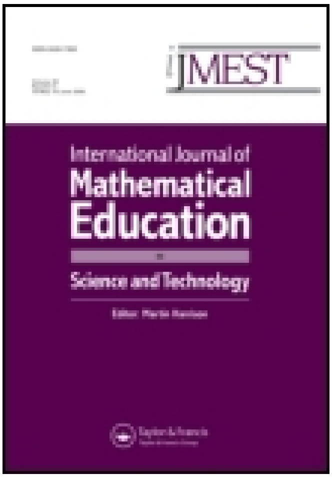

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Second-order van der Pol plane analysis

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A technique is developed, called second-order van der Pol plane analysis, for finding and classifying approximate initial conditions that lead to harmonic solutions to differential equations of the form

$$\ddot{x} + x = \varepsilon f(x, \dot{x}, t)$$

where $f(x, \dot{x}, t)$ is periodic in the variable t. The classical first-order van der Pol plane analysis may incorrectly classify critical points as centres, suggesting a harmonic solution for these initial conditions. This new technique correctly classifies all critical values. An approximate boundary curve is found in the $x\,\dot{x}$-phase plane that separates bounded solutions from unbounded ones for the forced soft spring equation

$$\ddot{x} + x + \varepsilon x^3 = F\cos\omega t$$

All of this suggests computer laboratory problems and student research problems, as the technique is general and can be applied to many classical nonlinear differential equations.

1. Introduction

The question of when a differential equation has a periodic solution is one that has occupied the best efforts of many great analysts. This effort has been driven in part by the obvious practical importance in technological applications. Only in a few cases can direct methods determine the presence of a periodic solution.

Nonlinear mechanical and electronic vibrations is an important field of study. Unfortunately the models of such vibrations often do not permit linearizations as approximations as linear vibrations are completely understood and the range of effects is limited. However, in nonlinear systems, fundamentally different motions can occur; for example, forced oscillations often exhibit harmonic and subharmonic behaviour.

In this article, we investigate finding periodic solutions to equations of the form

$$\ddot{x} + x = \varepsilon f(x, \dot{x}, t)$$

where the function $f(x, \dot{x}, t)$ is assumed to be periodic in the variable t. Generally speaking, there is no difficulty in finding an accurate numerical solution for any

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given set of initial conditions $x(0)$ and $\dot{x}(0)$. Moreover, the solutions, and hence trajectories in the phase plane, are continuous functions of the initial conditions. If we are lucky, solving such a differential equation for a given pair of initial conditions produces a phase plane trajectory which is nearly periodic. Trial-and-error solving using 'nearby' initial conditions (we call this 'tweaking') can focus in on a periodic trajectory. Once this has been found, a highly accurate Fourier series approximate solution can be generated (see [1]). The difficulty here is knowing where to initially start for a *nearly periodic trajectory*.

Thus the heart of the problem is to find initial conditions which lead to a periodic solution. One successful approach was begun by the Dutch electrical engineer Balthazar van der Pol (1889–1959). This has led to the employment of the so called (first-order) van der Pol plane and the calculation of approximate candidates for the initial conditions leading to a harmonic solution (a periodic solution of the same period as the forcing). We review this approach by discussing briefly harmonic averaging, harmonic balance, and the (first-order) van der Pol plane analysis.

We introduce a new approach to finding harmonic solutions to such differential equations. This approach produces exactly the same critical values as do the techniques listed in the previous paragraph. The traditional method employing (first-order) van der Pol plane analysis is shown to falsely classify critical values. This new approach remedies this defect. We also indicate how to produce an approximate stability boundary in the phase plane, separating bounded solutions from unbounded solutions.

All of this is illustrated through the famous undamped forced soft spring equation, one form of Duffing's equation,

$$\ddot{x} + x + \varepsilon x^3 = F \cos \omega t$$

a surprisingly simple equation which seems to embody all the good features of a nonlinear equation as well as all of the bad. Since this example has a physical interpretation, the reader will not be mislead if the independent variable t is thought of as time and the values of x as being displacements. Then the values of $\dot{x}$ and $\ddot{x}$ have the interpretations as velocity and acceleration, as is common.

Virtually effortless numerical and symbolic computation using a computer algebra system, such as *Mathematica*, make the investigations discussed herein within the grasp of the beginning student. Moreover, the techniques discussed are appropriate for application to other equations than the Duffing equations. This provides ample opportunity for students to investigate a number of issues on other important equations including the forced van der Pol equation. In this article, we only sought harmonic solutions. But subharmonic solutions are also important, and it is of interest to investigate the modifications of this approach to finding solutions having period twice or three times that of the forcing.

A homotopy approach [2] for finding initial conditions leading to periodic solutions is very effective. The second-order van der Pol plane analysis is attractive because it gives more information about all solutions to the equation. Both can be used together to find harmonic solutions, as the critical values from the van der Pol plane analysis can be used as starting values in the homotopy algorithm. This avoids the somewhat tedious 'tweaking' to find the correct initial conditions.

2. Harmonic averaging

The Dutch electrical engineer van der Pol developed a method to find approximate harmonic solutions to equations of the form

$$\ddot{x} + x = \varepsilon f(x, \dot{x}, t) \tag{2.1}$$

where $f(x, \dot{x}, t)$ is periodic in t of period $P = 2\pi/\omega$ and it is assumed that ε is small. This *method of averaging* is reminiscent of the *variation of parameters* technique commonly taught in beginning courses. One seeks a solution of period P and thus assumes the solution has a first-order Fourier series approximation of the form

$$x(t) = a\cos\omega t + b\sin\omega t \tag{2.2}$$

Numerical evidence from a variety of equations suggests that this is the correct first-order approximation and the expected constant term is zero. Imitating variation of parameters, we let $a = a(t)$ and $b = b(t)$ and assume that over the interval $[0, P]$ both $a(t)$ and $b(t)$ are slowly varying.

Differentiating, we have

$$\dot{x}(t) = -\omega a\sin\omega t + \omega b\cos\omega t + \dot{a}\cos\omega t + \dot{b}\sin\omega t$$

Imitating variation of parameters again, we set

$$\dot{a}\cos\omega t + \dot{b}\sin\omega t = 0 \tag{2.3}$$

Differentiating again, we have

$$\begin{aligned}\ddot{x}(t) &= -\omega^2 a\cos\omega t - \omega^2 b\sin\omega t - \omega\dot{a}\sin\omega t + \omega\dot{b}\cos\omega t\\ &= -(a\cos\omega t + b\sin\omega t) + \varepsilon f(a\cos\omega t + b\sin\omega t,\ -\omega a\sin\omega t + \omega b\cos\omega t, t)\end{aligned}$$

