Constrained Minimization of δ (Con't)

B. Iterative Sol'n

- Algorithm:

  1. "Guess" b ... This is BPE of the unknown

  2. Solve $Ku = b$ "Forward System" "Model Eqns"

  3. Evaluate $\delta = d - Su$

  4. Solve for λ

     $$K^T \lambda = 2 S^T W_\delta \delta$$

     "Adjoint System" Structurally, transpose of Forward

  5. Evaluate $\frac{\partial \Omega^+}{\partial b} = 2 W_b b - \lambda$

IF zero (or below some tolerance) stop, otherwise adjust b and repeat 2-5 until convergence.

- Step 5 is key to efficiency and differentiates various methods, e.g. Steepest Gradient Descent, Conjugate Gradient Method.

- Computationally intensive parts, Steps 2 + 4 repetitive sol'n of sparse, well-conditioned FE matrices, involving K, K^T. Single factorization at beginning is typical, followed by 2 backsubstitutions per iteration, so Step 5 key

Gradient Descent

- If $\nabla\Omega^+ \neq 0$, want a way to update current estimate of b, so $\nabla\Omega^+$ gets smaller

$$b_{k+1} = b_k + \Delta b$$

i.e. find Δb at each iteration to reduce $\nabla\Omega^+$

- Write $\Delta b = \alpha\,\partial b$, i.e. as magnitude α, direction $\underbrace{\partial b}_{\text{unit vector}}$

then updating b involves 2 questions,
1. What direction to go in? (∂b)
2. How far to go in that direction? (α)

- Start w/ question 2, find α
  - Given ∂b, search along a line in b-space with idea of minimizing Ω^+ along this line by selecting optimal α
  - Note effect of ∂b ... defines direction of u, ∂u which in turn determines direction of misfit, $\partial \delta$

$$\partial u = K^{-1}\partial b$$
$$\partial \delta = -S\,\partial u$$

Algebra linear in α

$$\Delta b = \alpha\,\partial b$$
$$\Delta u = K^{-1}\Delta b = \alpha\,\partial u$$
$$\Delta \delta = \alpha\,\partial \delta$$

then Ω^+ at the new position...

$$\Omega^+_{k+1} = (\delta_k + \Delta\delta)^T W_\delta (\delta_k + \Delta\delta) + (b_k + \Delta b)^T W_b (b_k + \Delta b)$$

$$= \Omega^k + \Delta\delta^T W_\delta \delta_k + \delta_k^T W_\delta \Delta\delta + \Delta\delta^T W_\delta \Delta\delta$$
$$+ \Delta b^T W_b b_k + b_k^T W_b \Delta b + \Delta b^T W_b \Delta b$$

Assuming W_δ, W_b symmetric

$$\Delta\Omega^+ = 2\alpha\,\partial\delta^T W_\delta \delta_k + \alpha^2 \partial\delta^T W_\delta \partial\delta$$
$$+ 2\alpha\,\partial b^T W_b b_k + \alpha^2 \partial b^T W_b \partial b$$

Want $\dfrac{\Delta\Omega^+}{\partial\alpha} = 0$; $\quad 2\,\partial\delta^T W_\delta \delta_k + 2\alpha\,\partial\delta^T W_\delta \partial\delta$
$$+ 2\,\partial b^T W_b b_k + 2\alpha\,\partial b^T W_b \partial b = 0$$

$$\alpha = -\frac{\partial b^T W_b b_k + \partial\delta^T W_\delta \delta_k}{\partial b^T W_b \partial b + \partial\delta^T W_\delta \partial\delta}$$

Optimal α for any direction ∂b.

- Now for ∂b, choose it as $-\dfrac{\partial\Omega^+}{\partial b}$, i.e. parallel to negative gradient, i.e. direction of maximum decrease ... This is Method of Steepest Descent ... can be slow, method has no "memory" of previous directions used, directions can repeat or nearly repeat themselves.

Conjugate Gradient Descent

- Computes a sequence of gradients

$$\frac{\partial \Omega}{\partial b}_k = g_k$$

and a sequence of directions

$$\partial b_k = h_k$$

such that

$$h_{k+1} = -g_k + \gamma_{k+1} h_k$$

where scalar γ is recomputed at each step

$$\gamma_{k+1} = \frac{(g_{k+1} - g_k)^T g_{k+1}}{g_k^T g_k}$$

- Method is started with $h_1 = -g_1$ (i.e. steepest decent for 1st step)

- Once started, prior directions used in setting the next. α selected as in SGD.

- Summary:
  - Each iteration has forward and adjoint model sol'n
  - $\nabla \Omega^+$ used to set the next direction
  - Ω^+ is minimized which includes quadratic norms of δ and b.

BPE of b

Forward Model for u

$Ku = b$

Compute Misfit

$\delta = d - Su$

Adjoint Model for λ

$K^T \lambda = 2 S^T W_\delta \delta$

Evaluate $\nabla \Omega^+$ for current b

$g = 2 W_b b - \lambda$

IF $|g| < Tol$ Stop

Set Direction of Descent

Steepest

$\partial b = -g$

OR

Conjugate Gradient

$$\gamma_{k+1} = \frac{(g_{k+1} - g_k)^T g_{k+1}}{g_k^T g_k}$$

$\partial b_{k+1} = -g_{k+1} + \gamma_{k+1} \partial b_k$

Project in that Direction

$\partial u = K^{-1} \partial b$

$\partial \delta = -S \partial u$

Set Step Size

$$\alpha = -\frac{\partial b^T W_b b_k + \partial \delta^T W_\delta \delta_k}{\partial b^T W_b \partial b + \partial \delta^T W_\delta \partial \delta}$$

Assemble Increments

$b_{k+1} = b_k + \alpha \partial b$

$u_{k+1} = u_k + \alpha \partial u$

$\delta_{k+1} = \delta_k + \alpha \partial \delta$