A Logic of Directions

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Abstract

We propose a logic of directions for points (LD) over 2D Euclidean space, which formalises primary direction relations east (E), west (W), and indeterminate east/west (I_{ew}), north (N), south (S) and indeterminate north/south (I_{ns}). We provide a sound and complete axiomatisation of it, and prove that its satisfiability problem is NP-complete.

1 Introduction

This work is motivated by the problem of matching spatial objects represented in different geospatial datasets and verifying the consistency of matching relations. A matching relation states that a spatial object in one dataset is the same as or part of a spatial object in the other dataset. In different datasets, the same real world object is usually represented using different geometries or coordinates. Previously, we proposed a number of qualitative spatial logics (a logic of NEAR and FAR for buffered points, a logic of NEAR and FAR for buffered geometries and a logic of Part and Whole for buffered geometries) which were developed to reason about distance relations between spatial objects from different datasets, tolerating slight differences in their geometric representations [Du *et al.*, 2013; Du and Alechina, 2016]. These spatial logics have been used to validate matching relations regarding the distance relations between spatial objects. The intuition is that two spatial objects which are definitely close in one dataset cannot be matched to two spatial objects which are definitely far away in the other dataset. However, these spatial logics do not cover the direction aspect, which is an important dimension of spatial relations. In this work, we propose a new spatial logic for validating matching relations with respect to direction relations between spatial objects. Using the relations defined in the new logic, the following intuition can be formalised: if a spatial object a is definitely to the east of a spatial object b in one dataset, then the spatial object corresponding to a in the other dataset cannot be definitely to the west of the spatial object corresponding to b.

Consider the case where every spatial object is represented as a single point. We assume that the distance between every pair of corresponding points from different datasets is less than or equal to a positive real number σ. σ is referred to as a *margin of error*. The value of σ can be determined empirically by comparing two geospatial datasets representing the same objects, and finding the largest 'distortion' which exists between any pair of objects. With respect to a point p, if a point q is within the bounding box of the σ-buffer of p (the σ-buffer contains exactly all the points within σ distance of p), then q is considered to be too close to talk about its exact direction. We say that q is not to the north, not to the south, not to the east and not to the west of p. In the logic of NEAR and FAR for buffered points [Du *et al.*, 2013], two points are *NEAR*, if their distance is within 2σ; two points are *FAR*, if their distance is greater than 4σ. A gap is left between *NEAR* and *FAR* so that two points are not *NEAR* and not *FAR*, if their distance is greater than 2σ and within 4σ. Similar to the way in which the relations *NEAR* and *FAR* were defined, we will leave some gaps or indeterminate regions between definite directions like definitely east and definitely west. E.g. for two points p, q with x coordinates x_p, x_q, we can define the three relations definitely east, not east and not west, and definitely west, as $x_p - x_q > 3\sigma$ (p is definitely to the east of q), $-\sigma \leq x_p - x_q \leq \sigma$ (p is not to the east and not the west of q) and $x_p - x_q < -3\sigma$ (p is definitely to the west of q) respectively. Instead of introducing a constant 3, we introduce another parameter $\tau > 1$ to represent gaps or indeterminate ranges or regions. The parameter τ is referred to as the level of indeterminacy in directions. For points p, q, if x_p and x_q are within $\tau\sigma$ distance, then the direction relation between points p, q are not definitely east nor definitely west. Following this initial idea, with respect to a central point $p = (0, 0)$, we divide the 2D Euclidean space into 25 totally or partially bounded regions (see Figure 1). Points in different regions have different direction relations with the central red point p. E.g. for any point q in region 1, q is definitely to the north and definitely to the west of p. The question is how to define the 25 different direction relations formally and provide a sound and complete axiomatisation to reason with them.

Several qualitative spatial or temporal calculi have been developed for formalizing and reasoning about direction or ordering relations [Aiello *et al.*, 2007; Ligozat, 2012]. These include the point calculus [Vilain and Kautz, 1986] which defines three ordering relations $<$ (less than), $>$ (greater than) and eq (equal) for points in a 1D Euclidean space, Allen's calculus [Allen, 1983], the cardinal direction calculus (CDC) which extends the point calculus to 2D Euclidean

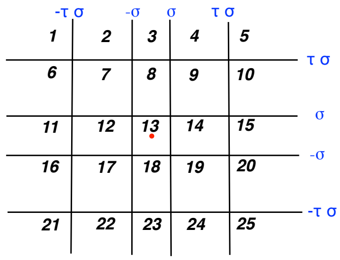

Figure 1: The 2D Euclidean space is divided into 25 totally or partially bounded regions. The red dot in region 13 is the central point $p = (0, 0)$.

	dW	sW	nEW	sE	dE
dN	$dNdW$	$dNsW$	$dNnEW$	$dNsE$	$dNdE$
sN	$sNdW$	$sNsW$	$sNnEW$	$sNsE$	$sNdE$
nNS	$nNSdW$	$nNSsW$	$nNSnEW$	$nNSsE$	$nNSdE$
sS	$sSdW$	$sSsW$	$sSnEW$	$sSsE$	$sSdE$
dS	$dSdW$	$dSsW$	$dSnEW$	$dSsE$	$dSdE$

Table 1: 25 jointly exhaustive and pairwise disjoint direction relations. Each entry in the table corresponds to the spatially corresponding entry in Figure 1, e.g. $nNSsW$ corresponds to entry 12.

space [Ligozat, 1998], the rectangle algebra [Balbiani *et al.*, 1998], the $2n$-star calculi which generalize the cardinal direction calculus by introducing a variable n referring to the granularity or the degree of refinement for defining direction relations [Renz and Mitra, 2004], and cardinal direction relations between regions [Goyal and Egenhofer, 1997; Skiadopoulos and Koubarakis, 2004; Skiadopoulos and Koubarakis, 2005]. Beside these formalisms where direction or ordering relations are defined using binary relations, there exist several spatial formalisms which define direction relations using ternary relations. These spatial formalisms include the $\mathcal{LR}$ calculus [Scivos and Nebel, 2004], the flip-flop calculus [Ligozat, 1993], the double-cross calculus [Freksa, 1992], the 5-intersection calculus [Billen and Clementini, 2004], etc., where relations like left, right, after, between, before, etc. are defined.

In this paper, we propose a logic of directions for points (LD) over 2D Euclidean space for defining and reasoning about the direction relations shown in Figure 1. Differing from the cardinal direction calculus, in the logic LD, we define direction relations with respect to the margin of error σ for tolerating slight differences in geometric representations in different geospatial datasets/maps, and the level of indeterminacy in directions τ. Over Euclidean spaces, there exist some sound and complete axiomatisations for spatial formalisms [Szczerba and Tarski, 1979; Balbiani *et al.*, 2007; Tarski, 1959; Tarski and Givant, 1999; Trybus, 2010]; however, none of them considers direction relations. Here we provide a sound and complete axiomatisation for the spatial logic LD which formalises direction relations between points. Some spatial logics, which can encode directions, are undecidable, e.g. the compass logic [Marx and Reynolds, 1999] and SpPNL [Morales *et al.*, 2007]. The satisfiability problem of some spatial logics (e.g. Cone [Montanari *et al.*, 2009] and SOSL [Walega and Zawidzki, 2019]) are PSPACE-complete. Here we show that the satisfiability problem of LD is NP-complete.

The logic LD could be used for checking consistency of *sameAs* matches between two real world geospatial datasets (e.g. Ordnance Survey of Great Britain and OpenStreetMap data) regarding direction information. A sound and complete axiomatisation of LD is an important and useful tool for developing an automated reasoner and performing automated *axiom pinpointing* [Baader and Peñaloza, 2010] for debugging matches between geospatial objects, as was done, for example, in [Du *et al.*, 2015] for the logic of Part and Whole for buffered geometries.

2 A Logic of Directions For Points

We present a logic of directions for points (LD), which defines six primary direction relations: east (E), west (W), and indeterminate east/west (I_{ew}), north (N), south (S) and indeterminate north/south (I_{ns}). LD is a family of logics LD^τ parameterised by a level of indeterminacy parameter τ.

Let A be a finite set of individual names. The language $L(LD, A)$ (we omit A for brevity below) is defined as

$$\phi, \psi := E(a,b) \mid W(a,b) \mid I_{ew}(a,b) \mid N(a,b) \mid S(a,b) \mid \mid I_{ns}(a,b) \mid \neg\phi \mid \phi \wedge \psi$$

where $a, b \in A$, $\phi \vee \psi =_{def} \neg(\neg\phi \wedge \neg\psi)$, $\phi \to \psi =_{def} \neg(\phi \wedge \neg\psi)$, $\phi \leftrightarrow \psi =_{def} (\phi \to \psi) \wedge (\psi \to \phi)$, $\perp =_{def} \phi \wedge \neg\phi$.

We interpret $L(LD)$ over 2D Euclidean models based on the 2D Euclidean space $\mathbb{R}^2$. Models of LD^τ are called τ-models.

Definition 1 (2D Euclidean τ-model of LD^τ)**.** *A 2D Euclidean τ-model M is a tuple $(\mathcal{I}, \sigma, \tau)$, where $\mathcal{I}$ is an interpretation function which maps each individual name in A to an element of $\mathbb{R}^2$, $\sigma \in \mathbb{R}_{>0}$ is a margin of error, and $\tau \in \mathbb{N}_{>1}$ refers to the level of indeterminacy in directions. The notion of $M \models_{LD} \phi$ (a formula ϕ of LD is true in τ-model M) is defined as follows:*

$M \models_{LD} E(a,b)$ *iff* $x_a - x_b > \sigma$;

$M \models_{LD} W(a,b)$ *iff* $x_a - x_b < -\sigma$;

$M \models_{LD} I_{ew}(a,b)$ *iff* $-\tau\sigma \leq x_a - x_b \leq \tau\sigma$;

$M \models_{LD} N(a,b)$ *iff* $y_a - y_b > \sigma$;

$M \models_{LD} S(a,b)$ *iff* $y_a - y_b < -\sigma$;

$M \models_{LD} I_{ns}(a,b)$ *iff* $-\tau\sigma \leq y_a - y_b \leq \tau\sigma$;

$M \models_{LD} \neg\phi$ *iff* $M \not\models_{LD} \phi$;

$M \models_{LD} \phi \wedge \psi$ *iff* $M \models_{LD} \phi$ *and* $M \models_{LD} \psi$,

where $a, b \in A$, $\mathcal{I}(a) = (x_a, y_a)$, $\mathcal{I}(b) = (x_b, y_b)$, ϕ, ψ are formulas in $L(LD)$.

τ is defined as a natural number rather than a real in order to facilitate the proof of Lemma 19. In practice, an integer τ is always likely to be sufficiently expressive.

The notions of τ-validity and τ-satisfiability of LD formulas in 2D Euclidean τ-models are standard. An $L(LD)$ formula is τ-satisfiable if it is true in some 2D Euclidean τ-model. An $L(LD)$ formula ϕ is τ-valid ($\models^\tau_{LD} \phi$) if it is true in all 2D Euclidean τ-models (hence if its negation is not τ-satisfiable). The logic LD^τ is the set of all τ-valid formulas of $L(LD)$.

As shown by Lemma 1 below, σ is a scaling factor.

Lemma 1. *For every $\tau \in \mathbb{N}_{>1}$, $\sigma_1, \sigma_2 \in \mathbb{R}_{>0}$, if an $L(LD)$ formula ϕ is true in a 2D Euclidean τ-model $M = (\mathcal{I}, \sigma_1, \tau)$, then it is true in a 2D Euclidean τ-model $M' = (\mathcal{I}', \sigma_2, \tau)$ such that $\mathcal{I}(a) = (x_a, y_a)$ iff $\mathcal{I}'(a) = (\frac{x_a\sigma_2}{\sigma_1}, \frac{y_a\sigma_2}{\sigma_1})$.*

The proof is by straightforward verification of truth conditions in Definition 1.

We introduce the following definitions as 'syntactic sugar'.

Definition 2.

definitely east $dE(a, b) =_{def} E(a, b) \wedge \neg I_{ew}(a, b)$

somewhat east $sE(a, b) =_{def} E(a, b) \wedge I_{ew}(a, b)$

neither east nor west $nEW(a, b) =_{def} \neg E(a, b) \wedge \neg W(a, b)$

somewhat west $sW(a,b) =_{def} W(a,b) \wedge I_{ew}(a,b)$

definitely west $dW(a,b) =_{def} W(a,b) \wedge \neg I_{ew}(a,b)$

definitely north $dN(a, b) =_{def} N(a, b) \wedge \neg I_{ns}(a, b)$

somewhat north $sN(a, b) =_{def} N(a, b) \wedge I_{ns}(a, b)$

neither north nor south $nNS(a, b) =_{def} \neg N(a, b) \wedge \neg S(a, b)$

somewhat south $sS(a, b) =_{def} S(a, b) \wedge I_{ns}(a, b)$

definitely south $dS(a,b) =_{def} S(a,b) \wedge \neg I_{ns}(a,b)$

The definitions of definite or somewhat direction relations have $\tau \in \mathbb{N}_{>1}$ as a parameter. By Definitions 1 and 2, $M \models_{LD} dE(a, b)$ iff $(x_a - x_b) \in (\tau\sigma, \infty)$; $M \models_{LD} sE(a, b)$ iff $(x_a - x_b) \in (\sigma, \tau\sigma]$. Let us call $(\tau\sigma, \infty)$ the range of $dE(a, b)$, $(\sigma, \tau\sigma]$ the range of $sE(a, b)$. As τ decreases, the range of $dE(a, b)$ becomes wider, the range of $sE(a, b)$ becomes narrower. If τ is allowed to be 1, then $dE(a, b) \equiv E(a, b)$ and $sE(a, b) \equiv \perp$. τ plays a similar role in defining other definite or somewhat direction relations.

There exist $5 \times 5 = 25$ jointly exhaustive and pairwise disjoint relations, which can be defined using the primary relations in the logic LD. The 25 direction relations are shown in Table 1. Each of them is defined as a conjunction of one of the relations dW, sW, nEW, sE, dE and one of the relations dN, sN, nNS, sS, dS. These 25 direction relations correspond to the 25 regions shown in Figure 1. For instance, with respect to the central point p, for any point q in region 2, we have $dNsW(q, p)$ (q is definitely to the north and somewhat to the west of p).

Similar to the logic LD, we could define a logic over 3D or higher Euclidean space. If we only use east and west (or north and south), we get a logic $LD1$ over 1D Euclidean space. The soundness, completeness, decidability and complexity results can be obtained similarly. The point calculus and the Cardinal Direction Calculus can be seen as a special case of $LD1$ and LD respectively, if σ is allowed to be 0. Finally, we observe that there exist different (from LD) extensions of the point calculus and Allen's calculus, for example, introducing the concept of granularity [Cohen-Solal *et al.*, 2015]; a granularity is defined as a sequence of sets of time points where the natural order of the time points are preserved.

3 A Complete Axiomatisation for LD

Here we will first describe some results for systems of linear inequalities that are used later in the proofs. Then for each level of indeterminacy τ, we present an axiomatisation (a set of axioms) of LD^τ, and prove soundness and completeness of the axiomatisation.

3.1 Deciding Linear Inequalities by Computing Loop Residues

We recap the definitions from [Shostak, 1981]. Let S be a set of linear inequalities of the form $ax + by \leq c$, where x, y are real variables and a, b, c are reals. Without loss of generality, we assume one of the variables in S, denoted as v_0, is special, appearing only with coefficient zero. It is called the 'zero variable'. All other variables in S have nonzero coefficients.

The graph for S, denoted as G, is constructed as follows. G contains a vertex for each variable in S and an edge for each inequality, where each vertex is labelled with its associated variable and each edge is labelled with its associated inequality. For example, the edge labelled with $ax + by \leq c$ connects the vertex labelled with x and the vertex labelled with y.

Let P be a path through G, given by a sequence $v_1, \dots, v_{n+1}$ of vertices and a sequence $e_1, \dots, e_n$ of edges, $n \geq 1$. The triple sequence for P is

$$(a_1, b_1, c_1), (a_2, b_2, c_2), \dots, (a_n, b_n, c_n)$$

where for each $i \in [1, n]$, $a_i v_i + b_i v_{i+1} \leq c_i$ is the inequality labelling e_i. A path is a *loop* if its first and last vertices are the same. A loop is *simple* if its intermediate vertices are distinct. P is *admissible* if for $i \in [1, n-1]$, b_i and a_{i+1} have opposite signs (one is strictly positive and the other is strictly negative). Definitions and results that follow apply to admissible paths.

The *residue inequality* of an admissible path P is defined as the inequality obtained from P by applying transitivity to the inequalities labelling its edges. The *residue* r_p of P is defined as the triple (a_p, b_p, c_p),

$$(a_p, b_p, c_p) = (a_1, b_1, c_1) * (a_2, b_2, c_2) * \cdots * (a_n, b_n, c_n)$$

where $(a_1, b_1, c_1), \dots, (a_n, b_n, c_n)$ is the triple sequence for P and $*$ is the binary operation on triples defined by

$$(a, b, c) * (a', b', c') = (kaa', -kbb', k(ca' - c'b))$$

where $k = a'/|a'|$. The *residue inequality* of P is $a_P x + b_P y \leq c_P$, where x, y are the first and last vertices of P.

Lemma 2. *[Shostak, 1981] Any point (i.e. assignment of reals to variables) that satisfies the inequalities labelling on admissible path P also satisfies the residue inequality of P.*

Let P be an admissible loop with initial vertex x. By Lemma 2, any point satisfying the inequalities along P also satisfies $a_P x + b_P x \leq c_P$. If $a_P + b_P = 0$ and $c_p < 0$, then the residue inequality of P is false, and P is called an *infeasible loop*.

Let G be the graph for S. A *closure* G' of G is obtained by adding, for each simple admissible loop P (modulo permutation and reversal) of G, a new edge labelled with the residue inequality of P. A graph is *closed* if it is a closure of itself.

Theorem 1. *[Shostak, 1981] Let S be a set of linear inequalities of the form $ax+by \leq c$, where x, y are real variables and a, b, c are real number constants; let G be a closed graph for S. Then S is satisfiable iff G has no simple infeasible loop.*

Theorem 1 is about inequalities of the form $ax + by \leq c$ only. It was extended to include both strict and non-strict inequalities [Shostak, 1981]. We say an admissible path is *strict* if one or more of its edges is labelled with a strict inequality, i.e. an inequality of the form $ax + by < c$. Then a strict admissible loop P with residue (a_P, b_P, c_P) is infeasible, if $a_P + b_P = 0$ and $c_P \leq 0$. Corollary 1 is stated for the case where inequalities are of the form $x - y \leq c$ or $x - y < c$. Lemma 3 is provided to help readers understand Corollary 1. It follows from the definition of closed graph.

