3

Hill on the Motion of the Lunar Perigee

Of Hill's two innovative papers on the lunar theory, the first, "On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon" (Cambridge, MA: John Wilson, 1877, 28pp; reprinted in *Acta Mathematica 8* (1886), pp. 1–36) was by far the most esoteric in its subject matter and hyper-refined in the methods it employed. The second paper introduces the new lunar theory in a more pedestrian and reader-friendly way, as the reader will discover in our later section on "Hill's Variation Curve." The first paper must have made a stunning impression on those readers who were prepared to appreciate it; it is a blockbuster of a paper, astonishing in what and how it achieves. We shall attempt to make its essential steps understandable for readers with a moderate amount of training in algebra and the calculus.

The first paper was initially published privately at the author's expense. The second paper, "Researches in the Lunar Theory," was published in the first three issues of the first volume of the *American Journal of Mathematics* in 1878. Most of Hill's earlier papers, and a few later ones (up to 1881), were published in *The Analyst*, a recreationally oriented American journal of pure and applied mathematics published from 1874 to 1883;[46] for his lunar papers, however, Hill apparently did not consider *The Analyst* a suitable vehicle. The first volume of the *American Journal of Mathematics*, in which Hill published his second paper, did not exist when Hill completed his first paper. He surely knew that he had achieved something important, and must have wanted to see it quickly in print. He opted for a private printing of 200 copies; this had the advantage of giving him control of the distribution. As noted earlier, John Couch Adams received a copy shortly after the article appeared – no doubt sent by Hill.

In opening his essay of 1877, Hill remarks that lunar theorists since the publication of Newton's *Principia* have been puzzled to account for the lunar perigee's motion, simply because they could not conceive that terms of the second and higher

[46] Parshall and Rowe, *The Emergence of the American Mathematical Community 1876–1900*, 51, 85.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_3,

orders with respect to the disturbing force produced more than half of it. Nor, he asserts, has the problem yet been satisfactorily solved:

> The rate of motion of the lunar perigee is capable of being determined from observation with about a thirteenth of the precision of the rate of mean motion in longitude. Hence if we suppose that the mean motion of the moon, in the century and a quarter which has elapsed since Bradley began to observe, is known within $3''$, it follows that the motion of the perigee can be got to within about 500,000th of the whole. None of the values hitherto computed from theory agrees as closely as this with the value derived from observation.

The perigee moves about $40°40'$ per year; hence in the 125 years since Bradley it has moved about $5085°$. Hill is asserting that this total motion can be determined by observation with a precision of about $3 \times 13 = 39$ arc-seconds $= 0°.010833$, which is approximately the 500,000th part of the whole. Lunar theorists had not yet come near to achieving so precise a determination.

> Hence I propose, in this memoir, to compute the value of this quantity, so far as it depends on the mean motions of the sun and moon, with a degree of accuracy that shall leave nothing further to be desired.

It is only part of the motion of the lunar perigee that Hill is here aiming to calculate, for the complete motion of the perigee depends in some measure on the eccentricities of the Moon's and Earth's orbits, and on the inclination of the Moon's orbit to the ecliptic. But the part Hill will be calculating – the part dependent on the constant **m** – will prove to be the main part. Indeed, the value Hill will obtain from his calculation will differ from the observational value by no more than 1/70th of the latter. Think of it! – a discrepancy of 1/70th instead of the one-half that Euler, Clairaut, and d'Alembert were initially confronted with. With the result of Hill's calculation in hand, it will no longer be a wild surmise that the Moon's path is more nearly approximated by the Variation curve than by any Earth-focused ellipse. And this curve is totally definable in terms of the small parameter **m**, the cause of all the problems of slow convergence that had stymied the earlier investigators.

The mathematical development in Hill's paper of 1877 assumes that the lunar inequalities depending solely on the parameter **m** – except for the motion of the apse – have already been obtained. In other respects, Hill no doubt intends his paper of 1877 to be self-contained, but his explanations here are remarkable for their concision. We can promise the reader that certain concepts presented here with the briefest characterization – the Jacobian integral, for instance, and the potential function Ω – will in our resume of the second paper be more fully explained.

Hill begins by presenting the differential equations in the form[47]

$$\frac{d^2x}{dt^2} = \frac{\partial\Omega}{\partial x}, \quad \frac{d^2y}{dt^2} = \frac{\partial\Omega}{\partial y}. \tag{I.1}$$

[47] In our numbering of equations, "I" stands for the paper of 1877, and "II" for the paper of 1878.

These are the equations Jacobi started from, in the paper of 1836 introducing the Jacobian integral. The variables x and y are the Moon's rectangular coordinates with respect to the Earth's center. Ω is the potential function, so that $\partial\Omega/\partial x$ and $\partial\Omega/\partial y$ express the net forces exerted on the Moon in the x- and y-directions. Hill leaves unspecified the terms of which Ω consists, and proceeds at once to the integral. As integrating factors ("Eulerian multipliers," he calls them) he proposes

$$F = \frac{dx}{dt} + n'y, \quad G = \frac{dy}{dt} - n'x,$$

where n' is the angular motion of the Sun about the Earth or of the Earth about the Sun, here taken to be uniform and circular. The first equation of (I.1) is to be multiplied by F, and the second by G; the resulting equations are then added together. The result (which Hill does not write out) is

$$\frac{d^2x}{dt^2}\frac{dx}{dt} + \frac{d^2y}{dt^2}\frac{dy}{dt} - n'\left(\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) = \frac{\partial\Omega}{\partial x}\frac{dx}{dt} + \frac{\partial\Omega}{\partial y}\frac{dy}{dt} - n'\left(x\frac{\partial\Omega}{\partial y} - y\frac{\partial\Omega}{\partial x}\right). \tag{I.2}$$

Note that the third term on the right is identically equal to the third term on the left, by (I.1). The time-integral of (I.2), Hill then claims, is

$$\frac{dx^2 + dy^2}{2dt^2} - n'\left(\frac{xdy - ydx}{dt}\right) = \Omega + C, \tag{I.3}$$

where C is the constant of integration.

That the left-hand side of (I.3) is the integral of the left-hand side of (I.2) is easily verified. On the right-hand side of (I.2) the first two terms give the indirect dependence of Ω on t through the variables x and y. Assuming that Ω depends in addition on t directly, we should have

$$\frac{d\Omega}{dt} = \frac{\partial\Omega}{\partial x}\frac{dx}{dt} + \frac{\partial\Omega}{\partial y}\frac{dy}{dt} + \frac{\partial\Omega}{\partial t}.$$

Then, for the right-hand side of (I.3) to be the integral of the right-hand side of (I.2), we must have

$$\frac{\partial\Omega}{\partial t} = n'y\frac{\partial\Omega}{\partial x} - n'x\frac{\partial\Omega}{\partial y}.$$

The latter equation can be verified if the terms of which Ω is composed are known. We find them, not in Hill, but in Jacobi:

$$\frac{1}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] - n'\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = \frac{M}{(x^2+y^2)^{1/2}}$$
$$+ m'\left[\frac{1}{[(x - a'\cos n't)^2 + (y - a'\sin n't)^2]^{1/2}} - \frac{x\cos n't + y\sin n't}{a'^2}\right] + \text{const.} \tag{I.4}$$

(We have omitted the terms involving the variable z, since Hill confines the orbit to the x-y plane.) In (I.4) a' is the Earth-Sun distance, M is the Earth's mass, and m' the Sun's mass. The Moon is assumed to be without mass. Taking the right-hand side of (I.4) as an expression of $\Omega +$ *const.*, we find by a straightforward calculation that

$$\frac{\partial\Omega}{\partial t} = n'\left(y\frac{\partial\Omega}{\partial x} - x\frac{\partial\Omega}{\partial y}\right),$$

as required.

Assuming that (I.1) and (I.3) have together been solved for the Variation orbit, Hill now proposes to investigate the effect of small departures from that solution. The Variation curve, as we shall see in our resume of Hill's second paper, is an oval symmetrical with respect to the rotating x- and y-coordinate axes, with origin at the Earth's center. Let x_0 and y_0 be the variables for the Variation orbit, and let the (I.1) and (I.2) be written with x and y thus distinguished by subscript 0.[48] Hill is asking what happens to the orbit when increments ∂x and ∂y are added, respectively, to x_0 and y_0 in the differential equations.

