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A SELF-ADAPTIVE CO-ORDINATE TRANSFORMATION FOR EFFICIENT NUMERICAL EVALUATION OF GENERAL BOUNDARY ELEMENT INTEGRALS

J. C. F. TELLES

Coppe-Univ. Federal do Rio de Janeiro, Programa de Engenharia Civil, Caixa Postal 68506, 21944, Rio de Janeiro, Brazil

SUMMARY

Almost all general purpose boundary element computer packages include a curved geometry modelling capability. Thus, numerical quadrature schemes play an important role in the efficiency of programming the technique. The present work discusses this problem in detail and introduces efficient means of computing singular or nearly singular integrals currently found in two-dimensional, axisymmetric and three-dimensional applications. Emphasis is given to a new third degree polynomial transformation which was found greatly to improve the accuracy of Gaussian quadrature schemes within the near-singularity range. The procedure can easily be implemented into existing BE codes and presents the important feature of being self-adaptive, i.e. it produces a variable lumping of the Gauss stations toward the singularity, depending on the minimum distance from the source point to the element. The self-adaptiveness of the scheme also makes it inactive when not useful (large source distances) which makes it very safe for general usage.

INTRODUCTION

It has now been widely accepted that in a general application boundary element (BE) computer package the isoparametric (usually quadratic) elements have to be included for both potential and elasticity problems capable of handling two-dimensional, axisymmetric and three-dimensional analyses.[1–3] However, the general implementation of the BE technique requires, for the creation of the influence matrices, computation of surface integrals with integrands of $O(\ln 1/r)$, $O(1/r)$ and $O(1/r^2)$, where r is the distance from a source point. Since the possibility of having curved elements to integrate normally implies that analytical integration is ruled out, current BE codes rely almost entirely on numerical quadrature schemes to perform singular or quasi-singular integrals. For this reason, much research work has already been published on this subject.[4–13] Here, the compromise between accuracy and computer time is the key factor when choosing the most appropriate scheme. Though the singular element integrals (i.e. those that contribute to the main diagonal of the influence matrices) are a major concern for the BEM, the efficiency of the technique is greatly dependent on the number of integration points employed for the nearly singular integrals, such as those over the elements in the neighbourhood of the singular element. In this case, the $(1/r)$ behaviour is still dominant in the integrands, which are now regular but require a large number of integration points to provide accurate enough results. Up to now, the best general procedure available has been[4,7,8] subdividing the quasi-singular element into a certain number of subelements and performing standard Gaussian type quadrature over each separate

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subelement at a time. A minimum r based automatic selection of the number of integration points is also included. This procedure benefits from the fact that it increases the concentration of points near to the minimum r location over the element. But it presents a drawback which is immediately apparent if the total number of points is applied over the element without any subdivision: the highest degree of the polynomial which can be integrated exactly is now found to increase considerably. Nevertheless, the behaviour of the integrand is such that, in the close range, subdivision does pay off. An ideal procedure, however, would be to concentrate the points without such a drastic decrease in the maximum degree of the allowable integrand. The present work is mainly concerned with this subject and presents an alternative scheme which was found to be considerably more efficient than element subdivision for general boundary element applications. The idea is based on a non-linear co-ordinate transformation which automatically lumps the points towards the minimum source distance position without the disadvantage of subdividing the element. The scheme can easily be implemented into existing BE codes which use Gauss integration. Some examples are also presented, and comparisons with standard and recently modified subdivision routines are also included to demonstrate the efficiency of the procedure.

BOUNDARY ELEMENT INTEGRALS

In the case of elastostatic problems, without body forces, the direct formulation of the boundary element technique leads to equations of the form[14]

$$c_{ij}(\xi)u_j(\xi) + \int_\Gamma p^*_{ij}(\xi, x)u_j(x)\,d\Gamma(x) = \int_\Gamma u^*_{ij}(\xi, x)p_j(x)\,d\Gamma(x), \xi, x \in \Gamma \tag{1}$$

where the integral on the left is in the Cauchy principal value sense and the following notation is used: Γ is the boundary of the body, ξ is the source point, x is a field point, c_{ij} is a coefficient which depends on the boundary geometry at ξ, u_j and p_j are the displacement and traction components and u^*_{ij} and p^*_{ij} are the fundamental displacement and traction at x in the j direction due to a unit load at ξ in the i direction.

For potential problems satisfying the Laplace equation the above equation is modified by substituting the kernel tensors and vector components by their scalar counterparts: u is the potential and p is the derivative of u with respect to the outward normal to Γ. In any case, after discretization, the following integrals have to be computed over each element Γ_j:

$$\mathbf{h} = \int_{\Gamma_j} \mathbf{p}^* \Phi^T d\Gamma \tag{2}$$

$$\mathbf{g} = \int_{\Gamma_j} \mathbf{u}^* \Phi^T d\Gamma \tag{3}$$

where Φ represents the interpolation functions, which are normally written in terms of the homogeneous co-ordinate system η_1, $\eta_2 (|\eta_1|, |\eta_2| \leqslant 1)$. Therefore, a suitable co-ordinate transformation gives

$$\mathbf{h} = \int_{-1}^{1}\int_{-1}^{1} \mathbf{p}^* \Phi^T G\,d\eta_1\,d\eta_2 \tag{4}$$

$$\mathbf{g} = \int_{-1}^{1}\int_{-1}^{1} \mathbf{u}^* \Phi^T G\,d\eta_1\,d\eta_2 \tag{5}$$

in which $G(\eta_1, \eta_2)$ is the Jacobian, possibly non-linear, of the transformation. The efficient numerical computation of the integrals (4) and (5) is the subject of the present work. Since the Jacobian and the interpolation functions are regular (usually polynomials), the behaviours of the kernel matrices $\mathbf{p}^*$ and $\mathbf{u}^*$ are greatly dominant in the integrals, particularly because of their typical dependence on the distance between ξ and x, herein called r. For three- and two-dimensional problems one has $\mathbf{p}^* = O(1/r^2)$ and $O(1/r)$, respectively, and $\mathbf{u}^* = O(1/r)$ and $O(\ln(1/r))$ near to ξ.

