Non Periodic Problems: Chebyshev Bases

Important Facts: $T_k(x)$

- Recursion: $T_{k+1}(x) = 2x T_k(x) - T_{k-1}(x)$

- Boundaries: $T_k(\pm 1) = (\pm 1)^k$

  $T_k'(\pm 1) = (\pm 1)^k k^2$

- Orthogonality: $\displaystyle\int_{-1}^{1} \frac{T_k(x) T_\ell(x)}{\sqrt{1-x^2}}\, dx = c_k \frac{\pi}{2} \delta_{k\ell}$

  $$c_k = \begin{cases} 2 & k=0 \\ 1 & k \geq 1 \end{cases}$$

Continuous Transform Pair:

$$u(x) = \sum_{k=0}^{\infty} \hat{u}_k T_k(x).$$

$$\hat{u}_k = \frac{2}{\pi c_k} \int_{-1}^{1} u(x) T_k(x) \frac{1}{\sqrt{1-x^2}}\, dx$$

Relation to Fourier Series:

$$x = \cos\theta \qquad \theta \in [0, \pi]$$

$$u(\cos\theta) \equiv \tilde{u}(\theta) = \sum_{k=0}^{\infty} \hat{u}_k \cos k\theta$$

- Quadrature Points / Weights

Chebyshev - Gauss - Lobatto

$$x_j = \cos\frac{\pi j}{N} \;;\; w_j = \begin{cases} \frac{\pi}{2N} & j = 0, N \\ \frac{\pi}{N} & 1 \le j \le N-1 \end{cases}$$

– Discrete Transform Pair:

$$\tilde{u}_k = \frac{1}{\gamma_k} \sum_{j=0}^{N} u_j P_{kj} w_j \qquad \text{(General)}$$

$$= \frac{2}{\bar{c}_k \pi} \sum_{j=0}^{N} u_j \frac{\pi}{\bar{c}_j N} \cos k\left(\cos^{-1}\left[\cos\frac{\pi j}{N}\right]\right)$$

$$= \frac{2}{N\bar{c}_k} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} u_j \cos k\frac{\pi j}{N} \qquad \text{for } k = 0, 1, \ldots N$$

$$= \sum_{j=0}^{N} C_{kj} u_j \qquad \text{where } C_{kj} = \frac{2}{N\bar{c}_k\bar{c}_j}\cos\frac{\pi jk}{N}$$

$$\bar{c}_j = \begin{cases} 2 & j = 0, N \\ 1 & 1 \le j \le N-1 \end{cases}$$

Alternate view:

$$\hat{u}_k = \frac{2}{\pi c_k}\int_{-1}^{1} u \cos k\theta \underbrace{w\,dx}_{d\theta} \approx \underbrace{\frac{2}{\pi c_k}\sum_{j=0}^{N} u_j \cos\frac{k\pi j}{N}\frac{\pi}{N c_j}}_{}$$

N+1 pt Quadrature Rule using Gauss-Lobatto pts

$$u_j = \sum_{k=0}^{N} \tilde{u}_k \cos\frac{\pi j k}{N} = \sum_{k=0}^{N} \underbrace{(C^{-1})_{jk}}_{=\cos\frac{\pi j k}{N}} \tilde{u}_k \quad \text{for } j=0,1,\ldots N$$

- Aliasing Error:

Recall: $u = \sum_{i=0}^{\infty} \hat{u}_i \phi_i$

$$(u, \phi_k)_N = \sum_{i=0}^{\infty} \hat{u}_i (\phi_i, \phi_k)_N = \sum_{i=0}^{N} \hat{u}_k (\phi_k \phi_k)_N + \sum_{i=N+1}^{\infty} \hat{u}_i (\phi_i, \phi_k)_N$$

$$\tilde{u}_k = \frac{1}{\gamma_k} \sum_{j=0}^{N} u_j \phi_{kj} w_j \qquad \gamma_k = \sum_{j=0}^{N} \phi_{kj}^2 w_j$$

$$= \sum_{i=0}^{N} \hat{u}_i \langle \phi_i \phi_k \rangle + \sum_{i=N+1}^{\infty} \hat{u}_i (\phi_i, \phi_k)_N$$

$$\tilde{u}_k \gamma_k = \hat{u}_k \gamma_k + \sum_{i=N+1}^{\infty} \hat{u}_i (\phi_i, \phi_k)_N$$

$$\tilde{u}_k = \hat{u}_k + \underbrace{\frac{1}{\gamma_k} \sum_{i>N} \hat{u}_i (\phi_i, \phi_k)_N}$$ aliasing error

For Chebyshev: $$(T_k, T_l)_N = \begin{cases} (T_k, T_k)_N & l = 2mN \pm k \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{u}_k = \hat{u}_k + \sum_{\substack{j = 2mN \pm k \\ j > N}} \hat{u}_j$$

kth mode depends on all Chebyshev modes which alias $T_k(x)$ on the grid (same as Fourier!)

