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MEMORY AND OPERATIONS COUNT SCALING OF COUPLED FINITE-ELEMENT AND BOUNDARY-ELEMENT SYSTEMS OF EQUATIONS

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SUMMARY

Memory and operations count scaling for the solution of coupled finite-element and boundary-element systems of equations is considered. Three algebraic approaches for solving the hybrid set of equations are studied using both direct and iterative matrix methods. Results show that once the matrix solution technique is chosen, a single hybrid algebra emerges as a clear favourite. Interestingly, the most computationally attractive hybrid algebra under assumptions of direct solution becomes the least desirable approach when iterative methods are applied. The analysis has been carried out using a range of mesh-dependent parameters which have been empirically derived through practical experience; however, the scaling expressions presented are valid for any mesh once these critical parameters have been determined.

INTRODUCTION

Mathematical problems with unbounded domains arise in numerous physical contexts. In many instances they also consist of regions of geometric complexity and/or material heterogeneity. Numerical methods which combine the finite-element method (FEM) with the boundary-element method (BEM)[1–7] are ideally suited for these types of situations. Such hybrid schemes or hybrid-element methods (HEM) incorporate the advantages of the FEM in dealing with geometric and material complexity with those of the BEM in dealing with boundaries at infinity. For example, in the electromagnetic wave context a HEM can provide a formal solution to the problem of an electrically heterogeneous, arbitrarily shaped body irradiated by a detached source.[6,7]

It has been shown that in cases where the body consists of severe geometric/material complexity, the HEM is computationally more attractive than a pure BEM treatment of the problem due to the appearance of multiple interior boundaries enclosing homogeneous subdomains.[6] Unfortunately, however, hybrid-element methods have computational overheads which tend to compromise the efficiencies normally associated with the pure FEM. Thus, it is important to choose an algebraic approach which conserves computational resources when using the HEM. While the concept of coupling finite- and boundary-element methods routinely appears in textbooks (e.g. see References 8–10), analysis of the computational economies associated with various HEM strategies has not been carefully investigated.

In this paper three algebraically equivalent hybrid schemes that have appeared in the literature are studied with respect to their computational costs. Scaling arguments for memory

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requirements and operations count are compared when both direct and iterative matrix solution techniques are considered. Where assumptions are needed to arrive at approximate scaling expressions, experience with practical meshes is brought to bear.[11] The implementations investigated herein are intended to be non-optimized in the sense that they are assumed to use generic matrix manipulation algorithms that are readily available in most mathematical software libraries.

THE HYBRID STRATEGIES

In this Section the three algebraically equivalent hybrid strategies are outlined. The prototype problem geometry is shown in Figure 1. To facilitate the discussion the following quantities are defined:

N_{I} = number of nodes inside the body (excluding the boundary)
N_{b} = number of nodes on the boundary of the body (excluding the interior)
N_{s} = number of nodes on the surface of the source
E = the problem unknown
F = the natural boundary condition quantity on E.

F is unknown on the body surface, and may or may not be specified on the surface of the source. Without loss of generality, it is assumed that E_{s} – the problem variable on the source – is known. Treatment of other boundary conditions can be found in Reference 11.

Hybrid method I

Hybrid method I (HEM1) begins by writing the FEM and BEM systems of equations as:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} E_{\mathrm{I}} \\ E_{\mathrm{b}} \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} \begin{Bmatrix} 0 \\ F_{\mathrm{b}} \end{Bmatrix} \tag{1a}$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} F_{\mathrm{b}} \\ F_{\mathrm{s}} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} E_{\mathrm{b}} \\ E_{\mathrm{s}} \end{Bmatrix} \tag{1b}$$

Subscripts on E and F indicate their nodal locations consistent with the subscripting on N above – 'I' is inside the body, 'b' is on the body surface, and 's' is on the source surface. The subscripts on the submatrices are for reference purposes only.

HEM1 proceeds by inverting the left-side matrix in (1b):

$$\begin{Bmatrix} F_{\mathrm{b}} \\ F_{\mathrm{s}} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^{-1} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} E_{\mathrm{b}} \\ E_{\mathrm{s}} \end{Bmatrix} = \begin{bmatrix} G_{11}, & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{Bmatrix} E_{\mathrm{b}} \\ E_{\mathrm{s}} \end{Bmatrix} \tag{2a}$$

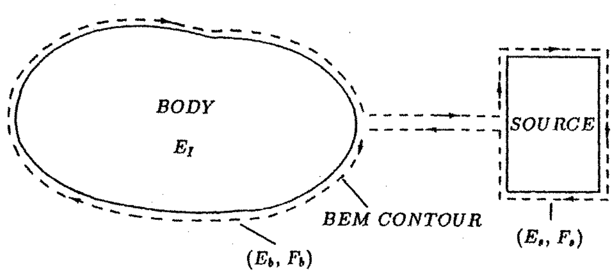

Figure 1. Prototype problem geometry for hybrid-element methods

where $G = C^{-1}D$, solving for F_b and multiplying by submatrix B from (1a):

$$[B]F_b = [B]([G_{11}]E_b + [G_{12}]E_s) \tag{2b}$$

This expression is substituted into (1a) and rearranged to produce

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - [B][G_{11}] \end{bmatrix} \begin{Bmatrix} E_I \\ E_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ [B][G_{12}]E_s \end{Bmatrix} \tag{2c}$$

Algebraic system (2c) is then solved for E. If F is desired, it can be computed from (2a) once E_b is known. Examples of the use of HEM1 can be found in References 12–14.

