Dartmouth College
Thayer School of Engineering

Signal Processing

Practice Problems #4

1. *Finding system function and impulse response from difference equation.* Consider a stable LTI system with input $x[n]$ and output $y[n]$. The input and output satisfy the difference equation

$$y[n-1] - \frac{10}{3}y[n] + y[n+1] = x[n]$$

   (a) Plot the poles and zeros of the system function in the z-plane.

   (b) Determine the impulse response $h[n]$.

Solution:

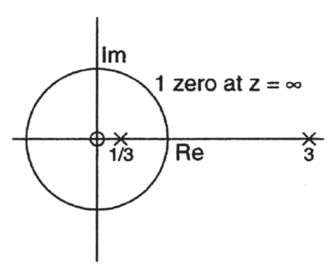
(a) First find the system function by taking the z-transform of both sides of the difference equation while using the time shift property:

$$z^{-1} - \frac{10}{3}Y(z) + zY(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z^{-1} - \frac{10}{3} + z} = \frac{z}{z^2 - \frac{10}{3}z + 1}$$
$$= \frac{z}{(z-\frac{1}{3})(z-3)}$$

Finding the partial fractions from a computing program:

$$H(z) = \frac{-\frac{1}{8}}{z-\frac{1}{3}} + \frac{\frac{9}{8}}{z-3}$$



(b)

$$H(z) = \frac{-\frac{1}{8}z^{-1}}{1-\frac{1}{3}z^{-1}} + \frac{\frac{9}{8}z^{-1}}{1-3z^{-1}}$$

Since this system is stable, the region of convergence must include the unit circle (but can't include the poles):

$$\tfrac{1}{3} \leq |z| \leq 3$$

Now, the inverse z-transform for each partial fraction is from a z-transform table and using the time shifting property:

$$h[n] = -\tfrac{1}{8}\left(\tfrac{1}{3}\right)^{n-1} u[n-1] - \tfrac{9}{8}(3)^{n-1}u[-n]$$

2. *Pole-zero plot characteristics and LTI systems.* Many properties of a discrete-time sequence $h[n]$ or an LTI system with impulse response $h[n]$ can be discerned from a pole-zero plot of $H(z)$. In this problem, we are only concerned with causal systems. Cleary describe the z-plane characteristic that corresponds to each of the following properties.
   (a) Real-valued impulse response
   (b) Finite impulse response
   (c) $h[n] = h[2\alpha - n]$ where 2α is an integer
   (d) Minimum phase
   (e) All-pass

Solution:

(a) Real-valued impulse response: Poles will be real-valued or in complex conjugate pairs. Zeros will be real-valued or in complex conjugate pairs.

(b) Finite impulse response: All poles are at the origin.

(c) $h[n] = h[2\alpha - n]$ where 2α is an integer: Causality with this symmetry property implies a finite-length $h[n]$ that can only be non-zero between time zero and time 2α. Therefore, all poles are at the origin and there at most 2α zeros.

(d) Minimum phase: All poles and zeros are inside the unit circle. This also implies that the inverse system is stable and causal.

(e) All-pass: Each pole is inside the unit circle and paired with a conjugate reciprocal zero.

3. *Minimum-phase systems.* For the following system function, specify a minimum-phase system function $H_{min}(z)$ such that the frequency-response magnitudes of the two systems

are equal: $|H(e^{j\omega})| = |H_{min}(e^{j\omega})|$. Note that a scale factor may be introduced when the pole or zero is reflected inside the unit circle.

$$H(z) = \frac{(1+3z^{-1})(1-\frac{1}{2}z^{-1})}{z^{-1}(1+\frac{1}{3}z^{-1})}$$

Solution: Reflect all poles and zeros outside the unit circle to their conjugate reciprocal locations to move them inside the unit circle. Find the scale factor to ensure magnitudes are the same. The zero at $z=-3$ is reflected to its conjugate reciprocal at $z=-\frac{1}{3}$. Finding the magnitude of both systems at zero frequency shows that the minimum phase system needs a gain of 3 to ensure they are equal.

$$H_{min}(z) = 3\frac{(1+\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}{z^{-1}(1+\frac{1}{3}z^{-1})}$$

The factors for the pole and zero at $z=-\frac{1}{3}$ cancel. Also, the factor $1/z^{-1}$ does not contribute to the frequency response magnitude so it can be removed.

$$H_{min}(z) = 3(1-\tfrac{1}{2}z^{-1})$$

4. *Linear-phase and generalized linear-phase systems.* Consider the class of linear-phase discrete-time filters whose frequency response has the form

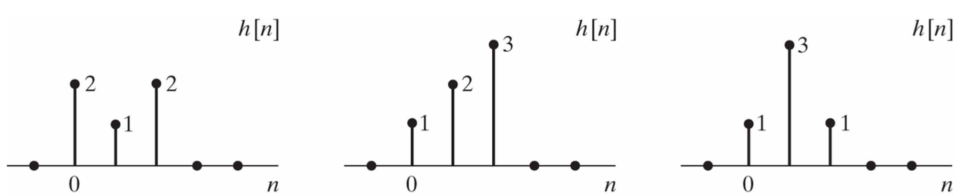
$$H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\alpha\omega}$$

where $|H(e^{j\omega}|$ is a real and nonnegative function of ω and α is a real constant. Also consider the class of generalized linear-phase filters of the form

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega+j\beta}$$

where $A(e^{j\omega})$ is a real (could be negative) function of ω, and α, β are real constants.

For each of the filters with impulse response in the figure below, determine whether it is a generalized linear phase filter. If it is, then find $A(e^{j\omega})$, α, and β. For these filters, indicate whether or not it also meets the more stringent requirement for being a linear phase filter.



Solution:

(a) Impulse response, $h[n]$, is symmetric about $n = 1$.

$$\begin{aligned} H(e^{j\omega}) &= 2 + e^{-j\omega} + 2e^{-2j\omega} \\ &= e^{-j\omega}(2e^{j\omega} + 1 + 2e^{-j\omega}) \\ &= [1 + 4\cos(\omega)]\, e^{-j\omega} \end{aligned}$$

Therefore, $A(\omega) = 1 + 4\cos(\omega)$, $\alpha = 1$ and $\beta = 0$. This is generalized linear phase but not linear phase since $A(\omega)$ is not always positive.

(b) Not generalized linear phase since is does not have even nor odd symmetry.

(c) Impulse response, $h[n]$, is symmetric (even) about $n = 1$.

$$\begin{aligned} H(e^{j\omega}) &= 1 + 3e^{-j\omega} + e^{-2j\omega} \\ &= e^{-j\omega}(e^{j\omega} + 3 + e^{-j\omega}) \\ &= [3 + 2\cos(\omega)]\, e^{-j\omega} \end{aligned}$$

Therefore, $A(\omega) = 3 + 2\cos(\omega)$, $\alpha = 1$ and $\beta = 0$. This is generalized linear phase and linear phase.