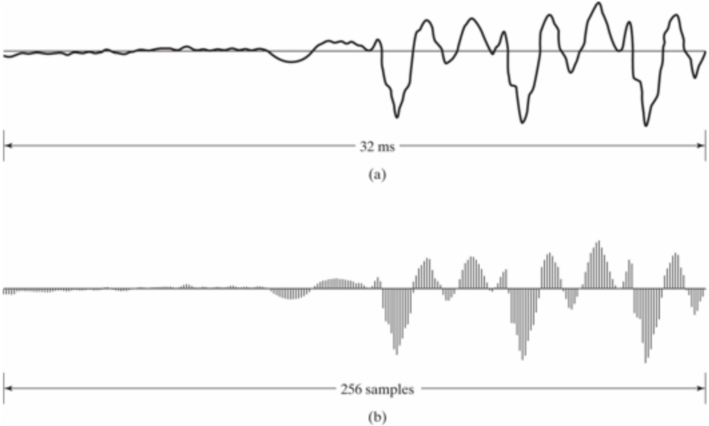
Classification of Discrete-Time Systems

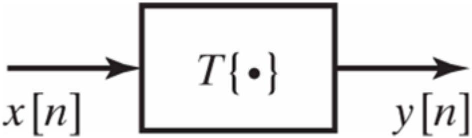
Continous-Time Speech Signal and Sequence of Samples

$$x_a(t) = \sum_{i=1}^{N} A_i(t) \sin\left[2\pi F_i(t)t + \theta_i(t)\right]$$



$$x[n] = x_a(nT), \text{ where } T = 125\mu s$$

Discrete-Time System



Memoryless System versus a System With Memory

Examples of Memoryless Systems:

$$\begin{aligned} y[n] &= ax[n] \\ y[n] &= nx[n] + bx^3[n] \end{aligned}$$

Examples of Systems with Memory:

$$\begin{aligned} y[n] &= x[n] + 3x[n-1] \\ y[n] &= \sum_{k=0}^{n} x(n-k) \\ y[n] &= \sum_{k=0}^{\infty} x(n-k) \end{aligned}$$

Linear Systems versus Nonlinear Systems

If $y_1[n]$ and $y_2[n]$ are the responses of a system when $x_1[n]$ and $x_2[n]$ are the respective inputs, then the system is linear if and only if the additivity property holds:

$$T\left\{x_1[n] + x_2[n]\right\} = T\left\{x_1[n]\right\} + T\left\{x_2[n]\right\} = y_1[n] + y_2[n]$$

and the scaling property holds:

$$T\left\{ax[n]\right\} = aT\left\{x[n]\right\} = ay[n]$$

Together, these two properties comprise the principle of superposition:

$$T\left\{ax_1[n] + bx_2[n]\right\} = aT\left\{x_1[n]\right\} + bT\left\{x_2[n]\right\}$$

Time-Invariant versus Time-Variant Systems

A relaxed system is time-invariant (or shift invariant) if and only if

$$x[n] \longrightarrow y[n]$$

implies that

$$x[n-k] \longrightarrow y[n-k]$$

Time-Invariant versus Time-Variant Systems

Examples of time-invariant systems:

$$
\begin{aligned}
y[n] &= x[n] - x[n-1] \\
y[n] &= \sum_{k=-\infty}^{n} x[k]
\end{aligned}
$$

Examples of time-varying systems:

$$
\begin{aligned}
y[n] &= nx[n] \\
y[n] &= x[-n]
\end{aligned}
$$

Causal versus Noncausal Systems

A system is causal if, for every choice of n_0, the output sequence value at the time index $n = n_0$ depends only on the input sequence value for $n \leq n_0$.

Example of noncausal system:

$$y[n] = x[n+1] - x[n]$$

Example of a causal system:

$$y[n] = x[n] - x[n-1]$$

Stable versus Unstable Systems

The input, $x[n]$, is bounded if there exists a fixed positive finite value B_x such that

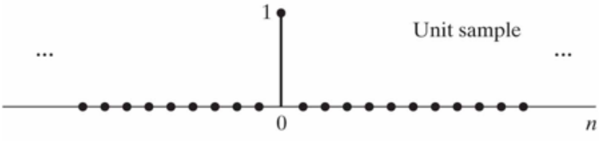
$$|x[n]| \leq B_x < \infty, \quad \text{for all } n$$

Stability requires that, for every bounded input, there exists a fixed positive finite value B_y such that

$$|y[n]| \leq B_y < \infty, \quad \text{for all } n$$

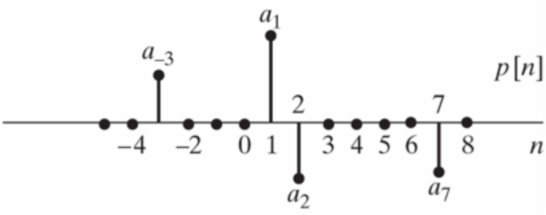
The Convolution Sum

Unit Sample Sequence (Unit Impulse)



$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

Signals represented as a sum of scaled, delayed impulses



$$p[n] = a_{-3}\delta[n+3] + a_1\delta[n-1] + a_2\delta[n-2] + a_7\delta[n-7]$$

Any sequence can be represented as a sum of scaled, delayed impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Impulse Response for LTI Systems

$$
\begin{aligned}
y[n] &= T\{x[n]\} \\
&= T\left\{ \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \right\}
\end{aligned}
$$

Now, using the superposition principle:

$$
y[n] = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h_k[n]
$$

When we let $h_k[n]$ be the response of the system to the input $\delta[n-k]$.

