Chebyshev Polynomials

- the second "most famous" spectral expansion

- can use on nonperiodic functions and still retain exponential convergence rates (can use Fourier w/ nonperiodic, but lose faster convergence, i.e. return to algebraic convergence)

- Chebyshev polynomial expansion essentially a Fourier cosine series in disguise:

$$u(x) = \sum_{k=0}^{\infty} \hat{u}_k T_k(x)$$

let $x = \cos\theta \Rightarrow u(\cos\theta) \equiv \bar{u}(\theta)$

then $\bar{u}(\theta) = \sum_{k=0}^{\infty} \hat{u}_k \cos k\theta$

So Chebyshev polynomials are nothing but cosine functions after a change of variable

i.e. a Chebyshev series for u corresponds to a cosine series for $\bar{u}$

Implications:

- if $u(x)$ is infinitely differentiable on $[-1,1]$, $\bar{u}(\theta)$ is infinitely differentiable and <u>periodic</u> in <u>all</u> derivatives on $[0, 2\pi]$ ... same integration-by-parts argument for Fourier Series holds... Chebyshev coefficients guaranteed to decay <u>faster</u> than <u>algebraically</u> $\Rightarrow$ exponential!
  Does not require $u(x)$ periodic <u>nor</u> does it matter what $u(x)$ does outside $[-1,1]$.

- have an orthogonality relationship on $[-1,1]$

  if $u(x) = \sum_{k=0}^{\infty} \hat{u}_k T_k(x)$

  then $\hat{u}_k = \frac{2}{\pi c_k} \int_{-1}^{1} u(x)\, T_k(x)\, w(x)\, dx$

  where $w(x) = \frac{1}{\sqrt{1-x^2}}$; $c_k = \begin{cases} 2 & \text{if } k=0 \\ 1 & \text{if } k \geq 1 \end{cases}$

- do have Gibbs phenomenon with an "interior" (i.e. in $[-1,1]$) discontinuity ($O(1/N)$ error as in Fourier) but <u>not</u> from nonperiodic values at ± 1

- A number of other orthogonal polynomials are also possible ... e.g. Legendre on [-1,1]

$$u(x) = \sum_{k=0}^{\infty} \hat{u}_k L_k(x), \quad \hat{u}_k = (k+1/2)\int_{-1}^{1} u(x) L_k(x)\,dx$$

- Both (Chebyshev, Legendre, others) are eigenfunctions of the "Sturm-Liouville Problem" (eigenvalue) with certain coefficient forms ... all have spectral accuracy for expansions of infinitely smooth functions

- Can generalize to any orthogonal basis

$$\phi_k(x) \Rightarrow \langle \phi_k \phi_\ell \rangle = 0 \quad k \neq \ell$$

then $$u(x) = \sum_k \hat{u}_k \phi_k(x)$$

$$\langle u \phi_\ell \rangle = \sum_k \hat{u}_k \langle \phi_k \phi_\ell \rangle$$

$$= \hat{u}_\ell \langle \phi_\ell \phi_\ell \rangle$$

so

$$\left.\begin{aligned} \hat{u}_\ell &= \frac{\langle u \phi_\ell \rangle}{\langle \phi_\ell \phi_\ell \rangle} \\ u(x) &= \sum_k \hat{u}_k \phi_k(x) \end{aligned}\right] \text{Transform Pair}$$

where $\langle u \phi_\ell \rangle \equiv \int u \, \phi_\ell \, w \, dx$

$\int$ ← domain of Sturm-Liouville Problem

w ← determined from Sturm-Liouville

- On computer, interested in Discrete Polynomial Transform:

$\phi_0(x), \phi_1(x) \ldots \phi_N(x)$ an orthogonal set

constant linear $\sim x^N$

Interpolating polynomial is formed such that

$$I_N u(x_j) = u(x_j) \quad j = 0, 1, \ldots N \quad (N+1 \text{ pts})$$

$$I_N u(x) = \sum_{k=0}^{N} \underbrace{\tilde{u}_k}_{\text{Discrete Polynomial Coefficients}} \phi_k(x)$$

$$\left\{ \tilde{u}_k = \underbrace{\left(\frac{1}{\sum_{j=0}^{N} \phi_k^2(x_j) w_j} \right)}_{\text{Normalizing Factor} \equiv \gamma_k} \underbrace{\sum_{j=0}^{N} u(x_j)\phi_k(x_j) w_j}_{} \right.$$

