HIGH SCHOOL MATHEMATICAL CONTEST IN MODELING OUTSTANDING PAPERS

HiMCM 1999

The contest offers students the opportunity to compete in a team setting using applied mathematics in the solving of real-world problems.

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HiMCM Problem

Major thoroughfares in big cities are usually highly congested. Traffic lights are used to allow cars to cross the highway or to make turns onto side streets. During commuting hours, when the traffic is much heavier than on any cross street, it is desirable to keep traffic flowing as smoothly as possible. Consider a 2 mile stretch of a major thoroughfare with cross streets every city block. Build a mathematical model that satisfies both the commuters on the thoroughfare as well as those on the cross streets trying to enter the thoroughfare as a function of the traffic lights. Assume there is a light at every intersection along your 2 mile stretch.

First, you may assume the city blocks are of constant length. You may then wish to generalize to blocks of variable length.

Editor's Note

It is our pleasure to bring this special HiMCM section of *Consortium* to our readers. We hope that it will encourage participation in HiMCM and be helpful to team members and advisors. Because of space considerations, the four outstanding papers that appear here have been edited to remove redundancies and shorten wording as much as possible. In particular, since summaries tend to repeat analyses found elsewhere in the papers, we have included only one to serve as an example.

Contest Director's Article

William P. Fox

Department of Mathematics
Francis Marion University
Florence, SC
Wfox@fmarion.edu

The High School Mathematical Contest in Modeling (HiMCM) is a positive first derivative of the Mathematical Contest in Modeling (MCM). The efforts of COMAP, the National Science Foundation (NSF), INFORMS, and the HIMCM advisory board culminated in the first HiMCM contest during 1999. "All students should have access to mathematics competitions that provide an atmosphere which generates enthusiasm for learning mathematics which encourages creativity, and rewards excellence," says a report of the NCTM (National Council of Teachers of Mathematics)/MAA (Mathematical Association of America) Task Force on Mathematics Competitions. This statement expresses one of the group's visions to provide a mathematical modeling competition that is fun, creative, challenging, and empowering.

The purpose of the HiMCM is to realize the above vision. COMAP has provided its experience in designing and executing the highly successful MCM to design, train, and (in the near future) smoothly transition a leadership team for administering the high school competition in mathematical modeling. This leadership team should involve NCTM, AMATYC, MAA, SIAM, and INFORMS.

This year's first contest consisted of 30 high schools from among our fifty states. The students accomplished what the vision asked. These student teams submitted highly creative and mathematically correct solutions to an open-ended modeling problem. All student teams and their advisors are congratulated for their efforts. The composition of the teams (89 males and 26 females) shows that this competition is for both boys and girls with an interest in mathematics. Additionally, the style of the submission showed to the judges that the teams had "fun" with the problem and their written solution. The teams were extremely creative. The pilot competition was a success!

There is no place to go but up. Our major objective for the competition for the millennium (2000) is to run three regional contests, then send the best papers forward to a "national/state" committee for further judging. This would mimic what might be done at the local and state levels of whatever hierarchy the eventual governing body feels appropriate. The regional sites planned are: the West Coast, the Middle U.S., and the Southeast. It is planned to have at least fifty high schools from each region to participate in the contest. High schools representing urban, rural, public, private, parochial, magnet, etc. will be asked to participate.

Each site will judge their submissions and rate as "successful participant," "honorable mention," "meritorious," and "outstanding." The best papers from each site will be forwarded to a national judging. Random samples of papers judged at each level will be forwarded to evaluate the consistency of the regional judging. The regional sites will perform several key functions such as training judges, tailoring software used in judging, document and test protocols, providing advisor workshops, and allowing the transition team to "shadow" for their 2001 operation.

These are exciting times for our high school students. Mathematics is more than learning skills and operations. Mathematics is a language that involves our daily lives. Applying the mathematical principles that one learns is the key to future successes. Understanding mathematical assumptions, connections, manipulating data, creativity, and using mathematics to support difficult decisions are all outputs of the modeling competition. Students learn confidence by tackling ill-defined problems and generating their own solutions. The students learn the importance of team building and team effort to solve problems. Applied mathematics is not an individual support—it is a team support.

Advisors need only to be a motivator and a facilitator. Allow your students to be creative and imaginative. Let them practice and use mathematics to be a problem solver. The best problems really have many possible solutions.

As contest director for the millennium contest, let me encourage all high school mathematics faculty to get involved, encourage your students, and open the doors to success. ❑

Illinois Mathematics and Science Academy

Advisor: Ronald Vavrinek

Team Members: Kevin Costello, Andrew Price, Keith Winstein, Terry Koo

ASSUMPTIONS:

1. All vehicles are identical. Their specifications are that they:
   a. Have the same (constant) acceleration, approximately 10.35 ft/s^2.
   b. Have the same length, approximately 185.9 in.
   c. Have the same stopping distance from 60 mph, equal to 144 ft.
2. All side streets have three 12 ft. wide lanes (one for each direction plus a left turn lane). The thoroughfare has five lanes at an intersection (two for each direction plus a left turn lane).
3. The speed limit on the thoroughfare is 60 mph.
4. At any time, a vehicle travels as fast as is safely possible. This entails accelerating if below the speed limit, keeping constant velocity if at the speed limit, and stopping at red lights. In addition, cars follow each other at the closest possible distance as given by the 2-second rule. However, if a driver observes a yellow light and knows that he can pass the intersection before it turns red, then he goes through the intersection.
5. The linear motion of cars on the thoroughfare is independent of whether or not cars are turning off of (or onto) the thoroughfare.

6. When turning, cars decelerate to 15 mph, and execute the turn at 15 mph. If they are making the turn from rest, they accelerate to 15 mph.

EXPLANATION OF ASSUMPTIONS:

1. Real life traffic moves as a stream without distinction of the cars. The values in 1a and 1b are an average of 12 sedans in a *Popular Mechanics* review. Over time, the deviations between cars should approach this average. For stopping distance we chose the largest value—had we picked a smaller one, cars with a greater stopping distance would not stop until they were in the middle of an intersection.
2. This is a standard side street in our area. The 12 ft width is from a study by a contractor about an idealized road.
3. We need a speed limit, and 60 mph seems reasonable.
4. Most people prefer safety over an extra 5–10 mph, especially in busy streets where speed is limited by traffic. Without this assumption, cars could do anything. As far as going the maximum speed, there is no reason for drivers *not* to wish to get to their destinations as quickly as possible.
5. Because the thoroughfare consists of two lanes in each direction and a left turn lane, it will accommodate all actions: the left turn lane allows a left turn without impeding others, and the rightmost of the two normal lanes allows a right turn without impeding cars continuing forward—these cars can use the center lane. Thus, cars turning off the thoroughfare do not affect the straight motion of others. Also, cars turning onto the thoroughfare will not change the motion of those on it. If cars are turning left onto the thoroughfare, cars on the thoroughfare will be stopped at a red light, and if a driver wants to turn right off a side road onto the thoroughfare, he is prudent, and turns only if he will not disrupt oncoming traffic.
6. Assumption 4 says drivers are prudent. In driver's education, we learned that 15 mph is the safe turning speed—thus drivers would not exceed 15 mph in turns.

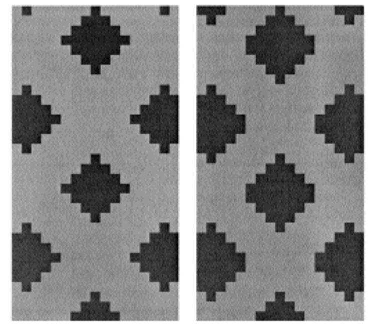
ANALYSIS AND MODEL DESIGN:

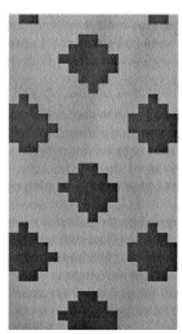
The natural question is, "what does flowing as smoothly as possible" mean? We measure "smoothness" as *rate of flow*, or how many cars travel along the thoroughfare per lane per hour. If this were our only criterion, we could optimize by making the thoroughfare lights always green. However, we must satisfy drivers on side streets.

We pursued two models. One was based on a "wave" pattern, and the other is a genetic algorithm that iteratively approaches a solution based on initial conditions.

