Sparse Regularization via Convex Analysis

Ivan Selesnick

Electrical and Computer Engineering
Tandon School of Engineering
New York University
Brooklyn, New York, USA

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Convex or non-convex: Which is better for inverse problems?

Benefits of convex optimization:

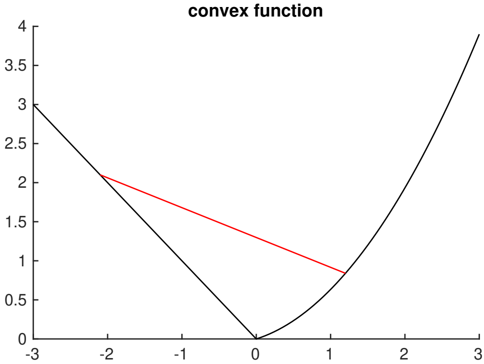
1. Absence of suboptimal local minima
2. Continuity of solution as a function of input data
3. Algorithms guaranteed to converge to a global optimum
4. Regularization parameters easier to set

But convex regularization tends to *under-estimate* signal values.

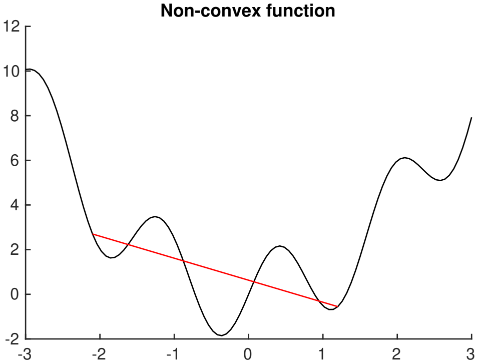
Non-convex regularization often performs better!

> Can we design non-convex sparsity-inducing penalties that maintain the convexity of the cost function to be minimized?

Convex function



Non-Convex function



Goal

Goal: Find a sparse approximate solution to a linear system $y = Ax$.

Minimize a cost function:

$$J(x) = \frac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_1$$

or

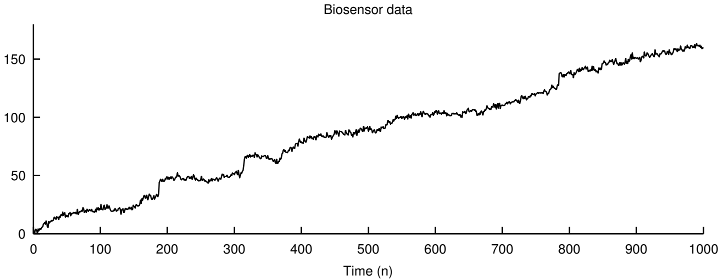
$$F(x) = \frac{1}{2}\|y - Ax\|_2^2 + \lambda\,\psi(x)$$

Question: How to define ψ ?

Let us allow ψ to be non-convex such that F is convex.

This is the *Convex Non-Convex* (CNC) approach.

Biosensor Signal



Linear Filter

Given noisy data $y \in \mathbb{R}^N$, perform smoothing via:

$$\hat{x} = \arg\min_{x \in \mathbb{R}^N} \left\{ \sum_{n=0}^{N-1} |y(n) - x(n)|^2 + \lambda \sum_{n=1}^{N-1} |x(n) - x(n-1)|^2 \right\}$$

which can be written

$$\hat{x} = \arg\min_{x \in \mathbb{R}^N} \left\{ \|y - x\|_2^2 + \lambda \|Dx\|_2^2 \right\}$$

where

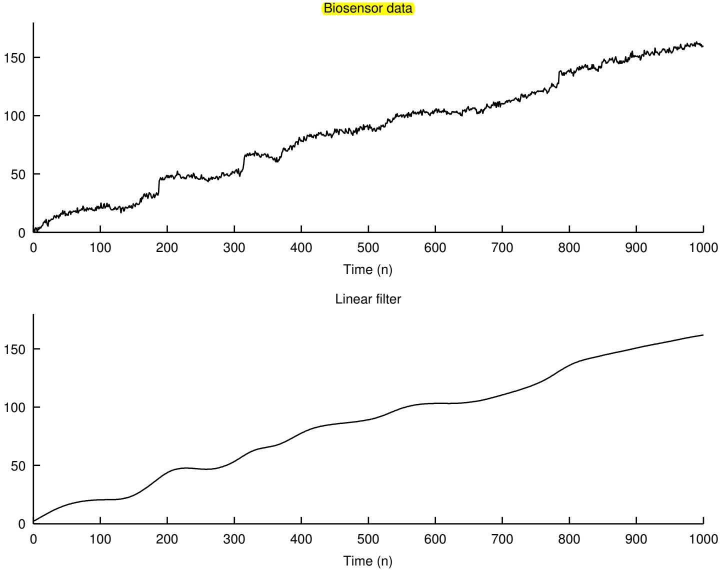
$$\|x\|_2^2 := \sum_n |x(n)|^2$$

$$D = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}$$

Solution:

$$\hat{x} = (I + \lambda D^{\mathsf{T}} D)^{-1} y$$

Biosensor Signal



Total Variation Denoising (Nonlinear Filter)

Given noisy data $y \in \mathbb{R}^N$, perform smoothing via:

$$\hat{x} = \arg\min_{x \in \mathbb{R}^N} \left\{ \frac{1}{2} \sum_{n=0}^{N-1} |y(n) - x(n)|^2 + \lambda \sum_{n=1}^{N-1} |x(n) - x(n-1)| \right\}$$

which can be written

$$\hat{x} = \arg\min_{x \in \mathbb{R}^N} \left\{ \frac{1}{2} \|y - x\|_2^2 + \lambda \|Dx\|_1 \right\}$$

where

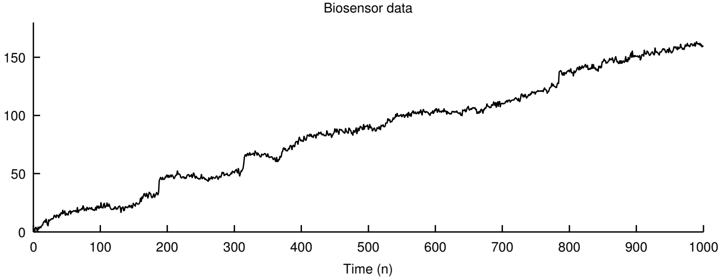
$$\|x\|_2^2 := \sum_n |x(n)|^2, \quad \|x\|_1 := \sum_n |x(n)|$$

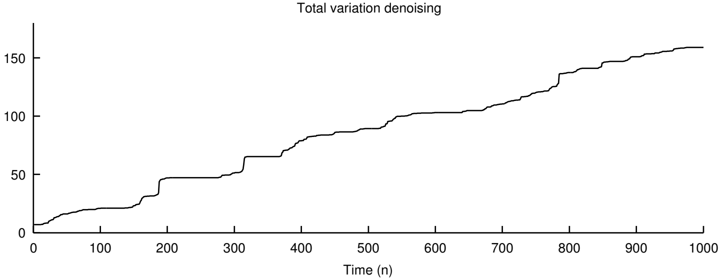
$$D = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}$$

Solution? *No closed form solution.*

Use iterative algorithm ... but note the cost function is not differentiable!

