CSC 3190 Introduction to Discrete Mathematics and Algorithms

Assignment 2

Due Date : Nov 15, 2020 (Sunday) 5:00pm

1 Prove the followings by induction:
  (a) $11^n - 6$ is divisible by 5 for all $n \geq 1$.
  (b) $3^n + 7^n - 2$ is divisible by 8 for all $n \geq 1$.
  (c) The number of subsets of $S = \{1, 2, \ldots, n\}$ with even cardinality is 2^{n-1}, for all $n \geq 1$.

2. Use generating function to solve the following:
  (a) In how many ways can 25 identical donuts be distributed to four men so that each man gets at least three but no more than seven donuts?
  (b) Find the number of ways to select 14 balls from a jar containing 100 red balls, 100 blue balls and 100 green balls so that no fewer than 3 and no more than 10 blue balls are selected. Assume that the order in which the balls are drawn does not matter.

3. Prove that at least one of the real numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

4. Show the following:
  (a) $3n^2 - 7n + 1 = \Theta(n^2)$
  (b) $n! = O(n^n)$
  (c) $2^n = O(n!)$

5. Write the pseudo-code of a recursive algorithm to find the maximum of a finite sequence of numbers. Analyze the complexity of your algorithm.

6. Consider the following algorithm for sorting n numbers for $n \geq 2$.
  (i) Use $2n$-3 comparisons to determine the largest and second largest of the n numbers.
  (ii) Recursively, sort the remaining n-2 numbers.
  Write pseudo-codes for the above algorithm (including both part (i) and (ii)), and analyze its complexity (big-O).

7. A Double Tower of Hanoi contains $2n$ disks of n different sizes, two of each size. As usual, we can move only one disk at a time, without putting a larger one over a smaller one. Assume that disks of equal size are indistinguishable from each other:
  (a) Write a pseudo-code to solve the above problem recursively.
  (b) What is the complexity of your algorithm?

8. (a) Devise a recursive algorithm to compute a^{2^n}, where a is a real number and n is a positive integer (Hint: Use the equality $a^{2^{n+1}} = (a^{2^n})^2$). What is the complexity of your algorithm?
  (b) Repeat the above with an iterative algorithm.

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(b) $3^n + 7^n - 2$ is divisible by 8 for all $n \geq 1$.
(c) The number of subsets of $S = \{1, 2, \ldots, n\}$ with even cardinality is 2^{n-1}, for all $n \geq 1$.

a) let $P(n)$ be $11^n - 6 = 5r$

$P(1) = 5$ which is true because $5/5 = 1$

Assume $P(K)$ is true

$\therefore P(k+1) = 11^{k+1} - 6$

$$
\begin{aligned}
11^{k+1} - 6 &= 11^k \cdot 11 - 6 \\
&= 11^k \cdot (5+6) - 6 \\
&= 11^k \cdot (5) + 11^k \cdot 6 - 6 + 30 - 30 \\
&= 11^k \cdot (5) + 6(11^k - 1 - 5) + 30 \\
&= 5(11^k) + 6(11^k - 6) + 30 \\
&= 5(11^k) + 6(5r) + 5(6) \\
&= 5(11^k + 6r + 6) \text{ which is divisible by } 5
\end{aligned}
$$

$\therefore 11^n - 6$ is divisible by 5 for all $n \geq 1$ by induction

b) let $P(n)$ be $3^n + 7^n - 2 = 8r$

$P(1) = 3 + 7 - 2 = 8$ which is divisible by 8

Assume $P(K)$ is true,

$P(k+1) = 3^{k+1} + 7^{k+1} - 2$

$$
\begin{aligned}
&= 3(3^k + 7^k) + 7^k \cdot 4 - 2 - 2 \cdot 2 + 2 \cdot 2 \\
&= 3(3^k + 7^k - 2) + 7^k \cdot 4 + 4 \\
&= 3(3^k + 7^k - 2) + 8(7^k \cdot \tfrac{1}{2} + \tfrac{1}{2}) \\
&= 3(8r) + 8(7^k \cdot \tfrac{1}{2} + \tfrac{1}{2}) \\
&= 8(3r + 7^k \cdot \tfrac{1}{2} + \tfrac{1}{2}) \text{ which is divisible by } 8
\end{aligned}
$$

