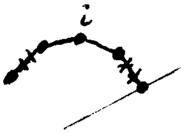
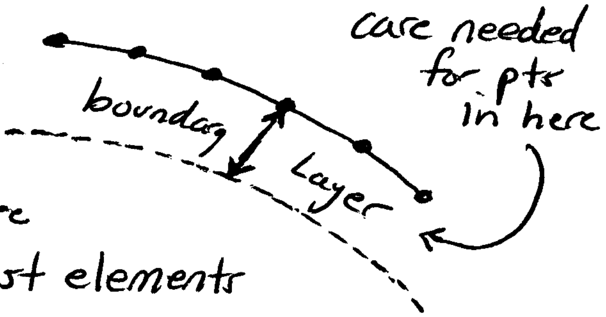
- Computing Interior Values
  - Second step of procedure after boundary info produced
  - Numerical integration of known quantities... easy, essentially repeats the assembly procedure with u + $\frac{\partial u}{\partial n}$ known on Γ
  - <u>No</u> singular integrations, but must watch out for "<u>nearly</u>" singular integrands when interior point gets "close" to the boundary

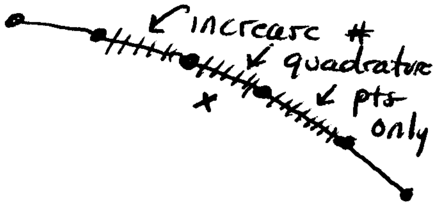
also in neighboring elements in step A



  - Rule of thumb: if interior pt is within one boundary segment's length to the boundary (i.e. inside the "boundary layer") need to worry about loss of accuracy due to "near-singular" integrands



Strategies: a) Simplest - increase the # Gauss pts in nearest elements to improve accuracy....
only in neighboring elements, others remain w/ low # Gauss pts...
can be costly, may require very many pts



b.) use special integration rule ... most common idea $\Rightarrow$ variable transformation to weaken or cancel out the singularity with the Jacobian, tends to "bunch" Gauss pts near the singularity (where needed most)

Can use these strategies in Step A (i.e. computation of missing boundary info) as well, especially when have higher order basis function interpolation (i.e. when analytic integration not so easy)

$\boxed{\text{Appendix A.5}}$

- Can compute derivatives of u on the interior

e.g. $$u_i = \oint \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, ds$$

then $$\frac{\partial u_i}{\partial x_i} = \oint \frac{\partial u}{\partial n} \frac{\partial G_i}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\frac{\partial G_i}{\partial n}\right) u \, ds$$

$$\frac{\partial u_i}{\partial y_i} = \oint \frac{\partial u}{\partial n} \frac{\partial G_i}{\partial y_i} - \frac{\partial}{\partial y_i}\left(\frac{\partial G_i}{\partial n}\right) u \, ds$$

For Laplace ... $G_i = \frac{-1}{2\pi} \ln r_i \; ; \; r_i = \left((x-x_i)^2 + (y-y_i)^2\right)^{1/2}$

then $$\frac{\partial G_i}{\partial x_i} = \frac{-1}{2\pi} \frac{1}{\left((x-x_i)^2+(y-y_i)^2\right)^{1/2}} \frac{1}{2}\left((x-x_i)^2 + (y-y_i)^2\right)^{-1/2}(-2(x-x_i))$$

$$= \frac{1}{2\pi}\left(\frac{(x-x_i)}{(x-x_i)^2+(y-y_i)^2}\right)$$

$\frac{\partial G}{\partial x_i} = \frac{\partial G_i}{\partial r_i}\frac{\partial r_i}{\partial x} \left(-\frac{1}{2\pi r_i}\right) \hat{r}_i \cdot \hat{x}$

same for $\frac{\partial G_i}{\partial y_i} \Rightarrow \frac{1}{2\pi}\left(\frac{y-y_i}{(x-x_i)^2+(y-y_i)^2}\right)$

