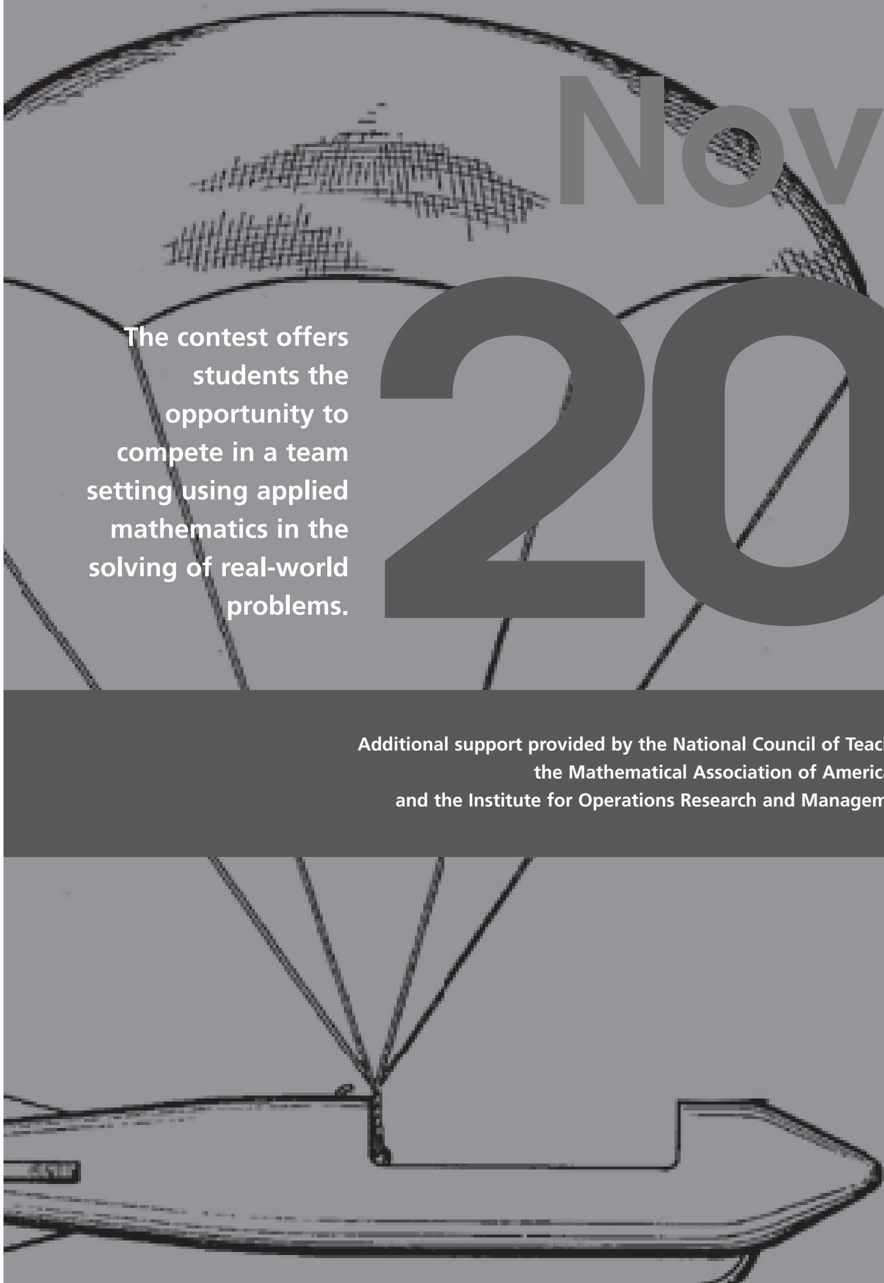
HIGH SCHOOL MATHEMATICAL CONTEST IN MODELING
OUTSTANDING PAPERS

HiMCM

November 2006



The contest offers students the opportunity to compete in a team setting using applied mathematics in the solving of real-world problems.

Additional support provided by the National Council of Teachers of Mathematics (NCTM), the Mathematical Association of America (MAA), and the Institute for Operations Research and Management Sciences (INFORMS).

Editor's Comments

This is our ninth HiMCM Special Issue. Since space does not permit printing all of the five National Outstanding papers, this special section includes the summaries from three papers and abridged versions of two. We emphasize that the selection of these two does not imply that they are superior to the other Outstanding papers. We also wish to emphasize that the papers were not written with publication in mind. Given the 36 hours that the teams had to work on the problems and prepare their papers, it is remarkable how much was accomplished and how well written many of the papers are. ❑

Contest Director's Article

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The High School Mathematical Contest in Modeling (HiMCM) completed its ninth year in excellent fashion. The mathematical and modeling ability of students and faculty advisors is very evident in the professional submissions and work being done. The contest is still moving ahead, growing with a positive first derivative, and consistent with our positive experiences from previous HiMCM contests.

This year the contest consisted of 276 teams (an increase of 23 teams over last year) from over 50 institutions. These institutions were from twenty-two states (increase of 4) and from the Hong Kong International School, China, and Costa Rica. This year, we again charged a registration fee of $50.

The teams accomplished the vision of our founders by providing *unique* and *creative mathematical* solutions to complex open-ended real-world problems. This year the students had a choice of two problems.

Problem A

A parachute is made from thin, lightweight fabric, support tapes and suspension lines. The lines are usually gathered through cloth loops or metal connector links at the ends of several strong straps called risers. The risers in turn are attached to the harness containing the load.

DEPLOYMENT SYSTEMS

Freefall deployed parachutes are pulled out of their containers by a smaller parachute called a pilot chute.

A way of deploying a parachute directly after leaving the aircraft is the static line. One end of the static line is attached to the aircraft, and the other to the deployment system of the parachute container.

ROUND PARACHUTES

Round parachutes, which are pure drag devices (i.e., they provide no lift like the ram-air types), are used in military, emergency and cargo applications. These have large dome-shaped canopies made from a single layer of cloth. Some skydivers call them "jellyfish 'chutes" because they look like dome-shaped jellyfish. Rounds are rarely used by skydivers these days. The first round parachutes were simple, flat circulars, but suffered from instability, so most modern round parachutes are some sort of conical or parabolic.

Some round parachutes are steerable, but not to the extent of the ram-air parachutes. An example of a steerable round is provided in the picture of the paratrooper's canopy; it is not ripped or torn but has a "T-U cut". This kind of cut allows air to escape from the back of the canopy, providing the parachute with limited forward speed. This gives the jumpers the ability to steer the parachute and to face into the wind to slow down the horizontal speed for the landing.

ANNULAR & PULL DOWN APEX PARACHUTES

A variation on the round parachute is the pull down apex parachute invented by a Frenchman named LeMoigne—referred to as a Para-Commander-type canopy in some circles, after the first model of the type. It is a round parachute, but with suspension lines to the canopy apex that apply load there and pull the apex closer to the load, distorting the round shape into a somewhat flattened or lenticular shape.

Often these designs have the fabric removed from the apex to open a hole through which air can exit, giving the canopy an annular geometry. They also have decreased horizontal drag due to their flatter shape, and when combined with rear-facing vents, can have considerable forward speed around 10 mph (15 km/h).

RIBBON AND RING PARACHUTES

Ribbon and ring parachutes have similarities to annular designs and they can be designed to open at speeds as high as Mach 2 (two times the speed of sound). These have a ring-shaped canopy, often with a large hole in the center to release the pressure. Sometimes the ring is broken into ribbons connected by ropes to leak air even more. The large leaks lower the stress on the parachute so it does not burst when it opens.

Often a high speed parachute slows a load down and then pulls out a lower speed parachute. The mechanism to sequence the parachutes is called a "delayed release" or "pressure detent release" depending on whether it releases based on time, or the reduction in pressure as the load slows down.

RAM-AIR PARACHUTES

Most modern parachutes are self-inflating "ram-air" airfoils known as a Para foil that provide control of speed and direction similar to Para gliders. Para gliders have much greater lift and range, but parachutes are designed to handle, spread and mitigate the stresses of deployment at terminal velocity. All ram-air Para foils have two layers of fabric; top and bottom, connected by airfoil-shaped fabric ribs. The space between the two fabric layers fills with high pressure air from vents that face forward on the leading edge of the airfoil. The fabric is shaped and the parachute lines trimmed under load such that the ballooning fabric inflates into an airfoil shape.

RESERVES

Paratroopers and sports parachutists carry two parachutes. The primary parachute is called a main parachute, the second, a reserve parachute. The jumper uses the reserve if the main parachute fails to operate correctly.