Computing Discrete-Time Convolution

The Convolution Sum

The response $y[n]$ of an LTI system as a function of the input signal $x[n]$ and the impulse response h[n] is called the convolution sum:

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \end{aligned}$$

Step to Compute the Convolution Sum

Let's find the output of the LTI system at one particular time instant $n = n_0$:

$$y[n_0] = \sum_{k=-\infty}^{\infty} x[k]h[n_0 - k]$$

1. *Folding.* Fold $h[k]$ about $k = 0$ to obtain $h[-k]$.
2. *Shifting.* Shift $h[-k]$ by n_0 to the right (left) if n_0 is positive (negative) to obtain $h[n_0 - k]$
3. Multiplication. Multiply the two sequences $x[k]$ and $h[n_0 - k]$ to obtain $x[k]h[n_0 - k]$.
4. *Summation.* Sum all the values of the product sequence to obtain the output at time $n = n_0$.

Repeat steps 2 through 4 for all possible time shifts to find the response over all time.

Convolution Sum Example

Example: The impulse response of a linear time-invariant system is $h[n] = \left\{1, \underset{\uparrow}{2}, 1, -1\right\}$. Determine the response of the system to the input signal $x[n] = \left\{\underset{\uparrow}{1}, 2, 3, 1\right\}$.

$$h[-k] = -1, 1, \underset{\uparrow}{2}, 1$$

$$h[-1-k] = -1, 1, 2, \underset{\uparrow}{1} \quad \text{where } n_0 = -1$$

$$x[k] = \underset{\uparrow}{1}, 2, 3, 1$$

$$x[k]h[-1-k] = ..., 0, 0, 0, \underset{\uparrow}{1}, 0, 0, 0, ...$$

$$y[-1] = \sum_{k=-\infty}^{\infty} x[k]h[-1-k] = 1$$

Convolution Sum Example (Continued)

Let the shift be $n_0 = 0$ to find output $y[0]$:

$$x[k] = \underset{\uparrow}{1}, 2, 3, 1$$

$$h[0-k] = -1, 1, \underset{\uparrow}{2}, 1$$

$$x[k]h[0-k] = \ldots, 0, 0, 0, \underset{\uparrow}{2}, 2, 0, 0, \ldots$$

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = 4$$

Convolution Sum Example (Continued)

Let the shift be $n_0 = 1$ to find output $y[1]$:

$$x[k] = \underset{\uparrow}{1}, 2, 3, 1$$

$$h[1-k] = -1, \underset{\uparrow}{1}, 2, 1$$

$$x[k]h[1-k] = ..., 0, 0, 0, \underset{\uparrow}{1}, 4, 3, 0, ...$$

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = 8$$

Convolution Sum Example (Continued)

Let the shift be $n_0 = 2$ to find output $y[2]$:

$$\begin{aligned} x[k] &= \underset{\uparrow}{1}, 2, 3, 1 \\ h[2-k] &= \underset{\uparrow}{-1}, 1, 2, 1 \\ x[k]h[2-k] &= ..., 0, 0, 0, \underset{\uparrow}{-1}, 2, 6, 1, 0... \end{aligned}$$

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = 8$$

Convolution Sum Example (Continued)

Let the shift be $n_0 = 3$ to find output $y[3]$:

$$x[k] = \underset{\uparrow}{1}, 2, 3, 1$$

$$h[3-k] = \underset{\uparrow}{0}, -1, 1, 2, 1$$

$$x[k]h[3-k] = ..., 0, 0, 0, \underset{\uparrow}{0}, -2, 3, 2, 0, 0, ...$$

$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k] = 3$$

Convolution Sum Example (Continued)

Let the shift be $n_0 = 4$ to find output $y[4]$:

$$\begin{aligned} x[k] &= \underset{\uparrow}{1}, 2, 3, 1 \\ h[4-k] &= \underset{\uparrow}{0}, 0, -1, 1, 2, 1 \\ x[k]h[4-k] &= ..., 0, 0, \underset{\uparrow}{0}, 0, -3, 1, 0, 0, ... \end{aligned}$$

$$y[4] = \sum_{k=-\infty}^{\infty} x[k]h[4-k] = -2$$

Convolution Sum Example (Continued)

Let the shift be $n_0 = 5$ to find output $y[5]$:

$$x[k] = \underset{\uparrow}{1}, 2, 3, 1$$

$$h[5-k] = \underset{\uparrow}{0}, 0, 0, -1, 1, 2, 1$$

$$x[k]h[5-k] = ...0, \underset{\uparrow}{0}, 0, 0, -1, 0, 0, ...$$

$$y[5] = \sum_{k=-\infty}^{\infty} x[k]h[5-k] = -1$$

Convolution Sum Example (Continued)