This yields the pair of equations

$$\begin{aligned}\dot{a}\cos\omega t + \dot{b}\sin\omega t &= 0\\ -\omega\dot{a}\sin\omega t + \omega\dot{b}\cos\omega t &= A(a, b, t)\end{aligned} \tag{2.4}$$

where to simplify matters, we have set

$$\begin{aligned}A(a, b, t) = (\omega^2 - 1)a\cos\omega t + (\omega^2 - 1)b\sin\omega t\\ + \varepsilon f(a\cos\omega t + b\sin\omega t,\ -\omega a\sin\omega t + \omega b\cos\omega t, t)\end{aligned}$$

This pair (2.4) has a unique solution for $\dot{a}$ and $\dot{b}$:

$$\begin{aligned}\dot{a} &= -\frac{A(a, b, t)}{\omega}\sin\omega t\\ \dot{b} &= \frac{A(a, b, t)}{\omega}\cos\omega t\end{aligned} \tag{2.5}$$

Now, however, unlike variation of parameters, we cannot integrate to find a and b since the right-hand sides of solution (2.5) are generally quite complicated. On the other hand, the right-hand sides are periodic in t of period P, so if they do not vary much over the interval $[0, P]$ and ε is small, then $a(t)$ and $b(t)$ are slowly varying as we assumed at the outset. Thus one can argue that the complicated right-hand sides can be replaced by their average values over one period. This gives rise to the equations

$$\begin{aligned} \dot{a} &= -\frac{1}{2\pi}\int_0^P A(a,b,t)\sin\omega t\,\mathrm{d}t \\ \dot{b} &= \frac{1}{2\pi}\int_0^P A(a,b,t)\cos\omega t\,\mathrm{d}t \end{aligned} \tag{2.6}$$

This pair of equations is more useful than the pair (2.5), as they are autonomous since the variable t has been integrated out.

Let (a_0, b_0) be a critical value of the system (2.6), so that $a(t)=a_0$ and $b(t)=b_0$ are solutions to the system (2.5). Then

$$x(t) = a_0\cos\omega t + b_0\sin\omega t$$

is a first-order approximation to a periodic solution of period P (a harmonic solution).

2.1. *Duffing's equation*

Let us apply the method of averaging to the undamped Duffing equation

$$\ddot{x} + x + \varepsilon x^3 = F\cos\omega t \tag{2.7}$$

where $F > 0$ and ε is assumed to be small. We guess that the solution $x(t)$ has the form in (2.2). Differentiating once, we set

$$\dot{a}\cos\omega t + \dot{b}\sin\omega t = 0$$

and upon differentiating a second time we obtain the equation

$$\begin{aligned} \ddot{x}(t) &= (\omega\dot{b} - \omega^2 a)\cos\omega t - (\omega\dot{a} - \omega^2 b)\sin\omega t \\ &= -(a\cos\omega t + b\sin\omega t) - \varepsilon(a\cos\omega t + b\sin\omega t)^3 + F\cos\omega t \end{aligned}$$

Using the trigonometric identity for $\cos^3\omega t$ and $\sin^3\omega t$, this becomes

$$\begin{aligned} \ddot{x}(t) &= (\omega\dot{b} - \omega^2 a)\cos\omega t - (\omega\dot{a} + \omega^2 b)\sin\omega t \\ &= \left\{-a - \frac{3\varepsilon}{4}a(a^2+b^2) + F\right\}\cos\omega t + \left\{-b - \frac{3\varepsilon}{4}a(a^2+b^2)\right\}\sin\omega t \\ &\quad - \frac{\varepsilon}{4}a(a^2-3b^2)\cos 3\omega t - \frac{\varepsilon}{4}b(3a^2-b^2)\sin 3\omega t \end{aligned}$$

This gives rise to the pair of equations (2.4) where

$$\begin{aligned} A(a,b,t) = {} & \left\{a(\omega^2-1) - \frac{3\varepsilon}{4}a(a^2+b^2) + F\right\}\cos\omega t + \left\{b(\omega^2-1) - \frac{3\varepsilon}{4}b(a^2+b^2)\right\} \\ & \sin\omega t - \frac{\varepsilon}{4}\left\{a(a^2-3b^2)\cos 3\omega t + b(3a^2-b^2)\sin 3\omega t\right\} \end{aligned}$$

and subsequently, to the equations (2.5). Applying van der Pol's averaging, it follows that

$$\begin{aligned} \dot{a} &= \frac{1}{8\omega}\{4b(1-\omega^2) + 3\varepsilon b(a^2+b^2)\} \\ \dot{b} &= \frac{1}{8\omega}\{4a(\omega^2-1) + 4F - 3\varepsilon a(a^2+b^2)\} \end{aligned}$$

The critical values for this autonomous system are solutions to the simultaneous equations

$$\begin{aligned} b\{4(1-\omega^2)+3\varepsilon(a^2+b^2)\} &= 0 \\ a\{4(1-\omega^2)+3\varepsilon(a^2+b^2)\} &= 4F \end{aligned} \tag{2.8}$$

In solving the simultaneous equations (2.8), if $b \neq 0$, then it follows that $F = 0$ contrary to assumption; hence $b = 0$ and critical values are of the form $(a_0, 0)$ where a_0 is root of the cubic equation

$$3\varepsilon a^3 + 4(1-\omega^2)a - 4F = 0 \tag{2.9}$$

We shall see precisely these critical values arise in three more ways.

3. Harmonic balance

In order to bolster the approach above, we consider again the undamped Duffing equation (2.7) and approximate a harmonic solution by

$$x(t) = a\cos\omega t + b\sin\omega t$$

where a and b are constants. Computing,

$$\ddot{x}(t) = -a\omega^2\cos\omega t - b\omega^2\sin\omega t$$

and

$$\begin{aligned} x^3(t) = {}& \frac{3}{4}a(a^2+b^2)\cos\omega t + \frac{3}{4}b(a^2+b^2)\sin\omega t \\ & + \frac{1}{4}a(a^2-3b^2)\cos 3\omega t + \frac{1}{4}b(3a^2-b^2)\sin 3\omega t \end{aligned} \tag{3.1}$$

If we assume that the coefficients of the higher frequency terms are small and thus may be neglected, then upon substitution into equation (2.7) and equating coefficients, we have

$$\begin{aligned} a\left\{(1-\omega^2)+\frac{3\varepsilon}{4}(a^2+b^2)\right\} &= F \\ b\left\{(1-\omega^2)+\frac{3\varepsilon}{4}(a^2+b^2)\right\} &= 0 \end{aligned} \tag{3.2}$$

and these equations are equivalent to the equations (2.8).

4. The classical van der Pol plane

In this section we briefly develop the approach used by several authors including Grimshaw [3, Chapter 9] and Jordan and Smith [4, Chapter 7]. The method is quite general but we shall only consider the undamped Duffing's equation (2.7).

