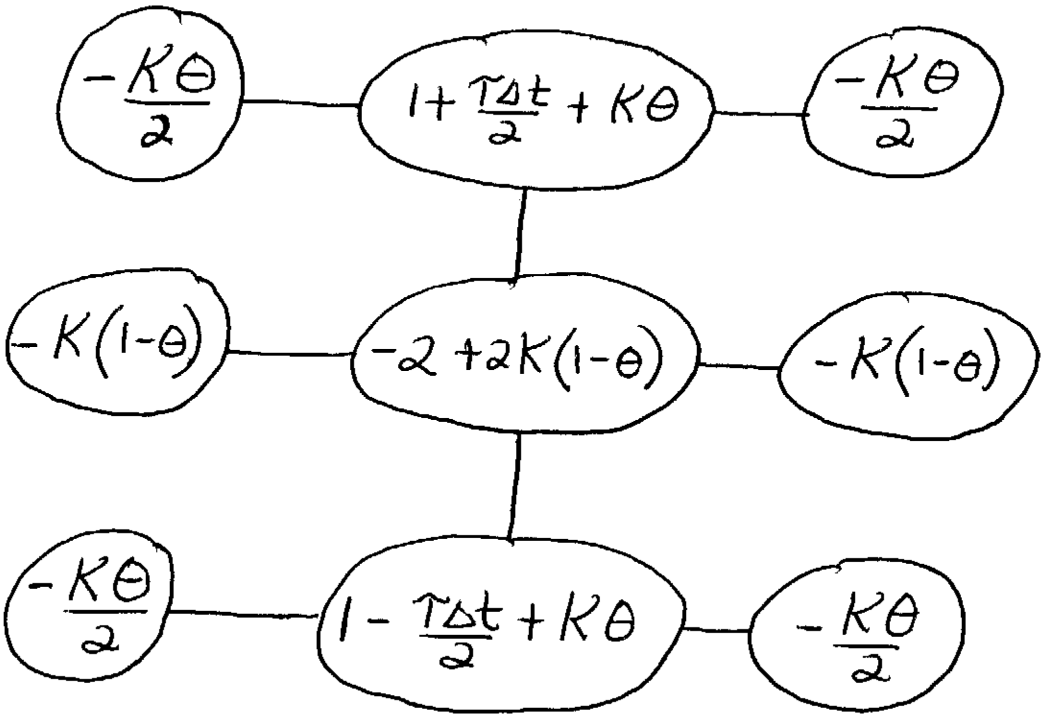
Hyperbolic Time-Stepping Strategies (Con't)

2nd Order Eqn (Con't)

b.) Implicit

$$\left(u_i^{\ell+1} - 2u_i^{\ell} + u_i^{\ell-1}\right) + \frac{\tau\Delta t}{2}\left(u_i^{\ell+1} - u_i^{\ell-1}\right) - K\delta_x^2\left[\theta\left(\frac{u_i^{\ell+1}+u_i^{\ell-1}}{2}\right) + (1-\theta)u_i^{\ell}\right] = 0$$



- Sol'n of system of Eqn's per time-step
- Centered in $(x,t) \Rightarrow O(h^2+\Delta t^2)$
- Stability: $\theta \geq 1/2$ Unconditional

  $K < \frac{1}{1-2\theta}$ otherwise
- Shortest waves propagate + damp (phase distortions possible)

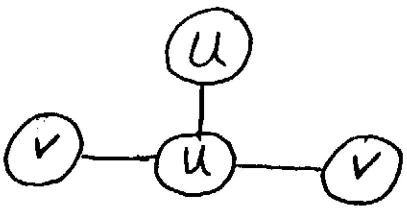
Coupled 1st Order Eqn's

$$\frac{\partial u}{\partial t} + \tau u - c_1 \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} - c_2 \frac{\partial u}{\partial x} = 0$$

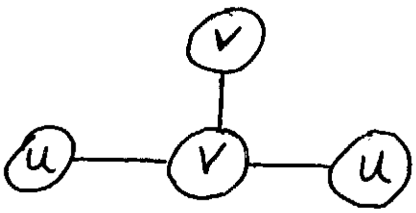
a.) Explicit :

- $$u_i^{\ell+1} - u_i^{\ell} + \Delta t\, \tau u_i^{\ell} - \frac{c_1 \Delta t}{2h}\left(v_{i+1}^{\ell} - v_{i-1}^{\ell}\right) = 0$$



