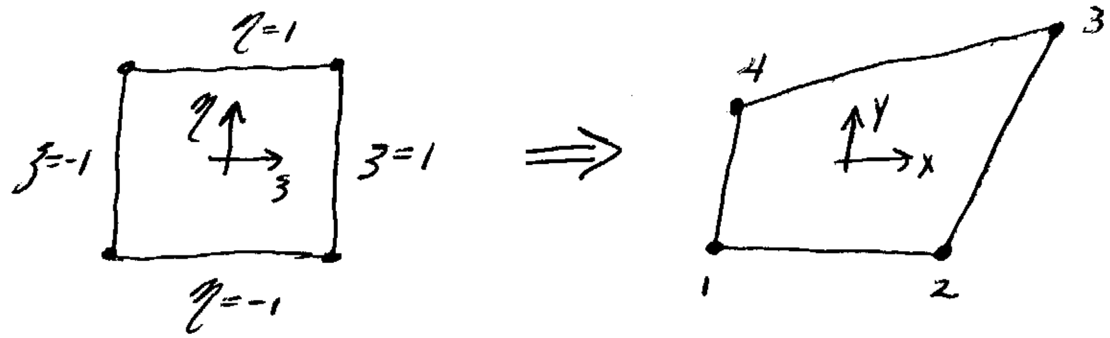
Deformed Bilinear Element



Map normalized element ("parent element") onto the deformed quadrilateral

$$x = x(\xi,\eta) = \sum_{i=1}^{4} x_i \phi_i(\xi,\eta)$$
$$y = y(\xi,\eta) = \sum_{i=1}^{4} y_i \phi_i(\xi,\eta)$$

"Shape functions" define "shape" of the quadrilateral

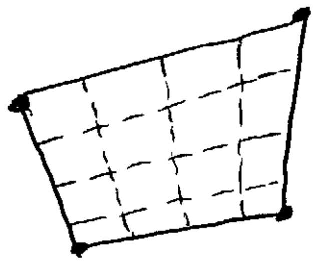
"Isoparametric" (geometry same as unknown); "Subparametric" (geometry less than unknown), "Superparametric" (geometry greater than unknown)

- Corners check
- along sides; e.g. top side ... $\eta=1$ constant

$$\phi_i = A_i + B_i \xi \qquad \left(\text{since } \phi_i(\xi,\eta) = \frac{(1+\xi_i\xi)(1+\eta_i\eta)}{4}\right)$$

$$\left.\begin{aligned} \frac{dx}{d\xi} &= \sum x_i \frac{d\phi_i}{d\xi} = \text{constant} \\ \frac{dy}{d\xi} &= \sum y_i \frac{d\phi_i}{d\xi} = \text{constant} \end{aligned}\right\} \frac{dx}{dy} = \text{constant}$$

- along any $\left.\begin{array}{l}\eta = \text{constant} \\ \xi = \text{constant}\end{array}\right\} \frac{dx}{dy}$ constant



Lines of constant ξ, η

Derivatives :

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{[J]\ \text{Jacobian Matrix}} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}$$

Jacobian of the Transformation is Det(J)

$$|J| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}$$

when positive, Transformation can be inverted (in principle) $\Rightarrow$ $\begin{array}{l}\xi = \xi(x,y) \\ \eta = \eta(x,y)\end{array}$ Usually not needed so . . .

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

$$\frac{\partial f}{\partial x} = \frac{1}{|J|} \left[\frac{\partial y}{\partial \eta} \frac{\partial f}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial f}{\partial \eta} \right]$$

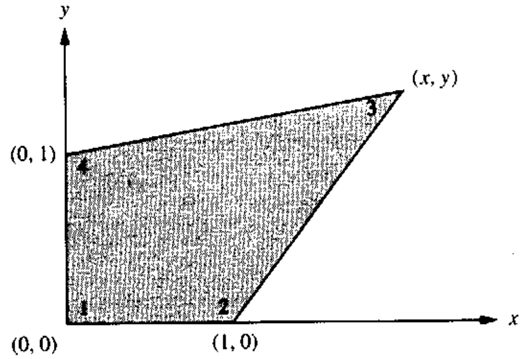
$$\frac{\partial f}{\partial y} = \frac{1}{|J|} \left[-\frac{\partial x}{\partial \eta} \frac{\partial f}{\partial \xi} + \frac{\partial x}{\partial \xi} \frac{\partial f}{\partial \eta} \right]$$