Hybrid method II

Hybrid method II (HEM2) also begins by writing the FEM and BEM system of equations as in (1), but instead of inverting the left side of (1b), the left side of (1a) is inverted†:

$$\begin{Bmatrix} E_I \\ E_b \end{Bmatrix} = \begin{bmatrix} AI_{11} & AI_{12} \\ AI_{21} & AI_{22} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} \begin{Bmatrix} 0 \\ F_b \end{Bmatrix} \tag{3a}$$

where $AI = A^{-1}$. Equation (3a) provides the relationship between E_b and F_b:

$$E_b = [AI_{22}][B]F_b \tag{3b}$$

which can be substituted into (1b)

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} F_b \\ F_s \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} [AI_{22}][B]F_b \\ E_s \end{Bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{Bmatrix} F_b \\ E_s \end{Bmatrix} \tag{3c}$$

and rearranged to produce

$$\begin{bmatrix} C_{11} - H_{11} & C_{12} \\ C_{21} - H_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} F_b \\ F_s \end{Bmatrix} = \begin{Bmatrix} [H_{12}]E_s \\ [H_{22}]E_s \end{Bmatrix} \tag{3d}$$

which can be solved for F. Once F is known, system of equations (1a) can be used to recover E. Examples of the use of HEM2 can be found in References 11, 15 and 16.

Hybrid method III

Hybrid method III (HEM3) attempts to avoid matrix inversion by combining equations (1a) and (1b) as a single large algebraic system

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & -B & 0 \\ 0 & -D_{11} & C_{11} & C_{12} \\ 0 & -D_{21} & C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} E_I \\ E_b \\ F_b \\ F_s \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ [D_{12}]E_s \\ [D_{22}]E_s \end{Bmatrix} \tag{4}$$

In this case F is computed in the same step as E. Examples of the use of HEM3 can be found in References 7, 8 and 14.

† In practice, only a portion of the inverse is actually needed.

DIRECT SOLUTION

In this Section, the memory and operations count scaling for the three hybrid methods is examined assuming all matrix solutions are performed with direct techniques. Where appropriate, memory economizing strategies are considered, specifically banded and profile forms.[11,17] Memory usages are assumed to be governed by the largest matrices that are required to be manipulated by the given algebraic approaches.

To simplify the operations count analysis, the following assumptions have been made concerning an $N \times M$ matrix (for a full matrix $M = N$ and for a banded matrix $M = NB$,

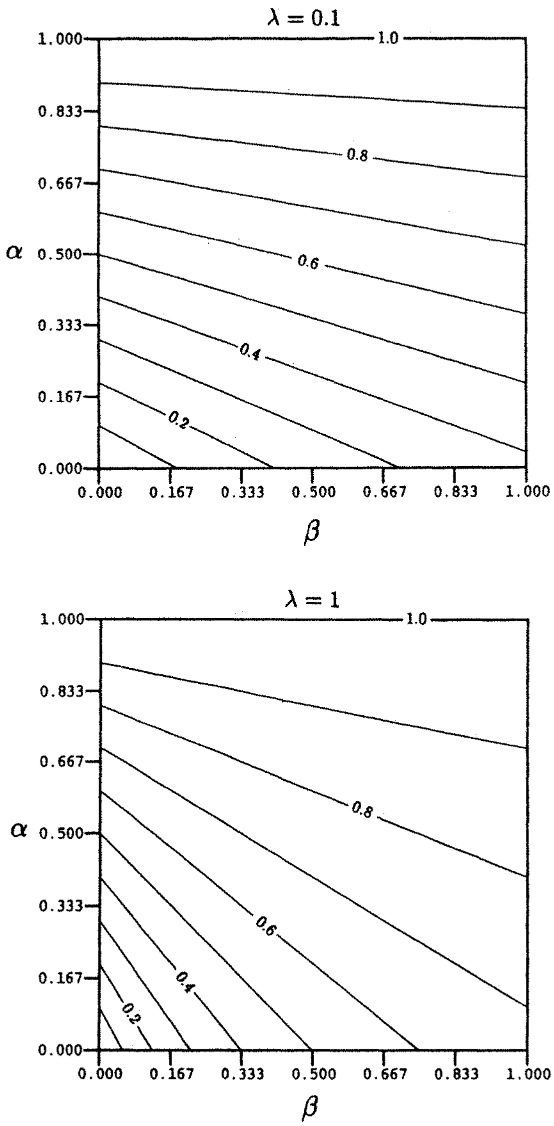

Figure 2. Contours of storage requirement ratio of HEM2 to HEM1 as a function of α and β for $\lambda = 0{\cdot}1$ and $\lambda = 1{\cdot}0$ when direct solution techniques are assumed

where NB is the bandwidth):

Manipulation	Operations
LU decomposition	NM^2
Back substitution	NM
Matrix inversion	$NM^2 + N^2M$

Addition/substraction is assumed negligible relative to multiplication. Since in some of the hybrid schemes computation of F occurs after E is known whereas in others F is computed prior to or along with E, the operations count will be given in terms of the complete solution to the problem – both E and F. In general, it is also assumed and implied by Figure 1 that $N_I \gg N_b$ and $N_b \approx N_s$.