Let the shift be $n_0 = 6$ to find output $y[6]$:

$$x[k] = \underset{\uparrow}{1}, 2, 3, 1$$

$$h[3-k] = \underset{\uparrow}{0}, 0, 0, 0, -1, 1, 2, 1$$

$$x[k]h[6-k] = ..., 0, 0, 0, \underset{\uparrow}{0}, 0, 0, , 0, 0, ...$$

$$y[6] = \sum_{k=-\infty}^{\infty} x[k]h[6-k] = 0$$

Convolution Sum Example (Continued)

Putting all of these outputs together:

$$\begin{aligned} y[-1] &= 1 \\ y[0] &= 4 \\ y[1] &= 8 \\ y[2] &= 8 \\ y[3] &= 3 \\ y[4] &= -2 \\ y[5] &= -1 \end{aligned}$$

$$y[n] = \left\{ ...0, 0, 1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1, 0, 0, ... \right\}$$

The Convolution Sum: A Commutative Operation

The convolution operation is commutative... it doesn't matter which sequence is folded and shifted:

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]x[n-k]
\end{aligned}
$$

Analytical Evaluation of the Convolution Sum

Example: Determine the output of this relaxed LTI system.

$$h[n] = a^n u[n], \quad |a| < 1 \qquad \text{with input} \qquad x[n] = u[n]$$

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} a^k u[k]u[n-k] \\
&= \sum_{k=0}^{n} a^k \\
&= \sum_{k=1}^{n+1} a^{k-1} = \frac{1-a^{n+1}}{1-a} \qquad \text{for } n > 0
\end{aligned}$$

Response of LTI system to arbitrary inputs: the convolution sum

$$h(n) \equiv T\left[\delta[n]\right]$$

Using time invariance, the response to $\delta[n-k]$ is now $h[n-k]$:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \qquad \text{for all } n$$

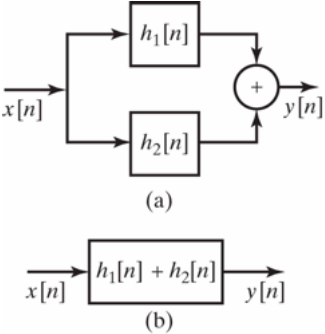
This equation is known as the convolution sum, and we represent convolution by the notation:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

Properties of Linear-Time Invariant Systems

Convolution Distributes Over Addition

$$x[n] * \big(h_1[n] + h_2[n]\big) = x[n] * h_1[n] + x[n] * h_2[n]$$



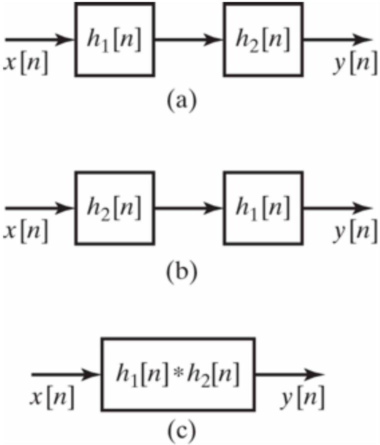
Two Systems Can Cascade In Either Order

Convolution satisfies the associative property:

$$y[n] = \big(x[n] * h_1[n]\big) * h_2[n] = x[n] * \big(h_1[n] * h_2[n]\big)$$

Using commutative property, order doesn't matter:

$$y[n] = x[n] * \big(h_2[n] * h_1[n]\big) = \big(x[n] * h_2[n]\big) * h_1[n]$$



LTI Systems Are Stable If and Only If $h[n]$ is Absolutely Summable

LTI systems are BIBO stable if and only if output sequence $y[n]$ is bounded for every input $x[n]$.

The output is bounded if the impulse response is absolutely summable and therefore satisfies:

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

LTI Systems Are Causal if $h[n] < 0$ for $n < 0$

Let's look at the output at time $n = n_0$ using convolution:

$$
\begin{aligned}
y[n_0] &= \sum_{k=-\infty}^{\infty} h[k]x[n_0 - k] \\
&= \sum_{k=0}^{\infty} h[k]x[n_0 - k] + \sum_{k=\infty}^{-1} h[k]x[n_0 - k] \\
&= [h[0]x[n_0] + h[1]x[n_0 - 1] + h[2]x[n_0 - 2] + \ldots] \\
&\quad + [h[-1]x[n_0 + 1] + h[-2]x[n_0 + 2] + \ldots]
\end{aligned}
$$

For output to depend only on the present and past inputs:

$$h[n] = 0, \quad n < 0$$

Modified Convolution Formula for Causal Systems

Since $h[n] = 0$ for $n < 0$, we can modify the limits on the summation of the convolution formula:

$$
\begin{aligned}
y[n] &= \sum_{k=0}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{n} x[k]h[n-k]
\end{aligned}
$$

Finite Impulse Response (FIR) & Infinite Impulse Response (IIR) Systems

- FIR: Systems with a finite-duration impulse response

$$h[n] = 0, \quad n < 0 \text{ and } N \geq M$$

$$y[n] = \sum_{k=0}^{M-1} h[k]x[n-k]$$

- IIR: Systems with infinite-duration impulse response

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k] \quad \text{(assuming causality)}$$

Using Constant-Coefficient Difference Equations to Characterize LTI Systems

General Form of Linear-Constant-Coefficient Difference Equations

Important class of LTI systems are those in which the input, $x[n]$, and output, $y[n]$, satisfy a linear constant-coefficient difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

N is the order of the difference-equation or the order of the system.

