HIGH SCHOOL MATHEMATICAL CONTEST IN MODELING

OUTSTANDING PAPERS

HiMCM

November 2002

The contest offers students the opportunity to compete in a team setting using applied mathematics in the solving of real-world problems.

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Editor's Comments

This is our fifth HiMCM Special Issue. Since space does not permit printing all of the ten National Outstanding papers, this special section includes the summaries from eight of the papers and edited versions of two. We emphasize that the selection of these two does not imply that they are superior to the other Outstanding papers. They were chosen because they are representative and fairly short. They have received light editing, primarily for brevity. We also wish to emphasize that the papers were not written with publication in mind; the contest does not allow time to revise and polish. Given the 36-hour time limit, it is remarkable how well written many of the papers are.

We appreciate the outstanding work of students and advisors and the efforts of our contest directors and judges. Their dedication and commitment have made HiMCM a big success. We also wish to note that this special section takes the place of our regular HiMCM column, which is edited by contest director Bill Fox. HiMCM Notes will return in the next issue.

Contest Director's Article

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The High School Mathematical Contest in Modeling (HiMCM) completed its fifth year in excellent fashion. The growth of students, faculty advisors, and the contest judges is very evident in the professional submissions and work being done. The contest is still moving ahead, growing in a positive first derivative, and consistent with our positive experiences from previous HiMCM contests.

This year the contest consisted of 349 teams submitting papers (a growth of over 64% from last year) from thirty-six states, the Hong Kong International School, Belgium, China, and Department of Defense schools. Thus, our contest continues to attract an international audience. The teams accomplished the vision of our founders by providing *unique* and *creative* mathematical solutions to complex open-ended, real-world problems. This year the students had a choice of two problems:

Problem A: School Busing

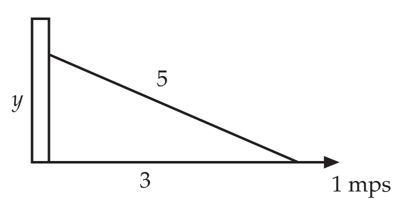
Consider a school where most of the students are from rural areas, and must be bussed. The buses might pick up all the students and then go to the elementary school and then continue from that school to pick up more students for the high school.

A clear alternative would be to have separate buses for each school even though they would then need to trace over the same routes. There are, or course, restrictions on time (no student should be in the bus more than an hour), drivers, equipment, money, etc.

How can you set up school bus routes to optimize budget dollars while balancing the "time on bus" for various school groups? Build a mathematical model that could be used by various rural and perhaps urban school districts. How would you test the model prior to implementation? Prepare a short article to the school board explaining your model, its assumptions, and its results.

Problem B: The Falling Ladder

A ladder 5 meters long is leaning against the vertical wall, with its foot on a rug on the floor. Initially, the foot of the ladder is 3 meters from the wall. The rug is pulled out, and the foot of the ladder moves away from the wall at a constant rate of 1 meter per second. Build a mathematical model or models for the motion of the ladder. Use your model (or models) to find the velocity at which the top of the ladder hits the floor and the distance the top of the ladder will be from the wall at the moment that it hits the ground.



Commendation: All students and their advisors are congratulated for their varied and creative mathematical efforts. The composition of the 349 teams consisted of 196 submitting solutions to the B problem and 153 submitting solutions to the A problem. The thirty-six continuous hours to work on the problem again provided (in our opinion) a vastly improved quality of the papers. Teams are commended for the overall quality of work.

Again, many of these teams had female participation, showing this competition is for both male and female students. This year there were approximately 805 males and 641 females participating. There were 67 all-female teams out of the 349 total teams. Teams again proved to the judges that they had fun with their chosen problems, demonstrating research initiative and creativity in their solutions. The 5th-year effort was deemed a success!

Judging: We ran three regional sites in December 2002 and January 2003. Each site judged papers for both problems A and B. The papers judged at each regional site were not from their respective region. Papers were judged as Outstanding, Meritorious, Honorable Mention, and Successful Participant. All regional finalist papers for the Regional Outstanding award were brought to the National Judging. The national judging chooses the "best of the best" as National Outstanding. The National Judges commend the regional judges for their efforts and found the results were very consistent. We feel that this regional structure provides a good prototype for the future of the contest's structure as it continues to grow exponentially.

JUDGING RESULTS:

National & Regional Combined Results

Problem	**National Outstanding***	Regional Outstanding	Meritorious	Honorable Mention	Successful Participant	Total
A	**4**	12	27	78	32	153
B	**6**	17	39	66	68	196
Total	**10**	29	66	144	100	349

GENERAL JUDGING COMMENTS:

The judge's commentaries provide comments on solutions to the specific problems. As contest director and head judge for these problems, I would like to speak generally about team solutions from a judge's point of view. Papers need to be very coherent, concise, and clear. Students need to restate the problem in their own words (not rewrite our problem statement) so that the judges can determine the individual (or team's) focus of the paper. For example, in the bussing problem, a restatement would include the actual rural area that the team used to model the situation. Thus, a restatement includes a refining or tailoring of the problem to a specific team. Papers that explain the development of their model, assumptions, and its solutions and then support the solution mathematically generally do quite well. Modeling assumptions need to be listed and justified but should include only those that come to bear on the team's solution (that can be part of simplifying the model). Laundry lists of assumptions that are never referred to in the context of the model development are not considered relevant and deter from the paper's quality. The model needs to be clearly developed and all variables that are used need to be defined. "Thinking outside of the box" is also an ingredient considered important by judges. This varies from problem to problem but usually includes model extensions or sensitivity analysis of the solution to the teams' inputs. A clear conclusion and answers to specific scenario questions are all key components. The strengths and weakness section is where the modeling team can reflect on its solution. Attention to detail and proofreading the paper prior to final submission are also very important regardless of the problem, since the judges look for clarity and style.

CONTEST FACTS:

- Wide range of schools/teams competed including teams from China, Netherlands, and Hong Kong
- 62.7%, or 219 teams of the 349 with female members. 64 of these teams were all female.
- 37.3% were all male.
- There were 805 males and 641 females on teams.
- There were 39 Outstanding awards out of the 349 entries (11.2%).
- 36 states participated in the contest.

THE FUTURE:

The contest, which attempts to give the under-represented an opportunity to compete and achieve success in mathematics endeavors, appears well on its way in meeting this important mission.

We continue to strive to grow. Again, any school/team will be allowed to enter the contest, as there will be no restrictions on the numbers of schools entering. A regional judging structure will be established based on the response of teams to compete in the contest.

Again, these are exciting times for our high school students. Mathematics continues to be more than learning skills and operations. Mathematics is a language that involves our daily lives. Applying the mathematical principles that one learns is the key to future success. The ability to recognize problems, formulate a mathematical model, solve, compute with technology, communicate and reflect on one's work are keys to success. Your ability to use technology aggressively to discover, experiment, analyze, resolve, and communicate results are also keys to success in the future. Students learn confidence by tackling ill-defined problems, working together to generate a solution. Through team-building and team effort solutions are built. Applying mathematics is a team sport.

Advisors need only be a motivator and facilitator. Allow students to be *creative* and *imaginative*. It is not the technique used but the process that discovers how assumptions drive the techniques that is fundamental. Let the students practice to be problem solvers. Let me invite all high school mathematics faculty to get involved: Encourage your students, make mathematics relevant, and open the doors to success. Mathematical modeling is an art and a science. Teach your students through modeling to think critically, communicate efficiently, and be confident, competent problem solvers for the new century.

CONTEST DATES:

Mark your calendars now, for the next HiMCM will be held from 7–24 November 2003. Registrations of teams are due by 25 October, 2003. Papers must be postmarked by 26 November, and mailed directly to COMAP. Teams will have a consecutive 36-hour block within this window to complete the problem. Teams can register via the worldwide Web at www.comap.com.

HiMCM Judges' Commentary

Problem A: The School Busing Problem

The judges were impressed with the creativity, quality of the analysis, and the writing by the teams modeling the School Busing Problem. This was the "purer" of the two modeling problem choices. This problem was ill-defined in many ways, requiring teams to step back and insure that they defined the problem aspect that they were trying to solve. The judges commented that the statement of assumptions with justification, the style of presentation, and the depth of analysis were all very good. The better papers offered a diversity of solutions.

It was imperative to pick a rural location for the model-testing and verification parts of the modeling process. Doing this provided teams with an opportunity to restate the problem in more manageable terms. This allowed successful teams to move from the general ill-defined problem to an attackable, specific one.

Some papers decided to simulate the school busing process. Simulation is a good mathematical modeling tool. However, a simulation is of little value without a step-by-step algorithm designed around assumptions based on a firm foundation. Successful teams provided a guide to their algorithm(s)—a step-by-step procedure for the judges to follow. (The code that teams attach is only read for the teams reaching the final rounds.) The results of any simulation need to be well explained and sensitivity analysis performed.

For example, consider a flip of a coin. Here is the algorithm:

INPUT:	Random number, number of trials
OUTPUT:	Heads or tails
Step 1:	Initialize all counters
Step 2:	Generate a random number between 0 and 1.
Step 3:	Choose an interval for heads, like [0.0.5]. If the random number falls in this interval, the flip is a heads. Otherwise the flip is a tails.
Step 4:	Record the result as a heads or a tails.
Step 5:	Count the number of trials and increment: Count = Count + 1

One of the items that discriminated the better papers was the discussion of the requirement to minimize costs of transporting the students. Another was an analysis of the routes and population density of the rural area chosen. Data were needed to operate most models developed. Teams were able to obtain valid data from the Internet. Teams varied in the reliability of the data they incorporated. Verification of models or model testing was also an important discriminator. Some tested their models to see if they made common sense. Others compared their predictions with historical results they were able to obtain.

