Introduction (Lynch Chap. 1)

- Definitions:

- ODE - Diff Eqn w/ one independent variable
- PDE - Diff Eqn w/ more than one independent variable
- Order - highest derivative that appears
- Degree - power of highest derivative

e.g. $$\frac{d^2u}{dt^2} + u\frac{du}{dt} - \left(\frac{du}{dt}\right)^3 = t^3$$ ODE, 2nd order, 1st degree

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = f(x,t)$$ PDE, 1st order, 1st degree

- Homogeneous Diff Eqn - no term involving only independent variables (or a constant)

e.g. $$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0$$ PDE, 2nd Order, 1st degree, homogeneous

otherwise Inhomogeneous

$$\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial t} + xt^3u = xe^{-t}$$

- Linear – dependent variable and derivatives (thereof) appear only to 1ST (or zero) power <u>and</u> no products of dependent variable (and its derivatives)

- Quasilinear – highest order derivative appears linearly, i.e. must be degree 1. includes all 1ST degree nonlinearities.

- Nonlinear – anything else

e.g. $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 0$

$$x^3 \frac{\partial^2 u}{\partial x \partial t} + e^{-t} \frac{\partial u}{\partial x} + u = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

- IVP – ODE where conditions imposed at a <u>single</u> value of independent variable

- BVP – ODE where conditions imposed at <u>more than one</u> value of independent variable

Ex:

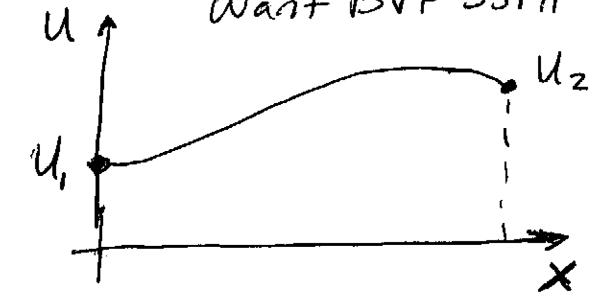
Consider the ODE

$$\frac{d^2 u}{dx^2} = f(x)$$

- Need 2 conditions
- u and/or $\frac{du}{dx}$ possible
- IVP: u, $\frac{du}{dx}$ at same point
- BVP: one condition at two different points

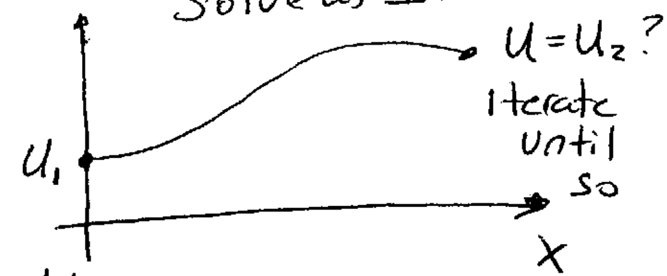
Equivalence between IVP and BVP $\Rightarrow$ Shooting

Want BVP Sol'n



$\frac{du_1}{dx}$ unknown

Solve as IVP



$\frac{du_1}{dx}$ guess

- PDEs may be either ... can be IVP in one independent variable (e.g. in time) and BVP in another (e.g. in space)

No shooting analog for PDEs
- in time always IVP (can't constrain the future)
- in space (multi-D) can't go from 1 pt to many pts.

- Types of Systems

"Distributed" (or Continuous) = PDE

"Lumped": Continuous in time, discrete in space (i.e. algebraic approx to continuum in space)
= Coupled ODEs

"Discrete": Continuum converted to algebraic in both space + time

Classification of BCs

Type I = u given "Dirichlet"

Type II = $\frac{\partial u}{\partial n}$ given "Neumann"
Why n? IF Type I $\frac{\partial u}{\partial t}$ known

Type III $au + b\frac{\partial u}{\partial n}$ given "Mixed," "Robbins"

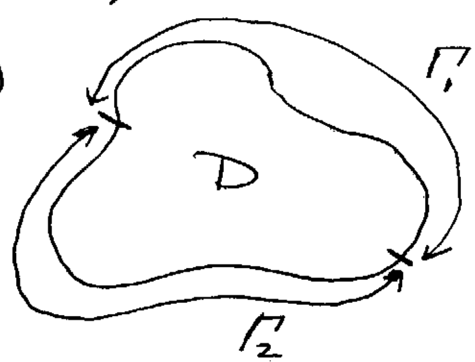
Uniqueness Proof: Poisson Eqn ... $\nabla^2 u = f$

- Consider 2 sol'ns (u_1, u_2)

$\nabla^2 u_1 = f$ on D $\nabla^2 u_2 = f$

$u_1 = a$ on Γ_1 $u_2 = a$

$\frac{\partial u_1}{\partial n} = b$ on Γ_2 $\frac{\partial u_2}{\partial n} = b$



Let $u_3 = u_2 - u_1$, then
$$\nabla^2 u_3 = 0 \quad \text{on } D$$
$$u_3 = 0 \quad \text{on } \Gamma_1$$
$$\frac{\partial u_3}{\partial n} = 0 \quad \text{on } \Gamma_2$$

so $\oint u_3 \nabla u_3 \cdot \hat{n}\, ds = 0 = \iint \nabla \cdot (u_3 \nabla u_3)\, dA$

