Advanced Particle Physics 2025 Problem Set 1

1 Constant Force and Relativistic Motion

Consider a particle subject to a constant force, F, along some given direction. Newton's second law states

$$\frac{dp}{dt} = F, \tag{1}$$

where p is the 3-momentum of the particle. Assume that the force is applied at time $t_0 = 0$.

1) Show that

$$v(t) = \frac{Ft/m}{\sqrt{1 + (Ft/m)^2}}, \tag{2}$$

where $v(t)$ is the particle velocity at time $t > 0$ and m is the rest mass. Assume that the particle is at rest at $t = 0$.

2) Find the position of the particle, $x(t)$, for all $t > 0$.

3) Suppose the particle under influence of a constant force is racing a light ray. Give the particle a head start, i.e. place it at position $x_0 > 0$ for $t = 0$, while the light ray starts from $x = 0$ at $t = 0$. At what time, t, will the light ray catch the particle?

2 Four-vector Gymnastics

In this problem we study four-vectors which we define as x^μ where $\mu = 0, 1, 2, 3$. In explicit form one may write $x^\mu = (ct, x, y, z)^T$. Typically, we think of these as column vectors which explains the transposition. More generally, one writes the components as $x^\mu = (x^0, x^1, x^2, x^3)^T$. If we consider the same vector in a different inertial frame we will use the notation $x'^\mu = (x'^0, x'^1, x'^2, x'^3)^T$.

1) Show that if we do a boost by velocity v along the x direction, then the transformation of x^μ into the moving frame is done by the following linear

transformation

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}, \tag{3}$$

where c is the speed of light and $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the usual gamma factor. Use this to show that

$$\left(x'^0\right)^2 - \left(x'^1\right)^2 - \left(x'^2\right)^2 - \left(x'^3\right)^2 = \left(x^0\right)^2 - \left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2. \tag{4}$$

2) Argue that *all* Lorentz transformations can be written as some 4x4 matrix which we will denote by $L^\mu{}_\nu$, where $\mu, \nu = 0, 1, 2, 3$, and that we can write the transformation law above as

$$x'^\mu = \sum_{\nu=0}^{3} L^\mu{}_\nu x^\nu \equiv L^\mu{}_\nu x^\nu, \tag{5}$$

where we indicate that we will now drop the explicit sum in such expressions using the rule that whenever the same index appears up and down in an expression, then there is a sum over that index. This is the famous Einstein summation convention. It is terribly convenient as it saves a lot of space. It takes a bit of time to get used to, but once you do it simplifies many things as we will now explore.

3) Now we introduce the so-called *metric tensor*, which is defined by $g_{\mu\nu}$ with components $g_{00} = 1$, $g_{11} = -1$, $g_{22} = -1$, $g_{33} = -1$, and zero otherwise. It is sometimes more convenient to write it as a matrix

$$g_{\mu\nu} = g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{6}$$

It it also useful to know the inverse of the metric, $g^{\mu\nu}$. In the case considered here with ± 1 on the diagonal, it has the same entries as $g_{\mu\nu}$, i.e. $g^{00} = 1$, $g^{11} = -1$, $g^{22} = -1$, $g^{33} = -1$. Show that

$$g_{\mu\nu} x'^\mu x'^\nu = \left(x'^0\right)^2 - \left(x'^1\right)^2 - \left(x'^2\right)^2 - \left(x'^3\right)^2 \tag{7}$$

Use this to conclude that $g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} x^\mu x^\nu$. Typically, one defines the so-called *contraction* of x'^μ with itself as

$$x'^\mu x'_\mu \equiv \left(x'^0\right)^2 - \left(x'^1\right)^2 - \left(x'^2\right)^2 - \left(x'^3\right)^2. \tag{8}$$

Use this definition and the previous result, to conclude that we may define

$$x^{'}_{\mu} = g_{\mu\nu} x^{'\nu}, \tag{9}$$

and that we can therefore use the metric tensor to move an index on four-vectors up and down. Argue that for any two four-vectors a^{μ}, b^{μ}, we have $a^{'\mu} b^{'}_{\mu} = a^{\mu} b_{\mu}$, i.e. contractions of four-vectors do not change under Lorentz transformations.