Geometrically... $|J|$ represents relationship between $dA(\xi,\eta) = d\xi\, d\eta$ and $dA(x,y)$ into which it is mapped: $dA(x,y) = |J|\, dA(\xi,\eta)$
$= |J|\, d\xi\, d\eta$

Computing $|J|$: $[J] \Rightarrow \frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \xi} \sum_1^4 x_i \phi_i$
$= \sum x_i \frac{\partial \phi_i}{\partial \xi}$ etc

$$[J] = \begin{bmatrix} \sum x_i \frac{\partial \phi_i}{\partial \xi} & \sum y_i \frac{\partial \phi_i}{\partial \xi} \\ \sum x_i \frac{\partial \phi_i}{\partial \eta} & \sum y_i \frac{\partial \phi_i}{\partial \eta} \end{bmatrix}$$

Consider the quadrilateral element:

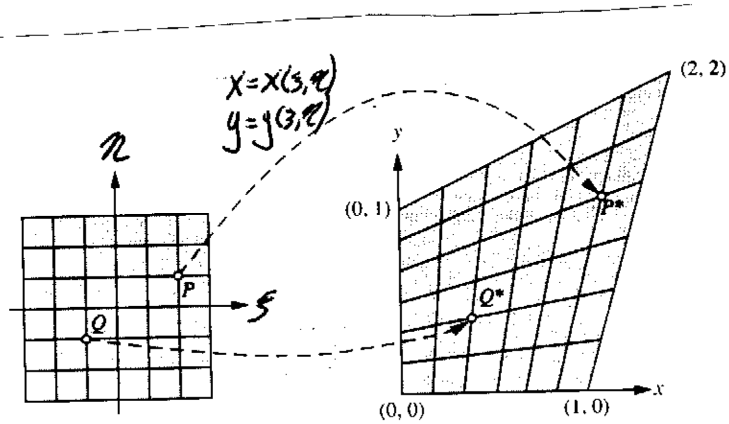


Let x_3, y_3 be variable and study the transformation for some choices of (x_3, y_3)

Pick $x = y = 2$, then one can show...

$$x = \frac{(1+\xi)(3+\eta)}{4} \qquad \left(x = \sum_{i=1}^{4} x_i \phi_i\right)$$

$$y = \frac{(1+\eta)(3+\xi)}{4} \qquad \left(y = \sum_{i=1}^{4} y_i \phi_i\right)$$



Jacobian in this case : $\underbrace{|J|}$ = $\dfrac{4+\xi+\eta}{8}$

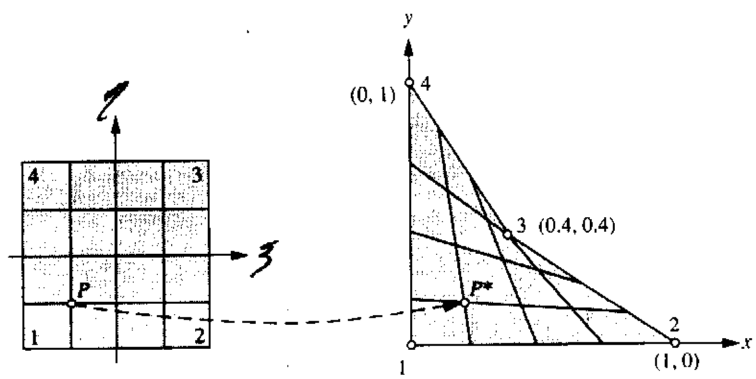
$$\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}$$

