4 Finite Axiomatisability of LD Depends on τ

In this section, we will show that whether LD has a finite sound and complete axiomatisation depends on the value of τ: if $\tau = 2$ or $\tau = 3$, then there exists a finite sound and complete axiomatisation; if $\tau > 3$, then LD is not finitely axiomatisable.

4.1 When $\tau = 2$

The following calculus LD^2 is sound and complete for LD^2.

AS 0 All tautologies of classical propositional logic

AS 1 $\neg W(a,a)$;

AS 2 $E(a,b) \leftrightarrow W(b,a)$;

AS 3 $I_{ew}(a,b) \rightarrow I_{ew}(b,a)$;

AS 4 $I_{ew}(a,b) \leftrightarrow (\neg dE(a,b) \wedge \neg dW(a,b))$;

AS 5 $W(a,b) \wedge W(b,c) \rightarrow W(a,c)$;

AS 6 $\neg dE(a,b) \wedge W(b,c) \rightarrow \neg E(a,c)$;

AS 7 $W(a,b) \wedge \neg dE(b,c) \rightarrow \neg E(a,c)$;

AS 8 $dW(a,b) \wedge \neg E(b,c) \rightarrow W(a,c)$;

AS 9 $\neg E(a,b) \wedge dW(b,c) \rightarrow W(a,c)$;

AS 10 $W(a,b) \wedge \neg E(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 11 $\neg E(a,b) \wedge W(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 12 $dW(a,b) \wedge \neg dE(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 13 $\neg dE(a,b) \wedge dW(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 14 $\neg S(a,a)$;

AS 15 $N(a,b) \leftrightarrow S(b,a)$;

AS 16 $I_{ns}(a,b) \rightarrow I_{ns}(b,a)$;

AS 17 $I_{ns}(a,b) \leftrightarrow (\neg dN(a,b) \wedge \neg dS(a,b))$;

AS 18 $S(a,b) \wedge S(b,c) \rightarrow S(a,c)$;

AS 19 $\neg dN(a,b) \wedge S(b,c) \rightarrow \neg N(a,c)$;

AS 20 $S(a,b) \wedge \neg dN(b,c) \rightarrow \neg N(a,c)$;

AS 21 $dS(a,b) \wedge \neg N(b,c) \rightarrow S(a,c)$;

AS 22 $\neg N(a,b) \wedge dS(b,c) \rightarrow S(a,c)$;

AS 23 $S(a,b) \wedge \neg N(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{S, dS, \neg N, \neg dN\}$;

AS 24 $\neg N(a,b) \wedge S(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{S, dS, \neg N, \neg dN\}$;

AS 25 $dS(a,b) \wedge \neg dN(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{S, dS, \neg N, \neg dN\}$;

AS 26 $\neg dN(a,b) \wedge dS(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{S, dS, \neg N, \neg dN\}$;

MP Modus ponens: $\phi,\ \phi \rightarrow \psi \vdash \psi$.

5 Decidability and Complexity of LD

We show that for every $\tau \in \mathbb{N}_{>1}$, the satisfiability problem for LD^τ is NP-complete.

Lemma 5. *For every $\tau \in \mathbb{N}_{>1}$, let S be a set of linear inequalities obtained by applying the 'τ-σ-translation' function over $L(LD)$ formulas as shown in Definition 3, where $\sigma = 1$; n be the number of variables in S, $n > 0$. If S is satisfiable, then it has a solution where for every variable, a rational number $t \in [-n\tau, n\tau]$ is assigned to it and the binary representation size of t is polynomial in n and τ.*

Proof. Take an arbitrary $\tau \in \mathbb{N}_{>1}$. By Definition 3, every linear inequality in S is of the form $x_1 - x_2 \leq c$ or $x_1 - x_2 < c$, where x_1, x_2 are real variables and c is a real number constant. Let G be a graph for S. By Corollary 1, S is satisfiable iff G has no simple infeasible loop. The construction of a solution of S is by extending the proof of Theorem 1 [Shostak, 1981] (pp. 777 and 778), which is for non-strict inequalities only, to include both strict and non-strict inequalities. If G has no simple infeasible loop, a solution of S can be constructed as follows. Let $v_1, \dots, v_{n-1}$ be the variables of S other than v_0 (the zero variable). We construct a sequence $\hat{v}_0, \hat{v}_1, \dots, \hat{v}_{n-1}$ of reals (a solution of S) and a sequence $G_0, G_1, \dots, G_{n-1}$ of graphs inductively:

1. Let $\hat{v}_0 = 0$ and $G_0 = G$.
2. If $\hat{v}_i$ and G_i have been determined for $0 \leq i < j < n$, let

$\sup_j = \min\{\frac{c_P}{a_P} \mid P$ is an admissible path from v_j to v_0 in G_{j-1} and $a_P > 0\}$

$\inf_j = \max\{\frac{c_P}{b_P} \mid P$ is an admissible path from v_0 to v_j in G_{j-1} and $b_P < 0\}$

where $\min \emptyset = \infty$ and $\max \emptyset = -\infty$. The range of $\hat{v}_j$ is obtained as follows.

- If there is an admissible path P from v_j to v_0 in G_{j-1} such that the residue inequality of P is $a_P v_j < c_P$, where $a_P > 0$, and $\frac{c_P}{a_P} = \sup_j$, then $\hat{v}_j < \sup_j$, otherwise, $\hat{v}_j \leq \sup_j$.
- If there is an admissible path P from v_0 to v_j in G_{j-1} such that the residue inequality of P is $b_P v_j < c_P$, where $b_P < 0$, and $\frac{c_P}{b_P} = \inf_j$, then $\hat{v}_j > \inf_j$, otherwise, $\hat{v}_j \geq \inf_j$.

Instead of letting $\hat{v}_j$ be any real number in the range [Shostak, 1981], we assign a value to $\hat{v}_j$ thus:

- if there exists an integer within the range of $\hat{v}_j$, we assign an integer to $\hat{v}_j$;
- otherwise, the range of $\hat{v}_j$ is of the form $\inf_j < \hat{v}_j < \sup_j$. Let $\hat{v}_j = \frac{\inf_j + \sup_j}{2}$.

Let G_j be obtained from G_{j-1} by adding two new edges from v_j to v_0, labelled $v_j \leq \hat{v}_j$ and $v_j \geq \hat{v}_j$ respectively.

To ensure that $\hat{v}_j$ and G_j are well defined, we need the following two claims:

1. For $1 \leq j < n$, the range of $\hat{v}_j$ is not empty.
2. For $0 \leq j < n$, G_j has no simple infeasible loop.

We prove them by induction on j, similar to the proof presented in [Shostak, 1981].

Base case $j = 0$. 1 holds vacuously; 2 holds since $G_0 = G$.