Reserve parachutes were introduced in World War II by the US Army paratroopers, and are now almost universal. For human jumpers, only emergency bail-out rigs have a single parachute and these tend to be of round design on older designs while modern PEPs (i.e., P124A/Aviator) contain large, docile ram-air parachutes.

DEPLOYMENT

Reserve parachutes usually have a ripcord deployment system, but most modern main parachutes used by sports parachutists use a form of hand deployed pilot chute. A ripcord system pulls a closing pin (sometimes multiple pins) which releases a spring-loaded pilot chute and opens the container, the pilot chute is propelled into the air stream by its spring then uses the force generated by passing air to extract a deployment bag containing the parachute canopy, to which it is attached via a bridle. A hand deployed pilot chute, once thrown into the air stream, pulls a closing pin on the pilot chute bridle to open the container, then the same force extracts the deployment bag. There are variations on hand deployed pilot chutes but the system described is the more common throw-out system. Only the hand deployed pilot chute may be collapsed automatically after deployment by a kill line reducing the in-flight drag of the pilot chute on the main canopy. Reserves on the other hand do not retain their pilot chutes after deployment. The reserve deployment bag and pilot chute are not connected to the canopy in a reserve system, this is known as a free bag configuration and the components are often lost during a reserve deployment. Occasionally a pilot chute does not generate enough force to either pull the pin or extract the bag, causes may be that the pilot chute is caught in the turbulent wake of the jumper (the “burble”), the closing loop holding the pin is too tight, or the pilot chute is generating insufficient force, this effect is known as “pilot chute hesitation” and if it does not clear it can lead to a total malfunction requiring reserve deployment.

Paratroopers’ main parachutes are usually deployed by static lines which release the parachute yet retain the deployment bag which contains the parachute without relying on a pilot chute for deployment. In this configuration the deployment bag is known as a direct bag system and the deployment is rapid, consistent and reliable. This kind of deployment is also used by student skydivers going through a static line progression, a kind of student program.

Using the modeling process, build a mathematical model for the opening of the parachutes discussed above. We are concerned with how the parachute inflates. Use your model to explain how the geometry of the folding of the parachute affects the inflation and then discuss how we might affect the rate of inflation of the parachute.

Problem B

A south sea island chain has decided to transform one of their islands into a resort. This roughly circular island, about 5 kilometers across, contains a mountain that covers the entire island. The mountain is approximately conical, is about 1000 meters high at the center, appears to be sandy, and has little vegetation on it. It has been proposed to lease some fire-fighting ships and wash the mountain into the harbor. It is desired to accomplish this as quickly as possible.

Build a mathematical model for washing away the mountain. Use your model to respond to the questions below.

- How should the stream of water be directed at the mountain, as a function of time?
- How long will it take using a single fire-fighting ship?
- Could the use of 2 (or 3, 4, etc.) fire-fighting ships decrease the time by more than a factor of 2 (or 3, 4, etc.)?
- Make a recommendation to the resort committee about what do.

Commendation: All students and advisors are congratulated for their varied and creative mathematical efforts. Of the 276 teams, 28 submitted solutions to the A problem and 248 to the B problem. The thirty-six continuous hours to work on the problem provided for quality papers; teams are commended for the overall quality of their work.

Many teams had female members. There were 378 female participants on the 276 teams. There were 1048 total participants, so the females made up approximately 36% of the total participation, showing this competition is for both genders. Teams again proved to the judges that they had “fun” with their chosen problems, demonstrating research initiative and creativity in their solutions. This year’s effort was a success!

Judging: We ran three regional sites in December 2006. Each site judged papers for problem B. The west region also judged all of problem A since there were only 28 papers. The papers judged at each regional site may or may not have been from their respective region. Papers were judged as Outstanding, Meritorious, Honorable Mention, and Successful Participant. All finalist papers for the Regional Outstanding award were sent to the National Judging. For example, eight papers may be discussed at a Regional Final and only four selected as Regional Outstanding but all eight papers are judged for the National Outstanding. Papers receive the higher of the two awards. The national judging chooses the “*best of the best*“ as National Outstanding. The National Judges commended the Regional Judges for their efforts and found the results were very consistent. We feel that this regional structure provides a good structure for the future as the contest continues to grow.

Judging Results:

NATIONAL & REGIONAL COMBINED RESULTS

Problem	**National Outstanding***	Regional Outstanding Only	Meritorious	Honorable Mention	Successful Participant	Total
A	**1**	4	7	9	7	**28**
B	**4**	20	49	134	41	**248**
Total	**5**	24	56	143	48	**276**

General Judging Comments: The judges’ commentaries provide specific comments on the solutions to each problem. As contest director and head judge, I would like to speak generally about solutions from a judge’s point of view. Papers need to be coherent, concise, and clear. Students need to restate the problem in their own words so that the judges can determine the focus of the paper. Papers that explain the development of the model, assumptions, and its solutions and then support the findings mathematically generally do quite well. Modeling assumptions need to be listed and justified, but only those that come to bear on the solution (that can be part of simplifying the model). Laundry lists of assumptions that are never referred to in the context of the model development are not considered relevant and deter from the paper's quality. The mathematical model needs to be clearly developed, and all variables that are used need to be well defined. Thinking outside of the "box" is also considered important by judges. This varies from problem to problem but usually includes model extensions or sensitivity analysis of the solution to the team's inputs. Students need to attempt to validate their model even if by numerical example or intuition. A clear conclusion and answers to specific scenario questions are all key components. The strengths and weakness section is where the team can reflect on their solution and comment on the model's strengths and weaknesses. Attention to detail and proofreading the paper prior to final submission are also important since the judges look for clarity and style.

CONTEST FACTS:

Facts from the 9th Annual Contest:

- Wide range of schools/teams competed including forty teams from Hong Kong (19), China (16), and Costa Rica (5).

- The 276 teams represented 51 institutions.

- There were 1048 student participants, 670 (64%) male and 378 (36%) female.

- Schools from twenty-two states participated in this year's contest.

THE FUTURE:

The contest, which attempts to give the under-represented an opportunity to compete and achieve success in mathematics, appears well on its way in meeting this important mission.

We continue to strive to improve the contest and we want the contest to grow. Again, any school/team can enter, as there will be no restrictions on the number of schools or the numbers of teams from a school. A regional judging structure will be established based on the number of teams.

These are exciting times for our high school students. Mathematics continues to be more than learning skills and operations. Mathematics is a language that involves our daily lives. Applying the mathematical principles that one learns is the key to future success. The ability to recognize problems, formulate a mathematical model, use technology, and communicate and reflect on one's work are keys to success. Students learn confidence by tackling ill-defined problems and working together to generate a solution. Applying mathematics is a team sport!

Advisors need only be motivators and facilitators. They should encourage students to be creative and imaginative. It is not the technique used but the process that discovers how assumptions drive the techniques that is fundamental. Let students practice to be problem solvers. Let me encourage all high school mathematics faculty to get involved, encourage your students, make mathematics relevant, and open the doors to success.

Mathematical modeling is an art and a science. Teach your students through modeling to think critically, communicate effectively, and be confident, competent problem solvers for this new century.

CONTEST DATES:

Mark your calendars early: the next HiMCM will be held in November 2007. Registrations are due in October 2007. Teams will have a consecutive 36-hour block within the contest window to complete the problem. Teams can register via the worldwide Web at www.comap.com.

HiMCM Judges' Commentary

Problem A

Only 28 (10%) out of 276 teams attempted Problem A. Those that tried this problem fell into three categories: experimental modelers (teams that cut out miniature parachutes and experimentally generated data), geometric modelers using either folds or parachute shapes, and modelers using only physics equations to build a model. The latter were not successful in answering the questions about the effect of the shape on the inflation rate. The experimental modelers created unique approaches and easily would have had Outstanding papers if they had been better written and better explained. Thus, the Regional and National Outstanding were those who examined the shapes and measured the effect of the shape on the inflation process.

Although the judges expected folds to be the prominent consideration, the problem wording was vague enough to allow the parachute's shape to be used in lieu of the folding technique.

Teams appeared to search the Internet and cut and paste many equations and figures into their papers. In most cases, it was clear that students did not understand the mathematics presented.

Overall the student papers in this problem were better than those in the other problem. The modeling contest format was also more closely adhered to in the papers. However, most summaries and letters were poorly written. It seemed that they could have been written before the model, analysis, and conclusions were completed. The summaries need to contain the results of the model.

One of the items that discriminated the better papers was careful modeling of the impact of geometric features of the parachute on the inflation rates.

Problem B

A majority (almost 90%) of teams chose Problem B. It was concise and students obviously thought it was the easier of the two problems.