Difference Equation Example: The Accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Let's show that the input and output satisfy a difference equation...

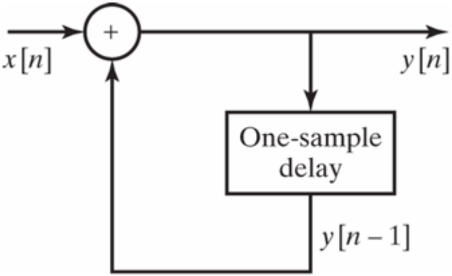
$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

Also,

$$y[n-1] = \sum_{k=-\infty}^{n-1} x[k]$$

So, that: $\quad y[n] = x[n] + y[n-1]$

or, equivalently: $\quad y[n] - y[n-1] = x[n]$



General Recursive System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

We can find the output using the recursive formula:

$$y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k],$$

N is the order of the difference-equation or the order of the system. If system is relaxed, or initially at rest, then the system will be linear, time-invariant, and causal.

Nonrecursive Systems

If $N = 0$, no recursion is required:

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k],$$

This is convolution! We can find the impulse response of this system by setting the input to an impulse, $x[n] = \delta[n]$:

$$h[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) \delta[n-k]$$

Therefore, the impulse response is

$$h[n] = \begin{cases} \left(\frac{b_k}{a_0}\right), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Any FIR system can be computed nonrecursively.

Frequency-Domain Representation

Complex Exponentials as Eigenfunctions for LTI Systems

Let's use a complex exponential sequence as the input:

$x[n] = e^{j\omega n}$ for $-\infty < n < \infty$

We can find the output using the convolution sum:

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{j\omega n}e^{-j\omega k}
\end{aligned}
$$

$$
\boxed{y[n] = H\left(e^{j\omega}\right)e^{j\omega n}}
$$

where, $H\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$ is the frequency response

Example: Response to Sinusoidal Input In LTI Systems

Let's use sinusoidal signal as an input to an LTI system:

$$x[n] = A\cos\left(\omega_0 n + \theta\right) = \frac{A}{2}e^{(j\omega_0 n+\theta)} + \frac{A}{2}e^{(-j\omega_0 n-\theta)}$$

Response to first complex exponential:

$$y_1[n] = H\left(e^{j\omega_0}\right)\frac{A}{2}e^{(j\omega_0 n+\theta)}$$

Response to second complex exponential:

$$y_2[n] = H\left(e^{-j\omega_0}\right)\frac{A}{2}e^{(-j\omega_0 n-\theta)}$$

Example: Response to Sinusoidal Input In LTI Systems (Continued)

Here is the total response to a complex exponential:

$$y[n] = H\left(e^{j\omega_0}\right)\frac{A}{2}e^{(j\omega_0 n+\theta)} + H\left(e^{-j\omega_0}\right)\frac{A}{2}e^{(-j\omega_0 n-\theta)}$$

If we restrict the impulse response, $h[n]$, to be real, then $H\left(e^{-j\omega_0}\right) = H^*\left(e^{j\omega_0}\right)$.
Let $\phi = \angle H\left(e^{j\omega_0}\right)$, then $H\left(e^{j\omega_0}\right) = \left|H\left(e^{j\omega_0}\right)\right|\angle\phi$ and
$H\left(e^{-j\omega_0}\right) = \left|H\left(e^{j\omega_0}\right)\right|\angle - \phi$

Therefore, the output of an LTI system, with a real $h[n]$, to a sinusoidal input is:

$$\boxed{y[n] = A\left|H\left(e^{j\omega_0}\right)\right|\cos\left(\omega_0 + \theta + \phi\right)}$$

Concept of Frequency Response in Discrete-Time LTI Systems

Note that the complex exponential sequence

$$\left\{e^{j\omega n}\right\}, \qquad -\infty < n < \infty$$

is indistinguishable from

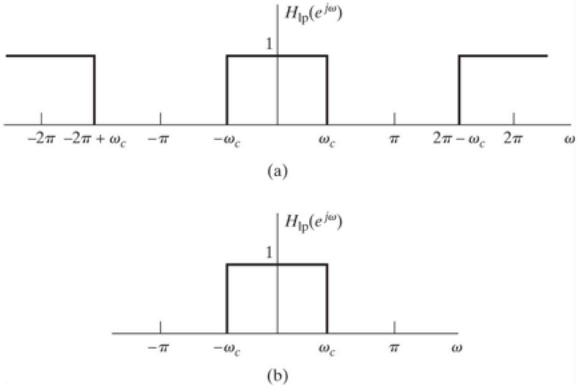
$$\left\{e^{j(\omega+2\pi n)}\right\}, \qquad -\infty < n < \infty$$

