# Stability

- Given bounded ICs, BCs + forcing get bounded sol'n to FD equations

- Two views....

  (a) Lax-Richtmeyer : at a fixed time, $T$, sol'n of FD equations remain bounded as $k \to 0$ (assuming $h$ related to $k$ such that $h \to 0$ as $k \to 0$)

  (b) Practical approach : $h, k$ are fixed and sol'n propagated forward from $t=0$ to $t = jk$ ... then stability defined in terms of boundedness as $j \to \infty$ for $k$ fixed

- Two approaches to Stability analysis

  1.) Matrix Methods ... cast FD propagation in form $u^{l+1} = Au^{l} + b^{l}$ and study properties of $A$

  2.) Fourier Method (Von Neumann) ... examine the propagation of Fourier components by the FD molecule

# Fourier Analysis Supplement

- Recall Fourier Series ... valid for any $f(x)$ continuous on $[0, \ell]$

$$f(x) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell}$$

$$= \sum_{n=0}^{\infty} C_n \left( \frac{e^{j\frac{n\pi x}{\ell}} + e^{-j\frac{n\pi x}{\ell}}}{2} \right) + B_n \left( -j \left( \frac{e^{j\frac{n\pi x}{\ell}} - e^{-j\frac{n\pi x}{\ell}}}{2} \right) \right)$$

$$= \sum_{n=-\infty}^{\infty} A_n e^{j\sigma_n x} \quad \text{where} \quad \sigma_n = \frac{n\pi}{\ell} = \frac{2\pi}{L_n}$$

$$\text{wavelength} = \frac{2\ell}{n}$$

$$A_0 = C_0/2$$

$$A_n = \frac{1}{2}(C_n - jB_n) \quad n>0$$

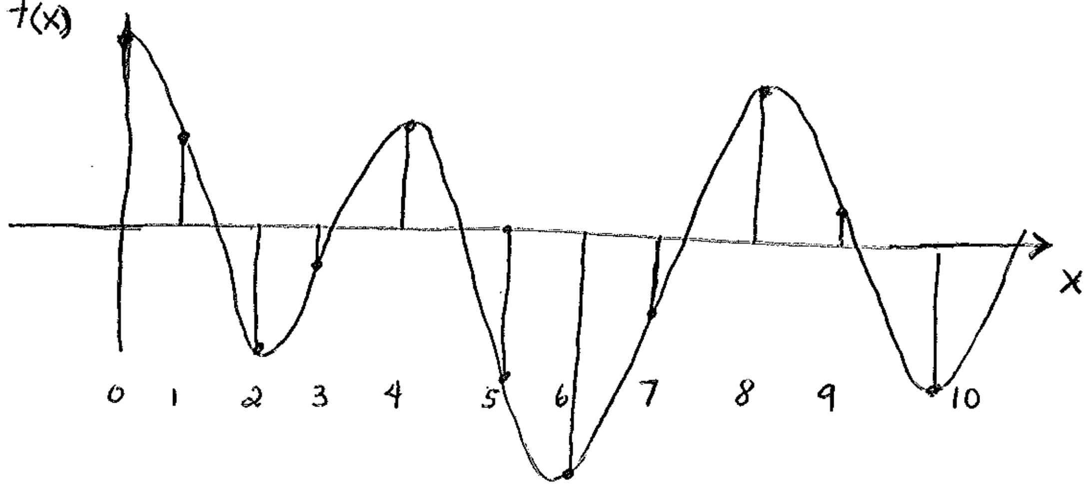
$$A_n = \tfrac{1}{2}(C_{-n} + jB_{-n}) \quad n<0$$

- on a discrete set of sample points

$$x_i = ih$$

$$\Rightarrow f(x_i) = \sum_{n=-\infty}^{\infty} A_n e^{j\sigma_n h i}$$

e.g.




$$f(x_i) = \cos(\pi/6\, i) + \frac{1}{4}\cos(\pi/4\, i) + 2\cos(\pi/2\, i)$$

- Express $u_i^0 = f(x_i)$ as sum of Fourier modes

Stability ... ask do these modes stay bounded as $u_i^0 \to u_i^1 \to u_i^2 \ldots u_i^{\ell} \to u_i^{\ell+1}$ ???

