# Homework 9
# Section 3.3

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3. **Show that** $f : R \to R^*$ **with** $R^* = \{(r, r) | (r, r) \in R \times R, r \in R\}$ **given by** $f(a) = (a, a)$**is an isomorphism.**

   To show that $f$ is an isomorphism, we must show that it is a bijective homomorphism, or that it meets the following four criteria:

   (a) $f(a + b) = f(a) + f(b)$ ($f$ preserves addition)

   (b) $f(a * b) = f(a) * f(b)$ ($f$ preserves multiplication)

   (c) $f(a) = f(b) \implies a = b$ ($f$ is injective)

   (d) For any $c$ in the range, there exists some $f(a) = c$. ($f$ is surjective)

   We shall check each criteria individually. First, to show that $f$ preserves addition we observe

$$\begin{aligned} f(a + b) &= (a + b, a + b) & a, b \in R \\ &= (a, a) + (b, b) \\ &= f(a) + f(b) \end{aligned}$$

   Therefore, $f$ preserves addition. By a similar fashion, let us check if $f$ preserves multiplication.

$$\begin{aligned} f(a * b) &= (a * b, a * b) & a, b \in R \\ &= (a, a) * (b, b) \\ &= f(a) * f(b) \end{aligned}$$

So $f$ preserves multiplicaton and is therefore a homomorphism. To check that $f$ is injective, consider

$$\begin{aligned} f(a) = c \qquad f(b) &= c \qquad\qquad a, b \in R \qquad c \in R^* \\ f(a) &= (a, a) \\ c &= (a, a) \\ f(b) &= (b, b) \\ c &= (b, b) \\ (a, a) &= (b, b) \\ a &= b \end{aligned}$$

Which shows that $f$ is injective. Finally, we check if $f$ is surjective. Let

$$\begin{aligned} (c, c) &\in R^* \\ c &\in R \\ f(c) &= (c, c) \end{aligned}$$

Thusly, $f$ meets the four criteria and we may conclude that $f$ is an isomorphism $\blacksquare$

8. **Let** $\mathbb{Q}(\sqrt{2}) = \{r + s\sqrt{2} | r, s \in \mathbb{Q}\}$**. Prove that** $f : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ **given by** $f(r + s\sqrt{2}) = r - s\sqrt{2}$ **is an isomorphism.**

   Following the structure in problem 3, we shall check that this problem also meets the same four criteria. First, to show that $f$ preserves addition we observe

$$\begin{aligned} f(a + b) &= f(a_1 + a_2\sqrt{2} + b_1 + b_2\sqrt{2}) \qquad a, b \in \mathbb{Q}(\sqrt{2}) \\ &= f((a_1 + b_1) + (a_2 + b_2)\sqrt{2}) \\ &= (a_1 + b_1) - (a_2 + b_2)\sqrt{2} \\ &= a_1 + b_1 - a_2\sqrt{2} - b_2\sqrt{2} \\ &= a_1 - a_2\sqrt{2} + b_1 - b_2\sqrt{2} \\ &= f(a) + f(b) \end{aligned}$$

   Therefore, $f$ preserves addition. By a similar fashion, let us check if $f$

preserves multiplication.

$$\begin{aligned}
f(a*b) &= f(a_1 + a_2\sqrt{2} * b_1 + b_2\sqrt{2}) \qquad a,b \in \mathbb{Q}(\sqrt{2})\\
&= f(a_1 * b_1) + (a_1 * b_2\sqrt{2}) + (a_2\sqrt{2} * b_1) + (a_2\sqrt{2} * b_2\sqrt{2})\\
&= f((a_1 * b_1 + 2 * a_2 * b_2) + (a_1 * b_2 + a_2 * b_1)\sqrt{2})\\
&= (a_1 * b_1 + 2 * a_2 * b_2) - (a_1 * b_2 + a_2 * b_1)\sqrt{2}\\
&= (a_1 * b_1) - (a_2\sqrt{2} * b_1) - (a_1 * b_2\sqrt{2}) + (a_2\sqrt{2} * b_2\sqrt{2})\\
&= (a_1 - a_2\sqrt{2}) * (b_1 - b_2\sqrt{2})\\
&= f(a) * f(b)
\end{aligned}$$