As in the method of averaging, we assume the solution of equation (2.7) to have the form

$$x(t) = a(t)\cos\omega t + b(t)\sin\omega t$$

where $a(t)$ and $b(t)$ are assumed to be slowly varying over the interval $[0, P]$. We substitute this back into the equation again, but this time we ignore not only the terms involving $\cos 3\omega t$ and $\sin 3\omega t$ as we did for harmonic balance, but additionally ignore the second derivatives $\ddot{a}$ and $\ddot{b}$. Differentiating, we have

$$\dot{x}(t) = (\dot{a} + \omega b)\cos\omega t + (\dot{b} - \omega a)\sin\omega t$$

and (neglecting $\ddot{a}$ and $\ddot{b}$)

$$\ddot{x}(t) = (-\omega^2 a + 2\omega\,\dot{b})\cos\omega t + (-2\omega\,\dot{a} - \omega^2 b)\sin\omega t$$

From equation (3.1), neglecting $\cos 3\omega t$ and $\sin 3\omega t$,

$$x^3(t) = \frac{3}{4}a(a^2 + b^2)\cos\omega t + \frac{3}{4}b(a^2 + b^2)\sin\omega t$$

Substituting into equation (2.7), we obtain

$$\left[2\omega\,\dot{b} - a\left\{(\omega^2 - 1) - \frac{3\varepsilon}{4}(a^2 + b^2)\right\}\right]\cos\omega t$$

$$+\left[-2\omega\,\dot{a} - b\left\{(\omega^2 - 1) - \frac{3\varepsilon}{4}(a^2 + b^2)\right\}\right]\sin\omega t = F\cos\omega t$$

We match the coefficients of the left-hand side with those of the right-hand side (harmonic balance), and obtain the autonomous system:

$$\begin{aligned} \dot{a} &= -\frac{b}{2\omega}\left\{(\omega^2 - 1) - \frac{3\varepsilon}{4}(a^2 + b^2)\right\} \equiv A(a, b) \\ \dot{b} &= \frac{a}{2\omega}\left\{(\omega^2 - 1) - \frac{3\varepsilon}{4}(a^2 + b^2)\right\} + \frac{F}{2\omega} \equiv B(a, b) \end{aligned} \tag{4.1}$$

The ab-plane for the system (4.1) is called the *(first-order) van der Pol plane*. Critical values for this system are solutions to the simultaneous equations $A(a,\ b) = 0$ and $B(a,\ b) = 0$. Setting these equations to zero and simplifying, we again have a pair of equations equivalent to the equations (2.8) and (3.2):

$$\begin{aligned} b\left\{(\omega^2 - 1) - \frac{3\varepsilon}{4}(a^2 + b^2)\right\} &= 0 \\ a\left\{(\omega^2 - 1) - \frac{3\varepsilon}{4}(a^2 + b^2)\right\} + F &= 0 \end{aligned} \tag{4.2}$$

As before, b is forced to be 0, and the critical values are of the form $(a,\ 0)$ where a is a solution to the cubic equation

$$\frac{3\varepsilon}{4}a^3 + a(1 - \omega^2) = F \tag{4.3}$$

Critical values $(a,\ 0)$ are classified by the eigenvalues of the linearization matrix

$$\begin{bmatrix} \dfrac{\partial A}{\partial a}(a, 0) & \dfrac{\partial A}{\partial b}(a, 0) \\ \dfrac{\partial B}{\partial a}(a, 0) & \dfrac{\partial B}{\partial b}(a, 0) \end{bmatrix} \tag{4.4}$$

4.1. A misclassification

The critical values (a, 0) obtained from equation (4.3) are intended to yield approximate initial conditions for harmonic solutions to equation (2.7). To see how this works we consider the following example.

Example 4.1. Find a harmonic solution to the undamped Duffing equation

$$\ddot{x} + x - \frac{x^3}{6} = \frac{1}{3}\cos\frac{3}{5}t \tag{4.5}$$

Equation (4.5) represents a forced soft spring. In this case there are three real roots to the cubic equation (4.3):

$$\begin{aligned} a_1 &= -2 \cdot 48831 \\ a_2 &= 0 \cdot 554052 \\ a_3 &= 1 \cdot 934256 \end{aligned} \tag{4.6}$$

Not all three critical values lead to harmonic solutions as a classification of the associated eigenvalues of the linearization matrix for the system (4.1) shows. This linearization matrix at a critical value $(a, 0)$ has the form (4.4). This eigenvalue analysis suggests that both $(a_1, 0)$ and $(a_2, 0)$ are centres as the eigenvalues are pure imaginary and that $(a_3, 0)$ is a saddle point as the eigenvalues are complex having positive and negative real parts. Representative paths in the ab-plane for the system (4.1) with $\varepsilon = -1/6$, $F = 1/3$ and $\omega = 3/5$ are shown in figure 4.1.

Remember that these critical values are only approximate candidates for the initial conditions for a harmonic solution. Upon solving equation (4.5) initially using starting values $x(0) = a_i$ and $\dot{x}(0) = 0$, by 'tweaking' the starting values, we were able to find a very nice harmonic solution for initial conditions quite near $(a_2, 0)$. Indeed, setting $x(0) = 0 \cdot 5507900342$ and $\dot{x}(0) = 0$, a very nice harmonic solution was generated accurate to 10 decimal places over the interval $[0,\ 5P] \approx [0,\ 60]$. We hesitate to generate numerical solutions over long intervals, particularly for nonlinear equations, as numerical roundoff error, truncation error, and algorithm error accumulate with increasing t. We generated a Fourier series solution using the technique discussed in [1] that verified the 10 decimal place accuracy.

We were unable to find any bounded solutions of equation (4.5) for initial conditions near $(a_1, 0)$ and thus the classification of $(a_1, 0)$ as a centre is incorrect. The behaviour of the numerical solutions having initial conditions near $(a_3, 0)$ suggested that $(a_3, 0)$ is correctly classified as a saddle point.

In Figure 4.1, a path passing through $(a_3, 0)$ is shown slightly thicker. This path is a separatrix as it separates the plane into regions of distinct behaviour. In particular, inside the loop of this separatrix all paths are closed curves enclosing the centre (a_2, 0) suggesting periodic oscillatory behaviour.

