2D (space) Problems $\Rightarrow \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

Two-level molecule:

$$\frac{u_{ij}^{l+1} - u_{ij}^{l}}{\Delta t} = \frac{\delta^2}{h^2}\left(u_{ij}^{l+1}\theta + (1-\theta)u_{ij}^{l}\right) \Rightarrow \delta^2 = \delta_x^2 + \delta_y^2$$

$$\left(u_{ij} - r\theta\delta^2 u_{ij}\right)^{l+1} = \underbrace{\left[u_{ij} + (1-\theta)r\delta^2 u_{ij}\right]^{l}}_{g_{ij}^{l}}$$

$$-r\theta\left(u_{i,j-1} + u_{i,j+1} + u_{i+1,j} + u_{i-1,j}\right)^{l+1} + (1+4r\theta)u_{ij}^{l+1} = g_{ij}^{l}$$

Spatial part $\Rightarrow$ same as elliptic $\Rightarrow$ Pentadiagonal

Has all the attractive features $\Rightarrow$ Diagonally Dominant

$\therefore$ Jacobi, GS, SOR converge for all r, θ

Consider an Iteration within a time-step

$\downarrow$

to solve the matrix equation that results at each time-level, $l \rightarrow l+1$

Notation: - drop the "ℓ" superscript
- add an iteration index k

So within a time-step $\Rightarrow$ i.e. in going from $\ell \rightarrow \ell+1$ we must solve:

$$\text{Jacobi} \Rightarrow (1+4r\theta)u_{ij}^{k+1} = g_{ij} + r\theta(u_{ij-1}^k + u_{ij+1}^k, u_{i+1j}^k + u_{i-1j}^k)$$

Fixed during iteration; has u_{ij}^{ℓ} in it $\Rightarrow$ This is the "right-hand-side"

GS, SOR $\Rightarrow$ natural extensions

Line Iterative methods possible too!

ADI:

Rearrange difference equation ...

$$\left[\delta^2 u_{ij} - \frac{1}{r\theta}u_{ij}\right]^{\ell+1} = \left[-\frac{1}{r\theta}u_{ij} - \frac{(1-\theta)}{\theta}\delta^2 u_{ij}\right]^{\ell} = \frac{-1}{r\theta}g_{ij}^{\ell}$$

$$\Rightarrow \left[\delta^2 u_{ij} - \underbrace{\frac{1}{r\theta}}_{\text{plays role of } fh^2 \text{ in previous ADI stuff}} u_{ij}\right]^{\ell+1} = -\frac{1}{r\theta}g_{ij}^{\ell}$$

Drop "ℓ" superscript...

Step 1:

$$-\omega u_{ij}^{k+1/2} + \left(\delta_x^2 - \frac{1}{2r\theta}\right) u_{ij}^{k+1/2} = -\left(\delta_y^2 - \frac{1}{2r\theta}\right) u_{ij}^{k} - \frac{1}{r\theta} g_{ij}^{\ell} - \omega u_{ij}^{k}$$

Step 2:

$$-\omega u_{ij}^{k+1} + \left(\delta_y^2 - \frac{1}{2r\theta}\right) u_{ij}^{k+1} = -\left(\delta_x^2 - \frac{1}{2r\theta}\right) u_{ij}^{k+1/2} - \frac{1}{r\theta} g_{ij}^{\ell} - \omega u_{ij}^{k+1/2}$$

As before ... 2 step process
repeat until convergence
Intermediate results (i.e. output from "Step 1")
are thrown away

Key: Do this for every time-step

Have 2 issues: 1) Stability (convergence) of Iterative Matrix sol'n $\Rightarrow$ ADI needs $\omega > 0$

2) Stability of time-stepping algorithm (e.g. $\theta \geq 1/2$ unconditional Stability)

Note: Can also use ADI as a time-stepping scheme!

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \ldots \text{ write as}$$

Step 1: $$\frac{u_{ij}^{l+1} - u_{ij}^{l}}{\Delta t} = \frac{1}{h^2}\left(\delta_x^2 u_{ij}^{l+1} + \delta_y^2 u_{ij}^{l}\right)$$

Step 2: $$\frac{u_{ij}^{l+2} - u_{ij}^{l+1}}{\Delta t} = \frac{1}{h^2}\left(\delta_x^2 u_{ij}^{l+1} + \delta_y^2 u_{ij}^{l+2}\right)$$

Rewrite: $\frac{1}{r} = \frac{h^2}{\Delta t}$

$$\left(\delta_x^2 - \frac{1}{r}\right) u_{ij}^{l+1} = \left(-\delta_y^2 - \frac{1}{r}\right) u_{ij}^{l}$$

$$\left(\delta_y^2 - \frac{1}{r}\right) u_{ij}^{l+2} = \left(-\delta_x^2 - \frac{1}{r}\right) u_{ij}^{l+1}$$

- $\frac{1}{r}$ plays role of ω : looks like ADI iteration for $\delta^2 u = 0$; with $\omega = \frac{1}{r} = \frac{h^2}{\Delta t}$