Some facets of solutions the judges noted that they would like to see in future contests included: annotation of computer programs if included, careful definition of inputs required and outputs generated by computer programs, an increase in the use of graphical displays with interpretations, demonstrations with very simple problems before employing the same logic with a detailed computer simulation, increased documentation of sources used—careful annotation of material used from references with an explanation of how equations that are incorporated from various references follow the assumptions that the modelers are making. And perhaps most importantly, judges desire a careful explanation of the model design; getting from the assumptions to the model.

The judges commend the teams for a truly outstanding job on a difficult, open-ended problem.

Problem B: The Falling Ladder

This problem relied heavily on physical laws. Many teams began by considering this problem as an easy related rates problem. Through related rates, a velocity equal to infinity was calculated. That devastation alone would be catastrophic! Therefore, other modeling techniques needed to be considered. Judges expected to see some experimentation, followed by simple modeling assumptions such as free fall. Many teams found a solid reference on the Internet, and it appeared that many teams attempted to change numbers without really understanding the original article. Judges were disappointed in students believing that transforming the original article's data to our problem completed the modeling process. Modeling is more than that!

For these teams, the judges expected one of the two approaches:

Cold Start:

Teams must list valid modeling assumptions with justifications that led to the equations used in the article. This might require finding the references from the original article and reviewing them also. Teams should verify the base case (our problem) but extend to more reality with discussions or models with drag forces.

Hot Ending:

Teams that merely copied the equations and modified the values to our problem were still eligible for Outstanding awards. The teams using this approach would then be expected to perform analysis: a value added, or "what if?", analysis of the model. This could have included the effect of making changes in the parameters, the impact of changes in the assumptions, or a refinement of the basic model, or perhaps the consideration of special cases.

The judges wanted teams and advisors to know that this is a modeling contest and therefore all the elements of good modeling need to be present in the students' submissions.

GENERAL COMMENT FROM JUDGES:

Summaries:

These are still, for the most part, one of weakest parts of team submissions. These should be complete in ideas, not details. They should include the bottom line and the key ideas used. They should include the particular questions addressed and their answers.

Restatement of the Problem:

Problem restatements are important for teams to move from the general case to the specific case. They allow teams to refine many aspects of their thinking to give their model uniqueness and a creative touch.

Assumptions/Justifications:

Teams should list only those assumptions that are vital to the building and simplifying of their mathematical model. Every assumption should have a justification with it. Variables chosen need to be listed with notation and be well defined.

Model:

Teams need to show a clear link between the assumptions they listed and the building of their model or models.

Model Testing:

Model testing is not the same as checking arithmetic. Teams need to compare results or attempt to verify (even with common sense) their results.

Teams that use simulation must provide a clear step-by-step algorithm. Lots of runs and related analysis are required when using a simulation to model a problem. Sensitivity analysis is also expected to see how sensitive a simulation is to the model's key parameters.

Conclusions:

This section deals with more than just results. Conclusions might also include speculations, extensions to the model, and generalizations of the model. Teams should ask themselves what other questions would be interesting if they had more time and then tell the judges about their ideas.

Strengths and Weaknesses:

Teams should be open and honest here. What could the team have done better?

References:

Teams may use references to assist in modeling the problem. However, they must also note the source of their assistance. It is still required of the team to show how the model was built and why it is the model chosen for this problem.

Problem A Summary: Godwin High School, Richmond VA

Advisor: Ann Sebrell

Team Members: Victor Fedorov, Ankit Jain, Randall Reed, Jeffrey Shutterly

To optimize a transportation budget for a school system, one must first contemplate the variables affecting expenditures, in this case the purchase price of buses, gasoline, maintenance and operating cost, and drivers' wages. All of these were combined into the cost equation, $C = x\left(a + b + 180g\frac{y}{m} + 180w\frac{y}{s}\right)$, where C = total cost for the first year (when the buses would be purchased), x = number of buses, a = cost for a new bus, b = annual operating cost, g = cost of gas per gallon, y = distance traveled, m = miles per gallon, w = hourly wage paid to drivers, and s = average speed of the bus. Research was conducted and real data found to assign values to most of these variables, excluding number of buses and distance traveled, the two areas which we needed to optimize. For our model, we decided that the number of buses would be the principal consideration in the total cost, since the costs for purchasing and maintaining more buses far exceed the expense of paying a given number of drivers for more work. Our goal was to pick up a given number of students using the fewest possible buses traveling a minimum distance. We began with a simple model, evenly distributing the population among seven congruent, conical sectors. Next, we complicated the model by dividing the area into concentric circular zones, based on bus capacity. Also, we used a variation with two campuses for the high school and elementary school, to increase versatility. Finally, our last and most realistic model used an actual rural school district circumscribed by a circle. Once again, circles radiated from the center, the location of the school, using the inverse normal to calculate the radius for a circle to contain a given number of students (the bus capacity). For all three models, the outermost region had superfluous bus stops, because of sparse populations. To account for this disparity, the region was still divided into the necessary number of bus stops, but in planning could be rotated to merge as many students as possible into as few stops as feasible.

Problem A Summary: Goldberg Homeschool

Advisor: David Goldberg

Team Members: Benny Goldberg, Taylor McLemore, Brendan Schwartz

Our Group was drawn to the school bus problem because of the creative latitude within the scenario. Initially, it was apparent that there was no obvious answer, but rather an opportunity to create a totally unique solution to the problem at hand. We attempted to create a school transportation system that is fiscally effective and logistically efficient.

The model attempts to be fiscally effective by creating the most efficient bus routes, bus stops, and hours paid to drivers. Our model is based on a nine-sector grid system. Each of the nine sectors is thirty-six square kilometers and these nine sectors combine to create the school district. The grid system is based upon assumptions of a 9:3:1 ratio among elementary, middle, and high schools respectively. It also depends on a student population that is equally distributed throughout the district by age and density. Each level of school (elementary, middle, or high school) is located in a central location with respect to assigned school sectors. The next step was planning a system for the fewest amounts of stops in the most convenient locations. By using knights' moves (one unit of movement in one direction and two in the other) the most efficient series of stops were established. Even though knights' moves could not perfectly distribute bus stops to service an entire sector, only a few extra stops had to be added. No student is required to walk more than one kilometer to reach a bus stop or school. In an effort to remain fiscally responsible, we tried to minimize the number of busing waves. We decided on having two busing waves, where elementary school students are delivered to their assigned schools, and then buses retrace their routes to pick up middle and high school students. To further elaborate the efficiency of this system, our summary contains the fiscal plans for financing and maintaining the infrastructure of the school transportation system.

We believe that our model satisfies the stipulations that the scenario set forth and moreover, is an appropriate method for solving the rural and urban busing problem. We approached the problem with the express intent of creating a fiscally effective and logistically efficient mathematical model.

Problem A Summary: Hong Kong International School

Advisor: William Stork

Team Members: Christopher Lee, Lowell Ling, Alex Shum, Gary Yung

Problem A asked us to devise a mathematical model to plan out a busing scheme for a school where students are predominantly from rural regions and are heavily reliant on school bus transportation.

The three main assumptions made to address this problem are:

1) The school is located in a rural region

2) Students are evenly distributed throughout the district

3) The High School and Elementary School are juxtaposed, therefore all buses need only travel to a single location

To solve this problem we made no attempts to model exact road systems or diagram particular bus routes, rather we focused more on how to efficiently use buses.

We discussed three models to examine the question.

Model 1:

A fairly basic bus scheme. One fleet of buses is used to first transport the high school students to school, then that same fleet goes back to bring in the elementary school students.

Model 2:

This is a slightly more sophisticated scheme. Two fleets of buses are used to increase time efficiency while minimizing effects on costs. The two fleets begin moving at different times.

Model 3:

This is our recommended option. It involves using two fleets of buses both beginning at the same time but from different locations. It involves having some buses collect students from both schools and others specialize in students from a particular school.

To model the situation, some constants were held:

1) Bus capacity

2) Bus velocity

3) Population density of students

To test the model, we used real-world school information to see if values were impressive.

Case Studies are attached as a part of the project.

Problem A Paper: Frontier Regional School, South Deerfield MA

Advisor: Pat Taylor

Team Members: Lauren Krolick, Julia Shelkey, Tamara Sobek-Rosnick

RESTATING THE PROBLEM:

One clear option for the school is to have buses pick up the elementary school students and drop them off before going around again to collect the high school students. Another alternative is to have separate buses for each of the schools so that the students arrive at school at the same times. The school obviously has financial restrictions, and the students should not have to ride the bus for more than an hour. Develop a bus route that offers a sound financial plan for the school and a reasonable riding time for the students. Describe how this route could be tested before it is instituted and write an article to the school board explaining the model.

ASSUMPTIONS:

1. No students are physically challenged.
2. The students live in one town.
3. The schools start one hour apart from each other if 4 buses are used.
4. The buses are parked at the appropriate school.
5. The buses travel at a constant speed.
6. It takes a student 30 seconds to board and be seated on a bus.
7. Buses can only drive on roads and students walk only on roads.
8. Students can only get to school by walking, taking the bus, or driving.

Due to the lack of information given, some assumptions had to be made. We attempted to make all of our assumptions as accurate as possible by researching information and taking polls. For our first scenarios we kept the assumptions very basic and then elaborated on the factor that would effect a competent and realistic bus system.

GOAL:

We want to get the students from their homes to the school and back in the most cost effective and practical way.

