Three Classic Definitions

1.) Convergence: $u_i^{\ell} \longrightarrow U(x_i, t_\ell)$ as $h, k \longrightarrow 0$

Exact sol'n to FD eqns — Exact Sol'n to PDE

- requires Discretization error $\longrightarrow 0$ as mesh lengths $\longrightarrow 0$

2.) Consistency: $L_i \longrightarrow L$ as $h, k \longrightarrow 0$

FD molecule — PDE operator

- Weaker than convergence, but easier to show
- requires Truncation error $\longrightarrow 0$ as mesh lengths $\longrightarrow 0$

3.) Stability: u_i^{ℓ} bounded for bounded BCs, ICs + forcing

Generally... $(2) + (3) \longrightarrow (1)$

Lax Equivalence Theorem: Given a properly posed initial boundary-value problem and a FD approx to it that is <u>consistent</u>, then <u>stability</u> is the necessary and sufficient condition for <u>convergence</u>

Convergence:

- Examine in terms of "discretization error"

$$\mathcal{U}_i^{\ell} - u_i^{\ell} \equiv \epsilon_i^{\ell}$$

- Depends on h, k mesh lengths, FD molecule

- Difficult to investigate ... usually in terms of derivatives (i.e. Truncation error) we don't know

- Lax Equivalence saves the day!

Try it once ... Simplest scenario $\Rightarrow$ Euler explicit

Strategy: write out difference Eqns w/ exact Sol'n to PDE keeping equality by retaining Truncation error terms ("ξ trick")

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2} \Rightarrow$$

$\ell' \in [\ell, \ell+1]$ $\qquad$ $i' \in [i-1, i+1]$

$$\frac{U_i^{\ell+1} - U_i^{\ell}}{k} - \frac{k}{2}\frac{\partial^2 U_i^{\ell'}}{\partial t^2} = D\left[\frac{U_{i-1}^{\ell} - 2U_i^{\ell} + U_{i+1}^{\ell}}{h^2} - \frac{h^2}{12}\frac{\partial^4 U_{i'}^{\ell}}{\partial x^4}\right]$$

Now subtract FD approximation to get difference eqn in ϵ_i^{ℓ} ...

$$U_i^{l+1} - U_i^l = rU_{i-1}^l - 2rU_i^l + rU_{i+1}^l + \frac{k^2}{2}\frac{\partial^2 U_i^{l'}}{\partial t^2} - \frac{Dkh^2}{12}\frac{\partial^4 U_{i'}^l}{\partial x^4}$$

$$u_i^{l+1} - U_i^l = ru_{i-1}^l - 2ru_i^l + ru_i^l$$

$$\epsilon_i^{l+1} - \epsilon_i^l = r\epsilon_{i-1}^l - 2r\epsilon_i^l + r\epsilon_{i+1}^l + \frac{k}{2}\left(k\frac{\partial^2 U_i^{l'}}{\partial t^2} - \frac{Dh^2}{6}\frac{\partial^4 U_{i'}^l}{\partial x^4}\right)$$

Want to show $\epsilon_i^l \to 0$ as $h,k \to 0$

$$\epsilon_i^{l+1} = r\epsilon_{i-1}^l + (1-2r)\epsilon_i^l + r\epsilon_{i+1}^l + \frac{k}{2}\left(k\frac{\partial^2 U_i^{l'}}{\partial t^2} - \frac{Dh^2}{6}\frac{\partial^4 U_{i'}^l}{\partial x^4}\right)$$

IF $(1-2r) > 0$ ... i.e. $r < 1/2$ (Recall we said this was required for stability)

and we let: $\epsilon^l = \max\limits_{\text{all } i} |\epsilon_i^l|$

$$A \geq \max_{\text{all } i, \text{all } l}\left\{ \frac{\left|\frac{\partial^2 U_i^{l'}}{\partial t^2}\right|}{2}, \frac{D}{12}\left|\frac{\partial^4 U_{i'}^l}{\partial x^4}\right| \right\}$$

then

$$|\epsilon^{l+1}| \leq |\epsilon^l| + A(k+h^2)k$$

but $\epsilon^0 = 0$ ... IC's match everywhere exactly!

$$\epsilon^1 \leq A(k+h^2)k$$

$$\epsilon^2 \leq \epsilon^1 + A(k+h^2)k = 2A(k+h^2)k$$

$$\vdots$$

$$\epsilon^l \leq A(k+h^2)lk = A(k+h^2)T$$

Can make ϵ^{ℓ} arbitrarily small as $h,k \to 0$ provided $r < 1/2$ and T finite

Aside: Common to try to cancel the leading error in the time discretization w/ leading error in space discretization .... leads to "improved" accuracy but practical utility questionable due to very small step sizes usually required

e.g. For Euler explicit, let's expand the truncation errors by an additional term:

$$\frac{k}{2}\frac{\partial^2 U_i^{\ell}}{\partial t^2} + \frac{k^2}{3!}\frac{\partial^3 U_i^{\ell'}}{\partial t^3} \quad \left\{ \begin{array}{l} \text{Error from Forward Diff} \\ \text{expression for } \frac{\partial U}{\partial t} \end{array} \right\}$$

$$-\frac{2Dh^2}{4!}\frac{\partial^4 U_i^{\ell}}{\partial x^4} - \frac{2h^4 D}{6!}\frac{\partial^6 U_{i'}^{\ell}}{\partial x^6} \quad \left\{ \begin{array}{l} \text{Error from Central Diff} \\ \text{expression for } \frac{\partial^2 U}{\partial x^2} \end{array} \right\}$$

IF $k = \frac{h^2}{6D}$, then

$$\underbrace{\frac{h^2}{12D}\left[\frac{\partial^2 U_i^{\ell}}{\partial t^2} - D^2\frac{\partial^4 U_i^{\ell}}{\partial x^4}\right]}_{=0 \text{ since } \frac{\partial}{\partial t}\left[\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2}\right] = \frac{\partial^2 U}{\partial t^2} = D\frac{\partial^2}{\partial x^2}\left(\frac{\partial U}{\partial t}\right) = D^2\frac{\partial^4 U}{\partial x^4}} + O(k^2 + h^4) \leftarrow \text{Improved accuracy!!}$$

Consistency

- FD molecule consistent w/ PDE if Truncation error $\to 0$ as $h, k \to 0$

- easy when we replace PDE w/ standard FD expressions since already know error terms

e.g. $\dfrac{\partial U}{\partial t} = D\dfrac{\partial^2 U}{\partial x^2} \Rightarrow \dfrac{\Delta U_i^\ell}{k} + O(k) = D\dfrac{\delta_x^2 U_i^\ell}{h^2} + O(h^2)$

