Block Iterative Methods

- "Point" methods ... each calculation modifies soln at single point ... u_{ij}^{l+1} can be computed by itself $\Rightarrow$ "explicit"

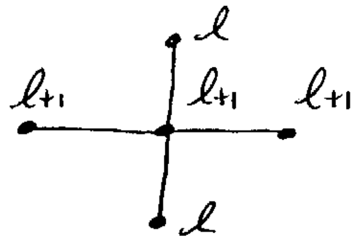
- "Block" methods ... groups of components ~~of~~ $\underset{\sim}{u}^{l+1}$ are computed simultaneously ... involves sol'n of system of equations $\Rightarrow$ individual components defined in terms of other components of the same Block $\Rightarrow$ "Implicit"

- Basic tradeoff between block size and iteration count ... Bigger the block $\Rightarrow$ fewer iterations but cost per block increases ... limiting case: 1 Block $\Rightarrow$ 1 "iteration" $\Rightarrow$ This is a direct soln !!

- Redefinition of explicit method as implicit one typically increases convergence rate

- Easy to do for Jacobi, Gauss-Siedel, SOR

- "Single line" Jacobi

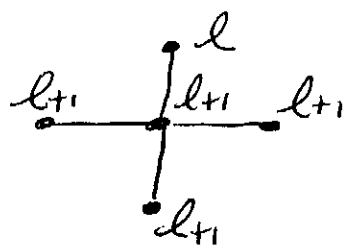


$$\underbrace{\beta_0 u_{ij}^{l+1} - \beta_1 u_{i+1,j}^{l+1} - \beta_2 u_{i-1,j}^{l+1}} = \beta_3 u_{i,j+1}^{l} + \beta_4 u_{i,j-1}^{l}$$

Tridiagonal system $\Rightarrow$ M eqn's in M unknowns for M nodes in "i" direction

- update sol'n line by line ... but must solve system to get $u_{i,j}^{l+1}$ $i = 1, 2, \dots M$ <u>for each</u> j

- "Single line" Gauss-Siedel (row wise ordering)



$$\beta_0 u_{ij}^{l+1} - \beta_1 u_{i+1,j}^{l+1} - \beta_2 u_{i-1,j}^{l+1} = \beta_3 u_{i,j+1}^{l} + \beta_4 u_{i,j-1}^{l+1}$$

SOR: $u_{ij}^{l+1} = \omega \bar{u}_{ij}^{l+1} + (1-\omega) u_{ij}^{l}$

$\leftarrow$ GS sol'n

$$u_{ij}^{l+1} - \omega\left[\frac{\beta_1}{\beta_0}u_{i+1,j}^{l+1} + \frac{\beta_2}{\beta_0}u_{i-1,j}^{l+1}\right] = (1-\omega)u_{ij}^{l} + \omega\left[\frac{\beta_3}{\beta_0}u_{i,j+1}^{l} + \frac{\beta_4}{\beta_0}u_{i,j-1}^{l+1}\right]$$

- Convergence Rates for Block Iterative Methods ... gain must compensate for extra effort to solve tridiagonal ... problem dependent

- For single line blocks ... Laplace on square w/ $h = \pi/(N+1)$ i.e. $0 < x < \pi$, $0 < y < \pi$

	Point	Line
Jacobi	$h^2/2$	h^2
G-S	h^2	$2h^2$
SOR (optimal)	$2h$	$\sqrt{2}(2h)$

$$\omega_{opt}^{line} = \frac{2}{1+\sqrt{1-\rho_{J_{line}}^2}}$$

- same as ω_{opt}^{point}
- replace $\rho_{J_{point}}$ w/ $\rho_{J_{line}}$

- Can extend to more lines ... i.e. 2 line etc.

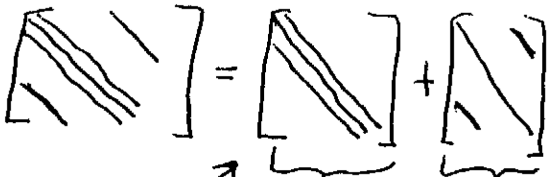
ADI: Alternating Direction Implicit

- Line methods discussed above ... always go in same direction ... Convergence often improved by changing directions!

- We think of a "single interation" as a 2 step process $\Rightarrow$ first implicit in row direction (i.e. x) followed by implicit in column direction (i.e. y)

e.g. Difference form: $\delta_x^2 U_{ij} + \delta_y^2 U_{ij} + f h^2 U_{ij} = g h^2$

- δ_x^2 or δ_y^2 can be tridiagonal, but not simultaneously! $\Rightarrow$ $\delta_x^2 + \delta_y^2 =$

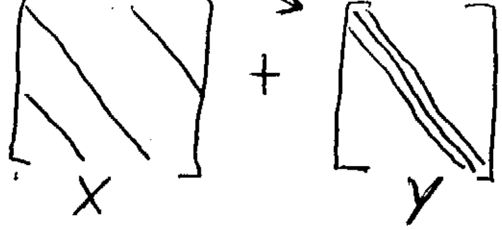


X Y

Number consecutively across row

Or

=



X Y

Number consecutively along a column

- We take advantage of this by writing FD equation

$$\delta_x^2 U_{ij} + \frac{fh^2}{2} U_{ij} = -\delta_y^2 U_{ij} - \frac{fh^2}{2} U_{ij} + gh^2$$

