Electromagnetic Field Example
(FE Soln to Maxwell Equations)

- For time-harmonic fields: $\nabla \times \underset{\sim}{E} = i\omega\mu \underset{\sim}{H}$ $\quad (i=\sqrt{-1})$

$$\nabla \times \underset{\sim}{H} = \underset{\sim}{J} - i\omega\epsilon \underset{\sim}{E}$$
$$\nabla \cdot \epsilon E = \rho$$
$$\nabla \cdot \mu H = 0$$

Common to approach through potentials:

Since $\underset{\sim}{H}$ is "divergenceless": $H = \frac{1}{\mu} \nabla \times A$ ← "Vector potential"

then $\nabla \times (\underbrace{\underset{\sim}{E} - i\omega \underset{\sim}{A}}) = 0$

can be equated to $\nabla \Phi$ since $\nabla \times \nabla \Phi = 0$

so $\underset{\sim}{E} = i\omega \underset{\sim}{A} - \nabla \Phi$ ← "scalar potential"

Not unique representation, since can add constant to Φ without changing $\underset{\sim}{E}$ or $\underset{\sim}{A}$; also can add scalar function to Φ without changing $\underset{\sim}{E}$, but will change $\underset{\sim}{A}$

- Now $\nabla \times \left(\frac{\nabla \times \underset{\sim}{E}}{\mu} \right) = i\omega \nabla \times \underset{\sim}{H} = i\omega (\underset{\sim}{J} - i\omega\epsilon \underset{\sim}{E})$

$$\nabla \times \frac{1}{\mu} \nabla \times \underset{\sim}{E} - \omega^2 \epsilon \underset{\sim}{E} = i\omega \underset{\sim}{J}$$
$$\nabla \cdot \epsilon \underset{\sim}{E} = \rho$$

which become

$$\nabla \times \frac{1}{\mu} \nabla \times \underset{\sim}{A} - \omega^2 \epsilon \left(\underset{\sim}{A} - \frac{1}{i\omega} \nabla \Phi \right) = \underset{\sim}{J}$$

$$\nabla \cdot \epsilon \left(\underset{\sim}{A} - \frac{1}{i\omega} \nabla \Phi \right) = \frac{\rho}{i\omega}$$

- Now in 2D case we have replaced vector E which has two components (i.e. E_x, E_y) with a 3 component system (A_x, A_y, Φ)... something arbitrary; can specify a relationship between $\underset{\sim}{A}$ and Φ for uniqueness

e.g. One choice: $\nabla \cdot \epsilon \underset{\sim}{A} = i\omega\epsilon^2\mu\Phi$ "Lorentz gauge" for heterogeneous media

Tells (specifies) divergence of $\underset{\sim}{A}$... curl of $\underset{\sim}{A}$ specified as $\underset{\sim}{H}$; $\underset{\sim}{A}$ now uniquely defined since both divergence and curl dictated (Helmholtz theorem)

- Plug in and we get

$$\nabla \times \frac{1}{\mu} \nabla \times \underset{\sim}{A} - \epsilon \nabla \frac{1}{\epsilon^2\mu} \nabla \cdot \epsilon \underset{\sim}{A} - \omega^2 \epsilon \underset{\sim}{A} = \underset{\sim}{J}$$

$$- \nabla \cdot \epsilon \nabla \Phi - \omega^2 \epsilon^2 \mu \Phi = \rho$$

Produces uncoupled "Helmholtz-like" equations for vector and scalar potentials (i.e. equations that reduce to Helmholtz forms in homogeneous regions

- Possible to choose BCs such that $\Phi = 0$ everywhere in which case $\underset{\sim}{E} = i\omega \underset{\sim}{A}$

So our equation for $\underline{E}$ becomes

$$\nabla \times \frac{1}{\mu} \nabla \times \underset{\sim}{E} - \epsilon \nabla \frac{1}{\epsilon^2 \mu} \nabla \cdot \epsilon \underset{\sim}{E} - \omega^2 \epsilon \underset{\sim}{E} = i\omega \underset{\sim}{J}$$

- lets look at FEM sol'n to this equation ...