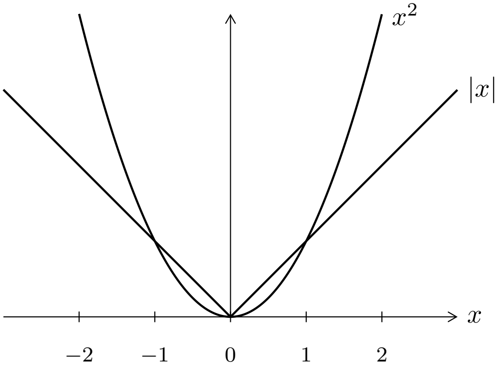
Biosensor Signal





Two Penalties

The ℓ_1 norm induces sparsity unlike the the sum of squares.



Combine Quadratic and Sparse Regularization

$$\arg \min_{u,v \in \mathbb{R}^N} \left\{ \frac{1}{2} \|y - u - v\|_2^2 + \lambda_1 \|Du\|_1 + \frac{\lambda_2}{2} \|Dv\|_2^2 \right\}$$

$$\hat{x} = u + v$$

Combine Quadratic and Sparse Regularization

$$\arg\min_{u,v\in\mathbb{R}^N}\left\{\frac{1}{2}\|y-u-v\|_2^2+\lambda_1\|Du\|_1+\frac{\lambda_2}{2}\|Dv\|_2^2\right\}$$

Solving for v gives

$$v=(I+\lambda_2 D^{\mathsf{T}}D)^{-1}(y-u)$$

$$x=v+u=(I+\lambda_2 D^{\mathsf{T}}D)^{-1}(y+\lambda_2 D^{\mathsf{T}}Du)$$

Substituting v back in to the cost function:

$$J(u)=\frac{\lambda_2}{2}(y-u)^{\mathsf{T}}D^{\mathsf{T}}(I+\lambda_2 DD^{\mathsf{T}})^{-1}D(y-u)+\lambda_1\|Du\|_1$$

or

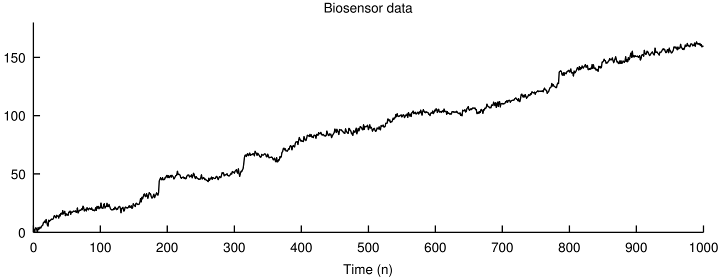
$$J(u)=\frac{\lambda_2}{2}\|R^{-1}D(y-u)\|_2^2+\lambda_1\|Du\|_1$$

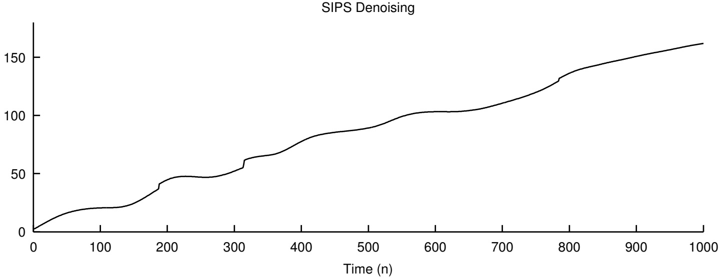
$$RR^{\mathsf{T}}=I+\lambda_2 DD^{\mathsf{T}}\qquad (R\text{ is a banded matrix})$$

Since x depends on Du, not u directly, define $g=Du$. So we need to minimize

$$F(g)=\frac{\lambda_2}{2}\|R^{-1}Dy-R^{-1}g\|_2^2+\lambda_1\|g\|_1$$

Biosensor Signal

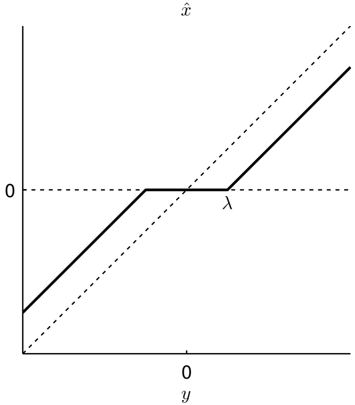




Scalar case

$$\hat{x} = \arg\min_{x} \left\{ \frac{1}{2}(y - x)^2 + \lambda |x| \right\}$$

$\Longrightarrow$



Non-convex scalar penalty functions (alternatives to ℓ_1 norm)

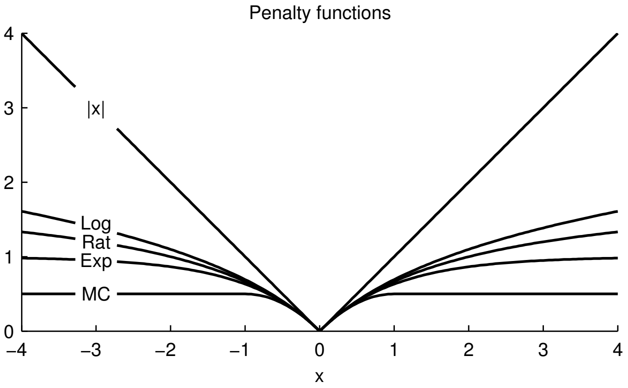
Log	$\phi_a(x) = \dfrac{1}{a}\log(1 + a\|x\|)$
Rat	$\phi_a(x) = \dfrac{\|x\|}{1 + a\|x\|/2}$
Exp	$\phi_a(x) = \dfrac{1}{a}\left(1 - \mathrm{e}^{-a\|x\|}\right)$
MC	$\phi_a(x) = \begin{cases} \|x\| - \frac{a}{2}x^2, & \|x\| \leqslant 1/a \\ \frac{1}{2a}, & \|x\| \geqslant 1/a \end{cases}$

The penalties are parameterized such that

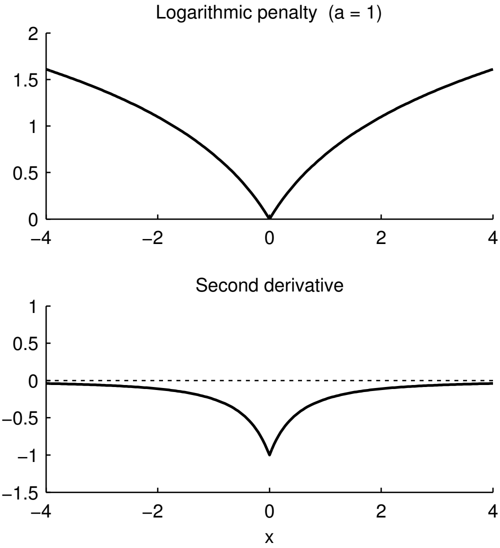
$$\phi_a'(0^+) = 1$$
$$\phi_a''(0^+) = -a.$$

Non-convex scalar penalty functions

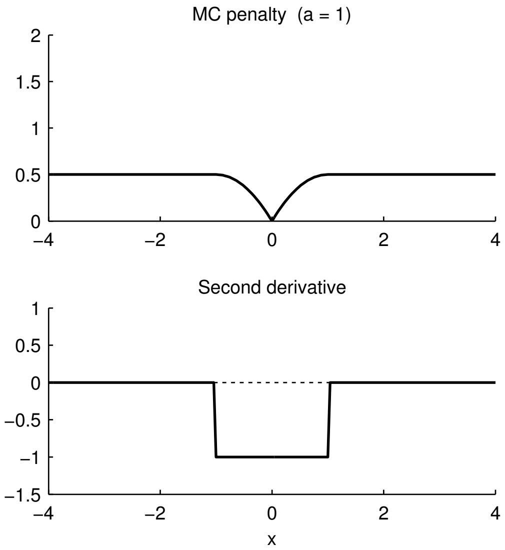
Penalty functions with $a = 1$.



