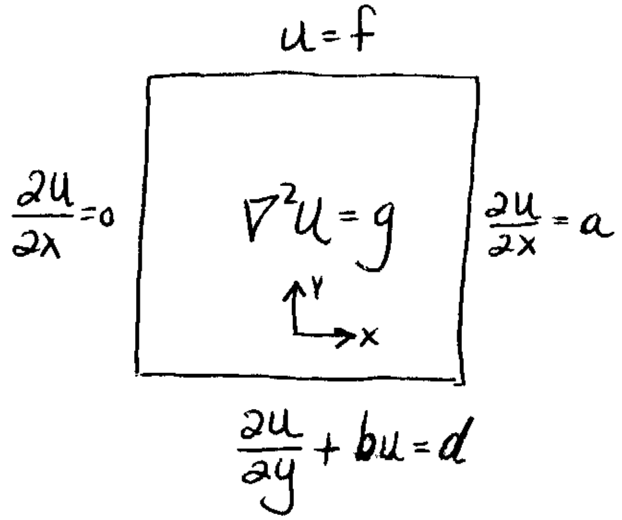
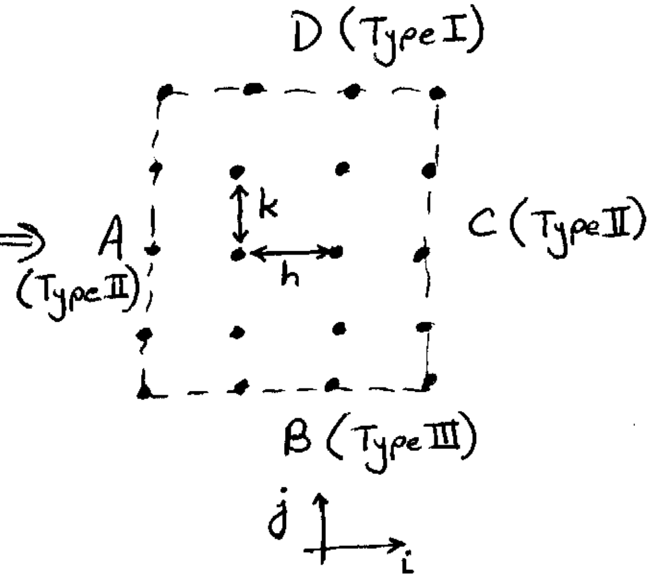
Elliptic Equations

Ex.: Consider:



$\Longrightarrow$



PDE: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g$$

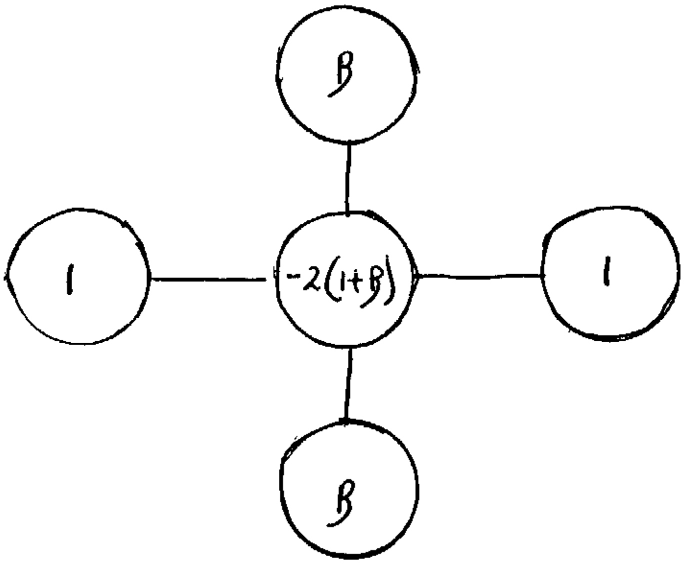
want second-order, centered FD expressions:

$$\Rightarrow \quad \frac{\delta_x^2 u_{ij}}{h^2} + \frac{\delta_y^2 u_{ij}}{k^2} = g_{ij}$$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = g_{ij}$$