- then solve for left side assuming right side is known,
- give ourselves ω tuning factor to speed up
- Repeat in the y direction writing

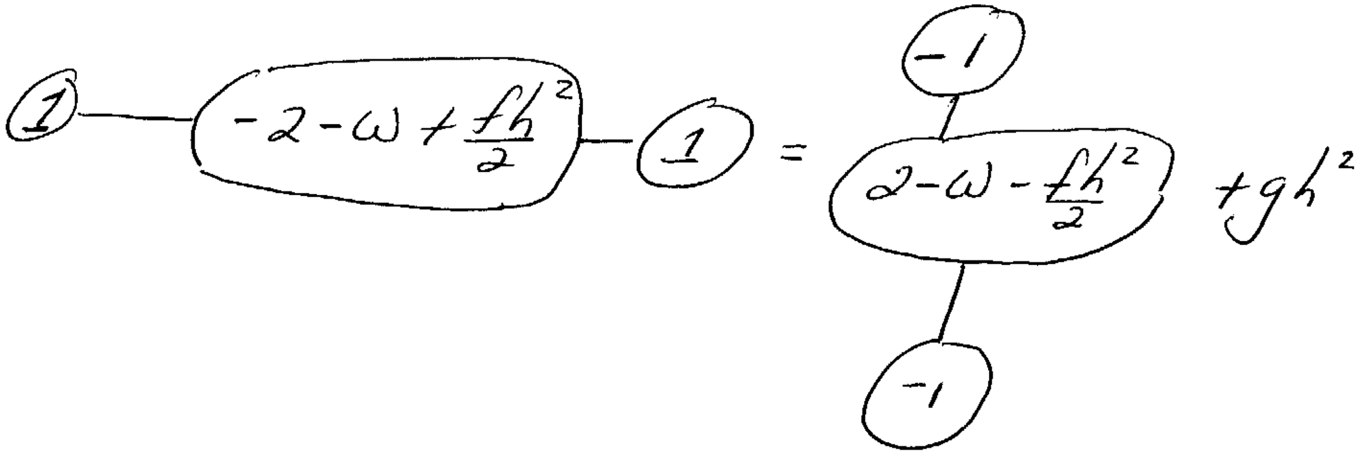
$$\delta_y^2 U_{ij} + \frac{fh^2}{2} U_{ij} = -\delta_x^2 - \frac{fh^2}{2} U_{ij} + gh^2$$

Always dealing w/ Tridiagonal... complete iteration involves a sweep in both directions

e.g. Step 1: Implicit in x

$$\left(\delta_x^2 + \frac{fh^2}{2}\right) U_{ij}^{l+1/2} - \omega U_{ij}^{l+1/2} = -\left(\delta_y^2 + \frac{fh^2}{2}\right) U_{ij}^{l} - \omega U_{ij}^{l} + gh^2$$

$\omega \equiv$ Iteration parameter



Step 2: Implicit in y

$$\left(\delta_y^2 + \frac{fh^2}{2}\right)U_{ij}^{l+1} - \omega U_{ij}^{l+1} = -\left(\delta_x^2 + \frac{fh^2}{2}\right)U_{ij}^{l+1/2} - \omega U_{ij}^{l+1/2} + gh^2$$

$$\begin{array}{c} 1 \\ | \\ -2-\omega+\frac{fh^2}{2} \\ | \\ 1 \end{array} = -1 \;—\; 2-\omega-\frac{fh^2}{2} \;—\; -1 \quad + gh^2$$

- Two Step Procedure ... Solve Tridiagonal Systems
  ... Intermediate Results not considered as valid

Convergence Rate:

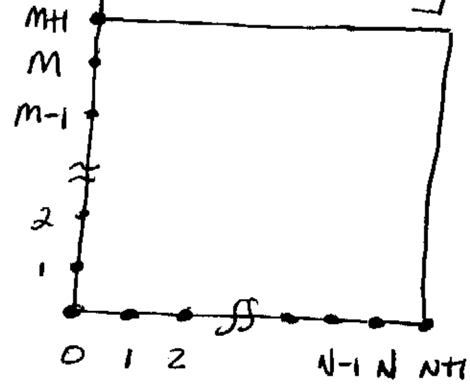
- Key is $\rho(G_{ADI})$ ... can cast this scheme in form $U^{l+1} = GU^l + c$ then study spectrum of G

- Complicated but some results are known
  - Convergence only for $\omega > 0$

- Optimal ω for Laplace on square w/ Type I BCs

$$\omega_{opt}^2 = \left[\left(\left(-\frac{fh}{2}\right)^2 + 4\sin^2\frac{\pi}{2R}\right)\left(\left(-\frac{fh}{2}\right)^2 + 4\cos^2\frac{\pi}{2R}\right)\right]$$

$R = \max(M+1, N+1)$ $\Rightarrow$



- ADI convergence $\approx$ SOR when using optimal ω for both

- Can improve significantly by generating a sequence of ω parameters $\Rightarrow$ i.e. ω^l

- optimal sequence difficult, but some approaches for estimates are available