For HEM1, the largest matrix that needs manipulating is (2c) since the bandwidth typically grows relative to (1a) due to the insertion of boundary information from (2b). Essentially, the bandwidth of HEM1 is governed by the number of nodes on the boundary of the body – N_b. For HEM2 and HEM3 the largest matrices are (1a) and (4), respectively. By defining $N_s = \lambda N_b$, $NB_1 = \beta N$, $NB_2 = \alpha NB_1$ and $NB_3 = (1 + \lambda) NB_1$, where NB_i is the bandwidth of the ith hybrid method, $N = N_I + N_b$, and $\alpha, \beta \leqslant 1$, the storage estimates for HEM1, HEM2 and HEM3 become:

$$S_{\mathrm{HEM1}} \approx N * NB_1 + 2(N_b + N_s)^2 = \left(1 + \frac{(1+\lambda)^2}{2}\beta\right)\beta N^2 \tag{5a}$$

$$S_{\mathrm{HEM2}} \approx N * NB_2 + 2(N_b + N_s)^2 = \left(\alpha + \frac{(1+\lambda)^2}{2}\beta\right)\beta N^2 \tag{5b}$$

$$S_{\mathrm{HEM3}} \approx (N + N_b + N_s) * NB_3 = \left((1+\lambda) + \frac{(1+\lambda)^2}{2}\beta\right)\beta N^2 \tag{5c}$$

Figures 2 and 3 show contours of the storage requirement ratio of HEM2 and HEM3 to HEM1

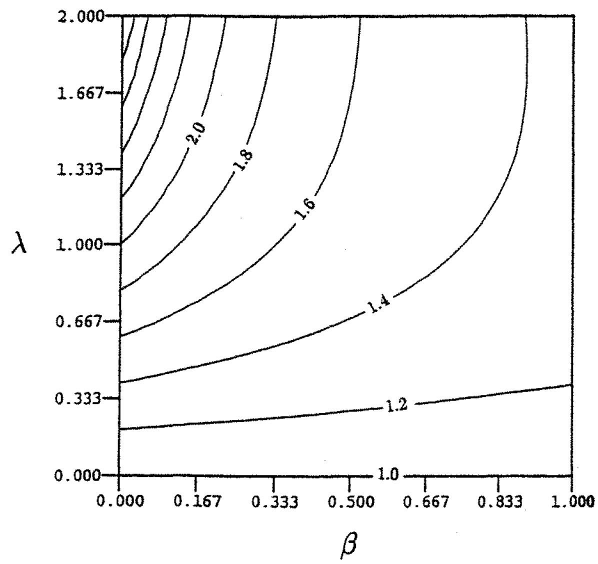

Figure 3. Contours of storage requirement ratio of HEM3 to HEM1 as a function of λ and β when direct solution techniques are assumed

as a function of parameters α, β and λ. For practical meshes it has been observed that $0{\cdot}1 < \beta < 0{\cdot}3$, $0{\cdot}3 < \alpha < 0{\cdot}7$ and $\lambda \approx 1$.[11] Using these values as guidelines in Figures 2 and 3, one finds that HEM2 requires the least storage (slightly more than half that of HEM1) and HEM3 the most storage (nearly twice that of HEM1). Note that in the limiting case of $N_s \ll N_b$ or $\lambda \rightarrow 0$ (i.e. the source small relative to the size of the body), HEM1 and HEM3 have essentially the same storage requirement, which is still about twice that needed for HEM2.

Estimations of operations count for the hybrid methods are obtained by following the algebraic steps outlined in the preceding Section and accounting for each using the matrix manipulation approximations given above. Tables I, II and III show these procedures for HEM1, HEM2 and HEM3, respectively. The operations count expressions developed in the tables can be truncated to their dominant terms:

$$O_{\mathrm{HEM1}} \approx \left(1 + \frac{3(1+\lambda)^3}{8}\beta\right)\beta^2 N^3 \tag{6a}$$

$$O_{\mathrm{HEM2}} \approx \left(\alpha^2 + \frac{\alpha}{2} + \frac{(1+\lambda)+(1+\lambda)^3}{8}\beta\right)\beta^2 N^3 \tag{6b}$$

$$O_{\mathrm{HEM3}} \approx \left((1+\lambda)^2 + \frac{(1+\lambda)^3}{2}\beta\right)\beta^2 N^3 \tag{6c}$$

Figures 4 and 5 show contours of the operations count ratio of HEM2 and HEM3 to HEM1 as a function of α, β and λ. Again taking representative values for α, β and λ operations count favours HEM2 over both HEM1 ($\approx$factor of 2) and HEM3 ($\approx$factor of 6). While the operations count for HEM2 is always less than that for HEM3, it can exceed the HEM1 operations count, but only when α is large ($>0{\cdot}8$). For this to occur over a large range of β values, λ must also be small ($<0{\cdot}5$).