5. The second-order van der Pol plane

In this section, we repeat the development in the preceding section, only this time, we do not neglect the second derivatives $\ddot{a}$ and $\ddot{b}$. We again assume the solution to equation (2.7) has the form

$$x(t) = a(t)\cos\omega t + b(t)\sin\omega t$$

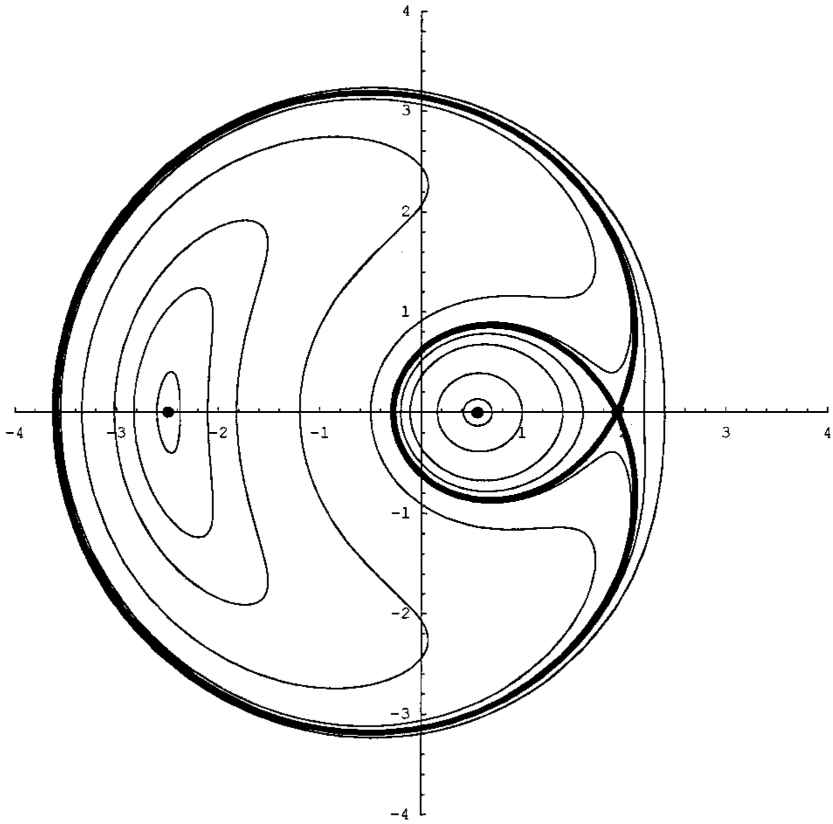

Figure 4.1. The first-order van der Pol plane.

and upon differentiating, we have

$$\dot{x}(t) = (\dot{a} + \omega b)\cos\omega t + (\dot{b} - \omega a)\sin\omega t$$

and

$$\ddot{x}(t) = (\ddot{a} - \omega^2 a + 2\omega\dot{b})\cos\omega t + (\ddot{b} - 2\omega\dot{a} - \omega^2 b)\sin\omega t$$

As before when calculating x^3, we shall neglect the terms involving $\cos 3\omega t$ and $\sin 3\omega t$. When we substitute into equation (2.7), we obtain

$$\left[\ddot{a} + 2\omega\dot{b} - a\left\{(\omega^2 - 1) - \frac{3\varepsilon}{4}(a^2 + b^2)\right\}\right]\cos\omega t + \left[\ddot{b} - 2\omega\dot{a} - b\left\{(\omega^2 - 1) - \frac{3\varepsilon}{4}(a^2 + b^2)\right\}\right]\sin\omega t = F\cos\omega t$$

Matching the coefficients and simplifying, it follows that

$$\begin{aligned} \ddot{a} &= -2\omega\dot{b} + a\left\{(\omega^2 - 1) - \frac{3\varepsilon}{4}(a^2 + b^2)\right\} + F \\ \ddot{b} &= 2\omega\dot{a} + b\left\{(\omega^2 - 1) - \frac{3\varepsilon}{4}(a^2 + b^2)\right\} \end{aligned} \tag{5.1}$$

We turn this coupled second-order system into a coupled first-order system by setting $\dot{a} = \alpha$ and $\dot{b} = \beta$, and the system becomes

$$\begin{aligned} \dot{a} &= \alpha \\ \dot{\alpha} &= -2\omega\beta + a\left\{(\omega^2 - 1) - \frac{3\varepsilon}{4}(a^2 + b^2)\right\} + F \\ \dot{b} &= \beta \\ \dot{\beta} &= 2\omega\alpha + b\left\{(\omega^2 - 1) - \frac{3\varepsilon}{4}(a^2 + b^2)\right\}. \end{aligned} \tag{5.2}$$

The critical values to this system of first-order equations automatically have $\alpha = 0$ and $\beta = 0$ and thus the critical values are of the form $(a, 0, b, 0)$ where a and b are solutions to the pair of equations:

$$\begin{aligned} a\left\{(\omega^2 - 1) - \frac{3\varepsilon}{4}(a^2 + b^2)\right\} + F = 0 \\ b\left\{(\omega^2 - 1) - \frac{3\varepsilon}{4}(a^2 + b^2)\right\} = 0 \end{aligned} \tag{5.3}$$

which are the equations (4.2) and equivalent to equations (2.8) and (3.2). Thus b is forced to be 0 and critical values are of the form $(a, 0, 0, 0)$ where a is a root of the cubic

$$\frac{3\varepsilon}{4}a^3 + a(1 - \omega^2) = F \tag{5.4}$$

The ab-plane for the system (5.2) is called the *second-order van der Pol plane*.

5.1. A correct classification

One advantage of considering this second-order system is the correct classification of all three critical values in Example 4.1. The linearization matrix for the system (5.2) at a critical value $(a_0, 0, 0, 0)$ is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ A_1 & 0 & 0 & -2\omega \\ 0 & 0 & 0 & 1 \\ 0 & 2\omega & B_1 & 0 \end{bmatrix}$$

where

$$\begin{aligned} A_1 &= \omega^2 - 1 - \frac{9\varepsilon}{4}a_0^2 \\ B_1 &= \omega^2 - 1 - \frac{3\varepsilon}{4}a_0^2 \end{aligned}$$

The eigenvalues of this matrix classify the critical points.

Returning to Example 4.1, the three critical values are of the form $(a_i, 0, 0, 0)$ for $i = 1,2,3$, see (4.6). The four eigenvalues of the linearization matrix for a_1 are complex having the same imaginary part and both positive and negative real parts, suggesting the critical value $(a_1, 0, 0, 0)$ is a node not a centre. The four eigenvalues of the linearization matrix for a_2 are pure imaginary suggesting a centre as before. The four eigenvalues for a_3 are mixed: two real eigenvalues—one positive and the other negative, and two pure imaginary eigenvalues suggesting $(a_3, 0, 0, 0)$ is a saddle point.

A second-order van der Pol plane portrait is shown in figure 5.1. The path shown passing through $(a_3, 0)$ and forming a loop containing the origin and the centre $(a_2, 0)$ is a separatrix. Paths with starting points inside the loop represent bounded oscillatory periodic solutions for $a(t)$ and $b(t)$. All other paths are unbounded.

In this figure, seven representative paths $(a(t), b(t))$ are shown.

