Boundary Elements
An Introductory Course
SECOND EDITION

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Why Boundary Elements?

Engineers who have been exposed to finite elements may ask themselves why it is necessary to produce yet another computational technique. The answer is that finite elements have been proved to be inadequate or inefficient in many engineering applications and what is perhaps more important the method is in many cases cumbersome to use and hence difficult to integrate in Computer Aided Engineering systems. Finite Element Analysis is still a comparatively slow process due to the need to define or redefine meshes in the piece or domain under study.

Boundary elements [1] have emerged as a powerful alternative to finite elements particularly in cases where better accuracy is required due to problems such as stress concentration or where the domain extends to infinity. The most important features of boundary elements however is that it only requires discretization of the surface rather than the volume. Hence boundary element codes are easier to use with existing solid modellers and mesh generators. This advantage is particularly important for designing as the process usually involves a series of modifications which are more difficult to carry out using finite elements. Meshes can easily be generated and design changes do not require a complete remeshing.

This point is illustrated in figure 1 by two views of a turbine blade section, one discretized using a finite element code and the other with boundary elements. Notice the presence of a series of cooling ducts in the blade whose size, position and number have to be reviewed during the design process. Such a variation creates difficulties for finite elements as some elements may easily become distorted or have bad dimension ratios. The boundary element mesh instead is easy to modify. Figure 1 describes a two dimensional application; these problems are of course compounded for finite elements when working in three dimensions.

Boundary element meshes, especially three dimensional ones can easily be linked to CAE systems as the structure is defined using only the boundary. The discretization process is even simpler when using discontinuous elements, which are not admissible in finite elements. The mesh shown in figure 2 represents the surface discretization of one eighth of a problem, i.e. a cylinder with a cylindrical perforation across. Notice that the use of elements which sometimes do not meet at corners and are consequently discontinuous in terms of their variables, facilitates the meshing. In addition there is no need to use elements on the planes of symmetry.

Figure 3 describes a turbine blade and its base. Notice that discontinuous elements allow for a simple mesh grading. The reason why these elements are possible in boundary elements is explained in some of the chapters in this book. From the user's point of view they offer many advantages in terms of alterations of meshes and general versatility. Figure 4 shows some Von Mises surface stresses concentration at the root of the blade [2].

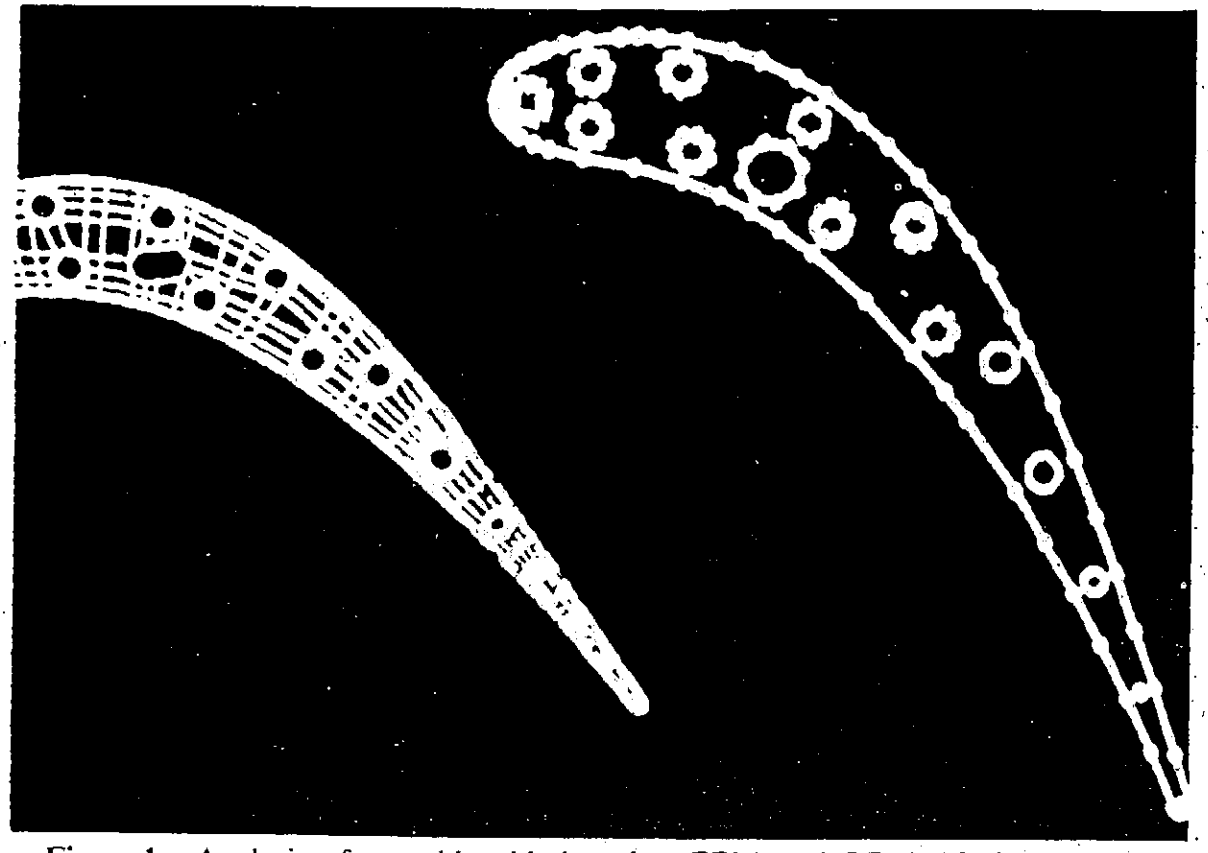


Figure 1. Analysis of a turbine blade using FEM and BEM. Notice that a variation in the configuration of cooling elements creates difficulties for the FE code (from a colour original)

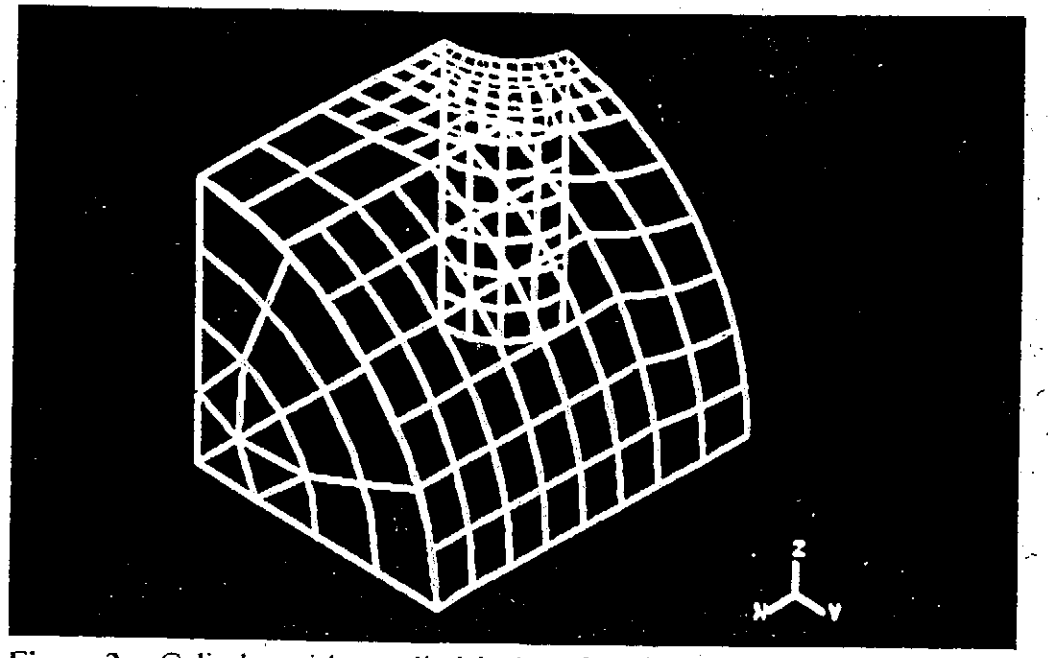


Figure 2. Cylinder with a cylindrical perforation. The boundary element mesh represents discretization of one eighth of the problem

More complex three dimensional structures such as the complete crankshaft model shown in figure 5 can be discretized relatively easily using a combination of continuous and discontinuous elements [2]. The model shown in the figure consists of approximately 2,000 surface boundary elements and each throw of the shaft has been represented by a boundary element zone (similar to a finite element substructure or super element) thus making a total of approximately 10,000 degrees of freedom. As only the surface of the shaft has been represented using elements the modelling time is quite rapid and the element mesh can be automatically created from a boundary model originating in a CAD system. The solution of problems of this size can nowadays be easily accomplished on the new generation of powerful engineering workstations. (This model was run on an IBM RS6000). This example demonstrates that computer time is no longer a primary concern in boundary element analysis, particularly as it is anticipated further increases in performance for workstations and all other computers within the next few years.

The model in figure 5 allows the user to analyse the overall behaviour of the crankshaft under different loadings. Afterwards part of the crankshaft can be

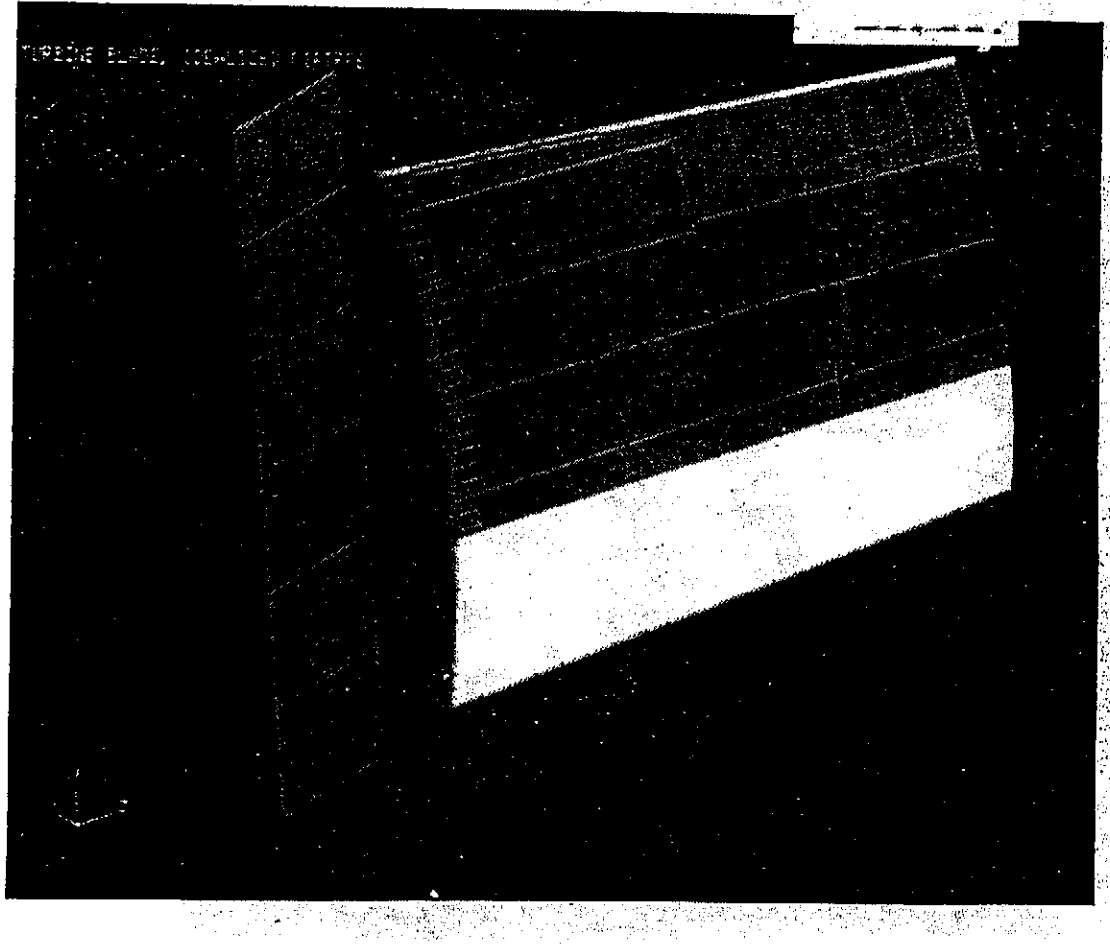


Figure 3. Discretization of a turbine blade using discontinuous elements (from a colour original)

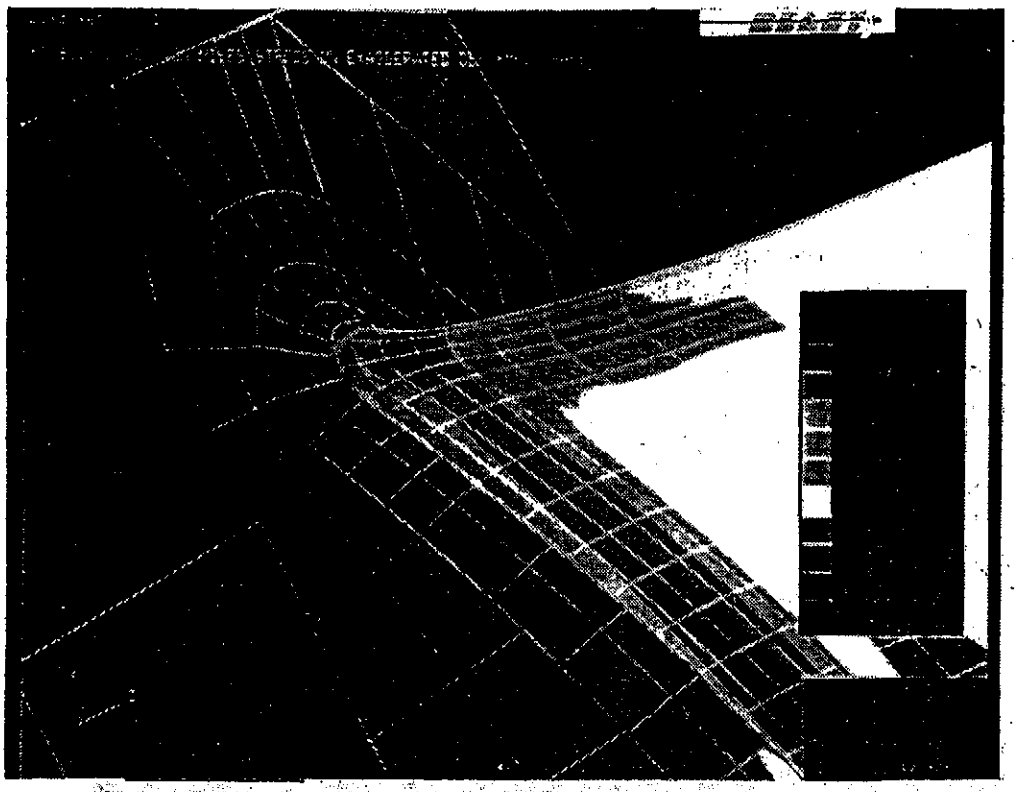


Figure 4. Von Mises surface stresses at the root of the blade (from a colour original)

