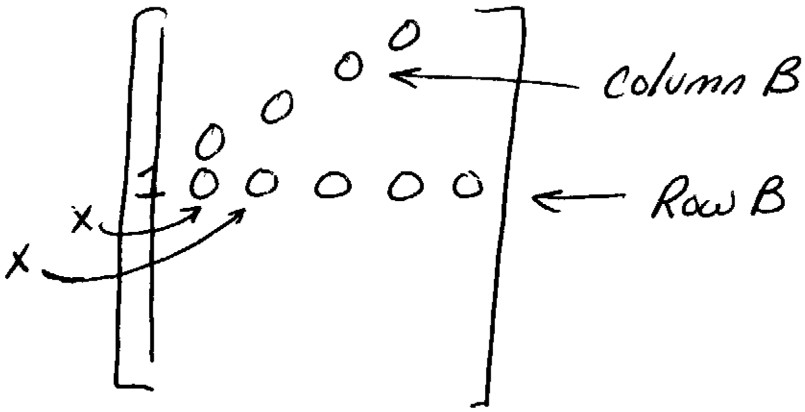
Galerkin removed BC in place

Symmetry Destroyed! Must remove Column B also!
(only worry about if taking advantage of symmetric matrix storage)

$$\begin{bmatrix} \ddots & 0 & \\ & 0 & \\ & 0 & \\ & 0 & \\ 0\;0\;0\;0 & 1 & 0\;0\;0\;0\;0 \\ & 0 & \\ & 0 & \\ & 0 & \\ & 0 & \ddots \end{bmatrix} \begin{Bmatrix} \\ \\ \\ \\ u_B \\ \\ \\ \\ \\ \end{Bmatrix} = \begin{Bmatrix} \\ \\ \\ \\ 5 \\ \\ \\ \\ \\ \end{Bmatrix} - \begin{Bmatrix} x \\ x \\ x \\ x \\ x \\ 0 \\ x \\ x \\ x \\ x \\ x \end{Bmatrix}$$

$5 * A_{i,B}$

Think about above operation in Banded Symmetric Mode !! Tricky



Importance of banded Storage !

grid in x-y 20 x 20 $\Rightarrow$ 400 nodes

Full storage $16 \times 10^4 = 160K$ numbers

Banded Storage : Half-Bandwidth ≈ 20

$400 \times (41) \simeq 16K$ (~Factor of 10)

Symmetric $\longrightarrow$ 8K

Available matrix solvers

"Generic" (Skeletal) FE Program

1. Load Input (mesh): $X(i), Y(i)$ $\{$ (X, y) coordinate of <u>Global</u> Node i "hw5.nodes.dat"

$i = 1$ TO Total Nodes

$IN(\ell, j)$ $\{$ <u>Global</u> node #'s in each element ℓ "hw5.elems.dat"

$\ell = 1$ TO Total#el

$j = 1$ to #nodes/el

2. Element Assembly:

Do $L = 1$ to #el

Call ElementMatrix$(\underbrace{L, IN, X, Y}_{\text{Inputs}}, \underbrace{Ae, be}_{\text{outputs: element matrix}})$

Do $I = \underline{1}$ to # nodes/el

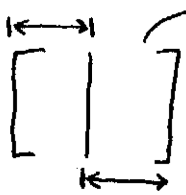
$\underline{II} = IN(L, I)$ ⟸ map local weighting function # to Global ... matrix row

"Domain" Integration terms only!! ⟹ $b(II) = b(II) + be(I)$

Do $J = 1$ to # nodes/el

$JJ = IN(L, J)$ ⟸ map local basis function to Global ... matrix column

$JB = \underbrace{(\text{Half BW} + 1)} + JJ - II$ ⟸ column shift for banded form



$A(II, JB) = A(II, JB) + Ae(I, J)$

END J

END I

END L

3.) Apply BCs ... for each Boundary node "B"

Type I : Do J = 1 to MDIM $\Leftarrow$ Full Bandwidth of A

A(B,J) = 0.0

END J

A(B, Half BW+1) = 1.0

b(B) = BV $\Leftarrow$ Boundary value specified

Type II/III : Assemble $\oint (\,)\,\phi_B\, ds$ ... local; only nearest neighbors contribute; integrations over "boundary elements" (segments); same as 1D element formulas; Type III must remember to add contribution to A