Note that even if a path $(a(t), b(t))$ is periodic, say of period L, the function

$$x(t) = a(t)\cos\omega t + b(t)\sin\omega t$$

need not be periodic. It will be periodic precisely when L is commensurable with $P = 2\pi/\omega$; that is, L/P is a rational number. Thus the approximate solution $x(t)$

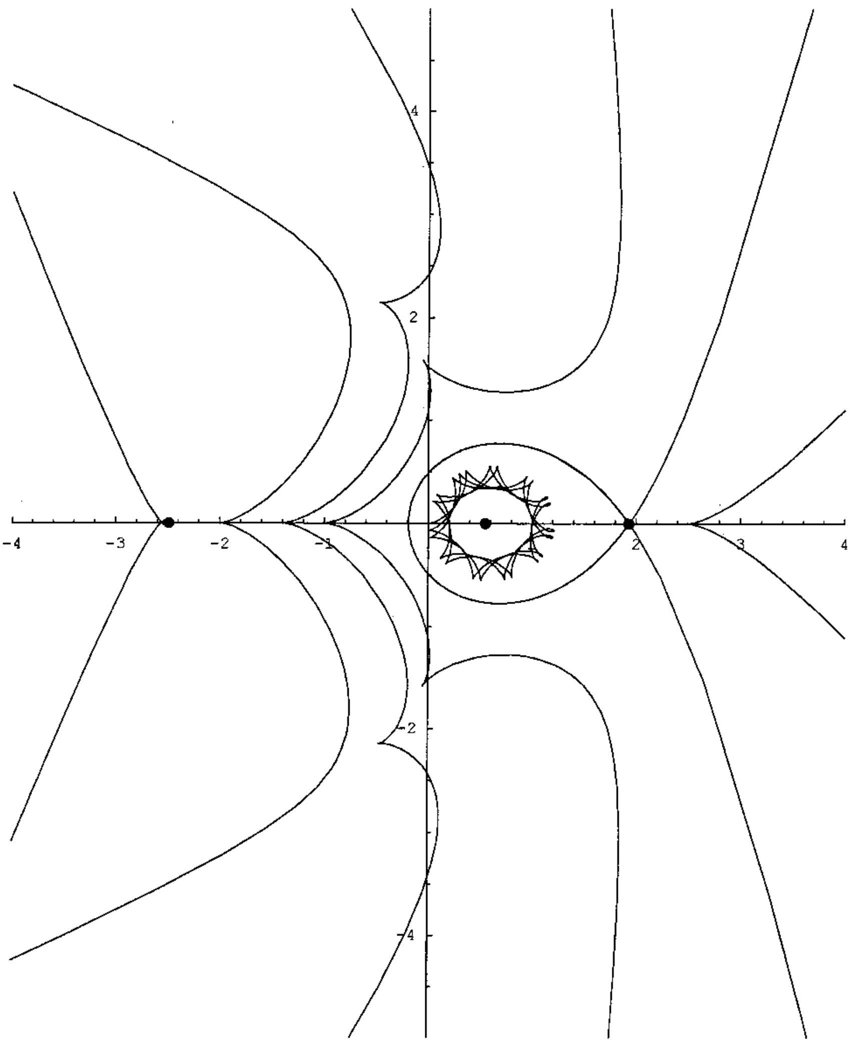

Figure 5.1. The second-order van der Pol plane.

may well be bounded, but need not be periodic and thus the true solution, being reasonably close to the approximation, may behave in the same way.

6. Stability boundary

The second-order van der Pol plane is useful in determining an approximate stability boundary in the $x\,\dot{x}$-phase plane. For example, if we consider the equation of Example 4.1 again,

$$\ddot{x} + x - \frac{x^3}{6} = \frac{1}{3}\cos\frac{3}{5}t$$

it is easy to see that for certain initial conditions trajectories become unbounded very quickly, while for others the trajectory may be well behaved for quite some time before abruptly becoming unbounded. And yet for other initial conditions we have, if not periodic trajectories, at least bounded trajectories over very long time intervals. It would seem to be a fundamental question to ask:

> *For what pairs of initial values is the solution to the initial value problem bounded?*

By trial and error, we produced a boundary in the phase plane (shown in figure 6.1) which appears as a closed curve: bounded solutions arise from initial conditions within the curve and unbounded ones arise from outside.

We call this curve the *stability boundary*.

The points shown were determined in the following way. We solved the initial value problems by trial-and-error for a variety of initial conditions over the time interval [0, 200] at working precision 16 (machine default). At this precision, solutions are produced rapidly and the trial-and-error is not particularly tedious. For example, the solution produced with $x(0) = 1{\cdot}893807$ and $\dot{x}(0) = 0$ is bounded, but that produced with $x(0) = 1\cdot 893808$ and $\dot{x}(0) = 0$ is unbounded. We select the point $(1\cdot 893808,\ 0)$ to be a point on the stability boundary. Similarly, the solution produced with $x(0) = -\,0\cdot 2171518$ and $\dot{x}(0) = 0$ is bounded, and that produced with $x(0) = -0\cdot 2171519$ and $\dot{x}(0) = 0$ is unbounded. We choose the point $(-\,0\cdot 2171519,\ 0)$ to be a point on the stability boundary. These two points are the extremities of the stability curve on the x-axis. Stepping between these two x-values and 'tweaking' the $\dot{x}(0)$ values to find the threshold between bounded and unbounded solutions, in a similar fashion, we produced the other boundary points shown in the figure.

The curve passing through these points was empirically derived as one loop of the Lemniscate of Bernoulli $\rho^2 = a^2\ \cos 2\theta$ (in polar coordinates) translated so that the vertex of the loop occurs at $(1\cdot 893807,\ 0)$. This is an accident, as for other values of ε, F, and ω the stability boundary has a different shape. However, this curve is useful in displays.

In figure 6.2, the separatrix loop shown in figure 5.1 passing through the point $(a_3,\ 0)$ is dashed and overlaid with the stability curve of figure 6.1. This loop is slightly larger and encloses the curve passing through the boundary points. This clearly demonstrates that we can interpret the separatrix loop as an approximation to the stability boundary. This separatrix loop certainly quickly gives us an indication of where to seek bounded solutions and 'tweaking' initial conditions

