A

Helmholtz Problem: $\nabla^2 u + k^2 u = 0$

$$\nabla^2 G_i + k^2 G_i = -\delta(\underset{\sim}{x} - \underset{\sim}{x}_i)$$

General Boundary expression:

$$\langle \nabla^2 u + k^2 u, G_i \rangle = \langle -\nabla u \cdot \nabla G_i \rangle + \langle k^2 u G_i \rangle + \oint \frac{\partial u}{\partial n} G_i \, ds$$

$$\langle \nabla^2 G_i + k^2 G_i, u \rangle = \langle -\nabla G_i \cdot \nabla u \rangle + \langle k^2 G_i u \rangle + \oint \frac{\partial G_i}{\partial n} u \, ds$$

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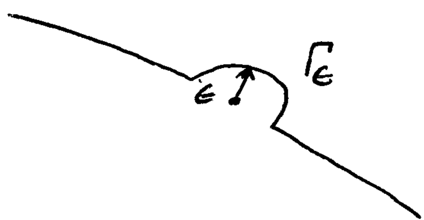
$$u_i = \oint \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, ds \quad \text{as expected}$$

2D: $G_i = \frac{i}{4} H_0^{(1)}(kr) = \left(J_0(kr) + i Y_0(kr)\right)\frac{i}{4}$




Look at: $\lim\limits_{\epsilon \to 0} \int_{\Gamma_\epsilon} \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, ds$

$\Rightarrow G_i = \frac{i}{4} H_0'(k\epsilon)$, so $\lim\limits_{\epsilon \to 0} \frac{i}{4} H_0'(k\epsilon) = \lim\limits_{\epsilon \to 0} \frac{i}{4}\left(J_0(k\epsilon) + i Y_0(k\epsilon)\right)$

$$= \frac{i}{4}\left(1 + \frac{i2}{\pi} \ln(k\epsilon)\right)$$

then

$$\lim_{\epsilon \to 0} \int_0^{\pi} \frac{\partial u}{\partial n} G_i \, \epsilon \, d\theta = \lim_{\epsilon \to 0} \frac{i}{4} \int_0^{\pi} \frac{\partial u}{\partial n}\left(1 + \frac{i2}{\pi} \ln(k\epsilon)\right) \epsilon \, d\theta = \underline{\underline{0}}$$

$$\frac{\partial G_i}{\partial n} = \frac{\partial G_i}{\partial \epsilon} \underbrace{\frac{\partial \epsilon}{\partial n}}_{1} = \frac{i}{4}\left(\frac{i2}{\pi} \frac{1}{\epsilon}\right)$$

then

$$\lim_{\epsilon\to 0}\int_0^{\pi} u_i \frac{\partial G_i}{\partial \theta}\epsilon\, d\theta = \int_0^{\pi} -\left(\frac{1}{2\pi}\right)\frac{u_i}{\epsilon}\epsilon\, d\theta = -\frac{u_i}{2}$$

$$u_i = \oint \frac{\partial u}{\partial n}G_i - \frac{\partial G_i}{\partial n}u\, ds + \frac{u_i}{2} \Rightarrow \frac{u_i}{2} = \oint \frac{\partial u}{\partial n}G_i - \frac{\partial G_i}{\partial n}u\, ds$$

Also:

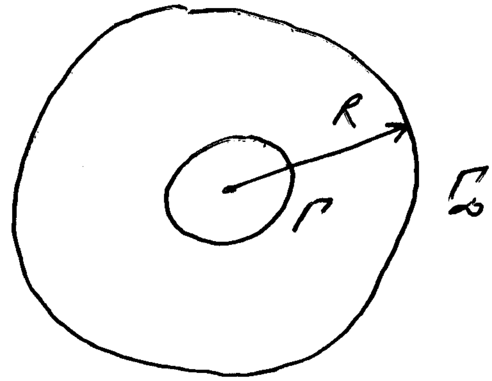

$$\lim_{\epsilon\to 0}\underbrace{\iint_{\Omega_\epsilon}\nabla\cdot\nabla G_i\, d\Omega_e}+\iint_{\Omega_\epsilon}k^2 G_i\, d\Omega_e = -1$$

$$\lim_{\epsilon\to 0}\left[\int_{\Gamma_\epsilon}\frac{\partial G_i}{\partial n}\,d\Gamma_\epsilon = \int_0^{2\pi}\left(-\frac{1}{2\pi\epsilon}\right)\epsilon\, d\theta\right] = -1$$

$\frac{i2}{\pi}$

$$\lim_{\epsilon\to 0}\left[\iint_{\Omega_\epsilon}k^2 G_i\, d\Omega_e = \frac{i}{4}\int_0^{\epsilon}\int_0^{2\pi}k^2\left[\ln(k\epsilon)+1\right]\epsilon\, d\theta\right] = 0$$

# Infinite Domains

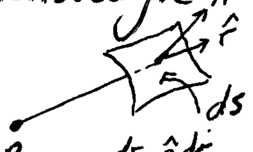



## 3D

$$\frac{\alpha_i u_i}{4\pi} = \oint \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, ds$$

Solid angle

$$= \int_{\Gamma} \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, d\Gamma + \int_{\Gamma_\infty} \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, d\Gamma_\infty$$




$d\tilde{s} = \hat{n}\, ds \qquad d\Omega = \frac{d\tilde{s} \cdot \hat{r}}{r^2} = \frac{d\tilde{s} \cdot \tilde{r}}{r^3} \Rightarrow \Omega = \int_S \frac{\tilde{r} \cdot d\tilde{s}}{r^3}$

Look at $\displaystyle \lim_{R \to \infty} \int_{\Gamma_\infty} \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, d\Gamma_\infty$

3D Helmholtz: $\displaystyle G_i = \frac{e^{ikr}}{4\pi r}$

$$\frac{\partial G_i}{\partial n} = \left( \frac{-e^{ikr}}{4\pi r^2} + \frac{ike^{ikr}}{4\pi r} \right) \hat{r} \cdot \hat{n}$$

$$= \frac{e^{ikr}}{4\pi r} \left( ik - \frac{1}{r} \right) \hat{r} \cdot \hat{n}$$

So
$$\int_{\Gamma_\infty} \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, d\Gamma_\infty = \int_0^{2\pi} \int_0^{\pi} \left\{ \frac{\partial u}{\partial R} \left( \frac{e^{ikR}}{4\pi R} \right) - \frac{e^{ikR}}{4\pi R} \left( ik - \frac{1}{R} \right) u \right\} R^2 \sin\theta \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{e^{ikr}}{4\pi} \left\{ \left( \frac{\partial u}{\partial R} R - iku R \right) - u \right\} \sin\theta \, d\theta \, d\varphi$$

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$$\lim_{R\to\infty} \iint \frac{e^{ikr}}{4\pi} \left\{ \left( \frac{\partial u}{\partial R} - iku \right) R - u \right\} \sin\theta \, d\theta \, d\varphi$$

requires $\underbrace{\lim_{R\to\infty} R\left(\frac{\partial u}{\partial R} - iku\right) = 0}$ and $\lim_{R\to\infty} u = 0$

Sommerfeld Radiation Condition

excludes incoming waves and guarantees unique solution to problem ... is satisfied by the Green's function (unbounded space)

$$\text{e.g. } \lim_{R\to\infty} R\left(\frac{\partial G}{\partial R} - ikG\right) = 0 \quad \Rightarrow \lim_{R\to\infty} \frac{-e^{ikr}}{4\pi R} = 0$$

$$-\frac{e^{ikr}}{4\pi R^2} + \frac{ike^{ikR}}{4\pi R} \qquad \frac{e^{ikR}}{4\pi R}$$

Note, in 2D ... Sommerfeld condition is

$$\lim_{R\to\infty} \sqrt{R}\left(\frac{\partial u}{\partial R} - iku\right) = 0$$