(by Divergence Theorem)

$$0 = \iint \nabla \cdot (u_3 \nabla u_3)\, dA \underset{\text{chain rule}}{=} \iint \underbrace{\nabla u_3 \cdot \nabla u_3}_{\text{always} \geq 0}\, dA + \iint u_3 \overset{0}{\cancel{\nabla^2 u_3}}\, dA$$

$\therefore$ $\nabla u_3 = 0$ everywhere on D

so u_3 = constant on D

= 0 if Γ_1 exists (i.e. some Type I BCs on boundary)

(if only Γ_2, then sol'n unique up to a constant

Conclude: need Type I or Type II for uniqueness

Same holds for Type III

So need 1 (<u>and only 1</u>) BC from the Type I, II or III selection on each (and all) points of a closed boundary to get unique sol'n.

Classification of PDEs

- Theory of characteristics ... (see Lapidus & Pinder, chapter 1 for detailed discussion)

- PDE becomes ODE along "characteristic curve" so only need ICs to propagate along a characteristic But

- Sol'n can't be found anywhere else ... i.e. can't find sol'n away from or "off" the characteristic curve ... don't want to specify BCs along a characteristic but on curves which cut across characteristics.

- For 2nd order PDEs (most important in Engineering/Physics)

$$AU_{xx} + BU_{xt} + CU_{tt} + DU_t + EU_x = R$$

$\dfrac{\partial^2 U}{\partial x^2}$

General form w/ A, B, C, D, E functions of x, t, U_x, U_t, U and R function of x, t, U (Quasilinear!)

Can show if

$B^2 - 4AC > 0$ Hyperbolic Eqn

$B^2 - 4AC = 0$ Parabolic Eqn

$B^2 - 4AC < 0$ Elliptic Eqn

(Recall from analytic geometry the quadratic form. $Ax^2 + Bxt + Ct^2 = Dx + Et + R$
describes hyperbola if $B^2 - 4AC > 0$
parabola if $B^2 - 4AC = 0$
ellipse if $B^2 - AC < 0$)

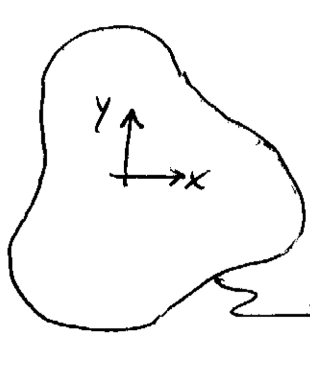
<u>Classical Examples:</u>

- $\nabla^2 U = 0$, $\nabla^2 U = f$ Laplace/Poisson Eqn

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \quad A = 1,\ B = 0,\ C = 1,$$

$$B^2 - 4AC < 0 \quad \text{Elliptic}$$

Elliptic Eqn – equilibrium problems, no time dependence. All space dimensions equivalent in 2nd derivatives. Change in boundary data "instantly" alters soln in entire domain



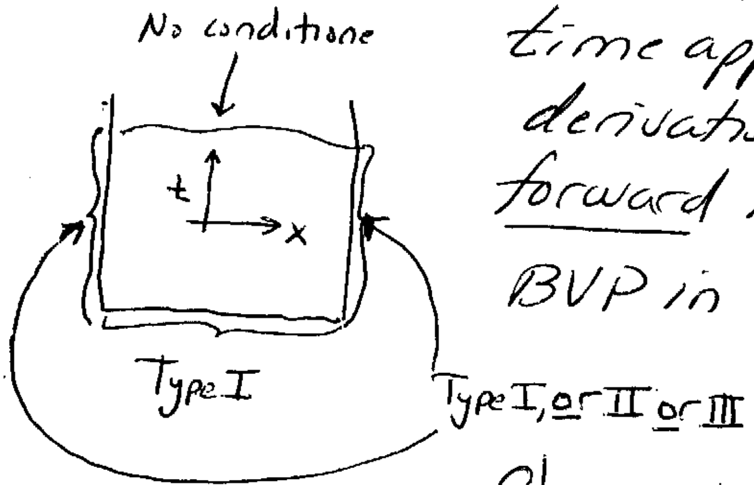
$\rightarrow$ must specify Type I, II or III over a closed boundary.

- $\nabla^2 U - \frac{\partial U}{\partial t} = 0$; Diffusion Equation

  $A=1,\ B=0,\ C=0,\ D=1$

  $B^2 - 4AC = 0$ Parabolic

Parabolic Eqn – propagation problems, time appears (usually) w/ 1ST derivative. Sol'n progresses forward in time (not backwards). BVP in space, IVP in time



(space same as Elliptic)

Change in BC data effects interior solution later in time

- $\nabla^2 U - \frac{\partial^2 U}{\partial t^2} = 0$ Wave Equation

  $A=1,\ B=0,\ C=-1$

  $B^2 - 4AC > 0$ Hyperbolic

Hyperbolic Eqn – like parabolic but 2nd derivative in time. Space same as elliptic, BVP in Space, IVP in time but need 2 conditions, Type I and Type II