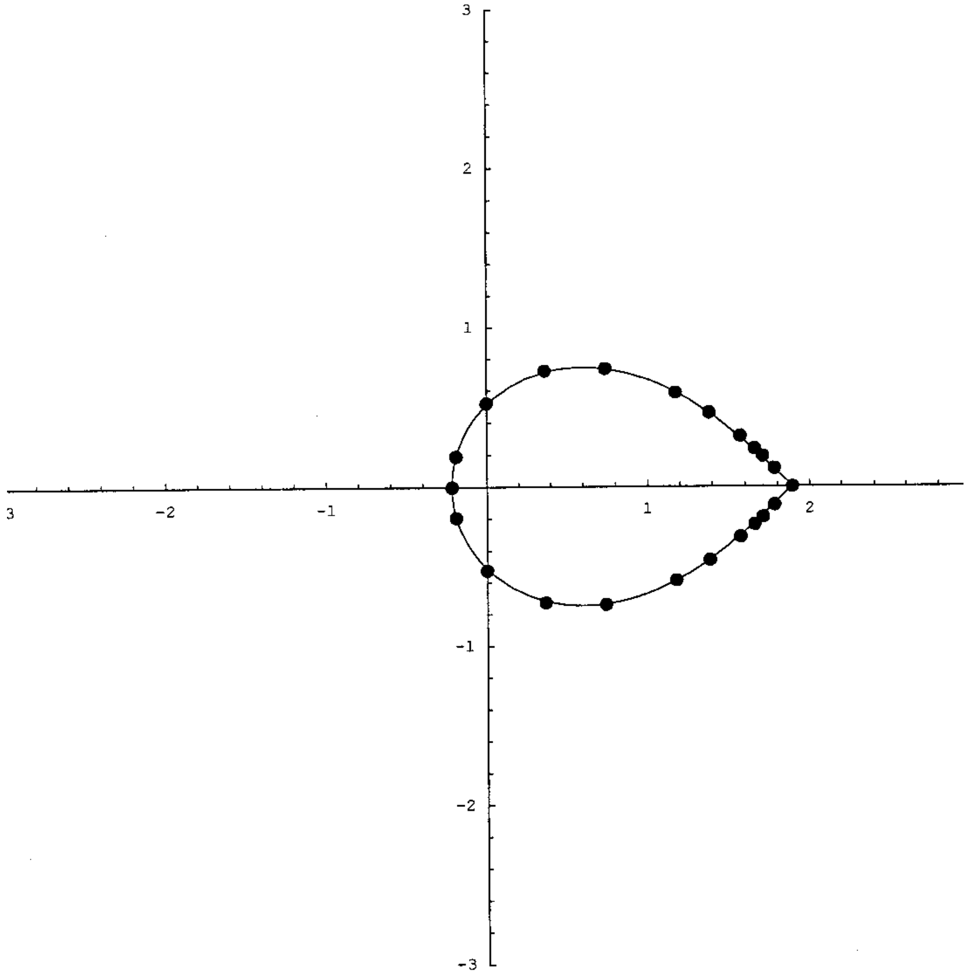

Figure 6.1. The stability boundary for $\ddot{x} + x - x^3/6 = 1/3\cos(3/5)t$.

near by (and solving numerically at higher precisions) can produce the stability boundary if a more accurate boundary is needed.

7. Critical point characterization

One must wonder if the example above with $\omega = 3/5$ is representative. There are other cases to consider as the cubic equation (5.4) could possibly have only one real root.

Example 7.1. Find a harmonic solution to the undamped Duffing equation

$$\ddot{x} + x - \frac{x^3}{6} = \frac{1}{3}\cos 2t \tag{7.1}$$

For this value of $\omega = 2$, there is a single real root to the cubic equation (5.4), $a_4 = -0 \cdot 11105404321227794$. The critical value $(a_4, 0, 0, 0)$ is classified as a centre in both the first- and second-order van der Pol planes. However, for this example, not all paths in the second-order van der Pol plane are bounded and there is no

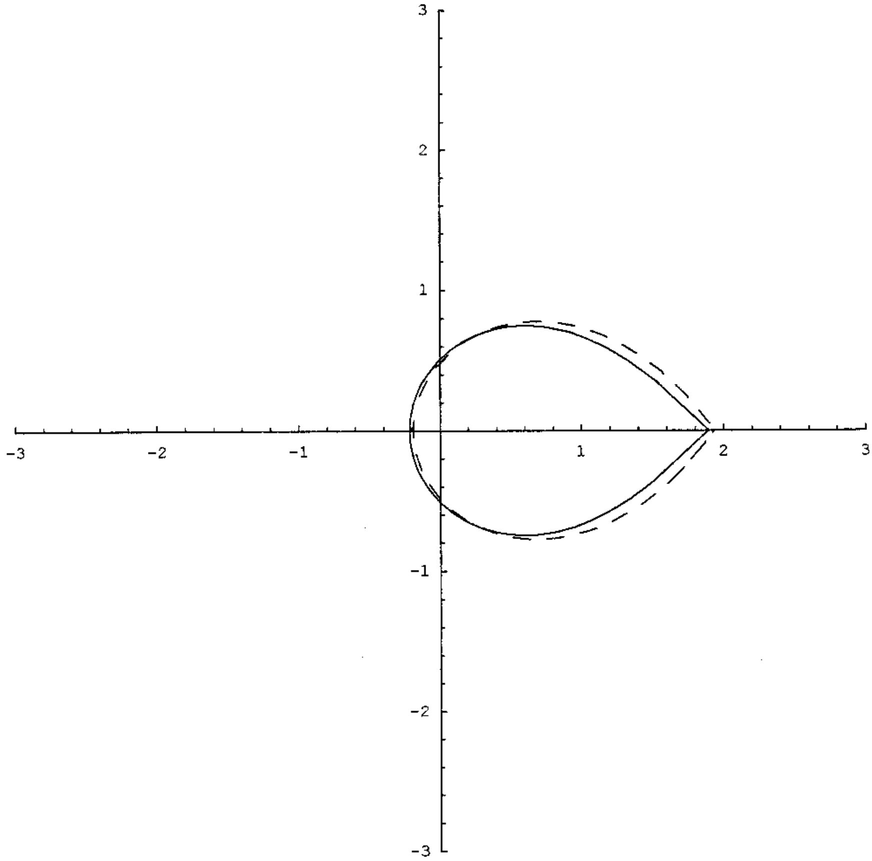

Figure 6.2. The stability boundary curve and the loop of the separatrix shown dashed.

separatrix to guide us; some paths can be quite pleasing suggesting periodic behaviour while others become unbounded (figure 7.1). All paths shown in this figure were produced with initial values of the form $(a, 0, 0, 0)$.

The initial values $x(0) = a_4$, $\dot{x}(0) = 0$, produces quite a nice harmonic solution whose trajectory is shown in figure 7.2.

In figure 7.3, we plot the contour $f(a, \omega) = 0$ of the function of two variables

$$f(a, \omega) = 3\varepsilon a^3 + 4a(1 - \omega^2) - 4F$$

This contour shows how the critical values arise as roots of the cubic equation (5.4).

Horizontal lines corresponding to fixed ω-values intersect this contour in either one or three points; see figure 7.4.

