Homework 5

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17. **Suppose that in a sequence of independent Bernoulli trials, each with probability p, the number of failures up to the first success is counted. What is the frequency funciton for this random variable?**

    Let X be the number of failures up to the first success. Then we see that when $X = k$, there must be k failures and 1 success, or in other words:

    $$P(X = k) = p(1-p)^k \qquad k = 0, 1, 2, \ldots$$

19. **Find the expression for the CDF of a geometric random variable.**

    The CDF of a geometric random variable for some threshold n could be found by calculating the value of the geometric distribution for $k = 0, 1, 2, \ldots, n$.

    Thus:

    $$\begin{aligned}\sum_{k=0}^{n} p(1-p)^{k-1} &= p(1-p)^0 + p(1-p)^1 + \cdots + p(1-p)^n \\ &= 1 - (1-p)^k\end{aligned}$$

20. **If X is a geometric random variable with $p = 0.5$, for what value of k is $P(X \leq k) \approx 0.99$?**

    To calculate $P(X \leq k)$ we can use the CDF of the geometric distribution function found above, namely

    $$P(X \leq k) = 1 - (1-p)^k$$

    we can see that when $p = 0.5$

$$
\begin{aligned}
P(X \leq k) &= 1 - (0.5)^k \\
0.99 &\approx 1 - (0.5)^k \\
-0.01 &\approx 0.5^k \\
log_{0.5}(-0.01) &\approx k
\end{aligned}
$$

Or rather, that $P(X \leq k) \approx 0.99$ occurs when $k = log_{0.5}(-0.01)$.

22. **Three identical fair coins are tossed simultaneously until all three show the same face. What is the probability that they are thrown more than 3 times?**

$$
\begin{aligned}
P(x) = P(\text{all 3 heads}) &= P(h_1) * P(h_2) * P(h_3) \\
&= 0.5 * 0.5 * 0.5 = 0.0625
\end{aligned}
$$

$$
\begin{aligned}
P(X > 3) &= 1 - P(X \leq 3) \qquad X = \text{Number of times coins thrown} \\
&= 1 - (P(X = 1) + P(X = 2) + P(X = 3)) \\
&= 1 - ((0.0625) + (0.0625)(1 - 0.0625) + (0.0625)(1 - 0.0625)^2) \\
&= 1 - (0.0625 + 0.05859375 + 0.05493164062) \\
&= 1 - 0.1760253906 \\
&= 0.8239746094
\end{aligned}
$$

26. **Given the setup, what is the probability that the professor will never be trapped ($X = 0$)? That he will be trapped once ($X = 1$)? Twice ($X = 2$)?**

We can approximate the value by a Poisson distribution, with

$$
\begin{aligned}
X &= \text{The number of times the professor is trapped.} \\
p &= 1/10,000 \\
n &= 5 * 52 * 10 = 2600 \\
\therefore \lambda &= np = 0.26
\end{aligned}
$$

And we approximate X as follows

$$
\begin{aligned}
P(X = k) &= \frac{\lambda^k}{k!} e^{-\lambda} \\
P(X = 0) &= \frac{0.26^0}{0!} e^{-0.26} = e^{-0.26} \\
P(X = 1) &= \frac{0.26^1}{1!} e^{-0.26} = 0.26 e^{-0.26} \\
P(X = 2) &= \frac{0.26^2}{2!} e^{-0.26} \\
&= \frac{0.0676}{2} e^{-0.26} = 0.0338 e^{-0.26}
\end{aligned}
$$

27. **Suppose a rare disease has an incidence of 1 in 1000. Find the probability of k cases in a population for $k = 0, 1, 2$.**

    We can approximate the value by a Poisson distribution. Let X be the number of cases, then given

$$
\begin{aligned}
p &= 1/1000 \\
n &= 100{,}000
\end{aligned}
$$

    we can calculate X by the formula

$$
\begin{aligned}
P(X = k) &= \frac{\lambda^k}{k!} e^{-\lambda} \qquad \lambda = np = 100 \\
P(X = 0) &= \frac{100^0}{0!} e^{-100} = e^{-100} \\
P(X = 1) &= \frac{100^1}{1!} e^{-100} = 100 e^{-100} \\
P(X = 2) &= \frac{100^2}{2!} e^{-100} \\
&= \frac{10000}{2} e^{-100} = 5000 e^{-100}
\end{aligned}
$$

32. **For what value of k is the Poisson frequency function maximized?**

    The Poisson frequency function

$$
\begin{aligned}
P(X = k) &= \frac{\lambda^k}{k!} e^{-\lambda} \qquad \lambda = np \\
&= \frac{\lambda^k e^{-\lambda}}{k!}
\end{aligned}
$$

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