SINGULAR INTEGRALS

When the source point is located within the element limits, expressions (4) and (5) become singular and the former is to be computed in the principal value sense. Though this is true in general terms, only the elasticity applications need special consideration. This is because the $\mathbf{p}^*$ kernel in potential theory is thoroughly multiplied by $\partial r/\partial n$ (n is the outward normal) which actually reduces the order of the singularity when x approaches ξ over the surface element;[14] consequently, $\mathbf{p}^*$ and $\mathbf{u}^*$ present integrable singularities of the same order. Here, special quadrature schemes for $O(1/r)$ integrals related to certain particular geometric configurations can be used.[5,6,11] In addition, there are well established[4,14] indirect means of computing the principal values, together with the corresponding c_{ij} coefficients, by simply applying equation (1) to represent rigid body translations. Nevertheless, in some problems it is more accurate to calculate expression (4) than to proceed otherwise. For instance, in the case of the so-called discontinuous elements (these are elements with nodal points located inside the element and not on its borders) which always give $c_{ij} = \delta_{ij}/2$ (δ_{ij} is the Kronecker delta symbol).

For general purpose programs, the best procedure seems to be to write expressions (4) and (5) in terms of a cylindrical co-ordinate system (ρ, θ) based at ξ and integrate over the triangular domains therein formed (Figure 1), i.e.

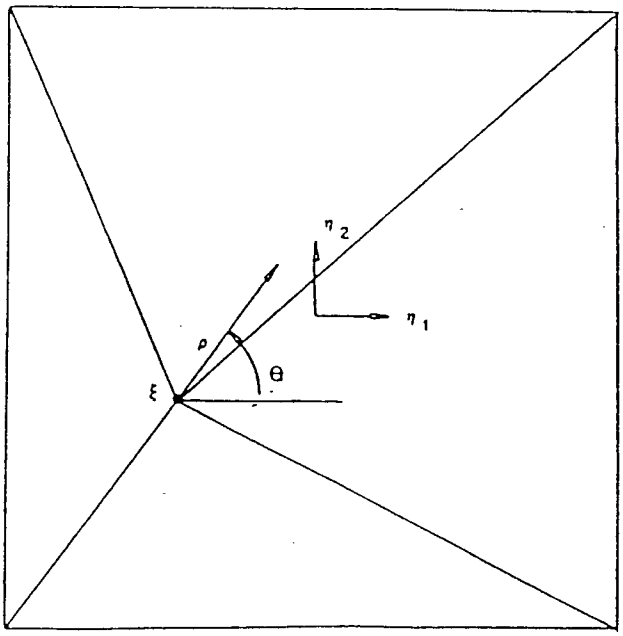

Figure 1. Rectangular element subdivided into triangles

$$\mathbf{h} = \sum^{\text{triangles}} \int_{\theta_1}^{\theta_2} \int_0^{F(\theta)} \mathbf{p}^* \boldsymbol{\Phi}^{\mathrm{T}} G \rho \, \mathrm{d}\rho \, \mathrm{d}\theta \tag{6}$$

$$\mathbf{g} = \sum^{\text{triangles}} \int_{\theta_1}^{\theta_2} \int_0^{F(\theta)} \mathbf{u}^* \boldsymbol{\Phi}^{\mathrm{T}} G \rho \, \mathrm{d}\rho \, \mathrm{d}\theta \tag{7}$$

The benefit of the above expressions is twofold: first the presence of ρ in the integrand smoothes out the $O(1,r)$ kernels and allows for standard Gaussian quadrature with respect to ρ; secondly the singularity of the principal values become restricted to an $O(1/r)$ behaviour over this one-dimensional integration path. Therefore, the only problem that remains is how to integrate the $O(1/r)$ integral with respect to ρ. Here, an efficient finite part numerical integration procedure can be employed. One-dimensional integration points and weights for Gaussian type finite part quadrature have been presented by Kutt[15] for different orders of singularity. For the case studied here, these points have been partially reproduced elsewhere[14] and should be used as follows:

$$\int_0^b \frac{f(y)}{y} \mathrm{d}y \cong \sum_{i=1}^{n} f(b\zeta_i) w_i + f(0) \ln(b) \tag{8}$$

where ζ_i is the co-ordinate of the ith integration point, w_i is the associated weighting factor and n is the total number of points. Notice that the last term on the right is due to the non-standard unit size scaling of the finite part integral and that $f(y)$ now plays the role of $\mathbf{p}^* \boldsymbol{\Phi}^{\mathrm{T}} G \rho y$ (i.e. the integrand in expression (6) is multiplied and divided by y).

A peculiarity of this scheme is that the first integration point always lies outside the integration limit ($y_1 < 0$), but this is of no consequence provided one keeps in mind that $\rho = |y|$.

The procedure indicated above has been found to produce very accurate results with a small number of integration points. For instance, in the case of quadratic (nine nodes) isoparametric Lagrangian quadrilateral elements (discontinuous) n should vary between 3 and 5 to produce acceptable accuracy for any practical application. In addition, though not strictly needed, the same integration routine has also been used for $\mathbf{g}$. This is always desirable in order to save the computer time spent in calculating common variables.

For potential problems a finite part procedure is not needed; standard Gauss quadrature is therefore preferred. A number of points varying between 3 and 5, depending on $F(\theta)$, is still required for general practical applications. In all cases, integration with respect to θ is regular and standard quadrature should be used.

In the case of two-dimensional applications, expression (8) should be used directly for computing $\mathbf{h}$, but the element may have to be subdivided into two parts if $|\eta(\xi)| < 1$. Alternatively, the procedure indicated in Reference 16 could be used. The calculation of the coefficients of the matrix $\mathbf{g}$ can be carried out in two different ways. The first consists of separating the kernel into a regular part (to be integrated by standard Gauss quadrature) and a $\ln(1/r)$ dependent part which can be integrated by a special numerical scheme with $\ln(1/y)$ weighting coefficients.[14] This procedure, in addition to requiring element partition if $|\eta(\xi)| < 1$, can be too cumbersome in some cases. Especially for axisymmetric and related problems in which the logarithmic singularity is hidden into special functions included in the fundamental solution. The second alternative, which is further explored in the next section, relies on a suitable co-ordinate transformation whose Jacobian smoothes out the singularity. This alternative is explained in what follows.