Truncation error is $O(k+h^2) \to 0$ as $k, h \to 0$

So FD molecule:

$$\begin{array}{ccccc} & & 1 & & \\ & & | & & \\ -r & — & 2r-1 & — & -r \end{array} = 0$$

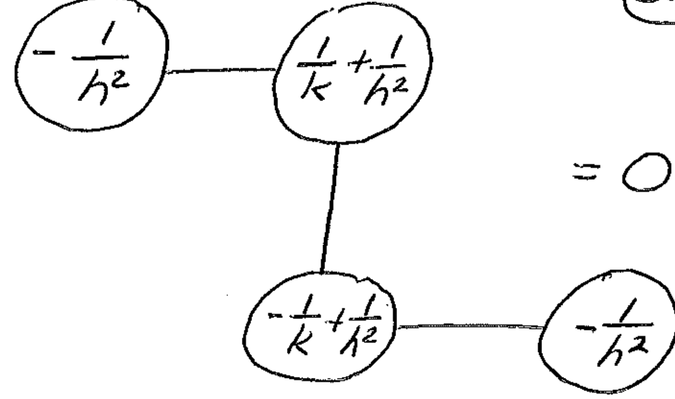
consistent w/ $\dfrac{\partial U}{\partial t} = D\dfrac{\partial^2 U}{\partial x^2}$

- Sometimes molecules not readily recognized as standard FD expressions

- Must work "Backwards"... Truncation error measures amount by which U_i^ℓ does not satisfy FD equations

- For consistency, want FD molecule of $U_i^\ell \to 0$ as $h, k \to 0$

e.g. Given FD molecule



Is it consistent w

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \ ?$$

FD Eqn: $$\frac{1}{h^2}\left(U_i^{l+1} - U_{i-1}^{l+1}\right) + \underbrace{\frac{1}{k}\left(U_i^{l+1} - U_i^{l}\right)} + \frac{1}{h^2}\left(U_i^{l} - U_{i+1}^{l}\right)$$

We recognize this!

$$\frac{\partial U_i^l}{\partial t} + \frac{k}{2}\frac{\partial^2 U_i^{l'}}{\partial t^2}$$

Other terms we don't see so clearly... plug back in Talyor Series for each:

$$U_i^{l+1} = U_i^l + k\frac{\partial U_i^l}{\partial t} + \frac{k^2}{2}\frac{\partial^2 U_i^l}{\partial t^2} + \frac{k^3}{3!}\frac{\partial^3 U_i^l}{\partial t^3} + \dots$$

$$-U_{i-1}^{l+1} = -U_i^l - k\frac{\partial U_i^l}{\partial t} + h\frac{\partial U_i^l}{\partial x} + kh\frac{\partial^2 U_i^l}{\partial t\,\partial x} - \frac{k^2}{2}\frac{\partial^2 U_i^l}{\partial t^2} - \frac{h^2}{2}\frac{\partial^2 U_i^l}{\partial x^2} + \dots$$

$$U_i^l = U_i^l$$

$$-U_{i+1}^l = -U_i^l - h\frac{\partial U_i^l}{\partial x} - \frac{h^2}{2}\frac{\partial^2 U_i^l}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 U_i^l}{\partial x^3} - \frac{h^4}{4!}\frac{\partial^4 U_i^l}{\partial x^4} + \dots$$

$$\underbrace{\frac{k^3}{3!}\frac{\partial^3 U_i^l}{\partial t^3} + kh\frac{\partial^2 U_i^l}{\partial t\,\partial x} - h^2\frac{\partial^2 U_i^l}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 U_i^l}{\partial x^3} + \dots}$$

gets multiplied by $1/h^2$

Put all together:

$$\underbrace{\frac{\partial U_i^l}{\partial t} - \frac{\partial^2 U_i^l}{\partial x^2}}_{\text{Here is the PDE we want at } (x_i, t_l)} + \frac{k}{2}\frac{\partial^2 U_i^l}{\partial t^2} - \frac{h}{3!}\frac{\partial^3 U_i^l}{\partial x^3} + \frac{k}{h}\frac{\partial^4 U_i^l}{\partial x^4} + \frac{k^3}{6h^2}\frac{\partial^3 U_i^l}{\partial t^3} = 0$$

Vanish as $k, h \to 0$

Could be trouble

Should vanish as long as $k \to 0$ at least as fast as h

- Consider $k = \alpha h$ where α = constant

$$\underbrace{\frac{\partial U_i^l}{\partial t} - \frac{\partial^2 U_i^l}{\partial x^2} + \alpha \frac{\partial^4 U_i^l}{\partial x^4} = 0}$$

Inconsistent w/ $\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0$!

molecule converges to this PDE $\Rightarrow O(k+h)$

- Consider $k = \alpha h^2$ where α = constant

$$\frac{\partial U_i^l}{\partial t} - \frac{\partial^2 U_i^l}{\partial x^2} = 0 \Rightarrow \text{Consistent w/ PDE}$$

$O(k+h)$

Stability

- Given bounded ICs, BCs + forcing get bounded sol'n to FD equations

- Two views....

  (a) Lax-Richtmeyer : at a fixed time, T, sol'n of FD equations remain bounded as $k \to 0$ (assuming h related to k such that $h \to 0$ as $k \to 0$)

  (b) Practical approach : h, k are fixed and sol'n propagated forward from $t=0$ to $t = jk$ ... then stability defined in terms of boundedness as $j \to \infty$ for k fixed

- Two approaches to Stability analysis

  1.) Matrix Methods ... cast FD propagation in form $u^{l+1} = Au^{l} + b^{l}$ and study properties of A

  2.) Fourier Method (Von Neumann)... examine the propagation of Fourier components by the FD molecule

Fourier Analysis Supplement

- Recall Fourier Series ... valid for any $f(x)$ continuous on $[0,\ell]$

$$f(x) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell}$$

$$= \sum_{n=0}^{\infty} C_n\left(\frac{e^{j\frac{n\pi x}{\ell}} + e^{-j\frac{n\pi x}{\ell}}}{2}\right) + B_n\left(-j\left(\frac{e^{j\frac{n\pi x}{\ell}} - e^{-j\frac{n\pi x}{\ell}}}{2}\right)\right)$$

$$= \sum_{n=-\infty}^{\infty} A_n e^{j\sigma_n x} \quad \text{where} \quad \sigma_n = \frac{n\pi}{\ell} = \frac{2\pi}{L_n}$$