- Must examine all possible $\sigma h$ values !!

Why ? Since may not need them all to represent specific ICs !

Answer : IF not then Stability dependent on ICs ... i.e. problem dependent for same governing equation ... not very useful

- More importantly: Rounding errors introduced are differenced by same molecule as $U_i^l$

e.g. $U_i^l + \epsilon_i^l$ = Computer sol'n to FD equations

$\uparrow$ Exact sol'n to FD equations

Euler Explicit:

$$U_i^{l+1} + \epsilon_i^{l+1} = r\left(U_{i-1}^l + \epsilon_{i-1}^l\right) - (2r-1)\left(U_i^l + \epsilon_i^l\right) + r\left(U_{i+1}^l + \epsilon_i^l\right)$$

$$U_i^{l+1} = r\,U_{i-1}^l - (2r-1)U_i^l + r\,U_{i+1}^l$$

$$\epsilon_i^{l+1} = r\,\epsilon_{i-1}^l - (2r-1)\,\epsilon + r\,\epsilon_{i+1}^l$$

- Need to make sure Rounding errors once introduced remain bounded

Recall... Fourier Series is an approach to analytic sol'n of PDEs

e.g. $$\frac{\partial U}{\partial t} - D\frac{\partial^2 U}{\partial x^2} = 0 \quad ; \text{ w/ } \quad \begin{aligned} U(x,0) &= G(x) \\ U(0,t) &= f(t) \\ U(l,t) &= g(t) \end{aligned}$$

Substitute in Fourier Series w/ time-dependent coefficients: $u(x,t) = \sum_{n=-\infty}^{\infty} A_n(t) e^{j\sigma_n x}$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \underbrace{\left[ \frac{dA_n}{dt} + D\sigma_n^2 A_n(t) \right]} e^{j\sigma_n x} = 0$$

Only way to satisfy requires this to vanish

$$\frac{dA_n}{dt} + D\sigma_n^2 A_n = 0 \qquad \text{1ST Order ODE in } t$$

We know sol'n is $A_n(t) = C_n e^{-D\sigma_n^2 t}$

$$\therefore \ u(x,t) = \sum_{n=-\infty}^{\infty} C_n e^{-D\sigma_n^2 t} e^{j\sigma_n x}$$

Deteemine from BCs

Deteemine from ICs

- Do same thing for Discrete System (i.e. Difference equations)

# Von Neumann (Fourier) Stability Analysis

- Idea... expand the spatial distribution of IC's (i.e. sol'n at some point in time) as Fourier Series

$$U_i^0 = \sum_n A_n e^{j\sigma_n x_i}$$

Have $U(x,0)$; need to find $A_n$'s such that $U(x_i,0) = U_i^0$

- Examine how each term in sum is propagated as $l = 1, 2, \ldots$ (in general $t_l$ to $t_{l+1}$) by FD molecule

- Stability... FD molecule must not allow any term in sum (i.e. Fourier mode) to grow as sol'n is advanced in time

- Sufficient to look at general form of single term and consider all possible $\sigma$ values

  - single term due to linearity
  - all $\sigma$ values due to Round-off
  - Don't care about $A_n$'s $\Rightarrow$ Want $\dfrac{U_i^{l+1}}{U_i^l}$

