Technical Note

Why use double nodes in BEM?

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In this paper we present an alternative approach to the double nodes concept used at corner points in the boundary element formulation, of the solution of boundary value problems for partial differential equations. In this approach, no double nodes are used and no BIE are considered at corners.

Key Words: complete set of orthonormal vectors, tangential derivatives, gradients, scalar field, displacements, tractions

INTRODUCTION

It is well known that there are boundary value problems for partial differential equations in which the number of unknowns exceeds the number of mutually independent boundary integral equations (BIE), in the boundary element formulation of the solution of these problems. Such a situation occurs at corners if the continuous boundary elements[1] are employed and the secondary fields (such as the normal derivative of the primary scalar field or tractions in elasticity) are unknown at the corner point on two or more boundary elements. The double (or multiple) nodes concept (see e.g. Refs 1–3) is employed almost exclusively for the treatment of this problem. In that approach the system of the BIE is supplemented by additional equations considered at multiple nodes.

After short thinking everybody finds that it is not necessary to use the double nodes concept, since two noncolinear tangent vectors at a corner point in two dimensions (or noncoplanar tangent vectors in the case of three dimensional problems) can be used successfully in the definition of a complete set of orthonormal vectors. Hence, also the gradient vector at the corner point can be expressed in terms of the tangential derivatives defined on the elements intersecting in the corner point[4,5]. This alternative technique is described in the present paper for a scalar field (for instance the temperature field) and displacements in the theory of elasticity for both the two and three dimensional problems.

TWO DIMENSIONAL PROBLEMS

Consider a boundary value problem for a scalar field $\theta(\vec{x})$ satisfying the Poisson equation. It is well known[1] that the solution of this problem can be expressed in terms of the boundary values of the field $\theta(\vec{\eta})$ and its normal derivative $q(\vec{\eta})$. Let the nodal point $\vec{\zeta}^b$ be the intersection of the elements Γ_q and $\Gamma_{q'}$ at which the normal is discontinuous (see Fig. 1).

Since the tangent vectors τ_i^{nq} and $\tau_i^{1q'}$ taken at the last and first nodal point of the elements Γ_a and $\Gamma_{q'}$, respectively, are noncolinear, they can be employed in the definition of a complete set of orthonormal vectors at $\vec{\zeta}^b$. Hence, the gradient $\theta_{,i}(\vec{\zeta}^b)$ can also be expressed in terms of the tangential derivatives

$$\frac{\partial\theta}{\partial\vec{\tau}}(\vec{\eta}^{nq}) \quad \text{and} \quad \frac{\partial\theta}{\partial\vec{\tau}}(\vec{\eta}^{1q'})$$

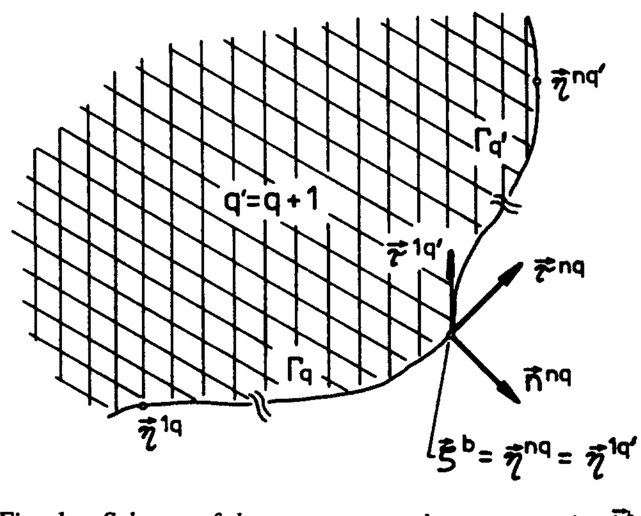

Fig. 1. Scheme of the geometry at the corner point $\vec{\zeta}^b$ in two dimensions

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Let the complete set of orthonormal vectors at $\vec{\zeta}^b$ be $(\vec{\tau}^{nq}, \vec{n}^{nq})$ where $\vec{n}^{nq}$ is the unit outward normal vector at $\vec{\eta}^{nq}$. In order to express $\vec{n}^{nq}$ in terms of $\vec{\tau}^{1q'}$, we define the vector $\vec{g}^b$ as

$$g_i^b = \epsilon_{ijk}\tau_j^{nq}\tau_k^{1q'} = \delta_{i3}g^b, \quad g^b = \tau_1^{nq}\tau_2^{1q'} - \tau_2^{nq}\tau_1^{1q'} \tag{1}$$

