Ultraharmonic and Subharmonic Resonance in an Oscillator†

by

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Summary: Different order subharmonic and ultraharmonic resonance curves for as oscillator with a cubic non-linearity are obtained theoretically by van der Pol's method. Their distinctive features are discussed. Experimental results obtained with the help of an electronic differential analyser are compared with the theoretical curves.

1. Introduction

A self-excited oscillator may exhibit harmonic, ultraharmonic or subharmonic resonance when it is subjected to a periodic signal.[1-5] In harmonic resonance which occurs when the frequency of the external signal is approximately equal to that of the oscillator, the free oscillation is suppressed and the oscillator executes only the forced oscillation. On the other hand in ultraharmonic or subharmonic resonance the free oscillation is not suppressed but its amplitude is altered and the frequency modified so as to be equal to the multiple or submultiple of that of the external signal. In ultraharmonic resonance the modification is due to the ultraharmonic of the forced oscillation having a frequency nearly equal to that of the oscillator. Its amplitude is exclusively determined by that of the external signal for small non-linearities in the oscillator and hence the general characteristics of ultraharmonic resonance are similar to those of harmonic resonance, there being variations only in details. However, in subharmonic resonance it is the difference frequency signal having the frequency approximately equal to that of the oscillator which modifies the free oscillation. Since the amplitude of the difference frequency signal is determined by the amplitude of both the free and forced oscillation, the characteristics of subharmonic resonance are quite distinct from the harmonic or ultraharmonic resonance.

Harmonic resonance curves have been studied in detail by several workers but ultraharmonic and subharmonic resonance curves have received comparatively less attention. In the present paper different order ultraharmonic and subharmonic resonance curves of an oscillator with a cubic non-linearity are obtained following the method used by van der Pol. The criteria for determining the stability of the resonance condition are deduced and the details of the characteristic features of the resonance curves are studied. Experimental data obtained with the help of a differential analyser are also discussed.

2. Theoretical Derivation of Resonance Curves

Theoretical study of the resonance characteristics of a self-excited oscillator subjected to an external signal involves the solution of its describing differential equation. The exact form of the equation is dependent on the circuit arrangement of the oscillator. However, the representative differential equation may be taken to be of the form

$$\ddot{x} - a\dot{x} + \mu f(x, \dot{x}) + \omega_0^2 x = E \sin \omega t \qquad \ldots\ldots\ldots(1)$$

The oscillator has the frequency ω_0 and initial negative damping coefficient a. $\mu f(x, \dot{x})$ is the non-linearity in the oscillator. The external periodic signal is assumed to be sinusoidal having the frequency ω and amplitude E.

There is no analytical method for obtaining the accurate solution of the above equation. However, if a and μ are assumed to be small, approximate solutions may be obtained starting with a periodic solution and choosing its phase and amplitude so as to satisfy the equation. The starting solution has to be chosen from physical considerations.

In general, the oscillator is expected to execute a forced oscillation of frequency ω and a

† Manuscript received 9th March 1959. (Paper No. 509.)

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U.D.C. No. 621.373.4

free oscillation the frequency of which will be close to ω_0. When ω is approximately equal to $n\omega_0$, where n is an integral ratio, it is observed that the free oscillation frequency is modified to ω/n. The locking of the frequency of free oscillation to an integral multiple of that of the external signal is the resonance phenomenon and the range of the external frequency over which it occurs is the zone of resonance.

Thus when $\omega \cong \omega_0/n$, the starting periodic solution may be taken to be

$$x = A\sin(\omega t+\phi)+A_n\sin(n\omega t+\phi_n) \qquad \ldots\ldots\ldots(2)$$

The first term represents the forced oscillation and the second the modified free oscillation. In general, A, ϕ, A_n and ϕ_n are time varying functions. But there may be equilibrium states for which they attain constant values.

