Review of LTI Systems: Impulse Response, Difference Equation, System Function, and Frequency Response

Convolution Sum, System Function, and Frequency Response

LTI systems can be completely characterized in the time domain by its impulse response, $h[n]$. Given and input, $x[n]$, the output is found with the convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The z-transform of the output of an LTI system is related to the z-transform of the impulse response and input:

$$Y(z) = H(z)X(z)$$

Since the system function, $H(z)$, is the z-transform of the impulse response, it also completely characterizes an LTI system. The frequency response is $H(z)$ evaluated on the unit circle (assuming ROC includes $z = e^{j\omega}$):

$$H(e^{j\omega})$$

Linear Difference Equations and System Function

$$y[n] = \underbrace{-\sum_{k=1}^{N} a_k y[n-k]}_{\text{past outputs}} + \underbrace{\sum_{k=0}^{M} b_k x[n-k]}_{\text{current and past inputs}}$$

$$\Longrightarrow \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad (a_0 = 1)$$

$$\Longrightarrow \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Longrightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Poles and Zeros of the System

Factored form of the system function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{a_0} \frac{\prod_{k=0}^{M} \left(1 - c_k z^{-1}\right)}{\prod_{k=0}^{N} \left(1 - d_k z^{-1}\right)} = \frac{b_0}{a_0} \frac{z^{N-M} \prod_{k=0}^{M} (z - c_k)}{\prod_{k=0}^{N} (z - d_k)}$$

- roots of the numerator polynomial are zeros of the system
- roots of the denominator polynomial are poles of the system
- causal and stable system has poles inside the unit circle (with ROC outward)

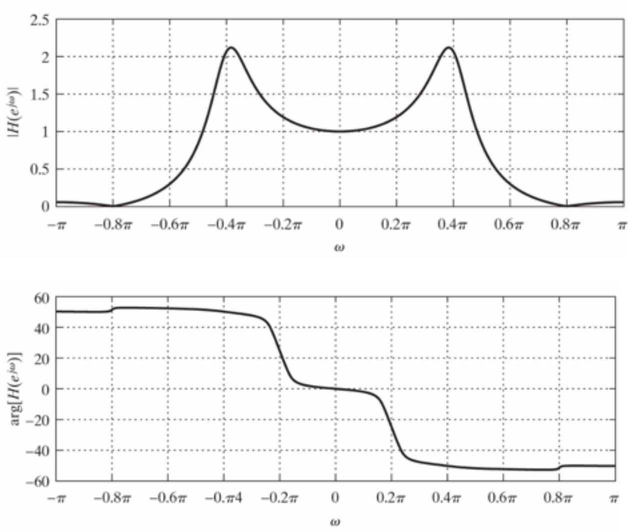
Frequency Response

Since a causal an stable system has poles inside the unit circle, the system function evaluated on the unit circle is the frequency response:

$$H(z)\Big|_{z=e^{j\omega}} = H\left(e^{j\omega}\right) = \frac{\displaystyle\sum_{k=0}^{M} b_k e^{-j\omega k}}{\displaystyle\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

- Real coefficients $\{a_k, b_k\} \implies$ implies a real impulse response $h[n]$
- Real $h[n] \implies$ hermitian frequency response $H\left(e^{j\omega}\right)$ (real part even, imaginary part odd).
- Magnitude response: $\left|H\left(e^{j\omega}\right)\right|$ has even symmetry.
- Phase response: $\arg H\left(e^{j\omega}\right)$ or $\angle H\left(e^{j\omega}\right)$ has odd symmetry

Example: Magnitude and Phase of Frequency Response



Frequency Response: Phase Response and Group Delay

Phase Angle

Relationship between the Fourier transform of the system input and output:

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

Since, in general, the Fourier transform is complex number at each frequency, we often investigate the magnitude and phase:

$$|Y(e^{j\omega})| = |H(e^{j\omega})||X(e^{j\omega})|$$
$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

Note that the phase angle is not uniquely defined, since any integer of 2π can be added to the phase without affecting the complex number.

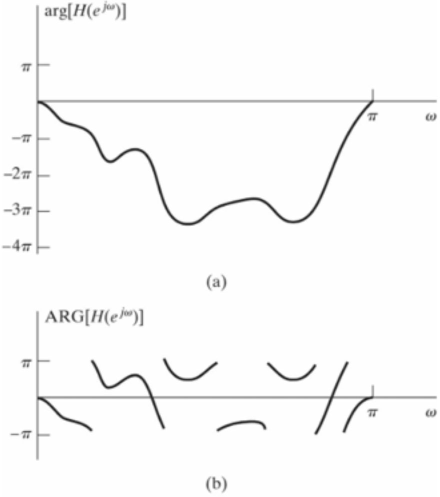
Phase Angle: Wrapped and Unwrapped

The continuous phase curve is known as *unwrapped*.

The principal, or *wrapped*, phase of the of $H(e^{j\omega})$ is:

$$-\pi < \angle H(e^{j\omega}) \leq \pi$$

Notice that this phase response is nonlinear.



Linear Phase Response

If the input to a LTI system is

$$x[n] = \cos(\omega_1 n) + \cos(\omega_2 n)$$

and the phase response is

$$\angle H(e^{j\omega}) = -\omega n_d$$

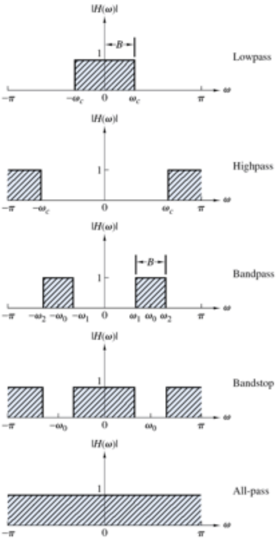
Then, ignoring the magnitude response, the output is:

$$\cos(\omega_1 n - \omega_1 n_d) + \cos(\omega_2 n - \omega_2 n_d) = \cos(\omega_1 (n - n_d)) + \cos(\omega_2 (n - n_d))$$

where each component is delayed by n_d samples.