The frequency response for a discrete-time system is a periodic function of ω with period 2π:

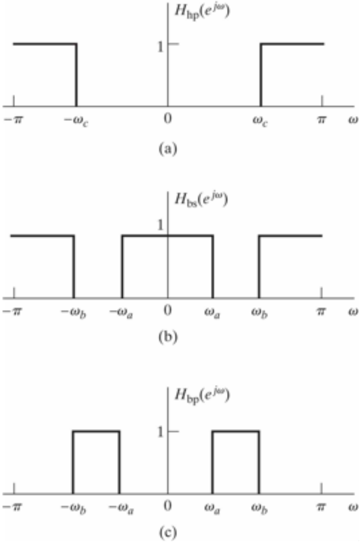
$$H\left(e^{j(\omega+2\pi k)}\right) = H\left(e^{j\omega}\right) \quad \text{for } k \text{ an integer}$$

Frequency Response of Ideal Low Pass Filters

Low Frequencies: close to zero (or an even multiple of π)
High Frequencies: close to $\pm\pi$ (or an odd multiple of π)



Frequency Response of Ideal Filters



Using The Discrete-Time Fourier Transform to Represent Sequences

Discrete-Time Fourier Transform (DTFT) Pair

The Fourier transform of a finite-energy discrete-time signal $x[n]$ is defined as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$X(\omega + 2\pi k) = X(\omega)$$

The inverse Fourier transform is

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Fourier Transform is a Complex-Valued Function

In rectangular form:

$$X(\omega) = X_R(\omega) + jX_I(\omega)$$

In polar form:

$$X(\omega) = |X\omega)|\angle X(\omega)$$

(note that the phase is not unique since any integer multiple of 2π can be added to the angle at any frequency and have the same result)

Impulse Response and Frequency Response

The frequency response is the discrete-time Fourier transform of the impulse response:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

The impulse response of a system can be obtained by using the inverse Fourier transform:

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)e^{j\omega n}d\omega$$

Symmetry Properties of the DTFT: Even and Odd Sequences

Conjugate-symmetric sequence: $x_e[n] = x_e^*[-n]$ (called "even" if a real sequence)
Conjugate-antisymmetric sequence: $x_o[n] = -x_0^*[-n]$ (called "odd" if a real sequence)

Any sequence can be expressed as a sum of conjugate-symmetric and conjugate-antisymmetric sequence:

$$x[n] = x_e[n] + x_o[n]$$

where,

$$x_e[n] = \frac{1}{2}\left(x[n] + x^*[-n]\right) = x_e^*[-n]$$
$$x_o[n] = \frac{1}{2}\left(x[n] - x^*[-n]\right) = -x_o^*[-n]$$

Symmetry Properties of the DTFT: Even and Odd Functions

A Fourier transform can be decomposed into a sum of conjugate-symmetric and conjugate-antisymmetric functions:

$$X(\omega) = X_e(\omega) + X_o(\omega)$$

where,

$$X_e(\omega) = \frac{1}{2}\left[X(\omega) + X^*(-\omega)\right] = X_e^*(-\omega)$$
$$X_o(\omega) = \frac{1}{2}\left[X(\omega) - X^*(-\omega)\right] = -X_o^*(-\omega)$$

Table for Symmetry Properties of the DTFT

TABLE 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

Time Shifting and Frequency Shifting Theorem of the DTFT

If,

$$x[n] \longleftrightarrow X(\omega)$$

then,

$$x[n-n_d] \longleftrightarrow e^{-j\omega n_d} X(\omega) \qquad \text{Time-Shifting Theorem}$$

and

$$e^{j\omega_0 n} x[n] \longleftrightarrow X(\omega-\omega_0) \qquad \text{Frequency-Shifting Theorem}$$

Convolution Theorem of the DTFT

If,

$$x[n] \longleftrightarrow X(\omega)$$

and

$$h[n] \longleftrightarrow H(\omega)$$

and

$$y[n] = \sum_{-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n] \qquad \text{Convolution Sum in the Time Domain}$$

then

$$Y(\omega) = X(\omega)H(\omega) \qquad \text{Multiplication in the Frequency Domain}$$

Fourier Transform Theorems and Properties

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

Common DTFT Pairs

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\|a\| < 1)$	$\dfrac{1}{1 - ae^{-j\omega}}$
5. $u[n]$	$\dfrac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n] \quad (\|a\| < 1)$	$\dfrac{1}{(1 - ae^{-j\omega})^2}$
7. $\dfrac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\|r\| < 1)$	$\dfrac{1}{1 - 2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\dfrac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \|\omega\| < \omega_c, \\ 0, & \omega_c < \|\omega\| \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$

