Spectral Approximation of Functions

- Idea: expand sol'n in terms of set of orthogonal functions with goal of achieving spectral accuracy (can get this due to rapid convergence of such expansions in approximating smooth functions)

- Can be done for both periodic and nonperiodic functions... but must choose basis functions carefully.

- Expansion in an orthogonal basis

$$u = \sum_{k=-\infty}^{\infty} a_k \phi_k(x)$$

introduces a linear transformation between u and a_k ... if complete, can invert ...

i.e. can express functions through their values in physical space <u>or</u> through coefficients in transformed space

- But a_k's depend on <u>all</u> values of u in physical space ... typically can't compute these exactly

Instead: compute a finite number of approximate a_k's using u at a finite number of positions, typically the sample positions of high accuracy quadrature formulas

$\Rightarrow$ This Gives Rise to: "discrete transform" between values of u at quadrature points and the set of approximate (discrete) a_k's

- The finite series (discrete transform) is actually the interpolant of u at the quadrature points

- IF spectral accuracy can be maintained in replacing complete transform w/ discrete transform, the interpolant series can be used instead of the truncated series (of the complete transform) to approximate a function

- Want to examine those orthogonal systems that guarantee spectral accuracy

    Most Famous: Fourier System

Fourier Spectral Method

$$u = \sum_k \hat{u}_k \phi_k(x)$$

$\hookrightarrow \phi_k(x) = e^{ikx}$ $\quad 0 \le x \le 2\pi$

k = integer

complex conjugate

then $$\int_0^{2\pi} \phi_k(x)\,\phi_l^*(x)\,dx = 2\pi\,\delta_{kl}$$

So $$\int_0^{2\pi} u\,\phi_l^*\,dx = \sum \hat{u}_k \int_0^{2\pi} \phi_k\,\phi_l^*\,dx = 2\pi\,\hat{u}_l$$

i.e. $$\underbrace{\hat{u}_k}_{\text{"Fourier Coefficients" of } u} = \frac{1}{2\pi}\int_0^{2\pi} u\,\phi_k^*(x)\,dx = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} u\,e^{-ikx}\,dx}_{\text{"Fourier Transform" of } u}$$

$$u = \sum_{k=-\infty}^{\infty} \hat{u}_k \phi_k(x) \Rightarrow \text{"Fourier Series" of } u$$

Want to know:

(i) when does it converge

(ii) how fast

Basic Question ... $P_N u = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{u}_k e^{ikx}$ $\left.\begin{matrix}\text{Finite} \\ \text{Truncated}\end{matrix}\right\}$ FS

how does this approximate u as N gets large?

Key Results:

(i) u continuous, periodic, bounded on $[0, 2\pi]$ then Fourier Series is uniformly convergent

$$\max_{x \in [0,2\pi]} |u(x) - P_N u(x)| \to 0 \quad \text{as } N \to \infty$$

(ii) u is bounded on $[0, 2\pi]$, $P_N u(x)$ converges pointwise to $\frac{u(x^+) + u(x^-)}{2}$ for $x \in [0, 2\pi]$

(iii) u is continuous, periodic, Fourier Series does not necessarily converge at every pt on $[0, 2\pi]$

(iv) Fourier Series convergent in the mean

$$\int_0^{2\pi} |u(x) - P_N u(x)|^2 dx \to 0, \quad N \to \infty$$

(v). Speed of Convergence

$$\|u\|^2 = \langle u u^* \rangle = \int_0^{2\pi} \left(\sum_{k=-\infty}^{\infty} \hat{u}_k \phi_k\right)\left(\sum_{m=-\infty}^{\infty} \hat{u}_m^* \phi_m^*\right) dx$$

$$= \sum_k \sum_m \hat{u}_k \hat{u}_m^* \int_0^{2\pi} \phi_k \phi_m^* dx$$

$$= 2\pi \sum_{k=-\infty}^{\infty} |\hat{u}_k|^2 \quad \text{"Parseval's Identity"}$$