Differentiating (2) twice:

$$\dot{x} = \omega A\cos(\omega t+\phi)+\dot{A}\sin(\omega t+\phi)+ +A\dot{\phi}\cos(\omega t+\phi)+n\omega A_n\cos(n\omega t+\phi_n)+ +\dot{A}_n\sin(n\omega t+\phi_n)+A_n\dot{\phi}_n\cos(n\omega t+\phi_n)$$

$$\ddot{x} = -\omega^2 A\sin(\omega t+\phi)+2\omega\dot{A}\cos(\omega t+\phi)- -2\omega A\dot{\phi}\sin(\omega t+\phi)-n^2\omega^2 A_n\sin(n\omega t+\phi_n)+ +2n\omega\dot{A}_n\cos(n\omega t+\phi_n)-2n\omega A_n\dot{\phi}_n\sin(n\omega t+\phi_n).$$

Second order terms like $\ddot{A}$, $\ddot{\phi}$ have been neglected since a and μ are assumed to be small.

Substituting in eqn. (1), neglecting second order terms and equating the co-efficients of like terms on the two sides:

$$2\omega A\dot{\phi} = -E\cos\phi+(\omega_0^2-\omega^2)A+a_1 \qquad \ldots\ldots\ldots(3a)$$

$$2\omega\dot{A} = -E\sin\phi+a\omega A-b_1 \qquad \ldots\ldots\ldots(3b)$$

$$2n\omega A_n\dot{\phi}_n = (\omega_0^2-n^2\omega^2)A_n+a_n \qquad \ldots\ldots\ldots(3c)$$

$$2n\omega\dot{A}_n = an\omega A_n-b_n \qquad \ldots\ldots\ldots(3d)$$

where

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi}\mu\,\mathrm{f}(\quad)\sin(\omega t+\phi)\,\mathrm{d}(\omega t)$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi}\mu\,\mathrm{f}(\quad)\cos(\omega t+\phi)\,\mathrm{d}(\omega t)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\mu\,\mathrm{f}(\quad)\sin(n\omega t+\phi_n)\,\mathrm{d}(n\omega t)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\mu\,\mathrm{f}(\quad)\cos(n\omega t+\phi_n)\,\mathrm{d}(n\omega t)$$

$$\mathrm{f}(\quad) = \mathrm{f}\,[A\sin(\omega t+\phi)+A_n\sin(n\omega t+\phi_n),\ \omega A\cos(\omega t+\phi)+n\omega A_n\cos(n\omega t+\phi_n)]$$

In the equilibrium state $\dot{A} = \dot{\phi} = \dot{A}_n = \dot{\phi}_n = 0$. Hence

$$E\cos\phi = (\omega_0^2-\omega^2)A+a_1 \qquad \ldots\ldots\ldots(4a)$$

$$E\sin\phi = a\omega A-b_1 \qquad \ldots\ldots\ldots(4b)$$

$$(\omega_0^2-n^2\omega^2)A_n = -a_n \qquad \ldots\ldots\ldots(4c)$$

$$an\omega A_n = b_n \qquad \ldots\ldots\ldots(4d)$$

For ultra or subharmonic resonance $(\omega_0^2-\omega^2)$. A is much larger than the other terms in eqns. (4a) and (4b); hence A is approximately given by $\dfrac{E}{\omega_0^2-\omega^2}$. Thus for obtaining A_n and ϕ_n, only the last two equations have to be solved. Plots of A_n, obtained on solving the equations, against $\dfrac{\omega_0^2-n^2\omega^2}{n\omega}$ give the theoretical resonance curves. All parts of the resonance curves do not represent physical solution since for a physical solution it is required that the equilibrium state $\dot{A}_n = \dot{\phi}_n = 0$ be stable. $\dot{A}$, $\dot{\phi}$ may be assumed to be zero to the first order of approximation.