If *linear phase*, then each frequency component will have identical delay.

Ideal Frequency Characteristics: Constant-Gain and Linear Phase



Ideal response for frequency selective filters, constant-gain and *linear phase*:

$$H(e^{j\omega}) = \begin{cases} Ce^{-j\omega n_0}, & \omega_1 < \omega < \omega_2 \\ 0, & \text{otherwise} \end{cases}$$

Signal at the filter output:

$$\begin{aligned} Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\ &= CX(e^{j\omega})e^{-j\omega n_0}, \qquad \omega_1 < \omega < \omega_2 \end{aligned}$$

Using the scaling and time-shifting properties of the Fourier transform, the time-domain output is:

$$y[n] = Cx[n - n_0]$$

Group Delay

The *group delay* of a filter is the derivative of the phase with respect to frequency and has the units of delay:

$$\tau(\omega) = -\frac{d}{d\omega}\left\{\angle(H(e^{j\omega})\right\}$$

If the phase response is linear:

$$\angle H(e^{j\omega}) = -\omega n_0$$

Then the the group delay is constant and all frequencies have same time delay:

$$\tau(\omega) = n_0$$

Nonlinearity of the phase results in time dispersion.

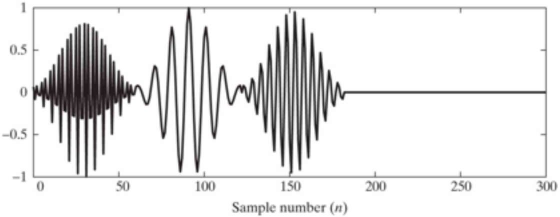
Example: Attenuation and Group Delay of Frequency Selective Filter

$$x_1[n] = w[n]\cos(0.2\pi n)$$
$$x_2[n] = w[n]\cos(0.4\pi n - \pi/2)$$
$$x_3[n] = w[n]\cos(0.8\pi n + \pi/5)$$

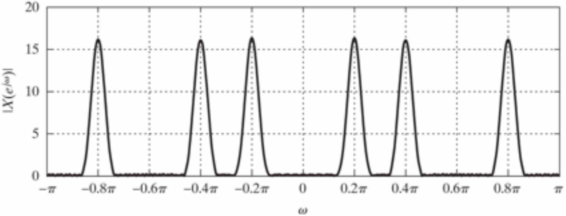
Filter input:

$$x[n] = x_3[n] + x_1[n-M-1] + x_2[n-2M-2]$$

with $M = 60$



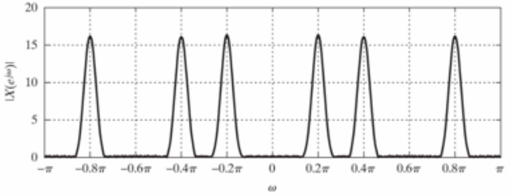
(a) Waveform of signal $x[n]$



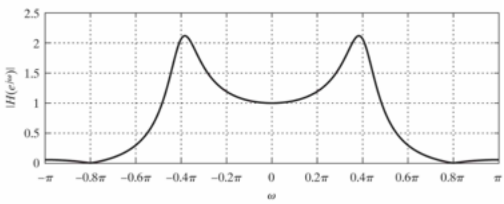
(b) Magnitude of DTFT of $x[n]$

Example: Attenuation and Group Delay of Frequency Selective Filter

Input Spectrum:



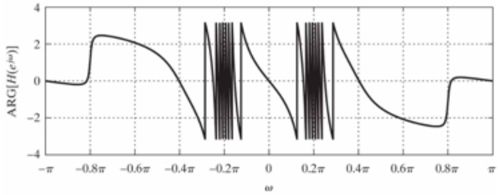
Filter Magnitude Response:



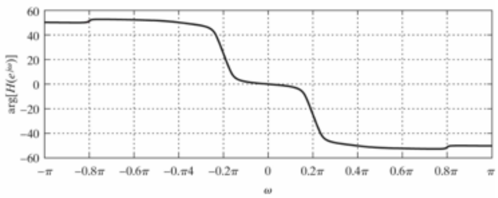
What frequency components exist in the output signal?

Example: Attenuation and Group Delay of Frequency Selective Filter

Filter Phase Response (principal phase, wrapped):

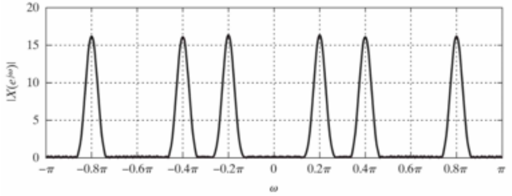


Filter Phase Response (continuous phase, unwrapped):

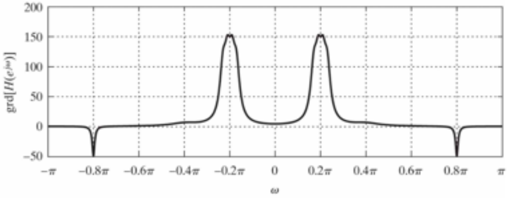


Example: Attenuation and Group Delay of Frequency Selective Filter

Input Spectrum:



Filter Group Delay:

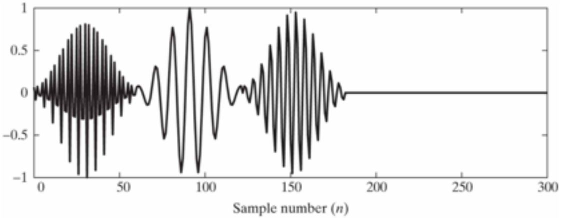


(a) Group delay of $H(z)$

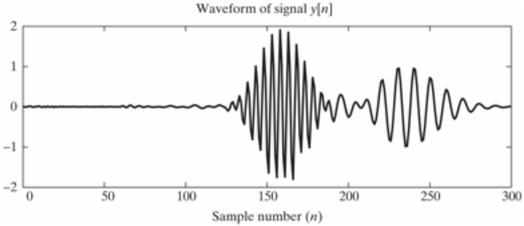
What is the delay for each frequency component in the output signal?