If all cars on the thoroughfare are going in one direction, an ideal solution is a "wave" of lights. By this we mean that if it takes a car moving at the speed limit *m* seconds to go from one intersection to the next, light changes are staggered by *m* seconds, and so a car can travel the entire stretch without stopping. What's more, between the green light waves it is possible to have waves of red lights allowing cars turning onto the thoroughfare to ride the next wave. The problem is that creating waves in only one direction creates trouble in the other direction. The question is: "How can we create two sets of green light waves, going in opposite directions?"

A solution is two waves of green lights travelling in opposite directions that overlap each other. This has one major problem: there can be stoplights which either are never red (as viewed by drivers traveling on the thoroughfare; this notation will be used hereafter), or have only a very short red light. In the diagrams at right, the colored blocks represent the status of a light (green or red). Time increases from top to bottom, and the height of each block is approximately 7.5 seconds (the time it takes a car to travel between intersections). There are 17 columns in each picture, representing 17 stoplights in a two-mile stretch. As one looks down a column of blocks, one can see the red-green pattern that that light will take over time. However, in the left diagram, which represents a simple crisscross pattern, one can find stoplights that are only red for one block at a time. This means that the side roads at this particular intersection only have a green light for 7.5 seconds, not enough for any feasible traffic flow. Therefore, we modified the model by adding another row of "blocks" beneath each red diamond, as shown on the right, making the minimum green light for a side road about 15 seconds. However, the thickness of the crisscross diagonal paths of green, which are analogous to the carrying capacity of the thoroughfare, is reduced by 1/7. The alternative is far worse, and increasing the thickness of the green crisscross paths reduces the ratio of loss in carrying capacity. There are, however, graver problems. For instance, varying the thickness of the "waves" of green and red lights can create lights that are always green.





An example of this behavior might be for crisscrossed green paths of thickness 6 and 7, as shown at left. As you can see, there are areas where the light is always green. In fact, in the 17 stoplight example, any time there is a green strip of even-numbered thickness, there will be such lights. These are costly to correct since they require the addition of at least two layers of red. Only when both green paths are of odd thickness is one additional layer required. Thus, we will use only paths of odd thickness.

The next task is to determine the shape of the model as a dependence on different traffic flows. Since we must assume that traffic is relatively uniform, as many cars come onto the thoroughfare as leave it again. However, there may be more traffic in one direction. For instance, the east end of the thoroughfare might lead to a highway to a suburb, while the west end might lead to office buildings. Therefore, most morning traffic will be from west to east as office workers enter the city, and in the opposite direction in the afternoons. Call the ratio of eastward traffic flow vs. westward traffic flow r. We must adjust the ratio of the thickness of the diagonal green bars so that the eastward green wave is r times as thick as the westward wave. Let the thickness of the eastward wave be a and the thickness of the westward wave be b. Because the thickness of the green and red wave relates directly to the period of the green and red cycles, it is important to keep the value of $a + b$, which is directly proportional to the period of a complete cycle, reasonable. A good target is 16, which corresponds to a two-minute cycle. We must also keep both a and b odd, to minimize the loss in efficiency due to corrections. This table relates approximations of a and b to values of r:

r	a	b
>5:1	17	0
5:1	15	3
4:1	13	3
3:1	13	5
2:1	11	5
3:2	11	7
1:1	7	7
2:3, etc.	read b in reverse	read a in reverse

We can use these values for a and b to deal with any ratio of traffic flow, and can plot a time/stoplight status graph. After correcting, we can then read down each column to determine the most efficient sequence of red and green lights.

Of course, we must compensate for a few things. First, the time it takes cars to move between blocks is not constant. Assuming blocks are each 1/8 mile (660 ft) and travel 60 mph = 88 ft/s, a car takes 7.5 seconds to travel 1 block. However, if it is accelerating from 0 to 60, it must conform to the motion equation,

$$x = \frac{1}{2}at^2 + v_0 t$$

which is $x = 5.175t^2$ in this case. It must do this for 8.6 seconds, during which time it goes 382.7 ft., a distance traveled in 4.3 seconds at 88 ft/s. This is a loss of 4.3 seconds, so a moving car will finish 4.3 seconds ahead of one that started from stop on any block longer than 382.7 ft. In our situation this occurs only at the beginning. We can compensate by having the first time stagger between lights as 11.8 seconds and ending the first green light in a cycle (on both ends) 4.3 seconds early.

We need to account for yellow lights and turns. If a car is traveling 60 mph, we do not want the driver to choose between stopping in the middle of the intersection or accelerating through it. Stopping distance is 144 ft, and a car closer than 144 ft to the intersection should glide through at constant velocity. In the worst case, the car travels 144 ft to the intersection plus 3 lanes at 12 ft/lane, or 180 ft. Since the car travels 88 ft/s, the yellow light must be 180/88 = 2.05 plus reaction time of about 0.75 s, so we make the yellow light 3 s. This will be taken out of the end of the green light.

Right turns can be done at any time, but left turns only while no other cars are passing through the intersection. So we include 2 short left turn times in our model, one after the green light and one after the red. The longest a car must turn through is 6 lanes (72 ft), and it accelerates from rest to 15 mph = 22 ft/sec. At 10.35 ft/s^2, this can be done in 2.13 seconds, during which time the car goes 10.87 ft. The car then moves 61 ft at 22 ft/s in 4.07 s, for a total of 6.20 s. Add 0.75 reaction time and it takes 6.95 s from light to turned. Since the traffic on the side street is light, we assume only 1 car wishes to turn in each direction during any cycle and make the turn light 7.5 s.

RESULTS AND TESTING

To get a numerical value we need to do calculations involving density. The 2-second rule says that a car in motion should stay at least 2 seconds behind the car in front. Thus, it should be at least 2s*88ft/s = 176 feet, which we add to the length of a car to get a car every 191.5 ft, or every 2.18 seconds. However, red lights create zones of no cars. In general, there are $a - 3$ "good" 7.5 second zones (a zones originally, $a - 1$ once we draw in extra red zones, 2 more for the turn lanes), out of $a + b$ total, so traffic on the left is 1car/2.18s*$(a - 1)/(a + b)$*3600s/1hr = 1651*$(a - 3)/(a + b)$ capacity. The other direction is calculated similarly. Note that $a = 17$, $b = 0$ does not seem to give any waves to one direction, which is acceptable since traffic in that direction is small and can go against the wave, though it may stop a few times.

We attempted to test the model with a genetic algorithm simulation, but did not complete it due to time constraints. Preliminary results indicate confirmation.

STRENGTHS AND WEAKNESSES

Strengths

1. The greatest strength is that the model can be integrated with other streets in the city since it does not depend on the two-mile length.
2. Once started, a car will not stop until it reaches the end of the stretch. There are a few paths that enter green and stop at a red light, but this will not happen when the stretch is integrated with other roads because a vehicle entering the stretch will do so while in a good wave of cars.
3. Our model is easily adjustable. If the ratio of cars changes, we need only change a and b to calculate new light times. If the speed limit changes, only the numbers would change. The yellow light might need to be longer, and the length of each stretch would change and the time delay between them. As long as the cross streets are symmetric, the model can be applied to other patterns, but instead of 7.5 s, we use whatever time it takes to go between intersections.
4. No one waits more than 2 minutes to turn onto the thoroughfare (in the extreme case where a:b > 5). In some cases (7:7 for example) the time is 1 minute.
5. The model helps enforce speed limits. A car exceeding the limit gets to a light before it turns green. Once in front of a wave, the fastest speed *is* the speed limit.

Weaknesses

1. Many cars do not have much time to turn or drive. At 7.5 s, a driver who doesn't notice the light may have to wait for it to change again.
2. At maximum capacity, traffic jams are likely. However, we think jams would make some drivers avoid the thoroughfare, bringing it back to "good" levels.
3. A car traveling a bit slower than we assume may be caught in a red light if it is in the back of wave.

REFERENCES

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Oldham, Scott. "Best Kept Secrets." http://popularmechanics.com/popmech/auto2/9902AUCTP.html February 1999.

Chesterfield County Math & Science H.S.

Advisor: Diane Leighty

Team members: Nicolas Creehan, Michael Bennett, Justin Morgan, Beth Reid

BACKGROUND:

We counted the blocks on a given stretch for several cities. The result was 27 and ranged from 20 to 40.