The increments ∂x and ∂y will destroy the symmetry, making the Moon's path eccentric with respect to the Earth's center, so as to have perigee(s) and apogee(s). For in the absence of perturbation, the Moon would move in a circle or else in an ellipse with a center eccentric to the Earth's center. The eccentricity can be expected to remain when solar perturbation supervenes. Newton, as mentioned earlier, had thought in terms of somehow melding the properties of the Variation and those of the ellipse, but he lacked a legitimate mathematical technique for doing this. Hill, with Euler's guidance, is setting out to combine the effects as determined by their defining parameters, **e** and **m**. This he can easily do, using the exponential expression of sines and cosines. It is a matter of adding exponents.

"Let us suppose," Hill writes, "...that it is desired to get [the inequalities] which are multiplied by the simple power of [the eccentricity]." Given this statement, the reader may be surprised to find that the eccentricity **e** does not figure as a quantity in the calculations of the paper we are examining. But the increments δx, δy *do* produce eccentricity. Hill's remark means that the increments are small enough so that their squares and their product can be neglected. Given eccentricity, there will be a perigee and an apogee, and solar perturbation will cause these points of the orbit to move forward. Hill aims in this paper to determine that motion, insofar as it depends on **m**. Such a determination is prerequisite for determining the mean anomaly in the resulting orbit, and hence for determining the inequalities proportional to **e**.

To arrive at differential equations for δx and δy, Hill first substitutes x_0 and y_0, then $x_0 + \delta x$ and $y_0 + \delta y$, for x and y in the two equations of (I.1), then takes the difference of the corresponding equations so as to eliminate x_0 and y_0. The result is

[48] This notation is due to Brouwer and Clemence, *Methods of Celestial Mechanics*, 350ff.

$$\frac{d^2\delta x}{dt^2} = \left(\frac{\partial^2\Omega}{\partial x^2}\right)_0 \delta x + \left(\frac{\partial^2\Omega}{\partial x\partial y}\right)_0 \delta y,$$

$$\frac{d^2\delta y}{dt^2} = \left(\frac{\partial^2\Omega}{\partial y^2}\right)_0 \delta y + \left(\frac{\partial^2\Omega}{\partial x\partial y}\right)_0 \delta x.$$

The zero subscripts indicate that the partial derivatives are to be evaluated using the variables of the Variation orbit. If with Hill we put

$$H = \left(\frac{\partial^2\Omega}{\partial x^2}\right)_0, \quad J = \left(\frac{\partial^2\Omega}{\partial x\partial y}\right)_0, \quad K = \left(\frac{\partial^2\Omega}{\partial y^2}\right)_0,$$

the equations take the form

$$\frac{d^2\delta x}{dt^2} = H\delta x + J\delta y, \frac{d^2\delta y}{dt^2} = K\delta y + J\delta x. \tag{I.5}$$

Next, Hill carries out the analogous operation on (I.3), discarding terms in which δx and δy are squared or multiply each other; the result is

$$\frac{dx_0}{dt}\frac{d(\delta x)}{dt} + \frac{dy_0}{dt}\frac{d(\delta y)}{dt} - n'\left(x_0\frac{d(\delta y)}{dt} - y_0\frac{d(\delta x)}{dt} + \frac{dy_0}{dt}\delta x - \frac{dx_0}{dt}\delta y\right)$$
$$= \left(\frac{\partial\Omega}{\partial x}\right)_0 \delta x + \left(\frac{\partial\Omega}{\partial y}\right)_0 \delta y + \delta C.$$

According to Hill, δC if developed in ascending powers of the eccentricity is found to contain only even powers of **e**; therefore in the approximation we are here exploring, we shall have $\delta C = 0$. Also, in accordance with (I.1), the first-order partial derivatives of Ω are

$$F\frac{d(\delta x)}{dt} + G\frac{d(\delta y)}{dt} - \frac{dF}{dt}\delta x - \frac{dG}{dt}\delta y = 0. \tag{I.6}$$

This equation, Hill observes, is identically satisfied by the solution $\delta x = F$ and $\delta y = G$. The same solution satisfies equations (I.5), giving

$$\frac{d^2F}{dt^2} = HF + JG, \quad \frac{d^2G}{dt^2} = KG + JF. \tag{I.5a}$$

This solution, being composed of terms having the same argument as the Variation, tells us nothing about an orbit incorporating the increments $\delta x, \delta y$. To obtain the latter orbit, Hill proposes a solution of the form $\delta x = F\rho, \delta y = G\sigma$, where ρ and σ are new variables. The use of F and G in this manner – a well-known technique – will enable Hill to reduce the order of his final differential equation. Introducing these new variables into (I.5) and (I.6), and making use of (I.5a), he finds

$$F\frac{d^2\rho}{dt^2} + 2\frac{dF}{dt}\frac{d\rho}{dt} + JG(\rho - \sigma) = 0,$$

$$G\frac{d^2\sigma}{dt^2} + 2\frac{dG}{dt}\frac{d\sigma}{dt} + JF(\sigma - \rho) = 0,$$

$$F^2\frac{d\rho}{dt} + G^2\frac{d\sigma}{dt} = 0.$$

Deriving from the first of these equations an expression for σ, he substitutes it into the third equation, and so obtains

$$\frac{d^3\rho}{dt^3} + \frac{d}{dt}\left[\ln\frac{F^3}{JG}\right]\frac{d^2\rho}{dt^2} + \left[\frac{J(F^2+G^2)}{FG} + \frac{JG}{F}\frac{d}{dt}\left(\frac{2}{JG}\frac{dF}{dt}\right)\right]\frac{d\rho}{dt} = 0. \quad \text{(I.7)}$$

Hill's final move is to introduce the substitution

$$\frac{d\rho}{dt} = \sqrt{\frac{JG}{F}}w. \quad \text{(I.8)}$$

This yields, after algebraic reductions, the differential equation

$$\frac{d^2w}{dt^2} + \theta w = 0, \quad \text{(I.9)}$$

where θ can be expressed by

$$\theta = \frac{J(F^2+G^2)}{FG} + \frac{d^2 \cdot \ln(JFG)}{2dt^2} - \left[\frac{d \cdot \ln(JFG)}{2dt}\right]^2. \quad \text{(I.10)}$$

(I.10) shows that θ depends solely on the variables x_0, y_0, their time-derivatives, and the derivative J of Ω with respect to them. Interchanging F and G leaves θ unchanged; thus if ρ had been eliminated instead of σ, a formally identical equation would have resulted. According to Hill,

> ...we arrive always at the same value of θ, no matter what variables have been used to express the original differential equations. From this we may conclude that θ depends only on the relative position of the Moon with reference to the Sun, and that it can be developed in a periodic series of the form
>
> $$\theta = \theta_0 + \theta_1 \cos 2\tau + \theta_2 \cos 4\tau + \cdots,$$
>
> in which τ denotes the mean angular distance of the two bodies.

Here θ_0, θ_1, etc., are constants, and $\tau = v(t - t_0)$, in which v is the frequency of the moon's synodic motion, and t_0 the time of the moon's conjunction with the Sun.

In this passage Hill does not explain why θ should be an infinite series of cosines, nor why the arguments of the cosines should be the even multiples of τ. Could not the function $\cos\tau$, for instance, be included in θ, since it gives the same value whenever the Moon is at the same angular distance from the Sun? The reason θ must be an infinite series of cosines with arguments that are *even* multiples of τ can be elucidated as follows. F and G are linear functions of $x_0, y_0, dx_0/d\tau, dy_0/d\tau$; the latter variables are given by infinite series of sinusoidal functions, with arguments that are *odd* multiples of τ. These representations of x_0 and y_0 were chosen initially (as explained in Hill's paper of 1878) in order to obtain a periodic orbit. Hence, in (I.10), F^2, G^2, and FG are infinite series of sinusoidal functions with arguments that are *even* multiples of τ (exponential expression of the sines and cosines makes this obvious). Moreover, J, the mixed partial derivative of Ω which multiplies F^2, G^2, and FG in (I.10), can also be expressed by an infinite series in which the arguments of the sinusoidal functions are even multiples of τ. Thus θ is represented by an infinite series of sinusoidal terms, in which the arguments are necessarily even multiples of τ.

