Forcing Terms

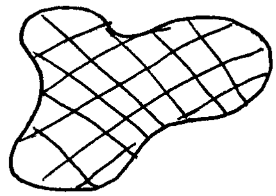
- Easily handled ... end up as RHS contribution involving domain integration:

$$[A]\{u\} = [B]\left\{\frac{\partial u}{\partial n}\right\} + \{F\}$$

where $f_i = \langle -f G_i \rangle$ e.g. when $\nabla^2 u = f$

- when f is distributed ... divide domain into subregions and numerically integrate

e.g.



need something like FE grid, then $\langle f G_i \rangle$ is easy to assemble using element structure for local integration

- Considerable effort has gone into transforming $\langle f G_i \rangle$ to boundary since the need for $\langle \rangle$ destroys some of the discretization advantages of the BEM

- Most results are for Laplace operator (i.e. only specialized results exist) e.g. $\nabla^2 u = f$ where $\nabla^2 f = 0$

From Green's Second Identity:

$$\langle f \nabla^2 w - w \cancelto{0}{\nabla^2 f} \rangle = \oint f \frac{\partial w}{\partial n} - w \frac{\partial f}{\partial n}$$

so $\langle f \nabla^2 w \rangle = \oint f \frac{\partial w}{\partial n} - w \frac{\partial f}{\partial n} \, ds$

But we want $\langle f G_i \rangle \Rightarrow$ Let $\nabla^2 w = G_i$

then the w needed must satisfy $\nabla^4 w = -\delta(\underline{x} - \underline{x}_i)$

Sol'n is known in this case:

$$w = -\frac{r^2}{8\pi}(\ln r - 1) \qquad 2D$$

$$= \frac{1}{8\pi} r \qquad 3D$$

$$\therefore \langle f G_i \rangle = \underbrace{\oint f \frac{\partial w_i}{\partial n} - w_i \frac{\partial f}{\partial n} \, ds}_{\text{Boundary only}}$$

Also w_i <u>not</u> singular at i

- More generalized approach (Pina 1990)

works for "<u>homogeneous functions</u>", $f(\underset{\sim}{x})$

- $f(\underset{\sim}{x})$ is "homogeneous of degree α" if

$$f(\lambda \underset{\sim}{x}) = \lambda^{\alpha} f(\underset{\sim}{x})$$

e.g. $f(\underset{\sim}{x}) = x^2 y^3$, $f(\lambda \underbrace{\underset{\sim}{x}}_{(x,y)}) = (\lambda x)^2 (\lambda y)^3$

$= \lambda^5 x^2 y^3$

- $\nabla f(\lambda \underset{\sim}{x}) \cdot \underset{\sim}{x} = \alpha \lambda^{\alpha-1} f(\underset{\sim}{x})$

$N \equiv$ problem dimension

1D $\Rightarrow N=1$

2D $\Rightarrow N=2$

etc.

Now $\nabla \cdot (f \underset{\sim}{x}) = \overbrace{\nabla f \cdot \underset{\sim}{x}}^{\alpha f \text{ (since } \lambda=1)} + f \overbrace{\nabla \cdot \underset{\sim}{x}}^{N}$

$= (\alpha + N) f$

so $\langle \nabla \cdot (f \underset{\sim}{x}) \rangle = \oint f \underset{\sim}{x} \cdot \hat{n} \, ds = (\alpha + N) \langle f \rangle$

<u>or</u>
$$\boxed{\langle f \rangle = \frac{1}{\alpha + N} \oint f \underset{\sim}{x} \cdot \hat{n} \, ds}$$

For Laplacian operator we have $\langle f L_N r \rangle$

which needs to be re-expressed as boundary integrals

Try the same approach...

$$\nabla\cdot(f\,\mathrm{Ln}\,r\,\underset{\sim}{x}) = f\underbrace{\nabla\cdot(\mathrm{Ln}\,r\,\underset{\sim}{x})}_{\underbrace{\frac{1}{r}\nabla r\cdot\underset{\sim}{x}}_{1} + \mathrm{Ln}\,r\underbrace{\nabla\cdot x}_{N}} + \overbrace{\nabla f\cdot\underset{\sim}{x}}^{\alpha f}\,\mathrm{Ln}\,r$$

$$= f + (N+\alpha) f\,\mathrm{Ln}\,r$$

this is what we need

$$\langle\nabla\cdot(f\,\mathrm{Ln}\,r\,\underset{\sim}{x})\rangle = \oint f\,\mathrm{Ln}\,r\,\underset{\sim}{x}\cdot\hat{n}\,ds = \underbrace{\langle f\rangle}_{\frac{1}{\alpha+N}\oint f\underset{\sim}{x}\cdot\hat{n}\,ds} + (N+\alpha)\langle f\,\mathrm{Ln}\,r\rangle$$

$$\therefore\ \boxed{\begin{aligned}\langle f\,\mathrm{Ln}\,r\rangle &= \frac{1}{N+\alpha}\oint f\,\mathrm{Ln}\,r\,\underset{\sim}{x}\cdot\hat{n}\,ds - \oint\frac{f\,\underset{\sim}{x}\cdot\hat{n}}{(\alpha+N)^2}\,ds \\ &= \frac{1}{N+\alpha}\left[\oint\left[\mathrm{Ln}\,r - \frac{1}{(N+\alpha)}\right]f\,\underset{\sim}{x}\cdot\hat{n}\,ds\right]\end{aligned}}$$

Note: Common case $f=$ constant $\Rightarrow \alpha = 0$

$$\langle\mathrm{Ln}\,r\rangle = \frac{1}{N}\left[\oint\left(\mathrm{Ln}\,r - \frac{1}{N}\right)\underset{\sim}{x}\cdot\hat{n}\,ds\right]$$

e.g. $N=2$ (2D), $f = x^m y^n \Rightarrow \alpha = m+n$

plug in and integrate around <u>boundary only</u>