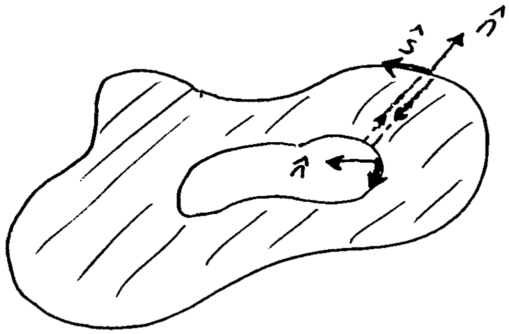
Holes, Heterogeneities, Infinite Domains

Keys: (i) $\hat{n}$ always points outward when viewed from "inside" the region of interest (2D has direction sense to $\oint$)

(ii) Use freespace Green's function appropriate for each region and integrate over <u>all</u> its boundaries

Examples:

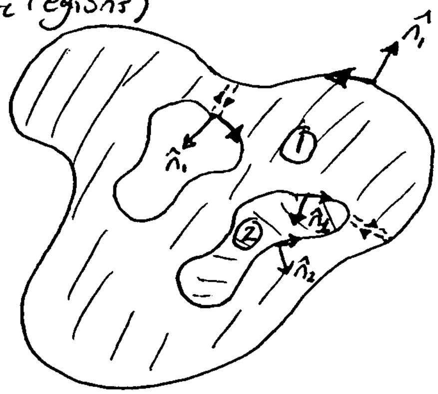


"Hole"

Convention: $\hat{n} \times \hat{s}$ is "out-of-the-page" by right-hand-rule

Concept: make one continuous integration around the boundary containing the object of interest by "cutting across" the object as needed ("back and forth" portions cancel identically! Since $\hat{n}$ is in opposite directions)

Heterogeneity (2 separate regions)

