Dartmouth College
Thayer School of Engineering

Signal Processing

Practice Problems #2

1. *Finding the statistical averages of a random variable.* Assume a random variable is uniformly distributed (equally likely) over a finite range of $a \le x \le b$ with a pdf given by

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

Find the mean, mean-squared value, and the variance of this random variable.

Solution:

(a) mean

$$m_x = \int_a^b x \frac{1}{b-a}\, dx = \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right) = \frac{a+b}{2}$$

(b) mean-squared value

$$E[X^2] = \int_a^b x^2 \frac{1}{b-a}\, dx = \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(a^2+ab+b^2)}{(b-a)3} = \frac{a^2+ab+b^2}{3}$$

(c) variance

$$\begin{aligned} \sigma_x^2 = E[X^2] - m_x^2 &= \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4} \\ &= \frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12} \\ &= \frac{a^2-2ab+b^2}{12} = \frac{(b-a)^2}{12} \end{aligned}$$

2. *Autocorrelation of white noise.* Let $e[n]$ denote a white-noise sequence, and let $s[n]$ denote a sequence that is uncorrelated with $e[n]$. Show that the sequence

$$y[n] = s[n]e[n]$$

is white. In other words, show that

$$E\{y[n]y[n+m]\} = A\delta[m]$$

where A is a constant.

Solution: The autocorrelation of $y[n]$:

$$\begin{aligned} E\{y[n]y[n+m]\} &= E\{(s[n]e[n]s[n+m]e[n+m]\} \\ &= E\{(s[n]s[n+m]\}E\{e[n]e[n+m]\} \quad \text{since } s[n] \text{ and } e[n] \text{ are uncorrelated} \\ &= E\{(s[n]s[n+m]\}\sigma_e^2\delta[m] \end{aligned}$$

This autocorrelation of $y[n]$ is only nonzero at lag $m = 0$ so $y[n]$ is a white random process where $A = E\{s^2[n]\}$.

3. *Autocorrelation functions and their Fourier transforms.* Consider a random signal

$$x[n] = s[n] + e[n]$$

where both $s[n]$ and $e[n]$ are independent zero-mean stationary random signals with autocorrelation functions $\phi_{ss}[m]$ and $\phi_{ee}[m]$ respectively. Determine the expressions for $\phi_{xx}[m]$ and its power spectral density $\Phi_{xx}(e^{j\omega})$.

Solution:

$$\begin{aligned} \phi_{xx}[m] &= E\{x[n]x[n+m]\} \\ &= E\{(s[n]+e[n])(s[n+m]+e[n+m])\} \\ &= E\{(s[n]s[n+m])\} + E\{(s[n]e[n+m])\} + E\{(e[n]s[n+m])\} + E\{(e[n]e[n+m])\} \\ &= \phi_{ss}[m] + E\{s[n]\}E\{e[n+m]\} + E\{e[n]\}E\{s[n+m]\} + \phi_{ee}[m] \\ &= \phi_{ss}[m] + 2E\{s[n]\}E\{e[n]\} + \phi_{ee}[m] \\ &= \phi_{ss}[m] + \phi_{ee}[m] \end{aligned}$$

Taking the Fourier transform, provides the power spectral density of $x[n]$:

$$\Phi_{xx}(e^{j\omega}) = \Phi_{ss}(e^{j\omega}) + \Phi_{ee}(e^{j\omega})$$

4. *Autocorrelation and Power Spectral Density.* Consider a random process $y[n]$ that is the output of the LTI system with frequency response:

$$H(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$

The input to this system is $x[n]$ which is a real zero-mean stationary white-noise process with $E\{x^2[n]\} = \sigma_x^2$

(a) Determine the power spectral density of the output.

(b) Determine the autocorrelation function of the output as a function of the deterministic autocorrelation sequence of the system impulse response, $c_{hh}[\ell]$.

Solution:

(a) The power spectral density of the output of a LTI system with an input that is a real-valued wide-sense stationary discrete-time random process:

$$\Phi_{yy}(e^{j\omega}) = \left|H(e^{j\omega})\right|^2 \Phi_{xx}(e^{j\omega})$$

Since the input is a zero-mean stationary white-noise process with $E\{w^2[n]\} = \sigma_w^2$ the autocorrelation of the input is only nonzero at lag zero and the power spectral density of the input is a constant:

$$\begin{aligned} \phi_{xx}[m] &= \sigma_x^2 \delta[m] \\ \Phi_{xx}(e^{j\omega}) &= \sigma_x^2 \end{aligned}$$

We can now determine the power spectral density of the output:

$$\begin{aligned} \Phi_{yy}(e^{j\omega}) &= \left|H(e^{j\omega})\right|^2 \Phi_{xx}(e^{j\omega}) \\ &= \left|H(e^{j\omega})\right|^2 \sigma_x^2 \\ &= H(e^{j\omega})H^*(e^{j\omega})\sigma_x^2 \\ &= \left(\frac{1}{1-0.5e^{-j\omega}}\right)\left(\frac{1}{1-0.5e^{j\omega}}\right)\sigma_x^2 \\ &= \sigma_x^2\left(\frac{1}{1-0.5e^{-j\omega}-0.5e^{j\omega}+1/4}\right) \\ &= \sigma_x^2\left(\frac{1}{1.25-\cos(\omega)}\right) \end{aligned}$$

(b) The impulse response is the inverse Fourier transform of the frequency response:

$$h[n] = \mathcal{F}^{-1}\left(\frac{1}{1-0.5e^{j\omega}}\right) = 0.5^n u[n]$$

The output power spectral density is

$$\begin{aligned} \phi_{yy}[m] &= \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell]c_{hh}[\ell] \\ \phi_{yy}[m] &= \sum_{\ell=-\infty}^{\infty} \sigma_x^2\delta[m-\ell]c_{hh}[\ell] \\ &= \sigma_x^2 c_{hh}[m] \end{aligned}$$

Note that the deterministic autocorrelation of the input sequence is

$$
\begin{aligned}
c_{hh}[m] &= h[m] * h[-m] \\
&= (0.5^n u[n]) * \big(0.5^{-n} u[-n]\big)
\end{aligned}
$$