Hyperbolic Equations

- Classic 2nd Order Form ... elliptic in space
  2nd derivative in time

$$\frac{\partial^2 u}{\partial t^2} - \nabla \cdot c^2 \nabla u = f\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \ldots\right)$$

- Prototype Equation to study:

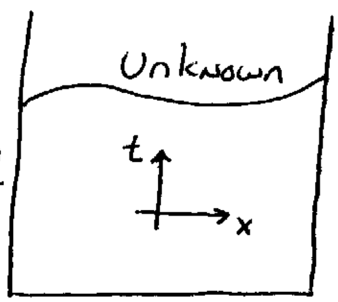
$$\frac{\partial^2 u}{\partial t^2} + \tau \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{"Telegraph Eqn"}$$

c – "wave speed" ; τ – dissipation or "loss factor"

$\tau = 0 \Rightarrow$ "Wave Equation

- Recall BCs + ICs needed:

u or $\frac{\partial u}{\partial x}$ or $a\frac{\partial u}{\partial x} + bu$



u or $\frac{\partial u}{\partial x}$ or $a\frac{\partial u}{\partial x} + bu$

u and $\frac{\partial u}{\partial t}$

- Can always decompose into Coupled set of $\underline{1^{ST}}$ order PDEs :

(1) $$\frac{\partial u}{\partial t} = c_1 \frac{\partial V}{\partial x}$$

(2) $$\frac{\partial V}{\partial t} + \tau V = c_2 \frac{\partial u}{\partial x}$$

$$\Rightarrow \underbrace{\frac{\partial^2 V}{\partial t^2} + \tau \frac{\partial V}{\partial t} - c^2 \frac{\partial^2 V}{\partial x^2} = 0}_{\text{Differentiate (2) in time + substitute (1)}}$$

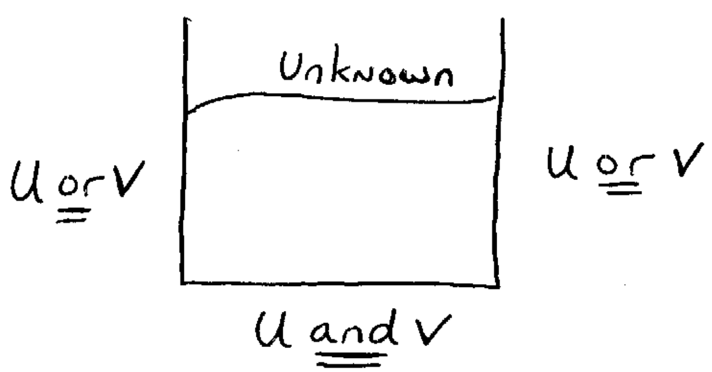
$c^2 \equiv c_1 c_2$

$\underline{\underline{\text{or}}}$: Differentiate (1) in time + substitute (2)

$$\frac{\partial^2 u}{\partial t^2} = c_1 \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial t}\right) = -c_1 \tau \underbrace{\frac{\partial V}{\partial x}}_{\frac{1}{c_1}\frac{\partial u}{\partial t}} + c^2 \frac{\partial^2 u}{\partial x^2}$$

so: $$\frac{\partial^2 u}{\partial t^2} + \tau \frac{\partial u}{\partial x} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

- BCs + ICs for coupled system...



- Many physical problems can be formulated in terms of Conservation Laws

1^{ST} Order Coupled PDEs are Conservation Statements ... when combined w/ Constitutive Relation ... lead to 2nd Order PDE

e.g. Electrical Transmission Line

$$\frac{\partial Q}{\partial t} + \frac{\partial I}{\partial x} = 0$$

Charge Conservation (mass Conservation)

Current

Charge/unit length

Voltage

$$L\frac{\partial I}{\partial t} + IR + \frac{\partial V}{\partial x} = 0$$

Force Balance (momentum Conservation)