The letters of recommendations to the resort committee for the most part were either absent or were poorly written. Some were excellent, and we could tell the students enjoyed writing them. They needed to be in English and quickly get to the point. The letters were supposed to explain how the teams planned to wash the mountain away so the resort could be built in a timely manner. Many letters read like technical reports or were too vague to be helpful.

Teams examined one ship and clearly stated the impossibility of one ship to do the task. Most teams failed to continue to find the number of ships required to make the time line feasible. Teams that felt that renting more ships was too cost prohibitive failed to allow the resort committee to make that judgment. Teams were to present the facts.

We found many of the assumptions poor in that they were too unrealistic. We usually gave the teams the benefit of doubt with their assumptions.

COMMENT ABOUT COMPUTER GENERATED SOLUTIONS:

Many papers used computer code. Computer codes used to implement mathematical expressions can be a good modeling tool. However, the judges expect to see an algorithm or flow chart from which the code was developed. Successful teams provided some explanation or guide to their algorithm(s)—a step-by-step procedure for the judges to follow. The code may only be read for the papers that reach the final rounds. The results of any simulation need to be well explained and sensitivity analysis preformed. For example, consider a flip of a fair coin. Here is the algorithm:

INPUT: Random number, number of trials

OUTPUT: Heads or tails

Step 1: Initialize all counters.

Step 2: Generate a random number between 0 and 1.

Step 3: Choose an interval for heads, like [0.0.5]. If the random number falls in this interval, the flip is a head. Otherwise the flip is a tail.

Step 4: Record the result as a head or a tail.

Step 5: Count the number of trials and increment: Count = Count + 1.

An algorithm such as this is expected in the body of the paper with the code as an appendix.

COMMENTS ABOUT GRAPHS:

Judges found many graphs that were not labeled nor explained. Many graphs did not appear to convey information used by the teams. All graphs need a verbal explanation of what the team expects the reader (judge) to gain (or see) from the graph. Legends, labels, and points of interest need to be clearly visible and understandable, even if hand written. Graphs taken from articles should be referenced and annotated.

General Comments from Judges:

Summaries: These are still, for the most part, the weakest parts of papers. These should be written after the solution is found. They should contain results and not details. They should include the "bottom-line" and the key ideas used in obtaining the solution. They should include the particular questions addressed and their answers. Teams should consider a brief three paragraph approach: a *restatement of the problem* in their own words, a short description of *their method and solution* to the problem (without giving any mathematical expressions), and the *conclusions* providing the numerical answers in context.

Restatement of the Problem: Problem restatements are important for teams to move from the general case to the specific case. They allow teams to refine their thinking to give their model uniqueness and a creative touch.

Assumptions/Justifications: Teams should list only those assumptions that are vital to the building and simplifying of their mathematical model. Assumptions should not be a reiteration of facts given in the problem description. Assumptions are variables (issues) acting or not acting on the problem. Every assumption should have a justification. We do not want to see "smoke screens" in the hopes that some items listed are what judges are looking to see. Variables chosen need to be listed with notation and be well defined.

Model: Teams need to show a ***clear link*** between the assumptions they listed and the building of their model or models. It is required of the team to show how the model was built and why it is the model chosen.

Model Testing: Model testing is not the same as testing arithmetic. Teams need to compare results or attempt to verify (even with common sense) their results.

Teams that use a computer simulation must provide a clear step-by-step algorithm. Lots of runs and related analysis are required when using a simulation. Sensitivity analysis should be done in order to see how sensitive the simulation is to the model's key parameters.

Conclusions: This section deals with more than just results. Conclusions might also include speculations, extensions, and generalizations. This is where all scenario specific questions should be answered. Teams should ask themselves what other questions would be interesting if they had more time and then tell the judges about their ideas.

Strengths and Weaknesses: Teams should be open and honest here. What could the team have done better?

References: Teams may use references to assist in their modeling. However, they must also ***reference the source*** of their assistance. Teams are reminded that only ***inanimate resources*** may be used. Teams cannot call upon real estate agents, bankers, hotel managers, or any other real person to obtain information related to the problem. References cited where used and not just listed in the back of the paper.

Adherence to Rules: Teams are reminded that detailed rules and regulations are posted on the COMAP site. Teams are reminded that they may use only ***inanimate sources*** to obtain information. Teams are reminded that the ***36-hour time limit is a consecutive 36 hours.***

MathModels.org

It is highly recommended that participants in this contest as well as future participants take a look at the new modeling Website, www.mathmodels.org, which has a wealth of information and resources.

Problem B Summary: New York Interschool

Advisor: Eric Zahler

Team Members: Janelle Duah, Lee Evangelakos, Sylvie Polsky, Cathy Spriggs

Our goal was to create a model of the most effective way to use water to erode a sandy mountain on a beautiful island in the South Sea. This mountain is roughly conical, with a radius of 2500 m and a height of 1000 m. We made various assumptions about the reaction of sand to the force of water and then set to work to determine where we should spray the water to minimize the time it takes to flatten the entire cone.

Once we set our conditions we tried two separate strategies to demolish the cone. We began with the simplest method, always spraying water at the top of the cone and eroding it layer by layer. This only takes into account the erosion directly caused by water. Our second solution is much more innovative; it levels the mountain in roughly half the time the simpler solution takes. In our second solution we looked further into the natural process of erosion by taking landslides into account. Land erodes when the earth below it has been removed. Any erosion of the side of the cone would cause some of the sand above it to fall too. Having already assigned a value to the direct erosion of sand by water for our first model, we assigned a reasonable variable to the amount of erosion due to the effect of a landslide. We then set up a model that maximizes erosion by ensuring that it is constantly occurring both directly and indirectly: sand lower on the mountain falls when it is hit by water and sand higher on the mountain slides away when the sand beneath it falls.

We concluded that the most effective way to demolish the mountain is to attack portions of the cone at a time, thereby obliterating it in mathematically equal sections. For each section, we sprayed the cone always at the midpoint of its height, ensuring both the landslide and the simple erosion effects at all times. If one fireboat were to do this continually, our model would take only 3.459 years. This may seem like a lot, but spraying the cone all at once from the top takes 5.203 years. Predictably, if more fireboats are used they will take less time to flatten the mountain. This was especially evident in our second model because the additional fireboats dramatically increase the rate of erosion and cut down on the time the boats need to travel around the island. We suggest twelve, or even more boats. It takes twelve fireboats only a little over 30 weeks to completely erode the mountain (compare this value to the 3.459 years it would take one boat). In short, we suggest that the owners of the island employ as many fireboats as possible to erode the island in sections, aiming their streams at the middle of the conical sections.

Problem B Summary: Illinois Mathematics and Science Academy

Advisor: Steven Condie

Team Members: Hon Lung Chu, Dennis Kriventsov, Francisco Saldana, Caleb Wang

Our task is to create a mathematical model that would simulate fire-fighting ships washing away a sand island mountain. We divide the problem into three categories and approach each individually. First, we devise the most efficient method of firing the water as a function of time. We use the Los Angeles Fire Department's Warner Lawrence, as our base model. We assumed that the mountain is perfectly conical, comprised of mostly sand, and that the water cannon radius and chemistry of the water are constant.

We explain that simply aiming the cannon constantly tangent to the circular mountain base as the ship moves around the base is extremely inefficient. Each revolution of the ship removes one ring of sand from the outer edge of the mountain. Thus, we try to aim farther into the mountain, forming a chord where our water stream intersects the base. This allows the ship to remove a larger ring of sand per revolution. Our model optimizes this technique by increasing the aiming angle (Φ) to the maximum allowed by the force provided by the water. This technique is our base model.

The second part of our model shows that using multiple ships working on the same section of the mountain produce is more efficient than individual ships working on separate parts. By working on the same part simultaneously, a second ship following the first closely can have a greater Φ, thereby removing a greater amount of sand per revolution. Φ is dependent on the kinetic coefficient of friction and the density of the sand, both of which are lowered for the second ship by the first. This is because once the original sand is blasted away, the sand that falls from above to replace it is looser—less dense, and easier to blast. Using this knowledge, we create a second model that illustrates multiple boats working together in this fashion and examine different ways of arranging these boats. We show that pairs of boats that used the aforementioned behavior are more efficient then larger boat chains using that behavior (e.g., having the second boat follow the first, the third follow the second, and so on); our model shows that the most efficient situation occurs when having squadrons of boat-pairs work on separate sections rather than having one long chained line of boats work on the same section.

The fourth section of our model takes a pragmatist's/tinker's perspective. During this step, we consider adding multiple pumps on each ship, model multiple ships with multiple pumps, and compute the cost efficiency of each ship. We build upon our first model to increase the pumping power of the ship. Within our base model, we define that the ship's pumping power comes from one pump. Our new model was able to vary the number of pumps and the number of ships to give us an accurate estimate of the time needed to accomplish the task using the aiming motion in the base model. We tested various combinations of pumps and ships, along with considerations in the price and logistics, and arrived at our conclusion.