Lemma 3. *[Shostak, 1981] Let S be a set of linear inequalities of the form $x - y \leq c$ or $x - y < c$, where x, y are real variables and c is a real number constant. Then the graph for S is closed.*

Corollary 1. *[Litvintchouk and Pratt, 1977; Pratt, 1977; Shostak, 1981] Let S be a set of linear inequalities of the form $x - y \leq c$ or $x - y < c$, where x, y are real variables and c is a real number constant; G be a graph for S. The set S is not satisfiable iff G has a simple infeasible loop.*

3.2 Axiomatising LD

The calculus below (which we will also refer to as LD^τ) is sound and complete for LD^τ (for any τ). Here, a and b are meta variables which may be instantiated by any individual name. There are 13 axiom schemas (AS 0 to AS 12) and one inference rule.

AS 0 All tautologies of classical propositional logic

AS 1 $\neg W(a, a)$;

AS 2 $E(a, b) \leftrightarrow W(b, a)$;

AS 3 $I_{ew}(a, b) \rightarrow I_{ew}(b, a)$;

AS 4 $I_{ew}(a, b) \leftrightarrow (\neg dE(a, b) \wedge \neg dW(a, b))$;

AS 5 For any $n \in \mathbb{N}_{>1}$:
$R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0) \rightarrow \perp$, where for every i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$, and $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$;

AS 6 For any $n \in \mathbb{N}_{>0}$:
$R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_n) \rightarrow W(a_0, a_n)$, where for every i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$, and $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$;

AS 7 $\neg S(a, a)$;

AS 8 $N(a, b) \leftrightarrow S(b, a)$;

AS 9 $I_{ns}(a, b) \rightarrow I_{ns}(b, a)$;

AS 10 $I_{ns}(a, b) \leftrightarrow (\neg dN(a, b) \wedge \neg dS(a, b))$;

AS 11 For any $n \in \mathbb{N}_{>1}$:
$R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0) \rightarrow \perp$, where for every i such that $1 \leq i \leq n$, $R_i \in \{S, dS, \neg N, \neg dN\}$, and $number(S) + \tau * number(dS) = number(\neg N) + \tau * number(\neg dN)$;

AS 12 For any $n \in \mathbb{N}_{>0}$:
$R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_n) \rightarrow S(a_0, a_n)$, where for every i such that $1 \leq i \leq n$, $R_i \in \{S, dS, \neg N, \neg dN\}$, and $number(S) + \tau * number(dS) > number(\neg N) + \tau * number(\neg dN)$;

MP Modus ponens: ϕ, $\phi \rightarrow \psi \vdash \psi$.

In AS 5, 6, 11 and 12, n is the number of conjuncts in the antecedent of an axiom, $number(\alpha)$ denotes the number of occurrences of α in $R_1, \ldots, R_n$. In AS 5 and AS 11, $n > 1$ because at least two conjuncts are required to make an equality like $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$ true. For AS 5, suppose that $n = 4$, $number(W)$, $number(dW)$, $number(\neg E)$ and $number(\neg dE)$ are all equal to 1, then an axiom satisfying this is $W(a_0, a_1) \wedge \neg dE(a_1, a_2) \wedge \neg E(a_2, a_3) \wedge dW(a_3, a_0) \rightarrow \perp$ (the order of the appearance of $W, dW, \neg E, \neg dE$ does not matter).

The notion of τ-derivability $\Gamma \vdash^\tau_{LD} \phi$ in the LD^τ calculus is standard. An $L(LD)$ formula ϕ is τ-derivable if $\vdash^\tau_{LD} \phi$. Γ is τ-inconsistent if for some formula ϕ it τ-derives both ϕ and $\neg\phi$ (otherwise it is τ-consistent).

Theorem 2. *For every $\tau \in \mathbb{N}_{>1}$, the LD^τ calculus is sound and complete for 2D Euclidean τ-models, i.e. $\vdash^\tau_{LD} \phi \Leftrightarrow \models^\tau_{LD} \phi$ (every τ-derivable formula is τ-valid and every τ-valid formula is τ-derivable).*

For every $\tau \in \mathbb{N}_{>1}$, the proof of soundness (every LD τ-derivable formula is τ-valid) is by an easy induction on the length of the derivation of ϕ. By truth definitions of the direction relations (Definition 1), AS 1-12 are valid and modus ponens preserves validity.

In the rest of this section, we prove completeness. We will actually prove that for every $\tau \in \mathbb{N}_{>1}$, *if a finite set of $L(LD)$ formulas Σ is τ-consistent, then there is a 2D Euclidean τ-model satisfying it*. Any finite set of formulas Σ can be rewritten as a formula ψ that is the conjunction of all the formulas in Σ. Σ is τ-consistent iff ψ is τ-consistent ($\nvdash^\tau_{LD} \neg\psi$). If there is a 2D Euclidean τ-model M satisfying Σ, then M satisfies ψ, hence $\not\models^\tau_{LD} \neg\psi$. Therefore, by showing that 'if Σ is τ-consistent, then there exists a 2D Euclidean τ-model satisfying it', we show that 'if $\nvdash^\tau_{LD} \neg\psi$, then $\not\models^\tau_{LD} \neg\psi$'. This shows that $\nvdash^\tau_{LD} \phi \Rightarrow \not\models^\tau_{LD} \phi$ and by contraposition we get completeness.

First, we will show that the truth conditions of any set of $L(LD)$ formulas can be expressed as a set of inequalities of the form $x_1 - x_2 \leq c$ or $x_1 - x_2 < c$.

Lemma 4. *An $L(LD)$ formula of the form $(\neg)E(a, b)$, $(\neg)W(a, b)$, $(\neg)dE(a, b)$, $(\neg)dW(a, b)$, $(\neg)N(a, b)$, $(\neg)S(a, b)$, $(\neg)dN(a, b)$, $(\neg)dS(a, b)$ is τ-satisfiable iff an expression of the form $x_1 - x_2 \leq c$ or $x_1 - x_2 < c$ is satisfiable.*

Proof. Definition 3 shows how to translate such formulas to corresponding inequalities. The translation can be easily verified to correspond to the truth definitions in Definition 1. □

Definition 3 (τ-σ-translation)**.** *The 'τ-σ-translation' function $tr(\tau, \sigma)$ is defined as follows:*

$tr(\tau,\sigma)(E(a,b)) = (x_b - x_a < -\sigma)$;
$tr(\tau,\sigma)(W(a,b)) = (x_a - x_b < -\sigma)$;
$tr(\tau,\sigma)(dE(a,b)) = (x_b - x_a < -\tau\sigma)$;
$tr(\tau,\sigma)(dW(a,b)) = (x_a - x_b < -\tau\sigma)$;
$tr(\tau,\sigma)(N(a,b)) = (y_b - y_a < -\sigma)$;
$tr(\tau,\sigma)(S(a,b)) = (y_a - y_b < -\sigma)$;
$tr(\tau,\sigma)(dN(a,b)) = (y_b - y_a < -\tau\sigma)$;
$tr(\tau,\sigma)(dS(a,b)) = (y_a - y_b < -\tau\sigma)$;
$tr(\tau,\sigma)(\neg\phi) = \neg(tr(\phi))$, *where* $\neg(z_1 - z_2 < c) = (z_2 - z_1 \leq -c)$.

The completeness theorem below is proven by rewriting a consistent $L(LD)$ formula ϕ into disjunctive normal form, where each disjunct ϕ_i is τ-satisfiable, iff a set of linear inequalities S_i is satisfiable, iff the graphs of S_i have no simple infeasible loop (Corollary 1 of Theorem 1). We proceed by contradiction, supposing every such graph has a simple infeasible loop P. From P we can obtain $L(LD)$ formulas as conjuncts in ϕ_i. Applying the axioms, we show $\perp$ is τ-derivable from every ϕ_i, thus $\perp$ is τ-derivable from ϕ, which contradicts that ϕ is τ-consistent.

Theorem 3. *For every $\tau \in \mathbb{N}_{>1}$, if a finite set of $L(LD)$ formulas Σ is τ-consistent, then there is a 2D Euclidean τ-model satisfying it.*

Proof. Take an arbitrary $\tau \in \mathbb{N}_{>1}$. Suppose a finite set of $L(LD)$ formulas Σ is τ-consistent. We obtain Σ' by rewriting every $I_{ew}(a,b)$ in Σ as $\neg dE(a,b) \wedge \neg dW(a,b)$, every $I_{ns}(a,b)$ in Σ as $\neg dN(a,b) \wedge \neg dS(a,b)$. By AS 4 and AS 10, Σ and Σ' are logically equivalent. Σ' can be rewritten as a formula ϕ that is the conjunction of all the formulas in Σ'. We rewrite the $L(LD)$ formula ϕ into disjunctive normal form $\phi_1 \vee \cdots \vee \phi_n$ $(n > 0)$. Then every literal is of one of the forms $E(a,b)$, $W(a,b)$, $dE(a,b)$, $dW(a,b)$, $N(a,b)$, $S(a,b)$, $dN(a,b)$, $dS(a,b)$, or their negations. Then ϕ is satisfiable in a 2D Euclidean τ-model, iff at least one of its disjuncts ϕ_i is τ-satisfiable. We obtain a set of inequalities S_i by translating every literal in a disjunct ϕ_i as in Definition 3. Then the inequalities in S_i are of the form $x_a - x_b < c$, $x_a - x_b \leq c$, $y_a - y_b < c$ or $y_a - y_b \leq c$, where x_a, x_b, y_a, y_b are real variables and c is a real constant. We call variables like x_a, x_b x variables and variables like y_a, y_b y variables. Divide S_i into two sets S_i^x and S_i^y, such that S_i^x and S_i^y contain all the inequalities involving x variables and y variables respectively. By Corollary 1 of Theorem 1, ϕ_i is τ-satisfiable iff the graph G_i^x of S_i^x has no simple infeasible loop and the graph G_i^y of S_i^y has no simple infeasible loop. To show there is a 2D Euclidean τ-model satisfying Σ, it is sufficient to show there exists a disjunct ϕ_i such that the graph G_i^x of S_i^x has no simple infeasible loop and the graph G_i^y of S_i^y has no simple infeasible loop.

We prove this by contradiction. Suppose for every disjunct ϕ_i, the graph G_i^x of S_i^x has a simple infeasible loop (Case 1) or the graph G_i^y of S_i^y has a simple infeasible loop (Case 2). We present the proof for Case 1. Case 2 is similar.

If G_i^x has a simple infeasible loop P, then P is either strict or non-strict. Let m denote the sum of the constants c around the loop P. Based on the definition of infeasible loop, if P is strict, then $m \leq 0$; otherwise, $m < 0$. By Definition 3, if a strict inequality $x_a - x_b < c$ is in S_i^x, then c is equal to $-\sigma$ or $-\tau\sigma$; if a non-strict inequality $x_a - x_b \leq c$ is in S_i^x, then c is equal to σ or $\tau\sigma$, where τ, σ are positive numbers (hence $c > 0$). If P is non-strict, then all the inequalities in it are of the form $x_a - x_b \leq c$ where $c > 0$ and the sum of such c is positive. This contradicts the fact that $m < 0$ for non-strict infeasible loops. Therefore P is strict, hence $m \leq 0$. We consider the two cases where $m = 0$ and $m < 0$ separately.

1. If $m = 0$, then the sum of the constants around the loop P is equal to 0. Without loss of generality, let us assume P consists of vertices $xa_0, xa_1, ..., xa_{n-1}, xa_0$. Since P is admissible, the linear inequalities in P are of the form $(xa_0 - xa_1)?c_1, ..., (xa_{n-1} - xa_0)?c_n$, where ? is $\leq$ or $<$, and for every i such that $1 \leq i \leq n$, c_i is σ, $-\sigma$, $\tau\sigma$ or $-\tau\sigma$. Then we translate the linear inequalities in P to formulas as follows. We translate every linear inequality of the form $x_a - x_b < -\sigma$ to $W(a,b)$; every $x_a - x_b < -\tau\sigma$ to $dW(a,b)$; every $x_a - x_b \leq \sigma$ to $\neg E(a,b)$; every $x_a - x_b \leq \tau\sigma$ to $\neg dE(a,b)$. In this way, from P we obtain a sequence of formulas of the form $R_1(a_0,a_1), ..., R_n(a_{n-1},a_0)$, where for every i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the sum of the constants around P is equal to 0, $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$ and $n \geq 2$. By AS 5, $R_1(a_0,a_1) \wedge ... \wedge R_n(a_{n-1},a_0) \rightarrow \perp$. By Definition 3, for every occurrence of $W(a,b)$ in $R_1(a_0,a_1) \wedge ... \wedge R_n(a_{n-1},a_0)$, it or $E(b,a)$ is a conjunct in ϕ_i; similarly, for every occurrence of $dW(a,b)$, it or $dE(b,a)$ is a conjunct in ϕ_i; for every occurrence of $\neg E(a,b)$, it or $\neg W(b,a)$ is a conjunct in ϕ_i; for every occurrence of $\neg dE(a,b)$, it or $\neg dW(b,a)$ is a conjunct in ϕ_i. By AS 2, $W(a,b) \leftrightarrow E(b,a)$. By Definition 2, AS 2 and AS 3, $dW(a,b) \leftrightarrow dE(b,a)$. Therefore, $\perp$ is τ-derivable from ϕ_i.

2. If $m < 0$, then the sum of the constants around the loop P is negative. In the same way described above, from P we obtain a sequence of formulas of the form $R_1(a_0,a_1), ..., R_n(a_{n-1},a_0)$, where for every i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the sum of the constants around the loop P is negative, $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$ and $n \geq 1$. By AS 6, $R_1(a_0,a_1) \wedge ... \wedge R_n(a_{n-1},a_0) \rightarrow W(a_0,a_0)$. By AS 1, $W(a_0,a_0) \rightarrow \perp$. Following the same argument above, $\perp$ is τ-derivable from ϕ_i.

In each case, $\perp$ is τ-derivable from ϕ_i. Thus every disjunct ϕ_i is not τ-consistent, hence ϕ is not τ-consistent. This contradicts the fact that Σ is τ-consistent. □

4 Finite Axiomatisability of LD Depends on τ

In this section, we will show that whether LD has a finite sound and complete axiomatisation depends on the value of τ: if $\tau = 2$ or $\tau = 3$, then there exists a finite sound and complete axiomatisation; if $\tau > 3$, then LD is not finitely axiomatisable.

4.1 When $\tau = 2$

The following calculus LD^2 is sound and complete for LD^2.

AS 0 All tautologies of classical propositional logic

AS 1 $\neg W(a,a)$;

AS 2 $E(a,b) \leftrightarrow W(b,a)$;

AS 3 $I_{ew}(a,b) \rightarrow I_{ew}(b,a)$;

AS 4 $I_{ew}(a,b) \leftrightarrow (\neg dE(a,b) \wedge \neg dW(a,b))$;

AS 5 $W(a,b) \wedge W(b,c) \rightarrow W(a,c)$;

AS 6 $\neg dE(a,b) \wedge W(b,c) \rightarrow \neg E(a,c)$;

AS 7 $W(a,b) \wedge \neg dE(b,c) \rightarrow \neg E(a,c)$;

AS 8 $dW(a,b) \wedge \neg E(b,c) \rightarrow W(a,c)$;

AS 9 $\neg E(a,b) \wedge dW(b,c) \rightarrow W(a,c)$;

AS 10 $W(a,b) \wedge \neg E(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 11 $\neg E(a,b) \wedge W(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 12 $dW(a,b) \wedge \neg dE(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 13 $\neg dE(a,b) \wedge dW(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 14 $\neg S(a,a)$;

AS 15 $N(a,b) \leftrightarrow S(b,a)$;

AS 16 $I_{ns}(a,b) \rightarrow I_{ns}(b,a)$;

AS 17 $I_{ns}(a,b) \leftrightarrow (\neg dN(a,b) \wedge \neg dS(a,b))$;

AS 18 $S(a,b) \wedge S(b,c) \rightarrow S(a,c)$;

AS 19 $\neg dN(a,b) \wedge S(b,c) \rightarrow \neg N(a,c)$;

AS 20 $S(a,b) \wedge \neg dN(b,c) \rightarrow \neg N(a,c)$;

AS 21 $dS(a,b) \wedge \neg N(b,c) \rightarrow S(a,c)$;

AS 22 $\neg N(a,b) \wedge dS(b,c) \rightarrow S(a,c)$;

AS 23 $S(a,b) \wedge \neg N(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{S, dS, \neg N, \neg dN\}$;

AS 24 $\neg N(a,b) \wedge S(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{S, dS, \neg N, \neg dN\}$;

AS 25 $dS(a,b) \wedge \neg dN(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{S, dS, \neg N, \neg dN\}$;

AS 26 $\neg dN(a,b) \wedge dS(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{S, dS, \neg N, \neg dN\}$;

MP Modus ponens: $\phi,\ \phi \rightarrow \psi \vdash \psi$.

Note that the set of axioms above contains duplicates AS 6 and AS 7. By using AS 2, we can obtain one from the other. Remove duplicates?

$\neg dE(a,b) \wedge W(b,c) \rightarrow \neg E(a,c)$;
$\neg dE(a,b) \wedge W(b,c) \wedge E(a,c) \rightarrow \bot$;
$\neg dE(a,b) \wedge E(c,b) \wedge W(c,a) \rightarrow \bot$;
$W(c,a) \wedge \neg dE(a,b) \rightarrow \neg E(c,b)$;
$W(a,b) \wedge \neg dE(b,c) \rightarrow \neg E(a,c)$;

AS 8 and AS 9 are duplicated as well

$dW(a,b) \wedge \neg E(b,c) \rightarrow W(a,c)$;
$dW(a,b) \wedge \neg E(b,c) \wedge \neg W(a,c) \rightarrow \bot$;
$dW(a,b) \wedge \neg W(c,b) \wedge \neg E(c,a) \rightarrow \bot$;
$\neg E(c,a) \wedge dW(a,b) \rightarrow W(c,b)$;
$\neg E(a,b) \wedge dW(b,c) \rightarrow W(a,c)$;

AS 10.1
$W(a,b) \wedge \neg E(b,c) \wedge W(c,d) \rightarrow W(a,d)$;

AS 10.2
$W(a,b) \wedge \neg E(b,c) \wedge dW(c,d) \rightarrow dW(a,d)$;

$\neg E(a,b) \wedge dW(b,c) \rightarrow W(a,c)$ (AS 9)
$W(a,b) \wedge W(b,c) \rightarrow dW(a,c)$

$W(a,b) \wedge \neg E(b,c) \wedge dW(c,d) \wedge \neg dW(a,d) \rightarrow \bot$;

$W(a,b) \wedge \neg E(b,c) \wedge dW(c,d) \wedge \neg dE(d,a) \rightarrow \bot$;

AS 10.3
$W(a,b) \wedge \neg E(b,c) \wedge \neg E(c,d) \rightarrow \neg E(a,d)$;

$\neg E(a,b) \wedge \neg E(b,c) \rightarrow \neg dE(a,c)$;
Then AS 7

$W(a,b) \wedge \neg E(b,c) \wedge \neg E(c,d) \wedge E(a,d) \rightarrow \bot$;
$W(a,b) \wedge \neg E(b,c) \wedge \neg E(c,d) \wedge W(d,a) \rightarrow \bot$;
same as AS 11.1.

