

Klein's Paradox for the Klein-Gordon Equation

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Here, we consider a relativistic spinless particle (mass m , energy E) incident from the left ($x < 0$) on a potential step:

$$V(x) = \begin{cases} 0 & \text{for } x < 0, \\ V_0 & \text{for } x > 0. \end{cases}$$

1. Klein-Gordon Equation for a Potential Step

The Klein-Gordon equation for a particle in a potential $V(x)$ is:

$$\left(\left(i \frac{\partial}{\partial t} - V \right)^2 - \nabla^2 + m^2 \right) \psi = 0.$$

Assuming the energy-preserving solution $\psi(x, t) = \phi(x)e^{-iEt/\hbar}$, the equation becomes:

$$-\frac{d^2\phi}{dx^2} + [(E - V(x))^2 - m^2]\phi = 0.$$

2. Wavefunction Solutions

a) Argue that for $x < 0$, the general solution is:

$$\phi_{x<0} = Ae^{ik_1x} + Be^{-ik_1x},$$

where $k_1 = \sqrt{E^2 - m^2}$. Identify the incident and reflected waves. Hint: Use that the conserved current is $j = -i(\phi^*\nabla\phi - \phi\nabla\phi^*)$.

b) Show that for $x > 0$, the solution depends on the value of V_0 :

– If $(E - V_0)^2 > m^2$, the solution is oscillatory:

$$\phi_{x>0} = Ce^{ik_2x},$$

where $k_2 = \sqrt{(E - V_0)^2 - m^2}$.

– Otherwise, the solution is exponential:

$$\phi_{x>0} = Ce^{-\kappa x},$$

where $\kappa = \sqrt{m^2 - (E - V_0)^2}$.

3. Boundary Conditions and Scattering Coefficients

At $x = 0$, the wavefunction and its derivative must be continuous:

$$\phi_{x<0}(0) = \phi_{x>0}(0), \quad \left. \frac{d\phi_{x<0}}{dx} \right|_{x=0} = \left. \frac{d\phi_{x>0}}{dx} \right|_{x=0}.$$

- a) Apply the boundary conditions to find expressions for the reflection coefficient $R = |B/A|^2$ and transmission coefficient $T = |C/A|^2$. You can solve the resulting system of equations using your favorite software.
- b) Plot the transmission and reflection coefficients as a function of V_0 . Interpret the behavior of the system in the limit $V_0 \rightarrow \infty$.