Dartmouth College
Thayer School of Engineering

Signal Processing

Practice Problems #3

1. *Sampling a continuous-time signal.* The following continuous-time input signal and corresponding discrete-time output signal from an ideal continuous-time to discrete-time converter. Specify a choice of the sampling period, T, that is consistent with $x_c(t)$. Indicate whether your choice of T is unique. If not, specify a second possible choice of T consistent with the signals.

$$x_c(t) = \sin(10\pi t) \qquad x[n] = \sin(\pi n/4)$$

Solution: Pick T such that

$$x[n] = x_c(nT) = c_c(10\pi nT) = \sin(\pi n/4)$$

Therefore, $T = 1/40$ seconds. There are multiple possible choices for the sample period and another one is $T = 9/40$ which can be found by using $\sin(\pi n/4) = \sin(\pi n/4 + 2\pi n)$.

2. *Sampling a continuous-time signal.* The sequence

$$x[n] = \cos\left(\frac{\pi}{4}n\right), \quad -\infty < n < \infty,$$

wss obtained by sampling the continuous-time signal

$$x_c(t) = \cos\left(\Omega_0 t\right), \quad -\infty < n < \infty,$$

at a sampling rate of 1000 samples/s. What are two positive values of Ω_0 that could have resulted in the sequence $x[n]$?

Solution: The mapping between discrete-time frequency is $\omega = \Omega T$ (or $f = \frac{F}{F_s}$), where T is the sample period and F_s is the sample rate. The signal frequency original continuous-time frequency is

$$\Omega_0 = \frac{\omega}{T} = \frac{\pi}{4}1000 = 250\pi.$$

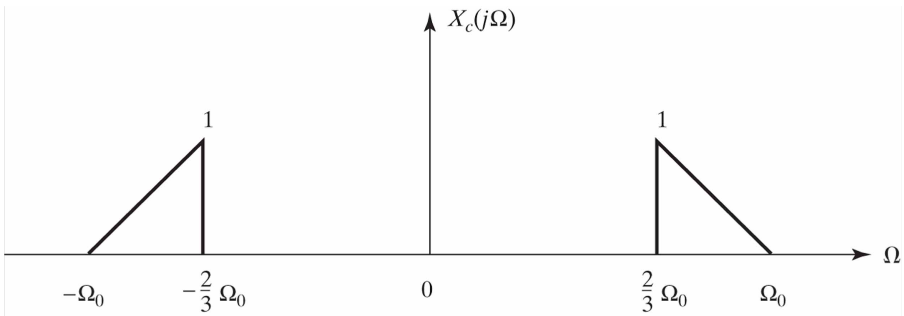
In discrete-time signals, frequencies 2π apart are indistinguishable from each other so that another possible value for the original continuous-time frequency is

$$\Omega_0 = \frac{\omega}{T} = \left(2\pi + \frac{\pi}{4}\right) 1000 = 2250\pi.$$

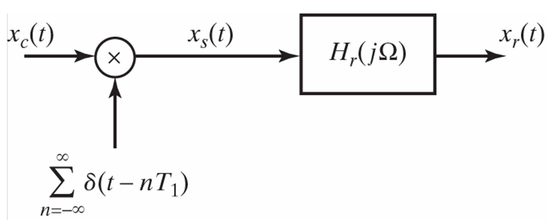
In general, for any integer k, these are possible solutions:

$$\Omega_0 = \frac{\omega}{T} = \left(2\pi k + \frac{\pi}{4}\right) 1000$$

3. *Frequency domain representation of sampling: periodic repetition of spectrum.* Consider a continuous-time signal $x_c(t)$ with (continuous-time) Fourier transform $x_c(j\Omega)$ shown in the figure below.



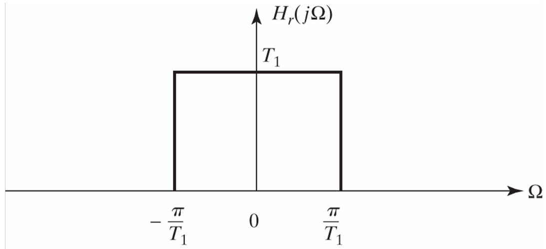
A continuous-time signal, $x_r(t)$, is obtained through the process of multilpying $x_c(t)$ by an impulse train with period T_1 and then passing it through a low-pass filter with a frequency response $H_r(j\Omega)$ as shown in the figure below.