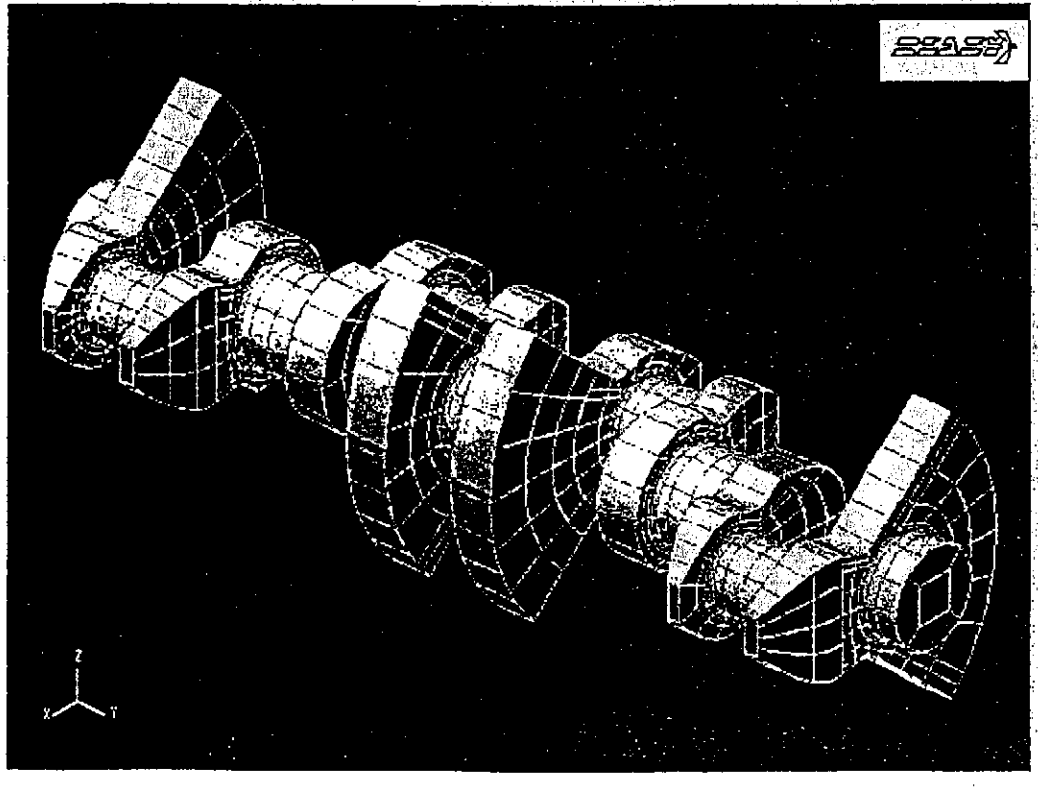


Figure 5. Complete crankshaft discretized into boundary elements

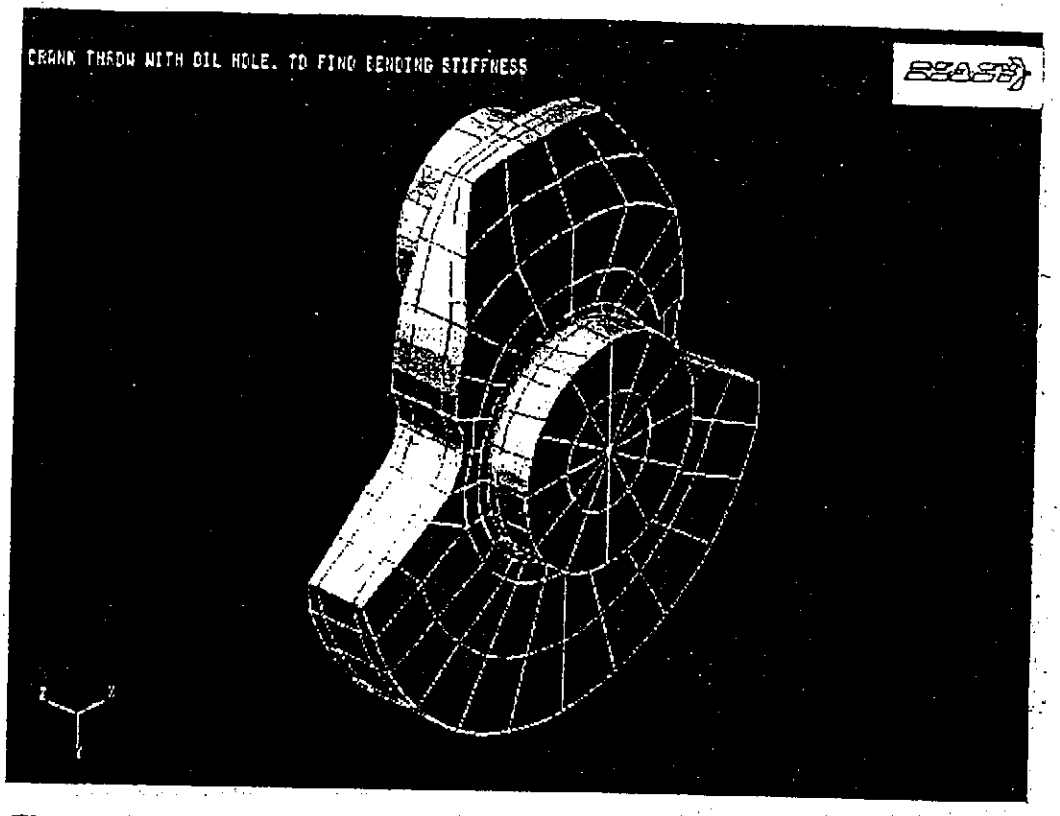


Figure 6. Solid view of the crank throw

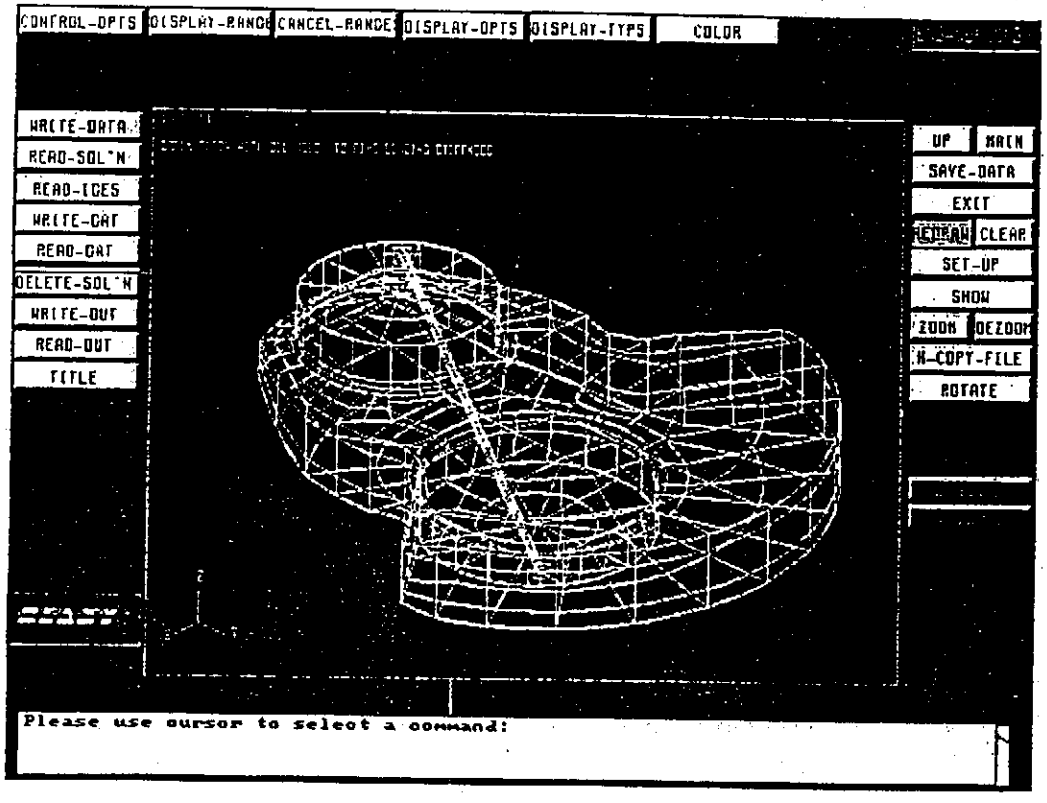


Figure 7. Wire frame view of crank showing oil passing through the crank

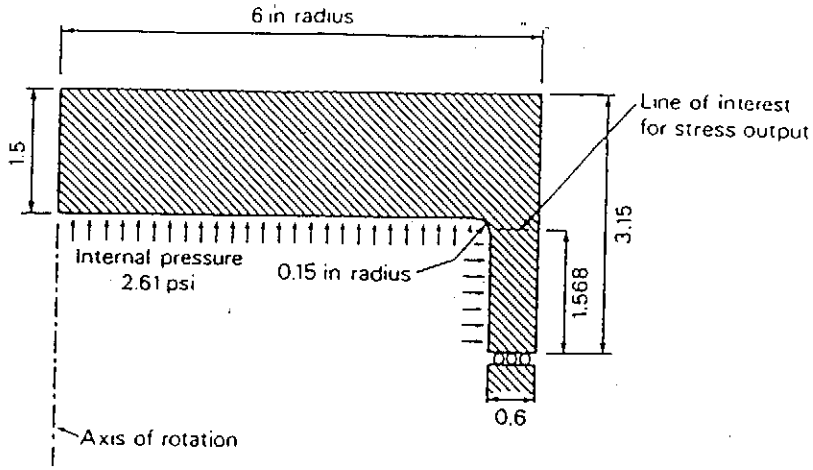
studied in more detail to find the effect of oil holes, fillets, etc. on the stress distribution. Figure 6 shows a model for part of a crankshaft discretized with a finer mesh than the one in figure 5. This crank throw has an oil hole as shown in the wire-frame diagram in figure 7. The interesting feature of this problem is that small details can be easily represented without causing a massive increase in the cost of the analysis. Note that the elements describing the oil hole in figure 7 only describe its surface and do not intersect the elements on the exterior part of the model except where the oil hole penetrates the surface.

It is evident from these examples that boundary elements are an ideal tool for engineering design mainly because it is easy to generate the data required to run a problem and carry out the modifications needed to achieve an optimum design. With computer costs declining while engineers' time becomes (or should become!) more expensive, the saving in engineers' time is of primary importance. (Also, engineers need relief from the dreary task of preparing finite element data.) More important still, any tool that can shorten the 'turn around' time taken by the analysis and design can bring forward the completion date of a project.

The future of BEM in engineering is promising and will continue to be so as long as the developers do not alienate the users by producing codes which are unreliable or cumbersome to use. Most of the advantages of BEM are related to its more complex mathematical foundations. This provides a high degree of versatility and accuracy in well-written codes but can have disastrous consequences in the case of poorly written BEM codes. The BEM is more susceptible to errors when the appropriate numerical techniques are not used and it is then important for developers to understand properly the theory of the method.

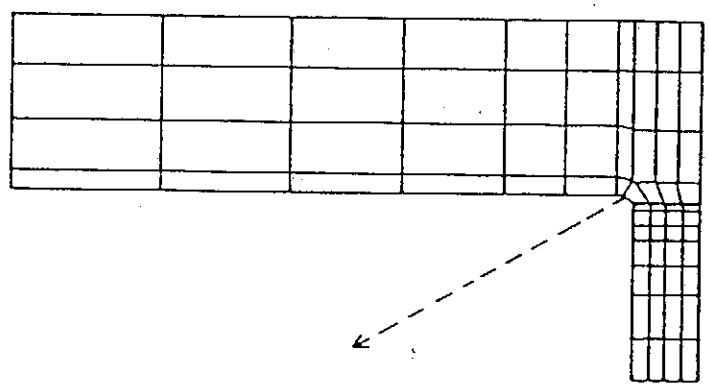
Although better computational performance is important in BEM, particularly for three dimensional problems, improvements in CPU times should not come at the expense of precision and accuracy. For instance, applying coarse numerical integration techniques to BEM codes can result in large savings in computer codes and give reasonable results in many cases. For other cases however the solution may be of very poor accuracy or give non-convergent results. This makes such codes unreliable.