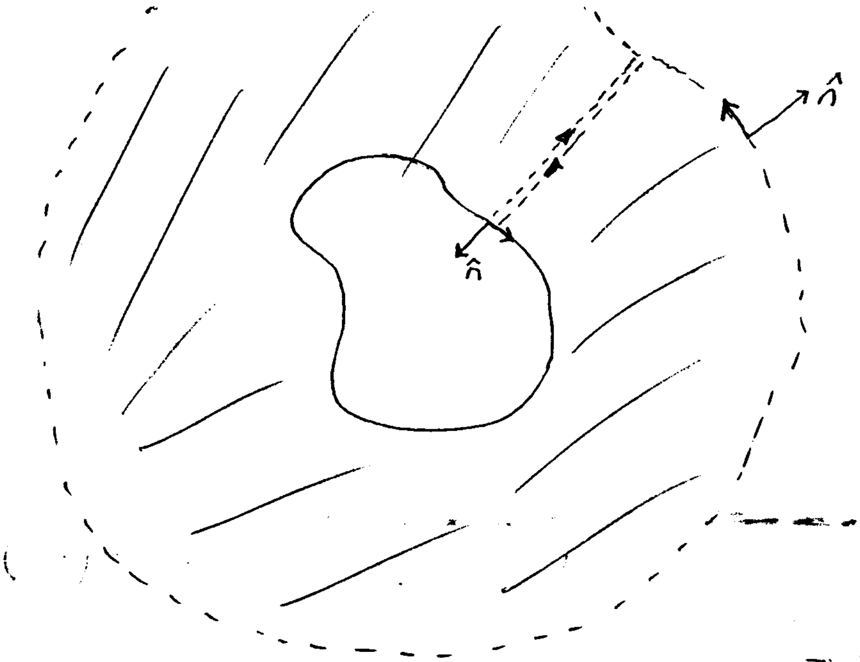


Two region problem w/ hole

Now have region ① with $\hat{n}_1$, region ② with $\hat{n}_2$

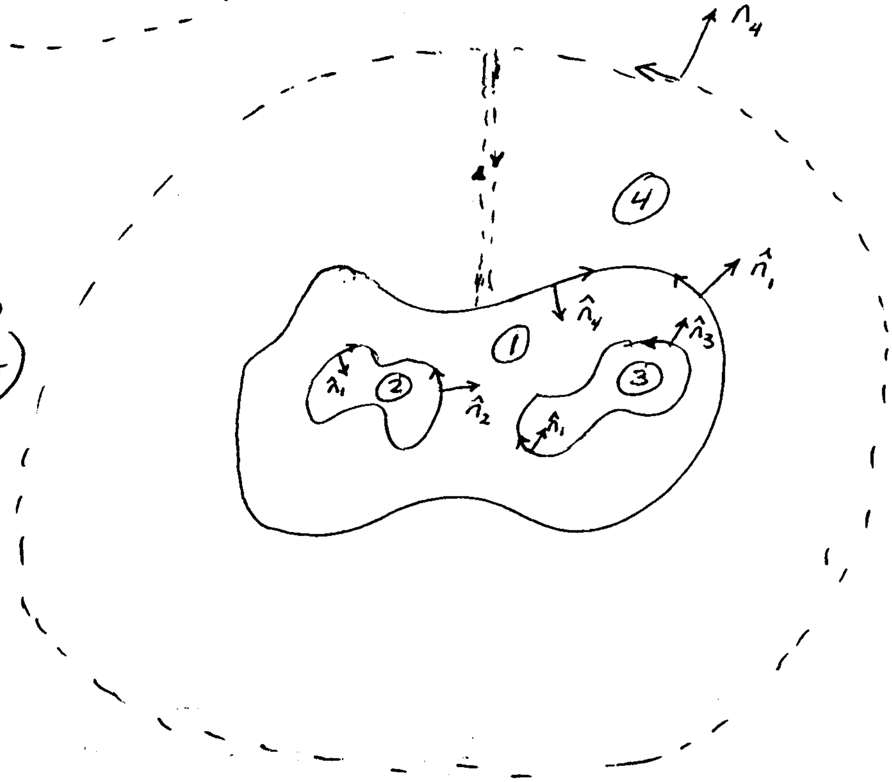
$\hat{n}_1 = -\hat{n}_2$ <u>but</u> $G_1 \neq G_2$; $\frac{\partial G_1}{\partial n} \neq \frac{\partial G_2}{\partial n}$

so <u>will</u> get contributions from <u>both</u> paths!



Infinite domain

Heterogeneity w/
Infinite domain
(4 separate regions)

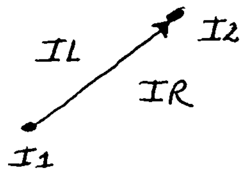


Need to expand our notion of a boundary element

Now is an item which separates two regions

Expand the Incidence list to allow this new view

2D: (plane)

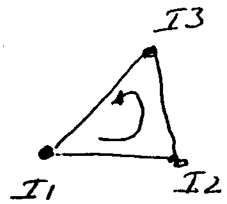


- Directed line segment (as before!)
- Divides the space in half: Left, right, ("Inside, outside")

- Incidence list: L, I1, I2, IL, IR

(By convention let the space "outside" the entire problem be region "0")

3D: (volume)

