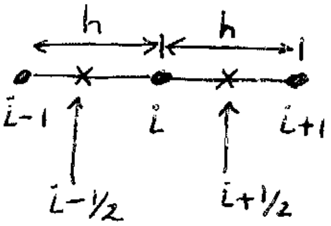
General Elliptic Eqn

$$\underbrace{\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)} + \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g$$

Common to see this term differenced as...

$$\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) = \frac{1}{h}\left[a_{i+1/2}\left(\frac{u_{i+1}-u_i}{h}\right) - a_{i-1/2}\left(\frac{u_i-u_{i-1}}{h}\right)\right]$$

where:

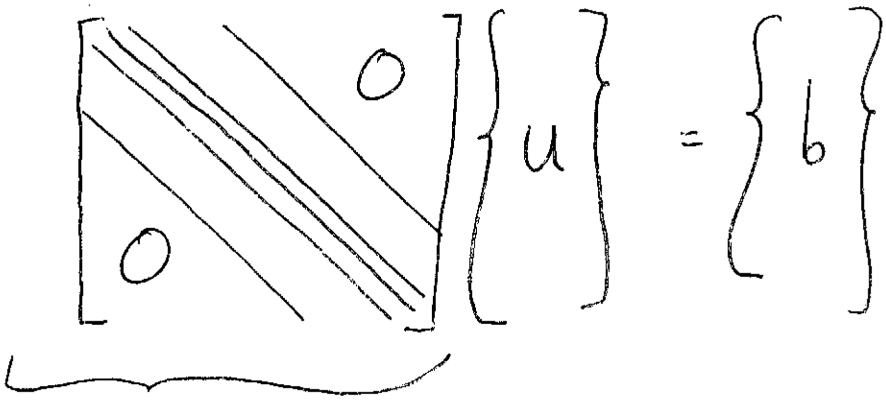


coefficient evaluated "in between" nodes

Molecule:

$$\begin{array}{ccccc}
 & & c_{i,j+1/2} + e_{ij}\frac{h}{2} & & \\
 & & | & & \\
a_{i-1/2,j} + -\frac{d_{ij}h}{2} & — & \begin{matrix} -a_{i+1/2,j} - a_{i-1/2,j} \\ -c_{i,j+1/2} - c_{i,j-1/2} \\ + f_{ij}h^2 \end{matrix} & — & a_{i+1/2,j} + \frac{d_{ij}h}{2} \\
 & & | & & \\
 & & c_{i,j-1/2} - e_{ij}\frac{h}{2} & &
\end{array} = g_{ij}h^2$$

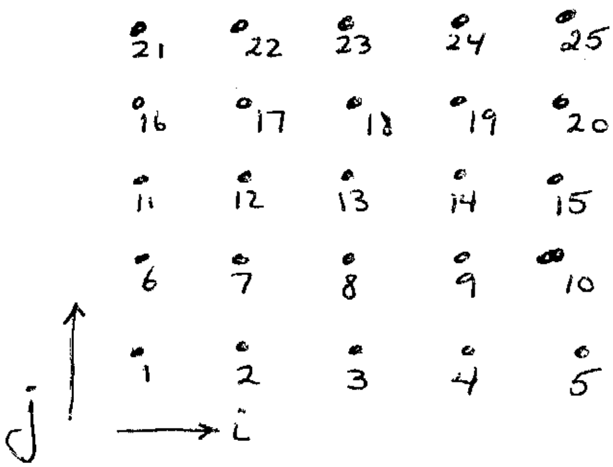
Coupled set of eqn's can be put in matrix form:



"Pentadiagonal"