Then,

$$n_i^{nq} = \epsilon_{ik3}\tau_k^{nq} = \frac{1}{g^b}\epsilon_{ikj}g_j^b\tau_k^{nq} = \frac{1}{g^b}(A^b\tau_i^{nq} - \tau_i^{1q'}) \tag{2}$$

where A^b stands for the scalar product

$$A^b = \tau_k^{nq}\tau_k^{1q'}$$

Now the gradient vector at $\vec{\zeta}^b$ becomes

$$\partial_i'\Big|_{\vec{\zeta}^b} \equiv \frac{\partial}{\partial\eta_i}\Big|_{\vec{\zeta}^b} = (n_i^{nq}n_j^{nq} + \tau_i^{nq}\tau_j^{nq})\partial_j'\Big|_{\vec{\zeta}^b}$$
$$= v_i^b\frac{\partial}{\partial\vec{\tau}}\Big|_{\vec{\eta}^{nq}} + w_i^b\frac{\partial}{\partial\vec{\tau}}\Big|_{\vec{\eta}^{1q'}} \tag{3}$$

in which

$$v_i^b = n_i^{1q'}/g^b, \quad w_i^b = -n_i^{nq}/g^b \tag{4}$$

Thus,

$$\theta_{,i}(\vec{\zeta}^b) = v_i^b\frac{\partial\theta}{\partial\vec{\tau}}(\vec{\eta}^{nq}) + w_i^b\frac{\partial\theta}{\partial\vec{\tau}}(\vec{\eta}^{1q'}) \tag{5}$$

When we use the n-th order polynomial approximation within the boundary elements, then

$$\eta_i\Big|_{\Gamma_q} = \sum_{a=1}^{n}\eta_i^{aq}N^a(\xi), \quad \xi\in\langle -1, 1\rangle$$
$$\theta(\vec{\eta})\Big|_{\Gamma_q} = \sum_{a=1}^{n}\theta(\vec{\eta}^{aq})N^a(\xi) \tag{6}$$

and

$$\tau_k(\vec{\eta})\,|_{\Gamma_q} = h_k^q(\xi)/h^q(\xi)$$

where

$$h_k^q(\xi) \equiv \frac{\partial\eta_k}{\partial\xi}\Big|_{\Gamma_q} = \sum_{a=1}^{n}\eta_k^{aq}N'^a(\xi), \quad h^q = \sqrt{h_k^q h_k^q}$$

Hence,

$$\frac{\partial\theta}{\partial\vec{\tau}}(\vec{\eta}^{nq}) = \frac{1}{h^q(1)}\sum_{a=1}^{n}\theta(\vec{\eta}^{aq})N'^a(1)$$
$$\frac{\partial\theta}{\partial\vec{\tau}}(\vec{\eta}^{1q'}) = \frac{1}{h^{q'}(-1)}\sum_{a=1}^{n}\theta(\vec{\eta}^{aq'})N'^a(-1) \tag{7}$$

Since the gradient is a continuous function at a corner, we may write

$$q(\vec{\eta}^{nq}) = n_i^{nq}\theta_{,i}(\vec{\zeta}^b)$$
$$q(\vec{\eta}^{1q'}) = n_i^{1q'}\theta_{,i}(\vec{\zeta}^b) \tag{8}$$

Concluding we may say that if the normal derivatives are unknown on both the elements Γ_q and $\Gamma_{q'}$, they can be computed at the corner point $\vec{\zeta}^b = \Gamma_q \cap \Gamma_{q'}$ simply from two noncolinear tangential derivatives taken at the nodal points $\vec{\eta}^{nq}$ and $\vec{\eta}^{1q'}$.

