Option pricing with random differential equations (RODEs)

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Abstract

SABR approximation does not guarantee arbitrage free prices, as a consequence researches have been focusing on developing SABR models that yield positive density function. Instead of improving SABR models, the paper presents a model that prices European options while being consistent with the static and dynamic market smile

1 Introduction

2 European option pricing

The financial market is defined by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, N)$, and an undiscounted call option on the reference rate R_T with strike K and expiry T is defined as follows

$$c(t, K) = E^N[(R_T - K)^+ \,|\mathcal{F}_t] \tag{1}$$

The forward rate F_t tied to the market rate R_T is

$$F_t = E^N[R_T|\mathcal{F}_t], \quad 0 \leq t \leq T \tag{2}$$

where F_0 is generated by building the interest yield curves. In practice, R_T can be a δ-tenor Libor rate fixed at time T and N is the $(T + \delta)$-forward measure, or a swap rate fixed at time T and N its natural annuity measure.

By construction, the random function $(F_t)_{0\leq t\leq T}$ is a $\mathcal{F}$-martingale , and is often modeled using a driftless stochastic process with initial value F_0. Traditionally, Black (3) and Bachelier-or Normal- (4) models are used, they respectively assume a log-normal and normal dynamics. Nowadays, the models are widely used as quotation format, their dynamics being inconsistent with the market observations.

$$dF_t = \sigma F_t dW_t \tag{3}$$

$$dF_t = \sigma dW_t \tag{4}$$

(3) and (4) are replaced by more sophisticated models such as the infamous SABR model.

2.1 SABR models

The Hagan et al SABR models for a forward rate F_t are introduced as follows :

$$\begin{aligned} dF_t &= \alpha_t C(F_t) dW_t \\ d\alpha_t &= \nu\alpha_t \left(\rho dW_t + \sqrt{1-\rho^2} dZ_t\right) \\ \alpha_0 &= \alpha \\ dW_t dZ_t &= 0 \end{aligned} \tag{5}$$

with C an arbitrary functional form. The most common choice for C is $C(F) = F^{\beta}$, which defines the classic SABR.

The model has the abilities to calibrate to the market smile through the approximated formula relating implied volatilities and SABR parameters .

For an option whose initial forward rate is F, with exercise date T, strike K, the normal volatility is given by the following formula

$$
\begin{aligned}
\sigma_N(F,K) &= \frac{\alpha\,(F-K)}{\int_K^F \frac{du}{C(u)}} \cdot \frac{\zeta}{x\,(\zeta)} \cdot \left[1 + \left[g\alpha^2 + \frac{1}{4}\rho\nu\alpha\frac{C(F)-C(K)}{F-K} + \frac{2-3\rho^2}{24}\nu^2\right] T\right] \\
\zeta &= \frac{\nu}{\alpha}\int_K^F \frac{du}{C(u)} \\
x(\zeta) &= log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\right) \\
g &= \frac{log\left(\frac{\sqrt{C(F)C(K)}}{F-K}\int_K^F \frac{du}{C(u)}\right)}{\left(\int_K^F \frac{du}{C(u)}\right)^2}
\end{aligned}
\tag{6}
$$

Hagan approximation is known to fail to replicate the actual SABR dynamics for long-term maturities, as well as generating arbitrage-full prices.

2.2 Terminal distribution

Using the definition of the forward rate F_t, (1) can be written as follows

$$c(t,K) = E^N\left[(F_T - K)^+ \,|\mathcal{F}_t\right] \tag{7}$$

Similarly to linear instrument pricing, the whole trajectory of F is not needed to price options, only the distribution of F_T at time T must be known. In this paper, F_T is modeled directly without introducing a stochastic differential equation.

The distribution of R_T is written as a function of n independent random variables $\{U_i\}_{i=1}^n$

$$R_T = G(U_1, U_2, ..., U_n) \tag{8}$$

Under this model, the price of a call option at time 0 is

$$c(0,K) = \int_{\mathbb{R}^n} \left[G(u_1,u_2,...,u_n) - K\right]^+ \prod_{i=1}^n f_{U_i}(u_i)du_i \tag{9}$$

with f_{U_i} the density function of the variable U_i.

