FD Solutions to Parabolic PDEs : Review

- BCs in space same as Elliptic, ICs in time; "open boundary" in Time

- Convergence : $U_i^\ell \longrightarrow U(x_i, t_\ell)$ as $h, k \longrightarrow 0$
- Consistency : $L_i \longrightarrow L$ as $h, k \longrightarrow 0$ ... i.e. Truncation error terms vanish
- Stability : $|U_i^\ell| < M < \infty$ for all i, ℓ
  Two views; Lax-Richtmyer, practical

- Stability + Consistency = Convergence (Lax)

- Spatial discretization same as Elliptic

- Time discretization: "Time-stepping"... Similar to iteration in matrix sol'n for Elliptic case <u>except</u> each new U is an approximate answer at a given time instant

- Generally $\underline{\underline{2}}$ time-levels involved due to <u>first</u> derivative in time

- Explicit schemes ... pointwise propagation; solution runs ahead of BCs; Stability constraints; $\frac{D\Delta t}{h^2} \equiv r$ is key factor

- Implicit schemes ... intrinsically more stable; need to solve system of equations to advance sol'n <u>one</u> time-level

1D in space ⇒ Tridiagonal; 2D in space pentadiagonal
<u>or</u> use iterative matrix sol'n methods as in Elliptic;
natural use of ij indexing; system is diagonally dominant!

- ADI as time-stepper: Alternate time-level evaluation of
  x + y derivatives; Fully implicit in derivatives tangential
  to "sweep" direction while fully explicit in derivatives
  normal to "sweep" direction ... tridiagonal systems only
  ... 2 step process: one rowwise followed by one columnwise
  sweep advances sol'n <u>one</u> time-level; looks like ADI
  as sol'n to algebraic system where $\frac{1}{r}$ is ω ⇒ unconditionally
  stable; looks like CN time-stepping w/ $\Delta t_{ADI} = \frac{\Delta t_{CN}}{2}$;
  dynamics differ $O(\Delta t^2)$; Need care in BCs at intermediate
  level in order to maintain accuracy

- Fourier Analysis ... leads to <u>2</u> types of information
  Stability + Accuracy

  - relate all space-time points in molecule to u_i^ℓ
  - obtain numerical "amplification factor" as function
    of σh "dimensionless wavenumber" (i.e. $\gamma_0 = f(\sigma h)$)
  - Stability: $|\gamma_0| \le 1$ Stable

    $-1 \le \gamma_0 \le 0$ bounded oscillations in time
    i.e. $2\Delta t$ wrinkles in sol'n
    at fx space pt as function
    of t

  - $|\gamma_0| > 1$ Unstable

- Consider all possible σh supportable on a discrete mesh $\Rightarrow$ $0 < \sigma h \leq \pi$

- must do this because ICs can have broad spectral content ... e.g. discontinuities or sharp transitions in slope; Potential for rounding errors

- shortest wavelengths usually worst offenders

- Accuracy ... compare numerical propagation w/ nature of the analytic propagation; consider all possible σh ... shortest waves most misbehaved ... can examine differences at single time-step, but more typical to consider propagation over "characteristic time" ... common to use one analytic time constant $\Rightarrow$ "Propagation Factor"; can speak of "underdamped" $T > 1$ and "overdamped" $T < 1$ numerical solutions