- 1ST Weight PDE + integrate as usual:

$$\left\langle \nabla \times \frac{1}{\mu} \nabla \times \underset{\sim}{E}\, \phi_i \right\rangle - \left\langle \epsilon \nabla \frac{1}{\epsilon^2 \mu} \nabla \cdot \epsilon \underset{\sim}{E}\, \phi_i \right\rangle - \left\langle \omega^2 \epsilon \underset{\sim}{E}\, \phi_i \right\rangle = \left\langle i\omega \underset{\sim}{J}\, \phi_i \right\rangle$$

- Look for vector identities to reduce 2nd derivatives to 1ST derivatives plus a boundary term:

e.g. $\nabla \times \frac{1}{\mu}\left(\nabla \times \underset{\sim}{E}\, \phi_i\right) = \phi_i \nabla \times \frac{1}{\mu} \nabla \times \underset{\sim}{E} + \nabla \phi_i \times \frac{1}{\mu} \nabla \times \underset{\sim}{E}$

So: $\left\langle \nabla \times \frac{1}{\mu} \nabla \times E\, \phi_i \right\rangle = \left\langle -\nabla \phi_i \times \frac{1}{\mu} \nabla \times E \right\rangle + \oint \hat{n} \times \frac{1}{\mu} \nabla \times E\, \phi_i\, ds$

Also

$$\nabla\left(\frac{1}{\epsilon^2 \mu} \nabla \cdot \epsilon \underset{\sim}{E}\, \phi_i\right) = \phi_i \nabla \frac{1}{\epsilon^2 \mu} \nabla \cdot \epsilon \underset{\sim}{E} + \nabla \phi_i \frac{1}{\epsilon^2 \mu} \nabla \cdot \epsilon E$$

So:

$$\left\langle \epsilon \nabla \frac{1}{\epsilon^2 \mu} \nabla \cdot \epsilon \underset{\sim}{E}\, \phi_i \right\rangle = \left\langle -\nabla \phi_i \frac{1}{\epsilon \mu} \nabla \cdot \epsilon \underset{\sim}{E} \right\rangle + \oint \hat{n}\, \frac{\nabla \cdot \epsilon E}{\epsilon \mu} \phi_i\, ds$$

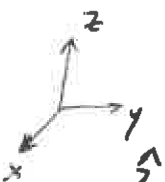
- Combining this all together gives...

$$\left\langle \frac{1}{\mu} \nabla \times \underset{\sim}{E} \times \nabla \phi_i \right\rangle + \left\langle \nabla \phi_i \frac{1}{\epsilon \mu} \nabla \cdot \epsilon \underset{\sim}{E} \right\rangle - \left\langle \omega^2 \epsilon \underset{\sim}{E}\, \phi_i \right\rangle$$

$$= \left\langle i\omega J \phi_i \right\rangle - \oint \hat{n} \times \frac{1}{\mu} \nabla \times E\, \phi_i\, ds + \oint \hat{n}\, \frac{\nabla \cdot \epsilon E}{\epsilon \mu} \phi_i\, ds$$

• Now expand into x, y components: (2D in x-y plane)

$$\nabla \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ E_x & E_y & 0 \end{vmatrix} = \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$



$$\hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \times \left(\frac{\partial \phi_i}{\partial x}\hat{x} + \frac{\partial \phi_i}{\partial y}\hat{y}\right) = \hat{y}\left(\frac{\partial E_y}{\partial x}\frac{\partial \phi_i}{\partial x} - \frac{\partial E_x}{\partial y}\frac{\partial \phi_i}{\partial x}\right)$$
$$+ \hat{x}\left(\frac{\partial E_x}{\partial y}\frac{\partial \phi_i}{\partial y} - \frac{\partial E_y}{\partial x}\frac{\partial \phi_i}{\partial y}\right)$$

$$\nabla \phi_i \frac{1}{\epsilon\mu} \nabla \cdot \epsilon E = \hat{x}\frac{\partial \phi_i}{\partial x}\left(\frac{1}{\epsilon\mu}\left(\frac{\partial}{\partial x}\epsilon E_x + \frac{\partial}{\partial y}\epsilon E_y\right)\right)$$
$$+ \hat{y}\frac{\partial \phi_i}{\partial y}\left(\frac{1}{\epsilon\mu}\left(\frac{\partial}{\partial x}\epsilon E_x + \frac{\partial}{\partial y}\epsilon E_y\right)\right)$$