The function G is chosen such as it replicates the market smile, the market backbone , last but not least, it must verify

$$F_0 = \int_{\mathbb{R}^n} G(u_1,u_2,...,u_n) \prod_{i=1}^n f_{U_i}(u_i)du_i \tag{10}$$

In this paper, G will be directed by a system of random ordinary differential equations (henceforth RODEs).

3 Random ordinary differential equations

Let $(\Omega, \mathcal{F}, N)$ denotes a probability space, and let $h : \mathbb{R}^d \times \mathbb{R}^n \longrightarrow \mathbb{R}^d$ be a continuous function.

A random ordinary differential equation in $\mathbb{R}^d$ is defined as follows,

$$\begin{aligned} \frac{dX}{dt} &= h(X, U(\omega)), \quad X \in \mathbb{R}^d \\ U(\omega) &:= (U_1(\omega), ..., U_n(\omega)) \end{aligned} \tag{11}$$

where $\{U_i\}_{i=1}^n$ are independent random variables.

For almost every realization $\omega \in \Omega$, the RODE (11) is also an autonomous ordinary differential equation (ODE)

$$\frac{dX}{dt} = H_\omega(X) := h(X, U(\omega)), \quad X \in \mathbb{R}^d \tag{12}$$

For the sack of clarity, we can remove the realizations of nil probability from Ω, hence (12) is valid for all realizations. The existence and uniqueness theorems for ODEs apply to (12) for a given realization ω.

For a given realization ω, we define $R_T(\omega) = g(X_T(\omega))$, where $X_T(\omega)$ is the solution of (12) at time T, the function $g : \mathbb{R}^d \longrightarrow \mathbb{R}$ verifies

$$E(g(X_T)) < \infty.$$

3.1 One-dimension Gaussian RODE

We define a family of one-dimension Gaussian RODE as follows

$$\begin{aligned} \frac{dX(t, u(\omega))}{dt} &= \alpha(u(\omega))\sigma_X(X(t, u(\omega))) \\ X(0, u(\omega)) &= x_0 \quad a.s \\ \omega &\in \Omega \\ u &\sim \mathcal{N}(0, 1) \\ 0 &\leq t \leq T \end{aligned} \tag{13}$$

with σ_X strictly positive and differentiable, and the linear map $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ almost surely.

Let $\omega \in \Omega$, using $H(t, X(t, u(\omega)))$, the solution of the following ODE

$$\begin{aligned} \frac{dH(t, X)}{dX} &= \sigma_X(X) \\ 0 &\leq t \leq T \end{aligned} \tag{14}$$

it can be proven that $X(T, u(\omega))$ is the solution of the following ODE

$$\frac{X(T, u(\omega))}{d\alpha(u(\omega))} = \sigma_X(X(T, u(\omega)))T \tag{15}$$

Solving the ODE (15) is more efficient than solving (13).

Furthermore, it is worth noting that $X(T, u(\omega))$ is an increasing function of α.

3.2 European call option price

Before using (9) to price European options, the realizations of U must be generated. The set of realizations are by nature concentrated around 0, however, the ODE (15) is solved instead of (13) , hence small increments of α are needed.

The function α is defined as

$$\alpha : x \mapsto \frac{x}{\sqrt{T}}$$

The stage grid $\{u_i\}_{i=1}^N$ of N realizations of U as

$$u_i = u_{min} + \frac{u_{max} - u_{min}}{N} i$$

N is chosen large enough to comply with (15) requirement, the ODE solver uses the step $\frac{u_{max}-u_{min}}{N}$. Similarly $u_{min} := u_{max}$ and u_{max} are large enough, verifying

$$\mathbb{P}(U < u_{min}) = \mathbb{P}(U > u_{max}) = \epsilon$$

with ϵ very small.

Finally, the realization $u = 0$ must be included in the state grid as $X(T, 0) = x_0 := F_0$.

The ODE(15) is solved using a 4th order Runge-Kutta method with 2 steps: the first step covers the states $[0, y_{max}]$, the second covers $[y_{min}, 0]$.

As stated in the previous section, the function $u \mapsto X(T, u)$ is strictly increasing, therefore a monotonic cubic spline interpolator can be used to generate the missing points between the ones generated by the ODE.