The impulse train sampling causes a periodic repetition of the spectrum, or the superposition of scaled versions of the spectrum shifted by integer multiples of $2\pi/T_1$:

$$X_s(j\Omega) = \tfrac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left[j\left(\Omega - k\tfrac{2\pi}{T_1}\right)\right]$$

The lowpass filter recovers the original spectrum. The ideal frequency response of this low pass filter is shown here:



(a) Draw a sketch of the spectrum, $X_s(j\Omega)$, of the signal after sampling by the impulse train but before the lowpass filter. Assume the sample period T_1 was chosen to avoid aliasing: $\Omega_s \geq 2\Omega_0$ or equivalently $T_1 \leq \pi/\Omega_0$.

(b) Since this is a bandpass signal (no frequency content near dc, or zero frequency) we can sample below the nyquist rate if sample period is determined carefully to avoid overlap of the aliased replicas in the spectrum. Draw a sketch of the spectrum, $X_s(j\Omega)$, of the signal after sampling by the impulse train but before the lowpass filter if the reduced sample rate leads to a larger sample period is $\frac{1.5\pi}{\Omega_0} \leq T_2 \leq \frac{2\pi}{\Omega_0}$

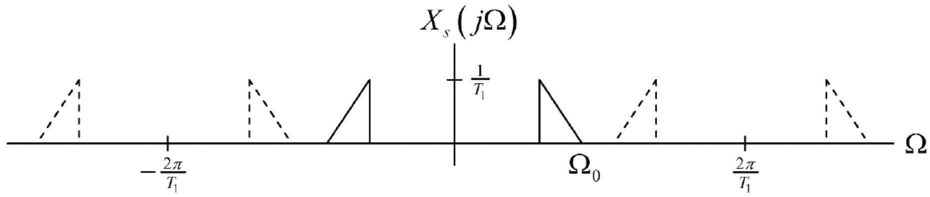
(c) Draw a sketch of the spectrum, $X_s(j\Omega)$, of the signal after sampling by the impulse train but before the lowpass filter if the reduced sample rate leads to a larger sample period is $T_2 = \frac{3\pi}{\Omega_0}$

Solution:

A. The impulse-train signal $x_s(t)$ has spectrum $X_s(j\Omega)$ given by

$$X_s(j\Omega) = \tfrac{1}{T} \sum_{k=-\infty}^{\infty} X\left[j\left(\Omega - k\tfrac{2\pi}{T_1}\right)\right].$$

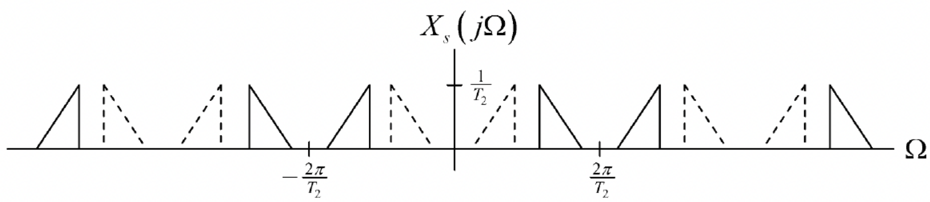
An example is shown below.



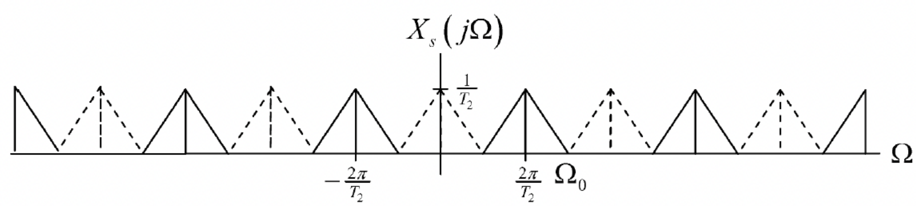
We will have $x_r(t) = x_c(t)$ provided $T_1 \le \frac{\pi}{\Omega_0}$.

B. We will have $x_o(t) = x_c(t)$ under any of the following circumstances:

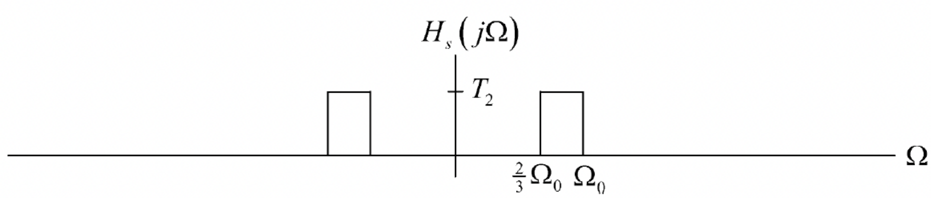
1. As illustrated above, $T_2 \le \frac{\pi}{\Omega_0}$.
2. As illustrated below, $\frac{1.5\pi}{\Omega_0} \le T_2 \le \frac{2\pi}{\Omega_0}$.