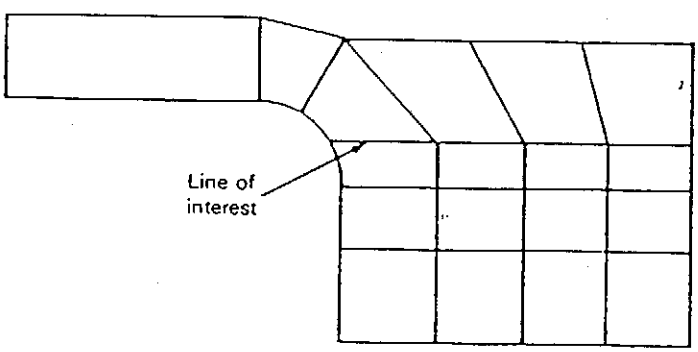
Another important advantage of BEM over FEM is when analysing problems with stress (or flux) concentration. Many such studies have now been carried out and they tend to demonstrate the high accuracy of boundary elements for problems such as re-entry corners, stress intensity problems and even fracture mechanics applications. It is not our intention in this introduction to review all these studies but rather to point out the difference in results that can be obtained using one or the other numerical method. As an illustration the finite element solutions found along a line in the neighbourhood of a re-entry corner (figure 8) of a pressure vessel is shown in figure 9. The problem was also analysed using a photo-elastic model and boundary elements. Results for a finite element mesh consisting of approximately 500 degrees of freedom (69 elements) and using eight nodes elements are compared against BEM solutions obtained using only 20 elements. It is evident from the figures that while the 69 elements finite element results show lack of equilibrium in the domain as well as on the boundary, reasonably accurate solutions were obtained using boundary elements. It was only



Dimensions in inches

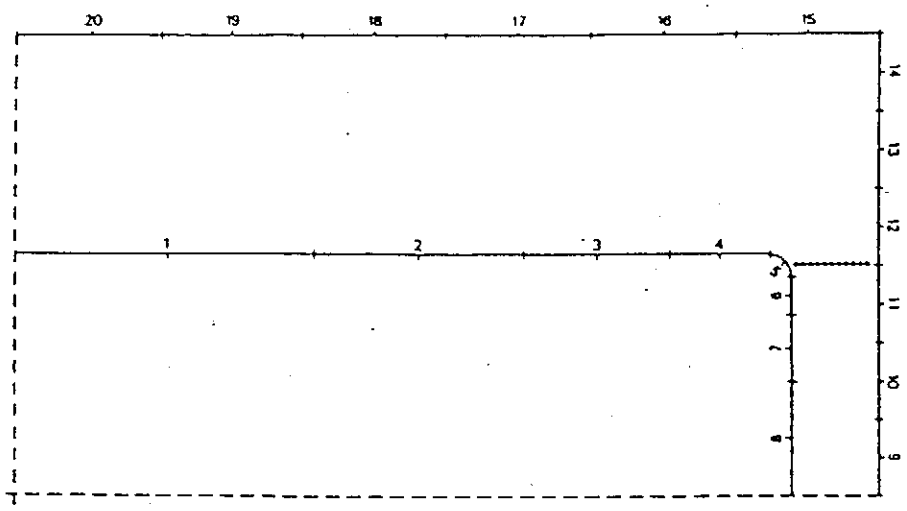
Geometry of the region under consideration





69 finite elements mesh
(with mid side nodes)
approximately 500 degrees of
freedom

Figure 8. Re-entry corner in pressure vessel



20 boundary elements mesh.
120 degrees of freedom

Figure 8 *continued*

when using a very refined finite element mesh that the FE results were in agreement with the boundary element and photo-elastic model solution i.e for results obtained using 240 elements (and 1,500 degrees of freedom.) For a full discussion of these results the reader is directed to references [3] to [5].

The development of more powerful hardware specially parallel and vector processing computers favour the use of BEM. These computers are better suited to deal with the fully populated matrices and the type of operations which are characteristics of boundary elements. The applications shown in figure 10 demonstrate the use of boundary elements for solving a non-linear problem, i.e. the contact analysis of a connecting rod. The model in this case was generated on a CAD system and the geometry automatically passed to the BEASY system and meshed. Notice that only one quarter of the rod needs to be discretized due to symmetry. The solution of this analysis is shown in figure 11 where contact surface gap between the pin and connecting rod is clearly seen.

Problems other than stress or temperature analysis can be solved using boundary elements. Typical applications include torsion, diffusion, seepage, fluid flow and electrostatics. Corrosion engineers have used the method to design better cathodic protection systems for offshore structures, ships and pipelines. Many of these structures are basically three dimensional and the region of interest extends to infinity. Consequently they could not be effectively analysed before the development of boundary elements. Early attempts to use finite differences or finite elements to solve these problems met with little success. For these cases the computer model has to represent the potential field around the structure, representing the shielding effect of the structural geometry and the effect of the different materials involved. Unlike a structural model the cathodic protection model is concerned with the seawater around the structure and the interface between the seawater and the structure. Hence the use of FEM to analyse the problem would require the subdivision of the seawater surrounding the structures which is a Herculean task.

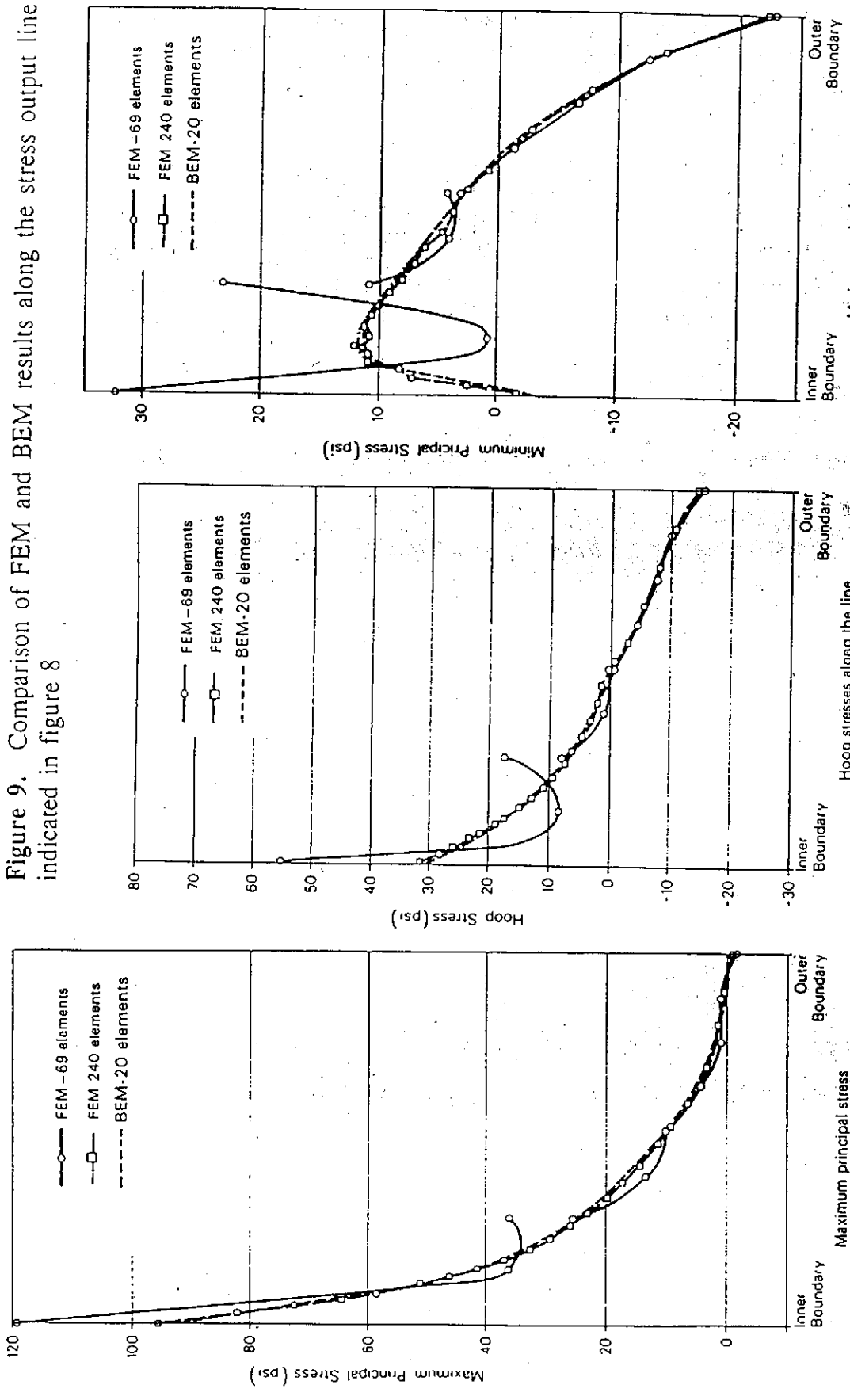


Figure 9. Comparison of FEM and BEM results along the stress output line indicated in figure 8

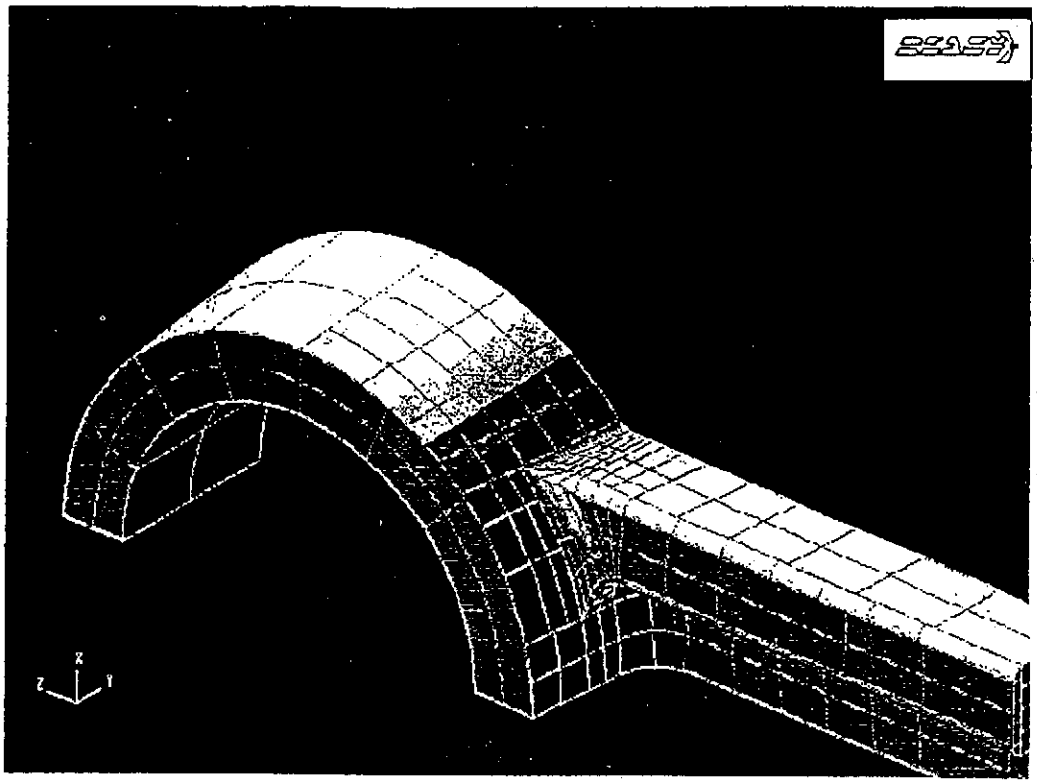


Figure 10. Quarter model of a connecting rod

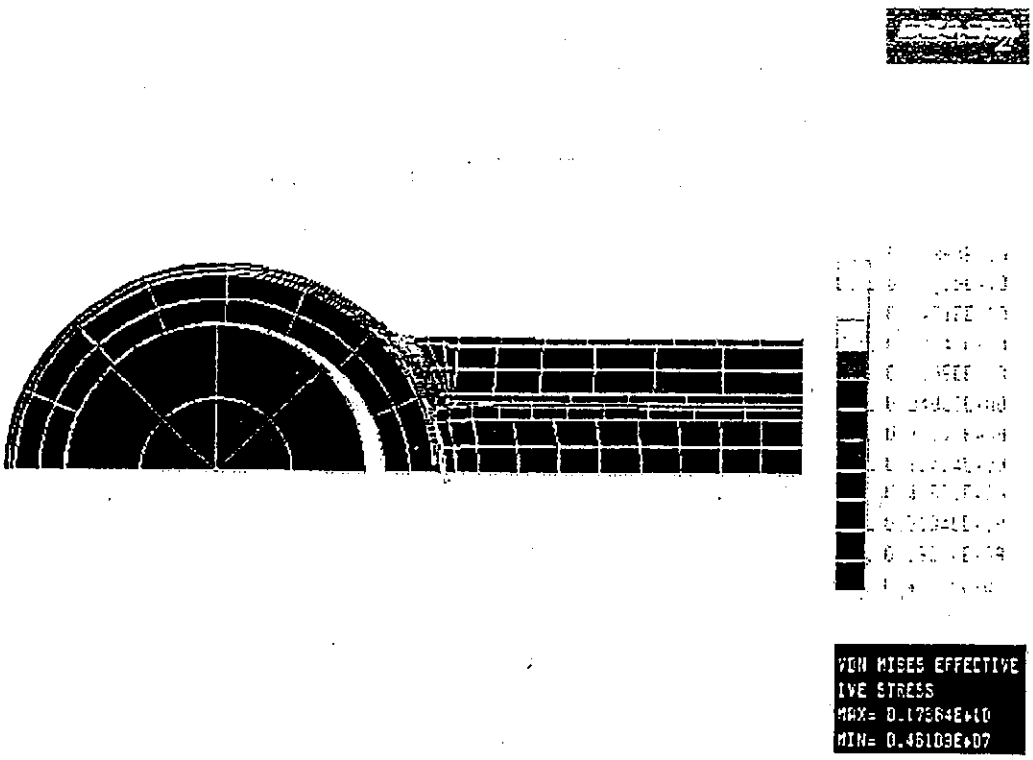


Figure 11. Final contact configuration after load application showing effective Von Mises stress distribution

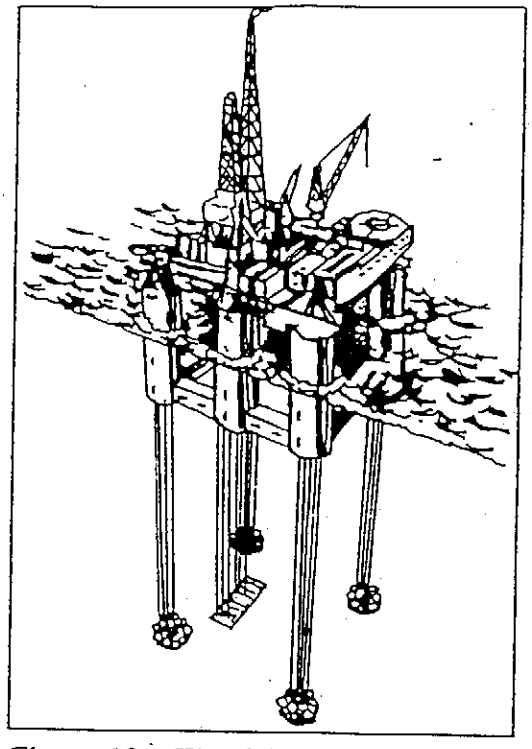


Figure 12. The CONOCO Hutton TLP (Tension Leg Platform)