The function G introduced in (8) is then defined as

$$\begin{aligned} G :& x \mapsto \hat{G}(x) + F_0 - \int_{\mathbb{R}} \hat{G}(u) e^{-\frac{1}{2}u^2} du \\ \hat{G} :& x \mapsto \mathit{Spline}\left(\{u_i\}_{i=1}^N, \{X(T, u_i)\}_{i=1}^N, x\right) \end{aligned}$$

where $\mathit{Spline}(U, V, x)$ is a monotonic interpolator that connects the points (U, V) and return the value at the point x

This approach guarantees arbitrage-free prices, however the essence of this paper is the construction of the function σ_X is a way that it matches both static and dynamics market smile.

4 Construction of the volatility function σ_X

Whilst giving enough control of the wings, the SABR models suffers from the approximation, not only the actual smile model is off compare to the approximated one, especially for long maturities, but the implied density function has several spikes as well as negative values, which is synonyms of arbitrage prices.

The objective of this section is to introduce a SABR like model that does not violate these properties. In Doust, Andreasen and Belland, and several recent papers on the SABR model, local volatility versions of the SABR model are investigated, usually through short-maturity expansions. The advantage of using local volatility models is the possibility to use forward Dupire PDE to get arbitrage-free prices, but in the detriment of using expensive PDE schemes, against asymptotic formulas.

In Andreasen and Huge, they reduce the time computation by using a one-step version of the forward Dupire PDE is used to guarantees arbitrage-free prices, but requires a few adjustments on the local volatility to maintain the initial dynamics.

In the section , we build a local volatility model that shares some of the SABR features, provides a better cpntrol of the backbone, an handle the multiple regimes in the market, and providing calibration flexibilities on the wings.

Taking a look at the SABR equation (1), we can get some intuition of the model dynamics:

$$\alpha_t C(F_t) = \alpha_0 C(F_0) + C(F_0)(\alpha_t - \alpha_0) + \alpha_0 C'(F_0)(F_t - F_0) + o((\alpha_t - \alpha_0)^2 + (F_t - F_0)^2)$$

We can write

$$d\alpha_t = \frac{\nu\rho}{C(F_t)} dF_t + \alpha_t \nu \sqrt{1 - \rho^2} dZ_t$$

Using the following first-order approximation , we have

$$\alpha_t - \alpha_0 = \frac{\nu\rho}{C(F_0)}(F_t - F_0) + \alpha_0 \nu \sqrt{1-\rho^2} Z_t$$

The volatility precess α_t has two orthogonal sources of randomness, a rate components and a change of pure volatility.

Replacing the formula to the XX , we have

$$\alpha_t C(F_t) = \alpha_0 C(F_0) + (\nu\rho + \alpha_0 C'(F_0))(F_t - F_0) + (\alpha_0 C(F_0)\nu\sqrt{1-\rho^2} Z_t) + o((\alpha_t - \alpha_0)^2 + (F_t - F_0)^2)$$

We define

$$\sigma_0 := \sigma_{ATM}(F_0) = \alpha_0 C(F_0)$$

σ_0 can be seen as the ATM normal volatility by freezing time to zero in (1),

The first part $\sigma_0 + (\nu\rho + \alpha_0 C'(F_0))(F_t - F_0)$ of $\alpha_t C(F_t)$ is linear in F_t, therefore the level and the skew of the smile are mainly driven by respectively σ_0 and $(\nu\rho + \frac{d\sigma_{ATM}(F_0)}{dF_0})$. The convexity is handled by $(\sigma_{ATM}(F_0)\nu\sqrt{1-\rho^2} Z_t) + o((\alpha_t - \alpha_0)^2 + (F_t - F_0)^2)$.

In this expansion, we can confirm the empirical observations that controls the smile : $\nu\rho$ controls the skew and $\nu\sqrt{1-\rho^2}$ controls the convexity. A more intuitive way of looking at the SABR model is to look at the couple $\left(\hat{\rho} := \nu\rho, \hat{\nu} := \nu\sqrt{1-\rho^2}\right)$ instead of (ρ, ν).

$$\alpha_t C(F_t) = \sigma_0 + (\hat{\rho} + \frac{d\sigma_{ATM}(F_0)}{dF_0})(F_t - F_0) + \sigma_0 \hat{\nu} Z_t + o((\alpha_t - \alpha_0)^2 + (F_t - F_0)^2)$$

No doubt that $(\hat{\rho}, \hat{\nu})$ alters the smile dynamics, however, one cannot use $(\hat{\rho}, \hat{\nu})$ to control it ,indeed, the sole purpose of ρ and ν is to match the smile at a given point in time, hence a reduced legitimacy of the extra stochastic factor Z.

