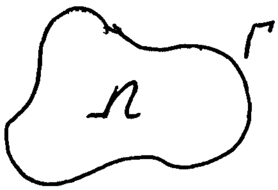
Transient Problems w/ BEM

- Prototype problem: diffusion eqn

$$k\nabla^2 u = \frac{\partial u}{\partial t} \qquad \text{BC's} \quad u \text{ or } \frac{\partial u}{\partial n} \text{ on } \Gamma \text{ for all } t$$



IC's u_0 for all $\underset{\sim}{x}$ at t_0

Approaches

1.) Eliminate time variable through Transformation
⇒ usually Fourier or Laplace

⇒ appeals to our analytical notions,
⇒ but inversion done numerically, usually requires
Some idea of expected behavior of sol'n
in order to select appropriate values of
transform parameter.

Best way to do it?

e.g. Try Laplace ... assume BCs, k { Constant in time

Recall: $\mathcal{L}\{u(\underset{\sim}{x},t)\} = \bar{u}(\underset{\sim}{x},\lambda) = \int_0^\infty u(x,t)e^{-\lambda t}\,dt$

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = \lambda\bar{u} - u_0$$

IC's

Integration-by-parts: $\mathcal{L}\left(\frac{\partial u}{\partial t}\right) = \int_0^\infty \frac{\partial u}{\partial t} e^{-\lambda t} dt$

$$\underbrace{u e^{-\lambda t}\Big|_0^\infty}_{u_0} - (-\lambda)\underbrace{\int_0^\infty u e^{-\lambda t} dt}_{\bar{u}}$$

Then $\mathcal{L}\left\{ \nabla^2 u - \frac{1}{k}\frac{\partial u}{\partial t} \right\} = 0$

$$\Rightarrow \quad \nabla^2 \bar{u} - \frac{\lambda}{k}\bar{u} + \frac{1}{k} u_0 = 0$$

Looks like elliptic type PDEs we've solved already

ICs become forcing term (known)

Also must Transform BCs:

on Γ: $\bar{u} = \frac{u}{\lambda}$

or

$\frac{\partial \bar{u}}{\partial n} = \frac{\partial u}{\partial n} \, 1/\lambda$

$$\Rightarrow \bar{u} = \int_0^\infty u e^{-\lambda t} dt$$

$$= u \int_0^\infty e^{-\lambda t} dt = \frac{u}{\lambda}$$

constant in t, easy

If time evolution of BCs is complex, this could get ugly!! Simple functions w/ known transforms OK

All we need to know is Green's function for

$$\nabla^2 G_i - \frac{\lambda}{k} G_i = -\delta(\underset{\sim}{x} - \underset{\sim}{x}_i)$$

Turns out to be:

$$G_i = \frac{1}{2\pi} K_0\left(\sqrt{\lambda/k}\, r\right) \qquad 2D$$

modified Bessel function of 2nd kind order zero

$$G_i = \frac{(\lambda/k)^{1/4}}{\sqrt{r}\,(2\pi)^{3/2}} K_{1/2}\left(\sqrt{\lambda/k}\, r\right) \qquad 3D$$

often see Green's function for $k\nabla^2 G_i - \lambda G_i = -\delta(\underline{x}-\underline{x}_i)$ only have to divide results by k e.g. 2D: $G_i = \frac{1}{2\pi k} K_0\left(\sqrt{\frac{\lambda}{k}}\, r\right)$

So Boundary expression is (2D)

$$\frac{\alpha_i}{2\pi}\bar{u}_i = \oint \frac{\partial \bar{u}}{\partial n} G_i - \frac{\partial G_i}{\partial n}\bar{u}\, ds + \left\langle \frac{1}{k} u_0 G_i \right\rangle$$

Solve as before... take "i" to discretized boundary

Can show K_0 has proper form near singularity $\Rightarrow$ $\lim_{z\to 0} K_0(z) = -\ln z$ just like Laplace... everything follows!

Solution, however, is in transformed domain

Note: IC's give rise to integration over domain ... try to transform to boundary when possible as we did before e.g. u_0 satisfies $\nabla^2 u_0 = 0$

Remaining step ... Inverse Transform ....
essentially a curve fitting process

e.g. assume u at any point represented by finite series: $u(\underset{\sim}{x}, t) = \sum_{n=1}^{N} a_n(\underset{\sim}{x}) e^{-b_n(\underset{\sim}{x}) t}$

then $$\bar{u}(\underset{\sim}{x}, \lambda) = \sum_{n=1}^{N} \frac{a_n(\underset{\sim}{x})}{\lambda + b_n(\underset{\sim}{x})}$$

Choose a sequence of λ's, Solve for $\bar{u}(\underset{\sim}{x}, \lambda)$ using BEM, then have set of relations for coefficients $a_n(\underset{\sim}{x})$, $b_n(\underset{\sim}{x})$ which can be solved

Similarly for $\frac{\partial \bar{u}}{\partial n}(\underset{\sim}{x}, \lambda)$ and $\frac{\partial u}{\partial n}(\underset{\sim}{x}, t)$

2.) Finite - Difference in Time
(Time- Independent Green's Function)

Write $\frac{\partial u}{\partial t} = \frac{u^{k+1} - u^k}{\Delta t}$ then we have

$$\nabla^2 u - \frac{1}{k}\frac{\partial u}{\partial t} \Rightarrow \nabla^2 u^{k+1} - \frac{1}{k}\left(\frac{u^{k+1} - u^k}{\Delta t}\right) = 0$$

$$\Rightarrow \nabla^2 u^{k+1} - \frac{1}{k\Delta t} u^{k+1} + \frac{1}{k\Delta t} u^k = 0$$

but this equation has exactly same form we just saw ... where u^k acts as IC's for solution at u^{k+1} $\Rightarrow \frac{1}{\Delta t}$ plays role of λ

