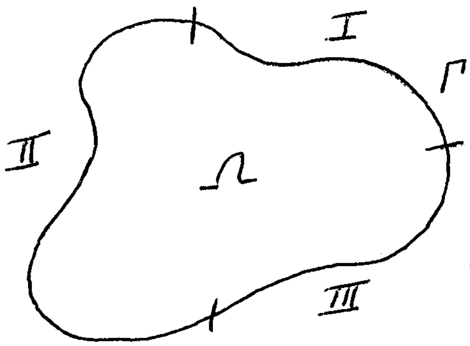
Multi-D BEM

Ex.: Poisson Eqn: $\nabla^2 u = f$

BC's: Type I or II or III on closed surface

u , $\frac{\partial u}{\partial n}$, $au + b\frac{\partial u}{\partial n}$



Domain Ω
Boundary Γ

Green's Fctn: $\nabla^2 G_i = -\delta(r_i)$

Be careful about sign convention

$$\langle \nabla^2 u, G_i \rangle = -\langle \nabla u \cdot \nabla G_i \rangle + \oint \frac{\partial u}{\partial n} G_i \, ds = \langle f G_i \rangle$$

$$\langle \nabla^2 G_i, u \rangle = -\langle \nabla G_i \cdot \nabla u \rangle + \oint \frac{\partial G_i}{\partial n} u \, ds = \langle -\delta_i u \rangle$$

Subtract: $$\langle \delta_i u \rangle = \underbrace{\oint \left(\frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \right) ds}_{\text{Homog. Sol'n (effects of BCs)}} - \underbrace{\langle f G_i \rangle}_{\text{Particular Sol'n (effects of forcing)}}$$

Same expression as before!

Note: Use of $\nabla^2 G_i = \delta(r_i)$ changes sign of all rhs terms

Sol'n method: 2-step Procedure

a.) Compute "missing" BC

b.) Use a.) to get interior sol'n

Step a.) - strategy is to employ G_i on boundary (i.e. move "i" to boundary)

So we discretize the boundary of the domain of interest

- Divide boundary into elements (boundary elements)

- Elements are basis of
  - i) local interpolation
  - ii) integration

  $\Big\}$ just like FEM

- with N (boundary) nodes:

  2N variables : u_i, $\frac{\partial u_i}{\partial n}$

  N BCs given

  N Green's Fctns on bdry (1/node)

This is a system of N eqn's in N unknowns!

(each G_i generates an equation ... in effect, G_i plays the role of weighting function in WR scheme of things)

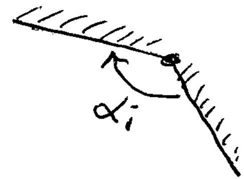
- On boundary: $u(s) = \sum u_j \phi_j(s)$

$$\frac{\partial u}{\partial n}(s) = \sum \frac{\partial u}{\partial n}_j \phi_j(s)$$

(2D BEM uses 1D bases... can have linear, quadratic, etc. ... same as 1D FEM)

- Since "i" moves to boundary which is discrete

$$\langle \delta_i u \rangle \Rightarrow \frac{\alpha_i}{2\pi} u_i$$



α_i is "interior angle"

So $\alpha = \pi$ on "smooth boundary" (straight)

$\alpha = 2\pi$ at interior pt

So the discretized boundary equations are:

$$\frac{\alpha_i}{2\pi} u_i = \sum_j \frac{\partial u_j}{\partial n} \oint G_i \phi_j(s) - \sum_j u_j \oint \frac{\partial G_i}{\partial n} \phi_j(s) - \langle f G_i \rangle$$

for $i = 1, 2 \ldots N$ (boundary nodes) produces

$$\underset{N \times N}{[A]} \underset{N \times 1}{\{u\}} = \underset{N \times N}{[B]} \underset{N \times 1}{\left\{\frac{\partial u}{\partial n}\right\}} + \underset{N \times 1}{\{F\}}$$