BASIC PREPARATIONS:

We decided to make our town a rectangle 6.4 x 6.0 miles (32 x 30 squares) because it represents a small town very well. We decided to split the rectangle into quadrants (I, II, III & IV) and to have one bus serve the students in each quadrant (two busses when separate elementary and high school buses are used).

Our Hypothetical Town: Pleasantville

- Pleasantville has a population of 1765.
- 176 students attend the elementary school.
- 88 students attend the high school.

Note on the Scenario Illustrations:
Each dot, either solid or not, represents two houses, each having one student.

DERIVATION OF THE RANDOMIZATION:

Because in a rural area house locations are not centralized, the placement of the houses in Pleasantville is important.

We achieved randomization with a Microsoft Excel formula. This formula (=RAND()*(31-0)+0)) chooses a random number between 0 and 31; and was entered in Al. In cell B1 the formula =INT(A1) was entered. This takes only the integer value of the value in A1. We created two columns of values with these formulas (for a total of four columns). (Except the second two columns were on an open range of 0–33.)

We dragged the formula down 132 values so that we would have one point for every two students. (This was for simplicity since it is very difficult to represent 264 houses on one 8.5 x 11 piece of paper.)

These two columns formed the *x*- and the *y*-coordinates for a scatter plot. We added gridlines to form squares to represent square miles.

To determine which houses housed elementary school students and which housed high students, we had the computer pick random points. (This same method was used to determine which high school houses housed drivers.)

CALCULATIONS:

We used national statistics to determine the percentage of high school and elementary students.

Fact: There are 288,525,000 people in the United States

Fact: There are 31,500,000 elementary school students.

Fact: There are 15,700,000 high school students.

Conclusion: Elementary students are 10.0% of the population, and high school students are 5.44%.

DESCRIPTION OF SCENARIOS:

Scenario A is the most basic; it is the constant for scenarios B-E. The four elementary school buses and the four high-school buses pick up students at the same time. The high school and elementary school are adjacent. All students ride a bus, and all students are picked up at their houses.

Scenario B improves on scenario A in one aspect: in real life, the schools are unlikely to be adjacent.

Scenario C improves on scenario A in another aspect: some high school students drive to school. We assumed that 15% of high school students have a license based on a survey of students in high school. We also determined that 75% of people with driver's licenses have cars. From these numbers we calculated that 0.563% of the total population are high school students who drive to school. From this we determined that 10 students in Pleasantville drive to school.

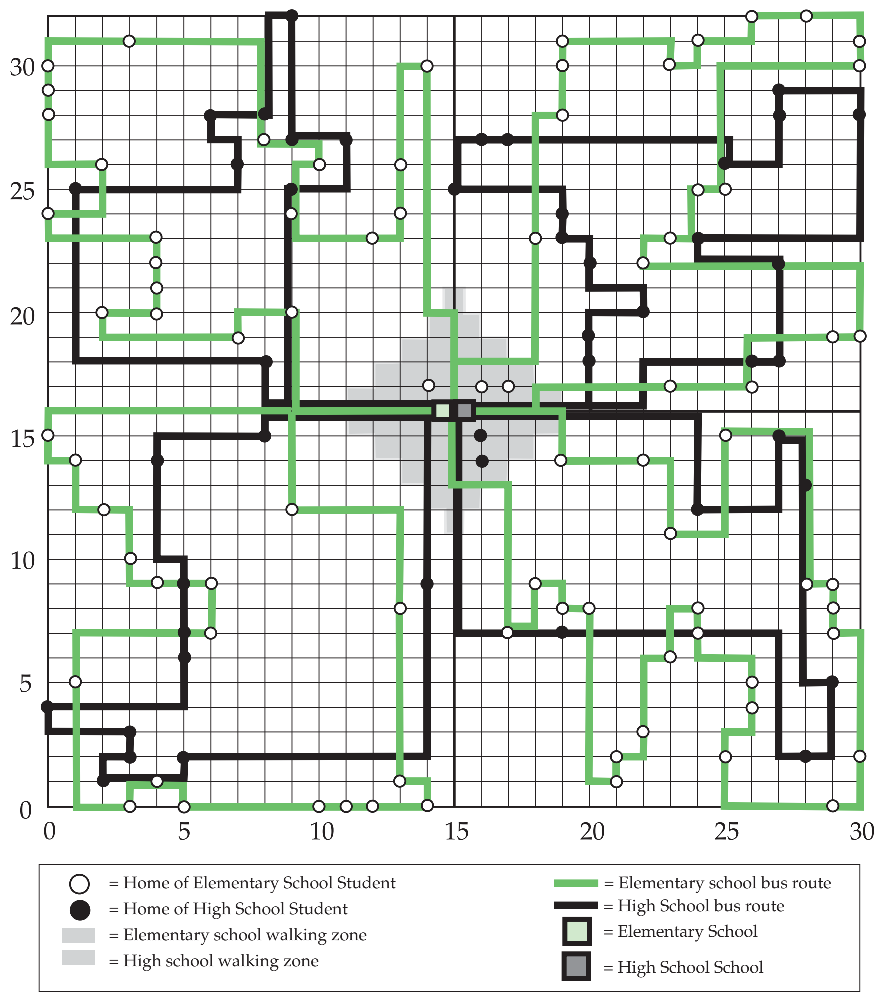
Scenario D improves upon scenario A in another aspect: four buses gather all of the students, first the elementary students, then the high school students.

Scenario E improves upon scenario A in yet another aspect: not all students get picked at their own homes. In many areas, students walk to bus stops close to their homes. We decided where the bus stops would be by looking at the visual aids and deciding that students could walk up to 0.6 of a mile (three boxes) to a stop. We placed bus stops where necessary to maximize the number of people who could go there and to minimize the total number of stops.

Scenario F is the most complex scenario. The improvements of scenarios B-E are combined to get a cost and time effective model for the bus system. The high school and elementary school are not next to each other, some high school students drive themselves to school, students are picked up at communal bus stops, and four buses make two rounds.

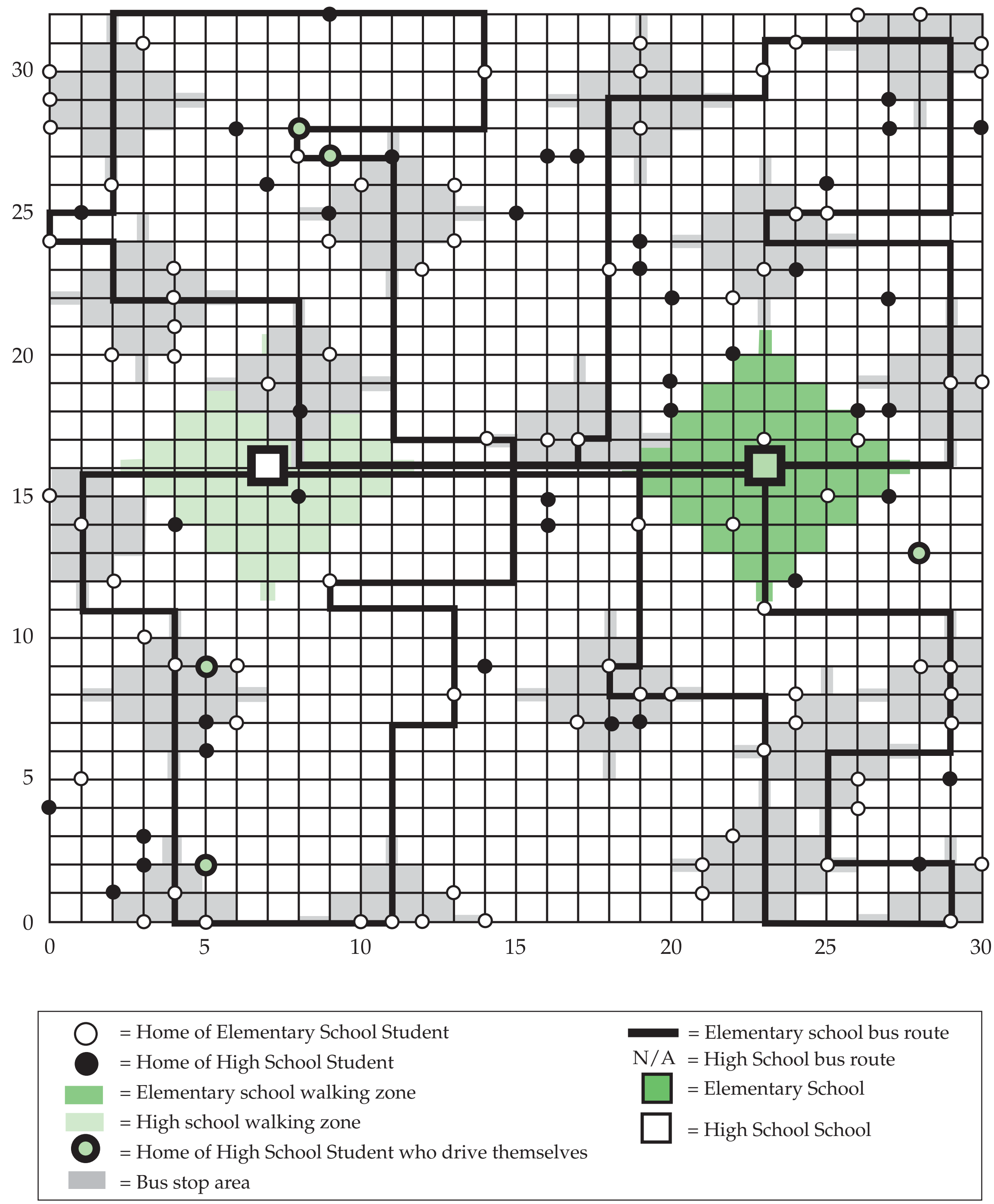
These are three samples of the eight hand-drawn scenarios.