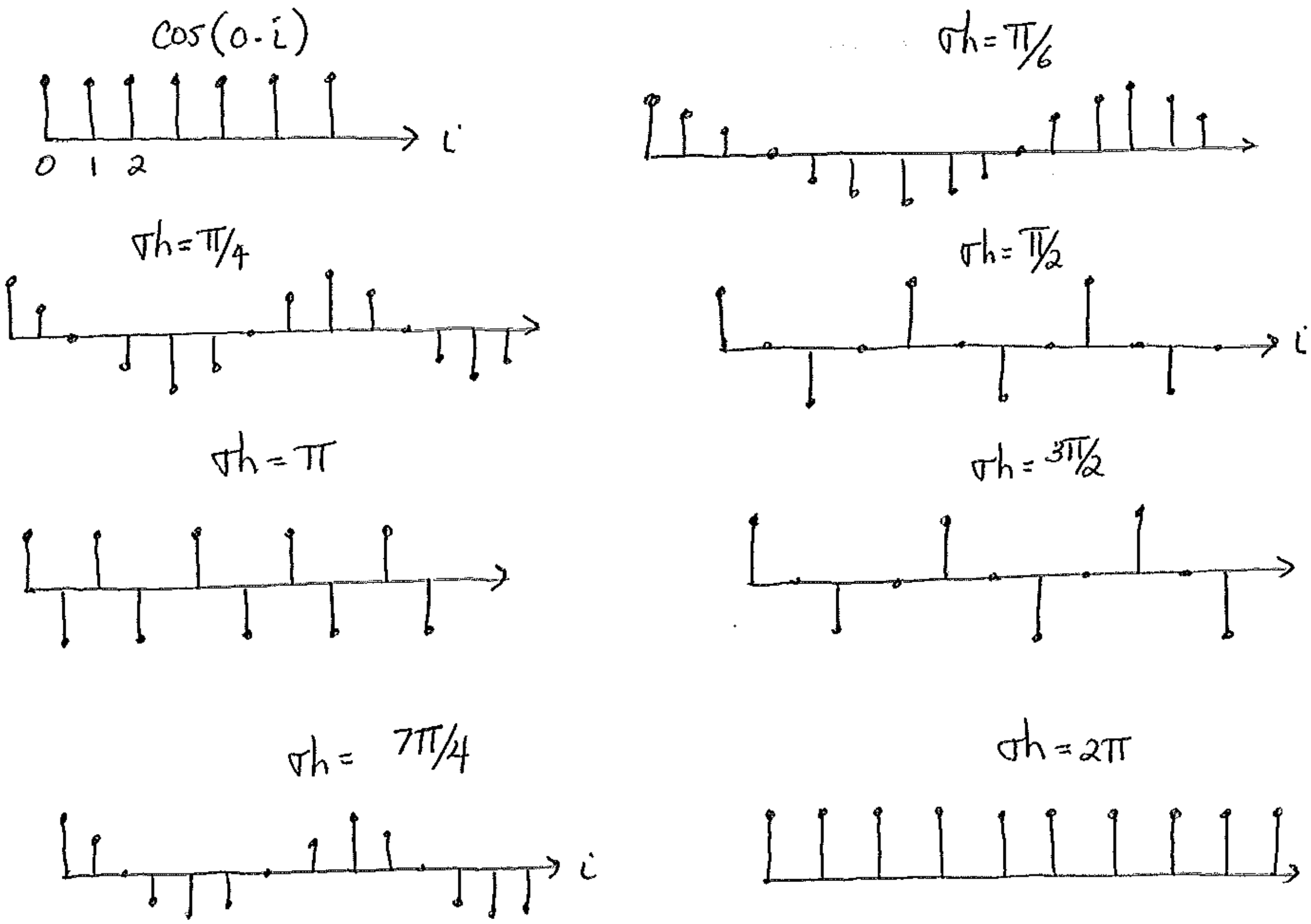
- $\sigma h$ is key quantity $\Rightarrow$ "Dimensionless wavenumber"

  $$u_i = e^{j\sigma x_i} = e^{j\sigma h i}$$

- $0 \leq \sigma h \leq \pi$ ... most rapid variation on a mesh is node-to-node oscillation