Consider the integral

$$I = \int_{-1}^{1} f(\eta)\,d\eta \tag{9}$$

in which $f(\eta)$ is singular at a point $\bar{\eta}$.

If one chooses a second-degree relation

$$\eta(\gamma) = a\gamma^2 + b\gamma + c \tag{10}$$

such that the following requirements are met:

$$\left.\begin{aligned} \left.\frac{d\eta}{d\gamma}\right|_{\bar{\eta}} &= 0 \\ \eta(1) &= 1 \\ \eta(-1) &= -1 \end{aligned}\right\} \tag{11}$$

Then the following solution is obtained:

$$\begin{aligned} a &= -c \\ b &= 1 \\ c &= \frac{\bar{\eta} \pm \sqrt{(\bar{\eta}^2 - 1)}}{2} \end{aligned} \tag{12}$$

in which the condition $|\bar{\eta}| \geq 1$ is readily found necessary to avoid complex roots.

Therefore, provided that $|\bar{\eta}| = 1$, expression (9) can be written as

$$I = \int_{-1}^{1} f\left[(1-\gamma^2)\frac{\bar{\eta}}{2} + \gamma\right](1 - \gamma\bar{\eta})\,d\gamma \tag{13}$$

The above transformation can be used to calculate integrals with a logarithmic singularity at one of the extremities. Its main advantage is that since the Jacobian cancels the singularity, standard Gaussian integration can be employed without the need to separate the regular part from the singular term in the kernel matrices.

In order to illustrate this procedure, consider the following example:

$$I = \int_{-1}^{1} \ln(1-\eta)\,d\eta = -0{\cdot}613705639$$

A standard 10-point Gauss integration routine gives as a result $I \cong -0{\cdot}6022$, i.e. an error of about 2 per cent. However, if expression (13) is used instead, $I \cong -0{\cdot}61387$ is obtained (an error of 0·03 per cent) for the same number of points.

For the case in which $|\bar{\eta}| < 1$ one can always split the integration path at $\bar{\eta}$ and apply the transformation over each side of the singularity.

There is, however, another non-linear transformation which always remains valid (without partition) for any position of the singularity, i.e. a third-degree relation of the form

$$\eta = a\gamma^3 + b\gamma^2 + c\gamma + d \tag{14}$$

In this case, in addition to conditions (11), a further requirement is found necessary:

$$\left.\frac{d^2\eta}{d\gamma^2}\right|_{\bar{\eta}} = 0 \tag{15}$$

Condition (15) implies that the Jacobian of the new transformation has to have a minimum at $\bar{\eta}$ (though at this stage a maximum could be equally possible).

A solution to this problem is given by

$$\begin{aligned} a &= 1/Q \\ b &= -3\bar{\gamma}/Q \\ c &= 3\bar{\gamma}^2/Q \\ d &= -b \\ Q &= 1 + 3\bar{\gamma}^2 \end{aligned} \tag{16}$$

where $\bar{\gamma}$ is simply the value of γ which satisfies $\eta(\bar{\gamma}) = \bar{\eta}$; this parameter can be calculated by

$$\bar{\gamma} = \sqrt[3]{(\bar{\eta}\eta^* + |\eta^*|)} + \sqrt[3]{(\bar{\eta}\eta^* - |\eta^*|)} + \bar{\eta} \tag{17}$$

where $\eta^* = \bar{\eta}^2 - 1$. Thus, in this case, expression (9) becomes

$$I = \int_{-1}^{1} f\{[(\gamma - \bar{\gamma})^3 + \bar{\gamma}(\bar{\gamma}^2 + 3)]/(1 + 3\bar{\gamma}^2)\}\, 3(\gamma - \bar{\gamma}^2)/(1 + 3\bar{\gamma}^2)\, \mathrm{d}\gamma \tag{18}$$

The application of equation (18) to evaluate the previous example gives, using 10 Gauss points, $I \cong -0{\cdot}61370105$ (an error of 0·0007 per cent).

As a second example, let us consider

$$I = \int_{-1}^{1} \ln|0{\cdot}3 + \eta|\, \mathrm{d}\eta = -1{\cdot}908598917$$

Standard Gaussian quadrature without any transformation gives (10 integration points) $I \cong -1{\cdot}7106$, which means an error of about 10 per cent. The same number of points, but using expression (18), produces $I \cong -1{\cdot}90328$ and the error has now dropped to 0·3 per cent.

The non-linear co-ordinate transformations presented here produce simple and efficient means of computing the coefficients of the matrix **g** for two-dimensional and related logarithmic singularity integrals. In practical terms, however, the Kutt scheme can be used if also needed for explicit evaluation of the matrix **h** (thus saving computation of common variables).

An interesting feature of the above transformations is that they automatically provide a great concentration of points near to the singularity, particularly in the case of the third degree procedure which does not require any element splitting in all cases. This fact has called the attention of the present author to the possibility of using such schemes for quasi-singular element integrals, i.e. integrals over elements located near to, but not containing, the source point ξ. This matter will be taken further in the next section.

QUASI-SINGULAR INTEGRALS

The computation of expressions (2) and (3) ($\xi \notin \Gamma_j$) in current boundary element programs with a curved geometry modelling capability, has been almost entirely based on Gaussian integration routines coupled with automatic selection of numbers of points.[2-4,8,9,17] This selection is based on the relative minimum distance from the element to the source point ξ. Such schemes increase the number of integration points as the minimum distance becomes smaller and start to subdivide the element into a number of subelements for critical configurations (e.g. distances smaller than one quarter of a typical element side).

The advantage of element subdivision is that it produces a higher concentration of points over the subelements near to ξ. But it suffers from the fact that the highest polynomial degree which can be integrated exactly now becomes attached to the local number of points selected for each subelement. For instance, a one-dimensional quadrature which is subdivided into three subelements and uses 2, 3 and 4 Gauss points, respectively, can integrate exactly polynomial integrands of degrees 3, 5 and 7 over each part $(2n-1)$, whereas if nine points are used to integrate over the complete path, a 17th-degree polynomial is allowable. For this reason, element subdivision should not be pushed too far and only becomes efficient in the close range.