Analogically in the theory of elasticity one can compute the components of the traction vectors at the corner point as

$$t_i(\vec{\eta}^{nq}) = n_j^{nq}\sigma_{ij}(\vec{\zeta}^b)$$
$$t_i(\vec{\eta}^{1q'}) = n_j^{1q'}\sigma_{ij}(\vec{\zeta}^b) \tag{9}$$

where the stress tensor components are expressed in terms of the tangential derivatives of displacements by

$$\sigma_{ij}(\vec{\zeta}^b) = \mu\Bigg\{v_j^b\frac{\partial u_i}{\partial\vec{\tau}}(\vec{\eta}^{nq}) + w_j^b\frac{\partial u_i}{\partial\vec{\tau}}(\vec{\eta}^{1q'})$$
$$+ v_i^b\frac{\partial u_j}{\partial\vec{\tau}}(\vec{\eta}^{nq}) + w_i^b\frac{\partial u_j}{\partial\vec{\tau}}(\vec{\eta}^{1q'})$$
$$+\frac{2\nu}{1-2\nu}\delta_{ij}\Bigg[v_b^k\frac{\partial u_k}{\partial\vec{\tau}}(\vec{\eta}^{nq})$$
$$+ w_k^b\frac{\partial u_k}{\partial\vec{\tau}}(\vec{\eta}^{1q'})\Bigg]\Bigg\} \tag{10}$$

$$\frac{\partial u_i}{\partial\vec{\tau}}(\vec{\eta}^{nq}) = \frac{1}{h^q(1)}\sum_{a=1}^{n}u_i(\vec{\eta}^{aq})N'^a(1)$$
$$\frac{\partial u_i}{\partial\vec{\tau}}(\vec{\eta}^{1q'}) = \frac{1}{h^{q'}(-1)}\sum_{a=1}^{n}u_i(\vec{\eta}^{aq'})N'^a(-1) \tag{11}$$

THREE DIMENSIONAL PROBLEMS

The same problem, though technically more complex, can be solved also in three dimensions. Now, one can define on each boundary element S_p two noncolinear vectors as

$$h_i^p(\xi_1, \xi_2) \equiv \frac{\partial\eta_i}{\partial\xi_1}\Big|_{S_p} = \sum_{a=1}^{n}\eta_i^{ap}N_{,1}^a(\xi_1, \xi_2)$$
$$k_i^p(\xi_1, \xi_2) \equiv \frac{\partial\eta_i}{\partial\xi_2}\Big|_{S_p} = \sum_{a=1}^{n}\eta_i^{ap}N_{,2}^a(\xi_1, \xi_2) \tag{12}$$

Consider two boundary elements S_p and $S_{p'}$ on which the corner nodal point $\vec{\zeta}^b$ lies, with $\vec{\zeta}^b = \vec{\eta}^{dp} = \vec{\eta}^{cp'}$. Denoting the local coordinates of this point respectively on S_p and $S_{p'}$ as (ξ_1^d, ξ_2^d) and (ξ_1^c, ξ_2^c), we may define the following vectors

$$h_i^{dp} = h_i^p(\xi_1^d, \xi_2^d), \quad k_i^{dp} = k_i^p(\xi_1^d, \xi_2^d)$$

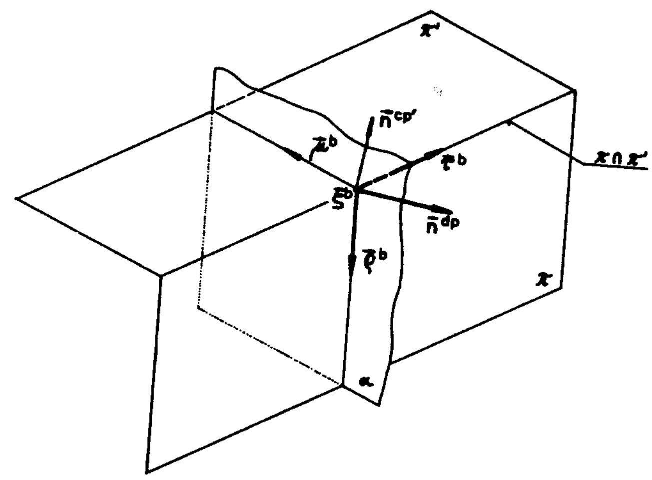

Fig. 2. Scheme of the geometry at the corner point $\vec{\zeta}^b$ in three dimensions

$$h_i^{cp'} = h_i^{p'}(\xi_1^c, \xi_2^c), \quad k_i^{cp'} = k_i^{p'}(\xi_1^c, \xi_2^c)$$
$$g_k^{dp} \equiv \epsilon_{kij} h_i^{dp} k_j^{dp}, \quad g^{dp} = \sqrt{g_i^{dp} g_i^{dp}}, \; n_i^{dp} \equiv g_i^{dp}/g^{dp}$$
$$g_k^{cp'} \equiv \epsilon_{kij} h_i^{cp'} k_j^{cp'}, \quad g^{cp'} = \sqrt{g_i^{cp'} g_i^{cp'}}, \; n_i^{cp'} \equiv g_i^{cp'}/g^{cp'} \tag{13}$$