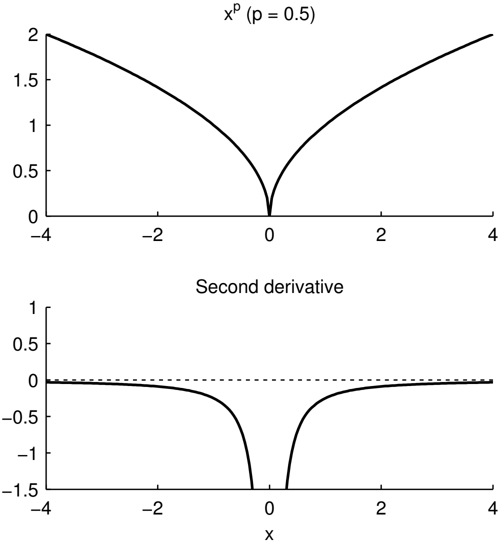
Logarithmic penalty



MC penalty


MC penalty (a = 1)
2
1.5
1
0.5
0
–4
–2
0
2
4
Second derivative
1
0.5
0
–0.5
–1
–1.5
–4
–2
0
2
4
x

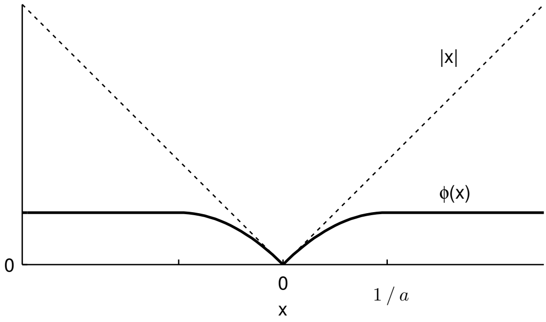
ℓ_p penalty, $0 < p < 1$ *(precluded)*



Scalar MC penalty

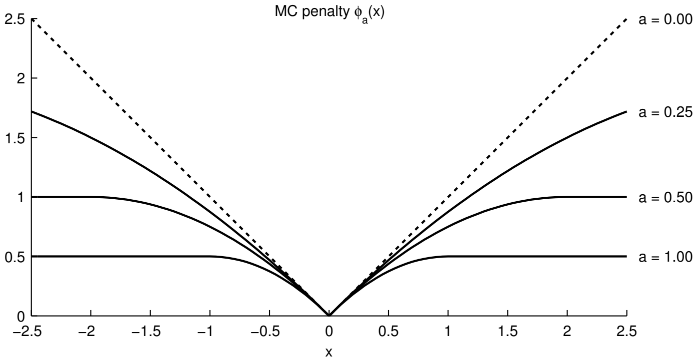
We consider henceforth only the minimax-concave (MC) penalty function

$$\phi_a(x) = \begin{cases} |x| - \frac{a}{2}x^2, & |x| \leqslant 1/a \\ \frac{1}{2a}, & |x| \geqslant 1/a \end{cases}$$



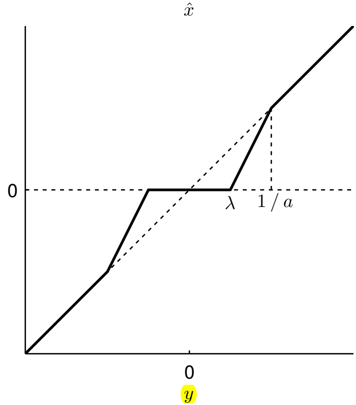
Scalar MC penalty

The parameter $a \geqslant 0$ controls the non-convexity of ϕ_a.



Scalar case using MC penalty

$$\hat{x} = \arg\min_{x} \left\{ \frac{1}{2}(y - x)^2 + \lambda\phi_a(x) \right\}$$



$\hat{x}$ is a continuous function of y when $a < \lambda$.

Scalar case using MC penalty

Consider

$$f(x) = \frac{1}{2}(y - x)^2 + \lambda\phi_a(x).$$

For what values 'a' is f a convex function?

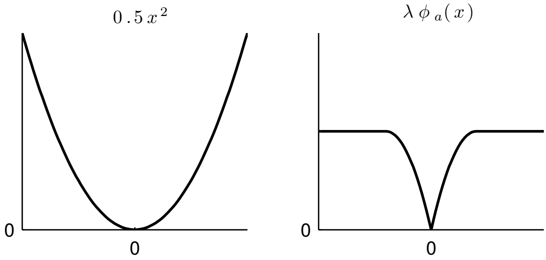
$$f(x) = \underbrace{\frac{1}{2}x^2 + \lambda\phi_a(x)}_{f_0(x)} + \underbrace{\left[\frac{1}{2}y^2 - yx\right]}_{\text{convex in } x}.$$

It is sufficient to consider the convexity of

$$f_0(x) = \frac{1}{2}x^2 + \lambda\phi_a(x).$$

Scalar case using MC penalty

$$f_0(x) = \frac{1}{2}x^2 + \lambda\phi_a(x)$$



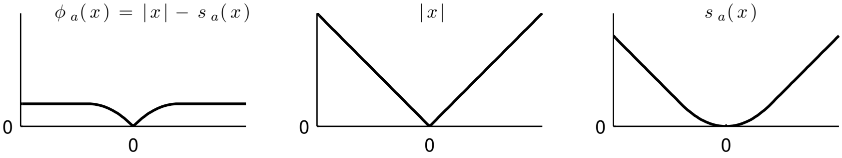
Is f_0 convex?

ϕ_a is *not differentiable*. So we can not simply check that the second derivative of f_0 is positive . . .

Scalar case using MC penalty

Let us write

$$\phi_a(x) = |x| - s_a(x)$$



We see the Huber function:

$$s_a(x) = \begin{cases} \frac{a}{2}x^2, & |x| \leqslant 1/a \\ |x| - \frac{1}{2a}, & |x| \geqslant 1/a. \end{cases}$$

Scalar case using MC penalty

Writing ϕ_a as

$$\phi_a(x) = |x| - s_a(x),$$

we have

$$\begin{aligned} f(x) &= \frac{1}{2}(y-x)^2 + \lambda\phi_a(x) \\ &= \frac{1}{2}(y-x)^2 + \lambda\Big[|x| - s_a(x)\Big] \\ &= \underbrace{\frac{1}{2}x^2 - \lambda s_a(x)}_{g(x)} + \underbrace{\Big[\lambda|x| + \frac{1}{2}y^2 - yx\Big]}_{\text{convex in } x} \end{aligned}$$

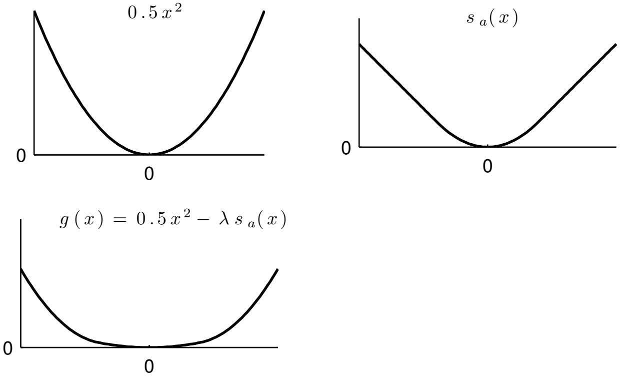
g convex $\implies$ f convex

Note that g is *differentiable* unlike f.

Is g *convex*? It depends on a and λ.