then $\|u - P_N u\| = \| \sum_{k=-\infty}^{\infty} \hat{u}_k \phi_k - \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{u}_k \phi_k \|$

$$= \| \sum_{|k| \gtrsim \frac{N}{2}} \hat{u}_k \phi_k \| \quad \left(\sum_{|k| \gtrsim \frac{N}{2}} \equiv \sum_{\substack{k < -\frac{N}{2} \\ k \geq \frac{N}{2}}} \right)$$

$$= \left(2\pi \sum_{|k| \gtrsim \frac{N}{2}} |\hat{u}_k|^2 \right)^{1/2}$$

So $\int_0^{2\pi} |u - P_N u|^2 dx = \int_0^{2\pi} \left(\sum_{k \gtrsim N/2} \hat{u}_k \phi_k \right)^2 dx \leq \int_0^{2\pi} \sum_{k \gtrsim N/2} |\hat{u}_k \phi_k|^2 dx$

$$|u - P_N u|^2 \leq \sum_{k \gtrsim \frac{N}{2}} |\hat{u}_k|^2 \underbrace{|\phi_k|^2}_{\text{unity}}$$

$$|u - P_N u| \leq \sum_{k \gtrsim N/2} |\hat{u}_k|$$

Conclude: size of error created by replacing u w/ N term truncated Fourier Series depends on how fast $u_k \to 0$

IF u is continuously differentiable on $[0, 2\pi]$

$$\hat{u}_k = \frac{1}{2\pi} \int_0^{2\pi} u e^{-ikx} dx$$

$$= -\frac{1}{2\pi i k} u e^{-ikx} \Big|_0^{2\pi} + \frac{1}{2\pi i k} \int_0^{2\pi} u' e^{-ikx} dx$$

$$= \frac{1}{2\pi i k}\left(u(2\pi) - u(0)\right) + \frac{1}{2\pi i k} \underbrace{\int_0^{2\pi} u' e^{-ikx}\,dx}_{-\frac{1}{ik}\left(u'(2\pi)-u'(0)\right) + \frac{1}{ik}\int_0^{2\pi} u'' e^{-ikx}\,dx}$$

which is $O(k^{-1})$

Now if u' is also continuously differentiable on $[0, 2\pi]$, then $\frac{1}{2\pi}\int_0^{2\pi} u' e^{-ikx}\,dx$ is the kth Fourier coefficient of u' which decays $O(k^{-1})$

which means $\hat{u}_k$ is $O(k^{-2})$ if $u(2\pi) = u(0)$
(i.e. u is periodic)

- Can continue this argument... $\hat{u}_k$ is $O(k^{-3})$ if u'' is continously differentiable and

$$u(2\pi) = u(0)$$
$$u'(2\pi) = u'(0)$$
$$\vdots$$

Conclude: if u is m times continuously differentiable on $[0, 2\pi]$ and if u^{j} is periodic for all $j \le m-2$, then

$$\hat{u}_k = O(k^{-m})$$

<u>Hence</u>: the k^{th} Fourier coefficient of a function which is infinitely differentiable and periodic with all its derivatives on $[0, 2\pi]$ decays <u>faster</u> than any negative power of k

"Spectral Accuracy"

Discrete Fourier Expansion

- Fourier coefficients of arbitrary function not known in closed form, must be approximated; also need practical way to recover physical space info from transformed space knowledge ... key is DFT

let $x_j = \frac{2\pi j}{N}$ $\quad j = 0, 1, \ldots N-1 \Rightarrow$ "nodes" "knots" "grid pts"

$$\underbrace{\tilde{u}_k}_{} = \frac{1}{N}\sum_{j=0}^{N-1} u(x_j)\, e^{-ikx_j} \qquad -\frac{N}{2} \le k \le \frac{N}{2} - 1$$

discrete Fourier coefficients ...