So $f$ preserves multiplicaton and is therefore a homomorphism. To check that $f$ is injective, consider

$$\begin{aligned}
f(a) = c \qquad f(b) &= c \qquad\qquad a,b,c \in \mathbb{Q}(\sqrt{2})\\
f(a) &= a_1 - a_2\sqrt{2}\\
c &= a_1 - a_2\sqrt{2}\\
f(b) &= b_1 - b_2\sqrt{2}\\
c &= b_1 - b_2\sqrt{2}\\
a_1 - a_2\sqrt{2} &= b_1 - b_2\sqrt{2}\\
a_1 &= b_1 \qquad a_2 = b_2
\end{aligned}$$

Which shows that $f$ is injective. Finally, we check if $f$ is surjective. Let

$$\begin{aligned}
c_1 + c_2\sqrt{2} &\in \mathbb{Q}(\sqrt{2})\\
c_1 + c_2\sqrt{2} + 0 - 2c_2\sqrt{2} &\in \mathbb{Q}(\sqrt{2})\\
c_1 - c_2\sqrt{2} &\in \mathbb{Q}(\sqrt{2})\\
f(c_1 - c_2\sqrt{2}) &= c_1 + c_2\sqrt{2}
\end{aligned}$$

Thusly, $f$ meets the four criteria and we may conclude that $f$ is an isomorphism ■

9. **If $f : \mathbb{Z} \to \mathbb{Z}$ is an isomorphism, prove that $f$ is the identity map.**

   Recall that $f(0) = 0$ for all rings. Furhtermore, since $f$ is an isomorphism, it preserves addition. Therefore

$$\begin{aligned}
f(0) &= 0 \\
f(1) = f(0+1) = f(0) + f(1) &= 0 + 1 = 1 \\
f(2) = f(1+1) = f(1) + f(1) &= 1 + 1 = 2 \\
f(3) = f(2+1) = f(2) + f(1) &= 2 + 1 = 3 \\
&\vdots \\
f(n) = f(n-1) + f(1) = f(n-1) + f(1) &= n - 1 + 1 = n
\end{aligned}$$

15. **Let $f : R \to S$ be a homomorphism of rings. if $r$ is a zero divisor in $R$, is $f(r)$ a zero divisor in $S$?**

    Yes. Observe

    $$\begin{aligned}
    r * x &= 0_R \\
    f(r) * f(x) &= f(0_R) \\
    &= 0_S
    \end{aligned}$$

    Therefore $f(r)$ is a zero divisor of $f(x)$ in $S$ ■

17. **Show that the complex conjugation function $f : \mathbb{C} \to \mathbb{C}$ given by $f(a + ib) = a - ib$ is a bijection.**

    Let $a + ib$ and $c + id$ be arbitrary elements in $\mathbb{C}$. Then if

    $$\begin{aligned}
    f(a + ib) = x - yi \qquad\qquad x - yi &\in \mathbb{C} \\
    f(c + id) &= x - yi \\
    x = a &= cy = b = d \\
    a &= cb = d
    \end{aligned}$$

    Therefore $f$ is injective. Now consider again some arbitrary $x - yi$ in $\mathbb{C}$.

    $$x - yi = f(x + yi)$$

    And since $x+yi$ must also be in $\mathbb{C}$, $f$ is surjective. Since $f$ is both injective and surjective, $f$ is bijective ■

19. **Show that $S = \{0, 4, 8, 12, 16, 20, 24\}$ is a subring of $\mathbb{Z}_{28}$. Then prove that the map $f : \mathbb{Z}_7 \to S$ given by $f([x]_7) = [8x]_{28}$ is an isomorphism.**

    To show that $S$ is a subring of $\mathbb{Z}_{28}$, we must show that it is closed under the subtraction and multiplication operators of $\mathbb{Z}_{28}$. Then, to show that $f$ is an isomorphism, we must prove the same four qualities shown in the first problem.

