Explicit Schemes on FE

- "Mass lumping" sum terms in the mass matrix and place result on diagonal i.e. "diagonilize" Mass Matrix ⇒ Inversion is trivial ⇒ explicit scheme
- Controversial... works well, but no strong rationale ... Stiffness matrice not treated this way ... only mass matrix

e.g. 1-D Diffusion: $(\theta = 0)$

$$\underbrace{\frac{h}{6}\left[u_{i-1} + 4u_i + u_{i+1}\right]^{k+1}}_{\downarrow\; u_i^{k+1}} = \underbrace{\frac{h}{6}\left[u_{i-1} + 4u_i + u_{i+1}\right]^{k}}_{\downarrow\; u_i^{k}} + \frac{K\Delta t}{h}\left[u_{i-1} - 2u_i + u_{i+1}\right]^k$$

$$u_i^{k+1} = u_i^k + \frac{K\Delta t}{h^2}\left[u_{i-1} - 2u_i + u_{i+1}\right]^k$$

No matrices!

Stability? ⇒ same as FD on uniform mesh

- More systematic approach "Integral lumping"

Treat all integrations (i.e. matrices) equivalently

View "lumping" as a quadrature approximation

e.g. in 1-D $\displaystyle\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \frac{f(x_i)+f(x_{i+1})}{2}\,h$

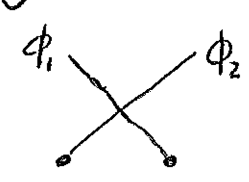
(Trapezoidal Rule)

$$\int_{x_i}^{x_{i+1}} f(x)\,dx = \int_{-1}^{1} f(\underbrace{x(\xi)}_{x=\sum_1^2 x_i\phi_i})\underbrace{\frac{h}{2}}_{|J|}\,d\xi \approx \frac{h}{2}\Big[f(\underbrace{x(-1)}_{x_i}) + f(\underbrace{x(1)}_{x_{i+1}})\Big]$$

Same!

Nodal quadrature rule ⇒ "place guass points at nodes", sufficient as long as we can at least integrate a constant ⇒ i.e. get Area of element

e.g. $\langle \phi_i \phi_j \rangle = \frac{h}{2}\left[\phi_i(-1)\phi_j(-1) + \phi_i(1)\phi_j(1)\right]$



$$A^e = \begin{bmatrix} \frac{h}{2} & 0 \\ 0 & h/2 \end{bmatrix}$$ diagonal!

Note: "Gauss pt weights" are unity

e.g. 1-D diffusion eqn

get same thing here since $\frac{\partial \phi_i}{\partial x}, \frac{\partial \phi_j}{\partial x}$ are constant on an element

$$h u_i^{k+1} = h u_i^k + \frac{K\Delta t}{h}\left[u_{i-1} - 2u_i + u_{i+1}\right]^k$$

$$u_i^{k+1} = u_i^k + \frac{K\Delta t}{h^2}\left[u_{i-1} - 2u_i + u_{i+1}\right]^k$$

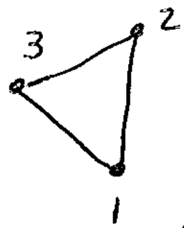
Works in Multi-D situations as well

e.g. Linear triangle:

$$M_{ij}^e = \underbrace{2A^e}_{|J|} \int \underbrace{\phi_i \phi_j}_{g_{ij}} \, dA \approx 2A^e \sum_{k=1}^{3} g_{ij}(\xi_k)\, w_k$$

w_k's such that if g_{ij} constant still get A^e

$\Rightarrow w_k = 1/6$



$$[M]^e = \begin{bmatrix} A^e/3 & 0 & 0 \\ 0 & A^e/3 & 0 \\ 0 & 0 & A^e/3 \end{bmatrix}$$

$$2A^e\left[\phi_1(1)\phi_1(1)w_1 + \underbrace{\phi_1(2)\phi_1(2)}_{0} w_2 + \underbrace{\phi_1(3)\phi_1(3)}_{0} w_3\right]$$

$$2A^e\left[w_1\phi_1(1)\underbrace{\phi_2(1)}_{0} + w_2\underbrace{\phi_1(2)}_{0}\phi_2(2) + \underbrace{\phi_1(3)\phi_2(3)}_{0}w_3\right]$$

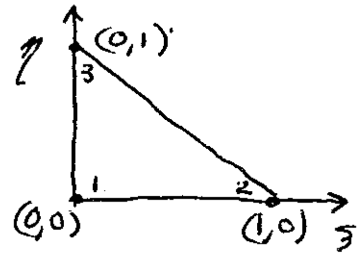
etc

⋮

Diagonal!

