Homework 5
Sections 2.2, 2.3

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SECTION 2.2

4. Solve $x^4 = [1]$ in $\mathbb{Z}_5$

$$x^4 = [1] \qquad \mathbb{Z}_5$$

$$\begin{aligned}[0]^4 &= [0] * [0] * [0] * [0] \\ &= 0 * 0 * 0 * 0 = 0 = [0]\end{aligned}$$

$$\begin{aligned}[1]^4 &= [1] * [1] * [1] * [1] \\ &= 1 * 1 * 1 * 1 = 1 = [1]\end{aligned}$$

$$\begin{aligned}[2]^4 &= [2] * [2] * [2] * [2] \\ &= 2 * 2 * 2 * 2 = 2 \\ &= 16 = 3 * 5 + 1 = [1]\end{aligned}$$

$$\begin{aligned}[3]^4 &= [3] * [3] * [3] * [3] \\ &= 3 * 3 * 3 * 3 = 3 \\ &= 81 = 16 * 5 + 1 = [1]\end{aligned}$$

$$\begin{aligned}[4]^4 &= [4] * [4] * [4] * [4] \\ &= 4 * 4 * 4 * 4 = 4 \\ &= 256 = 51 * 5 + 1 = [1]\end{aligned}$$

Therefore, $[1], [2], [3], [4]$ are all solutions.

8. Solve $x^3 + x^2 = [2]$ in $\mathbb{Z}_{10}$

$$x^3 + x^2 = [2] \qquad \mathbb{Z}_1 0$$

$$\begin{aligned}
[0]^3 + [0]^2 &= [0] * [0] * [0] + [0] * [0] \\
&= [0]
\end{aligned}$$

$$\begin{aligned}
[1]^3 + [1]^2 &= [1] * [1] * [1] + [1] * [1] \\
&= [1] + [1] \\
&= [2]
\end{aligned}$$

$$\begin{aligned}
[2]^3 + [2]^2 &= [2] * [2] * [2] + [2] * [2] \\
&= [8] + [4] \\
&= [2]
\end{aligned}$$

$$\begin{aligned}
[3]^3 + [3]^2 &= [3] * [3] * [3] + [3] * [3] \\
&= [27] + [9] \\
&= [6]
\end{aligned}$$

$$\begin{aligned}
[4]^3 + [4]^2 &= [4] * [4] * [4] + [4] * [4] \\
&= [64] + [16] \\
&= [0]
\end{aligned}$$

$$\begin{aligned}
[5]^3 + [5]^2 &= [5] * [5] * [5] + [5] * [5] \\
&= [125] + [25] \\
&= [0]
\end{aligned}$$

$$\begin{aligned}
[6]^3 + [6]^2 &= [6] * [6] * [6] + [6] * [6] \\
&= [216] + [36] \\
&= [2]
\end{aligned}$$

$$
\begin{aligned}
[7]^3 + [7]^2 &= [7] * [7] * [7] + [7] * [7] \\
&= [343] + [49] \\
&= [2]
\end{aligned}
$$

$$
\begin{aligned}
[8]^3 + [8]^2 &= [8] * [8] * [8] + [8] * [8] \\
&= [512] + [64] \\
&= [6]
\end{aligned}
$$

$$
\begin{aligned}
[9]^3 + [9]^2 &= [9] * [9] * [9] + [9] * [9] \\
&= [729] + [81] \\
&= [0]
\end{aligned}
$$

Therefore $[1], [2], [6], [7]$ are all solutions.

14. Solve the following.

    a. $x^2 + x = [0]$ in $\mathbb{Z}_5$

$$
\begin{aligned}
&[0]^2 + [0] = [0] * [0] + [0] = [0] \\
&[1]^2 + [1] = [1] * [1] + [1] = [1] + [1] = [2] \\
&[2]^2 + [2] = [2] * [2] + [2] = [4] + [2] = [1] \\
&[3]^2 + [3] = [3] * [3] + [3] = [9] + [3] = [2] \\
&[4]^2 + [4] = [4] * [4] + [4] = [16] + [4] = [0]
\end{aligned}
$$

    Therefore, $[0], [4]$ are solutions.

    b. $x^2 + x = [0]$ in $\mathbb{Z}_6$

$$
\begin{aligned}
&[0]^2 + [0] = [0] * [0] + [0] = [0] \\
&[1]^2 + [1] = [1] * [1] + [1] = [1] + [1] = [2] \\
&[2]^2 + [2] = [2] * [2] + [2] = [4] + [2] = [0] \\
&[3]^2 + [3] = [3] * [3] + [3] = [9] + [3] = [0] \\
&[4]^2 + [4] = [4] * [4] + [4] = [16] + [4] = [2] \\
&[5]^2 + [5] = [5] * [5] + [5] = [25] + [5] = [0]
\end{aligned}
$$

    Therefore, $[0], [2], [3], [5]$ are solutions.

    c. If p is prime, prove that the only solutions of $x^2 + x = [0]$ in $\mathbb{Z}_p$ are $[0]$ and $[p - 1]$.
    By Theorem 2.8 and the fact that

$$\begin{aligned} x^2 + x &= [0] \\ x(x+1) &= [0] \end{aligned}$$

we know that either $x = [0]$ or $x + 1 = [0]$ in $\mathbb{Z}_p$. Solving for x this shows us that the solutions to $x^2 + x = [0]$ must be $x = [0]$ or

$$\begin{aligned} x &= [0] - 1 \\ &= [p] - [1] \\ &= [p-1] \end{aligned}$$

and so the statement holds $\blacksquare$

16.a. Find all $[a]$ in $\mathbb{Z}_5$ for which the equation $[a] * x = [1]$ has a solution.