AS 10.4
$W(a,b) \wedge \neg E(b,c) \wedge \neg dE(c,d) \rightarrow \neg dE(a,d)$;

$W(a,b) \wedge \neg E(b,c) \wedge \neg dE(c,d) \wedge dE(a,d) \rightarrow \bot$;
$W(a,b) \wedge \neg E(b,c) \wedge \neg dE(c,d) \wedge dW(d,a) \rightarrow \bot$;
same as AS 13.1.

AS 11.1
$\neg E(a,b) \wedge W(b,c) \wedge W(c,d) \rightarrow W(a,d)$;

add $W(a,b) \wedge W(b,c) \rightarrow dW(a,c)$
then use AS 9
same as 10.3.

AS 11.2
$\neg E(a,b) \wedge W(b,c) \wedge dW(c,d) \rightarrow dW(a,d)$;
$\neg E(a,b) \wedge W(b,c) \wedge dW(c,d) \wedge \neg dW(a,d) \rightarrow \bot$;
$\neg E(a,b) \wedge W(b,c) \wedge dW(c,d) \wedge \neg dE(d,a) \rightarrow \bot$;
same as AS 12.3.

AS 11.3
$\neg E(a,b) \wedge W(b,c) \wedge \neg E(c,d) \rightarrow \neg E(a,d)$;
same as AS 10.1

AS 11.4
$\neg E(a,b) \wedge W(b,c) \wedge \neg dE(c,d) \rightarrow \neg dE(a,d)$;
$W(b,c) \wedge \neg dE(c,d) \rightarrow \neg E(b,d)$ (AS 7)
$\neg E(a,b) \wedge \neg E(b,d) \rightarrow \neg dE(a,d)$

$\neg E(a,b) \wedge W(b,c) \wedge \neg dE(c,d) \wedge dE(a,d) \rightarrow \bot$;
$\neg E(a,b) \wedge W(b,c) \wedge \neg dE(c,d) \wedge dW(d,a) \rightarrow \bot$;

same as AS 13.3.

AS 12.1

$dW(a,b) \wedge \neg dE(b,c) \wedge W(c,d) \to W(a,d)$

$\neg dE(b,c) \wedge W(c,d) \to \neg E(b,d)$ (AS 6)

$dW(a,b) \wedge \neg E(b,d) \to W(a,d)$ (AS 8)

AS 12. 2.

$dW(a,b) \wedge \neg dE(b,c) \wedge dW(c,d) \to dW(a,d)$

$dW(a,b) \wedge \neg dE(b,c) \wedge dW(c,d) \wedge \neg dW(a,d) \to \perp$

$dW(a,b) \wedge \neg dE(b,c) \wedge dW(c,d) \wedge \neg dE(d,a) \to \perp$

same as AS 13.4.

AS 12. 3.

$dW(a,b) \wedge \neg dE(b,c) \wedge \neg E(c,d) \to \neg E(a,d)$

$dW(a,b) \wedge \neg dE(b,c) \wedge \neg E(c,d) \wedge E(a,d) \to \perp$

$dW(a,b) \wedge \neg dE(b,c) \wedge \neg E(c,d) \wedge W(d,a) \to \perp$

$\neg dE(a,b) \wedge \neg E(b,c) \wedge W(c,d) \wedge dW(d,a) \to \perp$

same as AS 11.2

AS 12.4.

$dW(a,b) \wedge \neg dE(b,c) \wedge \neg dE(c,d) \to \neg dE(a,d)$

$dW(a,b) \wedge \neg dE(b,c) \wedge \neg dE(c,d) \wedge dE(a,d) \to \perp$

$dW(a,b) \wedge \neg dE(b,c) \wedge \neg dE(c,d) \wedge dW(d,a) \to \perp$

AS 13.1

$\neg dE(a,b) \wedge dW(b,c) \wedge W(c,d) \to W(a,d)$

same as 10.4.

AS 13.2

$\neg dE(a,b) \wedge dW(b,c) \wedge dW(c,d) \to dW(a,d)$

same as AS 12.4.

AS 13.3

$\neg dE(a,b) \wedge dW(b,c) \wedge \neg E(c,d) \to \neg E(a,d)$

$dW(b,c) \wedge \neg E(c,d) \to W(b,d)$ (AS 8)

$\neg dE(a,b) \wedge W(b,d) \to \neg E(a,d)$ (AS 6)

AS 13.4

$\neg dE(a,b) \wedge dW(b,c) \wedge \neg dE(c,d) \to \neg dE(a,d)$

same as AS 12.2

This ends up with the following set of axioms for east and west:

AS 1 $\neg W(a,a)$;

AS 2 $E(a,b) \leftrightarrow W(b,a)$;

AS 3 $I_{ew}(a,b) \to I_{ew}(b,a)$;

AS 4 $I_{ew}(a,b) \leftrightarrow (\neg dE(a,b) \wedge \neg dW(a,b))$;

AS 5 $W(a,b) \wedge W(b,c) \to dW(a,c)$;

AS 6 $W(a,b) \wedge \neg dE(b,c) \wedge W(c,a) \to \perp$;

AS 7 $\neg E(a,b) \wedge dW(b,c) \wedge \neg E(c,a) \to \perp$;

AS 8 $W(a,b) \wedge \neg E(b,c) \wedge W(c,d) \wedge \neg E(d,a) \to \perp$;

AS 9 $W(a,b) \wedge \neg E(b,c) \wedge \neg dE(c,d) \wedge dW(d,a) \to \perp$;

AS 10 $\neg E(a,b) \wedge W(b,c) \wedge dW(c,d) \wedge \neg dE(d,a) \to \perp$;

AS 11 $dW(a,b) \wedge \neg dE(b,c) \wedge dW(c,d) \wedge \neg dE(d,a) \to \perp$

AS 12 $dW(a,b) \wedge \neg dE(b,c) \wedge \neg dE(c,d) \wedge dW(d,a) \to \perp$

Theorem 4. *For $\tau = 2$, the LD^τ calculus is sound and complete for 2D Euclidean τ-models, i.e. $\vdash^\tau_{LD} \phi \Leftrightarrow \models^\tau_{LD} \phi$ (every τ-derivable formula is τ-valid and every τ-valid formula is τ-derivable).*

For $\tau = 2$, the proof of soundness (every LD τ-derivable formula is τ-valid) is by an easy induction on the length of the derivation of ϕ. By truth definitions of the direction relations (Definition 1), AS 1-26 are valid and modus ponens preserves validity.

In the rest of this section, we prove completeness. We will actually prove that for $\tau = 2$, *if a finite set of $L(LD)$ formulas Σ is τ-consistent, then there is a 2D Euclidean τ-model satisfying it*. By contraposition we get completeness.

In a sequence of formulas of the form $R_1(a_0,a_1),...,R_n(a_{n-1},a_0)$, where for every i such that $1 \le i \le n$, $R_i \in \{W, dW, \neg E, \neg dE\}$, we refer to $R_j(a_{j-1},a_j)$ and $R_{j+1}(a_j,a_{j+1})$, where $1 \le j < n$, as neighbours. $R_1(a_0,a_1)$ and $R_n(a_{n-1},a_0)$ are also referred to as neighbours.

Lemma 5. *Let F_n denote a formula of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_n(a_{n-1},a_0)$, where $n \in \mathbb{N}_{>1}$, for every i such that $1 \le i \le n$, $R_i \in \{W, dW, \neg E, \neg dE\}$, and $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$. Then there exist $R_s(a,b)$, $R_t(b,c)$, such that they are conjuncts in F_n, $R_s(a,b)$ and $R_t(b,c)$ are neighbours and one of the following cases holds:*

Case 1 *$R_s \in \{W, dW\}$ and $R_t \in \{\neg E, \neg dE\}$*

Case 2 *$R_s \in \{\neg E, \neg dE\}$ and $R_t \in \{W, dW\}$.*

Proof. Let us prove by contradiction. Suppose for every pair of $R_s(a,b), R_t(b,c)$, if they are conjuncts in F_n and $R_s(a,b)$ and $R_t(b,c)$ are neighbours, then neither Case 1 nor Case 2 holds, this is, they are both in $\{W, dW\}$ or both in $\{\neg E, \neg dE\}$. If there exists a conjunct $R_i(p,q)$ in F_n, $R_i \in \{W, dW\}$, then its neighbours are in $\{W, dW\}$ as well, hence $R_1, \ldots, R_n$ are all in $\{W, dW\}$. This contradicts $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$. Otherwise, there exists a conjunct $R_i(p,q)$ in F_n, $R_i \in \{\neg E, \neg dE\}$. Then its neighbours are in $\{\neg E, \neg dE\}$, hence $R_1, \ldots, R_n$ are all in $\{\neg E, \neg dE\}$. This contradicts $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$. ☐

Lemma 6. *Let F_n denote a formula of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_n(a_{n-1},a_0)$, where $n \in \mathbb{N}_{>1}$, for every i such that $1 \le i \le n$, $R_i \in \{W, dW, \neg E, \neg dE\}$, and $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$. Then for $\tau = 2$, any $n \in \mathbb{N}_{>1}$, $\perp$ can be derived from F_n.*

Proof. Let us prove by mathematical induction.

Base case When $n = 2$, since $R_i \in \{W, dW, \neg E, \neg dE\}$, $\tau = 2$, and $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, then $\{R_1, R_2\} = \{W, \neg E\}$ or $\{R_1, R_2\} = \{dW, \neg dE\}$. If $\{R_1, R_2\} = \{W, \neg E\}$, then by AS 2, $\perp$ can be derived. Otherwise, by AS 2 and AS 3, $dE(a,b) \leftrightarrow dW(b,a)$, hence $\perp$ can be derived.

Inductive step Suppose $\perp$ can be derived from $F_2, \ldots, F_n$, we will show $\perp$ can be derived from F_{n+1}. Since $n + 1 > 1$, by Lemma 5, there exist $R_s(a,b), R_t(b,c)$, such that they are conjuncts in F_{n+1}, $R_s(a,b)$ and $R_t(b,c)$ are neighbours and one of the following cases holds:

Case 1 $R_s \in \{W, dW\}$ and $R_t \in \{\neg E, \neg dE\}$
Case 2 $R_s \in \{\neg E, \neg dE\}$ and $R_t \in \{W, dW\}$.

Let us proceed by cases. Since $n+1 > 2$ and individual names involved in the $(n+1)$ formulas form a circle, every formula has two neighbours. Let $R_k(c,d)$ denote the other neighbour of $R_t(b,c)$.

1. If R_s is W and R_t is $\neg E$, then
   - if R_k is W, then by AS 8, $W(a,b) \wedge \neg E(b,c) \wedge W(c,d) \rightarrow W(a,d)$;
   - if R_k is dW, then by AS 7, $\neg E(b,c) \wedge dW(c,d) \rightarrow E(d,b)$; by AS 2, $E(d,b) \rightarrow W(b,d)$; by AS 5, $W(a,b) \wedge W(b,d) \rightarrow dW(a,d)$. Hence, $W(a,b) \wedge \neg E(b,c) \wedge dW(c,d) \rightarrow dW(a,d)$;
   - if R_k is $\neg E$, then by AS 7, $\neg E(b,c) \wedge \neg E(c,d) \rightarrow \neg dW(d,b)$; by Definition 2, AS 2 and AS 3, $\neg dW(d,b) \rightarrow \neg dE(b,d)$; by AS 6, $W(a,b) \wedge \neg dE(b,d) \rightarrow \neg W(d,a)$; by AS 2, $\neg W(d,a) \rightarrow \neg E(a,d)$. Hence, $W(a,b) \wedge \neg E(b,c) \wedge \neg E(c,d) \rightarrow \neg E(a,d)$.
   - if R_k is $\neg dE$, then by AS 9, $W(a,b) \wedge \neg E(b,c) \wedge \neg dE(c,d) \rightarrow \neg dW(d,a)$; by Definition 2, AS 2 and AS 3, $\neg dW(d,a) \rightarrow \neg dE(a,b)$. Hence, $W(a,b) \wedge \neg E(b,c) \wedge \neg dE(c,d) \rightarrow \neg dE(a,d)$.

   Hence in each case, $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d) \rightarrow R_k(a,d)$. We replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_k(a,d)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \le i \le m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of W and the number of $\neg E$ are both reduced by 1 and the number of dW and the number of $\neg dE$ are unchanged, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. Since $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, $m \ge 2$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.
2. If R_s is W and R_t is $\neg dE$, then by AS 6, $W(a,b) \wedge \neg dE(b,c) \rightarrow \neg W(c,a)$; by AS 2, $\neg W(c,a) \rightarrow \neg E(a,c)$. Hence, $R_s(a,b) \wedge R_t(b,c) \rightarrow \neg E(a,c)$. We replace $R_s(a,b) \wedge R_t(b,c)$ in F_{n+1} with $\neg E(a,c)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \le i \le m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of W and the number of $\neg dE$ are both reduced by 1, the number of $\neg E$ is increased by 1, the number of dW is unchanged and $\tau = 2$, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 1 = n > 1$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.
3. If R_s is dW and R_t is $\neg E$, then by AS 7, $dW(a,b) \wedge \neg E(b,c) \rightarrow E(c,a)$; by AS 2, $E(c,a) \rightarrow W(a,c)$. Hence, $R_s(a,b) \wedge R_t(b,c) \rightarrow W(a,c)$. We replace $R_s(a,b) \wedge R_t(b,c)$ in F_{n+1} with $W(a,c)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \le i \le m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of dW and the number of $\neg E$ are both reduced by 1, the number of W is increased by 1, the number of $\neg dE$ is unchanged and $\tau = 2$, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 1 = n > 1$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.
4. If R_s is dW and R_t is $\neg dE$, then
   - if R_k is W, then by AS 6, $\neg dE(b,c) \wedge W(c,d) \rightarrow \neg W(d,b)$; by AS 2, $\neg W(d,b) \rightarrow \neg E(b,d)$; by AS 7, $dW(a,b) \wedge \neg E(b,d) \rightarrow E(d,a)$; by AS 2, $E(d,a) \rightarrow W(a,d)$. Hence $dW(a,b) \wedge \neg dE(b,c) \wedge W(c,d) \rightarrow W(a,d)$.
   - if R_k is dW, then by AS 11, $dW(a,b) \wedge \neg dE(b,c) \wedge dW(c,d) \rightarrow dE(d,a)$; by Definition 2, AS 2 and AS 3, $dE(d,a) \rightarrow dW(a,d)$. Hence, $dW(a,b) \wedge \neg dE(b,c) \wedge dW(c,d) \rightarrow dW(a,d)$.
   - if R_k is $\neg E$, then by AS 10, $dW(a,b) \wedge \neg dE(b,c) \wedge \neg E(c,d) \rightarrow \neg W(d,a)$; by AS 2, $\neg W(d,a) \rightarrow \neg E(a,d)$. Hence, $dW(a,b) \wedge \neg dE(b,c) \wedge \neg E(c,d) \rightarrow \neg E(a,d)$.
   - if R_k is $\neg dE$, then by AS 12, $dW(a,b) \wedge \neg dE(b,c) \wedge \neg dE(c,d) \rightarrow \neg dW(d,a)$; by AS 2, $\neg dW(d,a) \rightarrow \neg dE(a,d)$. Hence, $dW(a,b) \wedge \neg dE(b,c) \wedge \neg dE(c,d) \rightarrow \neg dE(a,d)$.

   Hence in each case, $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d) \rightarrow R_k(a,d)$. We replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_k(a,d)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \le i \le m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of dW and the number of $\neg dE$ are both reduced by 1 and the number of W and the number of $\neg E$ are unchanged, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. Since $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, $m \ge 2$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.
5. If R_s is $\neg E$ and R_t is W, then
   - if R_k is W, then by AS 5, $W(b,c) \wedge W(c,d) \rightarrow dW(b,d)$; by AS 7, $\neg E(a,b) \wedge dW(b,d) \rightarrow E(d,a)$; by AS 2, $E(d,a) \rightarrow W(a,d)$. Hence, $\neg E(a,b) \wedge W(b,c) \wedge W(c,d) \rightarrow W(a,d)$.
   - if R_k is dW, then by AS 10, $\neg E(a,b) \wedge W(b,c) \wedge dW(c,d) \rightarrow dE(d,a)$; by Definition 2, AS 2 and AS 3, $dE(d,a) \rightarrow dW(a,d)$. Hence, $\neg E(a,b) \wedge W(b,c) \wedge dW(c,d) \rightarrow dW(a,d)$.

- if R_k is $\neg E$, then by AS 8, $\neg E(a,b) \wedge W(b,c) \wedge \neg E(c,d) \rightarrow \neg W(d,a)$; by AS 2, $\neg W(d,a) \rightarrow \neg E(a,d)$. Hence, $\neg E(a,b) \wedge W(b,c) \wedge \neg E(c,d) \rightarrow \neg E(a,d)$.
- if R_k is $\neg dE$, then by AS 6, $W(b,c) \wedge \neg dE(c,d) \rightarrow \neg W(d,b)$; by AS 2, $\neg W(d,b) \rightarrow \neg E(b,d)$; by AS 7, $\neg E(a,b) \wedge \neg E(b,d) \rightarrow \neg dW(d,a)$; by AS 2, $\neg dW(d,a) \rightarrow \neg dE(a,d)$. Hence, $\neg E(a,b) \wedge W(b,c) \wedge \neg dE(c,d) \rightarrow \neg dE(a,d)$.

Hence in each case, $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d) \rightarrow R_k(a,d)$. We replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_k(a,d)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of $\neg E$ and the number of W are both reduced by 1 and the number of dW and the number of $\neg dE$ are unchanged, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. Since $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, $m \geq 2$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.

