ECON511 Simulation Project

THE COST OF NEGLECTION OF RESIDUAL CROSS-CORRELATION IN COMPUTING IMPULSE RESPONSE FUNCTIONS OF A BIVARIATE VAR(1) SYSTEM WITH WHITE NOISE ERROR TERMS

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DATA GENERATING PROCESS (DGP)

The series y_{1t} and y_{2t} have $y_{10} = 0$ and $y_{20} = 0$. This is set for simplicity and to initialize the Data Generating Process.

The series are pre-determined as stationary through setting the coefficients such that they satisfy the stationarity condition.

The DGP is implemented for both Reduced Form VAR (RFVAR) and Adjusted-RFVAR (ARFVAR henceforth), which takes care of the residual cross-correlation. The objective is to observe the 'cost' of neglecting error cross-correlation by comparing the Impulse Response Functions (IRFs) of the two VAR forms. The problem is that IRFs are partial derivatives of a series with respect to a shock, which presume that these shocks are not correlated. We want to observe how much bias cross-correlated errors add to IRFs. Note that, by construction, the 2 series have no residual autocorrelation.

For simplicity, the VAR system consists of 2 variables and 2 equations. The model may resemble what Stock and Watson (2001) conducts with 3 variables (Inflation, Unemployment and Interest Rate) within a VAR with 3 equations. As mentioned, for simplicity, this exercise works with only 2 variables (i.e., 2 equations) within a VAR system.

The seed of random number generation is set to "123" in order to get the same results for the same parameters, and not regenerate randomness.

Needless to say, the specification of nuisance parameters may differentiate the results obtained post-simulation. Thus, there will be 4 different models from which data will be generated:

1) VAR with no intercept term and no time trend
2) VAR with an intercept term (set to 21) but no time trend
3) VAR with no intercept term but a time trend with drift 0.01
4) VAR with an intercept term (set to 21) and a time trend with drift 0.01

In addition to this,4 levels of error cross-correlation are simulated for comprehensiveness. The levels are 0, 0.3, 0.6 and 0.9, where the first can be interpreted as no, the second as low, the third as moderate and the fourth as high residual cross-correlation. Therefore, considering every combination of parameter and residual cross-correlation specification, there are 12 different configurations to examine.

Furthermore, the coefficients of the variables on the right hand side (RHS) of the VAR equations have to be specified. For stationarity purposes, they are set such that their absolute value is less than 1:

```
In [5]: # Coefficients for the VAR equations (AR(1))
        coefficients = np.array([[0.6, -0.4],   # Coefficients for lagged y1
                                 [0.3, 0.8]])   # Coefficients for lagged y2
```

Next, the variance-covariance matrix of the residuals have to be specified. For simplicity, the variances of both of the residuals are set to 1. This allows to interpret covariance directly as correlation, since the correlation formula is:

$$Correlation = \frac{Cov\ (x, y)}{\sigma x * \sigma y}$$

Where the denominator is the product of the standard deviation of the two series (in this case the series of the residuals), which are both equal to 1.

Last but not least, the residual cross-correlations are set to nonzero values to examine the 'cost' of residual cross-correlation. Hence, we have the following residual covariance-variance matrix:

```
# Generate covariance matrix with the specified cross-correlation
covariance_matrix = np.array([[1.0, correlation],
                              [correlation, 1.0]])
```

Note that the residuals are independent and identically (i.i.d.) White Noise (WN) distributed with constant mean 0 and variance 1 with no auto-correlation (identically distributed):

```
# Generate VAR errors with cross-correlation
mean = np.zeros(K)
errors = np.random.multivariate_normal(mean, covariance_matrix, size=T)
```

However, this does not imply that there is no residual cross-correlation. In fact, it is specified that cross-correlation exists despite the mentioned residual structure.