Example: DTFT of a Signal

Determine the Fourier transform of the sequence

$$x[n] = a^n u[n-5]$$

Let's rewrite this sequence as $x[n] = a^5 a^{n-5} u[n-5]$. Using a table of Fourier transform pairs, we can find the Fourier transform of a new sequence:

$$x_1[n] = a^5 a^n u[n] \longleftrightarrow X_1(\omega) = \frac{a^5}{1 - ae^{-j\omega}}$$

We can obtain the original sequence by delaying this new sequence, $x_1[n]$, by five samples so using the time-shifting property:

$$X(\omega) = e^{-j5\omega} X_1(\omega) = \frac{a^5 e^{-j5\omega}}{1 - ae^{-j\omega}}$$

The z-Transform

Definition of the z-Transform

The z-tranform of a sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

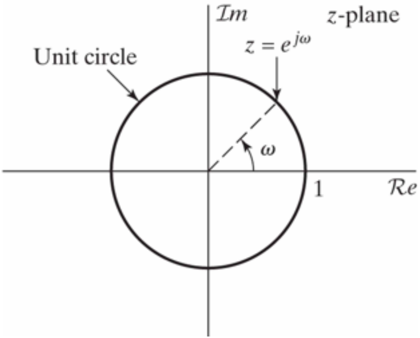
where z is a complex variable. Notice the close relationship to the Fourier transform:

$$X\left(\omega\right) = X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

The Fourier transform is $X(z)$ with $z = e^{j\omega}$ (evaluated on the unit circle)

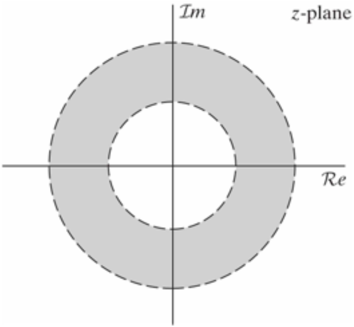
The Unit Circle in the Complex z-Plane

The z-transform is a function of a complex variable and is often viewed on the complex z-plane:



Region of Convergence

For a given sequence, $x[n]$, the set of values of z for which the z-transform power series converges is called the *region of convergence* (ROC).



Example: Finding the $z-$Transform

Find the z-Transform of the signal $x[n] = a^n u[n]$, where a denotes a real or complex number.

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right) z^n$$

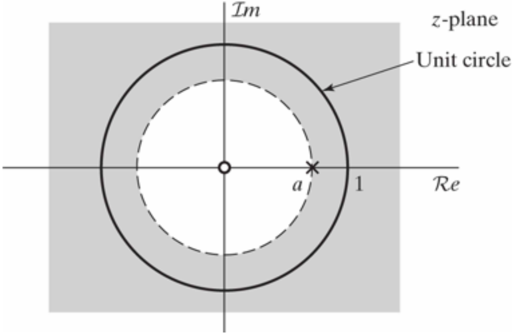
ROC: range of z when $|az^{-1}| < 1$ or, $|z| > |a|$

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right) z^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad ROC : |z| > |a|$$

Example: Finding the $z-$Transform (Continued)

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad ROC: |z| > |a|$$

$$X(z) = \frac{P(z)}{Q(z)} \quad \text{(zeros are roots of numerator and poles are roots of denominator)}$$



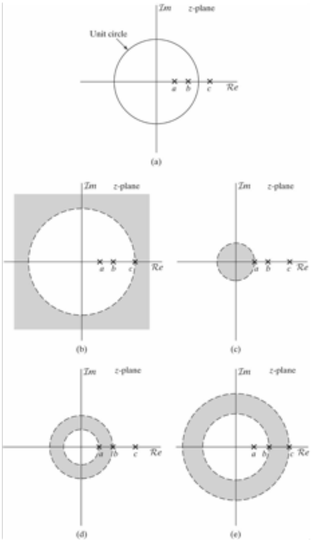
Common z-Transform Pairs

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Properties of the Region of Convergence (ROC)

- Fourier transform converges if and only if ROC contains unit circle
- ROC cannot contain any poles
- Finite-duration sequence: ROC is entire z-plane except possibly $z = 0$ or $z = \infty$
- Right-sided sequence: ROC extends outward from outermost pole
- Left-sided sequence: ROC extends inward from innermost pole
- Two-sided sequence: ROC is a ring bounded by poles



The Inverse z-Transform

Inverse z-Transform

Determine the inverse z-transform from the following complex contour integral:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} \, dz$$

Less formal but sufficient procedures:

- Power Series Expansion
- Partial Fraction Expansion and Table Lookup

Power Series Expansion

Given a z-transform $X(z)$ with its corresponding ROC, we can expand $X(z)$ into a power series of the form

$$X(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n}$$

Then, for all n,

$$x[n] = c_n$$

Example: Finite Length Sequence

Given the following z-transform, by the sequence, $x[n]$:

$$X(z) = z^2\left(1 - \tfrac{1}{2}z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right) \qquad ROC : |z| > 0$$