Hill does not say a word about what the new variable w represents.

George Howard Darwin, in his lectures on Hill's lunar theory,[49] points out that it is a common procedure in dynamics to consider "free oscillations" about a steady state (free oscillations are contrasted to forced oscillations, produced by an external force). If the Variation orbit is taken as the steady state, then the obvious oscillations to consider are those normal and tangential to the Variation curve. Let δp and δs represent these oscillations. If φ is the inclination of the outward normal of the Variation curve to the x-axis, then

$$\delta x = \delta p \cos\varphi - \delta s \sin\varphi.$$

$$\delta y = \delta p \sin\varphi + \delta s \cos\varphi.$$

The sine and cosine of the angle φ are furnished by the relations $dx_0/d\tau = -V\sin\varphi$, $dy_0/d\tau = V\cos\varphi$, where V is the orbital speed:

$$V = \left[\left(\frac{dx_0}{d\tau}\right)^2 + \left(\frac{dy_0}{d\tau}\right)^2\right]^{1/2}.$$

By the foregoing equivalences Darwin eliminates δx and δy from (I.5) and (I.6) in favor of δp and δs. He also eliminates x_0, y_0, and their derivatives, using initial terms from Hill's infinite series for x_0 and y_0 (he stops at terms in which the multiple of τ is 3). He thus obtains the following approximate differential equation for δp:

$$\frac{d^2\delta p}{d\tau^2} + \delta p\left[1 + 2\underline{m} - \frac{1}{2}\underline{m}^2 - 15\underline{m}^2\cos 2\tau\right] = 0.$$

This resembles Hill's equation for w, except that θ here consists of a constant and only one sinusoidal term, rather than an infinite series of such terms. A similar equation can be obtained for δs.

If θ were a constant, (I.9) would describe simple harmonic motion, with a solution of the form $w = A\cos ft$ where f is a frequency equal to $\sqrt{\theta}$. But θ consists of an initial constant (θ_0), plus an infinity of terms that vary, $\sum_{i=1}^{\infty}\theta_i\cos 2i\tau$. We shall

[49] G. H. Darwin, "Hill's Lunar Theory," *Scientific Papers of George Howard Darwin, 5* (Cambridge: Cambridge University Press, 1916), 27ff.

find that $\theta_0 > \theta_1 > \theta_2 > \cdots$, with θ_0 a good deal larger than its successors (it is more than ten times θ_1). Can the simple harmonic solution based on setting $\theta = \theta_0$ serve as a first approximation in a sequence of successive approximations leading to the final solution? This idea, Darwin shows, leads into in a cul-de-sac.

Darwin puts Hill's equation in the form

$$\frac{d^2 w}{dt^2} + (\theta_0 + 2\theta_1 \cos 2\tau + 2\theta_2 \cos 4\tau + \cdots)w = 0,$$

and takes

$$w = A\cos[t\sqrt{\theta_0} + \varepsilon]$$

as the first approximation. Substituting this expression in the term multiplied by θ_1, and neglecting θ_2, θ_3, etc., he obtains the equation

$$\frac{d^2 w}{dt^2} + \theta_0 w + A\theta_1\{\cos[t(\sqrt{\theta_0} + 2) + \varepsilon] + \cos[t(\sqrt{\theta_0} - 2) + \varepsilon]\} = 0.$$

Solving this by the usual rules he obtains the second approximation:

$$w = A\left\{\cos[t\sqrt{\theta_0} + \varepsilon] + \frac{\theta_1 \cos[t(\sqrt{\theta_0} + 2) + \varepsilon]}{4(\sqrt{\theta_0} + 1)} - \frac{\theta_1 \cos[t(\sqrt{\theta_0} - 2) + \varepsilon]}{4(\sqrt{\theta_0} - 1)}\right\}.$$

If this value of w is substituted into the terms of the differential equation having the coefficients θ_1 and θ_2, terms in $\cos[t(\surd\theta_0 + 4) + \varepsilon)]$ and $\cos[t(\surd\theta_0 - 4) + \varepsilon]$ are produced; and so are terms in $\cos[t\surd\theta_0 + \varepsilon]$—a term of exactly the same kind as that assumed for the first approximation. As a consequence, in the next stage of the approximation a secular term having the form $Ct\sin[t\surd\theta_0 + \varepsilon]$ arises. Such a term would come to dominate the solution and there falsify it.

A remedy would seem to be to start over again, using a first approximation of the form $w = A\cos[ct + \varepsilon]$, where c differs slightly from $\surd\theta_0$. But the process of successive approximations still circles back on itself, generating terms that modify the values of terms ostensibly determined earlier in the process. Evidently we are in need of a procedure that is holistic in the sense of taking account from the start of all the terms that can significantly influence the solution.

Since the reduction of Θ, in the form previously given, namely $(\theta_0 + \theta_1 \cos 2\tau + \theta_2 \cos 4\tau + \cdots)$, presents difficulties, Hill proposes to derive another form from differential equations in terms of coordinates expressing the relative position of the moon to the sun. He introduces rectangular coordinates x and y, rotating in the plane of the ecliptic with constant angular speed, in such a way that the axis of x passes constantly through the center of the sun. He adopts the imaginary variables

$$u = x + y\sqrt{-1}, \quad s = x - y\sqrt{-1},$$

and puts $\varepsilon^{\tau\sqrt{-1}} = \zeta$, where ε is the basis of natural logarithms. In addition he introduces the operator $D = -\frac{d}{d\tau}\sqrt{-1}$, so that

$$D(a\zeta^v) = va\zeta^v.$$

He makes the parameter **m** to be, as in our earlier discussion, the ratio of the synodic month to the sidereal year, or $\mathbf{m} = n'/(n - n')$. With μ as the sum of the masses of the earth and the moon, he puts $\kappa = \mu/(n - n')^2$. Finally, he defines the potential function by

$$\Omega = \frac{\kappa}{\sqrt{us}} + \frac{3}{8}\mathrm{m}^2(u+s)^2.$$

With these preliminaries, he can now derive differential equations of the moon's motion. A step-by-step derivation of the differential equations will be given in our resume of Hill's second paper, using the Lagrangian algorithm for extracting equations from the expressions for the potential function and the kinetic energy.

In the exposition of his first paper, Hill merely gives the result:

$$D^2u + 2\mathrm{m}Du + 2\frac{\partial\Omega}{\partial s} = 0,$$

$$D^2s - 2\mathrm{m}Ds + 2\frac{\partial\Omega}{\partial u} = 0. \tag{I.11}$$

Multiplying the first of these by Ds, the second by Du, adding the products and integrating the resulting equation, he obtains the Jacobian integral:

$$DuDs + 2\Omega = 2C. \tag{I.11a}$$

Subjecting (I.11) and (I.lla) to the operation δ yields the three equations

$$D^2\delta u + 2\mathrm{m}D\delta u + 2\left(\frac{\partial^2\Omega}{\partial u\partial s}\right)_0 \delta u + 2\left(\frac{\partial^2\Omega}{\partial s^2}\right)_0 \delta s = 0,$$

$$D^2\delta s - 2\mathrm{m}D\delta s + 2\left(\frac{\partial^2\Omega}{\partial u\partial s}\right)_0 \delta s + 2\left(\frac{\partial^2\Omega}{\partial u^2}\right)_0 \delta u = 0,$$

$$DuD\delta s + DsD\delta u + 2\left(\frac{\partial\Omega}{\partial u}\right)_0 \delta u + 2\left(\frac{\partial\Omega}{\partial s}\right)_0 \delta s = 0. \tag{I.12}$$

