Stability Analysis

- Bounded solution to FD equations given bounded ICs, BCs and forcing

- Lax-Richtmyer $\Rightarrow$ at fixed "t" sol'n of FD equations bounded as $k \to 0$ (assuming h related to k such that $h \to 0$ as $k \to 0$)

- Alternate view ... in practise h, k constant and sol'n propagated forward from $t=0$ to $t=jk$ ... then stability defined in terms of boundedness as $j \to \infty$ for k fixed.

- Two approaches to stability analysis

  1. Matrix Methods
  2. Fourier Method (Von Neumann)

Matrix Methods

- Well-covered in Smith (pp 47-67)
- Bare essentials here ...

In Lax-Richtmeyer view:

- Cast FD scheme as $\underset{\sim}{u}^{\ell+1} = A\underset{\sim}{u}^{\ell} + \underset{\sim}{b}^{\ell}$
- Sufficient to show $\|A\| \leq 1$ (PDE decreasing w/ time)
- Stability in this light plus consistency yields convergence
- Note: if $\|A\| \leq 1$ implies $\rho(A) \leq 1$

  $\Rightarrow$ $\rho(A) \leq \|A\|$, but $\rho(A) \leq 1$ does not guarantee $\|A\| \leq 1$

In fixed Mesh lengths view:

- sufficient to show $\rho(A) < 1$

- guarantees boundedness, but not convergence (unless $\rho(A) \le \|A\| < 1$) ... some subtleties here

- Same starting point $\Rightarrow \underset{\sim}{u}^{\ell+1} = A\underset{\sim}{u}^{\ell} + \underset{\sim}{b}^{\ell}$
expand $\underset{\sim}{\epsilon}$ in terms of eigenvectors of A

Useful facts $\Rightarrow$ Gerschgorin Theorems (see text)

- $\rho(A) \le \|A\|_\infty$ (max absolute row sum)

- Stability $\Rightarrow \|A\|_\infty \le 1$

e.g. Euler explicit: $u_i^{\ell+1} = r u_{i-1}^{\ell} + (1-2r)u_i^{\ell} + r u_{i+1}^{\ell}$

$$A = \begin{bmatrix} 1-2r & r & & \\ r & 1-2r & r & 0 \\ & r & 1-2r & r \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix} \Rightarrow \|A\|_\infty = |r| + |1-2r| + |r|$$

Need $\|A\|_\infty \le 1 \Rightarrow$ if $1-2r > 0$ then $\|A\|_\infty \le 1$
if $1-2r < 0$ then $\|A\|_\infty > 1$

$\therefore$ Need $r \le 1/2$ for Stability