To validate our system, we created a stability test to show that our model will withstand variations in each of the constants assumed, such as pump radius and sand density. Under this system, we successfully harness the energy of the water and direct it towards leveling the mountain. We believe that this model answers all three questions posed by the problem and ultimately provides a logical and realistic proposal to the resort committee.

Problem B Summary: Hempstead High School

Advisor: Karen Weires

Team Members: Claire Funke, Chase Liaboe, Sarah Longfield, Timothy Williams

For the purpose of discovering the best method of leveling a sand mountain with fire-fighting ships, several variables and constants had to be identified. Factors involving distances, angles of elevation, and equipment were defined as variables or constants depending on their input to the problem. Also, assumptions concerning other outside factors, including the number of water dispensing devices on each ship, firing distance, and cost of leasing ships, were made for the purpose of simplifying data collection. The authors justified their assumptions by noting that without these assumptions, no experiment could be performed and no accurate results could be determined.

Data were collected through the use of a scaled experiment. The variables were tested via the constants; previously described assumptions were taken into account. The analysis of the experiment correlated directly to the conclusion concerning the practical application. The experiment was one of the model's greatest strengths and provided the basic information for the model.

In order to make evaluations and computations more effective and reliable, several computer programs were written. With these programs, the authors were able to compute several important values to a greater degree of accuracy, which were necessary in analyzing the data.

After converting the experimental data to a practical application, the data were manipulated to find the optimum solution. By analyzing both time and cost benefits, the most efficient number of ships to lease was discovered. Numerous data tables and graphs were used to present the authors' conclusions.

Problem B Paper: Evanston Township High School

Advisor: Peter DeCraene

Team Members: Jacob Belser, Nell Elliott, Benjamin Simon, Jack Wadden

1 INTRODUCTION

Before designing a model, we made calculations in order to have a rough estimate to test our results. In doing so, we concluded that the mountain has too large a volume to be easily washed away by fire-fighting ships or any other method.

To provide a frame of reference: the world's largest earth-mover is 311 feet tall and 705 feet long, weighs over 45,500 tons, requires 5 operators, took 5 years and $1 million to design and build, and is capable of moving 76,465 m^3 of material per day.[1] In order to move the sand mountain, this machine would need 85,017 days—234 years. The world's largest dump truck has a capacity of 290,000 kg per load.[2] The volume of the sand mountain is equal to 43,034,492 of these loads. If an amount of sand equal to one load of this truck were removed from the mountain every minute, it would take 82 years to remove it all.

The world's most powerful fire-fighting ship, the *Warner Lawrence* of the Los Angeles Fire Department, can pump 140,000 liters of water per minute, and the most powerful of its eight jets has a reach of almost 200 meters. Using this ship, it takes approximately 137 years to pump enough water to completely saturate the sand mountain. (Based on our experimentally determined saturation values, this is about 3.71 km^3 of water.) However, in order to wash away the cone requires a jet stream that can reach halfway across the island, a distance of 2,500 meters—thirteen times farther than current technologies allow.

The above calculations provided a time frame for the model and showed that it is unrealistic to try to remove the mountain by any method. However, we determined that it is possible to remove enough of the sand cone with fire-fighting ships to create a flat rim on which to build the resort. Our result is a function for the time necessary to clear the sand based on the amount of water pumped per day, saturation point of the island's sand, the desired width of the rim, and the sand's angle of repose.

2 METHODS

2.1 ASSUMPTIONS (See Section 4 for more information)

- The island's base is a circular rock with radius 2.5 km.
- The sand forms a cone with uniform density, radius 2.5 km, and height 1 km.
- The amount of water needed to wash away a particular volume of sand is constant for the entire mountain, and the rate at which water can be pumped is therefore directly proportional to the amount of sand that is cleared.
- The sand's repose angle is constant throughout the mountain.
- The time needed to wash away the desired rim is a function of:
  - The rate at which water is pumped, v (km^3/day)
  - The amount of water needed to saturate an amount of sand, k (L water/L sand)
  - The width of desired rim, w (km)
  - The sand's angle of repose, θ (radians)

See **Figure 2.1**.

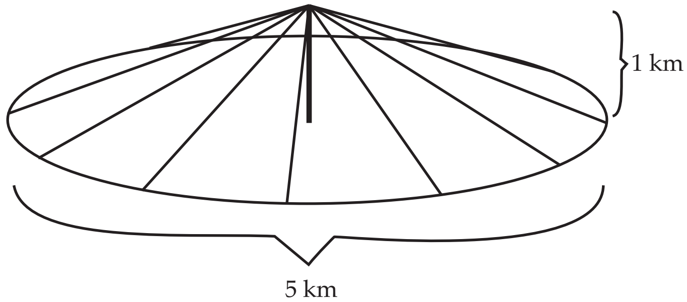

Figure 2.1: Diagram of island and conical sand mountain.

2.2 REPOSE ANGLE

Sand's maximum angle of stability with the horizontal is known as the angle of repose.[4] When sand is disrupted and any of the resulting angles exceed the repose angle, the sand will collapse until the angles of the new formation are all less than or equal to the repose angle.

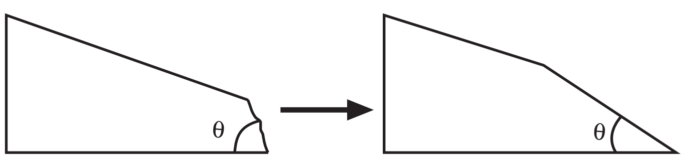

Figure 2.2: Diagram of the avalanching and repose angle of sand.

As seen in Figure 2.2, removing a section of the sand formation creates an angle (θ) with the horizontal that is too large to keep the structure stable. The sand will avalanche until the angle with the horizontal is equal to the repose angle. This phenomenon is important to our model; it allows us to predict the formation of the island's sand once the rim is cleared. The sand's original conical shape becomes a tent-like formation, as shown in Figure 2.3.

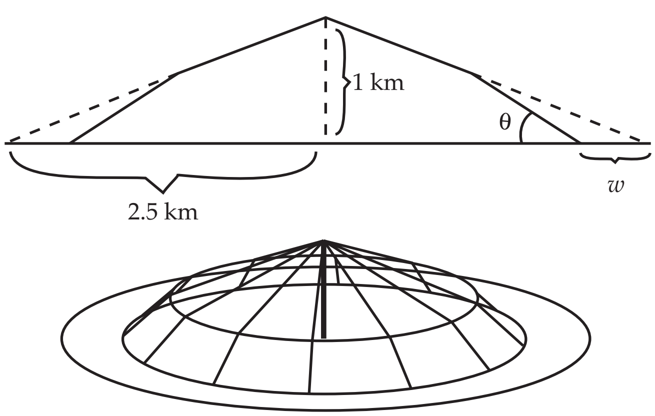

Figure 2.3: Diagram of resultant sand formation after avalanching.

2.3 CALCULATION OF VOLUME OF SAND REMOVED

In order to calculate the time needed to clear the rim, it is necessary to first calculate the amount of sand to be removed. We found an equation for the height of the original cone in terms of distance from the island's center: $h = 1 - \frac{2r}{5}$. We then found an equation for the height of the collapsed formation in terms of the repose angle and the rim width: $h = \left(\frac{5}{2} - r - w\right)\tan(\theta)$.

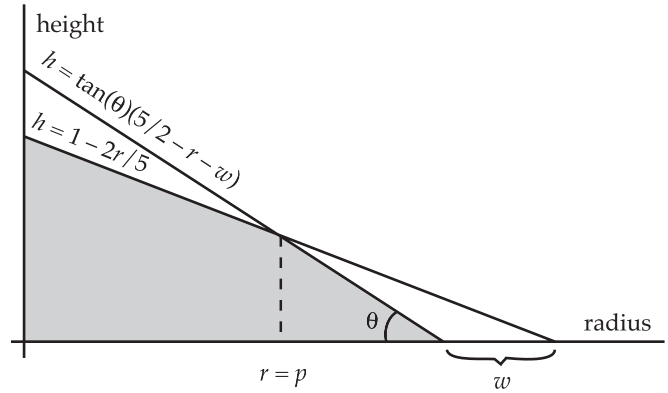

Figure 2.4:
Graph of heights versus radii of original and resulting mountains.

