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THE EVALUATION OF DOMAIN INTEGRALS IN THE BOUNDARY ELEMENT METHOD

H. L. PINA
Instituto Superior Técnico, Lisbon, Portugal

SUMMARY

In the application of the boundary element method (BEM) source terms give rise to domain integrals. To keep in the spirit of this method these integrals should be transformed into boundary integrals. Unfortunately this is possible only in some circumstances. In this paper we develop further an idea of Vijayakumar and Cormack (1987) to obtain other cases where such transformation is possible

INTRODUCTION

Let us consider the BEM for potential problems in a domain Ω of $\mathbb{R}^N$ with a Lipschitzean boundary Γ. If for simplicity we place the origin of the Cartesan co-ordinate system at the collocation point, the source terms give rise to integrals of the form

$$I(f) = \int_\Omega f(x) \ln r(x) \, \mathrm{d}\Omega \tag{1}$$

for $N = 2$ and

$$I(f) = \int_\Omega \frac{f(x)}{r^\alpha(x)} \, \mathrm{d}\Omega \tag{2}$$

for $N \geqslant 2$.

Here the function f is supposed to be smooth and the exponent α is related to the dimension N of the space in such a way that the last integral (2) exists in the Cauchy principal-value sense. The symbol r stands for the Euclidean distance of point x from the origin which, as stated above, has been made coincident with the collocation point.

When f is harmonic, the transformation of integrals (1) and (2) into boundary integrals has been accomplished by the use of Green's second identity as in Reference 1. Stokes' theorem was employed in Reference 2 to transform certain integrals over triangles into integrals over the respective boundary. When $N = 2$ and the function f assumes the form $f(x, y) = x^m y^n / r^\alpha$ or $f(x, y) = x^m y^n \ln r$, Reference 3 presents a method, also based on Stokes' theorem, to reduce domain integrals to boundary integrals. The latter are then evaluated analytically for polygonal domains. When f is a homogeneous function, Reference 4 shows that the application of the divergence theorem permits a very elegant transformation of domain integrals into boundary integrals and compares the formula obtained with those derived by other authors, e.g. References 3 and 5. We study this method in more detail.

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THE CASE OF HOMOGENEOUS f

We say that a function $f: \mathbb{R}^N \to \mathbb{R}$ is homogeneous of degree α if

$$f(\lambda x) = \lambda^\alpha f(x), \quad \forall x \in \mathbb{R}^N, \quad \forall \lambda \in \mathbb{R} \tag{3}$$

Taking derivatives of both sides of this expression with respect to λ we get

$$\nabla f(\lambda x) \cdot x = \alpha \lambda^{\alpha-1} f(x) \tag{4}$$

Making $\lambda = 1$ we arrive at the Euler relation for homogeneous functions:

$$\nabla f(x) \cdot x = \alpha f(x) \tag{5}$$

Appealing to the following identity:

$$\nabla \cdot (f(x)x) = \nabla f(x) \cdot x + f(x) \nabla \cdot x \tag{6}$$

and to property (5), and noticing that $\nabla x = N$, we get

$$\nabla \cdot (f(x)x) = (N + \alpha) f(x) \tag{7}$$

Therefore, the divergence theorem yields

$$\int_\Omega \nabla \cdot (f(x)x) \, d\Omega = \int_\Gamma f(x) x \cdot n \, d\Gamma = (N + \alpha) \int_\Omega f(x) \, d\Omega \tag{8}$$

where n stands for the unit outward normal to the boundary Γ. If $N + \alpha \neq 0$ then it follows immediately that

$$\int_\Omega f(x) \, d\Omega = \frac{1}{N + \alpha} \int_\Gamma f(x) x \cdot n \, d\Gamma \tag{9}$$

This is the basic result of Reference 4, which provides a straightforward route to some of the formulas derived in Reference 3.

