Boundary Element Method (BEM)

Drawbacks to FEM

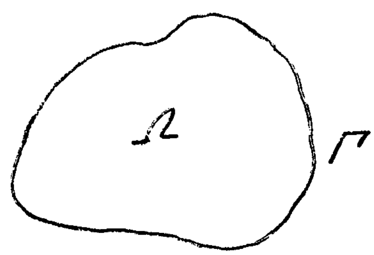
- Grid generation (domain elements a pain in the #!*)
- Open boundary problems (unbounded)

BEM addresses these issues (also reduction in problem dimension: domain vs boundary)

Boundary Formulations (Several approaches... lets look at a few)

I.

$\nabla^2 u = 0$ in Ω



Introduce weighting function $W(x,y)$

$$\langle \nabla^2 u, W \rangle = \langle -\nabla u \cdot \nabla W \rangle + \oint_\Gamma \nabla u \cdot \hat{n} W \, ds = 0$$

Weak form ... why?

a.) Continuity requirements on u reduced
b.) Natural BC's satisfied approximately
c.) PDE satisfied in average sense

Same beginning as FEM ... but now interchange roles of u + w: (w must be <u>twice</u> differentiable)

$$\langle \nabla^2 u, w \rangle = \langle -\nabla u \cdot \nabla w \rangle + \oint_\Gamma \frac{\partial u}{\partial n} w \, ds$$

$$\langle \nabla^2 w, u \rangle = \langle -\nabla w \cdot \nabla u \rangle + \oint_\Gamma \frac{\partial w}{\partial n} u \, ds$$

Subtract:

$$\langle (\nabla^2 u) w - (\nabla^2 w) u \rangle = \oint \left(\frac{\partial u}{\partial n} w - \frac{\partial w}{\partial n} u \right) ds$$

Green's 2nd Identity! ... actually "integration by parts" step is Green's 1st Identity

(i.e. $\langle \nabla^2 u, w \rangle + \langle \nabla u \cdot \nabla w \rangle = \oint \nabla u \cdot \hat{n} \, ds$)

Need u, w continuous + twice differentiable

- Key is to make LHS vanish...

  PDE says $\nabla^2 u = 0$

$$\langle -\nabla^2 w, u \rangle = \oint \left(\frac{\partial u}{\partial n} w - \frac{\partial w}{\partial n} u\right) ds$$

So we will need to choose w such that the problem reduces to the boundary only

- Also note ... well-posed BVP has
  u <u>or</u> $\frac{\partial u}{\partial n}$ specified on boundaries

  Both appear in boundary expression

  Strategy ... use boundary integral statement to compute the missing BC info ... How?

II. Alternate view ...

$$\langle \nabla^2 u, w \rangle = 0$$

Integrate by parts (Green's 1ST)

$$\langle \nabla^2 u, w \rangle = \langle -\nabla u \cdot \nabla w \rangle + \oint_\Gamma \frac{\partial u}{\partial n} w \, ds = 0$$

Do it again....

$$-\oint_\Gamma \frac{\partial w}{\partial n} u \, ds + \langle (\nabla^2 w), u \rangle + \oint \frac{\partial u}{\partial n} w \, ds = 0$$

$$\left(\text{Since } \nabla \cdot (u \nabla w) = \nabla u \cdot \nabla w + u \nabla^2 w \right)$$

so $\langle -\nabla^2 w, u \rangle = \oint \left(\frac{\partial u}{\partial n} w - \frac{\partial w}{\partial n} u \right) ds$

One way to think about things....

1 integration-by-parts $\Rightarrow$ leads to weak formulation for FEM

2 integrations-by-parts $\Rightarrow$ leads to weak form for BEM

Both FEM + BEM can be viewed as combination of WR process and integration by parts (applications of Green's theorems)

- In this light we might adopt the following definitions:

  a.) If approximate sol'n satisfies all bc's but not governing eqn in Ω, one has a pure "domain" method

  b.) If approximate sol'n satisfies goveening eqn, but not bc's, one has "boundary method"

  c.) If approximate sol'n satisfies neither goveening eqn or bc's, one has a "mixed" method.