Always positive! $\Rightarrow$ 1-to-1 mapping

Now Try $x = y = .4$, then we get

$$x = \frac{(1+\xi)(1.4-.6\eta)}{4}$$

$$y = \frac{(1+\eta)(1.4-.6\xi)}{4}$$



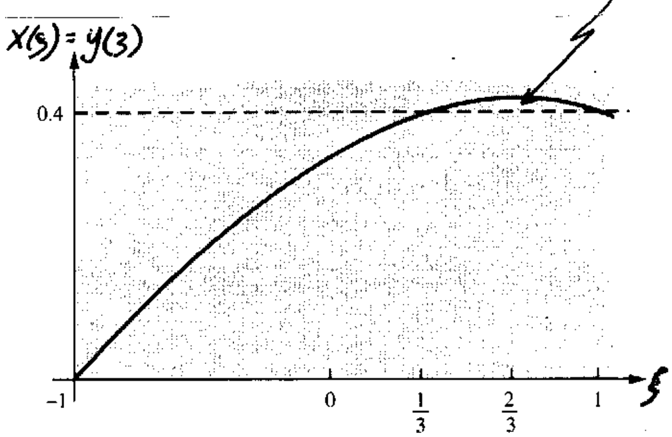
$$|J| = \frac{.8 - .6(\xi+\eta)}{8} \Rightarrow \text{Negative for } \xi+\eta > 4/3$$

Problems with Transformation $\Rightarrow$ Not Unique

e.g. Consider line $\xi = \eta$, then

$$x = y = (1+\xi)(.35 - .15\xi)$$

"outside" the element



So for $(\xi,\eta) = (1/3, 1/3)$ and $(\xi,\eta) = (1,1)$

Both mapped to same point $(x,y) = (.4,.4)$

Don't want interior angles to aproach $0°$ or π

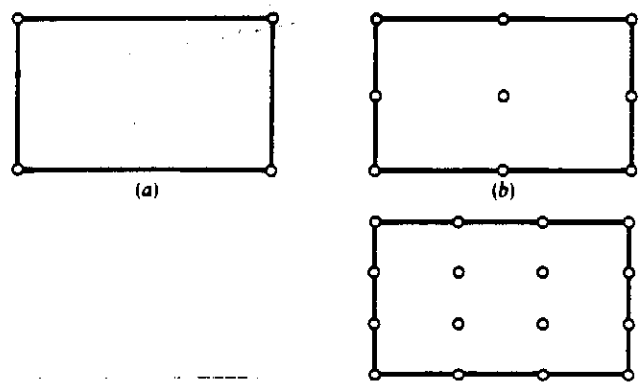
Higher-order Elements

a.) Lagrange Family $\quad \phi_j(\xi,\eta) = \dfrac{\prod_{j\neq i} (\xi-\xi_i)(\eta-\eta_i)}{\prod_{j\neq i} (\xi_j-\xi_i)(\eta_j-\eta_i)}$

"biquadratic"

"bicubic"

etc



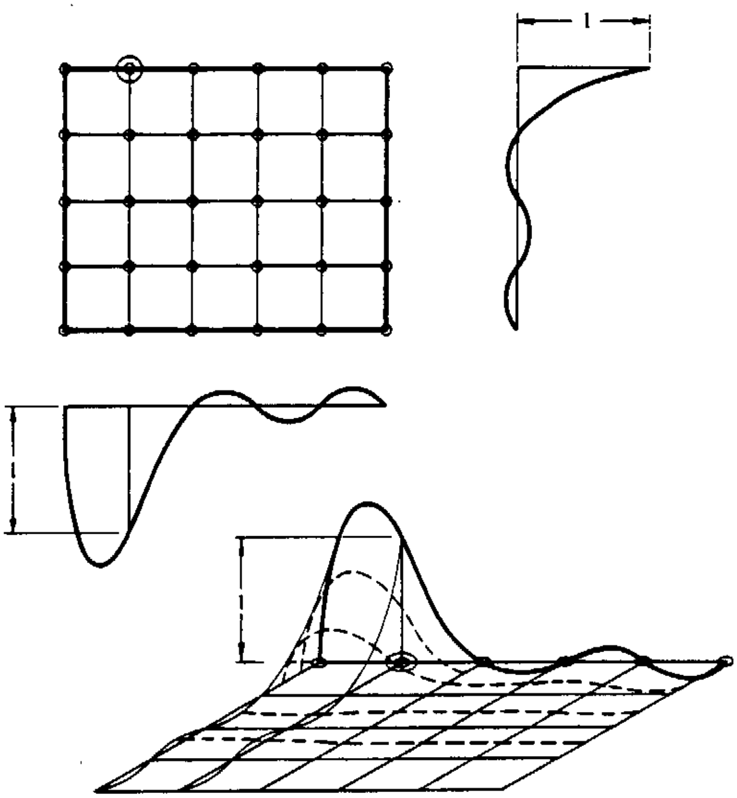
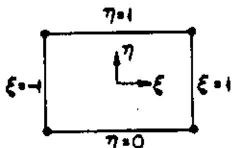
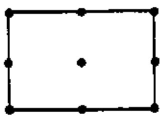
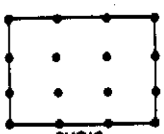
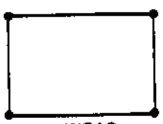