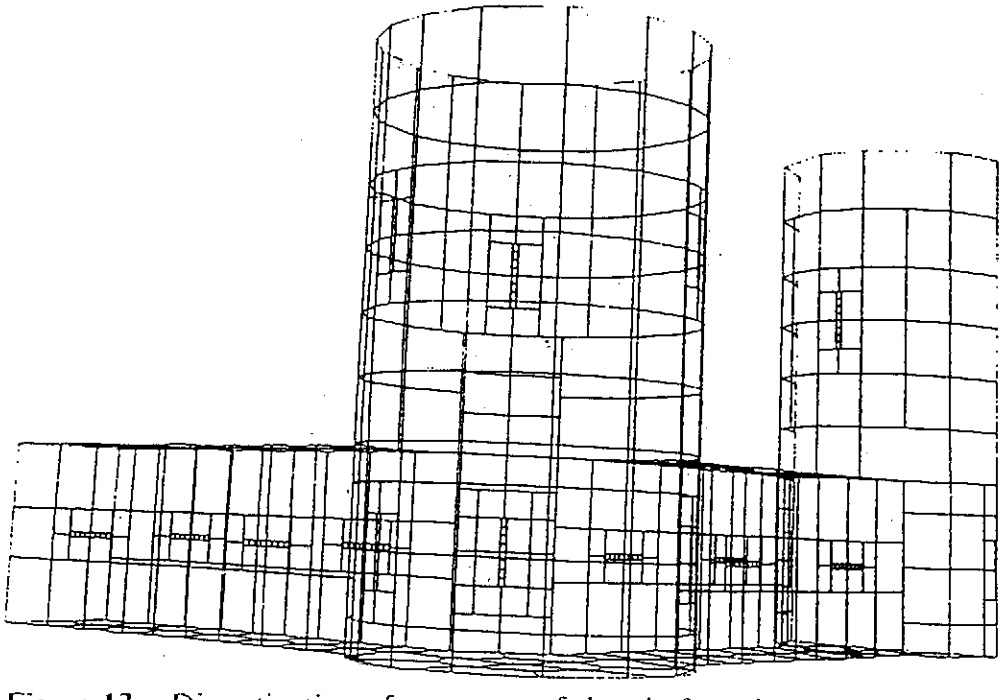


Figure 13. Discretization of a quarter of the platform into boundary elements

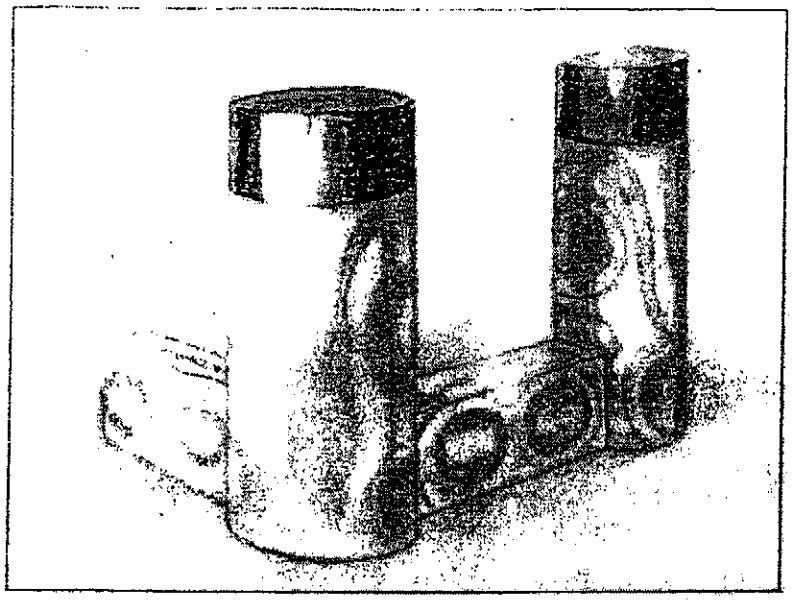


Figure 14. Model of the TLP showing contours of potentials

The use of boundary element method represents the only practical solution for this problem. The advantage of the method is that only the structure needs to be defined as the BEM automatically takes care of the field – i.e. the seawater – extending to infinity. Figure 12 shows the first three dimensional BEM cathodic protection application which was the study of the tension leg platform (TLP) built by CONOCO in the Hutton Field in the North Sea. Figure 13 shows the discretization of a quarter of the structure into the boundary elements used in the analysis and figure 14 the results obtained for the potentials on the surface for a particular configuration of the improved code system used. Since then the boundary element method has become the key to the successful and practical analysis of cathodic protection systems and further work has been carried out in this regard particularly at the Computational Mechanics Institute, Southampton, UK. A system is now available which allows the corrosion engineer to evaluate design options, look at problem areas, interpret experimental observations, optimize the design and predict with accuracy and confidence the degree of protection and life expectancy of a cathodic protection system.

The advances made in cathodic protection modelling using boundary elements are just one of the applications of the technique for systems extending to infinity. The method is nowadays extensively used in other problems with infinite or semi-infinite domains such as those occurring in geomechanics, ocean engineering, foundations, aerodynamics, flow through porous media and many others.

This brief introduction has attempted to point out the advantages of BEM for a wide variety of engineering problems and the reason why the method should be taught on an undergraduate as well as a graduate level. University courses should include the fundamentals of the method and provide workshops on applications while short courses with hands-on applications will help to bring the method to the attention of practising engineers. This book has been written to provide a

simple and up to date introduction to the method to help popularize the technique amongst engineers.

The future of BEM hinges on its acceptance by practising engineers, in particular as a design tool. Developers should aim to make the method more accessible to engineers by writing codes which are easy to use and by explaining the fundamentals of the method on the basis of engineering rather than mathematical concepts. This book has been written in a form that can be used as a textbook at undergraduate or graduate level and for the engineer in practice who wants to learn the fundamentals of the technique unaided. Of particular interest is the way in which the mathematics concepts are introduced and immediately applied in simple computer codes. These codes (4 for potential and 2 for elasticity) will facilitate the comprehension of BEM.

This book is based on the authors' many years experience as researchers and teachers of boundary elements. It is designed to teach in the most effective manner the fundamentals of the method rather than to attempt to demonstrate erudition on the subject. Many topics have been deliberately omitted to avoid confusing the reader. The essentials however are all here. It is now left to the reader to build on this knowledge.

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Chapter 1

Basic Concepts

1.1 Fundamental concepts

Consider a very simple differential equation applying in a one-dimensional domain x, from $x = 0$ to $x = 1$, i.e.,

$$\frac{d^2u}{dx^2} + \lambda^2 u - b = 0 \qquad \text{in } x \tag{1.1}$$

u is the function which governs the equation and we usually need to find it using a numerical technique which gives an approximate solution. λ^2 is a known positive constant and b is a known function of x.

The solution of equation (1.1) can be found by assuming a variation for u consisting of a series of known shapes (or functions) multiplied by unknown coefficients. These coefficients can then be found by forcing (1.1) to be satisfied at a series of points. This is the basis of the collocation (or point collocation to be precise) method and is essentially what one does when using finite differences. In finite elements instead the solution is found using the concept of distribution of error within the domain. This is somewhat a process of 'smoothing' and it is then not surprising that finite element solutions tend to have less 'noise' than finite difference ones.

The concept of distribution or weighting of a differential equation is not only valid for approximate solutions but it is a fundamental mathematical concept, which can be used in countless engineering applications. Engineers for instance are very familiar with the principle of virtual work which is usually formulated in terms of work done by internal and external forces. They are usually unaware however that the first 'demonstration' of the principle was proposed by Lagrange using the concepts of distributions, applying what are now called the 'Lagrangian' multipliers. These concepts are also essential to study the behaviour of the differential equations, and in particular the type of boundary conditions they require and which are consistent with them.

To understand what these concepts mean before proposing any approximation, one can consider another function w, arbitrary except for being continuous in the domain x and whose derivatives are continuous up to a required degree (the degree of continuity will vary with the problem as will be shown shortly). One can now multiply the whole of equation (1.1) by this w function and integrate on the domain x as follows:

$$\int_0^1 \left(\frac{d^2u}{dx^2} + \lambda^2 u - b \right) w \, dx = 0 \tag{1.2}$$

This operation is called an 'inner' product in mathematics and although does not imply any new concepts, allows us to investigate the properties of the governing equation. This is done by integrating by parts terms with derivatives in the above expression. In this case one can only 'manipulate' in this manner the first term, i.e. d^2u/dx^2, which gives

$$\int_0^1 \left\{ -\frac{du}{dx}\frac{dw}{dx} + (\lambda^2 u - b)w \right\} dx + \left[\frac{du}{dx} w \right]_0^1 = 0 \tag{1.3}$$

Notice that the integration by parts has produced two terms, one in the domain with first derivatives of u and w, and the other on the boundaries (which in this case are simply the two points $x = 0$, $x = 1$).

Furthermore, if the w function has sufficient degree of continuity one can integrate by parts again to obtain

$$\int_0^1 \left\{ u\frac{d^2w}{dx^2} + (\lambda^2 u - b)w \right\} dx + \left[\frac{du}{dx} w \right]_0^1 - \left[u\frac{dw}{dx} \right]_0^1 = 0 \tag{1.4}$$

Expression (1.4) is of course equivalent to (1.3) but here not only has one passed all derivatives to the newly defined w function but the two terms at $x = 0$ and $x = 1$ give us an insight into the boundary conditions required to solve the problem. In this case,

$$u \text{ or } \frac{du}{dx} \qquad \text{needs to be known at } x = 0 \text{ and } x = 1 \tag{1.5}$$

Notice that the w function which in principle was an arbitrary function with a certain degree of continuity can be made to satisfy certain boundary conditions if one wishes to do so. In the principle of virtual displacements for instance, arbitrary functions of this type are defined as virtual displacements but they are assumed to satisfy the homogeneous version of the displacement boundary conditions, i.e. they are set identically to zero at any points where the displacements are prescribed even if those displacements (represented by u) are not set to zero, i.e. $w \equiv 0$ on the parts of the boundary where u is given. This is done in order to eliminate terms of the type $\left[\frac{du}{dx} w \right]$ which give rise to a type of 'work' one does not wish to have. In general however one can assume that w and dw/dx can have values different from zero on the boundaries and this makes expression (1.4) more general.

The concept of an arbitrary function w used as a distribution function is related not only to virtual functions and consequently to virtual work but also to the idea of Lagrangian multipliers. These are functions of the w type defined in order to satisfy certain equations. They will be defined better in what follows.

Although equation (1.4) gives the user an insight into the type of boundary conditions required to solve the problem, these conditions have not yet been explicitly incorporated into the problem. In order to do so let us consider that the

boundary conditions are as follows:

$$\begin{aligned} u &= \bar{u} && \text{at } x = 0 \\ q &= \frac{du}{dx} = \bar{q} && \text{at } x = 1 \end{aligned} \tag{1.6}$$

where the derivatives of u are now defined as q and the terms with bars represent known values of the function and its derivatives. It is usual to call the first type of conditions in (1.6) 'essential' and those like q involving derivatives as 'natural'.

Substituting those values into (1.4) gives

$$\int_0^1 \left\{ u \frac{d^2w}{dx^2} + (\lambda^2 u - b)w \right\} dx + \{[\bar{q}w]_{x=1} - [qw]_{x=0}\} - \left\{ \left[u \frac{dw}{dx} \right]_{x=1} - \left[\bar{u} \frac{dw}{dx} \right]_{x=0} \right\} = 0 \tag{1.7}$$

It is now interesting to try to return to the original expression (1.2) by integrating by parts again, but this time passing the derivatives for w to u. The first integration gives,

$$\int_0^1 \left\{ -\frac{du}{dx}\frac{dw}{dx} + (\lambda^2 u - b)w \right\} dx + \left[u \frac{dw}{dx} \right]_{x=1} - \left[u \frac{dw}{dx} \right]_{x=0} + [\bar{q}w]_{x=1} - [qw]_{x=0} - \left[u \frac{dw}{dx} \right]_{x=1} + \left[\bar{u} \frac{dw}{dx} \right]_{x=0} = 0 \tag{1.8}$$

Notice that only the term in $[u\, dw/dx]_{x=1}$ disappears.

Furthermore the following expression results after carrying out a second integration,

$$\int_0^1 \left\{ \frac{d^2u}{dx} + (\lambda^2 u - b)w \right\} dx - \left[\frac{du}{dx} w \right]_{x=1} + \left[\frac{du}{dx} w \right]_{x=0} - \left[u \frac{dw}{dx} \right]_{x=0} + [\bar{q}w]_{x=1} - [qw]_{x=0} + \left[\bar{u} \frac{dw}{dx} \right]_{x=0} = 0 \tag{1.9}$$

Once again only one term disappears, in this case $[qw]_{x=0}$ – Notice that $q = du/dx$ as defined earlier. – Grouping the terms together one now arrives at an interesting expression, different from the original formula (1.4) i.e.

$$\int_0^1 \left\{ \frac{d^2u}{dx^2} w + (\lambda^2 u - b)w \right\} dx - [(q - \bar{q})w]_{x=1} + \left[(\bar{u} - u) \frac{dw}{dx} \right]_{x=0} = 0 \tag{1.10}$$

This expression implies that one is trying to enforce not only satisfaction of the

differential equation in x but the two boundary conditions. The w and dw/dx functions can be seen as Lagrangian multipliers.

Furthermore nothing has yet been said about approximations; the above expressions are valid for exact solutions as well. In other words the procedure describes a general tool for the investigation of differential equations.

1.2 The Poisson's Equation

An important equation in engineering analysis is the so-called Poisson equation which for two dimensions can be written as

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = b \qquad \text{in } \Omega \tag{1.11}$$

or

$$\nabla^2 u = b \qquad \text{in } \Omega \tag{1.12}$$

where $\nabla^2(\ ) = \dfrac{\partial^2(\ )}{\partial x_1^2} + \dfrac{\partial^2(\ )}{\partial x_2^2}$, is called the Laplace operator. x_1 and x_2 are the two coordinates and b is a known function of x_1, x_2. Ω is the domain on which the equation applies and is assumed to be bounded by Γ. The outward normal to the boundary is defined as n (figure 1.1).

The Poisson equation or its homogeneous form (i.e. $b = 0$) which is the Laplace equation, governs many types of engineering problems, such as seepage and aquifer analysis, heat conduction, diffusion processes, torsion, fluid motion and others. Consequently it is a very important equation in engineering analysis.