The main drawback of HEM1 is its enlarged bandwidth due to the dense subset of BEM

Table I. HEM1 operations count

Action	Equation	Operations
Invert **C**	2a	$2(N_b+N_s)^3 = \frac{(1+\lambda)^3}{4}\beta^3 N^3$
Multiply by **D**	2a	$(N_b+N_s)^3 = \frac{(1+\lambda)^3}{8}\beta^3 N^3$
Multiply by **B***	2b	$3N_b(N_b+N_s) = \frac{3(1+\lambda)}{4}\beta^2 N^2$
Decompose **A**	2c	$N \times NB_1^2 = \beta^2 N^3$
Back-substitute **A**	2c	$N \times NB_1 = \beta N^2$
Solve for **F**	2a	$(N_b+N_s)^2 = \frac{(1+\lambda)^2}{4}\beta^2 N^2$
Total count = $\left(\frac{3(1+\lambda)^3}{8}\beta^3 + \beta^2\right)N^3 + \left(\frac{(1+\lambda)(4+\lambda)}{4}\beta^2 + \beta\right)N^2$		

* **B** is tridiagonal with linear elements

Table II. HEM2 operations count

Action	Equation	Operations
Decompose **A**	3a	$N \times NB_2^2 = \alpha^2\beta^2 N^3$
Back-substitute N_b columns of **A**	3a	$N_b \times N \times NB_2 = \frac{\alpha\beta^2}{2} N^3$
Multiply by **B***	3b	$3N_b^2 = \frac{3}{4}\beta^2 N^2$
Multiply by **D**	3c	$(N_b + N_s)N_b^2 + (N_b + N_s)N_s = \frac{(1+\lambda)}{8}\beta^3 N^3 + \frac{\lambda(1+\lambda)}{4}\beta^2 N^2$
Decompose/back-substitute **C**	3d	$(N_b + N_s)^3 + (N_b + N_s)^2 = \frac{(1+\lambda)^3}{8}\beta^3 N^3 + \frac{(1+\lambda)^2}{4}\beta^2 N^2$
Multiply by **B***	3a	$3N_b = \frac{3}{2}\beta N$
Back-substitute **A**	3a	$N \times NB_2 = \alpha\beta N^2$
Total count $= \left(\frac{(1+\lambda) + (1+\lambda)^3}{8}\beta^3 + \left(\alpha^2 + \frac{\alpha}{2}\right)\beta^2\right) N^3 + \left(\frac{3 + (1+\lambda)(1+2\lambda)}{4}\beta^2 + \alpha\beta\right) N^2 + \frac{3}{2}\beta N$		

* **B** is tridiagonal with linear elements.

Table III. HEM3 operations count

Action	Equation	Operations
Decompose **A**	4	$(N + N_b + N_s) \times NB_3^2 = \left(\frac{(1+\lambda)^3}{2}\beta^3 + (1+\lambda)^2\beta^2\right) N^3$
Back-substitute **A**	4	$(N + N_b + N_s) \times NB_3 = \left(\frac{(1+\lambda)^2}{2}\beta^2 + (1+\lambda)\beta\right) N^2$
Total count $= \left(\frac{(1+\lambda)^3}{2}\beta^3 + (1+\lambda)^2\beta^2\right) N^3 + \left(\frac{(1+\lambda)^2}{2}\beta^2 + (1+\lambda)\beta\right) N^2$		

equations. Profile or skyline storage[11,17] can be effective when the sparsity pattern of the system of equations contains a small number of very long rows (or columns). Hence, profile storage has been examined to see whether HEM1 can be made more competitive relative to HEM2 while still offering the advantages of direct solution of the algebraic system.

The computational effort can be estimated as in the banded storage case by defining $\widetilde{NB}$ as the effective bandwidth of the coefficient matrix associated with the profile storage scheme:

$$\widetilde{NB} = \frac{2(\text{Profile} + N)}{N} \tag{7}$$

Then one can write $\widetilde{NB} = \delta NB$. Thus, both methods realize a reduction of δ in storage compared to banded form, and the operations count for matrix solution scales down by

$$\frac{O_{\text{PROFILE}}}{O_{\text{BANDED}}} \approx \frac{3\beta(1+\lambda)^3 + 8\delta^2}{3\beta(1+\lambda)^3 + 8} \tag{8a}$$

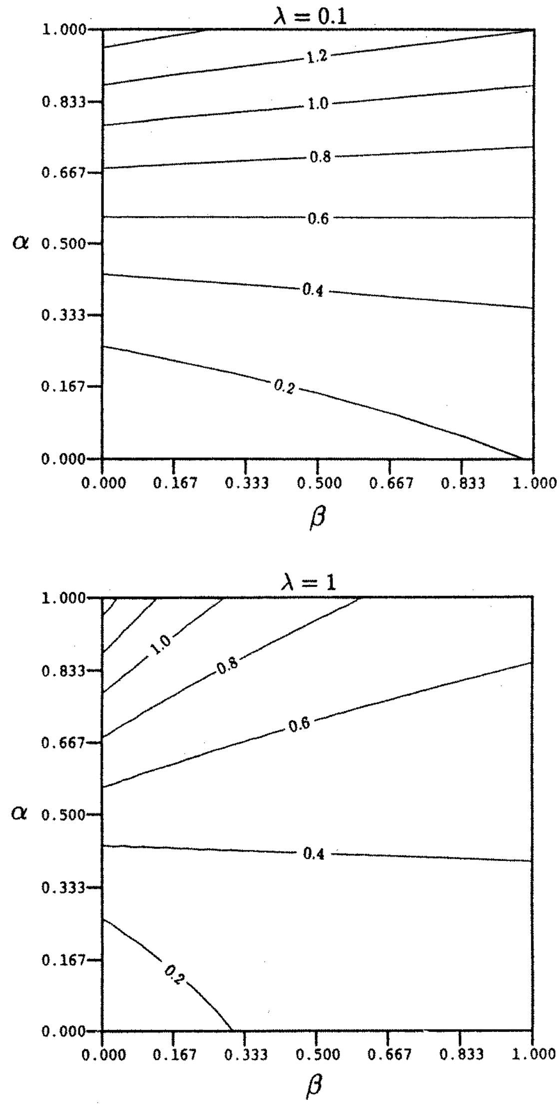

Figure 4. Same as Figure 2 for operations count ratio of HEM2 to HEM1

for HEM1 and

$$\frac{O_{\text{PROFILE}}}{O_{\text{BANDED}}} \approx \frac{8\alpha^2\delta^2 + 4\alpha\delta + \beta((1+\lambda)+(1+\lambda)^3)}{8\alpha^2 + 4\alpha + \beta((1+\lambda)+(1+\lambda)^3)} \tag{8b}$$