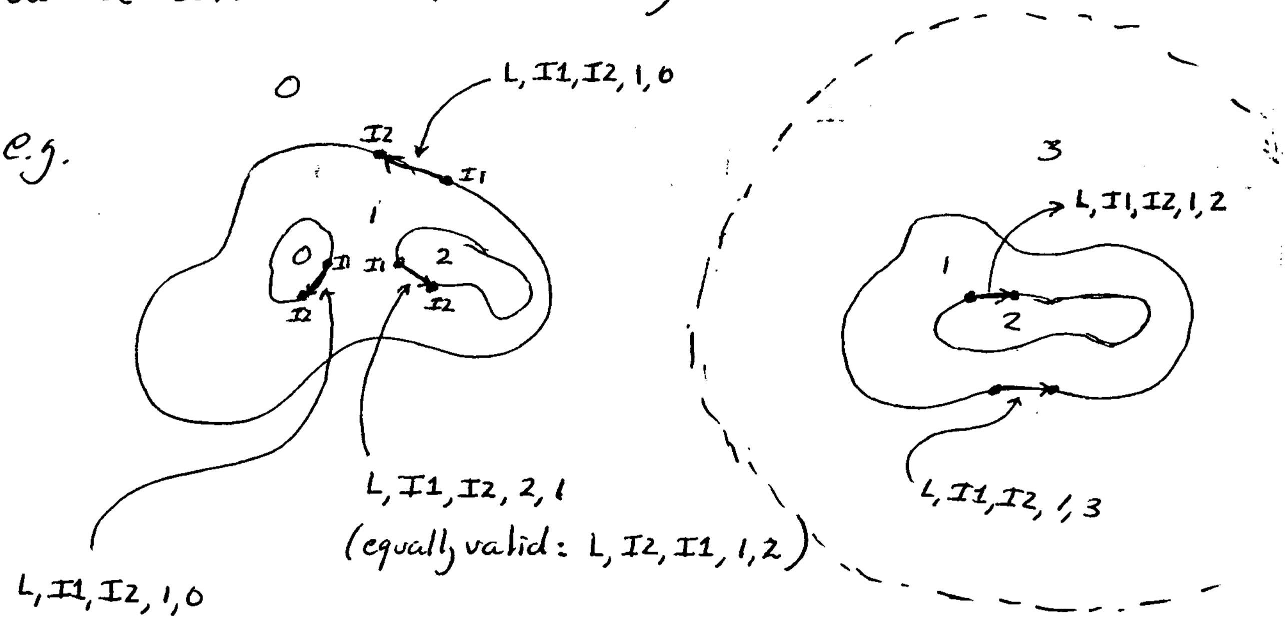


- Oriented Polygon
- Divides space in half: (Inside, outside)

- Incidence list: L, I1, I2, I3, IB, IT

Now can describe these more complicated objects easily as a collection of boundary elements

e.g.



Now look at some implementation details

<u>Holes:</u> Examine $\nabla^2 u = f$; $\nabla^2 G_i = -\delta_i$

$$\frac{\alpha_i}{2\pi} u_i = \oint \left(\frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u\right) ds - \langle f G_i \rangle$$

produces:

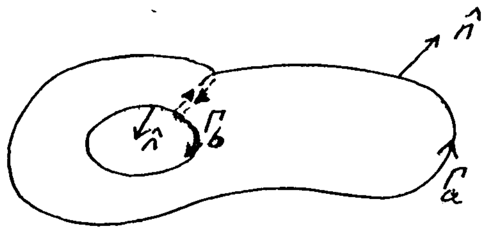
$$[A]\{u\} - [B]\left\{\frac{\partial u}{\partial n}\right\} = \{F\}$$

$$a_{ij} = \frac{\alpha_i}{2\pi}\delta_{ij} + \oint \frac{\partial G_i}{\partial n}\phi_j \, ds$$

$$b_{ij} = \oint G_i \phi_j \, ds$$

$$f_i = \langle -f G_i \rangle$$

Now:



$$\left[\begin{array}{c:c} A_{aa} & A_{ab} \\ \hdashline A_{ba} & A_{bb} \end{array}\right]\left\{\begin{array}{c} u_a \\ \hdashline u_b \end{array}\right\} - \left[\begin{array}{c:c} B_{aa} & B_{\,b} \\ \hdashline B_{ba} & B_{bb} \end{array}\right]\left\{\begin{array}{c} \frac{\partial u_a}{\partial n} \\ \hdashline \frac{\partial u_b}{\partial n} \end{array}\right\} = \left\{\begin{array}{c} F_a \\ \hdashline F_b \end{array}\right\}$$

No special care! Only need to make sure <u>$\hat{n}$ points outward</u>

Heterogeneities: $\nabla \cdot K \nabla u = f$

$$\nabla \cdot K \nabla G_i = -\delta_i$$

In this case: $G_i = -\frac{1}{2\pi K} \ln R_i \quad \frac{\partial G_i}{\partial n} = -\frac{1}{2\pi R_i K}(\hat{R}_i \cdot \hat{n})$

$$\frac{\alpha_i}{2\pi} u_i = \oint \Big(\underbrace{K\frac{\partial u}{\partial n}}_{\equiv q = \sum_j q_j \phi_j} G_i - K\frac{\partial G_i}{\partial n} u\Big) ds - \langle f G_i \rangle$$

then as before:

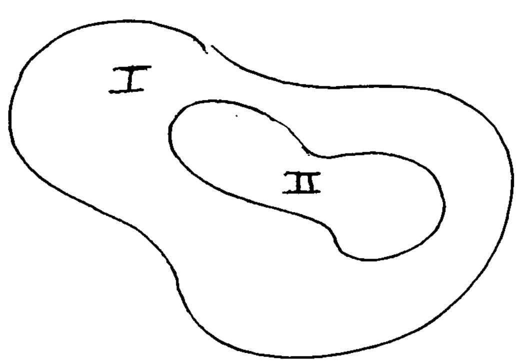
$$[A]\{u\} - [B]\{q\} = \{F\}$$

$$a_{ij} = \frac{\alpha_i}{2\pi}\delta_{ij} + \oint K \frac{\partial G_i}{\partial n} \phi_j \, ds$$

$$b_{ij} = \oint G_i \phi_j \, ds$$

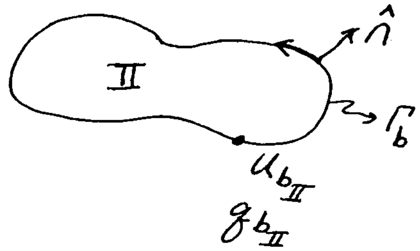
$$f_i = \langle -f G_i \rangle$$

Now the problem domain:



Key: Separate into 2 problems (one for each region)

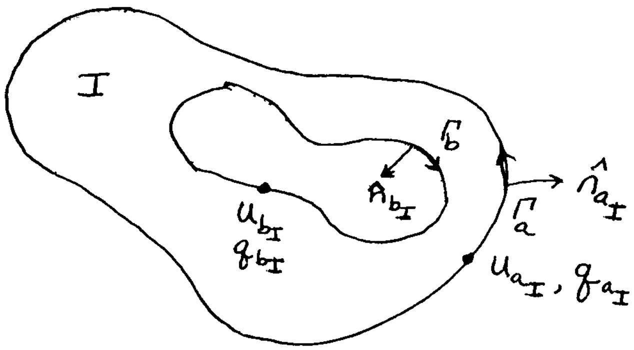
- Region II: easy... just like a simple region (single)



$$N_b \Big] \; [A_{bb}]_{II} \{u_b\}_{II} - [B_{bb}]_{II} \{q_b\}_{II} = \{F_b\}_{II}$$

Both $u_{b_{II}}$ and $q_{b_{II}}$ are unknown (have N_b equations)

- Region I: Same as "Hole" problem



$$\begin{array}{c} N_a \\ N_b \end{array} \left[\begin{array}{c|c} A_{aa} & A_{ab} \\ \hline A_{ba} & A_{bb} \end{array}\right]_I \left\{\begin{array}{c} u_a \\ \hline u_b \end{array}\right\}_I - \left[\begin{array}{c|c} B_{aa} & B_{ab} \\ \hline B_{ba} & B_{bb} \end{array}\right]_I \left\{\begin{array}{c} q_a \\ \hline q_b \end{array}\right\}_I = \left\{\begin{array}{c} F_a \\ \hline F_b \end{array}\right\}_I$$

Equations: $N_b + (N_a + N_b)$
Variables: $2 * (N_b + (N_a + N_b))$

BC's given: N_a

$\Rightarrow$ Need $2 * N_b$ more conditions to solve

Key: missing information comes from the "interface conditions" relating $(u_b, q_b)_I$ to $(u_b, q_b)_{II}$

Simplest Case: u, q continuous on Γ_b

Another $2 * N_b$ Conditions
$$\begin{cases} u_{b_I} = u_{b_{II}} \\ q_{b_I} = -q_{b_{II}} \Rightarrow K_I \dfrac{\partial u_b}{\partial n_I} = K_{II} \dfrac{\partial u_b}{\partial n_I} = \underbrace{-K_{II} \dfrac{\partial u_b}{\partial n_{II}}}_{q_{b_{II}}} \end{cases}$$

Combining Region II + I equations:

$$\begin{array}{c} N_{a_I} \\ N_{b_I} \\ N_{b_{II}} \end{array}
\left[\begin{array}{cc|c} A_{aa_I} & A_{ab_I} & 0 \\ A_{ba_I} & A_{bb_I} & \\ \hline 0 & & A_{bb_{II}} \end{array}\right]
\left\{\begin{array}{c} u_{a_I} \\ u_{b_I} \\ u_{b_{II}} \end{array}\right\}
-
\left[\begin{array}{cc|c} B_{aa_I} & B_{ab_I} & 0 \\ B_{ba_I} & B_{bb_I} & \\ \hline 0 & & B_{bb_{II}} \end{array}\right]
\left\{\begin{array}{c} q_{a_I} \\ q_{b_I} \\ q_{b_{II}} \end{array}\right\}$$

$$= \left\{\begin{array}{c} F_{a_I} \\ F_{b_I} \\ F_{b_{II}} \end{array}\right\}$$

Condense Matrix and Apply BCs (e.g. q_{a_I} known)

$$\underbrace{\begin{bmatrix} A_{aa_I} & A_{ab_I} & -B_{ab_I} \\ A_{ba_I} & A_{bb_I} & -B_{bb_I} \\ 0 & A_{bb_{II}} & B_{bb_{II}} \end{bmatrix}}_{A'} \underbrace{\begin{Bmatrix} u_{a_I} \\ u_{b_I} \\ q_{b_I} \end{Bmatrix}}_{X} = \underbrace{\begin{Bmatrix} F_{a_I} \\ F_{b_I} \\ F_{b_{II}} \end{Bmatrix} + \begin{bmatrix} B_{aa_I} \\ B_{ba_I} \\ 0 \end{bmatrix} \begin{Bmatrix} q_{a_I} \end{Bmatrix}}_{R}$$

$$A' \quad X = R$$

Inversion of $(N_a + 2*N_b)^2$ matrix

Run Time $\approx (N_a + 2*N_b)^3$

Solution produces $(u_a, u_b, q_b)_I$ simultaneously

Alternately... remove q_b's entirely ...