These equations still hold if δ is changed into D, since they then become the derivatives of (I.11) and (I.lla). Hence $\delta u = Du_0$, $\delta s = Ds$ constitute a particular solution of (I.12). This solution reveals nothing about the effect of the free oscillations δu, δs on the Variation orbit. As before, Hill uses the particular solution to reduce the order of the final solution. He adopts new variables v and w such that $\delta u = Du \cdot v$, $\delta s = Ds \cdot w$. When these are substituted into (I.12), and the second and third derivatives of u and s are eliminated by means of (I.11) and (I.lla), the result is

$$Du_0 \cdot D^2v - 2\left[2\left(\frac{\partial\Omega}{\partial s}\right)_0 + \mathrm{m}Du_0\right]Dv - 2\left(\frac{\partial^2\Omega}{\partial s^2}\right)_0 Ds_0 \cdot (v - w) = 0,$$

$$Ds_0 \cdot D^2w - 2[2\left(\frac{\partial\Omega}{\partial u}\right)_0 + \mathrm{m}Ds_0]Dw - 2\left(\frac{\partial^2\Omega}{\partial u^2}\right)_0 Du_0 \cdot (w - v) = 0,$$

$$Du_0Ds_0 \cdot D(v + w) - 2\left[\left(\frac{\partial\Omega}{\partial s}\right)_0 Ds_0 - \left(\frac{\partial\Omega}{\partial u}\right)_0 Du_0 + \mathrm{m}Du_0Ds_0\right](v - w) = 0. \tag{I.13}$$

Hill multiplies the first equation of (I.13) by Ds_0 and the second by Du_0 and takes their difference. The resulting equation, along with the third equation of (I.13), will be his basis for the solution of the problem. For brevity he writes

$$\Delta = \left(\frac{\partial\Omega}{\partial s}\right)_0 Ds_0 - \left(\frac{\partial\Omega}{\partial u}\right)_0 Du_0 + \mathrm{m}Du_0Ds_0,$$

and puts

$$\rho = v + w, \sigma = v - w.$$

His two equations then take the form

$$Du_0Ds_0 \cdot D\rho - 2\Delta \cdot \sigma = 0,$$

$$D[Du_0Ds_0 \cdot D\sigma] - 2\Delta \cdot D\rho - 2\left[\left(\frac{\partial^2\Omega}{\partial u^2}\right)_0 Du_0^2 + \left(\frac{\partial^2\Omega}{\partial s^2}\right)_0 Ds_0^2\right]\sigma = 0. \tag{I.14}$$

Eliminating $D\rho$ between the two equations of (I.14), he obtains an equation in which the single variable representing the free oscillation is σ:

$$D[Du_0Ds_0 \cdot D\sigma] - 2\left[\left(\frac{\partial^2\Omega}{\partial u^2}\right)_0 Du_0^2 + \left(\frac{\partial^2\Omega}{\partial s^2}\right)_0 Ds_0^2 + \frac{2\Delta^2}{Du_0Ds_0}\right]\sigma = 0.$$

To remove the term involving $D\sigma$, he makes the substitution

$$\sigma = \frac{w}{\sqrt{Du_0Ds_0}}.$$

The product Du_0Ds_0, be it noted, is the negative of the square of the speed in the Variation orbit. With this substitution, he obtains a differential equation for w:

$$D^2w = \theta w. \tag{I.15}$$

The coefficient θ can be put in the form

$$\frac{1}{Du_0Ds_0}\left[\left(\frac{\partial^2\Omega}{\partial u^2}\right)_0 Du_0^2 - 2\left(\frac{\partial^2\Omega}{\partial u\partial s}\right)_0 Du_0Ds_0 + \left(\frac{\partial^2\Omega}{\partial s^2}\right)_0 Ds_0^2\right]$$
$$+ 3\left(\frac{\Delta}{Du_0Ds_0}\right)^2 + \mathrm{m}^2.$$

The partial derivatives that appear here are determined from the formula:

$$\Omega = \frac{\kappa}{\sqrt{us}} + \frac{3}{8}\mathrm{m}^2(u+s)^2.$$

Also, $Du_0 Ds_0$ is replaced in accordance with the Jacobian integral by $2C - 2\Omega$. With these substitutions, θ becomes

$$\frac{\kappa}{(u_0 s_0)^{3/2}} + \frac{3}{8}\,\frac{\frac{\kappa}{(u_0 s_0)^{5/2}}[u_0 Ds_0 - s_0 Du_0]^2 + \mathrm{m}^2(Du_0 - Ds_0)^2}{C-\Omega} + \frac{3}{4}\left[\frac{\Delta}{C-\Omega}\right]^2 + \mathrm{m}^2.$$

This expression, Hill tells us, is suitable for development in infinite series, when the method of special values (harmonic analysis) is used. The quadrant from $\tau = 0°$ to $\tau = 90°$ is divided into a certain number of equal parts, say six, and from the values of u_0, s_0, Du_0, Ds_0 at the dividing points the corresponding values of θ are determined. From the latter, by a well-known process, the coefficients of the periodic terms of θ are determined. Hill thus obtains the following expression for θ:

$$\begin{aligned}\theta = {} & 1.15884\ 39395\ 96583 \\ & - 0.11408\ 80374\ 93807 \cos 2\tau \\ & + 0.00076\ 64759\ 95109 \cos 4\tau \\ & - 0.00001\ 83465\ 77790 \cos 6\tau \\ & + 0.00000\ 01088\ 95009 \cos 8\tau \\ & - 0.00000\ 00020\ 98671 \cos 10\tau \\ & + 0.00000\ 00000\ 12103 \cos 12\tau \\ & - 0.00000\ 00000\ 00211 \cos 14\tau. \end{aligned} \tag{I.16}$$

Hill also develops, with analytic ploys of considerable ingenuity, a literal formula for θ in terms of $\mathbf{m}$, accurate to the order of $\mathbf{m}^{10}$:

$$\begin{aligned}\theta = {} & 1 + 2\mathrm{m} - \frac{1}{2}\mathrm{m}^2 + \frac{3}{2}\mathrm{m}^2 a_1 + 54a_1^2 + (12 - 4\mathrm{m})a_1 a_{-1} + (6 - 4\mathrm{m})a_{-1}^2 \\ & + \left[(6 + 12\mathrm{m})a_1 + (6 + 8\mathrm{m})a_{-1} - \frac{3}{2}\mathrm{m}^2\right](\zeta^2 + \zeta^{-2}) \\ & + \left[\begin{array}{l} 20\mathrm{m}a_2 + (16 + 20\mathrm{m})a_{-2} - (9 + 40\mathrm{m})a_1^2 \\ +6a_1 a_{-1} + (7 + 4\mathrm{m})a_{-1}^2 - \frac{3}{2}\mathrm{m}^2(a_1 - a_{-1}) \end{array}\right](\zeta^4 + \zeta^{-4}). \end{aligned} \tag{I.17}$$

As this formula is not necessary to the central argument, we shall not examine its derivation.

We turn now to Hill's solution of (I.15). To begin with, he reformulates θ as a series of exponential terms. Thus, in place of the formula $\sum_{i=0}^{\infty} \theta_i \cos 2i\tau$ with the summation running from zero to infinity, he substitutes $\sum_{-\infty}^{+\infty} \theta_i \zeta^{2i}$, with the summation running from minus to plus infinity. In the latter formula we are to understand that $\theta_i = \theta_{-i}$. With the exception of θ_0, which retains its previous value, the new θ_i are the halves of the θ_i in the earlier formula. The symbol ζ stands as before for $\varepsilon^{\tau\sqrt{-1}}$, where ε is Hill's symbol for the base of natural logarithms. When the index i is negative, the exponent of ζ in Hill's new summation formula is negative. Thus Hill's new formula gives us the well-known exponential expression for the cosine:

$$2\cos 2i\tau = \varepsilon^{2i\tau\sqrt{-1}} + \varepsilon^{-2i\tau\sqrt{-1}}.$$

As the form of a possible solution of (I.15), Hill proposes

$$w = \sum_{-\infty}^{+\infty} b_i \zeta^{c+2i}. \tag{I.18}$$

Here c is the ratio of the synodic to the anomalistic month. Observation gives this constant as approximately 29.53/27.55, but the point now is to determine it from theory insofar as it depends on $\mathbf{m}$ alone. Under this restriction, c will give the rate at which w runs through its cycle, from perigee back to perigee. The b_i are also unknown constants, and a complete solution of (I.15) would require determining them; but Hill's aim in the present paper is solely to determine c.