- Now we have the statement...

$$\langle -\nabla^2 w, u \rangle = \oint \left(\frac{\partial u}{\partial n} w - \frac{\partial w}{\partial n} u \right) ds$$

to get a boundary method we choose w in one of 2 ways

  a.) w satisfies homogeneous PDE in Ω

i.e. $\nabla^2 w = 0$

so we get $\oint \left(\frac{\partial u}{\partial n} w - \frac{\partial w}{\partial n} u \right) ds = 0$

Trefftz's Method

b) w is a sol'n of PDE with special forcing such that it is still possible to reduce the problem to the boundary

$$\nabla^2 w = -\delta(\underset{\sim}{x} - \underset{\sim}{x}_i) \qquad \underset{\sim}{x}_i \equiv (x_i, y_i)$$

No BC's (Unbounded space)

w in this case referred to as "Green's Function" also called "fundamental sol'n" of governing PDE $\Rightarrow w(\underset{\sim}{x}, \underset{\sim}{x}_i) \equiv G(\underset{\sim}{x}, \underset{\sim}{x}_i) \equiv G_i(\underset{\sim}{x})$

So: $\langle -\nabla^2 w, u \rangle = \oint \frac{\partial u}{\partial n} w - \frac{\partial w}{\partial n} u \, ds$

$\langle \delta(\underset{\sim}{x} - \underset{\sim}{x}_i) u \rangle = \quad "$

$$u(\underset{\sim}{x}_i) = \oint \frac{\partial u}{\partial n} w_i - \frac{\partial w_i}{\partial n} u \, ds$$

or using Green's function notation

$$u_i = \oint \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, ds$$

Note... Given u and $\frac{\partial u}{\partial n}$ on Γ can compute u anywhere inside $\Rightarrow$ problem solved!

(but we know only u or $\frac{\partial u}{\partial n}$ is given)

Note forcing

Example: $\frac{d^2u}{dx^2} + x = 0$ $\quad u(0) = 0$

$u(1) = 0$

Exact $\Rightarrow u = \frac{x}{6} - \frac{x^3}{6}$

Trefftz Method: $\frac{d^2w}{dx^2} = 0 \Rightarrow w = a_1 x + a_2$

$\frac{dw}{dx} = a_1$

$$\left\langle \left(\frac{d^2u}{dx^2} + x \right) w \right\rangle = 0$$

(Formulate by integration-by-parts twice)

$$w \frac{du}{dx} \Big|_0^1 - \left\langle \frac{du}{dx} \frac{dw}{dx} \right\rangle + \langle x w \rangle = 0$$

$$w\frac{du}{dx}\Big|_0^1 - u\cancelto{0}{\frac{dw}{dx}}\Big|_0^1 + \cancelto{0}{\left\langle u\frac{d^2w}{dx^2}\right\rangle} + \langle Xw\rangle = 0$$

$$\int_0^1 X(a_1 x + a_2)\,dx + \frac{du(1)}{dx}(a_1 + a_2) - \frac{du(0)}{dx}a_2 = 0$$

$$\frac{a_1}{3} + \frac{a_2}{2} + \frac{du_1}{dx}(a_1 + a_2) - \frac{du_0}{dx}a_2 = 0$$

$$a_1\left(\frac{1}{3} + \frac{du_1}{dx}\right) + a_2\left(\frac{1}{2} - \frac{du_0}{dx} + \frac{du_1}{dx}\right) = 0$$

Must hold for arbitrary a_1 + a_2

$$\Rightarrow \quad \frac{du_1}{dx} = -1/3 \quad ; \quad \frac{du_0}{dx} = 1/6 \qquad \text{Exact!}$$

but No apparent way to get u in Ω!

<u>Green's Function</u> (or fundamental Sol'n) approach

- Formulate using governing PDE and PDE that Green's Function satisfies

$$\left\langle \left(\frac{d^2u}{dx^2}+x\right)G_i\right\rangle = 0 \Rightarrow \int_0^1 xG_i\,dx - \int_0^1 \frac{du}{dx}\frac{dG_i}{dx} + \frac{du}{dx}G_i\Big|_0^1$$

$$\left\langle\left(\frac{d^2G_i}{dx^2}+\delta(x-x_i)\right)u\right\rangle=0$$

$$\Rightarrow -\int_0^1 \frac{du}{dx}\frac{dG_i}{dx} + u\frac{dG_i}{dx}\Big|_0^1 + u_i$$

Forcing weighted by G simply gets added into Boundary statement

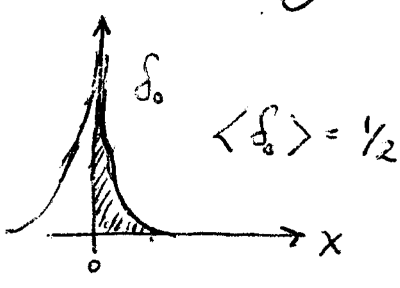
$$\int_0^1 xG_i\,dx + \underbrace{\left(\frac{du}{dx}G_i - \frac{dG_i}{dx}u\right)\Big|_0^1}_{\oint\left(\frac{\partial u}{\partial n}w - \frac{dw}{dn}u\right)ds} - u_i = 0$$