Matrix system

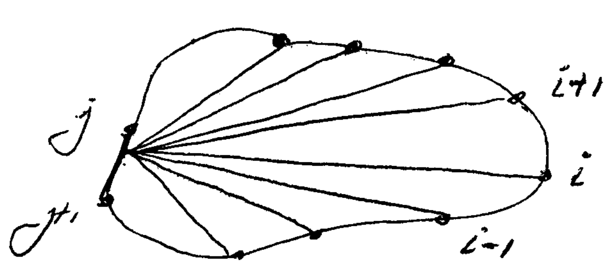
$$\left.\begin{aligned} a_{ij} &= \frac{\alpha_i}{2\pi}\delta_{ij} + \oint \frac{\partial G_i}{\partial n}\phi_j \\ b_{ij} &= \oint G_i \phi_j \end{aligned}\right\}$$

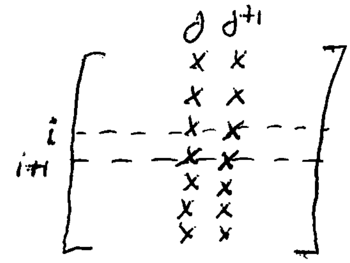
Homogeneous Problem only has boundary integrals

$f_i = \langle -f G_i \rangle \Rightarrow$ Inhomogeneous Problem has domain integrals

Features:

- G_i is singular around "i"; need special care
- G_i has global support $\therefore$ matrix is <u>full</u>
- matrix is nonsymmetric
- each element contributes something to each equation

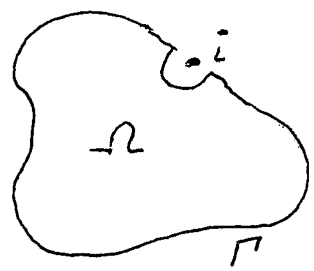




<u>Digression</u>!

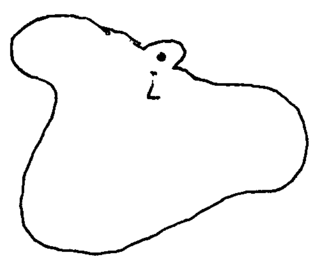
Look at boundary expression integrated around singularity "i" more closely ... do this from 2 perspectives:

A:

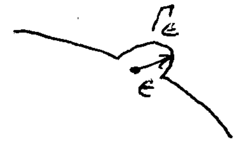


"i" is on the "boundary" but Ω is perturbed to exclude point "i", then let $\epsilon \to 0$

B:



Ω perturbed to include "i" within semi-circle perturbation, then let $\epsilon \to 0$



Case A: Since i excluded from Ω, on Ω $\nabla^2 G_i = 0$

So boundary expression becomes

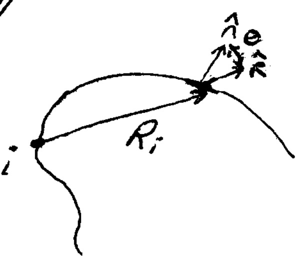
$$\oint \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, ds = 0$$

But $$\oint \frac{\partial u}{\partial n} G_i \, ds = \int_{\Gamma - \Gamma_\epsilon} \frac{\partial u}{\partial n} G_i \, ds + \lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} \frac{\partial u}{\partial n} G_i \, ds$$

$$\oint \frac{\partial G_i}{\partial n} u \, ds = \int_{\Gamma - \Gamma_\epsilon} \frac{\partial G_i}{\partial n} u \, ds + \lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} \frac{\partial G_i}{\partial n} u \, ds$$

To see what is going on... easiest to examine a specific G_i.

For Laplace in 2D: $G_i = -\dfrac{\ln(R_i)}{2\pi}$ $\quad R_i = |(x,y)_i - (x,y)_s|$



(also $\hat{n} \cdot \nabla G$) $\rightarrow$ $$\frac{\partial G_i}{\partial n} = \frac{\partial G}{\partial R}\frac{\partial R}{\partial n} = \frac{-1}{2\pi R_i} \hat{R} \cdot \hat{n}$$

$$= \frac{-1}{2\pi R_i} \cos\theta$$

So $$\oint \frac{\partial u}{\partial n} G_i \, ds = \int_{\Gamma - \Gamma_\epsilon} + \lim_{\epsilon \to 0} -\int_0^{\pi} \frac{\partial u}{\partial n} \frac{\ln \epsilon}{2\pi} \epsilon \, d\theta$$