ASSUMPTIONS:

We defined each intersection as shown in **Figure 1**. The thoroughfare is east-west. We assume that each side street is two-way and has the same traffic lane pattern. There are sensors at each left turn lane off the thoroughfare and at all lanes of a side street. They detect a car waiting for a green light.

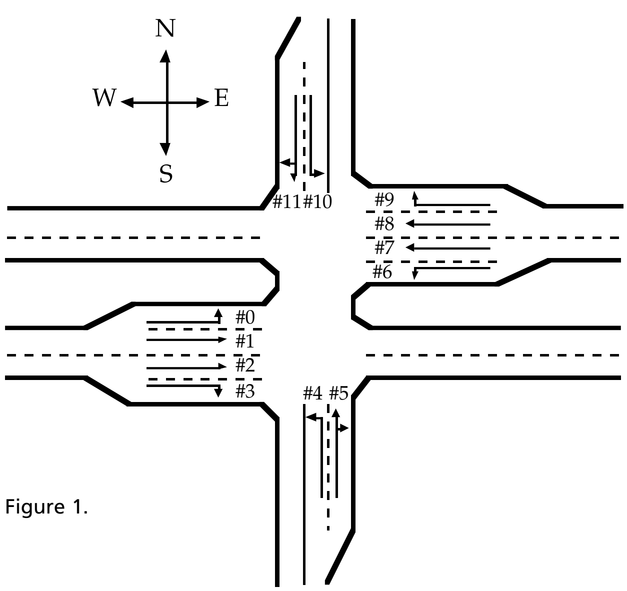


Figure 1.

We decided that right turns on red were illegal.

Our city is a "constant density" city. Therefore, a larger stretch between city blocks would increase traffic flow to the two bounding side streets. We have defined the average rate of cars entering the street as $(x_W + x_E)r_{N\text{-}S}/(2x_{AVG})$, where x_W is the distance to the west of the side-street, x_E is the distance to the east, $r_{N\text{-}S}$ is the rate of cars entering a standard length block, and x_{AVG} is the average block length. Therefore, the rate of cars entering a block is proportional to the length space feeding the block.

With this model, the average rate at which cars come to the side streets is inversely proportional to the number of streets on the stretch.

RESTATEMENT OF THE PROBLEM:

The logistics of traffic flow are complex, so we made several simplifying generalizations.

Our final goal was to determine the best traffic signal cycle. We assumed that the minimum delay before a light becomes green and the length of a non-thoroughfare green light are constant. It was our goal to determine a function for traffic light times in terms of flow rates on the thoroughfare and side streets.

CONJECTURES:

Because commuters often come from the outside the city, the flow in the morning would have more cars turning off the thoroughfare onto side streets. Afternoon would have the opposite. Since the length of each non-thoroughfare green light is constant, the optimum light lengths and delays can be determined by setting the number of cars turning onto the thoroughfare equal to the number of cars leaving the thoroughfare. Therefore neither morning nor afternoon has an advantage for passing cars through the system.

VARIABLES:

We have two main input parameters from which we derive everything else.

- $r_{N\text{-}S}$—the average number of cars that arrive on a side street (north-south) per second for a side street of average length (2 miles/number of side streets)
- $r_{E\text{-}W}$—the average number of cars that enter any intersection of the thoroughfare from either the east or west.

From these two variables we derived a third variable, P, the probability that a car approaching an intersection will leave the thoroughfare. This includes turning off the main road and crossing over it without actually travelling on it. P is useful in making the connection between $r_{E\text{-}W}$ and $r_{N\text{-}S}$.

HYPOTHESES:

We could not find data on traffic flows and light time cycles, so we hypothesized an average length of one light cycle to be about 70 seconds.

EQUATIONS GOVERNING P, AND THEIR DERIVATIONS:

It is possible to derive equations for the rate of cars making each of the three choices, from all four compass directions. For instance, if P is the probability to turn away from the thoroughfare, the rate of cars per second that want to turn away from the east or west is $P(r_{E\text{-}W})$, and the rate of cars that want to be in, say, lane 0, is $P(r_{E\text{-}W})/2$. The total rate of cars that are leaving the intersection going south is the rate turning right from west plus the rate turning left from east, or, $P(r_{E\text{-}W})/2 + P(r_{E\text{-}W})/2 = P(r_{E\text{-}W})$. Thus, $P(r_{E\text{-}W})$ is the amount of cars that have to come from the south onto the thoroughfare to make up for those cars lost, which can be split into half to accommodate east and left-bound cars. This is all mirrored by the southbound street, and so $P(r_{E\text{-}W})/2$ is the rate of cars in every turning lane in every direction.

For cars not turning, the rate approaching from the east or west minus the rate turning away is the total rate: $(r_{E\text{-}W}) - P(r_{E\text{-}W}) = (1 - P)*(r_{E\text{-}W})$.

The remaining possibility is cars crossing thoroughfare. P cars coming at the intersection do not leave via the thoroughfare, so the rate A of cars going straight is $P(A + P(r_{E\text{-}W})) = A$, which can be solved for $A = P^2(r_{E\text{-}W})/(1 - P)$.

GREEN LIGHT COMBINATIONS

To decide the most efficient way to cycle through the different combinations of green lights, we came up with all of the list in Appendix B. We chose the necessary ones and found the best circular order: 1, 5, 2, 4, 3, 6, and back to 1 again. In an average cycle, all of these states would be present for an amount of time proportional to the average number of cars that need to use each combination. We found the number of cars for each combo, which is based on P and $r_{E\text{-}W}$. For each combination, the lane with the most cars to pass through is the lane that limits its brevity. By finding the lane with the greatest relative amount of cars for each combination, we hoped to minimize the time that no traffic is moving and maximize the time that traffic is flowing.

We used two approaches to this problem. First, we assumed that all traffic stops when changing from one green light combination to another and improved our model to let traffic continue to flow when it does not need to stop. For instance, between green light combinations 1 and 2, lane 1 does not need to come to a halt. In both cases, we divided this greatest flow rate by the total flow rate, which is the sum of the six greatest flow rates. By repeating this for the six different green light combinations, we arrived at expressions that, when evaluated, returned the fraction of a full light cycle that the corresponding green light combination would be lit. For example, in the first and less efficient model, the total flow was $-1/(P - 1)$, and our limiting factor expression for the first green light combination is $(1 - P)$. Therefore, on average, the fraction of a light changing cycle that green light combination 1 is active is estimated as $(P - 1)^2$. All of these results are in Appendix B.

The graph in **Figure 2** gives a visual representation of how the overlapping data is better. The more efficient model is darker. The lines represent the six green light configurations, of which there are three unique expressions. They are a plot of the probability that each configuration is used in a cycle against the greater likelihood that more people would want to turn onto side roads. It is a graph of time fraction to P. The bold lines intersect sooner and are closer to each other than the fine lines are, an indication of better total efficiency.

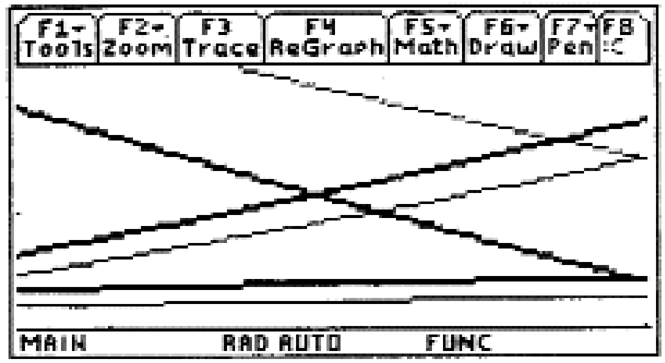


Figure 2. [0.38, 0.50] x [0.10, 0.33]

INTERESTING RELATIONSHIPS BETWEEN VARIABLES

We created an equation containing all three variables P, $r_{E\text{-}W}$, and $r_{N\text{-}S}$. First we added together the expressions that represent the output of lanes 4 and 5; this sum is equal to $r_{N\text{-}S}$. The equation is: $(r_{E\text{-}W}P) + (r_{E\text{-}W}P^2)/(1 - P) = r_{N\text{-}S}$. This equation makes more sense (and looks prettier) when solved for P: $P = r_{N\text{-}S}/(r_{N\text{-}S} + r_{E\text{-}W})$. It is then possible to substitute the right side of this equation into the ratio equations in Appendix B to get applicable data. The graph in **Figure 3**, shows how $r_{S\text{-}N}$ is in indirect proportion to $r_{E\text{-}W}$, when P is constant.

