Polynomial Basis/Weighting Functions

Why: - easy to differentiate/integrate
- represent a complete set for continuous functions ... Taylor Polynomial

- Lagrange Polynomials : $\phi_j(x) = \prod_{j \neq i} \frac{(x - x_i)}{(x_j - x_i)}$

  - Nth order polynomial through N+1 pts
  - easily automated
  - handy on uneven grids
  - zeros of $\phi_j(x)$ at $x = x_i \quad i \neq j$
  - $\phi_j(x_j) = 1 \Rightarrow u(x) = \sum_j c_j \phi_j \Rightarrow u(x_i) = c_i$

$$= \sum u_j \phi_j(x)$$

Coefficients are sol'n at nodes ... similar to FD, <u>but</u> have functional form specified in between.

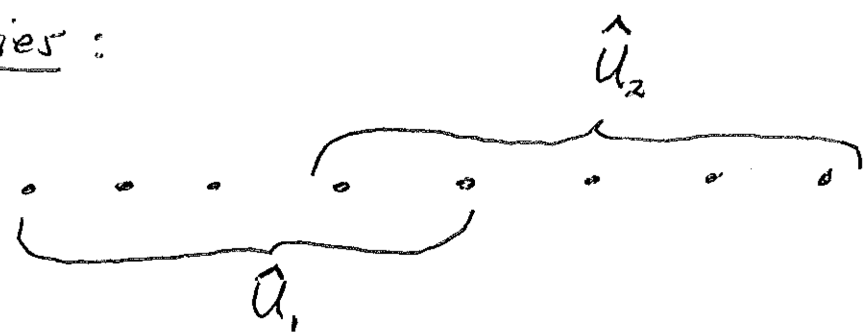
- $\sum \phi_j(x) = 1$ everywhere
- $\sum \frac{d\phi_j}{\partial x} = \frac{d}{dx} \sum \phi_j(x) = 0$ everywhere

- Global Polynomials:
  - Potential for disaster ... "polynomial wiggle"
  - Can have N zeros
  - sensitive to all u_j "Global Support"
  - Can have large variation between points

- Local Interpolation: Use subsets of nodes to represent sol'n over localized areas

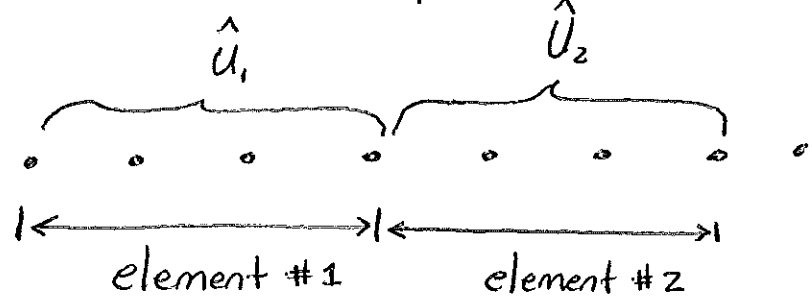
<u>Possibilities</u>:

a.)



- Overlapping subsets of nodes ... $\hat{u}$ nonunique or hard to implement

b.)



- Non-overlapping

- Define "element" as a collection of nodes
  ← is unit of local interpolation

- elements may be different... nodes need not be equally spaced
- elements are basic building blocks of FE methods
- Typically use same type of element throughout for programming ease (unless there is some reason not to do so)

- Continuity of $\hat{u}$: In 1D, if N+1 nodes/element
  - $\hat{u}$ is locally Nth order polynomial
  - on element interior... 1ST N derivatives continuous

<u>but</u>

- at element boundaries only $\hat{u}$ is continuous; $\frac{du}{dx}$ changes abruptly

"C^0" Continuity

- Higher order continuity
  e.g. "C^1" continuity ... need Hermite polynomial
  - Simplest local unit: Hermite cubic
  - $\frac{d\hat{u}}{dx}$ becomes "nodal parameter"

Example: $\dfrac{d^2u}{dx^2} + \underset{\substack{\uparrow \\ \text{constant}}}{f}u = g \qquad u(0) = 1, \quad \dfrac{du}{dx}(L) = 5$

Step 1: Generate Weighted Residual equation

$$\underbrace{\left\langle \frac{d^2u}{dx^2}, w_i \right\rangle}_{} + \langle fu, w_i \rangle = \langle g, w_i \rangle$$