- $\frac{\partial}{\partial x_i}\left(\frac{\partial G_i}{\partial n}\right)$ gets more detailed (but straightforward)

$$\frac{\partial}{\partial x_i}\left(\frac{\partial G_i}{\partial r_i}\frac{\partial r_i}{\partial n}\right) = \frac{\partial}{\partial x_i}\left(\frac{-1}{2\pi r_i}\right)\left(\frac{\partial r_i}{\partial n}\right) = \frac{\partial r_i}{\partial n}\frac{\partial}{\partial x_i}\left(\frac{-1}{2\pi r_i}\right) + \left(\frac{-1}{2\pi r_i}\right)\frac{\partial}{\partial x_i}\left(\frac{\partial r_i}{\partial n}\right)$$

$$\frac{\partial}{\partial x_i}\left(\frac{1}{2\pi r_i}\right) = \left(-\frac{1}{2}\left((x-x_i)^2+(y-y_i)^2\right)^{-3/2}\left(-2(x-x_i)\right)\right)\left(\frac{1}{2\pi}\right)$$

$$= \frac{x-x_i}{\left((x-x_i)^2+(y-y_i)^2\right)^{3/2}}\left(\frac{1}{2\pi}\right)$$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial r_i}{\partial n}\right) = \frac{\partial}{\partial x_i}\left(\frac{(yL(2)-yL(1)) * (x5-x_i) - (xL(2)-xL(1)) * (y5-y_i)}{\Delta S_e * r_i}\right)$$

$$= \frac{yL(1)-yL(2)}{\Delta S_e * r_i} + \frac{(yL(2)-yL(1)) * (x5-x_i) - (xL(2)-xL(1)) * (y5-y_i)}{\Delta S_e(-1/2)\left((x-x_i)^2+(y-y_i)^2\right)^{+3/2}}$$

$-2(x-x_i)$

$$= \frac{yL(1)-yL(2)}{\Delta S_e * r_i} + \frac{(yL(2)-yL(1)) * (x5-x_i) - (xL(2)-xL(1)) * (y5-y_i)}{\Delta S_e * \left((x-x_i)^2+(y-y_i)^2\right)^{3/2}}\left(x-x_i\right)$$

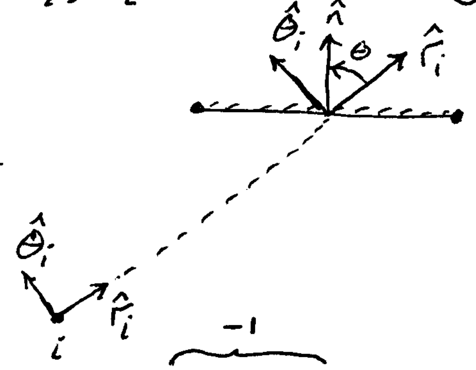
So, putting it all together:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial G_i}{\partial n}\right) = \frac{1}{2\pi}\left[\frac{x_i - x}{\left((x-x_i)^2+(y-y_i)^2\right)^{3/2}}\right]\frac{\partial r_i}{\partial n}$$

$$+ \frac{1}{2\pi r_i}\left[\frac{yL(2)-yL(1)}{\Delta S_E * r_i} + \frac{(x_i-x)}{(x-x_i)^2+(y-y_i)^2}\frac{\partial r_i}{\partial n}\right]$$

Same thing for $\frac{\partial}{\partial y_i}\left(\frac{\partial G_i}{\partial n}\right)$ . . . a little messy, but straightforward!

Alternately... use natural $\hat{r}_i, \hat{\theta}_i$ coordinate system



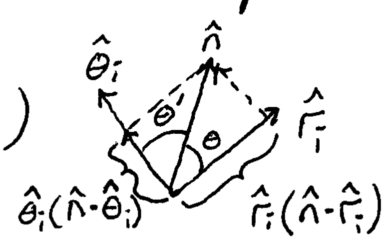
$$(\nabla U)_i = \oint \frac{\partial u}{\partial n}\nabla_i(G_i) - u\,\nabla_i\left(\frac{\partial G_i}{\partial n}\right) ds$$