6. If R_s is $\neg E$ and R_t is dW, then by AS 7, $\neg E(a,b) \wedge dW(b,c) \rightarrow E(c,a)$; by AS 2, $E(c,a) \rightarrow W(a,c)$. Hence, $R_s(a,b) \wedge R_t(b,c) \rightarrow W(a,c)$. We replace $R_s(a,b) \wedge R_t(b,c)$ in F_{n+1} with $W(a,c)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of $\neg E$ and the number of dW are both reduced by 1, the number of W is increased by 1, the number of $\neg dE$ is unchanged and $\tau = 2$, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 1 = n > 1$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.
7. If R_s is $\neg dE$ and R_t is W, then by AS 6, $\neg dE(a,b) \wedge W(b,c) \rightarrow \neg W(c,a)$; by AS 2, $\neg W(c,a) \rightarrow \neg E(a,c)$. Hence, $R_s(a,b) \wedge R_t(b,c) \rightarrow \neg E(a,c)$. We replace $R_s(a,b) \wedge R_t(b,c)$ in F_{n+1} with $\neg E(a,c)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of $\neg dE$ and the number of W are both reduced by 1, the number of $\neg E$ is increased by 1, the number of dW is unchanged and $\tau = 2$, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 1 = n > 1$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.
8. If R_s is $\neg dE$ and R_t is dW, then
   - if R_k is W, then by AS 9, $\neg dE(a,b) \wedge dW(b,c) \wedge W(c,d) \rightarrow E(d,a)$; by AS 2, $E(d,a) \rightarrow W(a,d)$. Hence, $\neg dE(a,b) \wedge dW(b,c) \wedge W(c,d) \rightarrow W(a,d)$.
   - if R_k is dW, then by AS 12, $\neg dE(a,b) \wedge dW(b,c) \wedge dW(c,d) \rightarrow dE(d,a)$; by Definition 2, AS 2 and AS 3, $dE(d,a) \rightarrow dW(a,d)$. Hence, $\neg dE(a,b) \wedge dW(b,c) \wedge dW(c,d) \rightarrow dW(a,d)$.
   - if R_k is $\neg E$, then by AS 7, $dW(b,c) \wedge \neg E(c,d) \rightarrow E(d,b)$; by AS 2, $E(d,b) \rightarrow W(b,d)$; by AS 6, $\neg dE(a,b) \wedge W(b,d) \rightarrow \neg W(d,a)$; by AS 2, $\neg W(d,a) \rightarrow \neg E(a,d)$. Hence, $\neg dE(a,b) \wedge dW(b,c) \wedge \neg E(c,d) \rightarrow \neg E(a,d)$.
   - if R_k is $\neg dE$, then by AS 11, $\neg dE(a,b) \wedge dW(b,c) \wedge \neg dE(c,d) \rightarrow \neg dW(d,a)$; by Definition 2, AS 2 and AS 3, $\neg dW(d,a) \rightarrow \neg dE(a,d)$. Hence, $\neg dE(a,b) \wedge dW(b,c) \wedge \neg dE(c,d) \rightarrow \neg dE(a,d)$.

   Hence in each case, $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d) \rightarrow R_k(a,d)$. We replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_k(a,d)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of $\neg dE$ and the number of dW are both reduced by 1 and the number of W and the number of $\neg E$ are unchanged, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. Since $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, $m \geq 2$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.

Therefore, in every case, $\perp$ can be derived from F_{n+1}.

Therefore, for $\tau = 2$, any $n \in \mathbb{N}_{>1}$, $\perp$ can be derived from F_n. □

Lemma 7. *Let F_n denote a formula of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_n(a_{n-1},a_0)$ where $n \in \mathbb{N}_{>0}$, for every i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$, and $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$. If there exists an R_i in F_n such that $R_i \in \{\neg E, \neg dE\}$, then there exist $R_s(a,b), R_t(b,c)$, such that they are conjuncts in F_n, $R_s(a,b)$ and $R_t(b,c)$ are neighbours and one of the following cases holds:*

Case 1 *$R_s \in \{W, dW\}$ and $R_t \in \{\neg E, \neg dE\}$*

Case 2 *$R_s \in \{\neg E, \neg dE\}$ and $R_t \in \{W, dW\}$.*

Proof. Let us prove by contradiction. Suppose there exists an R_i in F_n such that $R_i \in \{\neg E, \neg dE\}$. Further suppose for every pair of $R_s(a,b), R_t(b,c)$, if they are conjuncts in F_n and $R_s(a,b)$ and $R_t(b,c)$ are neighbours, then neither Case 1 nor Case 2 holds, this is, they are both in $\{W, dW\}$ or both in $\{\neg E, \neg dE\}$. Since there exists a conjunct $R_i(p,q)$ in F_n,

$R_i \in \{\neg E, \neg dE\}$, its neighbours are in $\{\neg E, \neg dE\}$ as well, hence $R_1, \ldots, R_n$ are all in $\{\neg E, \neg dE\}$. This contradicts $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$. $\square$

Lemma 8. *Let F_n denote a formula of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0)$ where $n \in \mathbb{N}_{>0}$, for every i such that $1 \le i \le n$, $R_i \in \{W, dW, \neg E, \neg dE\}$, and $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$. Then for $\tau = 2$, any $n \in \mathbb{N}_{>1}$, $\perp$ can be derived from F_n.*

Proof. Let us prove by mathematical induction.

Base case When $n = 1$, since $R_i \in \{W, dW, \neg E, \neg dE\}$, $\tau = 2$, and $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$, then $R_1 \in \{W, dW\}$. If R_1 is W, then by AS 1, $\perp$ can be derived. Otherwise, by the definition of dW (Definition 2) and AS 1, $\perp$ can be derived.

When $n = 2$, since $R_i \in \{W, dW, \neg E, \neg dE\}$, $\tau = 2$, and $number(W)+\tau*number(dW) > number(\neg E)+\tau * number(\neg dE)$, then $R_1, R_2 \in \{W, dW\}$ or $\{R_1, R_2\} = \{dW, \neg E\}$. If $R_1, R_2 \in \{W, dW\}$, then by the definition of dW (Definition 2), AS 5 and AS 1, $\perp$ can be derived. Otherwise, by AS 7, AS 2 and AS 1, $\perp$ can be derived.

Inductive step Suppose $\perp$ can be derived from $F_1, F_2, \ldots, F_n$, where $n \ge 2$, we will show $\perp$ can be derived from F_{n+1}. If every R_i in F_{n+1} is W or dW, then by the definition of dW (Definition 2), AS 5 and AS 1, $\perp$ can be derived from F_{n+1}.

Otherwise, there exists at least one R_i in F_{n+1} which is $\neg E$ or $\neg dE$. By Lemma 7, there exist $R_s(a, b)$, $R_t(b, c)$, such that they are conjuncts in F_{n+1}, $R_s(a, b)$ and $R_t(b, c)$ are neighbours and one of the following cases holds:

Case 1 $R_s \in \{W, dW\}$ and $R_t \in \{\neg E, \neg dE\}$

Case 2 $R_s \in \{\neg E, \neg dE\}$ and $R_t \in \{W, dW\}$.

Let us proceed by cases. Since $n+1 > 2$ and individual names involved in the $(n + 1)$ formulas form a circle, every formula has two neighbours. Let $R_k(c, d)$ denote the other neighbour of $R_t(b, c)$.

1. If R_s is W and R_t is $\neg E$, then by AS 10, $R_s(a, b) \wedge R_t(b, c) \wedge R_k(c, d) \rightarrow R_k(a, d)$. We replace $R_s(a, b) \wedge R_t(b, c) \wedge R_k(c, d)$ in F_{n+1} with $R_k(a, d)$, then we will obtain a formula F' of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_m(a_m, a_0)$, where for every i such that $1 \le i \le m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of W and the number of $\neg E$ are both reduced by 1 and the number of dW and the number of $\neg dE$ are unchanged, we have $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$, and $m = (n + 1) - 2 = n - 1$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.
2. If R_s is W and R_t is $\neg dE$, then by AS 7, $R_s(a, b) \wedge R_t(b, c) \rightarrow \neg E(a, c)$. We replace $R_s(a, b) \wedge R_t(b, c)$ in F_{n+1} with $\neg E(a, c)$, then we will obtain a formula F' of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_m(a_m, a_0)$, where for every i such that $1 \le i \le m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of W and the number of $\neg dE$ are both reduced by 1, the number of $\neg E$ is increased by 1, the number of dW is unchanged and $\tau = 2$, we have $number(W)+\tau*number(dW) > number(\neg E) + \tau * number(\neg dE)$, and $m = (n + 1) - 1 = n$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.
3. If R_s is dW and R_t is $\neg E$, then by AS 8, $R_s(a, b) \wedge R_t(b, c) \rightarrow W(a, c)$. We replace $R_s(a, b) \wedge R_t(b, c)$ in F_{n+1} with $W(a, c)$, then we will obtain a formula F' of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_m(a_m, a_0)$, where for every i such that $1 \le i \le m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of dW and the number of $\neg E$ are both reduced by 1, the number of W is increased by 1, the number of $\neg dE$ is unchanged and $\tau = 2$, we have $number(W)+\tau*number(dW) > number(\neg E) + \tau * number(\neg dE)$, and $m = (n + 1) - 1 = n$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.
4. If R_s is dW and R_t is $\neg dE$, then by AS 12, $R_s(a, b) \wedge R_t(b, c) \wedge R_k(c, d) \rightarrow R_k(a, d)$. We replace $R_s(a, b) \wedge R_t(b, c) \wedge R_k(c, d)$ in F_{n+1} with $R_k(a, d)$, then we will obtain a formula F' of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_m(a_m, a_0)$, where for every i such that $1 \le i \le m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of dW and the number of $\neg dE$ are both reduced by 1 and the number of W and the number of $\neg E$ are unchanged, we have $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$, and $m = (n + 1) - 2 = n - 1$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.
5. If R_s is $\neg E$ and R_t is W, then by AS 11, $R_s(a, b) \wedge R_t(b, c) \wedge R_k(c, d) \rightarrow R_k(a, d)$. We replace $R_s(a, b) \wedge R_t(b, c) \wedge R_k(c, d)$ in F_{n+1} with $R_k(a, d)$, then we will obtain a formula F' of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_m(a_m, a_0)$, where for every i such that $1 \le i \le m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of $\neg E$ and the number of W are both reduced by 1 and the number of dW and the number of $\neg dE$ are unchanged, we have $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$, and $m = (n + 1) - 2 = n - 1$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.
6. If R_s is $\neg E$ and R_t is dW, then by AS 9, $R_s(a, b) \wedge R_t(b, c) \rightarrow W(a, c)$. We re-

place $R_s(a,b) \wedge R_t(b,c)$ in F_{n+1} with $W(a,c)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \le i \le m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of $\neg E$ and the number of dW are both reduced by 1, the number of W is increased by 1, the number of $\neg dE$ is unchanged and $\tau = 2$, we have $number(W)+\tau*number(dW) > number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1)-1 = n$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.

7. If R_s is $\neg dE$ and R_t is W, then by AS 6, $R_s(a,b) \wedge R_t(b,c) \rightarrow \neg E(a,c)$. We replace $R_s(a,b) \wedge R_t(b,c)$ in F_{n+1} with $\neg E(a,c)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \le i \le m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of $\neg dE$ and the number of W are both reduced by 1, the number of $\neg E$ is increased by 1, the number of dW is unchanged and $\tau = 2$, we have $number(W)+\tau*number(dW) > number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1)-1 = n$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.
8. If R_s is $\neg dE$ and R_t is dW, then by AS 13, $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d) \rightarrow R_k(a,d)$. We replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_k(a,d)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \le i \le m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of $\neg dE$ and the number of dW are both reduced by 1 and the number of W and the number of $\neg E$ are unchanged, we have $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1)-2 = n-1$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.

Therefore, in every case, $\perp$ can be derived from F_{n+1}.

Therefore, for $\tau = 2$, any $n \in \mathbb{N}_{>1}$, $\perp$ can be derived from F_n. □

Similarly, we have Lemma 9 and Lemma 10. The proofs for Lemma 9 and Lemma 10 are omitted, since they are very similar to those for Lemma 6 and Lemma 8 respectively.

Lemma 9. *For $\tau = 2$, any $n \in \mathbb{N}_{>1}$, $\perp$ can be derived from $R_1(a_0,a_1) \wedge \cdots \wedge R_n(a_{n-1},a_0)$, where for every i such that $1 \le i \le n$, $R_i \in \{S, dS, \neg N, \neg dN\}$, and $number(S) + \tau * number(dS) = number(\neg N) + \tau * number(\neg dN)$.*

Lemma 10. *For $\tau = 2$, any $n \in \mathbb{N}_{>0}$, $\perp$ can be derived from $R_1(a_0,a_1) \wedge \cdots \wedge R_n(a_{n-1},a_0)$, where for every i such that $1 \le i \le n$, $R_i \in \{S, dS, \neg N, \neg dN\}$, and $number(S) + \tau * number(dS) > number(\neg N) + \tau * number(\neg dN)$.*

Theorem 5 is proved similarly to Theorem 3. The proof of Theorem 3 refers to AS 5, AS 6, AS 10 and AS 11 in the LD^τ calculus, which are stated for any $n \in \mathbb{N}_{>1}$ or for any $n \in \mathbb{N}_{>0}$. In the proof of Theorem 5 below, instead of using AS 5, AS 6, AS 10 and AS 11 in the LD^τ calculus, we refer to Lemma 6, Lemma 8, Lemma 9 and Lemma 10 respectively, which are all proved using axiom schemas in a finite axiomatisation of LD^2.

Theorem 5. *For $\tau = 2$, if a finite set of $L(LD)$ formulas Σ is τ-consistent, then there is a 2D Euclidean τ-model satisfying it.*

Proof. Suppose a finite set of $L(LD)$ formulas Σ is τ-consistent. We obtain Σ' by rewriting every $I_{ew}(a,b)$ in Σ as $\neg dE(a,b) \wedge \neg dW(a,b)$, every $I_{ns}(a,b)$ in Σ as $\neg dN(a,b) \wedge \neg dS(a,b)$. By AS 4 and AS 17, Σ and Σ' are logically equivalent. Σ' can be rewritten as a formula ϕ that is the conjunction of all the formulas in Σ'. We rewrite the $L(LD)$ formula ϕ into disjunctive normal form $\phi_1 \vee \cdots \vee \phi_n$ ($n > 0$). Then every literal is of one of the forms $E(a,b)$, $W(a,b)$, $dE(a,b)$, $dW(a,b)$, $N(a,b)$, $S(a,b)$, $dN(a,b)$, $dS(a,b)$, or their negations. Then ϕ is satisfiable in a 2D Euclidean τ-model, iff at least one of its disjuncts ϕ_i is τ-satisfiable. We obtain a set of inequalities S_i by translating every literal in a disjunct ϕ_i as in Definition 3. Then the inequalities in S_i are of the form $x_a - x_b < c$, $x_a - x_b \le c$, $y_a - y_b < c$ or $y_a - y_b \le c$, where x_a, x_b, y_a, y_b are real variables and c is a real constant. We call variables like x_a, x_b x variables and variables like y_a, y_b y variables. Divide S_i into two sets S_i^x and S_i^y, such that S_i^x and S_i^y contain all the inequalities involving x variables and y variables respectively. By Corollary 1 of Theorem 1, ϕ_i is τ-satisfiable iff the graph G_i^x of S_i^x has no simple infeasible loop and the graph G_i^y of S_i^y has no simple infeasible loop. To show there is a 2D Euclidean τ-model satisfying Σ, it is sufficient to show there exists a disjunct ϕ_i such that the graph G_i^x of S_i^x has no simple infeasible loop and the graph G_i^y of S_i^y has no simple infeasible loop.

We prove this by contradiction. Suppose for every disjunct ϕ_i, the graph G_i^x of S_i^x has a simple infeasible loop (Case 1) or the graph G_i^y of S_i^y has a simple infeasible loop (Case 2). We present the proof for Case 1. Case 2 is similar.

If G_i^x has a simple infeasible loop P, then P is either strict or non-strict. Let m denote the sum of the constants c around the loop P. Based on the definition of infeasible loop, if P is strict, then $m \le 0$; otherwise, $m < 0$. By Definition 3, if a strict inequality $x_a - x_b < c$ is in S_i^x, then c is equal to $-\sigma$ or $-\tau\sigma$; if a non-strict inequality $x_a - x_b \le c$ is in S_i^x, then c is equal to σ or $\tau\sigma$, where τ, σ are positive numbers (hence $c > 0$). If P is non-strict, then all the inequalities in it are of the form $x_a - x_b \le c$ where $c > 0$ and the sum of such c is positive. This contradicts the fact that $m < 0$ for non-strict infeasible loops. Therefore P is strict, hence $m \le 0$. We consider the two cases where $m = 0$ and $m < 0$ separately.

1. If $m = 0$, then the sum of the constants around the loop P is equal to 0. Without loss of generality, let us assume P consists of vertices $xa_0, xa_1, ..., xa_{n-1}, xa_0$. Since P is admissible, the linear inequalities in P are of the form $(xa_0 - xa_1)?c_1, ..., (xa_{n-1} - xa_0)?c_n$, where ? is $\le$ or $<$, and for every i such that $1 \le i \le n$, c_i is σ, $-\sigma$, $\tau\sigma$ or $-\tau\sigma$. Then we translate the linear inequalities in P to formulas as follows. We translate every linear

inequality of the form $x_a - x_b < -\sigma$ to $W(a,b)$; every $x_a - x_b < -\tau\sigma$ to $dW(a,b)$; every $x_a - x_b \leq \sigma$ to $\neg E(a,b)$; every $x_a - x_b \leq \tau\sigma$ to $\neg dE(a,b)$. In this way, from P we obtain a sequence of formulas of the form $R_1(a_0,a_1),...,R_n(a_{n-1},a_0)$, where for every i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the sum of the constants around P is equal to 0, $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$ and $n \geq 2$. By Lemma 6, $\perp$ can be derived from $R_1(a_0,a_1),...,R_n(a_{n-1},a_0)$. By Definition 3, for every occurrence of $W(a,b)$ in $R_1(a_0,a_1) \wedge ... \wedge R_n(a_{n-1},a_0)$, it or $E(b,a)$ is a conjunct in ϕ_i; similarly, for every occurrence of $dW(a,b)$, it or $dE(b,a)$ is a conjunct in ϕ_i; for every occurrence of $\neg E(a,b)$, it or $\neg W(b,a)$ is a conjunct in ϕ_i; for every occurrence of $\neg dE(a,b)$, it or $\neg dW(b,a)$ is a conjunct in ϕ_i. By AS 2, $W(a,b) \leftrightarrow E(b,a)$. By Definition 2, AS 2 and AS 3, $dW(a,b) \leftrightarrow dE(b,a)$. Therefore, $\perp$ is τ-derivable from ϕ_i.