Now $$\lim_{\epsilon \to 0} \epsilon \ln \epsilon = \lim_{\epsilon \to 0} \frac{\ln \epsilon}{1/\epsilon} = \frac{\infty}{\infty} \Rightarrow \text{L'Hospital!}$$

$$\lim_{\epsilon \to 0} \frac{1/\epsilon}{-1/\epsilon^2} = \lim_{\epsilon \to 0} -\frac{\epsilon^2}{\epsilon} = 0$$ No contribution!

likewise for

$$\oint \frac{\partial G_i}{\partial n} u \, ds = \int_{\Gamma-\Gamma_\epsilon} \frac{\partial G_i}{\partial n} u \, ds + \lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} \frac{\partial G_i}{\partial n} u \, ds$$

$$= \int_{\Gamma-\Gamma_\epsilon} + \lim_{\epsilon \to 0} -\int_0^{\pi} u \frac{1}{2\pi\epsilon}(-1)\,\epsilon \, d\theta$$

$\hat{n} = -\hat{R}$

$$= \int_{\Gamma-\Gamma_\epsilon} + \frac{u_i}{2\pi} \lim_{\epsilon \to 0} \int_0^{\pi} d\theta = \int_{\Gamma-\Gamma_\epsilon} + \frac{u_i}{2}$$

So... $$\oint \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, ds = \int_{\Gamma-\Gamma_\epsilon} \frac{\partial u}{\partial n} G_i - \int_{\Gamma-\Gamma_\epsilon} \frac{\partial G_i}{\partial n} u - \frac{u_i}{2} = 0$$

or $$\boxed{\frac{u_i}{2} = ⨍ \frac{\partial u}{\partial n} G_i - ⨍ \frac{\partial G_i}{\partial n} u \, ds}$$ $⨍ \equiv$ Principal Part Integration

This is exactly the case $\alpha = \pi$ (smooth boundary)

Get identical result for Case B:

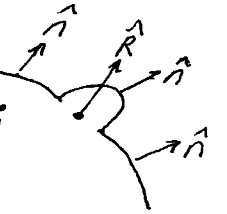
In this case we have

$$u_i = \oint \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, ds \quad \left(\text{since "}i\text{" is inside } \Omega\right)$$

i.e. $\langle \delta_i u \rangle = u_i$

Same arguments as above can be used to show. $\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} \frac{\partial u}{\partial n} G_i = 0$ and

$$\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} u \frac{\partial G_i}{\partial n} ds = -\frac{u_i}{2}$$ (since $\hat{R} = \hat{n}$:



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So $u_i = \oint \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, ds + \frac{u_i}{2}$

or $\frac{u_i}{2} = \oint \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, ds$ Identical!!

Now Back to Poisson Eqn Solution via BEM!!

Summarize: $\nabla^2 u = f$

BEM Formulation:

$$\frac{\alpha_i}{2\pi} u_i = \sum_j \frac{\partial u_j}{\partial n} \oint G_i \phi_j - \sum_j u_j \oint \frac{\partial G_i}{\partial n} \phi_j - \langle f G_i \rangle$$

where $G_i = -\frac{\ln(R_i)}{2\pi}$; $\frac{\partial G_i}{\partial n} = -\frac{1}{2\pi R_i} \hat{R}_i \cdot \hat{n}$, α_i = interior angle (radians)

Matrix System:

$$[A]\{u\} = [B]\left\{\frac{\partial u}{\partial n}\right\} + \{F\}$$

where

$$a_{ij} = \frac{\alpha_i}{2\pi} \delta_{ij} + \oint -\frac{\hat{n} \cdot \hat{R}_i}{2\pi R_i} \phi_j$$

$$b_{ij} = \oint -\frac{\ln R_i}{2\pi} \phi_j \, ds$$

$$f_i = \left\langle \frac{\ln R_i}{2\pi} f \right\rangle$$