The presence of $2i$ in the exponent of ζ in (I.17) is necessary because θ contains, besides the constant θ_0, terms of the form $\theta_i \cos 2i\tau$; such terms when multiplied by ζ^c will necessarily produce terms containing ζ^{c+2i}. Hence, for $D^2 w$ to be equal to θw as (I.15) requires, w must contain the factor ζ^{c+2i} from the start.

To solve (I.15), Hill uses the method of undetermined coefficients – his preferred method as it was Euler's. If we compute $D^2 w$ for a particular index j, we obtain $(c+2j)^2 b_j \zeta^{c+2j}$. The expression of θw on the right-hand side of the equation will contain all products of $\sum \theta_i \zeta^{2i}$ by $\sum b_i \zeta^{c+2(j-i)}$ such that the resulting exponent of ζ is $c+2j$. Using distinct indices in the two sums, we may write θw as follows:

$$\sum_{i=-\infty}^{+\infty} \theta_i \zeta^{2i} \times \sum_{k=-\infty}^{+\infty} b_k \zeta^{c+2k} = \sum_{i=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \theta_i b_k \zeta^{c+2(i+k)}.$$

The terms that we want from these products will be those in which $(i+k) = j$, in other words the sum $\sum_{i=-\infty}^{+\infty} \theta_i b_{j-i} \zeta^{c+2j}$. Although the number of terms in this sum is infinite, Hill's calculations have indicated that for large $|\pm i|$ the θ_i diminish sharply; hence the terms with i large should prove negligible. Hill moves the term $\theta_0 b_0 \zeta^{c+2j}$ from the right to the left side of the equation (changing its sign, of course), and replaces $(c+2j)^2 - \theta_0$ in each equation by the symbol $[j]$. Dividing ζ^{c+2j} out of each equation, he then gives explicitly

$$\begin{array}{llllll}
\cdots + [-2]b_{-2} & -\theta_1 b_{-1} & -\theta_2 b_0 & -\theta_3 b_1 & -\theta_4 b_2 & - \cdots = 0, \\
\cdots - \theta_1 b_{-2} & +[-1]b_{-1} & -\theta_1 b_0 & -\theta_2 b_1 & -\theta_3 b_2 & - \cdots = 0, \\
\cdots - \theta_2 b_{-2} & -\theta_1 b_{-1} & +[0]b_0 & -\theta_1 b_1 & -\theta_2 b_2 & - \cdots = 0, \\
\cdots - \theta_3 b_{-2} & -\theta_2 b_{-1} & -\theta_1 b_0 & +[1]b_1 & -\theta_1 b_2 & - \cdots = 0, \\
\cdots - \theta_4 b_{-2} & -\theta_3 b_{-1} & -\theta_2 b_0 & -\theta_1 b_1 & +[2]b_2 & - \cdots = 0.
\end{array}$$

For each $[j]$, where j is any positive or negative integer, there is an equation, and each equation contains an infinite number of terms. The equations are homogeneous, each term having one of the b's as linear factor. For a trivial solution, one could set all the b's equal to zero. Are non-trivial solutions possible?

If the equations were finite in number, containing a number of unknowns equal to the number of equations, it can be proved that a non-trivial solution would be possible if and only if the *determinant* of the equations were equal to zero. This determinant is composed of the coefficients. For the five terms of the five equations given above, it would be

$$\begin{vmatrix} [-2] & -\theta_1 & -\theta_2 & -\theta_3 & -\theta_4 \\ -\theta_1 & [-1] & -\theta_1 & -\theta_2 & -\theta_3 \\ -\theta_2 & -\theta_1 & [0] & -\theta_1 & -\theta_2 \\ -\theta_3 & -\theta_2 & -\theta_1 & [1] & -\theta_1 \\ -\theta_4 & -\theta_3 & -\theta_2 & -\theta_1 & [2] \end{vmatrix} \tag{I.19}$$

This determinant contains, in the bracketed quantities $[j]$, the unknown quantity c. Hence, were (I.19) the determinant in question, c might be determined in such a way as to make the determinant zero. As Hill puts it,

> ...we get a symmetrical determinant involving c, which, equated to zero, determines this quantity.

But, Hill's determinant is infinite. Do the rules for ordinary determinants apply? Hill believes they do, for he takes the infinite determinant as the limit of a sequence of finite determinants:

> The question of the convergence, so to speak, of a determinant, consisting of an infinite number of constituents, has nowhere, so far as I am aware, been discussed. All such determinants must be regarded as having a central constituent; when, in computing in succession the determinants formed from the 3^2, 5^2, 7^2, & c., constituents symmetrically situated with respect to the central constituent, we approach, without limit, a determinate magnitude, the determinant may be called convergent, and the determinate magnitude is its value.

In a trial computation, Hill writes out the 3×3 determinant at the center of (I.19), and sets it equal to zero:

$$\begin{vmatrix} [-1] & -\theta_1 & -\theta_2 \\ -\theta_1 & [0] & -\theta_1 \\ -\theta_2 & -\theta_1 & [1] \end{vmatrix} = [-1][0][1] - \theta_1^2\{[1] + [-1]\} - 2\theta_1^2\theta_2 + \theta_2^2[0] = 0.$$

He proposes to neglect terms of the order of $\mathbf{m}^5 = 0.000003454$. One such term is $-2\theta_1^2\theta_2$, which proves to be equal to -0.000002494. The final term, $\theta_2^2[0]$, with a provisional value of [0] calculated from the observational value of c, proves to be

-0.000000001, hence also negligible. Hill then puts the equation, with these terms deleted and the symbols $[j]$ replaced by what they signify, in the form

$$[(c^2+4-\theta_0)^2-16c^2][c^2-\theta_0]-2\theta_1^2[c^2+4-\theta_0]=0.$$

(Achieving this result takes a little doing.)

A nearly exact solution, Hill shows, can be obtained by means of two further deletions. To show that they are reasonable, we write out the equation in an expanded form, and, substituting the observational value of c, namely 1.071713598, compute the numerical value of each term; these values, with the factor 10^6 omitted, are placed below each term:

$$\begin{array}{ccccc} (c^2-\theta_0)^3 & +\,8(c^2-\theta_0)^2 & +\,16(1-c^2)(c^2-\theta_0) & -\,2\theta_1^2(c^2-\theta_0) & -\,8\theta_1^2=0. \\ -1.084 & 844.418 & 24422.213 & -\,66.862 & -\,26032.152 \end{array}$$

The first term is much the smallest, and the fourth is but 8% of the next larger term; Hill neglects both. The remaining terms reduce to

$$c^4-2c^2-\theta_0^2+2\theta_0-\theta_1^2=0.$$

The solution of this is

$$c=\sqrt{1+\sqrt{(\theta_0^2-1)^2-\theta_1^2}}=1.0715632.$$

The observational value is larger by 0.014%.

If Hill is on the right track, the calculated value should indeed err in the direction of smallness. For c is the ratio of the synodic month to the anomalistic month. If dw/dt is the mean sidereal rate of motion of the lunar perigee, and n, n' are the mean sidereal rates of motion of the Moon and the Sun, then this ratio can be expressed as

$$c=\frac{n-\frac{dw}{dt}}{n-n'}.$$

Thus c will come out larger if dw/dt is smaller. But the calculation has neglected the lunar orbit's inclination with respect to the ecliptic, and this inclination has the effect of diminishing the Sun's action on the Moon. The calculation therefore makes the Sun's action too great, hence gives too great a ratio of dw/dt to n, and thus too small a value for c.