We then used calculus to find the volume of the shaded region in Figure 2.4—the new formation of the mountain. This produced the series of integrals:

$$\text{volume} = \int_0^{2\pi}\int_0^{p}\int_0^{1-\frac{5r}{2}} r\,dh\,dr\,d\alpha + \int_0^{2\pi}\int_p^{\frac{5}{2}-w}\int_0^{\left(\frac{5}{2}-r-w\right)\tan(\theta)} r\,dh\,dr\,d\alpha \qquad (1)$$

where p is the x-coordinate of the intersection of the two lines in Figure 2.4.

Solving for this intersection gives:

$$p = \frac{5\left(2 - 5\tan(\theta) + 2w\tan(\theta)\right)}{2\left(2 - 5\tan(\theta)\right)} \qquad (2)$$

The amount of sand removed is equal to the original volume minus the evaluation of Equation 1. Simplification produced:

Sand Removed =

$$\frac{25\pi}{12} - \frac{\pi\left(100 - 4\tan(\theta)(125 - 30w^2 + rw^3) + 5\tan^2(\theta)(5-2w)^2(5+4w)\right)}{12\left(2 - 5\tan(\theta)\right)^2} \qquad (3)$$

On further analysis we concluded that if the repose angle is smaller than a specific value, or the width larger than a specific value, the resulting formation is not tent-shaped. Rather, it forms a new cone, as shown in Figure 2.5.

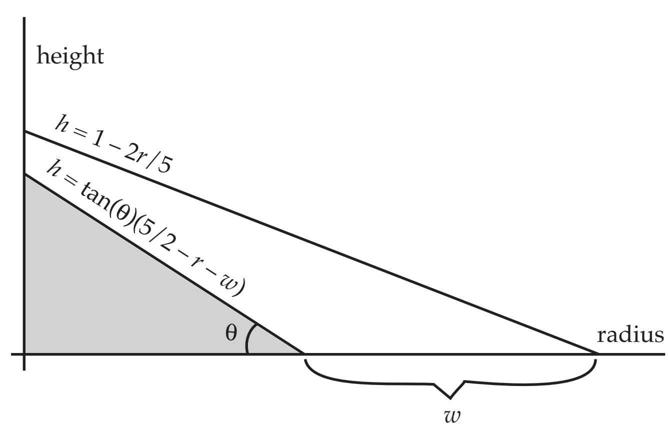

Figure 2.5:
Graph of heights versus radii of original and resulting mountains.

For this setup, the volume of the resulting cone is:

$$\int_0^{2\pi}\int_0^{\frac{5}{2}-w}\int_0^{\left(\frac{5}{2}-r-w\right)\tan(\theta)} r\,dh\,dr\,d\alpha \tag{4}$$

which results in the equation:

$$\text{Sand Removed} = \frac{25\pi}{12} - \frac{\pi}{3}\tan\theta\left(\frac{5}{2}-w\right)^3 \tag{5}$$

Equation 5 is correct when $p < 0$, which simplifies to when

$$w > \frac{5}{2} - \cot(\theta) \tag{6}$$

2.4 CALCULATION OF TIME NEEDED TO REMOVE SAND

Equations 3 and 5 give the volume of sand removed for a given rim width and repose angle. To calculate the amount of time needed to remove the sand, our final two factors must be accounted for. First, the amount of water needed to remove the given amount of sand must be calculated. This is accomplished by multiplying the appropriate equation (3 or 5) by the constant k (the volume of water needed per volume of sand). Finally, the amount of water is converted to time by dividing by v (the rate at which water is pumped, in km^3 per day).

Therefore, the equations for time (in days) are:

$$\text{Time} = \frac{k}{v}\left(\frac{25\pi}{12}-\frac{\pi\left(100-4\tan(\theta)(125-30w^2+rw^3)+5\tan^2(\theta)(5-2w)^2(5+4w)\right)}{12\left(2-5\tan(\theta)\right)^2}\right) \tag{7}$$

when $w < \frac{5}{2} - \cot(\theta)$

$$\text{Time} = \frac{k}{v}\left(\frac{25\pi}{12}-\frac{\pi}{3}\tan\theta\left(\frac{5}{2}-w\right)^3\right) \tag{8}$$

when $w > \frac{5}{2} - \cot(\theta)$.

3 RESULTS

As shown by Equations 7 and 8, time and k are linearly related; and time and v are inversely related. However, the relationships between time and θ, and time and w, are more complex. Figure 3.1 is a graph of time versus repose angle for $w = 0.2$ km, $k = 0.57$, and $v = 0.002$.

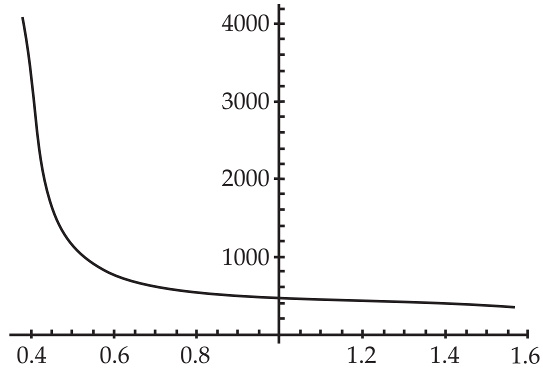

Figure 3.1: Graph of time (days) versus angle of repose (radians).

As Figure 3.1 shows, time decreases as repose angle increases because less sand will avalanche and be removed from the island. Figure 3.2 shows a graph of time versus rim width.

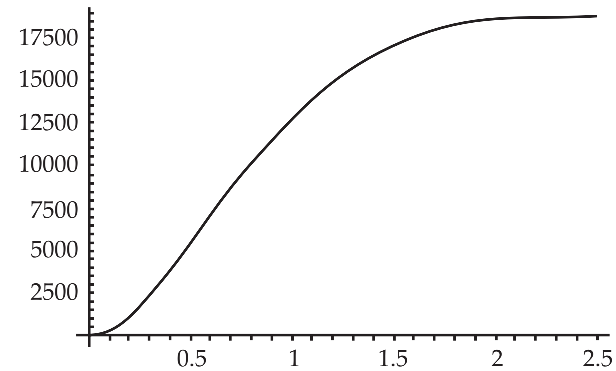

Figure 3.2: Graph of time (days) versus rim width (km).

As expected, the larger the rim, the more time it takes. Note that the inflection point in the graph occurs when $w = \frac{5}{2} - 2\cot(\theta)$— the point at which we switch from Equation 7 to Equation 8. (In Figure 3.2, $\theta = \frac{\pi}{6}$, $k = 0.57$, and $v = 0.002$; so the inflection point is at $w \approx 0.77$.)

According to our model, the stream of water should be directed at the sand cone so that the stream lands at the outer edge of the remaining formation (the point $(2.5 - w, 0)$ on our graphs). The boat will be moving continuously around the island, but in our model the speed of the boat is irrelevant. This is because we assume that all pumped water is used to saturate sand, and that k is constant. Therefore, only the amount of water affects time.

To estimate the time it takes one ship to clear the rim, we inputted typical values for each variable ($\theta = \frac{\pi}{6}$, $v = 0.0002$, $w = 0.15$, and $k = 0.57$) into Equation 7. With these values, the time needed is about 600 days.

Adding more ships only affects v. Because v and time are inversely related, changing the number of ships decreases time by the same factor as the number of ships.

Our recommendation to the resort committee is as follows:

- Determine the values of k and θ for the sand.
- Decide the desired value of w (the smallest possible w is most efficient).
- After deciding how many ships to use, use Equation 7 (or 8, if appropriate) to calculate the time needed to remove the sand.
- If the time is unsatisfactory, adjust w and/or v.

4 GENERALIZATIONS AND ASSUMPTIONS

4.1 ASSUMPTION 1

We assumed that the sand cone sits on a rock bed that is at or slightly above ocean level. This ensures that we do not wash away the entire island. In reality, we do not know the level at which the island's base changes from sand to rock, and thus cannot account for the composition of the island base.

4.2 ASSUMPTION 2

We also assumed that a solution of sand and water flows off the island as if it is an ideal liquid. In reality, saturated sand does not behave like a true liquid; rather, it behaves as a colloid, which is a suspension of one state of matter in another. When water is first 0added to sand, it is suspended between the sand grains, technically making a gel. At the saturation point, there is so much water that the grains become partially suspended in the water, creating a sol. Because grains of sand are much larger than water molecules, they tend to fall out of the colloid. Thus, the suspension is not uniformly dense and will not "carry" the grains of sand all the way to the ocean. Saturated sand has a repose angle of 12°, and liquids have repose angles near 0°.[6] Therefore, the water-sand mixture will settle on the island surface in the shape of a cone with a 12° base angle.

4.3 ASSUMPTION 3

We assumed that the mountain is composed of perfectly dry, uniformly dense sand. We also assumed that the "little vegetation" on the island does not affect our model. In reality, the presence of dune grasses and other vegetation with extensive root structures will change the repose angle.