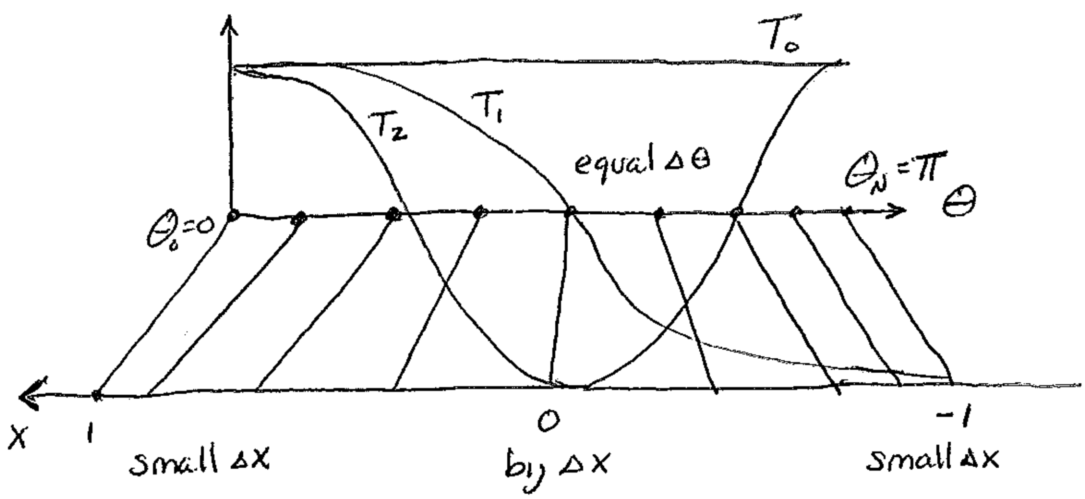
- Transforms:

$$T_k(x) = T_k(\theta(x)) \qquad \theta = \cos^{-1} x$$

$$T_k(\theta) = \cos k\theta$$

Lobatto Pts: $x_j = \cos\frac{\pi j}{N} \Rightarrow \theta_j = \cos^{-1} x_j = \frac{\pi j}{N}$

N+1 pts (exact for P^{2N-1})



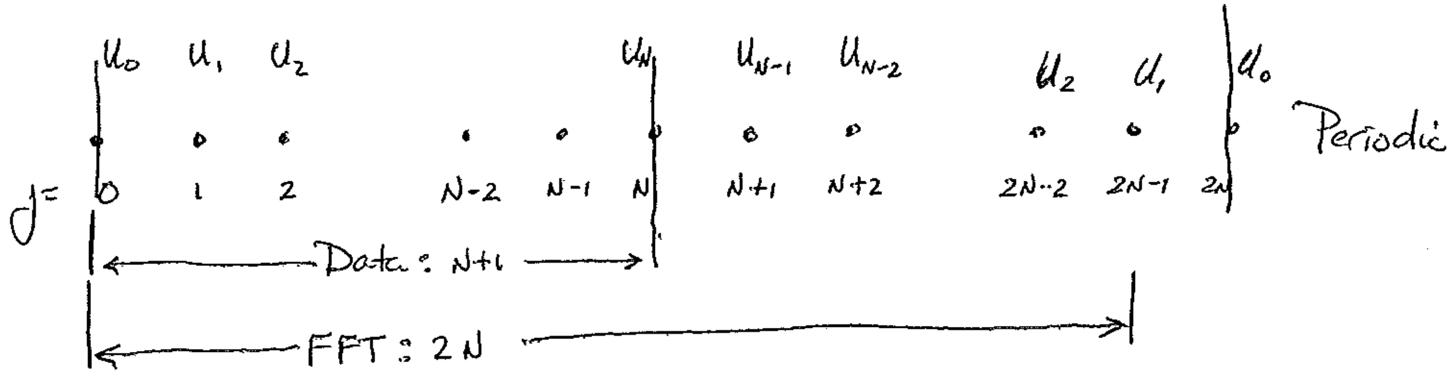
$$\tilde{u}_k = \frac{2}{N\bar{c}_k} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} u_j \cos\frac{\pi jk}{N} \qquad k = 0, 1, \ldots N$$

$$u_j = \sum_{k=0}^{N} \tilde{u}_k \cos\frac{\pi jk}{N} \qquad j = 0, \ldots N$$

$$\bar{c}_j = \begin{cases} 2 & j = 0, N \\ 1 & \text{otherwise} \end{cases}$$

Using the FFT for Chebyshev Transforms

- Double the mesh w/ symmetry



$$V_j \equiv u_j \qquad j = 0, 1, \dots N$$
$$\equiv u_{2N-j} \qquad j = N+1, N+2, \dots 2N-1$$