But now Hill commences his serious assault on the infinite determinant. He represents it by $D(c)$, and asks us to observe that $D(c)=D(-c)$: the determinant is an even function. Moreover, $D(c)=D(c\pm 2i)$, where i can be any integer: $D(c)$ is thus periodic. According to Hill:

> It will occur immediately to every one that the properties we have stated of the roots of $D(c)=0$ are precisely those of the transcendental equation
>
> $$\cos(\pi x)-a=0;$$

of which, if x_0 is one of the roots, the whole series of roots is represented by $\pm x_0 + 2i$. Hence we must necessarily have, identically,

$$D(c) = A[\cos(\pi c) - \cos(\pi c_0)],$$

A being some constant independent of c.

With a view to evaluating A, Hill introduces Euler's infinite product for $\cos(\pi c)$, namely, $\cos(\pi c) = \prod_{k=0}^{\infty}\left(1 - \frac{4c^2}{(2k+1)^2}\right)$. Hill may have read Euler's derivation of this formula, and of a parallel formula for $\sin(\pi c)$, in Euler's *Introductio in analysin infinitorum*, Lausanne, 1748.[50] When an approximation to $\cos(\pi c)$ is obtained from the first $(n + 1)$ factors in the foregoing infinite product, the highest power of c (namely c^{2n}) will have the coefficient

$$\frac{-4}{1^2} \times \frac{-4}{3^2} \times \frac{-4}{5^2} \times \cdots \times \frac{-4}{(2n+1)^2}.$$

Hill proposes to transform $D(c)$ so that in its expansion, computed to the same approximation as the formula for the infinite product giving $\cos(\pi c)$, the term containing the largest power of c will have this same coefficient.

The transformation consists in multiplying the row of $D(c)$ containing [0] by -4, the rows containing [1] and [-1] by $4/(4^2 - 1)$, and, in general, the rows containing $[j]$ and $[-j]$ by $4/[(4j)^2 - 1] = 4/(2j - 1)(2j + 1)$. A new determinant, $\nabla(c)$, thus arises, which has the same roots as $D(c)$, since multiplying an equation by a constant does not change its roots.

As a visual aid to the reader, I write out the central 5×5 sub-determinant of $\nabla(c)$:

$$\begin{vmatrix}
\frac{4}{63}[-2] & \frac{-4}{63}\theta_1 & \frac{-4}{63}\theta_2 & \frac{-4}{63}\theta_3 & \frac{-4}{63}\theta_4 \\
\frac{-4}{15}\theta_1 & \frac{4}{15}[-1] & \frac{-4}{15}\theta_1 & \frac{-4}{15}\theta_2 & \frac{-4}{15}\theta_3 \\
4\theta_2 & 4\theta_1 & -4[0] & 4\theta_1 & -4\theta_2 \\
\frac{-4}{15}\theta_3 & \frac{-4}{15}\theta_2 & \frac{-4}{15}\theta_1 & \frac{4}{15}[1] & \frac{-4}{15}\theta_1 \\
\frac{-4}{63}\theta_4 & \frac{-4}{63}\theta_3 & \frac{-4}{63}\theta_2 & \frac{-4}{63}\theta_1 & \frac{4}{63}[2]
\end{vmatrix}$$

The product of the five terms in its main diagonal, with the symbols $[j]$ replaced by what they signify, is

$$\frac{4}{7 \cdot 9}[(c-4)^2 - \theta_0] \times \frac{4}{3 \cdot 5}[(c-2)^2 - \theta_0] \times (-4)[c^2 - \theta_0]$$
$$\times \frac{4}{3 \cdot 5}[(c-2)^2 - \theta_0] \times \frac{4}{7 \cdot 9}[(c-4)^2 - \theta_0].$$

Note that the factors symmetrically placed on either side of the central factor have the same numerical coefficient. Evidently the coefficient of the highest power of c

50 See Leonhard Euler, *Opera Omnia*, I.8, 168–169.

in the above product is $(-4)\frac{4}{3^2} \times \frac{4}{5^2} \times \frac{4}{7^2} \times \frac{4}{9^2}$. For larger central sub-determinants of $\nabla(c)$, the new numerical factors added to the product will always have 4 in the numerator and the square of an odd number in the denominator: the pattern is the same as that for $\cos(\pi c)$. Therefore $A = 1$ and Hill can write

$$\nabla(c) = \cos(\pi c) - \cos(\pi c_0).$$

This equation holds for any value of c. Since $\cos(\pi c_0)$ is a constant independent of the value of c, we can determine its value by giving a particular value to c, for instance $c = 0$:

$$\cos(\pi c_0) = \cos(\pi c) - \nabla(c) = \cos(0) - \nabla(0) = 1 - \nabla(0).$$

Our aim is to find a value of c such that $\nabla(c) = 0$, and when $\nabla(c) = 0$, we shall also have $\cos(\pi c) - \cos(\pi c_0) = 0$. Hence, in this case $\cos(\pi c) = 1 - \nabla(0)$. It then follows that $\nabla(0) = 1 - \cos(\pi c)$, or

$$\nabla(0) = 2\sin^2\left(\frac{\pi c}{2}\right). \tag{I.20}$$

Therefore, if we knew the value of $\nabla(0)$, we could solve (I.20) for c. Note that in $\nabla(0)$, c has been set equal to zero, so that it does not occur, and $[j]$ where it appears in the main diagonal is reduced to $(2j)^2 - \theta_0$.

On the way to obtaining a numerical value for $\nabla(0)$, Hill introduces a new determinant, symbolized by $\square(0)$. He obtains it by dividing the terms in each row of $\nabla(0)$ by the term in that row that is in the main diagonal. Thus the central 5×5 sub-determinant in $\square(0)$ is

$$\begin{vmatrix} 1 & \frac{-\theta_1}{4^2-\theta_0} & \frac{-\theta_2}{4^2-\theta_0} & \frac{-\theta_3}{4^2-\theta_0} & \frac{-\theta_4}{4^2-\theta_0} \\ \frac{-\theta_1}{2^2-\theta_0} & 1 & \frac{-\theta_1}{2^2-\theta_0} & \frac{-\theta_2}{2^2-\theta_0} & \frac{-\theta_3}{2^2-\theta_0} \\ \frac{-\theta_2}{0^2-\theta_0} & \frac{-\theta_1}{0^2-\theta_0} & 1 & \frac{-\theta_1}{0^2-\theta_0} & \frac{-\theta_2}{0^2-\theta_0} \\ \frac{-\theta_3}{2^2-\theta_0} & \frac{-\theta_2}{2^2-\theta_0} & \frac{-\theta_1}{2^2-\theta_0} & 1 & \frac{-\theta_1}{2^2-\theta_0} \\ \frac{-\theta_4}{4^2-\theta_0} & \frac{-\theta_3}{4^2-\theta_0} & \frac{-\theta_2}{4^2-\theta_0} & \frac{-\theta_1}{4^2-\theta_0} & 1 \end{vmatrix}$$

Then $\nabla(0)$ is equal to $\square(0)$ multiplied by the product of the elements in the main diagonal of $\nabla(0)$.

The latter product, Hill states, is $1-\cos(\pi\sqrt{\theta_0}) = 2\sin^2\left(\frac{\pi}{2}\sqrt{\theta_0}\right)$. He proves this as follows:

> As, in the particular case, where θ_1, θ_2, etc., all vanish, the proper value of c is $\sqrt{\theta_0}$, it follows that the element of the determinant $\nabla(0)$, formed by the diagonal line of constituents involving θ_0, is
>
> $$1 - \cos(\pi\sqrt{\theta_0}) = 2\sin^2\left(\frac{\pi}{2}\sqrt{\theta_0}\right).$$

In effect, Hill is imagining the following operation as applied to (I.20). On the right-hand side, all the products of $\nabla(0)$ involving θ_i with i other than zero are to vanish; these terms are all and only those that are not in the main diagonal. The determinant $\nabla(0)$, then, is reduced to the product of the elements in the main diagonal. On the left-hand side of the equation, c reduces to $\sqrt{\theta_0}$, since that is the value of c when all the θ_i other than θ_0 vanish. Therefore the product of the elements in the diagonal of $\nabla(0)$ is $2\sin^2\left(\frac{\pi}{2}\sqrt{\theta_0}\right)$, and

$$\sin^2\left(\frac{\pi}{2}c\right) = \sin^2\left(\frac{\pi}{2}\sqrt{\theta_0}\right) \times \square(0). \tag{I.21}$$

Turning to the evaluation of $\square(0)$, Hill remarks that the product of the elements in the main diagonal is 1, and that all the other products in the expanded determinant are much smaller, since all elements of $\square(0)$ other than those in the main diagonal are much less than 1. To obtain these other products, Hill uses the procedure of exchanging columns. Whenever two columns of a determinant are exchanged, the resulting determinant has the same absolute value as the original determinant, but differs in sign, being negative if the original determinant was positive, and *vice versa.* A second exchange of columns reverses the sign again. The product of the elements in the diagonal of the new determinant is thus, when given the appropriate sign, one of the products in the expansion of the original determinant. The columns can be returned to their original positions, and two or more other columns exchanged in order to obtain another product in the expansion of the original determinant.