Scalar case using MC penalty

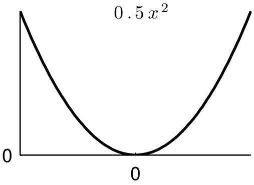
$$g(x) = \frac{1}{2}x^2 - \lambda s_a(x)$$

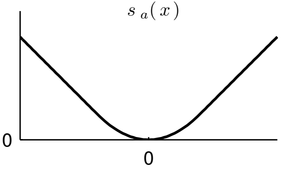


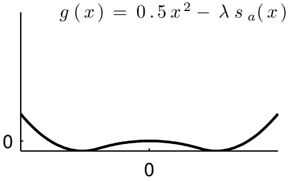
$a = 1$, $\lambda = 0.8$, g *is* convex.

Scalar case using MC penalty

$$g(x) = \frac{1}{2}x^2 - \lambda s_a(x)$$



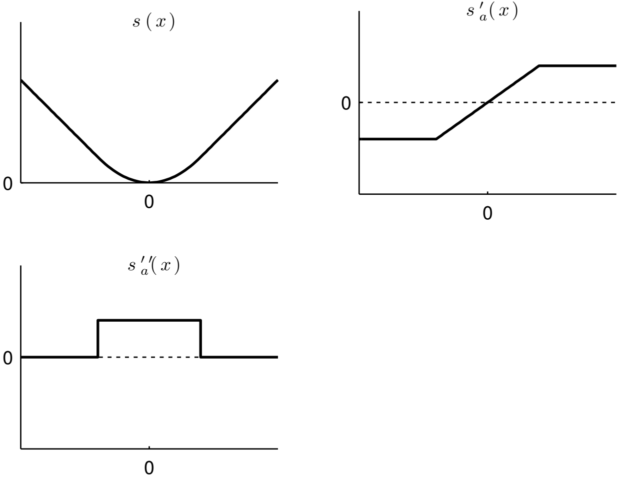




$a = 1$, $\lambda = 1.3$, g *is not* convex.

Scalar case using MC penalty

The Huber function is differentiable. But not twice differentiable.



Scalar case using MC penalty

$$g(x) = \frac{1}{2}x^2 - \lambda s_a(x)$$

When is g convex?

We can not check the second derivative of g because it is not twice differentiable (see previous page).

How can we ensure g (and hence f) is *convex*?

Huber function as an infimal convolution

The Huber function can be written as

$$s_a(x) = \min_v \left\{ \frac{a}{2}(x - v)^2 + |v| \right\}.$$

As infimal convolution

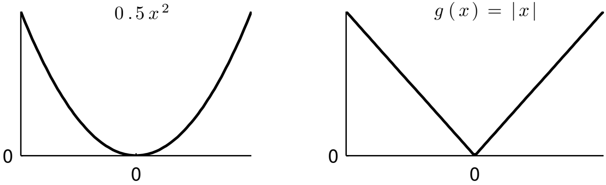
$$s_a(x) = \left(\frac{a}{2}(\,\cdot\,)^2 \,\square\, |\cdot| \right)(x)$$

where infimal convolution (Moreau-Yosida regularization) is defined as

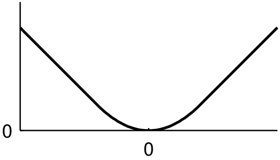
$$(f \,\square\, g)(x) := \min_v \{ f(v) + g(x - v) \}$$

Huber function as an infimal convolution

$$\left(\frac{a}{2}(\cdot)^2 \,\square\, |\cdot|\right)(x)$$

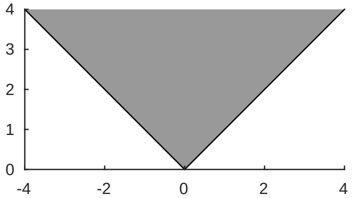


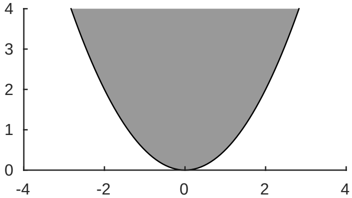
Huber function



Epigraph

The epigraph of a function is a set comprising points on and above the graph.





The epigraph of

$$\mathrm{epi}\{f \,\square\, g\} = \mathrm{epi}\{f\} + \mathrm{epi}\{g\}$$

Scalar case using MC penalty

The Huber function can be written as

$$s_a(x) = \min_v \left\{ \frac{a}{2}(x - v)^2 + |v| \right\}.$$

When is g convex?

$$\begin{aligned}
g(x) &= \frac{1}{2}x^2 - \lambda s_a(x) \\
&= \frac{1}{2}x^2 - \lambda \min_v \left\{ \frac{a}{2}(x - v)^2 + |v| \right\} \\
&= \frac{1}{2}x^2 - \lambda \min_v \left\{ \frac{a}{2}(x^2 - 2xv + v^2) + |v| \right\} \\
&= \frac{1}{2}x^2 - \lambda \frac{a}{2}x^2 + \underbrace{\lambda \max_v \Big\{ \underbrace{\frac{a}{2}(2xv - v^2) - |v|}_{\text{affine in } x} \Big\}}_{\text{convex in } x} \\
&= \frac{1}{2}(1 - a\lambda)\, x^2 + \text{convex function}
\end{aligned}$$

The function g is *convex* if $1 - a\lambda$ is non-negative, i.e.,

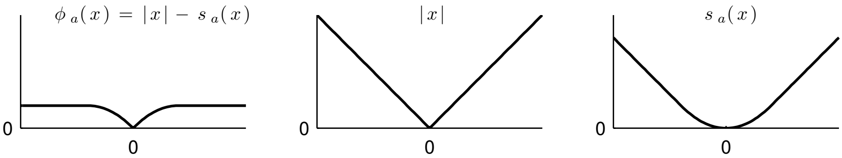
$$\boxed{a \leqslant 1/\lambda}$$

We do *not* need derivatives!

Scalar MC penalty

The MC penalty can be written as

$$\begin{aligned}\phi_a(x) &= |x| - s_a(x)\\ &= |x| - \underbrace{\min_v \left\{ \frac{a}{2}(x-v)^2 + |v| \right\}}_{\text{Huber function}}\\ &= |x| - \left(\frac{a}{2}(\,\cdot\,)^2 \,\Box\, |\cdot| \right)(x)\end{aligned}$$



Multivariate case

$$F(x) = \frac{1}{2}\|y - Ax\|_2^2 + \lambda\,\psi(x)$$

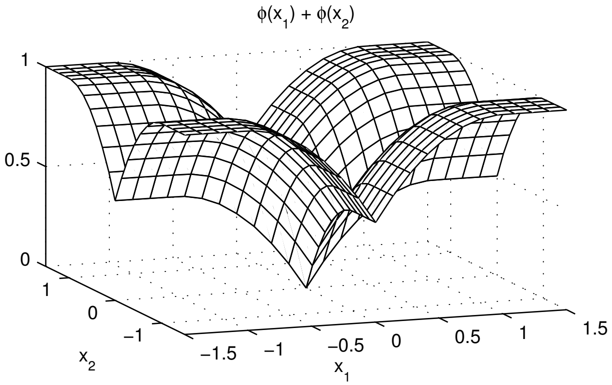
How can we set ψ so that F is convex and promotes sparsity of x?

We can generalize the scalar case . . .

