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# Ordinary Differential Equations and Dynamical Systems

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# Ordinary Differential Equations and Dynamical Systems

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*Dedicated to my father on the ocassion of his 90th birthday*

# Preface

Many years ago, I had my first opportunity to teach a graduate course on ordinary differential equations at UC, Santa Barbara. Not being a specialist, I sought advice and suggestions from others. In so doing, I had the good fortune of consulting with Manoussos Grillakis, who generously offered to share his lovely notes from John Mallet-Paret's graduate course at Brown University. These notes, combined with some of my own home cooking and spiced with ingredients from other sources, evolved over numerous iterations into the current monograph.

In publishing this work, my goal is to provide a mathematically rigorous introduction to the beautiful subject of ordinary differential equations to beginning graduate or advanced undergraduate students. I assume that students have a solid background in analysis and linear algebra. The presentation emphasizes commonly used techniques without necessarily striving for completeness or for the treatment of a large number of topics. I would half-jokingly subtitle this work as "ODE, as told by an analyst."

The first half of the book is devoted to the development of the basic theory: linear systems, existence and uniqueness of solutions to the initial value problem, flows, stability, and smooth dependence of solutions upon initial conditions and parameters. Much of this theory also serves as the paradigm for evolutionary partial differential equations. The second half of the book is devoted to geometric theory: topological conjugacy, invariant manifolds, existence and stability of periodic solutions, bifurcations, normal forms, and the existence of transverse homoclinic points and their link to chaotic dynamics. A common thread throughout the second part is the use of the implicit function theorem in Banach space. Chapter 5, devoted to this topic, serves as the bridge between the two halves of the book.

A few features (or peculiarities) of the presentation include:

– a characterization of the stable, unstable, and center subspaces of a linear operator in terms of its exponential,
– a proof of smooth dependence using the implicit function theorem,
– a simple proof of the Hartman-Grobman theorem by my colleague, Michael Crandall,
– treatment of the Hopf bifurcation using both normal forms and the Liapunov-Schmidt method,

- a treatment of orbital stability of periodic orbits using the Liapunov-Schmidt method, and
- a complete proof of the existence of transverse homoclinic points for periodic perturbations of Newton's Equation.

I am most grateful to Prof. Grillakis for sharing his notes with me, and I thank both Profs. Grillakis and Mallet-Paret for their consent to publish my interpretation of them. I also thank Prof. Michel Chipot, the Series Editor, for encouraging me to publish this monograph.

Santa Barbara, July 2013 Thomas C. Sideris

# Contents

# Chapter 1
# Introduction

The most general $n$th order ordinary differential equation (ODE) has the form

$$F(t, y, y', \ldots, y^{(n)}) = 0,$$

where $F$ is a continuous function from some open set $\Omega \subset \mathbb{R}^{n+2}$ into $\mathbb{R}$. An $n$ times continuously differentiable real-valued function $y(t)$ is a solution on an interval $I$ if

$$F(t, y(t), y'(t), \ldots, y^{(n)}(t)) = 0, \quad t \in I.$$

A necessary condition for existence of a solution is the existence of points $p = (t, y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+2}$ such that $F(p) = 0$. For example, the equation

$$(y')^2 + y^2 + 1 = 0$$

has no (real) solutions, because $F(p) = y_2^2 + y_1^2 + 1 = 0$ has no real solutions.

If $F(p) = 0$ and $\frac{\partial F}{\partial y_{n+1}}(p) \neq 0$, then locally we can solve for $y_{n+1}$ in terms of the other variables by the implicit function theorem

$$y_{n+1} = G(t, y_1, \ldots, y_n),$$

and so locally we can write our ODE as

$$y^{(n)} = G(t, y, y', \ldots, y^{(n-1)}).$$

This equation can, in turn, be written as a first order system by introducing additional unknowns. Setting

$$x_1 = y, \; x_2 = y', \; \ldots, \; x_n = y^{(n-1)},$$

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we have that

$$x_1' = x_2,\ x_2' = x_3,\ \ldots,\ x_{n-1}' = x_n,\ x_n' = G(t, x_1, \ldots, x_n).$$

Therefore, if we define $n$-vectors

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}, \quad f(t,x) = \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ G(t, x_1, \ldots, x_{n-1}, x_n) \end{bmatrix}$$

we obtain the equivalent first order system

$$x' = f(t,x). \tag{1.1}$$

The point of this discussion is that there is no loss of generality in studying the first order system (1.1), where $f(t,x)$ is a continuous function (at least) defined on some open region in $\mathbb{R}^{n+1}$.