Scenario A



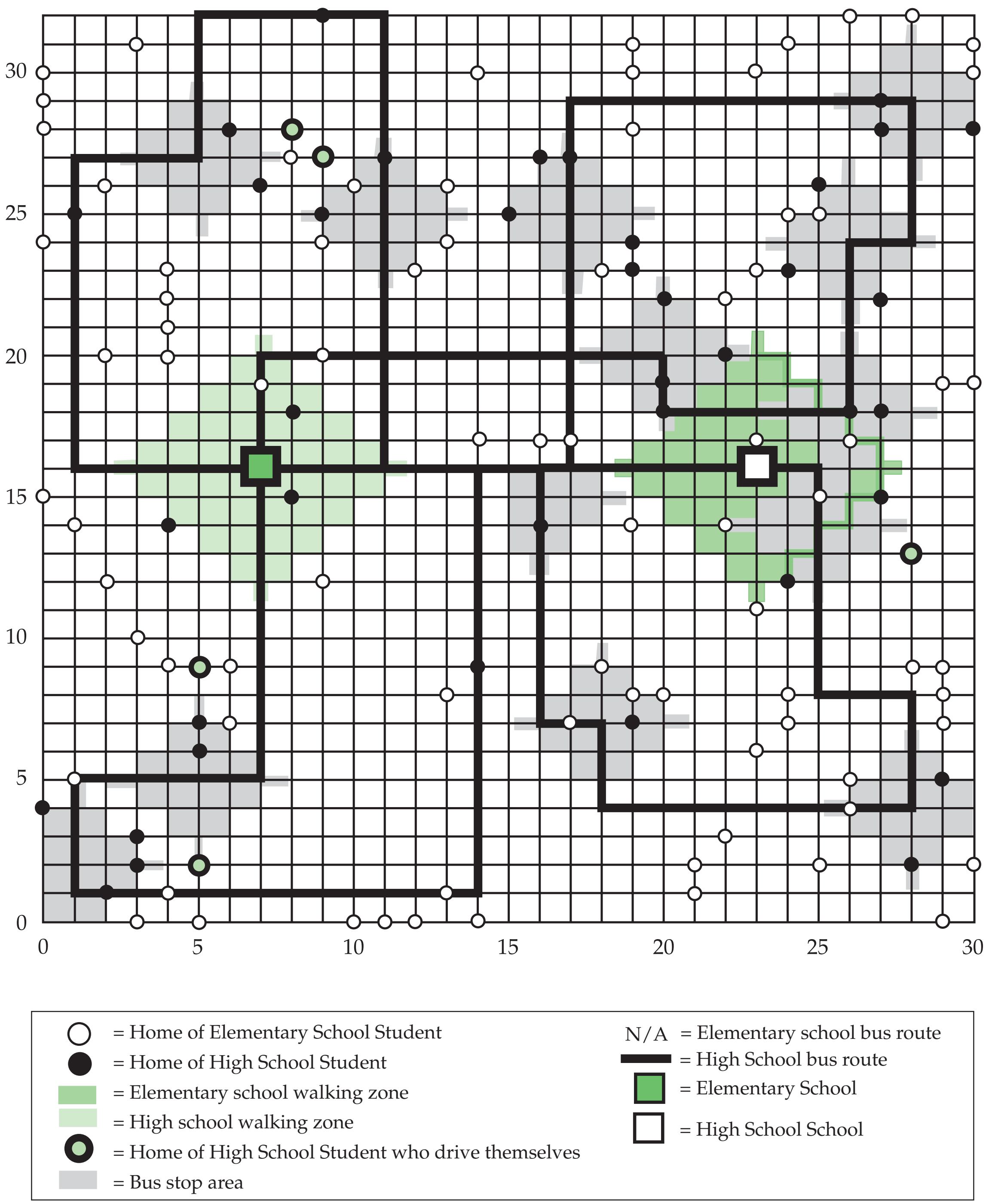
One side of a small square = 0.2 MILES

Scenario F1
Elementary school only



One side of a small square = 0.2 MILES

Scenario F2
High School only



One side of a small square = 0.2 MILES

CREATION OF THE TABLES:

The tables were created in Microsoft Excel. We did this to find the most cost efficient and most time efficient scenario.

The column "total elementary squares " is the number of sides of the squares that each bus must travel on its route.

The column "total miles traveled" is calculated by multiplying the previous column by 0.2 because one side of a box = 0.2 miles.

The column "elementary squares with student" is the number of sides of squares after a student has gotten on the bus.

The column "miles traveled with student" is calculated by multiplying the previous column by 0.2 because one side of a box = 0.2 miles.

The column "number of times bus stops" is calculated by counting the number of times the bus stopped to pick up students. Many times this is the number of students but when there are communal bus stops, this number decreases.

The "number of students on bus" is calculated by counting the number of houses whose children ride that bus and then multiplying by two because each house (dot) represents two houses each with one student.

"Time bus is traveling with student" is the calculation from the time that the first student gets on the bus until they get off, but only including the time the bus is moving. It was calculated by dividing the miles traveled (column 4) by 40. (We assume that the bus is traveling 40 mph except when it is loading or unloading students.) Then, that number is multiplied by 60 to get minutes.

"Time it takes for all kids to get on" is calculated by multiplying the number of students (column 6) by 0.25 minutes. This is done because we believe it takes a student 0.25 minutes (15 seconds) to board a bus.

"Time it takes at each bus stop" is calculated the same way as the previous column. We believe that it takes 15 seconds for the bus to open and close its doors.

"Time it takes to pick up all students" is the sum of the previous two columns, which summarizes the time that the bus is standing still.

"Student's maximum time on bus" is the sum of "time bus is traveling with student" and "time it takes to pick up all students." This summarizes the time from the moment the first student steps on a bus until the minute they leave it.

"Cost of other operating expenses" includes factors such as rust, damage to the body, wear and tear, and weather effects. The cost is put under every elementary scenario and under every high school scenario (except D and F because these scenarios use only 4 buses).

"Annual cost to school" is calculated by multiplying number of students on the bus, total miles traveled, number of days in a school year (180), and 0.13; and then adding the cost of other operating expenses if applicable. 0.13 is $0.13 per passenger per mile driven, per day, which we found through research. The total miles traveled change from scenario to scenario and quadrant to quadrant.

The "TOTALS" are found by summing of the annual cost for each quadrant.

Elementary School Bus Routes

		1 Total Elemen. Squares	2 Total Miles Traveled	3 Elemen. Squares with student	4 Miles Traveled with student	5 Number times Bus Stops	6 Number Students on Bus	7 Time (min) bus is traveling with student	8 Time (min) it takes for all kids to get on	9 Time it takes at each bus stop	10 Time (min) it takes to pick up all students	11 Student's Maximum Time on Bus	12 Cost of Other Operating Expenses (weather)	13 Annual Cost to School	
Scenario A	Quad I	81	16.2	71	14.2	36	36	21.3	9	9	18	39.3	$ 1,960.00	$ 15,606.88	TOTAL
	Quad II	86	17.2	71	14.2	40	40	21.3	10	10	20	41.3	$ 1,960.00	$ 18,059.20	$ 71,113.60
	Quad III	84	16.8	66	13.2	36	36	19.8	9	9	18	37.8	$ 1,960.00	$ 16,112.32	
	Quad IV	90	18	79	15.8	46	46	23.7	11.5	11.5	23	46.7	$ 1,960.00	$ 21,335.20	
Scenario B	Quad I	74	14.8	62	12.4	32	32	18.6	8	8	16	34.6	$ 1,960.00	$ 13,042.24	TOTAL
	Quad II	102	20.4	79	15.8	46	46	23.7	11.5	11.5	23	46.7	$ 1,960.00	$ 23,918.56	$ 72,330.40
	Quad III	100	20	76	15.2	36	36	22.8	9	9	18	40.8	$ 1,960.00	$ 18,808.00	
	Quad IV	78	15.6	66	13.2	40	40	19.8	10	10	20	39.8	$ 1,960.00	$ 16,561.60	
Scenario C	Quad I	81	16.2	72	14.4	36	36	21.6	9	9	18	39.6	$ 1,960.00	$ 15,606.88	TOTAL
	Quad II	86	17.2	71	14.2	40	40	21.3	10	10	20	41.3	$ 1,960.00	$ 18,059.20	$ 71,113.60
	Quad III	84	16.8	68	13.6	36	36	20.4	9	9	18	38.4	$ 1,960.00	$ 16,112.32	
	Quad IV	90	18	79	15.8	46	46	23.7	11.5	11.5	23	46.7	$ 1,960.00	$ 21,335.20	
Scenario D	Quad I	81	16.2	71	14.2	36	36	21.3	9	9	18	39.3	$ 1,960.00	$ 15,606.88	TOTAL
	Quad II	86	17.2	71	14.2	40	40	21.3	10	10	20	41.3	$ 1,960.00	$ 18,059.20	$ 55,506.72
	Quad III	84	16.8	66	13.2	36	36	19.8	9	9	18	37.8	$ 1,960.00	$ 16,112.32	
	Quad IV	90	18	79	15.8	46	46	23.7	11.5	11.5	23	46.7	$ 1,960.00	$ 21,335.20	
Scenario E	Quad I	70	14	60	12	10	36	18	9	2.5	11.5	29.5	$ 1,960.00	$ 13,753.60	TOTAL
	Quad II	70	14	55	11	8	38	16.5	9.5	2	11.5	28	$ 1,960.00	$ 14,408.80	$ 57,419.92
	Quad III	70	14	60	12	7	32	18	8	1.75	9.75	27.75	$ 1,960.00	$ 12,443.20	
	Quad IV	69	13.8	58	11.6	8	46	17.4	11.5	2	13.5	30.9	$ 1,960.00	$ 16,814.32	
Scenario F	Quad I	66	13.2	59	11.8	10	36	17.7	9	2.5	11.5	29.2	$ 1,960.00	$ 13,079.68	TOTAL
	Quad II	90	18	72	14.4	9	42	21.6	10.5	2.25	12.75	34.35	$ 1,960.00	$ 19,650.40	$ 61,650.64
	Quad III	83	16.6	59	11.8	6	34	17.7	8.5	1.5	10	27.7	$ 1,960.00	$ 15,166.96	
	Quad IV	63	12.6	49	9.8	8	40	14.7	10	2	12	26.7	$ 1,960.00	$ 13,753.60	