It is known[5] that the equilibrium state $\dot{A}_n = \dot{\phi}_n = 0$ is stable if

$$\frac{\partial\dot{A}_n}{\partial A_n}+\frac{\partial\dot{\phi}_n}{\partial\phi_n} < 0 \qquad \ldots\ldots\ldots(5a)$$

and

$$\frac{\partial\dot{A}_n}{\partial A_n}\cdot\frac{\partial\dot{\phi}_n}{\partial\phi_n}-\frac{\partial\dot{A}_n}{\partial\phi_n}\cdot\frac{\partial\dot{\phi}_n}{\partial A_n} > 0 \qquad \ldots\ldots\ldots(5b)$$

The differentials are to be evaluated at the equilibrium values of A_n and ϕ_n.

Details of the resonance curves may be obtained only when $\mu f(x, \dot{x})$ is known. Let it be considered that $\mu f(x, \dot{x}) = b\dot{x}^2+c\dot{x}^3$. It represents the type of non-linearity that is encountered in a properly adjusted triode oscillator. Evidently, in this case ultraharmonic resonance of orders 2 and 3 and subharmonic resonance of order $\frac{1}{2}$ and $\frac{1}{3}$ may be obtained.

2.1. *Ultraharmonic Resonance Curves*

Ultraharmonic resonance of order 3: It can be shown that when $n = 3$,

$$b_3 = \tfrac{3}{4}c[(3\omega A_3)^2+2(\omega A)^2+\tfrac{1}{3}\frac{(\omega A)^3}{(3\omega A_3)}\cos\Phi] \times(3\omega A_3) \qquad \ldots\ldots\ldots(6a)$$

$$a_3 = \tfrac{1}{4}c(\omega A)^3 \sin \Phi \qquad \text{.........(6b)}$$

where
$$\Phi = \phi_3 - 3\phi \qquad \text{.........(6c)}$$

On substituting in eqns. (3c) and (3d) and putting

$$\frac{4a}{3c} = \rho_0^2, \frac{3\omega A_3}{\rho_0} = \rho_3, \frac{\omega A}{\rho_0} = \rho \text{ and } \frac{\omega_0^2 - 9\omega^2}{a\ 3\omega} = \delta_3$$

$$2\dot{\rho}_3 = a[1 - \rho_3^2 - 2\rho^2 - \tfrac{1}{3}\frac{\rho^3}{\rho_3} \cos \Phi]\, \rho_3 \qquad \text{.........(7a)}$$

$$2\dot{\phi}_3 = a[\delta_3 + \tfrac{1}{3}\frac{\rho^3}{\rho_3} \sin \Phi] \qquad \text{.........(7b)}$$

The equilibrium value of ρ_3 is, therefore, given by

$$\left(\frac{\rho^3}{3\rho_3}\right)^2 = \delta_3^2 + (1 - 2\rho^2 - \rho_3^2)^2 \qquad \text{.........(8)}$$

The conditions of stability as obtained from the inequalities (5a) and (5b) are

$$\rho_3^2 + \rho^2 > 0{\cdot}5 \qquad \text{.........(9a)}$$

$$3\rho_3^4 - 4\rho_3^2(1 - 2\rho^2) + (1 - 2\rho^2)^2 + \delta_3^2 > 0 \qquad \text{.........(9b)}$$

The signal producing resonance has the amplitude $\frac{1}{3}\rho^3$ which is determined by E only. Curves for ultraharmonic resonance have, therefore, characteristics similar to those of harmonic resonance Three cases may be clearly distinguished. For $\rho < 0{\cdot}5864$, though ρ_3 is triple valued for the low values of δ_3, only the highest value is stable. As δ_3 is increased from zero ρ_3^2 decreases and becomes critical when the tangent to the resonance curve is vertical. For further increase in δ_3, $\frac{dA_3}{d\phi_3}$ has an unstable singularity, hence ultraharmonic resonance is not possible but limit cycles in the A_3-ϕ_3 plane may be exhibited. For $0{\cdot}5864 < \rho < 0{\cdot}6067$, two of the three possible values of δ_3 are stable and therefore jump phenomenon and the associated hysteresis[3] may be exhibited. For $\rho > 0{\cdot}6067$, resonance curves are single valued and stable harmonic resonance is exhibited up to a value of δ_3 up to which inequality (9a) is satisfied. However, for $\rho > 0{\cdot}707$ ultraharmonic oscillation is possible for all values of δ_3. In fact, it should be regarded as the ultraharmonic of the forced oscillation, which suppresses the free oscillation. The features noted above are illustrated by the resonance curves shown in Fig. 1 in which ρ_3^2 is plotted against δ_3 with ρ as parameter. The unstable parts of the resonance curves are shown by dotted lines. The lower branch of the curves for $\rho < 0{\cdot}58$ are not shown.

