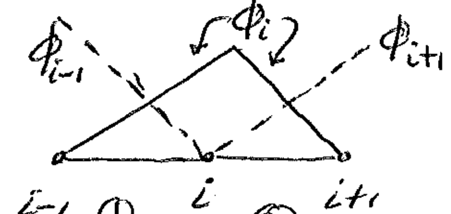
Stability: Same approach as FD...

Look at dispersion relation on uniform grid $\Rightarrow$ $u \sim e^{j\sigma x} e^{\alpha t}$; $\gamma \equiv \dfrac{u_i^{k+1}}{u_i^k} = e^{\alpha \Delta t}$

e.g. 1-D Diffusion Eqn $\Rightarrow$ Need to get FE difference equation at a node



$$\frac{\partial u}{\partial t} + K\frac{\partial^2 u}{\partial x^2} = 0 \quad \text{(homogeneous)}$$

$$\underbrace{\left\langle \frac{\partial \hat{u}}{\partial t}\phi_i \right\rangle}_{\sum \frac{du_j}{dt}\phi_j} = \langle \phi_{i-1}\phi_i \rangle^{①} \frac{u_{i-1}^{k+1} - u_{i-1}^{k}}{\Delta t} + \langle \phi_i \phi_i \rangle^{①+②} \frac{u_i^{k+1} - u_i^{k}}{\Delta t}$$

$$+ \langle \phi_{i+1}\phi_i \rangle^{②} \frac{u_{i+1}^{k+1} - u_{i+1}^{k}}{\Delta t}$$

$$= \left[\frac{h}{6}\left(u_{i-1} + 4u_i + u_{i+1}\right)^{k+1} - \frac{h}{6}\left(u_{i-1} + 4u_i + u_{i+1}\right)^{k}\right]\frac{1}{\Delta t}$$

$$\underbrace{\left\langle \frac{\partial \hat{u}}{\partial x}\frac{\partial \phi_i}{\partial x} \right\rangle}_{\sum u_j \frac{\partial \phi_j}{\partial x}} = \left\langle \frac{\partial \phi_{i-1}}{\partial x}\frac{\partial \phi_i}{\partial x} \right\rangle^{①} u_{i-1} + \left\langle \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_i}{\partial x} \right\rangle^{①+②} u_i$$

$$+ \left\langle \frac{\partial \phi_{i+1}}{\partial x}\frac{\partial \phi_i}{\partial x} \right\rangle^{②} u_{i+1}$$

$$= \frac{-u_{i-1}}{h} + \frac{u_i}{h} + \frac{u_i}{h} - \frac{u_{i+1}}{h} = -\left(\frac{u_{i-1} - 2u_i + u_{i+1}}{h}\right)$$

Galerkin Eqn at node i:

$$\frac{h}{6}\left[U_{i-1}+4U_i+U_{i+1}\right]^{k+1}-K\Delta t\theta\left[\frac{U_{i-1}-2U_i+U_{i+1}}{h}\right]^{k+1}$$

$$=\frac{h}{6}\left[U_{i-1}+4U_i+U_{i+1}\right]^{k}$$

$$+K\Delta t(1-\theta)\left[\frac{U_{i-1}-2U_i+U_{i+1}}{h}\right]^{k}$$

- Divide by h to see FD analogy
- $\frac{\partial^2 u}{\partial x^2}$ Same as FD
- $\frac{\partial u}{\partial t}$ Simpson's Rule $\Rightarrow$ FE "distributes" the $\frac{\partial}{\partial t}$ term $\Rightarrow$ formally can't have explicit method (FD: only $\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}$ distributed)

Now: $U_{i+1}-2U_i+U_{i-1} = (2\cos\sigma h-2)U_i = BU_i$

$U_{i+1}+4U_i+U_{i-1} = (2\cos\sigma h+4)U_i = AU_i$

So:

$$\frac{A}{6}U_i^{k+1}-\frac{K\Delta t\theta}{h^2}BU_i^{k+1}=\frac{1}{6}AU_i^{k}+\frac{K\Delta t(1-\theta)}{h^2}BU_i^{k}$$

$$\frac{K\Delta t}{h^2} \equiv r \quad ; \quad \gamma = \frac{u_i^{k+1}}{u_i^k}$$

$$\left(\frac{A}{6} - rB\theta\right)\gamma = \left(\frac{A}{6} + rB(1-\theta)\right)$$

$$\gamma = \frac{\frac{A}{6} + rB(1-\theta)}{\frac{A}{6} - rB\theta} \Rightarrow \text{Stability want}$$

$\gamma < 1$

$\gamma > -1$

$$\boxed{\begin{aligned} &\text{But} \quad 0 \le \sigma h \le \frac{2\pi}{L} h \le \pi \\ &\Rightarrow \quad 2 \le A \le 6 \\ &\quad\;\; -4 \le B \le 0 \end{aligned}}$$

$\frac{A}{6} - rB\theta > 0$ always $\Rightarrow \gamma < 1$ always ✓

$$\gamma > -1 \Rightarrow \quad \frac{A}{6} + rB(1-\theta) > rB\theta - \frac{A}{6}$$

$$r(1-2\theta)B > -\frac{A}{3}$$

for $\theta \geq 1/2$ $\quad B(1-2\theta) \geq 0 \Rightarrow$ Unconditional Stability