$\therefore 3^n + 7^n - 2$ is divisible by 8 by induction

$P(n)$ be sub set of S wher is $\binom{n}{0} + \binom{n}{2} + \ldots + \binom{n}{2m}$ where $2m \leq n$

c) For $n = 1$

$2^{1-1} = 1$ $\quad |S| = 1$ whose cardinality is even because ϕ is even

For $P(k)$

$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = (1+1)^n = 2n$

Since $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = (1-1)^n = 0$

$\therefore \binom{n}{0} + \binom{n}{2} + \ldots = \binom{n}{1} + \binom{n}{3} + \ldots$

Since both number in ~~odd~~ odd same with number in even element,
both of these no. $= \frac{1}{2} \cdot 2^n = 2^{n-1}$ when $n > 0$

Assum $P(K)$ is true $\quad = (1+1)^{k+1} = 2^{k+1}$

set of $(k+1) = \binom{k+1}{0} + \binom{k+1}{1} + \ldots + \binom{k+1}{k+1}$

for $P(k+1) = \binom{k+1}{0} + \ldots \binom{k+1}{k}$ if $k+1 \neq$ even & $P(k+1) = \binom{k+1}{0} + \ldots \binom{k+1}{k+1}$

$\therefore P(k+1) = \frac{1}{2} 2^{k+1} = 2^{(k+1)-1}$ for half of 2^{k+1} is odd element

which is true ~~a~~ for $P(K) = 2^{k-1}$

$\therefore$ It is true by math induction

2. Use generating function to solve the following:
   (a) In how many ways can 25 identical donuts be distributed to four men so that each man gets at least three but no more than seven donuts?
   (b) Find the number of ways to select 14 balls from a jar containing 100 red balls, 100 blue balls and 100 green balls so that no fewer than 3 and no more than 10 blue balls are selected. Assume that the order in which the balls are drawn does not matter.

3 to 10

11 to 4

a) $(x^3+x^4+x^5+x^6+x^7)^4$

$= (x^{14}+2x^{13}+3x^{12}+4x^{11}+5x^{10}+4x^9+3x^8+2x^7+x^6)^2$

$= \ldots + (4\cdot1+2\cdot3+3\cdot2+4\cdot1)x^{25}+\ldots$

$= \ldots + 20x^{25} + \ldots$

$\therefore$ There are 20 way

14 11

13 12

12 13

11 14

10 15

$\vdots$

b) First, determine the no. of way select blue ball.

Since order of ball drawn is not matter,

the series to representing blue ball:

$x^3+x^4+\ldots+x^{10}$

Since no restriction in drawing Red/green ball.

the series:

$1+x+x^2+\ldots$

$\therefore$ The total way of drawing ball:

$(x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10})(1+x+x^2+\ldots)^2$

$= x^3\left(\sum_{k=0}^{7} x^k\right)\left(\sum_{k=0}^{x=\infty} x^k\right)^2$

$= x^3\left(\frac{1-x^8}{1-x}\right)\left(\frac{1}{1-x}\right)^2$

$= x^3(1-x^8)(1-x)^{-3}$

$= x^3(1-x^8)\sum_{k=0}^{\infty}\binom{-3}{k}(-x)^k$

$= x^3\sum_{m=0}^{\infty}\binom{-3}{m}(-1)^m - x^{11}\sum_{k=0}^{\infty}\binom{-3}{k}(-1)^k x^k$

let $b_m = \binom{-3}{m}(-1)^m$ & $c_k = \binom{-3}{k}(-1)^k$

$= x^3\sum_{m=0}^{\infty} b_m x^m - x^{11}\sum_{k=0}^{\infty} c_k x^k$

To find coefficent of x^{12} (Since at most 11 blue ball), $m=11$ & $k=3$

$a_{12} = b_{11} - c_3 = 78 - 10 = 68$

3. Prove that at least one of the real numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

4. Show the following:
(a) $3n^2 - 7n + 1 = \Theta(n^2)$
(b) $n! = O(n^n)$
(c) $2^n = O(n!)$

3) Assume there are list $a_1, a_2, \ldots a_n$
assume that there are none of real no. greater/equal than average of no.

