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CSCI 3190 Introduction to Discrete Mathematics and Algorithms

Assignment 1

Due Date : October 11, 2020 5:00pm

1. Show that the following pairs of propositions are logically equivalent:
   (a) $p \to (q \wedge r)$ $\quad$ $(p \to q) \wedge (p \to r)$
   (b) $(p \vee q) \to r$ $\quad$ $(p \to r) \wedge (q \to r)$
   (c) $((p \to \neg p) \wedge (\neg p \to p))$ $\quad$ FALSE

2. Prove the following statement or find a counterexample to disprove it. All the variables have the same domain of the set of all integers.
   (a) $\forall y\, \exists x\, (y^2 = x)$
   (b) $\forall x\, \exists y\, (y^2 = x)$
   (c) $\exists x\, \exists y\, [(x - y = 5) \wedge (x + y = 1)]$

3. Prove the followings where P, Q and R are sets:
   (a) $(P \cup Q) - R = (P - R) \cup (Q - R)$
   (b) $(P - Q) \cup (P - R) = P - (Q \cap R)$

4. Let f, g and h be functions from $\boldsymbol{N}$ to $\boldsymbol{N}$, where $\boldsymbol{N}$ is the set of all natural numbers, i.e., 1, 2, 3, … and are defined as follows:

$$f(n) = 2n + 1$$
$$g(n) = n + 5$$
$$h(n) = \begin{cases} 0 & \text{when } n \text{ is even} \\ 1 & \text{when } n \text{ is odd} \end{cases}$$

   Determine $f \circ f$, $f \circ g$, $g \circ f$, $g \circ h$, $h \circ g$, $(f \circ g) \circ h$.

5. Prove the following tautological implication:
   (a) $(p \wedge q) \Rightarrow p$
   (b) $p \Rightarrow (p \vee q)$
   (c) $(p \wedge (p \to q) \wedge (s \vee r) \wedge (r \to \neg q)) \Rightarrow s$

6. If $|A| = 4$ and $|B| = 5$, how many function $f: A \to B$ can be formed? How many of them are invertible?

7. Are the following statements true? Prove or disprove:
   (a) The transitive closure of the symmetric closure of the reflexive closure of a relation is an equivalence relation.
   (b) The symmetric closure of the reflexive closure of the transitive closure of a relation is an equivalence relation.

8. Let R and S be relations on A. Determine whether the following statements are true. Prove or disprove:

   (a) If R and S are reflexive, $R \cup S$ is reflexive.
   (b) If R and S are symmetric, $R \cap S$ is symmetric.
   (c) If R is transitive, R^{-1} is transitive.
   (Note R^{-1} is the inverse of R defined as $(a, b) \in R^{-1}$ if and only if $(b, a) \in R$.)

9. Consider the relation R over the set of all positive integers I^+:

   $R = \{(a, b) \mid |a - b| \text{ is an even integer}\}$

   (a) Is R an equivalence relation? Prove your answer.
   (b) How many equivalence classes of R in I^+ are there? What are they?

Show that the following pairs of propositions are logically equivalent:

(a)	$p \to (q \wedge r)$	$(p \to q) \wedge (p \to r)$
(b)	$(p \vee q) \to r$	$(p \to r) \wedge (q \to r)$
(c)	$((p \to \neg p) \wedge (\neg p \to p))$	FALSE

a) $p \to (q \wedge r)$

$\equiv \sim p \vee (q \wedge r)$

$\equiv (\sim p \vee q) \wedge (\sim p \vee r)$

$\equiv (p \to q) \wedge (p \to r)$ //

b) $(p \vee q) \to r$

$\equiv \sim (p \vee q) \vee r$

$\equiv \sim p \wedge \sim q \vee r$

$\equiv (\sim p \wedge \sim q) \vee r$

$\equiv (\sim p \vee r) \wedge (\sim q \vee r)$

$\equiv (p \to r) \wedge (q \to r)$

c) $((p \to \sim p) \wedge (\sim p \to p))$

$\equiv ((\sim p \vee \sim p) \wedge (p \vee p))$

$\equiv ((\sim p) \wedge (p))$

Since when p is T, $\sim p$ is F

p is F, $\sim p$ is T

∴ It is impossible that p & $\sim p$ are both T,

$\equiv$ False

2. Prove the following statement or find a counterexample to disprove it. All the variables have the same domain of the set of all integers.
(a) $\forall y \exists x (y^2 = x)$
(b) $\forall x \exists y (y^2 = x)$
(c) $\exists x \exists y [(x - y = 5) \wedge (x + y = 1)]$