Here we extend these results to the case (1). To this end we need an auxiliary result which is interesting by itself. Starting from the identity

$$\nabla \cdot (\ln r(x)x) = \frac{1}{r(x)} \nabla r(x) \cdot x + \ln r(x) \nabla \cdot x \tag{10}$$

and using simple manipulations we obtain

$$\nabla \cdot (\ln r(x)x) = 1 + N \ln r(x) = \frac{1}{N} \nabla \cdot x + N \ln r(x) \tag{11}$$

Thus we have that

$$\ln r(x) = \frac{1}{N} \nabla \cdot \left[\left\{ \ln r(x) - \frac{1}{N} \right\} x \right] \tag{12}$$

and therefore, using the divergence theorem,

$$\int_\Omega \ln r(x) \, d\Omega = \frac{1}{N} \int_\Omega \nabla \cdot \left[\left(\ln r(x) - \frac{1}{N} \right) x \right] d\Omega = \frac{1}{N} \int_\Gamma \left[\ln r(x) - \frac{1}{N} \right] x \cdot n \, d\Gamma \tag{13}$$

(Note that expression (13) gives for $N = 1$ the familiar formula

$$\int_a^b \ln x \, dx = [(\ln x - 1)x|_a^b$$

For $N = 2$, expression (13) coincides with the formula that we can obtain employing Green's second identity for harmonic functions as in Reference 1.)

We now derive the following relations, where we make use of (5) and (11):

$$\begin{aligned}\nabla\cdot[f(x)\ln r(x)x] &= f(x)\nabla\cdot[\ln r(x)x] + (\nabla f(x)\cdot x)\ln r(x)\\ &= f(x)(1 + N\ln r(x)) + \alpha f(x)\ln r(x)\\ &= f(x) + (N+\alpha)f(x)\ln r(x)\end{aligned} \tag{14}$$

Integrating both sides of the last expression and employing the divergence theorem, we obtain

$$(N+\alpha)\int_\Omega f(x)\ln r(x)\, d\Omega = \int_\Gamma \nabla\cdot[f(x)\ln r(x)x]\, d\Gamma - \int_\Omega f(x)\, d\Omega \tag{15}$$

Substituting the last integral on the right-hand side of this expression by (9) we have, for $N+\alpha \neq 0$,

$$(N+\alpha)\int_\Omega f(x)\ln r(x)\, d\Omega = \int_\Gamma \left[f(x)\ln r(x)x\cdot n - \frac{1}{N+\alpha}f(x)x\cdot n\right] d\Gamma \tag{16}$$

and finally

$$\int_\Omega f(x)\ln r(x)\, d\Omega = \frac{1}{N+\alpha}\int_\Gamma \left[\ln r(x) - \frac{1}{N+\alpha}\right] f(x)x\cdot n\, d\Gamma \tag{17}$$

For $N = 2$ and $f(x) = x^m y^n$ we have $\alpha = m + n$, and (17) delivers the same expressions as found in Reference 3.

Note that expression (17) can be applied directly when f is a polynomial in N variables. In fact, each monomial is a homogeneous function, so we can apply (17) to each monomial in turn and, by the linearity of the integral, add together the individual contributions. Also, expression (17) can be applied to integrate in domain elements (cells) in order to obtain analytical expressions. We must then take Ω as the element domain and Γ as its boundary, as was done in Reference 3.

Integrals of the type

$$I(f) = \int_\Omega f(x)r^\beta(x)\ln r(x)\, d\Omega \tag{18}$$

can be dealt with easily by resorting to (17). It is sufficient to note that the function $r^\beta(x)$ is homogeneous of degree β, so if $f(x)$ is homogeneous of degree α their product $f(x)r^\beta(x)$ is homogeneous of degree $\alpha+\beta$. Applying (17) we obtain immediately that

$$\int_\Omega f(x)r^\beta(x)\ln r(x)\, d\Omega = \frac{1}{N+\alpha+\beta}\int_\Gamma \left[\ln r(x) - \frac{1}{N+\alpha+\beta}\right] r^\beta(x)f(x)x\cdot n\, d\Gamma \tag{19}$$

An example of application of this relation concerns the evaluation of domain integrals for the biharmonic equation governing, for instance, plate bending problems. In this case the fundamental solution involves the term $r^2 \ln r$,[6] so (19) yields, with $N = 2$ and $\beta = 2$, the expression

$$\int_\Omega f(x)r^2(x)\ln r(x)\, d\Omega = \frac{1}{4+\alpha}\int_\Gamma \left[\ln r(x) - \frac{1}{4+\alpha}\right] r^2(x)f(x)x\cdot n\, d\Gamma \tag{20}$$

CONCLUSIONS

We have shown that the method employed in References 4 and 3 are equivalent even when the integrand includes the kernels $\ln r$ or $r^{\beta} \ln r$. The first method is, however, more direct.

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