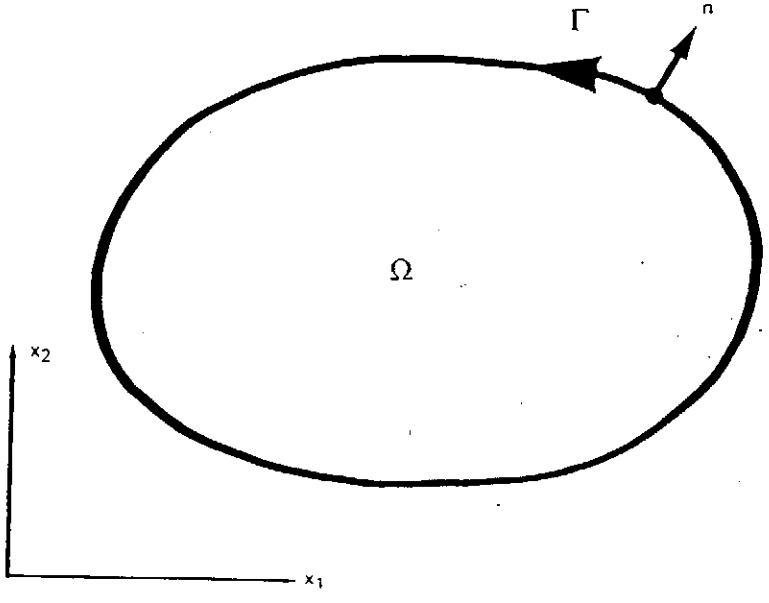


Figure 1.1 Domain under Consideration for Poisson Equation Basic Definitions

Here one can also introduce the idea of multiplying equation (1.12) by an arbitrary w function, continuous up to the second derivative. This gives,

$$\int_{\Omega} (\nabla^2 u - b) w \, d\Omega = 0 \tag{1.13}$$

Integrating by parts the terms in x_1 and x_2 gives

$$\int_{\Omega} \left(-\frac{\partial u}{\partial x_1}\frac{\partial w}{\partial x_1} - \frac{\partial u}{\partial x_2}\frac{\partial w}{\partial x_2} - bw \right) d\Omega + \int_{\Gamma} \frac{\partial u}{\partial n} w \, d\Gamma = 0 \tag{1.14}$$

In this case the integration by parts of the two terms produces the derivative of u with respect to the normal, i.e. $\partial u/\partial n$ which will later on be called q, i.e. $q = \partial u/\partial n$.

Integrating by parts again, one obtains,

$$\int_{\Omega} \left(\frac{\partial^2 w}{\partial x_1^2} u + \frac{\partial^2 w}{\partial x_2^2} u - bw \right) d\Omega + \int_{\Gamma} \frac{\partial u}{\partial n} w \, d\Gamma - \int_{\Gamma} u \frac{\partial w}{\partial n} d\Gamma = 0 \tag{1.15}$$

or

$$\int_{\Omega} \{(\nabla^2 w) u - bw\} \, d\Omega + \int_{\Gamma} \frac{\partial u}{\partial n} w \, d\Gamma - \int_{\Gamma} u \frac{\partial w}{\partial n} d\Gamma = 0 \tag{1.16}$$

Expression (1.16) is equal to (1.13) and hence one can write,

$$\int_{\Omega} (\nabla^2 u) w \, d\Omega = \int_{\Omega} (\nabla^2 w) u \, d\Omega + \int_{\Gamma} \frac{\partial u}{\partial n} w \, d\Gamma - \int_{\Gamma} u \frac{\partial w}{\partial n} d\Gamma = 0 \tag{1.17}$$

where the term in b has been eliminated as it appears on the two sides of the equation.

Equation (1.17) can also be expressed in the form known as the Green's theorem, i.e.

$$\int_{\Omega} \{(\nabla^2 u) w - (\nabla^2 w) u\} \, d\Omega = \int_{\Gamma} \left(\frac{\partial u}{\partial n} w - u \frac{\partial w}{\partial n} \right) d\Gamma \tag{1.18}$$

Although this theorem is in many cases given as the starting point for many engineering applications, including boundary element formulations, it is much more illuminating to use the concept of distribution as it illustrates the degree of continuity required of the functions and the importance of the accurate treatment of the boundary conditions. In this regard let us now consider that the Γ boundary of the Ω domain under study is divided into two parts, Γ_1 and Γ_2 ($\Gamma = \Gamma_1 + \Gamma_2$) such that,

$$\begin{aligned} u &= \bar{u} \qquad \text{on } \Gamma_1 \\ q &= \frac{\partial u}{\partial n} = \bar{q} \qquad \text{on } \Gamma_2 \end{aligned} \tag{1.19}$$

Hence equation (1.16) can now be written as,

$$\int_{\Omega} \{(\nabla^2 w)u - bw\}\, d\Omega + \int_{\Gamma_1} qw\, d\Gamma + \int_{\Gamma_2} \bar{q}w\, d\Gamma - \int_{\Gamma_1} \bar{u}\frac{\partial w}{\partial n}\, d\Gamma - \int_{\Gamma_2} u\frac{\partial w}{\partial n}\, d\Gamma = 0 \tag{1.20}$$

Once again one can integrate by parts to retrieve the original Laplacian $\nabla^2 u$ in order to see how the importance of the boundary conditions affect the equation. Integrating by parts once we have,

$$\int_{\Omega} \left\{ -\frac{\partial w}{\partial x_1}\frac{\partial u}{\partial x_1} - \frac{\partial w}{\partial x_2}\frac{\partial u}{\partial x_2} - bw \right\} d\Omega + \int_{\Gamma} \frac{\partial w}{\partial n} u\, d\Gamma + \int_{\Gamma_1} qw\, d\Gamma + \int_{\Gamma_2} \bar{q}w\, d\Gamma - \int_{\Gamma_1} \bar{u}\frac{\partial w}{\partial n}\, d\Gamma - \int_{\Gamma_2} u\frac{\partial w}{\partial n}\, d\Gamma = 0 \tag{1.21}$$

One can split the first integral on Γ into two terms (one on Γ_1 and the other on Γ_2), the second of which can be cancelled with the last integral in (1.21). This gives

$$\int_{\Omega} \left\{ -\frac{\partial w}{\partial x_1}\frac{\partial u}{\partial x_1} - \frac{\partial w}{\partial x_2}\frac{\partial u}{\partial x_2} - bw \right\} d\Omega + \int_{\Gamma_1} \frac{\partial w}{\partial n} u\, d\Gamma + \int_{\Gamma_1} qw\, d\Gamma + \int_{\Gamma_2} \bar{q}w\, d\Gamma - \int_{\Gamma_1} \bar{u}\frac{\partial w}{\partial n}\, d\Gamma = 0 \tag{1.22}$$

Integrating again by parts the following expression is obtained

$$\int_{\Omega} \{(\nabla^2 u)w - bw\}\, d\Omega - \int_{\Gamma} wq\, d\Gamma + \int_{\Gamma_1} \frac{\partial w}{\partial n} u\, d\Gamma + \int_{\Gamma_1} qw\, d\Gamma + \int_{\Gamma_2} \bar{q}w\, d\Gamma - \int_{\Gamma_1} \bar{u}\frac{\partial w}{\partial n}\, d\Gamma = 0 \tag{1.23}$$

The first integral in Γ can again be written as a summation of two integrals, one on Γ_1 and the other on Γ_2. The one on Γ_1 can be cancelled with the integral on Γ_1 of qw in the above equation. This gives

$$\int_{\Omega} \{(\nabla^2 u)w - bw\}\, d\Omega - \int_{\Gamma_2} wq\, d\Gamma + \int_{\Gamma_1} \frac{\partial w}{\partial n} u\, d\Gamma + \int_{\Gamma_2} \bar{q}w\, d\Gamma - \int_{\Gamma_1} \bar{u}\frac{\partial w}{\partial n}\, d\Gamma = 0 \tag{1.24}$$

This formula can be written as,

$$\int_{\Omega} \{(\nabla^2 u - b)w\}\, d\Omega - \int_{\Gamma_2} (q - \bar{q})w\, d\Gamma + \int_{\Gamma_1} (u - \bar{u}) \frac{\partial w}{\partial n}\, d\Gamma = 0 \tag{1.25}$$

Once again this expression shows that one is trying to satisfy a differential equation in the domain plus two types of boundary conditions, the 'essential' conditions $u = \bar{u}$ on Γ_1 plus the 'natural' conditions $q = \bar{q}$ on Γ_2. This is very much what has been shown in equation (1.10) with the only exception that the sign of the last term is different in both expressions. This is because in (1.10) the derivatives were taken with respect to x rather than with respect to the normal, as they are now.

1.3 Approximate Solutions

Although the previous sections have introduced the concept of distributions, the formulations apply irrespective of the type of solution one finds, i.e. they are valid for exact as well as approximate solutions. This section however will investigate what happens when the concept of an approximate solution is introduced in the formulation. In engineering practice the exact solution can only be known in a few simple cases and it is hence important to see how the solution behaves when one introduces an approximation. Let us consider now that the function u defines an approximate rather than the exact solution. In this case one can write for instance,

$$u = \alpha_1 \phi_1 + \alpha_2 \phi_2 + \dots \tag{1.26}$$

where α_i are unknown coefficients and the ϕ_i are a set of linearly independent functions which are known. α_i are generalized coefficients although in some cases they can be associated with nodal values of the variable under consideration. In general in engineering problems, one prefers to use nodal values as they have a clear physical meaning and this is done in finite elements, finite differences or the boundary element method. In such cases the approximation for u can be written as

$$\begin{aligned} u &= u_1 \phi_1 + u_2 \phi_2 + \dots \\ u &= \sum_{j=1}^{N} u_j \phi_j \end{aligned} \tag{1.27}$$

where ϕ_j are a set of linearly independent functions which are sometimes called interpolation functions. u_j are the nodal values of the field variable or its derivative (or more generally the nodal value of any variable with physical meaning directly related to u or its derivatives).

Introducing the approximation for u into the governing differential equation one finds that the equation is no longer identically satisfied except for the case in which (1.26) or (1.27) can represent the exact solution. This produces an error or residual function which will soon be defined.

For instance, introducing an approximate value of u into equation (1.1) one generally finds that

$$\frac{d^2u}{dx^2} + \lambda^2 u - b \neq 0 \qquad \text{in } x \tag{1.28}$$

The same will generally occur with the boundary conditions corresponding to this equation, i.e.

$$\begin{aligned} u - \bar{u} &\neq 0 \qquad \text{at } x = 0 \\ q - \bar{q} &\neq 0 \qquad \text{at } x = l \end{aligned} \tag{1.29}$$

One can now introduce the concept of an error function or residual which represents the errors occurring in the domain or on the boundary due to non-satisfaction of the above equations. The error function in the domain is called R and is given by

$$R = \frac{d^2u}{dx^2} + \lambda^2 u - b \tag{1.30}$$

and on the boundary one has,

$$\begin{aligned} R_1 &= u - \bar{u} \\ \text{and} \\ R_2 &= q - \bar{q} \end{aligned} \tag{1.31}$$

Although the above case is a particular and relatively simple equation the same occurs for any other problem. If one considers the Poisson's equation (1.12) for instance, the error function in the domain is

$$R = \nabla^2 u - b \qquad \text{in } \Omega \tag{1.32}$$

and the errors for the boundary conditions (equation (1.19)) are defined by

$$\begin{aligned} R_1 &= u - \bar{u} \qquad \text{on } \Gamma_1 \\ R_2 &= q - \bar{q} \qquad \text{on } \Gamma_2 \end{aligned} \tag{1.33}$$

The numerical methods used in engineering try to reduce these errors to a minimum by applying different techniques. This reduction is carried out by forcing the errors to be zero at certain points, regions or in a mean sense. This operation can be generally interpreted as *distributing* these errors. The way in which this distribution is carried out produces different types of error distribution techniques which, in general, force the integrals of the residuals weighted by a certain function to be zero. Because of this they are called weighted residual techniques.

1.4 Weighted Residual Techniques

The solution of the boundary value problem defined by equations (1.28) and (1.29), (1.32) and (1.33) or similar sets for other problems can be attempted by choosing an approximation for the function u. One can then have three types of method:

(i) If the assumed approximate solution identically satisfies all boundary conditions but not the governing equations in Ω, one has a purely 'domain' method.
(ii) If the approximate solution satisfies the field or governing equations but not the boundary conditions one has a 'boundary' method.
(iii) If the assumed solution satisfies neither the field equation nor the boundary conditions, one has a 'mixed' method.

Let us first assume that the functions ϕ_j which are defined to approximate u, satisfy all boundary conditions. One then has a residual R function in the domain as the field equations are generally not identically satisfied. The idea is now to make R as small as possible by setting its weighted residual equal to zero for various values of the weighting functions, ψ_j, such that

$$\int_\Omega R\psi_j \, d\Omega = 0 \qquad \text{in } \Omega \qquad j = 1, 2, \ldots, N \tag{1.34}$$

These functions have to be linearly independent.

Notice that another way of writing (1.34) in a form that is more compact and easy to operate with, is by defining a new function w, such that

$$w = \beta_1\psi_1 + \beta_2\psi_2 + \ldots + \beta_N\psi_N = \sum_{j=1}^{N} \beta_j\psi_j \tag{1.35}$$

where β_j are arbitrary coefficients. Hence equation (1.34) can now be written in a more compact form as,

$$\int_\Omega Rw \, d\Omega = 0 \qquad \text{in } \Omega \tag{1.36}$$

Different types of weighting functions ψ_j (or w) will define different approximate methods. Equation (1.34) or (1.35) will produce a system of algebraic equations from which the unknown values of the α_i or u_i coefficients used in u (equation (1.26) or (1.27)) can be obtained.

The approximation can always be improved by increasing the number of N functions used. (N is the number of terms in the approximate solution equal to the number of weighting functions required.)

Approximate methods based on equation (1.36) are called weighted residual methods and, given an approximate solution, the method will vary in accordance with the functions used as weighting functions. In what follows a few will be reviewed.