2. If $m < 0$, then the sum of the constants around the loop P is negative. In the same way described above, from P we obtain a sequence of formulas of the form $R_1(a_0,a_1),...,R_n(a_{n-1},a_0)$, where for every i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the sum of the constants around the loop P is negative, $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$ and $n \geq 1$. By Lemma 8, $\perp$ can be derived from $R_1(a_0,a_1) \wedge ... \wedge R_n(a_{n-1},a_0)$. Following the same argument above, $\perp$ is τ-derivable from ϕ_i.

In each case, $\perp$ is τ-derivable from ϕ_i. Thus every disjunct ϕ_i is not τ-consistent, hence ϕ is not τ-consistent. This contradicts the fact that Σ is τ-consistent. □

4.2 When $\tau = 3$

The following calculus LD^3 is sound and complete for LD^3.

AS 0 All tautologies of classical propositional logic

AS 1 $\neg W(a,a)$;

AS 2 $E(a,b) \leftrightarrow W(b,a)$;

AS 3 $I_{ew}(a,b) \rightarrow I_{ew}(b,a)$;

AS 4 $I_{ew}(a,b) \leftrightarrow (\neg dE(a,b) \wedge \neg dW(a,b))$;

AS 5 $W(a,b) \wedge W(b,c) \wedge W(c,d) \rightarrow dW(a,d)$;

AS 6 $\neg E(a,b) \wedge \neg E(b,c) \wedge \neg E(c,d) \rightarrow \neg dE(a,d)$;

AS 7 $\neg dE(a,b) \wedge W(b,c) \wedge W(c,d) \rightarrow \neg E(a,d)$;

AS 8 $W(a,b) \wedge W(b,c) \wedge \neg dE(c,d) \rightarrow \neg E(a,d)$;

AS 9 $W(a,b) \wedge \neg dE(b,c) \wedge W(c,d) \rightarrow \neg E(a,d)$;

AS 10 $dW(a,b) \wedge \neg E(b,c) \wedge \neg E(c,d) \rightarrow W(a,d)$;

AS 11 $\neg E(a,b) \wedge \neg E(b,c) \wedge dW(c,d) \rightarrow W(a,d)$;

AS 12 $\neg E(a,b) \wedge dW(b,c) \wedge \neg E(c,d) \rightarrow W(a,d)$;

AS 13 $R(a,b) \wedge W(b,c) \wedge \neg E(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 14 $W(a,b) \wedge \neg E(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 15 $R(a,b) \wedge \neg E(b,c) \wedge W(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 16 $\neg E(a,b) \wedge W(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 17 $R(a,b) \wedge dW(b,c) \wedge \neg dE(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 18 $dW(a,b) \wedge \neg dE(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 19 $R(a,b) \wedge \neg dE(b,c) \wedge dW(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 20 $\neg dE(a,b) \wedge dW(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 21 $W(a,b) \wedge R(b,c) \wedge \neg E(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 22 $\neg E(a,b) \wedge R(b,c) \wedge W(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 23 $dW(a,b) \wedge R(b,c) \wedge \neg dE(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 24 $\neg dE(a,b) \wedge R(b,c) \wedge dW(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 25 $W(a,b) \wedge W(b,c) \rightarrow W(a,c)$;

AS 26 $dW(a,b) \wedge \neg E(b,c) \rightarrow W(a,c)$;

AS 27 $\neg E(a,b) \wedge dW(b,c) \rightarrow W(a,c)$;

AS 28 $\neg dE(a,b) \wedge W(b,c) \rightarrow \neg dE(a,c)$;

AS 29 $W(a,b) \wedge \neg dE(b,c) \rightarrow \neg dE(a,c)$;

AS 30 $\neg S(a,a)$;

AS 31 $N(a,b) \leftrightarrow S(b,a)$;

AS 32 $I_{ns}(a,b) \rightarrow I_{ns}(b,a)$;

AS 33 $I_{ns}(a,b) \leftrightarrow (\neg dN(a,b) \wedge \neg dS(a,b))$;

AS 34 $S(a,b) \wedge S(b,c) \wedge S(c,d) \rightarrow dS(a,d)$;

AS 35 $\neg N(a,b) \wedge \neg N(b,c) \wedge \neg N(c,d) \rightarrow \neg dN(a,d)$;

AS 36 $\neg dN(a,b) \wedge S(b,c) \wedge S(c,d) \rightarrow \neg N(a,d)$;

AS 37 $S(a,b) \wedge S(b,c) \wedge \neg dN(c,d) \rightarrow \neg N(a,d)$;

AS 38 $S(a,b) \wedge \neg dN(b,c) \wedge S(c,d) \rightarrow \neg N(a,d)$;

AS 39 $dS(a,b) \wedge \neg N(b,c) \wedge \neg N(c,d) \rightarrow S(a,d)$;

AS 40 $\neg N(a,b) \wedge \neg N(b,c) \wedge dS(c,d) \rightarrow S(a,d)$;

AS 41 $\neg N(a,b) \wedge dS(b,c) \wedge \neg N(c,d) \rightarrow S(a,d)$;

AS 42 $R(a,b) \wedge S(b,c) \wedge \neg N(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 43 $S(a,b) \wedge \neg N(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 44 $R(a,b) \wedge \neg N(b,c) \wedge S(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 45 $\neg N(a,b) \wedge S(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 46 $R(a,b) \wedge dS(b,c) \wedge \neg dN(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 47 $dS(a,b) \wedge \neg dN(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 48 $R(a,b) \wedge \neg dN(b,c) \wedge dS(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 49 $\neg dN(a,b) \wedge dS(b,c) \wedge R(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 50 $S(a,b) \wedge R(b,c) \wedge \neg N(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 51 $\neg N(a,b) \wedge R(b,c) \wedge S(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 52 $dS(a,b) \wedge R(b,c) \wedge \neg dN(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 53 $\neg dN(a,b) \wedge R(b,c) \wedge dS(c,d) \rightarrow R(a,d)$, where $R \in \{W, dW, \neg E, \neg dE\}$;

AS 54 $S(a,b) \wedge S(b,c) \rightarrow S(a,c)$;

AS 55 $dS(a,b) \wedge \neg N(b,c) \rightarrow S(a,c)$;

AS 56 $\neg N(a,b) \wedge dS(b,c) \rightarrow S(a,c)$;

AS 57 $\neg dN(a,b) \wedge S(b,c) \rightarrow \neg dN(a,c)$;

AS 58 $S(a,b) \wedge \neg dN(b,c) \rightarrow \neg dN(a,c)$;

MP Modus ponens: $\phi,\ \phi \rightarrow \psi \vdash \psi$.

Theorem 6. *For $\tau = 3$, the LD^τ calculus is sound and complete for 2D Euclidean τ-models, i.e. $\vdash^\tau_{LD} \phi \Leftrightarrow \models^\tau_{LD} \phi$ (every τ-derivable formula is τ-valid and every τ-valid formula is τ-derivable).*

For $\tau = 3$, the proof of soundness (every LD τ-derivable formula is τ-valid) is by an easy induction on the length of the derivation of ϕ. By truth definitions of the direction relations (Definition 1), AS 1-58 are valid and modus ponens preserves validity.

In the rest of this section, we prove completeness. We will actually prove that for $\tau = 3$, *if a finite set of $L(LD)$ formulas Σ is τ-consistent, then there is a 2D Euclidean τ-model satisfying it.* By contraposition we get completeness.

Lemma 11. *Let F_n denote a formula of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0)$ where $n \in \mathbb{N}_{>2}$, for every i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$. For every $R_s(a,b)$, $R_t(b,c)$ and $R_k(c,d)$, if they are all conjuncts in F_n, $R_s(a,b)$ and $R_k(c,d)$ are neighbours of $R_t(b,c)$, then exactly one of the following cases holds:*

Case 1 *All of R_s, R_t, R_k are W*

Case 2 *All of R_s, R_t, R_k are $\neg E$*

Case 3 *One of R_s, R_t, R_k is W, one of them is $\neg E$, one of them is any of $\{W, dW, \neg E, \neg dE\}$*

Case 4 *One of R_s, R_t, R_k is dW, one of them is $\neg dE$, one of them is any of $\{W, dW, \neg E, \neg dE\}$*

Case 5 *One of R_s, R_t, R_k is dW, two of them are $\neg E$*

Case 6 *One of R_s, R_t, R_k is $\neg dE$, two of them are W*

Case 7 *All of R_s, R_t, R_k are dW*

Case 8 *All of R_s, R_t, R_k are $\neg dE$*

Case 9 *Two of R_s, R_t, R_k are dW, one of them is W or $\neg E$*

Case 10 *Two of R_s, R_t, R_k are $\neg dE$, one of them is W or $\neg E$*

Case 11 *One of R_s, R_t, R_k is dW, two of them are W*

Case 12 *One of R_s, R_t, R_k is $\neg dE$, two of them are $\neg E$*

Proof. Suppose $R_s(a,b)$, $R_t(b,c)$ and $R_k(c,d)$ are all conjuncts in F_n, $R_s(a,b)$ and $R_k(c,d)$ are neighbours of $R_t(b,c)$. Since $R_s, R_t, R_k \in \{W, dW, \neg E, \neg dE\}$, all the possible cases are listed below.

- $\{R_s, R_t, R_k\} = \{W\}$: Case 1 holds.
- $\{R_s, R_t, R_k\} = \{dW\}$: Case 7 holds.
- $\{R_s, R_t, R_k\} = \{\neg E\}$: Case 2 holds.
- $\{R_s, R_t, R_k\} = \{\neg dE\}$: Case 8 holds.
- $\{R_s, R_t, R_k\} = \{W, dW\}$: if two of R_s, R_t, R_k are W, Case 11 holds; otherwise, Case 9 holds.
- $\{R_s, R_t, R_k\} = \{W, \neg E\}$: Case 3 holds.
- $\{R_s, R_t, R_k\} = \{W, \neg dE\}$: if two of R_s, R_t, R_k are W, Case 6 holds; otherwise, Case 10 holds.
- $\{R_s, R_t, R_k\} = \{dW, \neg E\}$: if two of R_s, R_t, R_k are dW, Case 9 holds; otherwise, Case 5 holds.
- $\{R_s, R_t, R_k\} = \{dW, \neg dE\}$: Case 4 holds.
- $\{R_s, R_t, R_k\} = \{\neg E, \neg dE\}$: if two of R_s, R_t, R_k are $\neg E$, Case 12 holds; otherwise, Case 10 holds.
- $\{R_s, R_t, R_k\} = \{W, dW, \neg E\}$: Case 3 holds.
- $\{R_s, R_t, R_k\} = \{W, dW, \neg dE\}$: Case 4 holds.
- $\{R_s, R_t, R_k\} = \{W, \neg E, \neg dE\}$: Case 3 holds.
- $\{R_s, R_t, R_k\} = \{dW, \neg E, \neg dE\}$: Case 4 holds.

Therefore, for every possible case above, exactly one of Cases 1-12 holds. Hence for every $R_s(a,b)$, $R_t(b,c)$ and $R_k(c,d)$, if they are all conjuncts in F_n, $R_s(a,b)$ and $R_k(c,d)$ are neighbours of $R_t(b,c)$, then exactly one of Cases 1-12 holds. $\square$

Lemma 12. *Let F_n denote a formula of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0)$ where $n \in \mathbb{N}_{>1}$, for every i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$, and $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$. If $n > 2$, then there exist $R_s(a,b)$, $R_t(b,c)$ and $R_k(c,d)$, such that they are all conjuncts in F_n, $R_s(a,b)$ and $R_k(c,d)$ are neighbours of $R_t(b,c)$ and one of the following cases holds:*

Case 1 *All of R_s, R_t, R_k are W*

Case 2 *All of R_s, R_t, R_k are $\neg E$*

Case 3 *One of R_s, R_t, R_k is W, one of them is $\neg E$, one of them is any of $\{W, dW, \neg E, \neg dE\}$*

Case 4 *One of R_s, R_t, R_k is dW, one of them is $\neg dE$, one of them is any of $\{W, dW, \neg E, \neg dE\}$*

Case 5 *One of R_s, R_t, R_k is dW, two of them are $\neg E$*

Case 6 *One of R_s, R_t, R_k is $\neg dE$, two of them are W*

Proof. Let us prove by contradiction. Suppose for every $R_s(a,b)$, $R_t(b,c)$ and $R_k(c,d)$, if they are all conjuncts in F_n and $R_s(a,b)$ and $R_k(c,d)$ are neighbours of $R_t(b,c)$, then none of Cases 1-6 holds. By Lemma 11, one of Cases 1'-6' (which correspond to Cases 7-12 in Lemma 11) below holds:

Case 1' All of R_s, R_t, R_k are dW

Case 2' All of R_s, R_t, R_k are $\neg dE$

Case 3' Two of R_s, R_t, R_k are dW, one of them is W or $\neg E$

Case 4' Two of R_s, R_t, R_k are $\neg dE$, one of them is W or $\neg E$

Case 5' One of R_s, R_t, R_k is dW, two of them are W

Case 6' One of R_s, R_t, R_k is $\neg dE$, two of them are $\neg E$

In other words, take any $R_t(b,c)$ and its two neighbours $R_s(a,b)$ and $R_k(c,d)$ from F_n, one of Cases 1'-6' holds (**Prop. 1**). Let us proceed by cases:

Case 1' All of R_s, R_t, R_k are dW: since R_t, R_k are both dW, then by **Prop. 1**, for R_t, R_k and the other neighbour of R_k, Case 1' or Case 3' holds.

Case 2' All of R_s, R_t, R_k are $\neg dE$: since R_t, R_k are both $\neg dE$, then by **Prop. 1**, for R_t, R_k and the other neighbour of R_k, Case 2' or Case 4' holds.

Case 3' Two of R_s, R_t, R_k are dW, one of them is W or $\neg E$: since R_t, R_k are both dW, or $\{R_t, R_k\} = \{dW, W\}$, or $\{R_t, R_k\} = \{dW, \neg E\}$, by **Prop. 1**, for R_t, R_k and the other neighbour of R_k, one of odd cases (i.e. Case 1', Case 3', Case 5') holds.

Case 4' Two of R_s, R_t, R_k are $\neg dE$, one of them is W or $\neg E$: since R_t, R_k are both $\neg dE$, or $\{R_t, R_k\} = \{\neg dE, W\}$, or $\{R_t, R_k\} = \{\neg dE, \neg E\}$, by **Prop. 1**, for R_t, R_k and the other neighbour of R_k, one of even cases (i.e. Case 2', Case 4', Case 6') holds.

Case 5' One of R_s, R_t, R_k is dW, two of them are W: since R_t, R_k are both W, or $\{R_t, R_k\} = \{dW, W\}$, by **Prop. 1**, for R_t, R_k and the other neighbour of R_k, Case 3' or Case 5' holds.

Case 6' One of R_s, R_t, R_k is $\neg dE$, two of them are $\neg E$: since R_t, R_k are both $\neg E$, or $\{R_t, R_k\} = \{\neg dE, \neg E\}$, by **Prop. 1**, for R_t, R_k and the other neighbour of R_k, Case 4' or Case 6' holds.

Hence, F_n contains odd cases only, or it contains even cases only. If F_n contains odd cases only, then $number(\neg dE) = 0$ and $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$. Otherwise, $number(dW) = 0$ and $number(W) + \tau * number(dW) < number(\neg E) + \tau * number(\neg dE)$. Both contradict $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$. ☐

Lemma 13. *Let F_n denote a formula of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_n(a_{n-1},a_0)$ where $n \in \mathbb{N}_{>1}$, for every i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$, and $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$. Then for $\tau = 3$, any $n \in \mathbb{N}_{>1}$, $\perp$ can be derived from F_n.*

Proof. Let us prove by mathematical induction.

Base case When $n = 2$, since $R_i \in \{W, dW, \neg E, \neg dE\}$, $\tau = 3$, and $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, then $\{R_1, R_2\} = \{W, \neg E\}$ or $\{R_1, R_2\} = \{dW, \neg dE\}$. If $\{R_1, R_2\} = \{W, \neg E\}$, then by AS 2, $\perp$ can be derived. Otherwise, by AS 2, AS 3 and Definition 2, $dE(a,b) \leftrightarrow dW(b,a)$, hence $\perp$ can be derived.

Inductive step Suppose $\perp$ can be derived from any of F_2, $\dots$, F_n, we will show $\perp$ can be derived from F_{n+1}. Since $n+1 > 2$, by Lemma 12, there exist $R_s(a,b)$, $R_t(b,c)$ and $R_k(c,d)$, such that they are all conjuncts in F_{n+1}, $R_s(a,b)$ and $R_k(c,d)$ are neighbours of $R_t(b,c)$ and one of Cases 1-6 holds. Below we will show $\perp$ can be derived from F_{n+1} in each case.

Case 1 All of R_s, R_t, R_k are W: by AS 5, $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d) \rightarrow dW(a,d)$. We replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $dW(a,d)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of W is reduced by 3, the number of dW is increased by 1, the number of $\neg E$ and the number of $\neg dE$ are unchanged, and $\tau = 3$, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. Since $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, $m \geq 2$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.

Case 2 All of R_s, R_t, R_k are $\neg E$: by AS 6, $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d) \rightarrow \neg dE(a,d)$. We replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $\neg dE(a,d)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of $\neg E$ is reduced by 3, the number of $\neg dE$ is increased by 1, the number of W and the number of dW are unchanged, and $\tau = 3$, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. Since $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, $m \geq 2$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.

Case 3 One of R_s, R_t, R_k is W, one of them is $\neg E$, one of them is any of $\{W, dW, \neg E, \neg dE\}$:

1. if $\{R_t, R_k\} = \{W, \neg E\}$ and R_s is any of $\{W, dW, \neg E, \neg dE\}$, then by AS 13 and 15, $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d) \rightarrow R_s(a,d)$. We replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_s(a,d)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of W and the number of $\neg E$ are both reduced by 1, the number of dW and the number of $\neg dE$ are un-

changed, and $\tau = 3$, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. Since $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, $m \geq 2$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.