$\beta = \frac{h^2}{k^2}$, $\quad \Rightarrow \quad u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + \beta\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) = h^2 g_{ij}$

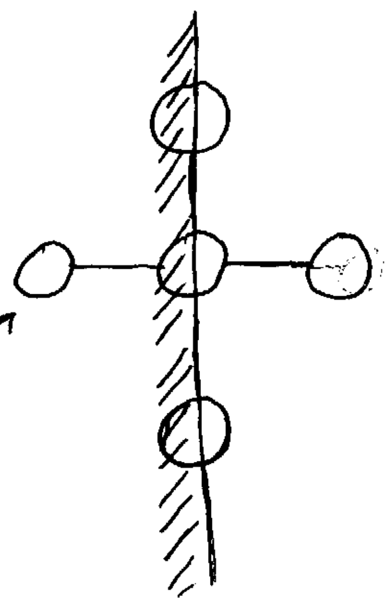
At each point on the grid, we have



$= g_{ij} h^2$ "computational molecule"

Valid at all <u>interior</u> nodes!, but what about boundaries? $\Rightarrow$ molecule "spills" over

e.g. Boundary A:

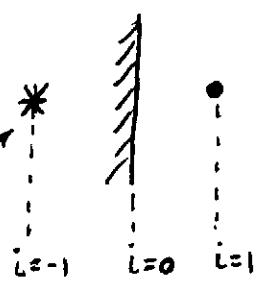


interior

outside the domain

Remove through BCs

$\left.\frac{\partial u}{\partial x}\right|_{i=0} = 0$ given $\Rightarrow$

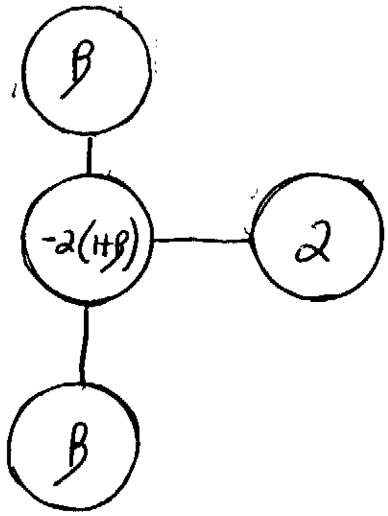


"Shadow node"

$\Rightarrow \dfrac{u_\bullet - u_*}{2h} = 0$

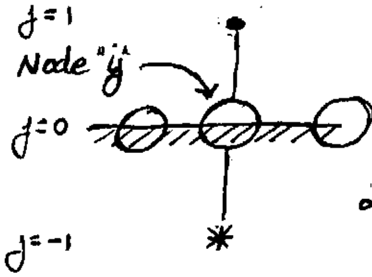
$\Rightarrow u_\bullet = u_*$

$\Rightarrow$ for nodes on Boundary A, the molecule becomes:



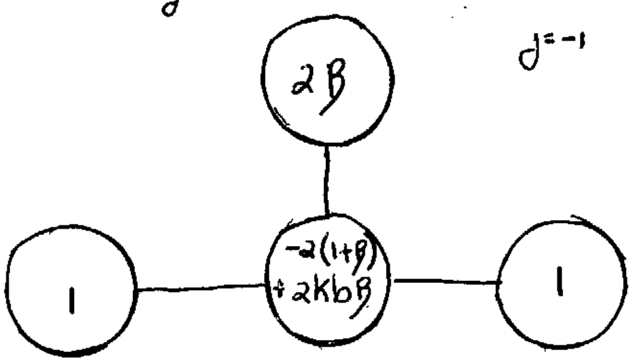
$= g_{ij} h^2$

Boundary B: $\left.\frac{\partial u}{\partial y}\right|_{j=0} + bu = d$



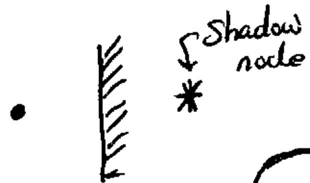
$\frac{U_\circ - U_*}{2k} + bU_{ij} = d$