(i) Subdomain Collocation

For this method the domain Ω is divided in M subdomains and the integral of the error in each of them is set to zero. The weighted functions are simply chosen as.

$$\psi_j = \begin{cases} 1 & \text{for } x \in \Omega_j \\ 0 & \text{for } x \notin \Omega_j \end{cases} \tag{1.37}$$

($\in$ indicates belonging to and Ω_j is the 'j' subdomain). Equation (1.34) becomes.

$$\int_{\Omega_j} R\, dx = 0; \qquad j = 1, 2, \ldots, N \tag{1.38}$$

(ii) Galerkin Method

In the case of Galerkin's method the weighting functions are the same as the appoximating functions, i.e.

$$\phi_j = \psi_j \tag{1.39}$$

hence equation (1.34) becomes.

$$\int_{\Omega} R\phi_j\, d\Omega = 0 \qquad j = 1, 2, \ldots, N \tag{1.40}$$

Using the same definition as in (1.35) this can be written as,

$$\int_{\Omega} Rw\, d\Omega = 0, \tag{1.41}$$

with,

$$w = \beta_1\phi_1 + \beta_2\phi_2 + \ldots + \beta_N\phi_N \tag{1.42}$$

This method is the starting point of many finite element formulations for which the symmetry of $\phi_j = \psi_j$ coupled to inherently symmetric field equations, lead to symmetric algebraic matrices.

(iii) Point Collocation Method

In this case N points $x_1, x_2, \ldots, x_N$ are chosen in the domain and the residual is set to zero at these points This operation can be interpreted as defining weighting functions in terms of Dirac deltas, i.e.

$$\psi_j = \Delta(x - x_j); \qquad j = 1, 2, \ldots, N \tag{1.43}$$

$\Delta(x - x_j)$ at point $x - x_j$ has an infinite value but is such that its integral gives unity, i.e.

$$\int_{\Omega} \Delta(x - x_j)\, d\Omega = 1; \qquad j = 1, 2, \ldots, N \tag{1.44}$$

The Dirac function can be interpreted as the limit of a regular function when its base tends to zero.

Hence equation (1.34) can now be written as,

$$\int_{\Omega} R\Delta(x - x_j)\, d\Omega = 0; \qquad j = 1, 2, \ldots, N \tag{1.45}$$

which simply says that the error function is zero at a series of points, i.e.

$$R|_{x = x_j} = 0; \qquad j = 1, 2, \ldots, N \tag{1.46}$$

The method consists of setting the residual or error function equal to zero at as many points as there are unknown coefficients in the approximate solution. The distribution of the collocation points is in principle arbitrary, but in practice better results are obtained if they are uniformly distributed.

Example 1.1

As an illustration of how to use weighted residuals, consider the following differential or field equation in the one dimensional domain x (where x varies from $x = 0$ to $x = 1$), i.e.

$$\frac{d^2u}{dx^2} + x = 0 \tag{a}$$

with homogeneous boundary conditions, i.e.

$$u = 0 \text{ at } x = 0 \text{ and } x = 1 \tag{b}$$

(Notice that equation (a) is a particular case of equation (1.1) when $\lambda = 0$ and $b = -x$.)

The exact solution of (a) can be found by integration and gives,

$$u_{\text{exact}} = \frac{x}{6} - \frac{x^3}{6} \tag{c}$$

Let us now attempt to solve (a) using the weighted residual techniques described above, starting by defining an approximate solution which satisfies the boundary conditions and can be written as

$$u = \alpha_1\phi_1 + \alpha_2\phi_2 + \ldots \tag{d}$$

One can use Hermitian polynomials for ϕ_j but since only two of them satisfy the homogeneous boundary conditions, only these two will be used, i.e.

$$u = \alpha_1\phi_1 + \alpha_2\phi_2 \tag{e}$$

where

$$\phi_1 = x - 2x^2 + x^3$$

$$\phi_2 = x^3 - x^2 \tag{f}$$

The residual or error function in this case is obtained by substituting (e) into equation (a) which gives,

$$\begin{aligned} R(x,\alpha_1,\alpha_2) &= \frac{d^2u}{dx^2} + x \\ &= \alpha_1 \frac{d^2\phi_1}{dx^2} + \alpha_2 \frac{d^2\phi_2}{dx^2} + x \\ &= \alpha_1(6x-4) + \alpha_2(6x-2) + x \end{aligned} \tag{g}$$

Let us now reduce (g) using the various techniques previously described.

(i) Subdomain Collocation

Consider the domain divided into 2 equal parts, one from 0 to $\frac{1}{2}$ and the other from $\frac{1}{2}$ to 1. In this case one can write,

$$\int_0^{1/2} R\,dx = \int_0^{1/2} [\alpha_1(6x-4) + \alpha_2(6x-2) + x]\,dx = 0$$

and

$$\int_{1/2}^{1} R\,dx = \int_{1/2}^{1} [\alpha_1(6x-4) + \alpha_2(6x-2) + x]\,dx = 0 \tag{h}$$

which produce the following system of equations

$$\begin{aligned} -1.2\alpha_1 - 0.25\alpha_2 + 0.125 &= 0 \\ 0.25\alpha_1 + 1.2\alpha_2 + 0.375 &= 0 \end{aligned} \tag{i}$$

from which one can obtain,

$$\alpha_1 = \tfrac{1}{6}; \qquad \alpha_2 = -\tfrac{1}{3} \tag{j}$$

Substituting (j) into (e) gives the following result

$$u = \frac{x}{6} - \frac{x^3}{6} \tag{k}$$

Notice that the exact solution (c) has been obtained since the assumed shapes of u are able to represent it.

(ii) Galerkin

In this case the weighting functions are,

$$\psi_1 = \phi_1$$
$$\psi_2 = \phi_2$$

and the weighted residual expressions are

$$\int_0^1 [\alpha_1(6x-4) + \alpha_2(6x-2) + x](x - 2x^2 + x^3)\,dx = 0$$
$$\int_0^1 [\alpha_1(6x-4) + \alpha_2(6x-2) + x](x^3 - x^2)\,dx = 0 \qquad \text{(l)}$$

which produces the following algebraic equations in α_1 and α_2.

$$-4\alpha_1 + \alpha_2 + 1 = 0$$
$$\alpha_1 - 4\alpha_2 - 1.5 = 0 \qquad \text{(m)}$$

This also results in

$$\alpha_1 = \tfrac{1}{6}, \qquad \alpha_2 = -\tfrac{1}{3}$$

(iii) Point Collocation

Here one forces the residual to be zero at a series of points. Consider in this case that R is zero at the two points $x = 0.25$ and $x = 0.75$. This gives

$$R|_{x=0.25} = -10\alpha_1 - 2\alpha_2 + 1 = 0$$
$$R|_{x=0.25} = 2\alpha_1 + 10\alpha_2 + 3 = 0 \qquad \text{(n)}$$

with the same results for α_1 and α_2, i.e.

$$\alpha_1 = \tfrac{1}{6}; \qquad \alpha_2 = -\tfrac{1}{3}$$

Notice that this case is rather trivial and the same results have been obtained for all the methods. In general this will not be true when the exact solution cannot be reproduced by the proposed value of u and one will find different results depending on the method used.

Example 1.2

Let us now study another equation using point collocation such that in this case we will obtain an approximate rather than the exact solution.

Consider the equation (1.1), with $\lambda^2 = 1$ and $x = -b$, i.e.

$$\frac{d^2u}{dx^2} + u + x = 0 \tag{a}$$

and the homogeneous boundary conditions, $u \equiv 0$ at $x = 0$ and $x = 1$.

The exact solution of (a) can be easily obtained by integration and gives

$$u = \frac{\sin x}{\sin 1} - x \tag{b}$$

Instead of using (b) we will try to approximate it defining a solution

$$u = a_1\phi_1 + a_2\phi_2 + a_3\phi_3 + \ldots \tag{c}$$

where the ϕ_i are terms of a polynomial in x, i.e.

$$\phi_1 = 1, \qquad \phi_2 = x, \qquad \phi_3 = x^2 \ldots \tag{d}$$

In order to satisfy the boundary conditions exactly, equation (c) has to give,

$$u \equiv 0 \text{ at } x = 0 \text{ and } x = 1 \tag{e}$$

which implies that,

$$\begin{aligned} &\text{at } x = 0 \rightarrow u = a_1 = 0 \\ &\text{at } x = 1 \rightarrow u = a_1 + a_2 + a_3 + \ldots = 0 \end{aligned} \tag{f}$$

Hence $a_1 \equiv 0$ and a_2 can be expressed in function of the other a_i parameter, i.e.

$$a_2 \equiv -(a_3 + a_4 + \ldots) \tag{g}$$

Substituting $a_1 \equiv 0$ and (g) into (c) one can write,

$$\begin{aligned} u &= a_3(x^2 - x) + a_4(x^3 - x) + a_5(x^4 - x) + \ldots \\ &= x(1 - x)(-a_3 - a_4) + x(1 - x)(-a_4)x + \ldots \end{aligned} \tag{h}$$

Defining now a new set of unknown parameters α_i such that,

$$\alpha_1 = -a_3 - a_4; \qquad \alpha_2 = -a_4 \ldots \tag{i}$$

one can write,

$$u = x(1 - x)(\alpha_1 + \alpha_2 x + \ldots) \tag{j}$$

This function satisfies the boundary conditions in u and has the degree of continuity required by the derivatives in equation (a), hence it is said to be 'admissible'. We will also see that the 'distance' between the approximate and exact solution decreases when the number of terms in (j) increases and this implies that the approximate formulation u is 'complete', i.e. tends to represent the exact solution better and better when the number of terms increases.

In order to apply the point collocation technique we will restrict ourselves to two terms in the (j) expression, i.e.

$$u = x(1-x)(\alpha_1 + \alpha_2 x) \tag{k}$$

Substituting this function into the governing equation (a) one finds the following residual, i.e.

$$R = \frac{d^2u}{dx^2} + u + x = (-2 + x - x^2)\alpha_1 + (2 - 6x + x^2 - x^3)\alpha_2 + x \tag{l}$$

Collocation can now be interpreted as setting $R \equiv 0$ at two points, say $x = \frac{1}{4}$ and $x = \frac{1}{2}$. This can also be expressed in terms of Dirac delta functions applied at these two points, i.e. the weighting function is,

$$w = \beta_1 \Delta_1 (x - \tfrac{1}{4}) + \beta_2 \Delta_2 (x - \tfrac{1}{2}) \tag{m}$$

The weighted residual integrals are represented by

$$\int_0^1 Rw\, dx = 0 \tag{n}$$

or simply,

$$R \equiv 0 \qquad \text{at } x = \tfrac{1}{4} \text{ and } x = \tfrac{1}{2} \tag{o}$$

Substituting these values of x into (l) one obtains two equations in α_1 and α_2. They can be written in matrix form as follows,

$$\begin{bmatrix} \frac{29}{16} & -\frac{35}{64} \\ \frac{7}{4} & \frac{7}{8} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{Bmatrix} \tag{p}$$

The solution of this system gives

$$\alpha_1 = \tfrac{6}{31}, \qquad \alpha_2 = \tfrac{40}{217} \tag{q}$$

The approximate value of u – equation (k) – can now be written as,

$$u = \frac{x(1-x)}{217}(42 + 40x) \tag{r}$$

Table 1.1 Results for Point Collocation

x	u (exact)	u (approximate)	R
0.10	0.018641	0.019078	−0.009953
0.30	0.051194	0.052258	+0.002027
0.50	0.069746	0.071428	+0.00000
0.70	0.065585	0.065806	−0.024884
0.90	0.030901	0.032350	−0.081474

Notice that the error function can now also be fully defined in terms of x, by substituting α_1 and α_2 into (l). This gives,

$$R = \tfrac{1}{21}(-4 + 19x - 2x^2 - 40x^3) \tag{s}$$

These results can be tabulated in table 1.1 where they are compared against the exact solution for u. Notice that the values of R are identically zero at $x = \frac{1}{4}$ and $x = \frac{1}{2}$ but that this does not mean that the solution for u is exact at those points.

Example 1.3

Let us apply Galerkin's technique to equation (1.1) for which $\lambda^2 = 1$ and $b = -x$ with homogeneous boundary conditions $u \equiv 0$ at $x = 0$ and $x = 1$. The approximate solution will be the same as in example 1.2., i.e.

$$u = \alpha_1 x(1 - x) + \alpha_2 x^2(1 - x) \tag{a}$$

which can be written as

$$u = \alpha_1 \phi_1 + \alpha_2 \phi_2 \tag{b}$$

where ϕ_1 and ϕ_2 are the shape functions ($\phi_1 = x(1 - x)$; $\phi_2 = x^2(1 - x)$). The residual is the same as previously, i.e.

$$\begin{aligned} R &= \frac{d^2u}{dx^2} + u + x \\ &= (-2 + x - x^2)\alpha_1 + (2 - 6x + x^2 - x^3)\alpha_2 + x \end{aligned} \tag{c}$$

The weighting function w in Galerkin is assumed to have the same shape function as the approximate solution (b), i.e.

$$w = \beta_1 \phi_1 + \beta_2 \phi_2 \tag{d}$$

The coefficients β_1 and β_2 are arbitrary.

The weighted residual statement is,

$$\int_0^1 Rw\,dx = 0 \tag{e}$$

which produces two integral expressions as β_1 and β_2 are arbitrary, i.e.

$$\int_0^1 R(\beta_1\phi_1 + \beta_2\phi_2)\,dx = 0 \tag{f}$$

or simply,

$$\int_0^1 R\phi_1\,dx = 0 \qquad \text{and} \qquad \int_0^1 R\phi_2\,dx = 0 \tag{g}$$

Substituting (c) and the functions ϕ_1 and ϕ_2 into (g) gives

$$\begin{aligned} &\int_0^1 [(-2 + x - x^2)\alpha_1 + (2 - 6x + x^2 - x^3)\alpha_2 + x][x(1-x)]\,dx = 0 \\ &\int_0^1 [(-2 + x - x^2)\alpha_1 + (2 - 6x + x^2 - x^3)\alpha_2 + x][x^2(1-x)]\,dx = 0 \end{aligned} \tag{h}$$

After integration this gives the following system

$$\begin{bmatrix} \frac{3}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{13}{105} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{12} \\ \frac{1}{20} \end{Bmatrix} \tag{i}$$

Notice that the matrix is symmetric because the equation is of an even order and the approximate and weighting functions are the same. Solving (i) gives

$$\alpha_1 = \tfrac{71}{369}; \qquad \alpha_2 = \tfrac{7}{41} \tag{j}$$

Substituting these values into (a) produces the approximate solution for u, i.e.

$$u = x(1-x)(\tfrac{71}{369} + \tfrac{7}{41}x) \tag{k}$$

One can also find the residual function R (equation c) which is now

$$R = \tfrac{1}{369}(-16 + 62x - 8x^2 - 63x^3)$$

The results for u and R are given in table 1.2 where they are compared against the exact solution of u. Notice that although the solution is overall more accurate than in the case of using the collocation technique, one now needs to carry out some integrations as shown in formula (h). This operation was not required for the case of point collocation.

Table 1.2 Results for Galerkin

x	u (exact)	u (approximate)	R
0.1	0.018641	0.018853	−0.026945
0.3	0.051194	0.051162	+0.000485
0.5	0.069746	0.069444	+0.013888
0.7	0.065582	0.065505	+0.005070
0.9	0.030901	0.031146	−0.034165

1.5 Weak Formulations

The fundamental integral statements of the boundary element and the finite element methods can be interpreted as a combination of a weighted residual statement and a process of integration by parts that reduces or 'weakens' the order of the continuity required for the u function.

If one returns to equation (1.12) with $b \equiv 0$ for simplicity, i.e.

$$\nabla^2 u = 0 \qquad \text{in } \Omega \tag{1.47}$$

one can write formula (1.25) as,

$$\int_\Omega (\nabla^2 u) w \, d\Omega - \int_{\Gamma_2} (q - \bar{q}) w \, d\Gamma + \int_{\Gamma_1} (u - \bar{u}) \frac{\partial w}{\partial n} \, d\Gamma = 0 \tag{1.48}$$

or in terms of residual functions,

$$\int_\Omega R w \, d\Omega - \int_{\Gamma_2} R_2 w \, d\Gamma + \int_{\Gamma_1} R_1 \frac{\partial w}{\partial n} \, d\Gamma = 0 \tag{1.49}$$

A special case of this equation is the case for which the function u exactly satisfies the 'essential' boundary conditions, $u = \bar{u}$ on Γ_1, which results in $R_1 \equiv 0$. In this case equation (1.49) becomes

$$\int_\Omega R w \, d\Omega = \int_{\Gamma_2} R_2 w \, d\Gamma \tag{1.50}$$

or,

$$\int_\Omega (\nabla^2 u) w \, d\Omega = \int_{\Gamma_2} (q - \bar{q}) w \, d\Gamma \tag{1.51}$$

A more usual form of this expression can be obtained by integrating by parts once which gives

$$-\int_\Omega \left(\frac{\partial u}{\partial x_1} \frac{\partial w}{\partial x_1} + \frac{\partial u}{\partial x_2} \frac{\partial w}{\partial x_2} \right) d\Omega = -\int_{\Gamma_2} \bar{q} w \, d\Gamma - \int_{\Gamma_1} q w \, d\Gamma \tag{1.52}$$

It should be pointed out that equation (1.52) could also be obtained by integrating by parts over the domain the weighted residual statement for $\nabla^2 u$ and then introducing the boundary conditions, i.e. starting with

$$\int_\Omega (\nabla^2 u) w \, d\Omega = 0 \tag{1.53}$$

one can integrate by parts once to produce the following expression,

$$-\int_\Omega \left(\frac{\partial u}{\partial x_1} \frac{\partial w}{\partial x_1} + \frac{\partial u}{\partial x_2} \frac{\partial w}{\partial x_2} \right) d\Omega + \int_\Gamma \frac{\partial u}{\partial n} w \, d\Gamma = 0 \tag{1.54}$$

Introducing then the corresponding boundary conditions in Γ ($\Gamma = \Gamma_1 + \Gamma_2$) results in equation (1.52).

The last term in equation (1.52) is usually forced to be identically equal to zero by the requirement that the w functions have to satisfy the Lagrangian version of the essential boundary conditions, or condition on Γ_1, i.e. $w \equiv 0$ on Γ_1. This gives a relationship well known in finite elements, i.e.

$$\int_\Omega \left(\frac{\partial u}{\partial x_1} \frac{\partial w}{\partial x_1} + \frac{\partial u}{\partial x_2} \frac{\partial w}{\partial x_2} \right) d\Omega = \int_{\Gamma_2} \bar{q} w \, d\Gamma \tag{1.55}$$

Equation (1.55) is usually interpreted in terms of virtual work or virtual power, by associating w with a virtual function. Notice that the integral on the left hand side is a measure of the internal virtual work and the one on the right the virtual work done by the external forces $\bar{q}$. Equation (1.55) is the starting point of most finite element schemes for Laplacian problems and is usually called a 'weak' variational formulation. The 'weakness' can be interpreted as due to two reasons, (i) the order of u function continuity has been reduced as its derivatives are now of a lower order (i.e. first rather than second order); (ii) satisfaction of the natural boundary conditions is done in an approximate rather than exact manner, which reduces the accuracy of boundary values of this variable. (Notice that R_2 is generally different from zero.)

The boundary element formulation can be interpreted as introducing a further formal step in the process of integration by parts on the derivatives of u, and consequently weakening the continuity requirements for u.

If one starts again from equation (1.48) and integrates by parts as before, the more complete expression obtained is as follows:

$$-\int_\Omega \left(\frac{\partial u}{\partial x_1} \frac{\partial w}{\partial x_1} + \frac{\partial u}{\partial x_2} \frac{\partial w}{\partial x_2} \right) d\Omega = -\int_{\Gamma_2} \bar{q} w \, d\Gamma - \int_{\Gamma_1} q w \, d\Gamma - \int_{\Gamma_1} (u - \bar{u}) \frac{\partial w}{\partial n} d\Gamma \tag{1.56}$$

Integrating again in order to eliminate all derivatives in u on the left hand side integral, one finds,

$$\int_\Omega (\nabla^2 w) u \, d\Omega = -\int_{\Gamma_2} \bar{q} w \, d\Gamma - \int_{\Gamma_1} q w \, d\Gamma + \int_{\Gamma_1} \bar{u} \frac{\partial w}{\partial n} d\Gamma + \int_{\Gamma_2} u \frac{\partial w}{\partial n} d\Gamma \tag{1.57}$$

This is the starting statement for the Boundary Element formulation of the Laplace equation. The same equation can be obtained starting from the integral of the weighted residual over the domain Ω (equation (1.53)), integrating by parts twice and *then* introducing the boundary conditions. The processes have already been shown from another field equation in formulae (1.13) to (1.16) and then (1.19) and (1.20), the only difference now being that b is zero.

Consider now equation (1.1) again to illustrate how a weak formulation can be used and the domain and boundary element statements are obtained. Let us start with equation (1.10) which was deduced from (1.1) by a process of integrations by parts and application of boundary conditions, i.e.

$$\int_0^1 \left\{ \frac{d^2u}{dx^2} w + (\lambda^2 u - b) w \right\} dx - [(q - \bar{q}) w]_{x=1} + \left[(u - \bar{u}) \frac{dw}{dx} \right]_{x=0} = 0 \quad (1.58)$$

which can also be expressed in a more compact form in function of residuals, i.e.

$$\int_0^1 Rw \, dx - [R_2 w]_{x=1} + \left[R_1 \frac{dw}{dx} \right]_{x=0} = 0 \quad (1.59)$$

The function u will now be assumed to satisfy exactly the 'essential' boundary conditions $u = \bar{u}$ at $x = 0$. In this case (1.58) becomes,

$$\int_0^1 \left[\frac{d^2u}{dx^2} w + (\lambda^2 u - b) w \right] dx = [(q - \bar{q}) w]_{x=1} \quad (1.60)$$

or in terms of (1.59), simply

$$\int_0^1 Rw \, dx = [R_2 w]_{x=1} \quad (1.61)$$

Integrating by parts equation (1.60) one can write

$$\int_0^1 \left\{ -\frac{du}{dx}\frac{dw}{dx} + (\lambda^2 u - b) w \right\} dx = [qw]_{x=0} - [qw]_{x=1} \quad (1.62)$$

If the weighting function w is forced to satisfy the homogeneous version of the essential boundary conditions at $x = 0$, equation (1.62) becomes;

$$\int_0^1 \left\{ -\frac{du}{dx}\frac{dw}{dx} + (\lambda^2 u - b) u \right\} dx = -[\bar{q} w]_{x=1} \quad (1.63)$$

which is analogous to equation (1.55) obtained for the Laplace field equation.

Notice that equation (1.59) can also be obtained by applying the boundary conditions into statement (1.3) and that this statement was simply obtained by integrating by parts weighted residual expression (1.2).

The Boundary Element type governing statement for the example under discussion is found by carrying out two consecutive integrations by parts of (1.10) and this gives the previously obtained formula (1.7), i.e.

$$\int_0^1 \left\{ u \frac{d^2 w}{dx^2} + (\lambda^2 u - b) w \right\} dx + \{ [\bar{q} w]_{x=1} - [q w]_{x=0} \} - \left\{ \left[u \frac{dw}{dx} \right]_{x=1} - \left[\bar{u} \frac{dw}{dx} \right]_{x=0} \right\} = 0 \tag{1.64}$$

This expression could also have been obtained by carrying out a double integration by parts of the weighted residual equation (1.2) and applying afterwards the boundary conditions.

It is worth noting that both in this one dimensional example and the two dimensional Laplace equations, a Finite Element type statement has been obtained after the first integration by parts (equations (1.3) and (1.14)), and Boundary Element type integral equation after the second integration (equations (1.4) and (1.15)).

Example 1.4

In order to understand the effect of weak formulations on the satisfaction of boundary conditions, we will now consider again equation (1.1) but assume that the boundary conditions are of two types, i.e.

$$\begin{aligned} &\text{at } x = 0 \rightarrow u = 0 && \text{('essential' condition)} \\ &\text{at } x = 1 \rightarrow q = \frac{du}{dx} = \bar{q} && \text{('natural' condition)} \end{aligned} \tag{a}$$

The expression previously used for the approximate values of u can not now be applied as the boundary conditions are different. Let us consider again the starting expression,

$$u = a_1 + a_2 x + a_3 x^2 + \ldots \tag{b}$$

and satisfy exactly the essential condition, at $x = 0$, i.e.

$$\text{at } x = 0; \qquad u = a_1 \equiv 0 \tag{c}$$

but not the natural condition.

Hence the approximate solution is now,

$$u = \alpha_1 x + \alpha_2 x^2 + \ldots \tag{d}$$

where $\alpha_1 = a_2$, $\alpha_2 = a_3 \ldots$

The residual will now be different from the one in the previous examples, i.e.

$$R = \frac{d^2u}{dx^2} + u + x = \alpha_1 x + \alpha_2(2 + x^2) + x \tag{e}$$

The weighted residual statement has to include now the natural boundary condition R_2 residual which is not identically satisfied, i.e.

$$\int_0^1 Rw\,dx = [R_2 w]_{x=1} \tag{f}$$

or in expanded form,

$$\int_0^1 \left(\frac{d^2u}{dx^2} + u + x\right) w\,dx = [(q - \bar{q})w]_{x=1} \tag{g}$$

One can now solve equation (g) in its present form or reduce the order of derivatives in the domain and the number of terms on the right hand side by integrating by parts the d^2u/dx^2 term. This gives

$$\int_0^1 \left\{\frac{du}{dx}\frac{dw}{dx} - (u + x)w\right\} dx = [\bar{q}w]_{x=1} - [qw]_{x=0} \tag{h}$$

Notice that in Galerkin the weighting function w has the same shapes as the approximation for u (equation (d)). Hence for two terms,

$$u = \alpha_1\phi_1 + \alpha_2\phi_2 \tag{i}$$

and

$$w = \beta_1\phi_1 + \beta_2\phi_2$$

where

$$\phi_1 = x \quad \text{and} \quad \phi_2 = x^2.$$

Substituting these values into (h) one finds,

$$\int_0^1 \left\{\left(\alpha_1\frac{d\phi_1}{dx} + \alpha_2\frac{d\phi_2}{dx}\right)\left(\beta_1\frac{d\phi_1}{dx} + \beta_2\frac{d\phi_2}{dx}\right) - (\alpha_1\phi_1 + \alpha_2\phi_2 + x)(\beta_1\phi_1 + \beta_2\phi_2)\right\} dx$$

$$= [\bar{q}(\beta_1\phi_1 + \beta_2\phi_2)]_{x=1} \tag{j}$$

or

$$\int_0^1 \{(\alpha_1 + 2\alpha_2 x)(\beta_1 + 2\beta_2 x) - (\alpha_1 x + \alpha_2 x^2 + x)(\beta_1 x + \beta_2 x^2)\}\,dx$$

$$= [\bar{q}(\beta_1 + \beta_2)] \tag{k}$$

As the β_1 and β_2 terms are arbitrary this implies satisfaction of the following two equations,

$$\int_0^1 \{(\alpha_1 + 2\alpha_2 x) - (\alpha_1 x^2 + \alpha_2 x^3 + x^2)\}\, dx = \bar{q}$$
$$\int_0^1 \{2(\alpha_1 x + 2\alpha_2 x^2) - (\alpha_1 x^3 + \alpha_2 x^4 + x^3)\}\, dx = \bar{q} \tag{l}$$

Integrating the above equations and writing the results in matrix form one finds,

$$\begin{bmatrix} \frac{2}{3} & \frac{3}{4} \\ \frac{3}{4} & \frac{17}{15} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{Bmatrix} \bar{q} + \frac{1}{3} \\ \bar{q} + \frac{1}{4} \end{Bmatrix} \tag{m}$$

The values of α_1 and α_2 are,

$$\alpha_1 = 0.9859 + 1.9864\bar{q}$$
$$\alpha_2 = -0.4319 - 0.4322\bar{q} \tag{n}$$

Notice that an error will now appear when we try to compute the value of q at $x = 1$, i.e.

$$q = \left[\frac{du}{dx}\right]_{x=1} = \alpha_1 + 2\alpha_2 = 0.1221 + 1.122\bar{q} \tag{o}$$

and hence this value will never be equal to the applied $\bar{q}$, i.e.

$$q = 0.1221 + 1.122\bar{q} \neq \bar{q} \tag{p}$$

This peculiar result is characteristic of weak formulations such as those used in finite elements. Because of this approximate satisfaction of the natural boundary conditions, f.e. solutions used in engineering practice tend to give poor results for surface fluxes or tractions. The resulting errors in many cases 'pollute' the results to such an extent that the finite element solutions are unreliable for many cases of stress or flux concentration except when using very fine meshes.

Results for u and R are given in table 1.3 for the case in which $\bar{q} = 0$. The exact solution is

$$u = \frac{\sin x}{\cos 1} - x \tag{q}$$

Table 1.3 Results for Weak Formulation and Galerkin Method

x	u (exact)	u (approximate	R
0.1	0.084773	0.094271	−0.669519
0.3	0.246953	0.256899	−0.306901
0.5	0.387328	0.384975	0.021175
0.7	0.492328	0.478499	0.314699
0.9	0.549794	0.537471	0.573671

1.6 Boundary and Domain Solutions

In section 4 the weighted residual technique was classified into boundary, domain and mixed methods. Boundary methods were defined as those for which the assumed approximate solution satisfies the governing or field equation in such a way that the only unknowns of the problem remain on the boundary. The satisfaction of the field equation may be of its homogeneous form or a special form with a singular right hand side.

In the process of double integration described earlier one had transferred the derivatives of the approximate solution u to the weighting function w and so the conditions previously imposed on the former apply now to the latter. A boundary method can be obtained by choosing a weighting function w in either of the following two ways, i.e.

(i) By selecting a function w which satisfies the governing equation in its homogeneous form, or

(ii) By using special types of functions which satisfy those equations in a way that it is still possible to reduce the problems to the boundary only. The best known of the functions applied as right hand side of the equation in the second method are the Dirac delta functions which give simply a value at a point when integrated over the domain.

It is important however to realize that other functions could also be proposed and may be very appropriate for other cases, provided that they can be reduced to the boundary.

We will now apply both techniques to our simple equation (1.1), i.e.

$$\frac{d^2u}{dx^2} + \lambda^2 u - b(x) = 0 \tag{1.65}$$

or its weighted residual statement,

$$\int_0^1 \left[u\left(\frac{d^2w}{dx^2} + \lambda^2 w\right) - bw \right] dx + \left[\frac{du}{dx} w\right]_0^1 - \left[u \frac{dw}{dx}\right]_0^1 = 0 \tag{1.66}$$

The *first approach* implies that a solution is known such that

$$\frac{d^2w}{dx^2} + \lambda^2 w \equiv 0 \tag{1.67}$$

without taking into account the actual boundary conditions of the problem.

Hence statement (1.66) reduces to

$$-\int_0^1 bw\,dx + \left[\frac{du}{dx} w\right]_0^1 - \left[u \frac{dw}{dx}\right]_0^1 = 0 \tag{1.68}$$

This approach is associated with the method called Trefftz.

The *second approach* is based on a function w such that

$$\frac{d^2w}{dx^2} + \lambda^2 w = -\Delta_i \tag{1.69}$$

where Δ_i indicates the Dirac function such that

$$\Delta_i \begin{cases} \text{singular at the } x_i \text{ point with } \int_{x_i-\varepsilon}^{x_i+\varepsilon} \Delta_i \, dx = 1 \\ = 0 \quad \text{at any other point} \end{cases} \tag{1.70}$$

Notice that in this case

$$\int_0^1 \left[u\left(\frac{d^2w}{dx^2} + \lambda^2 w\right)\right] dx = -\int_0^1 u\Delta_i \, dx = -u_i \tag{1.71}$$

where u_i represents the value of the function u at the point x_i. In this case equation (1.66) becomes,

$$-u_i - \int_0^1 bw \, dx + \left[\frac{du}{dx} w\right]_0^1 - \left[u \frac{dw}{dx}\right]_0^1 = 0 \tag{1.72}$$

When the x_i point is chosen on the boundary, then equation (1.72) gives a relationship between boundary variables.

The *second* approach is the one usually applied in boundary elements where the function w is called the 'fundamental' solution of the governing equation, or solution of (1.69). Notice that this solution is obtained without taking into consideration the boundary conditions of the problem.

Domain solutions are obtained from weighted residual statements when the assumed approximate solutions do not satisfy the governing equations One can return to equation (1.1) which after integration by parts gives the following statement,

$$\int_0^1 \left\{ -\frac{du}{dx}\frac{dw}{dx} + (\lambda^2 u - b)w \right\} dx + \left[\frac{du}{dx} w\right]_0^1 = 0 \tag{1.73}$$

This is a finite element type equation for which the last term can be found to be zero at the boundary points where $q = du/dx$ is unknown, by the requirement that $w \equiv 0$ there. Substituting an approximate solution u in terms of unknown coefficients and known weighting functions leads to a system of equations to solve the problem. Notice that in the case of finite elements the unknown function u is explicitly defined over all the domain.

Although the above remarks refer to the starting one dimensional equation (1.1) they also apply for the case of the Poisson equation (1.12) and the associated weighted residual statements (equations (1.14) and (1.16)). Similar considerations can be made for many other types of field equations.

Example 1.5

Let us now return to the same equation as defined in Example 1.1 and try to solve it using a weak formulation and considering boundary as well as domain techniques. The field equation is

$$\frac{d^2u}{dx^2} + x = 0 \tag{a}$$

with boundary conditions,

$$u = 0 \qquad \text{at } x = 0 \text{ and } x = 1 \tag{b}$$

(i) Boundary Solution. Homogeneous Approach

A weighting function which will satisfy the homogeneous version of equation (a), i.e.

$$\frac{d^2w}{dx^2} = 0 \tag{c}$$

is the simple function

$$w = a_1 x + a_2 \qquad \text{with } dw/dx = a_1 \tag{d}$$

Equation (1.68) can be written for the case $\lambda \equiv 0$ and $h = -x$ as,

$$\int_0^1 xw\,dx + [qw]_0^1 - \left[u\frac{dw}{dx}\right]_0^1 = 0 \tag{e}$$

After substituting above the boundary conditions (b) and the expressions for w and dw/dx as given by (d) equation (e) becomes

$$\int_0^1 x(a_1 x + a_2)\,dx + q_1(a_1 + a_2) - q_0 a_2 = 0 \tag{f}$$

As the above equation has to be satisfied for any arbitrary values of a_1 and a_2, it gives the following two expressions

$$\begin{aligned} q_1 &= -\int_0^1 x^2\,dx = -\tfrac{1}{3} \\ q_1 - q_0 &= -\int_0^1 x\,dx = -\tfrac{1}{2} \end{aligned} \tag{g}$$

and hence,

$$q_0 = \tfrac{1}{6} \tag{h}$$

These values of q at $x = 0$ and $x = 1$ which are now the problem unknowns, are in this case the exact values.

(ii) Boundary Solution. Singular Approach

The weighting function in this case is chosen such that

$$\frac{d^2w}{dx^2} + \Delta_i = 0 \tag{i}$$

A solution of equation (i) regardless of boundary conditions is

$$w = \begin{cases} x, & x \leqslant x_i \\ x_i, & x \geqslant x_i \end{cases} \tag{j}$$

Once the boundary conditions are applied, equation (1.72) becomes

$$-u_i + \int_0^1 xw\,dx + q_1 w_1 - q_0 w_0 = 0 \tag{k}$$

Taking into consideration that $w_0 \equiv 0$ and substituting the other values of w as given by (j) one finds,

$$u_i = \int_0^{x_i} x^2\,dx + \int_{x_i}^1 x_i x\,dx + q_1 x_i \tag{l}$$

or,

$$u_i = \frac{x_i}{2} - \frac{x_i^2}{6} + q_1 x_i \tag{m}$$

Notice that only one unknown (q_1) remains, since one of the boundary stresses disappeared because of the variation of the weighting function w. The value of q_1 can be determined by taking the coordinate $x_i = 1$, i.e.

$$u_1 = 0 = \tfrac{1}{2} - \tfrac{1}{6} + q_1 \tag{n}$$
$$\therefore\ q_1 = -\tfrac{1}{3}$$

which is the exact value in this case.

Any value of u inside the domain can be computed from (l), i.e.

$$u_{1/2} = \tfrac{3}{48} \tag{o}$$

which is also the exact solution.

If instead of the fundamental solution given by (j) one had chosen a fundamental solution that also satisfies the boundary conditions, then no unknowns would exist either in the domain or at the boundaries and the value of u at any point would be obtained by a single integration. Consider for instance the solution,

$$w = \begin{cases} (1 - x_i)x & x \leqslant x_i \\ (1 - x)x_i & x > x_i \end{cases} \tag{p}$$

This function satisfies (i) and the boundary conditions $w = 0$ at $x = 0$ and $x = 1$ ($w_0 = w_1 = 0$). Thus equation (k) gives

$$u_i = \int_0^1 xw\,dx = \int_0^{x_i} (1 - x_i)x^2\,dx + \int_{x_i}^1 x_i(1 - x)x\,dx \tag{q}$$

resulting in

$$u_i = \frac{x_i}{6} - \frac{x_i^2}{6} \tag{r}$$

which is the exact solution.

Fundamental solutions that satisfy the boundary conditions as well as the governing equations are called Green's functions.

(iii) Domain Solution

The weighted residual statement used here is the one resulting after one integration by parts has been carried out (equation (1.73)), i.e.

$$\int_0^1 \left\{ -\frac{du}{dx}\frac{dw}{dx} + xw \right\} dx + \left[\frac{du}{dx} w \right]_0^1 = 0 \tag{s}$$

The proposed approximate solution is the one in Example 1.1 which satisfies the boundary conditions, i.e.

$$u = \alpha_1\phi_1 + \alpha_2\phi_2 \tag{t}$$

with

$$\phi_1 = x - 2x^2 + x^3$$

$$\phi_2 = x^3 - x^2$$

$$\frac{d\phi_1}{dx} = 3x^2 - 4x + 1$$

$$\frac{d\phi_2}{dx} = 3x^2 - 2x$$

where ϕ_1 and ϕ_2 are the Hermitian cubic polynomials.

Substituting these values into (s) and using Galerkin, i.e.

$$\psi_1 = \phi_1; \qquad \psi_2 = \phi_2 \tag{u}$$

one can write,

$$\int_0^1 [\alpha_1(3x^2 - 4x + 1) + \alpha_2(3x^2 - 2x)](3x^2 - 4x + 1)\,dx$$

$$= \int_0^1 x(x - 2x^2 + x^3)\,dx \qquad \text{(w)}$$

$$\int_0^1 [\alpha_1(3x^2 - 4x + 1) + \alpha_2(3x^2 - 2x)](3x^2 - 2x)\,dx$$

$$= \int_0^1 x(x^3 - x^2)\,dx$$

which after integrating and solving also gives the exact solution, i.e.

$$\alpha_1 = \tfrac{1}{6}, \qquad \alpha_2 = -\tfrac{1}{3} \qquad \text{(x)}$$

It is worth noticing that if the approximation used for the 'weak' formulation is the same as the one used for the original domain weighted residual equation the result will be the same in all cases. The advantage of the 'weak' formulation is that the order of the derivatives of u is in this case reduced and hence the order of derivability required by the assumed approximate solution is also reduced.

1.7 Concluding Remarks

This chapter has presented the Boundary Element Method as a weighted residual technique. This approach permits relation of the method to other numerical techniques and gives an easy way of introducing boundary elements.

For simplicity one dimensional problems have been discussed throughout the chapter to present the relationship between different integral statements and also between approximate techniques. The presentation was then extended to potential problems governed by the Laplace or Poisson's equations, which will be used in the next chapter. Some authors prefer to deduce the boundary integral equations from Green's theorem instead. Notice that this theorem has also been presented here (equation (1.18)) where it was shown that it can be derived from Lagrangian multipliers or basic residual type statements.

Later on a similar approach will be discussed for elasticity problems as shown in Chapter 3. The beauty of weighted residuals is that they are simple to use and can be applied for a wide range of problems, including some very complex non-linear and time dependent cases which are not discussed in this book.

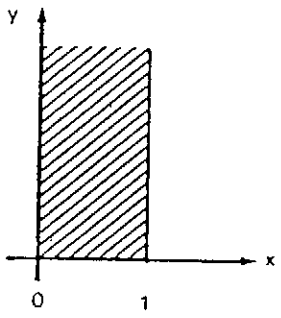
Exercises

1.1. Solve $\dfrac{d^2u}{dx^2} + u + x = 0$ with boundary conditions $u(0) = u(1) = 0$ using a trial function of the form $u = a_0 + a_1x + a_2x^2$ and point collocation for $x = 1/2$. Plot the solution and compare it with that of example 1.2 of the text and the exact solution given by equation (b) of that example.

1.2. Solve $\dfrac{d^2u}{dx^2} = \exp(u)$ from $x = 0$ to $x = 1$ with boundary conditions $u(0) = u(1) = 0$ using the same trial function and collocation point of exercise 1.1.

1.3. Solve $\nabla^2 u = 0$ in the plane domain $0 \leqslant x \leqslant 1$, $0 \leqslant y < \infty$ with boundary conditions

$$u(0, y) = u(1, y) = 0$$
$$u(x, \infty) = 0$$
$$u(x, 0) = x(1 - x)$$



using a trial function of the form $u = A(y)x(x - 1)$ and point collocation for $x = 1/2$, $0 \leqslant y < \infty$ as collocation point.

1.4. Solve exercise 1.3 using the Galerkin method with the integral and weighting function only along the x axis.

1.5. Solve the equation $\dfrac{d}{dx}\left[(1 + u)\dfrac{du}{dx}\right] = 0$ from $x = 0$ to $x = 1$ with boundary conditions $u(0) = 0$ and $u(1) = 1$ using $u = a_0 + a_1x + a_2x^2$ as trial function and subdomain collocation with only one subdomain (x from 0 to 1).

1.6. The equation of the vertical displacement of a cable suspended between two points is $\dfrac{d^2u}{dx^2} + p(x) = 0$ where $p(x)$ is the ratio between the distributed load and the horizontal force at the extremes.

Use the weak formulation and the homogeneous approach of boundary solution to compute the slope at the extremes for a cable that extends from $x = 0$ to $x = 1$ with boundary conditions $u(0) = u(1) = 0$. The function $p(x)$ is given by

$$p(x) = 0 \qquad 0 \leqslant x < 1/4$$
$$p(x) = 1 \qquad 1/4 \leqslant x < 3/4$$
$$p(x) = 0 \qquad 3/4 \leqslant x < 1$$

1.7. Using the same equation of exercise 1.6 and the singular approach of boundary solution compute the value of u at the mid-point.

1.8. The same as exercise 1.7 using a fundamental solution that also satisfies the boundary conditions (Green's function).

1.9. Solve the same equation of exercise 1.6 by means of a domain solution procedure and Galerkin. Use the following approximate solution

$$u = a_1 \sin \pi x + a_2 \sin 3\pi x$$