$\therefore$ Unconditionally Stable! if $r > 0$
(can get from Fourier analysis of elliptic iteration)

Look at this ADI scheme a little closer...

$$\left(\delta_x^2+\frac{1}{r}\right)\left[\left(\delta_x^2-\frac{1}{r}\right)u_{ij}^{\ell+1} = \left(-\delta_y^2-\frac{1}{r}\right)u_{ij}^{\ell}\right] \quad (A)$$

$$\left(\delta_x^2-\frac{1}{r}\right)\left[\left(\delta_x^2+\frac{1}{r}\right)u_{ij}^{\ell+1} = \left(-\delta_y^2+\frac{1}{r}\right)u_{ij}^{\ell+2}\right] \quad (B)$$

Subtract: $$0 = \left(\delta_x^2+\frac{1}{r}\right)\left(-\delta_y^2-\frac{1}{r}\right)u_{ij}^{\ell} - \left(\delta_x^2-\frac{1}{r}\right)\left(-\delta_y^2+\frac{1}{r}\right)u_{ij}^{\ell+2}$$

$$= \delta_x^2\delta_y^2\left(-u_{ij}^{\ell}+u_{ij}^{\ell+2}\right) + \frac{1}{r^2}\left(-u_{ij}^{\ell}+u_{ij}^{\ell+2}\right)$$

$$-\frac{1}{r}\left(\delta_x^2+\delta_y^2\right)\left(u_{ij}^{\ell}+u_{ij}^{\ell+2}\right)$$

Multiply by r^2:

$$r\left(\delta_x^2+\delta_y^2\right)\left(u_{ij}^{\ell}+u_{ij}^{\ell+2}\right) = \left(u_{ij}^{\ell+2}-u_{ij}^{\ell}\right)$$

$$+ r^2\delta_x^2\delta_y^2\left(u_{ij}^{\ell+2}-u_{ij}^{\ell}\right)$$

Divide by $2\Delta t$

$$\underbrace{\frac{1}{h^2}\left(\delta_x^2+\delta_y^2\right)\left(\frac{U_{ij}^{\ell}+U_{ij}^{\ell+2}}{2}\right) = \frac{U_{ij}^{\ell+2}-U_{ij}^{\ell}}{2\Delta t}} + \Delta t^2 \frac{\delta_x^2}{h^2}\frac{\delta_y^2}{h^2}\left(\frac{U^{\ell+2}-U^{\ell}}{2\Delta t}\right)$$

Exactly the C-N difference eqn's for $2\Delta t$!

$$\nabla^2 u + O(h^2) = \frac{\partial u}{\partial t} + O(\Delta t^2) + \Delta t^2\left[\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}\frac{\partial u}{\partial t} + O(h^2+\Delta t^2)\right]$$

Stability Analysis

- Natural extensions from 1-D case
- Fourier analysis $\Rightarrow$ decompose as $U = Ae^{\alpha t}e^{i\sigma x}e^{i\beta y}$

  then $U_{ij}^{\ell+1} = \gamma U_{ij}^{\ell}$; $\quad \delta^2 = (2\cos\sigma h - 2) + (2\cos\beta h - 2)$

e.g. Euler explicit:

$$U_{ij}^{\ell+1} - U_{ij}^{\ell} = r\delta^2 U_{ij}^{\ell}$$

$$\gamma - 1 = 2r\left\{ (\cos\sigma h - 1) + (\cos\beta h - 1) \right\}$$

$$\gamma = 1 - 2r\underbrace{\left[(1-\cos\sigma h) + (1-\cos\beta h)\right]}$$

longest waves in both x, y $\quad 0 \le [\quad\quad] \le 4 \quad$ Shortest waves in both x, y

$$1 - 2r(4) \geq -1 \Rightarrow 8r - 1 \le 1 \quad \Rightarrow \boxed{r \le 1/4}$$

Note: $1 - 2r(4) < 1$ always since $r > 0$

Also get an extra factor of 2 relative to 1D ... Recall 1D Euler requires $r \le 1/2$

Typical when increasing dimensionality ...

i.e. 1D $\rightarrow$ 2D $\rightarrow$ 3D

- Stability of ADI

$$\left(\delta_x^2 - \frac{1}{r}\right) u_{ij}^{\ell+1} = -\left(\delta_y^2 + \frac{1}{r}\right) u_{ij}^{\ell}$$

$$\left(\delta_y^2 - \frac{1}{r}\right) u_{ij}^{\ell+2} = -\left(\delta_x^2 + \frac{1}{r}\right) u_{ij}^{\ell+1}$$

$$\gamma_{ADI}^2 = \frac{u_{ij}^{\ell+2}}{u_{ij}^{\ell}} = \frac{\left(\delta_x^2 + \frac{1}{r}\right)\left(\delta_y^2 + 1/r\right)}{\left(\delta_x^2 - 1/r\right)\left(\delta_y^2 - 1/r\right)} = \frac{\left(r\delta_x^2 + 1\right)\left(r\delta_y^2 + 1\right)}{\left(r\delta_x^2 - 1\right)\left(r\delta_y^2 - 1\right)}$$

$$= \frac{\left[1 - 2r(1 - \cos\sigma h)\right]\left[1 - 2r(1 - \cos\beta h)\right]}{\left[1 + 2r(1 - \cos\sigma h)\right]\left[1 + 2r(1 - \cos\ h)\right]}$$