  - As $\sigma h$ increases from zero, $e^{j\sigma h i}$ has increasing rate of oscillation which peaks at $\sigma h = \pi$

e.g. $\cos(\sigma h i)$



- Define "Amplification factor" of the FD eqns

$$U_i^{l+1} = U_i^l \gamma_o$$

(Analytically we know $\underbrace{\gamma \equiv e^{\alpha \Delta t}}$)

$$U(x,t) = e^{\alpha t} e^{j\sigma x} \Rightarrow \frac{U(x,t+\Delta t)}{U(x,t)} = e^{\alpha \Delta t}$$

- Relate all (space, time) points in the FD molecule to pt $(i, l)$ using the defining relations

e.g. $U_i^{l+1} = \gamma_o U_i^l$

$$U_{i-1}^l = e^{-j\sigma h} U_i^l \Rightarrow e^{j\sigma(x_i - h)} = e^{-j\sigma h} e^{j\sigma x_i} = e^{-j\sigma h} U_i$$

$$U_{i+1}^{l+1} = e^{j\sigma h} \gamma_o U_i^l$$

etc

⋮

- Provides a relationship between $\gamma_o$ and $\sigma h$

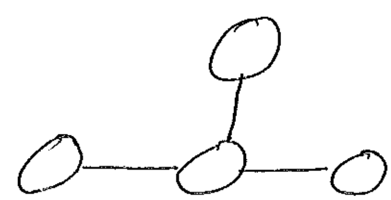
- Stability requires $-1 \le \gamma_o \le 1$ for all possible $\sigma h$ (i.e. $0 \le \sigma h \le \pi$)

- Bounded oscillations develop for $-1 \le \gamma_o < 0$

- Formally method only valid for
  - linear equations w/ constant coefficients
  - Uniform mesh
  - BCs at infinity

- Generally get same results as Matrix method (i.e. BCs effect stability in minor way relative to FD equations themselves)

e.g. Examine Euler Explicit



$$u_i^{\ell+1} - u_i^{\ell} = r\delta_x^2 u_i^{\ell}$$

$$= r\left(u_{i-1}^{\ell} - 2u_i^{\ell} + u_{i+1}^{\ell}\right)$$

$$\Rightarrow \quad (\gamma_0 - 1)u_i^{\ell} = r\left(e^{-j\sigma h} - 2 + e^{j\sigma h}\right)u_i^{\ell}$$

$$\gamma_0 = 1 - 2r(1-\cos\sigma h) \Rightarrow \text{Note: } \cos\sigma h = 1 - 2\sin^2\frac{\sigma h}{2}$$

$$\gamma_0 = 1 - 4r\sin^2\frac{\sigma h}{2}$$

$$\Rightarrow \text{For stability} \ldots |\gamma_0| < 1 \Rightarrow \underbrace{-1 \le 1 - 2r(1-\cos\sigma h) \le 1}_{\text{For } \underline{\underline{\text{all}}} \text{ possible } \sigma\text{'s}}$$

But $\quad 0 < \sigma h \le \pi \Rightarrow 0 < 1 - \cos\sigma h \le 2$

- Conclude $\gamma_0 < 1$

$\gamma_0$ can be negative $\Rightarrow$ Oscillation in $t$!

Negative when: $1 - 2r(1-\cos\sigma h) < 0$

$$2r(1-\cos\sigma h) - 1 > 0$$

$$\text{i.e. } r > \frac{1}{2(1-\cos\sigma h)}$$

- Conclude: $r > 1/4$ produces $\gamma_0 < 0$

$\Rightarrow$ shortest waves (i.e. highest frequency modes) oscillate ..... entirely a numerical artefact