High School Bus Routes

		1 Total High S. Squares	2 Total Miles Traveled	3 High S. Squares with student	4 Miles Traveled with student	5 Number times Bus Stops	6 Number Students on Bus	7 Time (min) bus is traveling with student	8 Time (min) it takes for all kids to get on	9 Time it takes at each bus stop	10 Time (min) it takes to pick up all students	11 Student's Maximum Time on Bus	12 Cost of Other Operating Expenses (weather)	13 Annual Cost to School	
Scenario A	Quad I	78	15.6	65	13	34	34	19.5	8.5	8.5	17	36.5	$ 1,960.00	$ 14,371.36	TOTAL
	Quad II	66	13.2	51	10.2	18	18	15.3	4.5	4.5	9	24.3	$ 1,960.00	$ 7,519.84	$ 36,462.88
	Quad III	64	12.8	56	11.2	22	22	16.8	5.5	5.5	11	27.8	$ 1,960.00	$ 8,549.44	
	Quad IV	62	12.4	49	9.8	14	14	14.7	3.5	3.5	7	21.7	$ 1,960.00	$ 6,022.24	
Scenario B	Quad I	94	18.8	78	15.6	34	34	23.4	8.5	8.5	17	40.4	$ 1,960.00	$ 16,917.28	TOTAL
	Quad II	54	10.8	39	7.8	16	16	11.7	4	4	8	19.7	$ 1,960.00	$ 6,003.52	$ 36,612.64
	Quad III	60	12	46	9.2	18	18	13.8	4.5	4.5	9	22.8	$ 1,960.00	$ 7,014.40	
	Quad IV	72	14.4	51	10.2	14	14	15.3	3.5	3.5	7	22.3	$ 1,960.00	$ 6,677.44	
Scenario C	Quad I	78	15.6	65	13	34	34	19.5	8.5	8.5	17	36.5	$ 1,960.00	$ 14,371.36	TOTAL
	Quad II	66	13.2	51	10.2	14	14	15.3	3.5	3.5	7	22.3	$ 1,960.00	$ 6,284.32	$ 33,280.48
	Quad III	62	12.4	54	10.8	18	18	16.2	4.5	4.5	9	25.2	$ 1,960.00	$ 7,182.88	
	Quad IV	62	12.4	49	9.8	12	12	14.7	3	3	6	20.7	$ 1,960.00	$ 5,441.92	
Scenario D	Quad I	78	15.6	65	13	34	34	19.5	8.5	8.5	17	36.5	NA	$ 12,411.36	TOTAL
	Quad II	66	13.2	51	10.2	18	18	15.3	4.5	4.5	9	24.3	NA	$ 5,559.84	$ 28,622.88
	Quad III	64	12.8	56	11.2	22	22	16.8	5.5	5.5	11	27.8	NA	$ 6,589.44	
	Quad IV	62	12.4	49	9.8	14	14	14.7	3.5	3.5	7	21.7	NA	$ 4,062.24	
Scenario E	Quad I	57	11.4	65	13	5	34	19.5	8.5	1.25	9.75	29.25	$ 1,960.00	$ 11,029.84	TOTAL
	Quad II	61	12.2	44	8.8	4	18	13.2	4.5	1	5.5	18.7	$ 1,960.00	$ 7,098.64	$ 31,539.52
	Quad III	61	12.2	53	10.6	5	22	15.9	5.5	1.25	6.75	22.65	$ 1,960.00	$ 8,240.56	
	Quad IV	49	9.8	48	9.6	3	14	14.4	3.5	0.75	4.25	18.65	$ 1,960.00	$ 5,170.48	
Scenario F	Quad I	72	14.4	53	10.6	5	34	15.9	8.5	1.25	9.75	25.65	NA	$ 11,456.64	TOTAL
	Quad II	52	10.4	37	7.4	4	12	11.1	3	1	4	15.1	NA	$ 2,920.32	$ 23,063.04
	Quad III	56	11.2	42	8.4	3	14	12.6	3.5	0.75	4.25	16.85	NA	$ 3,669.12	
	Quad IV	67	13.4	47	9.4	4	16	14.1	4	1	5	19.1	NA	$ 5,016.96	

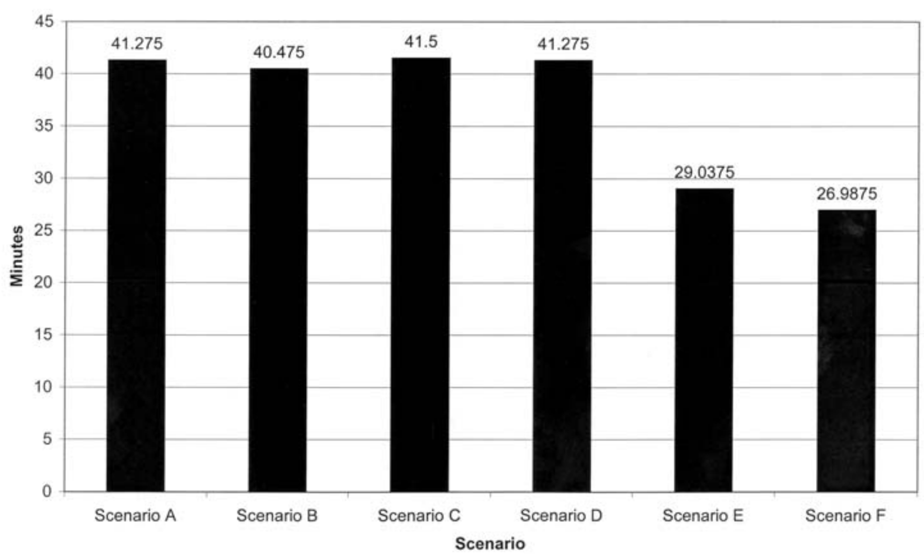
THE GRAPHS:

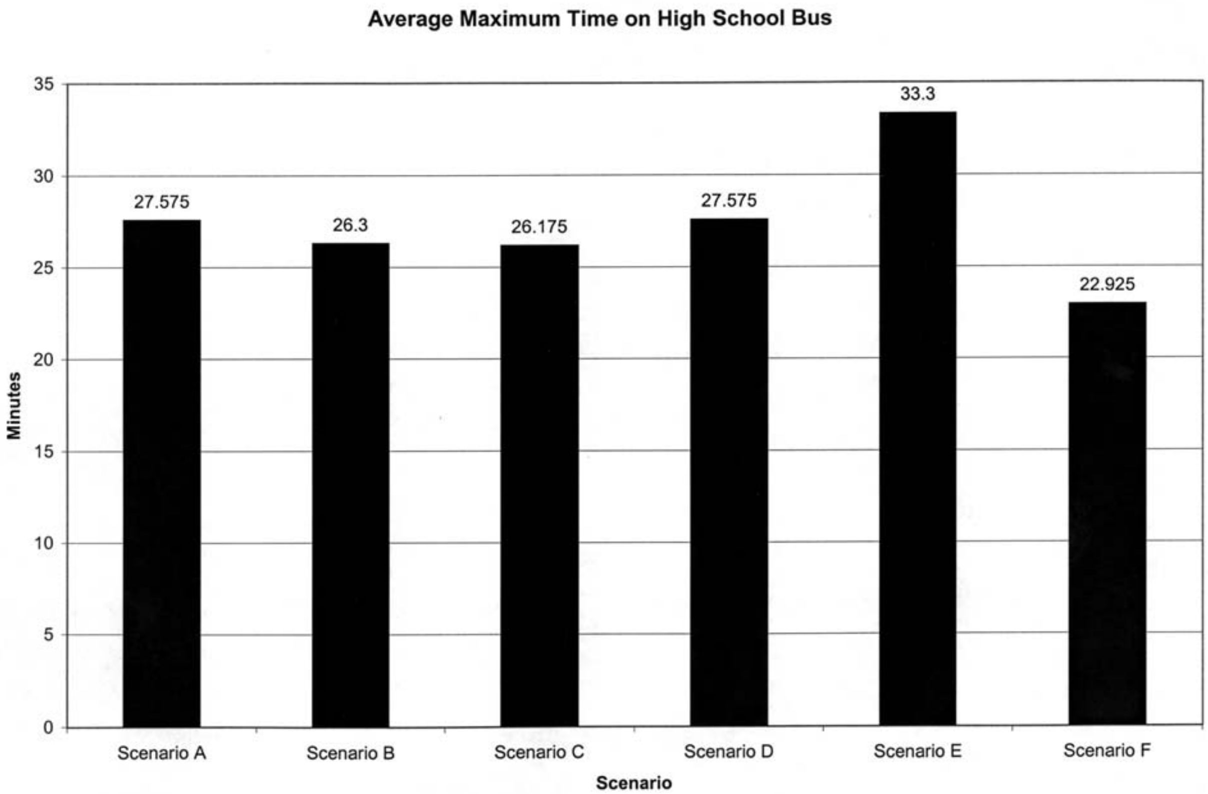
The "Total Cost to Schools" graph shows that Scenario D is the best cost wise. However, Scenario F (second best) is the most realistic because scenario D has the elementary school and the high school adjacent to each other.