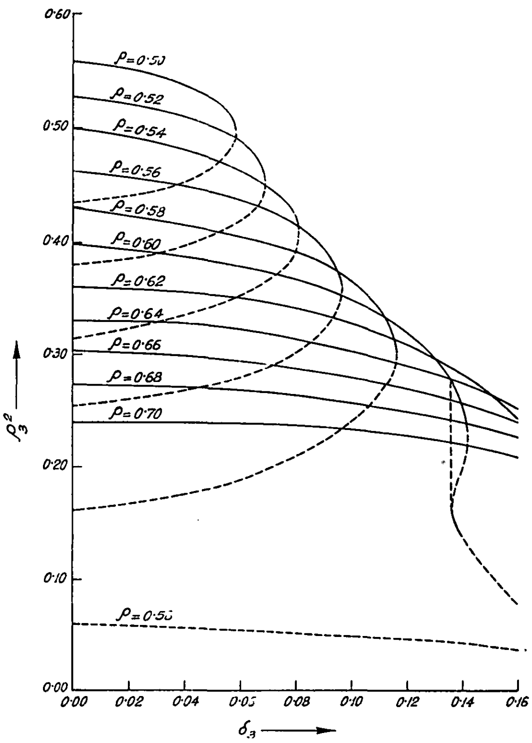

Fig. 1. Ultraharmonic resonance curves of order three.

It should be noted that the resonance curves in this case have the distinctive feature that ρ_3^2 decreases with increase in the external signal. This arises from the fact that with increase in E, ρ increases; thus though the signal producing resonance is increased, the damping for the ultraharmonic oscillation is also increased. It is also observed that ρ_3^2 is always less than 1 in the resonance condition; hence the power available for ultraharmonic oscillation is less than that for free oscillation.

Ultraharmonic resonance of order 2 : In this case the Fourier coefficients are:

$$b_2 = a\rho_0[\rho_2^2 + 2\rho^2 + \frac{2b}{3\rho_0 c}\,\frac{\rho^2}{\rho_2} \cos \Phi]\rho_2 \qquad \text{......(10a)}$$

$$a_2 = a\rho_0[\frac{2b}{3\rho_0 c}\,\rho^2 \sin \Phi] \qquad \text{.........(10b)}$$

where

$$\Phi = \phi_2 - 2\phi,\ \rho_2 = \frac{2\omega A}{\rho_0},\ \rho_0^2 = \frac{4a}{3c}, \frac{\omega A}{\rho_0} = \rho$$

Substituting in eqns. (3c) and (3d) and putting

$$\frac{\omega_0^2 - 4\omega^2}{a.\ 2\omega} = \delta_2$$

$$2\dot{\rho}_2 = a[1 - \rho_2^2 - 2\rho^2 - \frac{2b}{3\rho_0 c}\,\frac{\rho^2}{\rho_2}\cos\Phi]\,\rho_2 \qquad \text{.........(11a)}$$

$$2\dot{\phi}_2 = a[\delta_2 + \frac{2b}{3\rho_0 c}\,\frac{\rho^2}{\rho_2}\,\sin\Phi] \qquad \text{.........(11b)}$$