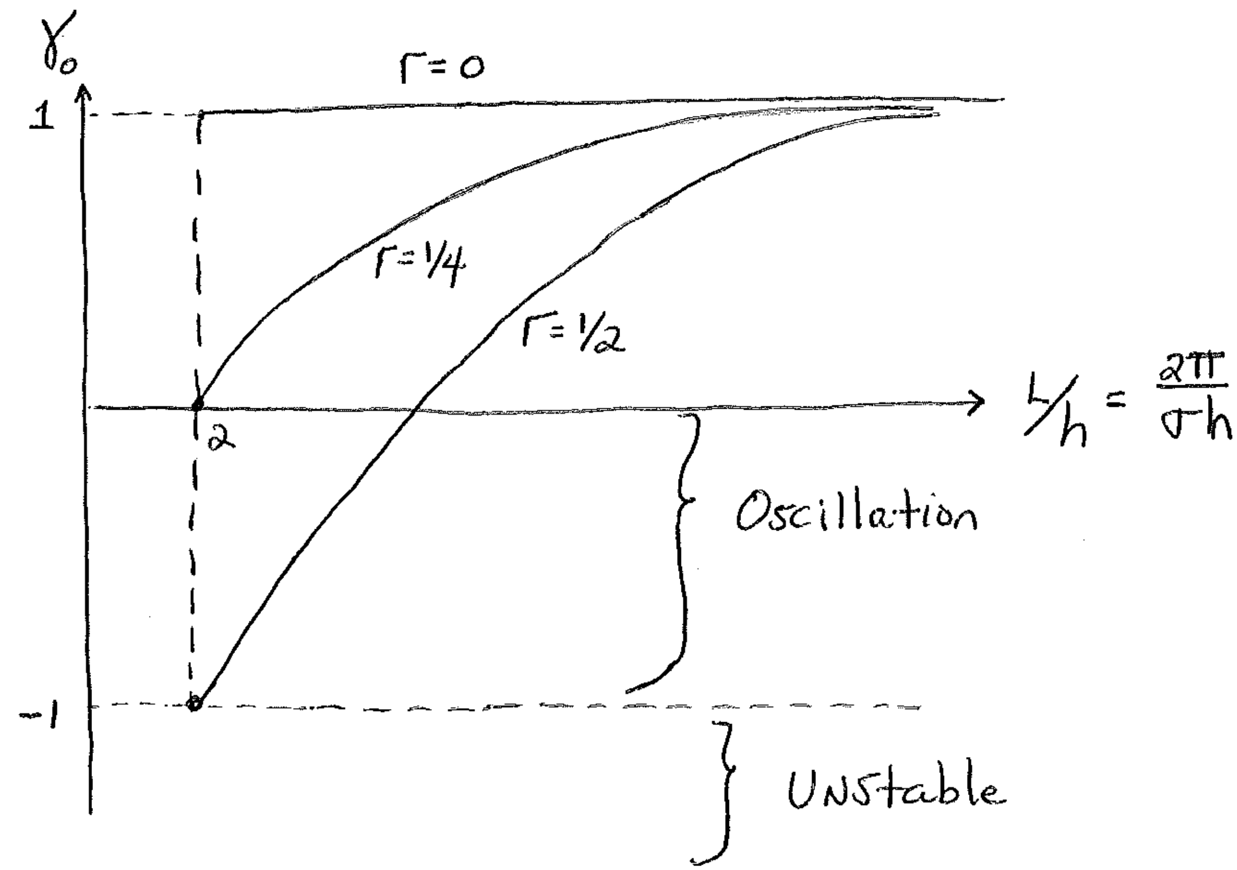
- Unstable when $\gamma_0 < -1$

$$1 - 2r(1-\cos\sigma h) < -1$$

$$2 < 2r(1-\cos\sigma h)$$

$$\frac{1}{1-\cos\sigma h} < r$$

i.e. $r > 1/2$ (Shortest waves have unstable oscillation)




Rule of Thumb... Short wavelengths are first to go

- Develop spurious oscillations
- Oscillations become fatal as k increases

What about accuracy? Can study

$$\frac{\text{Numerical aplification factor}}{\text{Analytical amplification factor}} \Rightarrow \frac{\gamma_0}{\gamma}$$