Matrix Assembly of FE Equations

- In FD common to proceed "molecule-by-molecule" each represents single difference eqn.

- In FE more natural to proceed "element-by-element" need to integrate over problem domain ⇒ union of nonoverlapping elements . . .

e.g. $[A]\{u\} = \{b\}$

where $a_{ij} = \left\langle -\frac{d\phi_j}{dx}\frac{d\phi_i}{dx} + f\phi_i\phi_j \right\rangle$; $b_i = \langle g\phi_i \rangle - \frac{du}{dx}\phi_i \Big|_0^L$

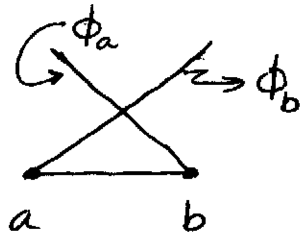
then $$\langle () \rangle = \int () \, dx = \underbrace{\sum_{\text{elements}} \int_e () \, dx}_{\substack{\text{sum of elements} \\ \text{comprises entire domain}}} = \sum_e \underbrace{\langle \ \rangle^e}_{\substack{\text{element} \\ \text{contribution} \\ \text{to inner product}}}$$

so $$[A] = \sum_e \underbrace{[A]^e}_{\substack{\text{"element matrix"... i.e. contains \underline{all} contributions} \\ \text{to } [A] \text{ for a given element}}}$$

likewise $$\{b\} = \sum_e \{b\}^e$$

- to get $[A]^e$ need to determine contributions from a general element

e.g. 1D linear (Chapeau)



- only 2 "alive" (i.e. nonzero) in any element

a Given row in [A]

- each weighting function constitutes a single eqn. in unknown coefficients ... 2 possible weighting functions: ϕ_a, ϕ_b

a given column in [A]

- each basis function has a corresponding unknown coefficient ... 2 possible basis functions: ϕ_a, ϕ_b

All possible combinations...

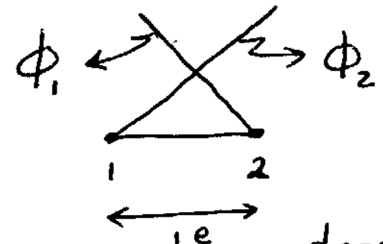
Weighting function (Row)	Basis function (Column)
ϕ_a	ϕ_a
ϕ_a	ϕ_b
ϕ_b	ϕ_a
ϕ_b	ϕ_b

So $[A]^e$ has only 4 nonzero coefficients: a^e_{aa}; a^e_{ab}; a^e_{ba}; a^e_{bb}

$$\begin{array}{c} a \rightarrow \\ b \rightarrow \end{array} \begin{bmatrix} a^e_{aa} & a^e_{ab} \\ a^e_{ba} & a^e_{bb} \end{bmatrix}^e$$
$$\quad\quad \begin{array}{cc} \uparrow & \uparrow \\ a & b \end{array}$$

So $[A]^e$ may be stored as a 2×2 submatrix where we use a "local" node numbering scheme

e.g.



$h^e \rightarrow$ depends on element

$$\Rightarrow \underbrace{\begin{bmatrix} a_{11}^e & a_{12}^e \\ a_{21}^e & a_{22}^e \end{bmatrix}}$$

Structure the same regardless of PDE for linear 1D element.... only details of coefficients differ

$$a_{11}^e = \left\langle -\frac{d\phi_1}{dx}\frac{d\phi_1}{dx} + f\phi_1\phi_1 \right\rangle^e = -\frac{1}{h^e} + \frac{fh^e}{3}$$

$$a_{12}^e = \left\langle -\frac{d\phi_2}{dx}\frac{d\phi_1}{dx} + f\phi_2\phi_1 \right\rangle^e = \frac{1}{h^e} + \frac{fh^e}{6}$$

$$a_{22}^e = \left\langle -\frac{d\phi_2}{dx}\frac{d\phi_2}{dx} + f\phi_2\phi_2 \right\rangle^e = -\frac{1}{h^e} + \frac{fh^e}{3}$$

$$a_{21}^e = \left\langle -\frac{d\phi_1}{dx}\frac{d\phi_2}{dx} + f\phi_1\phi_2 \right\rangle^e = \frac{1}{h^e} + \frac{fh^e}{6}$$

$$b_1^e = \langle g\phi_1 \rangle^e = \frac{gh^e}{2}$$

$$b_2^e = \langle g\phi_2 \rangle^e = \frac{gh^e}{2}$$

(assume g constant...and neglecting boundary term for the moment...)

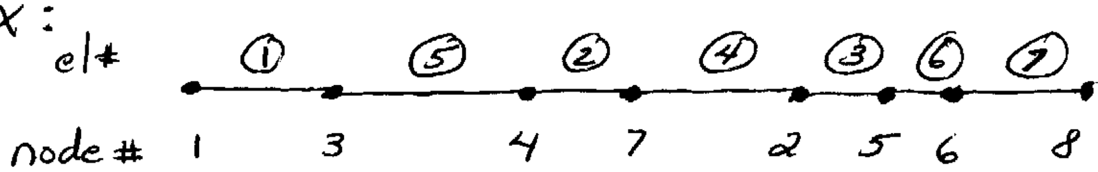
so

$$[A]^e = \begin{bmatrix} -\frac{1}{h^e} + \frac{fh^e}{3} & \frac{1}{h^e} + \frac{fh^e}{6} \\ \frac{1}{h^e} + \frac{fh^e}{6} & -\frac{1}{h^e} + \frac{fh^e}{3} \end{bmatrix} ; \quad \{b\}^e = \begin{Bmatrix} \frac{gh^e}{2} \\ \frac{gh^e}{2} \end{Bmatrix}$$

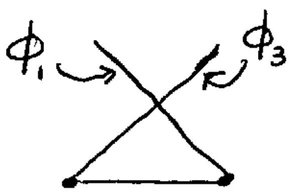
- Now need union of $[A]^e$'s ... i.e. need to assemble "Global" $[A]$ matrix and $\{b\}$ vector

- Achieved through "Incidence List"... relates local node #'s to global node #'s ... i.e. mapping of local element entries into Global matrix

ex:



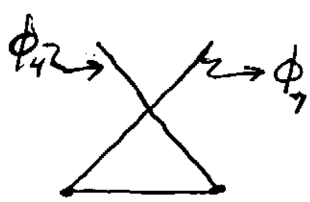
Element ① $\Rightarrow$



Global	1	3
Local	1	2

$a^e_{11} \longrightarrow a_{11}$
$a^e_{12} \longrightarrow a_{13}$
$a^e_{21} \longrightarrow a_{31}$
$a^e_{22} \longrightarrow a_{33}$
Local Global

Element ② $\Rightarrow$



Global	4	7
Local	1	2

$a^e_{11} \longrightarrow a_{44}$
$a^e_{12} \longrightarrow a_{47}$
$a^e_{21} \longrightarrow a_{74}$
$a^e_{22} \longrightarrow a_{77}$
Local Global

⋮

etc

Generally : $IN(e, J) = $ Global Node #

element # local node #

then for element # e

$$a^e_{11} \longrightarrow A(IN(e,1), IN(e,1))$$

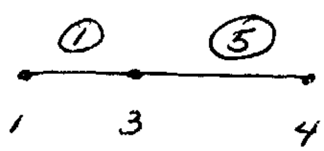
$$a^e_{21} \longrightarrow A(IN(e,2), IN(e,1))$$

$$\vdots$$

etc

always <u>add in</u> contributions to Global Matrix ... each entry in Global matrix may receive contributions from more than one element ... i.e. need union of elements

e.g. from our 1D example :



element assembly hits global node #3 twice ... once during el #1 <u>and</u> once during el #5

X ≡ contribution el 1

O ≡ Contribution el 5

$$\begin{array}{c} \\ \text{Row 1} \\ \text{Row 3} \\ \text{Row 4} \\ \\ \end{array} \underbrace{\begin{array}{ccc} \text{Col 1} & \text{Col 3} & \text{Col 4} \end{array} \\ \left[\begin{array}{cccc} x & x & & \\ x & \otimes & O & \\ & O & O & \\ & & & \\ \end{array}\right]}_{\text{Global } A}$$

this ~~entry~~ hit twice

want...

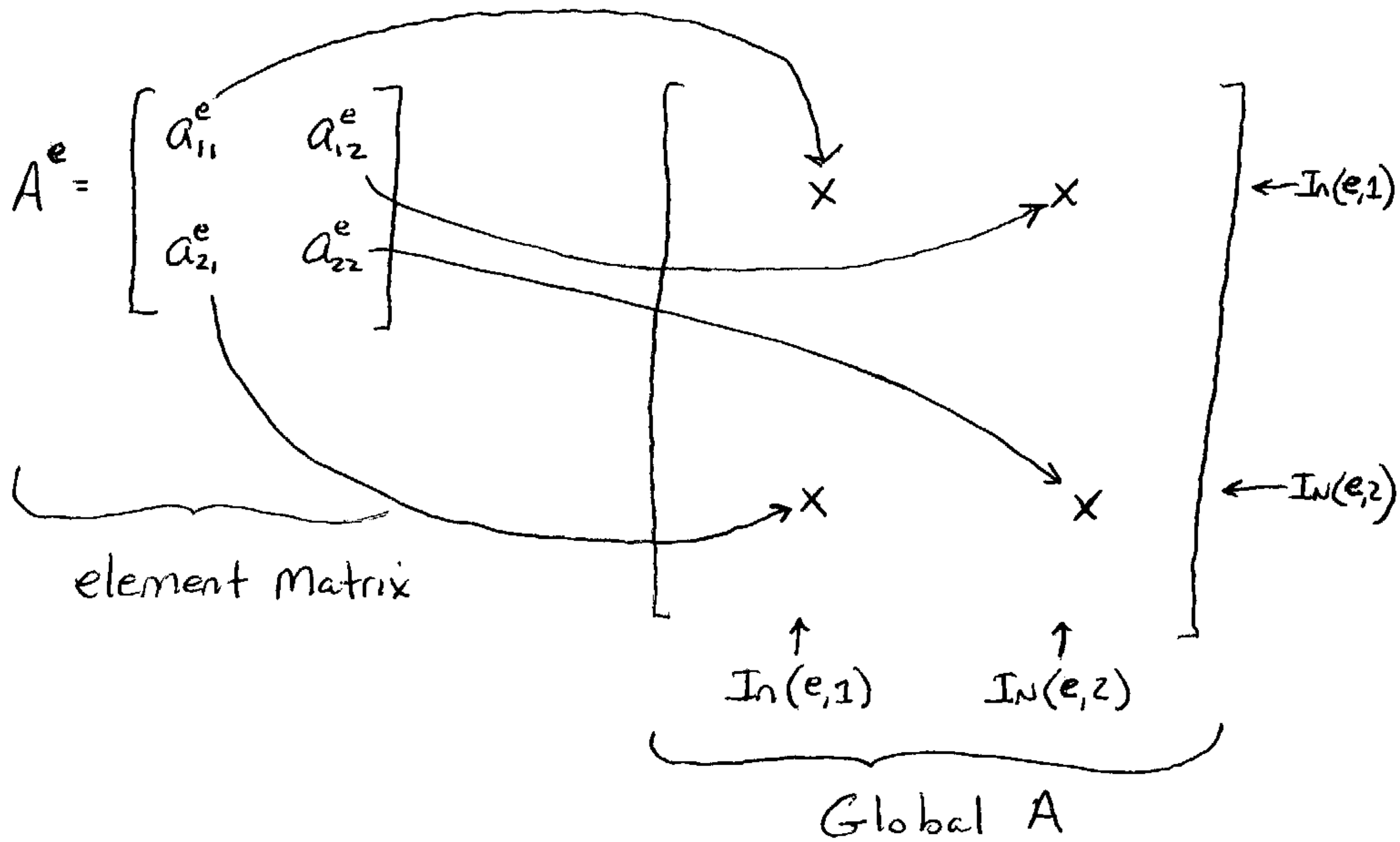
$$A(3,3) = X + O$$

- More generally we have

(e)

IN(e,1) IN(e,2) Global #s

1 2 local #s



- Size of element matrix (i.e. 2x2, 3x3, etc) depends on element type ... i.e. # nodes/element ... regardless of PDE

- Assembly of element matrix into global matrix depends on global node numbering ... regardless of PDE

- Entries in element matrix depend on PDE ... i.e. a^e_{11}, a^e_{12} etc.

- Problem domain depends only on # of elements

Recall Example... $\frac{d^2u}{dx^2} + fu = g$ $\quad u(0) = 1$, $\quad \frac{du}{dx}(l) = 5$

So far we have...