After multiplying the factors:

$$X(z) = z^2 - \tfrac{1}{2}z - 1 + \tfrac{1}{2}z^{-1}$$

Therefore,

$$x[n] = \left\{1, -\tfrac{1}{2}, \underset{\uparrow}{-1}, \tfrac{1}{2}\right\}$$

or,

$$x[n] = \delta[n+2] - \tfrac{1}{2}\delta[n+1] - \delta[n] + \tfrac{1}{2}\delta[n-1]$$

Example: Long Division

Determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} \qquad \text{ROC: } |z| > 1$$

Using long-division while eliminating the lowest power term of z^{-1} in each step:

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} = 1 + \tfrac{3}{2}z^{-1} + \tfrac{7}{4}z^{-2} + \tfrac{15}{8}z^{-3} + \tfrac{31}{16}z^{-4} + \ldots$$

By comparing with the definition of the z-tranform:

$$x[n] = \left\{ \underset{\uparrow}{1}, \tfrac{3}{2}, \tfrac{7}{4}, \tfrac{15}{8}, \tfrac{31}{16}, \ldots \right\}$$

Partial Fraction Expansion and Table Lookup

We want to express the function $X(z)$ as a linear combination

$$X(z) = \alpha_1 X_1(z) + \alpha_2 X_2(z) + ... + \alpha_K X_K(z)$$

Where $X_1(z), ..., X_K(z)$ have inverse transforms $x_1[n], ..., x_K[n]$ that can be found in a table. The inverse z-transform can be found using the linearity property:

$$x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n] + ... + \alpha_K x_K[n]$$

Useful when $X(z)$ is a rational function:

$$X(z) = \frac{B(Z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + ... + a_N z^{-N}}$$

Partial Fraction Expansion and Table Lookup (Continued)

Let's assume that $a_0 = 1$ (we can divide both numerator and denominator by a_0 if $a_0 \neq 1$)

$$X(z) = \frac{B(Z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + ... + b_M z^{-M}}{1 + a_1 z^{-1} + ... + a_N z^{-N}}$$

Proper rational function if $M < N$
Improper rational function if $M \geq N$

First, convert any improper rational function into a sum of individual terms plus a proper rational function by carrying out long division (with each polynomial written in reverse order) and stopping with the order of the remainder is less than the order of the denominator.

Partial Fraction Expansion and Table Lookup (Continued)

Let's assume $X(z)$ is now a proper rational function, where $M < N$

$$X(z) = \frac{B(Z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + ... + b_M z^{-M}}{1 + a_1 z^{-1} + ... + a_N z^{-N}}$$

We can eliminate the negative powers of z by multiplying the numerator, denominator by z^N:

$$X(z) = \frac{b_0 Z^N + b_1 z^{N-1} + ... + b_M z^{N-M}}{z^N + a_1 z^{N-1} + ... + a_N}$$

Since, $N > M$, when we divide through by z, this function is always proper and for *distinct* poles:

$$\frac{X(z)}{z} = \frac{b_0 Z^{N-1} + b_1 z^{N-2} + ... + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + ... + a_N} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + ... + \frac{A_N}{z - p_N}$$

Example: Using Partial Fraction Expansion

Determine the partial-fraction expansion of the proper function:

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$
$$= \frac{z^2}{z^2 - 1.5z + 0.5}$$

Poles are $p_1 = 1$ and $p_2 = 0.5$ (roots of the denominator), so:

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

Multiplying through by the the denominator term $(z-1)(z-0.5)$, we obtain:

$z = (z-0.5)A_1 + (z-1)A_2$ Solving for A_1 and A_2: $\dfrac{X(z)}{z} = \dfrac{2}{z-1} + \dfrac{-1}{z-0.5}$

Example: Using Partial Fraction Expansion (Continued)

$$\frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5}$$

$$\begin{aligned} X(z) &= \frac{2z}{z-1} - \frac{z}{z-0.5} \\ &= \frac{2}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}} \end{aligned}$$

Three possible regions of convergence:

- ROC: $|z| > 1$ causal / right-sided
  $\longrightarrow \quad x[n] = 2(1)^n u[n] - (0.5)^n u[n] = (2 - 0.5^n)u[n]$
- ROC: $|z| > 0.5$ anticausal / left-sided $\quad \longrightarrow \quad x[n] = [-2 + (0.5)^n]u[-n-1]$
- ROC: $0.5 < |z| < 1$ two-sided $\quad \longrightarrow \quad x[n] = -2(1)^n u[-n-1] - (0.5)^n u[n]$

Properties of the z-Transform

Linearity Property of the z-Transform

If

$$x_1[n] \longleftrightarrow X_1(z)$$

and

$$x_2[n] \longleftrightarrow X_2(z)$$

then

$$x[n] = a_1 x_1[n] a_2 x_2[n] \longleftrightarrow X(z) = a_1 X_1(z) + a_2 X_2(z)$$

The region of convergence of $X(z)$ is the intersection of the ROC of each of the individual z-transforms.

Time Shifting Property of the z-Transform

If

$$x[n] \longleftrightarrow X(z)$$

then

$$x[n-k] \longleftrightarrow z^{-k}X(z)$$

The ROC will be the same except for $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$.