From the above it is seen that a different procedure could be attempted not only to substitute for subdivision, but perhaps even to improve the efficiency of integrating over elements which do not originally require partitioning, but still take a large number of Gauss points (i.e. medium minimum distances). To this end, let us present some tests run with the polynomial transformations introduced in the last section.

In two recent publications Jun, Beer and Meek[8,9] have presented interesting means of controlling element subdivision based on error bound analysis. Their examples are taken here as bench-marks for the best performance via this element partitioning technique.

The first example is

$$I = \int_{-1}^{1} \frac{\mathrm{d}\eta}{(1{\cdot}1-\eta)^2} = 9{\cdot}52380952$$

The above quasi-singular integral has been computed in Reference 8 using nine combinations of subdivision (from two to six subelements) and the total number of Gauss points was found to vary between 12 and 18. The computation error obtained was also given for each case and was always within the range 0·04–0·4 per cent.

In order to apply the second-degree transformation one has to be aware that since $|\bar{\eta}| > 1$, there are two possible values of c in expressions (12). However, in order to ensure $|\eta(\gamma)| < 1$, only one of them can be used in the form

$$\int_{-1}^{1} f(\eta)\,\mathrm{d}\eta = \int_{-1}^{1} f[(1-\gamma^2)c+\gamma](1-2\gamma c)\,\mathrm{d}\gamma \tag{19}$$

where

$$c = \begin{cases} \dfrac{\bar{\eta}+\sqrt{\eta^*}}{2}, & \text{for } \bar{\eta} < -1 \\ \dfrac{\bar{\eta}-\sqrt{\eta^*}}{2}, & \text{for } \bar{\eta} > 1 \end{cases}$$

The application of expression (19) ($\bar{\eta} = 1{\cdot}1$) with 10 Gauss points gives as a result $I \cong 9{\cdot}5238059$ (i.e. an error of 0·00004 per cent) and if expression (18) is used instead $I \cong 9{\cdot}52380951$, which means an error of 10^{-7} per cent. This example clearly demonstrates the power of polynomial transformations for this kind of problem.

In another publication by Jun, Beer and Meek[9] a double exponential co-ordinate transformation has been studied. This procedure is based on the co-ordinate transformation presented in Reference 18 and seems to be effective for very small distances from the singular point; therefore element partitioning is still needed. An example which has been efficiently solved by this combined method is

$$I = \int_{-1}^{1} \frac{d\eta}{(1{\cdot}004 - \eta)^2} = 249{\cdot}500998$$

where a combination of subdivision, Gauss and double exponential transformation produced, with a total of 10 integration points, an error of 0·078 per cent. The same authors also present an error of 0·025 per cent obtained with 8 subdivisions and a total of 29 Gauss points (i.e. the double exponential formula was not used).

The application of the polynomial transformations to the above integral gives ($\bar{\eta} = 1{\cdot}004$ and 10 Gauss points) $I \cong 245{\cdot}59$ (an error of 1·6 per cent) for expression (19) and $I \cong 249{\cdot}434$ (an error of 0·026 per cent) for expression (18).

It is therefore seen that even in this very severe example, where a standard Gaussian quadrature (10 points) without transformation gives some 66 per cent error, the simple transformations presented here remain unbeatable, especially the third-degree polynomial.

Let us now examine an example with double integration.[9]

$$I = \int_{-1}^{1}\int_{-1}^{1} \frac{d\eta_1\, d\eta_2}{\sqrt{[(1{\cdot}004 - \eta_1)^2 + (1{\cdot}004 - \eta_2)^2]}} = 3{\cdot}3476318$$

In this case, the double exponential combination required 163 integration points to produce an error of 0·08 per cent and the pure element subdivision procedure of Reference 8 employed 208 points for an error of 0·01 per cent.

Here, applying a 6×6 scheme (a total of 36 points) with the polynomial transformation for each integration variable produced $I \cong 3{\cdot}478796$ (an error of 0·07 per cent) for second degree and $I \cong 3{\cdot}477516$ (an error of 0·03 per cent) for third degree.

An interesting characteristic of this example is that in this case element subdivision does not seem to improve efficiency; for instance, a standard 8×8 Gaussian scheme without any subdivision was found to produce an error of 0·06 per cent. Notice that the same cannot be said about the polynomial transformations.

In another example taken from Reference 8, the following typical integral was tested:

$$I = \int_{-1}^{1}\int_{-1}^{1} \frac{1}{r^2} d\eta_1\, d\eta_2$$

where

$$r = \sqrt{[(\bar{\eta} - \eta_1)^2 + (\bar{\eta} - \eta_2)^2]}$$

A comparison between the results obtained here and their computations using element subdivision is presented in Table I.

Table I. Computation errors for double $1/r^2$ integrals

α	$\bar{\eta}$	Integration procedure	Total number of points	Error (%)
		element subdivision	187	0·0007
1	1·02	2nd degree transformation	64	0·0006
		3rd degree transformation	64	0·0004
		element subdivision	58	0·002
3	1·2	2nd degree transformation	36	0·003
		3rd degree transformation	25	0·0055
		3rd degree transformation	36	0·0007

PARAMETER OPTIMIZATION

In the last section a good number of examples indicated that the third-degree polynomial transformation should be used in preference to element subdivision or even the second-degree transformation. The practical advantage of this procedure is that $\bar{\eta}$ can assume any value inside or outside the integration limits. For general application, however, an important point is still missing: how to efficiently apply the procedure when the source point is located outside the η axis at some distance D from it. Ideally, $\bar{\eta}$ should be taken as the co-ordinate of the nearest point to ξ over the η axis; notice that this point is to be found in the real space and *not* after the G transformation has taken place. In addition, since the distance D is not present in expression (18), the same transformation will be obtained for any possible distance. This undesirable behaviour is in sharp contrast with the self-adaptiveness of the procedure with respect to $\bar{\eta}$, i.e. it is easily seen that as $|\bar{\eta}|$ increases, both polynomial transformations become less pronounced until they degenerate into $\eta = \gamma$ as $|\bar{\eta}| \to \infty$.