Separable penalty *(precluded)*

Conventional penalty: additive (separable)

$$\phi(x) = \phi(x_1) + \phi(x_2)$$



Generalized Huber function

Let $B \in \mathbb{R}^{M \times N}$. We define the *generalized Huber function*

$$S_B(x) := \min_{v \in \mathbb{R}^N} \left\{ \frac{1}{2} \|B(x - v)\|_2^2 + \|v\|_1 \right\}.$$

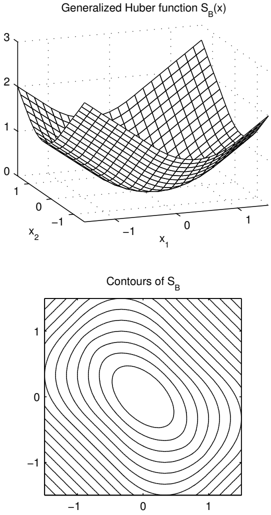
In the notation of infimal convolution, we have

$$S_B(x) := \left(\frac{1}{2} \|B \cdot\|_2^2 \,\Box\, \| \cdot \|_1 \right)(x).$$

Example 1. Generalized Huber function

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

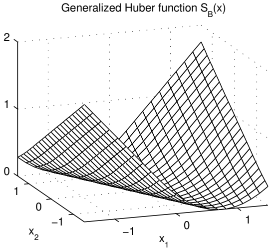
$$S_B(x) := \min_v \left\{ \tfrac{1}{2} \|B(x-v)\|_2^2 + \|v\|_1 \right\}$$

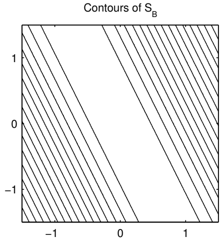


Example 2. Generalized Huber function

$$B = [1 \;\; 0.5]$$

$$S_B(x) := \min_v \left\{ \tfrac{1}{2}\|B(x-v)\|_2^2 + \|v\|_1 \right\}$$



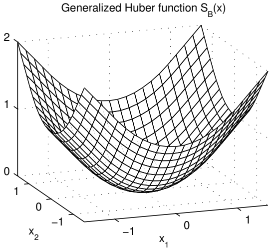


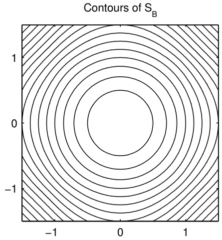
Example 3. Generalized Huber function

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S_B(x) := \min_v \left\{ \tfrac{1}{2} \|B(x - v)\|_2^2 + \|v\|_1 \right\}$$

If B is diagonal, then S_B is separable.





Generalized Huber function

The generalized Huber function is differentiable.

Its gradient is given by

$$\nabla S_B(x) = B^{\mathsf{T}}B\Big(x - \arg\min_{v \in \mathbb{R}^N} \Big\{\frac{1}{2}\|B(x-v)\|_2^2 + \|v\|_1\Big\}\Big).$$

Neither the generalized Huber function nor its gradient have simple closed form expressions. But we will still be able to use them . . .

When $B = I$ we recover a well known identity

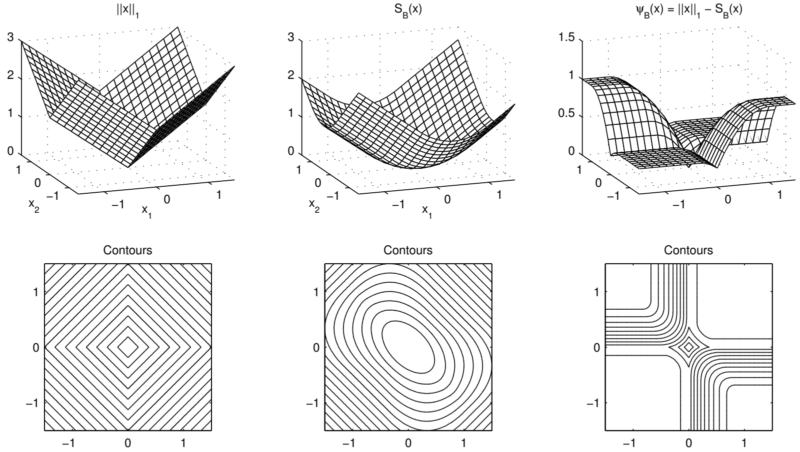
$$\nabla S_I(x) = x - \arg\min_{v \in \mathbb{R}^N} \Big\{\frac{1}{2}\|x-v\|_2^2 + \|v\|_1\Big\}$$

Generalized MC penalty

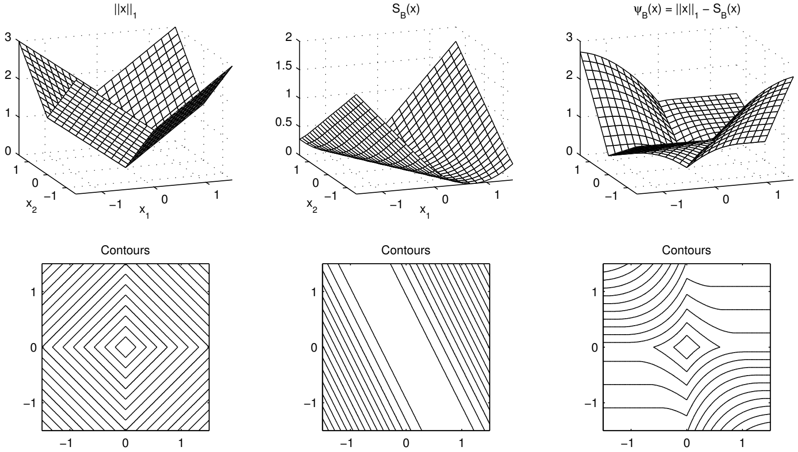
We define the *generalized MC (GMC)* penalty

$$\begin{aligned}\psi_B(x) &:= \|x\|_1 - S_B(x) \\ &:= \|x\|_1 - \min_{v \in \mathbb{R}^N} \left\{ \frac{1}{2} \|B(x-v)\|_2^2 + \|v\|_1 \right\}.\end{aligned}$$

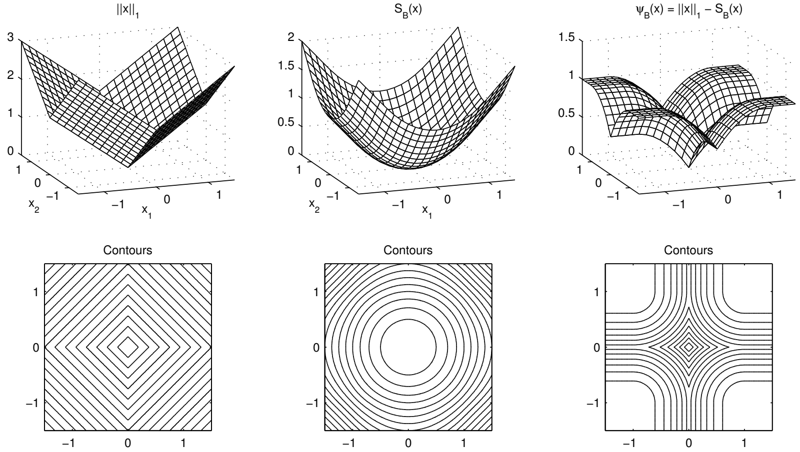
Example 1. Generalized MC penalty



Example 2. Generalized MC penalty



Example 3. Generalized MC penalty



Convexity

Theorem. The function

$$
\begin{aligned}
F(x) &= \frac{1}{2}\|y - Ax\|_2^2 + \lambda\,\psi_B(x) \\
&= \frac{1}{2}\|y - Ax\|_2^2 + \lambda\big[\|x\|_1 - S_B(x)\big] \\
&= \frac{1}{2}\|y - Ax\|_2^2 - \lambda\|x\|_1 - \lambda \min_{v \in \mathbb{R}^N} \big\{ \tfrac{1}{2}\|B(x - v)\|_2^2 + \|v\|_1 \big\}.
\end{aligned}
$$

is *convex* if

$$
\boxed{B^{\mathsf{T}} B \preccurlyeq \frac{1}{\lambda} A^{\mathsf{T}} A}
$$

even when ψ_B is *non-convex*.