Define: $r(1 - \cos\sigma h) \equiv R > 0$ always

$r(1 - \cos\beta h) \equiv S > 0$ always

$$\gamma_{ADI}^2 = \frac{(1-2R)(1-2S)}{(1+2R)(1+2S)} \Rightarrow \left|\gamma_{ADI}^2\right| < 1 \text{ always}$$

Compare w/ Crank-Nicholson ...

$$\left(\frac{r}{2}\delta^2 - 1\right) U_{ij}^{\ell+1} = \left(-\frac{r}{2}\delta^2 - 1\right) U_{ij}^{\ell}$$

$$\gamma_{C-N} = \frac{U_{ij}^{\ell+1}}{U_{ij}^{\ell}} = \frac{\left(\frac{r}{2}\delta^2 + 1\right)}{\left(1 - \frac{r\delta^2}{2}\right)}$$

$$= \frac{1 - r(1-\cos\sigma h) - r(1-\cos\beta h)}{1 + r(1-\cos\sigma h) + r(1-\cos\beta h)}$$

$$= \frac{1 - R - S}{1 + R + S}$$

$$\therefore \gamma_{ADI}^2 \sim \gamma_{C-N} \quad \Rightarrow \quad \gamma_{ADI}^2 = \frac{1 - 2R - 2S + 4RS}{1 + 2R + 2S + 4RS}$$

$$2\Delta t_{ADI} \sim \Delta t_{C-N}$$

- IF $R = 0$ (i.e. long wavelengths in $x \Rightarrow$ slowly varying in x)
  <u>or</u> $S = 0$ (i.e. long wavelengths in $y \Rightarrow$ slowly varying in y)

$$\gamma_{ADI}^2 = \gamma_{C-N}$$

- IF Not ... some differences; Dynamics differ by $O(\Delta t^2)$

Boundary Conditions:

- 2nd Step ... $\left(\delta_y^2 - \frac{1}{r}\right) u_{ij}^{\ell+2} = \left(-\delta_x^2 - \frac{1}{r}\right) u_{ij}^{\ell+1}$

  No problem since we know $u_{ij}^{\ell+2}$ at boundaries as given

- But what about 1ST Step when solving for $u_{ij}^{\ell+1}$?? ... What to use since $u_{ij}^{\ell+1}$ is the intermediate result ... not part of sol'n per say

- Key ... Resort back to the combined process

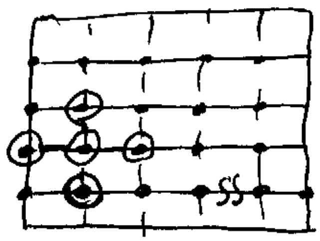
$$\left(\delta_x^2 - \frac{1}{r}\right) u_{ij}^{\ell+1} = \left(-\delta_y^2 - \frac{1}{r}\right) u_{ij}^{\ell}$$

$$\underline{\left(\delta_y^2 - \frac{1}{r}\right) u_{ij}^{\ell+2} = \left(-\delta_x^2 - \frac{1}{r}\right) u_{ij}^{\ell+1}}$$

$$-\frac{2}{r} u_{ij}^{\ell+1} = \left(\delta_y^2 - \frac{1}{r}\right) u_{ij}^{\ell+2} - \left(\delta_y^2 + 1/r\right) u_{ij}^{\ell}$$

so $$\boxed{u_{ij}^{\ell+1} = \left(\frac{1}{2} - \frac{r\delta_y^2}{2}\right) u_{ij}^{\ell+2} + \left(\frac{1}{2} + \frac{r\delta_y^2}{2}\right) u_{ij}^{\ell}}$$

So at boundary... say $i=0$, we compute the needed value in terms of $\underline{\underline{U_{0j}^{\ell+2} + U_{0j}^{\ell}}}$



Only need these since for step 1 we build Tridiagonal in $\underline{\underline{x}}$

δ_y part uses $U_{ij}^{\ell} \Rightarrow$ known

$$\Longrightarrow \quad U_{0,j}^{\ell+1} = \left(\frac{1}{2} - \frac{r\delta_y^2}{2}\right) U_{0j}^{\ell+2} + \left(\frac{1}{2} + \frac{r\delta_y^2}{2}\right) U_{0,j}^{\ell}$$

Known through BCs

Difference in y direction... along the boundary $\underline{\underline{\text{all}}}$ values known

Similar at $i=N$

Note: IF BC constant in time: $U_{0,j}^{\ell+2} = U_{0j}^{\ell} = U_{0j}$

$$U_{0,j}^{\ell+1} = \frac{1}{2} U_{0,j}^{\ell+2} + \frac{1}{2} U_{0,j}^{\ell} = U_{0,j}$$