4.4 ASSUMPTION 4

We assumed that currents disperse the sand-water mixture when it reaches the ocean. In reality, the sand will be deposited on any natural reefs that surround the island. This poses two problems. While the size of the island would increase dramatically, the coral reefs and ecosystems surrounding the island would be destroyed. This would be both a natural and economic disaster, as the island could lose a valuable tourist attraction. If this is unavoidable, further research can be conducted to evaluate the cost/benefit ratio of reef destruction.

4.5 ASSUMPTION 5

Our final and most significant assumption is that all of the pumped water removes sand in the constant ratio k. In reality, some water would flow off the rock, be dispersed into the air, or be absorbed by the mountain. This assumption allowed us to derive our time function and ignore the procedure by which water was directed at the mountain. However, it also causes us to underestimate time.

5 ALTERNATIVE METHODS

We considered and rejected three other methods. Of the two that use fire-fighting ships, our current model is the more practical. It is possible that using the third method in combination with ours will increase efficiency. However, the third method does not use fire-fighting ships, cannot be practically modeled, and cannot be used exclusively.

5.1 SATURATION FROM ABOVE

We could have used a system of pipes to saturate the mountain from above. We can assume that this saturation will cause the resulting sand-water solution to flow down the mountain and into the sea.

This idea is not feasible for two reasons. First, a huge volume of water is needed. Raising this water to a height of 1 km requires a significant amount of energy; although gravity will assist once the water reaches the top. The ship pumps are not strong enough to make this method work in a reasonable time frame.

Second, this method is unrealistic because of the properties of saturated sand. Even if it were possible to saturate the mountain in a viable amount of time, the sand would not immediately wash away. An excess of water is necessary to carry the sand away; in addition, the amount of time needed for sand to travel down the mountain and into the sea must be taken into account.

5.2 KINETIC ENERGY MODEL

We also rejected the idea of using the kinetic energy of water pumped from the ships to blast the sand away. This idea is flawed because the stream of water will not collide perfectly elastically with the sand. Instead, the small grains of sand will be enveloped by the stream; the properties of dry and wet sand differ enough to have a significant effect. This method also requires an unreasonable amount of water.

5.3 VIBRATIONS MODEL

Based on research by Evesque et al.[7], we propose that collapsing the mountain can be assisted by vibrations with low frequencies and high amplitudes. These vibrations will destabilize the mountain's base and decrease the repose angle, thus increasing the amount of sand that can be removed. These vibrations can be induced by various methods, including small explosions or loud speakers. The cost of producing such vibrations is insignificant compared with the cost of leasing a large fire-fighting ship.

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SECTION I: INTRODUCTION

Four types of parachutes are modeled in this paper: round, annular, ribbon-ring, and ram-air. The round design is the standard dome parachute that acts as a pure drag device. It ranges in size from about 25 to 35 feet (Zodiac). Lines are attached around the perimeter of the canopy.

The annular design differs from the round in two ways. First, it has lines attached to the apex of the canopy that pull on the apex, causing the canopy to have a flatter shape. Second, it has a section removed from the apex, allowing air to exit. Thus, pressure inside the chute is decreased, and the design is not as effective at slowing objects.

The ribbon-ring design is also similar to the round, but goes further than the annular to reduce pressure. Sections of the canopy are cut away, creating the appearance of a series of ribbons. This design may be deployed at high speeds without fear of bursting under the pressure.

The ram-air design is composed of two layers of fabric that are cut in a rectangular pattern when viewed from above. The two layers are ribbed together at various places, forming a series of cells containing air pockets. The orientation of the inflated cells creates an airfoil shape that adds lift. Once inflated, the canopy curves due to the tug of the lines, forming a parabolic shape when viewed from the front.

In order to understand the differences among these designs and the inflation mechanics, one must be familiar with the parts of the parachute and how they interact upon deployment. The actual fabric of the parachute that produces the drag is called the canopy. Coming down from the perimeter of the canopy are a series of ropes called lines that vary in length from 20 to 30 feet (Brain). The lines connect to straps called risers that are attached to the skydiver.

The canopy is folded and packed into a small deployment bag (about 18 inches long) called the d-bag (Galloway, 2006). The d-bag is packed into the diver's backpack, or container, along with a reserve chute. The last part of the system is a smaller chute, known as a pilot or drogue chute. It is 12 to 18 inches in diameter, and serves to pull the main canopy out of the container. It is connected to the canopy by a cord of nylon webbing about 20 feet long called the bridle.

The physical principle that regulates inflation is drag. For a falling parachute, drag is essentially the force of air resistance. Drag is proportional to the density of air, the object's cross-sectional surface area, and the square of the velocity at which the object is traveling. After deployment, parachute surface area increases, so drag force is greater than that of a person falling alone. The velocity of the diver decreases until drag and gravitational forces are equal, and then the diver travels at a constant velocity called the terminal velocity.

SECTION 2: RESTATEMENT OF PROBLEM

We divided the parachute deployment process into three phases: pilot chute deployment, d-bag deployment, and main chute release from the d-bag.

Pilot chute deployment consists first of a sub-phase of bridle elongation, during which no unfolding of the pilot chute occurs. After the bridle tautens, the pilot inflates to initiate a new drag force. After this, the d-bag is released from the container, and elongation of the lines begins. Once the lines are fully elongated, a pin on the d-bag is released. Next the main chute unfurls from the d-bag, and then it inflates, concluding the deployment process.

We are asked to model the final phase of main chute inflation. We must address the ways in which shape affects inflation rate for different types of chutes. That is, we want to model the rate of airflow from when the parachute is first deployed to when it is fully inflated.

SECTION 3: ASSUMPTIONS

We assume that the round chute's structure is that of a paraboloid, to which we may assign a function in three dimensions. We assume that the annular chute differs from the round only in height (not radius). We assume that the ribbon-ring design takes a shape similar to that of the round when fully inflated and that it has a surface area that is two-thirds that of the round. Finally, we consider the ram-air chute a parabolic cylinder. Although the chute has two surfaces that interact with air, we only consider interaction with one parabolic surface. We consider the time for unfolding of the short side to be negligible.

We consider the skydiver-parachute system an isolated system—no other objects interact with it. We assume that all airflow is upwards into the chute. We do not consider horizontal drag since this force has little effect on inflation, though it may affect other phases of deployment. We also consider air density constant (see Section 4 for density calculations). Air density is a function of altitude, which changes as a function of velocity. According to Galloway (2006), parachute inflation takes between two and five seconds after deployment, and skydivers are instructed to deploy chutes at an altitude of 1829 m. Data show that in the distance traveled by a skydiver beyond 1829 m at typical speeds, there are negligible changes in air density (Peterson, Strickland, 1996).

Our model is largely based on flux. In working with flux, we assume that air is constantly moving through the chute, not caught in it. Finally, we use an average value for each of certain values, such as mass or dimensions.

SECTION 4: MODEL ALGORITHM

In this section, we consider an integral for flux as a function of the force field acting on the parachute. We use Stewart (2003) for these equations. The force field is expressed as $\vec{F} = p \cdot \vec{v}$.

We consider p, the air density to be constant, while $\vec{v}$ is the velocity field. Assuming the velocity is constantly upwards, we can express velocity as a function of time and calculate it for different times. Since we assume the force and velocity to be vertical, $\vec{F} = R\hat{k}$, where $R = pv$. Flux, Φ, can be expressed as

$$\Phi = \iint_D R\,dA = \iint_D pv\,dA .$$

That is, flux is equal to the double integral of the constant air density multiplied by the velocity at which the air is moving through the chute over the two-dimensional projection of the chute onto a plane. As we proceed, we will keep in mind our search for v. Calculation of p comes later in this section.

SECTION 4-1: ROUND PARACHUTE

(all values denoted by subscript "*rd*")

The surface area (cross-sectional area) of the parachute, A_{rd}, is given by

$$A_{rd} = \frac{4\pi}{3}\left[\left(1+\frac{4h_{rd}}{a_{rd}}\right)^{\frac{3}{2}} - 1\right],$$

where h_{rd} is the maximum height of the top of the chute from the bottom, and a_{rd} is the expansion factor. (Derivations of surface area formulas are in Appendix III.)