So we solve system of equations:

$$\frac{\alpha_i}{2\pi} u_i^{k+1} = \oint \frac{\partial u^{k+1}}{\partial n} G_i - \frac{\partial G_i}{\partial n} u^{k+1} + \frac{1}{k\Delta t}\langle u^k G_i \rangle$$

where G_i satisfies $\nabla^2 G_i - \frac{1}{k\Delta t} G_i = -\delta(\underset{\sim}{x} - \underset{\sim}{x}_i)$

$$\Rightarrow G_i = \frac{1}{2\pi} K_0\left(\frac{r}{\sqrt{k\Delta t}}\right)$$

2D

$\Rightarrow$ Starting with ICs advance in time
must calculate interior values at
sufficient # interior points so that
u^{k+1} can serve as "IC's" for u^{k+2} etc...
i.e. need $\langle u^{k+1} G_i \rangle \frac{1}{k\Delta t}$ to get u^{k+2}

3) Time-dependent Fundamental Solutions (Green's Functions)

Green's function satisfies:

$$\nabla^2 G + \frac{1}{k}\frac{\partial G}{\partial t} = -\delta(\underset{\sim}{x}-\underset{\sim}{x}_i)\,\delta(t_F - t)$$

G is function of $\underset{\sim}{x}, t, \underset{\sim}{x}_i, t_F \Rightarrow G(\underset{\sim}{x}, t; \underset{\sim}{x}_i, t_F)$

represents the effect of unit point source at $\underset{\sim}{x}_i$ applied at time t_F on the location $\underset{\sim}{x}$ at time, t

Sol'n: $$G_{i,t_f} = \begin{cases} 0 & t > t_F \\ \dfrac{1}{4\pi\tau}\exp\left(-\dfrac{r^2}{4k\tau}\right) & \end{cases} \qquad \underline{\underline{2D}}$$

$\tau = t_F - t$

$$\lim_{t \to t_F} G_{i,t_f} = k\delta(\underset{\sim}{x} - \underset{\sim}{x}_i) \quad \underline{\text{or}} \quad \lim_{t \to t_F} \int_\Omega f(\underset{\sim}{x}, t)\, G_{i,t_F} = f(\underset{\sim}{x}_i, t_F)\, k$$

$$G_{i,t_f} = \begin{cases} 0 & t > t_F \\ \dfrac{1}{k^{1/2}(4\pi\tau)^{3/2}} \exp\left\{ -\dfrac{r^2}{4k\tau} \right\} & t < t_F \end{cases} \qquad \underline{\underline{3D}}$$

IF Laplace Transform these, get earlier time-independent Green's Functions !!

Now get weighted Residual form:

$$\int_{t_0}^{t_F} \left\langle \nabla^2 u - \frac{1}{k}\frac{\partial u}{\partial t}, G_{i,t_F} \right\rangle dt$$

$$= \int_{t_0}^{t_F} \left\{ -\langle \nabla u \cdot \nabla G_{i,t_F} \rangle - \frac{1}{k}\left\langle \frac{\partial u}{\partial t} G_{i,t_F} \right\rangle + \oint \frac{\partial u}{\partial n} G_{i,t_F}\, ds \right\} dt$$

$$\int_{t_0}^{t_F} \left\langle \nabla^2 G_{i,t_F} + \frac{1}{k}\frac{\partial G_{i,t_F}}{\partial t}, u \right\rangle$$

$$= \int_{t_0}^{t_F} \left\{ -\langle \nabla G_{i,t_F} \cdot \nabla u \rangle + \frac{1}{k}\left\langle \frac{\partial G_{i,t_F}}{\partial t} u \right\rangle + \oint \frac{\partial G_{i,t_F}}{\partial n} u\, ds \right\} dt$$

Subtract

$$\int_{t_0}^{t_F} \left\{ \left\langle \cancel{\nabla^2 u - \frac{1}{k}\frac{\partial u}{\partial t}}^{\,0}, G_{i,t_F} \right\rangle - \left\langle \nabla^2 G_{i,t_F} + \frac{1}{k}\frac{\partial G_{i,t_F}}{\partial t}, u \right\rangle \right\} dt$$

$$= \int_{t_0}^{t_F} \left\{ -1/k \underbrace{\left\langle \frac{\partial u}{\partial t} G_{i,t_F} + \frac{\partial G_{i,t_F}}{\partial t} u \right\rangle}_{\frac{\partial}{\partial t}(u G_{i,t_F})} + \oint \frac{\partial u}{\partial n} G_{i,t_F} - \frac{\partial G_{i,t_F}}{\partial n} u \, ds \right\} dt$$

so
$$\int_{t_0}^{t_F} -\left\langle \nabla^2 G_{i,t_F} + \frac{1}{k}\frac{\partial G_{i,t_F}}{\partial t}, u \right\rangle dt =$$

$$-\frac{1}{k}\left\langle u\, G_{i,t_F} \right\rangle \Big|_{t_0}^{t_F} + \int_{t_0}^{t_F} dt \oint \frac{\partial u}{\partial n} G_{i,t_F} - \frac{\partial G_{i,t_F}}{\partial n} u \, ds$$

Investigate singularity that occurs at $t = t_F$

Avoid ending integrations exactly at peak of delta function ... look at $\lim_{\epsilon \to 0} \int_{t_0}^{t_F - \epsilon} (\;)$

so
$$\lim_{\epsilon \to 0} \int_{t_0}^{t_F-\epsilon} -\left\langle \cancel{\nabla^2 G_{i,t_F} + \frac{1}{k}\frac{\partial G_{i,t_F}}{\partial t}}^{\,0}, u \right\rangle dt = 0$$

since $\nabla^2 G_{i,t_F} + \frac{1}{k}\frac{\partial G_{i,t_F}}{\partial t} = 0$ in Ω for $t < t_F$

$$-\frac{1}{k}\left\langle \underbrace{U_{t_F} G_{i,t_F}(t_F)}_{k\delta(\underset{\sim}{x}-\underset{\sim}{x}_i)} - U_{t_0} G_{i,t_F}(t_0) \right\rangle + \int_{t_0}^{t_F} dt \oint \frac{\partial U}{\partial n} G_{i,t_F} - \frac{\partial G_{i,t_F}}{\partial n} U \, ds = 0$$

$$\boxed{U_{i,t_F} = \frac{1}{k}\left\langle U_{t_0} G_{i,t_F}(t_0) \right\rangle + \int_{t_0}^{t_F}\oint \frac{\partial U}{\partial n} G_{i,t_F} - \frac{\partial G_{i,t_F}}{\partial n} U \, ds \, dt}$$

Valid for any point in Ω (i.e. $\underset{\sim}{x}_i$ in Ω)

<u>or</u>

could look at $\lim_{\epsilon \to 0} \int_{t_0}^{t_F+\epsilon} (\ ) \ldots$ in this case $G(\underset{\sim}{x}, t, \underset{\sim}{x}_i, t_F+\epsilon) = 0$

$$\int_{t_0}^{t_F+\epsilon} -\left\langle \underbrace{\nabla^2 G_{i,t_F} + \frac{1}{k}\frac{\partial G_{i,t_F}}{\partial t}}_{-\delta(\underset{\sim}{x}-\underset{\sim}{x}_i)\delta(t_F-t)}, U \right\rangle dt = -\frac{1}{k}\left\langle U_{t_F+\epsilon} \cancelto{0}{G_{i,t_F}(t_F+\epsilon)} - U_{t_0} G_{i,t_F}(t_0) \right\rangle + \int_{t_0}^{t_F} dt \oint \frac{\partial U}{\partial n} G_{i,t_F} - \frac{\partial G_{i,t_F}}{\partial n} U \, ds$$

$$\boxed{U_{i,t_F} = \left\langle \frac{1}{k} U_{t_0} G_{i,t_F}(t_0) \right\rangle + \int_{t_0}^{t_F} dt \oint \frac{\partial U}{\partial n} G_{i,t_F} - \frac{\partial G_{i,t_F}}{\partial n} U \, ds}$$

same as before! Must move to boundary to generate system of equations as before

- Two approaches to time-marching with time-dependent G_i

1. treat each time-step as new problem: need to compute internal values of u to act as IC's for next step

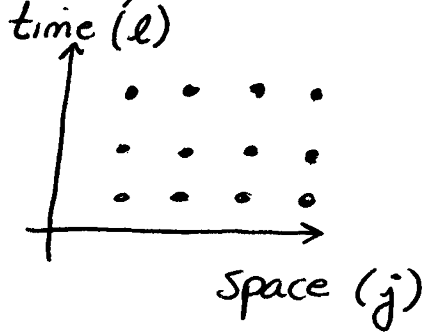
2. time integration always restarts from t_0, so despite the increasing # intermediate steps as t increases, internal values of u not needed (only u_0)

Look at approach 1.) First:

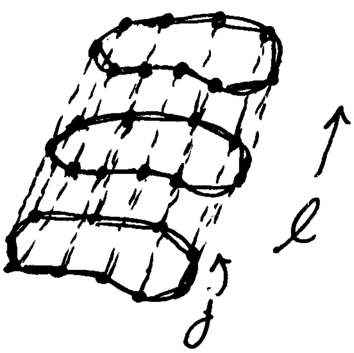
Expand $u(\underset{\sim}{x},t)$ as follows:

$$u(\underset{\sim}{x},t) = \sum_{\ell=1}^{F}\sum_{j=1}^{N} u_{j\ell}\,\phi_j(\underset{\sim}{x})\,\psi_\ell(t)$$

Think of discretization (nodes) in space-time
N nodes and F time steps



<u>or</u>



$\phi_j(\underset{\sim}{x}_i) = \delta_{ij}$

$\psi_\ell(t_k) = \delta_{\ell k}$

Substitute into general boundary expression

$$C_{iF}U_{iF} + \sum_{j=1}^{N}\sum_{l=1}^{F}\left(\oint_{\Gamma}\phi_j \int_{t_0}^{t_F}\frac{\partial G_{i,F}}{\partial n}\psi_l\,dt\,d\Gamma\right)U_{jl}$$

$$U_{iF} + \lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\int_{t_0}^{t_F}\frac{\partial G_{iF}}{\partial n}U_{iF}\,dt\,d\Gamma$$

$$= \sum_{j=1}^{N}\sum_{l=1}^{F}\left(\oint_{\Gamma}\phi_j \int_{t_0}^{t_F} G_{i,F}\,\psi_l\,dt\,d\Gamma\right)\frac{\partial U}{\partial n}_{jl}$$

$$+ \frac{1}{K}\sum_{m=1}^{M}U_{m,0}\int_{\Omega} G_{iF}\Big|_{t_0}\phi_m\,d\Omega$$

known IC
Interior divided into M "cells" or elements.

Now if we want to consider advancing only 1 Δt, reduce integral over time:

$$C_{iF}U_{i,F} + \sum_{j=1}^{N}\sum_{l=1}^{F}\left(\oint_{\Gamma}\phi_j \int_{t_{F-1}}^{t_F}\frac{\partial G_{i,F}}{\partial n}\psi_l\,dt\,d\Gamma\right)U_{jl}$$

$$= \sum_{j=1}^{N}\sum_{l=1}^{F}\left(\oint_{\Gamma}\phi_j \int_{t_{F-1}}^{t_F} G_{i,F}\,\psi_l\,dt\,d\Gamma\right)\frac{\partial U}{\partial n}_{jl}$$

$$+ \frac{1}{K}\sum_{m=1}^{M}U_{m,F-1}\int_{\Omega} G_{iF}\Big|_{t_{F-1}}\phi_m\,d\Omega$$

known, acts as IC's

try simplest $\psi(t)$ variation $\Rightarrow$ constant

e.g. $\psi_\ell(t) = \begin{cases} 1 & t_{\ell-1} < t \le t_\ell \\ 0 & \text{otherwise} \end{cases}$

Then time integrations can be done analytically!

$$\int_{t_{F-1}}^{t_F} \frac{\partial G_{iF}}{\partial n} \psi_F = \left(\frac{-r_i}{8\pi k} \frac{\partial r_i}{\partial n} \right) \int_{t_{F-1}}^{t_F} \frac{e^{-r_i^2/4k\tau_F}}{\tau_F^2} dt$$

$$\left(\text{Recall } G_{i,F} = \frac{1}{4\pi\tau_F} e^{-r_i^2/4k\tau_F} \Rightarrow \frac{\partial G_{iF}}{\partial n} = \frac{\partial G_{iF}}{\partial r} \frac{\partial r_i}{\partial n} \right.$$

$$\left. \Rightarrow \frac{-2r_i}{4k\tau_F} \left(\frac{1}{4\pi\tau_F} \right) \frac{\partial r_i}{\partial n} e^{-r_i^2/4k\tau_F} \right)$$

$$= \left(\frac{-r_i}{8\pi k} \frac{\partial r_i}{\partial n} \right) \underbrace{\int_{t_{F-1}}^{t_F} \frac{e^{-r_i^2/4k\tau_F}}{\tau_F^2} dt}_{I}$$

Let $s = \dfrac{r_i^2}{4k\tau_F} = \dfrac{r_i^2}{4k(t_F - t)} \Rightarrow t = t_F - \dfrac{r_i^2}{4ks}$

$$dt = \frac{r_i^2 ds}{4ks^2}, \quad s_{F-1} = \frac{r_i^2}{4k\Delta t_F}$$

$$s_F = \infty$$

$$I = \int_{\frac{r_i^2}{4k\Delta t_F}}^{\infty} \left(\frac{4k}{r_i^2} \right)^2 s^2 \frac{r_i^2}{4ks^2} e^{-s} ds$$

$$= \frac{4k}{r_i^2} e^{-r_i^2/4k\Delta t_F}$$

So we get:

$$\boxed{\int_{t_{F-1}}^{t_F} \frac{\partial G_{iF}}{\partial n} \psi_F \, dt = \frac{-1}{2\pi r_i} \frac{\partial r_i}{\partial n} e^{-r_i^2/4k\Delta t_F}}$$

Now ...

$$\int_{t_{F-1}}^{t_F} G_{iF} \psi_F \, dt = \int_{t_{F-1}}^{t_F} \frac{1}{4\pi\tau} e^{-r^2/4k\tau} dt$$

Same substitution: $\displaystyle \frac{1}{4\pi} \int_{\frac{r_i^2}{4k\Delta t_F}}^{\infty} \underbrace{\left(\frac{4ks}{r_i^2}\right)}_{\uparrow} \underbrace{\frac{r_i^2}{4ks^2} e^{-s} ds}_{dt}$

$$= \frac{1}{4\pi} \int_{\frac{r_i^2}{4k\Delta t_F}}^{\infty} \frac{e^{-s}}{s} ds$$

Now define exponential integral $E_1(s) = \int_s^{\infty} \frac{1}{z} e^{-z} dz$

$$= \boxed{\frac{1}{4\pi} E_1\left(\frac{r_i^2}{4k\Delta t_F}\right) = \int_{t_{F-1}}^{t_F} G_{iF} \psi_F \, dt}$$

So we have the matrix equation

$$[A]\{u_F\} = [B]\left\{\frac{\partial u}{\partial n}_F\right\} + [D]\{u_{F-1}\}$$

where

$$a_{ij} = c_{iF}\delta_{ij} - \frac{1}{2\pi}\oint_\Gamma \phi_j \left(\frac{1}{r_i}\frac{\partial r_i}{\partial n}\right) e^{-r_i^2/4k\Delta t_F}\, d\Gamma$$

$$b_{ij} = \frac{1}{4\pi}\oint_\Gamma \phi_j\, E_1\left(\frac{r_i^2}{4k\Delta t_F}\right) d\Gamma$$

$$d_{im} = \frac{1}{4\pi k\Delta t_F}\iint_\Omega \phi_m\, e^{-r_i^2/4k\Delta t_F}\, d\Omega$$

$\boxed{\text{Compute } C_{i,F} \text{ here, see next pg}}$

Note ... if $\Delta t_F = \Delta t$ (i.e. time-step constant) then coefficients are stationary ... can apply BC's to form overall matrix to be solved, decompose, backsubstitute at each time-step (only decompose once!!)

RHS changes at each time-step

Similar to FEM

Also note: must rework "analytic integrations when near spatial singularity

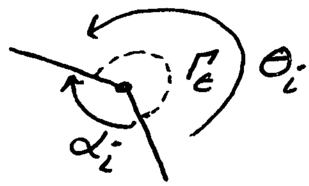
Can show $E_1(s)$ has $\ln s$ $s \to 0$ type singularity ... common to write

$$E_1(s) = -\ln s + f(s)$$

smooth function, integrate w/ normal quadrature

integrate analytically or w/ special quadrature

Compute $C_{i,F}$ term:



$$C_{i,F} = 1 + \lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} \int_{t_{F-1}}^{t_F} \frac{\partial G_{i,F}}{\partial n}\, dt\, d\Gamma_\epsilon$$

$$= 1 + \lim_{\epsilon \to 0} \int_0^{\theta_i} \epsilon\, d\theta \left(\frac{-1}{2\pi\epsilon} \underbrace{\frac{\partial r}{\partial n}}_{=1} \right) e^{-\epsilon^2/4k\Delta t_F}$$

$$= 1 - \frac{1}{2\pi} \lim_{\epsilon \to 0} \int_0^{\theta_i} e^{-\epsilon^2/4k\Delta t_F}\, d\theta = 1 - \frac{\theta_i}{2\pi} = \frac{2\pi - \theta_i}{2\pi} = \alpha_i/2\pi$$

So $\boxed{C_{i,F} = \alpha_i/2\pi}$

Can, of course, have higher order interpolation in time

Try Linear... e.g. $\psi_2 = \dfrac{t - t_{F-1}}{\Delta t_F}$; $\psi_1 = \dfrac{t_F - t}{\Delta t_F}$

then we get matrix equation (for advancing $1\ \Delta t$)

$$\underbrace{[A^1]}_{\text{integrations have }\psi_1}\overbrace{\{u_{F-1}\}}^{\text{known}} + \underbrace{[A^2]}_{\text{Integrations have }\psi_2}\{u_F\} = [B^1]\overbrace{\left\{\frac{\partial u}{\partial n}_{F-1}\right\}}^{\text{known}} + [B^2]\left\{\frac{\partial u}{\partial n}_F\right\} + [D]\{u_{F-1}\}$$

{workout integrations (pgs $\overset{[A]}{17\text{-}19}$, $\overset{[B]}{20\text{-}21}$ then come back to below}

$$a^1_{ij} = -\frac{1}{8\pi k \Delta t_F}\oint_\Gamma \phi_j\, r_i \frac{\partial r_i}{\partial n} E_1\left(\frac{r_i^2}{4k\Delta t_F}\right) d\Gamma$$

$$a^2_{ij} = c_{iF}\,\delta_{ij} - \frac{1}{2\pi}\oint \frac{1}{r_i} e^{\frac{-r_i^2}{4k\Delta t_F}} \frac{r_i}{4k\Delta t_F} E_1\left(\frac{r_i^2}{4k\Delta t_F}\right)\Bigg]\frac{\partial r_i}{\partial n}\phi_j\, d\Gamma$$

$$b^1_{ij} = \frac{1}{16\pi k\Delta t_F}\oint_\Gamma \phi_j\, r_i^2\, \Gamma\left(-1, \frac{r_i^2}{4k\Delta t_F}\right) d\Gamma$$

$$b^2_{ij} = \frac{1}{4\pi}\oint\left[E_1\left(\frac{r_i^2}{4k\Delta t_F}\right) - \frac{r_i^2}{4k\Delta t_F}\Gamma\left(-1, \frac{r_i^2}{4k\Delta t_F}\right)\right]\phi_j\, d\Gamma$$

d_{ij} same as before

Work out the time integrations:

$$\int_{t_{F-1}}^{t_F} \frac{\partial G_{iF}}{\partial n}\psi_1\,dt \quad \text{and} \quad \int_{t_{F-1}}^{t_F} \frac{\partial G_{iF}}{\partial n}\psi_2\,dt$$

$$\frac{1}{\Delta t_F}\int_{t_{F-1}}^{t_F}(t_F - t)\frac{\partial G_{iF}}{\partial n}\,dt = \frac{t_F}{\Delta t_F}\underbrace{\int_{t_{F-1}} \frac{\partial G_{iF}}{\partial n}\,dt}_{\text{already done}} - \frac{1}{\Delta t_F}\underbrace{\int_{t_{F-1}}^{t_F} t\,\frac{\partial G_{iF}}{\partial n}\,dt}_{\text{Need to do}}$$

$$\int_{t_{F-1}}^{t_F} t\,\frac{\partial G_{iF}}{\partial n}\,dt = -\frac{r_i}{8\pi k}\frac{\partial r_i}{\partial n}\int_{t_{F-1}}^{t_F}\frac{e^{-r_i^2/4k\tau_F}}{\tau_F^2}\,t\,dt$$

same substitution as before $s = \frac{r_i^2}{4k\tau_F}$, $t = t_F - \frac{r_i^2}{4ks}$

