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ENGS110
PS1

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Thayer School of Engineering

Signal Processing

Problem Set #1

Instructions. These graded problem sets will synthesize signal processing concepts and methods from the week. Prepare your assignment neatly on lined paper, engineering paper, by hand or on a digital device before capturing it to upload as one PDF document or as images as per the guidance on the web submission tool. Alternatively, you may use LaTeX to typeset your document. Overleaf is a popular online LaTeX editor for which Dartmouth has a license and you can sign in using your Dartmouth single sign-on (SSO): https://www.overleaf.com.

This assignment is worth 50 points. Show all of your work when solving problems and answer conceptual questions clearly and concisely. You may discuss these problems with others after you have made an initial start by yourself. List the names of any collaborators and acknowledge all sources and tools that you used outside the normal course readings (such as other books, websites, computing tools, etc.).

In this problem set, you will explore discrete-time signals and systems which provide a foundation for the signal processing topics to come later in this course. You will work with signals in the time domain and in the frequency domain using the discrete-time Fourier transform and the z-transform.

1. *Interesting aspect of discrete-time signals: periodicity.* Unlike continuous-time signals, discrete-time time complex exponential signals or real sinusoidal signals are only periodic if their frequency can be expressed as:

$$\omega_0 = \frac{2\pi k}{N}$$

where k is an integer and N is the integer period if common factors are cancelled. Determine which of the following signals is periodic. If a signal is periodic, determine its period, N.

   (a) $x[n] = e^{j(2\pi n/5)}$
   (b) $x[n] = \sin(\pi n/19)$
   (c) $x[n] = ne^{j\pi n}$
   (d) $x[n] = e^{jn}$

2. *Properties of discrete-time systems.* For each of the following systems, determine whether the systems is (1) stable, (2) causal, (3) linear, and (4) time-invariant.

(a) $y[n] = T(x[n]) = (\cos(\pi n))\, x[n]$

(b) $y[n] = T(x[n]) = \sum_{k=n-1}^{\infty} x[k]$

3. *Discrete-time convolution.* Given the input $x(n) = \{3, 1, 2\}$ and unit impulse response $h(n) = \{3, 2, 1\}$ of a discrete-time linear time-invariant (LTI) system, find the output using convolution without using a computer but feel free to check your answer with a computing tool.

4. *Using the discrete-time Fourier transform.* A LTI system has impulse response $h[n] = 5(-1/2)^n u[n]$. Use the Fourier transform to find the output of this system, $y[n]$, when the input is $x[n] = (1/3)^n u[n]$. We can avoid using convolution to find the output since convolution in time is equivalent to multiplying the Fourier transforms.

5. *Finding the z-transform of a sequence.* Determine the z-transform of the following signal.

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \geq 5 \\ 0, & n \leq 4 \end{cases}$$

6. *Inverse z-transform and region of convergence.* Determine all possible signals $x(n)$ (assuming the three possible ROCs) having the z-transform

$$X(z) = \frac{1 - 1.7z^{-1}}{1 - 2.05z^{-1} + z^{-2}}$$

7. *Discrete-time LTI systems: finding the output.* Consider an LTI system that is stable and for which $H(z)$, the z-transform of the impulse response, is given by

$$H(z) = \frac{3}{1 + \frac{1}{3}z^{-1}}$$

Suppose $x[n]$, the input to the system, is a unit step sequence, $x[n] = u[n]$.

(a) Determine the output $y[n]$ by evaluating the discrete convolution of $x[n]$ and $h[n]$.

(b) Determine the output $y[n]$ by computing the inverse z-transform of $Y(z)$, where $Y(z) = H(z)X(z)$.

8. *Discrete-time LTI systems: finding the system function.* We want to design a causal discrete-time LTI system with the property that if the input is

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

then the output is

$$y[n] = \left(\frac{1}{3}\right)^n u[n]$$

(a) Determine the system function $H(z)$ of a system that satisfies the condition provided and then determine the impulse response $h[n]$.

(b) Find the difference equation that characterizes this system from the system function.

(c) Determine if the system is stable from the location of the poles.