Convexity condition – proof

Write F as

$$\begin{aligned} F(x) &= \tfrac{1}{2}\|y - Ax\|_2^2 + \lambda\,\psi_B(x) \\ &= \tfrac{1}{2}\|y - Ax\|_2^2 + \lambda\big[\|x\|_1 - S_B(x)\big] \\ &= \Big[\tfrac{1}{2}\|Ax\|_2^2 + \tfrac{1}{2}\|y\|_2^2 - y^{\mathsf{T}}Ax\Big] + \lambda\|x\|_1 - \lambda S_B(x) \\ &= \underbrace{\tfrac{1}{2}\|Ax\|_2^2 - \lambda S_B(x)}_{G(x)} + \underbrace{\tfrac{1}{2}\|y\|_2^2 - y^{\mathsf{T}}Ax + \lambda\|x\|_1}_{\text{convex in } x} \end{aligned}$$

G convex $\implies$ F convex

Convexity condition – proof

Write G as

$$
\begin{aligned}
G(x) &= \tfrac{1}{2}\|Ax\|_2^2 - \lambda S_B(x) \\
&= \tfrac{1}{2}\|Ax\|_2^2 - \lambda \min_v \left\{ \tfrac{1}{2}\|B(x-v)\|_2^2 + \|v\|_1 \right\} \\
&= \tfrac{1}{2}\|Ax\|_2^2 - \lambda \min_v \left\{ \tfrac{1}{2}\|Bx\|_2^2 + \tfrac{1}{2}\|Bv\|_2^2 - v^{\mathsf{T}} B^{\mathsf{T}} Bx + \|v\|_1 \right\} \\
&= \tfrac{1}{2}\|Ax\|_2^2 - \tfrac{\lambda}{2}\|Bx\|_2^2 + \underbrace{\lambda \max_v \Big\{ \underbrace{v^{\mathsf{T}} B^{\mathsf{T}} Bx - \tfrac{1}{2}\|Bv\|_2^2 - \|v\|_1}_{\text{affine in } x} \Big\}}_{\text{convex in } x} \\
&= \tfrac{1}{2} x^{\mathsf{T}} \left(A^{\mathsf{T}} A - \lambda B^{\mathsf{T}} B \right) x + \text{ convex function}
\end{aligned}
$$

The function G (hence F) is *convex* if $A^{\mathsf{T}}A - \lambda B^{\mathsf{T}}B$ is positive semidefinite, i.e.

$$B^{\mathsf{T}}B \preccurlyeq \frac{1}{\lambda} A^{\mathsf{T}}A.$$

Convexity condition

A straightforward choice of B to satisfy

$$B^{\mathsf{T}}B \preccurlyeq \frac{1}{\lambda}A^{\mathsf{T}}A.$$

is

$$B = \sqrt{\frac{\gamma}{\lambda}}A$$

for some γ with $0 \leqslant \gamma \leqslant 1$.

Optimization Algorithm

We can use the Forward-Backward Splitting (FBS) algorithm

$$\begin{aligned}
F(x) &= \tfrac{1}{2}\|y - Ax\|_2^2 + \lambda\,\psi_B(x) \\
&= \tfrac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_1 - \lambda S_B(x) \\
&= \underbrace{\tfrac{1}{2}\|y - Ax\|_2^2 - \lambda S_B(x)}_{\text{convex, differentiable}} + \underbrace{\lambda\|x\|_1}_{\text{convex}} \\
&= f_1(x) + f_2(x)
\end{aligned}$$

The FBS algorithm is given by:

$$\begin{aligned}
w^{(i)} &= x^{(i)} - \mu\left[\nabla f_1(x^{(i)})\right] \\
x^{(i+1)} &= \arg\min_x \left\{\tfrac{1}{2}\|w^{(i)} - x\|_2^2 + \mu f_2(x)\right\} \\
&= \operatorname{prox}_{\mu f_2}(w^{(i)})
\end{aligned}$$

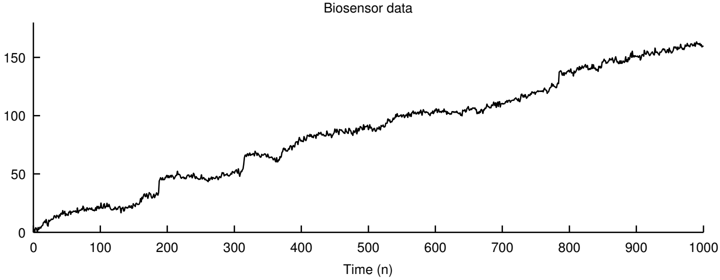
Optimization Algorithm

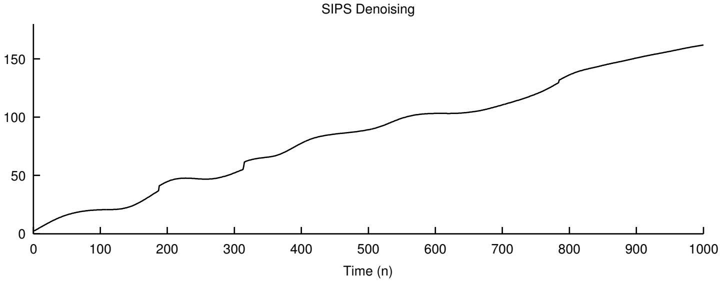
Let $0 \leqslant \gamma \leqslant 1$ and $B = \sqrt{\gamma/\lambda}\, A$. Then a minimizer of the objective function F is obtained by the iteration:

$$
\begin{aligned}
v^{(i)} &= \arg\min_{v \in \mathbb{R}^N} \left\{ \frac{\gamma}{2} \|A(x^{(i)} - v)\|_2^2 + \lambda \|v\|_1 \right\} \\
z^{(i)} &= \gamma A(x^{(i)} - v^{(i)}) \\
x^{(i+1)} &= \arg\min_{x \in \mathbb{R}^N} \left\{ \frac{1}{2} \|y + z^{(i)} - Ax\|_2^2 + \lambda \|x\|_1 \right\}
\end{aligned}
$$

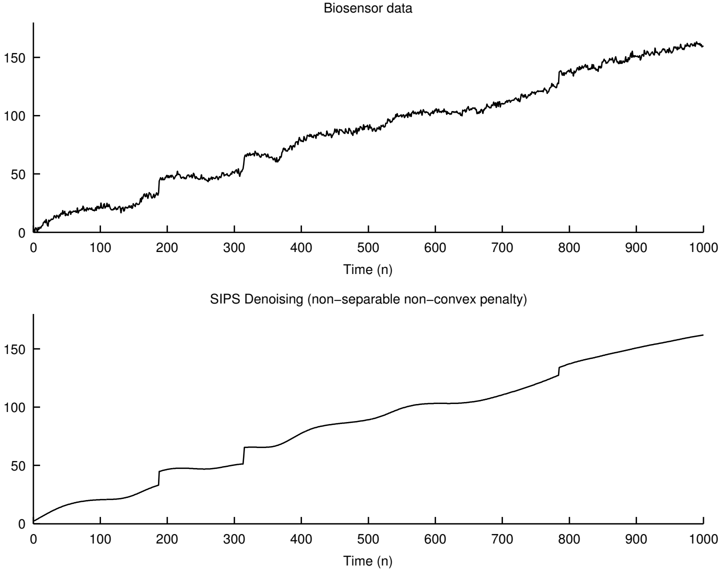
Interpretation: iteratively adjusted additive data perturbation of ℓ_1 norm regularized problem ...

