Difference Formulas for Cross-Derivatives

- 2D Taylor Series Approach

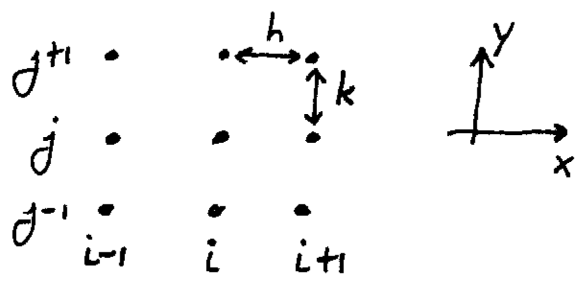
$$u(x+\Delta x, y+\Delta y) = u\Big|_{x,y} + \left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)u\Big|_{x,y}$$
$$+ \frac{1}{2!}\left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)^2 u\Big|_{x,y}$$
$$+ \frac{1}{3!}\left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)^3 u\Big|_{x,y} + \dots$$

where $\left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)^2 \equiv \Delta x^2\frac{\partial^2}{\partial x^2} + \Delta y^2\frac{\partial^2}{\partial y^2} + 2\Delta x\Delta y\frac{\partial^2}{\partial x\partial y}$

... Continue as in 1D case: write Taylor for all mesh points in terms of u, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ ... etc at point where derivative is desired (i.e. (i,j)); then mix together to get desired accuracy

- Easier: Operate on 1D Formulas

- Consider 2D mesh of pts: (equally spaced in x & y ... though $\Delta x \neq \Delta y$ necessarily)



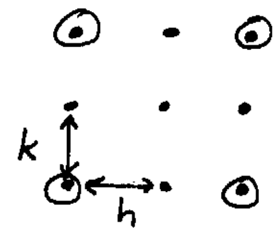
- Want to compute $\left.\dfrac{\partial^2 u}{\partial x \partial y}\right|_{(i,j)}$

$$= \left.\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\right|_{(i,j)} = \left.\frac{\partial F}{\partial x}\right|_{(i,j)} \quad \text{where } F \equiv \frac{\partial u}{\partial y}$$

But
$$\left.\frac{\partial F}{\partial x}\right|_{(i,j)} = \frac{F_{i+1,j} - F_{i-1,j}}{2h} = \frac{\frac{\partial u}{\partial y}_{i+1,j} - \frac{\partial u}{\partial y}_{i-1,j}}{2h}$$

$$= \frac{\dfrac{u_{i+1,j+1} - u_{i+1,j-1}}{2k} - \dfrac{u_{i-1,j+1} - u_{i-1,j-1}}{2k}}{2h}$$

$$= \frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4hk} \Rightarrow$$



Involves 4 corners!

What about leading error term?

Follows through from 1D expressions as well

From 1D: $$\frac{\partial u_j}{\partial y} = \frac{u_{j+1} - u_{j-1}}{2k} - \frac{k^2}{6}\frac{\partial^3 u_j}{\partial y^3} + \dots$$

then $$\frac{\partial}{\partial x}\left(\frac{\partial u_j}{\partial y}\right) = \frac{\frac{\partial u_{i,j+1}}{\partial x} - \frac{\partial u_{i,j-1}}{\partial x}}{2k} - \frac{k^2}{6}\frac{\partial^4 u_{ij}}{\partial x \partial y^3} + \dots$$

But $$\frac{\partial u_{i,j+1}}{\partial x} = \frac{u_{i+1,j+1} - u_{i-1,j+1}}{2h} - \frac{h^2}{6}\frac{\partial^3 u_{i,j+1}}{\partial x^3} + \dots$$

so leading error term is...

$$\frac{1}{2k}\left[-\frac{h^2}{6}\left(\frac{\partial^3 u_{i,j+1}}{\partial x^3} - \frac{\partial^3 u_{i,j-1}}{\partial x^3}\right)\right] - \frac{k^2}{6}\frac{\partial^4 u_{ij}}{\partial x \partial y^3}$$

$$= -\frac{h^2}{6}\underbrace{\left(\frac{\frac{\partial^3 u_{i,j+1}}{\partial x^3} - \frac{\partial^3 u_{i,j-1}}{\partial x^3}}{2k}\right)} - \frac{k^2}{6}\frac{\partial^4 u_{ij}}{\partial x \partial y^3}$$

$$G_{ij} \equiv \frac{\partial^3 u_{ij}}{\partial x^3}; \quad \frac{G_{i,j+1} - G_{i,j-1}}{2k} = \frac{\partial G_{ij}}{\partial y} + O(k^2)$$

$$= \underbrace{-\frac{h^2}{6}\frac{\partial^4 u_{ij}}{\partial x^3 \partial y} - \frac{k^2}{6}\frac{\partial^4 u_{ij}}{\partial x \partial y^3}}_{\text{Leading terms are } O(h^2+k^2)\ !!} + O(h^2k^2)$$