Homework 7
Section 3.2

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3.b. **Find all the idempotents in $\mathbb{Z}_{12}$.**

In $\mathbb{Z}_{12}$:

$$\begin{aligned} 0^2 &= 0 & 6^2 &= 0 \\ 1^2 &= 1 & 7^2 &= 1 \\ 2^2 &= 4 & 8^2 &= 4 \\ 3^2 &= 9 & 9^2 &= 9 \\ 4^2 &= 4 & 10^2 &= 4 \\ 5^2 &= 1 & 11^2 &= 1 \end{aligned}$$

Therefore $\{0, 1, 4, 9\}$ are the idempotents in $\mathbb{Z}_{12}$.

6.b. **If $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, find four solutions to the equation $AX = 0$.**

$$\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 8 & 8 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} 16 & 16 \\ -8 & -8 \end{bmatrix}$$

8. **Let R be a ring and b be a fixed element of R. Let $T = \{rb | r \in R\}$. Prove that T is a subring of R.**

To prove that T is a subring of R we must prove that

(a) T is a subset of R.

(b) T is closed under subtraction.

(c) T is closed under multiplication.

Since T only contains multiples of elements from R, and since R is closed under multiplicaiton, it is trivial to see that T is a subset of R. Let us then focus on the other two requirements.

Given some arbitrary $u, v \in T$, we can see that

$$
\begin{aligned}
u - v &= (rb) + -(sb) \qquad r, b, s \in R \\
&= (r - s)b \\
&= kb \qquad k = r - s
\end{aligned}
$$

Since R is a ring, it must be closed under subtraction, therefore $k \in R$ and by extension, $u - v \in T$, so T is closed under subtraction. Next we consider multiplication.

$$
\begin{aligned}
u * v &= rb * sb \qquad r, b, s \in R \\
&= (r * s)b \\
&= kb \qquad k = r * s
\end{aligned}
$$

and since R is closed under multiplicaiton, $kb \in R \implies u * v \in T$. Therefore, T must be a subring of R $\blacksquare$

13.a. **Let S and T be subrings of a ring R. Prove or disprove that $S \cap T$ is a subring.**

Let $s \in S$, $t \in T$ such that $s, t \in S \cap T$. It is obvious that if $s, t \in S \cap T$ then $s, t \in S$. Since S is a subring of R, we know that it must be closed under subtraction, and thus that $s - t \in S$. By similar logic, we know that $s, t \in T$ and so $s - t \in T$. Since $s - t \in S$ and $s - t \in T$, it is clear that $s - t \in S \cap T$. Therefore, $S \cap T$ is closed under subtraction.

By the same logic above, $s, t \in S \cap T$ implies that $s, t \in S, T$, leading to $s * t \in S, T$ and finally $s * t \in S \cap T$. This shows that $S \cap T$ is closed under multiplication, and according to Theorem 3.6 this means that $S \cap T$ is a subring of R $\blacksquare$

13.b. **Let S and T be subrings of a ring R. Prove or disprove that $S \cup T$ is a subring.**

Consider the example when $R = \mathbb{Z}$, $S = \{3k | k \in \mathbb{Z}\}$, $T = \{2j | j \in \mathbb{Z}\}$. Then let $s = 3(1)$ and $t = 2(1)$. We can see that $s \in S$ and $t \in T$ implies that $s, t \in S \cup T$, but $s + t = 5$, and since 5 is not a multiple of 2 or 3, $5 \notin S$ and $5 \notin T$, therefore $S \cup T$ is not necessarily closed under subtraction, and we cannot assume that $S \cup T$ is a subring of R $\blacksquare$

15.a. **If a and b are units in a ring R with identity, prove that ab is a unit whose inverse is $(ab)^{-1} = b^{-1} * a^{-1}$**

To show that ab is a unit, we must show that there exists some $x \in R$ such that $1 = abx$ in R. Since R is a ring with identity, we know that a^{-1} and b^{-1} are both in R, and since rings are closed under multiplication we know that the product of those two elements must also be in the ring. Therefore, let $x = b^{-1} * a^{-1}$ and we can see that:

$$\begin{aligned} abx &= 1 \\ a * b * b^{-1} * a^{-1} &= 1 \\ a * 1 * a^{-1} &= 1 \\ a * a^{-1} &= 1 \\ 1 &= 1 \end{aligned}$$

Which is true. Therefore, if a and b are units in a ring R with identity, ab is a unit whose inverse is $(ab)^{-1} = b^{-1} * a^{-1}$ ■

31.a. **Prove that $a + a = 0$ for every a in a boolean ring.**

Let R be a boolean ring, and let $a \in R$. Furthermore, $x * x = x$ implies that there is no x other than 0 such that $x * x = 0$, or rather that R is an integral domain. We can then see that

$$\begin{aligned} (a + a) &= (a + a)^2 \\ (a + a)^2 &= (a + a)(a + a) \\ (a + a) &= (a + a)(a + a) \\ 0 &= (a + a) \end{aligned}$$

Therefore $a + a = 0$ ■

31.b. **Prove that every boolean ring is commutative.**

Let R be a boolean ring, and let $a, b \in R$. Then

$$\begin{aligned} a + b &= (a + b)^2 \\ &= a^2 + ab + ba + b^2 \\ &= a + ab + ba + b \\ 0 &= ab + ba \\ ab &= -ba \end{aligned}$$

However, note that since R is a boolean ring

$$\begin{aligned}(-a) &= (-a)^2 \\ &= (-a)(-a) \\ &= a^2 \\ &= a\end{aligned}$$

Therefore $-ba$ from our original equation is equal to ba, and we see that

$$ab = ba$$

Or, in other words, that R is commutative $\blacksquare$