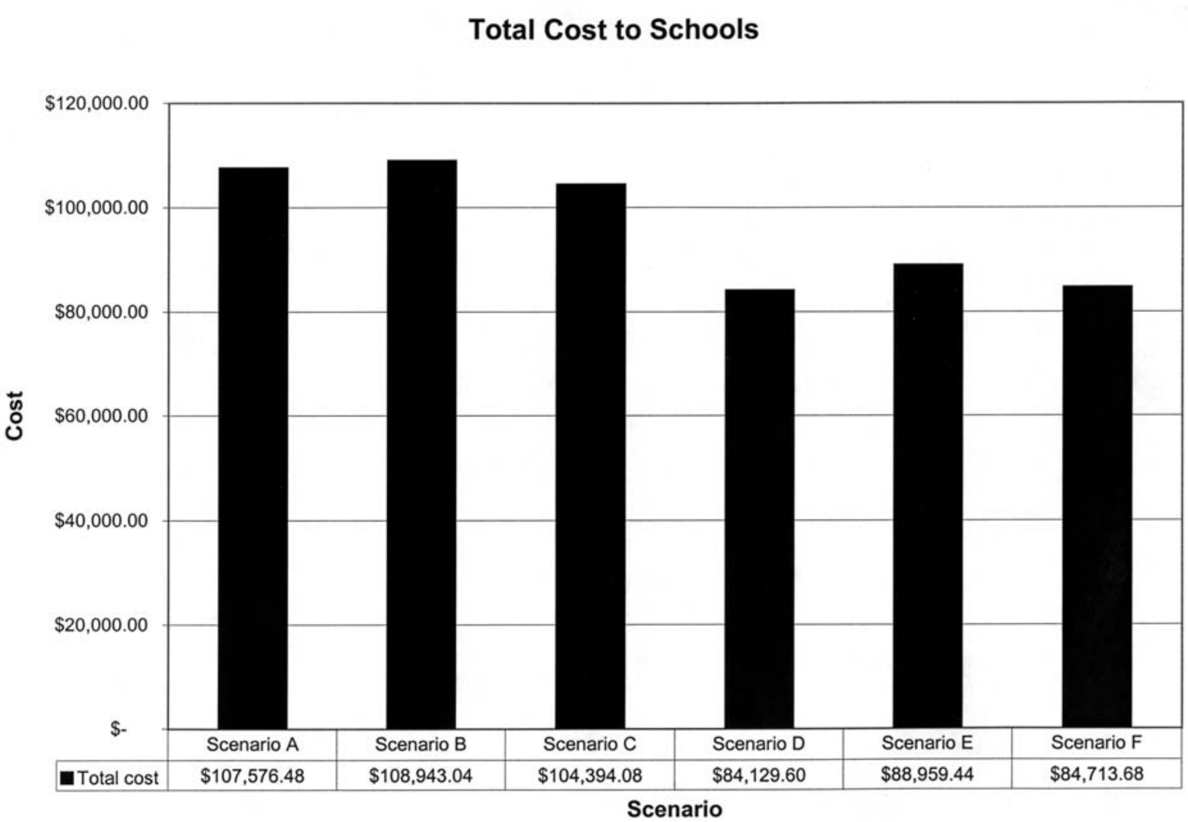
The "Average Maximum Time Spent on Elementary Bus" graph shows that Scenario F is the best.

The "Average Maximum Time Spent on High School Bus" graph shows that Scenario F is the best.

Average Maximum Time Spent on Elementary Bus




Average Maximum Time on High School Bus
Minutes
35
30
25
20
15
10
5
0
27.575
26.3
26.175
27.575
33.3
22.925
Scenario A
Scenario B
Scenario C
Scenario D
Scenario E
Scenario F
Scenario


Total Cost to Schools
Cost
$120,000.00
$100,000.00
$80,000.00
$60,000.00
$40,000.00
$20,000.00
$-
Scenario A
Scenario B
Scenario C
Scenario D
Scenario E
Scenario F
Total cost
$107,576.48
$108,943.04
$104,394.08
$84,129.60
$88,959.44
$84,713.68
Scenario

STRENGTHS AND WEAKNESSES

Scenario A Strengths

Every student gets picked up.

Students who live within a one-mile radius of school don't have to take a bus.

Each bus driver only has to pick up students in one quadrant for either high school or elementary school since there are eight bus drivers.

Scenario A Weaknesses

Eight buses go at the same time, causing costs to increase.

Students are on the buses longer since all are picked up at their houses.

The high school and elementary school are adjacent.

Since all students are picked up the cost/time increases.

Some students who live close to school are on the bus longest because they are picked up first.

Scenario B Strengths

Every student gets picked up.

Students who live within a one-mile radius of school don't have to take a bus.

There are two different school locations.

Each bus driver only has to pick up students in one quadrant for either high school or elementary school since there are eight bus drivers.

Scenario B Weaknesses

Students are on the buses longer since all are picked up at their houses.

Since all students are picked up the cost/time increases.

Eight buses go at the same time, causing costs to increase.

Some students who live close to school are on the bus longest because they are picked up first.

Scenario C Strengths

The scenario is more realistic because high school drivers are included.

Students who live within a one-mile radius of school don't have to take a bus.

Each bus driver only has to pick up students in one quadrant for either high school or elementary school since there are eight bus drivers.

Scenario C Weaknesses

Students are on the buses longer since all are picked up at their houses.

Eight buses go at the same time, causing costs to increase.

The high school and elementary school are adjacent.

Some students who live close to school are on the bus longest because they are picked up first.

Scenario D Strengths

Only four buses are used, which saves money.

Each student is picked up.

Students who live within a one-mile radius of school don't have to take a bus.

Scenario D Weaknesses

Bus drivers have to drive two routes.

Students are on the buses longer since all are picked up at their houses.

The high school and elementary school are adjacent.

Some students who live close to school are on the bus longest because they are picked up first.

Scenario E Strengths

The bus stops minimize the lengths of the routes.

The bus stops decrease costs.

The bus stops decrease the time students spend on the bus.

Students who live within a one-mile radius of school don't have to take a bus.

Scenario E Weaknesses

Eight buses go at the same time, causing costs to increase.

The high school and elementary school are adjacent.

Some students who live close to school are on the bus longest because they are picked up first.

Scenario F Strengths

The bus stops that minimize the lengths of the routes.

The bus stops decrease costs.

The bus stops decrease the time students spend on the bus.

Students who live within a one-mile radius of school don't have to take a bus.

There are two different school locations.

Only four buses are used, which saves money.

The scenario is more realistic because high school drivers are included.

Scenario F Weaknesses

Some students who live close to school are on the bus longest because they are picked up first.

There are only four bus drivers so each has to drive two routes.

TESTING OUR MODEL PRIOR TO IMPLEMENTATION

To test our model we used real data. By doing this we could determine if our model could be applied and implemented in real-life situations.

We searched the Internet and compiled data from Kentucky. We found that for 428,325 students, 9,572 buses were used. We set up a proportion. If 9,572 buses are used for 428,325 students then x = number of buses that should be used for 176 students (number of students in the elementary school).

$\frac{9572}{428325} = \frac{176}{x}$, so $x = 3.933$.

Thus, our model is realistic since our most efficient scenario uses 4 buses.

Dear Members of the School Board,

We are pleased to inform you of the results of our study on school transportation. We were asked to find an efficient way to transport high school and elementary school students in rural areas. Our cost- and time-effective model of transportation can be implemented in rural towns across the country.

Our best model takes into account many factors that influence busing. These include the fact that the high school and elementary school are not adjacent, a percentage of students drive to school, communal bus stops are used, and only four busses are used. By including these factors we can decrease the cost of transportation and the amount of time that each student must ride a bus.

To create a productive model we made several assumptions. These include; no students are physically challenged; the students live in one town; the schools start an hour apart if four buses are used; the buses are parked at the appropriate school; the buses travel at a constant speed; it takes students 30 seconds to board a bus and be seated; busses only drive on roads and students walk on roads; and students get to school by walking, taking the bus, or driving.

We created 6 scenarios, A-F. Scenario A is the most basic, with constants of the four variables, and is the control for the others. In scenarios B-E, one variable is changing and the other three are held constant. Scenario F is the quintessential model, which optimizes costs and is realistic about the community. We found that it will cost $84,713.68. This was the second best model financially, but the best model does not represent the community. A good model should have communal bus stops, allow students to drive to school, and have four busses run their routes twice to pick up the elementary school students first and then the high school students an hour later. The likelihood of the schools being adjacent to each other is low, so Scenario F is the best. We propose that the elementary school start one hour before the high school so the buses can make two runs to minimize the cost of busses.

In conclusion we recommend that the school system follow scenario F for financial purposes and for the well being of the students.