When two adjacent columns of $\square(0)$ are interchanged, the main diagonal of the resulting determinant will consist of l's except for two elements, each of which has θ_1 as its numerator. The denominators of these two elements have the form $(2i)^2-\theta_0$, where i is an integer which can be positive, negative, or zero; but i in the one element will differ from i in the other by 1. Following Hill, we symbolize $(2i)^2 - \theta_0$ by $\{i\}$. Then the product of the elements of the main diagonal of the new determinant will be $-\frac{\theta_1^2}{\{i\}\{i+1\}}$. To obtain the sum of all the terms of this type – they are infinite in number – requires evaluating the sum $-\theta_1^2\sum_{-\infty}^{+\infty}\frac{1}{\{i\}\{i+1\}}$. Hill develops a formula for doing this; we shall describe its derivation in a moment. In the particular case we are examining it yields

$$-\frac{\theta_1^2\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{4\sqrt{\theta_0}(\theta_0-1)} = +0.00180\ 46110\ 93422\ 7.$$

Recall that the first product in the determinant $\square(0)$ was equal to 1; the second product, we now see, is less than 0.2% of the first. Yet it is the largest among the remaining products, and in fact 10^4 times larger than any of the others. Hill undoubtedly felt secure about the "convergence" of this determinant.

To derive the formula used in the foregoing calculation, Hill considers the more general sum $\sum_{-\infty}^{+\infty}\frac{1}{\{i\}\{i+k\}} = \sum_{-\infty}^{+\infty}\frac{1}{[(2i)^2-\theta_0][2^2(i+k)^2-\theta_0]}$. Here k is to be understood as a fixed integer, later to be assigned the values 1, 2, or 3, while i remains the variable index of the terms in the summation. To convert each of the two factors in the denominator into the difference of two squares, Hill introduces the substitution

$4\theta^2 = \theta_0$, and then factors the factors.[51] The expression under the summation sign can thus be given the form

$$\frac{1}{16(\theta+i)(\theta-i)(\theta+i+k)(\theta-i-k)}.$$

This expression can be resolved into a sum of algebraically irreducible partial fractions:

$$\frac{1}{16}\left[\frac{A}{\theta+i}+\frac{B}{\theta-i}+\frac{C}{\theta+i+k}+\frac{D}{\theta-i-k}\right],$$

where A, B, C, and D are constants. These constants are determined by setting our two expressions equal, clearing them of fractions, then giving i in succession the values $-\theta$, $+\theta$, $-\theta-k$, $\theta-k$, so as to cause, each time, three of the four resulting terms to vanish. We thus obtain four equations for the four constants:

$$\begin{aligned} 2k\theta(2\theta-k)A &= 1,\\ -2k\theta(2\theta+k)B &= 1,\\ -2k\theta(2\theta+k)C &= 1,\\ 2k\theta(2\theta-k)D &= 1. \end{aligned}$$

But, Hill tells us, it is well known that[52]

$$\sum_{-\infty}^{+\infty}\frac{1}{\theta+i}=\sum_{-\infty}^{+\infty}\frac{1}{\theta-i}=\sum_{-\infty}^{+\infty}\frac{1}{\theta+i+k}=\sum_{-\infty}^{+\infty}\frac{1}{\theta-i-k}=\pi\cot\pi\theta.$$

Hence,

$$\begin{aligned} \sum_{-\infty}^{+\infty}\frac{1}{\{i\}\{i+1\}} &= \frac{1}{16}(A+B+C+D)\pi\cot\pi\theta\\ &= \frac{\pi\cot\pi\theta}{8\theta(4\theta^2-k^2)}\\ &= \frac{\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{4\sqrt{\theta_0}(\theta_0-k^2)}. \end{aligned} \tag{I.22}$$

For $k = 1, 2, 3$, respectively, (I.22) yields the coefficients for the products $\theta_1^2, \theta_2^2, \theta_3^2$; and thus Hill obtains the contributions of these three products to the value of $\square(0)$:

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51 In the article as printed in the *Collected Mathematical Works*, I, 265, this substitution is given incorrectly, as $4\theta = \theta_0$. It is given correctly in *Acta Mathematica*, VIII (1886), 30.

52 As indicated earlier, this formula was available to Hill in Euler's *Introductio in analysin infinitorum* and in the textbook of Briot & Bouquet, *Théorie des functions doublement périodiques*.

$$\frac{\pi \cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{4\sqrt{\theta_0}}\left[\frac{\theta_1^2}{1-\theta_0}+\frac{\theta_2^2}{4-\theta_0}+\frac{\theta_3^2}{9-\theta_0}\right].$$

Hill could, of course, have included the term for θ_4^2, but this has a value of 3×10^{-15}, and when this factor is multiplied by its coefficient and evaluated numerically, the result proves less than 10^{-15}. He chooses to limit the precision of his calculation to the fifteenth decimal.

Seven more terms, however, must be calculated to bring the overall precision to this level, and they require, in addition to the formula (I.22) given above, two other general formulas derived in the same manner:

$$\sum_{i=-\infty}^{+\infty}\frac{1}{\{i\}\{i+k\}\{i+k'\}}$$

$$= -\frac{1}{16}\frac{3\theta_0-(k^2-kk'+k^2)}{\sqrt{\theta_0}(\theta_0-k^2)(\theta_0-k^2)[\theta_0-(k-k')^2]}\pi \cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right), \tag{I.23}$$

$$\sum_{i=-\infty}^{+\infty}\frac{1}{\{i\}\{i+1\}\{i+k\}\{i+k+1\}}$$

$$-\frac{1}{32}\frac{5\theta_0-(k^2+1)}{\sqrt{\theta_0}(\theta_0-1)(\theta_0-k^2)[\theta_0-(k+1)^2][\theta_0-(k-1)^2]}\pi \cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right). \tag{I.24}$$

Consider first the product $\theta_1^2\theta_2$; it is obtained by first exchanging two adjacent columns, then exchanging one of these with its just acquired new neighbor. The term to be calculated is

$$+\frac{3\pi \cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{8\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)}\theta_1^2\theta_2.$$

Somewhat similarly, the product $\theta_1\theta_2\theta_3$ is obtained by first exchanging adjacent columns, then exchanging one of the exchanged columns with the column on the other side of the just acquired new neighbor. Its coefficient is obtained from formula (I.23) by substituting $k = 1, k' = 3$. The term to be calculated is thus

$$+\frac{(7-3\theta_0)\pi \cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{4\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)(9-\theta_0)}\theta_1\theta_2\theta_3.$$

To obtain $\theta_1^3\theta_3$, two adjacent columns are first exchanged, then one of them is moved by two further exchanges, to the right if it is on the right after the first exchange, to the left in the opposite case; thus the initial pattern *abcd* becomes *bcda* or *dabc*. The coefficient is obtained from formula (I.24) above by substituting $k = 2$. The term to be calculated is therefore

$$+\frac{5\pi \cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{16\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)(9-\theta_0)}\theta_1^3\theta_3.$$

The four remaining terms all require double or triple summations. Thus in the case of θ_1^4, we may start from the double summation

$$\sum_{k=2}^{+\infty}\sum_{i=-\infty}^{+\infty}\frac{1}{\{i\}\{i+1\}\{i+k\}\{i+k+1\}}.$$