1. *Interesting aspect of discrete-time signals: periodicity.* Unlike continuous-time signals, discrete-time time complex exponential signals or real sinusoidal signals are only periodic if their frequency can be expressed as:

$$\omega_0 = \frac{2\pi k}{N}$$

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(b) $x[n] = \sin(\pi n/19)$

(c) $x[n] = ne^{j\pi n}$

(d) $x[n] = e^{jn}$

(a) $\omega = \frac{2\pi}{5} = 2\pi \cdot \frac{1}{5}$, $1, 5 \in \mathbb{Z}$ $\therefore$ $\boxed{x[n] \text{ is periodic}, N = 5}$

(b) $\omega = \frac{\pi}{19} = \frac{2\pi}{38} = 2\pi \cdot \frac{1}{38}$, $1, 38 \in \mathbb{Z}$ $\therefore$ $\boxed{x[n] \text{ is periodic}, N = 38}$

(c) alhough $\omega = 2\pi \cdot \frac{1}{2} \Rightarrow$ periodic, because the signal does not repeat every period due to the n in front of the exponential, it leads me to conclude that the signal is $\boxed{\text{not periodic}}$

(d) $\omega = 1 = 2\pi \cdot \frac{1}{2\pi}$, $2\pi \notin \mathbb{Z}$ $\therefore$ signal is $\boxed{\text{not periodic}}$

2. *Properties of discrete-time systems.* For each of the following systems, determine whether the systems is (1) stable, (2) causal, (3) linear, and (4) time-invariant.

(a) $y[n] = T(x[n]) = (\cos(\pi n))\, x[n]$

(b) $y[n] = T(x[n]) = \sum_{k=n-1}^{\infty} x[k]$

(a)

$\max(|\cos(\pi n)|) = 1 \ \forall n$ $\therefore$ BI $\Rightarrow$ BO and sys is stable

only current time step affects output $\therefore$ causal

$$\begin{aligned} T\{a x_1[n] + b x_2[n]\} &= \cos(\pi n)\,(a x_1[n] + b x_2[n]) \\ &= a\cos(\pi n)\, x_1[n] + b\cos(\pi n)\, x_2[n] \\ &= aT\{x_1[n]\} + bT\{x_2[n]\} \end{aligned}$$

$\therefore$ T is linear

$$y[n] = \cos(\pi n)\, x[n], \quad \begin{aligned} y[n+1] &= \cos(\pi(n+1))\, x[n+1] \\ &= \cos(\pi n + \pi)\, x[n+1] \\ &= -\cos(\pi n)\, x[n+1] \end{aligned}$$

$\therefore$ system is not time invariant

(b)

$$y[n] = \sum_{k=n-1}^{\infty} x[k]$$

$\boxed{\text{if } \exists\, m : \forall\, k > m,\ x[k] = 0 \text{ then sys is stable, otherwise not stable}}$

non causal as $y[n]$ depends on future time steps

$$\begin{aligned} T\{a x_1[n] + b x_2[n]\} &= \sum_{k=n-1}^{\infty} a x_1[k] + b x_2[k] \\ &= a\sum_{k=n-1}^{\infty} x_1[n] + b\sum_{k=n-1}^{\infty} x_2[n] \\ &= aT\{x_1[n]\} + bT\{x_2[n]\} \end{aligned}$$ $\therefore$ linear

$$x[n] \Rightarrow y[n], \quad x[n-a] \Rightarrow \sum_{k=(n-a)-1}^{\infty} x[k] = y[n-a]$$

time invariant

3. *Discrete-time convolution.* Given the input $x(n) = \{3, 1, 2\}$ and unit impulse response $h(n) = \{3, 2, 1\}$ of a discrete-time linear time-invariant (LTI) system, find the output using convolution without using a computer but feel free to check your answer with a computing tool.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

n	...	-4	-3	-2	-1	0	1	2	3	4	...
x[n]	...	0	0	0	0	3	1	2	0	0	...
h[n]	...	0	0	0	0	3	2	1	0	0	...
h[-n]	...	0	0	1	2	3	0	0	0	0	...
h[1-n]	...	0	0	0	1	2	3	0	0	0	...
h[2-n]	...	0	0	0	0	1	2	3	0	0	...
h[3-n]	...	0	0	0	0	0	1	2	3	0	...
h[4-n]	...	0	0	0	0	0	0	1	2	3	...