22. **Let $\bar{\mathbb{Z}}$ denote the ring of integers with $a \oplus b = a + b - 1$ and $a \odot b = ab - (a+b) + 2$ operations. Prove that $\bar{\mathbb{Z}}$ is isomorphic to $\mathbb{Z}$.**

To show that $\bar{\mathbb{Z}}$ is isomorphic to $\mathbb{Z}$, we again must show that it is homomorphic and bijective. Let us begin by checking preservation of addition:

$$\begin{aligned} f(a \oplus b) &= f(a + b - 1) \\ &= f(a) + f(b) + f(-1) \end{aligned}$$

Next, preservation of multiplication:

$$\begin{aligned} f(a \odot b) &= f(ab - (a+b) + 2) \\ &= f(ab) - f(a+b) + f(2) \\ &= f(a) * f(b) - f(a) - f(b) + f(2) \end{aligned}$$

Next, injectivity:

27.a. **If $g : R \to S$ and $f : S \to T$ are homomorphisms, show that $f \circ g : R \to T$ is a homomorphism.**

To show that $f \circ g : R \to T$ is a homomorphism, we must show that it preserves addition and multiplication from $R \to T$. That is, that

$$\begin{aligned} f(g(x+y)) &= f(g(x)) + f(g(y)) \qquad x, y \in R \\ f(g(x*y)) &= f(g(x)) * f(g(y)) \end{aligned}$$

Indeed, since $f$ and $g$ are homomorphisms,

$$\begin{aligned} f(g(x+y)) &= f(g(x) + g(y)) \\ &= f(g(x)) + f(g(y)) \end{aligned}$$

$$\begin{aligned} f(g(x*y)) &= f(g(x) * g(y)) \\ &= f(g(x)) * f(g(y)) \end{aligned}$$

So $f \circ g : R \to T$ is a homomorphism $\blacksquare$

27.b. **If $f$ and $g$ are isomorphisms, show that $f \circ g$ is also an isomorphism.**

As shown in the previous problem, if $f$ and $g$ are homomorphisms, then $f \circ g$ is also a homomorphism. Since an isomorphism must be a homomorphism, all that remains is to show that if $f$ and $g$ are isomorphims, that $f \circ g$ is bijective.

Therefore, let us consider an arbitrary $a, b$ in the domain of $g$, and some $c$ in the range of $f$. Then

$$\begin{aligned} & f(g(a)) = cf(g(b)) = c \\ f(g(a)) = c \qquad & f(g(b)) = c \implies g(a) = g(b) \\ g(a) = c \qquad & g(b) = c \implies a = b \end{aligned}$$

So $f \circ g$ is injective. To check surjectivity, take $c$ as an arbitrary element in the range of $f$. Then since $g$ and $f$ are isomorphic,

$$\begin{aligned} c \in R(f) &\implies f(b) \\ f(b) \in R(g) &\implies g(a) \\ \therefore f(g(a)) = c & \end{aligned}$$

And we see that $f \circ g$ is surjective and therefore an isomorphism ∎

35.a. **Show that $E$ and $\mathbb{Z}$ are not isomorphic.**

$\mathbb{Z}$ is a ring with identity, and $E$ is not. Therefore, the two cannot be isomorphic.

35.b. **Show that $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $M(\mathbb{R})$ are not isomorphic.**

$\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is a commutative ring, and $M(\mathbb{R})$ is not, so they cannot be isomorphic.

35.c. **Show that $\mathbb{Z}_4 \times \mathbb{Z}_{14}$ and $\mathbb{Z}_{16}$ are not isomorphic.**

There are $4 * 14 = 56$ elements in $\mathbb{Z}_4 \times \mathbb{Z}_{14}$, but only 16 in $\mathbb{Z}_{16}$. Therefore the two cannot be isomorphic.