Example: Attenuation and Group Delay of Frequency Selective Filter

Filter Input:



Filter Output:



More Insight into LTI Systems:
Rational System Functions, Inverse Systems

System Function: Algebraic and Factored Forms

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

- Each factor $(1 - c_k z^{-1})$ in the numerator contributes a zero at $z = c_k$ (and pole at $z = 0$)
- Each factor $(1 - d_k z^{-1})$ in the denominator contributes a pole at $z = d_k$ (and zero at $z = 0$)

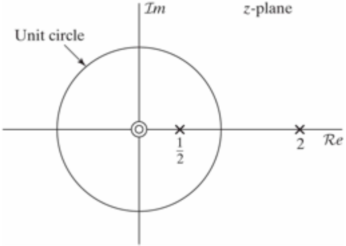
Example: ROC, Stability, Causality

Given LTI system with difference equation:

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n]$$

Has the system function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{5}{2}z^{-1} + z^{-2}} = \frac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$



- Three possible choices for region of convergence, but if the system is causal, then ROC is outside of outermost pole ($|z| > 2$).
- However, for system to be stable, the ROC must include the unit circle ($\frac{1}{2} < |z| < 2$).

Inverse Systems

For a given system function, $H(z)$, the corresponding inverse system, $H_i(z)$, has an overall effective system of unity when the two systems are cascaded:

$$G(z) = H(z)H_i(z) = 1$$

Which implies:

$$H_i(z) = \frac{1}{H(z)} \qquad H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})} \qquad g[n] = h[n] * h_i[n] = \delta[n]$$

- Not all systems have inverses, for those that do the poles of $H_i(z)$ are the zeros of $H(z)$ and vice versa
- The regions of convergence of $H(z)$ and $H_i(z)$ must overlap
- If $H(z)$ is causal, its ROC is $|z| > \max |d_k|$ and ROC for $H_i(z)$ must overlap

Example: Inverse System

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}} \qquad \text{ROC: } |z| > 0.9$$

Then, the inverse system is

$$H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}}$$

$H_i(z)$ as one pole at 0.5 so it has two possibilities of ROC. The only one that overlaps with the $|z| > 0.9$ is $|z| > 0.5$ and the inverse z-transform gives the impulse response of the inverse system:

$$h_i[n] = (0.5)^n u[n] - 0.9(0.5)^{n-1} u[n-1]$$

Note that this inverse system is causal and stable.

Example: Another Inverse System

$$H(z) = \frac{z^{-1} - 0.5}{1 - 0.9z^{-1}} \qquad \text{ROC: } |z| > 0.9$$

Then, the inverse system is

$$H_i(z) = \frac{1 - 0.9z^{-1}}{z^{-1} - 0.5} = \frac{-2 + 1.8z^{-1}}{1 - 2z^{-1}}$$

Since both possible ROC of the inverse system, $|z| < 2$ and $|z| > 2$, overlap the ROC of the original system, both lead to valid inverse systems:

$$\text{ROC: } |z| < 2 \qquad h_{i1} = 2(2)^n u[-n-1] - 1.8(2)^{n-1} u[-n] \qquad \text{stable and noncausal}$$

$$\text{ROC: } |z| > 2 \qquad h_{i1} = -2(2)^n u[n] + 1.8(2)^{n-1} u[n-1] \qquad \text{unstable and causal}$$

Causal and Stable Inverse Systems

If an LTI system is causal and stable, when will its corresponding inverse system be causal and stable?

- If $H(z)$ is causal with zeros at c_k then its inverse will be causal if:

$$H_i(z) \text{ ROC is } |z| > \max |c_k|$$

- If the inverse system is stable, then ROC for $H_i(z)$ must include unit circle:

$$\max |c_k| < 1$$

- Therefore, a LTI causal and stable system has causal and stable inverse if both the poles and the zeros of $H(z)$ are inside the unit circle. These systems are called *minimum-phase systems*.

More Insight into LTI Systems: Finite Impulse Response (FIR)

Finite Impulse Response (FIR) Systems

$H(z)$ has no poles except at $z = 0$ and is a polynomial in z^{-1} (since $N = 0$)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \implies \sum_{k=0}^{M} b_k z^{-k}$$

So the impulse response is the b coefficients:

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

The difference equation is the same as the convolution sum:

$$y[n] = \sum_{k=0}^{M} b_k n[n-k]$$

Example: FIR System Pole-Zero Plot, Difference Equation

Consider truncation of the impulse response from the IIR system:

$$G(z) = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

$$h[n] = \begin{cases} a^n, & o \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

The system function of this FIR system is

$$H(z) = \sum_{n=0}^{M} a^n z^{-n} = \sum_{n=0}^{M} (az^{-1})^n = \frac{1 - (az^{-1})^{M+1}}{1 - az^{-1}} = \frac{z^{-M-1}\left(z^{M+1} - a^{M+1}\right)}{z^{-1}\left(z - a\right)}$$

$$= \frac{z^{M+1} - a^{M+1}}{z^M \left(z - a\right)} \qquad \text{poles: } z = 0 \text{ and } z = a$$

