Probability Models and Random Variables

Deterministic and Random Signals

Deterministic signal: specified by a explicitly by a formula or implicitly by difference equation driven by deterministic signal:

$$
\begin{aligned}
x[n] &= \cos \omega_0 n \\
y[n] &= -ay[n-1] + bu[n] \\
y[n] &= -ax[n]y[n-1]
\end{aligned}
$$

A *random* or *stochastic* signal is governed by probability.

Probability Theory

- *experiment:* two or more possible outcomes
- *event:* combination of one or more outcomes
- *probability:* every event assigned a number between 0 and 1 called the probability of the event; probabilities of all possible events must sum to one
- *random variable:* function that maps each outcome of an experiment to a number
- *random process:* sequence of random variables
- *realization:* set of measurements of a random process (or a *sample function*)
- *ensemble:* set of all possible realizations

Probability Density Function (pdf)

Let $\mathbf{x}$ be a continuous random variable, this is the cumulative distribution function (CDF):

$$P_{\mathbf{x}}(x) = P(\mathbf{x} \leq x)$$

Note that $P_{\mathbf{x}}(-\infty) = 0$ and $P_{\mathbf{x}}(\infty) = 1$.

The *probability density function* (pdf), $p_{\mathbf{x}}(x)$, is defined as

$$p_{\mathbf{x}}(x) = \frac{dP_{\mathbf{x}}(x)}{dx}$$

The distribution function is an integral of the probability density function:

$$P_{\mathbf{x}}(x) = \int_{-\infty}^{x} p_{\mathbf{x}}(\xi)\ d\xi$$

Probability Density Function (pdf) (Continued)

The pdf has unit area,

$$\int_{-\infty}^{\infty} p_{\mathbf{x}}(\xi)\ d\xi = P_{\mathbf{x}}(\infty) = 1.$$

The probability that an observed random variable is in the interval $[a, b)$ is

$$P(a \leq \mathbf{x} < b) = \int_{a}^{b} p_{\mathbf{x}}(\xi)\ d\xi = P_{\mathbf{x}}(b) - P_{\mathbf{x}}(a)$$

Example PDF: Uniform Distribution

$$p_{\mathbf{x}}(x) = \begin{cases} 1/(b-a), & a \leq x < b \\ 0, & \text{otherwise} \end{cases} \qquad E\{\mathbf{x}\} = \frac{b+a}{2} \quad \mathrm{var}\{\mathbf{x}\} = \frac{1}{12}(b-a)^2$$

Averages of Random Variables

The *expected value*, or *mean*, is formally defined:

$$m_{\mathbf{x}} = E\left\{\mathbf{x}\right\} = \int_{-\infty}^{\infty} x \; p_{\mathbf{x}}(x) \; dx$$

Function of a random variable:

$$E\left\{g(\mathbf{x})\right\} = \int_{-\infty}^{\infty} g(\xi) \; p_{\mathbf{x}}(\xi) \; d\xi$$

Variance (average squared distance from the mean)

$$\mathrm{var}\{\mathbf{x}\} = \sigma_{\mathbf{x}}^2 = \int_{-\infty}^{\infty} (x - m_{\mathbf{x}})^2 \; p_{\mathbf{x}}(x) \; dx = E\left\{\mathbf{x}^2\right\} - m_{\mathbf{x}}^2$$

$\sigma_{\mathbf{x}}$ is the standard deviation or the root mean square value

Expected Value and Variance Properties

Let a and b be (nonrandom) constants. Then:

$$\begin{aligned} E\left\{a\mathbf{x}\right\} &= aE\left\{\mathbf{x}\right\} \\ \mathrm{var}\left\{a\mathbf{x}\right\} &= a^2\mathrm{var}\left\{\mathbf{x}\right\} \\ E\left\{\mathbf{x}+a\right\} &= E\left\{\mathbf{x}\right\}+a \\ \mathrm{var}\left\{\mathbf{x}+a\right\} &= \mathrm{var}\left\{\mathbf{x}\right\} \end{aligned}$$