We know $u(0)$, $u(1)$ ... but not $\frac{du}{dx}(0)$, $\frac{du}{dx}(1)$

How do we get them?... put G_i on the two boundary points $\Rightarrow$ 2 Eqn's + 2 unknowns

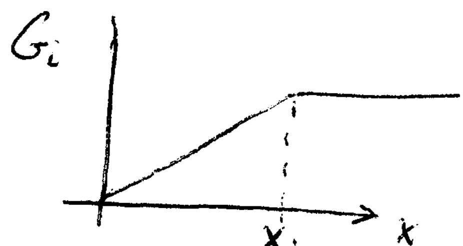
On the boundary ...

$$\left\langle \frac{d^2G_0}{dx^2}u\right\rangle = -\frac{u_0}{2}$$

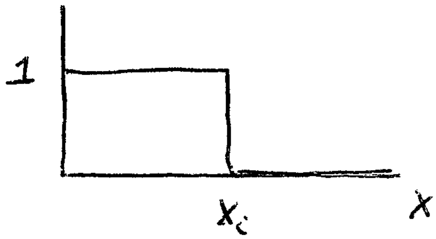


$$\left\langle \frac{d^2G_1}{dx^2}u\right\rangle = -\frac{u_1}{2}$$

Now $G_i = \begin{cases} x & x \le x_i \\ x_i & x > x_i \end{cases}$



$\frac{dG_i}{dx} \Rightarrow$



so $\frac{d^2G_i}{dx^2}$ singular at x_i ✓

Also $\Delta\left(\frac{dG_i}{dx}\right) = -1 \Rightarrow \int_{x_i-\epsilon}^{x_i+\epsilon} \frac{d^2G_i}{dx^2} = -\int_{x_i-\epsilon}^{x_i+\epsilon} \delta(x-x_i)$

$$\left.\frac{dG}{dx}\right|_{x_i+\epsilon} - \left.\frac{dG}{dx}\right|_{x_i-\epsilon} = -1$$

$$0 - 1 = -1 \quad ✓$$

So $$\frac{u_0}{2} = \int_0^1 x G_0\, dx + \left.\left(\frac{du}{dx}G_0 - \frac{dG_0}{dx}u\right)\right|_0^1$$

$$\frac{u_1}{2} = \int_0^1 x G_1\, dx + \left.\left(\frac{du}{dx}G_1 - \frac{dG_1}{dx}u\right)\right|_0^1$$

Formally 2 equations in 2 unknowns

In this case things simplify $\Rightarrow u(0) = u(1) = 0$

$G_0 = 0$

eqn #2 $\Rightarrow G_1 = x\ ,\ x \leq 1 \Rightarrow \frac{dG_1}{dx} = 1$

$$\Rightarrow\quad 0 = \int_0^1 x^2 dx + \left.\frac{du_1}{dx}x\right|_0^1 - \overset{0}{\left.\cancel{u}\right|_0^1}$$

$$\frac{du_1}{dx} = -1/3$$

Interior Solution . . .

$$u_i = \int_0^{x_i} x^2 dx + \int_{x_i}^{1} x_i x \, dx + \left(\frac{du}{dx} G_i - \frac{dG_i}{dx} u\right)\Big|_0^1$$

$$= \frac{x_i^3}{3} + \frac{x_i}{2} - \frac{x_i^3}{2} + \frac{du_1}{dx} x_i$$

$$= \frac{x_i}{2} - \frac{x_i^3}{6} + \left(-1/3\right) x_i = \frac{x_i}{6} - \frac{x_i^3}{6}$$

$$\Rightarrow u(x) = \frac{x}{6} - \frac{x^3}{6} \qquad \text{exact!}$$

Note: G_i satisfies BC at $x=0 \Rightarrow$ so don't need the value $\frac{du}{dx}(0)$ to solve for u_i

In general can "cook up" G to satisfy BC's but difficult to do for arbitrary boundaries (typically not done in BEM)

e.g. $G_i = \begin{cases} (1-x_i)x & x \le x_i \\ (1-x)x_i & x > x_i \end{cases}$

$$\text{then} \quad u_i = \int_0^1 x G_i \, dx + \left(\cancelto{0}{\frac{du}{dx} G_i} - \cancelto{0}{\frac{dG_i}{dx} u} \right) \Bigg|_0^1$$

$$= \int_0^{x_i} (1-x_i) x^2 \, dx + \int_{x_i}^1 x(1-x) x_i \, dx$$

$$= (1-x_i) \frac{x_i^3}{3} + x_i \left[\frac{x^2}{2} - \frac{x^3}{3} \right] \Bigg|_{x_i}^1$$

$$= \frac{x_i}{6} - \frac{x_i^3}{6}$$