2. if $\{R_s, R_k\} = \{W, \neg E\}$ and R_t is any of $\{W, dW, \neg E, \neg dE\}$, then by AS 21 and 22, $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d) \rightarrow R_t(a,d)$. We replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_t(a,d)$, then we will obtain a formula F' of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_m(a_m, a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of W and the number of $\neg E$ are both reduced by 1, the number of dW and the number of $\neg dE$ are unchanged, and $\tau = 3$, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. Since $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, $m \geq 2$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.

3. if $\{R_s, R_t\} = \{W, \neg E\}$ and R_k is any of $\{W, dW, \neg E, \neg dE\}$, then by AS 14 and 16, $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d) \rightarrow R_k(a,d)$. We replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_k(a,d)$, then we will obtain a formula F' of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_m(a_m, a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of W and the number of $\neg E$ are both reduced by 1, the number of dW and the number of $\neg dE$ are unchanged, and $\tau = 3$, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. Since $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, $m \geq 2$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.

Case 4 One of R_s, R_t, R_k is dW, one of them is $\neg dE$, one of them is any of $\{W, dW, \neg E, \neg dE\}$:

1. if $\{R_t, R_k\} = \{dW, \neg dE\}$ and R_s is any of $\{W, dW, \neg E, \neg dE\}$, then by AS 17 and 19, $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d) \rightarrow R_s(a,d)$. We replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_s(a,d)$, then we will obtain a formula F' of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_m(a_m, a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of dW and the number of $\neg dE$ are both reduced by 1, the number of W and the number of $\neg E$ are unchanged, and $\tau = 3$, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. Since $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, $m \geq 2$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.

2. if $\{R_s, R_k\} = \{dW, \neg dE\}$ and R_t is any of $\{W, dW, \neg E, \neg dE\}$, then by AS 23 and 24, $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d) \rightarrow R_t(a,d)$. We replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_t(a,d)$, then we will obtain a formula F' of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_m(a_m, a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of dW and the number of $\neg dE$ are both reduced by 1, the number of W and the number of $\neg E$ are unchanged, and $\tau = 3$, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. Since $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, $m \geq 2$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.

3. if $\{R_s, R_t\} = \{dW, \neg dE\}$ and R_k is any of $\{W, dW, \neg E, \neg dE\}$, then by AS 18 and 20, $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d) \rightarrow R_k(a,d)$. We replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_k(a,d)$, then we will obtain a formula F' of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_m(a_m, a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of dW and the number of $\neg dE$ are both reduced by 1, the number of W and the number of $\neg E$ are unchanged, and $\tau = 3$, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. Since $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, $m \geq 2$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.

Case 5 One of R_s, R_t, R_k is dW, two of them are $\neg E$: by AS 10-12, $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d) \rightarrow W(a,d)$. We replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $W(a,d)$, then we will obtain a formula F' of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_m(a_m, a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of dW is reduced by 1, the number of $\neg E$ are reduced by 2, the number of W is increased by 1, the number of $\neg dE$ is unchanged, and $\tau = 3$, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. Since $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, $m \geq 2$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.

Case 6 One of R_s, R_t, R_k is $\neg dE$, two of them are W: by AS 7-9, $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d) \rightarrow \neg E(a,d)$. We replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $\neg E(a,d)$, then we will obtain a formula F' of the form

$R_1(a_0, a_1) \wedge \cdots \wedge R_m(a_m, a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of $\neg dE$ is reduced by 1, the number of W are reduced by 2, the number of $\neg E$ is increased by 1, the number of dW is unchanged, and $\tau = 3$, we have $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. Since $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, $m \geq 2$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.

Therefore, in every case, $\perp$ can be derived from F_{n+1}.

Therefore, for $\tau = 3$, any $n \in \mathbb{N}_{>1}$, $\perp$ can be derived from F_n. □

Lemma 14. *Let F_n denote a formula of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0)$ where $n \in \mathbb{N}_{>0}$, for every i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$, and $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$. Then for $\tau = 3$, any $n \in \mathbb{N}_{>0}$, $\perp$ can be derived from F_n.*

Proof. Let us prove by mathematical induction.

Base case When $n = 1$, since $R_i \in \{W, dW, \neg E, \neg dE\}$, $\tau = 3$, and $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$, then $R_1 \in \{W, dW\}$. If R_1 is W, then by AS 1, $\perp$ can be derived. Otherwise, by the definition of dW (Definition 2) and AS 1, $\perp$ can be derived.
When $n = 2$, since $R_i \in \{W, dW, \neg E, \neg dE\}$, $\tau = 3$, and $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$, then $R_1, R_2 \in \{W, dW\}$ or $\{R_1, R_2\} = \{dW, \neg E\}$. If $R_1, R_2 \in \{W, dW\}$, then by the definition of dW (Definition 2), AS 25, AS 1, $\perp$ can be derived. Otherwise, by AS 26, AS 27 and AS 1, $\perp$ can be derived.

Inductive step Suppose $\perp$ can be derived from any of $F_1, \ldots, F_n$, we will show that $\perp$ can be derived from F_{n+1}. If every R_i in F_{n+1} is W or dW, then by the definition of dW (Definition 2), AS 25 and AS 1, $\perp$ can be derived. Otherwise, there exists at least one R_i which is $\neg E$ or $\neg dE$. By Lemma 7, there exist $R_s(a, b), R_t(b, c)$, such that they are conjuncts in F_{n+1}, $R_s(a, b)$ and $R_t(b, c)$ are neighbours and one of the following cases holds:

Case 1 $R_s \in \{W, dW\}$ and $R_t \in \{\neg E, \neg dE\}$

Case 2 $R_s \in \{\neg E, \neg dE\}$ and $R_t \in \{W, dW\}$.

Let us proceed by cases. Since $n+1 > 2$ and individual names involved in the $(n+1)$ formulas form a circle, every formula has two neighbours. Let $R_k(c, d)$ denote the other neighbour of $R_t(b, c)$.

1. If R_s is W and R_t is $\neg E$, then by AS 14, $R_s(a, b) \wedge R_t(b, c) \wedge R_k(c, d) \rightarrow R_k(a, d)$. We replace $R_s(a, b) \wedge R_t(b, c) \wedge R_k(c, d)$ in F_{n+1} with $R_k(a, d)$, then we will obtain a formula F' of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_m(a_m, a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of W and the number of $\neg E$ are both reduced by 1 and the number of dW and the number of $\neg dE$ are unchanged, we have $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.
2. If R_s is W and R_t is $\neg dE$, then by AS 29, $R_s(a, b) \wedge R_t(b, c) \rightarrow R_t(a, c)$. We replace $R_s(a, b) \wedge R_t(b, c)$ in F_{n+1} with $R_t(a, c)$, then we will obtain a formula F' of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_m(a_m, a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of W is reduced by 1, the number of $\neg E$, the number of dW and the number of $\neg dE$ are unchanged, we have $number(W) + \tau * number(dW) \geq number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 1 = n$. If $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, then by Lemma 13, $\perp$ can be derived from F'. Otherwise, by inductive hypothesis, $\perp$ can be derived from F'. Hence, in either case, $\perp$ can be derived from F_{n+1}.
3. If R_s is dW and R_t is $\neg E$, by AS 26, $R_s(a, b) \wedge R_t(b, c) \rightarrow W(a, c)$. We replace $R_s(a, b) \wedge R_t(b, c)$ in F_{n+1} with $W(a, c)$, then we will obtain a formula F' of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_m(a_m, a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the the number of dW and the number of $\neg E$ are both reduced by 1, the number of W is increased by 1, the number of $\neg dE$ is unchanged and $\tau = 3$, we have $number(W) + \tau * number(dW) \geq number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 1 = n$. If $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, then by Lemma 13, $\perp$ can be derived from F'. Otherwise, by inductive hypothesis, $\perp$ can be derived from F'. Hence, in either case, $\perp$ can be derived from F_{n+1}.
4. If R_s is dW and R_t is $\neg dE$, then by AS 18, $R_s(a, b) \wedge R_t(b, c) \wedge R_k(c, d) \rightarrow R_k(a, d)$. We replace $R_s(a, b) \wedge R_t(b, c) \wedge R_k(c, d)$ in F_{n+1} with $R_k(a, d)$, then we will obtain a formula F' of the form $R_1(a_0, a_1) \wedge \cdots \wedge R_m(a_m, a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of dW and the number of $\neg dE$ are both reduced by 1 and the number of W and the number of $\neg E$ are unchanged, we have $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.
5. If R_s is $\neg E$ and R_t is W, then by AS 16, $R_s(a, b) \wedge R_t(b, c) \wedge R_k(c, d) \rightarrow R_k(a, d)$. We replace $R_s(a, b) \wedge R_t(b, c) \wedge R_k(c, d)$ in F_{n+1}

with $R_k(a,d)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of $\neg E$ and the number of W are both reduced by 1 and the number of dW and the number of $\neg dE$ are unchanged, we have $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.

6. If R_s is $\neg E$ and R_t is dW, by AS 27, $R_s(a,b) \wedge R_t(b,c) \rightarrow W(a,c)$. We replace $R_s(a,b) \wedge R_t(b,c)$ in F_{n+1} with $W(a,c)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the the number of dW and the number of $\neg E$ are both reduced by 1, the number of W is increased by 1, the number of $\neg dE$ is unchanged and $\tau = 3$, we have $number(W) + \tau * number(dW) \geq number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 1 = n$. If $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, then by Lemma 13, $\perp$ can be derived from F'. Otherwise, by inductive hypothesis, $\perp$ can be derived from F'. Hence, in either case, $\perp$ can be derived from F_{n+1}.
7. If R_s is $\neg dE$ and R_t is W, then by AS 28, $R_s(a,b) \wedge R_t(b,c) \rightarrow R_s(a,c)$. We replace $R_s(a,b) \wedge R_t(b,c)$ in F_{n+1} with $R_s(a,c)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of W is reduced by 1, the number of $\neg E$, the number of dW and the number of $\neg dE$ are unchanged, we have $number(W) + \tau * number(dW) \geq number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 1 = n$. If $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$, then by Lemma 13, $\perp$ can be derived from F'. Otherwise, by inductive hypothesis, $\perp$ can be derived from F'. Hence, in either case, $\perp$ can be derived from F_{n+1}.
8. If R_s is $\neg dE$ and R_t is dW, then by AS 20, $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d) \rightarrow R_k(a,d)$. We replace $R_s(a,b) \wedge R_t(b,c) \wedge R_k(c,d)$ in F_{n+1} with $R_k(a,d)$, then we will obtain a formula F' of the form $R_1(a_0,a_1) \wedge \cdots \wedge R_m(a_m,a_0)$, where for every i such that $1 \leq i \leq m$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the number of $\neg dE$ and the number of dW are both reduced by 1 and the number of W and the number of $\neg E$ are unchanged, we have $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$, and $m = (n+1) - 2 = n - 1$. By inductive hypothesis, $\perp$ can be derived from F', hence $\perp$ can be derived from F_{n+1}.

Therefore, in every case, $\perp$ can be derived from F_{n+1}. Therefore, for $\tau = 3$, any $n \in \mathbb{N}_{>0}$, $\perp$ can be derived from F_n. ◻

Similarly, we have Lemma 15 and Lemma 16. The proofs for Lemma 15 and Lemma 16 are omitted, since they are very similar to those for Lemma 13 and Lemma 14 respectively.

Lemma 15. *For $\tau = 3$, any $n \in \mathbb{N}_{>1}$, $\perp$ can be derived from $R_1(a_0,a_1) \wedge \cdots \wedge R_n(a_{n-1},a_0)$, where for every i such that $1 \leq i \leq n$, $R_i \in \{S, dS, \neg N, \neg dN\}$, and $number(S) + \tau * number(dS) = number(\neg N) + \tau * number(\neg dN)$.*

Lemma 16. *For $\tau = 3$, any $n \in \mathbb{N}_{>0}$, $\perp$ can be derived from $R_1(a_0,a_1) \wedge \cdots \wedge R_n(a_{n-1},a_0)$, where for every i such that $1 \leq i \leq n$, $R_i \in \{S, dS, \neg N, \neg dN\}$, and $number(S) + \tau * number(dS) > number(\neg N) + \tau * number(\neg dN)$.*

Theorem 7 is proved similarly to Theorem 3. The proof of Theorem 3 refers to AS 5, AS 6, AS 10 and AS 11 in the LD^τ calculus, which are stated for any $n \in \mathbb{N}_{>1}$ or for any $n \in \mathbb{N}_{>0}$. In the proof of Theorem 7 below, instead of using AS 5, AS 6, AS 10 and AS 11 in the LD^τ calculus, we refer to Lemma 13, Lemma 14, Lemma 15 and Lemma 16 respectively, which are all proved using axiom schemas in a finite axiomatisation of LD^3.

Theorem 7. *For $\tau = 3$, if a finite set of $L(LD)$ formulas Σ is τ-consistent, then there is a 2D Euclidean τ-model satisfying it.*

Proof. Suppose a finite set of $L(LD)$ formulas Σ is τ-consistent. We obtain Σ' by rewriting every $I_{ew}(a,b)$ in Σ as $\neg dE(a,b) \wedge \neg dW(a,b)$, every $I_{ns}(a,b)$ in Σ as $\neg dN(a,b) \wedge \neg dS(a,b)$. By AS 4 and AS 33, Σ and Σ' are logically equivalent. Σ' can be rewritten as a formula ϕ that is the conjunction of all the formulas in Σ'. We rewrite the $L(LD)$ formula ϕ into disjunctive normal form $\phi_1 \vee \cdots \vee \phi_n$ $(n > 0)$. Then every literal is of one of the forms $E(a,b)$, $W(a,b)$, $dE(a,b)$, $dW(a,b)$, $N(a,b)$, $S(a,b)$, $dN(a,b)$, $dS(a,b)$, or their negations. Then ϕ is satisfiable in a 2D Euclidean τ-model, iff at least one of its disjuncts ϕ_i is τ-satisfiable. We obtain a set of inequalities S_i by translating every literal in a disjunct ϕ_i as in Definition 3. Then the inequalities in S_i are of the form $x_a - x_b < c$, $x_a - x_b \leq c$, $y_a - y_b < c$ or $y_a - y_b \leq c$, where x_a, x_b, y_a, y_b are real variables and c is a real constant. We call variables like x_a, x_b x variables and variables like y_a, y_b y variables. Divide S_i into two sets S_i^x and S_i^y, such that S_i^x and S_i^y contain all the inequalities involving x variables and y variables respectively. By Corollary 1 of Theorem 1, ϕ_i is τ-satisfiable iff the graph G_i^x of S_i^x has no simple infeasible loop and the graph G_i^y of S_i^y has no simple infeasible loop. To show there is a 2D Euclidean τ-model satisfying Σ, it is sufficient to show there exists a disjunct ϕ_i such that the graph G_i^x of S_i^x has no simple infeasible loop and the graph G_i^y of S_i^y has no simple infeasible loop.

We prove this by contradiction. Suppose for every disjunct ϕ_i, the graph G_i^x of S_i^x has a simple infeasible loop (Case 1) or the graph G_i^y of S_i^y has a simple infeasible loop (Case 2). We present the proof for Case 1. Case 2 is similar.

If G_i^x has a simple infeasible loop P, then P is either strict or non-strict. Let m denote the sum of the constants c around the loop P. Based on the definition of infeasible loop, if P is strict, then $m \leq 0$; otherwise, $m < 0$. By Definition 3, if a strict inequality $x_a - x_b < c$ is in S_i^x, then c is equal to $-\sigma$ or $-\tau\sigma$; if a non-strict inequality $x_a - x_b \leq c$ is in S_i^x, then c is equal to σ or $\tau\sigma$, where τ, σ are positive numbers (hence $c > 0$). If P is non-strict, then all the inequalities in it are of the form $x_a - x_b \leq c$ where $c > 0$ and the sum of such c is positive. This contradicts the fact that $m < 0$ for non-strict infeasible loops. Therefore P is strict, hence $m \leq 0$. We consider the two cases where $m = 0$ and $m < 0$ separately.

1. If $m = 0$, then the sum of the constants around the loop P is equal to 0. Without loss of generality, let us assume P consists of vertices $xa_0, xa_1, ..., xa_{n-1}, xa_0$. Since P is admissible, the linear inequalities in P are of the form $(xa_0 - xa_1)?c_1, ..., (xa_{n-1} - xa_0)?c_n$, where ? is $\leq$ or $<$, and for every i such that $1 \leq i \leq n$, c_i is σ, $-\sigma$, $\tau\sigma$ or $-\tau\sigma$. Then we translate the linear inequalities in P to formulas as follows. We translate every linear inequality of the form $x_a - x_b < -\sigma$ to $W(a, b)$; every $x_a - x_b < -\tau\sigma$ to $dW(a, b)$; every $x_a - x_b \leq \sigma$ to $\neg E(a, b)$; every $x_a - x_b \leq \tau\sigma$ to $\neg dE(a, b)$. In this way, from P we obtain a sequence of formulas of the form $R_1(a_0, a_1), ..., R_n(a_{n-1}, a_0)$, where for every i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the sum of the constants around P is equal to 0, $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$ and $n \geq 2$. By Lemma 13, $\perp$ can be derived from $R_1(a_0, a_1), ..., R_n(a_{n-1}, a_0)$. By Definition 3, for every occurrence of $W(a, b)$ in $R_1(a_0, a_1) \wedge ... \wedge R_n(a_{n-1}, a_0)$, it or $E(b, a)$ is a conjunct in ϕ_i; similarly, for every occurrence of $dW(a, b)$, it or $dE(b, a)$ is a conjunct in ϕ_i; for every occurrence of $\neg E(a, b)$, it or $\neg W(b, a)$ is a conjunct in ϕ_i; for every occurrence of $\neg dE(a, b)$, it or $\neg dW(b, a)$ is a conjunct in ϕ_i. By AS 2, $W(a, b) \leftrightarrow E(b, a)$. By Definition 2, AS 2 and AS 3, $dW(a, b) \leftrightarrow dE(b, a)$. Therefore, $\perp$ is τ-derivable from ϕ_i.
2. If $m < 0$, then the sum of the constants around the loop P is negative. In the same way described above, from P we obtain a sequence of formulas of the form $R_1(a_0, a_1), ..., R_n(a_{n-1}, a_0)$, where for every i such that $1 \leq i \leq n$, $R_i \in \{W, dW, \neg E, \neg dE\}$. Since the sum of the constants around the loop P is negative, $number(W) + \tau * number(dW) > number(\neg E) + \tau * number(\neg dE)$ and $n \geq 1$. By Lemma 14, $\perp$ can be derived from $R_1(a_0, a_1) \wedge ... \wedge R_n(a_{n-1}, a_0)$. Following the same argument above, $\perp$ is τ-derivable from ϕ_i.