For $\theta < 1/2$ $\quad r B(2\theta - 1) < \frac{A}{3}$

$$r < \frac{A}{3B(2\theta-1)}$$

Shortest waves worst case $\Rightarrow B = -4,\ A = 2$

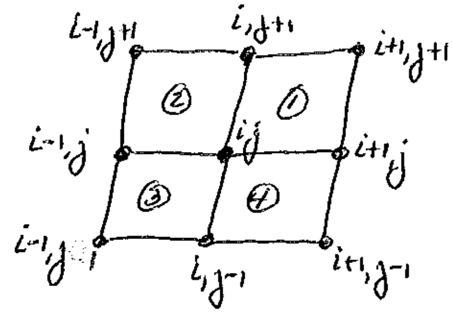
$$\boxed{r < \frac{1}{6(1-2\theta)} \quad \text{for } \theta < 1/2}$$

Recall FD result : $r < \dfrac{1}{2(1-2\theta)}$

FE factor of 3 more sensitive to short wave instability

FD: put $A = 6$ in previous formulas (Removes the distribution in space of $\frac{\partial}{\partial t}$)

What about 2D?



e.g. $\sum U_j \langle \phi_i \phi_j \rangle$; $i \to ij$

$j \to ij;\ i+1,j;\ i+1,j+1;$ etc 9 Terms to deal w/

$$j \to i+1,j \ ; \ U_{i+1,j}\langle \phi_{ij}\, \phi_{i+1,j}\rangle = U_{i+1,j}\left[\langle \phi_{ij}\, \phi_{i+1,j}\rangle_{①} + \langle \phi_{ij}\, \phi_{i+1,j}\rangle_{④}\right]$$

on ①: $$\langle \phi_{ij}\, \phi_{i+1,j}\rangle = \frac{A}{4}\int_{-1}^{1}\int_{-1}^{1} \frac{(1-\xi)(1-\eta)}{4}\,\frac{(1+\xi)(1-\eta)}{4}\, d\xi\, d\eta$$

$$= \frac{A}{4(16)}\int_{-1}^{1}\int_{-1}^{1}(1-\eta)^2(1-\xi^2)\, d\xi\, d\eta = \frac{A}{4(16)}\int_{-1}^{1}(1-\eta)^2 \underbrace{\left[\xi - \frac{\xi^3}{3}\right]_{-1}^{1}}_{4/3} d\eta$$

$$= \frac{A}{3(16)}\int_{-1}^{1}(1-\eta)^2\, d\eta = \frac{A}{3(16)}\underbrace{\frac{(1-\eta)^3}{3}}_{8/3}\Bigg|_{1}^{-1} = \frac{A}{18}$$

on ④: $$\langle \phi_{ij}\, \phi_{i+1,j}\rangle = \frac{A}{4}\int_{-1}^{1}\int_{-1}^{1} \frac{(1-\xi)(1+\eta)}{4}\,\frac{(1+\xi)(1+\eta)}{4}\, d\xi\, d\eta$$

$$= \frac{A}{4(16)}\int_{-1}^{1}(1+\eta)^2 \underbrace{\int_{-1}^{1}(1-\xi^2)\, d\xi}_{4/3}\, d\eta = \frac{A}{3\,16}\,\frac{(1+\eta)^3}{3}\Bigg|_{-1}^{1} = \frac{A}{18}$$

$$\therefore\ U_{i+1,j}\langle \phi_{i,j}\, \phi_{i+1,j}\rangle = \frac{A}{9}\, U_{i+1,j}$$

$$j \to i+1, j+1 \; ; \quad U_{i+1,j+1}\langle \phi_{ij}\,\phi_{i+1,j+1}\rangle = U_{i+1,j+1}\langle \phi_{ij}\,\phi_{i+1,j+1}\rangle_{1}$$

$$\langle \phi_{ij}\,\phi_{i+1,j+1}\rangle_{1} = \frac{A}{4}\int_{-1}^{1}\int_{-1}^{1} \frac{(1-\xi)(1-\eta)}{4}\,\frac{(1+\xi)(1+\eta)}{4}\,d\xi\,d\eta$$

$$= \frac{A}{(16)4}\int_{-1}^{1}(1-\eta^2)\,d\eta\int_{-1}^{1}(1-\xi^2)\,d\xi = \frac{A}{16(4)}\left(\tfrac{4}{3}\right)\left(\tfrac{4}{3}\right) = \underline{\underline{\frac{A}{36}}}$$

Can continue ..., but easier to recognize

$$\sum U_j \langle \phi_i\,\phi_j\rangle = \sum U_j \iint \phi_i(x,y)\,\phi_j(x,y)\,dx\,dy$$

$$= \sum U_j \int \underbrace{\underbrace{\left[\int \phi_i(x,y)\,\phi_j(x,y)\,dx\right]}_{\frac{h}{6}\left[U_{i-1} + 4U_i + U_{i+1}\right]_j}}_{F_j} dy$$

$$\Rightarrow \int F_j\,dy = \frac{h}{6}\left[F_{j-1} + 4F_j + F_{j+1}\right]$$

$$= \frac{h}{6}\left[\frac{h}{6}\left(U_{i-1,j-1} + 4U_{i,j-1} + U_{i+1,j-1}\right)\right] + \frac{4h}{6}\left[\frac{h}{6}\left(U_{i-1,j} + 4U_{i,j} + U_{i+1,j}\right)\right]$$

$$+ \frac{h}{6}\left[\frac{h}{6}\left(U_{i-1,j+1} + 4U_{i,j+1} + U_{i+1,j+1}\right)\right]$$

So $u_{i+1,j}$ term has coefficient $\frac{4h^2}{36} = \frac{A}{9}$

$u_{i+1,j+1}$ term has coefficient $\frac{h^2}{36} = A/36$

Checks w/ previous calculation!

- Can do derivative terms the same way

key is x-derivatives are constants w/ respect to x, but linear in y

likewise y-derivatives are constant in y, but linear in x