FFT (using +, 2N)

$$\tilde{V}_k = \sum_{j=0}^{2N-1} V_j e^{i\frac{\pi j}{N}k} = \sum_{j=1}^{N-1} u_j \underbrace{\left(e^{ik\frac{\pi j}{N}} + e^{-ik\frac{\pi j}{N}}\right)}_{2\cos k\frac{\pi j}{N}} + \underbrace{u_0 e^{i\frac{\pi k}{N}(0)} + u_N e^{i\frac{\pi N}{N}k}}_{1 * \cos \pi k}$$

$$\boxed{\tilde{V}_k = \sum_{j=0}^{N} \frac{2}{\bar{c}_j} u_j \cos\frac{\pi j}{N}k} \Rightarrow \boxed{\tilde{u}_k = \frac{1}{N\bar{c}_k}\tilde{V}_k \quad k = 0, 1, \dots N}$$

Inverse Transform: Use same (+, 2N) FFT

Need: $$u_j = \sum_{k=0}^{N} \tilde{u}_k \cos\frac{\pi jk}{N} = \sum_{k=0}^{N} \frac{2}{\bar{c}_k} \underbrace{\left(\frac{\bar{c}_k}{2}\tilde{u}_k\right)}_{\tilde{w}_k} \cos\frac{\pi jk}{N}$$

$$\left.\begin{aligned} \tilde{w}_k &\equiv \frac{\bar{c}_k}{2}\tilde{u}_k & k &= 0, 1, \dots N \\ &\equiv \tilde{w}_{2N-k} & k &= N+1, \dots 2N-1 \end{aligned}\right\} \text{FFT}(+, 2N) \Rightarrow w_j$$

$$u_j = w_j \quad j = 0, 1, \dots N$$

- Differentiation:

Trig Identity: $2\sin\theta\cos k\theta = \sin(k+1)\theta - \sin(k-1)\theta$

$$2\cos k\theta = \left[\frac{(k-1)\sin(k-1)\theta}{k-1} - \underbrace{\frac{(k+1)\sin(k+1)}{(k+1)}}_{\frac{1}{k+1}\frac{d}{d\theta}(\cos(k+1)\theta)}\right]\left(\frac{-1}{\sin\theta}\right)$$

$$\boxed{2T_k(x) = \frac{T'_{k+1}(x)}{k+1} - \frac{T'_{k-1}(x)}{k-1}} \qquad k > 1$$

$$2T_k(x) = \frac{T'_{k+1}}{k+1} \qquad k = 1$$

$$u = \sum_{k=0}^{N} \tilde{u}_k T_k \;;\quad u' = \sum_{1}^{N} \tilde{u}_k T_k' = \sum_{0}^{N-1} \tilde{u}_k^{(1)} T_k$$

want to express $\tilde{u}_k^{(1)}$ in terms of $\tilde{u}_k$

$$\sum_{1}^{N} \tilde{u}_k T_k' = \tilde{u}_0^{(1)} T_0 + \underbrace{\sum_{1}^{N-1} \tilde{u}_k^{(1)} T_k}_{\sum_{1}^{N-1} \frac{\tilde{u}_k^{(1)}}{2}\left[\frac{T'_{k+1}}{k+1} - \frac{T'_{k-1}}{k-1}\right]}$$

$$= \underbrace{\tilde{u}_0^{(1)} T_0}_{\substack{T_0 \equiv T_1' \\ \tilde{u}_0^{(1)} = c_0 \frac{u_0^{(1)}}{2}}} + \sum_{2}^{N} \frac{\tilde{u}_{k-1}^{(1)}}{2} \frac{T_k'}{k} - \underbrace{\sum_{0}^{N-2} \frac{\tilde{u}_{k+1}^{(1)}}{2} \frac{T_k'}{k}}$$