$$\begin{aligned} [a] = [0] &\implies [0]x = [1] && \text{No Solution} \\ [a] = [1] &\implies [1]x = [1] && \text{Solution at x} = [1] \\ [a] = [2] &\implies [2]x = [1] && \text{Solution at x} = [3] \\ [a] = [3] &\implies [3]x = [1] && \text{Solution at x} = [2] \\ [a] = [4] &\implies [4]x = [1] && \text{Solution at x} = [4] \end{aligned}$$

Therefore, $[a] * x = [1]$ has solutions at values $[1], [2], [3], [4]$.

SECTION 2.3

1.b. Find all the units in $\mathbb{Z}_8$.

As discussed in class, Units and Zero Divisors partition the nonzero elements of $\mathbb{Z}_n$, so every nonzero element of $\mathbb{Z}_8$ must be either a Zero Divisor or a Unit. We then find that:

$$\begin{aligned} &[0] \text{ is not a nonzero element.} \\ &[1] = [1] * [1] \implies [1] \text{ is a Unit.} \\ &[0] = [4] * [2] \implies [2] \text{ is a Zero Divisor.} \\ &[1] = [3] * [3] \implies [3] \text{ is a Unit.} \\ &[0] = [2] * [4] \implies [4] \text{ is a Zero Divisor.} \\ &[1] = [5] * [5] \implies [5] \text{ is a Unit.} \\ &[0] = [4] * [6] \implies [6] \text{ is a Zero Divisor.} \\ &[7] = [7] * [7] \implies [7] \text{ is a Unit.} \end{aligned}$$

Therefore, $[1], [3], [5], [7]$ are all units in $\mathbb{Z}_8$.

2.b. Find all the Zero Divisors in $\mathbb{Z}_8$.

By the previous problem, $[2], [4], [6]$ are all Zero Divisors.

4.c. How many solutions does $[6]x = [4]$ have in $\mathbb{Z}_9$?

$$
\begin{aligned}
[6][0] &= [0] \\
[6][1] &= [6] \\
[6][2] &= [3] \\
[6][3] &= [0] \\
[6][4] &= [6] \\
[6][5] &= [3] \\
[6][6] &= [0] \\
[6][7] &= [6] \\
[6][8] &= [3]
\end{aligned}
$$

Therefore, $[6]x = [4]$ has no solutions in $\mathbb{Z}_9$.

8. a. Give three examples of equations in the form $[a]x = [b]$ in $\mathbb{Z}_{12}$ that have no nonzero solutions.

$$[2]x = [1] \tag{1}$$
$$[3]x = [1] \tag{2}$$
$$[4]x = [1] \tag{3}$$

b. For each example above, does the equation $[a]x = [0]$ have a nonzero solution?

Yes.

$$
\begin{aligned}
[2][6] &= [0] \\
[3][4] &= [0] \\
[4][3] &= [0]
\end{aligned}
$$

15. Use Exercise 13 to solve the following equations:

a. $[15]x = [9]$ in $\mathbb{Z}_{18}$

Given Exercise 13, we know that for $a, b, n \in \mathbb{Z}$, $n > 0$, the solutions to $[a]x = [b]$ in $\mathbb{Z}_n$ are given by the set:

$$
\begin{gathered}
\{[ub_1 + 0n_1], [ub_1 + n_1], \ldots, [ub_1 + (d-1)n_1]\} \\
d = au + nv \qquad u, v \in \mathbb{Z} \\
b = db_1 \qquad n = dn_1 \qquad a = da_1
\end{gathered}
$$

Applying this property to our problem, we see that

$$\begin{aligned} n = 18 \qquad a &= 15 \qquad b = 9 \\ dn_1 = 18 \qquad da_1 &= 15 \qquad db_1 = 9 \\ d &= 15u + 18v \\ d &= 3(5u + 6v) \\ d = 3 \qquad u &= -1 \qquad v = 1 \\ 18 = 3n_1 &\implies n_1 = 6 \\ 15 = 3a_1 &\implies a_1 = 5 \\ 9 = 3b_1 &\implies b_1 = 3 \end{aligned}$$

Therefore, $[(-1)(3)], [(-1)(3) + 6], [(-1)(3) + (2)6]$ are all solutions. Simplifying the terms, we find the solutions to be $[15], [3], [9]$.

b. $[25]x = [10]$ in $\mathbb{Z}_{65}$

Following the above, we have $n = 65$, $a = 25$, $b = 10$. Therefore:

$$\begin{aligned} d &= 25u + 65v \\ &= 5(5u + 7v) \\ &= 5 \qquad u = 3 \qquad v = -2 \end{aligned}$$

So then

$$\begin{aligned} 5n_1 = 65 &\implies n_1 = 7 \\ 5a_1 = 25 &\implies a_1 = 5 \\ 5b_1 = 10 &\implies b_1 = 2 \end{aligned}$$

and

$$\begin{aligned} [3(2) + 0(7)] &= [6] \\ [3(2) + 1(7)] &= [13] \\ [3(2) + 2(7)] &= [20] \\ [3(2) + 3(7)] &= [27] \\ [3(2) + 4(7)] &= [34] \end{aligned}$$

are our solutions.