a) $\forall y \exists x (y^2 = x)$ for all integer y, there exsit integer x such that $y^2 = x$

for every integer y, $y^2 = y \times y$ $\therefore x = y \times 2$
Since the multiple of two integer must be integer
$\therefore$ The statment is true

b) $\forall x \exists y (y^2 = x)$
It implies for all integer x, there exist integer y such that $y^2 = x$
For $y^2 = x$, $y = \sqrt{x}$
Since x can be positive or negative
If x = negative, square root of x is not real number
$\therefore$ statement is false

c) $\exists x \exists y [(x-y=5) \wedge (x+y=1)]$
Implies for some integer x there are some integer y such that both $(x-y=5)$ and $(x+y=1)$ be true

For $x-y=5$
$x = y+5$
$y = x-5$

for $x+y=1$
$x = 1-y$
$y = 1-x$

For $x = y+5$ and $x = 1-y$

$y+5 = 1-y$
$y = 3$

when $y=3$
$x = -2$

Since there are one y and one x fullfill the statement,
The statement is true

3. Prove the followings where P, Q and R are sets:

(a) $(P \cup Q) - R = (P - R) \cup (Q - R)$

(b) $(P - Q) \cup (P - R) = P - (Q \cap R)$

a) $(P \cup Q) - R =$

Let $x \in (P \cup Q) - R$

By definition of difference

$x \in P \cup Q$ and $x \notin R$

By definition of Union

$x \in P$ or $x \in Q$

since $x \in P$ and $x \notin R$, by definition of difference

$x \in P - R$

since $x \in Q$ and $x \notin R$, by definition of difference

$x \in Q - R$

since $x \in P - R$ or $x \in Q - R$, by definition of Union

$x \in (P - R) \cup (Q - R)$

which is R.H.S

b) $(P - Q) \cup (P - R)$

Let $x \in (P - Q) \cup (P - R)$

By definition of Union

$x \in P - Q$ or $x \in P - R$

By definition of difference

$(x \in P$ and $x \notin Q)$ or $(x \in P$ and $x \notin R)$

since $x \notin Q$ or $x \notin R$, it's $\sim Q \vee \sim R$

which is equal to $\sim (Q \wedge R)$

$\therefore x \notin (Q \cap R)$

Since $x \in P$ or $x \in P$, which is $p \vee p \equiv p$

$\therefore x \in P$

$x \in P$ and $x \notin (Q \cap R)$ By difference

$x \in P - (Q \cap R)$

which is R.H.S.

4. Let f, g and h be functions from $\boldsymbol{N}$ to $\boldsymbol{N}$, where $\boldsymbol{N}$ is the set of all natural numbers, i.e., 1, 2, 3, … and are defined as follows:

$$f(n) = 2n + 1$$
$$g(n) = n + 5$$
$$h(n) = \begin{cases} 0 & \text{when } n \text{ is even} \\ 1 & \text{when } n \text{ is odd} \end{cases}$$

Determine $f \circ f$, $f \circ g$, $g \circ f$, $g \circ h$, $h \circ g$, $(f \circ g) \circ h$.

