Von Neumann (Fourier) Stability Analysis

- Idea... represent ICs w/ Fourier Series, then study how the Fourier modes are propagated by the FD eqn.

  (Analogous to expressing "initial guess" as eigenvector expansion in Matrix Methods)

- Stability: FD molecule must not allow any Fourier mode to increase as FD sol'n advanced from t_ℓ to $t_{\ell+1}$

- Formally, method only valid for
  - linear equations w/ constant coefficients
  - uniform mesh
  - BCs at infinity

Generally, get same results as Matrix method (i.e. BCs effect stability in minor way relative to FD equations themselves)

- Because of linearity ... look at each mode separately

- Assume sol'n of form $u = e^{\alpha t} e^{j\sigma x}$ $(j = \sqrt{-1})$

and require $\dfrac{u(t+\Delta t)}{u(t)} \le 1$

$$\underbrace{\frac{u(t+\Delta t)}{u(t)}} = \frac{e^{\alpha(t+\Delta t)} e^{j\sigma x}}{e^{\alpha t} e^{j\sigma x}} = e^{\alpha \Delta t} \equiv \gamma$$

"amplification factor"

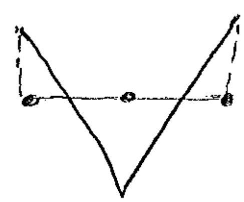
- "wavenumber" $\sigma = \dfrac{2\pi}{L}$ ← wavelength

  → # of wavelengths in distance 2π

on mesh w/ spacing h + BCs at ∞ ...

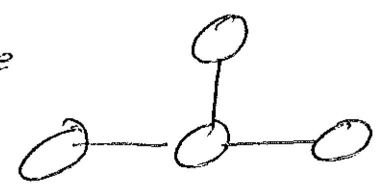
$$2h \le L < \infty$$

$$0 < \sigma \le \pi/h$$



"highest" frequency

Examine Euler Explicit Scheme



$$u_i^{\ell+1} - u_i^{\ell} = r\,\delta_x^2 u_i^{\ell}$$

Now... $u_i^{\ell} = e^{\alpha(k\ell)} e^{j\sigma(ih)}$

$$u_i^{\ell+1} = e^{\alpha(\ell+1)k} e^{j\sigma(ih)} = e^{\alpha k}\left(e^{\alpha(k\ell)} e^{j\sigma(ih)}\right)$$

$$= \gamma_0\, u_i^{\ell}$$

amplification factor for discrete system $\rightarrow \gamma_0$

$$u_{i-1}^{\ell} = e^{\alpha(k\ell)} e^{j\sigma h(i-1)} = u_i^{\ell} e^{-j\sigma h}$$

$$u_{i+1}^{\ell} = u_i^{\ell} e^{j\sigma h}$$

Substituting into FD eqn: $u_i^{\ell+1} - u_i^{\ell} = r\left(u_{i-1}^{\ell} - 2u_i^{\ell} + u_{i+1}^{\ell}\right)$

$$u_i^{\ell}(\gamma_0 - 1) = u_i^{\ell}\, r\left(e^{-j\sigma h} - 2 + e^{j\sigma h}\right)$$

$$\gamma_0 = 1 - 2r(1 + \cos\sigma h) \Rightarrow$$ Note: $\cos\sigma h = 1 - 2\sin^2\frac{\sigma h}{2}$

$$\gamma_0 = 1 - 4r\sin^2\frac{\sigma h}{2}$$

For stability... $|\gamma_0| \leq 1 \Rightarrow -1 \leq 1 - 2r(1 - \cos\sigma h) \leq 1$

for all possible σ's

Recall: $0 < \sigma \leq \pi/h \Rightarrow 0 < \sigma h \leq \pi$

$$0 < 1 - \cos\sigma h \leq 2$$

- Conclude $\gamma_0 < 1$

γ_0 can be negative $\Rightarrow$ Oscillation in t!