This problem can be studied a little further. Recalling conditions (11), it is seen that since the integral is now regular, there is no need to impose a zero value of the Jacobian at $\bar{\eta}$. Therefore, this equation can be written as follows:

$$J(\bar{\gamma}) = \left.\frac{\mathrm{d}\eta}{\mathrm{d}\gamma}\right|_{\bar{\eta}} = \bar{r} \tag{20}$$

where $\bar{r}$ is a free parameter which can be taken as a function of D.

In this case, the complete transformation becomes dependent on $\bar{r}$:

$$\begin{aligned}
\eta &= a\gamma^3 + b\gamma^2 + c\gamma + d \\
J &= 3a\gamma^2 + 2b\gamma + c \\
a &= (1-\bar{r})/Q \\
b &= -3(1-\bar{r})\bar{\gamma}/Q \\
c &= (\bar{r} + 3\bar{\gamma}^2)/Q \\
d &= -b \\
Q &= 1 + 3\bar{\gamma}^2
\end{aligned}$$

$$\bar{\gamma} = \sqrt[3]{[-q + \sqrt{(q^2+p^3)}]} + \sqrt[3]{[-q - \sqrt{(q^2+p^3)}]} + \frac{\bar{\eta}}{1+2\bar{r}} \tag{21}$$

$$q = \frac{1}{2(1+2\bar{r})}\left[\left(\bar{\eta}(3-2\bar{r}) - \frac{2\bar{\eta}^3}{1+2\bar{r}}\right)\frac{1}{1+2\bar{r}} - \bar{\eta}\right]$$

$$p = \frac{1}{3(1+2\bar{r})^2}[4\bar{r}(1-\bar{r}) + 3(1-\bar{\eta}^2)]$$

From the above expressions it can be seen that if $\bar{r} = 1$, the transformation degenerates into $\eta = \gamma$, $J = 1$. Therefore, one should always keep $0 \leqslant \bar{r} \leqslant 1$.

In order to define an explicit relation between $\bar{r}$ and D, the most practical approach seems to be finding an $\bar{r}$ for a given D which produces the minimum integration error in the least-squares sense. Herein, this optimization has been carried out for integrands of orders $1/r$ and $1/r^2$, since these two types are found in three-dimensional applications. However, other types of functions (e.g. $\ln(1/r)$, $(1/r^3)$) can also be attempted.

Let us assume a fixed D value; for a given number of Gauss points K and a known $\bar{\eta}$, the integration error can be defined as

$$\varepsilon(D,\bar{\eta},K,\bar{r})=\sum_{k=1}^{K}\left(\frac{J_k}{r_k^2}w_k\right)-I \tag{22}$$

where $I=\int_{-1}^{1}(1/r^2)\,d\eta$ is calculated analytically.

Keeping in mind that the optimization should be particularly effective for any position within the most sensitive range of $\bar{\eta}$ (i.e. $|\bar{\eta}|\leqslant 1$), one can choose a certain number L of discrete values of $\bar{\eta}$ in this interval and generate the function

$$v(D,K,\bar{r})=\sum_{l=1}^{L}\varepsilon^2(D,\bar{\eta}_l,K,\bar{r}) \tag{23}$$

where $L=5$ and $\bar{\eta}_1=0{\cdot}1,\bar{\eta}_2=0{\cdot}3,\ldots,\bar{\eta}_5=0{\cdot}9$ have been used.

A practical removal of the implicit dependence on the particular K employed can now be obtained by simply adding up v for a sufficiently large number of different K values. In this way, a general error function is created of the form

$$V(D,\bar{r})=\sum_{i=1}^{N}v(D,K_i,\bar{r}) \tag{24}$$

where in the present work $N=13$ was used with K_i being sequentially valued from 2 up to 6 and then approximately evenly spaced until a maximum of 32 integration points.

The optimization is now reduced to finding the minimum of V with respect to $\bar{r}$. Hence

$$\frac{\partial V}{\partial \bar{r}}=0 \tag{25}$$

which means that one has to solve the equation

$$\sum_{i=1}^{N}\sum_{l=1}^{L}\left\{\left[\sum_{k=1}^{K_i}\left(\frac{J_{kl}}{r_{kl}^2}w_k\right)-I_l\right]\left[\sum_{k=1}^{K_i}\left(\frac{1}{r_{kl}^2}\frac{\partial J_{kl}}{\partial \bar{r}}-\frac{\alpha J_{kl}}{r_{kl}^{\alpha+1}}\frac{\partial r_{kl}}{\partial \bar{r}}\right)w_k\right]\right\}=0 \tag{26}$$

Equation (26) has been solved by a regula falsi type method for a number of D values, starting from $D=0{\cdot}05$ (which is small enough for practical problems) up to a large value. The computations were carried out for $\alpha=1$ and $\alpha=2$, and in every case a root was found within the expected range $0\leqslant\bar{r}\leqslant 1$ without any convergence problems. The results are depicted in Figure 2, where it can be seen that $\bar{r}\to 1$ as $D\to\infty$, i.e. the scheme now becomes fully self-adaptive and the transformation vanishes as the source point moves away from the element.

In order to implement the complete polynomial transformation procedure into existing BE codes one now needs a simple curve fitting exercise, but it is desirable to have the same transformation for the **h** and **g** matrices so that they can be calculated within the same integration loop with a minimum number of operations. Here, the following expressions were found to work well for both matrices in three-dimensional applications:

$$\begin{aligned}
&\bar{r}=0{\cdot}85+0{\cdot}24\ln(D), && 0{\cdot}05\leqslant D\leqslant 1{\cdot}3\\
&\bar{r}=0{\cdot}893+0{\cdot}0832\ln(D), && 1{\cdot}3\leqslant D\leqslant 3{\cdot}618\\
&\bar{r}=1, && 3{\cdot}618\leqslant D
\end{aligned} \tag{27}$$

where the above curve lies between the $\alpha=1$ and $\alpha=2$ optimum lines of Figure 2 in the most important range.

