Dartmouth College
Thayer School of Engineering

Signal Processing

Problem Set #3

1. *Sampling a continuous-time signal.* The continuous-time signal

$$x_c(t) = \sin(20\pi t) + \cos(40\pi t)$$

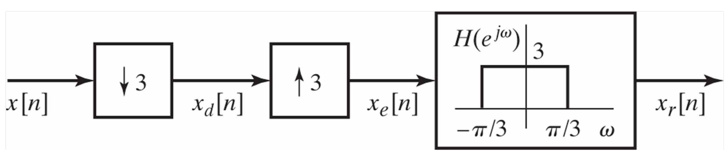
is sampled with a sampling period T to obtain the discrete-time signal, by replacing t with nT,

$$x[n] = \sin\left(\frac{\pi n}{5}\right) + \cos\left(\frac{2\pi n}{5}\right).$$

   (a) Determine a choice for T consistent with this information.

   (b) Is your choice of T in part (a) unique? If so, explain why. If not, specify another choice of T consistent with the information given.

2. *Changing the sample rate, avoiding aliasing.* Consider the system shown in the figure below.

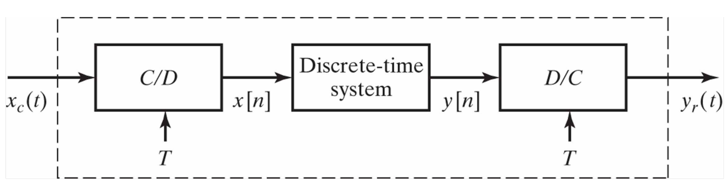


For each of the following input signals, $x[n]$, indicate whether the output, $x_r[n] = x[n]$.

   (a) $x[n] = \cos(\pi n/4)$

   (b) $x[n] = \cos(\pi n/2)$

3. *Frequency mapping between continuous-time and discrete-time.* Consider the system shown below, for discrete-time processing of continuous-time signals. The discrete-time system is an ideal lowpass filter with cutoff frequency $\pi/8$ radians/s.



   (a) If $x_c(t)$ is bandlimited to 5 kHz, what is the maximum value of the sample period, T that will avoid aliasing in the C/D converter?

(b) If $1/T = 10$ kHz, what will the cutoff frequency, in Hertz, of the effective continuous-time filter be?

(c) Repeat part (b) for $1/T = 20$ kHz.

4. *Analog-to-digital conversion: quantization levels and step-size.* A digital communication link carries binary-coded words representing samples of an input signal

$$x_a(t) = 3\cos(600\pi t) + 2\cos(1800\pi t)$$

(a) What is the Nyquist rate for the signal $x_a(t)$ (the minimum sample frequency in hertz to avoid aliasing).

(b) If the communication link is operated at 10,000 bits/sec and each input sample is quantized into 1024 different voltage levels, what is the actual sampling frequency, in hertz, used in this system?

(c) What are the frequencies of the resulting discrete-time signal $x[n]$ in radians/sample, ω.

(d) What is the step-size (resolution), Δ, of the quantizer? You can assume the signal is matched to the full range of of the quantizer.

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(a)

$$20\pi nT = \frac{\pi n}{5}, \quad \boxed{T = 1/100}$$

$$x[n] = \sin(20\pi nT) + \cos(40\pi nT)$$

$$= \sin\left(\frac{\pi}{5}n\right) + \cos\left(\frac{2\pi}{5}n\right)$$

(b)

no it is not unique

$$20\pi nT = \frac{\pi n}{5} + 2\pi n, \quad \boxed{T = \frac{11}{100}}$$

$$x[n] = \sin(20\pi nT) + \cos(40\pi nT)$$

$$= \sin\left(\frac{220\pi n}{100}\right) + \cos\left(\frac{440\pi n}{100}\right)$$

$$= \sin\left(\frac{\pi}{5}n + 2\pi n\right) + \cos\left(\frac{2\pi}{5}n + 4\pi n\right)$$

$$= \sin\left(\frac{\pi}{5}n\right) + \cos\left(\frac{2\pi}{5}n\right)$$

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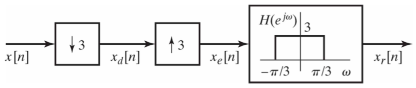

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(a)

$x_d[n] = x[3n] = \cos\left(\frac{3\pi n}{4}\right)$ does not alias

$$x_e[n] = x_d[n/3] = \begin{cases} \cos\left(\frac{\pi n}{4}\right) & , |n| = 0, L, 2L, \ldots \\ 0 & , \text{otherwise} \end{cases}$$

$x_r[n] = \cos\left(\frac{\pi n}{4}\right)$ due to low pass filter interpolation

$x_r[n] = \cos\left(\frac{\pi n}{4}\right)$ $\boxed{x[n] = x_r[n]}$

(b)

$x_d[n] = \cos\left(\frac{3\pi}{2} n\right)$ which aliases to $\cos\left(\frac{\pi}{2} n\right)$

$$x_e[n] = \begin{cases} \cos\left(\frac{\pi}{6} n\right) & , |n| = 0, L, 2L, \ldots \\ 0 & , \text{otherwise} \end{cases}$$

$x_r[n] = \cos\left(\frac{\pi}{6} n\right)$ by LPF $\boxed{x[n] \neq x_r[n]}$

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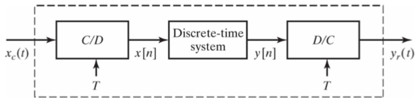

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(c) Repeat part (b) for $1/T = 20$ kHz.

(a)

$F_N = 5\,kHz \qquad F_S \geq 2F_N \qquad T = 1/F_S$

$$\boxed{T = \frac{1}{10000}\,s}$$

(b)

$$\omega_c = \frac{\pi}{8}, \quad \Omega_c = \frac{\omega_c}{T} = \frac{10000\pi}{8} = 1250\pi$$

$$F_c = \frac{\Omega_c}{2\pi} \qquad \boxed{F_c = 625\,Hz}$$

(c)

$$\Omega_c = \frac{20000\pi}{8} = 2500\pi$$

$$\boxed{F_c = 1250\,Hz}$$

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ⓐ

$F_N = 900\ \text{Hz}$, $F_S \geq 2F_N$

$$\boxed{F_S \geq 1800\ \text{Hz}}$$

ⓑ

$1024 = 2^{10}$ therefore ADC uses 10 bits per sample

$$100{,}000\ \frac{\text{bits}}{\text{sec}} \cdot \frac{1\ \text{sample}}{10\ \text{bits}} = 10{,}000 \text{ samples per sec}$$

$$\boxed{F_S = 10\ \text{kHz}}$$

ⓒ

$x_c(t) = 3\cos(2\pi F_1 t) + 2\cos(2\pi F_2 t)$, $F_1 = 600\ \text{Hz}$, $F_2 = 1800\ \text{Hz}$

$\Omega = 2\pi F$, $\Omega_1 = 1200\pi$, $\Omega_2 = 3600\pi$

$\omega = \Omega T$, $T = \frac{1}{F_S}$, $\boxed{\omega_1 = \frac{6\pi}{50} \ , \ \omega_2 = \frac{9\pi}{25}}$

ⓓ

$x_c(t) \in [-5, 5]$, therefore signal is 10V peak to peak

$$\text{step size} = \frac{10\text{V}}{1024\ \text{steps}} = \boxed{9.766\ \text{mV/step}}$$