Differentiation

- Simple in transformed space

$$u = \sum_{k=-\infty}^{\infty} \hat{u}_k e^{ikx} \Rightarrow u' = \underbrace{\sum_{k=-\infty}^{\infty} ik\hat{u}_k e^{ikx}}_{\text{Fourier Series of } u'}$$

then $(P_N u)' = P_N u'$

i.e. truncation and differentiation commute

- In the discrete case

given u_j at $x_j = \frac{2\pi j}{N}$, transform to $\tilde{u}_k$

then $$(I_N u)'_j = \underbrace{\sum_{k=-N/2}^{N/2-1} ik\tilde{u}_k e^{ikx_j}}_{(D_N u)_j}$$

$(D_N u)_j \equiv$ grid values of the derivative of discrete Fourier Series

Conclude: $\underbrace{D_N u}_{\text{"Fourier collocation derivative"}} = (I_N u)' \neq \underbrace{P_N u'}_{\text{Fourier Galerkin derivative}}$

But $(I_N u)' \neq I_N u'$

i.e. Interpolation and differentiation do <u>not</u> commute

<u>However</u>

$(I_N u)' - I_N u'$ is of the same order as the truncation error for the derivative

$$u' - P_N u'$$

$\therefore$ Collocation differentiation is spectrally accurate

– Fourier Collocation Differentiation

Can be represented as a matrix system:

$$I_N u = \sum_{k=-N/2}^{N/2-1} \tilde{u}_k \phi_k \qquad (\phi_k = e^{ikx})$$

$$(I_N u)' = \sum_{k=-N/2}^{N/2-1} ik\tilde{u}_k \phi_k$$

$$\hookrightarrow \tilde{u}_k = \frac{1}{N}\sum_{j=0}^{N-1} u_j e^{-ikx_j} \Rightarrow \text{uses } \underline{\text{all}}\ u_j$$

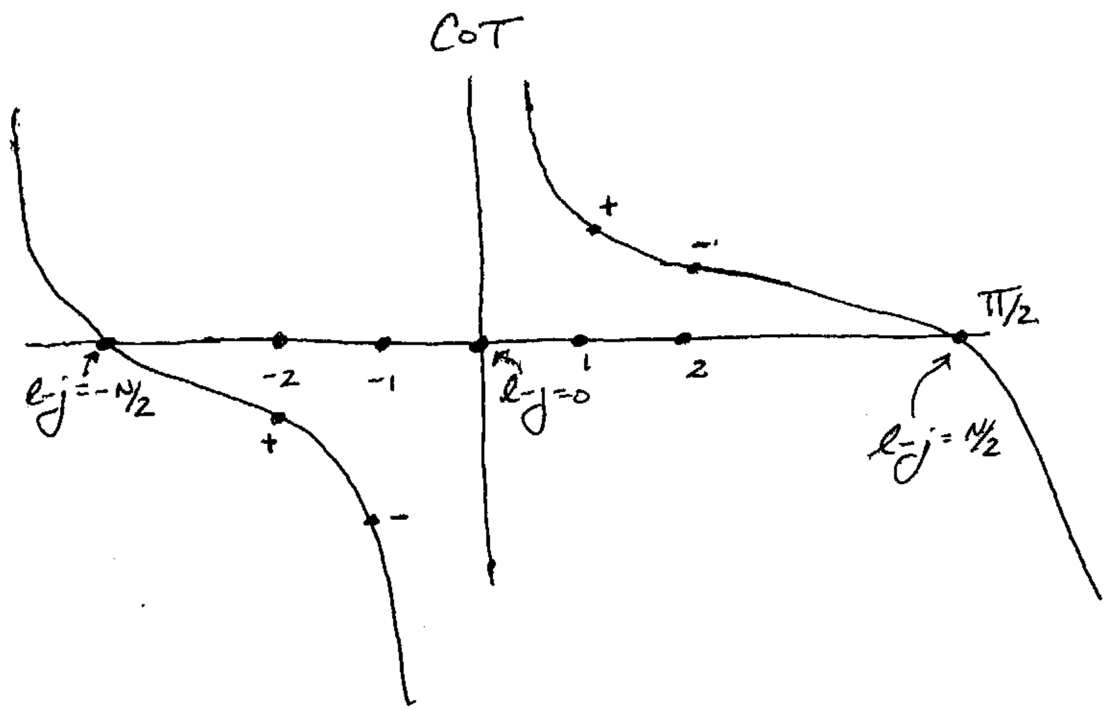
So $(D_N u)_l$ = derivative at physical node l
uses all $\tilde{u}_k$; uses (linearly) all u_j

$$= \sum_{j=0}^{N-1} \underbrace{(D_N)_{lj}}_{\text{effect of } u_j \text{ on } (D_N u) \text{ at node } l} u_j$$

where $\displaystyle (D_N)_{\ell j} = \frac{1}{N}\sum_{k=-N/2}^{N/2-1} ik\, e^{2ik(\ell-j)\pi/N}$

has a closed form sum:

$$(D_N)_{\ell j} = \begin{cases} \frac{1}{2}(-1)^{\ell+j}\cot\left(\frac{(\ell-j)\pi}{N}\right) & \ell \neq j \\ 0 & \ell = j \end{cases}$$



- Symmetric
- Zero at center
- Unbounded as $N \to \infty$
- Compare w/ FD

$O(h^2)$		-1	0	1		h
$O(h^4)$	1	-8	0	8	-1	$h/2$