Suppose a random variable $\mathbf{z}$ has zero mean and unit variance. Then the random variable $\mathbf{x} = a\mathbf{z} + b$ has mean and variance:

$$\begin{aligned} E\left\{\mathbf{x}\right\} &= E\left\{a\mathbf{z}+b\right\} = aE\left\{\mathbf{z}\right\}+b = b \\ \mathrm{var}\left\{\mathbf{x}\right\} &= E\left\{(a\mathbf{z}+b-b)^2\right\} = a^2E\left\{\mathbf{z}^2\right\} = a^2 \end{aligned}$$

Normal, or Gaussian, Distribution

The standard normal distribution,

$$p_{\mathbf{z}}(z) = \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}$$

can be shown to have zero mean and unit variance. The random variable $\mathbf{x} = \sigma\mathbf{z} + \mu$ has mean μ and variance σ^2. The pdf of this random variable is also normal or Gaussian:

$$p_{\mathbf{x}}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty < x < \infty$$

Physical Interpretations

Assume the random process $x[n]$ is a physical signal, with units of volts.

- The expected value, $m_{\mathbf{x}}$, is the baseline, or "DC level", measured in volts.
- The standard deviation, σ_x, is the RMS value, in volts. This is the value measured by an AC voltmeter.
- The variance, σ_x^2, has units of volts^2. Recall V^2/R has units of power (watts). Thus, we may interpret σ_x^2 as the AC power dissipated in a 1Ω resistor.

Signal-to-Noise Ratio

We use power to compare the strengths of a desired signal and a background noise with the signal power-to-noise power ratio, or SNR:

$$\mathrm{SNR} = \frac{P_{\mathrm{signal}}}{P_{\mathrm{noise}}} = \frac{\sigma^2_{\mathrm{signal}}/R_{\mathrm{load}}}{\sigma^2_{\mathrm{noise}}/R_{\mathrm{load}}} = \frac{\sigma^2_{\mathrm{signal}}}{\sigma^2_{\mathrm{noise}}}$$

or, in decibels,

$$\mathrm{SNR}_{\mathrm{dB}} = 10\,\log_{10}\left(\frac{\sigma^2_{\mathrm{signal}}}{\sigma^2_{\mathrm{noise}}}\right)\,\mathrm{dB}$$

SNR is also defined as a ratio of RMS voltages,

$$\mathrm{SNR} = \frac{\sigma_{\mathrm{signal}}}{\sigma_{\mathrm{noise}}} \implies \mathrm{SNR}_{\mathrm{dB}} = 20\,\log_{10}\left(\frac{\sigma_{\mathrm{signal}}}{\sigma_{\mathrm{noise}}}\right)\mathrm{dB}$$

Random Signals: Statistical Averages

Discrete-Time Random Process

- Random process is an indexed family of random variables $\{\mathbf{x}_n\}$
- Set of all possible sequences is an *ensemble*, or the random process
- Each sample value of $x[n]$ governed by a probability law
- probability density function: $p_{\mathbf{x}_n}(x_n, n)$
- joint probability density: $p_{\mathbf{x}_n,\mathbf{x}_m}(x_n, n, x_m, m)$

Stationary Random Process: When all probability distributions are invariant to translation of time axis:

$$p_{\mathbf{x}_n,\mathbf{x}_m}(x_n, n, x_m, m) = p_{\mathbf{x}_{n+k},\mathbf{x}_{m+k}}(x_{n+k}, n+k, x_{m+k}, m+k)$$

This is *stationary in the strict sense* as the joint PDFs of sets of random variables are identical, even as each random variable in the second set were displaced in time from the first set by amount k.