A fundamental question that we will address is the existence and uniqueness of solutions to the initial value problem (IVP)

$$x' = f(t,x), \quad x(t_0) = x_0,$$

for points $(t_0, x_0)$ in the domain of $f(t,x)$. We will then proceed to study the qualitative behavior of such solutions, including periodicity, asymptotic behavior, invariant structures, etc.

In the case where $f(t,x) = f(x)$ is independent of $t$, the system is called *autonomous*. Every first order system can be rewritten as an autonomous one by introducing an extra unknown. If

$$z_1 = t,\ z_2 = x_1,\ \ldots,\ z_{n+1} = x_n,$$

then from (1.1) we obtain the equivalent autonomous system

$$z' = g(z), \quad g(z) = \begin{bmatrix} 1 \\ f(z) \end{bmatrix}.$$

Suppose that $f(x)$ is a continuous map from an open set $U \subset \mathbb{R}^n$ into $\mathbb{R}^n$. We can regard a solution $x(t)$ of an autonomous system

$$x' = f(x), \tag{1.2}$$

as a curve in $\mathbb{R}^n$. This gives us a geometric interpretation of (1.2). If the vector $x'(t) \neq 0$, then it is tangent to the solution curve at $x(t)$. The Eq. (1.2) tells us what the value of this tangent vector must be, namely, $f(x(t))$. So if there is one and only one solution through each point of $U$, we know just from the Eq. (1.2) its tangent direction at every point of $U$. For this reason, $f(x)$ is called a *vector field* or *direction field* on $U$.

The collection of all solution curves in $U$ is called the *phase diagram* of $f(x)$. If $f \neq 0$ in $U$, then locally, the curves are parallel. Near a point $x_0 \in U$ where $f(x_0) = 0$, the picture becomes more interesting.

A point $x_0 \in U$ such that $f(x_0) = 0$ is called, interchangably, a *critical point*, a *stationary point*, or an *equilibrium point* of $f$. If $x_0 \in U$ is an equilibrium point of $f$, then by direct substitution, $x(t) = x_0$ is a solution of (1.2). Such solutions are referred to as *equilibrium* or *stationary* solutions.

To understand the phase diagram near an equilibrium point we are going to attempt to approximate solutions of (1.2) by solutions of an associated *linearized system*. Suppose that $x_0$ is an equilibrium point of $f$. If $f \in C^1(U)$, then Taylor expansion about $x_0$ yields

$$f(x) \approx Df(x_0)(x - x_0),$$

when $x - x_0$ is small. The linearized system near $x_0$ is

$$y' = Ay, \quad A = Df(x_0).$$

An important goal is to understand when $y$ is a good approximation to $x - x_0$. Linear systems are simple, and this is the benefit of replacing a nonlinear system by a linearized system near a critical point. For this reason, our first topic will be the study of linear systems.

# Chapter 2
# Linear Systems

## 2.1 Definition of a Linear System

Let $f(t, x)$ be a continuous map from an open set in $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$. A first order system

$$x' = f(t, x)$$

will be called *linear* when

$$f(t, x) = A(t)x + g(t).$$

Here $A(t)$ is a continuous $n \times n$ matrix valued function and $g(t)$ is a continuous $\mathbb{R}^n$ valued function, both defined for $t$ belonging to some interval in $\mathbb{R}$.

A linear system is *homogeneous* when $g(t) = 0$. A linear system is said to have *constant coefficients* if $A(t) = A$ is constant.

In this chapter, we shall study linear, homogeneous systems with constant coefficients, i.e. systems of the form

$$x' = Ax,$$

where $A$ is an $n \times n$ matrix (with real entries).

## 2.2 Exponential of a Linear Transformation

Let $V$ be a finite dimensional normed vector space over $\mathbb{R}$ or $\mathbb{C}$. $L(V)$ will denote the set of linear transformations from $V$ into $V$.

T. C. Sideris, *Ordinary Differential Equations and Dynamical Systems*,
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**Definition 2.1.** Let $A \in L(V)$. Define the operator norm

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|.$$

Properties of the operator norm:

- $\|A\| < \infty$, for every $A \in L(V)$.
- $L(V)$ with the operator norm is a finite dimensional normed vector space.
- Given $A \in L(V)$, $\|Ax\| \leq \|A\|\|x\|$, for every $x \in V$, and $\|A\|$ is the smallest number with this property.
- $\|AB\| \leq \|A\|\|B\|$, for every $A, B \in L(V)$.