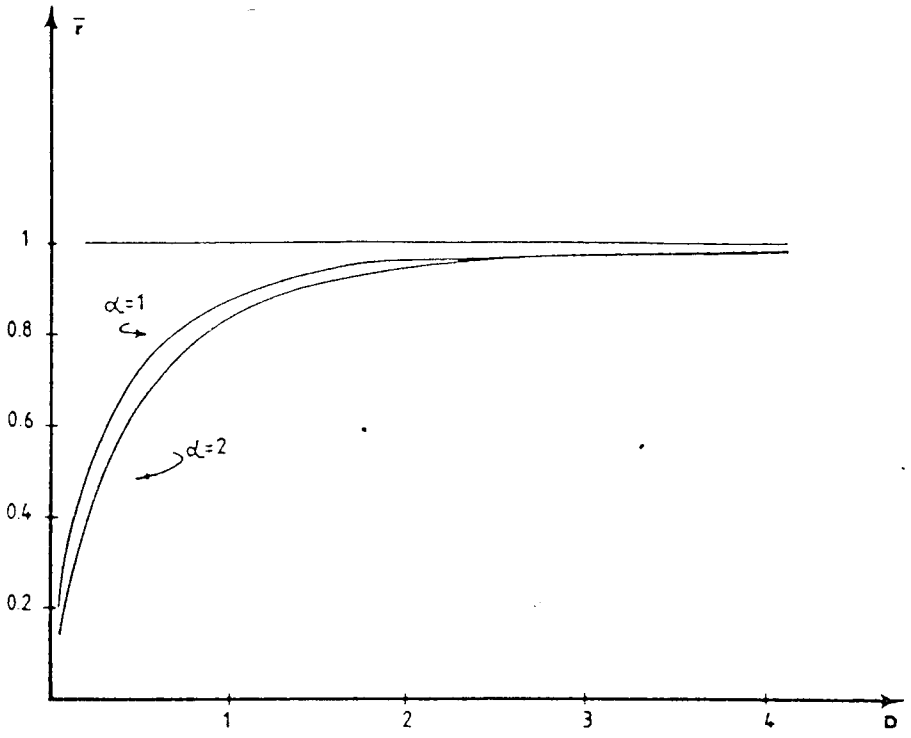

Figure 2. Optimized parameter $\bar{r}$ plotted against distance D

Some remarks are perhaps now due; in order to fully benefit from the optimization it is apparently better to keep calculating the distances D between ξ and each of the many η_1, η_2 one-dimensional paths of integration which are spread out over the element. By doing this, a radiating curved pattern of integration point positions is readily obtained, typical of well adjusted numerical quadrature schemes.[11] In this case the transformation itself varies depending on the real space position of each Gauss point. But this procedure would be cumbersome, especially if $|\bar{\eta}_1|$ and $|\bar{\eta}_2|$ are less than unity. This is because the distances of the η_1 integration paths are dictated by the η_2 co-ordinates in the other direction, and vice versa; thus the problem becomes non-linear. In practice, however, the total integration error is greatly dominated by the η_1 and η_2 integration paths nearest to ξ. Consequently, it is advisable to calculate the minimum distance from ξ to the element surface and keep it constant for the complete integration. This alternative was tested in many problems and the accuracy was found *not* to deteriorate at all, the only difference being that now the points are concentrated towards ξ under straight line patterns. In addition, when either $\bar{\eta}_1$ or $\bar{\eta}_2$, or both, has an absolute value greater than one, it may be cumbersome to find their actual values, since this would require extrapolation over the element surface. In this case, comparative tests have also indicated that one can avoid extrapolation (i.e. set the respective value to $+1$ or -1) and compensate for it by taking D as corresponding to the distance from the element border.

Therefore, for general implementation of the present scheme one simply has to identify the $\bar{\eta}_1$, $\bar{\eta}_2$ co-ordinates of the closest point located over the element and calculate its real space distance R_{min} from ξ (these operations are already included in most BE codes for selecting the number of integration points). The relative distance D to apply in each direction is then calculated by the formulae