for HEM2. Experience has shown that $\delta \approx 0{\cdot}7$ for HEM1 and δ is only marginally larger for HEM2 when the bandwidth and profile reduction routines in References 18 and 19 are used on practical meshes.[11] With representative values of $\lambda \approx 1$, $\alpha \approx 0{\cdot}5$, $\beta \approx 0{\cdot}2$ and $\delta \approx 0{\cdot}7$, the operations count scale factors are $\approx 0{\cdot}7$ for both methods as well, indicating that there is little or no gain in computational effort with HEM1 relative to HEM2 if the profile storage strategy is adopted.

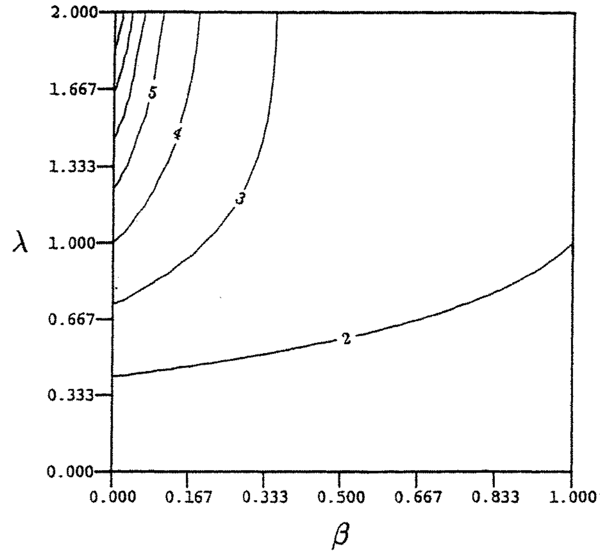

Figure 5. Same as Figure 3 for operations count ratio of HEM3 to HEM1

ITERATIVE SOLUTION

In this Section, the memory and operations count scaling is examined assuming all sparse matrix solutions are performed with iterative techniques. Under these conditions only the non-zero coefficients of the various matrices need to be stored. Again, one can define an equivalent bandwidth for the sparse storage scheme: $\widetilde{NB} = \gamma NB$, where $\gamma < 1$. The storage estimates for the hybrid methods then become:

$$S_{\mathrm{HEM1}} \approx N * \widetilde{NB}_1 + 2(N_b + N_s)^2 = \left(\gamma_1 + \frac{(1+\lambda)^2}{2}\beta\right)\beta N^2 \tag{9a}$$

$$S_{\mathrm{HEM2}} \approx N * \widetilde{NB}_2 + 2(N_b + N_s)^2 = \left(\alpha\gamma_2 + \frac{(1+\lambda)^2}{2}\beta\right)\beta N^2 \tag{9b}$$

$$S_{\mathrm{HEM3}} = S_{\mathrm{HEM2}} \tag{9c}$$

Experience with practical meshes suggests that $\gamma \approx 0{\cdot}1 \rightarrow 0{\cdot}2$ for the main matrices in HEM1 and HEM2, with γ always smaller for HEM2, typically $\gamma \leqslant 0{\cdot}1$.[11] Thus, HEM3 becomes competitive in terms of memory requirements (equal to that of HEM2), and relative to HEM1 has the storage ratio plotted in Figure 6 when $\gamma_1 = 0{\cdot}2$ and $\gamma_2 = 0{\cdot}1$. Assuming typical values of α and β, this ratio is approximately $0{\cdot}6$ or less provided $\lambda \leqslant 1$, but can grow above $0{\cdot}8$ if λ becomes large. As expected all three methods show a reduction in storage compared to their direct solution counterparts – roughly $0{\cdot}5$ for HEM1 and HEM2 and $0{\cdot}2$ for HEM3.

The operations count can be estimated by assuming that solution of a system of size N requires $\approx N * \widetilde{NB} * N_{\mathrm{it}}$ operations†, where N_{it} is the number of iterations needed to reach

† e.g. various conjugate direction methods have operation counts of this form with small constants of proportionality.

some convergence criterion. The operations counts for the three hybrid strategies then become:

$$O_{\mathrm{HEM1}} \approx \left(\frac{\gamma_1 N_{\mathrm{it}}}{\beta N} + \frac{3(1+\lambda)^3}{8}\beta\right)\beta^2 N^3 \tag{10a}$$

$$O_{\mathrm{HEM2}} \approx \left(\frac{\gamma_2 \alpha}{2} N_{\mathrm{it}} + \frac{(1+\lambda)+(1+\lambda)^3}{8}\beta\right)\beta^2 N^3 \tag{10b}$$

$$O_{\mathrm{HEM3}} \approx \frac{N_{\mathrm{it}}}{N}\left(\frac{\gamma_2 \alpha}{\beta} + \frac{(1+\lambda)^2}{4}\right)\beta^2 N^3 \tag{10c}$$

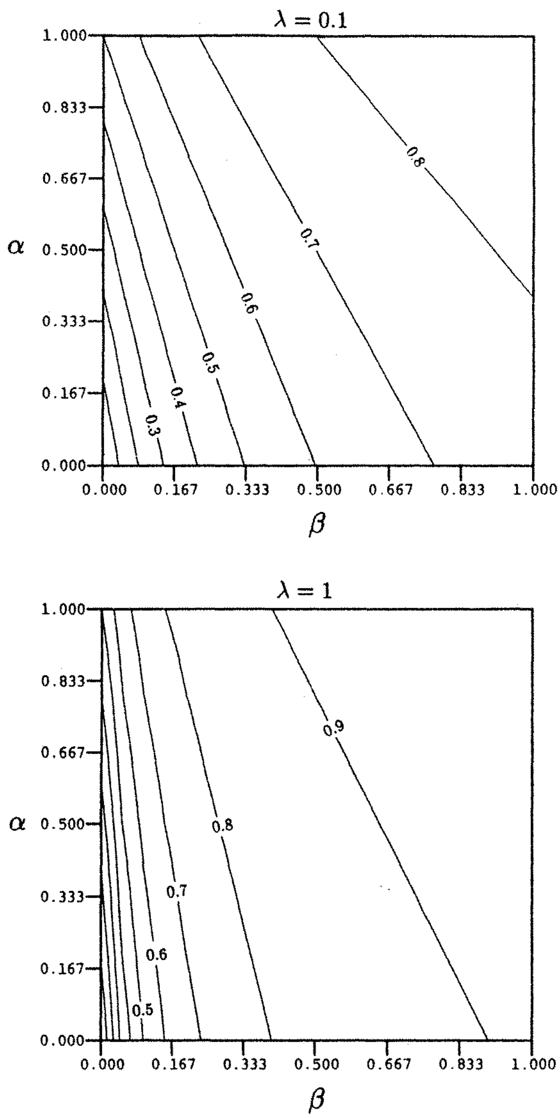

Figure 6. Contours of storage requirement ratio of HEM3 to HEM1 as a function of α and β for $\lambda = 0{\cdot}1$ and $\lambda = 1{\cdot}0$ when iterative solution techniques are assumed

Comparison of expressions (10) is more difficult because of the lack of a definitive relationship between N_{it} and N. One conservative possibility would be direct proportionality: $N_{\text{it}} = \tau N$. Under these circumstances, HEM2 clearly loses out for large N since its operations count becomes $\mathcal{O}(N^4)$ relative to $\mathcal{O}(N^3)$ for HEM1 and HEM3. For moderate N, operations count ratios of HEM2 to HEM1 are shown in Figure 7. When typical parameter values of α, β, λ and γ are assumed, the operations count for HEM2 is consistently greater than that for HEM1 unless N is small (<250), and it can be significantly larger (ratio > 5) as N increases. Under favourable (to HEM2) conditions of $\alpha = 0 \cdot 1$, $\beta = 0.5$ and $\lambda = 2$, operations count ratios less than unity can readily occur unless τ is relatively large ($> 0 \cdot 5$).

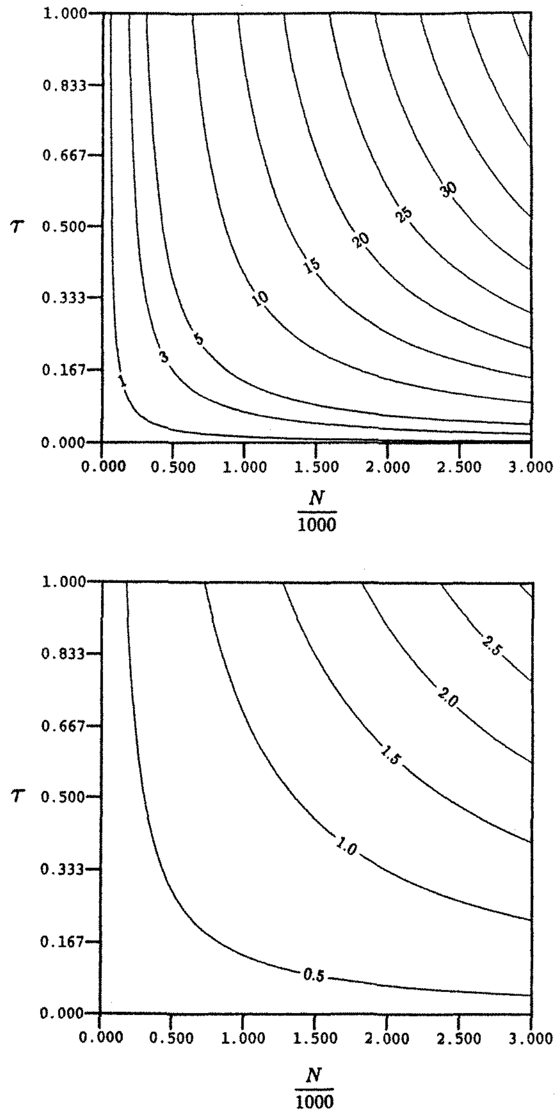

Figure 7. Contours of operations count ratio of HEM2 to HEM1 as a function of τ and N when iterative solution techniques are assumed. Typical case (top): $\alpha = 0 \cdot 5$, $\beta = 0 \cdot 2$, $\lambda = 1$, $\gamma_1 = 0 \cdot 2$, $\gamma_2 = 0 \cdot 1$; favourable (to HEM2) case (bottom): $\alpha = 0 \cdot 1$, $\beta = 0 \cdot 5$, $\lambda = 2$, $\gamma_1 = 0 \cdot 2$, $\gamma_2 = 0 \cdot 1$

The operations count for HEM3 is compared to HEM1 using the proportionality assumption in Figure 8. Results show that under a wide range of parameter selections the HEM3 algebra consistently requires fewer operations, i.e. approximately 0·3 times that of HEM1 for typical values. Only in the extreme case of $\alpha \approx 1$, $\beta \approx 0{\cdot}1$, $\lambda \approx 2$ and $\tau > 0{\cdot}8$ does the operations count of HEM3 exceed that of HEM1, and even then their ratio is very near unity ($\approx 1{\cdot}1$).

In the best possible scenario, N_{it} would approach an asymptotic value independent of N. With this assumption both HEM1 and HEM2 lose out relative to HEM3 for large N due to their $\mathcal{O}(N^3)$ operations count. Figures 9 and 10 show operations count ratios of HEM2 and

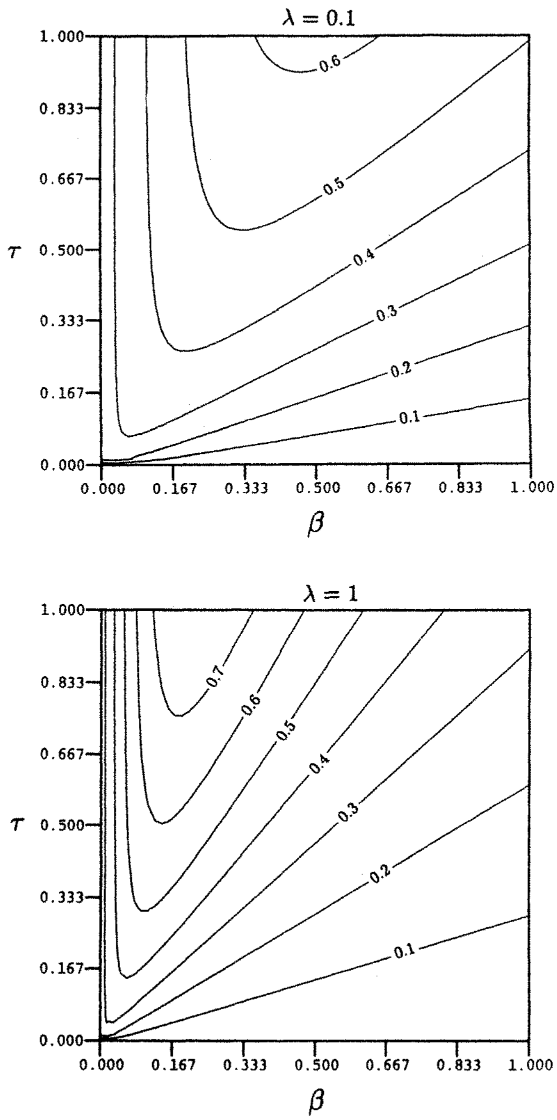

Figure 8. Contours of operations count ratio of HEM3 to HEM1 as a function of τ and β for $\lambda = 0{\cdot}1$ and $\lambda = 1{\cdot}0$ when iterative solution techniques are assumed

HEM3 to HEM1 as a function of N and N_{it}. For typical parameter values, the HEM2 operations count ratio is greater than unity for $N > 250$ and $N_{it} > 15$. This ratio can become quite large (>5 for $N \geqslant 1000$ and $N_{it} \geqslant 15$) when α is large ($\approx 0{\cdot}7$) and λ is small ($\approx 0{\cdot}1$), but it can also be less than unity ($\approx 0{\cdot}4$) when α is small ($\approx 0{\cdot}1$) and λ is large (≈ 2). The operations count for HEM3, on the other hand, is consistently less than that for HEM1. Using

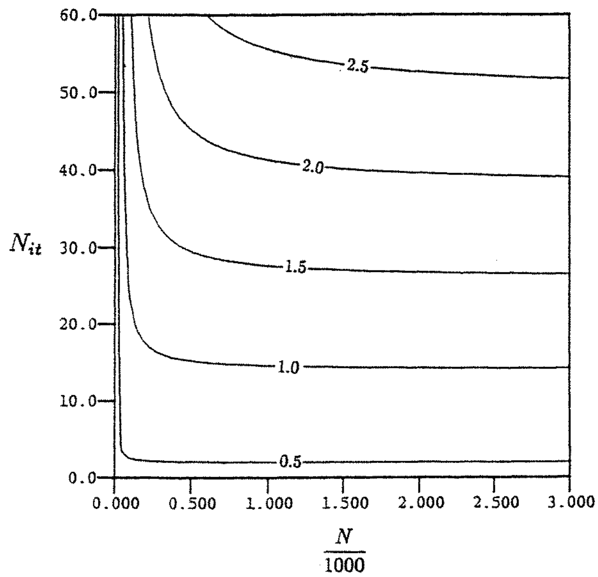

Figure 9. Contours of operations count ratio of HEM2 to HEM1 as a function of N_{it} and N. Typical case shown: $\alpha = 0{\cdot}5$, $\beta = 0{\cdot}2$, $\lambda = 1$, $\gamma_1 = 0{\cdot}2$, $\gamma_2 = 0{\cdot}1$