Negative when: $1 - 2r(1 - \cos\sigma h) < 0$

$$2r(1 - \cos\sigma h) - 1 > 0$$

$$\text{i.e. } r > \frac{1}{2(1 - \cos\sigma h)}$$

- Conclude: $r > 1/4$ produces $\gamma_0 < 0$

$\Rightarrow$ Shortest waves (i.e. highest frequency modes) oscillate..... entirely a numerical artefact

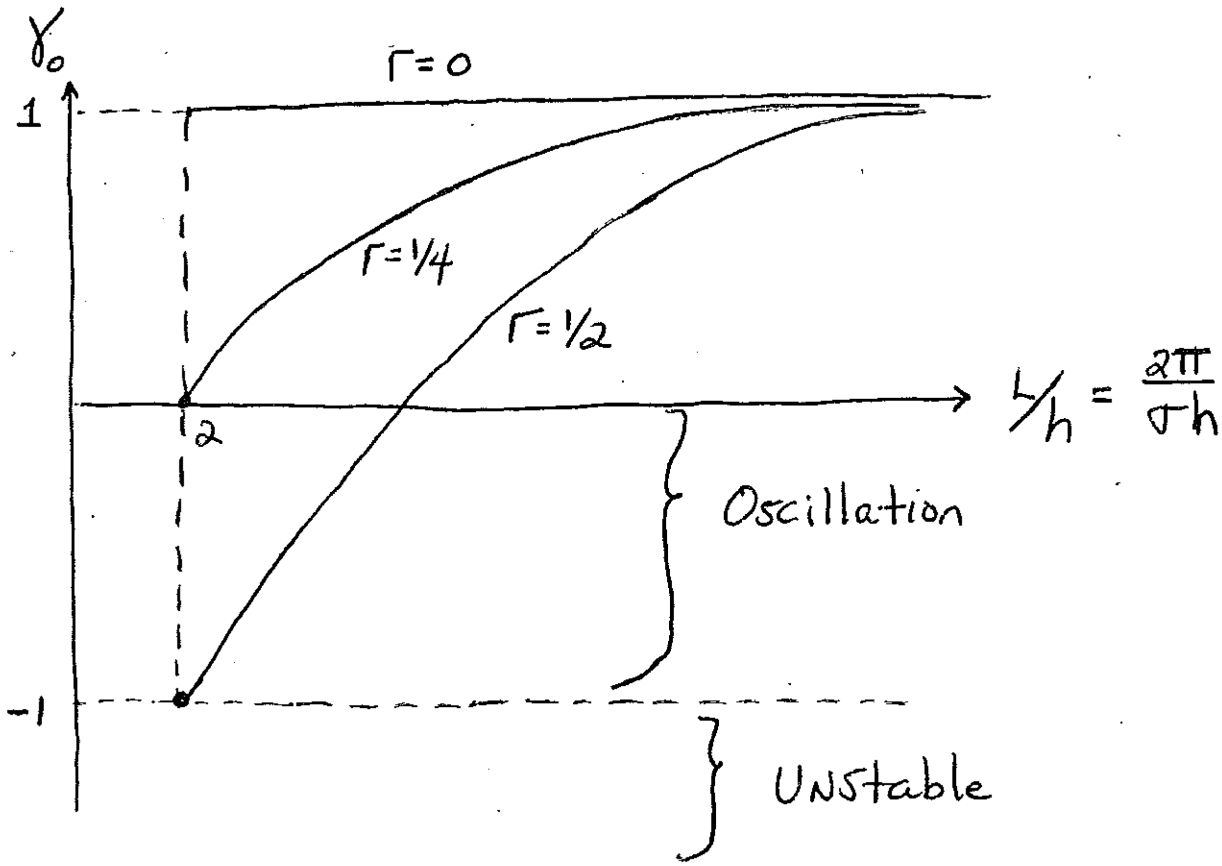
- Unstable when $\gamma_0 < -1$

$$1 - 2r(1 - \cos\sigma h) < -1$$

$$2 < 2r(1 - \cos\sigma h)$$

$$\frac{1}{1 - \cos\sigma h} < r$$

i.e. $r > 1/2$ (Shortest waves have unstable oscillation)



Rule of Thumb... Short wavelengths are first to go

- Develop spurious oscillations
- Oscillations become fatal as k increases

What about accuracy? Can study

$$\frac{\text{Numerical aplification factor}}{\text{Analytical amplification factor}} \Rightarrow \frac{\gamma_0}{\gamma}$$