3. As illustrated below, $T_2 = \frac{3\pi}{\Omega_0}$.

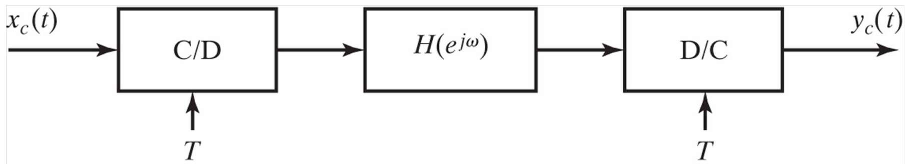


The frequency response of the filter that is needed to recover $x_c(t)$ is shown below.



4. *Discrete-time processing of continuous-time signals, frequency mapping.* The system in the figure below is used to filter continuous time music signals using a sampling rate of

16 kHz.



$H(e^{j\omega})$ is an ideal lowpass filter with a cutoff of $\pi/2$. If the input has been bandlimited such that $X_c(j\Omega) = 0$ for $|\Omega| > \Omega_c$, how should Ω_c be chosen so that the overall system is linear time-invariant (LTI)?

Solution: To avoid aliasing when sampling, we generally sample at the Nyquist rate, or twice the maximum frequency. Using this assumption, the input should be bandlimited to 8 kHz. However, in this system, the discrete-time filter is a lowpass filter so that we can have some aliased components as long as those are rejected by the filter. The filter cutoff is at normalized frequency of $\frac{\pi}{2}$ (or $f = \frac{1}{4}$) which can be mapped back to the frequency of the continuous-time system:

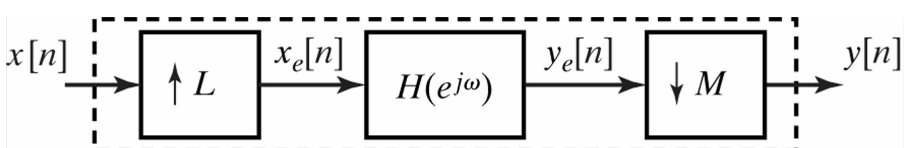
$$w = \Omega T$$
$$f = \frac{F}{F_s}$$

A discrete-time filter cutoff frequency of $f = \frac{1}{4}$ corresponds to a frequency of the continuous-time signal of

$$f = \frac{F}{F_s}$$
$$\frac{1}{4} = \frac{F}{16} \text{ kHz}$$
$$F = 4 \text{ kHz}$$

Since the signal is sampled at 16 kHz, the spectrum repeats every 16 kHz so as long as the bandwidth is less than 16 kHz - 4 kHz = 12 kHz the aliased components will not be in the passband of the lowpass filter. Therefore, for this overall system to be LTI, the input should be bandlimited to 12 kHz.

5. *Multirate signal processing.* Consider the discrete-time system shown below where L and M are positive integers.



The output of the upsampling block is

$$x_e[n] = \begin{cases} x[n/L], & n = kL \\ 0, & \text{otherwise} \end{cases}$$

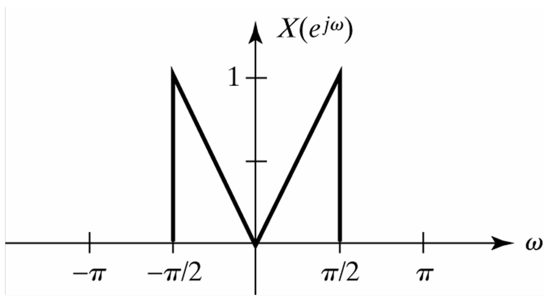
The output of the downsampling block is

$$y[n] = y_e[nM]$$

The frequency response of the discrete-time of the discrete-time filter is

$$H(e^{j\omega}) = \begin{cases} M, & |\omega| \le \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\omega| \le \pi \end{cases}$$

Assume that $X(e^{j\omega}$, the DTFT of $x[n]$, is real and is shown in the figure below.

