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Chapter 4.2

1. Project the vector b onto the line through a. Check that e is perpendicular to a.

a. (a a^T) / (a^T a) b

([1, 1, 1][1 1 1] / [1 1 1][1, 1, 1])([1, 2, 2])

$$1/3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$1/3 \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$$

$$p = \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix}$$

$$e = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix}$$

$$e = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\begin{aligned} e(a^T) &= [-2/3, 1/3, 1/3][1, 1, 1] \\ &= -2/3 + 1/3 + 1/3 \\ &= 0 \end{aligned}$$

b. ([-1, -3, -1][-1 -3 -1] / [-1 -3 -1][-1, -3, -1])([1, 3, 1])

$$1/11 \begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$1/11 \begin{bmatrix} 11 \\ 33 \\ 11 \end{bmatrix}$$

$$p = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$e = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = 0$$

$$e(a^T) = 0$$

3. For each of the above, find P, verify P^2 = P, compute p by Pb.

a. (a a^T) / (a^T a) b

([1, 1, 1][1 1 1] / [1 1 1][1, 1, 1])([1, 2, 2])

$$P = 1/3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P^2 = 1/9 \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = P$$

$$1/3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$1/3 \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$$

$$p = \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix}$$

b. ([-1, -3, -1][-1 -3 -1] / [-1 -3 -1][-1, -3, -1])([1, 3, 1])

$$P = 1/11 \begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}$$

$$P^2 = 1/121 \begin{bmatrix} 11 & 33 & 11 \\ 33 & 99 & 33 \\ 11 & 33 & 11 \end{bmatrix} = P$$

$$1/11 \begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$1/11 \begin{bmatrix} 11 \\ 33 \\ 11 \end{bmatrix}$$

$$p = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

```
11. Project b onto the column space of A by solving A^T(Ax) = A^T(b)
and p = Ax. Find e = b - p.
 a. A^T(A)= [  1  1 ]
            [  1  2 ]

    (A^T(A))^-1= 1/1[  2 -1 ]
                    [ -1  1 ]

    * A^T = [  2 -1 ][  1  0  0 ]
            [ -1  1 ][  1  1  0 ]

    * b   = [  1 -1  0 ][  2 ]
            [  0  1  0 ][  3 ]
                        [  4 ]

    x^    = [ -1 ]
            [  3 ]

    p = [  1  1 ][ -1 ]
        [  0  1 ][  3 ]
        [  0  0 ]

      = [  2 ]
        [  3 ]
        [  0 ]

        [  2 ]   [  2 ]   [  0 ]
    e = [  3 ] - [  3 ] = [  0 ]
        [  4 ]   [  0 ]   [  4 ]

 b. A^T(A)= [  2  2 ]
            [  2  3 ]

    (A^T(A))^-1= 1/2[  3 -2 ]
                    [ -2  2 ]

    * A^T = 1/2 [  3 -2 ][  1  1  0 ]
                [ -2  2 ][  1  1  1 ]

    * b   = 1/2 [  1  1 -2 ][  4 ]
                [  0  0  2 ][  4 ]
                            [  6 ]

    x^    = 1/2 [ -4 ] = [ -2 ]
                [ 12 ]   [  6 ]

    p = [  1  1 ][ -2 ]
        [  1  1 ][  6 ]
        [  0  1 ]

      = [  4 ]
        [  4 ]
        [  6 ]
```

```
        [  4 ]   [  4 ]   [  0 ]
    e = [  4 ] - [  4 ] = [  0 ]
        [  6 ]   [  6 ]   [  0 ]
```

12.

21. Multiply P by itself, show that P^2 = P.

```
    P = A(A^T(A))^-1 A^T
    P^2 = A(A^T(A))^-1 A^T A(A^T(A))^-1 A^T
        = A(A^T(A))^-1 I A^T
        = A(A^T(A))^-1 A^T
        = P
```

Explain why P(Pb)=Pb: The vector Pb is in the column space, so the projection is always itself.

22. Prove that P is symmetric by computing P^T.

```
    P^T = ( A (A^T(A))^-1 A^T )^T
        = (A^T)^T ((A^T(A))^-1)^T (A)^T
        = A (A^T(A))^-1 A^T
        = P
```

Chapter 4.3

1. With b = { 0, 8, 8, 20 } while t = { 0, 1, 3, 4 }, set up and solve the normal equations
A^T(Ax) = A^T(b).

```
    { C + 0D =  0 }
    { C + 1D =  8 }
    { C + 3D =  8 }
    { C + 4D = 20 }

        [  1  0 ]
    A = [  1  1 ]
        [  1  3 ]
        [  1  4 ]

    A^T=[  1  1  1  1 ]
        [  0  1  3  4 ]

    x = [ C ]
        [ D ]

    b = [  0 ]
        [  8 ]
        [  8 ]
        [ 20 ]

     (A^T(A))  = [ 4  8 ]
                 [ 8 26 ]
```

$${}^{-1} * A^T = 1/40\begin{bmatrix} 26 & -8 \\ -8 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}$$

$$* \ b = 1/40\begin{bmatrix} 26 & 18 & 2 & -6 \\ -8 & -4 & 4 & 8 \end{bmatrix}\begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$x = 1/40\begin{bmatrix} 40 \\ 160 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$1 + 4t = b$ is the closest approximation.

For the best straight line in figure 4.9a, find its four heights pi and four
errors ei. What is the minimum value $E = e1^2 + e2^2 + e3^2 + e4^2$?

$p1 = 1 + 4(0) = 1$
$p2 = 1 + 4(1) = 5$
$p3 = 1 + 4(3) = 13$
$p4 = 1 + 4(4) = 17$

$e1 = 0 - 1 = -1$
$e2 = 8 - 5 = 3$
$e3 = 8 - 13 = -5$
$e4 = 20 - 17 = 3$

The minimum value for E is $1 + 9 + 25 + 9 = 44$.

2. With b = { 0, 8, 8, 20 } while t = { 0, 1, 3, 4 }, write down the four equations $Ax = b$.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

Change the measurements to p = { 1, 5, 13, 17 } and find an exact solution.

As shown in problem 1, $1 + 4t = p$ is an exact solution, when $C = 1$ and $D = 4$.

3. Check that e = b - p = [-1, 3, -5, 3] is perpendicular to both columns of the same matrix A.

```
    A1^T = [  1  1  1  1 ]
    A2^T = [  0  1  3  4 ]

    A1^T(e) = 6 - 6 = 0
    A2^T(e) = 3 - 15 + 12 = 0
```

What is the shortest distance ||e|| from b to the column space of A?

```
    ||e|| = sqrt(1 + 9 + 25 + 9)
          = sqrt(44)
```

6. Project b = [0, 8, 8, 20] onto the line through a = [1, 1, 1, 1]. Find x = a^T(b) / a^T(a) and p = xa. Check that e = b - p is perpendicular to a, and find the shortest distance ||e|| from b to the line through a.

```
    x = a^T(b) / a^T(a)
      = [ 1 1 1 1 ][ 0, 8, 8, 20] / [ 1 1 1 1][ 1, 1, 1, 1 ]
      = 36 / 1
      = 36

    p = xa
      = 36[ 1, 1, 1, 1 ]
      = [ 36, 36, 36, 36 ]

    e = b - p
      = [ 0, 8, 8, 20 ] - [ 36, 36, 36, 36 ]
      = [ -36, -28, -28, -16 ]
```