The equilibrium values of ρ_2 are given by

$$\left(\frac{2b}{3\rho_0 c}\,\frac{\rho^2}{\rho_2}\right)^2 = \delta_2^2 + (1 - \rho_2^2 - 2\rho^2)^2 \qquad \text{.........(12)}$$

The equilibrium is stable if

$$\rho_2^2 + \rho^2 > 0{\cdot}5 \qquad \text{.........(13a)}$$

and

$$3\rho_2^4 - 4\rho_2^2(1 - 2\rho^2) + (1 - 2\rho^2)^2 + \delta_2^2 > 0 \qquad \text{......(13b)}$$

Plots of ρ_2^2 against δ_2 with ρ as a parameter for $\frac{2b}{3\rho_0 c} = 1$ are shown in Fig. 2. The characteristics are identical to those of the previous case. However, the signal causing the resonance has the amplitude $\frac{2b}{3\rho_0 c}\rho^2$ instead of $\frac{\rho^3}{3}$. Hence the zone of resonance as well as the maximum power at the ultraharmonic frequency for a fixed value of ρ may be controlled by adjusting the asymmetry in the non-linearity.

2.2. *Subharmonic Resonance Curves*

Subharmonic resonance of order 1/3: On evaluating a_n and b_n for $n = \frac{1}{3}$ and substituting

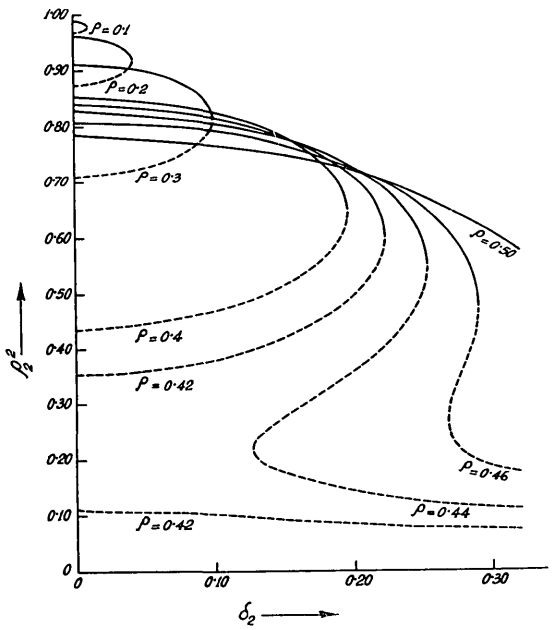

Fig. 2. Ultraharmonic resonance curves of order two.

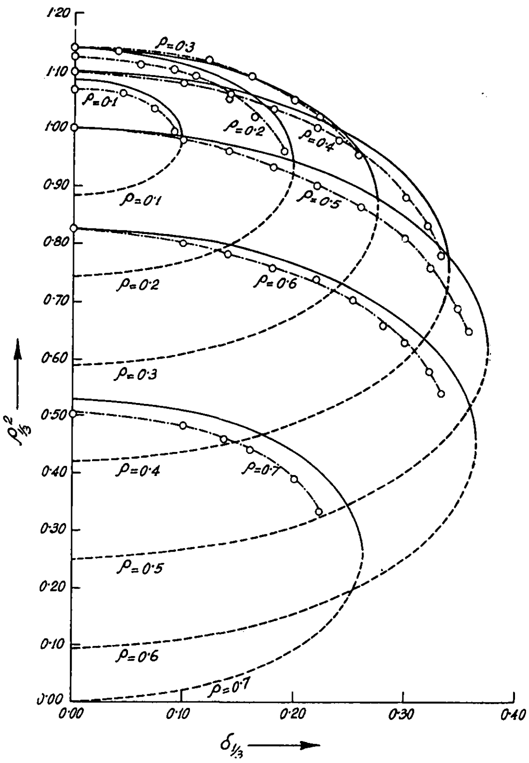

Fig. 3. Subharmonic resonance curves of order one-third.