Inductance/unit length

Resistance/unit length

Capacitance per unit length

$$CV = Q$$

Constitutive Relation

$$\Rightarrow \underbrace{\begin{gathered} C\frac{\partial V}{\partial t} + \frac{\partial I}{\partial x} = 0 \\ L\frac{\partial I}{\partial t} + IR + \frac{\partial V}{\partial x} = 0 \end{gathered}}_{\text{Coupled } 1^{ST} \text{ Order}} \Rightarrow \underbrace{\frac{\partial^2 V}{\partial t^2} + \frac{R}{L}\frac{\partial V}{\partial t} - \frac{1}{LC}\frac{\partial^2 V}{\partial x^2} = 0}_{\text{Second Order}}$$

Time-Stepping Strategies

- Consider 2nd Order System First

- Obvious approach... replace $\frac{\partial^2 U}{\partial t^2}$ w/ Centered 2nd order approximation... requires $\underline{\underline{3}}$ levels in time!

a.) Explicit: $$\frac{\delta_t^2 U_i^\ell}{\Delta t^2} + \tau \frac{\delta_t U_i^\ell}{2\Delta t} - \frac{c^2 \delta_x^2 U_i^\ell}{h^2} = 0$$

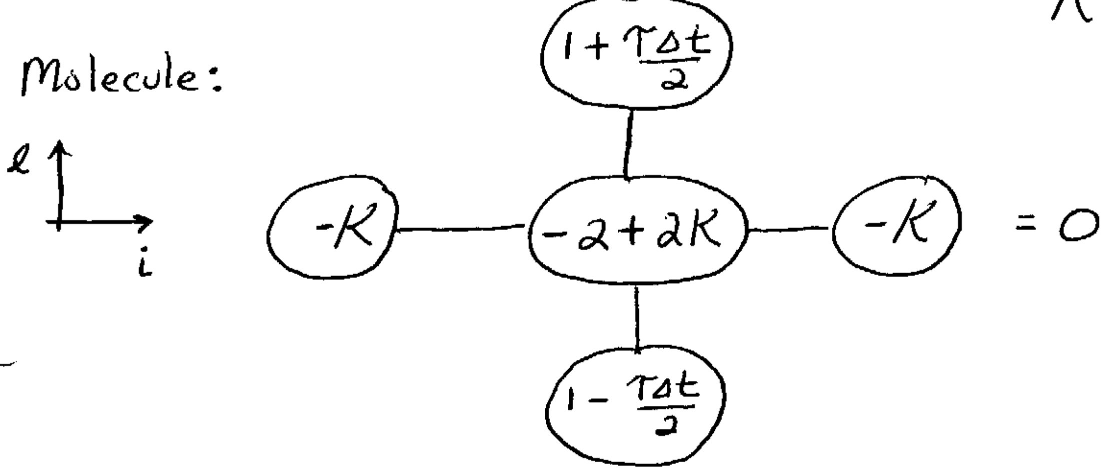
$$\frac{U_i^{\ell+1} - 2U_i^\ell + U_i^{\ell-1}}{\Delta t^2} + \frac{\tau}{2\Delta t}\left(U_i^{\ell+1} - U_i^{\ell-1}\right) - \frac{c^2}{h^2}\left(U_{i-1}^\ell - 2U_i^\ell + U_{i+1}^\ell\right) = 0$$

multiply through by Δt^2:

$$U_i^{\ell+1} - 2U_i^\ell + U_i^{\ell-1} + \frac{\tau\Delta t}{2}\left(U_i^{\ell+1} - U_i^{\ell-1}\right) - \underbrace{\frac{c^2\Delta t^2}{h^2}}_{K \equiv \frac{c^2\Delta t^2}{h^2}}\left(U_{i-1}^\ell - 2U_i^\ell + U_{i+1}^\ell\right) = 0$$