Example: Using the Time Shifting Property of the z-Transform

Determine $x[n]$ if

$$X(z) = z^{-1}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right) \qquad \text{ROC: } |z| > \tfrac{1}{4}$$

The factor z^{-1} is associated with a time shift of one sample to the right.
First, let's find the inverse transform of

$$\frac{1}{1-\frac{1}{4}z^{-1}} \qquad \text{ROC: } |z| > \tfrac{1}{4} \qquad \text{from table: } \left(\tfrac{1}{4}\right)^{n} u[n]$$

and shift this time-domain sequence one sample to the right:

$$x[n] = \left(\tfrac{1}{4}\right)^{n-1} u[n-1]$$

Convolution Property of the z-Transform

If

$$x_1[n] \longleftrightarrow X_1(z)$$

and

$$x_2[n] \longleftrightarrow X_2(z)$$

then

$$x[n] = x_1[n] * x_2[n] \longleftrightarrow X(z) = X_1(z)X_2(z)$$

Example: Using the Convolution Property

Compute the convolution, $x[n] = x_1[n] * x_2[n]$, of the following two signals:

$$x_1[n] = \{1, -2, 1\}$$

$$x_2[n] = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & elsewhere \end{cases}$$

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$

$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

$$X(z) = X_1(z)X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

$$x[n] = \{\underset{\uparrow}{1}, -1, 0, 0, 0, 0, -1, 1\}$$

Summary of Properties of the z-Transform

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Analysis of LTI Systems in the z-Domain

System Function of a Linear Time-Invariant System

An LTI system can be represented as the convolution of the input and impulse response:

$$y[n] = x[n] * h[n]$$

From the convolution property of the z-transform:

$$Y(z) = X(z)H(z)$$

where the System Function is

$$H(z) = \frac{Y(z)}{X(z)}$$

$h[n] \longleftrightarrow H(z)$ are a z-transform pair:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

Finding the System Function From the Difference Equation

With zero input prior to $n = 0$ and at initial rest (zero initial conditions), then a causal LTI system is defined by this constant coefficient difference equation (assuming $a_0 = 1$):

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

We can find the system function, $H(z)$, using taking the z-transform of both sides and applying the time-shifting property:

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z)\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) = X(z)\left(\sum_{k=0}^{M} b_k z^{-k}\right)$$

Finding the System Function From the Difference Equation (Continued)

Solving for the system function:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- This is rational system function, called a pole-zero system, with N poles and M zeros
- Since this is a causal system, ROC is outward from the pole farthest from the origin
- If all poles are inside the unit circle, then system is stable and has a frequency response
- Due to the presence of poles, this system is an infinite-impulse response (IIR) system

Special Case: *All-Zero System*

If $a_k = 0$ for $1 \leq k \leq N$ then the $H(z)$ reduces to:

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k}$$

- Contains M zeros
- Contains M-th order pole at the origin, $z = 0$
- Since poles at the origin are considered trivial, this is an *all-zero system*
- Has a finite impulse response (the b_k coefficients)
- Called a FIR system or a moving average (MA) system

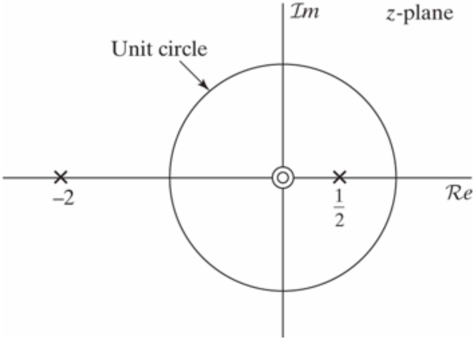
Special Case: *All-Pole System*

If $b_k = 0$ for $1 \leq k \leq M$ then the $H(z)$ reduces to:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{b_0}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 z^N}{\sum_{k=0}^{N} a_k z^{N-k}}$$

- Contains N poles (locations determined by parameters a_k)
- Contains N-th order zero at the origin, $z = 0$
- Since zeros at the origin are considered trivial, this is called an *all-pole system*
- Has a infinite impulse response (due to the presence of poles)
- Called an IIR system

Stability, Causality, and the ROC



LTI System with impulse reponse $h[n]$
$H(z)$ is the System Function with pole-zero plot

Three possible ROCs:

- $|z| < \frac{1}{2}$
- $\frac{1}{2} < |z| < 2$
  - stable but non-causal
- $|z| > 2$
  - causal but not stable

Example: Finding the System Function and Impulse Response from the Difference Equation

Determine the system function and the unit sample response (impulse response) of the system described by the difference equation:

$$y[n] = \tfrac{1}{2}y[n-1] + 2x[n]$$

By determining the z-transform of the difference equation, we obtain

$$Y(z) = \tfrac{1}{2}z^{-1}Y(z) + 2X(z)$$

Solving for the the system function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - \frac{1}{2}z^{-1}} \quad \text{ROC: } |z| > \tfrac{1}{2} \quad \longleftrightarrow \quad h[n] = 2\left(\tfrac{1}{2}\right)^n u[n]$$