Homework 10
Section 4.1

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3.b. **List all polynomials with degree less than 3 in $\mathbb{Z}_3[x]$.**

The following is a list of all polynomials of the form

$$a_2x^2 + a_1x + a_0 \qquad a_i \in \mathbb{Z}_3$$

1. $[0]x^2 + [0]x + [0] = [0]$
2. $[0]x^2 + [0]x + [1] = [1]$
3. $[0]x^2 + [0]x + [2] = [2]$
4. $[0]x^2 + [1]x + [0] = [1]x$
5. $[0]x^2 + [1]x + [1] = [1]x + [1]$
6. $[0]x^2 + [1]x + [2] = [1]x + [2]$
7. $[0]x^2 + [2]x + [0] = [2]x$
8. $[0]x^2 + [2]x + [1] = [2]x + [1]$
9. $[0]x^2 + [2]x + [2] = [2]x + [2]$
10. $[1]x^2 + [0]x + [0] = [1]x^2$
11. $[1]x^2 + [0]x + [1] = [1]x^2 + [1]$

12. $[1]x^2 + [0]x + [2] = [1]x^2 + [2]$
13. $[1]x^2 + [1]x + [0] = [1]x^2 + [1]x$
14. $[1]x^2 + [1]x + [1]$
15. $[1]x^2 + [1]x + [2]$
16. $[1]x^2 + [2]x + [0] = [1]x^2 + [2]x$
17. $[1]x^2 + [2]x + [1]$
18. $[1]x^2 + [2]x + [2]$
19. $[2]x^2 + [0]x + [0] = [2]x^2$
20. $[2]x^2 + [0]x + [1] = [2]x^2 + [1]$
21. $[2]x^2 + [0]x + [2] = [2]x^2 + [2]$
22. $[2]x^2 + [1]x + [0] = [2]x^2 + [1]x$
23. $[2]x^2 + [1]x + [1]$
24. $[2]x^2 + [1]x + [2]$
25. $[2]x^2 + [2]x + [0] = [2]x^2 + [2]x$
26. $[2]x^2 + [2]x + [1]$
27. $[2]x^2 + [2]x + [2]$

5.a. **Find polynomials** $q(x)$ **and** $r(x)$ **such that** $3x^4 - 2x^3 + 6x^2 - x + 2 = (x^2 + x + 1)q(x) + r(x)$**, and** $r(x) = 0$ **or deg** $r(x) <$ **deg** $g(x)$ **in** $\mathbb{Q}[x]$**.**

$$
\begin{array}{r|l}
 & \quad\quad\quad\quad 3x^2 - 5x + 8 \\
\hline
x^2 + x + 1 & 3x^4 - 2x^3 + 6x^2 - x + 2 \\
 & -3x^4 - 3x^3 - 3x^2 \\
\cline{2-2}
 & \quad -5x^3 + 3x^2 - x \\
 & \quad 5x^3 + 5x^2 + 5x \\
\cline{2-2}
 & \quad\quad 8x^2 + 4x + 2 \\
 & \quad\quad -8x^2 - 8x - 8 \\
\cline{2-2}
 & \quad\quad\quad -4x - 6
\end{array}
$$

Therefore

$$
\begin{aligned}
q(x) &= 3x^2 - 5x + 8 \\
r(x) &= -4x - 6
\end{aligned}
$$

5.c. **Find polynomials** $q(x)$ **and** $r(x)$ **such that** $2x^4 + x^2 - x + 1 = (2x - 1)q(x) + r(x)$**, and** $r(x) = 0$ **or deg** $r(x) <$ **deg** $g(x)$ **in** $\mathbb{Z}_5[x]$**.**

6.c. **Are all polynomials of degree $\leq k$, where k is a fixed positive integer, a subring of $R[x]$? Justify your answer.**

No, since polynomials of degree $\leq k$ are not closed under multiplication. Consider $(x+1)$ and $(x-1)$ as elements of a subset of $\mathbb{R}[x]$ with $k=1$. Then

$$(x+1)(x-1) = x^2 - 1$$

which is of degree $k = 2$, and hence, polynomials of degree $\leq k$ are not a subring of $\mathbb{R}[x]$.

6.d. **Are all polynomials in which the odd powers of x have zero coefficients a subring of $R[x]$? Justify your answer.**

Yes. Since subtraction of polynomials is done by coefficient, it is impossible to create a nonzero coefficient in an odd power of x, since $0-0$ is always 0. Therefore, the subring is closed under subtraction. In a like manner, since multiplicaiton of two even numbers will always result in an even number, it is impossible to multiply any two even powers of x and obtain an odd power of x. Since our polynomials are multiplied term-by-term, and all of the terms have even powers of x, there is no way for the multiplicaiton of even powered polynomials to result in a polynomial with any odd powers of x. Therefore, polynomials in which the odd powers of x have zero coefficients are a subring of $R[x]$.

6.e. **Are all polynomials in which the even powers of x have zero coefficients a subring of $R[x]$? Justify your answer.**

Yes. By symmetry of the above, since subtraction of polynomials is done by coefficient, it is impossible to create a nonzero coefficient in an even power of x with only odd inputs, and the subring is closed under subtraction. In a like manner, it is impossible to multiply any two odd powers of x and obtain an even power of x, and so there is no way for the multiplicaiton of odd powered polynomials to result in a polynomial with any even powers of x. Therefore, polynomials in which the even powers of x have zero coefficients are a subring of $R[x]$.

11. **Show that $1 + 3x$ is a unit in $\mathbb{Z}_9$.**

To show that $1 + 3x$ is a unit in $\mathbb{Z}_9$ we must find some value k such that

$$(1+3x)k = 1$$

in $\mathbb{Z}_9$. Consider $(1-3x)$:

$$\begin{aligned}(1+3x)(1-3x) &= 1 - 3x + 3x - 9x^2 \\ &= 1 + 0 - 0x^2 \\ &= 1\end{aligned}$$

Therefore, $k = 1 - 3x$ and $1 + 3x$ is a unit in $\mathbb{Z}_9$.