$f \circ f = f(f(n))$
$= 2(2n+1) + 1$
$= 4n+3$ //

$f \circ g = f(g(n))$
$= 2(n+5) + 1$
$= 2n + 6$

$g \circ f = g(f(n))$
$= (2n+1) + 5$
$= 2n + 6$

$g \circ h = g(h(n))$
when n = odd
$g \circ h = (1) + 5 = 6$
when n = even
$g \circ h = (0) + 5 = 5$

$h \circ g = h(n+5)$
when n = odd
$h \circ g = 0$
when n = even
$h \circ g = 1$

$(f \circ g) \circ h$

$= f(g(n)) \circ h$
$= (2(h(n)) + 6)$

when n = odd
$(f \circ g) \circ h = 8$
when n = even
$(f \circ g) \circ h = 6$

5. Prove the following tautological implication:

(a) $(p \wedge q) \Rightarrow p$

(b) $p \Rightarrow (p \vee q)$

(c) $(p \wedge (p \rightarrow q) \wedge (s \vee r) \wedge (r \rightarrow \neg q)) \Rightarrow s$

a) $(p \wedge q) \Rightarrow p$

$\equiv \sim(p \wedge q) \vee p$

$\equiv \sim p \vee \sim q \vee p$

$\equiv (\sim p \vee p) \vee \sim q$

$\equiv T \vee \sim q$

$\equiv T$

b) $p \Rightarrow (p \vee q)$

$\equiv \sim p \vee (p \vee q)$

$= (\sim p \vee p) \vee q$

$= T \vee q$

$= T$

c)

	p	q	r	s	p -> q	s v r	r -> ~q	LHS	LHS=> S
3	0	0	0	0	1	0	1	0	1
4	0	0	0	1	1	1	1	0	1
5	0	0	1	0	1	1	1	0	1
6	0	0	1	1	1	1	1	0	1
7	0	1	0	0	1	0	1	0	1
8	0	1	0	1	1	1	1	0	1
9	0	1	1	0	1	1	0	0	1
10	0	1	1	1	1	1	0	0	1
11	1	0	0	0	0	0	1	0	1
12	1	0	0	1	0	1	1	0	1
13	1	0	1	0	0	1	1	0	1
14	1	0	1	1	0	1	1	0	1
15	1	1	0	0	1	0	1	0	1
16	1	1	0	1	1	1	1	0	1
17	1	1	1	0	1	1	0	1	1
18	1	1	1	1	1	1	0	0	1

6. If $|A| = 4$ and $|B| = 5$, how many function $f: A \to B$ can be formed? How many of them are invertible?

7. Are the following statements true? Prove or disprove:
   (a) The transitive closure of the symmetric closure of the reflexive closure of a relation is an equivalence relation.
   (b) The symmetric closure of the reflexive closure of the transitive closure of a relation is an equivalence relation.

6) The number of function $f: A \to B$ can be formed

$= 5^4 = 625$

The function is invertible only if $f: A \to B$, $f^{-1}: B \to A$
Since $|A| \neq |B|$, there are no function are invertible

7) a)

example
Let $x = \{(a_1, a_2), (a_2, a_3)\}$ be reation on set $A = \{a_1, a_2, \dots, a_n\}$
Then reflexive closure of $x = \{(a_1, a_1), (a_2, a_2) \dots, (a_n, a_n), x\}$

Then Symmetric closure of reflexive closure of x
$= \{$ (relexive closure of x), (remaining symmetric set), (orginal x) $\}$
(a_2, a_1) (a_3, a_2)

Since symmetric closure of reflexive cl-sure of x won't make it not reflexive, ∴ its both reflexive and symmetric

Then transitive closure
$= \{$ (reflexive closure), (symmetric set), (orginal x), (transitive closure) $\}$

Since closure is transitive if $(a, b) \in R$, $(b, c) \in R$, (a, c) also in R
and when closure is already symmetric, all (a, b) have oppositive (b, a)
and all (b, c) have oppositive (c, b)
Then the closure of transitive is still symmetric because both (a, c) and (c, a) are included.
And It is reflexive because all $(a_1, a_1) \dots (a_n, a_n)$ are included in closure

So it is equivalence relation

7h) let $x = \{(a_1, a_2); (a_2, a_3\}$ in set $\{a_1, a_2, a_3\}$

transitive closure of x

$= \{(a_1, a_3), (a_2, a_3)\}$

reflexive closure of transive closure of x

$= \{(a_1, a_3), (a_2, a_3), (a_1, a_1), (a_2, a_2), (a_3, a_3)\}$

symmetric closure of reflexive closure of transitive closure of x

$= \{(a_1, a_3), (a_2, a_3), (a_1, a_1), (a_2, a_2) (a_3, a_3),$

$(a_3, a_1), (a_2, a_3)\}$

Since it is not transitive (lack of (a_2, a_1) (a_3, a_1))

Then there are one counter example and it is false relation.