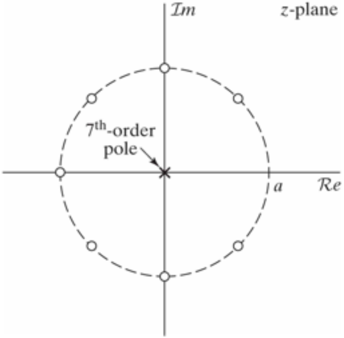
Example: FIR System Pole-Zero Plot, Difference Equation (Continued)

$$H(z) = \frac{z^{M+1} - a^{M+1}}{z^M \left(z - a\right)}$$

Finding zeros: $z^{M+1} = a^{M+1}$

zeros: $z_k = ae^{j2\pi k/(M+1)}$, k = 0,1,...,M

Pole at $z = a$ cancelled by $z_0 = a$



Two equivalent difference equations for this system:

$$y[n] = \sum_{k=0}^{M} a^k x[n-k]$$

$$H(z) = \frac{1 - (az^{-1})^{M+1}}{1 - az^{-1}} \implies y[n] - ay[n-1] = x[n] - a^{M+1}x[n-M-1]$$

Frequency Response From Rational System Functions: Impact of Pole and Zero Locations

Frequency Response, Phase Response, Magnitude Response

The frequency response is the system function, $H(z)$, evaluated on the unit circle:

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} \qquad H(e^{j\omega}) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - d_k e^{-j\omega})}$$

Magnitude and the magnitude-squared function:

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M} |1 - c_k e^{-j\omega}|}{\prod_{k=1}^{N} |1 - d_k e^{-j\omega}|} \qquad |H(e^{j\omega})|^2 = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M} (1 - c_k e^{-j\omega})(1 - c_k^* e^{j\omega})}{\prod_{k=1}^{N} (1 - d_k e^{-j\omega})(1 - d_k^* e^{j\omega})}$$

Gain in dB and Phase Response

Gain in dB:

$$20\log_{10}\left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20\log_{10}\left|1 - c_k e^{-j\omega}\right| - \sum_{k=1}^{M} 20\log_{10}\left|1 - d_k e^{-j\omega}\right|$$

Phase response:

$$\angle H(e^{j\omega}) = \angle\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M} \angle\left[1 - c_k e^{-j\omega}\right] - \sum_{k=1}^{N} \angle\left[1 - d_k e^{-j\omega}\right]$$

- Magnitude in dB and phase are a sum of the contributions from each of the poles and zeros of the system function
- Understanding the impact of a single pole/zero location on frequency response provides insight into higher-order systems

Frequency Response: 1st-Order Systems

Single factor of $H(e^{j\omega})$: $(1 - re^{j\theta}e^{-j\omega})$ the pole or zero has radius r and angle θ

Magnitude-squared of factor:

$$|1 - re^{j\theta}e^{-j\omega}|^2 = (1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{j\omega}) = 1 + r^2 - 2r\cos(\omega - \theta)$$

Contribution to gain in dB (*where* $+$ *for a zero and* $-$ *for a pole*):

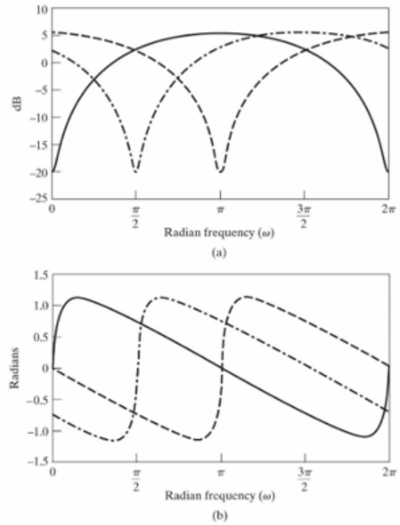
$$\pm 20\log_{10}|1 - re^{j\theta}e^{-j\omega}| = \pm 10\log_{10}\left[1 + r^2 - 2r\cos(\omega - \theta)\right]$$

Contribution to phase:

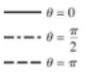
$$\pm\angle\left[1 - re^{j\theta}e^{-j\omega}\right] = \pm\arctan\left[\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right]$$

- Note how magnitude in dB and phase change based on pole/zero: r and θ

Example: Single Zero at Different Angles



Single Zero at $r = 0.9$ and $\theta = 0, \pi/2,$ or π



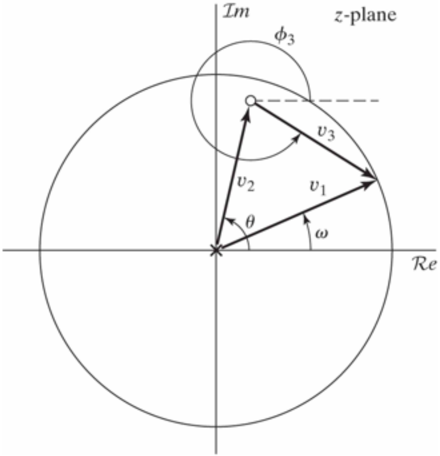
Log magnitude response (minimized at $\omega = \theta$):

$$+10 \log_{10} \left[1 + r^2 - 2r\cos(\omega - \theta)\right]$$

Phase Response (zero at $\omega = \theta$):

$$+\arctan\left[\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right]$$

Vector Diagram Perspective of Frequency Response



$$H(z) = (1 - re^{j\theta}z^{-1}) = \frac{z - re^{j\theta}}{z} \qquad r < 1$$

Geometric interpretation:

$$\vec{v_1} = e^{j\omega} \quad \text{location on unit circle}$$

$$\vec{v_2} = re^{j\theta} \quad \text{zero}$$

$$\vec{v_3} = \vec{v_1} - \vec{v_2} = e^{j\omega} - re^{j\theta}$$

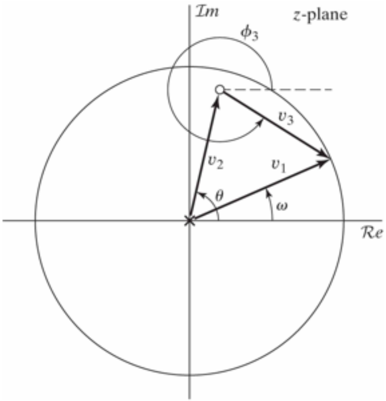
Magnitude of $H(z)$:

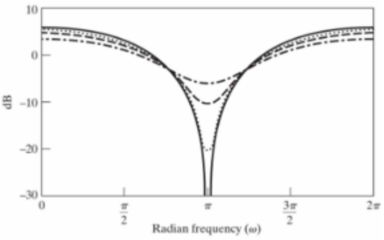
$$|1 - re^{j\theta}e^{-j\omega}| = |e^{-j\omega}(e^{j\omega} - re^{j\theta})| = \left|\frac{e^{j\omega} - re^{j\theta}}{e^{j\omega}}\right| = \frac{|\vec{v_3}|}{|\vec{v_1}|}$$

Phase: $\angle(1 - re^{j\theta}e^{-j\omega}) = \angle\vec{v_3} - \angle\vec{v_1} = \phi_3 - \omega$

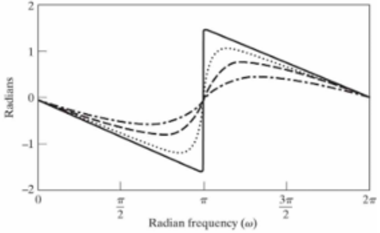
Example: Single Zero at Different Radii with $\theta = \pi$

Magnitude of $H(z)$ $|1 - re^{j\theta}e^{-j\omega}| = |\vec{v_3}|$
Phase: $\phi_3 - \omega$





(a)

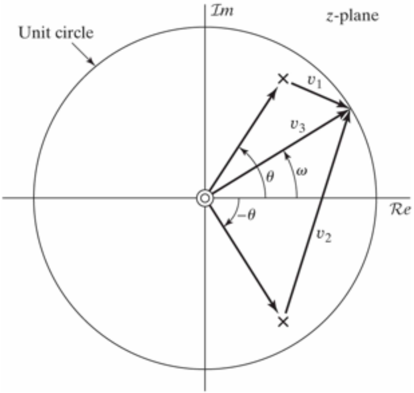


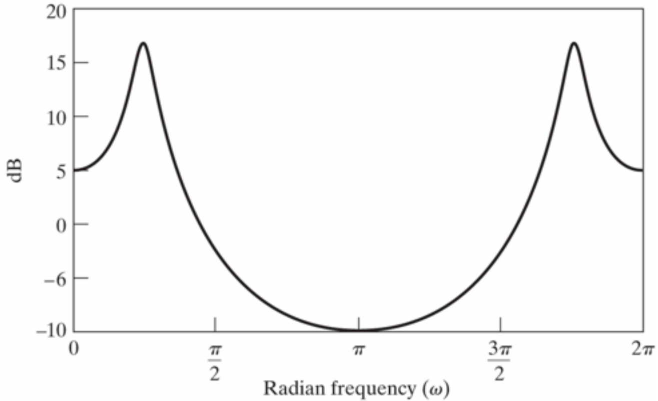
(b)

Example: Second-Order IIR System

$$|H(e^{j\omega})| = \frac{|v_3||v_3|}{|v_1||v_2|}$$

Complex-conjugate pair or poles:
$r = 0.9, \theta = \pi/4$





(a)

Magnitude-Phase Relationships and All-Pass Systems

Exploring Possible System Functions

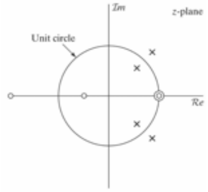
If given the square of the magnitude of the system frequency response, what are possible choices of the system function?

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H(z)H^*(1/z^*)|_{z=e^{j\omega}}$$
$$\Longrightarrow \; h[n] * h^*[-n]$$

Therefore, $|H(e^{j\omega})|^2$ is the evaluation of this z-transform on the unit circle:

$$C(z) = H(z)H^*(1/z^*) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1-c_k z^{-1})(1-c_k^* z)}{\prod_{k=1}^{N}(1-d_k z^{-1})(1-d_k^* z)}$$

- For each pole d_k of $H(z)$ there is a pole of $C(z)$ at $(d_k^*)^{-1}$
- For each zero c_k of $H(z)$ there is a zero of $C(z)$ at $(c_k^*)^{-1}$



Different Systems With Same Magnitude Response

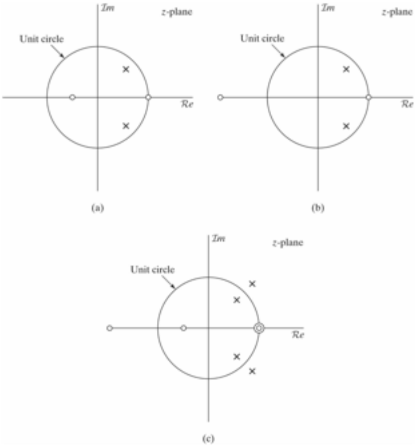


Figure (a): $H_1(z)$ Figure (b): $H_2(z)$
Figure (c): $C(z)=H_1(z)H_1^*(1/z^*)=H_2(z)H_2^*(1/z^*)$