$dt = \frac{r_i^2}{4ks^2}$, $s_{F-1} = \frac{r_i^2}{4k\Delta t_F}$

$s_F = \infty$

$$= \frac{-r_i}{8\pi k}\frac{\partial r_i}{\partial n}\int_{s_{F-1}}^{s_F}\left(t_F - \frac{r_i^2}{4ks}\right)\frac{r_i^2}{4ks^2}\left(\frac{4ks}{r_i^2}\right)^2 e^{-s}\,ds$$

$$= \frac{-r_i}{8\pi k}\frac{\partial r_i}{\partial n}\left[\frac{4k}{r_i^2}\,t_F\int_{s_{F-1}}^{s_F} e^{-s}\,ds - \int_{s_{F-1}}^{s_F}\frac{e^{-s}}{s}\,ds\right]$$

$$\boxed{-\frac{t_F}{2\pi r_i}\frac{\partial r_i}{\partial n}e^{-r_i^2/4k\Delta t_F}+\frac{r_i}{8\pi k}\frac{\partial r_i}{\partial n}E_1\left(\frac{r_i^2}{4k\Delta t_F}\right)=\int_{t_{F-1}}^{t_F} t\,\frac{\partial G_{iF}}{\partial n}\,dt}$$

$$\therefore \int_{t_{F-1}}^{t_F}\frac{\partial G_{iF}}{\partial n}\psi_1\,dt=-\frac{t_F}{\Delta t_F 2\pi r_i}\frac{\partial r_i}{\partial n}e^{-r_i^2/4k\Delta t_F}+\frac{t_F}{\Delta t_F 2\pi r_i}\frac{\partial r_i}{\partial n}e^{-r_i^2/4k\Delta t_F}$$

$$-\frac{r_i}{8\pi k\Delta t_F}\frac{\partial r_i}{\partial n}E_1\left(\frac{r_i^2}{4k\Delta t_F}\right)$$

$$\Rightarrow \boxed{\int_{t_{F-1}}^{t_F}\frac{\partial G_{iF}}{\partial n}\psi_1\,dt=\frac{-r_i}{8\pi k\Delta t_F}\frac{\partial r_i}{\partial n}E_1\left(\frac{r_i^2}{4k\Delta t_F}\right)}$$

$$\int_{t_{F-1}}^{t_F}\frac{\partial G_{iF}}{\partial n}\psi_2\,dt=\frac{1}{\Delta t_F}\int_{t_{F-1}}^{t_F} t\,\frac{\partial G_{iF}}{\partial n}\,dt-\frac{t_{F-1}}{\Delta t_F}\int_{t_{F-1}}^{t_F}\frac{\partial G_{iF}}{\partial n}\,dt$$

$$=\frac{-t_F}{2\pi r_i\Delta t_F}\frac{\partial r_i}{\partial n}e^{-r_i^2/4k\Delta t_F}+\frac{r_i}{8\pi k\Delta t_F}\frac{\partial r_i}{\partial n}E_1\left(\frac{r_i^2}{4k\Delta t_F}\right)+\frac{t_{F-1}}{\Delta t_F 2\pi r_i}\frac{\partial r_i}{\partial n}e^{-r_i^2/4k\Delta t_F}$$

$$=\boxed{-\frac{1}{2\pi r_i}\frac{\partial r_i}{\partial n}e^{-r_i^2/4k\Delta t_F}+\frac{r_i}{8\pi k\Delta t_F}\frac{\partial r_i}{\partial n}E_1\left(\frac{r_i^2}{4k\Delta t_F}\right)=\int_{t_{F-1}}^{t_F}\frac{\partial G_{iF}}{\partial n}\psi_2\,dt}$$

What about $C_{i,F-1}$ and $C_{i,F}$ in this case?

$$C_{i,F-1} = \lim_{\epsilon\to 0} \int_{\Gamma_\epsilon} \epsilon\, d\theta \int_{t_{F-1}}^{t_F} \frac{\partial G_{iF}}{\partial n} \psi_{F-1}\, dt$$

$$= \lim_{\epsilon\to 0} \int_0^\theta \epsilon \left(\frac{-\epsilon}{8\pi k \Delta t_F}\right) \underbrace{E_1\left(\frac{\epsilon^2}{4k\Delta t_F}\right)}_{\text{goes as } \ln \epsilon} dt = \underline{\underline{0}}$$

$$C_{i,F} = \lim_{\epsilon\to 0} \int_{\Gamma_\epsilon} \epsilon\, d\theta \int_{t_{F-1}}^{t_F} \frac{\partial G_{iF}}{\partial n} \psi_F\, dt$$

$$= 1 + \lim_{\epsilon\to 0} \int_{\Gamma_\epsilon} \epsilon \left(\frac{-1}{2\pi\epsilon}\right) e^{-\epsilon_i^2/4\pi k \Delta t_F} + \cancelto{0}{\frac{\epsilon^2}{8\pi k \Delta t_F} E_1\left(\frac{\epsilon^2}{4k\Delta t_F}\right)} d\theta$$

$$1 - \frac{\theta}{2\pi} = \frac{2\pi - \theta}{2\pi} = \frac{\alpha_i}{2\pi}$$

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Also need $\int_{t_{F-1}}^{t_F} G_{iF}\,\psi_1\,dt$ and $\int_{t_{F-1}}^{t_F} G_{iF}\,\psi_2\,dt$

$$\int_{t_{F-1}}^{t_F} \frac{t_F - t}{\Delta t_F} G_{iF}\,dt = \frac{t_F}{\Delta t_F}\underbrace{\int_{t_{F-1}}^{t_F} G_{iF}\,dt}_{\text{done}} - \frac{1}{\Delta t_F}\underbrace{\int_{t_{F-1}}^{t_F} t\,G_{iF}\,dt}_{\text{Need}}$$

So: $$\int_{t_{F-1}}^{t_F} t\,G_{iF}\,dt = \frac{1}{4\pi}\int_{t_{F-1}}^{t_F} \frac{t}{\tau_F} e^{-\frac{r^2}{4k\tau_F}}\,dt$$

$$= \frac{1}{4\pi}\int_{s_{F-1}}^{s_F}\left(t_F - \frac{r_i^2}{4ks}\right)\left(\frac{r_i^2}{4ks^2}\right)\frac{4ks}{r_i^2}e^{-s}\,ds$$

$$= \frac{1}{4\pi}\int_{s_{F-1}}^{s_F}\frac{t_F}{s}e^{-s}\,ds - \frac{1}{4\pi}\frac{r_i^2}{4k}\int_{s_{F-1}}^{s_F}\frac{e^{-s}}{s^2}\,ds$$

$$\boxed{= \frac{t_F}{4\pi}E_1\left(\frac{r_i^2}{4k\Delta t_F}\right) - \frac{r_i^2}{16\pi k}\Gamma\left(-1, \frac{r_i^2}{4k\Delta t_F}\right) = \int_{t_{F-1}}^{t_F} t\,G_{iF}\,dt}$$