$\therefore \quad a_1 < \frac{a_1 + a_2 + \ldots + a_n}{n}$

$a_2 < \frac{a_1 + a_2 + \ldots + a_n}{n}$

$a_n < \frac{a_1 + a_2 + \ldots + a_n}{n}$

Add all & we can get,

$a_1 + a_2 + \ldots + a_n < a_1 + a_2 + \ldots + a_n$

Since the inequality can't be true, and there is contradiction that none of number greater/equal to average of number,
$\therefore$ There are at least one number greater/equal to average of these number

The prove is by contradiction

4) a)

$3n^2 - 7n + 1 \le 3n^2 + 7n + 1$
$\le 3n^2 + 7n^2 + n^2$
$\le 11n^2$ for every integer $n \ge$
$\therefore C = 11 \quad N = 1$ By definition,
$3n^2 - 7n + 1$ in $O(n^2)$

$3n^2 - 7n + 1$
$2n^2 + 1 + n^2 - 7n$
For $n^2 - 7n > 0$
$n_1 > 7$
For $c_1 = 2$,
Then for all $n \ge n_1$
$3n^2 - 7n + 1$
$= 2n^2 + (n^2 - 7n) + 1$
$\ge 2n^2 + 1 \quad (\because n > 7)$
$\ge 2n^2 = cn^2$
By definit, $3n^2 - 7n + 1$ in $\Omega(n^2)$
$\therefore$ By $O(n^2)$ & $\Omega(n^2)$
$\therefore 3n^2 - 7n + 1$ in $\Theta(n^2)$

b) For $n = 1$
$n! = 1 \quad n^n = 1 \quad \therefore 1 \le 1$ is true

$n! = n \cdot (n-1)(n-2) \ldots (3)(2)(1)$
$n^n = (n)(n)(n) \ldots (n)(n)(n)$
$\because$ Both have n item & $n > n - r$ where $r \ge 1$
$\therefore n^n \ge n!$ is true for all $n \ge 1$
$\therefore n! \in O(n^n)$

c) For $n = 1$
$2^1 = 2 \quad n! = 2$
$\therefore 2 \le 2$ is true
$2^n = 2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2$
$n! = n \cdot (n-1) \cdot \ldots \cdot 1$
Since Both have n case but $n - r > 2$ when $n \ge 4$
$n! \ge 2^n$ when $n \ge 4$
$\therefore 2^n = O(n!)$

5. Write the pseudo-code of a recursive algorithm to find the maximum of a finite sequence of numbers. Analyze the complexity of your algorithm.

6. Consider the following algorithm for sorting n numbers for $n \geq 2$.
   (i) Use $2n$-3 comparisons to determine the largest and second largest of the n numbers.
   (ii) Recursively, sort the remaining n-2 numbers.
   Write pseudo-codes for the above algorithm (including both part (i) and (ii)), and analyze its complexity (big-O).

5)
```
Max(a1, a2, ... an : integer n≥1)
if n = 1
  max = a1
if n > 1
  for i:=2 to n
    if max < ai
    then max := ai

return max
```

Complexity $= O(n-1)$

6)
```
Find (a1, a2, ... an : integer n≥1)
If a1 > a2
  Max = a1  Second = a2
Else
  Max = a2, Second = a1

For i := 3 to n
 if ai > Max
   Second = Max  Max = ai
else
   if ai > Second
   second = ai
return Max, Second
```

Complexity $= O(n)$

```
(ii) int Find ( (a1, a2, ... an), index , Max, Second )
  if index = n
     return Max, Second

  int temp = a index
   if temp > Second
        if temp > Max
           Find ((a1, a2, ... an) , index+1, temp, Second
        else
           Find (a1, a2, ..., an) , index+1, Max, temp)
  return Find ((a1, a2, ... an), index+1, Max, Second )
```

$O(n)$