Figure 2 shows two nonparallel planes π and π' which represent respectively the tangent planes to boundary elements S_p and $S_{p'}$ at $\vec{\zeta}^b$. The plane α is orthogonal to the intersection of planes π and π'.

Furthermore, we define the vectors

$$\tau_i^b \equiv \nu_i^b/\nu^b, \quad \nu_i^b \equiv \epsilon_{ijk} n_j^{cp'} n_k^{dp}, \quad \nu^b = \sqrt{\nu_i^b \nu_i^b} \tag{14}$$

$$\sigma_i^b \equiv \epsilon_{ijk} \tau_j^b n_k^{dp} \tag{15}$$

$$\mu_i^b \equiv \epsilon_{ijk} n_j^{cp'} \tau_k^b \tag{16}$$

$$x_i^b \equiv \epsilon_{ijk} \sigma_j^b \mu_k^b, \quad x^b = \sqrt{x_k^b x_k^b} \tag{17}$$

It can be seen that vectors $\vec{n}^{dp}$, $\vec{n}^{cp'}$, $\vec{\sigma}^b$, and $\vec{\mu}^b$ lie in plane α, while $\vec{\tau}^b$ is orthogonal to this plane. The plane π contains the vectors $\vec{\sigma}^b$ and $\vec{\tau}^b$, while the plane π' contains the vectors $\vec{\mu}^b$ and $\vec{\tau}^b$. Hence,

$$\tau_i^b = x_i^b/x^b \tag{18}$$

$$n_i^{dp} = \epsilon_{ijk} \sigma_j^b \tau_k^b \tag{19}$$

Substituting the definition of $n_j^{cp'}$ into equation (14), we obtain

$$\tau_i^b = k_i^{cp'} A^b - h_i^{cp'} B^b \tag{20}$$

with

$$A^b = h_k^{cp'} n_k^{dp}/\nu^b g^{cp'}, \quad B^b = k_j^{cp'} n_j^{dp}/\nu^b g^{cp'}$$

Similarly, substituting the definition of n_k^{dp} into equation (15), we obtain

$$\sigma_i^b = h_i^{dp} C^b - k_i^{dp} D^b \tag{21}$$

with

$$C^b = k_i^{dp} \tau_i^b/g^{dp}, \quad D^b = h_i^{dp} \tau_i^b/g^{dp}$$

Finally, substituting equations (18) and (17) into (19), we may write

$$n_i^{dp} = \sigma_i^b E^b - \mu_i^b F^b \tag{22}$$

with

$$E^b = \mu_j^b \sigma_j^b/x^b, \quad F^b = 1/x^b$$

From (13) and (14), we have

$$\mu_i^b = k_i^{cp'} G^b - h_i^{cp'} H^b \tag{23}$$

where

$$G^b = h_j^{cp'} \tau_j^b/g^{cp'}, \quad H^b = k_j^{cp'} \tau_j^b/g^{cp'}$$

Taking into account equations (21) and (23), we may rewrite equation (22) as

$$n_i^{dp} = h_i^{dp} C^b E^b - k_i^{dp} D^b E^b + h_i^{cp'} H^b F^b - k_i^{cp'} G^b F^b \tag{24}$$

Thus, we have found a triad of orthonormal vectors ($\vec{\tau}^b$, $\vec{\sigma}^b$, $\vec{n}^{dp}$) at $\vec{\zeta}^b$, which can be expressed in terms of the components of vectors $\vec{h}^{dp}$, $\vec{k}^{dp}$, $\vec{h}^{cp'}$, and $\vec{k}^{cp'}$.