n	y[n]	
0	9	= 9
1	6+3	9
2	3+2+6	11
3	1+4	5
4	2	2

$$\boxed{y[n] = \{9, 9, 11, 5, 2\}}$$

4. *Using the discrete-time Fourier transform.* A LTI system has impulse response $h[n] = 5(-1/2)^n u[n]$. Use the Fourier transform to find the output of this system, $y[n]$, when the input is $x[n] = (1/3)^n u[n]$. We can avoid using convolution to find the output since convolution in time is equivalent to multiplying the Fourier transforms.

$$y[n] = h[n] * x[n] \quad , \; Y(\omega) = H(\omega)\, X(\omega)$$

$$H(\omega) = \frac{5}{1-(-1/2)e^{-j\omega}} = \frac{5}{1+1/2\, e^{-j\omega}}$$

$$X(\omega) = \frac{1}{1 - 1/3 e^{-j\omega}}$$

$$Y(\omega) = \frac{5}{(1-1/3\, e^{-j\omega})(1+1/2\, e^{-j\omega})}$$ let $x = e^{-j\omega}$

$$= \frac{A}{(1-1/3\, x)} + \frac{B}{(1+1/2\, x)}$$

$$5 = A(1+1/2x) + B(1-1/3x)$$

$$= (A+B) + x(A/2 - B/3)$$

$A+B=5$

$3A = 2B$

$A=2, B=3$

$$Y(\omega) = \frac{2}{1-e^{-j\omega}/3} + \frac{3}{1+e^{-j\omega}/2}$$

$$\boxed{y[n] = (2(1/3)^n + 3(-1/2)^n)\, u[n]}$$

5. *Finding the z-transform of a sequence.* Determine the z-transform of the following signal.

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \geq 5 \\ 0, & n \leq 4 \end{cases}$$

$$x[n] = (1/2)^n u[n-5]$$

$$= (1/2)^5 (1/2)^{n-5} u[n-5)$$

$$\boxed{X(z) = (1/2)^5 \frac{z^{-5}}{1 - 1/2 z^{-1}}}$$

6. *Inverse z-transform and region of convergence.* Determine all possible signals $x(n)$ (assuming the three possible ROCs) having the z-transform

$$X(z) = \frac{1 - 1.7z^{-1}}{1 - 2.05z^{-1} + z^{-2}}$$

$$X(z) = \frac{1-1.7z^{-1}}{(z^{-1}-1.25)(z^{-1}-0.8)}$$

$$= \frac{A}{z^{-1}-1.25} + \frac{B}{z^{-1}-0.8}$$

$$1-1.7z^{-1} = A(z^{-1}-0.8) + B(z^{-1}-1.25)$$

$$= (-0.8A - 1.25B) + z^{-1}(A+B)$$

$$\begin{bmatrix} -0.8 & -1.25 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ -1.7 \end{bmatrix}$$

$$A = -2.5$$
$$B = 0.8$$

$$X(z) = -\frac{2.5}{z^{-1}-1.25} + \frac{0.8}{z^{-1}-0.8}$$

$$= \frac{2.5}{1.25 - z^{-1}} - \frac{0.8}{0.8 - z^{-1}}$$

$$= \frac{2}{1 - \frac{z^{-1}}{1.25}} - \frac{1}{1 - \frac{z^{-1}}{0.8}}$$

$$= \frac{2}{1 - 4/5\, z^{-1}} - \frac{1}{1 - 5/4\, z^{-1}}$$

$$\boxed{\begin{aligned} |z| > 5/4 &: x[n] = \left(2(4/5)^n - (5/4)^n\right) u[n] \\ |z| < 4/5 &: x[n] = \left(-2(4/5)^n + (5/4)^n\right) u[-n-1] \\ 4/5 < |z| < 5/4 &: x[n] = 2(4/5)^n u[n] + (5/4)^n u[-n-1] \end{aligned}}$$

7. *Discrete-time LTI systems: finding the output.* Consider an LTI system that is stable and for which $H(z)$, the z-transform of the impulse response, is given by

$$H(z) = \frac{3}{1+\frac{1}{3}z^{-1}}$$

Suppose $x[n]$, the input to the system, is a unit step sequence, $x[n] = u[n]$.