Common to Integrate by parts

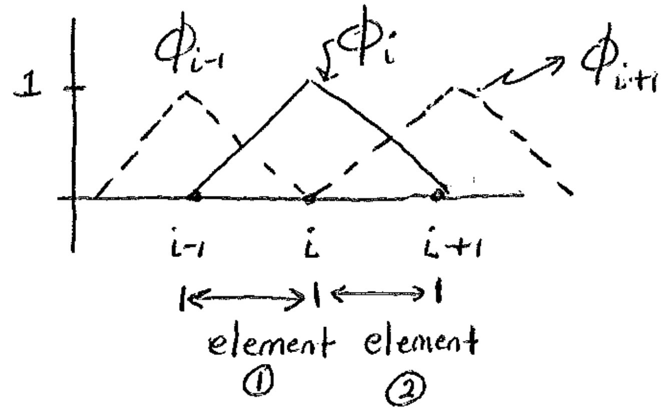
$$\frac{du}{dx} w_i \Big|_0^L - \left\langle \frac{du}{dx} \frac{dw_i}{dx} \right\rangle + \langle fu, w_i \rangle = \langle g, w_i \rangle$$

Why ... reduces continuity requirements needed on ϕ_j, w_i ... integrand may be piecewise discontinuous with finite discontinuities.

Another sense in which the approach is "weak"

Step 2: Choose Basis for $\hat{u}$

- Try simplest Lagrange polynomial $\Rightarrow$ Linear
- C^0 continuity ... OK for this PDE
- has local support



"Chapeau"
"Hat"
"Roof-top"

Step 3: Choose weighting Function; $w_i = \phi_i$; Galerkin

Step 4: Assemble matrices (evaluate coefficients)

- each row of matrix is $\langle R, \phi_i \rangle = 0$

But $\phi_i = 0$ over most of X (local)

$\phi_i \neq 0$ only on 2 elements ...

$$\int (\,)\,\phi_i\,dx = \int_{①} (\,)\,\phi_i\,dx + \int_{②} (\,)\,\phi_i\,dx$$

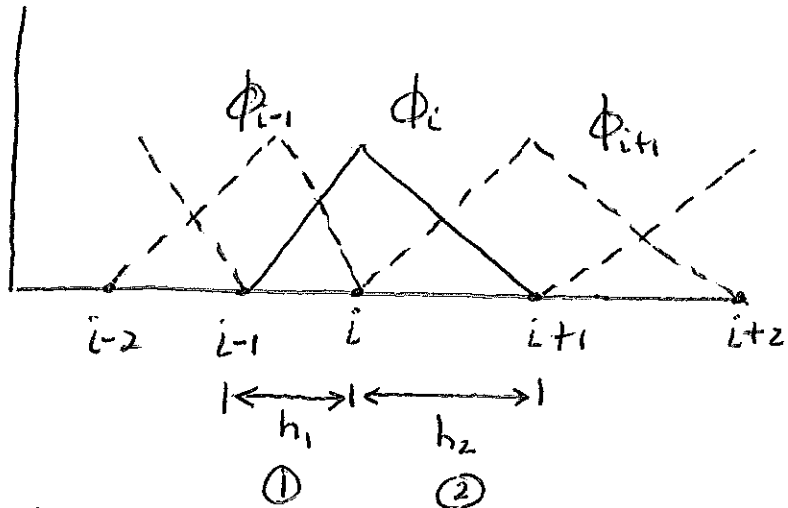
$$\Rightarrow -\left\langle \frac{du}{dx}\frac{dw_i}{dx} + f u w_i \right\rangle = \langle g, w_i \rangle - \frac{du}{dx} w_i \Big|_0^L$$

$$-\sum_{j=1}^{N} u_j \left\langle \frac{d\phi_j}{dx}\frac{d\phi_i}{dx} + f\phi_j\phi_i \right\rangle = \langle g, \phi_i \rangle - \frac{du}{dx} w_i \Big|_0^L$$