$2kbU_{ij} - 2kd + U_\circ = U_*$

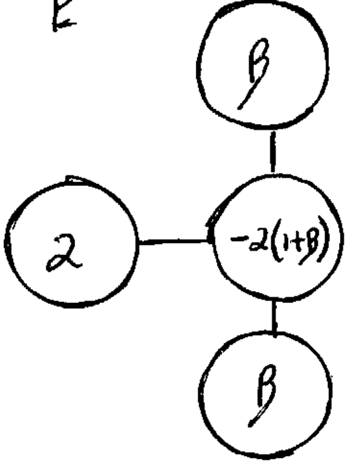


$= g_{ij} h^2 + 2kd\beta$

Boundary C:



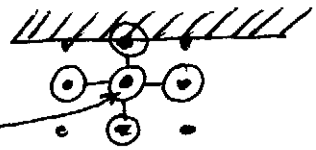
$\frac{U_* - U_\circ}{2h} = a \Rightarrow U_* = U_\circ + 2ah$



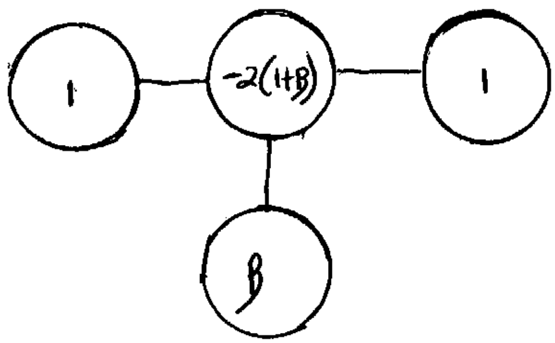
$= g_{ij} h^2 - 2ah$

Boundary D: Type 1 condition ... don't use PDE

Stop one node short of boundary

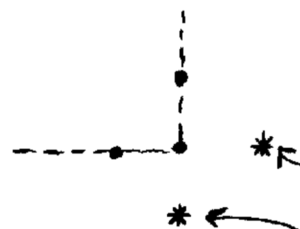


Molecule for this node:



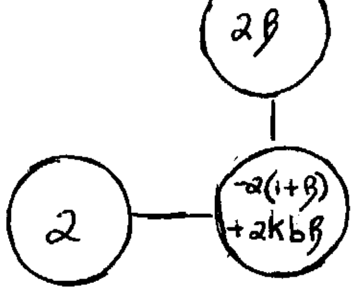
$= g_{ij}h^2 + -\beta f_z$

Corners: e.g.



shadow node for boundary C

shadow node for boundary B



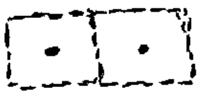
$= g_{ij}h^2 - 2ah + 2k\beta d$

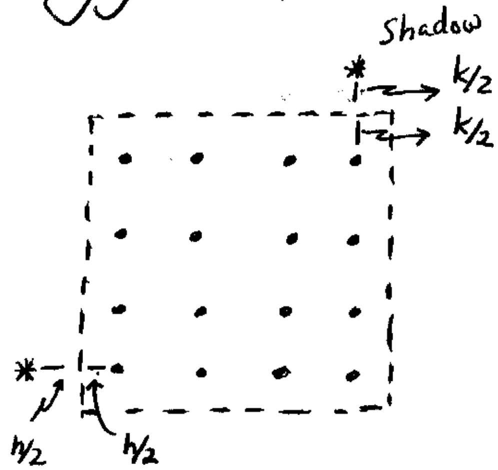
Basic Rule: Type 1 BC do not use PDE

Type II, III, Use PDE plus BC together (i.e. use BC to eliminate shadow node)

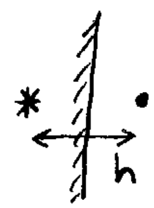
Alternate strategy ... place nodes $h/2$ from boundary

Think of nodes as centers of "cells"