Statistical (Ensemble) Averages: Mean, Variance

Average / Mean (expected value):

$$m_{\mathbf{x}_n} = E\{\mathbf{x}_n\} = \int_{-\infty}^{\infty} x p_{\mathbf{x}_n}(x,n)dx$$

For uncorrelated (or linearly independent) random variables:

$$E\{\mathbf{x}_n \mathbf{y}_m\} = E\{\mathbf{x}_n\}E\{\mathbf{y}_m\}$$

Mean-square value or average power (average of $|\mathbf{x}_n|^2$):

$$E\{|\mathbf{x}_n|^2\} = \int_{-\infty}^{\infty} |x|^2 p_{\mathbf{x}_n}(x,n)dx$$

Variance (mean-square value of $\mathbf{x}_n - m_{x_n}$):

$$\sigma_{\mathbf{x}_n}^2 = E\{|\mathbf{x}_n - m_{x_n}|^2\} = E\{|\mathbf{x}_n|^2\} - |m_{x_n}|^2$$

Statistical (Ensemble) Averages: Autocorrelation and Cross-correlation

Autocorrelation, measure of dependence between random process at different times:

$$\phi_{xx}[n,m] = E\{\mathbf{x}_n \mathbf{x}_m^*\} = \int_{\infty}^{\infty}\int_{\infty}^{\infty} x_n x_m^* p_{\mathbf{x}_n,\mathbf{x}_m}(x_n, n, x_m, m)\, dx_n dx_m$$

Autocovariance:

$$\gamma_{xx}[n,m] = E\{(\mathbf{x}_n - m_{x_n})(\mathbf{x}_m - m_{x_m})^*\} = \phi_{xx}[n,m] - m_{x_n} m_{x_m}^*$$

Cross-correlation, measure of dependence between two different random signals:

$$\phi_{xy}[n,m] = E\{\mathbf{x}_n \mathbf{y}_m^*\}$$

Cross-Covariance:

$$\gamma_{xy} = E\{(\mathbf{x}_n - m_{x_n})(\mathbf{y}_m - m_{y_m})^*\} = \phi_{xy}[n,m] - m_{x_n} m_{y_m}^*$$

Examples With Random Signals

Continuous Sinusoidal Signal With Random Amplitude

Let $x(t) = A\cos(2\pi t)$ where A is a random variable. Find the mean, autocorrelation, and autocovariance of $x(t)$.

$$m_{\mathbf{x}_t} = E\{A\cos(2\pi t)\} = E\{A\}\cos(2\pi t)$$

$$\begin{aligned}\phi_{xx}(t_n, t_m) &= E\{\mathbf{x}_{t_n}\mathbf{x}_{t_m}\} = E\{A\cos(2\pi t_n)A\cos(2\pi t_m)\} \\ &= E\{A^2\}\cos(2\pi t_n)\cos(2\pi t_m)\end{aligned}$$

$$\begin{aligned}\gamma_{xx}(t_n, t_m) &= \phi_{xx}(t_n, t_m) - m_{\mathbf{x}_{t_n}} m_{\mathbf{x}_{t_m}} \\ &= \left[E\{A^2\} - E\{A\}^2\right]\cos(2\pi t_n)\cos(2\pi t_m) \\ &= \mathrm{var}[A]\cos(2\pi t_n)\cos(2\pi t_m)\end{aligned}$$

Note: Mean is time-varying and correlation/covariance depends on times t_n and t_m

Continuous Sinusoidal Signal With Random Phase

If amplitude, A, and frequency, ω, are fixed quantities and θ is a uniformly distributed random variable in the interval $(0, 2\pi)$, here is the random signal:

$$\mathbf{x}_t = x(t) = A\cos(\omega t + \theta)$$

$$m_{\mathbf{x}_t} = E\{\mathbf{x}_t\} = E\{A\cos(\omega t + \theta)\} = \int_0^{2\pi} A\cos(\omega t + \theta)\left(\frac{1}{2\pi}\right) d\theta = 0 = m_{\mathbf{x}}$$

The autocorrelation of the random process (also autocovariance since zero mean):

$$\begin{aligned}
\phi_{xx}(t_n, t_m) &= E\{\mathbf{x}_{t_n}\mathbf{x}_{t_m}\} = E\{A\cos(\omega t_n + \theta)A\cos(\omega t_m + \theta)\} \\
&= \frac{A^2}{2\pi}\int_0^{2\pi} \frac{1}{2}\left[\cos(\omega(t_n - t_m)) + \cos(\omega(t_n + t_m) + 2\theta)\right] d\theta = \frac{A^2}{2}\cos(\omega(t_n - t_m))
\end{aligned}$$

$$\phi_{xx}(\tau) = \frac{A^2}{2}\cos(\omega\tau) \qquad \text{where } \tau = t_n - t_m \qquad \text{(only depends on time difference)}$$