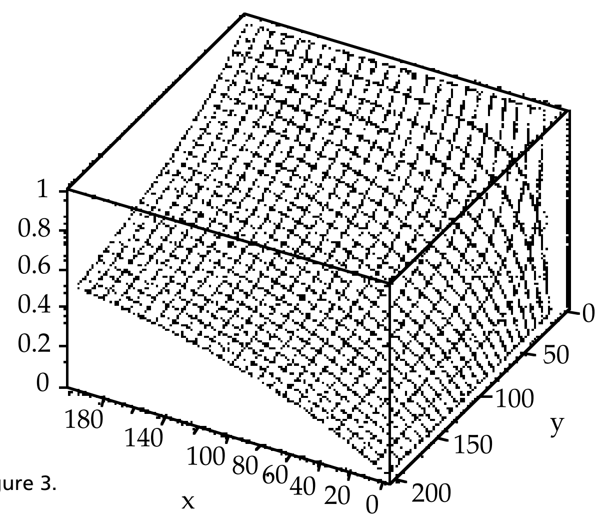


Figure 3.

DETERMINING THE OPTIMUM TIME PER LIGHT CYCLE

Several factors impact the problem. First, synchronization of lights must be avoided. This is because if cars get backed up at one light and the next light simultaneously turns green, there will not be many cars to pass through the second green light. Synchronization is difficult to control because the length of a light cycle can vary depending on whether cars are on the sensor when the cycle is allowed to move on to a non-straight-thoroughfare. In a city with differing block lengths, the probabilities of cars coming from side streets vary and change the optimum time per cycle for that particular signal. However, in a 2-mile-stretch with essentially equal probabilities, synchronization can be avoided by slightly varying the optimum time per light cycle.

To maximize traffic flow, one must find the length of the cycle at which the losses in traffic flow due to frequent delays (due to changing positions of a green light) is balanced by the increased flow rate of traffic when the light cycle is shorter. Because light cycles are dependent on the flows produced by the cycle of the previous signal, the best way to attack the problem is with a simulation.

We produced a simulation based on the calculated probabilities described above for a given set of flow rates. It calculates the state of the thoroughfare for each second of an hour-long period. It randomly generates cars at each entrance to the thoroughfare based on the given flow rates. It kept track of how much time each car spent in the system and how far it traveled, and reported these values when it left the system. Efficiency was measured by the combined average speed of all the cars exiting the system. By changing the time cycles, we hoped to find an optimal cycle length. Because the program handled all combinations of green light cycles, we could verify the theoretical calculations.

TESTING THE MODEL:

The easiest way to verify our model is to choose a major thoroughfare and take measurements. This would also give us a better concept of the length of an entire light cycle, which presented a problem for us in our modeling.

STRENGTH AND WEAKNESSES:

Because we attempted to include so many details in our computer model, in the given time we did not completely debug it. Numerical values for the length of time cycles were not produced. Because we never found data measured from actual operating traffic lights, we cannot be certain of the accuracy of our model.

The computer program would have been a versatile tool for thoroughfares with varying block lengths. All the factors included in the computer model, such as flow rates, turning probabilities, and light cycles, are easily adjustable. We were also thorough in our approximations by using kinematic equations to determine the rate at which cars pass through the intersection. Although our model seemed complicated, we made assumptions and generalizations so it can be applied to any major thoroughfare. We provided for a range in the number of blocks and adjusted for a “constant density” city. We also used the constant density assumption to account for variable length blocks. Most importantly, we allowed the flow rates to be parameters for our model of traffic light cycles.

APPENDIX A

Assumptions used in calculations:

- Cars are 15 feet long and have 5 feet between them when stopped and 15 feet while moving.
- There is a 2 second delay between the time a light turns red and the next green light for another lane.
- The speed limit on the thoroughfare is 35 mph.
- The average acceleration of a vehicle is 10.3 ft/s^2, meaning a car takes 5 seconds to reach 35 mph from rest.
- A driver has a 0.5 second reaction time before beginning acceleration after the car in front moves. This assumption allows for the following distance to increase from 5 to 15 ft as the cars begin to move.
- Each intersection is approximately 40 ft long for cars going through the thoroughfare and 80 ft for cars crossing it.
- Kinematic equations were used to verify that one car passes through the intersection every second.

Justification:

We modeled the maximum number of cars to go through a green light of t seconds with the equation:

$D = \frac{1}{2}at^2$ (kinematic equation assuming constant acceleration)

$$(20 \text{ ft})*n_{max} = \tfrac{1}{2}*(10.3 \text{ ft/s}^2)*(t - (0.5(n_{max} - 1)))^2$$

This equation is based on an acceleration of 0 to 35 mph in 5 seconds and a 0.5 second reaction time.

The average intersection is 328 ft and could therefore hold about 16 cars. The results of the above equation can be approximated by having a car pass through the intersection every second ($n_{max} = t$).

Furthermore, the equation ($n_{max} = t$) models incoming cars with an average velocity of 20.5 mph. This is reasonable because cars ahead of the already moving cars have not yet accelerated to the max speed of 35 mph.

Therefore we have simplified this model to the equation $n_{max} = t$ where n is the number of cars and t is the time passed in seconds of a green light crossing any intersection.

APPENDIX B

A list of all of the green light combinations.

Combination 1: Lights are green for lanes 1, 2, 3, 7, 8, 9

Combination 2: 3, 4, 5

Combination 3: 9, 10, 11

Combination 4: 5, 11

Combination 5: 0, 1, 2, 3, and 11 right turn only

Combination 6: 6, 7, 8, 9, and 5 right turn only

Combination 7: 0, 6, 11 right turn only, and 5 right turn only

With no overlap, the total limiting rate was $(-1/(P-1))$

Combination 1: $(1-P)/(-1/(P-1)) = (P-1)^2$

Combination 2: $(P/2)/\ (-1/(P-1)) = -P(P-1)/2$

Combination 3: $(P/2)/\ (-1/(P-1)) = -P(P-1)/2$

Combination 4: $(P^2/(1-P))/(-1/(P-1)) = P^2$

Combination 5: $(P/2)/\ (-1/(P-1)) = -P(P-1)/2$

Combination 6: $(P/2)/\ (-1/(P-1)) = -P(P-1)/2$

Combination 7: not used—not necessary

With overlap, the total limiting rate was $(-1/(P-1)) - (P/2)$

Combination 1: $(1-3P/2)/((-1/(P-1)) - (P/2)) = (P-1)(3P-2)/(P^2-P+2)$

Combination 2: $(P/2)/((-1/(P-1)) - (P/2)) = -P(P-1)/(P^2-P+2)$

Combination 3: $(P/2)/((-1/(P-1)) - (P/2)) = -P(P-1)/(P^2-P+2)$

Combination 4: $(P^2/(1-P))/((-1/(P-1)) - (P/2)) = 2P^2/(P^2-P+2)$

Combination 5: $(P/2)/((-1/(P-1)) - (P/2)) = -P(P-1)/(P^2-P+2)$

Combination 6: $(P/2)/((-1/(P-1) - (P/2)) = -P(P-1)/(P^2-P+2)$

Combination 7: not used—not necessary

Westminster Schools

Advisor: Landy Godbold

Team members: Jana Dopson, Michael Miller, Jordan Thomas, Conor Tochilin

SUMMARY

Our first step was to simplify the problem to one traffic light. We decided the best model would be one that minimized the time drivers on both the thoroughfare and cross streets would wait on a red light. We calculated the percentage of red light time for thoroughfare drivers that minimized the time drivers waited for the light. The equation was $R = V/(H + V)$, where R is red light percentage, V is vertical traffic, and H is horizontal traffic. Then we used formulas for average time waiting on light to calculate the minimum number of cycles to discharge all traffic that came into an intersection during a cycle:
$T = HV/(2H + 2V)$, where T is the number of cycles.