Solving for a_{rd} we have: $\frac{r_{rd}^2}{a_{rd}} = h_{rd}$ where r_{rd} is the radius of the round parachute. With values obtained from Zodiac Aerosafety Systems,

$$\frac{5.35^2}{a_{rd}} = 5.35$$

$$a_{rd} = 5.35$$

We now substitute into the equation for surface area.

$$A_{rd} = \frac{4\pi}{3}\left[\left(1+\frac{4(5.35)}{5.35}\right)^{\frac{3}{2}} - 1\right]$$

$$A_{rd} = 42.64\text{m}^2$$

We then solve for equivalent velocity, which is given by (Meade, 1998):

$$v(t) = \frac{m_h g}{k_{rd}}\left(e^{\frac{-k_{rd}t}{m_h}} - 1\right)$$

where m_h is the mass of the skydiver, g is acceleration due to gravity, and k_{rd} is the air resistance constant given by

$$k_{rd} = \frac{1}{2}(C_p + C_h)p(A_{rd} + A_h)$$

where C_p is the drag coefficient for any chute moving through air, C_h is the drag coefficient of a skydiver (values obtained from Drag Coefficient, 2006), p is the density of dry air, and A_h is the cross-sectional area of the diver.

We have this equation for p (Density of Air, 2006): $P = \frac{PM}{RT}$ where P is atmospheric (air) pressure, M is molar mass of dry air, R is the ideal gas constant (in L·atm/mol·K), and T is air temperature. Pressure and temperature are both functions of the altitude, h, and the chute opens at 1829 m (Galloway, 2006).

$$P = P_0\left(1+\frac{Lh}{T_0}\right)^{-\frac{gM}{RL}}$$

$$T = T_0 + Lh$$

where P_0 is the atmospheric pressure at sea level, L is the temperature lapse rate, R is the ideal gas constant in joules/mol·K, and T_0 is the standard temperature at sea level.

$$P = 101.32\left(1+\frac{(-0.0065)(1829)}{288.15}\right)^{\frac{(9.8)(0.02896)}{(8.314)(0.0065)}}$$

$$P = 81.211kPa$$

$$T = 288.15 + (-0.0065)(1829)$$

$$T = 276.26K$$

Values for pressure and temperature are substituted into the equation for p:

$$p = \frac{(81.211)(0.02896)}{(0.0821)(276.26)} = 0.103$$

This value is assumed to be the same for all parachutes.

The cross-sectional area of a person, A_h, is given by:

$$Ah = h_h \cdot w_h$$

where h_h is the person's average height, and w_h is the person's average width.

$$A_h = (1.778)(0.4572) = 0.8129 \text{ m}^2$$

Solve the equation for k_{rd}:

$$k_{rd} = \frac{1}{2}(C_p + C_h)p(A_{rd} + A_h)$$

$$k_{rd} = \frac{1}{2}(0.75 + 1.2)(0.103)(42.64 + 0.8129) = 4.363$$

Now, all the values to find the function of velocity in terms of time are known. So they are substituted into the svelocity function.

$$v(t) = \frac{(77.01)(9.8)}{4.363}\left(e^{\frac{-4.363t}{77.01}} - 1\right) = 172.98(e^{-0.057t} - 1)$$

SECTION 4-2: ANNULAR PARACHUTE

(all values denoted by subscript "*an*")

The surface area (cross-sectional area) of the parachute, A_{an}, is given by

$$A_{an} = \left(\frac{4\pi}{3}\left(\left(1+\frac{4h_{an}}{a_{an}}\right)^{\frac{3}{2}} - 1\right)\right) - \left(\pi\left(\frac{r_{an}}{6}\right)^2\right)$$

where h_{an} is the maximum height of the top of the parachute from the bottom r_{an} is the radius of the parachute, and a_{an} is the expansion factor.

As in Section 4.1, we solve for aan using values obtained from Zodiac Aerosafety Systems:

$$a_{an} = \frac{(5.35)^2}{3.21} = 8.92$$

Substituting into the equation above, we have

$$A_{an} = \left(\frac{4\pi}{3}\left(\left(1+\frac{4(3.21)}{8.92}\right)^{\frac{3}{2}} - 1\right)\right) - \left(\pi\left(\frac{5.35}{6}\right)^2\right) = 9.273\text{m}^2$$

Assuming p is the same as in Section 4-1, we find k_{an}:

$$k_{an} = \frac{1}{2}(0.75 + 1.2)(0.103)(9.273 + 0.8129) = 1.013$$

We find the function for velocity in terms of time as in Section 4.1:

$$v(t) = \frac{(77.01)(9.8)}{1.013}\left(e^{\frac{-1.013t}{77.01}} - 1\right) = 745.01(e^{-0.0132t} - 1)$$

SECTION 4-3: RIBBON-RING PARACHUTE

The surface area (cross-sectional area) of the parachute, A_{rn}, is given by

$$A_{rn} = \left(-\sum_{n=1}^{7}\frac{32\pi}{3a_{rn}^2}\left(\frac{a_{rn}^2}{12}\left(1+\frac{4\left(\frac{2n\sqrt{a_{rn}h_{rn}}}{15}\right)^2}{a_{rn}^2}\right)^{\frac{3}{2}} - \frac{a_{rn}^2}{12}\left(1+\frac{4\left(\frac{2n\sqrt{a_{rn}h_{rn}}-1}{15}\right)^2}{a_{rn}^2}\right)^{\frac{3}{2}}\right)\right) + \left(\frac{4\pi}{3}\left(\left(1+\frac{4h_{rn}}{a_{rn}}\right)^{\frac{3}{2}} - 1\right)\right)$$

where h_{rn} is the maximum height of the top of the parachute from the bottom, r_{rn} is the radius of the parachute, and a_{rn} is the expansion factor.

As in Sections 4.1 and 4.2, we find a_{rn} and substitute into the above equation:

$$a_{rn} = \frac{(5.35)^2}{5.35} = 5.35$$

$$A_{rn} = \left(-\sum_{n=1}^{7}\frac{32\pi}{3(5.35)^2}\left(\frac{(5.35)^2}{12}\left(1+\frac{4\left(\frac{2n\sqrt{(5.35)(5.35)}}{15}\right)^2}{5.35^2}\right)^{\frac{3}{2}} - \frac{5.35^2}{12}\left(1+\frac{4\left(\frac{2n\sqrt{(5.35)(5.35)}-1}{15}\right)^2}{5.35^2}\right)^{\frac{3}{2}}\right)\right) + \left(\frac{4\pi}{3}\left(\left(1+\frac{4(5.35)}{5.35}\right)^{\frac{3}{2}} - 1\right)\right) = 29.68\text{m}^2$$

Assuming p is the same as in Section 4-1, we find k_{rn}:

$$k_{rn} = \frac{1}{2}(0.75 + 1.2)(0.103)(29.677 + 0.8129) = 3.061$$

We find the function for velocity in terms of time as in Section 4.1:

$$v(t) = \frac{(77.01)(9.8)}{3.061}\left(e^{\frac{-3.061t}{77.01}} - 1\right) = 246.55(e^{-0.0397t} - 1)$$

SECTION 4-4: RAM-AIR PARACHUTE

(all values denoted by subscript "rm")

The surface area (cross-sectional area) of the parachute, A_{rm}, is given by

$$A_{rm} = y_{rm}\left|\frac{1}{a_{rm}}\right|\left[a_{rm}^2 \cdot \ln\left(\frac{\sqrt{4x_{rm}^2 + a_{rm}^2} + 2x_{rm}}{|a_{rm}|}\right) + 2x_{rm}\sqrt{4x_{rm}^2 + a_{rm}^2}\right]$$

where y_{rm} is one-half the length of the chute's short side, x_{rm} is one-half the length of the chute's long side, and a_{rm} is the expansion factor.

To find a_{rm}, we have

$$\frac{x_{rm}^2}{a_{rm}} = h_{rm} \text{ or } a_{rm} = \frac{x_{rm}^2}{h_{rm}}$$

where h_{rm} is the maximum height of the arc made by the parachute.

From the Zodiac Aerosafety Systems, we have these values for a typical ram-air chute: $y_{rm} = 1.22$m, $x_{rm} = 6.67$m, and $h_{rm} = 1.91$m.

Thus, $a_{rm} = \frac{6.67^2}{1.91} = 23.29$.

Substituting these values into the equation above, we have

$$a_{rm} = 1.22\left|\frac{1}{223.29}\right|\left[(23.29)^2 \cdot \ln\left(\frac{\sqrt{4(6.67)^2 + 23.29^2} + 2(6.67)}{|23.29|}\right) + 2(6.67)\sqrt{4(6.67)^2 + 23.29^2}\right] = 17.13\text{m}^2$$

Assuming p is the same as in Section 4-1, we find k_{rm}:

$$k_{rm} = \frac{1}{2}(0.75 + 1.2)(0.103)(17.13 + 0.8129) = 1.802$$

We find the function for velocity in terms of time as in Section 4.1:

$$v(t) = \frac{(77.01)(9.8)}{1.802}\left(e^{\frac{-1.802t}{77.01}} - 1\right) = 418.8(e^{-0.023t} - 1)$$

SECTION 5: CONCLUSIONS

The most prominent weakness of our model is that it does not take into account effects from horizontal or downward winds. To improve the model, we could add horizontal components to the velocity field vector.

