A review of the projection matrix

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This file gives a brief review of the theoretical foundation of the projection matrix in the cone beam CT reconstruction.

1 Theory

1.1 Most simple case

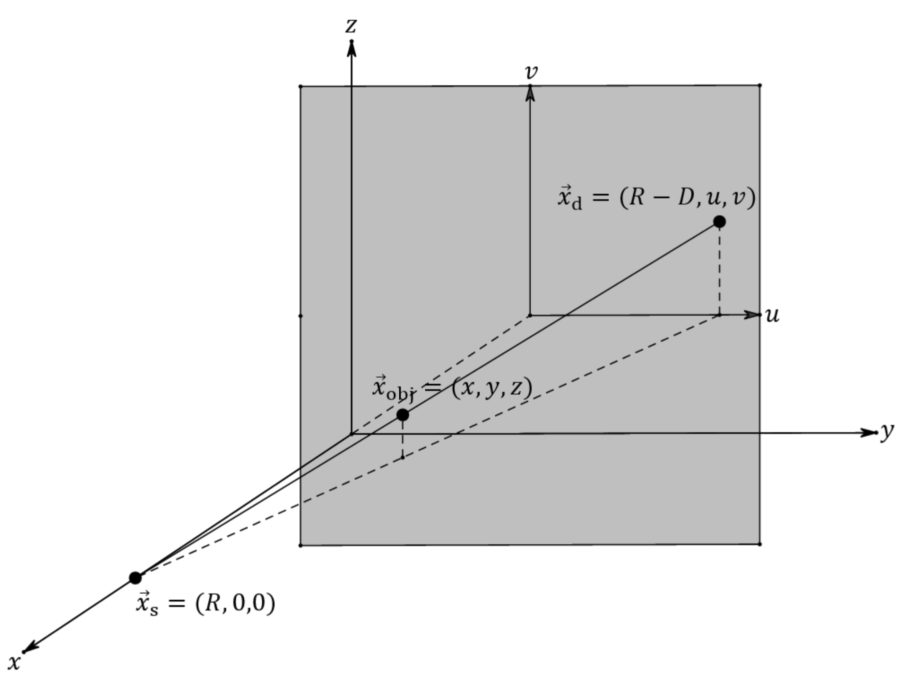

Figure 1: An example of the calculation of the projection matrix.

Generally speaking, the projection matrix (p-matrix) is used to relate the coordinates in the flat panel detector plane to the coordinates in a three-dimensional real world coordinate system. One can actually derive such relationship even without knowing what the projection matrix is. An example is shown in Fig. 1. The x-ray source is place at $\vec{x}_{\mathrm{s}} = (R, 0, 0)$. A point in the detector plane is denoted as $\vec{x}_{\rm d} = (R-D, u, v)$. A point in the image object is at $\vec{x}_{\rm obj} = (x, y, z)$. What we want are the relationships between (u, v) and (x, y, z)

$$\begin{cases} u = u(x, y, z); & (1) \\ v = v(x, y, z). & (2) \end{cases}$$

Since these three points are on a straight line, in which case

$$\vec{x}_{\rm d} - \vec{x}_{\rm s} = M(\vec{x}_{\rm obj} - \vec{x}_{\rm s}), \tag{3}$$

where M is the magnification factor. The equations for all the components are

$$\begin{cases} -D = M(x - R); & (4) \\ u = My; & (5) \\ v = Mz. & (6) \end{cases}$$

The solution for the above equations is straightforward

$$\begin{cases} u = \dfrac{Dy}{R-x}; & (7) \\ v = \dfrac{Dz}{R-x}. & (8) \end{cases}$$

One can tell that the relationship between (u, v) and (x, y, z) are non-linear. In that case, the relationship can not be represented by a "matrix". Then where the projection "matrix" come from? Actually a little modification of the variables are made to make the relationship linear. First, we try to find the inverse of the magnification factor M

$$\frac{1}{M} = \frac{R-x}{D}, \tag{9}$$

which is a linear function. Then u and v are replaced by $\frac{u}{M}$ and $\frac{v}{M}$,respectively. In that case,

$$\frac{u}{M} = y; \tag{10}$$

$$\frac{v}{M} = z. \tag{11}$$

Therefore, one can derive that

$$\begin{cases} \dfrac{u}{M} = y; & (12) \\ \dfrac{v}{M} = z; & (13) \\ \dfrac{1}{M} = \dfrac{R-x}{D}. & (14) \end{cases}$$

Or in a matrix form

$$\begin{bmatrix} \frac{u}{M} \\ \frac{v}{M} \\ \frac{1}{M} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{D} & 0 & 0 & \frac{R}{D} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}. \tag{15}$$

This is the most simple case of a "projection matrix".