wavelength $= \frac{2\ell}{n}$

$A_0 = C_0/2$

$A_n = \frac{1}{2}(C_n - jB_n) \quad n>0$

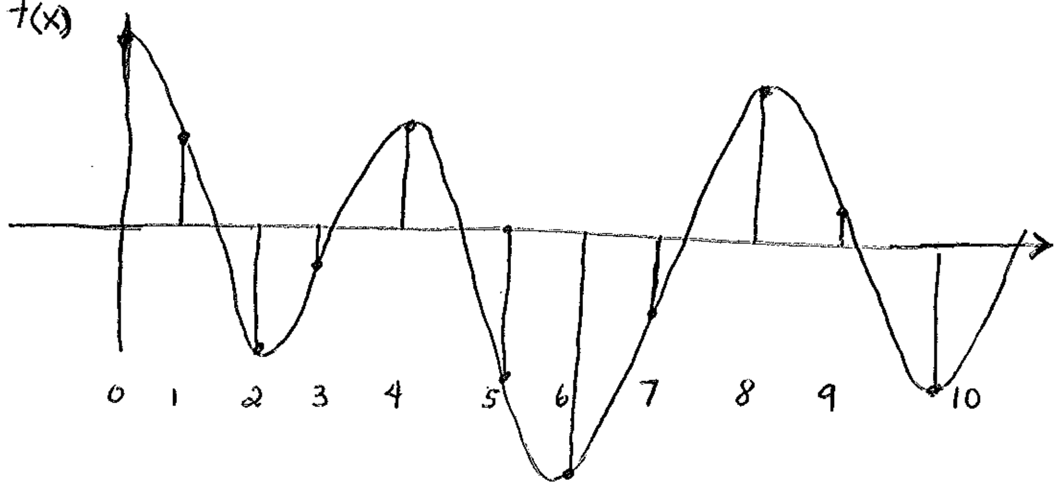
$A_n = \frac{1}{2}(C_{-n} + jB_{-n}) \quad n<0$

- on a discrete set of sample points

$X_i = ih$

$$\Rightarrow f(x_i) = \sum_{n=-\infty}^{\infty} A_n e^{j\sigma_n h i}$$

e.g.



$$f(x_i) = \cos(\pi/6\, i) + \frac{1}{4}\cos(\pi/4\, i) + 2\cos(\pi/2\, i)$$

- Express $U_i^0 = f(x_i)$ as sum of Fourier modes

Stability ... ask do these modes stay bounded as $U_i^0 \to U_i^1 \to U_i^2 \ldots U_i^l \to U_i^{l+1}$???

- Must examine all possible σh values !!

Why? Since may not need them all to represent specific ICs!

Answer: IF not then Stability dependent on ICs ... i.e. problem dependent for same governing equation ... not very useful

- More importantly: Rounding errors introduced are differenced by same molecule as U_i^ℓ

e.g. $U_i^\ell + \epsilon_i^\ell =$ Computer sol'n to FD equations

U_i^ℓ ← Exact sol'n to FD equations

Euler Explicit:

$$U_i^{\ell+1} + \epsilon_i^{\ell+1} = r\left(U_{i-1}^\ell + \epsilon_{i-1}^\ell\right) - (2r-1)\left(U_i^\ell + \epsilon_i^\ell\right) + r\left(U_{i+1}^\ell + \epsilon_i^\ell\right)$$

$$U_i^{\ell+1} = r\,U_{i-1}^\ell - (2r-1)U_i^\ell + r\,U_{i+1}^\ell$$

$$\epsilon_i^{\ell+1} = r\,\epsilon_{i-1}^\ell - (2r-1)\epsilon + r\,\epsilon_{i+1}^\ell$$

- Need to make sure Rounding errors once introduced remain bounded

Recall... Fourier Series is an approach to analytic sol'n of PDEs

e.g. $\dfrac{\partial U}{\partial t} - D\dfrac{\partial^2 U}{\partial x^2} = 0$; w/ $U(x,0) = G(x)$

$U(0,t) = f(t)$

$U(\ell,t) = g(t)$

Substitute in Fourier Series w/ time-dependent coefficients: $u(x,t) = \sum_{n=-\infty}^{\infty} A_n(t)\, e^{j\sigma_n x}$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \underbrace{\left[\frac{dA_n}{dt} + D\sigma_n^2 A_n(t)\right]}\, e^{j\sigma_n x} = 0$$

Only way to satisfy requires this to vanish

$$\frac{dA_n}{dt} + D\sigma_n^2 A_n = 0 \qquad \text{1ST Order ODE in } t$$

We know soln is $A_n(t) = C_n e^{-D\sigma_n^2 t}$

$$\therefore\ u(x,t) = \sum_{n=-\infty}^{\infty} C_n\, e^{-D\sigma_n^2 t}\, e^{j\sigma_n x}$$

Deteemine from BCs

Deteemine from ICs

- Do same thing for Discrete System (i.e. Difference equations)

Von Neumann (Fourier) Stability Analysis

- Idea... expand the spatial distribution of IC's (i.e. sol'n at some point in time) as Fourier Series

$$u_i^0 = \sum_n A_n e^{j\sigma_n x_i}$$

Have $u(x,0)$; need to find A_n's such that $u(x_i,0) = u_i^0$

- Examine how each term in sum is propagated as $\ell = 1, 2, \ldots$ (in general t_ℓ to $t_{\ell+1}$) by FD molecule

- Stability... FD molecule must not allow any term in sum (i.e. Fourier mode) to grow as sol'n is advanced in time

- Sufficient to look at general form of single term and consider all possible σ values
    - single term due to linearity
    - all σ values due to Round-off
    - Don't care about A_n's $\Rightarrow$ Want $\dfrac{u_i^{\ell+1}}{u_i^\ell}$