Next we found a formula for optimum length of a light cycle. We used a submodel that used acceleration and velocity data to model the flow traffic after a light changed from red to green. The formula was $N = 0.625G$, where N is the number of cars that get through an intersection from a stop, and G is the length of the green light in seconds.

We incorporated this equation with the equations for R and T to an equation for optimum length of a light cycle:
$F = (V/0.625 + H/0.625)/[HV/(2H + 2V)]$.

We were able to extrapolate this model to a system with multiple traffic lights. By using the relationship between number of cars and optimum T, we showed that the number of cars waiting at the second light is $2(H + V)/H$. The flow between lights is optimized when the first light turns green as the last person in line at the second light begins to move. Because our model allows a delay time of 1 second between cars, the first light should be $2(V + H)/H$ second behind the second. To account for two-way traffic, half the lights should be staggered from left to right in this manner and the other half from right to left. In addition, if block length in car lengths is less than or equal to the number of cars on that block, lights would have to be adjusted accordingly. Variations on block length would mean that light patterns should be dictated by shortest block length.

AN ENLIGHTENED LOOK AT TRAFFIC

We develop several mathematical models, expressing the length of a light cycle, the total delay time, and the number of cars per block of highway in terms of the number of cars per light cycle arriving in both the horizontal and vertical directions.

ASSUMPTIONS

1. the junction is never blocked in any way
2. all vehicles are 5 m in length, with identical average velocities and initial velocities
3. there is a 2 m gap between stationary cars
4. it takes one second for a car to begin moving after the car in front of it starts moving
5. turning cars have a negligible impact on traffic flow
6. stopping time has a negligible impact on traffic flow

A SIMPLIFIED MODEL

Our simplification involves one traffic light and blocks one mile in length. Our goal is to minimize delay at lights. This would maximize the amount of time that cars are going at peak speed. Because this problem is a function of traffic lights, we decided to minimize the number of traffic light cycles required to empty the interchange of all vehicles stopped during a given cycle.[1] We defined the following variables:

T: total delay time (in cycles) of all cars arriving in a given cycle

H: number of cars/cycle arriving in the horizontal directions

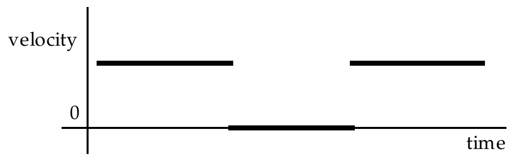
[1] This was a similar method as that used by Packel in *Discrete and System Models*, Vol. 3, Chapter 5 New York: Springer-Verlag, 1976.

V: number of cars/cycle arriving in the vertical directions

R: fraction of cycle in which light is red in the horizontal direction

1-R: fraction of cycle in which light is red in vertical direction

We assume that slowdown and speedup times are negligible. A velocity time graph for a car driving up to a light, stopping, and starting again might look like this:



To minimize the total time cars are stopped we calculate the total stopping time. The number of cars stopped in the horizontal direction is HR, and the average wait is R/2 cycles. For the vertical direction, these quantities are V(1 –R) and (1 – R)/2, respectively. The total number of cycles of the delay for all cars is:

$T = HR*R/2 + V(1 - R)(1 - R)/2 = HR^2/2 + V(1 - R)^2/2$

T is a quadratic function of R and is minimized when $R = V/(H + V)$.

Plugging this into the original equation gives $T_{min} = HV/(2H + 2V)$

Now we have expressions for optimum red light percentage and optimum number of traffic cycles to clear interchange in terms of H and V. H and V are in cars/cycle, which is awkward since we want to find the best cycle length, for which something like cars/sec is better. In the formula for T_{min}, units don't cancel, making the value some scaling factor times the optimum number of cycles per second. This makes T_{min} alone a meaningless value, but if it can be applied in such a way that the units cancel, it can still help us.

We next found a formula for the optimum cycle based on H and V. The optimum cycle length is the time it takes to get all cars through the interchange divided by the optimum number of cycles, which is T_{min}. To figure out how long it would take all cars stopped during a cycle to get through the intersection, we set up a submodel[2]. We need to include acceleration time in the submodel because it is more significant. Our assumptions:

a. the junction is not blocked in any way

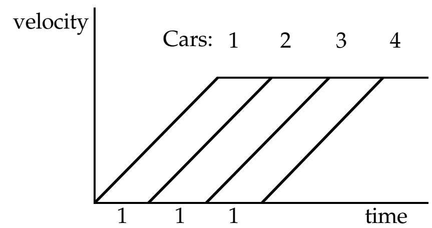
b. all vehicles 5 m in length, and initially at rest

c. there is a 2 m gap between stationary cars

A car accelerates uniformly until the speed limit (30 mph, about 15 meters/s) is reached. It achieves 0 to 60 mph in 10 seconds, which gives an acceleration of 2.68 m/s^2. We round down to 2.0 m/s^2 since drivers don't achieve max acceleration in interchanges. It would take 7.5 s for a driver to reach max speed of 15 m/s. A velocity graph for the first car is on the right.

[2] Edwards, Dilwyn, Mike Hamson, editors. *Guide to Mathematical Modeling*, Example 2.25 Boca Raton: CRC Press, 1989.

Velocity graphs for several cars look like this:



Distance traveled is the area under the graph. Therefore, after 4 seconds, we have:

car 1 has gone 16 m

car 2 has gone 9 m

car 3 has gone 4 m

car 4 has gone 1 m

After 7.5 seconds, some cars have reached a maximum speed of 15 m/s and the formula is:

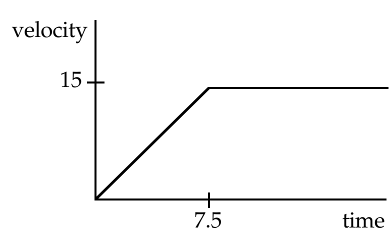
$d = T^2$ if $T < 7.5$

$d = 7.5^2 + 15(T - 7.5)$ if $T > 7.5$

where d = distance traveled and T = number of seconds after light change.

Based on assumptions about car length and distance between cars, we made a chart of time since light for each car and that car's position in relation to the beginning of the interchange. A sample chart appears below:

Car	Time in Motion	Net Position (= d – initial position)
1	15	168.75
2	14	146.75
3	13	124.75
4	12	102.75
5	11	80.75
6	10	58.75
7	9	36.75
8	8	14.75
9	7	–7
10	6	–27



For a 15 second green light, 9 cars go through because 8 have passed it and number 9 is too close to stop. We calculated how many cars go through for various light lengths:

Green Light Length (sec)	Number of Cars Through
0	0
1	1
2	1
3	2
4	2
5	3
6	3
7	4
8	4
9	5
10	6
11	6
12	7
13	8
14	8
15	9
16	10
17	10
18	11
19	12
20	12
21	13

These data are very linear, and least-squares regression gives N = 0.625G, where G = length of green light (sec) and N = number of cars that get through interchange.

We return to our original question: finding the time that it takes all cars to regain speed (i.e. make it through the interchange). In the horizontal distance the number of stopped cars is RH, so the following equations can be generated:

$$RH = 0.625G$$

$$RH = 0.625(1 - R)t_h$$

$$V(1 - R) = 0.625Rt_v$$

$$t = t_h + t_v$$

$$t = RH/[0.625(1 - R)] + V(1 - R)/(0.625R)$$

where G = length of green light in horizontal direction, t = time to clear all cars (sec), t_h = time for horizontal cars to clear, and t_v = time for vertical cars to clear.

This quantity divided by the optimum number of cycles [HV/(2H + 2V)] is the optimum cycle length. Since we want our answer in terms of H and V, we substituted R = V/(H+V) and got F = (V/0.625 + H/0.625)/[HV/(2H + 2V)], where F = time for full cycle (sec).

