Dartmouth College
Thayer School of Engineering

**Signal Processing**

# Practice Problems #1

1. Given the following two discrete-time systems, determine whether the system is stable, causal, linear, and time-invariant.
   (a) $y[n] = x[-n-2]$
   (b) $y[n] = \cos(x[n])$

**Solution:**

(a) $y[n] = x[-n-2]$

i. Stability: If the input is bounded and a constant, $x[n] = C$, then the output will also be this constant $C$ so therefore the output is bounded and this system is stable.

ii. Causality: This system is not causal since the output at $n = -2$ depends on a future input: $y[-2] = x[-(-2)-2] = x[0]$.

iii. Linearity: If $y_1[n] = x_1[-n-2]$ and $y_2[n] = x_2[-n-2]$ then let's check the superposition principle:

$$\begin{aligned} T\{ax_1[n] + bx_2[n]\} &= ax_1[-n-2] + bx_2[-n-2] \\ &= ay_1[n] + by_2[n] \end{aligned}$$

Therefore, this system is linear.

iv. Time-invariance: The input $x_1[n]$ gives the output $y_1[n] = x_1[-n-2]$ and if we delay this output by $k$, we get $x_1[-(n-k)-2]$ which is the same output we obtain if the input is shifted by $k$, $T\{x_1[n-k]\}$ so this system is time-invariant.

(b) $y[n] = \cos[x[n]]$

i. Stability: The system output is always between -1 and 1 so the system is stable.

ii. Causality: The system is causal because it always depends on the current input $x[n]$ and not future inputs.

iii. Linearity: If $y_1[n] = T\{x_1[n]\} = \cos(x_1[n]$ and $y_2[n]) = T\{x_2[n]\} = \cos(x_2[n])$ then let's check the superposition principle:

$$\begin{aligned} T\{ax_1[n] + bx_2[n]\} &= \cos\left(a\cos(x_1[n]) + b\cos(x_2[n])\right) \\ &\neq ay_1[n] + by_2[n] \end{aligned}$$

Therefore, this system is not linear.

iv. Time-invariance: The input $x_1[n]$ gives the output $y_1[n] = \cos(x_1[n])$ and if we delay this output by $k$, we get $\cos(x_1[n-k]$ which is the same output we obtain if the input is shifted by $k$ so this system is time-invariant.

2. Determine the convolution sum of the following pair of discrete-time signals in the time-domain.

$$\{\underset{\uparrow}{2}, 1, 3\} * \{1, \underset{\uparrow}{4}, 2\}$$

**Solution:**

a) The convolution of the input and impulse response provides the ouput of an LTI System: $y(n) = x(n) * h(n)$.

b) 1. fold one sequence: $\{3, 1, \underset{\uparrow}{2}\}$

repeat for different $n$

2. Shift left by 1 so $n=-1$
3. multiply to get product sequence
4. add to get $y(-1)$

| | | | |
|---|---|---|---|
| $n=-1$ | 312 / 142 | $\{0,0,2,0,0\}$ | $y(-1)=2$ |
| $n=0$ | 312 / 142 | $\{0,1,8,0\}$ | $y(0)=9$ |
| $n=1$ | 312 / 142 | $\{3,4,4\}$ | $y(1)=11$ |
| $n=2$ | 312 / 142 | $\{0,12,2,0\}$ | $y(2)=14$ |
| $n=3$ | 312 / 142 | $\{0,0,6,0,0\}$ | $y(3)=6$ |
| $n=4$ | 312 / 142 | no overlap | $y(4)=0$ |