in eqns. (3c) and (3d), one gets

$$2\dot{\rho}_{\frac{1}{3}} = a[1 - \rho_{\frac{1}{3}}^2 - 2\rho^2 - \rho\,\rho_{\frac{1}{3}}\cos\Phi]\,\rho_{\frac{1}{3}} \qquad \text{........(14a)}$$

$$2\dot{\phi}_{\frac{1}{3}} = a[\delta_{\frac{1}{3}} - \rho\,\rho_{\frac{1}{3}}\sin\Phi] \qquad \text{.........(14b)}$$

where

$$\Phi = \phi - 3\phi_{\frac{1}{3}}\,,\ \delta_{\frac{1}{3}} = \frac{\omega_0^2 - \omega^2/9}{a\ \omega/3}\,,\ \rho_{\frac{1}{3}} = \frac{\omega/3\ A_{\frac{1}{3}}}{\rho_0}$$

and, as before, $\rho_0^2 = \frac{4a}{c}\,,\ \rho = \frac{\omega A}{\rho_0}$

Equilibrium values of ρ are given by

$$(\rho\,\rho_{\frac{1}{3}})^2 = \delta_{\frac{1}{3}}^2 + (1 - \rho_{\frac{1}{3}}^2 - \rho^2)^2 \qquad \text{.........(15)}$$

$$\rho_{\frac{1}{3}}^2 = 1 - \frac{3}{2}\rho^2 \pm \sqrt{\rho^2 - \frac{7}{4}\rho^4 - \delta_{\frac{1}{3}}^2} \qquad \text{........(16)}$$

Conditions of stability of the equilibrium are

$$\rho_{\frac{1}{3}}^2 + \rho^2 > 0{\cdot}5 \qquad \text{.........(17a)}$$

$$\sqrt{\rho^2 - \frac{7}{4}\rho^4 - \delta_{\frac{1}{3}}^2} \pm (1 - \frac{3}{2}\rho^2) > 0 \qquad \text{.........(17b)}$$

Plots of $\rho_{\frac{1}{3}}^2$ against $\delta_{\frac{1}{3}}$ with ρ as a parameter are shown in Fig. 3. For a fixed value of ρ, $\rho_{\frac{1}{3}}$ decreases with increase in $\delta_{\frac{1}{3}}$ till the tangent to the curve becomes vertical. With further increase in $\delta_{\frac{1}{3}}$ subharmonic oscillation vanishes.

and for $\rho^2 > 0{\cdot}5$ the oscillation occurs only at the external frequency while for $\rho^2 < 0{\cdot}5$ a combination oscillation for which the phase and amplitude vary over a cycle is executed. It should be noted that outside the zone of resonance $\frac{dA_{\frac{1}{3}}}{d\varphi_{\frac{1}{3}}}$ does not have any singularity. Hence no limit cycle in the $A_{\frac{1}{3}}$ -$\phi_{\frac{1}{3}}$ plane is possible and the resultant oscillation is the combination of the free and harmonic ones. Resonance curves in this case have also the distinctive feature that the zone of resonance at first increases with increase in E, attains a maximum value and then decreases to zero. With increase in E, ρ increases; the increase of ρ at the same time decreases $\rho_{\frac{1}{3}}$. Hence the amplitude of the resonating signal which is equal to $\rho\rho_{\frac{1}{3}}{}^2$, at first increases but ultimately decreases to zero, the decreasing effect of $\rho_{\frac{1}{3}}$ offsetting the increasing effect of ρ.

Thus, it is evident that subharmonic resonance may be exhibited only within a limited range of ρ; for ultraharmonic or harmonic resonance there is no such limit. Further, in subharmonic resonance there is a maximum value of the zone of resonance which is $0{\cdot}286a$ and is dependent only on the initial damping. It may be seen that in the other two cases the zone of resonance increases with increase in the external signal. Hysteresis or the jump within the zone of resonance is also not possible in the subharmonic resonance of order 1/3. But sudden jump from a finite value of $\rho_{\frac{1}{3}}$ to zero on the border of the zone of resonance is possible, the jump being reversible. Maximum power available at the subharmonic frequency is greater than that for free oscillation for $\rho < 0{\cdot}5$.

