Review: Use of FFT

"Correct" Spectrum:

desired transform pair

$$\tilde{u}_k = \frac{1}{N}\sum_{j=0}^{N-1} u_j e^{-ikx_j} \qquad -\frac{N}{2} \le k \le \frac{N}{2}-1$$

$$x_j = \frac{2\pi j}{N}$$

$$u_j = \sum_{k=-\frac{N}{2}}^{N/2-1} \tilde{u}_k e^{ikx_j}$$

At the nodes . . .

e^{ikx_j} identical for the pairs $k = (-\frac{N}{2} \longleftrightarrow \frac{N}{2})$

$(-\frac{N}{2}+1 \longleftrightarrow \frac{N}{2}+1)$

$\vdots \qquad \vdots$

$(-1 \longleftrightarrow N-1)$

So an equivalent transform pair:

This is what FFT does

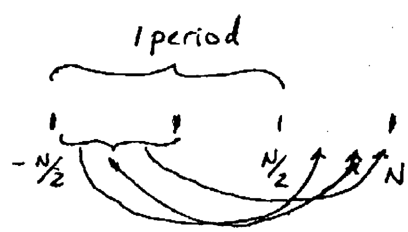
$$\tilde{\tilde{u}}_k = \frac{1}{N}\sum_{j=0}^{N-1} u_j e^{-ikx_j} \qquad 0 \le k \le N-1$$

$$x_j = \frac{2\pi j}{N}$$

$$u_j = \sum_{k=0}^{N-1} \tilde{\tilde{u}}_k e^{ikx_j}$$

$\tilde{u}_k$ (correct) ; $\tilde{\tilde{u}}_k$ (aliased)

$$\boxed{\begin{array}{ll} \tilde{\tilde{u}}_k = \tilde{u}_k & 0 \le k \le \frac{N}{2}-1 \\ \tilde{\tilde{u}}_k = \tilde{u}_{k-N} & \frac{N}{2} \le k \le N-1 \end{array}}$$



This part of spectrum is aliased over to [N/2, N]

Derivatives

$$\frac{\partial u}{\partial x} = \underbrace{\frac{\partial}{\partial x}\left(\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{u}_k e^{ikx} \right)}_{\text{TRUE}} \neq \underbrace{\frac{\partial}{\partial x}\left(\sum_{k=0}^{N-1} \tilde{\tilde{u}}_k e^{ikx} \right)}_{\text{aliased}}$$

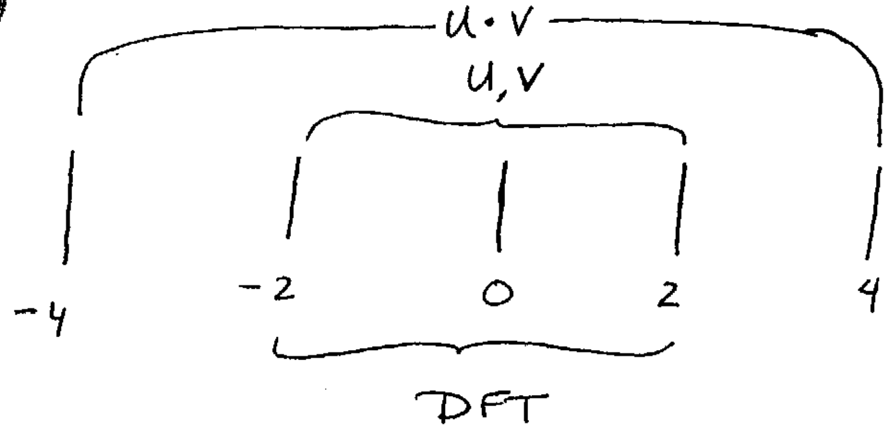
$$\frac{\partial u}{\partial x} = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} ik\,\tilde{u}_k e^{ikx} = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{w}_k e^{ikx} \quad \text{where } \underline{\tilde{w}_k = ik\,\tilde{u}_k}$$