4) More generally, we may consider the general transformations of two four-vectors a^{μ} and b^{μ}, $a^{'\mu} = L^{\mu}{}_{\nu} a^{\nu}$ and $b^{'\mu} = L^{\mu}{}_{\nu} b^{\nu}$. Show that

$$a^{'\mu} b^{'}_{\mu} = L^{\mu}{}_{\nu} g_{\mu\rho} L^{\rho}{}_{\sigma} a^{\nu} b^{\sigma}, \tag{10}$$

where there is a sum over $\mu, \nu, \rho, \sigma = 0, 1, 2, 3$. Use this conclude that we must have

$$L^{\mu}{}_{\nu} g_{\mu\rho} L^{\rho}{}_{\sigma} = g_{\nu\sigma}. \tag{11}$$

5) We can also define the inverse Lorentz transformation. Typically for a boost along a single axis, one just needs to take $v \to -v$, but more generally we must write it as

$$x^{\mu} = \left(L^{-1}\right)^{\mu}{}_{\nu} x^{'\nu}. \tag{12}$$

Use the Lorentz transformation of $x^{'\mu}$ above, to show that

$$\left(L^{-1}\right)^{\mu}{}_{\nu} L^{\nu}{}_{\rho} = \delta^{\mu}_{\rho}, \tag{13}$$

where δ^{μ}_{ρ} is the so-called Kronecker delta which is 1 if $\mu = \rho$ and zero otherwise.

6) Consider again the relation

$$L^{\mu}{}_{\nu} g_{\mu\rho} L^{\rho}{}_{\sigma} = g_{\nu\sigma}. \tag{14}$$

Multiply this relation on both sides by $\left(L^{-1}\right)^{\sigma}{}_{\delta}$, where $\sigma, \delta = 0, 1, 2, 3$, then perform a sum over σ and use this to show that

$$L_{\sigma\nu} = \left(L^{-1}\right)_{\nu\sigma}, \tag{15}$$

and hence conclude that

$$\left(L^{-1}\right)^{\mu}{}_{\nu} = L_{\nu}{}^{\mu}. \tag{16}$$

This shows that a Lorentz transformation and its inverse can be related by the metric tensor.

7) Use the fact that Lorentz transformations are linear, to show that

$$\frac{\partial x^{'\rho}}{\partial x^{\mu}} = L^{\rho}{}_{\mu} \quad \text{and} \quad \frac{\partial x^{\rho}}{\partial x^{'\mu}} = \left(L^{-1}\right)^{\rho}{}_{\mu}. \tag{17}$$

Now we define the differential four-vector operator as a natural combination of gradient and time-derivative in the following way

$$\frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right), \tag{18}$$

where the gradient is with respect to $\boldsymbol{x}$. Show that

$$\frac{\partial}{\partial x'^\mu} = \left(L^{-1}\right)^\rho{}_\mu \frac{\partial}{\partial x^\rho}. \tag{19}$$

Use this relation to conclude that this four-vector gradient transforms as a four-vector with Lorentz index down and that a good short-hand notation for it is therefore

$$\partial_\rho \equiv \frac{\partial}{\partial x^\rho}. \tag{20}$$

8) Use the fact that the metric tensor can move an index to define $\partial^\rho = g^{\rho\nu}\partial_\nu$. Show that

$$\partial'^\rho = L^\rho{}_\nu \partial^\nu, \tag{21}$$

and hence conclude that this operator transformations as a four-vector with index up. Show explicitly that

$$\partial^\rho = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right). \tag{22}$$

Furthermore, show that

$$\partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial^2 t} - \boldsymbol{\nabla}^2, \tag{23}$$

and that this operator is invariant under Lorentz transformations.

9) We now consider the Klein-Gordon Lagrangian for a field in three spatial dimensions, $u(t, \boldsymbol{x})$, and assume that the fields extends over all of space. The Lagrangian is therefore of the form

$$L = \int \mathrm{d}^3\boldsymbol{x} \left[\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{c^2}{2}\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} u - \frac{m^2 c^4}{2\hbar^2} u^2\right], \tag{24}$$

where the integral is over all of space. Show that

$$\left(\partial^\mu u\right)\left(\partial_\mu u\right) = \frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 - \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} u. \tag{25}$$

Use this to show that the Lagrangian density is given by

$$\mathcal{L} = \frac{c^2}{2}\left(\partial^\mu u\right)\left(\partial_\mu u\right) - \frac{m^2 c^4}{2\hbar^2} u^2 \tag{26}$$

10) Show that the Euler-Lagrange equations for the Lagrangian density yields the equation of motion

$$\left(\frac{\partial^2}{\partial t^2} - c^2\boldsymbol{\nabla}^2\right) u = -\frac{m^2c^4}{\hbar^2}u. \tag{27}$$

Show now that

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu u\right)} = c^2\partial^\mu u, \tag{28}$$

and use this to conclude that the Euler-Lagrange equation can be elegantly written in the form

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu u\right)}\right) = \frac{\partial \mathcal{L}}{\partial u}. \tag{29}$$

3 Lorentz transformations and space-like intervals

Here we consider a particle moving in one spatial dimension for simplicity, all results can be easily generalized to three spatial dimensions.