Biosensor Signal

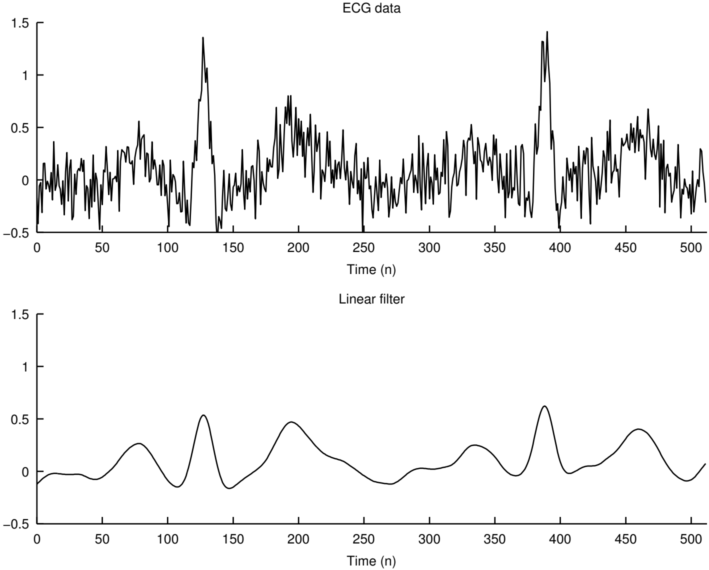




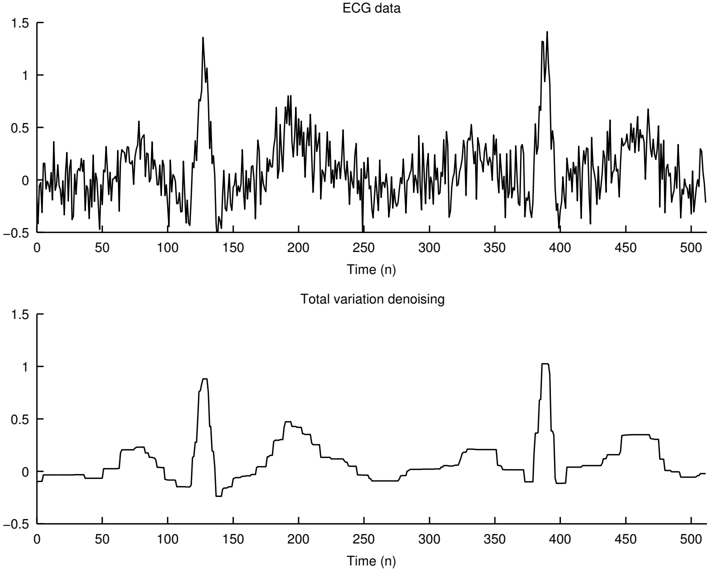
Biosensor Signal



ECG Signal

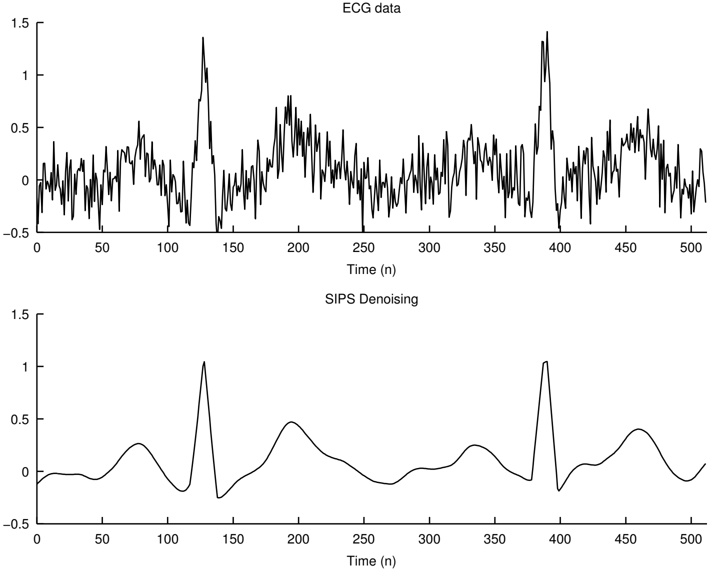

ECG data
1.5
1
0.5
0
−0.5
0 50 100 150 200 250 300 350 400 450 500
Time (n)
Linear filter
1.5
1
0.5
0
−0.5
0 50 100 150 200 250 300 350 400 450 500
Time (n)

ECG Signal

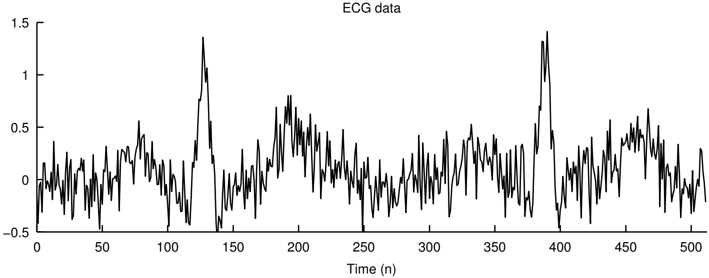


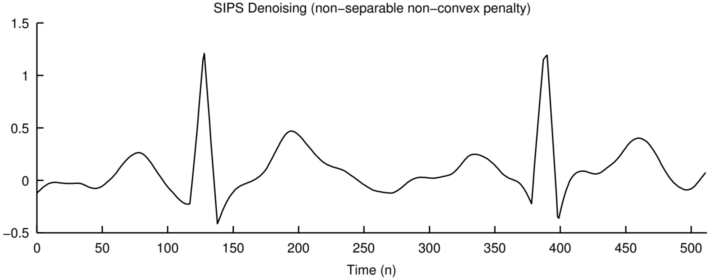
ECG data
1.5
1
0.5
0
−0.5
0
50
100
150
200
250
300
350
400
450
500
Time (n)
Total variation denoising
1.5
1
0.5
0
−0.5
0
50
100
150
200
250
300
350
400
450
500
Time (n)

ECG Signal


ECG data
1.5
1
0.5
0
−0.5
0
50
100
150
200
250
300
350
400
450
500
Time (n)
SIPS Denoising
1.5
1
0.5
0
−0.5
0
50
100
150
200
250
300
350
400
450
500
Time (n)

ECG Signal


ECG data
1.5
1
0.5
0
−0.5
0
50
100
150
200
250
300
350
400
450
500
Time (n)


SIPS Denoising (non−separable non−convex penalty)
1.5
1
0.5
0
−0.5
0
50
100
150
200
250
300
350
400
450
500
Time (n)

Generalizations (TV, nuclear norm, etc)

Consider the function

$$\mathcal{J}(x) := \frac{1}{2}\|Ax - b\|_2^2 + \lambda\,\mathcal{R}(x) \qquad (1)$$

where $\mathcal{R}$ is a convex regularizer of the form

$$\mathcal{R}(x) = \Phi(G(Lx))$$

where L is a linear operator, G is possibly nonlinear, and Φ promotes sparsity.