which is equivalent to

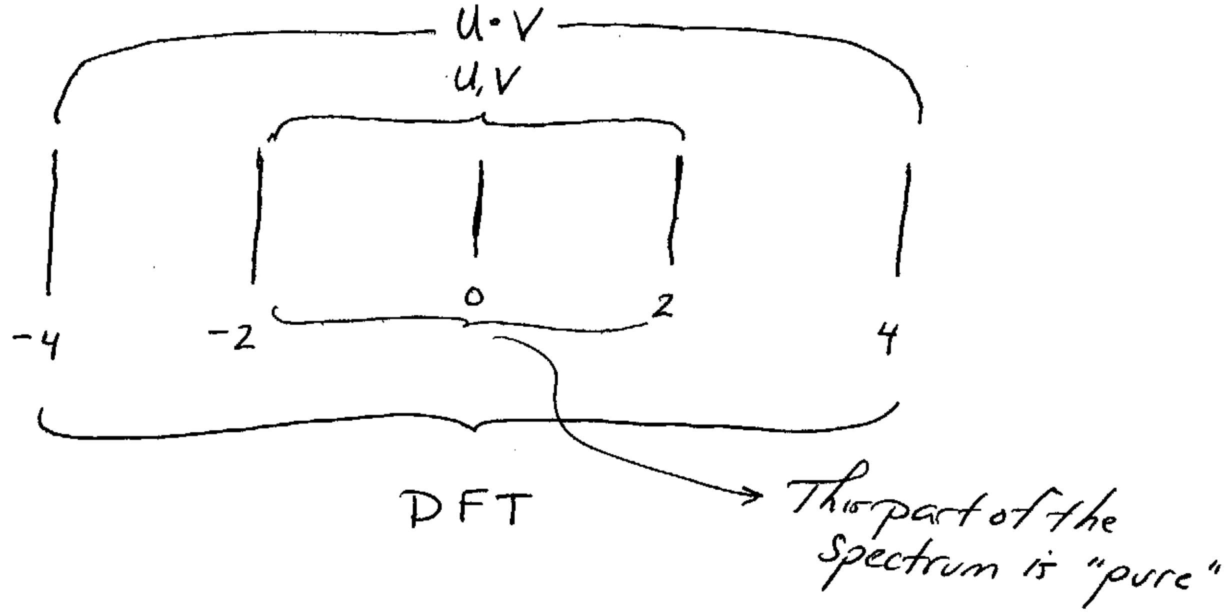
$$\frac{\partial u}{\partial x} = \sum_{k=0}^{N-1} \tilde{\tilde{w}}_k e^{ikx}$$

with
$$\boxed{\begin{aligned} \tilde{\tilde{w}}_k &= \tilde{w}_k = ik\,\tilde{u}_k = ik\,\tilde{\tilde{u}}_k && 0 \le k \le \tfrac{N}{2}-1 \\ \tilde{\tilde{w}}_k &= \tilde{w}_{k-N} = i(k-N)\,\tilde{u}_{k-N} = i(k-N)\,\tilde{\tilde{u}}_k && \tfrac{N}{2} \le k \le N-1 \end{aligned}}$$