Extend top to N
w/ $u_N^{(1)} = u_{N+1}^{(1)} \equiv 0$
ignore $\frac{T_0'}{0} = 0$

$$\sum_1^N \tilde{u}_k T_k' = \sum_1^N c_{k-1} \frac{\tilde{u}^{(1)}_{k-1}}{2} \frac{T_k'}{k} - \sum_1^N \frac{u^{(1)}_{k+1}}{2} \frac{T_k'}{k}$$

So Term by Term:

$$2k\tilde{u}_k = c_{k-1}\tilde{u}^{(1)}_{k-1} - \tilde{u}^{(1)}_{k+1}$$

Rearrange:

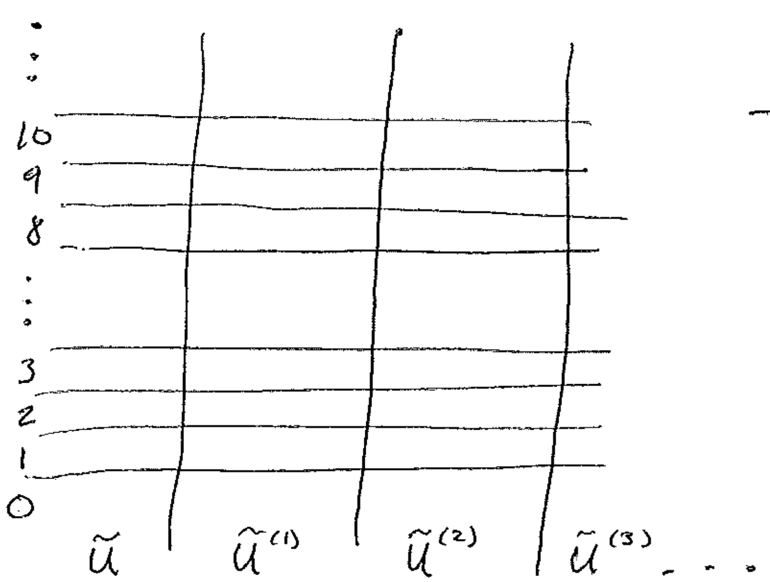
$$c_k \tilde{u}^{(1)}_k = \tilde{u}^{(1)}_{k+2} + 2(k+1)\tilde{u}_{k+1} \qquad 0 \le k \le N-1$$

Work from top down: $\left(\tilde{u}^{(1)}_k = 0 \quad k \geq N\right)$

More generally

$$c_k \tilde{u}^{(q)}_k = \tilde{u}^{(q)}_{k+2} + 2(k+1)\tilde{u}^{(q-1)}_{k+1}$$

e.g. $q=2$ for $\frac{d^2u}{dx^2} = \sum_0^{N-2} \tilde{u}^{(2)}_k T_k(x)$



Top → Bottom
left → right

– Can also have derivative expressed through a matrix multiply:

$$\mathcal{D}_N u_\ell = \sum_{j=0}^{N} (\mathcal{D}_N)_{\ell j}\, u_j \qquad \ell = 0, 1, \dots N$$

where

$$(\mathcal{D}_N)_{\ell j} = \begin{cases} \dfrac{\bar{c}_\ell\, (-1)^{\ell+j}}{\bar{c}_j\, (x_\ell - x_j)} & \ell \neq j \\[2ex] \dfrac{-x_j}{2(1-x_j^2)} & 1 \leq \ell = j \leq N-1 \\[2ex] \dfrac{2N^2+1}{6} & \ell = j = 1 \\[2ex] -\left(\dfrac{2N^2+1}{6}\right) & \ell = j = N \end{cases}$$

Example: $\dfrac{\partial u}{\partial t} = K \dfrac{\partial^2 u}{\partial x^2}$ $\qquad u(-1) = a$

$\qquad u(1) = b$

$$u = \sum_{k=0}^{N} \hat{u}_k(t)\, T_k(x)$$

$$\frac{\partial u}{\partial x} = \sum_{k=0}^{N} \hat{u}_k^{(1)}\, T_k(x)$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{k=0} \hat{u}_k^{(2)}\, T_k(x)$$

$$\hat{u}_k \longrightarrow \hat{u}_k^{(1)} \longrightarrow \hat{u}_k^{(2)}$$

$$\hat{u}_{N+1}^{(1)} = \hat{u}_N^{(1)} = 0 \Rightarrow c_k \hat{u}_k^{(1)} = \hat{u}_{k+2}^{(1)} + 2(k+1)\hat{u}_{k+1} \quad k = N-1,\ N-2,\ \vdots$$

$$\hat{u}_{N+1}^{(2)} = \hat{u}_N^{(2)} = \hat{u}_{N-1}^{(2)} = 0 \Rightarrow c_k \hat{u}_k^{(2)} = \hat{u}_{k+2}^{(2)} + 2(k+1)\hat{u}_{k+1}^{(1)} \quad k = N-2,\ N-3,\ \vdots$$