**Definition 2.2.** A sequence $\{A_n\}$ in $L(V)$ converges to $A$ if and only if

$$\lim_{n \to \infty} \|A_n - A\| = 0.$$

With this notion of convergence, $L(V)$ is complete.

All norms on a finite dimensional space are equivalent, so $A_n \to A$ in the operator norm implies componentwise convergence in any coordinate system.

**Definition 2.3.** Given $A \in L(V)$, define $\exp A = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} A^k$.

**Lemma 2.1.** *Given $A, B \in L(V)$, we have the following properties:*

1. $\exp At$ *is defined for all* $t \in \mathbb{R}$, *and* $\|\exp At\| \leq \exp \|A\||t|$.
2. $\exp(A + B) = \exp A \ \exp B = \exp B \ \exp A$, *provided* $AB = BA$.
3. $\exp A(t + s) = \exp At \ \exp As = \exp As \ \exp At$, *for all* $t, s \in \mathbb{R}$.
4. $\exp At$ *is invertible for every* $t \in \mathbb{R}$, *and* $(\exp At)^{-1} = \exp(-At)$.
5. $\dfrac{d}{dt} \exp At = A \exp At = \exp At \ A$.

*Proof.* The exponential is well-defined because the sequence of partial sums

$$S_n = \sum_{k=0}^{n} \frac{1}{k!} A^k$$

is a Cauchy sequence in $L(V)$ and therefore converges. Letting $m < n$, we have that

$$\begin{aligned} \|S_n - S_m\| &= \| \sum_{k=m+1}^{n} \frac{1}{k!} A^k \| \\ &\leq \sum_{k=m+1}^{n} \frac{1}{k!} \|A^k\| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=m+1}^{n} \frac{1}{k!}\|A\|^k \\
&= \frac{1}{(m+1)!}\|A\|^{m+1} \sum_{k=0}^{n-m-1} \frac{(m+1)!}{(k+m+1)!}\|A\|^k \\
&\leq \frac{1}{(m+1)!}\|A\|^{m+1} \sum_{k=0}^{\infty} \frac{1}{k!}\|A\|^k \\
&= \frac{1}{(m+1)!}\|A\|^{m+1} \exp\|A\|.
\end{aligned}
$$

From this, we see that $S_n$ is Cauchy. It also follows that $\|\exp A\| \leq \exp\|A\|$.

To prove proeperty (2), we first note that when $AB = BA$ the binomial expansion is valid:

$$(A+B)^k = \sum_{j=0}^{k} \binom{k}{j} A^j B^{k-j}.$$

Thus, by definition

$$
\begin{aligned}
\exp(A+B) &= \sum_{k=0}^{\infty} \frac{1}{k!}(A+B)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} A^j B^{k-j} \\
&= \sum_{j=0}^{\infty} \frac{1}{j!} A^j \sum_{k=j}^{\infty} \frac{1}{(k-j)!} B^{k-j} \\
&= \sum_{j=0}^{\infty} \frac{1}{j!} A^j \sum_{\ell=0}^{\infty} \frac{1}{\ell!} B^{\ell} \\
&= \exp A \ \exp B.
\end{aligned}
$$

The rearrangements are justified by the absolute convergence of all series.

Property (3) is a consequence of property (2), and property (4) is an immediate consequence of property (3).

Property (5) is proven as follows. We have

$$
\begin{aligned}
&\|(\Delta t)^{-1}[\exp A(t+\Delta t)\exp At] - \exp At \ A\| \\
&\qquad = \|\exp At\{(\Delta t)^{-1}[\exp A\Delta t - I] - A\}\| \\
&\qquad = \left\| \exp At \sum_{k=2}^{\infty} \frac{(\Delta t)^{k-1}}{k!} A^k \right\|
\end{aligned}
$$

$$\leq \|\exp At\| \left\| A^2 \Delta t \sum_{k=2}^{\infty} \frac{(\Delta t)^{k-2}}{k!} A^{k-2} \right\|$$
$$\leq |\Delta t| \|A\|^2 \exp \|A\|(|t| + |\Delta t|).$$

This last expression tends to 0 as $\Delta t \to 0$. Thus, we have shown that $\frac{d}{dt} \exp At = \exp At\ A$. This also equals $A \exp At$ because $A$ commutes with the partial sums for $\exp At$ and hence with $\exp At$ itself. □