In each case, $\perp$ is τ-derivable from ϕ_i. Thus every disjunct ϕ_i is not τ-consistent, hence ϕ is not τ-consistent. This contradicts the fact that Σ is τ-consistent. □

4.3 When $\tau > 3$, LD is Not Finitely Axiomatisable

In this section, we will show that for every $\tau > 3$, LD is not finitely axiomatisable.

Lemma 17. *For every $\tau \in \mathbb{N}_{>1}$, every integer $n > 1$, the following formula A_n is an axiom:* $W(a_0, a_1) \wedge W(a_1, a_2) \wedge dW(a_2, b_1) \wedge dW(b_1, b_2) \wedge \cdots \wedge dW(b_{n-1}, b_n) \wedge \neg E(b_n, c_1) \wedge \neg E(c_1, c_2) \wedge \neg dE(c_2, d_1) \wedge \neg dE(d_1, d_2) \wedge \cdots \wedge \neg dE(d_{n-1}, d_n) \rightarrow \perp$, *where* $d_n = a_0$.

Proof. In A_n, $number(W) = 2$, $number(dW) = n$, $number(\neg E) = 2$ and $number(\neg dE) = n$. Hence $number(W) + \tau * number(dW) = number(\neg E) + \tau * number(\neg dE)$. By AS 5 in the LD^τ calculus, A_n is an axiom. □

Lemma 18. *For every $\tau \in \mathbb{N}_{>3}$, every integer $n > 1$, every integer $m > 1$, $m \neq n$, the following two axioms are independent:*

1. $A_n = (W(a_0, a_1) \wedge W(a_1, a_2) \wedge dW(a_2, b_1) \wedge dW(b_1, b_2) \wedge \cdots \wedge dW(b_{n-1}, b_n) \wedge \neg E(b_n, c_1) \wedge \neg E(c_1, c_2) \wedge \neg dE(c_2, d_1) \wedge \neg dE(d_1, d_2) \wedge \cdots \wedge \neg dE(d_{n-1}, d_n) \rightarrow \perp)$, *where* $d_n = a_0$
2. $A_m = (W(a_0, a_1) \wedge W(a_1, a_2) \wedge dW(a_2, b_1) \wedge dW(b_1, b_2) \wedge \cdots \wedge dW(b_{m-1}, b_m) \wedge \neg E(b_m, c_1) \wedge \neg E(c_1, c_2) \wedge \neg dE(c_2, d_1) \wedge \neg dE(d_1, d_2) \wedge \cdots \wedge \neg dE(d_{m-1}, d_m) \rightarrow \perp)$, *where* $d_m = a_0$.

Proof. Without loss of generality, let us suppose $m > n$. We will show that there exists a graph model where A_n is false and A_m is true and there there exists a graph model where A_m is false and A_n is true.

For every conjunct in the antecedent of A_n, we translate it into a linear inequality by Definition 3. Then we obtained a sequence of linear inequalities S_1: $x_{a_0} - x_{a_1} < -\sigma$, $x_{a_1} - x_{a_2} < -\sigma$, $x_{a_2} - x_{b_1} < -\tau\sigma$, $x_{b_1} - x_{b_2} < -\tau\sigma$, $\ldots$, $x_{b_{n-1}} - x_{b_n} < -\tau\sigma$, $x_{b_n} - x_{c_1} \leq \sigma$, $x_{c_1} - x_{c_2} \leq \sigma$, $x_{c_2} - x_{d_1} \leq \tau\sigma$, $x_{d_1} - x_{d_2} \leq \tau\sigma$, $\ldots$, $x_{d_{n-1}} - x_{a_0} \leq \tau\sigma$. We construct a graph $G_1 = (V_1, E_1)$ for S_1 as shown in Section 3.1. Then G_1 contains a vertex for each variable in S_1 and an edge for each inequality, where each vertex is labelled with its associated variable and each edge is labelled with its associated inequality. It is clear that the linear inequalities in S_1 form a loop P_1 and the sum of constants around P_1 is equal to 0. By Definition 3, G_1 is a model of $\neg A_n$ (A_n is false in G_1).

Similarly, we translate the antecedent of A_m into a sequence of linear inequalities S_2 by Definition 3. Then we construct a graph $G_2 = (V_2, E_2)$ for S_2 as shown in Section 3.1: $x'_{a_0} - x'_{a_1} < -\sigma$, $x'_{a_1} - x'_{a_2} < -\sigma$, $x'_{a_2} - x'_{b_1} < -\tau\sigma$, $x'_{b_1} - x'_{b_2} < -\tau\sigma$, $\ldots$, $x'_{b_{m-1}} - x'_{b_m} < -\tau\sigma$, $x'_{b_m} - x'_{c_1} \leq \sigma$, $x'_{c_1} - x'_{c_2} \leq \sigma$, $x'_{c_2} - x'_{d_1} \leq \tau\sigma$, $x'_{d_1} - x'_{d_2} \leq \tau\sigma$, $\ldots$, $x'_{d_{m-1}} - x'_{a_0} \leq \tau\sigma$. The linear inequalities in S_2 form a loop P_2 and the sum of constants around P_2 is equal to 0. By Definition 3, G_2 is a model of $\neg A_m$ (A_m is false in G_2).

To show A_m is true in G_1 (G_1 is not a model of $\neg A_m$), it is sufficient to show that it is impossible to define an interpretation function f from V_2 to V_1 such that f preserves all the linear inequalities in S_2. Suppose such a function f exists. Since $\tau > 3$, from $x_{a_0} - x_{a_1} < -\sigma$, $x_{a_1} - x_{a_2} < -\sigma$ in S_1, we cannot obtain $x_{a_0} - x_{a_2} < -\tau\sigma$. In order to preserve the linear inequalities of the form $x'_{b_j} - x'_{b_{j+1}} < -\tau\sigma$ in S_2, f should map $x'_{b_j}, x'_{b_{j+1}}$, where $x'_{b_0} = x'_{a_2}$, $j \in [0, m)$, to x_{b_g}, x_{b_h} respectively, where $g < h$, and $g, h \in [0, n]$. Since $m > n$, the number of $x'_{b_0} \ldots x'_{b_m}$, is larger than the number of $x_{b_0} \ldots x_{b_n}$. Hence at least two of $x'_{b_0} \ldots x'_{b_m}$, saying x'_{b_s} and x'_{b_t}, will be mapped to the same x_{b_k} in $x_{b_0} \ldots x_{b_n}$. Thus, $f(x'_{b_s}) - f(x'_{b_t}) = 0$. However, the linear inequality

$x'_{b_s} - x'_{b_t} < c$ (or $x'_{b_t} - x'_{b_s} < c$), where $c \leq -\tau\sigma < 0$, cannot be preserved by f. Hence A_m is true in G_1.

To show A_n is true in G_2 (G_2 is not a model of $\neg A_n$), it is sufficient to show that it is impossible to define an interpretation function f from V_1 to V_2 such that f preserves all the linear inequalities in S_1. Suppose such a function exists. Since $\tau > 3$, from $x'_{a_0} - x'_{a_1} < -\sigma$, $x'_{a_1} - x'_{a_2} < -\sigma$ in S_2, we cannot obtain $x'_{a_0} - x'_{a_2} < -\tau\sigma$. Since $\tau > 3$, from $x'_{d_{m-1}} - x'_{a_0} \leq \tau\sigma$, $x'_{a_0} - x'_{a_1} < -\sigma$, $x'_{a_1} - x'_{a_2} < -\sigma$, we cannot obtain $x'_{d_{m-1}} - x'_{a_2} < \sigma$. In order to preserve the linear inequalities of the form $x_{b_j} - x_{b_{j+1}} < -\tau\sigma$ in S_1, f should map $x_{b_j}, x_{b_{j+1}}$, where $x_{b_0} = x_{a_2}$ and $j \in [0, n)$, to x'_{b_g}, x'_{b_h} respectively, where $g < h$ and $g, h \in [0, m]$. Let $x_{b_0} = x_{a_2}$, $x_{c_0} = x_{b_n}$. Since $m > n$, there exist $x_{a_s}, x_{a_{s+1}}$ such that $x_{a_s} - x_{a_{s+1}} < -\sigma$, and $f(x_{a_s}) - f(x_{a_{s+1}}) < c'$, where $c' < -\sigma$; or there exist $x_{b_s}, x_{b_{s+1}}$ such that $x_{b_s} - x_{b_{s+1}} < -\tau\sigma$, and $f(x_{b_s}) - f(x_{b_{s+1}}) < c'$, where $c' < -\tau\sigma$; or there exist $x_{c_s}, x_{c_{s+1}}$ such that $x_{c_s} - x_{c_{s+1}} \leq \sigma$, and $f(x_{c_s}) - f(x_{c_{s+1}}) < c'$, where $c' < \sigma$. Since $x_{a_0} - x_{c_2} < (-2\sigma - n\tau\sigma + 2\sigma) = -n\tau\sigma$ (obtained from S_1), $f(x_{a_0}) - f(x_{x_{c_2}}) < c'$, where $c' < -n\tau\sigma$. Since $x_{x_{c_2}} - x_{a_0} \leq n\tau\sigma$ (obtained from S_1), if f exists, $f(x_{x_{c_2}}) - f(x_{a_0}) \leq c''$, where $c'' \leq n\tau\sigma$, this contradicts the fact that $f(x_{a_0}) - f(x_{x_{3n}}) < c'$, where $c' < -n\tau\sigma$. Hence A_n is true in G_2. □

Theorem 8. *There is no axiom $\mathcal{A}$ such that $\mathcal{A}$ is valid in 2D Euclidean models and $\mathcal{A}$ entails all the axioms A_n, where $\tau \in \mathbb{N}_{>3}$, $n \in \mathbb{N}_{>1}$.*

Proof. Suppose there exists an axiom $\mathcal{A}$ such that $\mathcal{A}$ is valid in 2D Euclidean models and $\mathcal{A}$ entails all the axioms A_n, where $n \in \mathbb{N}_{>1}$. Clearly, $\mathcal{A}$ is a formula over some finite number of individual names t. Without loss of generality, let us suppose $t > 2$. We construct a graph model G such that G satisfies $\neg A_n$ for some $2n + 4 > t$. Then we show that any LD property over at most t individual names which is true in G also true in some 2D Euclidean model. Hence all instances of $\mathcal{A}$ are true in G, because otherwise their negation would have been satisfiable in a 2D Euclidean model. Hence G satisfies all instances of $\mathcal{A}$ and $\neg A_n$: a contradiction with the assumption that $\mathcal{A}$ entails A_n.

For every conjunct in the antecedent of A_n, we translate it into a linear inequality by Definition 3. Then we obtained a sequence of linear inequalities S: $x_{a_0} - x_{a_1} < -\sigma$, $x_{a_1} - x_{a_2} < -\sigma$, $x_{a_2} - x_{b_1} < -\tau\sigma$, $x_{b_1} - x_{b_2} < -\tau\sigma, \ldots, x_{b_{n-1}} - x_{b_n} < -\tau\sigma$, $x_{b_n} - x_{c_1} \leq \sigma$, $x_{c_1} - x_{c_2} \leq \sigma$, $x_{c_2} - x_{d_1} \leq \tau\sigma$, $x_{d_1} - x_{d_2} \leq \tau\sigma, \ldots, x_{d_{n-1}} - x_{a_0} \leq \tau\sigma$. We construct a graph $G = (V, E)$ for S as shown in Section 3.1. Then G contains a vertex for each variable in S and an edge for each inequality, where each vertex is labelled with its associated variable and each edge is labelled with its associated inequality. It is clear that the linear inequalities in S form a loop and the sum of constants around the loop is equal to 0. By Definition 3, G is a model of $\neg A_n$.

By construction, G is over $2n + 4$ individual names. Take any t individual names $o_1, \ldots, o_t$ from G, where $2 < t < 2n + 4$. Without of loss of generality, let us suppose we have a sequence of linear inequalities S_t: $(o_1 - o_2)?c_1, \ldots, (o_{t-1} - o_t)?c_{t-1}, (o_t - o_1)?c_t$, where $? \in \{<, \leq\}$, each linear inequality is either a linear inequality in S or obtained from linear inequalities in S. For any linear inequality $(x - y)?c$ in S, if $c < 0$, then ? is $<$; if $c > 0$, then ? is $\leq$. Thus, if $(o_i - o_{i+1})?c_i$ and $c_i \leq 0$, then ? is $<$. Since the sum of constants around the loop defined by S is 0, we have $c_1 + \cdots + c_t = 0$. Hence $o_1, \ldots, o_t$ form an infeasible loop P, which cannot be realized in 1D Euclidean space. However, not all of the linear inequalities may be expressed precisely using LD formulas. For example, if $(o_i - o_{i+1}) < -2\sigma$, then $W(o_i, o_{i+1})$ holds, but $dW(o_i, o_{i+1})$ does not hold, since $\tau > 3$. Since we want to show 'any LD property over at most t individual names which is true in G also true in some 2D Euclidean model', we only need to guarantee $W(o_i, o_{i+1})$ to be true in some 1D Euclidean model, as the linear inequalities in S are over x-coordinates only. So we can replace $(o_i - o_{i+1}) < -2\sigma$ in S_t with $(o_i - o_{i+1}) < -\sigma$, which is the corresponding inequality of $W(o_i, o_{i+1})$. Then we only need to show the resulting loop can be realized in 1D Euclidean space.

The 'replacement' of linear inequalities in S_t is defined as follows. For any pair of individual names o_i, o_j,

Rule 1: if there is a linear inequality $(o_i - o_j) < c$ in S_t, where $c < -\tau\sigma$, then we replace it with $o_i - o_j < -\tau\sigma$ to preserve the property $dW(o_i, o_j)$;

Rule 2: if there is a linear inequality $(o_i - o_j) < c$ in S_t, where $-\tau\sigma < c < -\sigma$, then we replace it with $o_i - o_j < -\sigma$ to preserve the property $W(o_i, o_j)$;

Rule 3: if there is a linear inequality $(o_i - o_j) < c$ in S_t, where $-\sigma < c \leq 0$, then we replace it with $o_i - o_j \leq \sigma$ to preserve the property $\neg E(o_i, o_j)$;

Rule 4: if there is a linear inequality $(o_i - o_j)?c$ in S_t, where $? \in \{<, \leq\}$, $0 < c < \sigma$, then we replace it with $o_i - o_j \leq \sigma$ to preserve the property $\neg E(o_i, o_j)$;

Rule 5: if there is a linear inequality $(o_i - o_j) < \sigma$ in S_t, then we replace it with $o_i - o_j \leq \sigma$ to preserve the property $\neg E(o_i, o_j)$;

Rule 6: if there is a linear inequality $(o_i - o_j)?c$ in S_t, where $? \in \{<, \leq\}$, $\sigma < c < \tau\sigma$, then we replace it with $o_i - o_j \leq \tau\sigma$ to preserve the property $\neg dE(o_i, o_j)$;

Rule 7: if there is a linear inequality $(o_i - o_j) < \tau\sigma$ in S_t, then we replace it with $o_i - o_j \leq \tau\sigma$ to preserve the property $\neg dE(o_i, o_j)$;

Rule 8: if there is a linear inequality $(o_i - o_j)?c$ in S_t, where $? \in \{<, \leq\}$, $c > \tau\sigma$, then this property cannot be expressed using a single formula $R \in \{W, dW, \neg E, \neg dE\}$ nor a boolean combination of such R, except for $\top$ which is always true in a 2D Euclidean model. Hence we do not need to consider the case where $c > \tau\sigma$.

Since the sum of constants $c_1, \ldots, c_t$ around the loop P is equal to 0, P is strict. Note that after applying at least one of Rules 1-4 and 6, the sum of constants $c_1, \ldots, c_t$ around the loop P will always be increased. Then $c_1 + \cdots + c_t > 0$, P will not be infeasible any more, hence can be realized in 1D Euclidean space. Rules 5 and 7 only change $<$ to $\leq$, but do not change the constant value of the linear inequality, hence do not affect the sum of constants. Since P is strict and the constants involved in Rules 5 and 7 are both positive, if P is infeasible, then after applying Rule 5 or 7, then it is still infeasible.

Let us proceed by contradiction. Suppose none of the Rules 1-4 and 6 is applied. Then every linear inequality in S_t is the same as a corresponding inequality of R, where $R \in \{W, dW, \neg E, \neg dE\}$, or it is of the form $(o_i - o_j) < \sigma$ or $(o_i - o_j) < \tau\sigma$. Let $o_{t+1} = o_1$. Since $t < 2n + 4$, then there exists at least one pair of individual names o_i, o_{i+1}, where $i \in [1, t]$, such that $(o_i - o_{i+1})?c_i$ is not in S, but obtained from a sequence of at least two linear inequalities from S. Let us prove by cases. By Rule 8, we do not need to consider the case where $c > \tau\sigma$. Let us assume $c_i \leq \tau\sigma$.

