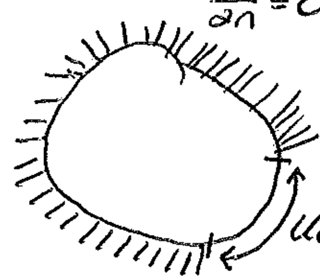
Time-Stepping Example

$$c\frac{\partial u}{\partial t} - \nabla\cdot K\nabla u = g$$



Galerkin:

$$\left\langle c\frac{d\hat{u}}{dt}, \phi_i \right\rangle + \left\langle K\nabla\hat{u}\cdot\nabla\phi_i \right\rangle = \langle g, \phi_i\rangle + \underbrace{\oint K\frac{\partial u}{\partial n}\phi_i\, ds}$$

Zero in every case except where $u(t)$ to
But don't need it since Galerkin discarded!

FD in time:

$$\left\langle c\left(\frac{\hat{u}^{k+1} - \hat{u}^{k}}{\Delta t}\right)\phi_i \right\rangle + \left\langle K\nabla\left(\theta\hat{u}^{k+1} + (1-\theta)\hat{u}^{k}\right)\cdot\nabla\phi_i \right\rangle$$

$$= \theta\langle g, \phi_i\rangle^{k+1} + (1-\theta)\langle g, \phi_i\rangle^{k}$$

Multiply by Δt, rearrange:

$$\underbrace{\langle c\hat{u}, \phi_i\rangle^{k+1} + \langle K\Delta t\theta\nabla\hat{u}\cdot\nabla\phi_i\rangle^{k+1}}_{\text{Matrix } [A]} = \langle c\hat{u}\phi_i\rangle^{k} - \langle K\Delta t(1-\theta)\nabla\hat{u}^{k}\cdot\nabla\phi_i\rangle$$

$$\underbrace{+\ \theta\Delta t\langle g\phi_i\rangle^{k+1} + (1-\theta)\Delta t\langle g, \phi_i\rangle^{k}}_{\text{RHS } \{b\}}$$

$$A_{ij} = \langle c\,\phi_j\,\phi_i\rangle + \langle K\Delta t\theta\,\nabla\phi_j\cdot\nabla\phi_i\rangle$$

$$= \langle c\,\phi_j\,\phi_i\rangle + \Delta t\theta\left\langle K\frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial x} + K\frac{\partial\phi_j}{\partial y}\frac{\partial\phi_i}{\partial y}\right\rangle$$

On linear triangles (constant coefficients on an element)

$$A_{ij}^e = c^e \frac{A^e}{\alpha} + \Delta t \theta K^e \left[\frac{\Delta y_i^e \Delta y_j^e}{4A^e} + \frac{\Delta x_i^e \Delta x_j^e}{4A^e} \right]$$

$\alpha = 12;\ i \neq j$
$\quad = 6;\ i = j$

$$b_i^e = \langle c \hat{u}^k \phi_i \rangle^e - \left\langle K \Delta t (1-\theta) \left(\frac{\partial \hat{u}^k}{\partial x} \frac{\partial \phi_i}{\partial x} + \frac{\partial \hat{u}^k}{\partial y} \frac{\partial \phi_i}{\partial y} \right) \right\rangle^e$$
$$+ \theta \Delta t \langle g^{k+1} \phi_i \rangle^e + (1-\theta) \Delta t \langle g^k \phi_i \rangle^e$$

$$\langle c \hat{u}^k \phi_i \rangle^e = c^e \left[u_1^k \langle \phi_1 \phi_i \rangle^e + u_2^k \langle \phi_2 \phi_i \rangle^e + u_3^k \langle \phi_3 \phi_i \rangle^e \right]$$
$$= c^e \frac{A^e}{12} \left[u_1 + u_2 + u_3 + u_i \right]^k$$

$$\langle K \Delta t (1-\theta) \cdots \rangle = K^e \Delta t (1-\theta) \left[\left(\frac{u_1 \Delta y_1^e + u_2 \Delta y_2^e + u_3 \Delta y_3^e}{2A^e} \right) \frac{\Delta y_i^e}{2} + \left(\frac{u_1 \Delta x_1^e + u_2 \Delta x_2^e + u_3 \Delta x_3^e}{2A^e} \right) \left(\frac{\Delta x_i^e}{2} \right) \right]^k$$

$$\langle g \phi_i \rangle^e = g^e \frac{A^e}{3}$$

Functional coefficients

$$\text{e.g. } \langle c\hat{u}\phi_i\rangle = \underbrace{u_1\langle\phi_1 c\phi_i\rangle^e} + u_2\langle\phi_2 c\phi_i\rangle^e + u_3\langle\phi_3 c\phi_i\rangle$$

$$u_1\Big[c_1\underbrace{\langle\phi_1\phi_1\phi_i\rangle^e}_{\begin{array}{l}\frac{2A2!}{5!};\ i\neq 1\\ \frac{2A3!}{5!};\ i=1\end{array}} + c_2\underbrace{\langle\phi_1\phi_2\phi_i\rangle^e}_{\begin{array}{l}\frac{2A1!}{5!};\ i=3\\ \frac{2A2!}{5!};\ i\neq 3\end{array}} + c_3\underbrace{\langle\phi_1\phi_3\phi_i\rangle^e}_{\begin{array}{l}\frac{2A1!}{5!};\ i=2\\ \frac{2A2!}{5!};\ i\neq 2\end{array}}\Big]$$

BC's as usual

Type I: Remove Galerkin equation (Row of Matrix "i" + RHS) Replace with

$u_i = BV$ (Boundary Value given)

Type II: If homogeneous: already done at outset

If inhomogeneous: add $\oint$ to RHS