## 2.3 Solution of the Initial Value Problem for Linear Homogeneous Systems

**Theorem 2.1.** *Let $A$ be an $n \times n$ matrix over $\mathbb{R}$, and let $x_0 \in \mathbb{R}^n$. The initial value problem*

$$x'(t) = Ax(t), \quad x(t_0) = x_0 \tag{2.1}$$

*has a unique solution defined for all $t \in \mathbb{R}$ given by*

$$x(t) = \exp A(t - t_0)\, x_0. \tag{2.2}$$

*Proof.* We use the method of the integrating factor. Multiplying the system (2.1) by $\exp(-At)$ and using Lemma 2.1, we see that $x(t)$ is a solution of the IVP if and only if

$$\frac{d}{dt}[\exp(-At)x(t)] = 0, \quad x(t_0) = x_0.$$

Integration of this identity yields the equivalent statement

$$\exp(-At)x(t) - \exp(-At_0)x_0 = 0,$$

which in turn is equivalent to (2.2). This establishes existence, and uniqueness. □

## 2.4 Computation of the Exponential of a Matrix

The main computational tool will be reduction to an elementary case by similarity transformation.

**Lemma 2.2.** *Let $A, S \in L(V)$ with $S$ invertible. Then*

$$\exp(SAS^{-1}) = S(\exp A)S^{-1}.$$

*Proof.* This follows immediately from the definition of the exponential together with the fact that $(SAS^{-1})^k = SA^kS^{-1}$, for every $k \in \mathbb{N}$. $\square$

The simplest case is that of a diagonal matrix $D = \mathrm{diag}[\lambda_1, \dots, \lambda_n]$. Since $D^k = \mathrm{diag}(\lambda_1^k, \dots, \lambda_n^k)$, we immediately obtain

$$\exp Dt = \mathrm{diag}(\exp \lambda_1 t, \dots, \exp \lambda_n t).$$

Now if $A$ is diagonalizable, i.e. $A = SDS^{-1}$, then we can use Lemma 2.2 to compute

$$\exp At = S \exp Dt \; S^{-1}.$$

An $n \times n$ matrix $A$ is diagonalizable if and only if there is a basis of eigenvectors $\{v_j\}_{j=1}^n$. If such a basis exists, let $\{\lambda_j\}_{j=1}^n$ be the corresponding set of eigenvalues. Then

$$A = SDS^{-1},$$

where $D = \mathrm{diag}[\lambda_1, \dots, \lambda_n]$ and $S = [v_1 \cdots v_n]$ is the matrix whose columns are formed by the eigenvectors. Even if $A$ has real entries, it can have complex eigenvalues, in which case the matrices $D$ and $S$ will have complex entries. However, if $A$ is real, complex eigenvectors and eigenvalues occur in conjugate pairs.

In the diagonalizable case, the solution of the initial value problem (2.1) is

$$x(t) = \exp At \; x_0 = S \exp Dt \; S^{-1} x_0 = \sum_{j=1}^{n} c_j \exp \lambda_j t \; v_j,$$

where the coefficients $c_j$ are the coordinates of the vector $c = S^{-1}x_0$. Thus, the solution space is spanned by the elementary solutions $\exp \lambda_j t \; v_j$.

There are two important situations where an $n \times n$ matrix can be diagonalized.

- $A$ is real and symmetric, i.e. $A = A^T$. Then $A$ has real eigenvalues and there exists an orthonormal basis of real eigenvectors. Using this basis yields an orthogonal diagonalizing matrix $S$, i.e. $S^T = S^{-1}$.
- $A$ has distinct eigenvalues. For each eigenvalue there is always at least one eigenvector, and eigenvectors corresponding to distinct eigenvalues are independent. Thus, there is a basis of eigenvectors.

An $n \times n$ matrix over $\mathbb{C}$ may not be diagonalizable, but it can always be reduced to Jordan canonical (or normal) form. A matrix $J$ is in Jordan canonical form if it is block diagonal

$$J = \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_p \end{bmatrix}$$

and each Jordan block has the form

$$B = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}.$$

Since $B$ is upper triangular, it has the single eigenvalue $\lambda$ with multiplicity equal to the size of the block $B$.