1. Generated weighted residual equation which is <u>one</u> row in $[A]\{u\} = \{b\}$ as

$$a_{ij} = \left\langle -\frac{d\phi_j}{dx}\frac{dw_i}{dx}\right\rangle + \left\langle f\phi_j w_i\right\rangle$$
$$b_i = \langle g w_i\rangle - w_i \frac{du}{dx}\Big|_0^l$$

2. Chosen a basis for approximate sol'n
   Chapeau functions : $u = \sum u_j \phi_j(x)$

3. Chosen weights w_i to be $\phi_i(x)$ (Galerkin)

4. Assembled <u>all</u> equations in matrix

$$\int (\,)\,dx = \sum_e \int_e (\,)\,dx \quad \text{for each eqn!}$$

<u>but</u> we consider all possible eqn's (rows) for each element... need only a <u>single</u> pass through elements

Left to do... Apply BC's !!

ex • • • ∫∫ • • •
1 2 N Node #

a.) Type I BC ... Satisfy $u_1 = 0$ exactly

- Similar to FD ... remove $\langle R, \phi_1 \rangle$ from Galerkin set of eqn's (analogous to <u>not</u> using PDE molecule on boundary in FD)

- One less unknown in algebraic system for each Type I boundary node

- Simplest to save room in matrix for $\langle R, \phi_1 \rangle$ ... i.e. assemble <u>all</u> elements as though all nodes are "interior" nodes... then <u>after</u> assembly but <u>prior</u> to matrix solution remove $\langle R, \phi_1 \rangle$ and insert boundary condition equation

- Save $\langle R, \phi_1 \rangle$ for later use $\Rightarrow$ Conservation !!

b.) Type II BC ... Satisfy "weakly" in $\{b\}$

- $\{b\}$ has term $\phi_i \frac{\partial u}{\partial x}\Big|_0^L$ ... involves BC info $\frac{\partial u}{\partial x}$ and ϕ_i on boundary

$$\Longrightarrow \quad \phi_i(L)\frac{\partial u}{\partial x}(L) - \phi_i(0)\frac{\partial u}{\partial x}(0)$$

- But for i as an interior node . . .

$$\phi_i(0) = \phi_i(L) = 0$$

- At Type I boundary . . . ϕ_1 removed

- At $i=N$: $\underbrace{\phi_N(L)}_{1}\underbrace{\frac{\partial u}{\partial x}(L)}_{5} - \underbrace{\phi_N(0)}_{0}\underbrace{\frac{\partial u}{\partial x}(0)}_{?}$

so $b_i = \langle g\,\phi_i\rangle$ at interior nodes

$$b_N = \langle g\,\phi_N\rangle - 5$$

c.) Type III BC . . . Similar to Type II approach

e.g. $\dfrac{\partial u}{\partial x} + \alpha u = 5$

then : $-\phi_N \dfrac{\partial u}{\partial x}\Big|_0^L = \alpha u_N - 5$ goes in $\{b\}$ at b_N

goes in $[A]$ at a_{nn}

i.e. $a_{nn} = \left\langle -\dfrac{d\phi_N}{dx}\dfrac{d\phi_N}{dx} + f\phi_N\phi_N\right\rangle - \alpha$

$$b_n = \langle g, \phi_N\rangle - 5$$

Summarize ...

Type I: "Throw away Galerkin Equation"; use BC data directly

Type II + III: retain Galerkin equation, apply BC directly in $\phi_i \frac{\partial u}{\partial x}\Big|_0^L$

In fact ... for Type I the discarded Galerkin equation is the equation for determining the unknown "flux" $\frac{du}{dx}$:

e.g. at Node 1 the equation we didn't use ...

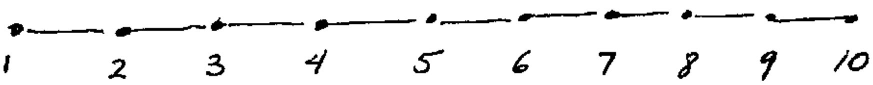
$$-\sum_j \underbrace{u_j}_{\text{known, once sol'n computed}} \underbrace{\left\langle \frac{\partial\phi_1}{\partial x}\frac{\partial\phi_j}{\partial x} + f\phi_1\phi_j \right\rangle}_{\text{Coefficients in 1ST Row of A before removed}} - \underbrace{\langle g\phi_1 \rangle}_{\text{Right-hand-side before BC data enforced}} = \overbrace{\phi_1(0)}^{1}\frac{\partial u}{\partial x}(0) - \cancel{\phi_1(L)}\frac{\partial u}{\partial x}(L)$$

Left-side of this relation is known ... Right side is the desired quantity !!

Banded Matrices in 1D:

- Linear elements
  - want to use natural ordering of node numbers ... leads to Tridiagonal system ... use Thomas algorithm

e.g.