Galerkin:

$$\left\langle \sum_\ell \frac{d\hat{u}_\ell}{dt} T_\ell T_k \right\rangle = \left\langle K \sum_\ell \hat{u}_\ell^{(2)} T_\ell(x) T_k \right\rangle \quad k = 0, 1, \ldots \underline{\underline{N-2}}$$

$$c_k \frac{\pi}{2} \frac{d\hat{u}_k}{dt} = K c_k \frac{\pi}{2} \hat{u}_k^{(2)} \qquad \text{for } \hat{u}_k, \quad k = 0, 1, \ldots N-2$$

BC's: $T_k(\pm 1) = (\pm)^k$

$$\left.\begin{aligned} u(-1) &= \sum_{k=0}^{N} (-1)^k \hat{u}_k = a \\ u(+1) &= \sum_{k=0}^{N} \hat{u}_k = b \end{aligned}\right] \text{ for top 2 } \hat{u}_k \text{ i.e. } \hat{u}_N, \hat{u}_{N-1}$$

"Tau" Method ... Delete Galerkin Eqn's w/ T_N, T_{N-1} in favor of BC enforcement

Collocation:

Start from x-domain

$$u_j \longrightarrow \hat{u}_k \longrightarrow \hat{u}_k^{(1)} \longrightarrow \hat{u}_k^{(2)} \longrightarrow \frac{\partial^2 u}{\partial x^2}_j$$

$$\frac{du}{dt}_j = \kappa \frac{d^2u}{dx^2}_j \qquad j = 1, 2, \ldots N-1$$

BC's $\quad u_j = a \quad j = 0$

$\quad\quad\quad\; = b \quad j = N$

Burger's Equation

$$\frac{\partial u}{\partial t} = \underbrace{-u\frac{\partial u}{\partial x}}_{\text{key is how to treat this term}} + \nu\frac{\partial^2 u}{\partial x^2}$$

as before: Galerkin ⇒ Convolution or Pseudospectral w/ dealiasing

Pseudospectral ⇒ $k \longrightarrow j \longrightarrow k$
(differentiate and compute in k-space)

Collocation ⇒ $j \longrightarrow k \longrightarrow j$
(differentiate in k-space
compute in j-space)

$$\frac{\partial \hat{u}}{\partial t} + \hat{w}_k - \nu \hat{u}_k^{(2)} = 0 \qquad k = 0, 1, \ldots N-2$$

$$\hat{w}_k = \frac{2}{\pi c_k} \int_{-1}^{1} \left(\sum_{l=0}^{N} \hat{u}_l T_l \right) \left(\sum_{i=0}^{N} \hat{u}_i^{(1)} T_i \right) T_k (1-x^2)^{-1/2} dx$$

$$w/ \quad \sum_{k=0}^{N} \hat{u}_k = b \ ; \quad \sum_{k=0}^{N} (-1)^k \hat{u}_k = a \quad \text{BC's}$$

What about Type II BC's ?

have $\hat{u}_k \quad k = 0, 1, \ldots N-2$
What are $\hat{u}_{N-1}, \hat{u}_N$?? $\Big\}$ In k-space.

$$\frac{du}{dx} = \sum \hat{u}_k T_k' \Rightarrow T_k'(\pm 1) = (\pm)^k k^2$$

$$\Rightarrow \sum_{k=0}^{N} \hat{u}_k (\pm)^k k^2 = \frac{du}{dx}(\pm 1)$$

$$\begin{bmatrix} (N-1)^2 & N^2 \\ (-1)^{N-1}(N-1)^2 & (-1)^N N^2 \end{bmatrix} \begin{Bmatrix} \hat{u}_{N-1} \\ \hat{u}_N \end{Bmatrix} = \begin{matrix} -\sum_{k=0}^{N-2} \hat{u}_k (+1)^k k^2 + \frac{du}{dx}(x=1) \\ -\sum_{k=0}^{N-2} \hat{u}_k (-1)^k k^2 + \frac{du}{dx}(x=-1) \end{matrix}$$

2 equations, 2 unknowns $\Rightarrow$ solve for $\hat{u}_{N-1}, \hat{u}_N$