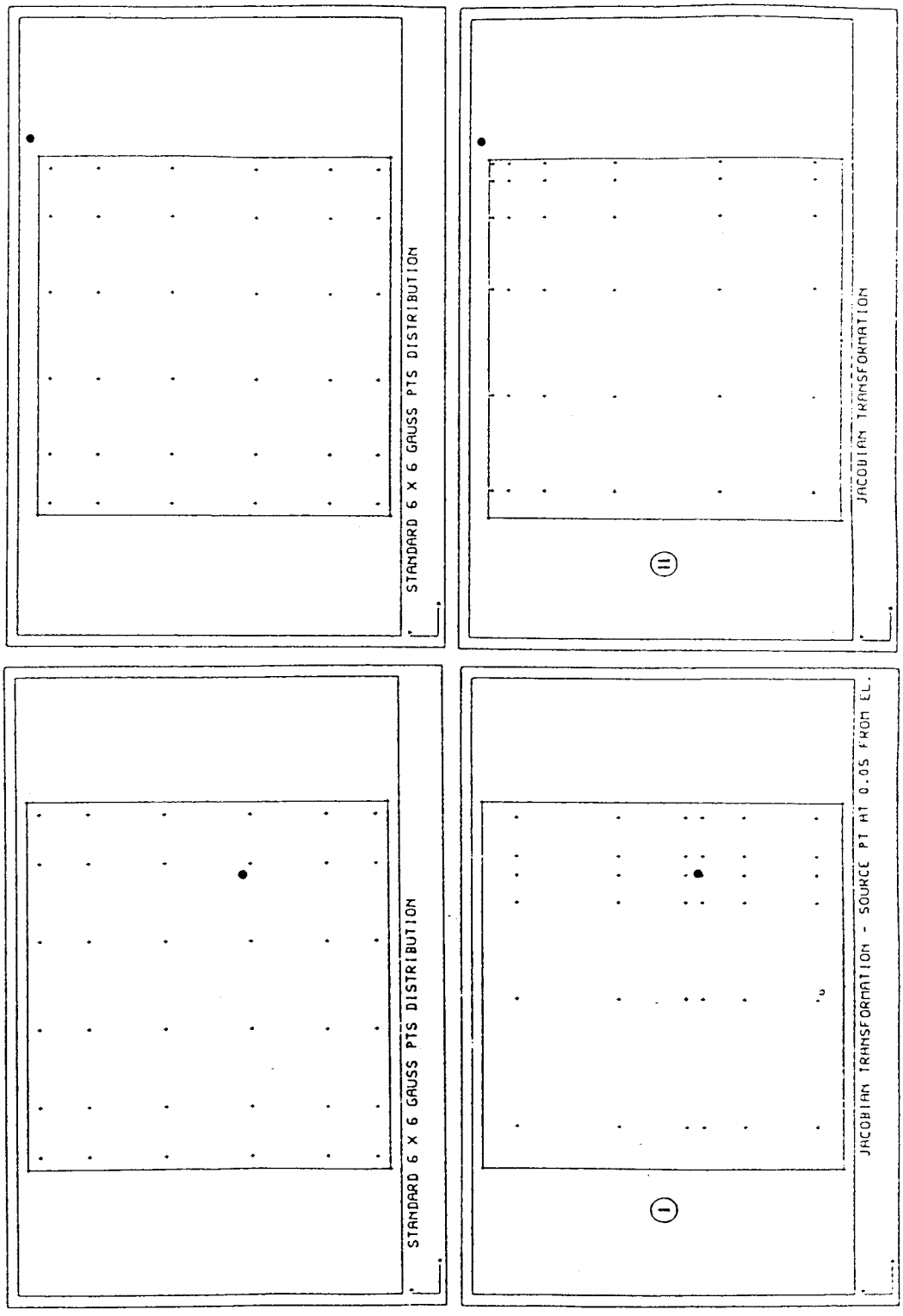

Figure 3(a). Lumping of Gauss points towards the source point (●): I—source point outside the element plane; II—source point on the element plane

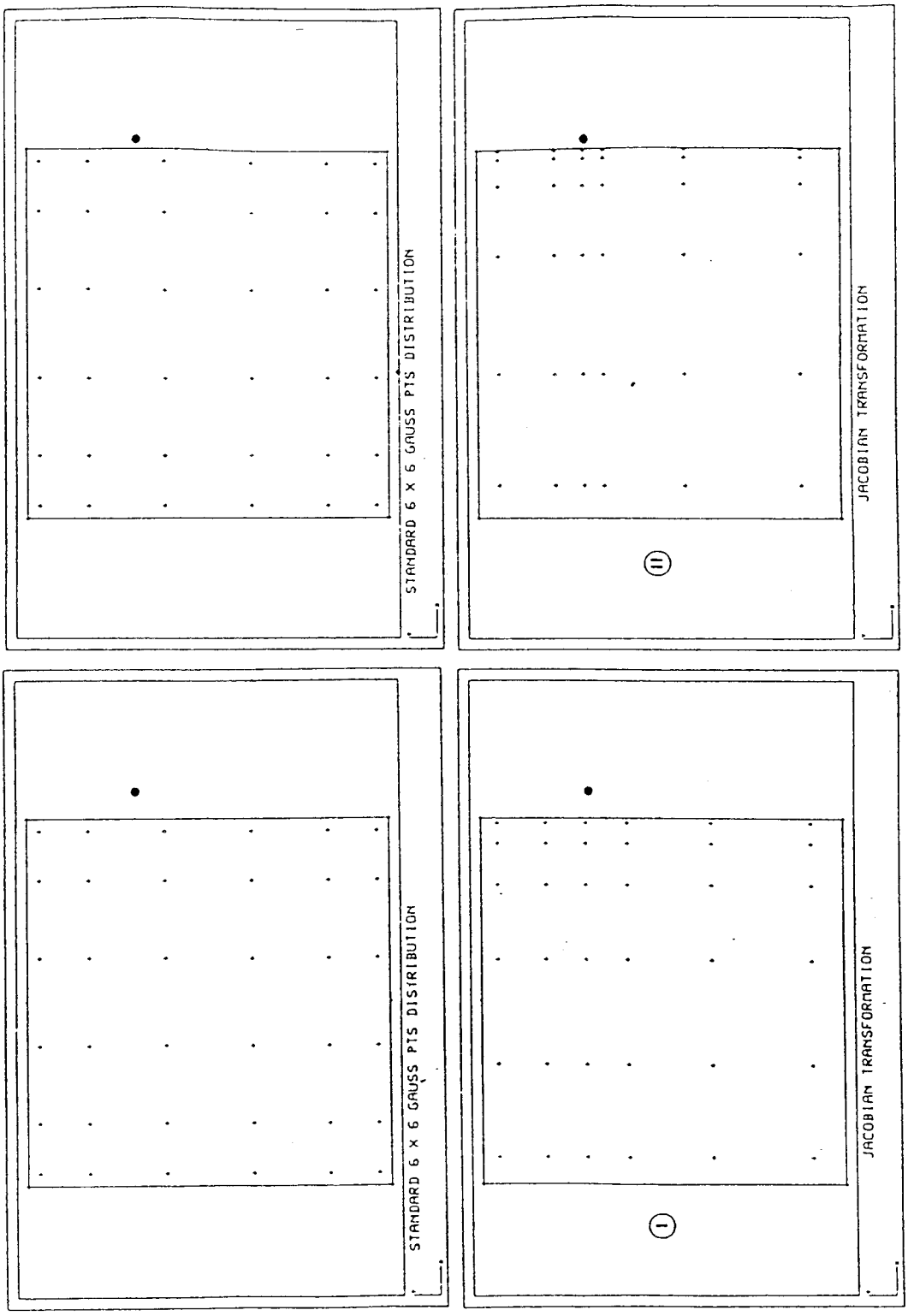

Figure 3(b). Typical configurations of Gauss points at two different distances R_{min}: I and II—source point on the element plane

$$D_1 = 2R_{min}/|x(1, \bar{\eta}_2) - x(-1, \bar{\eta}_2)|$$
$$D_2 = 2R_{min}/|x(\bar{\eta}_1, 1) - x(\bar{\eta}_1, -1)|$$

where the symbol $|\cdot - \cdot|$ means the distance between two points in the real space.

Typical 6×6 Gauss station distributions over a square element are presented in Figure 3, where the lumping of points produced by the transformation is clearly seen in comparison to standard Gaussian integration. Notice that, as the source point moves away from the element, the concentration becomes less pronounced until it naturally dies away and standard quadrature takes over.