Jacobian on Triangle:

"parent" element in ξ, η space:



$$\left.\begin{aligned} \phi_1 &= 1-\xi-\eta \\ \phi_2 &= \xi \\ \phi_3 &= \eta \end{aligned}\right\} \text{check} \Rightarrow \begin{aligned} &\phi_1(2)=0,\ \phi_1(3)=0 \\ &\phi_2(1)=0,\ \phi_2(3)=0 \\ &\phi_3(1)=0,\ \phi_3(2)=0 \end{aligned}$$

$$\sum \phi_i = 1 \ ; \ \sum \frac{\partial \phi_i}{\partial \xi} = 0 \ ; \ \sum \frac{\partial \phi_i}{\partial \eta} = 0$$

$$\frac{\partial \phi_1}{\partial \xi} = -1 \qquad \frac{\partial \phi_1}{\partial \eta} = -1$$

$$\frac{\partial \phi_2}{\partial \xi} = 1 \qquad \frac{\partial \phi_2}{\partial \eta} = 0$$

$$\frac{\partial \phi_3}{\partial \xi} = 0 \qquad \frac{\partial \phi_3}{\partial \eta} = 1$$

$$|J| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial x}{\partial \xi} = x_1 \frac{\partial \phi_1}{\partial \xi} + x_2 \frac{\partial \phi_2}{\partial \xi} + x_3 \frac{\partial \phi_3}{\partial \xi} = x_2 - x_1$$

$$\frac{\partial x}{\partial \eta} = x_1 \frac{\partial \phi_1}{\partial \eta} + x_2 \frac{\partial \phi_2}{\partial \eta} + x_3 \frac{\partial \phi_3}{\partial \eta} = x_3 - x_1$$

same for $\frac{\partial y}{\partial \xi} = y_2 - y_1$; $\frac{\partial y}{\partial \eta} = y_3 - y_1$

Then $|J| = (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)$

$$= (x_2 - x_1)\Delta y_2 + (x_3 - x_1)\Delta y_3$$

$$= x_2 \Delta y_2 + x_3 \Delta y_3 - x_1 y_3 + x_1 y_1 - x_1 y + x_1 y_2$$

$$= x_2 \Delta y_2 + x_3 \Delta y_3 + x_1 \Delta y_1$$

$$= 2A^e$$

Derivative terms remain the same since $\langle \underbrace{\frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x}}_{\text{constant}} \rangle^e$ and $\langle 1 \rangle^e$ w/

nodal quadrature gives exactly the Area of the element (i.e. same result produced by the "regular" quadrature

- What about Bilinear element (deformed)?

  - easiest to call Basis subroutine w/ "Gauss pts"

$$\left.\begin{aligned} (\xi_1, \eta_1) &= (-1,-1) \\ (\xi_2, \eta_2) &= (1,-1) \\ (\xi_3, \eta_3) &= (1,1) \\ (\xi_4, \eta_4) &= (-1,1) \end{aligned}\right\} \text{node positions in } (\xi, \eta)$$

  - Need to determine what the weights will be

  - Must be able to show $\langle 1 \rangle = \iint J \, d\xi \, d\eta = \text{Area}$ exact w/ nodal gauss pts
  - What about derivatives? change?

Hyperbolic PDEs ....

- easiest approach:

replace time derivatives w/ centered differences

e.g. $\frac{\partial^2 u}{\partial t^2} \longrightarrow \frac{u^{k+1} - 2u^k + u^{k-1}}{\Delta t^2}$

$u \longrightarrow \theta\left(\frac{u^{k+1} + u^{k-1}}{2}\right) + (1-\theta)u^k$

- Alternately as integration in time... both result in

$$[M]\left(\{u\}^{k+1} - 2\{u\}^k + \{u\}^{k-1}\right) + \left[\Delta t^2 K\right]\left(\frac{\theta}{2}\{u\}^{k+1} + \frac{\theta}{2}\{u\}^{k-1} + (1-\theta)\{u\}^k\right) = \{R\}^{k+1/2}$$

$$\underbrace{\left[M - \frac{\theta}{2}\Delta t^2 K\right]}_{[A]}\{u\}^{k+1} = \underbrace{\left[2M - \Delta t^2(1-\theta)K\right]}_{[B]}\{u\}^k - \underbrace{\left[M + \frac{\Delta t^2\theta}{2}K\right]}_{[C]}\{u\}^{k-1} + \{R\}^{k+1}$$

so:

$$[A]\{u\}^{k+1} = \underbrace{[B]\{u\}^{k} + [C]\{u\}^{k-1} + \{R\}^{k+1/2}}_{\text{old info... known!}}$$

- Need two-levels of u to proceed

Typically start system at rest

$$u^0 = u^{-1} = 0$$

Formally, must have $\frac{\partial u}{\partial t}(0)$ specified

How to enforce? ... use shadow node approach since are essentially using finite differences in the time dimension

$$\frac{\partial u}{\partial t}(0) = f(x,y) \Rightarrow \frac{u^1 - u^{-1}}{2\Delta t} = f(x,y)$$

$$u^{-1}(x,y) = -2\Delta t\, f(x,y) + u^1(x,y)$$

so $u_i^{-1} = u_i^1 - 2\Delta t f_i \Rightarrow \{u\}^{-1} = \{u\}^1 - 2\Delta t\{f\}$

$$[A - C]\{u\}^1 = [B]\{u\}^0 - 2\Delta t[C]\{f\} + \{R\}^{1/2}$$