Analytically we have $\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$

w/ $u = e^{\alpha t} e^{j\sigma x} \Rightarrow \alpha = -D\sigma^2$

- all modes decay (i.e. all $\alpha < 0$)
- longest waves decay slowest
- sol'n gets smoother over time

Now $\gamma = e^{\alpha \Delta t} = e^{-D\sigma^2 k} = e^{-r(\sigma h)^2}$

- but as $k \to 0$, $\gamma_0 \to \gamma \to 1$ don't learn much

Common to introduce characteristic time, T

and examine $\left(\frac{\gamma_0}{\gamma}\right)^N$ where $N = \frac{T}{k}$

(i.e. # time-steps to advance sol'n by T)

- Use time constant of σ in analytic sol'n

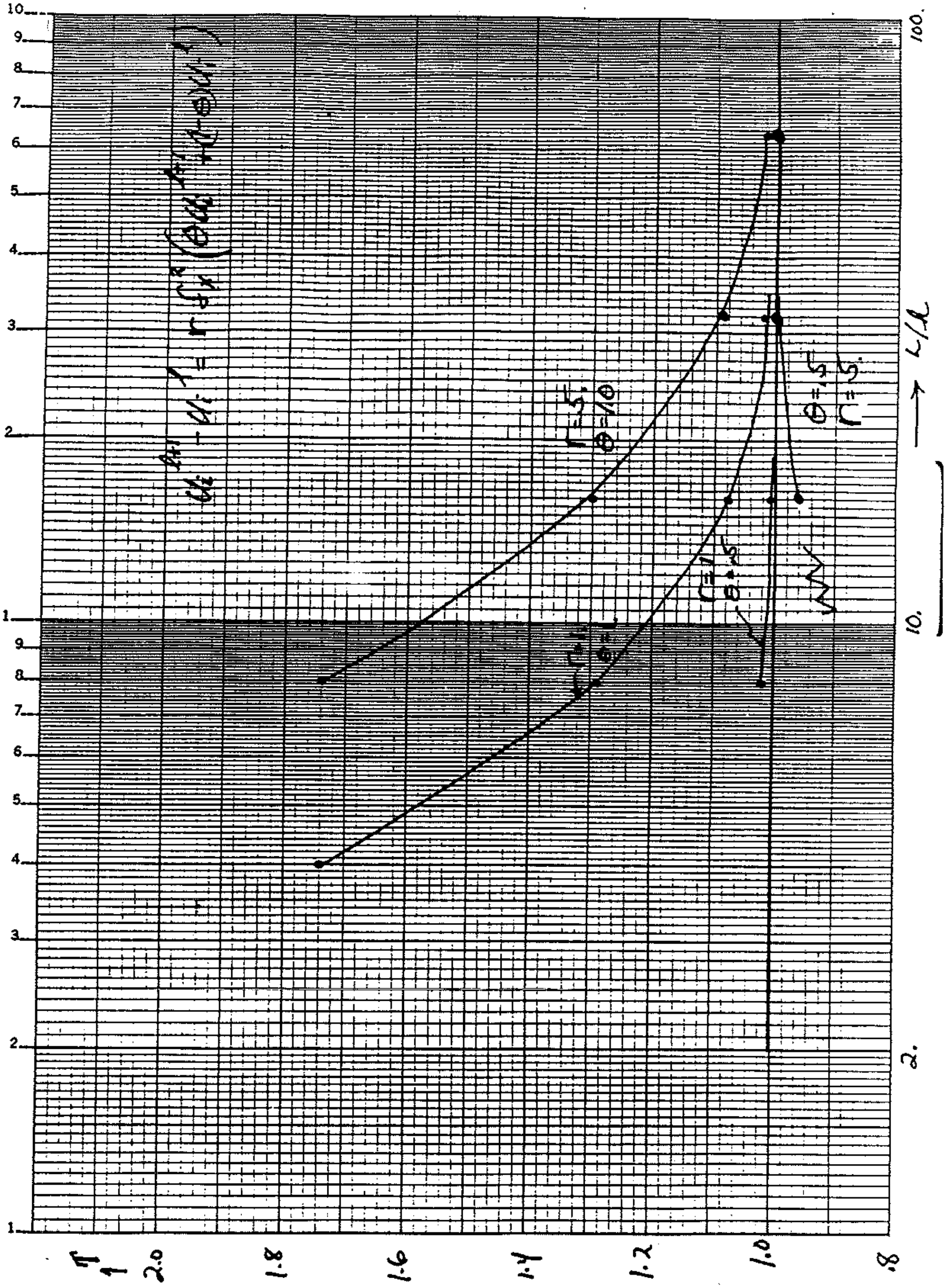
i.e. $T = \frac{1}{|\alpha|} = \frac{1}{D\sigma^2}$, $\therefore N = \frac{1}{D\sigma^2 k} = \frac{1}{r(\sigma h)^2}$

Define $\mathcal{T} = \left(\frac{\gamma_0}{\gamma}\right)^N = \frac{\gamma_0^{\frac{1}{r(\sigma h)^2}}}{\left(e^{-r(\sigma h)^2}\right)^{1/r(\sigma h)^2}} = \frac{\gamma_0^{1/r(\sigma h)^2}}{e^{-1}}$

"Propagation Factor"

Plot T vs $\sigma h = \frac{2\pi h}{L}$ for various r

$T = 1$ is perfect



Also note...

$$\gamma = e^{-r(\sigma h)^2} = 1 - r(\sigma h)^2 + \frac{(r(\sigma h)^2)^2}{2} - \frac{(r(\sigma h)^2)^3}{3!} + \dots$$

$$\gamma_0 = 1 - 2r(1 - \cos\sigma h)$$

$$= 1 - 2r\left[\frac{(\sigma h)^2}{2!} - \frac{(\sigma h)^4}{4!} + \frac{(\sigma h)^6}{6!} - \dots\right]$$

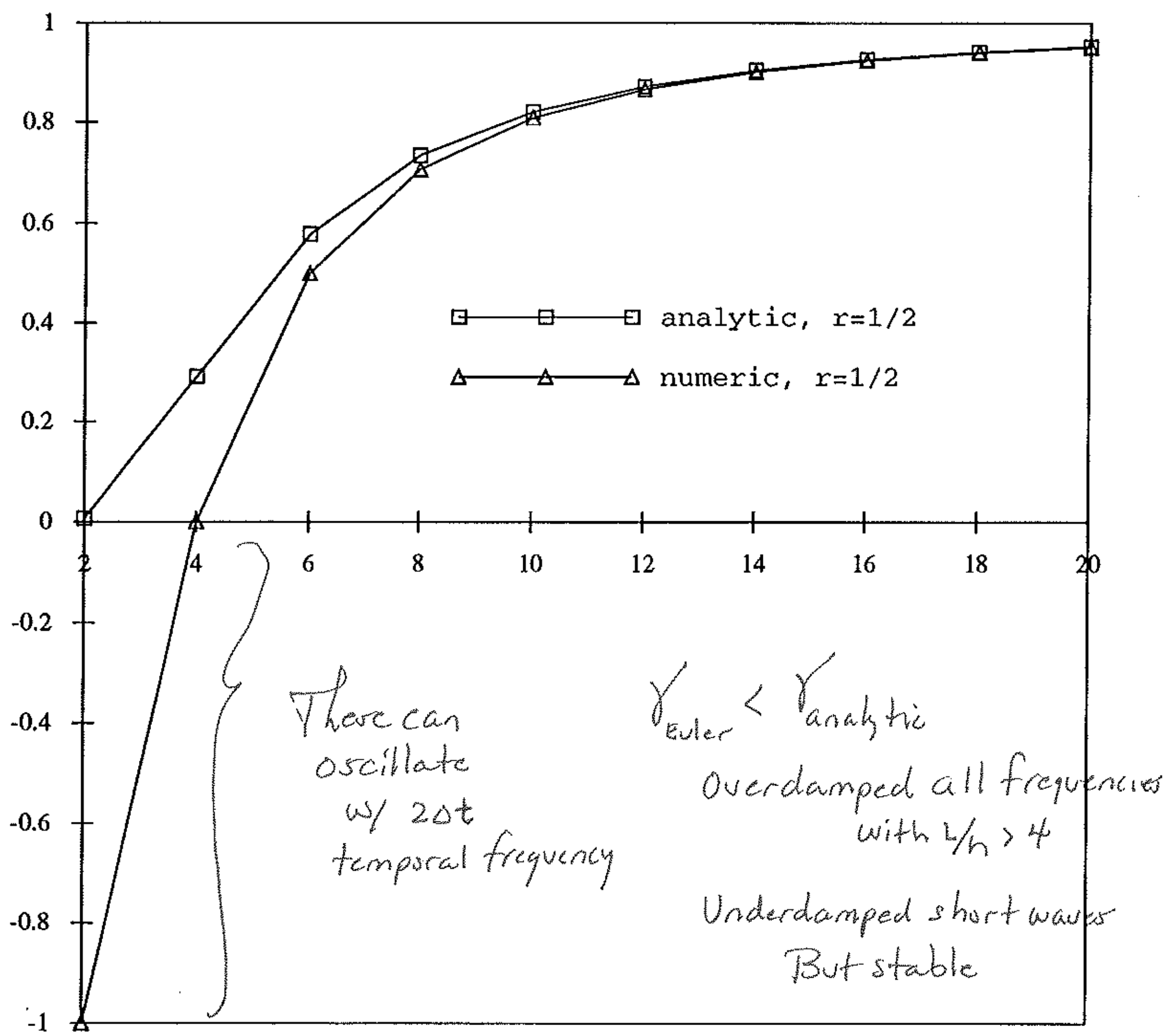
$$= 1 - r(\sigma h)^2 + \underbrace{\frac{r(\sigma h)^4}{12}}_{\text{leading error term}} - \frac{2r(\sigma h)^6}{6!} + \dots$$

But $\frac{r(\sigma h)^4}{12} = \frac{r^2(\sigma h)^4}{2}$ when $r = 1/6$

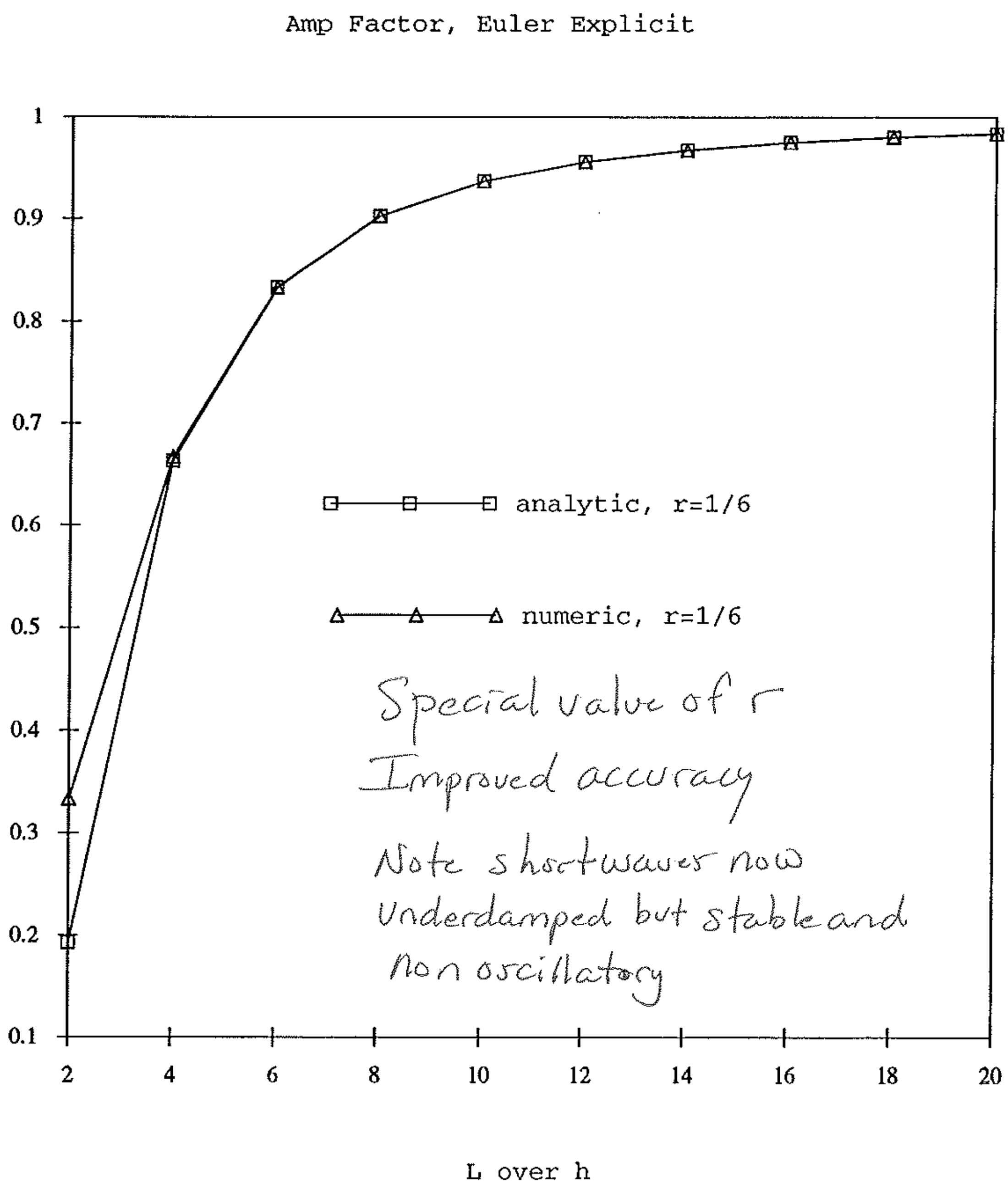
Error is "pushed back" one more term (we saw this earlier... Δt error just cancels leading Δx error)