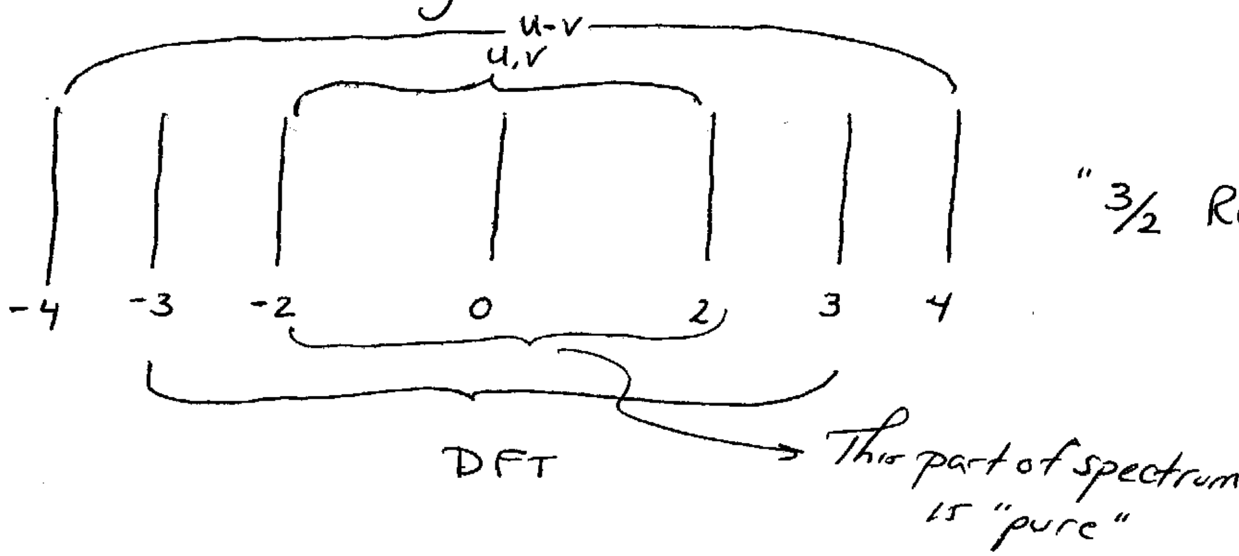
Dealiasing by oversampling (padding/truncation)



"Fully Aliased"



"Fully Dealiased"



"3/2 Rule"

Sample: u_j, v_j on M pts $\quad M \geq \frac{3N}{2} - 1$

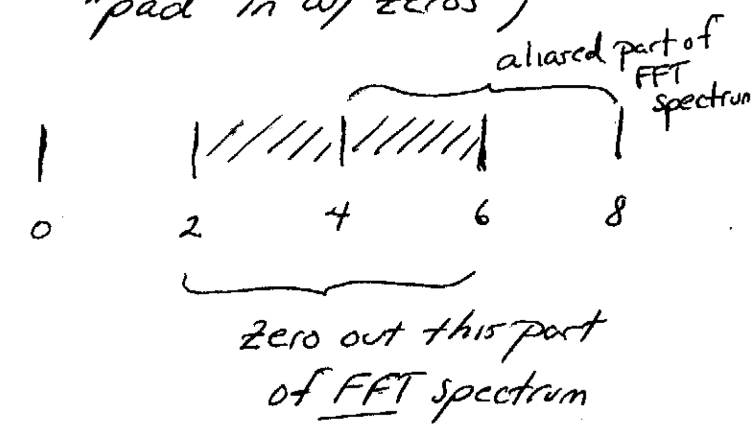
(in practice $M = 2N$ since FFTs need 2^p points)

Product: $u_j v_j = w_j$ (M pts)

Transform: w_k (Mpt FFT)

Truncate: w_k (to N nonzero values... "pad" in w/ zeros)