Cross-Correlation of Signal Plus Noise Process

We observe a random signal, $y[n]$, which has a desired signal, x[n], plus noise, $v[n]$:

$$y[n] = x[n] + v[n]$$

The cross-correlation between the observed and desired signal, assuming that the noise and desired signal are uncorrelated:

$$\begin{aligned}
\phi_{xy}[n,m] &= E\{\mathbf{x}_n \mathbf{y}_m\} = E\{\mathbf{x}_n(\mathbf{x}_m + \mathbf{v}_m)\} \\
&= E\{\mathbf{x}_n \mathbf{x}_m\} + E\{\mathbf{x}_n \mathbf{v}_m\} \\
&= \phi_{xx}[n,m] + E\{\mathbf{x}_n\}E\{\mathbf{v}_m\} \\
&= \phi_{xx}[n,m] + m_{\mathbf{x}_n} m_{\mathbf{v}_m}
\end{aligned}$$

Random Signals:
Wide-Sense Stationarity, Ergodicity, Power Density Spectrum

Statistical (Ensemble) Averages: Wide-Sense Stationary

Stationary random process: statistical properties invariant to shift of time origin.

- First order PDF and averages are independent of time
- Second order joint PDF's and averages depend only on time difference

Therefore, mean and variance is independent of n:

$$m_x = E\{\mathbf{x}_n\}$$
$$\sigma_x^2 = E\{|(\mathbf{x}_n - m_x)|^2\}$$

Autocorrelation is one-dimensional sequence, function of time-difference (or lag) m:

$$\phi_{xx}[n+m, n] = \phi_{xx}[m] = E\{\mathbf{x}_{n+m}\mathbf{x}_n^*\}$$

If the probability distributions are not time invariant but the above equations for the averages sill hold, the random process is *wide-sense stationary*.

Time Averages and Ergodicity

A random process is *ergodic* if averages can be obtained from a single realization which allows us to estimate ensemble averages using time averages.

$$\hat{m}_x = \frac{1}{L} \sum_{n=0}^{L-1} x[n]$$

$$\hat{\sigma}_x^2 = \frac{1}{L} \sum_{n=0}^{L-1} |x[n] - \hat{m}_x|^2$$

$$\langle x[n+m]x^*[n] \rangle_L = \frac{1}{L} \sum_{n=0}^{L-1} x[n+m]x^*[n]$$

We will assume ergodicity and wide-sense stationarity unless otherwise specified.

Fourier Transform Representation: Power Density Spectrum

Spectral characteristic of a random process is Fourier transform of autocorrelation function (Wiener-Khintchine theorem):

$$\Phi_{xx}(e^{j\omega}) = \sum_{m=-\infty}^{\infty} \phi_{xx}[m]e^{-j\omega m} \qquad \longleftrightarrow \qquad \phi_{xx}[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{xx}(e^{j\omega})e^{j\omega m}d\omega$$

If zero-mean process ($m_x = 0$) and define $P_{xx}(\omega) = \Phi_{xx}(e^{j\omega})$, then at zero lag ($m = 0$), the average power of the random process is:

$$E\{|x[n]|^2\} = \phi_{xx}[0] = \sigma_x^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{xx}(e^{j\omega})d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\omega)d\omega$$

- $P_{xx}(\omega)$ is the *power density spectrum*,
- If $P_{xx}(\omega)$ is a constant, the random process is a *white noise process*
- $P_{xx}(\omega)$ is always real valued, also even for real signals: $P_{xx}(\omega) = P_{xx}(-\omega)$

Correlation and Covariance Sequences

Wide-Sense Stationary...A Frequent Assumption

The autocorrelation and autocovariance are functions of the two observation times, n and $n+m$. This simplifies with two assumptions, which are frequently valid:

1. The mean $m_{\mathbf{x}_n}$ is constant.
2. The correlation and covariance depend only on the time difference m, called the *lag*, rather than on the instants n and $n+m$ separately.

$$
\begin{aligned}
\phi_{xx}[m] &= E\left\{\mathbf{x}_{n+m}\mathbf{x}_n\right\} \\
\gamma_{xx}[m] &= E\left\{\left(\mathbf{x}_{n+m} - m_{\mathbf{x}}\right)\left(\mathbf{x}_n - m_{\mathbf{x}}\right)\right\} \\
&= \phi_{xx}[m] - m_{\mathbf{x}}^2
\end{aligned}
$$

A signal with these properties is called *wide-sense stationary* (wss); we make this assumption moving forward.