The Akaike Information Criteria (AIC) for lag values of 1 to 5 are computed albeit the DGP specified the RFVAR model to be VAR(1). This is to ensure that the DGP does what it is supposed to do. The following is the AIC results for each of the 12 configurations (differing by nuisance parameters and level of residual cross-correlation):

Cross-correlation: 0

Correlation between error series in 'errors': 0.082

AR Form: Intercept=False, Trend=False

AIC values: [0.253, 0.295, 0.333, 0.420, 0.453]

AR Form: Intercept=True, Trend=False

AIC values: [0.222, 0.261, 0.291, 0.359, 0.420]

AR Form: Intercept=False, Trend=True

AIC values: [0.303, 0.363, 0.366, 0.457, 0.507]

AR Form: Intercept=True, Trend=True

AIC values: [0.317, 0.377, 0.401, 0.418, 0.489]

Cross-correlation: 0.3

Correlation between error series in 'errors': 0.354

AR Form: Intercept=False, Trend=False

AIC values: [0.253, 0.295, 0.333, 0.420, 0.453]

AR Form: Intercept=True, Trend=False

AIC values: [0.222, 0.261, 0.291, 0.359, 0.420]

AR Form: Intercept=False, Trend=True

AIC values: [0.303, 0.363, 0.366, 0.457, 0.507]

AR Form: Intercept=True, Trend=True

AIC values: [0.317, 0.377, 0.401, 0.418, 0.489]

Cross-correlation: 0.6

Correlation between error series in 'errors': 0.621

AR Form: Intercept=False, Trend=False

AIC values: [0.253, 0.295, 0.333, 0.420, 0.453]

AR Form: Intercept=True, Trend=False

AIC values: [0.222, 0.261, 0.291, 0.359, 0.420]

AR Form: Intercept=False, Trend=True

AIC values: [0.303, 0.363, 0.366, 0.457, 0.507]

AR Form: Intercept=True, Trend=True

AIC values: [0.317, 0.377, 0.401, 0.418, 0.489]

Cross-correlation: 0.9

Correlation between error series in 'errors': 0.922

AR Form: Intercept=False, Trend=False

AIC values: [0.253, 0.295, 0.333, 0.420, 0.453]

AR Form: Intercept=True, Trend=False

AIC values: [0.222, 0.261, 0.291, 0.359, 0.420]

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AIC values: [0.303, 0.363, 0.366, 0.457, 0.507]

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AIC is conventionally used to determine number of lags in VAR literature. The lower the AIC, the better the fit of the model given the lag value p (Lütkepohl, 1993, Chapter 4). As expected (and specified), the supermajority of the configurations have the minimum AIC when the number of lags is 1.

Thus, the equations (in scalar form) from which we generate data for the ARFVAR with cross-correlated errors/residuals is:

$$y_{1,t} = \phi_{11} * y_{1,t-1} + \phi_{12} * y_{2,t-1} + \varepsilon_{1t}$$

$$y_{2,t} = \phi_{21} * y_{1,t-1} + \phi_{22} * y_{2,t-1} + \varepsilon_{2t}$$

i: Series I (2 series in total)

t: time period t (100 simulated time periods in total)

$y_{i,t}$: Value of series I at time period t

$\phi_{i,j}$: Coefficient of the jth series' lagged variable (one lag) to explain series i

$\varepsilon_{i,t}$: Error/residual value at time t of series i (encapsulates only self-contemporaneous shock)

ARFVAR DGP&Cholesky Decomposition

The DGP of ARFVAR is similar to that of RFVAR, except the implementation of Cholesky Decomposition (CD) to make the errors orthogonal and thus removing error cross-correlation:

```
# Perform Cholesky decomposition
chol_decomp = np.linalg.cholesky(covariance_matrix)

# Transform errors to make them orthogonal
orthogonal_errors = np.dot(errors, np.linalg.inv(chol_decomp.T))
```

For simplicity, in both DGPs, I did not include contemporaneous shocks of a variable in the equation of another variable; only the contemporaneous shock of the variable itself is included.

However, the past shocks are still present on the RHS of each VAR equation through the lagged variable of the other series, indicating error cross-correlation.

Bear in mind that in practical applications where contemporaneous shocks of a variable is included in the equation of another variable, the CD requires an imposition of a '0 short run' restriction in the residual coefficient matrix to handle the 'identification problem'[1]. In particular, one of the shocks is assumed to not have any contemporaneous effect on another variable. For a bivariate system, the identification problem implies the nonuniqueness of the Cholesky Decomposition:

$$BB' = \Omega$$

Where **Ω is the variance-covariance** matrix. B is not unique since there are 3 equations and 4 unknowns. The rest of the equations to demonstrate the identification problem, and the solution of the problem are omitted since this is not the main objective of this simulation exercise.