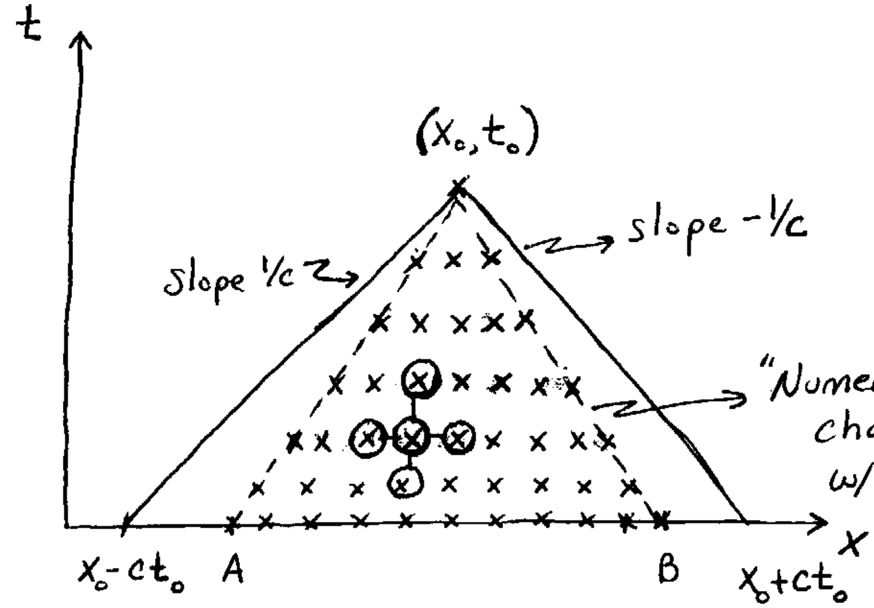
"Courant#"

Molecule:



- Features...
  - Centered in $x + t \Rightarrow \mathcal{O}(h^2 + \Delta t^2)$
  - No matrices
  - Pointwise propagation ... experience w/ Parabolic suggests stability constraint

- Recall Characteristic Family for Wave Eqn



- Numerically could propagate sol'n to (x_0, t_0) w/ only ICs along AB

- Analytically, we know ICs in $[x_0 - ct_0, A] + [B, x_0 + ct_0]$ influence sol'n at (x_0, t_0)

- Changes in ICs along these intervals never felt by numerical solution ... expect trouble!!
  Don't get convergence as $h, \Delta t \to 0$ for $\frac{\Delta t}{h} > 1/c$

- Courant, Friedrichs and Lewy Condition

  Convergence requires numerical characteristics (i.e. domain of dependence of FD) to have slopes less than $1/c$ (i.e. must include domain of dependence of PDE)

  $$\therefore \quad \frac{\Delta t}{h} < 1/c \Rightarrow \underbrace{\frac{c\Delta t}{h} < 1}$$

  This is essentially a stability requirement

- Lets look at formal stability analysis

  $$U_i^{l+1} = \gamma_0 U_i^l \; ; \; U_{i\pm 1}^l = U_i^l e^{\pm j\sigma h}$$

$$U_i^{l+1} - 2U_i^l + U_i^{l-1} + \frac{\tau\Delta t}{2}\left(U_i^{l+1} - U_i^{l-1}\right) + K\left(U_{i-1}^l - 2U_i^l + U_{i+1}^l\right) = 0$$

$$\gamma_0 - 2 + \frac{1}{\gamma_0} + \frac{\tau\Delta t}{2}\left(\gamma_0 - \frac{1}{\gamma_0}\right) - K\left(2\cos\sigma h - 2\right) = 0$$

$$\gamma_0^2\left(1 + \frac{\tau\Delta t}{2}\right) + \gamma_0\left(2K(1-\cos\sigma h) - 2\right) + \left(1 - \frac{\tau\Delta t}{2}\right) = 0$$

Need $|\gamma_0| \le 1$; requires $\frac{c}{a} \le 1$; $|b| \le a+c$

- $\frac{c}{a} \le 1 \Rightarrow \dfrac{1-\frac{\tau\Delta t}{2}}{1+\frac{\tau\Delta t}{2}} < 1$ always!