The units in this equation cancel. It is a ratio between V and H that matters, not the actual values. We chose to use cars/sec arriving at the interchange. Values for R and F can be obtained by plugging in values of V:

H	V	R	F
1	0.1	0.091	38.72
1	0.2	0.167	23.04
1	0.3	0.231	18.03
0.8	0.2	0.2	20
0.8	0.3	0.273	16.13
1.5	0.3	0.167	23.04
1.5	0.4	0.211	19.25
2	0.2	0.091	38.72
2	0.3	0.130	28.21

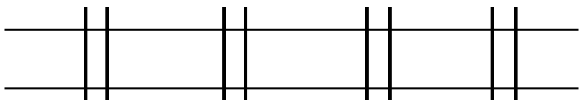
For realistic data, the percentage of time the light should be red for the horizontal drivers is between 0.1 and 0.27, averaging about 0.17. The total length of a light cycle should be between 20 and 40 seconds, averaging about 25 seconds for maximum efficiency (i.e. minimum time spent at light).

MULTIPLE TRAFFIC LIGHTS: A MORE ACCURATE MODEL

We assume that block lengths are constant and the number of cars entering from each side street is the same. We add the variable L for block length in car lengths.

For the most part, we treat each light as though it is the only traffic light on the road. We begin with the first light and use the same timing patterns derived earlier because they minimized the delay per car at the traffic light: $T_{optimum} = \frac{HV}{2(H+V)}$ and $R_{optimum} = \frac{V}{H+V}$. Our next goal is to figure out the pattern of traffic flow going into and out of light number 2. We assume the number of cars going straight on the crossroads is negligible. Therefore, we know that the number of cars that go through the first light is equal to the total number entering the first intersection per cycle divided by the average delay time in cycles: $\frac{H_1+V}{T} = H_2$ where H_1 = incoming horizontal traffic (per cycle) at light 1 and H_2 = incoming horizontal traffic (per cycle) at light 2. We are saying that all traffic entering the first light will enter the second light in the horizontal direction, at a rate proportional to the delay time. We can substitute $T_{optimum}$ for T in this equation, which simplifies to $\frac{2(H+V)^2}{HV}$. Then we determine that every car coming to the second intersection while it is red will have to stop. How many cars is this?

All we have to do is multiply the total incoming horizontal traffic by the optimal R and we find that the number of cars that stop at the second intersection is $\frac{2(H+V)}{H}$.



Green after: $\frac{3*2(\mathrm{H+V})}{\mathrm{H\ sec}}$ $\frac{2*2(\mathrm{H+V})}{\mathrm{H\ sec}}$ $\frac{2(\mathrm{H+V})}{\mathrm{H\ sec}}$ 0 sec

Figure. A

We now must find the best timing method that activates the lights sequentially so the most traffic gets through. We found that the best way to maximize traffic flow is to assume that all of the blocks have 2(H + V)/H cars and all signals are red. For the most cars to move, the last light on the road must be the first light to turn green. After this light has turned green, all the cars in that block must move before any of the cars in the block behind it move. Therefore, the most effective way to use the green light is to turn the next to last light on when the last car in the block ahead of it begins moving. To determine this time between lights, we multiply the 1 second it takes each car to start up after the car in front of it by the number of cars that must start up. Thus, it will take $\frac{2(H+V)}{H}$ seconds for all the cars that have stopped at the nth light to start moving and that is when the n – 1 light should turn green. This pattern continues down the row and because each cycle is the same length, the lights will turn red and then green again in the same intervals (Figure A). As a final note, because we are dealing with two-way traffic, a beneficial traffic light in one direction will be a hindrance in the other. Therefore, for the best possible traffic flow in both directions, lights should be staggered so that every other light obeys our model for that particular direction.

BLOCK LENGTH

The block length, L, must be greater than the number of cars that back up in that block when the light is red. In other words, when $\frac{2(H+V)}{H} \leq L$ we can decrease the light cycle time and lower the number of cars to an amount that does not cause a backup. This makes the traffic move more slowly, but it is more efficient overall because the green lights are used effectively. We can further generalize that with varying block lengths, the cycle length is a function of the minimum block length. In other words, the cycle time would still be the same for all of the traffic lights on the road, but the times would be based on the maximum time for the smallest block.

Here are some sample values that make sense in our scenario:

H	V	number of cars	Time delay
1	0.1	2.2	2.2
1	0.2	2.4	2.4
1	0.3	2.6	2.6
0.8	0.2	2.5	2.5
0.8	0.3	2.75	2.75
1.5	0.3	2.4	2.4
1.5	0.4	2.5	2.5
2	0.2	2.2	2.2
2	0.3	2.3	2.3

We can see that the number of cars waiting at the second light is small, about 2.5 per cycle of the first light, if the second light follows the same optimum patterns. This small number of cars should not, in most situations, be more than a stretch of road can contain.

CONSTRAINTS

We have assumed that slowdown and speedup times are negligible. This is a partial assumption because we included speedup times in our submodel. This means our model will be most accurate when traffic is not moving very fast because it will take less time to speed up and slow down. Otherwise, our model would be more efficient than reality because our cars are moving more effectively than those in real life do. We have also assumed that the most efficient way to move cars down the road is by using the same length cycle of red and green for each light. We believe this assumption is justified because if the lights did not have the same cycle length, the sequence would deteriorate and the lights would be going off and on at nearly unpredictable times. We also assumed that turning has a negligible impact on traffic flow. In real world applications it would have an effect however, in turning left from the main road onto a side street. Because of the nature of our scenario, a car is more likely to continue straight at an intersection than turn onto a cross street. Also, we have made assumptions on the nature of vehicles and their following distance. Because we are dealing with averages, these are safe assumptions, but in real life, varying car lengths and spaces between cars would affect traffic. A final assumption was to neglect the yellow light, which is a reasonable extension of green.

CONCLUSIONS AND APPLICATIONS

We have figured out how to time arbitrarily placed traffic lights on a stretch of road depending on the traffic entering from different directions. By modeling the different functions of traffic lights using mathematical equations we found formulas to portray the traffic flow. This allowed us to derive formulas to decide how long and when each traffic light should change. A traffic planner could use our equations to find the ideal light timing knowing only the traffic densities in the vertical and horizontal directions. The traffic planner could then test our model and take real-world samples of the data we arrived at mathematically. This would tell us if our model accurately portrays traffic or if we must further revise it. In addition, we included some information in our model, such as speed limit, which could be beneficial. If flow patterns can be derived from traffic densities and those densities can be derived from speed limits, planners could examine the affects of speed limit.

One of our model's strengths is that it works well with heavy traffic, because of our acceleration assumptions. A second strength is that it is built upon the empirical data of our submodel. Therefore, we know that at least one substantial portion of our model is followed in actual driving. A third strength of our model is that it gives easy ways to maximize that flow based only on horizontal and vertical traffic.

The first weakness of our model is the large number of assumptions. Fortunately, these assumptions are reasonable and some of them, such as those in the submodel, can be proven by data (see *Guide to Mathematical Modeling*). Another weakness is that our model works best for one-way traffic. Two-way traffic causes conflict between light timing when the model is based upon total traffic flow.

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Francis W. Parker School

Advisor: John Stadler

Team members: Oren Cass, Ani Ravi, Jon Tang, Sean Ting

INTRODUCTION

We have given ourselves an additional challenge of guaranteeing that a car can go from the beginning of our road to the end in 10 minutes.

VARIABLES AND ASSUMPTIONS

The Roads

1. This is a six-lane road, with a barrier separating the three lanes in each direction. Breaks in the barrier occur at intersections.

Reason for assumption: City streets often have three lanes in each direction. Three lanes allow for a "left turn lane," "right turn lane," and "straight lane." With fewer lanes, merging on and off the road is difficult, and their advantage over other streets is minimized. Also, with less than three lanes, the volume of cars is reduced.

More than three lanes are not necessarily practical (the same problem as with adding a second lane to a cross street). Similarly, a one-way street is easier to model but not as realistic or as interesting.

2. Cross streets are one-lane streets, except for every tenth one, which is a thoroughfare like the one that we are modeling.

Reason for assumption: Cross streets are often one-lane. However, it makes sense to place a major road after every nine small roads (rounded to ten from the New York City model[1] of "every eleventh road").

3. All cross streets are one-way.

Reason for assumption: Many cities use one-way streets because left turns are easier and, given the grid like structure of city streets, drivers can reach their destination without much inconvenience. One-way streets increase our model's efficiency and satisfy commuters on the thoroughfare and those on cross streets.

4. A city block is 300ft long.

Reason for assumption: We need a block length because the problem requires side roads every block. On main roads, Los Angeles city blocks are 330ft[2], Chicago city blocks are 350ft[3], and New York city blocks are 250ft[4]. 300ft fits nicely into this range.