Fig. 7.5 A typical shape function for a Lagrangian element

b.) Serendipity Family: No interior nodes

Table 4.1

Elements of the Lagrangian and Serendipity Families

Elements	Coordinate basis functions (let $\xi_0 = \xi\xi_i$, $\eta_0 = \eta\eta_i$)
	Lagrangian elements
 LINEAR	All Nodes $\Omega_i^e = \frac{1}{4}(1+\xi_0)(1+\eta_0)$
 QUADRATIC	Corner Nodes $\Omega_i^e = [\frac{1}{2}\xi_0(1+\xi_0)][\frac{1}{2}\eta_0(1+\eta_0)]$ Midside Nodes $\xi_i = 0,\quad \Omega_i^e = (1-\xi^2)[\frac{1}{2}\eta_0(1+\eta_0)]$ $\eta_i = 0,\quad \Omega_i^e = (1-\eta^2)[\frac{1}{2}\xi_0(1+\xi_0)]$ Midelement Nodes $\xi_i = 0,\quad \eta_i = 0,\quad \Omega_i^e = (1-\xi^2)(1-\eta^2)$
 CUBIC	Corner Nodes $\Omega_i^e = [\frac{1}{16}(1+\xi_0)(9\xi^2-1)][\frac{1}{16}(1+\eta_0)(9\eta^2-1)]$ Midside Nodes $\xi_i = \pm 1,\quad \eta_i = \pm\frac{1}{3},\quad \Omega_i^e = [\frac{1}{16}(1+\xi_0)(9\xi^2-1)][\frac{9}{16}(1-\eta^2)(9\eta_0+1)]$ $\xi_i = \pm\frac{1}{3},\quad \eta_i = \pm 1,\quad \Omega_i^e = [\frac{9}{16}(1-\xi^2)(9\xi_0+1)][\frac{1}{16}(1+\eta_0)(9\eta^2-1)]$ Midelement Nodes $\eta_i = \pm\frac{1}{3},\quad \xi_i = \pm\frac{1}{3},\quad \Omega_i^e = [\frac{9}{16}(1-\xi^2)(9\xi_0+1)][\frac{9}{16}(1-\eta^2)(9\eta_0+1)]$
	Serendipity elements
 LINEAR	All Nodes $\Omega_i^e = \frac{1}{4}(1+\xi_0)(1+\eta_0)$
 QUADRATIC	Corner Nodes $\Omega_i^e = \frac{1}{4}(1+\xi_0)(1+\eta_0)(\xi_0+\eta_0-1)$ Midside Nodes $\xi_i = 0,\quad \Omega_i^e = \frac{1}{2}(1-\xi^2)(1+\eta_0)$ $\eta_i = 0,\quad \Omega_i^e = \frac{1}{2}(1+\xi_0)(1-\eta^2)$
 CUBIC	Corner Nodes $\Omega_i^e = \frac{1}{32}(1+\xi_0)(1+\eta_0)[-10+9(\xi^2+\eta^2)]$ Midside Nodes $\xi_i = \pm 1,\quad \eta_i = \pm\frac{1}{3},\quad \Omega_i^e = \frac{9}{32}(1+\xi_0)(1-\eta^2)(1+9\eta_0)$ $\xi_i = \pm\frac{1}{3},\quad \eta_i = \pm 1,\quad \Omega_i^e = \frac{9}{32}(1+\eta_0)(1-\xi^2)(1+9\xi_0)$

Isoparametric ⇒ Curved-Sided

$$\left.\begin{array}{l} x = \sum x_i \phi_i(\xi,\eta) \\ y = \sum y_i \phi_i(\xi,\eta) \end{array}\right\} \text{mapping}$$

e.g. Quadratic Element: x = quadratic function of ξ or η

y = ''