Interpreting Autocorrelation $\phi_{xx}[m]$

- Autocorrelation is a measure of rate of change of a random process
- Autocorrelation at lag $m = 0$ gives the average power of the random signal:

$$\phi_{xx}[0] = E\{\mathbf{x}_n^2\}$$

- The covariance at zero lag, $\gamma_{xx}[0]$, is the variance, σ_x^2
  - this is also the average power when the mean is zero
- Autocorrelation sequence has even symmetry:

$$\phi_{xx}[m] = \phi_{xx}[-m]$$

- Autocorrelation is a maximum at lag $m = 0$:

$$|\phi_{xx}[m]| \leq \phi_{xx}[0]$$

White Noise Process (contains all frequencies)

White Noise: Sequence of uncorrelated random variables $x[n]$ with zero-mean and constant variance σ_x^2 has this autocorrelation function:

$$\phi_{xx}[m] = \sigma_x^2 \delta[m]$$

With power spectral density (PSD):

$$P_{xx}(\omega) = \Phi_{xx}(e^{j\omega}) = \sum_{m=-\infty}^{\infty} \phi_{xx}[m] e^{-j\omega m}$$

$$= \sum_{m=-\infty}^{\infty} \sigma_x^2 \delta[m] e^{-j\omega m} = \sigma_x^2$$

The PSD is constant, therefore all frequencies are present in the same amount.

Filtered Random Process

Let $\mathbf{x}$ be a white noise process so all the $\mathbf{x}_n$ are mutually independent and uncorrelated. Form a new random process by the difference equation

$$\mathbf{y}_n = \mathbf{x}_n + \mathbf{x}_{n-1}$$

$\mathbf{y}_1 = \mathbf{x}_1 + \mathbf{x}_0$, $\mathbf{y}_2 = \mathbf{x}_2 + \mathbf{x}_1$, etc. Calculate a few covariances:

$$\begin{aligned}
\gamma_{yy}[1,1] &= E\{\mathbf{y}_1\mathbf{y}_1\} = E\left\{(\mathbf{x}_1 + \mathbf{x}_0)(\mathbf{x}_1 + \mathbf{x}_0)\right\} = E\left\{\mathbf{x}_1\mathbf{x}_1 + \mathbf{x}_0\mathbf{x}_1 + \mathbf{x}_1\mathbf{x}_1 + \mathbf{x}_0\mathbf{x}_1\right\} \\
&= \sigma_{\mathbf{x}}^2 + 0 + \sigma_{\mathbf{x}}^2 + 0 = 2\sigma_{\mathbf{x}}^2 \\
\gamma_{yy}[1,2] &= E\left\{(\mathbf{x}_1 + \mathbf{x}_0)(\mathbf{x}_2 + \mathbf{x}_1)\right\} = E\left\{\mathbf{x}_1\mathbf{x}_2 + \mathbf{x}_0\mathbf{x}_2 + \mathbf{x}_1\mathbf{x}_1 + \mathbf{x}_0\mathbf{x}_1\right\} \\
&= 0 + 0 + \sigma_{\mathbf{x}}^2 + 0 = \sigma_{\mathbf{x}}^2 \\
\gamma_{yy}[1,3] &= E\left\{(\mathbf{x}_1 + \mathbf{x}_0)(\mathbf{x}_3 + \mathbf{x}_2)\right\} = E\left\{\mathbf{x}_1\mathbf{x}_3 + \mathbf{x}_0\mathbf{x}_3 + \mathbf{x}_1\mathbf{x}_2 + \mathbf{x}_0\mathbf{x}_2\right\} \\
&= 0 + 0 + 0 + 0 = 0 \\
\gamma_{yy}[2,3] &= E\left\{(\mathbf{x}_2 + \mathbf{x}_1)(\mathbf{x}_3 + \mathbf{x}_2)\right\} == E\left\{\mathbf{x}_2\mathbf{x}_3 + \mathbf{x}_1\mathbf{x}_3 + \mathbf{x}_2\mathbf{x}_2 + \mathbf{x}_1\mathbf{x}_2\right\} \\
&= 0 + 0 + \sigma_{\mathbf{x}}^2 + 0 = \sigma_{\mathbf{x}}^2
\end{aligned}$$