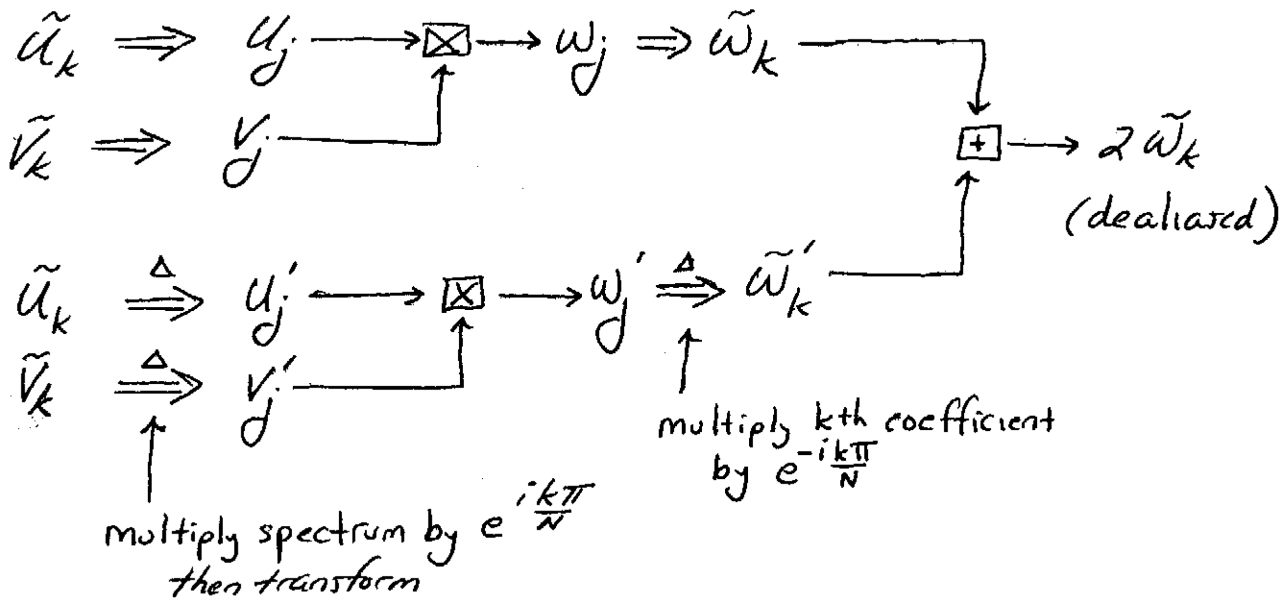
e.g.

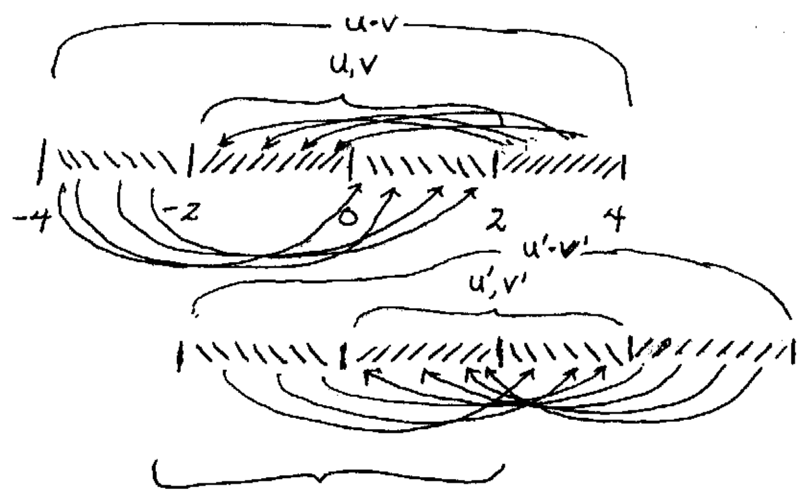


Integrate: $\dfrac{d\tilde{u}_k}{dt}$ (using only N wavenumbers) i.e. time advance only non-zero portion of spectrum

Transform Back: w_j (Mpts... IFFT the padded/truncated FFT spectrum)

Dealias by Phase Shifting:





This part of spectrum for u·v is "pure" due to cancellation when summed

Key: Pseudospectral Transform : 3 FFTs + N mutiplies

$$\tilde{u}_k \longrightarrow \left\{ \begin{array}{ccc} \tilde{u}_k & \xrightarrow{IFFT} & u_j \\ \tilde{v}_k & \xrightarrow{IFFT} & v_j \end{array} \right\} \; w_j = u_j v_j \xrightarrow{FFT} \tilde{w}_k$$

$$k\text{-space} \longrightarrow x\text{-space} \longrightarrow k\text{-space}$$

$$\tilde{v}_k \equiv \left(\widetilde{\frac{\partial u}{\partial x}}\right)_k \equiv ik\tilde{u}_k \; ; \; v_j \equiv \left(\frac{\partial u}{\partial x}\right)_j$$