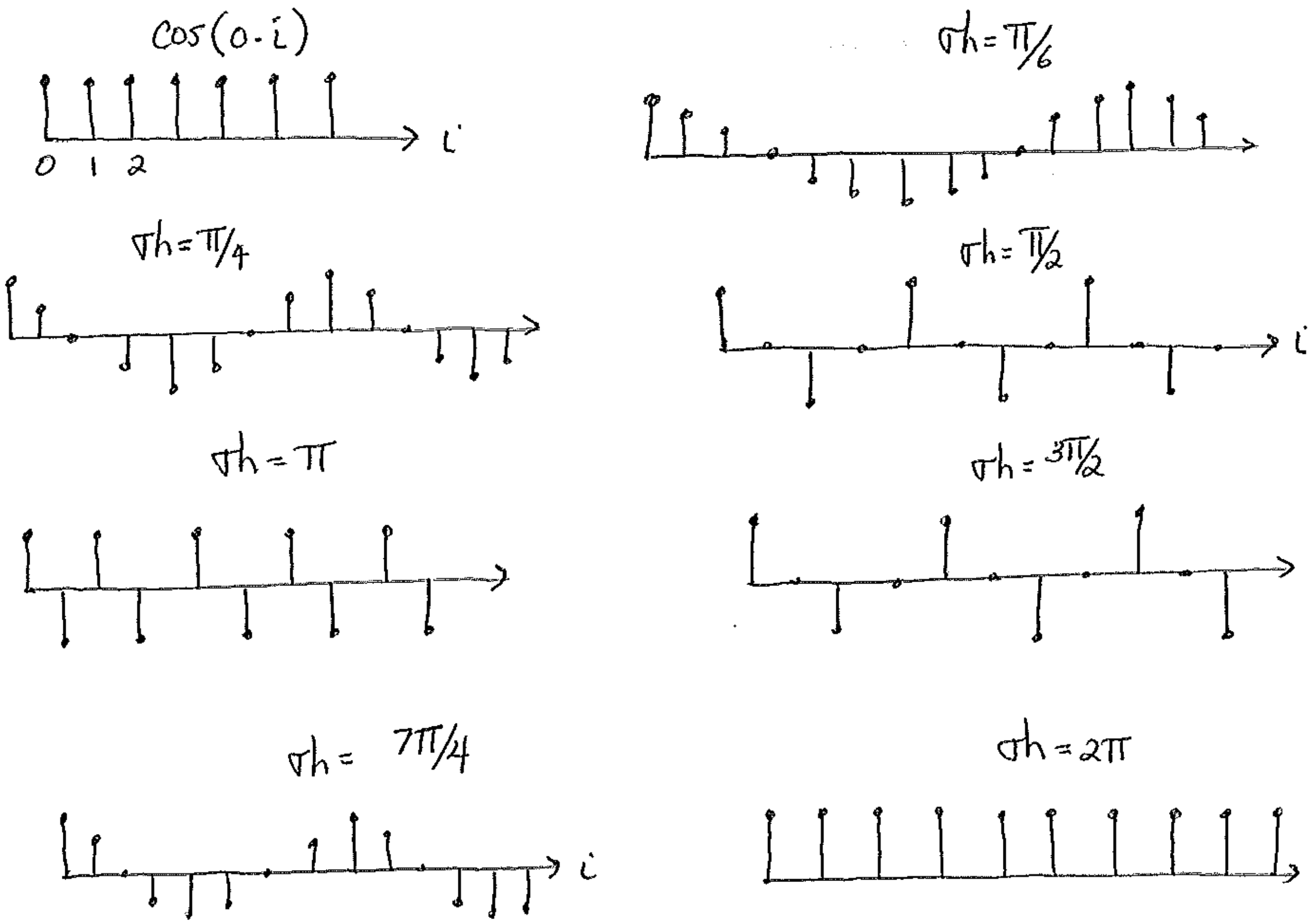
- σh is key quantity $\Rightarrow$ "Dimensionless wavenumber"

$$u_i = e^{j\sigma x_i} = e^{j\sigma h i}$$

- $0 \le \sigma h \le \pi$ ... most rapid variation on a mesh is node-to-node oscillation

  - As σh increases from zero, $e^{j\sigma h i}$ has increasing rate of oscillation which peaks at $\sigma h = \pi$

e.g. $\cos(\sigma h i)$



- Define "Amplification factor" of the FD eqns

$$U_i^{\ell+1} = U_i^{\ell}\gamma_0$$

$\left(\text{Analytically we know } \underbrace{\gamma \equiv e^{\alpha\Delta t}}_{}\right)$

$$U(x,t) = e^{\alpha t} e^{j\sigma x} \Rightarrow \frac{U(x,t+\Delta t)}{U(x,t)} = e^{\alpha\Delta t}$$

- Relate all (space, time) points in the FD molecule to pt (i,ℓ) using the defining relations

e.g. $U_i^{\ell+1} = \gamma_0 U_i^{\ell}$

$$U_{i-1}^{\ell} = e^{-j\sigma h} U_i^{\ell} \Rightarrow e^{j\sigma(x_i - h)} = e^{-j\sigma h} e^{j\sigma x_i} = e^{-j\sigma h} U_i$$

$$U_{i+1}^{\ell+1} = e^{j\sigma h}\gamma_0 U_i^{\ell}$$

etc
⋮

- Provides a relationship between γ_0 and σh

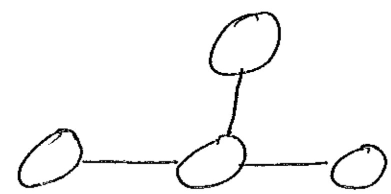
- Stability requires $-1 \le \gamma_0 \le 1$ for all possible σh (i.e. $0 \le \sigma h \le \pi$)

- Bounded oscillations develop for $-1 \le \gamma_0 < 0$

- Formally method only valid for
  - linear equations w/ constant coefficients
  - Uniform mesh
  - BCs at infinity

- Generally get same results as Matrix method (i.e. BCs effect stability in minor way relative to FD equations themselves)

e.g. Examine Euler Explicit



$$u_i^{l+1} - u_i^l = r\,\delta_x^2 u_i^l$$
$$= r\left(u_{i-1}^l - 2u_i^l + u_{i+1}^l\right)$$

$$\Rightarrow (\gamma_0 - 1)\,u_i^l = r\left(e^{-j\sigma h} - 2 + e^{j\sigma h}\right) u_i^l$$

$$\gamma_0 = 1 - 2r(1 - \cos\sigma h) \Rightarrow \text{Note: } \cos\sigma h = 1 - 2\sin^2\frac{\sigma h}{2}$$
$$\gamma_0 = 1 - 4r\sin^2\frac{\sigma h}{2}$$

$\Rightarrow$ For stability ... $|\gamma_0| < 1 \Rightarrow \underbrace{-1 \leq 1 - 2r(1-\cos\sigma h) \leq 1}_{\text{For \underline{all} possible } \sigma\text{'s}}$

But $0 < \sigma h \leq \pi \Rightarrow 0 < 1 - \cos\sigma h \leq 2$

- Conclude $\gamma_0 < 1$

γ_0 can be negative $\Rightarrow$ Oscillation in t!