There are several weaknesses in our interpretation of the ram-air chute. We assume that the short side has already unfolded before the long side begins to inflate. To improve on this, we might try to produce an equation for changing short side on top of the one for changing long side. The fact that the ram-air design has two surfaces is also not addressed.

For some of the other chutes, we fail to create a universal model. For example, we assume that one-third of the ribbon-ring chute's surface area has been removed, which is not always the case. To improve on this, we might try to create a generalized equation with a certain amount of area to be removed, which would only require a few changes in our formulas.

Furthermore, the resulting numerical values (for flux, time, etc.) in our model are not constant in all situations. We used average values, though our model does allow for these numbers. Thus, our model provides specific values, while also remaining versatile for input of different constants.

Despite flaws, our model has its strengths. We incorporate a fairly accurate model, and many of our assumptions are valid. Except for the ring-ribbon, all models yield inflation times between the expected 2 and 5 seconds.

EDITOR'S NOTE:

For brevity, we have omitted Appendix I (Images of Parachute Designs) and Appendix II (Variables and Physical Constants). We are including a portion of Appendix II (Derivation of Surface Area Formulas) and Appendix IV (Sensitivity Testing).

APPENDIX III: DERIVATION OF SURFACE AREA FORMULAS

The surface areas of the four chutes were derived from the surface area equation for $z = f(x, y)$:

$$A = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

The surface areas were derived by hand, converting to polar as needed. The ram-air parachute surface area was an exception and was found using a TI-89. In the following derivations, the depth of the parachute, h, is treated as a constant because dimensions were not available. Also, keeping the expansion variable, a, a constant was necessary for flux calculations.

ROUND PARACHUTE

The round parachute is approximated by the paraboloid equation

$$z = h_{rd} - \frac{r_{rd}^2}{a_{rd}} = h_{rd} - \frac{x_{rd}^2}{a_{rd}} - \frac{y_{rd}^2}{a_{rd}}$$

The limit of integration with respect to r is derived by letting $z = 0$.

$$0 = h_{rd} - \frac{r_{rd}^2}{a_{rd}}$$

$$h_{rd} = \frac{r_{rd}^2}{a_{rd}}$$

$$r_{rd} = \sqrt{a_{rd}h_{rd}}$$

$$A_{rd} = \iint_D \sqrt{1+\frac{4x_{rd}^2}{a_{rd}^2}+\frac{4y_{rd}^2}{a_{rd}^2}}\,dx\,dy = \iint_D r_{rd}\sqrt{1+\frac{4r_{rd}^2}{a_{rd}^2}}\,d\theta\,dr$$

RIBBON-RING PARACHUTE

The surface of this chute recalls the geometry of the round chute. Based on photos, we use 15 concentric rings. To account for the empty spaces along alternating rings, we estimate two-thirds of every other ring to be empty space. Therefore, we derive the surface area from 0 to $\frac{4\pi}{3}$ for alternating rings. Although the empty spaces are evenly spread along each ring, the symmetry of the paraboloid allows us to treat them as a unit without affecting flux calculations.

For the surface area derivation, we summed the empty spaces and subtracted the sum from the surface area of the round chute. The derivation yielded:

$$A_{rn} = \left(-\sum_{n=1}^{7}\frac{32\pi}{3a_{rn}^2}\left(\frac{a_{rn}^2}{12}\left(1+\frac{4\left(\frac{2n\sqrt{a_{rn}h_{rn}}}{15}\right)^2}{a_{rn}^2}\right)^{\frac{3}{2}} - \frac{a_{rn}^2}{12}\left(1+\frac{4\left(\frac{2n\sqrt{a_{rn}h_{rn}}-1}{15}\right)^2}{a_{rn}^2}\right)^{\frac{3}{2}}\right)\right)+\left(\frac{4\pi}{3}\left(\left(1+\frac{4h_{rn}}{a_{rn}}\right)^{\frac{3}{2}}-1\right)\right)$$

The equation reflects a similar step in the round chute equation:

$$\left[\left(\frac{16\pi}{a_{rd}^2}\right)\left(\frac{2}{3}\left(1+\frac{4r_{rd}^2}{a_{rd}^2}\right)^{\frac{3}{2}}\frac{a_{rd}^2}{8}\right)\right]_0^{\sqrt{ah}}$$

However, $\left(\frac{16\pi}{a_{rd}^2}\right)$ is here replaced by $\frac{32\pi}{3a_{rn}^2}$ because 2π, the limit of integration for θ, is replaced by $\frac{4\pi}{3}$. Also, the summation finds the empty area for rings $\frac{2n-1}{15}$ to $2n$, $n = 1$ to $n = 7$.

APPENDIX IV: MODEL SENSITIVITY TESTING

The vertical force, $F(t)$, is the product of the vertical velocity and the constant density. The flux, Φ, is the integration of $F(t)$ over the projection of the parachute onto the xy-plane. The change in volume is the product of flux and the time interval, 0.1, for all four chutes. The time begins when the chute begins inflating. For the round and ram-air models, we allow $a = 0.1$ to represent an initial, thin column of air. This value also determines the initial volume. For the annular chute, $a = 1$ represents the beginning of inflation because the central hole does not provide drag. For the ring-ribbon parachute, $a = 2$ reflects the larger gap of empty space in the middle. The multiple holes of the ring-ribbon chute are treated as one.

Time increases until the maximum a value is passed. The new volume is related to the next a value through these equations (with constants substituted):

$$a_{rd} = \frac{2V}{\pi(5.35)^2},\ a_{an} = \frac{2\left(V + 3.21\pi\left(\frac{5.35}{6}\right)^2\right)}{\pi(3.21)^2},\ a_{rn} = \frac{2\left(V + 5.35\pi\left(\frac{5.35}{3}\right)^2\right)}{\pi(5.35)^2},$$

$$a_{rm} = \frac{3V}{4.88(1.91)^{\frac{3}{2}}}$$

RAM AIR

Density	0.1037
Vertical Velocity	$v(t) = 418.8(e^{-0.023t} - 1)$
Vertical Force	$F(t) = -(0.1037)(418.8)(e^{-0.023t} - 1)$
Δt	0.1
Initial Volume	2.71568012

t	a	F(t)	Φ	ΔV	New Volume
0.1	0.1	0.099773205	0.21279	0.021279	2.736959086
0.2	0.10157326	0.199317195	0.428421	0.042842	2.779801207
0.3	0.10477804	0.298632497	0.651942	0.065194	2.844995368
0.4	0.10975035	0.397719636	0.888621	0.088862	2.933857451
0.5	0.11671342	0.496579137	1.144156	0.114416	3.048273084
0.6	0.12599419	0.595211523	1.424896	0.14249	3.19076266
0.7	0.13804853	0.693617314	1 .73809	0.173809	3.364571683
0.8	0.15349787	0.791797032	2.092192	0.209219	3.573790888
0.9	0.17318132	0.889751197	2.497213	0.249721	3.823512201
1	0.19822925	0.987480325	2.965165	0.296516	4.120028672
1.1	0.2301671	1 .084984935	3.510604	0.35106	4.471089068
1.2	0.27106248	1.182265542	4.15132	0.415132	4.8862211
1.3	0.32373451	1.279322661	4.909204	0.49092	5.377141523
1.4	0.39205383	1.376156805	5.811353	0.581135	5.958276855
1.5	0.48137566	1.472768486	6.891489	0.689149	6.647425733
1.6	0.59916959	1.569158216	8.191777	0.819178	7.466603433
1.7	0.75594282	1.665326504	9.765183	0.976518	8.443121699
1.8	0.96660463	1.76127386	11.67852	1.167852	9.610973757
1.9	1.25249948	1.85700079	14.01643	1.401643	11.01261663
2	1.64446189	1.952507801	16.88656	1.688656	12.70127264
2.1	2.18744554	2.047795398	20.42639	2.042639	14.74391198
2.2	2.94759785	2.142864086	24.8122	2.48122	17.22513183
2.3	4.02316534	2.237714366	30.27088	3.027088	20.25222018
2.4	5.56145021	2.332346742	37.09571	3.709571	23.96179084
2.5	7.78540695	2.426761714	45.66719	4.566719	28.52850995
2.6	11.035727	2.520959781	56.48107	5.648107	34.17661672
2.7	15.8380208	2.614941441	70.18575	7.018575	41.19519143
2.8	23.0110175	2.708707192	87.63276	8.763276	49.95846719
2.9	33.8423866	2.802257529	109.9449	10.99449	60.952956

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