Global A :

$$\begin{bmatrix} x & x & & & & & & & \\ x & x & x & & & & 0 & & \\ & x & x & x & & & & & \\ & & x & x & x & & & & \\ & & & x & x & x & & & \\ & & & & x & x & x & & \\ & 0 & & & & x & x & x & \\ & & & & & & x & x & x \\ & & & & & & & x & x \end{bmatrix}$$

Formally in Banded Form:

Bandwidth = 3

$$\begin{bmatrix} & x & x \\ x & x & x \\ x & x & x \\ x & x & x \\ x & x & x \\ x & x & x \\ x & x & x \\ x & x & x \\ x & x & x \\ x & x & \end{bmatrix}$$

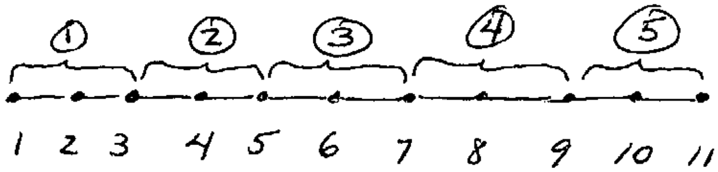
IS the MAX difference between node numbers in any element

Half-Bandwidth = 1

Bandwidth = 2* Half-BW + 1

main diagonal

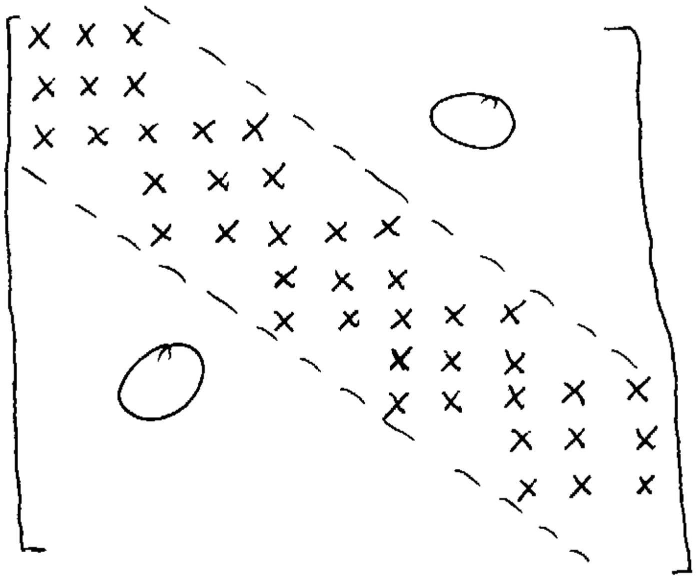
- *Quadratic elements*



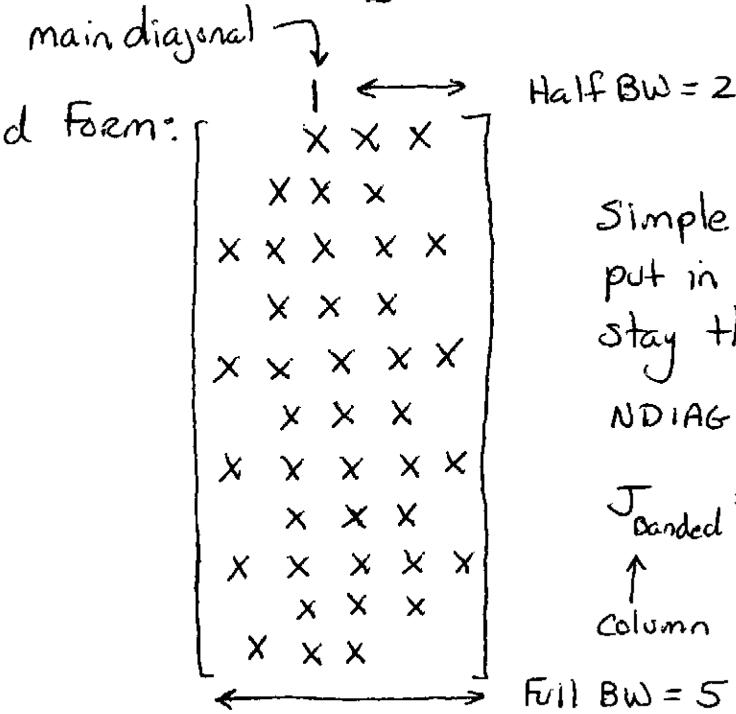
– In Banded form expect: Half BW = 2 (e.g. All elements have max difference of 2 between node #s)

$BW = 2(2) + 1 = 5$

– Full Matrix form:



– Banded Form:



Simple shift of columns to put in Banded form ... rows stay the same

NDIAG ≡ Half-BW + 1

$J_{Banded} = NDIAG + J_{Full} - I$

J_{Banded}: column; J_{Full}: column; I: row