From expression (27) it is seen that the lumping is effective up to an R_{min} value of about 1·8 of the reference element dimension. Therefore, since this distance is well outside the usual element subdivision range, the scheme also allows considerable savings over elements which would be otherwise integrated with simple Gauss quadrature, but using a large number of points. The saving, of course, increases as ξ approaches the element.

It has been verified in practice that (for quadratic isoparametric elements) starting from a distance where standard Gaussian routines require a 5×5-point scheme, the transformation can work with 4×4 in many cases. But this difference can increase up to a reduction of about three times fewer points in the close range when compared to the subdivision technique.

As an elastostatics test a unit cube with one discontinuous quadratic isoparametric Lagrangian element per face has been solved with element subdivision and the polynomial transformation. A reduction of nearly 50 per cent of the computer run time for the total computation of non-singular element integrals has been observed.

CONCLUSIONS

Efficient means of computing boundary element integrals have been presented in this work. Particular emphasis was given to a new co-ordinate transformation based on a complete thir degree polynomial relation which can be applied to element integrals currently found in two- ar. three-dimensional problems. Although originally designed to calculate one-dimensional integrals with logarithmic singularities, the procedure has been found to work equally well for general nearly singular element integrals. It removes the need for element subdivision with a great overall reduction in the number of integration points.

The transformation produces a lumping effect in the Gauss point positions, moving them towards the source point. It also presents an important self-adaptive characteristic which makes it inactive when not useful (i.e. for large source to element distance). In addition, the scheme only needs the local co-ordinates of the element point located at the nearest position to the source point and the minimum source to element distance. These parameters are already calculated in most BE codes for selecting the number of Gauss stations. Therefore, the procedure is very easy to implement in any existing BE package.

Finally, it should be mentioned that other non-linear co-ordinate transformations could also be attempted. Mustoe[7] discusses one such possibility which, like the double exponential formulae[9] does not seem to match the versatility of the third degree approach.

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REFERENCES

1. C. A. Brebbia, D. J. Danson and J. Baynham, 'BEASY a boundary element analysis system', in C. Brebbia (ed.), *Finite Element Systems, a Handbook*, Springer-Verlag, Berlin, 1985.
2. A. Chaudouet and M. Afzali, 'CASTOR 3D: three dimensional boundary element analysis computer code', in C. Brebbia, T. Futagami and M. Tanaka (eds), *Boundary Elements—Proc. 5th Int. Conf.*, Hiroshima Japan, Springer-Verlag/CMP, Berlin, 1983.
3. P. K. Banerjee, R. B. Wilson and N. Miller, 'Development of a large system for three-dimensional inelastic analysis', in T. Cruse (ed.), *Proc. ASME Conf. Advanced Topics in Boundary Element Analysis*, November 1985.
4. J. C. Lachat and J. O. Watson, 'Effective numerical treatment of boundary integral equations: a formulation for three-dimensional elastostatics', *Int. j. numer. methods eng.*, **10**, 273–289 (1976).
5. M. Cristescu and G. Loubignac, 'Gaussian quadrature formulas for functions with singularities in $1/r$ over triangles and quadrangles', in C. Brebbia (ed.), *Recent Advances in Boundary Element Methods*, Pentech Press, London, 1978.
6. H. L. G. Pina and J. L. M. Fernandes, 'Some numerical integration formulae over triangles and squares with a $1/r$ singularity', *Appl. Math. Modelling*, **5**, 209–211 (1981).
7. G. G. Mustoe, 'Advanced integration schemes over boundary elements and volume cells for two- and three-dimensional non-linear analysis', in P. Banerjee and S. Mukherjee (eds), *Developments in Boundary Element Methods—3*, Applied Science, London, 1984.
8. L. Jun, G. Beer and J. L. Meek, 'Efficient evaluation of integrals of order $1/r$, $1/r^2$, $1/r^3$ using Gauss quadrature', *Engineering Analysis*, **2**, 118–123, (1985).
9. L. Jun, G. Beer and J. L. Meek, 'The application of double exponential formulas in the boundary element method', in C. Brebbia and G. Maier (eds), *Boundary Elements VII—Proc. 7th Int. Conf.*, Como, Italy, Springer-Verlag/CMP, Berlin, Vol. 2, 1985.
10. C. H. Schwab and W. L. Wendland, '3-D B.E.M. numerical integration', in C. Brebbia and G. Maier (eds), *Boundary Elements VII—Proc. 7th Int. Conf.*, Como, Italy, Springer-Verlag/CMP, Berlin, Vol. 2, 1985.
11. M. H. Lean and A. Wexler, 'Accurate numerical integration of singular boundary element kernels over boundaries with curvature', *Int. j. numer. methods eng.*, **21**, 211–228 (1985)
12. H. B. Li, G. M. Han and H. A. Mang, 'A new method for evaluating singular integrals in stress analysis of solids by the direct boundary element method', *Int. j. numer. methods eng.*, **21**, 2071–2098 (1985).
13. M. H. Aliabadi, W. S. Hall and T. G. Phemister, 'Taylor expansion for singular kernels in the boundary element method', *Int. j. numer. methods eng.*, **21**, 2221–2236 (1985).
14. C. A. Brebbia, J. C. F. Telles and L. C. Wrobel, *Boundary Element Techniques: Theory and Applications in Engineering*, Springer-Verlag, Berlin, 1984.
15. H. R. Kutt, 'Quadrature formulae for finite part integrals', *Report WISK 178*, The National Institute for Mathematical Sciences, Pretoria, 1975.
16. R. Piessens, 'Numerical evaluation of Cauchy principal values of integrals', *BIT*, **10**, 476–480 (1970).
17. T. Higashimachi, N. Okamoto, Y. Ezawa, T. Aizawa and A. Ito, 'Interactive structural analysis system using the advanced boundary element method', in C. Brebbia, T. Futagami and M. Tanaka (eds), *Boundary Elements—Proc. 5th Int. Conf.*, Hiroshima, Japan, Springer-Verlag/CMP, Berlin, 1983.
18. H. Takahasi and M. Nori, 'Quadrature formulas obtained by variable transformation', *Numer. Math.*, **21**, 206–219 (1973).