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0, 0.3, 0.6 \; or \; 0.9 \\ 0, 0.3, 0.6 \; or \; 0.9 & 1 \end{bmatrix}$$

In python, np.linalg.cholesky() that takes the residual variance-covariance matrix as parameter automatically handles this identification problem to solve for b_{11}, b_{12}, b_{21} and b_{22}. Needless to say, b_{12} and $b_{21}$are equivalent.

The errors are orthogonalized through (linear transformation) modifying the RFVAR to the following ARFVAR model presented in matrix form:

$B^{-1} * y_t = B^{-1} * \emptyset * y_{t-1} + B^{-1} * \varepsilon_t$

t: time period t (100 simulated time periods in total)

y_t: Series vector$_{(2x1)}$ at time period t

B^{-1}: inverse of CD matrix$_{(2x2)}$

$\emptyset$: Coefficient matrix$_{(2x2)}$ of the 1-lagged series y_{t-1}

ε_t: Error/residual matrix$_{(2x1)}$ at time t (encapsulates only self-contemporaneous shock)

Similarly, the AIC values are checked just in case. The lower triangular matrix B computed by CD for each cross-correlation level is also provided:

Cross-correlation: 0

Correlation between error series in 'orthogonal_errors': 0.023

Lower Triangular Matrix B:

[[1.00, 0.00],

[0.00, 1.00]]

AR Form: Intercept=False, Trend=False

AIC values: [-0.045, -0.015, 0.001, 0.071, 0.054]

AR Form: Intercept=True, Trend=False

[1] https://www.aptech.com/blog/understanding-and-solving-the-structural-vector-autoregressive-identification-problem/

AIC values: [-0.069, -0.002, 0.013, 0.095, 0.104]

AR Form: Intercept=False, Trend=True

AIC values: [0.098, 0.108, 0.092, 0.161, 0.156]

AR Form: Intercept=True, Trend=True

AIC values: [0.086, 0.100, 0.172, 0.204, 0.193]

Cross-correlation: 0.3

Correlation between error series in 'orthogonal_errors': -0.038

Lower Triangular Matrix B:

[[1.00, 0.00],

[0.30, 0.95]]

AR Form: Intercept=False, Trend=False

AIC values: [0.119, 0.163, 0.218, 0.250, 0.325]

AR Form: Intercept=True, Trend=False

AIC values: [0.061, 0.130, 0.168, 0.216, 0.267]

AR Form: Intercept=False, Trend=True

AIC values: [0.379, 0.333, 0.383, 0.392, 0.475]

AR Form: Intercept=True, Trend=True

AIC values: [0.321, 0.329, 0.359, 0.394, 0.425]

Cross-correlation: 0.6

Correlation between error series in 'orthogonal_errors': -0.122

Lower Triangular Matrix B:

[[1.00, 0.00],

[0.60, 0.80]]

AR Form: Intercept=False, Trend=False

AIC values: [-0.293, -0.224, -0.151, -0.078, -0.018]

AR Form: Intercept=True, Trend=False

AIC values: [-0.269, -0.214, -0.153, -0.125, -0.036]

AR Form: Intercept=False, Trend=True

AIC values: [-0.138, -0.090, -0.013, 0.059, 0.093]

AR Form: Intercept=True, Trend=True

AIC values: [-0.137, -0.080, -0.007, 0.055, 0.117]

Cross-correlation: 0.9

Correlation between error series in 'orthogonal_errors': 0.167

Lower Triangular Matrix B:

[[1.00, 0.00],

[0.90, 0.44]]

AR Form: Intercept=False, Trend=False

AIC values: [-0.003, 0.061, 0.148, 0.216, 0.230]

AR Form: Intercept=True, Trend=False

AIC values: [0.030, 0.098, 0.170, 0.262, 0.317]

AR Form: Intercept=False, Trend=True

AIC values: [0.178, 0.211, 0.299, 0.335, 0.388]

AR Form: Intercept=True, Trend=True

AIC values: [0.162, 0.213, 0.286, 0.348, 0.427]

One trend to notice is that RFVAR AIC values are slightly lower than ARFVAR AIC values, indicating that ARFVAR (even without imposing any restrictions while doing the CD) fits the simulated data worse than RFVAR. As expected, just as in the RFVAR model, the AIC values for different number of lags indicate the optimal number of lags is 1.