Do this for each $i = 1, 2, \ldots N$; becomes system

$$[A]\{u\} = \{b\} \quad w/ \quad a_{ij} = \left\langle -\frac{d\phi_j}{dx}\frac{d\phi_i}{dx} + f\phi_i\phi_j \right\rangle$$

$$b_i = \langle g\,\phi_i \rangle - \frac{du}{dx}\phi_i \Big|_0^L$$



$$a_{ij} = \left\langle -\frac{d\phi_j}{dx}\frac{d\phi_i}{dx} + f\phi_i\phi_j \right\rangle$$ for $j = 1, 2, \ldots N$, but only $j = i-1, i, i+1$ contribute

$\phi_{i-2} = 0$ on 1 + 2

$\phi_{i+2} = 0$ on 1 + 2

So:

$$\begin{array}{lllll} & & \quad 1 & \quad 2 & \\ j=i-1: & \left\langle \frac{d\phi_{i-1}}{dx}\frac{d\phi_i}{dx} \right\rangle = & -\frac{1}{h_1}\frac{1}{h_1}h_1 & + \; 0 & = -1/h_1 \\ j=i: & \left\langle \frac{d\phi_i}{dx}\frac{d\phi_i}{dx} \right\rangle = & \frac{1}{h_1}\frac{1}{h_1}h_1 & + \; \frac{-1}{h_2}\frac{-1}{h_2}h_2 & = \frac{1}{h_1} + \frac{1}{h_2} \\ j=i+1: & \left\langle \frac{d\phi_{i+1}}{dx}\frac{d\phi_i}{dx} \right\rangle = & 0 & + \; \frac{1}{h_2}\frac{-1}{h_2}h_2 & = -1/h_2 \end{array}$$

$$\begin{array}{lllll} j=i-1: & \langle \phi_{i-1}\phi_i \rangle = & \frac{h_1}{6} & + \; 0 & = h_1/6 \\ j=i: & \langle \phi_i\phi_i \rangle = & h_1/3 & + \; h_2/3 & = \frac{h_1+h_2}{3} \\ j=i+1: & \langle \phi_{i+1}\phi_i \rangle = & 0 & + \; h_2/6 & = h_2/6 \end{array}$$

e.g. $$\langle \phi_{i-1}\, \phi_i \rangle = \int_{x_{i-1}}^{x_i} \left(\frac{x-x_i}{x_{i-1}-x_i}\right)\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right) dx$$

$$= -\frac{1}{h_1^2}\int_{x_{i-1}}^{x_i} (x-x_i)(x-x_{i-1})\, dx$$

$$= -\frac{1}{h_1^2}\int_{x_{i-1}}^{x_i} x^2 - x(x_i + x_{i+1}) + x_i x_{i-1}\, dx$$

$$= -\frac{1}{h_1^2}\left[\frac{x^3}{3} - \frac{x^2}{2}(x_i + x_{i+1}) + x\, x_i\, x_{i-1}\right]_{x_{i-1}}^{x_i}$$

W/o loss of generality ... pick $x_{i-1} = 0$

$$= -\frac{1}{h_1^2}\left[\frac{x_i^3}{3} - \frac{x_i^3}{2}\right] = \frac{1}{h_1^2}\left[\frac{h_1^3}{2} - \frac{h_1^3}{3}\right] = \frac{h_1}{6}$$

... Conclude we have exact integration/differentiation in our WR method ... approximation is in the assumption of linear variation of sol'n between nodes

Assembly of Row i :

$$\frac{1}{h_1} u_{i-1} + \left(\frac{1}{h_1} + \frac{1}{h_2}\right) u_i + \frac{1}{h_2} u_{i+1}$$

$$+ \frac{f h_1}{6} u_{i-1} + f\left(\frac{h_1 + h_2}{3}\right) u_i + \frac{f h_2}{6} u_{i+1}$$

$$= \underbrace{g_{i-1} \frac{h_1}{6} + g_i \left(\frac{h_1 + h_2}{3}\right) + g_{i+1} \frac{h_2}{6}}_{g = \sum g_k \phi_k}$$

Uniform h :

$$\underbrace{\frac{1}{h^2}\left(u_{i-1} - 2u_i + u_{i+1}\right)}_{\delta^2 u_i} + \frac{f}{6}\underbrace{\left(u_{i-1} + 4u_i + u_{i+1}\right)}$$

$$= \overbrace{\frac{1}{6}\left(g_{i-1} + 4g_i + g_{i+1}\right)}^{\text{Simpson's Rule}}$$

Similar to FD !!