$\therefore$

$$\hat{x}: \left\langle \frac{1}{\mu}\frac{\partial E_x}{\partial y}\frac{\partial \phi_i}{\partial y} + \frac{1}{\epsilon\mu}\frac{\partial \phi_i}{\partial x}\frac{\partial}{\partial x}\epsilon E_x \right\rangle + \left\langle \frac{1}{\epsilon\mu}\frac{\partial \phi_i}{\partial x}\frac{\partial}{\partial y}\epsilon E_y - \frac{1}{\mu}\frac{\partial E_y}{\partial x}\frac{\partial \phi_i}{\partial y} \right\rangle$$
$$- \left\langle \omega^2 \epsilon E_x \phi_i \right\rangle = \left\langle i\omega J_x \phi_i \right\rangle$$
$$- \hat{x} \cdot \left[\oint \hat{n} \times \frac{1}{\mu} \nabla \times E \phi_i \, ds \right.$$
$$\left. - \oint \hat{n} \frac{\nabla \cdot \epsilon E}{\mu\epsilon} \phi_i \, ds \right]$$

$$\hat{y}: \left\langle \frac{1}{\mu}\frac{\partial E_y}{\partial x}\frac{\partial \phi_i}{\partial x} + \frac{1}{\epsilon\mu}\frac{\partial \phi_i}{\partial y}\frac{\partial}{\partial y}\epsilon E_y \right\rangle + \left\langle \frac{1}{\epsilon\mu}\frac{\partial \phi_i}{\partial y}\frac{\partial}{\partial x}\epsilon E_x - \frac{1}{\mu}\frac{\partial E_x}{\partial y}\frac{\partial \phi_i}{\partial x} \right\rangle$$
$$- \left\langle \omega^2 \epsilon E_y \phi_i \right\rangle = \left\langle i\omega J_y \phi_i \right\rangle$$
$$- \hat{y} \cdot \left[\oint \hat{n} \times \frac{1}{\mu} \nabla \times E \phi_i \, ds \right.$$
$$\left. - \oint \hat{n} \frac{\nabla \cdot \epsilon E}{\mu\epsilon} \phi_i \, ds \right]$$

- Write $\underline{E}$ as sum of unknown coefficients times the known basis (Galerkin)

$$E_x = \sum_{j=1}^{N} E_{x_j} \phi_j$$

$$E_y = \sum_{j=1}^{N} E_{y_j} \phi_j$$

This leads to the system of equations $AE = b$ where for each (i,j) combination we have

$$A_{ij} = \begin{bmatrix} \left\langle \frac{1}{\mu}\left(\frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial x} + \frac{\partial\phi_j}{\partial y}\frac{\partial\phi_i}{\partial y}\right) - \omega^2\epsilon\,\phi_j\,\phi_i \right\rangle & \left\langle \frac{1}{\mu}\left(\frac{\partial\phi_j}{\partial y}\frac{\partial\phi_i}{\partial x} - \frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial y}\right)\right\rangle \\ \left\langle \frac{1}{\mu}\left(\frac{\partial\phi_i}{\partial y}\frac{\partial\phi_j}{\partial x} - \frac{\partial\phi_i}{\partial x}\frac{\partial\phi_j}{\partial y}\right)\right\rangle & \left\langle \frac{1}{\mu}\left(\frac{\partial\phi_j}{\partial y}\frac{\partial\phi_i}{\partial y} + \frac{\partial\phi_i}{\partial x}\frac{\partial\phi_j}{\partial x}\right) - \omega^2\epsilon\,\phi_j\,\phi_i \right\rangle \end{bmatrix}$$