with orthogonality relation

$$\frac{1}{N}\sum_{j=0}^{N-1} e^{ilx_j} = \begin{cases} 1 & \text{if } l = Nm, \; m = 0, \pm 1, \pm 2 \ldots \\ 0 & \text{otherwise} \end{cases}$$

and inversion formula

$$u(x_j) = \sum_{k=-\frac{N}{2}}^{N/2-1} \tilde{u}_k\, e^{ikx_j} \qquad j = 0, 1, \ldots N-1$$

then the polynomial

$$I_N u(x) = \sum_{k=-\frac{N}{2}}^{N/2} \tilde{u}_k e^{ikx}$$

is an $\frac{N}{2}$ degree trigonometric interpolant of u at the nodes

i.e. $I_N u(x_j) = u(x_j)$, $j=0,1,\dots N-1$

discrete Fourier series of $u(x)$

$\tilde{u}_k$'s only depend on N values of u_j at $x_j = \frac{2\pi j}{N}$

So DFT is the mapping of N numbers u_j, $j=0,1..$ and N numbers $\tilde{u}_k$, $k=-\frac{N}{2}..\frac{N}{2}-1$ (complex!)

Note: continuous Fourier coefficients $\left(\hat{u}_k = \frac{1}{2\pi}\int_0^{2\pi} u e^{-ikx}\right)$ of the interpolant $I_N u(x)$ are exactly $\tilde{u}_k$

$$\hat{u}_k = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{l=-\frac{N}{2}}^{N/2} \tilde{u}_l e^{ilx}\right) e^{-ikx} dx$$

$$= \frac{1}{2\pi}\sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{u}_l \underbrace{\int_0^{2\pi} e^{i(l-k)x} dx}_{=0 \text{ unless } l=k} = \frac{1}{2\pi}\int_0^{2\pi} \tilde{u}_k dx = \tilde{u}_k$$

So $\tilde{u}_k$ can be regarded as an approximation to $\hat{u}_k$ using trapezoidal rule integration to evaluate $\int_0^{2\pi} u(x) e^{-ikx} dx$!!

- Can express discrete coefficients in terms of exact Fourier coefficients as

$$\tilde{u}_k = \hat{u}_k + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \hat{u}_{k+mN}$$

i.e. kth mode of the trigonometric polynomial (interpolant) of u depends on not only the kth mode of u, but also on all modes of u which "alias" the kth one on the discrete grid

i.e. the $(k+mN)$th frequency "aliases" the kth frequency since it is indistinguishable at the nodes

$$\text{i.e. } \phi_{k+mN}(x_j) = \phi_k(x_j) \Rightarrow e^{ikx_j}$$

$$e^{i(k+mN)x_j} = e^{ikx_j} \underbrace{e^{iNmx_j}}_{e^{imN\frac{2\pi}{N}j} = 1}$$

then we can write

$$I_N u = \sum_{k=-\frac{N}{2}}^{N/2-1} \tilde{u}_k e^{ikx} = \sum_{k=-N/2}^{N/2-1} \Big(\hat{u}_k + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \hat{u}_{k+mN} \Big) e^{ikx}$$

$$= \underbrace{\sum_{k=-N/2}^{N/2-1} \hat{u}_k e^{ikx}}_{P_N u} + \underbrace{\sum_{k=-N/2}^{N/2-1} \left(\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \hat{u}_{k+mN} \right) e^{ikx}}_{R_N u}$$

error between interpolating polynomial and Truncated Fourier Series

"aliasing error"

Turns out that one can write

$$\|u - I_N u\|^2 = \|u - P_N u\|^2 + \|R_N u\|^2$$

$\therefore$ error due to interpolation always larger than error due to truncation of Fourier Series

But one key result (Kreiss and Oliger 1979)

... influence of aliasing errors on accuracy is of the same order as the truncation error

i.e. truncation errors and interpolation errors decay at same rate

- Interpolating polynomials exhibit similar convergence properties to those of truncated Fourier Series.

i.e. as $N \to \infty$

1.) If u continuous, periodic and bounded on $[0, 2\pi]$, $I_N u$ converges uniformly on $[0, 2\pi]$

2. If u is bounded on $[0, 2\pi]$, $I_N u$ is uniformly bounded on $[0, 2\pi]$ and converges pointwise to u at every continuity pt. for u