Homework 2
Section 1.2

Jonathan Petersen
A01236750

January 24th, 2016

---

3. **Prove that if $a|b$ and $b|c$, then $a|c$.**

   Since $a|b$, we know that

   $$b = am \qquad m \in \mathbb{Z}$$

   By a similar argument,

   $$c = bn \qquad n \in \mathbb{Z}$$

   Substituting $b = am$ in this equation, we find that

   $$\begin{aligned} c &= amn \\ c &= ak \qquad k = mn \end{aligned}$$

   Since $m, n \in \mathbb{Z}$, and since multiplication is closed over $\mathbb{Z}$, $k \in \mathbb{Z}$. Therefore it must be that, $a|c$ $\blacksquare$

8. **Prove that $gcd(n, n+1) = 1$ for every integer n.**

   Let x be the GCD of $n, n+1$. By the Linear Combination property, we know that x can be written as:

   $$\begin{aligned} x &= nu + (n+1)v \qquad u, v \in \mathbb{Z} \\ x &= nu + nv + v \end{aligned}$$

   If we then let $u = -1$ and $v = 1$, we find that:

   $$\begin{aligned} x &= -n + n + 1 \\ x &= 1 \\ 1 &= nu + (n+1)v \end{aligned}$$

   Or, in other words, that $gcd(n, n+1)|1$. As discussed in class, this shows that the GCD of $n, n+1$ must be 1, as 1 is the only divisor of 1 $\blacksquare$

11. **If $n \in \mathbb{Z}$, what are the possible values of**

   a. $gcd(n, n+2)$
      1, 2

   b. $gcd(n, n+6)$
      1, 2, 3

14. **Find the smallest positive integer in the given set.**

   a. $\{6u + 15v | u, v \in \mathbb{Z}\}$.
      Since $3|6$ and $3|15$, we see that

$$6u + 15v = 3(2u + 5v)$$

      Furthermore, to find the smallest positve integer in the set it is obvous that we must make $2u + 5v$ as small as possible, without becoming zero. The smallest possible positive integer would be $2u + 5v = 1$, and indeed if $u = -7, v = 3$ we find that

$$\begin{aligned} 3(2u + 5v) &= 3(2(-7) + 5(3)) \\ &= 3(-14 + 15) \\ &= 3(1) \end{aligned}$$

      Which must be the smallest value as shown above.

   b. $\{12r + 17s | r, s \in \mathbb{Z}\}$.
      By similar reasoning, if we can find r, s such that $12r + 17s = 1$, it must also be the smallest positive integer. And indeed, if we set $r = -7, s = 5$ we see that

$$\begin{aligned} 12r + 17s &= 12(-7) + 17(5) \\ &= -84 + 85 \\ &= 1 \end{aligned}$$

      Which must mean that 1 is the smallest positive integer in the set.

22. **If $gcd(a, c) = 1$ and $gcd(b, c) = 1$, prove that $gcd(ab, c) = 1$.**

   Given that $gcd(a, c) = 1$ and $gcd(b, c) = 1$, we know by the Linear Combination property that

$$\begin{aligned} 1 &= au_1 + cv_1 \\ 1 &= bu_2 + cv_2 \end{aligned}$$

and it then follows that

$$a = abu_2 + acv_2$$

which, by substitution, means that

$$\begin{aligned} 1 &= (abu_2 + acv_2) + cv_1 \\ &= abu_2 + c(av_2 + v_1) \\ &= abu_3 + cv_3 \qquad\qquad u_3 = u_2, v_3 = av_2 + v_1 \end{aligned}$$

By the converse of the Linear Combination property in the special case when considering a GCD of 1, this shows that if $gcd(a, c) = 1$ and $gcd(b, c) = 1$, then $gcd(ab, c) = 1$ ■

20. Extra Credit: Prove that $gcd(a, b) = gcd(a, b + at)$ for every $t \in \mathbb{Z}$.