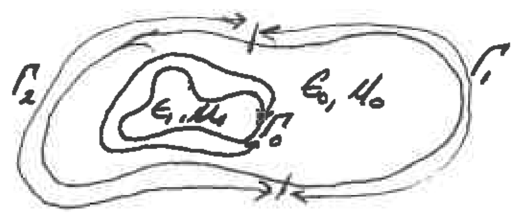
$$E_j = \begin{Bmatrix} E_{x_j} \\ E_{y_j} \end{Bmatrix}$$

$$b_i = \begin{Bmatrix} \langle i\omega J_x \phi_i \rangle + \hat{x}\cdot\left[\oint \hat{n}\frac{\nabla\cdot\epsilon E}{\mu\epsilon}\phi_i\,ds - \oint \hat{n}\times\frac{1}{\mu}\nabla\times E\,\phi_i\,ds\right] \\ \langle i\omega J_y \phi_i \rangle + \hat{y}\cdot\left[\oint \frac{\hat{n}\,\nabla\cdot\epsilon E}{\mu\epsilon}\phi_i\,ds - \oint \hat{n}\times\frac{1}{\mu}\nabla\times E\,\phi_i\,ds\right] \end{Bmatrix}$$

- As in Elasticity example, have 2 coupled PDEs to enforce at each node + 2 unknown coefficients to determine

- Also boundary data naturally supplied through $\oint$ term in normal/tangential framework; best to rotate equations + variables at boundary nodes to accommodate!

- Lets look at the boundary data that is appropriate for this equation



- On the outer boundary along Γ_1 we have:

  $\hat{n} \times \underset{\sim}{E}$ = known value $\longrightarrow$ Type I on tangential $\underset{\sim}{E}$

  $\nabla \cdot \epsilon \underset{\sim}{E} = 0 \longrightarrow$ Type II, used in normal equation

So at a node on Γ_1 we would...

(i) Rotate equations and variables into local (n, t) system

(ii) Remove the tangential Galerkin equation (i.e. row $2*B$, for boundary node B); place unity on the diagonal and known value on right-hand-side

(iii) Set the boundary integral $\oint \hat{n} \frac{\nabla \cdot \epsilon E}{\mu \epsilon} \phi_i \, ds = 0$ in the normal equation (i.e. on row $2*B-1$, we don't add anything additional to the RHS b vector)

- On the outer boundary along Γ_2 we have

$$i\omega\, n \times \underset{\sim}{H} = n \times \frac{1}{\mu} \nabla \times \underline{E} = \text{known value} \Rightarrow \text{Type II on tangential equations}$$

$$\hat{n} \cdot \epsilon \underset{\sim}{E} = \text{known value} \Rightarrow \text{Type I on normal component of } \underset{\sim}{E}$$

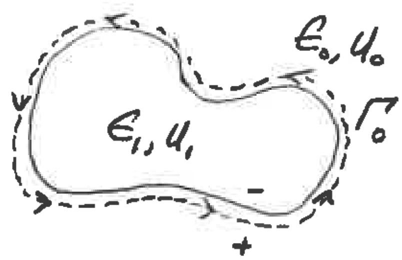
So at a node on Γ_2 we would

(i) Rotate equations and variables into local (n,t) system

(ii) Remove the normal Galerkin equation (i.e. row $2B-1$ for boundary node B); place unity on the diagonal and the known value on the right-hand-side

(iii) Set the boundary term in the tangential Galerkin equation.. i.e. compute $\oint n \times \frac{1}{\mu} \nabla \times E \, \phi_i \, ds$ since the integrand is known and add this value to the RHS b vector on row $2*B$ for boundary node B.

