Homework 8
Section 3.3

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11.b. **State one reason why $g : E \to E$ where E is the ring of even integers and $g(x) = 3x$ is not a homomorphism.**

g does not preserve multiplication on E. For example,

$$2 * 4 = 8 \not\Rightarrow 6 * 12 = 18$$

11.c. **State one reason why $h : \mathbb{R} \to \mathbb{R}$ and $h(x) = 2^x$ is not a homomorphism.**

h does not preserve the identity. The identity of $\mathbb{R}$ is 1, but $h(1) = 2$. By exensions, this implies that h does not preserve multiplication, as the identity is the only element of a ring such that $a * i = a$ for some a in the ring and i as the ring's identity.

12.c. **Is $g : \mathbb{Q} \to \mathbb{Q}$, $g(x) = \frac{1}{x^2+1}$ a homomorphism?**

No. g does not preserve the identity of $\mathbb{Q}$. The identity of Q is $i = \frac{1}{1}$, but $g(i) = \frac{1}{2}$.

12.d. **Is $h : \mathbb{R} \to M(\mathbb{R})$, $h(a) = \begin{bmatrix} -a & 0 \\ a & 0 \end{bmatrix}$ a homomorphism?**

No. Once more, h does not preserve the identity. The identity of $M(\mathbb{R})$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the identity of $\mathbb{R}$ is 1, but $h(1) = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$.

13.a. **Prove that $f : R \times S \to R$ given by $f(r, s) = r$ is an epimorphism.**

To prove that f is an epimorphism, we must prove that f is both surjective and a homomorphism. As follows:

Claim: f is surjective

Let b be an arbitrary element of the range space R. To show that f is surjective, we must find some a in $R \times S$ such that $f(a, s) = b$ for all b. But this is trivial, since we can simply let $a = b$, and we see that $f(b, s) = b$ for any value of b and s.

Claim: f is a homomorphism

To prove that f is a homomorphism, we must show that it preserves addition and multiplication. Observe

$$\begin{aligned} f((r_1, s_1) + (r_2, s_2)) &= f((r_1 + r_2, s_1 + s_2)) \\ &= r_1 + r_2 \\ &= f((r_1, s_1)) + f((r_2, s_2)) \end{aligned}$$

$$\begin{aligned} f((r_1, s_1) * (r_2, s_2)) &= f((r_1 r_2, s_1 s_2)) \\ &= r_1 r_2 \\ &= f((r_1, s_1)) * f((r_2, s_2)) \end{aligned}$$

Therefore, f is an epimorphism $\blacksquare$

13.c. **If both R and S are nonzero rings, prove that the homomorphisms $f(r, s) = r$ and $g(r, s) = s$ are not injective.**

Let a, b be arbitrary nonzero elements in R and S respectively. It is clear that $f(a, 0_S) = a$ and $f(a, b) = a$, but that $0_S \neq b$. Therefore, f cannot be injective. By a similar logic, $g(0_R, b) = b$ and $g(a, b) = b$, but $0_R \neq a$, so g likewise cannot be injective $\blacksquare$

25. **Let L be the ring of all matrices in $M(\mathbb{Z})$ of the form $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$. Show that the function $f : L \to \mathbb{Z}$ given by $f\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right) = a$ is an epimorphism but not an isomorphism.**

To prove that f is an epimorphism but not an isomorphism, we must prove that

(a) f is a homomorphism.

(b) f is surjective.

(c) f is not injective.

Claim: f is a homomorphism

Again, we must show that f preserves addition and multiplication to show that is a ring homomorphism.

$$
\begin{aligned}
f\left(\begin{bmatrix} a_1 & 0 \\ b_1 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ b_2 & c_2 \end{bmatrix}\right) &= f\left(\begin{bmatrix} a_1 + a_2 & 0+0 \\ b_1 + b_2 & c_1 + c_2 \end{bmatrix}\right) \\
&= a_1 + a_2 \\
&= f\left(\begin{bmatrix} a_1 & 0 \\ b_1 & c_1 \end{bmatrix}\right) + f\left(\begin{bmatrix} a_2 & 0 \\ b_2 & c_2 \end{bmatrix}\right)
\end{aligned}
$$

$$
\begin{aligned}
f\left(\begin{bmatrix} a_1 & 0 \\ b_1 & c_1 \end{bmatrix} * \begin{bmatrix} a_2 & 0 \\ b_2 & c_2 \end{bmatrix}\right) &= f\left(\begin{bmatrix} a_1 a_2 + 0b_2 & 0a_1 + 0c_2 \\ b_1 a_2 + c_1 b_2 & 0b_1 + c_1 c_2 \end{bmatrix}\right) \\
&= f\left(\begin{bmatrix} a_1 a_2 & 0 \\ b_1 a_2 + c_1 b_2 & c_1 c_2 \end{bmatrix}\right) \\
&= a_1 a_2 \\
&= f\left(\begin{bmatrix} a_1 & 0 \\ b_1 & c_1 \end{bmatrix}\right) * f\left(\begin{bmatrix} a_2 & 0 \\ b_2 & c_2 \end{bmatrix}\right)
\end{aligned}
$$

Therefore, f is a ring homomorphism.

<u>Claim: f is surjective</u>

Let x be an arbitrary element of $\mathbb{Z}$, and l be an arbitrary element of L. We can see that for any x, $f(l) = x$ if $a_l = x$. Therefore, f is surjective.

<u>Claim: f is not injective</u>

Using the above setup, we can see that there exist many ways for $f(l) = x$, one for each distinct value of b_l and c_l. One such example are the matrices

$$l_1 = \begin{bmatrix} x & 0 \\ 1 & 1 \end{bmatrix}$$

and

$$l_2 = \begin{bmatrix} x & 0 \\ 2 & 2 \end{bmatrix}$$

which are clearly distinct but have the same mapping. Therefore, f is not injective.

Thusly, it is clear that f is an epimorphism but not an isomorphism ■

31. **Let $f : R \to S$ be a ring homomorphism and T be a subring of S. Let $P = \{r \in R | f(r) \in T\}$. Prove that P is a subring of R.**

To prove that P is a subring of R, we must show that P is a subset of R and that P is closed under subtraction and multiplication. That is, we must show that

(a) $P \subseteq R$

(b) $p_1 - p_2 \in P$

(c) $p_1 * p_2 \in P$

By definition, every element of P is an element of R, so the first condition is trivial. Let us examine the second condition. By the definition of P, $p = f(r)$ for a given value p, and $f(r)$ is in T, a subring of S. Because f is a homomorphism

$$\begin{aligned} p_1 - p_2 &= f(r_1) - f(r_2) \\ &= f(r_1 - r_2) \end{aligned}$$

We can see from the definition of P that p is surjunctive, or that for every element $t \in T$, there must be some p such that $p = f(r) = t$. Since T is a subring of S, we know it is closed under subtraction, and therefore

$$f(r_1) - f(r_2) = f(r_1 - r_2)$$

implies that $f(r_1 - r_2) \in T$.

Combining these two facts, we can see that

$$\begin{aligned} p_1 - p_2 &= f(r_1 - r_2) \qquad f(r_1 - r_2) \in T \\ &= t \\ p_3 &= t \end{aligned}$$

or rather, that P is closed under subtraction.

Using similar proofs, we see that

$$\begin{aligned} p_1 * p_2 &= f(r_1) * f_(r_2) \\ &= f(r_1 * r_2) \\ &= t_3 \\ p_3 &= t_3 \end{aligned}$$

or that P is closed under multiplication. Therefore, P has all of the necessary and sufficent qualities to be a subring of R $\blacksquare$