Negative when: $1 - 2r(1-\cos\sigma h) < 0$

$$2r(1-\cos\sigma h) - 1 > 0$$

$$\text{i.e.} \quad r > \frac{1}{2(1-\cos\sigma h)}$$

- Conclude: $r > 1/4$ produces $\gamma_0 < 0$

$\Rightarrow$ Shortest waves (i.e. highest frequency modes) oscillate ..... entirely a numerical artefact

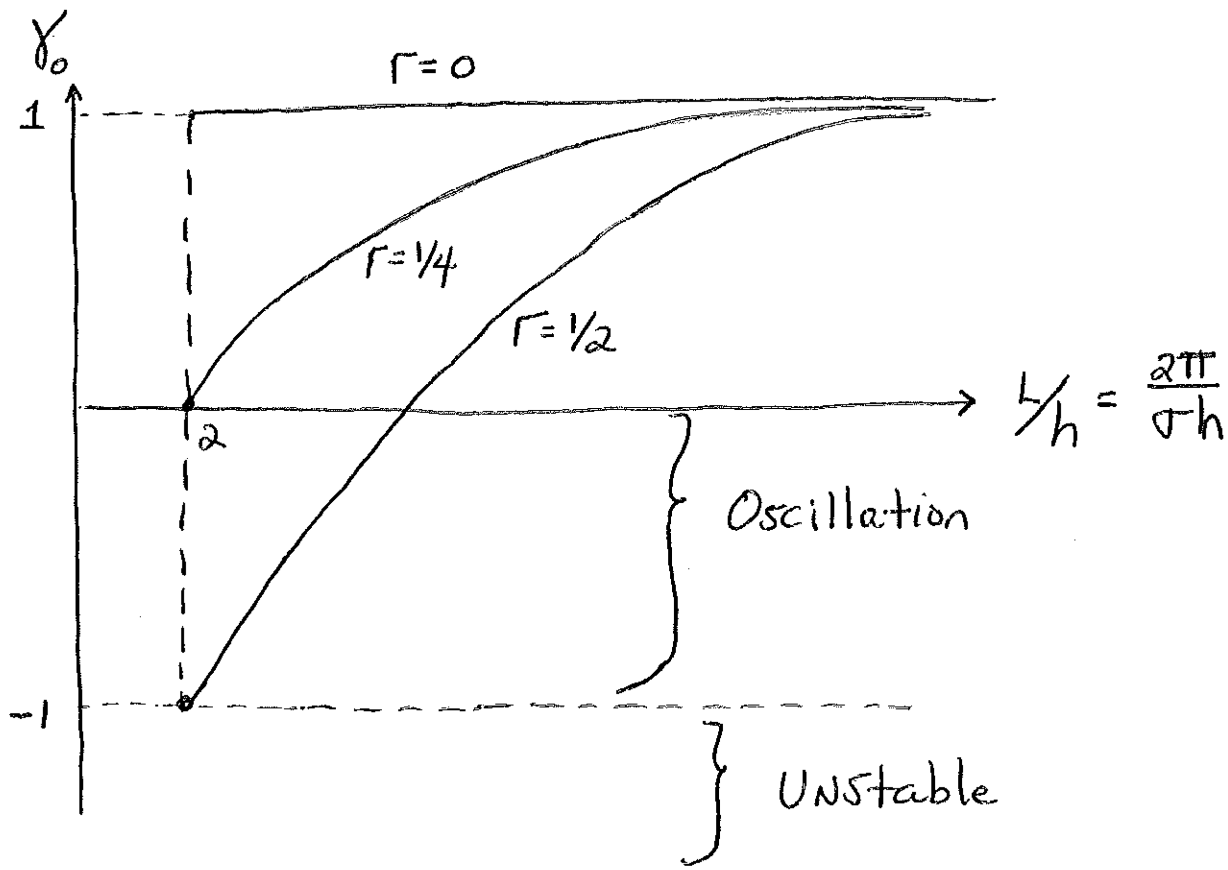
- Unstable when $\gamma_0 < -1$

$$1 - 2r(1-\cos\sigma h) < -1$$

$$2 < 2r(1-\cos\sigma h)$$

$$\frac{1}{1-\cos\sigma h} < r$$

i.e. $r > 1/2$ (Shortest waves have unstable oscillation)



Rule of Thumb... Short wavelengths are first to go

- Develop spurious oscillations
- Oscillations become fatal as k increases

What about accuracy? Can study

$$\frac{\text{Numerical aplification factor}}{\text{Analytical amplification factor}} \Rightarrow \frac{\gamma_0}{\gamma}$$

Analytically we have $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$

w/ $u = e^{\alpha t} e^{j\sigma x} \Rightarrow \alpha = -D\sigma^2$

- all modes decay (i.e. all $\alpha < 0$)
- longest waves decay slowest
- sol'n gets smoother over time

Now $\gamma = e^{\alpha \Delta t} = e^{-D\sigma^2 k} = e^{-r(\sigma h)^2}$

- but as $k \to 0$, $\gamma_0 \to \gamma \to 1$ don't learn much

Common to introduce characteristic time, T

and examine $\left(\frac{\gamma_0}{\gamma}\right)^N$ where $N = \frac{T}{k}$

(i.e. # time-steps to advance sol'n by T)

- Use time constant of σ in analytic sol'n

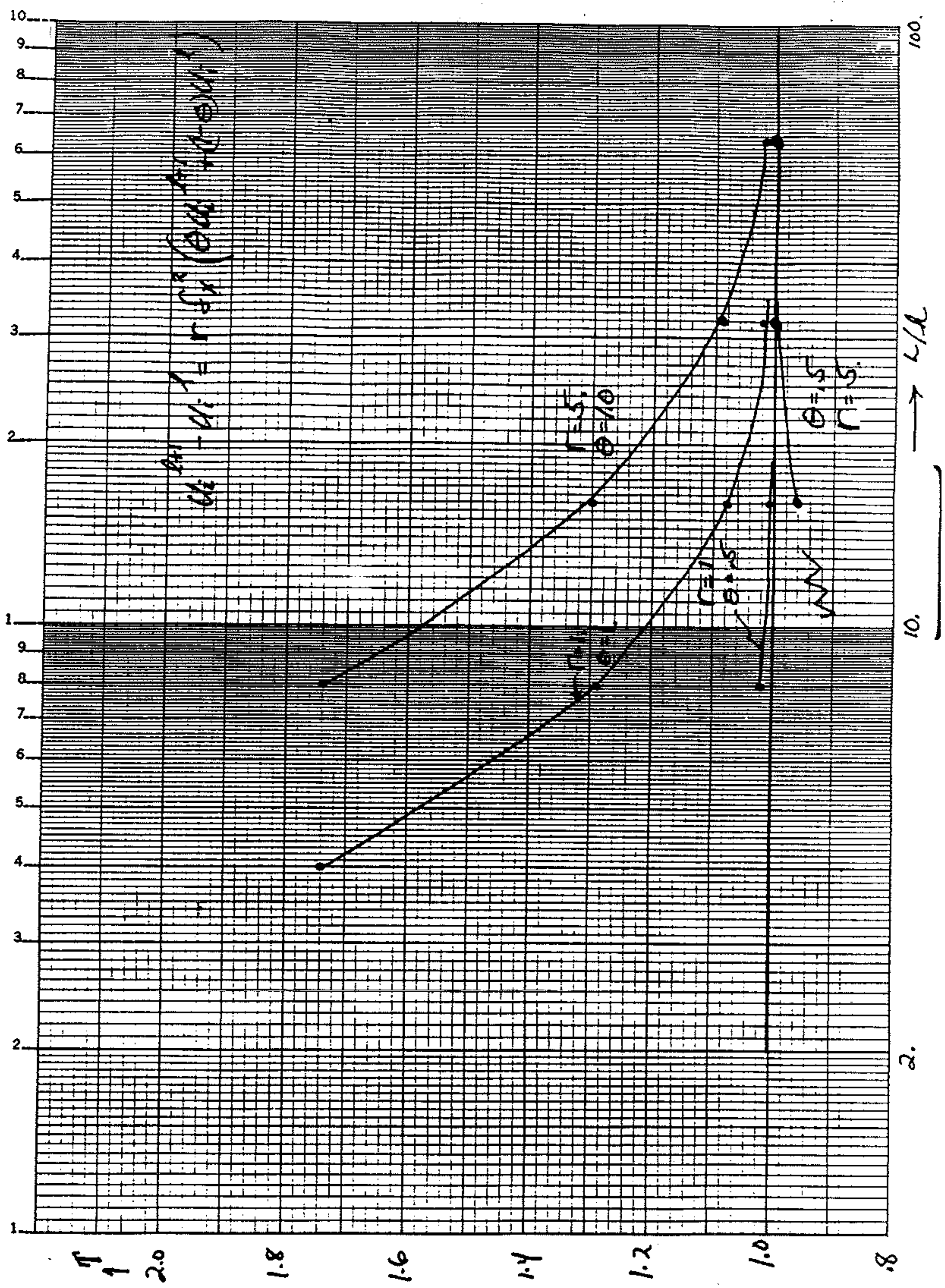
i.e. $T = \frac{1}{|\alpha|} = \frac{1}{D\sigma^2}$, $\therefore N = \frac{1}{D\sigma^2 k} = \frac{1}{r(\sigma h)^2}$