IRF COMPARISON

The IRFs for both RFVAR and ARFVAR (12 configurations for each, thus 24 plots in total) are plotted with their confidence intervals (CIs). The dotted curves surrounding the IRF estimations are the CIs, which indicate that with 95% confidence (i.e., significant level of 0.05) it can be said that our IRFs lie within these bands/intervals.

Let's interpret each RFVAR and ARFVAR configuration:

Impulse Response for Cross-correlation: 0, Intercept: False, Time Trend: False

Impulse Response for Cross-correlation: 0, Intercept: True, Time Trend: False

Impulse Response for Cross-correlation: 0, Intercept: False, Time Trend: True

Impulse Response for Cross-correlation: 0, Intercept: True, Time Trend: True

Conceivably, there is no significant difference for any of the configuration at error cross-correlation level 0, since CD gives a diagonal matrix. Thus, the RFVAR is just multiplied by a diagonal matrix, which does not change anything whatsoever.

Impulse Response for Cross-correlation: 0.3, Intercept: False, Time Trend: False

Very similar in terms of location and shape of the curve and also the width of the confidence intervals (CIs).

Impulse Response for Cross-correlation: 0.3, Intercept: True, Time Trend: False

Very similar.

Impulse Response for Cross-correlation: 0.3, Intercept: False, Time Trend: True

ARFVAR IRFs have less fluctuations and thus, the maximum and minimum IRF values, in absolute terms, are lower than those of RFVAR IRFs. The widths of the CIs are alike.

Impulse Response for Cross-correlation: 0.3, Intercept: True, Time Trend: True

There seems to be a slight upward bias in assesing the effect of the lagged shocks of the series 2 to the curent value of series 1.

Impulse Response for Cross-correlation: 0.6, Intercept: False, Time Trend: False

Since cross-correlation has been increased, the fluctuation difference is more visible than prior plots.There seems to be upward (in positive values) and downward (in negative values) bias on RFVAR IRF plots, upward bias slightly more significant.

Impulse Response for Cross-correlation: 0.6, Intercept: True, Time Trend: False

Aligned with the previous results, the plot implies that the addition of an intercept term as a nuisance parameter alleviates the difference of the IRFs of ARFVAR and RFVAR.

Impulse Response for Cross-correlation: 0.6, Intercept: False, Time Trend: True

Interestingly, all plots indicate that ARFVAR fluctuates much more and RFVAR is away from the ARFVAR values in absolute terms. The CIs of RFVAR IRFs are much wider than those of ARFVAR, indicating the unreliability of RFVAR for computing IRFs.

Impulse Response for Cross-correlation: 0.6, Intercept: True, Time Trend: True

As expected, the addition of intercept term mitigates the difference that has been observed in the case of Cross-correlation: 0.6, Intercept: False, Time Trend: True.

Impulse Response for Cross-correlation: 0.9, Intercept: False, Time Trend: False

The shocks of y_{2t} series to series y_{1t} plots indicate how the cross-correlation may be deceptive by inflating the absolute value of the shock coefficients, thereby giving them more weight in explaining the series on the LHS. The CIs of RFVAR IRFs are also much wider than those of ARFVAR.

Impulse Response for Cross-correlation: 0.9, Intercept: True, Time Trend: False

Aligned with the previous results, the plot implies that the addition of an intercept term as a nuisance parameter alleviates the difference of the IRFs of ARFVAR and RFVAR.

Impulse Response for Cross-correlation: 0.9, Intercept: False, Time Trend: True

Similar to prior results, all plots indicate that ARFVAR fluctuates much more and RFVAR is away from the ARFVAR values in absolute terms. The CIs of RFVAR IRFs are much wider than those of ARFVAR, indicating the unreliability of RFVAR for computing IRFs.

Impulse Response for Cross-correlation: 0.9, Intercept: True, Time Trend: True

The IRFs of ARFVAR take higher values in absolute terms. In particular, this can be inferred from the plots representing the impact of the shock of one series to the other series itself.

REFERENCES

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