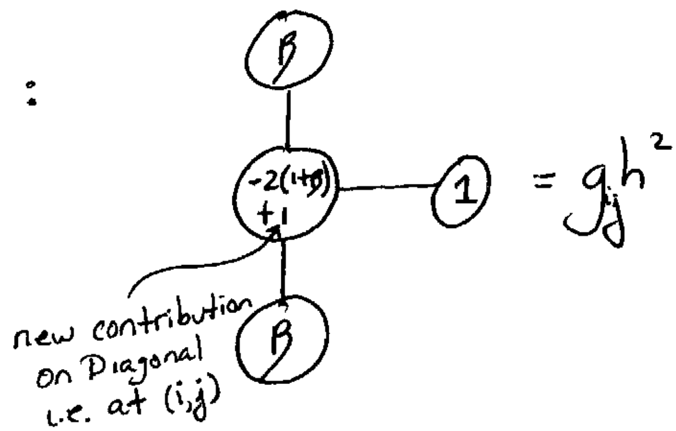


Type II condition: $\frac{\partial u}{\partial x} = 0$



$$\frac{u_\circ - u_*}{h} = 0 \Rightarrow u_\circ = u_*$$

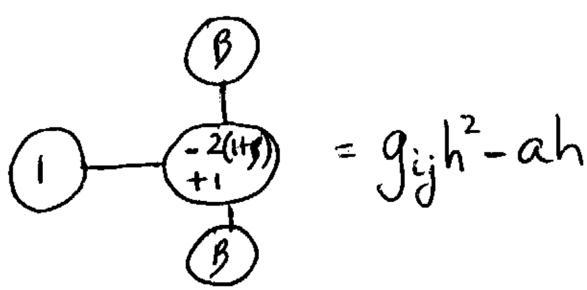
Molecule:



Type II condition: $\frac{\partial u}{\partial x} = a$

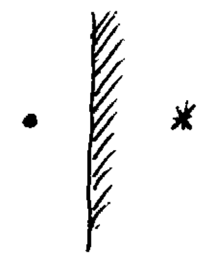
$$\frac{u_* - u_\circ}{h} = a \Rightarrow u_* = ah + u_\circ$$

Molecule



Not convenient when have u involved on boundary $\Rightarrow$ Type I or Type III

e.g.



$\frac{\partial u}{\partial x} + bu = a$

$$U_* = U_i + \frac{h}{2}\frac{\partial U_i}{\partial x} + \frac{\left(\frac{h}{2}\right)^2}{2!}\frac{\partial^2 U_i}{\partial x^2} + \dots$$

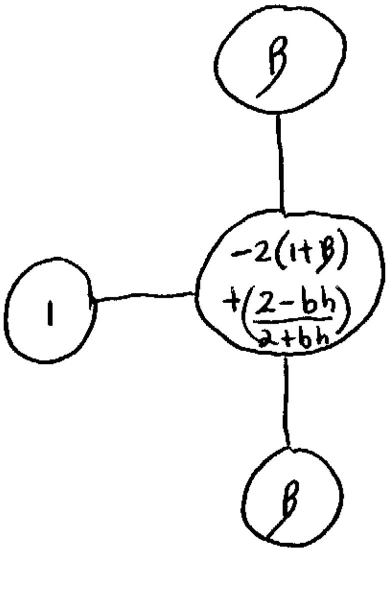
$$\frac{U_* - U_\circ}{h} + b\,U_{bound} = a \qquad \text{i.e. } U_\circ = U_i - \frac{h}{2}\frac{\partial U_i}{\partial x} + \frac{\left(\frac{h}{2}\right)^2}{2}\frac{\partial^2 U_i}{\partial x^2}$$

$$\frac{U_* + U_\circ}{2} = U_i + O(h^2)$$

Take as average of $U_\circ$ + U_* $\Rightarrow \frac{U_* + U_\circ}{2}$

$$\frac{U_* - U_\circ}{h} + b\left(\frac{U_* + U_\circ}{2}\right) = a \Rightarrow 2U_* - 2U_\circ + bhU_* + bhU_\circ = 2ah$$

