HW1

1, 3, 4, 6, 7, 8

Jonathan Peterson
A01236750
MATH 5710

1. a. The sample space is

$\{hhh, hht, hth, htt, thh, tht, tth, ttt\}$ h = heads, t = tails

b. A = at least two heads

$A = \{hhh, hht, hth, thh\}$

B = the first two tosses are heads.

$= \{hhh, hht\}$

C = The last toss is a tail

$= \{hht, htt, tht, ttt\}$

c. $A^c = \{htt, tht, tth, ttt\}$

$A \cap B = \{hhh, hht\}$

$A \cup B = \{hhh, hht, hth, thh\}$

3. $S = \{rrr, rrg, rrw, rgr, rgg, rgw, rwr, rwg, grr, grg, grw, ggr, ggw, gwr, gwg, wrr, wrg, wgr, wgg\}$

r = a red ball is drawn,
g = a green ball is drawn,
w = a white ball is drawn.

$A = \{rrr, rrg, rrw\}$ The first two draws are red
$B = \{ggr, ggw\}$ The first two draws are green.
$C = \{wrr, wrg, wgr, wgg\}$ The first color drawn is white.

$A \cup B = \{rrr, rrg, rrw, ggr, ggw\}$
$A \cup C = \{rrr, rrg, rrw, wrr, wrg, wgr, wgg\}$
$B \cup C = \{ggr, ggw, wrr, wrg, wgr, wgg\}$

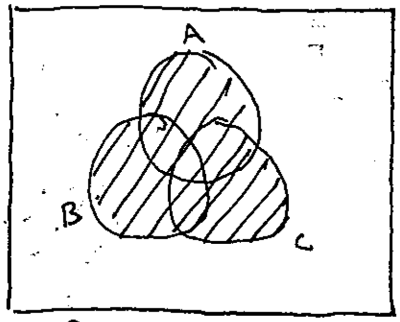
$\left.\begin{array}{l} A \cap B = \emptyset \\ A \cap C = \emptyset \\ B \cap C = \emptyset \end{array}\right\}$ A, B, C are disjoint events.

4.

$(A \cup B)^c$ (///) $A^c \cap B^c$ (✻) $(A \cap B)^c$ (///) $A^c \cup B^c$ (✻)

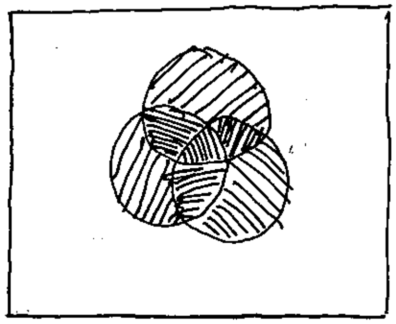
Jonathan Petersen

6.



$P(A \cup B \cup C)$
(///)

=



$P(A)+P(B)+P(C)-P(A\cap B)$
$-P(A\cap C)-P(B\cap C)$
$+P(B\cap A\cap C)$

~~(///): P(A) (|||): P(B) (\\\): P(C)~~

Let A, B, C be arbitrary events in S. It then follows that

$$\begin{aligned} P(A\cup B\cup C) &= P(A\cup B\cup C) \\ &= P(A\cup B)+P(C)-P(A\cup B\cap C) \\ &= P(A)+P(B)+P(C)-P(A\cap B)-P((A\cup B)\cap C) \end{aligned}$$

Assuming property D. ($P(A\cup B)=P(A)+P(B)-P(A\cap B)$).
Then, by the distributive laws,

$$P((A\cup B)\cap C) = P((A\cap C)\cup(B\cap C))$$

and further application of prop. D yields

$$\begin{aligned} P((A\cap C)\cup(B\cap C)) &= P\cancel{(C)+P(B)-P(C\cap B) \quad C=(A} \\ &= P(A\cap C)+P(B\cap C)-P((A\cap C)\cap(B\cap C)) \end{aligned}$$

leaving us overall with:

$$\begin{aligned} P(A\cup B\cup C) &= P(A)+P(B)+P(C)-P(A\cap B)-P((A\cup B)\cap C) \\ &= P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P((A\cap C)\cap(B\cap C)) \end{aligned}$$

By the associative law,

$$\begin{aligned} P((A\cap C)\cap(B\cap C)) &= P(A\cap C\cap B\cap C), \\ &= P(A\cap C\cap B) \end{aligned}$$

Therefore,

$$P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C) \quad \blacksquare$$

Jonathan Peterson

7. Given:

① $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for $A, B \in S$
② $P(S) = 1$
③ If $A \subseteq B$, $P(A) \leq P(B)$

Let A, B be events in S. It then follows from ① that:

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

and so long as $P(A \cup B) \leq 1$ then

$$P(A) + P(B) - P(A \cup B) \leq P(A) + P(B) - 1.$$

Since A, B are events in S, it is obvious that $A \subset S$ and $B \subset S$. Therefore:

$$\left.\begin{array}{l} P(A) \leq P(S) \\ P(B) \leq P(S) \end{array}\right\} \Rightarrow \begin{array}{l} P(A) \leq 1 \\ P(B) \leq 1 \end{array}$$

Furthermore, since $(A \cap B) \subset \{A, B\}$, it must be that

$P(A \cap B) \leq 1$, or

$P(A \cap B) \leq P(A) + P(B) - 1$ ∎

Jonathan Petersen

8. Proof by induction:

Base Case:

Let A, B be events in S. Assuming that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, ①
it then follows that

$$P(A \cup B) \leq P(A) + P(B)$$

since

$$P(A) + P(B) - P(A \cap B) \leq P(A) + P(B).$$

Inductive Step:

It then follows that if we let C be a collection of events such that

$$P(\bigcup_1^n C) \leq \sum_1^n (P(C))$$

and carry the same set of assumptions above,

$$P(\bigcup_1^n C) \cup D = P(C_1) + P(C_2) + \ldots + P(C_n) + P(D) - \text{All of the intersections of } C_1 \ldots C_n + D.$$

assuming D is another event in S. Since

$$P(C_1) + P(C_2) + \ldots + P(C_n) + P(D) - \text{intersections} \leq P(C_1) + P(C_2) + \ldots + P(C_n) + P(D),$$

it follows that including D in C will not violate the condition that

$$P(\bigcup_1^n C) \leq \sum_1^n (P(C)).$$

Therefore, if A_n is a collection of n events, we can see by inductive reasoning that if $A_0 = A$, $A_1 = B$, $A_n = C$, it follows that

$$P(\bigcup_{i=1}^n A_n) \leq \sum_{i=1}^n (P(A_i)) \blacksquare$$