"Incomplete Gamma Function"

$$\Gamma(a, z) = \int_z^{\infty}\frac{e^{-s}}{s^{1-a}}\,ds$$

Gamma function

$$= \Gamma(a) - \int_0^z \frac{e^{-s}}{s^{1-a}}\,ds$$

$$\int_0^{\infty}\frac{e^{-s}}{s^{1-a}}\,ds$$

then

$$\int_{t_{F-1}}^{t_F} G_{iF}\,\psi_1\,dt = \frac{t_F}{\Delta t_F\,4\pi} E_1\left(\frac{r_i^2}{4k\Delta t_F}\right) - \frac{t_F}{4\pi\Delta t_F} E_1\left(\frac{r_i^2}{4k\Delta t_F}\right)$$

$$+ \frac{r_i^2}{16\pi k\Delta t_F}\Gamma\left(-1, \frac{r_i^2}{4k\Delta t_F}\right)$$

$$\boxed{\int_{t_{F-1}}^{t_F} G_{iF}\,\psi_1\,dt = \frac{r_i^2}{16\pi k\Delta t_F}\Gamma\left(-1, \frac{r_i^2}{4k\Delta t_F}\right)}$$

$$\int_{t_{F-1}}^{t_F} G_{iF}\,\psi_2\,dt = \frac{1}{\Delta t_F}\int_{t_{F-1}} t\,G_{iF}\,dt - \frac{t_{F-1}}{\Delta t_F}\int_{t_F} G_{iF}\,dt$$

$$= \frac{t_F}{4\pi\Delta t_F} E_1\left(\frac{r_i^2}{4k\Delta t_F}\right) - \frac{r_i^2}{16\pi k\Delta t_F}\Gamma\left(-1, \frac{r_i^2}{4k\Delta t_F}\right) - \frac{t_{F-1}}{4\pi\Delta t_F} E_1\left(\frac{r_i^2}{4k\Delta t_F}\right)$$

$$\boxed{\int_{t_{F-1}}^{t_F} G_{iF}\,\psi_2\,dt = \frac{1}{4\pi} E_1\left(\frac{r_i^2}{4k\Delta t_F}\right) - \frac{r_i^2}{16\pi k\Delta t_F}\Gamma\left(-1, \frac{r_i^2}{4k\Delta t_F}\right)}$$

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Matrices only function of Δt_F. Also w/ Linear time-variation, $\Rightarrow$ time-step constant then have stationary matrices $\Rightarrow$ LU decompose only <u>1 time</u>. Note that we need initial values of $\frac{\partial u}{\partial n}$ as well in Linear time variation case!

Strategy II: To advance sol'n from t_F to t_{F+1} recompute $\int_{t_0}^{t_{F+1}}$ i.e. integrate over the entire time from IC's ... to get t_{F+2} $\Rightarrow \int_{t_0}^{t_{F+2}}$ etc. Idea is to avoid computation of domain integration (Recall Strategy I has matrix system $[A]\{u_F\} = [B]\{\frac{\partial u}{\partial n}_F\} + \underbrace{[D]}_{\text{involves } \langle\rangle!}\{u_{F-1}\}$

IF we consider ψ_ℓ as constant on $t_{\ell-1} < t \leq t_\ell$ and zero elsewhere then can write

$$c_i u_{iF} + \sum_{j=1}^{N}\sum_{\ell=1}^{F} u_{j\ell}\left(\oint_\Gamma \phi_j \int_{t_{\ell-1}}^{t_\ell} \frac{\partial G_{iF}}{\partial n}\, dt\, d\Gamma\right)$$

$$= \sum_{j=1}^{N}\sum_{\ell=1}^{F} \frac{\partial u}{\partial n}_{j\ell}\left(\oint_\Gamma \phi_j \int_{t_{\ell-1}}^{t_\ell} G_{iF}\, dt\, d\Gamma\right) + \frac{1}{K}\sum_{m=1}^{M} u_{m,0} \int_\Omega G_{iF}\big|_{t_0}\, \phi_m\, d\Omega$$

(Strategy II con't)

which we could write as a matrix equation

$$[A_{1F}]\{u_1\} + [A_{2F}]\{u_2\} + \cdots [A_{FF}]\{u_F\}$$

$$= [B_{1F}]\left\{\frac{\partial u_1}{\partial n}\right\} + [B_{2F}]\left\{\frac{\partial u_2}{\partial n}\right\} + \cdots [B_{FF}]\left\{\frac{\partial u_F}{\partial n}\right\}$$

$$+ [D]\{u_0\}$$

where $a_{ij}^{1F} = \oint_\Gamma \phi_j \int_{t_0}^{t_1} \frac{\partial G_{iF}}{\partial n} dt\, d\Gamma$

$$a_{ij}^{2F} = \oint_\Gamma \phi_j \int_{t_1}^{t_2} \frac{\partial G_{iF}}{\partial n} dt\, d\Gamma$$

$$\vdots$$

$$a_{ij}^{FF} = c_i \delta_{ij} + \oint \phi_j \int_{t_{F-1}}^{t_F} \frac{\partial G_{iF}}{\partial n} dt\, d\Gamma$$

$$b_{ij}^{1F} = \oint_\Gamma \phi_j \int_{t_0}^{t_1} G_{i,F}\, dt\, d\Gamma$$

$$b_{ij}^{2F} = \oint_\Gamma \phi_j \int_{t_1}^{t_2} G_{i,F}\, dt\, d\Gamma$$

$$\vdots$$

$$b_{ij}^{FF} = \oint_\Gamma \phi_j \int_{t_{F-1}}^{t_F} G_{i,F}\, dt\, d\Gamma$$