Sincerely,

The Research Team

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Problem B Summary: Maggie Walker Governor's School, Richmond VA

Advisor: John Barnes

Team Members: David Chiao, Lyric Doshi, Simon Helmore, Linda Westrick

In this paper, the descent of a leaning ladder being pulled by its base at a constant rate is modeled. First, two equations for the angular acceleration of the ladder are found: one as the ladder slides down the wall, and one as the ladder falls independently from the wall. The intersection of these equations is labeled as the critical point, the point at which the ladder leaves the wall because staying against the wall would require accelerating faster than gravity allows. Formulas are derived to determine how long it takes for the ladder to get to the critical point and how long it takes to hit the ground from that point. Additionally, formulas derived for angular position and velocity of the ladder at all intermediate times allow us to find the angular and tangential velocities at impact. The total falling time is used to find the position of the top of the ladder when it lands. With the given initial conditions of a 5 m ladder, 3 m from the wall, with its base moving away from the wall at 1 m/s, the ladder will hit the ground .065636 m away from the wall, with an angular velocity of 1.455772 rad/s and a translational velocity of 1 m/s away from the wall and 7.278858 m/s down. Although weaknesses exist in this model, it is fundamentally strong because its general form allows for adaptability to any initial conditions, and it produces accurate results at any instant in its descent.

Problem B Summary: Arkansas School for Mathematics and Sciences, Hot Springs AR

Advisor: Bruce Turkal

Team Members: Justin Hodges, Tyler House, Peter Smith, Paul Timko

We modeled a falling ladder using free fall physics equations. Research found that other methods modeled this problem by using pendulum motion. Since document results were found for this approach, we wanted to see if another way would work. We agreed that the ladder would lose contact with the wall at some point. Linear motion was used to describe the movement of the ladder while in contact with the wall. When the ladder left the wall its movement was described by using projectile motion kinematics equations. Related rates were used to find the velocity in the y direction. Much time was spent algebraically manipulating kinematics equations in search of equalities. The final velocity of the linear ladder movement down the wall was

related to the initial velocity of the ladder while in freefall. Parametric equations were derived and used to model the falling ladder in Java. The equations were used to find the break off point at which the ladder left the wall. Once this was found, the time the ladder fell from that point was used to find the final distance away from the wall. The final velocity was solved using the same equations. It was found that the ladder's final velocity was –6.526051 meters per second and the tip of the ladder was 0.3210020 meters from the wall.

Problem B Summary: Suncoast High School, Riviera Beach FL

Advisor: David Williams

Team Members: Curtis Fonger, Billy Jacobs, Dan Lehman, Piotr Nowak

Using logic and testing, we determined that the ladder would go down the wall, disconnect from the wall, and fall the rest of the way with pendulum-like behavior. Because the base of the ladder moves with a constant rate of 1 m/s, the distance between the base of the ladder and the wall can be found as a function of time. Using $x(t)$ and basic trigonometry, $\theta_n(t)$ can be found.

$\omega_n(t) \equiv \frac{d\theta_n(t)}{dt}$. Also, $\alpha_n(t) \equiv \frac{d\omega_n(t)}{dt}$. Now we have α_n as a function of t.

In physics, $\tau = r \times F = I\alpha$. The moment of rotational inertia of the ladder is $\frac{1}{3}ml^2$. τ_g (torque exerted by gravity) can be found using $\tau = r \times F$. The ladder is only going to stay on the wall if τ_g is great enough to keep the top of the ladder against the wall. $\tau_g + \tau_N = \tau_n(t)$ when the ladder is in contact with the wall (τ_N is the torque exerted by the normal force). As with all normal forces, $\tau_N \geq 0$. When $\tau_N < 0$, the ladder will no longer be in contact with the wall. This will occur when $\tau_g < \tau_N$ so we must solve $\tau_g = \tau_N$ for t. $t_l = 1.85550$. Also, we can find the angle and angular speed when it left the wall using previous equations.

After leaving the wall, the only torque acting on the ladder is from gravity. $\tau_a = r \times F = I\alpha_a$. $\alpha_a = -\frac{3g}{10}\cos\theta_a$. We used a computer program to determine t_a and ω_f. We got the following results: $t_a = 0.210828$ and $\omega_f = -1.45139$. We checked our work using the concept of energy.

To solve for x_t, we use the formula for the x position of the bottom of the ladder. Since the length of the ladder is 5 meters, $x_t = t + 3 - 5 = 2.06633 - 2 = 0.006633$. Thus, the answer to the second question is about 0.06633 meters. By basic physics laws, $\theta = \frac{v}{r}$, where r is the distance from the pivot to the point in question. Thus, $v = \omega r = (-1.45139{*}j){*}5 = -7.25695j$. j is a unit vector. In other words, the velocity of the top of the ladder as it hits the floor is about 7.25695 meters per second in the downward direction.

Problem B Summary: Brazoswood High School, Clute TX

Advisor: Deborah Sitka

Team Members: David Creswick, Richard Darst, Neil Strickland

Proceeding from the given situation, we apply calculus to the physics of rotational mown of rigid bodies. Combining these two methods is shown to generate a succinct projection of the ladder's mechanics. Our model is both superior to the typical calculus textbook solution and accurately relevant in modeling a wide range of similar pivot scenarios. This is in part because we initially restricted our calculations to analytical methods. To specifically apply our general solution to this ladder, we use the python programming language to independently calculate values. These numerical results partially verify the accuracy of our analytical model, which projected that:

The critical angle between ladder and ground (at which the ladder comes out of contact with the wall) is 0.24097 radians;

The speed at which the top of the ladder hits the ground is –5.92596 m/s;

The ladder hits the ground separated from the wall by a distance of 0.08926 m.

Problem B Summary: Illinois Mathematics and Science Academy

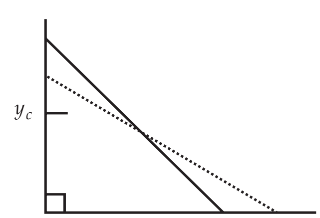
Advisor: Steven Condie

Team Members: Jeff Chang, Alex Garivaltis, David Xu

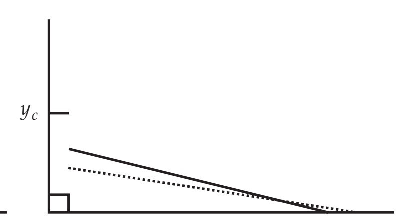
Our objective is to model the behavior of the ladder mathematically. We need to use our model to determine, at the time the ladder hits the ground, its vertical velocity and distance from the wall. By doing a simple experiment, we realize that as the ladder falls, the tip is in continuous contact with the wall until the ladder reaches a certain height, y_c, after which it disengages. (We use a pencil to represent the ladder, a piece of paper as the rug, and a textbook as the wall. The tabletop acts as the ground, and we pull the piece of paper away from the book at a constant rate.) This observation warrants the construction of two separate models.

Our first model describes the ladder's fall while the upper tip's height is greater than y_c. Using related rates, we derive a formula for $y(t)$, the height in meters of the tip of the ladder after t seconds:

$$y(t) = \sqrt{-t^2 - 6t + 16}$$



First Model: ladder remains in contact with the wall.



Second Model: ladder has ended contact with the wall.

Our second model describes the ladder's motion when the height of its tip is less than y_c. Let us refer to the angle made by the ladder and the ground as θ. By manipulating expressions for torque and Newton's Second Law of Rotation, we find angular acceleration coupled with a set of parametric equations representing the movement of the tip of the ladder as it descends:

equation (A) $\theta'' = -\dfrac{3g}{2L}\cos(\theta)$

equation (B) $\begin{cases} x(t) = x_c + t - 5\cos(\theta) \\ y(t) = 5\sin(\theta) \end{cases}$

We define θ_c as the value of θ for which the two models agree, that is, the angle at which the ladder separates from the wall. To calculate this θ_c, we work with an expression for the angular acceleration in terms of θ from each model. Letting the two be equal, the critical angle in this particular scenario is determined to be 0.241 radians.

Our next step is to find θ as a function of time. To do so, we have to solve the second order differential equation (A). We accomplish this with the program *Mathematica*, which incorporates the computed $\theta(t)$ function into the parametric equations seen in equation (B). We now have a sound model of the ladder's motion after its departure from the wall. With these parametric equations we are able to approximate the vertical velocity at which the ladder strikes the ground, –7.24 meters per second, and its final distance away from the wall, 0.066 meters.

Problem B Paper: Centennial High School, Ellicott City MD

Advisor: Carolyn Miller

Team Members: Max Chen, Curran Muhlberger, Henry Pao, Richard Peng

PROBLEM RESTATEMENT

Our model is designed to illustrate the motion of a falling ladder whose base moves from the wall at a constant velocity (Figure 1). The model determines the velocity of the ladder and the distance from the top of the ladder to the wall at the instant before the ladder hits the ground.

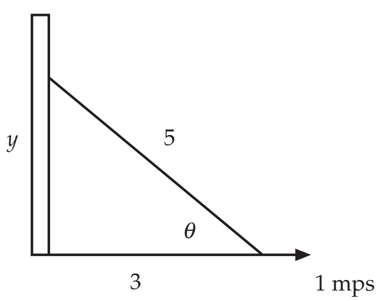


Figure 1.

ASSUMPTIONS AND VARIABLES

Gravity g is 9.8 m/s^2.

The center of gravity cg of the ladder is at its geometric center, halfway along its length and the ladder is considered to have uniform density

Air resistance is negligible.

The length of the ladder L is 5 m and the initial position of its base x_0 is 3 m.

The friction between the end of the ladder and the wall Fr_W and the normal force of the wall N_W act to restrict the angular acceleration provided by gravity to that provided by the geometric model.

The ground is assumed to be parallel to the tangent of the surface of the Earth.

$\theta \equiv$ the angle that the ladder forms with the ground.

$\omega \equiv$ the angular velocity of the ladder.

$\alpha \equiv$ the angular acceleration of the ladder.