Discrete Polynomial Transform

Inner Product (Discrete)

$$(u, v)_N \equiv \sum_{j=0}^{N} u_j v_j w_j$$

$$\underbrace{\tilde{u}_k = \hat{u}_k + \frac{1}{\gamma_k} \sum_{\ell > N} \underbrace{(\phi_\ell \phi_k)_N}_{\text{Not "orthogonal"}} \hat{u}_\ell}_{\text{aliasing error}}$$

But discrete inner product looks just like a quadrature formula involving $N+1$ Gauss pts

i.e. $\langle u\,v \rangle = \sum_{j=0}^{N} u_j v_j w_j$

exact provided uv is Polynomial degree $2N+1$

(Gauss Legendre pts)

then $\sum_{j=0}^{N} \phi_k^2(x_j) w_j \cong \langle \phi_k^2(x) \rangle$

$\sum_{j=0}^{N} u(x_j)\,\phi_k(x_j)\,w_j \approx \langle u\,\phi_k \rangle$

So $\tilde{u}_k \approx \dfrac{\langle u\,\phi_k \rangle}{\langle \phi_k\,\phi_k \rangle} \equiv \hat{u}_k$

Discrete Polynomial coefficients are an approximate version of the continuous polynomial coefficients when the x_j's are chosen as the Gauss pts on $[-1,1]$ ... i.e. correspond to coefficients obtained under numerical approximation to the continuum integrals.

Now the usual Gauss points (Gauss legendre) never hit ± 1, the endpoints of the orthogonality interval...

But $I_N u(x) = \sum_{k=0}^{N} \tilde{u}_k \phi_k(x)$ is also the interpolating polynomial for the approximate solution and we want this to match the boundary conditions imposed at one or both of the endpoints (i.e. ± 1)

$$I_N u(\pm 1) = u(\pm 1) = \sum_{k=0}^{N} \tilde{u}_k \phi_k(\pm 1)$$

$\therefore$ Need to find alternate quadrature formulas that include the endpoints!

- each constraint reduces the polynomial order exactly integrated

"Gauss-Radau" $-1 = x_0, x_1 \ldots x_N$ are $N+1$ roots of polynomial

$$q(x) = P_{N+1}(x) + a P_N(x)$$

$$\text{where } q(-1) = 0 \Rightarrow a = -\frac{P_{N+1}(-1)}{P_N(-1)}$$

and weights $w_0, w_1 \ldots w_N$ are the solution of the linear system

$$\sum_{j=0}^{N} (x_j)^k w_j = \int_{-1}^{1} x^k w(x)\,dx \qquad 0 \le k \le N$$

then $$\sum_{j=0}^{N} p(x_j) w_j = \int_{-1}^{1} p(x) w(x)\,dx$$ for

$p(x)$ a polynomial of degree $2N$ or less

"Gauss-Lobatto" Integration

$-1 = x_0, x_1 \ldots x_N = 1$ are the $N+1$ roots of

$$q(x) = P_{N+1}(x) + a P_N(x) + b P_{N-1}(x)$$

where a and b are chosen such that $q(-1) = q(1) =$

and $w_0, w_1 \ldots w_N$ is the solution of linear system

$$\sum_{j=0}^{N} x_j^k w_j = \int_{-1}^{1} x^k w(x)\,dx \qquad 0 \le k \le N$$

then $$\sum_{j=0}^{N} p(x_j) w_j = \int_{-1}^{1} p(x) w(x)\,dx$$ for

$p(x)$ a polynomial of degree $2N-1$

Example:

Chebyshev-Gauss-Lobatto:

$$x_j = \cos\frac{\pi j}{N} \qquad w_j = \begin{cases} \frac{\pi}{2N} & j=0,N \\ \frac{\pi}{N} & 1 \le j \le N-1 \end{cases}$$

then $$\tilde{u}_k = \frac{1}{\gamma_k}\sum_{j=0}^{N} u(x_j)\underbrace{T_k(x_j)}_{\cos(k\cos^{-1}x) \,=\, \cos\frac{k\pi j}{N}} w_j$$

$$= \sum_{j=0}^{N} C_{kj}\, u_j$$

where $$C_{kj} = \frac{1}{\pi/2\, c_k}\left(\frac{\pi}{N c_j}\right)\cos\frac{k\pi j}{N}$$

$$= \frac{2}{N c_k c_j}\cos\frac{k\pi j}{N} \quad \text{with } c_j = \begin{cases} 2 & j=0,N \\ 1 & 1 \le j \le N-1 \end{cases}$$