Coupling of Finite and Boundary Methods
(Hybrid Methods)

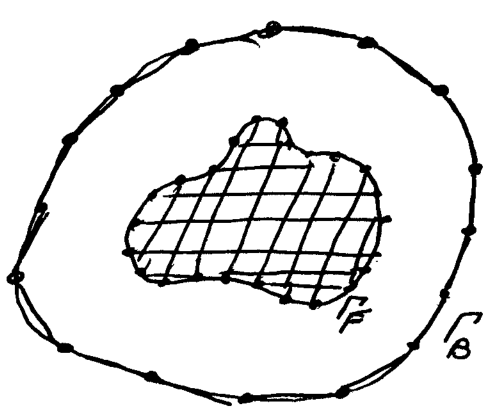
- Try to capitalize on strengths of FEM + BEM while avoiding their weaknesses when used alone

FEM strengths: heterogeneous, irregular, bounded domains, regular functions, sparse eqn sets
weaknesses: grid generation, unbounded problems

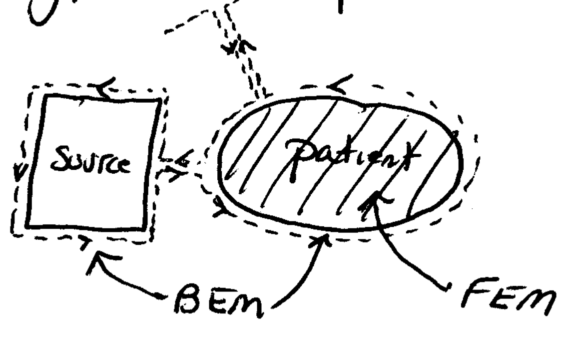
BEM strengths: homogeneous, unbounded domains, reduction of problem dimension, grid generation
weaknesses: heterogeneity, singular functions, full equation sets

- Lets look at Helmholtz eqn as prototype

$$\nabla^2 u + k^2 u = 0$$
$$\nabla^2 G_i + k^2 G_i = -\delta(\underset{\sim}{x} - \underset{\sim}{x}_i)$$



or Hyperthermia problem



FEM Formulation: $\langle () \rangle = \iint_{\Omega_F} () \, d\Omega_F$

$$\langle \nabla^2 u + k^2 u, \phi_i \rangle = \langle -\nabla u \cdot \nabla \phi_i \rangle + \langle k^2 u \phi_i \rangle + \oint \frac{\partial u}{\partial n} \phi_i \, ds = 0$$

Expand $u = \sum_{j=1}^{N_F+N_I} u_j \phi_j$, $\frac{\partial u}{\partial n} = \sum_{j=1}^{N_I} \phi_j \frac{\partial u}{\partial n}_j$

Defined on boundary only

$$\sum_{j=1}^{N_F+N_I} u_j \langle \nabla \phi_j \cdot \nabla \phi_i - k^2 \phi_j \phi_i \rangle = \sum_{j=1}^{N_I} \frac{\partial u}{\partial n}_j \oint \phi_j \phi_i \, ds$$

Matrix form: $[A]\{u\} = \{F\}$

$$a_{ij} = \langle \nabla \phi_j \cdot \nabla \phi_i - k^2 \phi_j \phi_i \rangle$$

$$f_i = \sum_{j=1}^{N_I} \frac{\partial u}{\partial n}_j \oint \phi_j \phi_i \, ds$$

requires $\frac{\partial u}{\partial n}_j$ or u_j known for all boundary nodes of the FEM grid (i.e. $j = 1, N_I$)

These cannot be specified in this problem!

partition FEM matrix:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{22} & A_{22} \end{bmatrix} \begin{Bmatrix} u_F \\ u_I \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & B_{22} \end{bmatrix} \begin{Bmatrix} 0 \\ \frac{\partial u}{\partial n}_I \end{Bmatrix}$$

BEM Formulation:

$$\frac{\alpha_i}{2\pi} u_i = \int_{\Gamma_B + \Gamma_F} \frac{\partial u}{\partial n} G_i - \frac{\partial G_i}{\partial n} u \, ds = \sum_{j=1}^{N_I} \frac{\partial u}{\partial n}_j \int_{\Gamma_F} \phi_j G_i \, d\Gamma_F$$

$$+ \sum_{j=1}^{N_B} \frac{\partial u}{\partial n}_j \int_{\Gamma_B} \phi_j G_i \, d\Gamma_B - \sum_{j=1}^{N_I} u_j \int_{\Gamma_F} \phi_j \frac{\partial G_i}{\partial n} \, d\Gamma_F$$

$$- \sum_{j=1}^{N_B} u_j \int \phi_j \frac{\partial G_i}{\partial n} \, d\Gamma_B$$