Filtered Random Process (Continued)

Filtered white noise:

$$\mathbf{y}_n = \mathbf{x}_n + \mathbf{x}_{n-1}$$

Autocorrelation of output:

$$\phi_{yy}[m] = \{0, \sigma_x^2, \underline{2\sigma_x^2}, \sigma_x^2, 0\}$$

Impulse response of filter:

$$h[n] = \{1, 1\}$$

Note:

$$h[n] * h[-n] = \{0, 1, 2, 1, 0\}$$

Compare:

$$\begin{aligned} \phi_{xx}[m] &= \sigma_x^2 \delta[m] \\ h[n] * h[-n] &= \{0, 1, 2, 1, 0\} \\ \phi_{yy}[m] &= \{0, \sigma_x^2, \underline{2\sigma_x^2}, \sigma_x^2, 0\} = \sigma_x^2 \delta[m] * \{0, 1, 2, 1, 0\} \end{aligned}$$

Discrete-Time Random Signals and LTI Systems

Mean of LTI System Output

- $h[n]$ impulse response of stable LTI system
- $x[n]$ is real-valued input that is sample sequence of wide-sense stationary discrete-time random process with mean m_x and autocorrelation $\phi_{xx}[m]$
- $y[n]$ is output of LTI system an also a random process

$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Find the mean of the output process:

$$m_y[n] = E\{y[n]\} = \sum_{k=-\infty}^{\infty} h[k]E\{x[n-k]\} = m_x \sum_{k=-\infty}^{\infty} h[k] = m_y$$

- Output is also constant. In terms of the frequency response: $m_y = H(e^{j0})m_x$

Autocorrelation Function of LTI System Output

$$
\begin{aligned}
\phi_{yy}[n, n+m] &= E\{y[n]y[n+m]\} \\
&= E\left\{\sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} h[k]h[r]x[n-k]x[n+m-r]\right\} \\
&= \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r]E\left\{x[n-k]x[n+m-r]\right\} \\
&= \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r]\phi_{xx}[m+k-r] \\
&= \phi_{yy}[m]
\end{aligned}
$$

- Output is also wide-sense stationary for LTI system with wide-sense stationary input

Autocorrelation Function of LTI System Output (Continued)

Making a substitution, $l = r - k$, the autocorrelation of the output is

$$\begin{aligned}
\phi_{yy}[m] &= \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r]\phi_{xx}[m+k-r] \\
&= \sum_{l=-\infty}^{\infty} \phi_{xx}[m-l] \boxed{\sum_{k=-\infty}^{\infty} h[k]h[l+k]} \\
&= \sum_{l=-\infty}^{\infty} \phi_{xx}[m-l] c_{hh}[l]
\end{aligned}$$

where $c_{hh}[l]$ is the (deterministic) autocorrelation sequence of $h[n]$ defined as

$$c_{hh}[l] = \sum_{k=-\infty}^{\infty} h[k]h[l+k] \qquad \text{(Note that this is equivalent to } h[n] * h[-n]\text{)}$$