Computing the exponential of a Jordan block is easy. Write

$$B = \lambda I + N,$$

where $N$ has 1's along the superdiagonal and 0's everywhere else. The matrix $N$ is nilpotent. If the block size is $d \times d$, then $N^d = 0$. We also clearly have that $\lambda I$ and $N$ commute. Therefore,

$$\exp Bt = \exp(\lambda I + N)t = \exp \lambda I t \exp Nt = \exp(\lambda t) \sum_{j=1}^{d-1} \frac{t^j}{j!} N^j.$$

The entries of $\exp Nt$ are polynomials in $t$ of degree at most $d - 1$.

Again using the definition of the exponential, we have that the exponential of a matrix in Jordan canonical form is the block diagonal matrix

$$\exp Jt = \begin{bmatrix} \exp B_1 t & & \\ & \ddots & \\ & & \exp B_p t \end{bmatrix}.$$

The following central theorem in linear algebra will enable us to understand the form of $\exp At$ for a general matrix $A$.

**Theorem 2.2.** *Let $A$ be an $n \times n$ matrix over $\mathbb{C}$. There exists a basis $\{v_j\}_{j=1}^n$ for $\mathbb{C}^n$ which reduces $A$ to Jordan normal form $J$. That is, if $S = [v_1 \cdots v_n]$ is the matrix whose columns are formed from the basis vectors, then*

$$A = SJS^{-1}.$$

*The Jordan normal form of $A$ is unique up to the permutation of its blocks.*

When $A$ is diagonalizable, the basis $\{v_j\}_{j=1}^n$ consists of eigenvectors of $A$. In this case, the Jordan blocks are $1 \times 1$. Thus, each vector $v_j$ lies in the kernel of $A - \lambda_j I$ for the corresponding eigenvalue $\lambda_j$.

In the general case, the basis $\{v_j\}_{j=1}^n$ consists of appropriately chosen *generalized eigenvectors* of $A$. A vector $v$ is a generalized eigenvector of $A$ corresponding to an eigenvalue $\lambda_j$ if it lies in the kernel of $(A - \lambda_j I)^k$ for some $k \in \mathbb{N}$. The set of generalized eigenvectors of $A$ corresponding to a given eigenvalue $\lambda_j$ is a subspace, $E(\lambda_j)$, of $\mathbb{C}^n$, called the generalized eigenspace of $\lambda_j$. These subspaces are invariant under $A$. If $\{\lambda_j\}_{j=1}^d$ are the distinct eigenvalues of $A$, then

$$\mathbb{C}^n = E(\lambda_1) \oplus \cdots \oplus E(\lambda_d),$$

is a direct sum, that is, every vector in $x \in \mathbb{C}^n$ can be uniquely written as a sum $x = \sum_{j=1}^d x_j$, with $x_j \in E(\lambda_j)$.

We arrive at the following algorithm for computing $\exp At$. Given an $n \times n$ matrix $A$, reduce it to Jordan canonical form $A = SJS^{-1}$, and then write

$$\exp At = S \exp Jt \; S^{-1}.$$

Even if $A$ (and hence also $\exp At$) has real entries, the matrices $J$ and $S$ may have complex entries. However, if $A$ is real, then any complex eigenvalues and generalized eigenvectors occur in conjugate pairs. It follows that the entries of $\exp At$ are linear combinations of terms of the form $t^k e^{\mu t} \cos \nu t$ and $t^k e^{\mu t} \sin \nu t$, where $\lambda = \mu \pm i\nu$ is an eigenvalue of $A$ and $k = 0, 1, \ldots, p$, with $p + 1$ being the size of the largest Jordan block for $\lambda$.

## 2.5 Asymptotic Behavior of Linear Systems

**Definition 2.4.** Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. Define the complex stable, unstable, and center subspaces of $A$, denoted $E_s^{\mathbb{C}}$, $E_u^{\mathbb{C}}$, and $E_c^{\mathbb{C}}$, respectively, to be the linear span over $\mathbb{C}$ of the generalized eigenvectors of $A$ corresponding to eigenvalues with negative, positive, and zero real parts, respectively.