- $H_1(z)$ and $H_2(z)$ are different systems with same magnitude response but different phase response
- Poles and zeros of $C(z)$ occur in conjugate reciprocal pairs (one inside / one outside unit circle, or both on unit circle)
- Determining $H(z)$ from $C(z)$: Casual system: poles can be identified from $C(z)$ (inside unit circle) but zeros can't be identified
- Unknown phase but finite choices due to possible locations of zeros

All-Pass Factors

Assume $H(z)$ can be factored:

$$H(z) = H_1(z) \frac{z^{-1} - a^*}{1 - az^{-1}}$$

Factors in this form are *all-pass factors* (unity magnitude) and they cancel in $C(z)$:

$$C(z) = H(z)H^*(1/z^*) = H_1(z)H_1^*(1/z^*)$$

Given $C(z)$, choices for $H(z)$ can be cascaded with an arbitrary number of all-pass factors if the total number of poles/zeros of $H(z)$ were unknown.

All-Pass Systems: Constant Gain or Attenuation

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}} \qquad\qquad |a| < 1$$

$$H_{ap}(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} = e^{-j\omega}\frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}}$$

Constant Magnitude (independent of frequency):

$$|H_{ap}(e^{j\omega})| = 1 \qquad\qquad \text{(log magnitude in dB is zero) :}$$

General form of system function (d_k are real poles and e_k are complex poles)

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

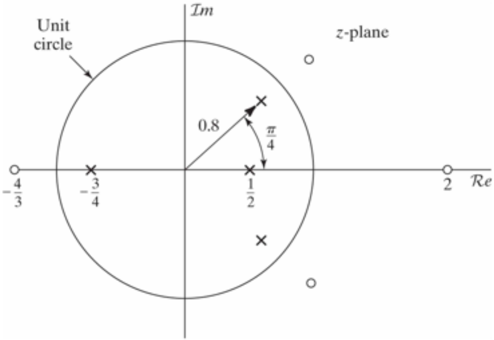
For stable/causal systems: $|d_k| < 1$ and $|e_k| < 1$; $M = N = 2M_c + M_r$ poles/zeros.

Typical Pole-Zero Plot for All-Pass System

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

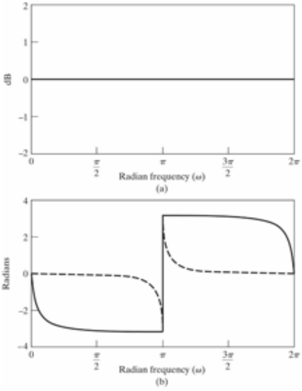
For stable/causal systems: $|d_k| < 1$ and $|e_k| < 1$; $M = N = 2M_c + M_r$ poles/zeros.

$M_r = 2$ and $M_c = 1$

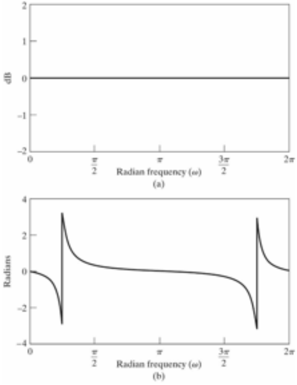


Example 1st and 2nd Order All-Pass Systems

Pole at 0.9 (solid), Pole at -0.9 (dashed)



Poles at $z = 0.9e^{\pm j\pi/4}$



- phase (unwrapped) is always nonpositive for $0 < \omega < \pi$

Minimum-Phase Systems

Minimum-Phase Systems

Introduction to Minimum Phase Systems

- Inverse Systems
  - $h[n] * h_{\text{inv}}[n] = \delta[n]$
  - Cascade of system and its inverse is the identity system: $H(z)H_{\text{inv}}(z) = 1$
- Restrict inverse system, $1/H(z)$, to be stable and causal
- Therefore, all zeros (not just poles) must be inside the unit circle
- A causal and stable LTI system with a causal and stable inverse is called a *minimum-phase system*

Given a magnitude-squared function of a minimum-phase system:

$$C(z) = H(z)H^*(1/z^*)$$

then, $H(z)$ is uniquely determined by choosing all poles/zeros inside unit circle.

Minimum-Phase and All-Pass Decomposition

Assume $H(z)$ has one zero at $z = 1/a^*$ outside the unit circle ($|a| < 1$), and all other poles/zeros are inside the unit circle:

$$H(z) = H_1(z)\left(z^{-1} - a^*\right)$$

Therefore, $H_1(z)$ is minimum-phase. We can continue to decompose with an all-pass factor:

$$H(z) = \underbrace{H_1(z)\left(1 - az^{-1}\right)}_{\text{minimum-phase}} \underbrace{\frac{\left(z^{-1} - a^*\right)}{\left(1 - az^{-1}\right)}}_{\text{all-pass}}$$

Repeat this for every zero outside the unit circle for any arbitrary rational system function.