$= 0$

- $$v_i^{\ell+1} - v_i^{\ell} - \frac{c_2 \Delta t}{2h}\left(u_{i+1}^{\ell} - u_{i-1}^{\ell}\right) = 0$$



$= 0$

Stability :

$$\begin{bmatrix} \gamma_0 - 1 + \Delta t\, \tau & -\frac{c_1 \Delta t}{h} j \sin \sigma h \\ -\frac{c_2 \Delta t}{h} j \sin \sigma h & \gamma_0 - 1 \end{bmatrix} \begin{Bmatrix} u_i^{\ell} \\ v_i^{\ell} \end{Bmatrix} = 0$$

$$\Rightarrow (\gamma_0 - 1 + \Delta t\, \tau)(\gamma_0 - 1) + K \sin^2 \sigma h = 0$$

$$\gamma_0^2 + \gamma_0(\Delta t\tau - 2) + 1 - \Delta t\tau + K\sin^2\sigma h = 0$$

Need $\frac{c}{a} \leq 1 \Rightarrow 1 - \Delta t\tau + K\sin^2\sigma h \leq 1$

$$K\sin^2\sigma h \leq \tau\Delta t \Rightarrow \boxed{K \leq \tau\Delta t}$$

$|b| \leq c + a \quad |\Delta t\tau - 2| \leq 1 - \Delta t\tau + K\sin^2\sigma h + 1$

worst case when $K\sin^2\sigma h \to 0$; then

$$|\Delta t\tau - 2| \leq 2 - \Delta t\tau \Rightarrow \Delta t\tau \leq 2$$

Conclude: $\boxed{K \leq \tau\Delta t \leq 2}$ ... Unconditional Instability when $\tau = 0$!!

– Examine short wave behavior... (Same at $\sigma h \to 0$)

$$\gamma_0 = \frac{2 - \Delta t\tau}{2} \pm \frac{\sqrt{(\Delta t\tau - 2)^2 - 4(1 - \Delta t\tau)}}{2}$$

$$= 1 - \frac{\Delta t\tau}{2} \pm \left(\frac{(\Delta t\tau)^2}{4}\right)^{1/2} \Rightarrow \left.\begin{array}{l}\gamma_0 = 1\\ \gamma_0 = 1 - \tau\Delta t\end{array}\right\}$$ Neither of these propagate

Undamped Parasite

Mimics physical propagation i.e Amplitude decays; $\tau\Delta t > 1 \Rightarrow 2\Delta t$ oscillations

For $\sigma h = \pi/2$

$$\gamma_0 = 1 - \frac{\Delta t \tau}{2} \pm \sqrt{\left(\frac{\Delta t \tau}{2}\right)^2 - K}$$

Amplitudes for both roots $< 1 \Rightarrow$ decay

phase distortions for $K > \left(\frac{\Delta t \tau}{2}\right)^2$

$\left(\frac{\Delta t \tau}{2}\right)^2 > K$ don't propagate only decay

- Common Strategy ... put τu term at $l+1$

i.e. $u_i^{l+1} - u_i^{l} + \Delta t \tau \underline{u_i^{l+1}} - \frac{c_1 \Delta t}{2h}\left(V_{i+1}^{l} - V_{i-1}^{l}\right) = 0$

then
$$\begin{bmatrix} \gamma_0(1+\tau\Delta t) - 1 & -c_1 \Delta t j \sin\sigma h \\ -c_2 \Delta t j \sin\sigma h & \gamma_0 - 1 \end{bmatrix}$$

$$\Rightarrow \gamma_0^2(1+\tau\Delta t) - (2+\tau\Delta t)\gamma_0 + 1 + K\sin^2\sigma h = 0$$

- $\frac{c}{a} \le 1 \Rightarrow \frac{1+K\sin^2\sigma h}{1+\tau\Delta t} \le 1$

so $1 + K\sin^2\sigma h \le 1 + \tau\Delta t \Rightarrow \boxed{K \le \tau\Delta t}$

same as before

- $|b| \le c + a \Rightarrow \quad 2 + \tau\Delta t \le 2 + \tau\Delta t + K\sin^2\sigma h$