Define $\Gamma = \left(\frac{\gamma_0}{\gamma}\right)^N = \frac{\gamma_0^{\frac{1}{r(\sigma h)^2}}}{\left(e^{-r(\sigma h)^2}\right)^{1/r(\sigma h)^2}} = \frac{\gamma_0^{1/r(\sigma h)^2}}{e^{-1}}$

"Propagation Factor"

Plot T vs $\sigma h = \frac{2\pi h}{L}$ for various r

T = 1 is perfect



Also note...

$$\gamma = e^{-r(\sigma h)^2} = 1 - r(\sigma h)^2 + \frac{(r(\sigma h)^2)^2}{2} - \frac{(r(\sigma h)^2)^3}{3!} + \dots$$

$$\gamma_0 = 1 - 2r(1-\cos\sigma h)$$

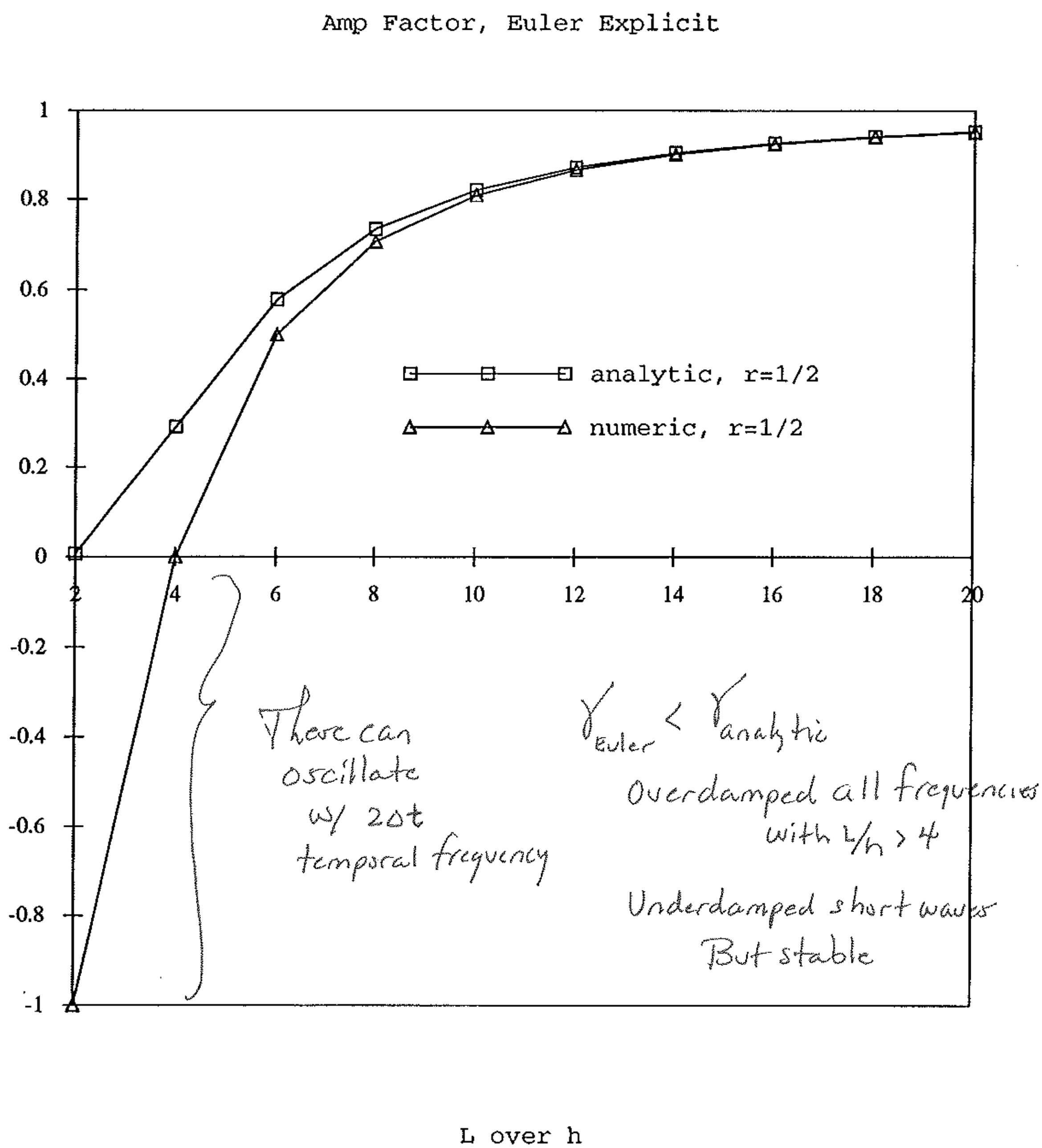
$$= 1 - 2r\left[\frac{(\sigma h)^2}{2!} - \frac{(\sigma h)^4}{4!} + \frac{(\sigma h)^6}{6!} - \dots\right]$$

