Spectral Methods

- view as weighted residual method
  involves basis (trial) and weighting (testing) functions

- choice of basis distinguishes Spectral methods
  from FE or FD

  - FE: local bases
    piecewise continuity
    non-differentiability

  - Spectral: "special functions"
    Global bases
    Complete differentiability
    Complete continuity

- choice of weighting functions
  - Galerkin (basis and weights the same)
  - Collocation (weights are translated
    Dirac delta functions
  - Tau (same spirit as Galerkin
    but BCs handled differently)

- Use of Spectral Methods governed by
  - Accuracy ... must be better than FE or FD for similar degrees of freedom
  - Efficiency ... must supply improved accuracy as efficiently as w/ conventional methods

- Key claim: "Spectral Accuracy"
  "Infinite Order Accuracy"
  "Exponential Convergence"

Sol'n converges to exact result faster than any finite power of the mesh spacing ... e.g. Δx^2, Δx^4 etc.

Example: $\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0$ $\quad 0 \le x \le 2\pi$

Periodic BCs

Arbitrary ICs

$$u^N(x,t) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} a_k(t)\, \phi_k(x)$$

basis functions

expansion coefficients

WR: $\int_0^{2\pi} \left(\frac{\partial u^N}{\partial t} - \frac{\partial u^N}{\partial x} \right) \psi_k \, dx = 0$

weighting functions

Most straightforward spectral Method ... trig polynomials

$$\phi_k(x) = e^{ikx} \;;\quad \psi_k = \frac{1}{2\pi} e^{-ikx} \qquad 2\pi \psi_k = \phi_k^*$$

Orthogonality built in $\Rightarrow \int_0^{2\pi} \phi_k(x)\, \psi_\ell(x)\, dx = \delta_{k\ell}$

$$\sum_k \frac{da_k}{dt} \int_0^{2\pi} \phi_k \psi_\ell \, dx = \sum_k a_k \, ik \int_0^{2\pi} \phi_k \psi_\ell \, dx \qquad \ell = -\frac{N}{2} \text{ to } \frac{N}{2}$$

$$\Rightarrow \boxed{\frac{da_k}{dt} = ik\, a_k \qquad k = -\frac{N}{2}, \dots \frac{N}{2} - 1}$$

Uncoupled set of ODEs, need $a_k(0)$ to solve

want $u^N(x,0) = u(x,0) = \sum_k a_k(0)\, \phi_k$

$$\int_0^{2\pi} u(x,0)\, \psi_\ell \, dx = \sum_k a_k(0) \int_0^{2\pi} \phi_k \psi_\ell \, dx$$

$$\underline{\underline{a_k(0) = \int_0^{2\pi} u(x,0)\, \psi_k \, dx}}$$

Compare w/ 2nd Order FD

$$\frac{dU_k}{dt} = \frac{U_{k+1} - U_{k-1}}{2h}$$

$$U(x) = U_k e^{ikx} \Rightarrow \begin{array}{l} U_{k+1} = U_k e^{ikh} \\ U_{k-1} = U_k e^{-ikh} \end{array}$$

$$U_{k+1} - U_{k-1} = U_k \left(e^{ikh} - e^{-ikh}\right) = U_k (2i \sin kh)$$

$$\frac{dU_k}{dt} = U_k \left(\frac{2i \sin kh}{2h}\right) = ik\left(\frac{\sin kh}{kh}\right) U_k$$

$$\frac{dU_k}{dt} = ik \underbrace{\left(\frac{\sin kh}{kh}\right)}_{\text{Discretization Error}} U_k$$

Ex: $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \Rightarrow U = \sum a_k(t) \phi_k(x)$

$$\phi_k = e^{ikx}, \quad \psi_k = \phi_k^*$$

$$\sum \frac{da_k}{dt} \int_0^{2\pi} \phi_k \psi_k \, dx = \sum a_k (ik)^2 \int_0^{2\pi} \phi_k \psi_k \, dx$$

Spectral $\Rightarrow$ $$\frac{da_k}{dt} = -k^2 a_k$$

2nd Order FD:

$$\frac{du_k}{dt} = \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} = \frac{\delta^2 u_k}{h^2}$$

$$\delta^2 = \left[e^{ikh} + e^{-ikh} - 2\right] = 2\cos kh - 2 = -4\sin^2\left(\frac{kh}{2}\right)$$

$$\frac{\delta^2}{h^2} = -\left[\frac{\sin^2\left(\frac{kh}{2}\right)}{\left(\frac{kh}{2}\right)^2}\right] k^2$$

$$\Rightarrow \quad \frac{du_k}{dt} = -k^2 \underbrace{\left[\frac{\sin^2\left(\frac{kh}{2}\right)}{\left(\frac{kh}{2}\right)^2}\right]}_{\text{Discretization Error}} u_k$$