1) Let the space-time coordinates of the particle be (t, x) in an inertial frame. Write down a Lorentz transformation that takes us to a different inertial frame that moves relative to the first inertial frame with velocity v. Use units where the speed of light is $c = 1$.

2) Consider two space-time points (t_1, x_1) and (t_2, x_2). Define $\Delta t = t_1 - t_2$ and $\Delta x = x_1 - x_2$. Show that $(\Delta t)^2 - (\Delta x)^2$ is invariant under Lorentz transformations.

3) Consider the case where $(\Delta t)^2 - (\Delta x)^2 < 0$. Show that there is a Lorentz transformation with velocity parameter $v < 1$ such that Δt becomes $-\Delta t$, i.e. the order of events is reserved. (Hint: From the condition on the space-time interval you can construct an allowed velocity, try the corresponding transformation).

4 Invariant Volume Element

Here we want to show that the volume element $\mathrm{d}^4x = \mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z$ is a Lorentz invariant, i.e. it is conversed under Lorentz transformations. When the Lagrangian density, $\mathcal{L}$ is Lorentz invariant, the invariance of the volume element makes the action, $S = \int \mathrm{d}^4 x \mathcal{L}$, Lorentz invariant which is what we want from any relativistic theory.

1) Argue that it is enough to show that the determinant of a Lorentz transformation, L, satisfies $|\det(L)| = 1$.

2) Reduce the problem to consider only a boost of velocity along the z-direction. Write down the matrix for the corresponding Lorentz transformation and calculate the determinant.

5 Lorentz Invariance of Phase Space Factors

In order to get more familiar with four-vectors and develop skills in handling such things, the electromagnetic field strength is a very good object to study. Here one can get familiar with relativistic invariants, Lorentz scalars, four-derivatives, and the details of classical field theory. This is discussed from section 3.3 and onward in the note by Chris Blair.

Here we want to show that the phase space factor

$$\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}}, \tag{30}$$

where $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$ is the energy and $\boldsymbol{p}$ is the 3-momentum, is Lorentz invariant.

1) Show that

$$\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} = \int \frac{d^4 p}{(2\pi)^4} (2\pi) \delta(p^2 - m^2), \tag{31}$$

where p is the 4-momentum and we assume that $p_0 > 0$. (Hint: Do the integral over p_0 and use the properties of delta functions of the form $\delta(f(x))$ that can be found in many books or on Wikipedia.)

2) Argue that this expression is now perfectly Lorentz-invariant.

6 Hamiltonian of the Electromagnetic Field

For electrodynamics the field is the four-vector $A^\mu = (V, \boldsymbol{A})$ also known as the four-potential or just electromagnetic potential for short (we are always talking about relativity anyways). The free Lagrangian for the EM field is given by

$$\mathcal{L}_{EM} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}, \tag{32}$$

where the antisymmetric field strength tensor (as it is properly called) has the form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{33}$$

The factor in front of the contracted field tensors depends on the units one uses in electrodynamics and often one uses $1/4$ instead of $1/16\pi$ (the 4π is a famous difference between typical SI units and Gaussian units). Via the Euler-Lagrange equations with A_μ (or equivalently A^μ) as the variable we would obtain the free Maxwell's equations from this Lagrangian density. It remains to be worked out what the corresponding Hamiltonian looks like. In relativistic field theory the transformation from the Lagrangian to the Hamiltonian for a four-vector field, A^μ is

$$\mathcal{H}_{EM} = \frac{\partial \mathcal{L}_{EM}}{\partial(\partial_0 A_\mu)} \partial_0 A_\mu - \mathcal{L}_{EM} \tag{34}$$

1) Show that the Hamiltonian can be written as

$$\mathcal{H}_{EM} = \frac{1}{4\pi} \left(F^{\mu 0} \partial_0 A_\mu + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right), \tag{35}$$

where the factor in front would be absent in SI units.

2) Next show that

$$F^{\mu 0} \partial_0 A_\mu = F^{\mu 0} \left[F_{0\mu} + \partial_\mu A_0 \right], \tag{36}$$

and then

$$F^{\mu 0} \left[F_{0\mu} + \partial_\mu A_0 \right] = \boldsymbol{E}^2 - F^{0i} \partial_i A_0, \tag{37}$$

where $\boldsymbol{E}$ is the usual E-field three-vector of EM and $i = 1, 2, 3$.

3) Combine 1) and 2) to show that

$$\mathcal{H}_{EM} = \frac{1}{4\pi} \left(\frac{1}{2} \left[\boldsymbol{E}^2 + \boldsymbol{B}^2 \right] + \boldsymbol{E}_i \frac{\partial V}{\partial x_i} \right), \tag{38}$$

where V is the usual potential known from electrodynamics.

4) For a free electromagnetic field we have no sources. Argue that we may then drop the second term in Eq. (38).