Arrange the eigenvalues of $A$ so that $\operatorname{Re}\lambda_1 \le \ldots \le \operatorname{Re}\lambda_n$. Partition the set $\{1, \ldots, n\} = I_s \cup I_c \cup I_u$ so that

$$\begin{aligned} \operatorname{Re}\lambda_j &< 0, & j &\in I_s \\ \operatorname{Re}\lambda_j &= 0, & j &\in I_c \\ \operatorname{Re}\lambda_j &> 0, & j &\in I_u. \end{aligned}$$

Let $\{v_j\}_{j=1}^n$ be a basis of generalized eigenvectors corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$. Then

$$\begin{aligned}\operatorname{span}\{v_j : j \in I_s\} &= E_s^{\mathbb{C}}\\ \operatorname{span}\{v_j : j \in I_c\} &= E_c^{\mathbb{C}}\\ \operatorname{span}\{v_j : j \in I_u\} &= E_u^{\mathbb{C}}.\end{aligned}$$

In other words, we have

$$E_s^{\mathbb{C}} = \oplus_{j\in I_s} E(\lambda_j), \quad E_c^{\mathbb{C}} = \oplus_{j\in I_c} E(\lambda_j), \quad E_u^{\mathbb{C}} = \oplus_{j\in I_u} E(\lambda_j).$$

It follows that $\mathbb{C}^n = E_s^{\mathbb{C}} \oplus E_c^{\mathbb{C}} \oplus E_u^{\mathbb{C}}$ is a direct sum. Thus, any vector $x \in \mathbb{C}^n$ is uniquely represented as

$$x = P_s x + P_c x + P_u x \in E_s^{\mathbb{C}} \oplus E_c^{\mathbb{C}} \oplus E_u^{\mathbb{C}}.$$

These subspaces are invariant under $A$.

The maps $P_s, P_c, P_u$ are linear projections onto the complex stable, center, and unstable subspaces. Thus, we have

$$P_s^2 = P_s, \quad P_c^2 = P_c, \quad P_u^2 = P_u.$$

Since these subspaces are independent of each other, we have that

$$P_s P_c = P_c P_s = 0, \ldots$$

Since these subspaces are invariant under $A$, the projections commute with $A$, and thus, also with any function of $A$, including $\exp At$.

If $A$ is real and $v \in \mathbb{C}^n$ is a generalized eigenvector with eigenvalue $\lambda \in \mathbb{C}$, then its complex conjugate $\bar{v}$ is a generalized eigenvector with eigenvalue $\bar{\lambda}$. Since $\operatorname{Re}\lambda = \operatorname{Re}\bar{\lambda}$, it follows that the subspaces $E_s^{\mathbb{C}}$, $E_c^{\mathbb{C}}$, and $E_u^{\mathbb{C}}$ are closed under complex conjugation. For any vector $x \in \mathbb{C}^n$, we have

$$\overline{P_s x} + \overline{P_c x} + \overline{P_u x} = \bar{x} = P_s\bar{x} + P_c\bar{x} + P_u\bar{x}.$$

This gives two representations of $\bar{x}$ in $E_s^{\mathbb{C}} \oplus E_c^{\mathbb{C}} \oplus E_u^{\mathbb{C}}$. By uniqueness of representations, we must have

$$P_s\bar{x} = \overline{P_s x}, \quad P_c\bar{x} = \overline{P_c x}, \quad P_u\bar{x} = \overline{P_u x}.$$

So if $x \in \mathbb{R}^n$, we have that

$$P_s x = \overline{P_s x}, \quad P_c x = \overline{P_c x}, \quad P_u x = \overline{P_u x}.$$

Therefore, the projections leave $\mathbb{R}^n$ invariant:

$$P_s : \mathbb{R}^n \to \mathbb{R}^n, \quad P_c : \mathbb{R}^n \to \mathbb{R}^n, \quad P_u : \mathbb{R}^n \to \mathbb{R}^n.$$

**Definition 2.5.** Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. Define the real stable, unstable, and center subspaces of $A$, denoted $E_s$, $E_u$, and $E_c$, to be the images of $\mathbb{R}^n$ under the corresponding projections:

$$E_s = P_s\,\mathbb{R}^n, \quad E_c = P_c\,\mathbb{R}^n, \quad E_u = P_u\,\mathbb{R}^n.$$

Equivalently, we could define $E_s = E_s^{\mathbb{C}} \cap \mathbb{R}^n$, etc.

We have that $\mathbb{R}^n = E_s \oplus E_c \oplus E_u$ is a direct sum. When restricted to $\mathbb{R}^n$, the projections possess the same properties as they do on $\mathbb{C}^n$.

The real stable subspace can also be characterized as the linear span over $\mathbb{R}$ of the real and imaginary parts of all generalized eigenvectors of $A$ corresponding to an eigenvalue with negative real part. Similar statements hold for $E_c$ and $E_u$.

We are now ready for the first main result of this section, which estimates the norm of $\exp At$ on the invariant subspaces. These estimates will be used many times.