MODEL

To model the motion of the falling ladder, we derived equations for angular acceleration, angular velocity, and the base angle of the ladder in terms of its weight, initial position, length, and center of gravity. We found the final distance of the base of the ladder from the wall to be 0.086 m and the instantaneous velocity of the end of the ladder the instant before hitting the ground to be 6.3 m/s. Friction between the wall and the top of the ladder is not a factor in determining the distance and velocity because Fr_W and N_W act to limit the angular acceleration of the ladder resulting from the torque provided by gravity to the angular acceleration provided by the ideal geometric model. Once the ideal angular acceleration of the ladder provided by the geometric model exceeds the maximum angular acceleration that can be provided by gravity alone, the ladder loses contact with the wall and falls as a result of gravity. Our geometric model of the falling ladder is based on the initial position of the ladder, which is analogous to a 3-4-5 right triangle. As the distance between the base of the ladder and the wall grows at a constant rate and the length of the ladder stays static, the distance between the floor and the top of the ladder decreases according to the equation $y = 4 - \sqrt{-t^2 - 6t + 16}$, and the angle between the ladder and the floor changes with time in accordance with the equation

$$\theta(t) = \arccos\left(\frac{X(t)}{L}\right) \tag{1.1}$$

where $X(t)$ is given by $X(t) = x_0 + t \cdot V(t)$ and $V(t)$ is given as 1 m/s.

Because $\omega = \lim\limits_{\Delta t \to 0} \dfrac{\Delta\theta}{\Delta t} = \dfrac{d\theta}{dt}$, the angular velocity of the falling end of the ladder can be determined by

$$\omega(t) = -\frac{V(t) + t\dfrac{dV}{dt}}{L\sqrt{1 - \dfrac{\left(x_0 + tV(t)\right)^2}{L^2}}} \tag{1.2}$$

Furthermore, since $\alpha \equiv \lim\limits_{\Delta t \to 0} \dfrac{\Delta\omega}{\Delta t} = \dfrac{d\omega}{dt}$, the angular acceleration of the falling end of the ladder can be given by

$$\alpha(t) = -\frac{\left(x_0 + tV(t)\right)\left(V(t) + t\dfrac{dV}{dt}\right)^2 + L^2\left(1 - \dfrac{\left(x_0 + tV(t)\right)^2}{L^2}\right)\left(2\dfrac{dV}{dt} + t\dfrac{d^2V}{dt^2}\right)}{L^3\left(1 - \dfrac{\left(x_0 + tV(t)\right)^2}{L^2}\right)^{3/2}} \tag{1.3}$$

By applying the assumptions, variables, and initial conditions already mentioned, the generalized equations 1.1, 1.2, and 1.3 become

$$\theta = \arccos\left(\frac{t+3}{5}\right) \tag{1.4}$$

$$\omega = \frac{-1}{\sqrt{16-6t-t^2}} \tag{1.5}$$

$$\alpha = \frac{(t+3)}{\left(16-6t-t^2\right)^{3/2}} \tag{1.6}$$

respectively.

However, eventually this ideal mathematical acceleration will exceed the value that can be provided by the torque of gravity. This acceleration is given by

$$\alpha = \frac{MgL\cos(\theta)}{d_{cg}I} \tag{1.7}$$

Where $I = \int pr^2 dV$. Assuming that the center of gravity of the ladder is located in the geometric center of the ladder and making use of the values given in the problem setup, equation 1.7 becomes

$$\alpha = \frac{-3(9.8)((t-1.855)+3)}{50}. \tag{1.8}$$

When the angular acceleration provided by the ideal geometric model exceeds the torque provided by gravity, 1.855s will have elapsed from the beginning of the problem (see Figure 2).

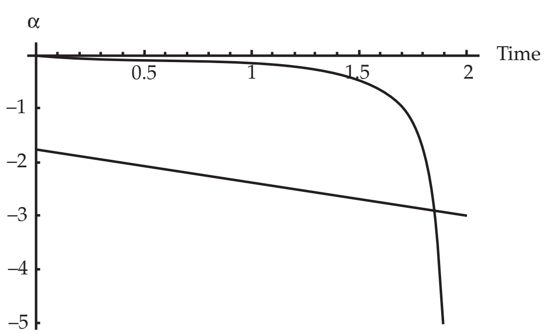


Figure 2.

After that, the end of the ladder will not be falling fast enough to complete the triangle formed by the constantly increasing distance between the base of the ladder and the wall and the fixed length of the ladder. Under these circumstances, the ladder will lose contact with the wall, and the only torque acting on the ladder will be provided by the ladder's weight. As t approaches 1.855s, the friction force of the wall approaches 0, leaving the angular velocity of the ladder to be –0.837 rad/sec (eq. 1.5, see Figure 3) This value provides the constant of integration needed when determining angular velocity from eq. 1.8. The resultant equation is

$$\omega(t) = -0.837 - \frac{-3(9.8)}{50}\left(\frac{(t-1.855)^2}{2} + 3(t-1.855)\right) \tag{1.9}$$

when provided with the initial conditions and assumptions given. The final step is to determine the angular position of the ladder in terms of t. By evaluating θ at t = 1.855, the required constant of integration becomes 0.241 rad (see Figure 4). Thus,

$$\theta(t) = 0.241 - 0.837(t-1.855) - \frac{9(9.8)}{100}(t-1.855)^2 - \frac{9.8}{100}(t-1.855)^3 \tag{1.10}$$

When the ladder has fallen and is on the ground, the base angle $\theta = 0$. According to eq. 1.10, the ladder will hit the ground at t = 2.086 s. Since $v = rw$, the final tangential velocity of the free end of the ladder is –6.3 m/sec and will be perpendicular to the ground. Since the horizontal position of the ladder is given by the equation $X(t) = 3 + t$, it is found that, where the ladder hits the: ground, its base is 5.026 m from the wall, leaving the free end a distance of 0.026 m from the wall.

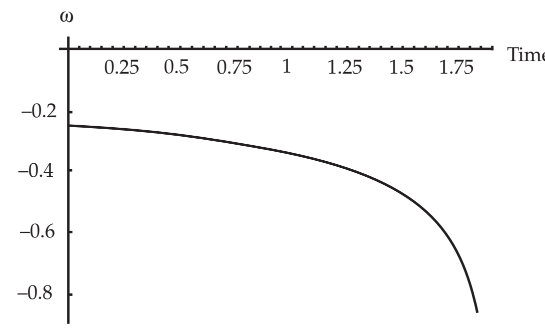


Figure 3.

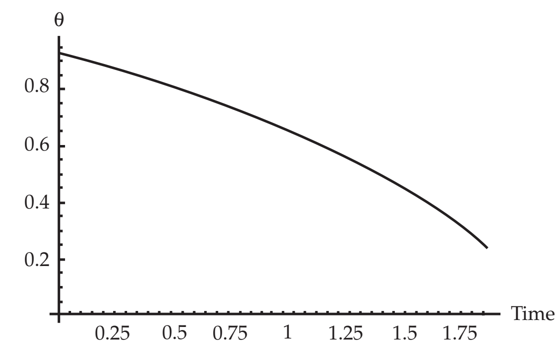


Figure 4.

Using equations 1.1, 1.2, and 1.3, a generalization of our model can be created for conditions beyond those that are given. Suppose the base of the ladder is moving away from the wall with a constant acceleration opposed to constant velocity. Assuming that the ladder begins at rest, $V(t)$ in equation 1.3 can be substituted with at. The solution would be derived, by examining the intersections of the graphs in Figure 5 and proceeding-with the same methods as described above, using the appropriate values for a.

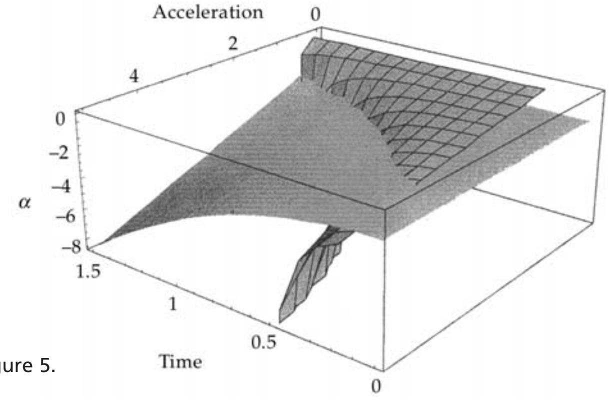


Figure 5.

Figure 5 shows the intersection of the graphs of angular acceleration provided by geometry and angular acceleration provided by the torque of gravity. The intersection of the graphs shows the points at which the ladder leaves the wall. This graph shows that critical angular acceleration increases in small amounts over time despite large changes in acceleration.

ANALYSIS AND CONCLUSIONS

Using our model we found that the distance at which the end of ladder lands from the wall is 0.086 m and that its instantaneous tangential velocity is 6.3 m/s. Our model shows the relationship among the angle the ladder forms with the ground, the angular velocity of the ladder, and the angular acceleration of the ladder at any given time while the ladder is falling. The equations used in our model were based on elementary physics equations and their appropriate derivatives and integrals with respect to time.

STRENGTHS AND WEAKNESSES

Strengths:

The model allows us to solve for the desired results.

The values correspond to real-life observations.

Weaknesses:

Model does not take into account the effects of friction on the vertical motion of the ladder.

Air resistance is neglected.

Model does not take into account properties of chaos that may apply to the situations.

The model does not take into account the differences in gravity resulting from a deviation of the slope of the ground from the tangent plane to the Earth.

SOURCES

Serway, Raymond A. *Physics for Scientists and Engineers, 3rd Ed. Updated Version*. Harcourt Brace College Publishers: Philadelphia, 1992.