Minimum-Phase and All-Pass Decomposition (Continued)

Any rational system can be factored into a minimum-phase system and an all-pass system:

$$H(z) = H_{min}(z)H_{ap}(z)$$

- $H_{min}(z)$ contains all poles/zeros of $H(z)$ that are inside unit circle
- $H_{min}(z)$ also contains conjugate reciprocals of zeros of $H(z)$ outside unit circle
- $H_{ap}(z)$ contains all zeros of $H(z)$ that are outside unit circle
- $H_{ap}(z)$ contains poles to cancel conjugate reciprocals zeros in $H_{min}(z)$

Example: Minimum-Phase/All-Pass Decomposition

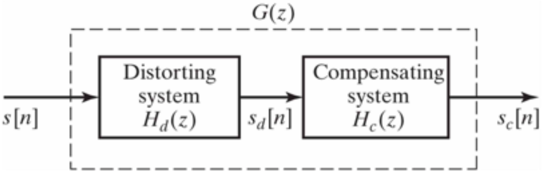
Find the minimum-phase all-pass decomposition of this stable and causal system:

$$H(z) = \frac{1 + 3z^{-1}}{1 + \frac{1}{2}z^{-1}} \qquad \text{pole at } z = -\tfrac{1}{2} \text{ and zero at } z = -3$$

- $H_{min}(z)$ contains all poles/zeros of $H(z)$ that are inside unit circle
- $H_{min}(z)$ also contains conjugate reciprocals of zeros of $H(z)$ outside unit circle
- $H_{ap}(z)$ contains all zeros of $H(z)$ that are outside unit circle
- $H_{ap}(z)$ contains all poles to cancel conjugate reciprocals zeros in $H_{min}(z)$

$$H(z) = \frac{1 + 3z^{-1}}{1 + \frac{1}{2}z^{-1}} = 3\left(\frac{z^{-1} + \frac{1}{3}}{1 + \frac{1}{2}z^{-1}}\right) = \underbrace{3\left(\frac{1 + \frac{1}{3}z^{-1}}{1 + \frac{1}{2}z^{-1}}\right)}_{H_{min}(z)}\underbrace{\left(\frac{z^{-1} + \frac{1}{3}}{1 + \frac{1}{3}z^{-1}}\right)}_{H_{ap}(z)}$$

Application: Frequency Response Compensation (Inverse Filters)



For a nonminimum phase system, we can form minimum phase system, $H_{d,min}(z)$:

$$H_d(z) = H_{d,min} H_{ap}(z)$$

Then the compensating filter is stable and causal:

$$H_c(z) = \frac{1}{H_{d,min}(z)} \qquad \text{where} \qquad H_{d,min}(z) = \frac{H_d(z)}{H_{ap}(z)}$$

Overall system function:

$$G(z) = H_d(z) H_c(z) = H_{ap}(z)$$

Property of Minimum-Phase Systems

Using minimum-phase and all-pass decomposition:

$$H(z) = H_{min}(z)H_{ap}(z) \implies \angle H(z) = \angle H_{min}(z) + \angle H_{ap}(z)$$

For all systems that have the same magnitude response, the system with all poles and zeros inside the unit circle has:

1. minimum phase-lag (*phase-lag* is negative of the phase function)
   - all-pass systems always have negative unwrapped phase curve
   - therefore, the all-pass system increases the negative of the phase (phase-lag)
   - leaving $H_{min}(z)$ to have minimum phase-lag
2. minimum group delay
   - all-pass systems always have a positive group delay
   - therefore, the all-pass system increases group delay
   - leaving $H_{min}(z)$ to have minimum group delay

Systems With Linear Phase

Ideal Delay System: Linear Phase and Constant Magnitude

Consider an LTI system with frequency response, an *ideal delay* system:

$$H(e^{j\omega}) = e^{-j\omega\alpha} \qquad |\omega| < \pi$$

where α is a real number but not necessarily an integer. For this system:

$$\begin{aligned} |H(e^{j\omega})| = 1 &\quad \text{constant magnitude response} \\ \angle H(e^{j\omega}) = -\omega\alpha &\quad \text{linear phase} \\ \tau_{gd} = \alpha &\quad \text{constant group delay} \end{aligned}$$

The inverse Fourier transform provides the impulse response:

$$h[n] = \frac{\sin\left(\pi(n-\alpha)\right)}{\pi(n-\alpha)} = \mathrm{sinc}\left(n-\alpha\right) \qquad -\infty < n < \infty$$

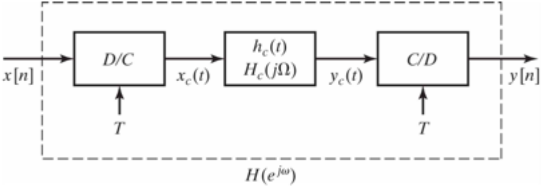
When, $\alpha = n_d$, and n_d is an integer, then

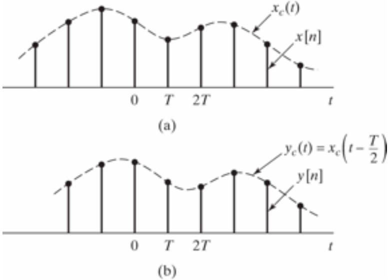
$$h[n] = \delta[n-n_d] \qquad \text{and} \qquad y[n] = x[n] * \delta[n-n_d] = x[n-n_d]$$

Ideal Delay System: Non-Integer Delay

What if the delay, α, is not an integer? Interpret as a continuous time system:

$$h_c(t) = \delta(t - \alpha T) \quad \text{and} \quad H_c(j\Omega) = e^{-j\Omega\alpha T}$$





Overall response: $H(e^{j\omega}) = e^{-j\omega\alpha} \qquad |\omega| < \pi$

Output: $y[n] = x_c(nT - \alpha T)$ *Time shift of* α *samples (even non-integer).*

Linear Phase With Non-constant Magnitude Response

A linear phase system has the frequency response

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega\alpha} \qquad |\omega| < \pi$$

where $A(e^{j\omega})$ is purely real and is called the *zero phase response*.