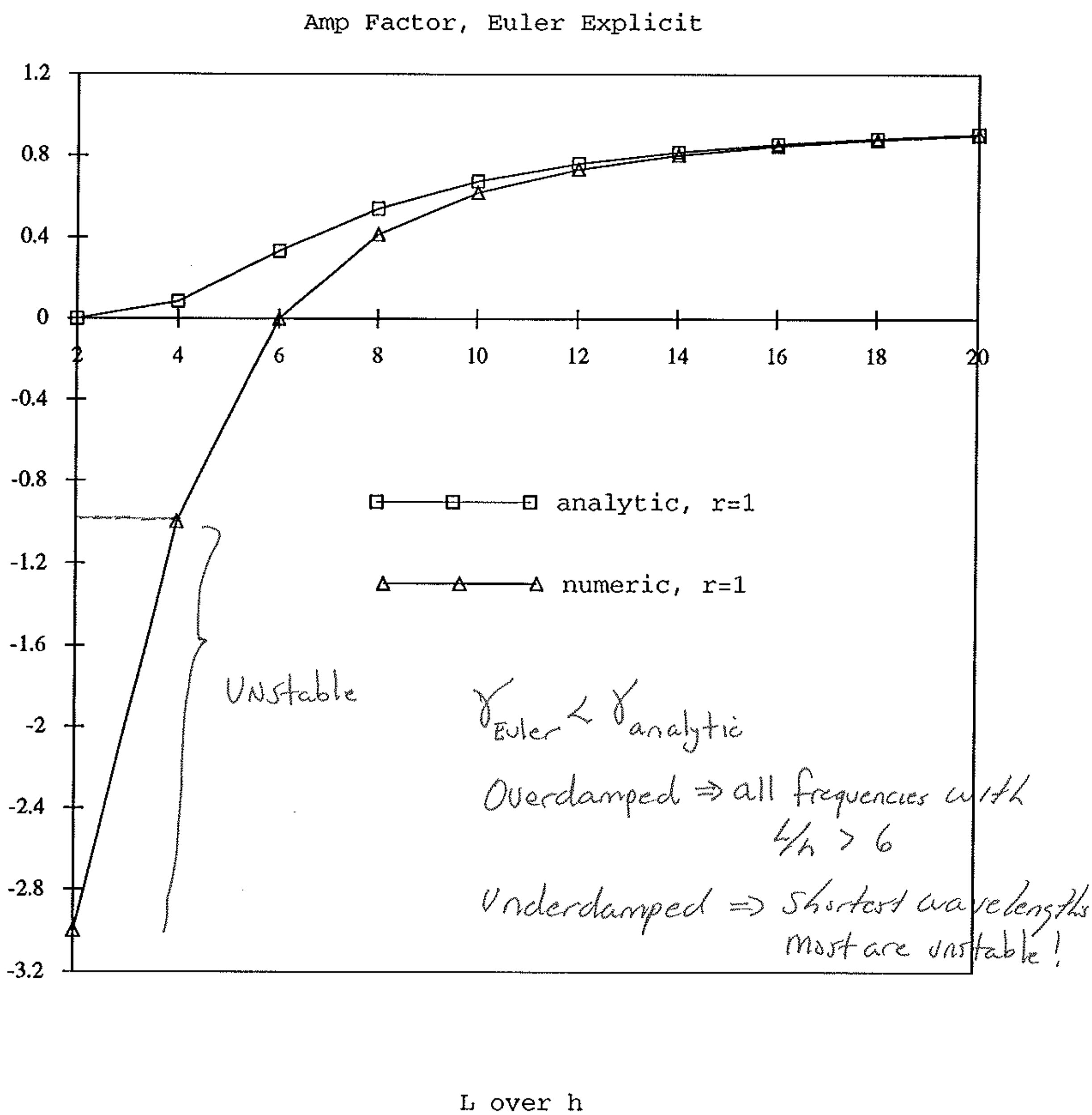
For Euler: $\gamma < \gamma_0 \Rightarrow$ Numerical sol'n underdamped (generally true... depends on r and σh)

Amp Factor, Euler Explicit



L over h


Amp Factor, Euler Explicit
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
2
4
6
8
10
12
14
16
18
20
analytic, r=1/6
numeric, r=1/6
Special value of r
Improved accuracy
Note shortwaves now
Underdamped but stable and
non oscillatory
L over h


Amp Factor, Euler Explicit
1.2
0.8
0.4
0
-0.4
-0.8
-1.2
-1.6
-2
-2.4
-2.8
-3.2
2
4
6
8
10
12
14
16
18
20
analytic, r=1
numeric, r=1
Unstable
$\gamma_{Euler} < \gamma_{analytic}$
Overdamped ⇒ all frequencies with $L/h > 6$
Underdamped ⇒ Shortest wavelengths most are unstable!
L over h

Can examine any scheme in this manner

e.g. "Richardson" $u_i^{l+1} - u_i^{l-1} = r\delta_x^2 u_i^l$

$$\gamma - \frac{1}{\gamma} = 2r(\cos\sigma h - 1)$$

$$\gamma^2 + 2r(1-\cos\sigma h)\gamma - 1 = 0$$ <u>2 Roots!</u>

- Stability for general quadratic

$$a\gamma^2 + b\gamma + c = 0 \Rightarrow |\gamma| \le 1$$

when $\frac{c}{a} \le 1$ <u>and</u> $|b| \le a + c$

In our case... $\frac{c}{a} = -1 \le 1$ always

$$|b| = 2r\underline{(1-\cos\sigma h)} \le 0$$

always positive!