Cars

1. The average car length is 15ft.

Reason for assumption: To calculate how many cars can fit on a given stretch, we need a car size. A Saturn sedan is 176.9 inches (approximately 14ft 9in) long[5]. Most cars fall into the "sedan range." 15 is a round number (and fits nicely into 300).

2. Moving cars remain one half car-length from the car in front for every 10 mph they are travelling. When stopped, cars remain one foot from the car in front.

Reason for assumption: For the same reason we need a car length, we need a distance between cars. The recommended distance is one car length for every 10 mph[6], but recommendations are seldom followed completely, especially at rush hour. When stopped, cars pack closely. One foot is a good average.

3. Cars accelerate from 0 to 30 mph in less than 10 seconds.

Reason for assumption: To calculate the number of cars that pass through an intersection during a green light, we need acceleration. A Nissan Maxima accelerates from 0 to 60 mph in less than 10 seconds[7]. A Maxima is not the most powerful car and not the least powerful, so all cars should meet our acceleration rate.

[1] http://maps.yahoo.com (See Figure 1). We have assumed that one of the major North-South roads (shorter space between blocks, like our model) best represents our model. Between the two major cross streets, ten smaller side streets intersect the main road.

[2] http://maps.yahoo.com (See Figure 2)

[3] http://maps.yahoo.com (See Figure 3)

[4] http://www.mapquest.com (See Figure 4)

[5] http://www.saturn.com

[6] Jon Tang's Driving instructor/Common knowledge of "road rules."

[7] http://www.nissan-usa.com

Traffic

1. At any intersection, an equal number of cars turn off and onto the main road.

Reason for assumption: Although not necessarily true at a given moment, it is true over time. This is dictated by the laws of probability. This assumption makes our job easier and is the best way to test our model. The alternative is to generate random numbers, which is messy and does not aid understanding of the model.

2. The model is only expected to work optimally at rush hour.

Reason for assumption: Another 24 hours could be dedicated to effectively modeling light traffic. But our model is designed for rush hour when high speeds cannot be hoped for, and convenience, ease, and avoidance of gridlock are priorities.

PRELIMINARY ANALYSIS

Most cities design traffic lights so the road that has five-sixths of the traffic has a green light five-sixths of the time. This is ineffective because cars on the main road often wait at red lights when there aren't cars on cross streets waiting to pass.

The solution is a "green-wave" model, where lights are timed so that, after stopping at a red light, a car (if travelling at the appropriate speed) passes each light while it is green. This system is effective for one-way roads. With cars from both directions, cars from one direction have a green-wave, but the others have a red-wave. This cannot be avoided because, if both roads use a green-wave, the lights cannot be coordinated.

Why do the lights have to be coordinated? The purpose of a red light on a main road is to allow cars on a cross street to cross. If the lights for cars travelling on the main road in opposite directions are not coordinated, two problems occur. First, when a road has a red light (not affecting the green-wave), the other side of the road does not necessarily have a red light (because it might affect *its* green-wave) and cross street cars never have a chance to cross. Second, if only one side has a red light, and cross street cars cannot cross, then that red light is wasted.

The actual green vs. red pattern of a traffic light is not the only problem. Left turns, for instance, require their own lanes, lights, and speeds. We have partially dealt with this problem by assuming that cross streets are all one way, but turning left on the main road, across traffic, still presents a problem. This is something that will have to be dealt with at the same time as the traffic pattern.

Our first goal was to get as many cars from one end of our main road to the other as quickly as possible, while not inconveniencing drivers on the cross streets. Second, we wanted cross street drivers to have as little wait as possible. Our third goal was to avoid as much "stop and go" driving as possible. Finally, we wanted a model feasible for a real city and somehow different from any model currently in use.

PART II: THE MODEL

Our model uses 'staggered lights'(The light turns green in one intersection a few seconds after the previous light turns green), in an effort to create a green-wave. A problem arises in that when the lights are properly staggered in one direction, the motorists going in the opposite direction face a series of red lights. This is unacceptable, so we found a way to stagger the lights in both directions.

Our model makes each light green half the time and red half the time. With proper timing, cars travelling in both directions ride a green wave while not interfering with cross street traffic.

THE EXPLANATION

Because we want red and green lights to alternate throughout the roadway, the time a light is green must equal the time it is red. Although this seems inefficient (the cross streets carry less traffic and therefore need less time on green), cars on the main road never encounter a red light. Timing is determined by block length and car speed. Additionally, the period should not be so short that a car cannot reach the next green light when starting at rest. Given these constraints, our goal is to maximize speed.

We want the light to change just as the traffic hits it, so the duration of each light is the time it takes traffic to travel from one light to the next. The light-changing interval is therefore represented by the equation speed (in fps) = 300/time (in seconds), where speed is the average speed of the cars and time is the period of the lights. We also need the time to be enough to accelerate from 0 and still make the light (given by $d = \frac{1}{2}at^2$), so we have the constraint of t > 8.257228, assuming a distance of 300 feet and acceleration of 4.4 fps^2 or 0–60 in 20 seconds. Thus, the minimum traffic light period is about 8.25 seconds, and the maximum speed is about 35 mph (4.4 fps * 8.25 seconds).

Two factors control the time interval for the traffic light change. First, we cannot allow more than 300ft worth of cars to pass through a green light at one time, or they will be caught in a red light. Using our assumption that a car takes up its own space, plus $\frac{1}{2}$ its length for every 10mph it travels, we calculate that at 10mph each car takes up approximately 22 feet. Therefore, 14 cars can fit in 300ft. Second, if 14 cars are to pass through each green light, then they must be able to accelerate from 0 to 10mph and have time to pass the light before it turns red. The 14th car, starting 224 feet from the light (14 * 16ft per car), travelling at 14 fps (10mph) will need more than 16 seconds to pass each light. With a speed limit of 10mph, the required traffic light interval is 16 seconds. This is inefficient (most efficient is 8.25 seconds).

The most efficient numbers are the following. If the speed limit is 20mph, then each car will take up 30 feet, and 10 cars can fit on a section of the road. Therefore, the first light (when it turns from red to green) has to allow 10 cars through. The tenth car will start approximately 150ft from the light. Accelerating at the maximum 4.4 fps^2, but not exceeding the speed limit by much, the car will reach the light in slightly less than 10 seconds (Appendix A.1). If the speed limit is higher, the time interval can be lower, but few cars will be able to travel through.

Having each light turn green just as the traffic hits raises another problem. Normally, when one has a red light and is nearing an intersection, one has to stop, which can cause a stop-and-go nightmare. Our solution is to design the red light to display a countdown instead of yellow.

The final issue is left turns. Our assumption that the cross streets are one-way helps since left turns off a cross street are as easy as right turns (because traffic on the main road is stopped). To handle left turns off the main road, we adapt our timing schedule. We create a left-turn-lane with room for about five cars, (10 cars in each of three lanes equal 30 cars, and the probability of more than 1/6 of the cars turning is low). When the main road gets a red light (every 10 seconds), those wishing to turn left can do so. This doesn't interrupt the main road's rhythm, because the light is only green for those getting off the road and so those going straight still have the alternating light pattern.

SUMMARY

Our model lets cars travel a green-wave after passing the first red light, at about 20mph, in groups of 30. Lights alternate every 10 seconds (i.e. if one light is red, the next is green, and after 10 seconds they switch). A car travels the road in about 6.5 minutes (see Appendix A.2).

PART III: REFLECTION

The main strength of our model lies in the timing of lights, which allows a car entering the road to stop only once at a red light, yet allows cars in cross streets a large amount of time to cross. The model handles left and right turns without slowing traffic. Our 2-mile stretch of road can accommodate approximately 10,000 (see Appendix A.3) cars per hour. In comparison, the central artery of Boston, I-93[8], can accommodate 18,750 cars an hour, but I-93 has four lanes, and our stretch has 35 traffic lights.

Another important advantage is the countdown timer. It allows cars to not slow down, while squeezing past the lights. Whereas a yellow light is sudden, the timer updates the driver, prevents running red lights, and reduces the number of accidents.

Our model does not account for pedestrians. Unfortunately, modifications to allow for pedestrains would upset the timing. Another problem is that only half the street has cars on it at a given time.