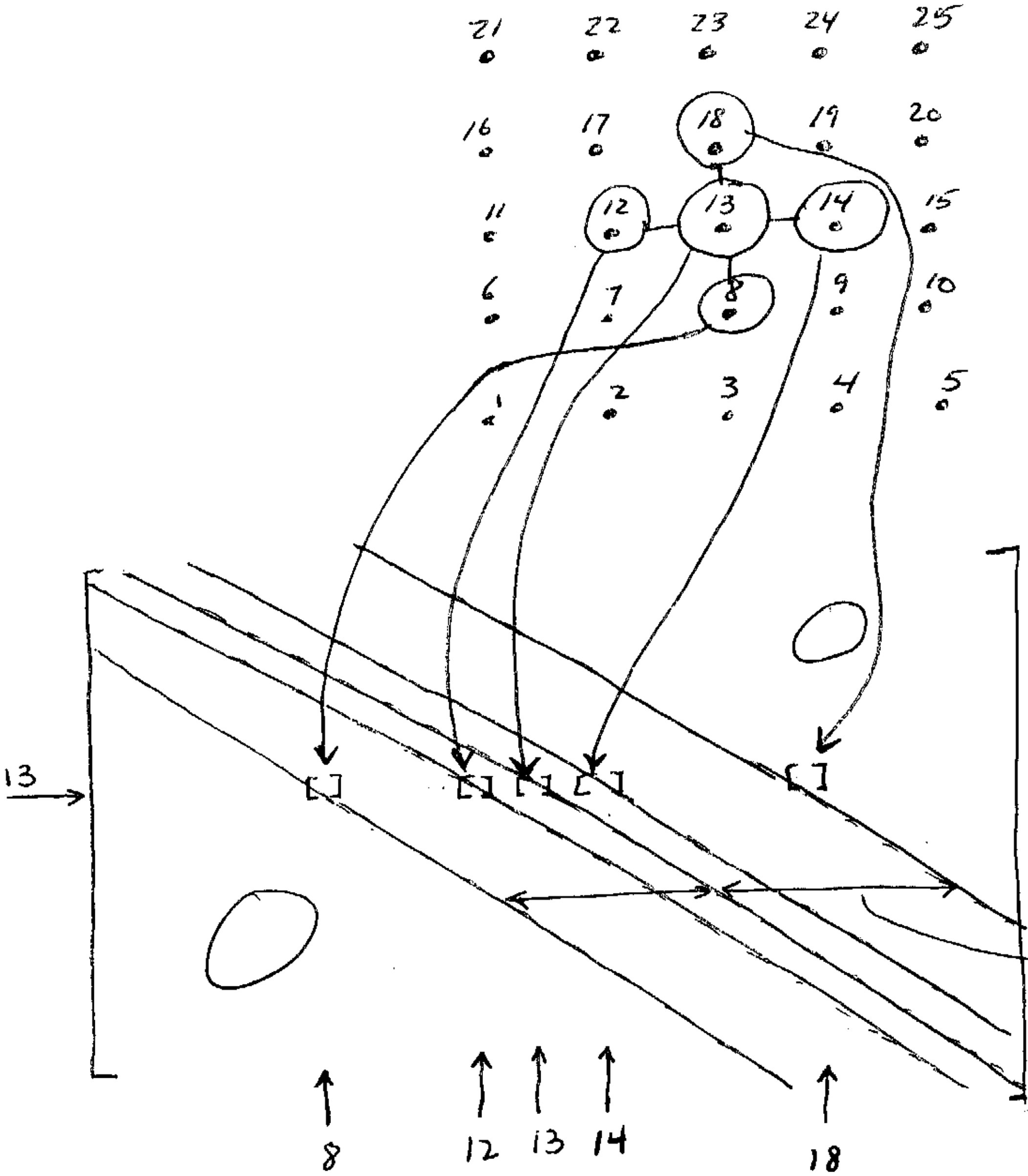
- Need a mapping between (i,j) template location and (n,m) matrix entry in A ... assign a unique number to each mesh point ... generates pentadiagonal structure provided some "natural" ordering is used

e.g.



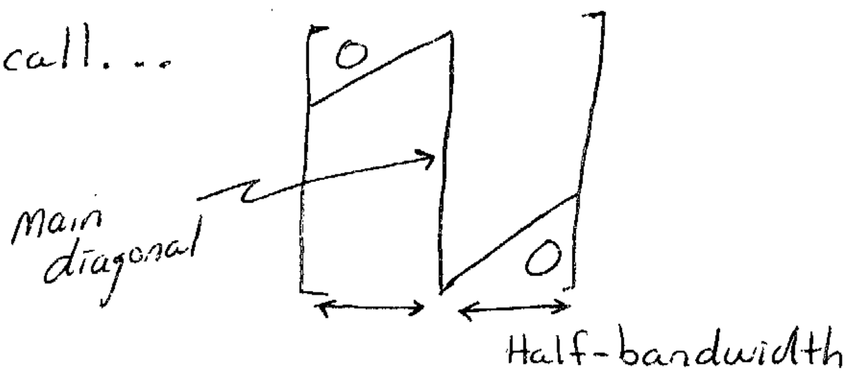
- each u_{ij} maps to a unique column in A
- each (i,j) template "center" maps to unique row in A

elliptic molecule
"centered" at 13
so this is eqn for row 13
Involves u_{ij} at
12, 13, 14, 8, 18
i.e. coefficients in template
go in these rows



Typical to store as banded matrix . . .

Recall . . .