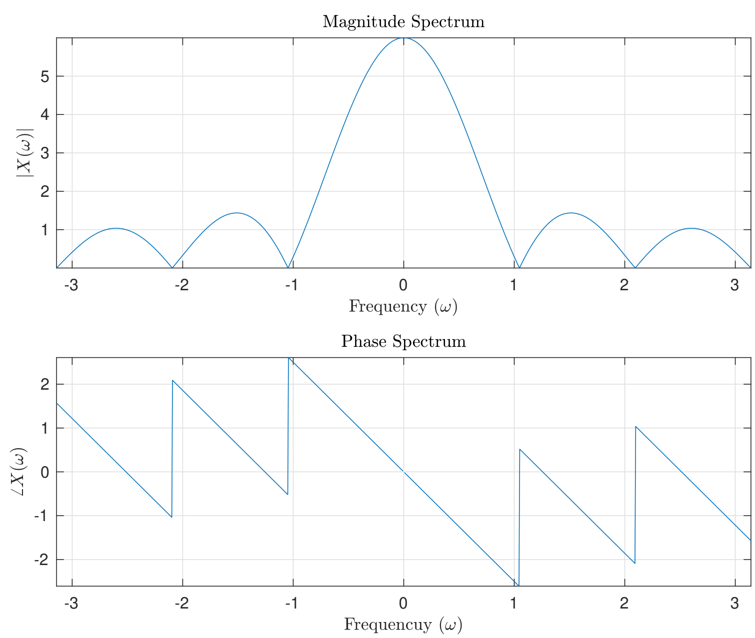
$$\boxed{\{2, \underset{\substack{\uparrow \\ n=0}}{9}, 11, 14, 6\}}$$

3. Determine the discrete-time Fourier transform of the following signals.
   (a) $x(n) = u(n) - u(n-6)$
   (b) $x(n) = (1/4)^n u(n+4)$

**Solution:**

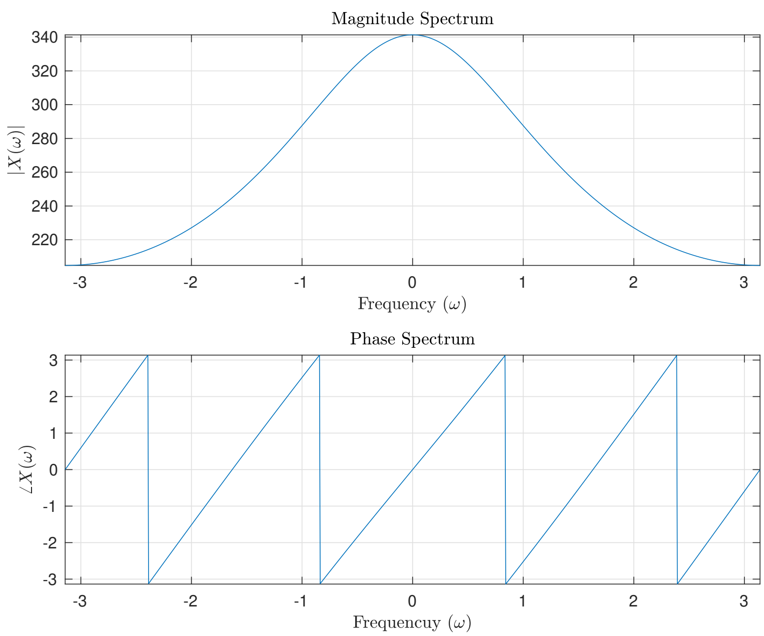
(a)

$$
\begin{aligned}
X(\omega) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\
&= \sum_{n=0}^{5} e^{-j\omega n} \\
&= \sum_{n=0}^{5} \left(e^{-j\omega}\right)^n \\
&= \frac{1 - e^{-j6\omega}}{1 - e^{-j\omega}}
\end{aligned}
$$



(b) $x(n) = (1/4)^n u(n+4)$

$$\begin{aligned}
X(\omega) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\
&= \sum_{n=-4}^{\infty} (1/4)^n e^{-j\omega n} \\
&\text{let } m = n+4 \text{ and therefore } n = m-4 \\
&= \sum_{m=0}^{\infty} (1/4)^{m-4} e^{-j\omega(m-4)} \\
&= 4^4 e^{j4\omega} \sum_{m=0}^{\infty} (1/4)^m \left(e^{-j\omega}\right)^m \\
&= \frac{4^4 e^{j4\omega}}{1 - \frac{1}{4} e^{-j\omega}}
\end{aligned}$$