Here the summation in which i runs from minus to plus infinity gives all interchanges of adjacent columns leading to the product θ_1^2; the summation in which k runs from 2 to plus infinity then gives all interchanges of adjacent columns capable of being combined with the former exchanges so as to yield the product θ_1^4. First we resolve the expression under the summation signs into partial fractions with respect to i as variable, and sum between the indicated limits; the result is

$$-\frac{1}{32}\frac{5\theta_0-(k^2+1)}{\sqrt{\theta_0}(\theta_0-1)(\theta_0-k^2)[\theta_0-(k+1)^2][\theta_0-(k-1)^2]}\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right).$$

This expression is then to be resolved into partial fractions with respect to k. The summation is most conveniently carried out, not from 2 to infinity, but from 0 to infinity, after which the values of the expression for $k=0$ and $k=1$ can be subtracted. We obtain

$$\frac{\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{32\sqrt{\theta_0}(1-\theta_0)^2}\left[\frac{\pi\cot(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{9}{2(4-\theta_0)}\right]\theta_1^4.$$

By analogous processes the remaining three summations are obtained:

$$\frac{3\pi cit\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{32\sqrt{\theta_0}(1-\theta_0)^2(4-\theta_0)}\left[\frac{\pi\cot(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{2}{4-\theta_0}+\frac{20}{3(9-\theta_0)}\right]\theta_1^4\theta_2;$$

$$\frac{\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{16\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)}\left[\frac{\pi\cot(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{2}{4-\theta_0}+\frac{10}{9-\theta_0}\right]\theta_1^2\theta_2^2;$$

$$\frac{\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{128\sqrt{\theta_0}(1-\theta_0)^3}\left\{\begin{array}{l}\left[-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{9}{2(4-\theta_0)}\right]\frac{\pi\cot(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{25}{8\theta_0}-\frac{1}{\theta_0^2}\\+\frac{2}{1-\theta_0}+\frac{4}{(1-\theta_0)^2}-\frac{9}{8(4-\theta_0)}+\frac{9}{(4-\theta_0)^2}-\frac{4}{9-\theta_0}-\frac{\pi^2}{3\theta_0}\end{array}\right\}\theta_1^6.$$

To obtain his value for $\square(0)$, Hill evaluated these several expressions numerically to sixteen decimal places – a task which included the computation of $\sqrt{\theta_0}$ and two cotangents to the same precision. Then, adding together these results and the main term of the determinant – which, we recall, was equal to unity – he obtained

$$1.00180\ 47920\ 21011\ 2.$$

This result had then to be introduced into (I.20):

$$\sin^2\left(\frac{\pi}{2}c\right)=\sin^2\left(\frac{\pi}{2}\sqrt{\theta_0}\right)\times 1.00180\ 47920\ 21011\ 2.$$

From this expression Hill derived:

$$c = 1.07158\ 32774\ 16016.$$

To check the accuracy of this result, Hill turned back to the equations from which he had derived the determinant $\nabla(0)$. Each of these equations can be expressed by the formula

$$[j]b_j - \sum_i \theta_{j-i} b_i = 0, \tag{I.25}$$

where $[j] = (c+2j)^2 - \theta_0$, and under the summation sign the term with index $i = j$ is omitted. Using his value for c, Hill computed $[j]$ for the following values:

$$\begin{aligned}
[0] &= -0.01055\ 32191\ 58933,\\
[-1] &= -0.29688\ 63288\ 2300,\\
[1] &= +8.37577\ 98905\ 1,\\
[-2] &= +7.41678\ 05615\ 1,\\
[2] &= +24.56211\ 3,\\
[-3] &= +23.13045,\\
[3] &= +48.85,\\
[-4] &= +46.8.
\end{aligned}$$

Now the central equation of the array – the equation in which [0] occurs – may be written

$$[0]b_0 - \sum_i \theta_{j-i} b_i = 0. \tag{I.26}$$

Hill sets about eliminating from (I.26), successively, the unknowns $b_{-1}, b_1, b_{-2}, b_2, b_{-3}, b_{-4}$, using in each case an equation of the form of (I.25) to eliminate the b having subscript j. Thus to eliminate b_{-1} he solves the equation

$$[-1]b_{-1} - \sum_i \theta_{-1-i} b_i = 0$$

or

$$b_{-1} = \frac{\sum_i \theta_{-1-i} b_i}{[-1]},$$

where in the summation on the right the term in which $i = -1$ is omitted. When this value of b_{-1} is substituted into (I.26), and the terms contributing to the coefficient of each b_i are collected, the result is

$$\left[[0] - \frac{\theta_1^2}{[-1]}\right] b_0 - \sum_i \left[\theta_{-i} + \frac{\theta_1 \theta_{i+1}}{[-1]}\right] b_i = 0.$$

The new coefficient of b_0, which Hill symbolizes by $[0]^{(-1)}$, turns out to be smaller than [0] in absolute value. Hill repeats this eliminative process for the b's we have listed, obtaining the following reductions in the coefficient of b_0:

$$\begin{aligned}
[0] &= -0.01055\ 32191\ 58933,\\
[0]^{(-1)} &= +0.00040\ 72723\ 11650,\\
[0]^{(-1,1)} &= +0.00001\ 50888\ 08423,\\
[0]^{(-2,-1,1)} &= +0.00000\ 00253\ 21700,\\
[0]^{(-2,-1,1,2)} &= +0.00000\ 00009\ 20420,\\
[0]^{(-3,-2,-1,1,2)} &= +0.00000\ 00000\ 03941,\\
[0]^{(-3,-2,-1,1,2,3)} &= +0.00000\ 00000\ 00155,\\
[0]^{(-4,-3,-2,-1,1,2,3)} &= +0.00000\ 00000\ 00008.
\end{aligned}$$

As the coefficient of b_0 decreases, so, proportionately, must the second term of (I.26), so that the sum of all terms adds to zero.

Can the 8×10^{-15} of $[0]^{(-4,-3,-2,-1,1,2,3)}$ be reduced further by carrying out further eliminations, for instance, of b_4, b_{-5}, etc.? Hill tells us that these further eliminations do not sensibly change the result. Rather than repeating the whole eliminative process with a lower value of c, it will be sufficient, he says, to subtract half of the residual from the value of c he has assumed. To understand this step, notice that $[0] = c^2 - \theta_0$, and that c exceeds 1 by only about 0.072. When the binomial $(c - 4 \times 10^{-15})$ is squared, the result will therefore be less than c^2, very nearly, by $2 \times 4 \times 10^{-15}$. Hence replacing the assumed value of c by $(c - 4 \times 10^{-15})$ will reduce the residual to zero. Hill's final value of c is thus

$$c = 1.07158\ 32774\ 16012.$$

In R.S. Woodward's obituary of Hill published in *The Astronomical Journal, 28* (1914), 161–162, it is stated that Hill made two exploratory trips into Canada, one into the Hudson Bay region and one into the Canadian Northwest.

> It was during journey through the latter territory that he worked out his famous solution of the problem involving an infinite determinant, a solution "aussi originale que hardi," as remarked by Poincaré.

What actually got worked out during the journey – initial steps, main ideas, final steps? – can only be guessed. The solution demanded extensive and complicated paper-and-pencil computations, most easily imagined as performed where paper was plentiful and filing facilities available. In concluding his memoir, Hill remarks:

> It may be stated that all the computations have been made twice, and no inconsiderable portion of them three times.

The value of c that Hill here obtained from gravitational theory differs from the observational value by only one part in 550. To obtain the implied value of dw/dt, the rate of motion of the lunar perigee, it is necessary to substitute in the equation

$$\frac{dw}{dt} = n - c(n - n').$$

With the observational value of c we obtain 0.00194419 radians/day; Hill's calculated value of c gives 0.001971441 radians/day, which exceeds the observational value by $1/72$nd part, or 1.4%. As we affirmed at the start of this section, no earlier computation of dw/dt had approached the observational value anywhere near so closely. A few of Hill's readers, like John Couch Adams, were able to recognize the significance and the wonder of this achievement.