<u>or</u> more compactly: $\sum_{\ell=1}^{F} A_{\ell F} u_F = \sum_{\ell=1}^{F} B_{\ell F} \frac{\partial u_F}{\partial n} + [D] u_0$

IC's are such that $U_0 = 0$, constant $\Rightarrow \nabla^2 U_0 = 0$
so can transform $\langle\,\rangle$ to $\oint$

Now if $U_{F-1}, U_{F-2} \ldots U_1$ and $\frac{\partial U}{\partial n}_{F-1}, \frac{\partial U}{\partial n}_{F-2} \ldots \frac{\partial U}{\partial n}_1$, are known, then this results in a set of equations for U_F and $\frac{\partial U}{\partial n}_F$!!

$$A_{FF} U_F - B_{FF} \frac{\partial U}{\partial n}_F = -\sum_{\ell=1}^{F-1} A_{\ell F} U_\ell + \sum_{\ell=1}^{F-1} B_{\ell F} \frac{\partial U}{\partial n}_\ell + [D] U_0$$

Requires the calculation of matrices

$$\left.\begin{array}{l} A_{1F}, A_{2F} \ldots A_{FF} \text{ and } B_{1F}, B_{2F} \ldots B_{FF} \\ \text{and to advance to } t_{F+1} \text{ we would need} \\ A_{1,F+1}, A_{2F+1} \ldots A_{F,F+1}\; A_{F+1,F+1} \text{ and } B_{1,F+1}, B_{2,F+1} \ldots \end{array}\right\}$$

Seems like a horrendous Task!! but not as bad as it may appear if Δt constant

Note the linear case follows similarly

Could write:

$2 \;\; \ell$
$1 \;\; \ell-1$

Basis #

$$\sum_{\ell=1}^{F} A^1_{\ell F} U_{F-1} + A^2_{\ell F} U_F = \sum_{\ell=1}^{F} B^1_{\ell F} \frac{\partial U}{\partial n}_{F-1} + B^2_{\ell F} \frac{\partial U}{\partial n}_{F-1} + D U_0$$

Now, the time integrations we do analytically ... recall we make change of integration $s = \frac{r_i^2}{4\pi\tau_F}$ $\Rightarrow \int_{t_{f-1}}^{t_f} (\,)\,dt \Rightarrow \int_{s_{f-1}}^{s_f} (\,)\,dt$

$$s_f = \frac{r_i^2}{4\pi(t_F - t_f)} \qquad s_{f-1} = \frac{r_i^2}{4\pi(t_F - t_{f-1})}$$

$\Leftarrow$ If $t_F = \Delta t_F F$ (i.e. equally spaced)

then $s_f = \dfrac{r_i^2}{4\pi \Delta t_F (F-f)}$

$$s_{f-1} = \frac{r_i^2}{4\pi \Delta t_F (F+1-f)}$$

So we repeat all but 1 of the integrations when we advance from t_{F-1} to t_F $\therefore$ only need to calculate 1 new A and 1 new B matrices at each time advance $\Rightarrow$ Need A_{1F} B_{1F}

e.g.

	t_{F-1}: $t_F - t_f$		New t_F		New t_{F+1}
t_0	$t_5 - t_0 = 5\Delta t$		$t_6 - t_0 = 6\Delta t$	t_0	$t_7 - t_0 = 7\Delta t$
	$f = 1$	Same	$f = 1$		
t_1	$t_5 - t_1 = 4\Delta t$		$t_6 - t_1 = 5\Delta t$	t_1	$t_7 - t_1 = 6\Delta t$
	$f = 2$	Same	2		
t_2	$t_5 - t_2 = 3\Delta t$		$t_6 - t_2 = 4\Delta t$	t_2	$t_7 - t_2 = 5\Delta t$
	3	Same	3		
t_3	$t_5 - t_3 = 2\Delta t$		$t_6 - t_3 = 3\Delta t$	t_3	$t_7 - t_3 = 4\Delta t$
	4	Same	4		
t_4	$t_5 - t_4 = \Delta t$		$t_6 - t_4 = 2\Delta t$	t_4	$t_7 - t_4 = 3\Delta t$
	5	Same	5		
$t_{F-1} \rightarrow t_5$	$t_5 - t_5 = 0$		$t_6 - t_5 = \Delta t$	t_5	$t_7 - t_5 = 2\Delta t$
			6		
t_F			$t_6 - t_6 = 0$	t_6	$t_7 - t_6 = \Delta t$
				t_7	$t_7 - t_7 = 0$

So matrices needed to advance to t_F:

$(A_{\ell F})$

nodes in time

location of Green's function Time Singularity

t_F:	1	2	3	4	...	F
1	A_{11}					
2	A_{12}	A_{22}				
3	A_{13}	A_{23}	A_{33}			
4	A_{14}	A_{24}	A_{34}	A_{44}		
⋮	⋮					
F	A_{1F}	A_{2F}	A_{3F}	A_{4F}	...	A_{FF}

Matrices needed to advance to t_F

This column contains matrices that must be calculated at each new t_F

Similarly for $B_{\ell F}$