From Region II:

$$\{q_{b_{II}}\} = [B_{bb}]_{II}^{-1} [A_{bb}]_{II} \{u_b\} - [B_{bb}]_{II}^{-1} \{F_b\}_{II}$$

then

$$\begin{bmatrix} A_{aa_I} & A_{ab_I} + B_{ab_I} B^{-1}_{bb_{II}} A_{bb_{II}} \\ A_{ba_I} & A_{bb_I} + B_{bb_I} B^{-1}_{bb_{II}} A_{bb_{II}} \end{bmatrix} \begin{Bmatrix} u_a \\ u_b \end{Bmatrix} = \begin{Bmatrix} F_{a_I} + B_{aa_I} q_{a_I} + B_{ab_I} B^{-1}_{bb_{II}} F_{b_{II}} \\ F_{b_I} + B_{ba_I} q_{a_I} + B_{bb_I} B^{-1}_{bb_{II}} F_{b_{II}} \end{Bmatrix}$$

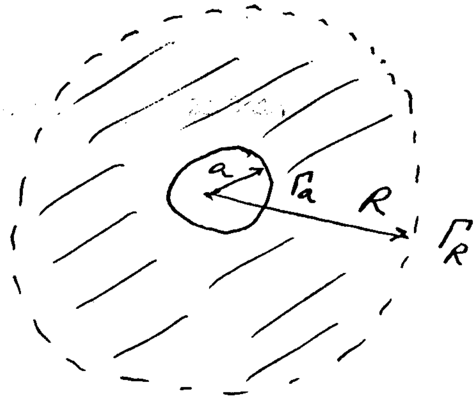
This system is $(N_a + N_b)^2$ in size and $(N_a + N_b)^3$ in Runtime

In this approach can view the internal boundary as "Type III-like", i.e. have a relationship between q_b and u_b ... only it is expressed as a matrix system ... i.e. q_b at a node is related to all nodal values of u_b

Infinite Domains

Key: get no contribution from integrations when boundaries are at infinity ... i.e. can ignore

- Examine for Laplace:



Assume u given on Γ_a then BEM in region exterior to Γ_a:

$$\frac{\alpha_i}{2\pi} u_i = \int_{\Gamma_a} \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, d\Gamma_a + \underbrace{\int_{\Gamma_R} \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, d\Gamma_R}_{\text{Look at this as } R \to \infty}$$

- Case 1: when "i" is on Γ_a

$$\lim_{R\to\infty} \int_{\Gamma_R} \underset{\uparrow \atop O(\ln R)}{\frac{\partial u}{\partial n} G_i} - \underset{\uparrow \atop O(1/R)}{\frac{\partial G_i}{\partial n}} u \, d\Gamma_R = \lim_{R\to\infty} \int_0^{2\pi} \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, R \, d\theta$$

General sol'n for 2D Laplace:

$$u(r,\varphi) = a_0 + b_0 \ln r + \sum_{n=1}^{\infty} e^{in\varphi}\left(a_n r^n + \frac{b_n}{r^n}\right) + \sum_{n=1}^{\infty} e^{-in\varphi}\left(c_n r^n + \frac{d_n}{r^n}\right)$$

Since u finite as $R\to\infty$, $a_n = c_n = b_0 = 0$

so u is $O(1/R)$; $\frac{\partial u}{\partial n}$ is $O(1/R^2)$

$$\therefore \lim_{R\to\infty} \underbrace{\int_0^{2\pi} O\left(\tfrac{1}{R^2}\right) O(\ln R) \, R\,d\theta}_{\lim_{R\to\infty} \frac{1}{R^2} R \ln R \Rightarrow \lim_{R\to\infty} \frac{\ln R}{R} = 0} - \underbrace{\int_0^{2\pi} O(1/R)\, O(1/R)\, R\,d\theta}_{\lim_{R\to\infty} O(1/R) = 0}$$

Conclude: get <u>No</u> contribution when "i" on Γ_a from Γ_R
$\therefore$ can ignore this integration

- Case 2: when "i" on Γ_R
don't care since can get system of N equations in N unknowns by discretizing only Γ_a

- Same strategy works in 3D, only enclose problem domain with sphere of radius R, then let $R\to\infty$