For example, total variation regularization is expressed as

$$\mathcal{R}(x) = \|G(Lx)\|_1 = \sum_i |g_i(Lx)|$$

with

$$g_i(Lx) = \begin{cases} \left(\begin{bmatrix} D_h \\ D_v \end{bmatrix} x\right)_i & \text{anisotropic TV} \\ \sqrt{(D_h x)_i^2 + (D_v x)_i^2} & \text{isotropic TV} \end{cases}$$

Generalizations (TV, nuclear norm, etc)

Suppose $\mathcal{R}$ satisfies

A1) $\mathcal{R}(\cdot) = \Phi(G(L\,\cdot))$ is convex and bounded from below by zero;

A2) $\Phi(G(\cdot))$ is a proper, lower semicontinuous and coercive function.

Consider the function

$$\mathcal{J}_B(x) := \frac{1}{2}\|Ax - b\|_2^2 + \lambda\,\mathcal{R}_B(x) \tag{2}$$

with non-convex regularizer

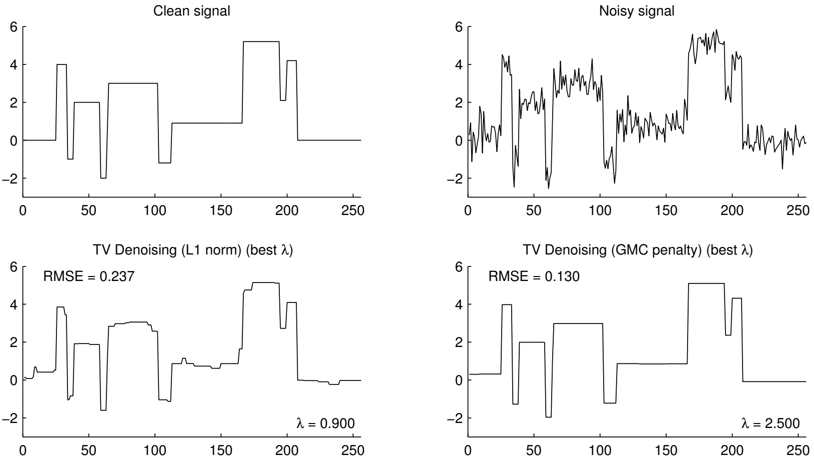
$$\mathcal{R}_B(x) := \mathcal{R}(x) - \min_{v\in\mathbb{R}^N}\left\{\tfrac{1}{2}\|B(x-v)\|_2^2 + \mathcal{R}(v)\right\}.$$

The function $\mathcal{J}_B(x)$ is convex (strongly convex) if

$$A^{\mathsf{T}}A - \lambda\,B^{\mathsf{T}}B \succcurlyeq 0 \;\; (\succ\; 0)\,.$$

Reference: A. Lanza, S. Morigi, I. Selesnick, and F. Sgallari. Sparsity-inducing Non-convex Non-separable Regularization for Convex Image Processing. 2017. Submitted.

Numerical Examples – 1D Total Variation denoising



Reference: I. Selesnick, A. Lanza, S. Morigi, and F. Sgallari. Total Variation Signal Denoising via Convex Analysis. 2017. Submitted.

Summary

- We show how to construct non-convex regularizers that preserve the convexity of functionals for sparse-regularized linear least-squares.
- Generalizes the ℓ_1 norm.
- Can be used in conjunction with other convex non-smooth regularizers (TV, nuclear norm, etc).
- Resulting regularizers are non-separable.
- Optimization implementable using proximal algorithms (as for ℓ_1 norm).

References

A. Lanza, S. Morigi, I. Selesnick, and F. Sgallari. Sparsity-inducing Non-convex Non-separable Regularization for Convex Image Processing. 2017. Submitted.

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A. Lanza, S. Morigi, I. Selesnick, and F. Sgallari. Nonconvex nonsmooth optimization via convex–nonconvex majorization–minimization. *Numerische Mathematik*, 136(2):343–381, 2017.

I. Selesnick. Sparse Regularization via Convex Analysis. *IEEE Trans. Signal Process.*, 65(17):4481–4494, September 2017.

I. Selesnick. Total Variation Denoising via the Moreau Envelope. *IEEE Signal Processing Letters*, 24(2):216–220, February 2017.

A. Lanza, S. Morigi, and F. Sgallari. Convex Image Denoising via Non-convex Regularization with Parameter Selection. *J. Math. Imaging and Vision*, 56(2):195–220, 2016.

A. Lanza, S. Morigi, and F. Sgallari. Convex Image Denoising via Non-Convex Regularization. In *Scale Space and Variational Methods in Computer Vision*. Volume 9087. 2015.

I. W. Selesnick, A. Parekh, and I. Bayram. Convex 1-D Total Variation Denoising with Non-convex Regularization. *IEEE Signal Processing Letters*, 22(2):141–144, February 2015.

Optimization Algorithm (saddle point algorithm)

With

$$\mathcal{J}_B(x) = \tfrac{1}{2}\|Ax - b\|_2^2 + \lambda\,\mathcal{R}_B(x)$$
$$\mathcal{R}_B(x) = \mathcal{R}(x) - \min_v \left\{ \tfrac{1}{2}\|B(x - v)\|_2^2 + \mathcal{R}(v) \right\}$$

we have

$$\begin{aligned}
\mathcal{J}_B(x) &= \tfrac{1}{2}\|Ax - b\|_2^2 + \lambda\mathcal{R}(x) - \lambda \min_v \left\{ \tfrac{1}{2}\|B(x - v)\|_2^2 + \mathcal{R}(v) \right\} \\
&= \tfrac{1}{2}\|Ax - b\|_2^2 + \lambda\mathcal{R}(x) + \lambda \max_v \left\{ -\tfrac{1}{2}\|B(x - v)\|_2^2 - \mathcal{R}(v) \right\}
\end{aligned}$$

and

$$\begin{aligned}
\hat{x} &= \arg\min_x \mathcal{J}_B(x) \\
&= \arg\min_x \left\{ \tfrac{1}{2}\|Ax - b\|_2^2 + \lambda\mathcal{R}(x) + \lambda \max_v \left\{ -\tfrac{1}{2}\|B(x - v)\|_2^2 - \mathcal{R}(v) \right\} \right\}
\end{aligned}$$

or a *saddle point* problem

$$(\hat{x}, \hat{v}) = \arg\min_x \max_v \left\{ \tfrac{1}{2}\|Ax - b\|_2^2 + \lambda\mathcal{R}(x) - \tfrac{\lambda}{2}\|B(x - v)\|_2^2 - \lambda\mathcal{R}(v) \right\}$$

Optimization Algorithm (saddle point algorithm)

This saddle-point problem is an instance of a monotone inclusion problem. Hence, the solution can be obtained using the forward-backward (FB) algorithm.

for $k = 0, 1, 2, \ldots$

$$w_k = x_k - \mu \left[A^{\mathsf{T}}(Ax_k - b) + \lambda\, B^{\mathsf{T}}B\,(v_k - x_k) \right]$$

$$u_k = v_k - \mu\lambda B^{\mathsf{T}}B(v_k - x_k)$$

$$x_{k+1} = \arg\min_{x \in \mathbb{R}^n} \left\{ \mathcal{R}(x) + \frac{1}{2\mu\lambda} \|x - w_k\|_2^2 \right\}$$

$$v_{k+1} = \arg\min_{v \in \mathbb{R}^n} \left\{ \mathcal{R}(v) + \frac{1}{2\mu\lambda} \|v - u_k\|_2^2 \right\}$$

end

This algorithm reduces to ISTA for $B = 0$ and $\mathcal{R}(x) = \|x\|_1$.