$K\sin^2\sigma h \ge 0$ always true

$\therefore \boxed{K \le \tau\Delta t \text{ only}}$ Improved Stability

- What about short wave problem:

$$\gamma_0 = \frac{2+\tau\Delta t}{2(1+\tau\Delta t)} \pm \frac{\left((2+\tau\Delta t)^2 - 4(1+\tau\Delta t)\right)^{1/2}}{2(1+\tau\Delta t)}$$

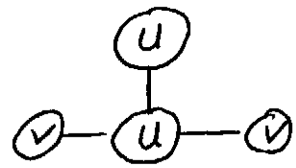
$$= \frac{2+\tau\Delta t \pm \tau\Delta t}{2(1+\tau\Delta t)} \Longrightarrow \quad \gamma_0 = 1, \quad \gamma_0 = \frac{1}{1+\tau\Delta t}$$

Same problem

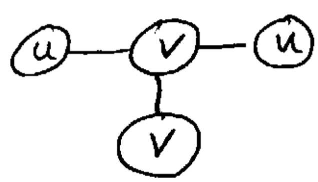
Undamped short wavelengths present!!

b.) Explicit-Implicit ... Sequential Sol'n 1st u, then V
(use latest u values to complete V)

$$U_{i+1}^{\ell+1} - U_i^{\ell} + \Delta t\, \tau U_i^{\ell} - C_1 \frac{\Delta t}{2h}\left(V_{i+1}^{\ell} - V_{i-1}^{\ell}\right) = 0$$



$$V_i^{\ell+1} - V_i^{\ell} - \frac{C_2\Delta t}{h}\left(U_{i+1} - U_{i-1}\right)^{\ell+1} = 0$$



$$\begin{bmatrix} \gamma_0 - 1 + \Delta t\,\tau & -\frac{C_1 \Delta t}{h} j \sin\sigma h \\ -\frac{C_2 \Delta t}{2h} j \sin\sigma h\, \underline{\underline{\gamma_0}} & \gamma_0 - 1 \end{bmatrix} \begin{Bmatrix} u_i^\ell \\ v_i^\ell \end{Bmatrix} = 0$$

$$\gamma_0^2 + \gamma_0 \left(\Delta t\,\tau - 2 + K \sin^2\sigma h \right) + 1 - \tau \Delta t = 0$$

- $\frac{c}{a} = \frac{1 - \tau\Delta t}{1} \leq 1$ always
- $|b| \leq a + c \Rightarrow |\tau\Delta t - 2 + K\sin^2\sigma h| \leq 2 - \tau\Delta t$

$$\Rightarrow \boxed{K \leq 4 - 2\tau\Delta t \ \underline{\underline{\text{and}}}\ \tau\Delta t \leq 2}$$

As before... if put τu_i term at $\ell+1$

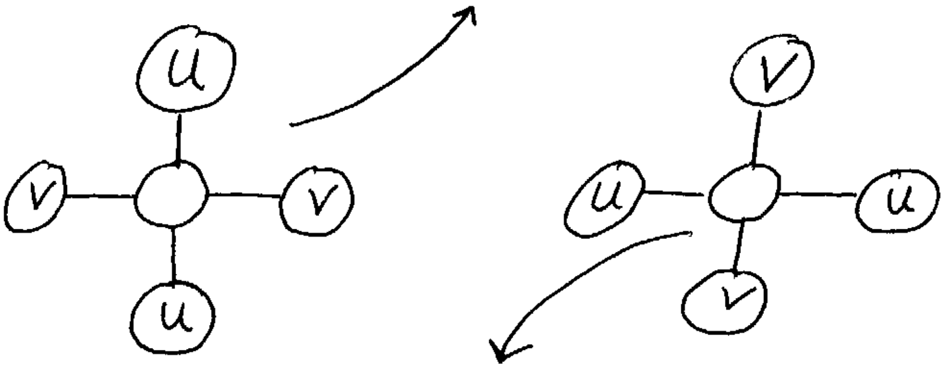
$$\Rightarrow \boxed{K \leq 4 - 2\tau\Delta t \text{ only}}$$

- Stable for $\tau = 0$
- short wave problem persists $\Rightarrow$ undamped

c.) Leapfrog . . .