- On an interface between distinct electrical properties i.e. along Γ_0 which is interior to our domain



$\underset{\sim}{E}$ must satisfy: $\hat{n} \times (\underset{\sim}{E}^+ - \underset{\sim}{E}^-) = 0$ (i.e. tangential $\underset{\sim}{E}$ continuous)

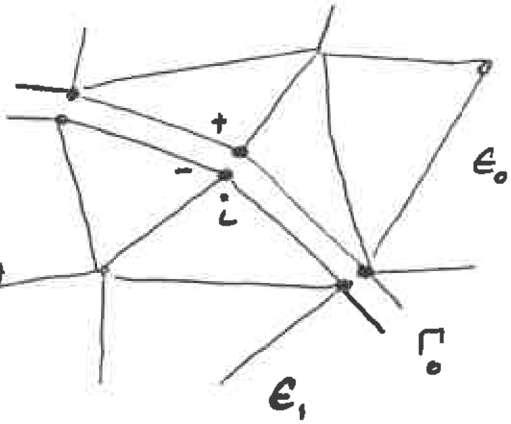
$\hat{n} \cdot (\epsilon_0 E^+ - \epsilon_1 E^-) = 0$ (i.e. normal $\underset{\sim}{E}$ discontinuous by ratio of ϵ's)

Want to enforce these constraints on our computed solution but how to do this??

Conceptually... Sever the mesh at an interface

two nodes exist !!
since two values of
$\underline{E}$ exist !!

Conclude: have 4 total unknowns
but have 2 scalar component
Maxwell equations plus
2 Interface conditions
$\therefore$ 4 eqn's for 4 unknowns



In (n,t) system easy to express interface conditions:

$$\begin{Bmatrix} E^+_{n_i} \\ E^+_{t_i} \end{Bmatrix} = \begin{bmatrix} \epsilon_1/\epsilon_0 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} E^-_{n_i} \\ E^-_{t_i} \end{Bmatrix}$$

Now the $\langle \text{Galerkin}_i \rangle = \langle \text{Galerkin}_i \rangle_- + \langle \text{Galerkin}_i \rangle_+$

Here coefficients multiply unknowns $\underline{E}_j^-$ for all j "connected" to i on side $-$

Here coefficients multiply $\underline{E}_j^+$ for all j "connected" to i on side $+$

But for node i, we only have room for 2 unknown coefficients in our column vector... i.e.
$$\begin{cases} 2*i-1 & (\text{for } \hat{x} \text{ component or } \hat{n} \text{ when rotated}) \\ 2*i & (\text{for } \hat{y} \text{ component or } \hat{t} \text{ when rotated}) \end{cases}$$

Must remove either $\underline{E}_j^-$ or $\underline{E}_j^+$ from the system of equations via interface relations

1st Rotate the equation set at interface node i

$$[R_i]\sum_j \underbrace{[A_{ij}^-]}_{\langle()\rangle_- \text{ only}}[R_j^T]\{E_j^-\}_{(n,t)} + [R_i]\sum_j \underbrace{[A_{ij}^+]}_{\langle()\rangle_+ \text{ only}}[R_j^T]\{E_j^+\}_{(n,t)} = 0$$

$$\{E_j^+\}_{(n,t)} = \underbrace{\begin{bmatrix} \epsilon_1/\epsilon_0 & 0 \\ 0 & 1 \end{bmatrix}}_{D_j^T}\begin{Bmatrix} E_{n_j}^- \\ E_{t_j} \end{Bmatrix}$$

$$\Rightarrow [R_i]\left[\sum_j\left([A_{ij}^-][R_j^T] + [A_{ij}^+][R_j^T][D_j^T]\right)\{E_j^-\}_{(n,t)}\right] = 0$$

To preserve symmetry must also premultiply by $[D]$
"in effect" the basis function for E_j now has a discontinuity in it of size ϵ_1/ϵ_0 at the interface; to have a Galerkin formulation the weighting function must also have this feature
Premultiplication by $[D]$ achieves this end..

Finally we have

$$\left[\left([R_i]\sum_j[A_{ij}^-][R_j^T] + [D_i][R_i]\sum_j[A_{ij}^+][R_j^T][D_j^T]\right)\{E_j^-\}_{(n,t)}\right] = 0$$

IF we had a Non zero Right-side for interface node it would be

$$[R_i]\{b_i^-\} + [D_i][R_i]\{b_i^+\}$$