**Theorem 2.3.** *Let $A$ an $n \times n$ matrix over $\mathbb{R}$. Define*

$$-\lambda_s = \max\{\operatorname{Re}\lambda_j : j \in I_s\} \quad \textit{and} \quad \lambda_u = \min\{\operatorname{Re}\lambda_j : j \in I_u\}.$$

*There is a constant $C > 0$ and an integer $0 \le p < n$, depending on $A$, such that for all $x \in \mathbb{C}^n$,*

$$\begin{aligned}
\|\exp At\; P_s x\| &\le C(1+t)^p e^{-\lambda_s t}\|P_s x\|, && t > 0\\
\|\exp At\; P_c x\| &\le C(1+|t|)^p \|P_c x\|, && t \in \mathbb{R}\\
\|\exp At\; P_u x\| &\le C(1+|t|)^p e^{\lambda_u t}\|P_u x\|, && t < 0.
\end{aligned}$$

*Proof.* We will prove the first of these inequalities. The other two are similar.

Let $\{v_j\}_{j=1}^n$ be a basis of generalized eigenvectors with indices ordered as above. For any $x \in \mathbb{C}^n$, we have

$$x = \sum_{j=1}^n c_j v_j, \quad \text{and} \quad P_s x = \sum_{j \in I_s} c_j v_j.$$

Let $S$ be the matrix whose columns are the vectors $v_j$. Then $S$ reduces $A$ to Jordan canonical form: $A = S(D+N)S^{-1}$, where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and $N^{p+1} = 0$, for some $p < n$. If $\{e_j\}_{j=1}^n$ is the standard basis, then $Se_j = v_j$, and so, $e_j = S^{-1}v_j$. We may write

$$\begin{aligned}
\exp At\; P_s x &= S\;\exp Nt\;\exp Dt\; S^{-1} P_s x\\
&= S\;\exp Nt\;\exp Dt\;\sum_{j\in I_s} c_j e_j
\end{aligned}$$

$$= S \exp Nt \sum_{j \in I_s} c_j \exp(\lambda_j t) e_j$$
$$\equiv S \exp Nt \; y.$$

Taking the norm, we have

$$\| \exp At \; P_s x \| \le \|S\| \| \exp Nt \| \|y\|.$$

Now, if $p + 1$ is the size of the largest Jordan block, then $N^{p+1} = 0$. Thus, we have that

$$\exp Nt = \sum_{j=0}^{p} \frac{t^k}{k!} N^k,$$

and so,

$$\| \exp Nt \| \le \sum_{j=0}^{p} \frac{|t|^k}{k!} \|N\|^k \le C_1 (1 + |t|)^p.$$

Next, we have, for $t > 0$,

$$\begin{aligned}
\|y\|^2 &= \left\| \sum_{j \in I_s} c_j \exp(\lambda_j t) e_j \right\|^2 \\
&= \sum_{j \in I_s} |c_j|^2 \exp(2 \operatorname{Re} \lambda_j t) \\
&\le \exp(-2\lambda_s t) \sum_{j \in I_s} |c_j|^2 \\
&= \exp(-2\lambda_s t) \|S^{-1} P_s x\|^2 \\
&\le \exp(-2\lambda_s t) \|S^{-1}\|^2 \|P_s x\|^2,
\end{aligned}$$

and so $\|y\| \le \|S^{-1}\| \exp(-\lambda_s t) \|P_s x\|$.

The result follows with $C = C_1 \|S\| \|S^{-1}\|$. □

*Remark 2.1.* Examination of the proof of Theorem 2.3 shows that the exponent $p$ in these inequalities has the property that $p + 1$ is the size of the largest Jordan block of the Jordan form of $A$. In fact, the value of this exponent can be made more precise by noting, for example, that $\exp At \; P_s = \exp AP_s t \; P_s$, and so in the first case, $p + 1$ may be taken to be the size of the largest Jordan block of $AP_s$, i.e. the size of the largest Jordan block of $A$ corresponding to eigenvalues with negative real part. Similar statements hold in the other two cases.

It will often be convenient to use the following slightly weaker version of Theorem 2.3.