1. If $(o_i - o_{i+1})?c_i$ is obtained from a sequence of at least two linear inequalities in S such that all the linear inequalities correspond to the same LD formula $R \in \{W, dW, \neg E, \neg dE\}$,
   (a) if R is W, then $c_i = -2\sigma$. Since $\tau > 3$, Rule 2 will be applied, which contradicts the assumption that none of the Rules 1-4 and 6 is applied.
   (b) if R is dW, then $-n\tau\sigma \leq c_i \leq -2\tau\sigma$. Thus, Rule 1 will be applied, which contradicts the assumption that none of the Rules 1-4 and 6 is applied.
   (c) if R is $\neg E$, then $c_i = 2\sigma$. Since $\tau > 3$, Rule 6 will be applied, which contradicts the assumption that none of the Rules 1-4 and 6 is applied.
   (d) if R is $\neg dE$, then $c_i > \tau\sigma$, which contradicts the assumption that $c_i \leq \tau\sigma$.
2. If $(o_i - o_{i+1})?c_i$ is obtained from a sequence of at least two linear inequalities in S such that all the linear inequalities correspond to two LD formulas R_1 and R_2 such that $R_1, R_2 \in \{W, dW, \neg E, \neg dE\}$ and $R_1 \neq R_2$:
   (a) if $R_1, R_2 \in \{W, dW\}$, then $c_i < -\tau\sigma$, hence Rule 1 will be applied, which contradicts the assumption that none of the Rules 1-4 and 6 is applied.
   (b) if $R_1, R_2 \in \{dW, \neg E\}$, since $\tau > 3$, $c_i < -\sigma$ and $c_i \neq -\tau\sigma$. Hence Rule 2 or 1 will be applied, which contradicts the assumption that none of the Rules 1-4 and 6 is applied.
   (c) if $R_1, R_2 \in \{\neg E, \neg dE\}$, then $c_i > \tau\sigma$, which contradicts the assumption that $c_i \leq \tau\sigma$.
   (d) if $R_1, R_2 \in \{\neg dE, W\}$, since $\tau > 3$, $\sigma < c_i < \tau\sigma$ or $c_i > \tau\sigma$. If $\sigma < c_i < \tau\sigma$, then Rule 6 will be applied, which contradicts the assumption that none of the Rules 1-4 and 6 is applied. Otherwise, $c_i > \tau\sigma$, which contradicts the assumption that $c_i \leq \tau\sigma$.
3. If $(o_i - o_{i+1})?c_i$ is obtained from a sequence of at least two linear inequalities in S such that all the linear inequalities correspond to three LD formulas R_1, R_2 and R_3 such that $R_1, R_2, R_3 \in \{W, dW, \neg E, \neg dE\}$ and R_1, R_2 and R_3 are all different:
   (a) if $R_1, R_2, R_3 \in \{W, dW, \neg E\}$, then all the linear inequalities $x_{a_2} - x_{b_1} < -\tau\sigma$, $x_{b_1} - x_{b_2} < -\tau\sigma$, $\ldots$, $x_{b_{n-1}} - x_{b_n} < -\tau\sigma$ are involved in the sequence. Since $\tau > 3$, $c_i < -\tau\sigma$. Hence Rule 1 will be applied, which contradicts the assumption that none of the Rules 1-4 and 6 is applied.
   (b) if $R_1, R_2, R_3 \in \{dW, \neg E, \neg dE\}$, then the linear inequalities $x_{b_n} - x_{c_1} \leq \sigma$, $x_{c_1} - x_{c_2} \leq \sigma$ are involved in the sequence. Since $\tau > 3$, $c_i > \tau\sigma$, or $\sigma < c_i < \tau\sigma$, or ($c_i < -\sigma$ and $c_i \neq -\tau\sigma$). If $\sigma < c_i < \tau\sigma$, then Rule 6 will be applied, which contradicts the assumption that none of the Rules 1-4 and 6 is applied. If $c_i < -\sigma$ and $c_i \neq -\tau\sigma$, then Rule 1 or Rule 2 will be applied, which contradicts the assumption that none of the Rules 1-4 and 6 is applied. Otherwise, $c_i > \tau\sigma$, which contradicts the assumption that $c_i \leq \tau\sigma$.
   (c) if $R_1, R_2, R_3 \in \{\neg E, \neg dE, W\}$, then all the linear inequalities $x_{c_2} - x_{d_1} \leq \tau\sigma$, $x_{d_1} - x_{d_2} \leq \tau\sigma$, $\ldots$, $x_{d_{n-1}} - x_{a_0} \leq \tau\sigma$ are involved in the sequence. Hence $c_i > \tau\sigma$, which contradicts the assumption that $c_i \leq \tau\sigma$.
   (d) if $R_1, R_2, R_3 \in \{\neg dE, W, dW\}$, then the linear inequalities $x_{a_0} - x_{a_1} < -\sigma$, $x_{a_1} - x_{a_2} < -\sigma$ are involved in the sequence. Since $\tau > 3$, $-\tau\sigma < c_i < -\sigma$, or $c_i < -\tau\sigma$, or $\sigma < c_i < \tau\sigma$, or $c_i > \tau\sigma$. If $-\tau\sigma < c_i < -\sigma$, then Rule 2 will be applied, which contradicts the assumption that none of the Rules 1-4 and 6 is applied. If $c_i < -\tau\sigma$, then Rule 1 will be applied, which contradicts the assumption that none of the Rules 1-4 and 6 is applied. If $\sigma < c_i < \tau\sigma$, then Rule 6 will be applied, which contradicts the assumption that none of the Rules 1-4 and 6 is applied. Otherwise, $c_i > \tau\sigma$, which contradicts the assumption that $c_i \leq \tau\sigma$.
4. If $(o_i - o_{i+1})?c_i$ is obtained from a sequence of at least two linear inequalities in S such that all the linear inequalities correspond to four LD formulas R_1, R_2, R_3 and R_4 such that $R_1, R_2, R_3, R_4 \in \{W, dW, \neg E, \neg dE\}$ and R_1, R_2, R_3 and R_4 are all different:
   (a) if all the linear inequalities in S corresponding to W and dW are involved in the sequence, then $c_i < -\tau\sigma$ since $t < 2n + 4$. Hence Rule 1 will be applied, which contradicts the assumption that none of the Rules 1-4 and 6 is applied.
   (b) if all the linear inequalities in S corresponding to dW and $\neg E$ are involved in the sequence, then $c_i < -\sigma$ and $c_i \neq -\tau\sigma$ since $t < 2n + 4$ and $\tau > 3$. Hence Rule 1 or 2 will be applied, which contradicts the assumption that none of the Rules 1-4 and 6 is applied.
   (c) if all the linear inequalities in S corresponding to $\neg E$ and $\neg dE$ are involved in the sequence, then $c_i > \tau\sigma$, since $t < 2n + 4$. This contradicts the assumption that $c_i \leq \tau\sigma$.
   (d) if all the linear inequalities in S corresponding to $\neg dE$ and W are involved in the sequence, then $\sigma < c_i < \tau\sigma$ or $c_i > \tau\sigma$, since $\tau > 3$ and $t < 2n + 4$. If $\sigma < c_i < \tau\sigma$, Rule 6 will be applied, which

contradicts the assumption that none of the Rules 1-4 and 6 is applied. Otherwise, $c_i > \tau\sigma$, which contradicts the assumption that $c_i \leq \tau\sigma$.

Therefore, in every case, a contradiction can be derived. Hence, in every case, at least one of the Rules 1-4 and 6 is applied. Since the sum of constants around the loop P becomes positive, the resulting loop can be realized in 1D Euclidean space. □

5 Decidability and Complexity of LD

We show that for every $\tau \in \mathbb{N}_{>1}$, the satisfiability problem for LD^τ is NP-complete.

Lemma 19. *For every $\tau \in \mathbb{N}_{>1}$, let S be a set of linear inequalities obtained by applying the 'τ-σ-translation' function over $L(LD)$ formulas as shown in Definition 3, where $\sigma = 1$; n be the number of variables in S, $n > 0$. If S is satisfiable, then it has a solution where for every variable, a rational number $t \in [-n\tau, n\tau]$ is assigned to it and the binary representation size of t is polynomial in n and τ.*

Proof. Take an arbitrary $\tau \in \mathbb{N}_{>1}$. By Definition 3, every linear inequality in S is of the form $x_1 - x_2 \leq c$ or $x_1 - x_2 < c$, where x_1, x_2 are real variables and c is a real number constant. Let G be a graph for S. By Corollary 1, S is satisfiable iff G has no simple infeasible loop. The construction of a solution of S is by extending the proof of Theorem 1 [Shostak, 1981] (pp. 777 and 778), which is for non-strict inequalities only, to include both strict and non-strict inequalities. If G has no simple infeasible loop, a solution of S can be constructed as follows. Let $v_1, \ldots, v_{n-1}$ be the variables of S other than v_0 (the zero variable). We construct a sequence $\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_{n-1}$ of reals (a solution of S) and a sequence $G_0, G_1, \ldots, G_{n-1}$ of graphs inductively:
1. Let $\hat{v}_0 = 0$ and $G_0 = G$.
2. If $\hat{v}_i$ and G_i have been determined for $0 \leq i < j < n$, let

$\sup_j = \min\{\frac{c_P}{a_P} \mid P$ is an admissible path from v_j to v_0 in G_{j-1} and $a_P > 0\}$

$\inf_j = \max\{\frac{c_P}{b_P} \mid P$ is an admissible path from v_0 to v_j in G_{j-1} and $b_P < 0\}$

where $\min \emptyset = \infty$ and $\max \emptyset = -\infty$. The range of $\hat{v}_j$ is obtained as follows.

- If there is an admissible path P from v_j to v_0 in G_{j-1} such that the residue inequality of P is $a_P v_j < c_P$, where $a_P > 0$, and $\frac{c_P}{a_P} = \sup_j$, then $\hat{v}_j < \sup_j$, otherwise, $\hat{v}_j \leq \sup_j$.
- If there is an admissible path P from v_0 to v_j in G_{j-1} such that the residue inequality of P is $b_P v_j < c_P$, where $b_P < 0$, and $\frac{c_P}{b_P} = \inf_j$, then $\hat{v}_j > \inf_j$, otherwise, $\hat{v}_j \geq \inf_j$.

Instead of letting $\hat{v}_j$ be any real number in the range [Shostak, 1981], we assign a value to $\hat{v}_j$ thus:

- if there exists an integer within the range of $\hat{v}_j$, we assign an integer to $\hat{v}_j$;
- otherwise, the range of $\hat{v}_j$ is of the form $\inf_j < \hat{v}_j < \sup_j$. Let $\hat{v}_j = \frac{\inf_j + \sup_j}{2}$.

Let G_j be obtained from G_{j-1} by adding two new edges from v_j to v_0, labelled $v_j \leq \hat{v}_j$ and $v_j \geq \hat{v}_j$ respectively.

To ensure that $\hat{v}_j$ and G_j are well defined, we need the following two claims:
1. For $1 \leq j < n$, the range of $\hat{v}_j$ is not empty.
2. For $0 \leq j < n$, G_j has no simple infeasible loop.

We prove them by induction on j, similar to the proof presented in [Shostak, 1981].
Base case $j = 0$. 1 holds vacuously; 2 holds since $G_0 = G$.
Inductive step Suppose the claim holds for $j-1$, $0 \leq j-1 < n-1$. We will show the claim holds for j.
For 1, suppose, to the contrary, that the range of $\hat{v}_i$ is empty. Then in G_{j-1}, there exist an admissible path P_1 from v_j to v_0, where $a_P > 0$, and an admissible path P_2 from v_0 to v_j, where $b_P < 0$. P_1 and P_2 forms an admissible loop. By the construction of the range of $\hat{v}_i$ described above, if this range is empty, then the admissible loop formed by P_1 and P_2 is infeasible, which contradicts the inductive hypothesis that G_{j-1} has no simple infeasible loop.
For 2, suppose G_j has a simple infeasible loop P. Since G_{j-1} has no such loop, and the loop formed by the two new edges added to G_{j-1} to obtain G_j is not infeasible, then P (or its reverse) is of the form $P'E$, where E is one of the two new edges (say the one labelled $v_j \leq \hat{v}_j$; the other case is handled similarly), and P' is a path from v_0 to v_j in G_{j-1}. Since P is infeasible, if P' is strict, $\hat{v}_j \leq \frac{c_{P'}}{b_{P'}}$, this contradicts that $\hat{v}_j > \inf_j$, since $\inf_j \geq \frac{c_{P'}}{b_{P'}}$; if P' is not strict, $\hat{v}_j < \frac{c_{P'}}{b_{P'}}$, this contradicts that $\hat{v}_j \geq \inf_j$, since $\inf_j \geq \frac{c_{P'}}{b_{P'}}$. Q.E.D.

Now it remains to show that $\hat{v}_j$ satisfies S. Let $ax + by \leq c$ be an inequality in S. We will show that $a\hat{x} + b\hat{y} \leq c$. We present the case where $a > 0$ and $b < 0$. The other cases are similar. Let E be the edge labelled $ax + by \leq c$ in G_{n-1}. Then, where E_1 is the edge labelled $\hat{x} \leq x$ in G_{n-1} and E_2 is the one labelled $y \leq \hat{y}$, $E_1 E E_2$ forms an admissible loop. Since G_{n-1} has no infeasible loop, $E_1 E E_2$ is feasible. Hence we have $a\hat{x} + b\hat{y} \leq c$. The proof for inequalities of the form $ax + by < c$ is similar.

By Definition 3, $-n\tau \leq c_P \leq n\tau$, $a_P = 1$ for $\sup_j$, $b_P = -1$ for $\inf_j$. Therefore, $\sup_j \leq n\tau$, $\inf_j \geq -n\tau$. Hence every $\hat{v}_j$ ($0 \leq j < n$) is a rational number in $[-n\tau, n\tau]$.

Now we will show that the representation size of $\hat{v}_j$ ($0 \leq j < n$) is polynomial in the size of n and τ. By the construction described above, $\hat{v}_j$ is either an integer in $[-n\tau, n\tau]$ or obtained by applying the 'average operation' $\hat{v}_j = \frac{\inf_j + \sup_j}{2}$. Since τ is a natural number and $\sigma = 1$, $\inf_1$ and $\sup_1$ are integers in $[-n\tau, n\tau]$. Also, since $0 < j < n$, the number of 'average operations' applied to obtain a $\hat{v}_j$ is at most n. Hence the largest denominator of the values of $\hat{v}_j$ is 2^n. Therefore, $\hat{v}_j$ can be represented in a binary notation (bits) of size $\log(2n\tau * 2^n)$, which is in $O(n + \log\tau)$. Hence the representation size of $\hat{v}_j$ is polynomial in n and τ. □

Definition 4. *Let ϕ be an $L(LD)$ formula. Its size $s(\phi)$ is defined as follows:*

- $s(R(a, b)) = 3$, *where* $R \in \{E, W, I_{ew}, N, S, I_{ns}\}$;

- $s(\neg\phi) = 1 + s(\phi)$;
- $s(\phi \wedge \psi) = 1 + s(\phi) + s(\psi)$,

where $a, b \in A$, ϕ, ψ are formulas in $L(LD)$.

The combined size of $L(LD)$ formulas in a set S is defined as the size of the conjunction of all formulas in S.

Theorem 9. *For every $\tau \in \mathbb{N}_{>1}$, the satisfiability problem for a finite set of $L(LD)$ formulas in a 2D Euclidean τ-model is NP-complete.*

Proof. Take an arbitrary $\tau \in \mathbb{N}_{>1}$. NP-hardness is from propositional logic being included in LD^τ. To prove that the satisfiability problem for each LD^τ is in NP, we show that if a finite set of $L(LD)$ formulas Σ is τ-satisfiable, then we can guess a 2D Euclidean τ-model for Σ and verify that this model satisfies Σ, both in time polynomial in the combined size of formulas in Σ and τ. Let s and n denote the combined size of formulas in Σ and the number of individual names in Σ respectively. By Definition 4, $n < s$. As σ is a scaling factor, if Σ is τ-satisfiable, it is τ-satisfiable in a model where $\sigma = 1$.

Following the proof of Theorem 3 (first paragraph), Σ is satisfiable in a 2D Euclidean τ-model, iff there exists an S_i such that its subsets S_i^x and S_i^y are both satisfiable, where S_i^x and S_i^y are sets of linear inequalities obtained by applying the τ-σ-translation function over $L(LD)$ formulas as shown in Definition 3. By Lemma 19, if S_i^x is satisfiable, then it has a solution where for every variable, a rational number $t \in [-n\tau, n\tau]$ is assigned to it and the representation size of t is in $O(n+\log\tau)$ (polynomial in n and τ). The same holds for S_i^y. Hence for every individual name in Σ, we can guess such a pair of rational numbers for it in $O(n + \log\tau)$. Thus we can guess a 2D Euclidean τ-model M for Σ in $O(n^2 + n\log\tau)$, in time polynomial in n and τ. To verify that M satisfies Σ, we need to check every formula in Σ. For any $R(a, b)$, where $R \in \{E, W, I_{ew}, N, S, I_{ns}\}$, $a, b \in A$, checking that $R(a, b)$ is true in M takes $O(n + \log\tau)$ time by Definition 1 and applying bit operations. Hense, checking all formulas in Σ takes time polynomial in s and τ. □

An alternative decidability/membership of NP proof could use reduction to a finite set of disjunctive linear relations (DLRs) [Jonsson and Bäckström, 1998] or a Q_{basic} formula [Kreutzmann and Wolter, 2014].

6 Conclusion and Future Work

We have introduced a new qualitative logic of directions LD for reasoning about directions in 2D Euclidean space. We have shown it to be sound and complete, and that its decidability is NP-complete. The logic incorporates a margin of error and a level of indeterminacy in directions, that allow it to be used to compare and reason about not perfectly aligned representations of the same spatial objects in different datasets (for example, hand sketches or crowd sourced digital maps). While there have been many spatial calculi previously proposed (as discussed in the introduction), LD is unique in allowing indeterminate directions which we believe are crucial in practice. Moreover, many previous spatial calculi have not been treated to the same theoretical analysis that we do here (i.e. the soundness, completeness and complexity results in this paper). In future work, we plan to combine the logics for qualitative distances [Du *et al.*, 2013; Du and Alechina, 2016] and qualitative directions, and develop reasoners for checking the consistency of matching relations automatically.

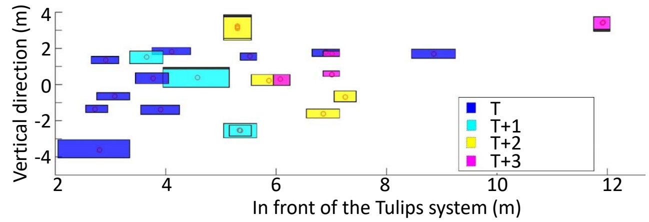

Figure 2: Detected events (rectangles) and their centroids (circles within the rectangles) at different times ahead of a TBM (from [Wei *et al.*, 2019]; best viewed in colour).

We also plan to experiment with the logic on actual data in a variety of possible application scenarios. One such scenario could be in spatial data fusion. E.g. consider Figure 2; this shows detections of possible 'events' (such as a karst or an anthropomorphic structure) ahead of a Tunnel Boring Machine (TBM) from sensors mounted on the front of the TBM at different times and spatial locations as the TBM advances through the ground. The detected events will typically appear at different absolute spatial locations because as the TBM advances the sensors are better able to detect and localise features – sensors only ever give approximate locations. The challenge is to determine which events at the different time points correspond. The relative positions/directions of the events can be represented using LD. (Of course LD is a logic of points, not regions, but for the purposes of this example we can use the centroid or, probably better, the end points, or just the nearest endpoint since that will have best signal.) In [Wei *et al.*, 2019] simple overlap is used to decide whether two events are the same or not. We hypothesize that it is possible to build a more nuanced system using LD. Events which are dE or dW of each other, may be regarded as discrete events; but if they are nEW then they are candidates to be the same event. By varying σ and τ different levels of tolerance and indeterminacy could be considered and presented to the TBM experts for further analysis and verification.

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