- With real-valued $h[n]$, then $|H(e^{j\omega})| = |A(e^{j\omega})|$ are even
- Since A is real, A is even and the inverse Fourier transform is real and even
- The impulse response $h[n]$ is real and even and time shifted by α
- If α is an integer: $h[n]$ is symmetric about $n = \alpha$
- If α is an integer $+ \frac{1}{2}$: $h[n]$ is symmetric about $n = \alpha + \frac{1}{2}$
- Both cases: if 2α is an integer then $h[n]$ has symmetry: $h[2\alpha - n] = h[n]$

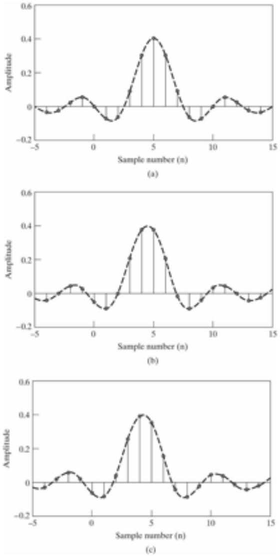
Linear phase and *causal*: must be FIR with $M = 2\alpha$ and
$h[0] = h[M]$, $h[1] = h[M-1]$, etc.

Example: Ideal Lowpass Filter With Linear Phase

Linear phase lowpass filter with cutoff ω_c:

$$h[n] = \frac{\sin\left(\omega_c(n - n_d)\right)}{\pi(n - n_d)}$$

- Figure (a): $\omega_c = 0.4\pi$, $\alpha = n_d = 5$
  $h[n]$ symmetric about $n = 5$
- Figure (b):
  $\omega_c = 0.4\pi$, $\alpha = n_d = 4.5$
  $h[n]$ symmetric about $\alpha = n_d = 4.5$
- Figure (c):
  $\omega_c = 0.4\pi$, $\alpha = n_d = 4.3$
  $h[n]$ not symmetric, still linear phase



Generalized Linear Phase

Generalized Linear Phase System

Broader class to include more linear phase systems (including odd symmetry):

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta}$$

- $A(e^{j\omega})$ is real (possible bipolar) function of ω
- α and β are constants
- Phase is the equation of a straight line $(-\alpha\omega + \beta)$
- Constant group delay, $\tau_{gd} = \alpha$

Necessary Conditions For Constant Group Delay

For generalized linear phase systems

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta}$$

Leads to necessary conditions on real-valued $h[n]$, and constants α, and β:

$$\sum_{n=-\infty}^{\infty} h[n] \sin\left[\omega(n-\alpha)+\beta\right] = 0, \quad \text{for all } \omega$$

This doesn't tell us how to find a linear phase system but we are interested in particular cases:

$$2\alpha = M = \text{ even or odd integer, and } \beta = 0, \pi \text{ or } \beta = \tfrac{\pi}{2}, \tfrac{3\pi}{2}$$

Causal Generalized Linear Phase Systems

$$\sum_{n=0}^{\infty} h[n] \sin\left[\omega(n-\alpha)+\beta\right] = 0, \quad \text{for all } \omega$$

Causality and symmetry conditions imply:

$$h[n] = 0, \qquad n < 0 \quad \text{and} \quad n > M$$

Symmetric or antisymmetric impulse response (even, odd):

$$h[n] = \begin{cases} h[M-n], & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}$$

where $\beta = 0$ or $\beta = \pi$

$$h[n] = \begin{cases} -h[M-n], & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$$H(e^{j\omega}) = jA_o(e^{j\omega})e^{-j\omega M/2} = A_o(e^{j\omega})(e^{j\omega M/2 + j\pi/2})$$

where $\beta = \frac{\pi}{2}$ or $\beta = \frac{3\pi}{2}$

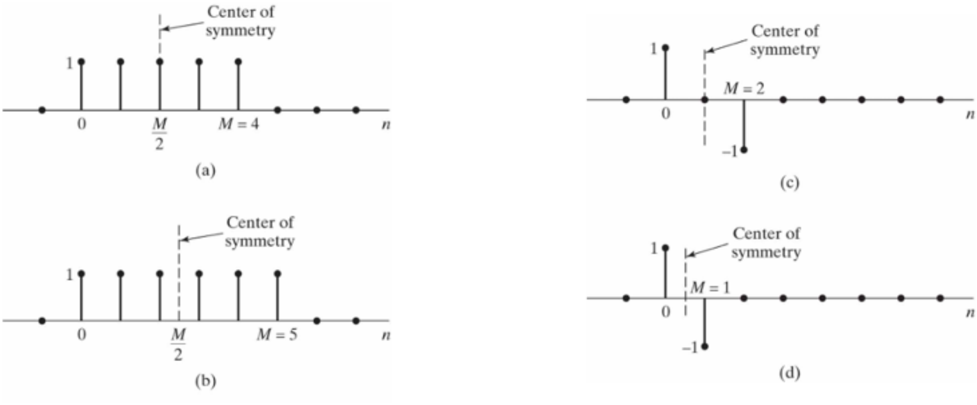
Four Types of FIR Generalized Linear Phase Systems

Type of symmetry and whether M is an even or odd integer lead to different expressions for the frequency response of FIR linear phase systems:

Type	I	II	III	IV
M	even	odd	even	odd
h symmetry	even	even	odd	odd
A symmetry	even	even	odd	odd
$A\left(e^{\pm j\pi}\right)$	$\neq 0$	0	0	$\neq 0$
α	integer	half-integer	integer	half-integer
β	0 or π	0 or π	$\frac{\pi}{2}$ or $\frac{3\pi}{2}$	$\frac{\pi}{2}$ or $\frac{3\pi}{2}$

Note: Frequency response constrained to zero at $\omega = \pi$ for types II and III.

Examples of FIR Linear Phase Impulse Response



Locations of Zeros for FIR Linear Phase Systems

Symmetric cases of the impulse response, $h[n]$, puts constraints on the zeros of FIR system. Here are some examples: