Section 2.5

7. If A has row 1 + row 2 = row 3, show that A is not invertible:

   a. Explain why $Ax = (1, 0, 0)$ cannot have a solution.

   If row 1 = 1 and row 2 = 0, then row 3 must follow suit and row 3 = 0 + 1 = 1. Because this is not the case in (1, 0, 0), there must be no solution.

   b. Which right sides (b_1, b_2, b_3) might allow a solution to $Ax = b$?

   There will be a solution in any case where $b_1 + b_2 = b_3$.

   c. What happens to row 3 in elimination?

   The entire row is eliminated, becoming [0, 0, 0 | 0].

12. If the product $C = AB$ is invertible (A and B are square), then A itself is invertible. Find a formula for A^{-1} that involves C^{-1} and B.

$C = AB$ where $(AB)^{-1}$ exists.
$C^{-1} = (AB)^{-1}$
$C^{-1} = (B^{-1})(A^{-1})$
$(B)(C^{-1}) = (B)(B^{-1})(A^{-1})$
$B(C^{-1}) = I(A^{-1})$
$A^{-1} = B(C^{-1})$

18. If B is the inverse of A^2, show that AB is the inverse of A.

$B = (A^2)^{-1}$
$B = (A^{-1})(A^{-1})$

$AB = A(A^{-1})(A^{-1})$
$AB = I(A^{-1})$
$AB = A^{-1}$

27. Invert these matrices A by the Gauss-Jordan method starting with $[A\ I]$.

a. $A = \begin{matrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{matrix}$ b. $A = \begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{matrix}$

a. $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad * \quad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad * \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad => \quad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \quad = \quad A^{-1}$

b. $$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \quad * \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \quad * \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{bmatrix} \quad * \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \quad * \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \quad * \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \quad * \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix} \quad = \quad A^{-1}$$

29. True or false (with a counterexample if false and a reason if true):

a. A 4 by 4 matrix with a row of zeroes is not invertible.

True, because at the pivot of the given row must be 0, and there must be no zero pivots in order to invert a matrix.

b. Every matrix with 1's down the main diagonal is invertible.

False. Consider $\begin{bmatrix} 1 & -1 & 2 & 1 \\ -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 2 & 1 \end{bmatrix}$ While there are 1's down the main diagonal, the matrix is still a singular case, as row 3 = row 1 + row 2 and row 4 = row 1.

c. If A is invertible, then A^{-1} and A^2 are invertible.

True.

Section 2.6

5. What matrix E puts A into triangular form EA = U? Multiply by $E^{-1} = L$ to factor A into LU:

$$A = \begin{matrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{matrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

6. What two elimination matrices E_{21} and E_{32} put A into upper triangular form $E_{32} * E_{21} * A = U$? Multiply by E_{32}^{-1} and E_{21}^{-1} to factor A into $LU = E_{21}^{-1} * E_{32}^{-1} * U$

$$A = \begin{matrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{matrix}$$

Section 2.7

1. Find A^T and A^{-1} and $(A^{-1})^T$ and $(A^T)^{-1}$ for

$$A = \begin{matrix} 1 & 0 \\ 9 & 3 \end{matrix}$$

$$A^T = \begin{bmatrix} 1 & 9 \\ 0 & 3 \end{bmatrix}$$

$$A^{-1} = 1 / (3 - 0) \begin{bmatrix} 3 & 0 \\ -9 & 1 \end{bmatrix}$$

$$= 1 / 3 \begin{bmatrix} 3 & 0 \\ -9 & 1 \end{bmatrix}$$

$$(A^T)^{-1} = 1 / (3 - 0) \begin{bmatrix} 3 & -9 \\ 0 & 1 \end{bmatrix}$$

$$(A^T)^{-1} = 1 / (3) \begin{bmatrix} 3 & -9 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{matrix} 1 & c \\ c & 0 \end{matrix}$$

$$A^T = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}$$

$$A^{-1} = 1 / (1 - c^2) \begin{bmatrix} 0 & -c \\ -c & 1 \end{bmatrix}$$

$$= 1 / (1 - c^2) \begin{bmatrix} 0 & -c \\ -c & 1 \end{bmatrix}$$

$$(A^T)^{-1} = 1 / (1 - c^2) \begin{bmatrix} 0 & -c \\ -c & 1 \end{bmatrix}$$

$$(A^T)^{-1} = 1 / (1 - c^2) \begin{bmatrix} 0 & -c \\ -c & 1 \end{bmatrix}$$

4. Show that $A^2 = 0$ is possible, but $A^T * A = 0$ is not possible (Unless A = zero matrix):

Consider $A = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} (0 + 0) & (0 + 0) \\ (0 + 0) & (0 + 0) \end{bmatrix}$. In other words,

In order for $(A^T)A = 0$ to be valid, there must exist some matrix such that the rows of A^T times the columns of A = 0 for every element. Or, since A^T is the transpose of A, there must exist some matrix A where the columns of A (Which are the rows of A^T) times the columns of A is zero. Obviously, this cannot be true unless the column of A is entirely 0, as there exists no number x other than 0 such that $x * x = 0$. Therefore, $(A^T)A$ must be nonzero unless A is a zero matrix.

7. True or False:

a. The block matrix $A = \begin{matrix} 0 & A \\ A & 0 \end{matrix}$ is automatically symmetric.

True.

b. If A and B are symmetric, then their product AB is symmetric.

False.

c. If A is not symmetric then A^-1 is not symmetric.

True.

d. When A, B, C are symmetric, The transpose of ABC = CBA.

True.

16. If A = A^T and B = B^T, which of the following are certainly symmetric?
a. A^2 - B^2

Symmetric.

b. (A + B)(A - B)

Symmetric.

c. ABA

Not certainly symmetric.

d. ABAB

Symmetric.