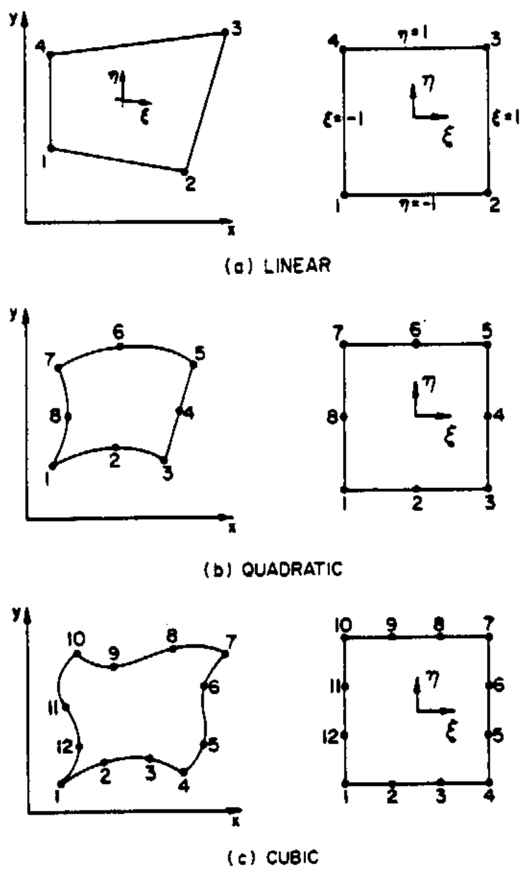
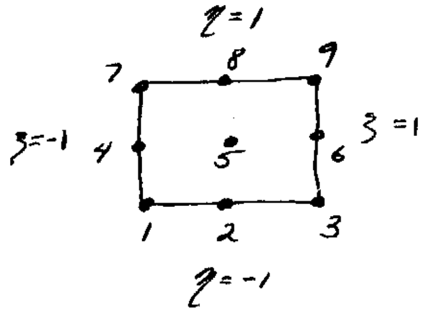


Fig. 4.13. (a) Linear, (b) quadratic and (c) cubic isoparametric elements in local ar coordinates.

e.g. Bi-quadratic element: 9 Nodes/element

Lagrange Family:

In x-y $\Rightarrow$ $\phi_i(x,y) = a_i + b_i x + c_i y + d_i xy + e_i x^2 + f_i y^2 + g_i x^2 y + h_i y^2 x + p_i x^2 y^2$



In ξ-η $\Rightarrow$ Corner node (lower left)

$$\frac{(1-\xi)(1-\eta)\xi\eta}{4} = \quad \phi_1(\xi,\eta) = \frac{(\xi-1)\xi\eta(\eta-1)}{4}$$

Midside: $\Rightarrow$ $\phi_2(\xi,\eta) = -\dfrac{(\xi-1)(\xi+1)\eta(\eta-1)}{2} = -\dfrac{(1-\xi)(1+\xi)\eta(1-\eta)}{2}$

Center: $\Rightarrow$ $\phi_5(\xi,\eta) = (\xi-1)(\xi+1)(\eta-1)(\eta+1)$

etc

⋮

Need $\dfrac{\partial\phi_i}{\partial\xi}, \dfrac{\partial\phi_i}{\partial\xi}$ $\Rightarrow$ easy differentiation

Comput

$$\frac{\partial\phi_i}{\partial x} = \frac{1}{|J|}\left[\frac{\partial\phi_i}{\partial\xi}\frac{\partial y}{\partial\eta} - \frac{\partial\phi_i}{\partial\eta}\frac{\partial y}{\partial\xi}\right]$$

$$\frac{\partial\phi_i}{\partial y} = \frac{1}{|J|}\left[-\frac{\partial\phi_i}{\partial\xi}\frac{\partial x}{\partial\eta} + \frac{\partial\phi_i}{\partial\eta}\frac{\partial x}{\partial\xi}\right]$$

$$|J| = \frac{\partial x}{\partial\xi}\frac{\partial y}{\partial\eta} - \frac{\partial y}{\partial\xi}\frac{\partial x}{\partial\eta}$$

All have same general form as bilinear element

Change is in mapping

$$x(\xi,\eta) = \sum_{i=1}^{9} x_i \phi_i(\xi,\eta)$$

$$y(\xi,\eta) = \sum_{i=1}^{9} y_i \phi_i(\xi,\eta)$$