4. Find all possible sequences, $x[n]$, with the $z$-transform, $X(z)$, given below.

$$X(z) = \frac{6}{1 + 0.75z^{-1} - 1.125z^{-2}}$$

**Solution:** Let's start by performing partial fraction expansion to break this rational function into simple fractions in which we can find in a $z$-transform table.

$$H(z) = \frac{6}{1 + 0.75z^{-1} - 1.125z^{-2}} = \frac{6z^2}{z^2 + 0.75z - 1.125}$$
$$\frac{H(z)}{z} = \frac{6z}{(z - \frac{3}{4})(z + \frac{3}{2})} = \frac{A_1}{(z - \frac{3}{4})} + \frac{A_2}{(z + \frac{3}{2})}$$

Multiplying through by the pole factors, we can solve for $A_1$ and $A_2$:

$$6z = A_1(z + \tfrac{3}{2}) + A_2(z - \tfrac{3}{4})$$

When $z = \frac{3}{4} \implies A_1 = 2$.
When $z = \frac{-3}{2} \implies A_2 = 4$.

$$H(z) = \frac{2z}{(z - \frac{3}{4})} + \frac{4z}{(z + \frac{3}{2})} = \frac{2}{(1 - \frac{3}{4}z^{-1})} + \frac{4}{(1 + \frac{3}{2}z^{-1})}$$

Now that we have simple fractions in which we can use a table to find the inverse $z$-transform, we will consider all three possible regions of convergence. Noticesthat the poles are at $z_{p1} = 0.75$ and $z_{p2} = -1.5$ and finding the inverse $z$-transform using a table:

- ROC: $|z| > 1.5$

$$h[n] = \left[2(0.75)^n + 4(-1.5)^n\right] u[n]$$

- ROC: $0.75 < |z| < 1.5$

$$h[n] = 2(0.75)^n u[n] + 4(-1.5)^n u[-n-1]$$

- ROC: $|z| < 0.75$

$$h[n] = \left[-2(0.75)^n - 4(-1.5)^n\right] u[-n-1]$$

5. A LTI system is characterized by the difference equation

$$y(n) = y(n-1) + y(n-2) + 2x(n) + x(n-1)$$

(a) Determine the system function $H(z)$.

(b) Plot the poles and zeros of $H(z)$ and indicate the region of convergence and whether or not the system is stable.

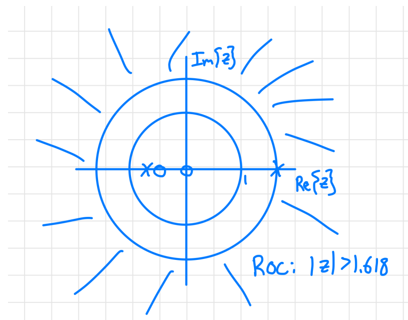
(c) From $H(z)$, determine the impulse response of the system.

**Solution:**

(a) The system function is found by taking the $z$-transform of the difference equation.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2+z^{-1}}{1-z^{-1}-z^{-2}} = \frac{2z^2+z}{z^2-z-1} = \frac{z(2z+1)}{(z+0.618)(z-1.618)}$$

(b) Zeros at $z = 0$ and $z = -0.5$. Poles at $z = -0.618$ and $z = 1.618$. Since the system is causal, the impulse response is a causal sequence and the ROC is $|z| > 1.618$.




(c) The impulse response, $h(n)$, is the inverse $z$-transform of the system function after expanding the system function into a sum of simple fractions.

$$\frac{H(z)}{z} = \frac{2z+1}{(z+0.618)(z-1.618)} = \frac{A_1}{(z+0.618)} + \frac{A_2}{(z-1.618)}$$

$$H(z) = \frac{0.1056z}{(z+0.618)} + \frac{1.8944z}{(z-1.618)} = \frac{0.1056}{1+0.618z^{-1}} + \frac{1.8944}{1-1.618^{-1}}, \quad |z| > |a|$$

$$h(n) = [.1056(-0.618)^n + 1.8944(1.618)^n]\, u(n)$$