(a) Determine the output $y[n]$ by evaluating the discrete convolution of $x[n]$ and $h[n]$.

(b) Determine the output $y[n]$ by computing the inverse z-transform of $Y(z)$, where $Y(z) = H(z)X(z)$.

(a)

$$h[n] = 3(-1/3)^n u[n]$$

$$h[n] = \{3, -1, 1/3, -1/9, \dots\}$$

$$x[n] = \{1, 1, 1, 1, \dots\}$$

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] = \{3, 2, 2\,1/3, 2\,2/9, 2\,7/27, \dots\} = \{3\sum_{k=0}^{n}(-1/3)^k\}}$$

(b)

$$X(z) = \frac{1}{1-z^{-1}} \qquad H(z) = \frac{3}{1+1/3\,z^{-1}}$$

$$Y(z) = H(z)X(z)$$

$$= \frac{3}{(1-z^{-1})(1+1/3\,z^{-1})}$$

$$= \frac{A}{(1-z^{-1})} + \frac{B}{(1+1/3\,z^{-1})}$$

$$3 = A(1+1/3\,z^{-1}) + B(1-z^{-1})$$

$$= (A+B) + (1/3A - B)z^{-1}$$

$$\begin{bmatrix} 1 & 1 \\ 1/3 & -1 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$A = 2.25,\ B = 0.75$

$$Y(z) = \frac{2.25}{1-z^{-1}} + \frac{0.75}{1+1/3\,z^{-1}}$$

$$\boxed{\begin{aligned} y[n] &= 2.25\,u[n] + 0.75(-1/3)^n u[n] \\ &= \{3, 2, 2\,1/3, 2\,2/9, 2\,7/27, \dots\} \end{aligned}}$$

same as above, but w/ z transform, it is easier to see that sys will converge to 2.25 as $n \to \infty$

8. *Discrete-time LTI systems: finding the system function.* We want to design a causal discrete-time LTI system with the property that if the input is

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] - \tfrac{1}{4}\left(\tfrac{1}{2}\right)^{n-1} u[n-1]$$

then the output is

$$y[n] = \left(\tfrac{1}{3}\right)^n u[n]$$

(a) Determine the system function $H(z)$ of a system that satisfies the condition provided and then determine the impulse response $h[n]$.

(b) Find the difference equation that characterizes this system from the system function.

(c) Determine if the system is stable from the location of the poles.

(a)

$$Y(z) = H(z)\,X(z)$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{1-\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$Y(z) = \frac{1}{1-\frac{1}{3}z^{-1}}$$

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}} = \frac{A}{1-\frac{1}{4}z^{-1}} + \frac{B}{1-\frac{1}{3}z^{-1}}$$

$$1-\tfrac{1}{2}z^{-1} = A(1-\tfrac{1}{3}z^{-1}) + B(1-\tfrac{1}{4}z^{-1})$$

$$= (A+B) + (-\tfrac{1}{3}A - \tfrac{1}{4}B)z^{-1} \qquad A = 3,\ B = -2$$

$$H(z) = \frac{3}{1-\frac{1}{4}z^{-1}} - \frac{2}{1-\frac{1}{3}z^{-1}}$$

$$\boxed{\begin{aligned} h[n] &= \left(3(\tfrac{1}{4})^n - 2(\tfrac{1}{3})^n\right)u[n] \\ &= \{1,\ \tfrac{1}{12},\ -\tfrac{5}{144},\ \dots\} \end{aligned}}$$

(b)

$$Y(z)(1-\tfrac{1}{4}z^{-1})(1-\tfrac{1}{3}z^{-1}) = X(z)(1-\tfrac{1}{2}z^{-1})$$

$$Y(z)(1-\tfrac{7}{12}z^{-1}+\tfrac{1}{12}z^{-2}) = X(z)(1-\tfrac{1}{2}z^{-1})$$

$$y[n] - \tfrac{7}{12}y[n-1] + \tfrac{1}{12}y[n-2] = x[n] - \tfrac{1}{2}x[n-1]$$

$$\boxed{y[n] = \tfrac{7}{12}y[n-1] - \tfrac{1}{12}y[n-2] + x[n] - \tfrac{1}{2}x[n-1]}$$

(c)

system is **stable** as poles are at $z = 1/3, 1/4$ which are within the unit circle