Since we are only interested in positive values of ω, starting with $\omega_0 = 0$ and slowly increasing, we see that we have three real roots to the cubic (5.4) as there are three points of intersection with the contour plot and the horizontal line $\omega = \omega_0$. In increasing order, we label these points (a_i, ω_0) and note that (a_1, ω_0) is always on the unbounded branch of the contour plot. The points (a_2, ω_0) and (a_3, ω_0) are on the bounded loop of the contour. By examining the eigenvalues of the linearization matrix, these points are classified exactly as they were in Example 4.1: $(a_1, 0, 0, 0)$ is an unstable node, $(a_2, 0, 0, 0)$ is a centre, and $(a_3, 0, 0, 0)$

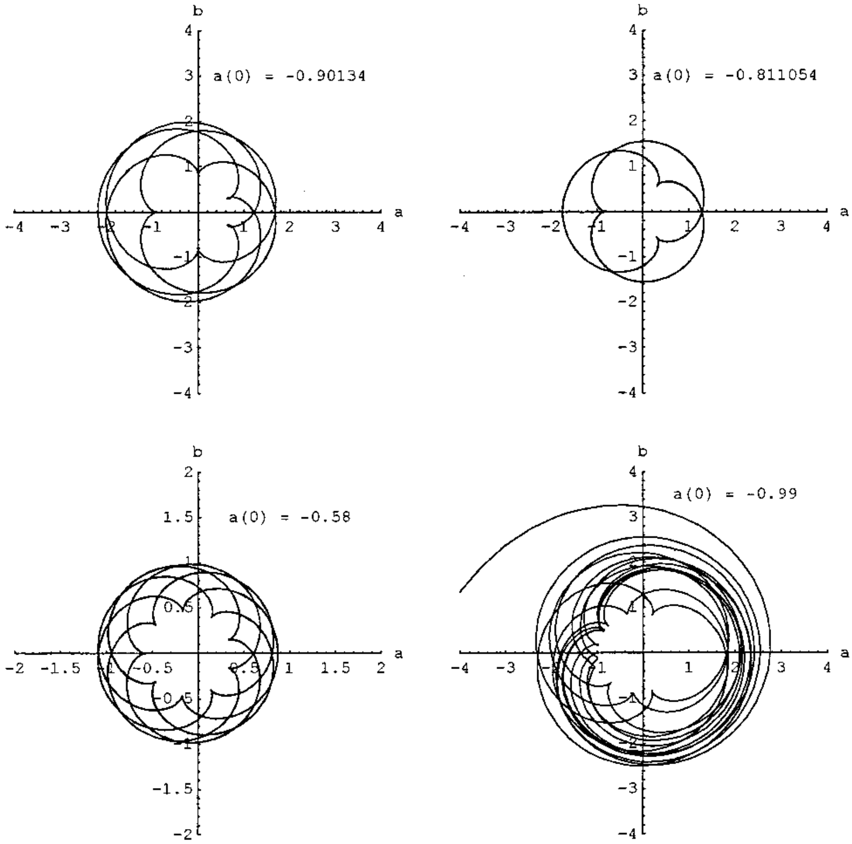

Figure 7.1. Paths in the second-order van der Pol plane.

is a saddle point. As the horizontal line $\omega = \omega_0$ approaches the top of the loop, the centre and the saddle points coalesce into a single double root. This happens for $\omega_0 = 0 \cdot 73872853728$, and a_2 and a_3 become $a_5 = 1 \cdot 10064$. The eigenvalues of the linearization matrix for the critical value $(a_5, 0, 0, 0)$ are pure imaginary, and hence we have a centre. The point a_1 has become $-2 \cdot 20128$ and is classified as a repelling node.

Hence a *centre − saddle → centre* bifurcation takes place at $\omega = 0 \cdot 73872853728$. Another bifurcation takes place at $\omega = 0 \cdot 7664117$. For $\omega_0 \leqslant 0 \cdot 7664117$, the corresponding unique critical value is classified as a repelling node; however, for $\omega_0 > 0 \cdot 7664117$, the critical value becomes a centre.

There are some more interesting features of the situation for this undamped Duffing's equation with $\varepsilon = -1/6$ and $F = 1/3$, moderately small values. For $\omega < 0 \cdot 73872853728$, there is a critical point which is a saddle point and gives rise to a separatrix in the second-order van der Pol plane: this is useful in approximating the region in the $x\ \dot{x}$-phase plane from which bounded solutions arise. If $\omega > 0 \cdot 73872853728$, then there is no such separatrix to guide us, but trial-and-error can produce such regions in the $x\ \dot{x}$-phase plane. For example, if $\omega = 0 \cdot 739$, the initial conditions $(x(0) \cdot 0)$ produce bounded solutions for $0 \cdot 965 \leqslant x(0) \leqslant 1 \cdot 156$, a very short interval. If $\omega > 0 \cdot 74$, we were unable to find any bounded solutions for $x(0) > 0$.

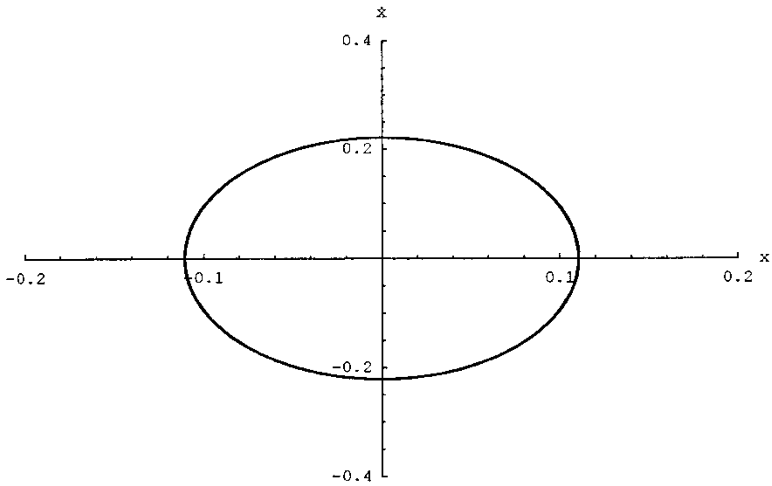

Figure 7.2. The harmonic trajectory for $x(0) = -0{\cdot}1110540$, $\dot{x}(0) = 0$.

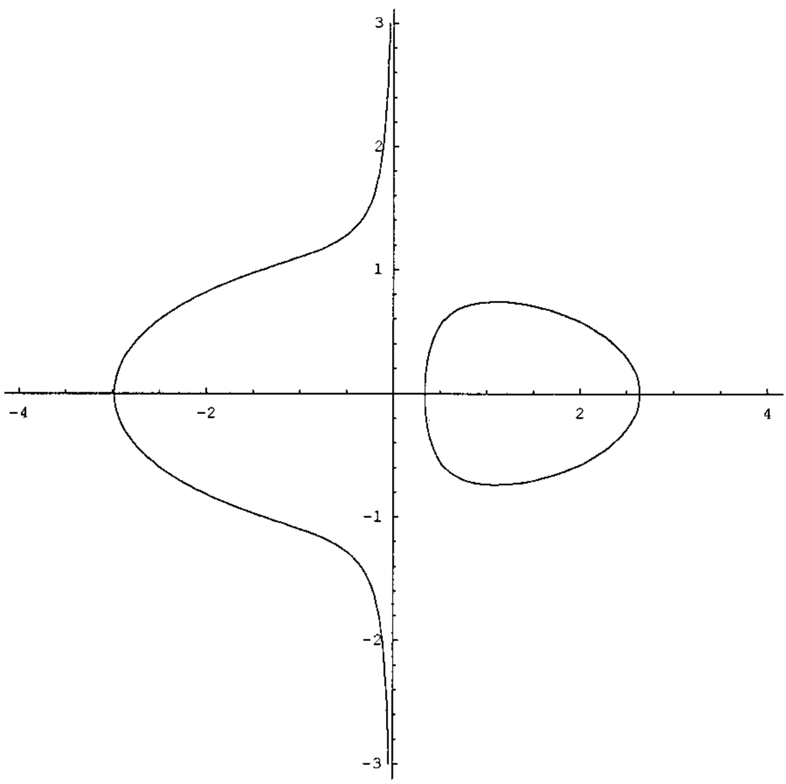

Figure 7.3. The cubic $f(a, \omega) = 0$.