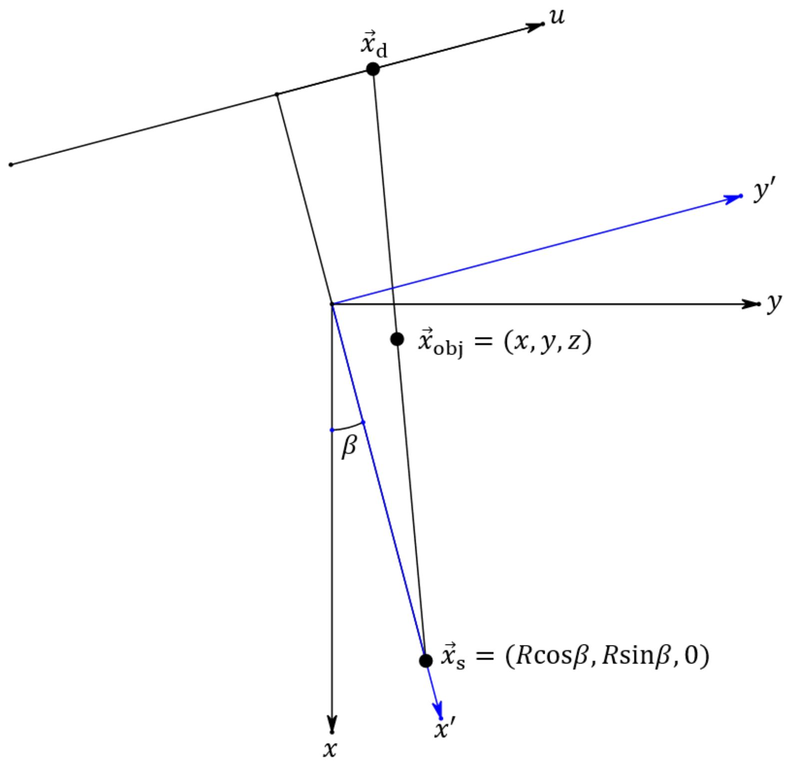

Figure 2: Calculation of the projection matrix when the source is rotated by β.

1.2 Case when the source is rotated

A little more complex case is when the x-ray source is rotated by an angle β along the z-axis to the position $\vec{x}_\mathrm{s} = (R\cos\beta, R\sin\beta, 0)$. Assume that the detector stays stationary with respect to the source. We can rotate the coordinate system by β to a new coordinate system $x' - y' - z'$. The coordinate of the point (x, y, z) in this coordinate system is

$$x' = x\cos\beta + y\sin\beta;\ y' = -x\sin\beta + y\cos\beta;\ z' = z. \tag{16}$$

Detailed derivation can be found later. The relationship can also be expressed in a matrix form

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{17}$$

Therefore, the p-matrix is given as

$$\begin{bmatrix} \frac{u}{M} \\ \frac{v}{M} \\ \frac{1}{M} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{D} & 0 & 0 & \frac{R}{D} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{D} & 0 & 0 & \frac{R}{D} \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 \\ -\sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{19}$$

$$= \begin{bmatrix} -\sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\cos\beta}{D} & -\frac{\sin\beta}{D} & 0 & \frac{R}{D} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}. \tag{20}$$

1.3 More general cases

The cases analyzed in the previous sections assume that the detector and the source forms a rigid body; the two base vectors (u and v) describing the detector pixel array are perpendicular to the line connecting the source and the isocenter; Also the v-axis is along the z-axis. These assumptions may not be true for a true cone beam system.

For a true cone beam system, the location of the origin of the detector is denoted as $\vec{x}_{\mathrm{do}}$. The unit vectors along the u- and v-axis are $\hat{e}_u$ and $\hat{e}_v$ respectively (the lengths of $\hat{e}_u$ and $\hat{e}_v$ does not have to be one; they should be equal to the detector pixel width and height). Then a point on the detector plane is given as $\vec{x}_{\mathrm{d}} = \vec{x}_{\mathrm{do}} + u\hat{e}_u + v\hat{e}_v$. Eq. (3) is still valid:

$$\vec{x}_{\mathrm{d}} - \vec{x}_{\mathrm{s}} = M(\vec{x}_{\mathrm{obj}} - \vec{x}_{\mathrm{s}}) \tag{21}$$

$$\Rightarrow \quad \frac{\vec{x}_{\mathrm{do}}}{M} + \frac{u}{M}\hat{e}_u + \frac{v}{M}\hat{e}_v - \frac{\vec{x}_{\mathrm{s}}}{M} = \vec{x}_{\mathrm{obj}} - \vec{x}_{\mathrm{s}} \tag{22}$$

$$\Rightarrow \quad \frac{\vec{x}_{\mathrm{do}} - \vec{x}_{\mathrm{s}}}{M} + \frac{u}{M}\hat{e}_u + \frac{v}{M}\hat{e}_v = \vec{x}_{\mathrm{obj}} - \vec{x}_{\mathrm{s}}. \tag{23}$$

If $\frac{1}{M}$, $\frac{u}{M}$ and $\frac{v}{M}$ are treated as variables, then Eq. (23) can be written in a matrix form

$$\begin{bmatrix} \hat{e}_u & \hat{e}_v & \vec{x}_{\mathrm{do}} - \vec{x}_{\mathrm{s}} \end{bmatrix} \begin{bmatrix} \frac{u}{M} \\ \frac{v}{M} \\ \frac{1}{M} \end{bmatrix} = \vec{x}_{\mathrm{obj}} - \vec{x}_{\mathrm{s}}. \tag{24}$$

Assume that $A = \begin{bmatrix} \hat{e}_u & \hat{e}_v & \vec{x}_{\mathrm{do}} - \vec{x}_{\mathrm{s}} \end{bmatrix}$. Then

$$\begin{bmatrix} \frac{u}{M} \\ \frac{v}{M} \\ \frac{1}{M} \end{bmatrix} = A^{-1}\vec{x}_{\mathrm{obj}} - A^{-1}\vec{x}_{\mathrm{s}} = \begin{bmatrix} A^{-1} & -A^{-1}\vec{x}_{\mathrm{s}} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \tag{25}$$

where $\vec{x}_{\mathrm{obj}}^T = (x, y, z)$. The projection matrix is $\begin{bmatrix} A^{-1} & -A^{-1}\vec{x}_{\mathrm{s}} \end{bmatrix}$.

1.4 Revisit of section 1.2

We will calculate the case in section 1.2 again based on the more general case. In section 1.2,

$$\begin{cases} \hat{e}_u = (-\sin\beta, \cos\beta, 0); & (26) \\ \hat{e}_v = (0, 0, 1); & (27) \\ \vec{x}_{\mathrm{do}} = (R - D)(\cos\beta, \sin\beta, 0); & (28) \\ \vec{x}_{\mathrm{s}} = R(\cos\beta, \sin\beta, 0). & (29) \end{cases}$$

Therefore

$$A = \begin{bmatrix} \hat{e}_u & \hat{e}_v & \vec{x}_{\mathrm{do}} - \vec{x}_{\mathrm{s}} \end{bmatrix} = \begin{bmatrix} -\sin\beta & 0 & -D\cos\beta \\ \cos\beta & 0 & -D\sin\beta \\ 0 & 1 & 0 \end{bmatrix}. \tag{30}$$

The inverse of A can be calculated as

$$A^{-1} = \begin{bmatrix} -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \\ -\frac{\cos\beta}{D} & -\frac{\sin\beta}{D} & 0 \end{bmatrix}. \tag{31}$$

Also

$$A^{-1}\vec{x}_{\mathrm{s}} = \begin{bmatrix} -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \\ -\frac{\cos\beta}{D} & -\frac{\sin\beta}{D} & 0 \end{bmatrix} \begin{bmatrix} R\cos\beta \\ R\sin\beta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{R}{D} \end{bmatrix} \tag{32}$$

The projection matrix is

$$\begin{bmatrix} A^{-1} & -A^{-1}\vec{x}_{\mathrm{s}} \end{bmatrix} = \begin{bmatrix} -\sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\cos\beta}{D} & -\frac{\sin\beta}{D} & 0 & \frac{R}{D} \end{bmatrix}, \tag{33}$$

which is the same with Eq. (20).

2 Experimental considerations

In a real experiment, a calibration phantom (e.g., helix phantom) is often used to calibrate the projection matrix for each view. For one view, we assume that we already acquire a set of (x_i, y_i, z_i) and (u_i, v_i). We will not dive into the experimental details of how we acquire such data set. What we want is to find the elements in the projection matrix P from the measured data set. Naively, we want to establish the equation:

$$\begin{bmatrix} \frac{u_i}{M_i} \\ \frac{v_i}{M_i} \\ \frac{1}{M_i} \end{bmatrix} = P \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix} \tag{34}$$

We can write down the equations explicitly with the elements of P

$$\frac{u_i}{M_i} = P_{11}x_i + P_{12}y_i + P_{13}z_i + P_{14} \tag{35}$$

$$\frac{v_i}{M_i} = P_{21}x_i + P_{22}y_i + P_{23}z_i + P_{24} \tag{36}$$

$$\frac{1}{M_i} = P_{31}x_i + P_{32}y_i + P_{33}z_i + P_{34} \tag{37}$$

The problem is that we do not have the measured M_i in the data set. One method to proceed is to replace $1/M_i$ with $(P_{31}x_i + P_{32}y_i + P_{33}z_i + P_{34})$ and insert it into the first two equations:

$$\begin{cases} u_i(P_{31}x_i + P_{32}y_i + P_{33}z_i + P_{34}) = P_{11}x_i + P_{12}y_i + P_{13}z_i + P_{14} \\ v_i(P_{31}x_i + P_{32}y_i + P_{33}z_i + P_{34}) = P_{21}x_i + P_{22}y_i + P_{23}z_i + P_{24} \end{cases} \tag{38}$$

$$\Rightarrow \quad \begin{cases} u_ix_iP_{31} + u_iy_iP_{32} + u_iz_iP_{33} + u_iP_{34} - x_iP_{11} - y_iP_{12} - z_iP_{13} - P_{14} = 0 \\ v_ix_iP_{31} + v_iy_iP_{32} + v_iz_iP_{33} + v_iP_{34} - x_iP_{21} - y_iP_{22} - z_iP_{23} - P_{24} = 0 \end{cases} \tag{39}$$

Clearly if we acquire multiple (x, y, z) and (u, v) pairs, we should be able to establish several equations to solve the elements of P. However, the problem is that such equations can have the all-zero solution; that is to say by setting all elements of P to be zero and Eqs. (39) are satisfied. Therefore, based on this naïve method, we may not be able to find the correct projection matrix.

What is the problem behind this all-zero solution? In Eq. (37), the left side is $1/M$, which can not be zero; while the right size will be zero if we have the all-zero solution. It should also be pointed that that without the physical meaning of the matrix A, Eqs. (39) indeed have infinite solutions of coefficients of P. For example, we have solved one matrix P; one can tell that $P' = \lambda P$ would lead to the same relationship between (x, y, z) and (u, v). If we revisit the derivation of the projection matrix, we can see that the elements of P should be $\begin{bmatrix} A^{-1} & -A^{-1}\vec{x}_\mathrm{s} \end{bmatrix}$, where $A = \begin{bmatrix} \hat{e}_u & \hat{e}_v & \vec{x}_\mathrm{do} - \vec{x}_\mathrm{s} \end{bmatrix}$. We need to make sure that matrix A from P should follow the physical dimensions of the detector and this is the constraint that P should not have infinite number of solutions.

Then how to solve the coefficient of the Pmatrix? By observing Eq. (37), one can tell that when the image object is placed at the origin, i.e., $(x, y, z) = (0, 0, 0)$, $1/M = P_{34}$. Therefore, P_{34} would definitely be non-zero. Here, we first assume P_{34} = 1, and get one solution of P, denoted as P_1. After that, since any $P' = \lambda P_1$ would fit Eqs. (39), we will pickup the correction λ value to make sure matrix A from P' should follow the physical dimensions of the detector. Here is the detailed work flow:

1. For each view, acquire the grouped (x_i, y_i, z_i), (u_i, v_i) data.

2. Set P_{34} in Eqs. (39) to 1 and solve P_1 using the least-square method. Details will be provided in the appendix.

3. Calculate matrix A from P_1: The first three columns of P_1 forms a square matrix; inverse the square matrix to get A.

4. The first column vector of A should $\hat{e}_u$, whose length is the detector pixel size along the horizontal direction. $L'_u = \sqrt{A_{11}^2 + A_{21}^2 + A_{31}^2}$ and $L_u = |\hat{e}_u|$. Calculate the ratio $\lambda = \frac{L'_u}{L_u}$.

5. Set $P' = \lambda P_1$ to get the solution of Pmatrix with physics meaning.

Appendix: Solving Eqs. (39) using least-square method

By setting $P_{3,4} = 1$, Eqs. (39) can be rewritten into a matrix form:

$$\begin{bmatrix} -x_1 & -y_1 & -z_1 & -1 & 0 & 0 & 0 & 0 & u_1x_1 & u_1y_1 & u_1z_1 \\ 0 & 0 & 0 & 0 & -x_1 & -y_1 & -z_1 & -1 & v_1x_1 & v_1y_1 & v_1z_1 \\ -x_2 & -y_2 & -z_2 & -1 & 0 & 0 & 0 & 0 & u_2x_2 & u_2y_2 & u_2z_2 \\ 0 & 0 & 0 & 0 & -x_2 & -y_2 & -z_2 & -1 & v_2x_2 & v_2y_2 & v_2z_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n & -y_n & -z_n & -1 & 0 & 0 & 0 & 0 & u_nx_n & u_ny_n & u_nz_n \\ 0 & 0 & 0 & 0 & -x_n & -y_n & -z_n & -1 & v_nx_n & v_ny_n & v_nz_n \end{bmatrix} \begin{bmatrix} P_{11} \\ P_{12} \\ P_{13} \\ P_{14} \\ P_{21} \\ P_{22} \\ P_{23} \\ P_{24} \\ P_{31} \\ P_{32} \\ P_{33} \end{bmatrix} = \begin{bmatrix} -u_1 \\ -v_1 \\ -u_2 \\ -v_2 \\ \cdots \\ -u_n \\ -v_n \end{bmatrix} \tag{40}$$

Assume

$$S = \begin{bmatrix} -x_1 & -y_1 & -z_1 & -1 & 0 & 0 & 0 & 0 & u_1x_1 & u_1y_1 & u_1z_1 \\ 0 & 0 & 0 & 0 & -x_1 & -y_1 & -z_1 & -1 & v_1x_1 & v_1y_1 & v_1z_1 \\ -x_2 & -y_2 & -z_2 & -1 & 0 & 0 & 0 & 0 & u_2x_2 & u_2y_2 & u_2z_2 \\ 0 & 0 & 0 & 0 & -x_2 & -y_2 & -z_2 & -1 & v_2x_2 & v_2y_2 & v_2z_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n & -y_n & -z_n & -1 & 0 & 0 & 0 & 0 & u_nx_n & u_ny_n & u_nz_n \\ 0 & 0 & 0 & 0 & -x_n & -y_n & -z_n & -1 & v_nx_n & v_ny_n & v_nz_n \end{bmatrix} \tag{41}$$

$$\vec{X} = \begin{bmatrix} P_{11} \\ P_{12} \\ P_{13} \\ P_{14} \\ P_{21} \\ P_{22} \\ P_{23} \\ P_{24} \\ P_{31} \\ P_{32} \\ P_{33} \end{bmatrix} \tag{42}$$

$$\vec{Y} = \begin{bmatrix} -u_1 \\ -v_1 \\ -u_2 \\ -v_2 \\ \cdots \\ -u_n \\ -v_n \end{bmatrix} \tag{43}$$

Then $\vec{X} \approx (S^T S)^{-1} S^T \vec{Y}$.