- $|b| \le a+c$

$$|2K(1-\cos\sigma h)-2| \le 2 \Rightarrow |K(1-\cos\sigma h)-1| \le 1$$

i.e. $\quad -1 \le K(1-\cos\sigma h)-1 \le 1$

Consider "left" inequality : $-1 \le K(1-\cos\sigma h)-1$

always true since $K>0$
$1-\cos\sigma h > 0$

"Right" inequality : $K(1-\cos\sigma h)-1 \le 1$

$$K \le \frac{2}{1-\cos\sigma h}$$

Most restrictive limit is stability restriction!
as usual ... short waves are potentially most unstable ... i.e. $\sigma h = \pi$

Conclude: $\boxed{K \le 1 \; ; \quad \frac{c^2\Delta t^2}{h^2} < 1 \text{ or } \frac{c\Delta t}{h} < 1}$

"Courant" Condition

- But what about accuracy ... Use Propagation Factor Concept: $T \equiv \left(\frac{\gamma_0}{\gamma}\right)^N$

analytic amplification

Characteristic Time Scale

- Need γ + N ... Lets Recall the nature of the analytic propagation for

$$\frac{\partial^2 u}{\partial t^2} + \tau \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Try soln of form: $u = e^{\alpha t} e^{j\sigma x}$

$$\alpha^2 + \tau\alpha - c^2(j\sigma)^2 = \alpha^2 + \tau\alpha + c^2\sigma^2 = 0$$

So $$\alpha = -\frac{\tau}{2} \pm \sqrt{\left(\tfrac{\tau}{2}\right)^2 - c^2\sigma^2}$$

$$= \underbrace{-\tau/2}_{\text{damping}} \underbrace{\pm j\sigma\sqrt{c^2 - \left(\frac{\tau}{2\sigma}\right)^2}}_{\text{propagating}}$$

i.e. $u = e^{-\tau/2\, t} \, e^{j\sigma\left(x \pm \underbrace{\sqrt{c^2 - \left(\frac{\tau}{2\sigma}\right)^2}}_{\text{Wave speed} \equiv c'} \, t\right)}$

- longest waves; $\sigma \to 0$ do not propagate (i.e. $\tau > 2c\sigma \Rightarrow \alpha$ is real)

- Damping Factor τ reduces wave speed to $\sqrt{c^2 - \left(\frac{\tau}{2\sigma}\right)^2}$ when $\tau = 0$; all waves propagate at same speed without decay

- Alteenately ... Consider Primitive Pair

$$\frac{\partial u}{\partial t} + \tau u - c_1 \frac{\partial v}{\partial x} = 0$$
$$\frac{\partial v}{\partial t} - c_2 \frac{\partial u}{\partial x} = 0$$
$$\Rightarrow \begin{bmatrix} \frac{\partial}{\partial t} + \tau & -c_1 \frac{\partial}{\partial x} \\ -c_2 \frac{\partial}{\partial x} & \frac{\partial}{\partial t} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assume: $\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix} e^{\alpha t} e^{j\sigma x}$

Then
$$\underbrace{\begin{bmatrix} \alpha + \tau & -c_1 j\sigma \\ -c_2 j\sigma & \alpha \end{bmatrix}}_{\det = 0 \text{ for Nontrivial Sol'n}} \begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\alpha^2 + \tau\alpha - c_1 c_2 (j\sigma)^2 = 0$$

$$\Rightarrow \alpha^2 + \tau\alpha + c^2\sigma^2 = 0$$

<u>Same</u> as <u>2nd Order</u> as expected !!

$$\text{- So} \quad u \sim e^{-\tau/2\, t} e^{j\sigma(x \pm c' t)}$$

$$\text{But} \quad \gamma_{analytic} \equiv \gamma = e^{\alpha \Delta t} = \frac{u(t+\Delta t)}{u(t)} = \frac{e^{-\tau/2\, t} e^{j\sigma(x \pm c't)}}{e^{-\tau/2\, t} e^{j\sigma(x \pm c't)}} e^{-\frac{\tau}{2}\Delta t} e^{j\sigma c' \Delta t}$$