In general, our model oversimplifies traffic planning. For instance, we assume that every time a car leaves the road, another one enters at the same point. From a statistical point of view, this is accurate. However, if one were to study the logistical minutia, problems would arise.

It appears that our model effectively deals with rush hour traffic, which raises a question. Why isn't it used now? Is there a problem that we don't see, is it something that no one has thought of, or is it already in use?

APPENDIX A: MATH

1. A car, accelerating at 4.4 fps^2, reaches 30 fps (20mph) after $6\frac{2}{3}$ seconds. At that point, the car has traveled 97 feet (distance = $\frac{1}{2}$ acceleration * time squared). Now, travelling at a constant rate of 30 fps, the car traverses the remaining 63 feet in 2 seconds. Thus the total travel time from 0 mph to passing the light at 20mph is about 8.7 seconds, leaving time for error in the ten second light change interval.

2. If a car arrives at the first light on the two mile stretch and is stopped, tenth from the front, and the light has just turned red (worst case scenario), then the car must wait 10 seconds, accelerate for 8.7 seconds (see Appendix A.1) and travel 2 miles at a speed of 20mph (6 minutes). Add an incidental problem, and the total time is 6.5 minutes.

3. At a given moment, there are 35 groups of cars on the road because there are 35 intersections, and a group of cars is always either approaching an intersection, or travelling on an intersection. There are 30 cars in each group (3 lanes * 10 cars per lane). Thus, a total of 1,050 cars are on the road. Every 6.5 minutes, a new set of cars will be on the road (see Appendix A.2). There are approximately 9 sets of 6.5 minutes in an hour. Therefore, 9*1050 = slightly more than 10000 cars per hour.

[8] Personal knowledge, from project done in the past.

Judge's Commentary

Patrick J. Driscoll

Department of Mathematical Sciences
U.S. Military Academy
West Point, NY 10996

Author Information: Pat Driscoll is an Academy Professor in the Department of Mathematical Sciences at USMA. He received his M.S. in both Operations Research and Engineering Economic Systems from Stanford University, and a Ph.D in Industrial & Systems Engineering from Virginia Tech. He is currently the program director for math electives at USMA. His research focuses on reformulation-linearization techniques in the context of linear and nonlinear optimization. Pat was the Head INFORMS judge for the HiMCM problem.

The modeling problem for this charter year HiMCM was simply posed and deceptively challenging: How should one go about setting the timing of a series of traffic lights on a major thoroughfare such that traffic flows smoothly and entering traffic from connecting side roads can also be accommodated? Every major city in the United States wrestles with this problem in response to changing populations, new construction, urban renewal, and periodic surges to the steady state traffic conditions. The City of New York recently announced their intention to adjust the signal timing of the cross-town traffic lights to increase cross-Manhattan traffic access. Their methodology is largely empirical, whereas the challenge set forth to the HiMCM teams was to develop an analytical model to accomplish a similar result.

This year's papers could best be characterized by both diversity and creativity. Various teams chose to conceptualize the changing of traffic lights as a wave that starts at one end and either proceeds to the other end of the thoroughfare or reflects at a point somewhere in the middle of the stretch of road being examined. Many others developed mathematical relationships from first principles of physics and applied these to a known city street layout obtained from a mapping service on the Internet. Others still recognized the probabilistic nature of the arrival of vehicles onto either the main thoroughfare or the connecting side streets and attempted to construct simulations that would test various light timing strategies. Several teams proceeded to collect data from actual traffic intersections and use this data to fit statistical models. Virtually all teams made use of computational technology in the form of either computers or calculators to assist them in their modeling efforts.

In judging this year's papers, several notable issues arose that are worthy of mention as general comments to provide teams with insights into how they might better tackle the HiMCM in the future.

At its core, the HiMCM represents a challenge in time management and organization. Several teams apparently found themselves without sufficient time remaining to resolve key issues related to these tasks. Should they attempt to develop a model to exactness, or employ a logical assumption and work with an approximate model? Should they attempt to introduce a number of different models for the given problem in the hope that 'more is better' or focus on developing a complete single model recognizing that it is 'quality, not quantity that counts?' Lastly, should they provide extensive mathematical derivations, computer code, and charts, or select only a subset of critical information and present a clear, concise, logical exposition based on this subset? Let's take these in order.

The answer to the first question is largely dependent upon the abilities of the team and the type of information the model they create has to provide access to. However, it is rarely advisable to develop an exact model for a situation in which available data can be considered estimates at best. All modeling efforts require the introduction of simplifying assumptions at some point. The better papers submitted this year clearly indicate what assumptions they used and why they were needed, as opposed to simply providing a list of assumptions included. As each assumption is added, however, the resulting model moves farther away from the physical reality of the problem, and any results obtained by using the model must be interpreted relative to these. It is, in fact, a careful examination of the model and its results relative to the assumptions made that provides a basis for a team to discuss their model's strengths and weaknesses.

As witnessed over time in the MCM, very few teams find success in pursuing a 'more is better' approach because the resulting paper contains such a shallow presentation of each modeling option that the judges are left with a question as to how well such a team really understood what it was doing. Predictably, this trend continued for several HiMCM papers this year as well. The HiMCM focus is on the modeling process from start to finish. The best papers started with a very nice executive summary and proceeded to evolve a focused, coherent, and balanced modeling thread running through their paper that enabled judges to understand how and why the team was doing what it was attempting to do. Few teams will find sufficient time within the contest limits to allow them to develop multiple modeling approaches that each reflect the same complete development, thereby spelling disaster for the 'more is better' approach.

The majority of good papers also represented a team strategy that, when faced with a decision to either present highly detailed mathematical analysis on only a portion of the problem, or an overall complete modeling effort, chose the latter. The exceptional papers contained a judicious amount of both elements woven together to answer the questions posed by the problem. Exactly how the balance was struck between the degree of inclusion of the two elements varied from paper to paper. However, it was clear that each team had chosen appropriate mathematical techniques to develop within the context of a complete modeling effort.

Information types such as computer code, extensive derivations, and non-critical graphs/charts are generally considered supporting information whose purpose is to convince a small specialized subset of the report readership (e.g., computer programmers, mathematicians) that the team understood what they were doing. This relegates information of these types to appendices of the paper, not the main body of exposition. The better papers this year contained tables, charts, pictures, and graphs that were judiciously selected to help clarify ideas and results to the reader. Key mathematical formulas were carefully developed leading to a final model, rather than just being ‘dropped in’ from some reference. Several of the weaker papers chose to present a volume of detail without an accompanying explanation, which is equivalent to saying to the judges: “Here, we did the work, now you figure it out.” This is never a good idea. Using a spell checker and carefully verifying any algebra used cannot be overemphasized.

A point worth mentioning is the difference between ‘evaluating a model’s results’ and ‘evaluating the strengths and weaknesses’ of a model. Both elements are critical to the modeling process. Simply put, the strengths and weaknesses of a model are identified by examining the model and its results in light of the simplifying assumptions made to create the model. Evaluating a model’s results consists of determining if the results make sense, and interpreting what these results mean with regard to the information sought. This latter action provides the basis for modifying the model until a team is satisfied with the model’s performance.

By and large, the exceptional papers provided conclusive evidence that their teams had dedicated a substantial amount of time thinking about the problem prior to starting their quest for supporting information. While the Internet does provide a seemingly limitless source of information, it can also act as a siren’s call to act before thinking, thus using up valuable time potentially pursuing dead ends.

Finally, the need for precise supporting documentation in the body of the final paper cannot be stressed enough. Exceptional papers all convey a clear link to verifiably credible information sources within the body of their paper. Lesser quality papers show a reliance on supporting information that fails to include necessary explanations of why certain facts are valid. Although the temptation to ‘cut-and-paste’ directly from sources such as the Internet is recognizably strong, doing so can often result in a paper that is predominantly statements of unsupported ‘facts’ rather than one demonstrating that the team has a clear understanding of the model. ❑

HiMCM School websites

Parker School:
www.parker.org

Westminster Schools:
www.westminster.net

Illinois Math and Science Academy:
www.imsa.edu

Chesterfield County Math & Science HS:
chesterfield.k12.va.us/Schools/Clover_Hill_HS/mathsci.htm