4) Solve (Call Banded Matrix solve: Solve.f
Solve.c)

5.) Output nodal values of solution and/or "derived quantities"

$$\text{e.g. } \nabla u = \frac{\partial u}{\partial x}\hat{x} + \frac{\partial u}{\partial y}\hat{y} \Rightarrow \nabla\hat{u} = \sum u_j \frac{\partial \phi_j}{\partial x}\hat{x} + \sum u_j \frac{\partial \phi_j}{\partial y}\hat{y}$$

On linear Triangles; $\frac{\partial \phi_j}{\partial x}, \frac{\partial \phi_j}{\partial y}$ are constants

... typical to compute the vector ∇u at element centroids

(i.e. $x_c = (x_1 + x_2 + x_3)/3 \Rightarrow$ average of local coordinates)

$y_c = (y_1 + y_2 + y_3)/3$

... then at element level

$$\nabla\hat{u}(x_c, y_c) = \sum_{j=1}^{3} u_j \frac{\partial \phi_j}{\partial x}\hat{x} + \sum_{j=1}^{3} u_j \frac{\partial \phi_j}{\partial y}\hat{y}$$

Conservation Properties of FE method

$$\nabla \cdot q = \sigma \qquad q = -K\nabla u \quad \Rightarrow \nabla \cdot K \nabla u = -\sigma$$

Flux — Source/Volume — PDE

Global Conservation for PDE solution:

$$\oint q \cdot \hat{n}\, ds = \langle \sigma \rangle = \iint \sigma\, dx\, dy = -\oint K\nabla u \cdot \hat{n}\, ds$$

Galerkin approximation: $u \sim \hat{u}$

$$\langle \nabla \cdot K \nabla \hat{u}, \phi_i \rangle = \langle -\sigma, \phi_i \rangle$$

$$\langle -K\nabla \hat{u} \cdot \nabla \phi_i \rangle + \oint K\nabla \hat{u} \cdot \hat{n}\, \phi_i\, ds = \langle -\sigma \phi_i \rangle$$

$\sum_i$ all Galerkin equations ... Recall $\Sigma \phi_i = 1$

$\Sigma \nabla \phi_i = \nabla \Sigma \phi_i = 0$

$$\langle -K\nabla \hat{u} \cdot \cancelto{0}{\nabla \sum_i \phi_i} \rangle + \oint K\nabla \hat{u} \cdot \hat{n}\, \Sigma \phi_i\, ds = \langle -\sigma \Sigma \phi_i \rangle$$

$$\oint K\nabla \hat{u} \cdot \hat{n}\, ds = \langle -\sigma \rangle$$

Exact Conservation (Global)

Priviso: Must use all the Galerkin eqn's

otherwise $\sum_i \phi_i \neq 1$ in boundary elements

summed over all Galerkin equations

Operationally: - Remove $\langle r, \phi_i \rangle$ from $[A]\{u\}=\{b\}$ when i is Type I BC

- after $\{u\}$ is obtained, evaluate $\langle r, \phi_i \rangle$ for $\oint q \cdot \hat{n} \phi_i \, ds$

★ - $\langle r, \phi_i \rangle$ is the equation for $\oint q \cdot \hat{n} \phi_i \, ds$ when "i" is Type I boundary

Example in 1-D:

$$K \frac{d^2 u}{dx^2} = -\sigma$$

$x=0$ at node 1; nodes 1, 2, ..., $N-1$, N; $x=L$ at node N ← u given as Type I

Galerkin at Node N:

$$\left\langle -K \frac{\partial \hat{u}}{\partial x} \frac{d\phi_N}{dx} \right\rangle + K \frac{\partial u}{\partial x} \phi_N \Big|_0^L = \left\langle -\sigma, \phi_N \right\rangle$$

$\frac{d\hat{u}}{dx}$ in last element

$$-K\left(\frac{u_N}{h} + \frac{u_{N-1}}{(-h)}\right) \quad \frac{1}{h}$$

Source "near" N

$$K \frac{\partial u}{\partial x}(L) = \underbrace{\langle -\sigma, \phi_N \rangle}_{\text{correction which conserves } \sigma} + K\left(\frac{u_N - u_{N-1}}{h}\right)$$