Take N as # Time steps to propagate one analytic wavelength

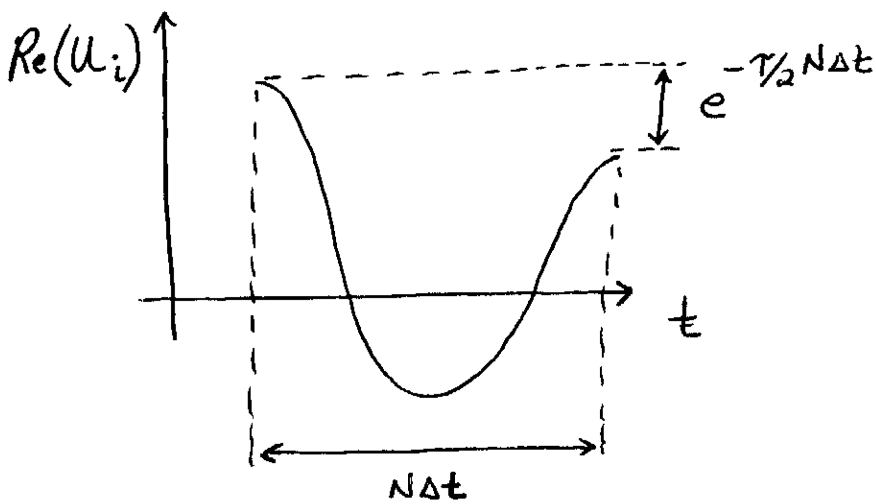
$$\text{So} \quad N\Delta t = \frac{L}{c'} = \frac{2\pi}{\sigma c'}$$

$$\text{Now} \quad (\gamma)^N = \left(e^{\alpha \Delta t}\right)^N = e^{-\tau/2\, \Delta t N} e^{j\sigma c' \Delta t N}$$

$$= e^{-\tau/2\, \Delta t N} \underbrace{e^{j2\pi}}_{1}$$

$(\gamma)^N$ is real ... $0 \leq \gamma^N \leq 1$ Pure Decay

then <u>u</u> at a fixed x value looks like



Numerical Sol'n : $(\gamma_o)^N = f\left(\begin{matrix}\text{Numerical Difference} \\ \text{Eqn}\end{matrix}\right)$

In general complex-valued :

$$(\gamma_o)^N = |\gamma_o|^N e^{j(\phi + 2\pi)}$$

But we can write $Arg(\gamma_o^N)$ as

$$\phi + 2\pi = \sigma \tilde{c}' \Delta t N = \sigma \tilde{c}' \left(\frac{2\pi}{\sigma c'}\right) = \frac{2\pi \tilde{c}'}{c'}$$

Numerical Wave speed

$$\therefore \quad \frac{\phi + 2\pi}{2\pi} = \frac{\tilde{c}'}{c'} \Rightarrow$$

$\phi > 0$ Numerical waves too fast

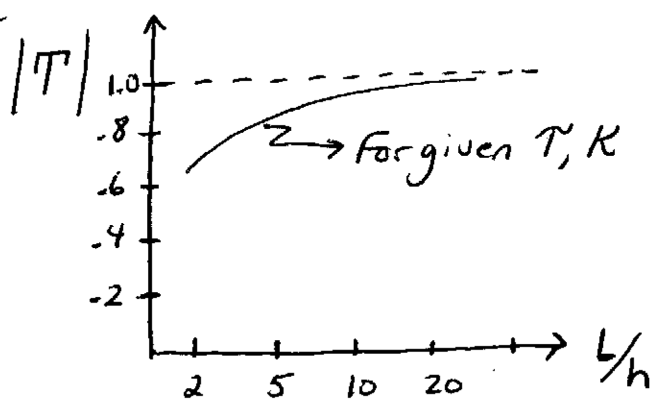
$\phi < 0$ too slow

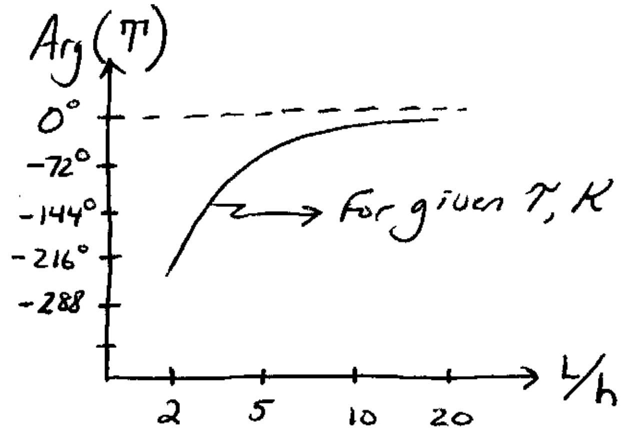
So $\mathcal{T} = \left(\dfrac{\gamma_0}{\gamma}\right)^N$