Now, the gradient vector at $\vec{\zeta}^b$ takes the form

$$\partial_i'\Big|_{\vec{\zeta}^b} \equiv \frac{\partial}{\partial \eta_i}\Big|_{\vec{\zeta}^b} = (\tau_i^b \tau_j^b + \sigma_i^b \sigma_j^b + n_i^{dp} n_j^{dp})\partial_j'\Big|_{\vec{\zeta}^b}$$
$$= (\nu_i^b h_j^{dp} \partial_j' + w_i^b k_j^{dp} \partial_j' + \alpha_i^b h_j^{cp'} \partial_j' + \beta_i^b k_j^{cp'} \partial_j')\Big|_{\vec{\zeta}^b} \tag{25}$$

in which

$$\nu_i^b = \sigma_i^b C^b + n_i^{dp} C^b E^b, \quad w_i^b = -\sigma_i^b D^b - n_i^{dp} D^b E^b$$
$$\alpha_i^b = -\tau_i^b B^b + n_i^{dp} H^b F^b, \quad \beta_i^b = \tau_i^b A^b - n_i^{dp} G^b F^b \tag{26}$$

Making use of equation (25), we may write for the gradient of a scalar field at $\vec{\zeta}^b$ the following expression

$$\theta_{,i}(\vec{\zeta}^b) = \sum_{a=1}^{n} \theta(\vec{\eta}^{ap})[\nu_i^b N^a_{,1}(\xi_1^d, \xi_2^d) + w_i^b N^a_{,2}(\xi_1^d, \xi_2^d)]$$
$$+ \sum_{a=1}^{n} \theta(\vec{\eta}^{ap'})[\alpha_i^b N^a_{,1}(\xi_1^c, \xi_2^c)$$
$$+ \beta_i^b N^a_{,2}(\xi_1^c, \xi_2^c)] \tag{27}$$

Similarly, one can write the gradient of displacements at $\vec{\zeta}^b$ as

$$u_{i,j}(\vec{\zeta}^b) = \sum_{a=1}^{n} u_i(\vec{\eta}^{ap})[\nu_j^b N^a_{,1}(\xi_1^d, \xi_2^d) + w_j^b N^a_{,2}(\xi_1^d, \xi_2^d)]$$
$$+ \sum_{a=1}^{n} u_i(\vec{\eta}^{ap'})[\alpha_j^b N^a_{,1}(\xi_1^c, \xi_2^c)$$
$$+ \beta_j^b N^a_{,2}(\xi_1^c, \xi_2^c)] \tag{28}$$

Once the gradients of displacements are known, we also know the stress tensor at the corner point

$$\sigma_{ij}(\vec{\zeta}^b) = \mu\left[u_{i,j}(\vec{\zeta}^b) + u_{j,i}(\vec{\zeta}^b) + \frac{2\nu}{1-2\nu}\,\delta_{ij}u_{k,k}(\vec{\zeta}^b)\right]$$

The normal derivatives of the scalar field θ and tractions at the points $\vec{\eta}^{dp}$ and $\vec{\eta}^{cp'}$ are given by

$$q(\vec{\eta}^{dp}) = n_i^{dp}\theta_{,i}(\vec{\zeta}^b), \quad q(\vec{\eta}^{cp'}) = n_i^{cp'}\theta_{,i}(\vec{\zeta}^b)$$

$$t_i(\vec{\eta}^{dp}) = n_j^{dp}\sigma_{ij}(\vec{\zeta}^b), \quad t_i(\vec{\eta}^{cp'}) = n_j^{cp'}\sigma_{ij}(\vec{\zeta}^b) \tag{29}$$

Usually double (or multiple) nodes are employed[2,3] at corners if the normal derivatives of a scalar field or tractions are unknown at the corner point on two or more boundary elements. This is so because the number of unknowns at such a corner exceeds the number of the mutually independent boundary integral equations which can be written at this point. To compute all the unknowns the system of the BIE is supplemented by the additional equations at multiple nodes. Now we have shown that the normal derivative of the scalar field or tractions in elasticity can be computed at such a corner point without solving any differential or integral equation. These unknowns can easily be expressed in terms of the tangential derivatives of the primary fields on two noncolinear or noncoplanar boundary elements in two and three dimensions, respectively. Thus, these unknowns can be computed by simple differentiation from the boundary data prescribed before solving the BIE. Consequently, no BIE are considered at such corner points. The present concept seems to be more efficient from the point of view of numerical computation and easier for programming.

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