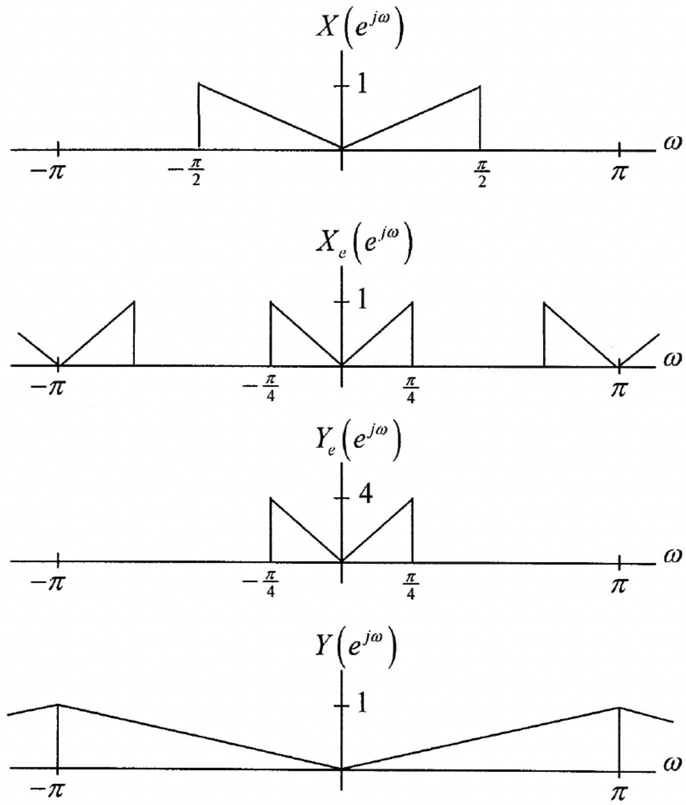


(a) Assume that $L = 2$ and $M = 4$. Sketch the the spectrum of the signals at each stage of the system: $X_e(e^{j\omega})$, $Y_e(e^{j\omega})$, and $Y(e^{j\omega})$. Label the salient amplitudes and frequencies.

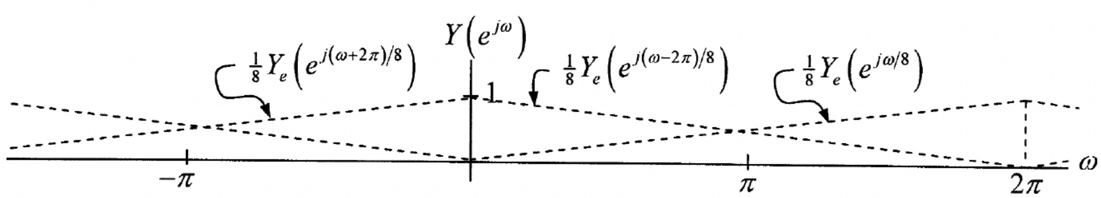
(b) Now assume that $L = 2$ and $M = 8$. Sketch the output spectrum, $Y(e^{j\omega})$. What can you infer from the spectrum about output in the time domain, $y[n]$?

Solution:

With $L=2$ and $M=4$,



With $L=2$ and $M=8$, $X_e(e^{j\omega})$ and $Y_e(e^{j\omega})$ remain as in part A, except that $Y_e(e^{j\omega})$ now has a peak value of 8. After expanding we have



We see that $Y(e^{j\omega})=1$ for all ω. Inverse transforming gives $y[n]=\delta[n]$ in this case.

6. *Analog-to-digital conversion: quantizer step-size.* How many bits are required in the A/D converter if the discrete-time signal

$$x(n) = 6.35\cos((\pi/10)n)$$

is quantized with a resolution (step size, Δ) of

(a) $\Delta = 0.1$

(b) $\Delta = 0.02$

Solution:

(a) $\Delta = 0.1$ If R is the range of the quantizer and the word length we use for each sample is $b+1$ bits, then the resolution of the quantizer is given by:

$$\Delta = \frac{\mathrm{R}}{2^{b+1}}$$

The range is the peak-to-peak value of our signal that goes from $+6.35$ to -6.35 so $R = 12.7$ and we have a resolution of $\Delta = 0.1$:

$$\Delta = 0.1 = \frac{\mathrm{R}}{2^{b+1}} = \frac{12.7}{2^{\mathrm{b}+1}}$$

so $2^{\mathrm{b}+1} = 12.7/0.1$ and $b+1 \geq \log_2(12.7/0.1)$ and $b+1 = 7$ bits A/D.

(b) $\Delta = 0.02$

$$\Delta = \frac{\mathrm{R}}{2^{b+1}}$$

The range is the peak-to-peak value of our signal that goes from $+6.35$ to -6.35 so $R = 12.7$ and we have a resolution of $\Delta = 0.1$:

$$\Delta = 0.02 = \frac{\mathrm{R}}{2^{b+1}} = \frac{12.7}{2^{\mathrm{b}+1}}$$

so $2^{\mathrm{b}+1} = 12.7/0.02$ and $b+1 \geq \log_2(12.7/0.02)$ and $b+1 = 10$ bits A/D.