DFT in Practice

$$x_j = \left(\frac{2\pi}{N}\right) j \equiv \Delta j$$

"sampling interval"

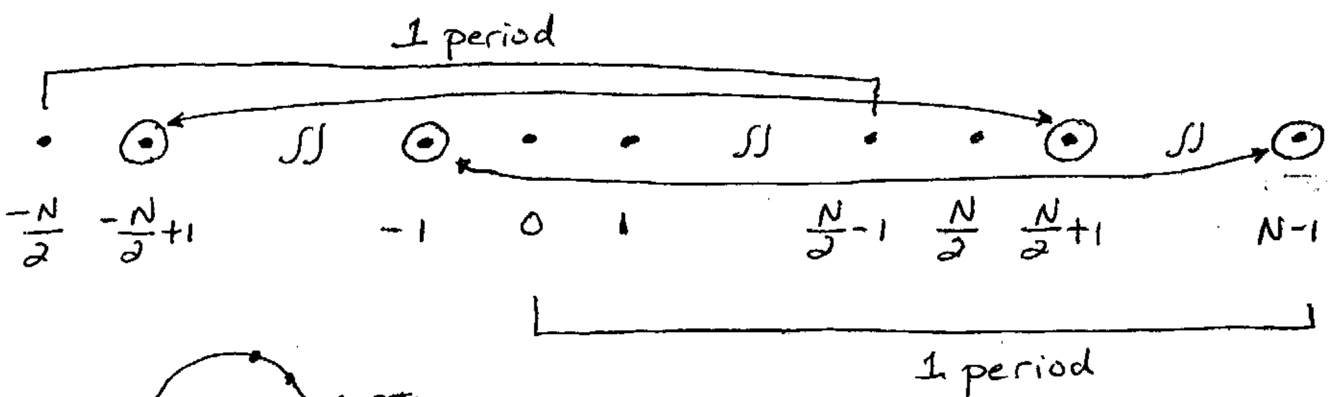
$$\tilde{u}_k = \frac{1}{N} \sum_{j=0}^{N-1} u_j \, e^{-ikx_j} \qquad \text{"Transform"}$$

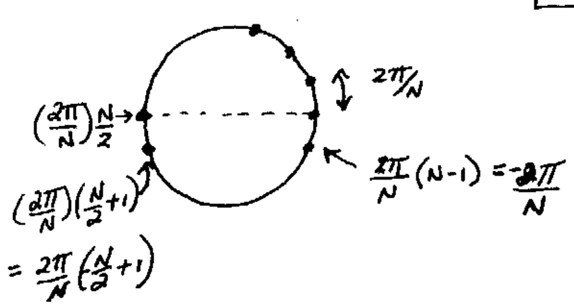
$$u_j = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{u}_k \, e^{ikx_j} \qquad \text{"Inverse Transform"}$$

Note essential similarity except: (i) exponent sign
(ii) $1/N$
(iii) limits

Because of periodicity Inverse Transform is same as

$$u_j = \sum_{k=0}^{N-1} \tilde{u}_k \, e^{ikx_j}$$





So have a single code for forward/inverse transform

$$\text{e.g.}\quad b_j = \underbrace{\sum_{k=0}^{N-1} a_k\, e^{SIGN * i k \Delta j}}$$

does both transform
(user supplies the $\frac{1}{N}$)

Then the discrete fourier coefficients, $\tilde{u}_k$ $k=0,1\ldots N-1$ correspond to real frequencies according to

$$k=0 \Rightarrow \text{zero frequency}$$

$$1 \le k \le \frac{N}{2}-1 \Rightarrow 0 < f < f_c$$

$$\frac{N}{2}+1 \le k \le N-1 \Rightarrow -f_c < f < 0$$

$$k = \frac{N}{2} \Rightarrow f = \pm f_c$$

where $f_c = \frac{1}{2\Delta}$ is the Nyquist critical frequency

DFT as Trapezoidal Rule approx. to FT:

$$\hat{u}_k = \frac{1}{2\pi}\int_0^{2\pi} u(x)\, e^{-ikx}\, dx$$

$$\approx \frac{1}{2\pi}\left[\frac{h}{2}\left(u_0 e^{-ikx_0} + u_N e^{-ikx_N} + 2\sum_{j=1}^{N-1} u_j e^{-ikx_j}\right)\right]$$

$$\approx \frac{1}{2\pi}\,\frac{2\pi}{2N}\left(2u_0 e^{-ikx_0} + 2\sum_{j=1}^{N-1} u_j e^{-ikx_j}\right)$$

$$\approx \frac{1}{N}\sum_{j=0}^{N-1} u_j e^{-ikx_j} = \tilde{u}_k$$

Gibbs Phenomenon

- occurs in truncated Fourier Series in neighborhood of discontinuity

- oscillatory behavior w/ max. amplitude nearest discontinuity tends to finite limit (1.08949)

- detailed derivation in text pp 45–53

- Similar behavior in $I_N u$ as for $P_N u$, although generally smaller (i.e. occurs in <u>both</u> truncation and interpolation)

- Ruins convergence rate, even away from discontinuity $\Rightarrow \mathcal{O}(1/N)$ ... i.e. slow decay of Fourier coefficients of a discontinuous function

- Typical to try to use smoothing functions; attenuate higher order (high frequency modes) coefficients

- Smoothing must be carefully choosen ... too strong and behavior is excessively smeared $\Rightarrow$ poor representation of true function $\Rightarrow$ $\mathcal{L}_N u = \sum_{k=-N/2}^{N/2-1} \sigma_k \hat{u}_k e^{ikx}$

Smoothed Fourier Series

smoothing factor