In Matrix Form

$$[C]\{u\} = [D]\left\{\frac{\partial u}{\partial n}\right\}$$

Where $\displaystyle c_{ij} = \frac{\alpha_i}{2\pi} \delta_{ij} + \int_{\Gamma_F} \phi_j \frac{\partial G_i}{\partial n} \, d\Gamma_F + \int_{\Gamma_B} \phi_j \frac{\partial G_i}{\partial n} \, d\Gamma_B$

$$d_{ij} = \int_{\Gamma_F} \phi_j G_i \, d\Gamma_F + \int_{\Gamma_B} \phi_j G_i \, d\Gamma_B$$

Partitioned Form:

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} u_I \\ u_B \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial n}_I \\ \frac{\partial u}{\partial n}_B \end{Bmatrix}$$

Now assume u_B is the given BC info...

then invert $[D]$, define $DI = D^{-1}$:

$$\begin{Bmatrix} \frac{\partial u}{\partial n}_I \\ \frac{\partial u}{\partial n}_B \end{Bmatrix} = \begin{bmatrix} DI_{11} & DI_{12} \\ DI_{21} & DI_{22} \end{bmatrix} \begin{bmatrix} C_{12} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} u_I \\ u_B \end{Bmatrix}$$

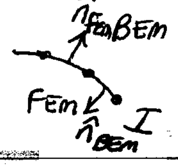
Top equation says...

$$\left\{ \frac{\partial u}{\partial n}_I \right\} = \left[DI_{11} C_{11} + DI_{12} C_{21} \right] \{u_I\} + \left[DI_{11} C_{12} + DI_{12} C_{22} \right] \{u_B\}$$

Relationship for $\frac{\partial u_I}{\partial n}$ in terms of u_I (already part of FEM algebra) and u_B (known!)

$$\therefore \begin{bmatrix} A_{11} & A_{12} \\ A_{22} & A_{22} + B_{22}\left[DI_{11} C_{11} + DI_{12} C_{21} \right] \end{bmatrix} \begin{Bmatrix} u_F \\ u_I \end{Bmatrix} =$$

because $\frac{\partial u}{\partial n}_I$ from FEM is $-\frac{\partial u_I}{\partial n}$ from BEM



$$= \begin{Bmatrix} 0 \\ -B_{22}\left[DI_{11}C_{12} + DI_{12}C_{22}\right]\{U_B\} \end{Bmatrix}$$

known

System of Equations becomes $[A']\{u\} = \{F'\}$

rows stays the same, but the Bandwidth may grow (usually does) ... minimum Half-BW becomes $\left(\begin{array}{c}\#\text{ Boundary nodes} \\ \text{on FEM Grid}\end{array} - 1\right)$ ... may be worse

Note that if we want to compute $\frac{\partial u}{\partial n}$ (flux) on the source ... we use the bottom equation once $[A']\{u\} = \{F'\}$ is solved, i.e.

$$\left\{\frac{\partial u_B}{\partial n}\right\} = \left[DI_{21}C_{11} + DI_{22}C_{21}\right]\{u_I\} + \left[DI_{21}C_{12} + DI_{22}C_{22}\right]\{u_B\}$$

Alternatively ... if $\frac{\partial u_B}{\partial n}$ is specified

$$\begin{bmatrix} D_{11} & -C_{12} \\ D_{21} & -C_{22} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial n}_I \\ u_B \end{Bmatrix} = \begin{bmatrix} C_{11} & -D_{12} \\ C_{21} & -D_{22} \end{bmatrix} \begin{Bmatrix} u_I \\ \frac{\partial u}{\partial n}_B \end{Bmatrix}$$

Invert LHS matrix and call it "DCI"

$$\begin{Bmatrix} \frac{\partial u}{\partial n}_I \\ u_B \end{Bmatrix} = \begin{bmatrix} DCI_{11} & DCI_{12} \\ DCI_{21} & DCI_{22} \end{bmatrix} \begin{bmatrix} C_{11} & -D_{12} \\ C_{21} & -D_{22} \end{bmatrix} \begin{Bmatrix} u_I \\ \frac{\partial u}{\partial n}_B \end{Bmatrix}$$

top equation closes FEM system, bottom used to compute u on remote boundary ...

$$\left\{\frac{\partial u}{\partial n}_I\right\} = \left[DCI_{11} C_{11} + DCI_{12} C_{21}\right]\left\{u_I\right\} - \left[DCI_{11} D_{12} + DCI_{12} D_{22}\right]\left\{\frac{\partial u}{\partial n}_B\right\}$$

So FEM system of equations:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} + B_{22}\left[DCI_{11} C_{11} + DCI_{12} C_{21}\right] \end{bmatrix} \begin{Bmatrix} u_F \\ u_I \end{Bmatrix}$$

again, due to $\hat{n}_{FEM} = -\hat{n}_{BEM}$

$$= \begin{Bmatrix} 0 \\ -B_{22}\left[DCI_{11} D_{12} + DCI_{12} D_{22}\right]\left\{\frac{\partial u}{\partial n}_B\right\} \end{Bmatrix}$$

Can approach the algebra differently:

- "Invert" FEM matrix and take what is needed and insert into BEM equations

  Advantage: don't alter sparsity (size) of either FEM or BEM matrices (first approach alters original FEM Bandwidth)

Conceptually ...