$$|\mathcal{T}| = \frac{|\text{Numerical Damping}|}{|\text{Analytic Damping}|}$$

$$\text{Arg}(\mathcal{T}) = \phi \,;\quad 1 + \frac{\phi}{2\pi} = \frac{\text{Numerical Wave Speed}}{\text{Analytic Wave Speed}}$$

- Plot $\mathcal{T}$ vs L/h and look at
  - deviations from unity in $|\mathcal{T}|$
  - deviations from $0°$ in $\text{Arg}(\mathcal{T})$





Note: $N = \dfrac{2\pi}{\Delta t\, \sigma c'} = \dfrac{2\pi}{\Delta t\, \sigma \sqrt{c^2 - \left(\frac{\tau}{2\sigma}\right)^2}}$

$$= \frac{2\pi}{\sqrt{K(\sigma h)^2 - \left(\frac{\tau \Delta t}{2}\right)^2}} \quad ; \quad K = \frac{c^2 \Delta t^2}{h^2}$$

$$\left(\text{i.e. } c^2 = \frac{K h^2}{\Delta t^2}\right)$$

Hyperbolic Eqn's (Con't)

- So what kind of errors do we expect w/ explicit scheme ...

- Experience tells us "short waves" are the most difficult to handle ... look at $\gamma_0(\sigma h = \pi)$

Recall: $\gamma_0^2\left(1+\frac{\tau\Delta t}{2}\right) + \gamma_0\left(2K(1-\cos\sigma h)-2\right) + \left(1-\frac{\tau\Delta t}{2}\right) = 0$

at $\sigma h = \pi$:

$$\gamma_0^2\left(1+\frac{\tau\Delta t}{2}\right) + \gamma_0\left(4K-2\right) + \left(1-\frac{\tau\Delta t}{2}\right) = 0$$

$$\gamma_0 = \frac{2-4K \pm \sqrt{(4K-2)^2 - 4\left(1-\left(\frac{\Delta t\,\tau}{2}\right)^2\right)}}{2\left(1+\tau\Delta t/2\right)}$$

$$= \frac{2-4K \pm j\left(4\left(1-\left(\frac{\Delta t\,\tau}{2}\right)^2\right) - (4K-2)^2\right)^{1/2}}{2\left(1+\tau\Delta t/2\right)}$$

generally get decay (Good News), but also phase distortion!!
(Bad news)

e.g. $K = 1/2$: $\gamma_0 = \pm j \left(\dfrac{\left(1 - \left(\frac{\Delta t\, \tau}{2}\right)^2\right)}{1 + \tau \Delta t/2} \right)^{1/2} = \pm j \left(\dfrac{1 - \frac{\Delta t \tau}{2}}{1 + \frac{\Delta t \tau}{2}} \right)^{1/2}$

$\therefore |\gamma_0| < 1$; $\mathrm{Arg}(\gamma_0) = \pm \pi/2$

$K = 1/4$: $\gamma_0 = \dfrac{1 \pm j\sqrt{4\left(1 - \left(\frac{\Delta t\, \tau}{2}\right)^2\right) - 1}}{2(1 + \tau \Delta t/2)}$

$\left.\begin{array}{l} \tau = .015 \\ h = 5 \\ \Delta t = .25 \end{array}\right\}$ $\gamma_0 = \dfrac{1 \pm j\sqrt{4(1 - (1.875E\text{-}03)^2 - 1}}{2(1 + 1.875E\text{-}03)}$

$= \dfrac{1 \pm j\, 1.7320}{2.00375} \Rightarrow |\gamma_0| < 1$

$\mathrm{Arg}(\gamma_0) \approx \pm \pi/3$

Amplitude damped; expect phase errors

Best to study $T(\sigma h) \Rightarrow \left(\dfrac{\gamma_0}{\gamma}\right)^N$