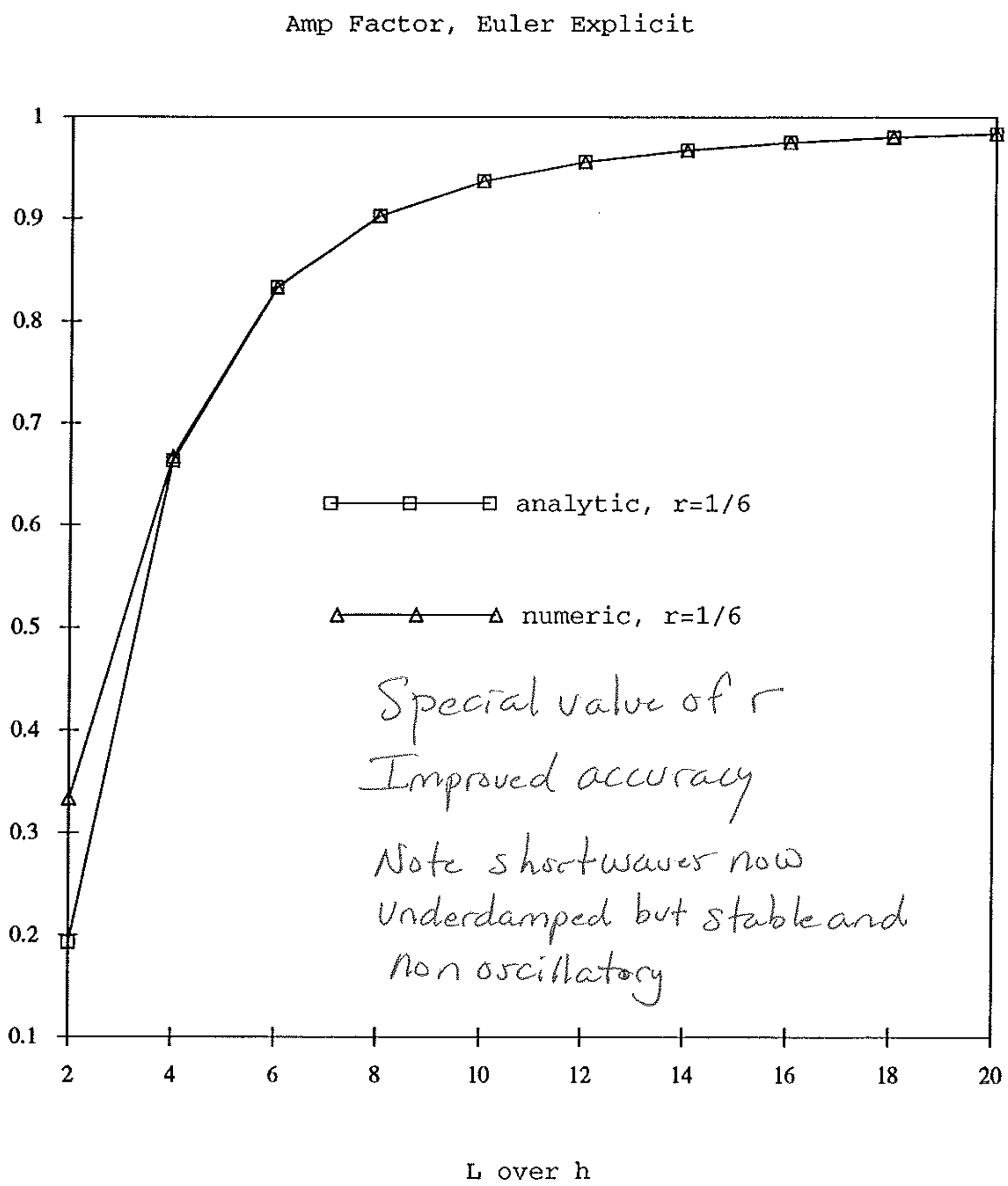
$$= 1 - r(\sigma h)^2 + \underbrace{\frac{r(\sigma h)^4}{12}}_{\text{leading error term}} - \frac{2r(\sigma h)^6}{6!} + \dots$$

But $\frac{r(\sigma h)^4}{12} = \frac{r^2(\sigma h)^4}{2}$ when $r = 1/6$

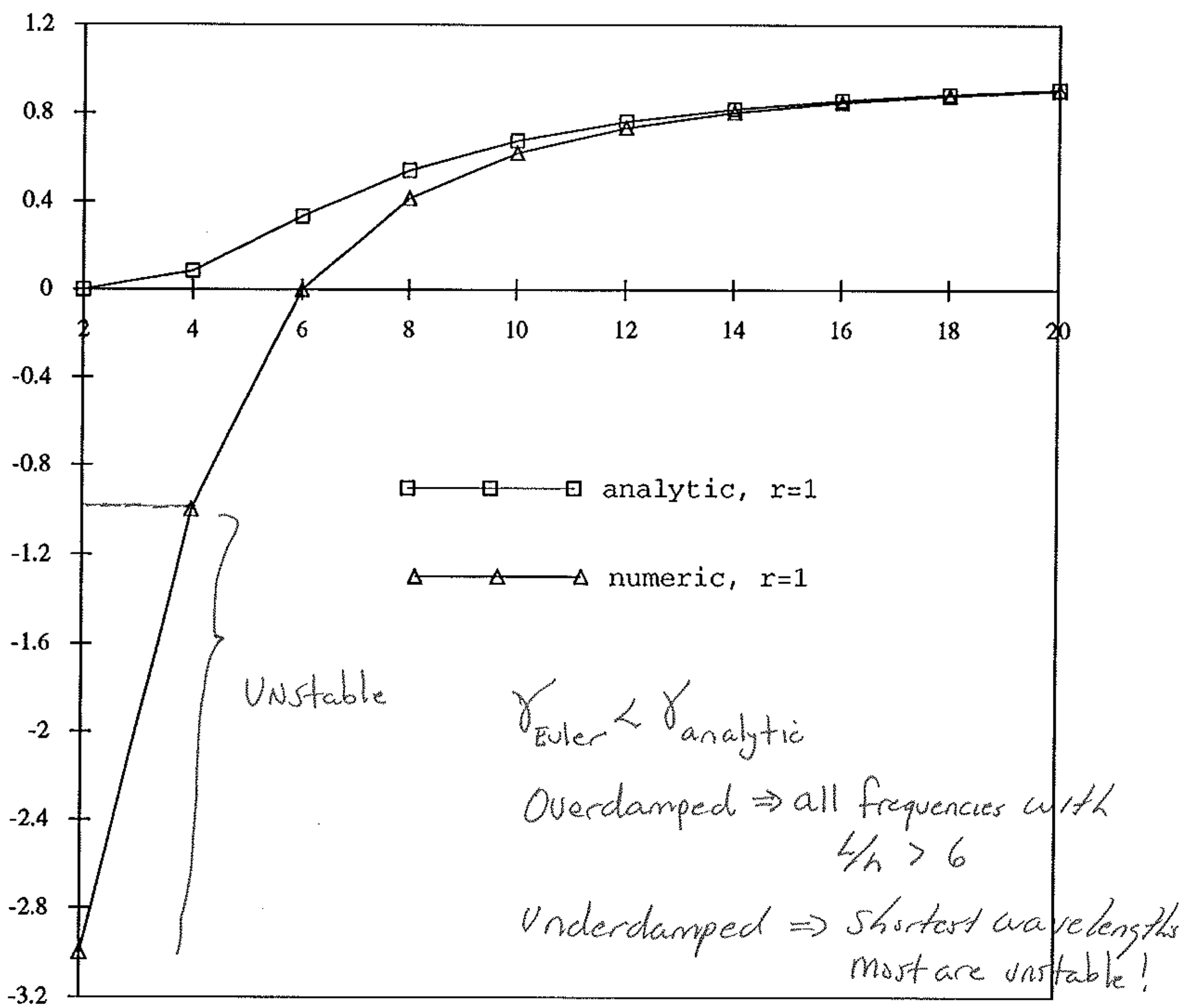
Error is "pushed back" one more term (we saw this earlier ... Δt error just cancels leading Δx error)