Using Fourier Transforms: Power Density Spectrum of Output Process

Since the autocorrelation of the output, is a convolution:

$$\phi_{yy}[m] = \sum_{l=-\infty}^{\infty} \phi_{xx}[m-l] c_{hh}[l]$$

we can represent this using Fourier transforms (we now assume $m_x = 0$):

$$\Phi_{yy}(e^{j\omega}) = C_{hh}(e^{j\omega}) \Phi_{xx}(e^{j\omega})$$

Since $C_{hh}(e^{j\omega}) = H(e^{j\omega}) H^*(e^{j\omega}) = |H(e^{j\omega})|^2$ we now have the power density spectrum of the output process:

$$\Phi_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2 \Phi_{xx}(e^{j\omega})$$

average power: $\quad E\{y^2[n]\} = \phi_{yy}[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|H(e^{j\omega})\right|^2 \Phi_{xx}(e^{j\omega})\, d\omega$

Using Fourier Transforms: Cross-Power Density Spectrum

Cross-correlation between the input and output of an LTI system:

$$\begin{aligned}
\phi_{yx}[m] &= E\{x[n]y[n+m]\} \\
&= E\left\{x[n]\sum_{k=-\infty}^{\infty} h[k]x[n+m-k]\right\} \\
&= \sum_{k=-\infty}^{\infty} h[k]\phi_{xx}[m-k]
\end{aligned}$$

Therefore, the cross-correlation is the convolution of the impulse response with the input autocorrelations sequence. This can be represented in the frequency domain:

$$\Phi_{yx}(e^{j\omega}) = H(e^{j\omega})\Phi_{xx}(e^{j\omega})$$

Examples of White Noise as Input to LTI Systems

Random Signals and LTI Systems in the Frequency Domain

Fourier transform of the autocorrelation and cross-correlation sequences:

$$\phi_{xx}[m] \overset{DTFT}{\longleftrightarrow} \Phi_{xx}(e^{j\omega})$$

$$\phi_{xy}[m] \overset{DTFT}{\longleftrightarrow} \Phi_{xy}(e^{j\omega})$$

Power spectral density of the output process:

$$\Phi_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2 \Phi_{xx}(e^{j\omega})$$

$$\text{average power:} \quad E\{y^2[n]\} = \phi_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H(e^{j\omega})\right|^2 \Phi_{xx}(e^{j\omega})\, d\omega$$

Cross-power spectral density of the output process:

$$\Phi_{yx}(e^{j\omega}) = H(e^{j\omega})\Phi_{xx}(e^{j\omega})$$

Example: Using White Noise to Represent Random Signals

The autocorrelation of a *white noise* signal is $\phi_{xx}[m] = \sigma_x^2 \delta[m]$ and the power spectrum is constant for all frequencies, ω. Let's assume the signal has zero mean:

$$\Phi_{xx}(e^{j\omega}) = \sigma_x^2 \qquad -\pi \leq \omega \leq \pi$$

If we have a system where $h[n] = a^n u[n]$ then the frequency response is:

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

The output of this system can represent any random signal with this power spectrum:

$$\begin{aligned}
\Phi_{yy}(e^{j\omega}) &= |H(e^{j\omega})|^2 \Phi_{xx}(e^{j\omega}) \\
&= \left| \frac{1}{1 - ae^{-j\omega}} \right|^2 \sigma_x^2 = \frac{\sigma_x^2}{1 + a^2 - 2a\cos(\omega)}
\end{aligned}$$

Example: Using White Noise to Identify An Unknown System

Using a zero-mean white-noise signal as an input to an LTI system, where $\phi_{xx}[m] = \sigma_x^2\delta[m]$ and $\Phi_{xx}(e^{j\omega}) = \sigma_x^2$, the cross-correlation between the output and input is:

$$\phi_{yx}[m] = \sum_{k=-\infty}^{\infty} h[k]\phi_{xx}[m-k] = \sum_{k=-\infty}^{\infty} h[k]\sigma_x^2\,\delta[m-k] = \sigma_x^2\,h[m]$$

Also,

$$\begin{aligned}\Phi_{yx}(e^{j\omega}) &= H(e^{j\omega})\Phi_{xx}(e^{j\omega}) \\ &= \sigma_x^2 H(e^{j\omega})\end{aligned}$$

By using white-noise as an input, and unknown system, $H(e^{j\omega})$ can be identified: cross-correlate input sequence with output sequence to obtain $\phi_{yx}[m]$; compute the Fourier transform and result is proportional to system frequency response.