However, for negative values of $x(0)$, bounded solutions could be found. Generally speaking, if $0 \cdot 73872853728 < \omega < 0 \cdot 7664117$, then a harmonic solution may be found for $x(0) < 0$. The interval along the x-axis from which bounded solutions arise is very short. For example, if $\omega = 0 \cdot 76$, a very nice harmonic

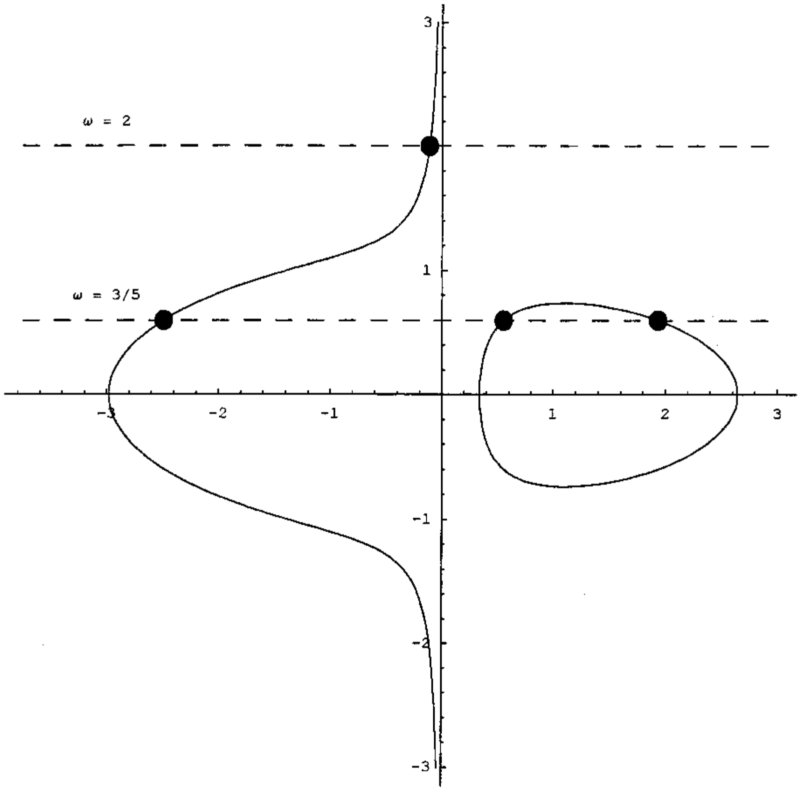

Figure 7.4. The critical values for Examples 4.1 and 7.1.

solution is found for $x(0) = -2 \cdot 1$ and $\dot{x}(0) = 0$. Unbounded solutions are found for $x(0) = -2 \cdot 04$ and for $x(0) = -2 \cdot 226$ (and $\dot{x}(0) = 0$). Thus the interval has length $0 \cdot 186$.

Finally, if $\omega = 0 \cdot 7664118$, just large enough to have the critical value become a centre, there is a very nice bounded solution for $x(0) = -2 \cdot 133290575$, $\dot{x}(0) = 0$. A periodic solution can be found near by 'tweaking' the $x(0)$ value. By trial-and-error, we produced a rather interesting stability boundary in the $x\,\dot{x}$-phase plane shown in figure 7.5.

8. Conclusions

The computer algebra system *Mathematica* was used for all the symbolic, numeric and graphical calculations in this article. Such systems make trial-and-error experimental investigations almost effortless. Moreover, the software permits us to keep rather than neglect some terms such as $\ddot{a}$ and $\ddot{b}$ above. By being able to keep these terms, we were able to correct a misclassification at no extra effort on our part.

This type of exploratory investigation opens a wide range of opportunities for computer laboratory experiments and student research. How does the analysis depend upon the size of the parameter ε? Can we repeat the analysis without neglecting sin $3\omega t$ and cos $3\omega t$? Another interesting project would be to fix ε and ω,

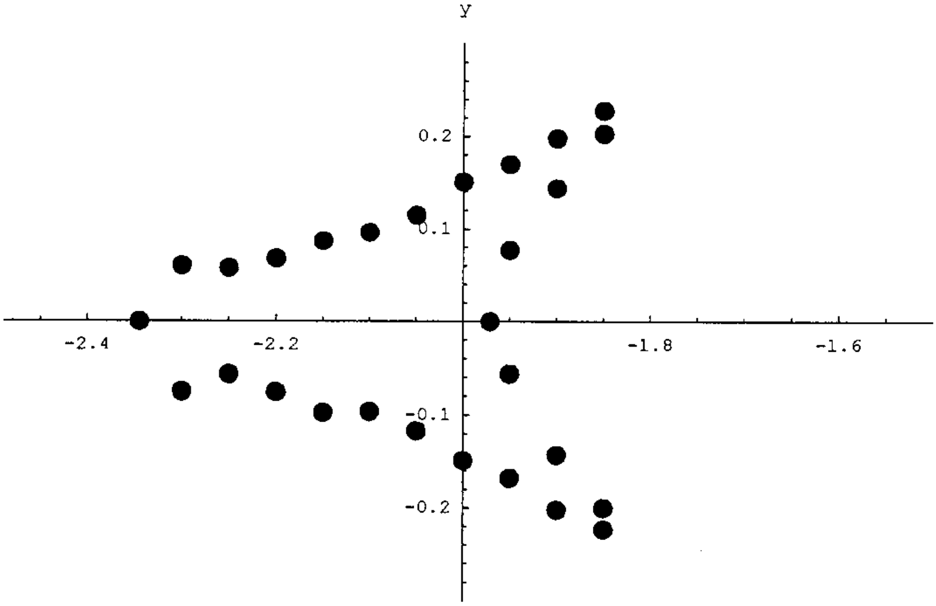

Figure 7.5. The stability boundary for $\ddot{x} + x - (x^3/6) = 1/3 \cos \omega t$ with $\omega = 0{\cdot}7664118$.

and to vary F. How do the roots to the cubic (5.4) behave? How do the harmonic solutions change? How does the stability boundary change?

Nonlinear vibration models are still important as a field of study in mechanics. Any computer technique that can simplify or expedite the investigations of these vibrational models should be of some interest to both mathematicians and engineers.

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