Half-bandwidth = maximum difference between node numbers "connected" through the template

- want to study properties of A ... are important for solving $Au=b$, especially iteratively

write out the molecule as ...

$$\beta_1 u_{i+1,j} + \beta_2 u_{i-1,j} + \beta_3 u_{i,j+1} + \beta_4 u_{i,j-1} - \beta_0 u_{ij} = g_{ij} h^2$$

where $\beta_1 = a_{i+1/2,j} + d_{ij}\frac{h}{2}$

$\beta_2 = a_{i-1/2,j} - d_{ij}\frac{h}{2}$

$\beta_3 = c_{i,j+1/2} + e_{ij}\frac{h}{2}$

$\beta_4 = c_{i,j-1/2} - e_{ij}\frac{h}{2}$

$\beta_0 = a_{i+1/2,j} + a_{i-1/2,j} + c_{i,j+1/2} + c_{i,j-1/2} - f_{ij}h^2$

- We note if $\underline{\underline{a>0,\ c>0,\ f\leq 0}}$

- then $\beta_0 > 0$ and $\beta_1 \to \beta_4$ can be made positive by choosing h small enough

  i.e. $0 < h < \min\left\{ \frac{2a_{i\pm 1/2,j}}{|d_{ij}|}, \frac{2c_{i,j\pm 1/2}}{|e_{ij}|} \right\}$

- $\beta_0 \geq \sum_{i=1}^{4} \beta_i$

Conclude: $Au = b$, A with elements α_{ij} has the properties

i) $\alpha_{ii} > 0$, $\alpha_{ij} \le 0$ for $i \ne j$ (or the reverse)

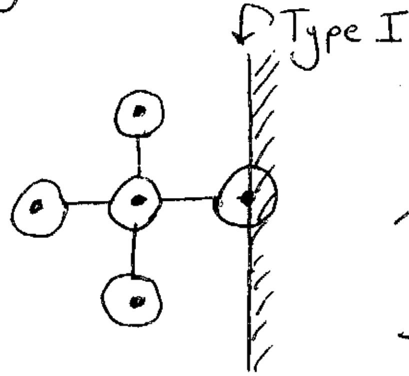
ii) $\alpha_{ii} \geq \sum_{i \ne j} |\alpha_{ij}|$

In ii) get strict inequality if $f_{ij} < 0$

<u>or</u>

get strict inequality for some "i" if $f_{ij} = 0$ and we have Type I BCs

e.g.



β_1 does not appear in matrix A

so clearly $\beta_0 > \sum_{i=2}^{4} \beta_i$

- A is diagonally dominant ... not "strict sense" though ... generally good news for iterative solvers

Can prove classical methods converge in this case (more later)

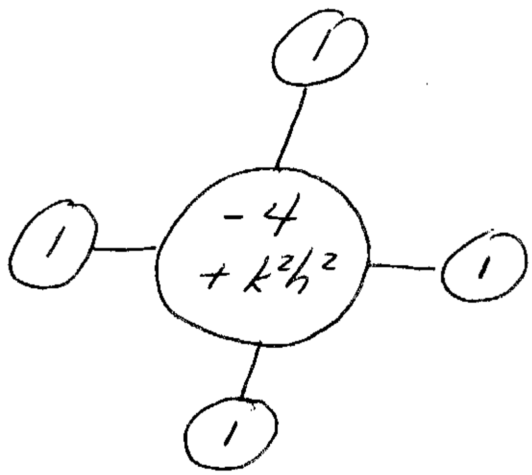
<u>Aside</u>: May seem restrictive requiring $a>0$, $c>0$, $f\le 0$

- but must have a + c same sign (Recall $b^2-4ac<0$) and most physical problems have positive coefficients

- $f<0$ is more restrictive (clearly $f=0$ is common!) are cases where $f>0$ ... e.g. Helmholtz eqn
$$\nabla^2 u + k^2 u = 0$$

$$FD \Rightarrow \delta_x^2 u_{ij} + \delta_y^2 u_{ij} + k^2h^2 u_{ij} = 0$$

2D molecule



Diag. Dominance lost as $h\to 0$ ... only have it when
$$k^2h^2 - 4 > 4 \Rightarrow k^2h^2 > 8 \Rightarrow \underbrace{\frac{2\pi}{\sqrt{8}}}_{2.2} > \underbrace{\lambda/h}_{\text{samples/wavelength}} \qquad \lambda \equiv \text{wavelength}$$

want: $h \le \lambda/20$ ← <u>very poor</u> resolution
$$(kh)^2 \le (2\pi/20)^2$$