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} u_F \\ u_I \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & B_{22} \end{bmatrix} \begin{Bmatrix} 0 \\ \frac{\partial u_I}{\partial n} \end{Bmatrix} \quad \text{FEM}$$

$$\begin{Bmatrix} u_F \\ u_I \end{Bmatrix} = \begin{bmatrix} AI_{11} & AI_{12} \\ AI_{21} & AI_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & B_{22} \end{bmatrix} \begin{Bmatrix} 0 \\ \frac{\partial u_I}{\partial n} \end{Bmatrix} \quad AI = A^{-1}$$

so $\{u_I\} = [AI_{22}\, B_{22}] \left\{ \frac{\partial u_I}{\partial n} \right\}$

BEM:

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} u_I \\ u_B \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial n}_I \\ \frac{\partial u}{\partial n}_B \end{Bmatrix}$$

Assume u_B is given BC info

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} -\left[AI_{22}\ B_{22}\right] \frac{\partial u_I}{\partial n} \\ u_B \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial n}_I \\ \frac{\partial u}{\partial n}_B \end{Bmatrix}$$

$$\begin{bmatrix} D_{11} + C_{11}\left[AI_{22}\ B_{22}\right] & D_{12} \\ D_{21} + C_{21}\left[AI_{22}\ B_{22}\right] & D_{22} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial n}_I \\ \frac{\partial u}{\partial n}_B \end{Bmatrix} = \begin{Bmatrix} C_{12}\, u_B \\ C_{22}\, u_B \end{Bmatrix}$$

IF $\frac{\partial u_B}{\partial n}$ is given...

$$\begin{bmatrix} -C_{11}\left[AI_{22}\ B_{22}\right] - D_{11} & C_{12} \\ -C_{21}\left[AI_{22}B_{22}\right] - D_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_I}{\partial n} \\ u_B \end{Bmatrix} = \begin{Bmatrix} D_{12} \frac{\partial u}{\partial n}_B \\ D_{22} \frac{\partial u}{\partial n}_B \end{Bmatrix}$$

Once we solve this system we have $\frac{\partial u}{\partial n}_I$ $(n \Rightarrow \hat{n}_{BEM})$ which is what we need to solve original FEM matrix equation

Conclude ... we don't insert anything into FEM Matrix equation (LHS) ⇒ sparsity is same

we do insert quantities into BEM, but it is already full ⇒ don't change storage

Note on "Inverting" A: (In general the inverse of matrix is full even if it is sparse)

We do this by 1.) 1st LU decomposing A (still Banded), which we need to do anyway in order to solve for interior FEM solution

2.) get inverse only of the columns corresponding to the boundary nodes of FEM Grid

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & \underbrace{A_{22}} \end{array}\right]$$

only get inverse of these columns and throw out top part, this gives AI_{22}

put "1" in the row of RHS column vector corresponding to the column of the inverse that we want and "0" everywhere else in RHS + back substitute ⇒ produces 1 column of A^{-1}

• Final approach ... Throw everything into 1 Big Matrix (Shoots the BW to #?!@, but no inversion needed)

$$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & -B_{22} \end{bmatrix} \begin{Bmatrix} u_F \\ u_I \\ \frac{\partial u_I}{\partial n} \end{Bmatrix} = 0 \qquad \text{FEM system}$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} u_I \\ u_B \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial n}_I \\ \frac{\partial u}{\partial n}_B \end{Bmatrix}$$

IF $\frac{\partial u}{\partial n}_B$ is known:

$$\begin{array}{c} \uparrow \\ N_F \\ + \\ 2N_I \\ + \\ N_B \\ \downarrow \end{array} \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & -B_{22} & 0 \\ 0 & C_{11} & D_{11} & C_{12} \\ 0 & C_{21} & D_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} u_F \\ u_I \\ \frac{\partial u_I}{\partial n} \\ u_B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ [D_{12}] \left\{ \frac{\partial u}{\partial n}_B \right\} \\ [D_{22}] \left\{ \frac{\partial u_B}{\partial n} \right\} \end{Bmatrix}$$

$$\longleftarrow N_F + 2N_I + N_B \longrightarrow$$

or if u_B is known:

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & -B_{22} & 0 \\ 0 & C_{11} & D_{11} & -D_{12} \\ 0 & C_{21} & D_{21} & -D_{22} \end{bmatrix} \begin{Bmatrix} u_F \\ u_I \\ \frac{\partial u}{\partial n}_I \\ \frac{\partial u}{\partial n}_B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -[C_{12}]u_B \\ -[C_{22}]u_B \end{Bmatrix}$$