Homework 12
Section 4.4, 4.5

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4.4.6.a. **Verify that every element of $\mathbb{Z}_3$ is a root of $x^3 - x \in \mathbb{Z}_3[x]$.**

$$\begin{aligned}
[0]_3: &\qquad [0]^3 - [0] = [0] - [0] = [0] \\
[1]_3: &\qquad [1]^3 - [1] = [1] - [1] = [0] \\
[0]_3: &\qquad [2]^3 - [2] = [6] - [2] = [0]
\end{aligned}$$

4.4.6.b. **Verify that every element of $\mathbb{Z}_5$ is a root of $x^5 - x \in \mathbb{Z}_5[x]$.**

$$\begin{aligned}
[0]_5: &\qquad [0]^5 - [0] = [0] - [0] = [0] \\
[1]_5: &\qquad [1]^5 - [1] = [1] - [1] = [0] \\
[2]_5: &\qquad [2]^5 - [2] = [32] - [2] = [0] \\
[3]_5: &\qquad [3]^5 - [3] = [243] - [3] = [0] \\
[4]_5: &\qquad [4]^5 - [4] = [1024] - [4] = [0]
\end{aligned}$$

4.4.6.c. **Make a conjecture about roots of $x^p - x \in \mathbb{Z}_p[x]$.**
$x^p - x \in \mathbb{Z}_p[x]$ has a root at every $z \in \mathbb{Z}_p$.

4.4.10. **Find a prime $p > 5$ such that $x^2 + 1$ is reducible in $\mathbb{Z}_p[x]$.**

$$x^2 + 1 = (x - 5)(x - 8) \qquad \text{in } \mathbb{Z}_{13}[x]$$

4.4.17. **Find a polynomial of degree 2 in $\mathbb{Z}_6[x]$ that has four roots in $\mathbb{Z}_6$. Does this contradict Corollary 4.17?**

... (Incomplete, but probably involves the zero divisors of $\mathbb{Z}_6$)

However, regardless of the polynomial found, it does not contradict Corollary 4.17, as Corollary 4.17 only holds when talking about fields, and $\mathbb{Z}_6[x]$ is not a field, since $6 = 2 * 3$ is not prime.

4.5.1.c. **Use the Rational Root Test to write $3x^5 + 2x^4 - 7x^3 + 2x^2$ as a product of irreducible polynomials in $\mathbb{Q}[x]$.**

Since $a_0 = 0$ in this polynomial, the Rational Root Test is ill-defined. However, it is clear that we can factor out the polynomial x^2 from the given polynomial to form

$$3x^5 + 2x^4 - 7x^3 + 2x^2 = (x^2)(3x^3 + 2x^2 - 7x + 2)$$

From here, we may apply the Rational Root Test on $3x^3 + 2x^2 - 7x + 2$ to find futher factors.

4.5.1.f. **Use the Rational Root Test to write $6x^4 - 31x^3 + 25x^2 + 33x + 7$ as a product of irreducible polynomials in $\mathbb{Q}[x]$.**

4.5.5.a. **Use Einstein's Criterion to show that $x^5 - 4x + 22$ is irreducible in $\mathbb{Q}[x]$.**

Let $p = 2$. Then p divides all of the coefficients of the given polynomial except for the leading coefficient. Futher, $p^2 = 4$, which does not divide the constant coefficient. Therefore, by Einstein's Criterion, the polynomial is irreducible in $\mathbb{Q}[x]$.

4.5.5.b. **Use Einstein's Criterion to show that $10 - 15x + 25x^2 - 7x^4$ is irreducible in $\mathbb{Q}[x]$.**

Let $p = 5$. Then p divides all of the coefficients of the given polynomial except for the leading coefficient. Futher, $p^2 = 25$, which does not divide the constant coefficient. Therefore, by Einstein's Criterion, the polynomial is irreducible in $\mathbb{Q}[x]$.

4.5.5.c. **Use Einstein's Criterion to show that $5x^11 - 6x^4 + 12x^3 + 36x - 6$ is irreducible in $\mathbb{Q}[x]$.**

Let $p = 6$. Then p divides all of the coefficients of the given polynomial except for the leading coefficient. Futher, $p^2 = 36$, which does not divide the constant coefficient. Therefore, by Einstein's Criterion, the polynomial is irreducible in $\mathbb{Q}[x]$.

4.5.6. **Show that there are infinitely many k such that $x^9 + 12x^5 - 21x + k$ is irreducible in $\mathbb{Q}[x]$.**

Observe that the only prime factor that 12 and 21 share is 3. Futhermore, 3 does not divide 1, the leading coefficient. Therefore, by Einstein's Criterion, our polynomial is irreducible for any value of k when $3 \mid k$ and

$3^2 = 9$ does not divide k. Since the set of all multiples of three that are not multiples of nine is infinite, there must be an infinite number of values of k that make our polynomial irreducible in $\mathbb{Q}[x]$.