8. Let R and S be relations on A. Determine whether the following statements are true. Prove or disprove:

(a) If R and S are reflexive, $R \cup S$ is reflexive.
(b) If R and S are symmetric, $R \cap S$ is symmetric.
(c) If R is transitive, R^{-1} is transitive.
(Note R^{-1} is the inverse of R defined as $(a, b) \in R^{-1}$ if and only if $(b, a) \in R$.)

9. Consider the relation R over the set of all positive integers I^+:

$R = \{(a, b) \mid |a - b| \text{ is an even integer}\}$

(a) Is R an equivalence relation? Prove your answer.
(b) How many equivalence classes of R in I^+ are there? What are they?

8 a) let $a \in A$

Since R is reflexive

$(a, a) \in R$

Since S is reflexive

$(a, a) \in S$

The union $R \cup S$ contain all element in R or in S

Then $(a, a) \in R \cup S$ (for every element $a \in A$)

$\therefore R \cup S$ are reflexive

b) Let x be $R \cap S$

$(a_1, a_2) \in x$

$(a_1, a_2) \in R \cap S$

The $(a_1, a_2) \in R$ or $(a_1, a_2) \in S$

since R and S are symmetric

then $(a_2, a_1) \in R$ or $(a_2, a_1) \in S$

Since x is $R \cap S$

$(a_2, a_1) \in x$

$\therefore R \cap S$ is symmetric

c) For R is transitive

By definition

$R \circ R = R$

Thus

$R^{-1} \circ R^{-1} = (R \circ R)^{-1}$

$\subseteq R^{-1}$

9. Consider the relation R over the set of all positive integers I^+:

$$R = \{(a, b) \mid |a - b| \text{ is an even integer}\}$$

(a) Is R an equivalence relation? Prove your answer.
(b) How many equivalence classes of R in I^+ are there? What are they?

a) For relation R,

R is an equivalence relation if and only if
R is reflexive, symmetric and transitive.

for all positive integer,
$|a-b|$ is an even integer, let $(a,b) \in A$

For reflexive, $a=b$, since $a-b=0$ which is even, ∴ Implies that R is reflexive

For symetric, if $a>b$, if $a-b=c$ is even,

$-b = c-a$
$b-a=-c$, since c is even
$|-c|$ is also even

if $b>a$, $|a-b|=c$
then
$|b-a| = c$ too
It implies R is symetric

For transitive, if there are $|a-b|$ = even no. let the even no. be $|x|$
and $|b-c|$ = even no. let the even no. be $|y|$

Then $a-c = a-b+b-c = x+y$

let x be $2n$ and y be $2m$,
$2n+2m = 2(n+m)$ which is even no. no matter n and m are +ve / -ve.

∴ No matter x or y is positive or negative, $|a-c|$ must be even no.

∴ R is transitive

∴ R is both transitive symetric and reflexive and R is equivalence relation

b) let $(a,b) \in I^+$ and let $n = a-b$

If $n = 0$, then the equivalence class are $[(1,1)] = \{(x,x) \in I^+\}$
like ~~(2,2)~~, $(1,1)$; $(2,2)$, $(3,3)$ ... (x,x)

If $n > 0$, then the equivalence class are $[1+n, 1] = \{(x+n, x \in I^+)\}$
like if $n = 2$, then $(3,1)$, $(4,2)$, ..., $(x+2, x)$ if $n = 4$, $(5,1)$, $(6,2)$ ... $(x+4, x)$

If $n < 0$, then the equivarlence class are $[1, 1+n] = \{(x, x+n \in I^+)\}$
like if $n = -2$, then $(1,3)$, $(2,4)$ ... $(x, x+2)$, if $n = 4$, $(1,5)$, $(2,6)$ ... $(x, x+4)$

There are 3 kind of equivalence class