Subharmonic resonance of order 1/2: On evaluating the Fourier coefficients and substituting in eqns. (3c) and (3d), one gets

$$2\dot{\rho}_{\frac{1}{2}} = a[1 - \rho_{\frac{1}{2}}{}^2 - 2\rho^2 - \frac{4b}{3\rho_0 c}\rho\cos\Phi]\,\rho_{\frac{1}{2}} \quad \text{.........(18a)}$$

$$2\dot{\phi}_{\frac{1}{2}} = a[\delta_{\frac{1}{2}} - \frac{4b}{3\rho_0 c}\sin\Phi] \quad \text{.........(18b)}$$

where $\Phi = \phi - 2\phi_{\frac{1}{2}}, \rho = \frac{\frac{1}{2}\omega A_{\frac{1}{2}}}{\rho_0}, \delta_{\frac{1}{2}} = \frac{\omega_0{}^2 - \frac{1}{4}\omega^2}{\frac{1}{2}a\omega},$

and $\rho_0{}^2 = \frac{4a}{3c}$, $\rho = \frac{\omega A}{\rho_0}$ as before.

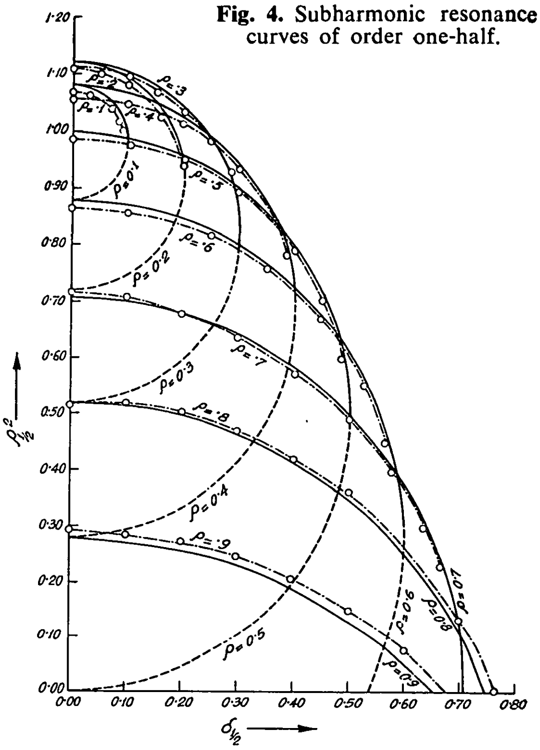

Fig. 4. Subharmonic resonance curves of order one-half.

The equilibrium values of $\rho_{\frac{1}{2}}$ are given by

$$\rho_{\frac{1}{2}}{}^2 = 1 - 2\rho^2 \pm \sqrt{\left(\frac{4b}{3\rho_0 c}\rho\right)^2 - \delta_{\frac{1}{2}}{}^2} \quad \text{.........(19)}$$

The conditions of stability are

$$\rho^2 + \rho_{\frac{1}{2}}{}^2 > 0{\cdot}5 \quad \text{.........(20a)}$$

$$\rho_{\frac{1}{2}}{}^2 + 2\rho^2 - 1 > 0 \quad \text{.........(20b)}$$

Plots of $\rho_{\frac{1}{2}}{}^2$ as obtained from eqn. (19) are shown in Fig. 4 for $\frac{4b}{3\rho_0 c} = 1$, the stable values being indicated by full lines.

The characteristics of the curves are similar to those for subharmonic resonance of order 1/3. For $\rho^2 < 0{\cdot}5$, with increase in $\delta_{\frac{1}{2}}$, $\rho_{\frac{1}{2}}{}^2$ decreases and subharmonic oscillation vanishes when the tangent to the curve becomes vertical. But for $\rho^2 > 0{\cdot}5$, the resonance curves do not have a vertical tangent, $\rho_{\frac{1}{2}}{}^2$ decreases gradually to zero with increase in $\delta_{\frac{1}{2}}$. The zone of resonance increases with increase in ρ, attains a maximum value and decreases to zero. Also, the resonance is possible only within a limited range of ρ. No jump phenomenon, however, is possible. The maximum zone of resonance is dependent not only on a but also on b/c.

3. Experimental Resonance Curves

Experimental observations for verifying the resonance curves shown in Figs. 1-4 have been made with the help of an electronic differential analyser. The set-up of the analyser for solving eqn. (1) is shown in Fig. 5. The non-linear function was generated by a diode function generator. Different values of δ were set by varying ω_0^2, the external signal frequency being kept fixed and so adjusted that for $\omega_0^2 = 1$; $\delta = 0$. For measuring ρ_n^2, subharmonic or ultraharmonic oscillations were separated from the forced oscillations by combining properly the outputs at A and B (Fig. 5) representing respectively $\ddot{x}$ and x. The mean square value was then obtained with the help of a thermocouple and a d.c. millivoltmeter.

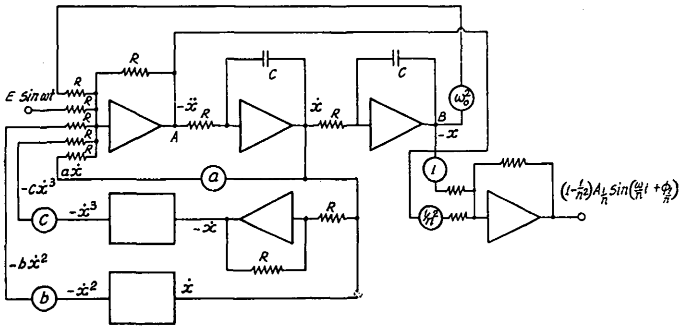

Fig. 5. Set-up of the Differential Analyser for obtaining the resonance curves.

Experimental points for subharmonic resonance are shown by circles on the theoretical curves of Figs. 3 and 4. It will be seen that the experimental results agree quite closely with the theoretical curves. The small discrepancy that exists may be accounted for by errors in the setting of ω_0^2.

On the other hand, ultraharmonic resonance curves obtained experimentally showed only rough agreement with the theoretical curves. A probable cause of the departure may be the effect of the sum and difference frequency terms, the contributions of which were neglected in deriving the theoretical curves. In ultraharmonic resonance the sum and difference frequencies are quite close to the frequency of free oscillation and therefore may have large effect. In subharmonic resonance, however, these frequencies are far removed from the frequency of free oscillation and have little effect.

4. Conclusion

The above analysis revealed the following distinguishing features of ultraharmonic and subharmonic resonance. The characteristic features of ultraharmonic resonance are: (a) zone of resonance increases with increase in the external signal; (b) jump and associated hysteresis effect may be exhibited; (c) amplitude of the ultraharmonic oscillation decreases with increase in the external signal; (d) the maximum power available for the ultraharmonic oscillation is less than that for free oscillation. Characteristic features of subharmonic resonance are: (a) subharmonic resonance occurs for a limited range of the amplitude of the external signal; (b) zone of resonance has a maximum value; (c) jump and hysteresis effect is not possible in the zone of resonance; (d) the amplitude of the sub-harmonic oscillation at first increases and then decreases with increase in the external signal; (e) maximum power available for subharmonic oscillations may be greater than that for free oscillation.

In addition it may be noted that the experimental subharmonic resonance curves agree closely with the theoretical curves while for ultraharmonic resonance the agreement is not so good.

5. Acknowledgments

The author is deeply indebted to Professor J. N. Bhar for his constant guidance during the progress of the work.

6. References

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