For Euler: $\gamma < \gamma_0 \Rightarrow$ Numerical sol'n underdamped (generally true ... depends on r and σh)


Amp Factor, Euler Explicit
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
2
4
6
8
10
12
14
16
18
20
analytic, r=1/2
numeric, r=1/2
There can oscillate w/ 2Δt temporal frequency
$\gamma_{Euler} < \gamma_{analytic}$
Overdamped all frequencies with L/h > 4
Underdamped short waves
But stable
L over h


Amp Factor, Euler Explicit
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
2
4
6
8
10
12
14
16
18
20
analytic, r=1/6
numeric, r=1/6
Special value of r
Improved accuracy
Note shortwaves now
Underdamped but stable and
non oscillatory
L over h

Amp Factor, Euler Explicit



L over h

Can examine any scheme in this manner

e.g. "Richardson" $u_i^{l+1} - u_i^{l-1} = r\delta_x^2 u_i^l$

$$\gamma - \frac{1}{\gamma} = 2r(\cos\sigma h - 1)$$

$$\gamma^2 + 2r(1-\cos\sigma h)\gamma - 1 = 0 \qquad \underline{2 \text{ Roots!}}$$

- Stability for general quadratic

$$a\gamma^2 + b\gamma + c = 0 \Rightarrow |\gamma| \le 1$$

when $\frac{c}{a} \le 1$ <u>and</u> $|b| \le a + c$

In our case... $\frac{c}{a} = -1 \le 1$ always

$$|b| = 2r\underline{(1-\cos\sigma h)} \le 0$$

always positive!

No value of r satisfies this constraint for all values of σh ... short waves are the biggest offenders as usual!

Unconditionally unstable!!

- Stability Analysis using Matrix Methods

• In Lax-Richtmyer view... if we have the scheme $u^{l+1} = Au^l + c^l$; A grows in size; need to show $\|A\| \leq 1$ guarantees stability (and $\therefore$ convergence for a consistent molecule)

• In practical view of fixed mesh lengths... if have a scheme of form $u^{l+1} = Au^l + c^l$ ... A has fixed size; sufficient to show $\rho(A) \leq 1$ to ensure boundedness

• Formally must have
$$\rho(A) \leq \|A\| \leq 1 \quad \text{as size of } A \longrightarrow \infty$$
to guarantee convergence for a consistent scheme (possible to have $\rho(A) \leq 1$ w/ $\|A\| > 1$)

e.g. Euler Explicit: $u_i^{l+1} = r u_{i-1}^l + (1-2r) u_i^l + r u_{i+1}^l$
w/ Type I BCs...

$$A = \begin{bmatrix} 1-2r & r & 0 & \cdots & & \\ r & 1-2r & r & 0 & \cdots & 0 \\ 0 & r & 1-2r & r & 0 & \cdots \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & \ddots & \ddots \end{bmatrix}$$

$\|A\|_\infty = |r| + |1-2r| + |r|$; Need $\|A\|_\infty \leq 1$

if $1-2r > 0$ the $\|A\|_\infty = 1$

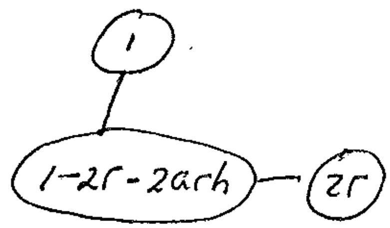
$1-2r < 0$ $\quad \|A\|_\infty = 4r - 1 > 1$ since $r > 1/2$

Conclude ... Need $r \le 1/2$ ... Same as Von Neumann

Now if we have derivative BCs ... e.g. Type III

$$\frac{\partial u}{\partial x} = au + b \quad \text{at } x=0 \text{ boundary}$$

... then molecule becomes



i.e. $u_{-1} = u_1 - 2ahu_0 - bh$

So A has the structure ...

$$\begin{bmatrix} 1-2r(1+ah) & 2r & & & \\ r & 1-2r & r & & \\ & r & 1-2r & r & \\ & & \ddots & \ddots & \ddots \\ & & & & \end{bmatrix}$$

- all but row 1 require $r \le 1/2$ for $\|A\|_\infty \le 1$
- must see if row 1 changes this restriction ...

we want $|1-2r(1+ah)| + |2r| \le 1$

Two cases to consider:

(a) $1-2r(1+ah) \ge 0$ (i.e. diagonal term positive)

then $|1-2r(1+ah)| + 2r = 1-2r(1+ah) + 2r \le 1$

$1 - 2rah \le 1$ always OK

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But for diagonal coefficient to be positive

$$2r(1+ah) \leq 1 \Rightarrow r \leq \frac{1}{2(1+ah)}$$

So if coefficient is positive, problem is stable and we need $r \leq \frac{1}{2(1+ah)}$ to achieve this

(b) $1-2r(1+ah) \leq 0$ (diagonal is negative)

then $|1-2r(1+ah)| + |2r| = 2r(1+ah) - 1 + 2r \leq 1$

$$2r(2+ah) \leq 2$$

$$r < \frac{1}{2+ah}$$

But $\frac{1}{2(1+ah)} < \frac{1}{2+ah}$ ... So we can maintain stability when diagonal turns negative provided $r \leq \frac{1}{2+ah}$

IF r gets any bigger... diagonal still negative, but $\|A\|_\infty > 1$

- Also note $\frac{1}{2+ah} < 1/2$ i.e. Stability restriction greater w/ Type III than Type I BCs!