The elements that gives us control of the dynamics is the choice of the functional form $C(F)$: it controls both backbone and skew dynamics.

The first-order analysis of the general SABR model yields a more intuitive local-volatility model $\sigma(F)$ defined as

$$\sigma(F_t) = \sigma_0 + \left(\hat{\rho} + \frac{d\sigma_0}{dF_0}\right)(F_t - F_0) + \hat{\nu}|F_t - F_0|$$

The convexity term $\hat{\nu}(F_t - F_0)$ comes from the Bachelier model , as a result of

$$F_t - F_0 = \sigma_0 Z_t$$

, and with the ad-hoc assumption of 100% correlation between W_t and Z_t. The absolute value is employed to guarantee that- ceteris paribus- $\sigma(F_t)$ is an increasing function of $\hat{\nu}$.

$\hat{\rho}$, $\hat{\nu}$ and σ_0 will locally control the shape of the smile. For respectively low and high strike controls, we introduce two parameters a_{min} and a_{max} that will reduce/increase the local volatility. Finally, for very high strikes, mainly for CMS replication, we use the parameter a_{CMS}.

Using formula XX, we have

$$\sigma^+ = \sigma(F_0 + \sigma_0\sqrt{T}) = \sigma_0 + \left(\hat{\rho} + \frac{d\sigma_0}{dF_0} + \hat{\nu}\right)\sigma_0\sqrt{T}$$
$$\sigma^- = \sigma(F_0 - \sigma_0\sqrt{T}) = \sigma_0 - \left(\hat{\rho} + \frac{d\sigma_0}{dF_0} - \hat{\nu}\right)\sigma_0\sqrt{T}$$

We define the cutoff parameter γ that corresponds to the number of standard deviations that leads the forward rate to the zone controlled by either a_{min} or a_{max}. The cutoff parameter γ_{cms} holds a similiar role and is tied to a_{cms}

Combined with formula XX, we have

$$\sigma^{++} = \sigma(F_0 + \gamma\sigma_0\sqrt{T}) = \sigma_0 + \gamma\left(\hat{\rho} + \frac{d\sigma_0}{dF_0} + (\hat{\nu} + a_{max})\right)\sigma_0\sqrt{T}$$
$$\sigma^{--} = \sigma(F_0 - \gamma\sigma_0\sqrt{T}) = \sigma_0 - \gamma\left(\hat{\rho} + \frac{d\sigma_0}{dF_0} - (\hat{\nu} + a_{min})\right)\sigma_0\sqrt{T}$$
$$\sigma^{cms} = \sigma(F_0 + \gamma_{cms}\sigma_0\sqrt{T}) = \sigma^{++} + a_{cms}\left(\gamma_{cms} - \gamma\right)\sigma_0\sqrt{T}$$

The intermediate points of the local volatility function are created by using interpolators. The three points (F_0, σ_0), $(F_0 + \sigma_0\sqrt{T}, \sigma^+)$ and $(F_0 - \sigma_0\sqrt{T}, \sigma^-)$ are interpolated using a second-order polynomials. $(F_0 + \sigma_0\sqrt{T}, \sigma^+)$ and $(F_0 + \gamma\sigma_0\sqrt{T}, \sigma^{++})$ are connected with a second-order polynomial, the first derivative is matched on the former point. The same thing is done for the points $(F_0 - \sigma_0\sqrt{T}, \sigma^-)$ and $(F_0 - \gamma\sigma_0\sqrt{T}, \sigma^{--})$.

$(F_0 + \gamma\sigma_0\sqrt{T}, \sigma^{++})$ and $(F_0 + \gamma_{cms}\sigma_0\sqrt{T}, \sigma^{cms})$ are connected using linear interpolation.

The local volatility function may be composed with another to ensure that it is positive.

References