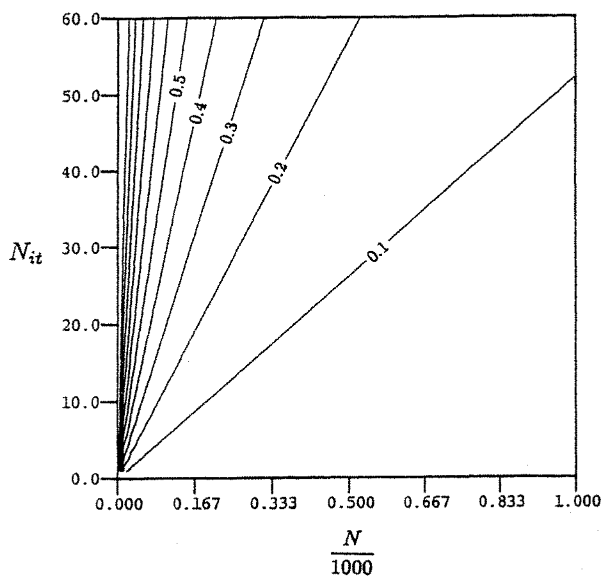

Figure 10. Same as Figure 9 for the operations count ratio of HEM3 to HEM1

typical parameter values for α, β and λ, the operations count ratio for these two algebras is approximately $0\cdot 1$ or less for $N \geqslant 500$ and $N_{\text{it}} \leqslant 20$. When α and λ are such that the operations count for HEM2 is less than that of HEM1, the operations count ratio for HEM3 to HEM1 is below $0\cdot 1$. Thus the operations count ratio for HEM3 to HEM2 is consistently less than unity even in the most favourable (to HEM2) scenarios.

It is informative to compare iterative with direct operations counts for the three hybrid algebras. The ratio of iterative to direct operations counts for HEM1 will be less than unity provided

$$\frac{\gamma_1 N_{\text{it}}}{\beta N} < 1 \tag{11a}$$

which should almost always be true since $\gamma_1 \approx \beta$ and $N_{\text{it}} \ll N$. The same can be said for HEM2 as long as

$$\gamma_2 N_{\text{it}} < 2\alpha + 1 \tag{11b}$$

This inequality will not always hold, but is likely to be true for $N_{\text{it}} < 20$ unless α is small, in which case $N_{\text{it}} < 10$ may be required. With HEM3 algebra, fewer operations are needed for an iterative approach than for a direct method if

$$\frac{\dfrac{N_{\text{it}}}{N}\left(\dfrac{\gamma_2\alpha}{\beta} + \dfrac{(1+\lambda)^2}{4}\right)}{(1+\lambda)^2 + \dfrac{(1+\lambda)^3}{2}\beta} < 1 \tag{11c}$$

Since $\gamma_2 \approx \beta$, this ratio will also be less than unity given that $N_{\text{it}} \ll N$.

CONCLUSIONS

Three hybrid-element algebras have been studied with respect to their computational costs. Scaling expressions for memory and operations count have been derived assuming both direct and iterative matrix solution techniques. When necessary, mesh-dependent parameters have been determined empirically from practical meshes. However, the expressions given are perfectly valid for other meshes once the parameters defined herein have been determined.

Results show that HEM2 is to be favoured when direct methods are applied. HEM2 has memory and operations counts which are about one-half those of its nearest competitor, HEM1. HEM1, on the other hand, is more attractive computationally than HEM3 under the direct solution scenario by approximately the same relative amounts. It has been observed that profile storage results in scale reduction factors of roughly $0\cdot 7$ for memory and operations counts compared to banded form for a given hybrid algebra. However, no significant inter-method gains in computational effort are found when using the profile storage strategy.

Iterative methods proved to be harder to compare because of the difficulty in quantifying the number of iterations needed to reach a solution. Nonetheless, under both conservative and optimistic assumptions, HEM3 is to be favoured when iterative solution techniques are applied to hybrid algebras. Comparing between methods showed that the memory requirements of HEM2 and HEM3 are identical and typically a little more than one-half of HEM1. However, in terms of operations count, HEM3 is the most economical, having a ratio of $0\cdot 1 \rightarrow 0\cdot 3$ with its nearest competitor, which was usually HEM1. HEM2 was consistently found to be the least competitive except in some extreme circumstances, where its operations count was approximately $2\cdot 5$ times less than HEM1, but still 4 times more than HEM3.

When compared to their direct solution counterparts, iterative approaches for the three hybrid algebras realized memory reductions of one-half or better in most cases (as low as $0{\cdot}2$ in some instances). Operations counts for iterative methods are, of course, sensitive to the number of iterations needed to reach convergence, but were found to be consistently less than direct solution equivalents for HEM1 and HEM3. HEM2, on the other hand, required convergence to be reached in 10–20 (or fewer) iterations for the operations count to be less than its direct solution approach.

Finally, it should be noted that the three hybrid method algebras were stable with respect to round-off errors, producing solutions that were essentially equivalent for the sets of FEM/BEM equations studies herein.

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