$$\nabla_i G_i = -\frac{\partial G_i}{\partial r}\hat{r}_i \quad \left(\frac{\partial}{\partial r_i}G(|r-r_i|)\right) = \frac{\partial G}{\partial r_i}\overbrace{\frac{\partial}{\partial r_i}|r-r_i|}^{-1}$$

$$\nabla_i\left(\frac{\partial G_i}{\partial n}\right) = \nabla_i\left(\frac{\partial G_i}{\partial r}\hat{r}_i\cdot\hat{n}\right) = \left(-\frac{\partial^2 G_i}{\partial r^2}\hat{r}_i\right)\hat{r}_i\cdot\hat{n} + \frac{\partial G_i}{\partial r}\nabla_i(\hat{n}\cdot\hat{r}_i)$$

$$\text{But } \nabla_i\underbrace{(\hat{n}\cdot\hat{r}_i)}_{\cos\theta} = \hat{\theta}_i\frac{1}{r}\frac{\partial\cos\theta}{\partial\theta} + \cancel{\hat{r}\frac{\partial}{\partial r}\cos\theta} = -\frac{\hat{\theta}\sin\theta}{r}$$

$$\text{Now } \hat{\theta}_i\sin\theta = \hat{\theta}_i\underbrace{(\hat{n}\cdot\hat{\theta}_i)}_{\cos\theta' = \cos(\frac{\pi}{2}-\theta) = \sin\theta} = \hat{n}_i - \hat{r}_i(\hat{n}_i\cdot\hat{r}_i)$$



So

$$(\nabla u)_i = \oint \frac{\partial u}{\partial n} \overbrace{\nabla_i G_i}^{-\frac{\partial G_i}{\partial r}\hat{r}_i} - u \underbrace{\nabla_i\left(\frac{\partial G_i}{\partial n}\right)} \, ds$$

$$-\frac{\partial^2 G_i}{\partial r^2}\hat{r}_i(\hat{r}_i \cdot \hat{n}) - \frac{\partial G_i}{\partial r}\frac{1}{r_i}\left(\hat{n}_i - \hat{r}_i(\hat{r}_i \cdot \hat{n})\right)$$

If have a forcing term in original PDE must add in $\langle f \nabla_i G_i \rangle$ term (if $\nabla^2 u = \underline{\underline{-}}f$)

(or add in $\underline{\underline{-}}\langle f \nabla_i G_i \rangle$ if $\nabla^2 u = \underline{\underline{+}}f$)

Computing Interior Value at (XI, YI)

$u_{int} = 0$

Loop over elements; $L = 1, NE$ $\Rightarrow \oint = \sum_e \int_e$

Load $\begin{Bmatrix} XL(I), YL(I) \\ JL(I) \end{Bmatrix}$ $I = 1, 2$

$\Delta S_e = ((XL(2) - XL(1))**2 + (YL(2) - YL(1))**2$ Element length

Loop over Gauss pts; $k = 1, m$

$Z = \xi(k)$

$\phi(1) = \frac{1-Z}{2}$ Define Basis

$\phi(2) = \frac{1+Z}{2}$

$XS = XL(1) * \phi(1) + XL(2) * \phi(2)$ $(x,y)_{Gauss\,pt}$

$YS = YL(1) * \phi(1) + YL(2) * \phi(2)$

$r_i = ((XS - XI)**2 + (YS - YI)**2)^{1/2}$

$$\frac{\partial r_i}{\partial n} = \frac{(YL(2) - YL(1)) * (XS - XI) - (XL(2) - XL(1)) * (YS - YI)}{\Delta S_e * r_i}$$

$G_i = -\text{LN}\, r_i$

$\frac{\partial G_i}{\partial n} = -\frac{1}{r_i} * \frac{\partial r_i}{\partial n}$

Loop over the knowns; $J = 1, 2$

$$u_{int} = u_{int} + \left(\frac{\partial u}{\partial n}(JL(J)) * \phi(J) * G_i - u(JL(J)) * \phi(J) * \frac{\partial G_i}{\partial n}\right) * \left(\frac{\Delta S_e}{2} * W_k\right)$$

END J Loop

END Gauss pt loop

END element Loop

Repeat for each interior point desired