$$U_* = \frac{2ah}{2+bh} + \left(\frac{2-bh}{2+bh}\right)U_\circ$$

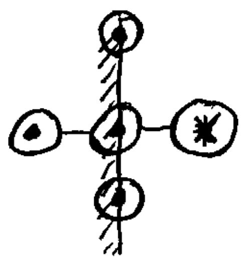


$$= g_{ij}h^2 - \frac{2ah}{2+bh}$$

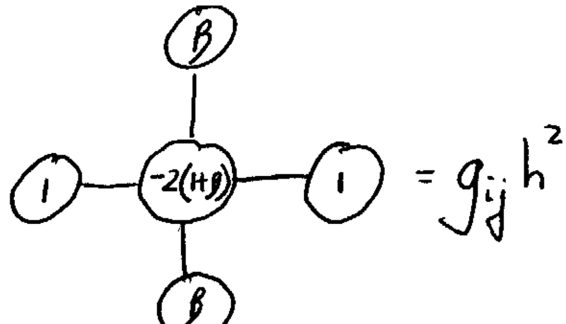
Same Strategy for Type I

Note that in case of Type I BC, we don't use the PDE at boundary node . . .
Stop assembling PDE "one node in" from boundary

This equation contains useful info, use it to determine flux information . . . after u is determined, can reconstruct $\nabla u \cdot \hat{n}$ via the unused boundary molecule



use



to solve for u_* (shadow node value)

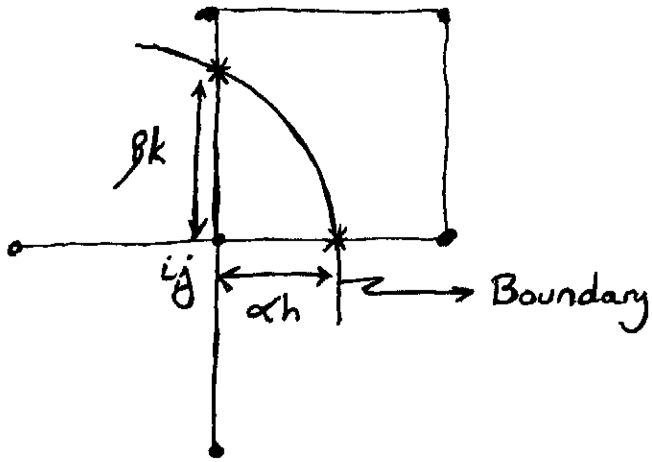
$$u_* = g_{ij} h^2 - \beta u_{i,j+1} - \beta u_{i,j-1} + 2(1+\beta) u_{ij} - u_{i-1,j}$$

known!

then $\dfrac{\partial u}{\partial x} = \nabla u \cdot \hat{n} = \dfrac{u_* - u_\bullet}{2h}$ ⇒ known value

$$\oint \nabla u \cdot \hat{n} \, ds = \int \nabla \cdot \nabla u \, dV = \int g \, dV$$ Conservation Statement

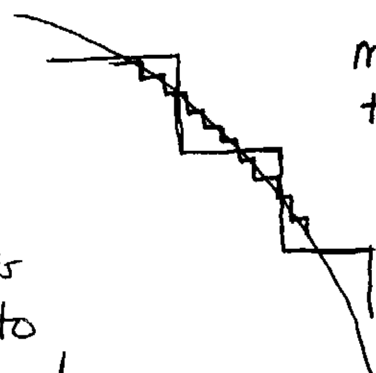
What about a situation like



Can write PDE molecule on uneven mesh (lose accuracy!)
OK if Type I BC ... Simply replace U_* with known value
but if Type II BC ... must approximate $\frac{\partial U_*}{\partial n}$ as difference expression involving internal points ... can get <u>very</u> messy

Alternatively ... "stair-step" the curved boundary

then everything on a uniform lattice and can proceed as usual



May require over-resolution to allow reasonable approx of BCs

Here, must translate BC's from physical boundary to mesh boundary, but once done can use standard algorithm