No value of r satisfies this constraint for all values of σh ... short waves are the biggest offenders as usual!

Unconditionally unstable!!

- Stability Analysis using Matrix Methods

• In Lax-Richtmyer view... if we have the scheme $u^{l+1} = Au^l + c^l$; A grows in size; need to show $\|A\| \leq 1$ guarantees stability (and ∴ convergence for a consistent molecule)

• In practical view of fixed mesh lengths... if have a scheme of form $u^{l+1} = Au^l + c^l$ ... A has fixed size; sufficient to show $\rho(A) \leq 1$ to ensure boundedness

• Formally must have
$$\rho(A) \leq \|A\| \leq 1 \quad \text{as size of } A \longrightarrow \infty$$
to guarantee convergence for a consistent scheme (possible to have $\rho(A) \leq 1$ w/ $\|A\| > 1$)

e.g. Euler Explicit: $u_i^{l+1} = r u_{i-1}^l + (1-2r) u_i^l + r u_{i+1}^l$
w/ Type I BCs...

$$A = \begin{bmatrix} 1-2r & r & 0 & \cdots & & \\ r & 1-2r & r & 0 & \cdots & 0 \\ 0 & r & 1-2r & r & 0 & \cdots \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & \ddots & \ddots \end{bmatrix}$$

$\|A\|_\infty = |r| + |1-2r| + |r|$; Need $\|A\|_\infty \leq 1$

if $1-2r > 0$ the $\|A\|_\infty = 1$

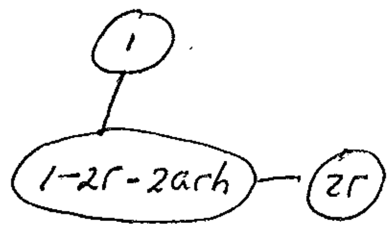
$1-2r < 0$ $\|A\|_\infty = 4r-1 > 1$ since $r > 1/2$

Conclude ... Need $r \le 1/2$ ... Same as Von Neumann (B)

Now if we have derivative BCs ... e.g. Type III

$$\frac{\partial U}{\partial x} = aU + b \quad \text{at } x=0 \text{ boundary}$$

... then molecule becomes



i.e. $U_{-1} = U_1 - 2ahU_0 - bh$

So A has the structure ...

$$\begin{bmatrix} 1-2r(1+ah) & 2r & & & \\ r & 1-2r & r & & \\ & r & 1-2r & r & \\ & & \ddots & \ddots & \ddots \\ & & & & \end{bmatrix}$$

- all but row 1 require $r \le 1/2$ for $\|A\|_\infty \le 1$
- must see if row 1 changes this restriction ...

we want $|1-2r(1+ah)| + |2r| \le 1$

Two cases to consider:

(a) $1-2r(1+ah) \ge 0$ (i.e. diagonal term positive)

then $|1-2r(1+ah)| + 2r = 1-2r(1+ah) + 2r \le 1$

$1 - 2rah \le 1$ always OK

C

But for diagonal coefficient to be positive

$$2r(1+ah) \leq 1 \Rightarrow r \leq \frac{1}{2(1+ah)}$$

So if coefficient is positive, problem is stable and we need $r \leq \frac{1}{2(1+ah)}$ to achieve this

(b) $1-2r(1+ah) \leq 0$ (diagonal is negative)

then $|1-2r(1+ah)| + |2r| = 2r(1+ah) - 1 + 2r \leq 1$

$$2r(2+ah) \leq 2$$

$$r < \frac{1}{2+ah}$$

But $\frac{1}{2(1+ah)} < \frac{1}{2+ah}$ ... So we can maintain stability when diagonal turns negative provided $r \leq \frac{1}{2+ah}$

IF r gets any bigger... diagonal still negative, but $\|A\|_\infty > 1$

- Also note $\frac{1}{2+ah} < 1/2$ i.e. Stability restriction greater w/ Type III than Type I BCs!