Note time level

$$U_i^{\ell+1} - U_i^{\ell-1} + 2\Delta t\,\tau\,U_i^{\ell-1} - \frac{2\Delta t}{2h}\,c_1\left(V_{i+1}^{\ell} - V_{i-1}^{\ell}\right) = 0$$



$$V_i^{\ell+1} - V_i^{\ell-1} - \frac{2\Delta t}{2h}\,c_2\left(U_{i+1}^{\ell} - U_{i-1}^{\ell}\right) = 0$$

$$\begin{bmatrix} \gamma_0^2 - 1 + 2\Delta t\,\tau & \left(-\frac{\Delta t}{h}\,c_1\,2j\sin\sigma h\right)\gamma_0 \\ \left(-\frac{\Delta t}{h}\,c_2\,2j\sin\sigma h\right)\gamma_0 & \gamma_0^2 - 1 \end{bmatrix} \begin{Bmatrix} U_i \\ V_i \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

let $\gamma_0^2 \equiv G$

$$(G - 1 + 2\Delta t\,\tau)(G-1) + 4K\sin^2\sigma h\,G = 0$$

$$\Rightarrow\; G^2 + G\left(-2 + 2\Delta t\,\tau + 4K\sin^2\sigma h\right) + \left(1 - 2\tau\Delta t\right) = 0$$

- Identical to Explicit-Implicit scheme quadratic
  ... except $2\Delta t$ appears where Δt did before

... so ...

Explicit-Implicit = 2 Steps of Leapfrog

$$\Delta t_{EI} \Leftrightarrow 2\Delta t_{Leap}$$

But intermediate Leapfrog : $\gamma = \pm\sqrt{G}$

spurious root!

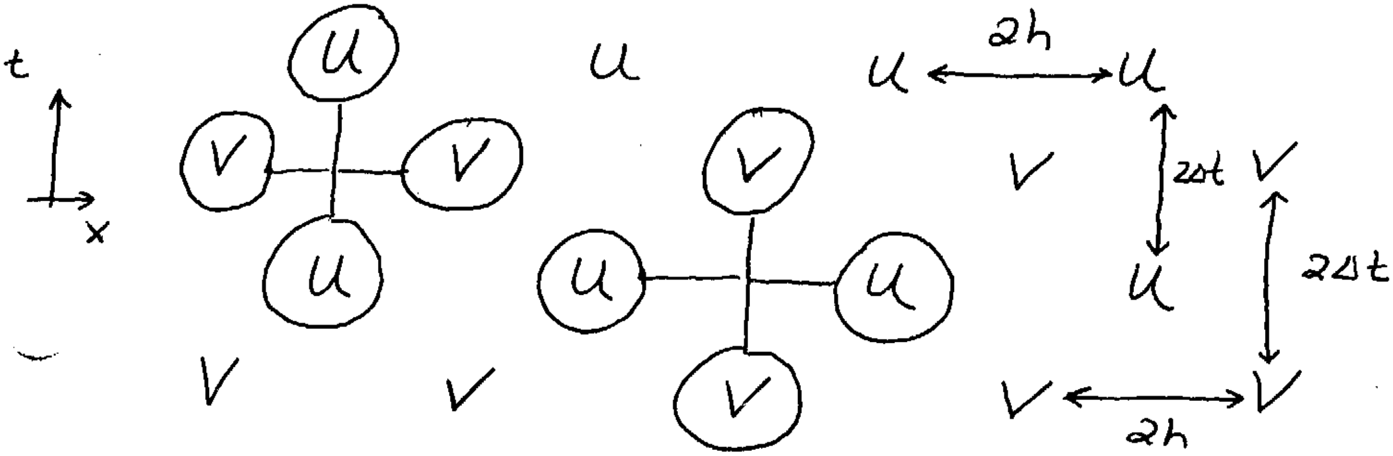
i.e. $\gamma = \pm 1$

$$\pm\left(\frac{1}{1+T\Delta t}\right)^{1/2}$$

Oscillations in time !!

Note: TU_i^{ℓ} (Centered) $\Rightarrow$ Unconditional Instability

- Cure for short wave problem: Space-Time Staggering:
  - u variables separated by $2h$
  - v variables likewise
  - same in time (separation by $2\Delta t$)
  - Only single variable at each space-time pt



- short waves in x gone ... effective node spacing now $2h$; wavelength spectrum truncated; i.e. shortest wavelength now $4h$ so $\frac{2\pi}{\sigma} \geq 4h \Rightarrow \sigma h \leq \pi/2$

- short waves in t gone ... Leapfrog without staggering in time produces $\gamma = \pm\sqrt{}$ ← oscillations in $2\Delta t$

These eliminated by making effective time-step $2\Delta t$