**Corollary 2.1.** *Let $A$ an $n \times n$ matrix over $\mathbb{R}$. Define $\lambda_s$ and $\lambda_u$ as in Theorem 2.3. Assume that $0 < \alpha < \lambda_s$ and $0 < \beta < \lambda_u$.*

*There is a constant $C > 0$ depending on $A$, such that for all $x \in \mathbb{C}^n$,*

$$\begin{aligned} \|\exp At\;\; P_s x\| &\leq Ce^{-\alpha t}\|P_s x\|, && t > 0 \\ \|\exp At\;\; P_u x\| &\leq Ce^{\beta t}\|P_u x\|, && t < 0. \end{aligned}$$

*Proof.* Notice that for any $\varepsilon > 0$, the function $(1+t)^p \exp(-\varepsilon t)$ is bounded on the interval $t > 0$. Thus, for any constant $0 < \alpha < \lambda_s$, we have that

$$(1+t)^p \exp(-\lambda_s t) = (1+t)^p \exp[-(\lambda_s - \alpha)t] \exp(-\alpha t) \leq C \exp(-\alpha t),$$

for $t > 0$. The first statement now follows from Theorem 2.3. The second statement is analogous.

**Corollary 2.2.** *Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. There is a constant $C > 0$ and an integer $0 \leq p < n$, depending on $A$, such that for all $x \in \mathbb{C}^n$,*

$$\begin{aligned} \|\exp At\;\; P_s x\| &\geq C(1+|t|)^{-p}e^{-\lambda_s t}\|P_s x\|, && t < 0 \\ \|\exp At\;\; P_u x\| &\geq C(1+t)^{-p}e^{\lambda_u t}\|P_u x\|, && t > 0. \end{aligned}$$

*Proof.* Write $P_s x = \exp(-At) \exp At\;\; P_s x$. Since the eigenvalues of $-A$ are the negatives of the eigenvalues of $A$, the stable and unstable subspaces are exchanged as well as the numbers $\lambda_s$ and $\lambda_u$. Since $\exp At\;\; P_s x$ belongs to the stable subspace of $A$ and hence to the unstable subspace of $-A$, we obtain from Theorem 2.3 that

$$\|P_s x\| \leq C(1+|t|)^p e^{\lambda_s t}\|\exp At P_s x\|, \quad t < 0,$$

which proves the first statement. The second statement is similarly proven. □

**Corollary 2.3.** *Let $A$ be an $n \times n$ matrix over $\mathbb{R}$.*

*If there are constants $C_1 > 0$, $0 \leq \alpha < \lambda_u$ such that*

$$\|\exp At\;\; P_u x\| \leq C_1 e^{\alpha t}, \quad \textit{for all} \quad t \geq 0, \tag{2.3}$$

*then $P_u x = 0$.*

*If there are constants $C_1 > 0$, $0 \leq \alpha < \lambda_s$ such that the bound*

$$\|\exp At\;\; P_s x\| \leq C_1 e^{-\alpha t}, \quad \textit{for all} \quad t \leq 0,$$

*then $P_s x = 0$.*

*Proof.* By Corollary 2.2 and (2.3), we have

$$\|P_u x\| \leq C(1+|t|)^p \mathrm{e}^{-\lambda_u t}\|\exp At\;\; P_u x\| \leq CC_1(1+|t|)^{p+d}\mathrm{e}^{(\alpha-\lambda_u)t},$$

for all $t \geq 0$. Send $t \to \infty$ to get $P_u x = 0$.

The second statement is proven in the same way. □

**Lemma 2.3.** *Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. If $x \in E_c$ and $x \neq 0$, then*

$$\liminf_{|t|\to\infty} \| \exp At\; x\| > 0.$$

*Proof.* Express $A$ in Jordan normal form as above,

$$A = S(D+N)S^{-1}.$$

The columns $\{v_j\}$ of $S$ comprise a basis of generalized eigenvectors for $A$.

Suppose that $y \in E_c$. Then $y = \sum_{j\in I_c} c_j v_j$, and so, $S^{-1}y = \sum_{j\in I_c} c_j e_j$. It follows that $\exp Dt\;\; S^{-1}y = \sum_{j\in I_c} c_j e^{\lambda_j t} e_j$, and since $\operatorname{Re}\lambda_j = 0$ for $j \in I_c$, we have that $\|\exp Dt\;\; S^{-1}y\| = \|c\| = \|S^{-1}y\|$. The general fact that $\|Sz\| \geq \|S^{-1}\|^{-1}\|z\|$, combines with the preceding to give

$$\|S \exp Dt\;\; S^{-1}y\| \geq \|S^{-1}\|^{-1}\|\exp Dt\;\; S^{-1}y\| \geq \|S^{-1}\|^{-1}\|S^{-1}y\|,$$

for all $y \in E_c$.

Using the Jordan Normal Form, we have

$$\begin{aligned}\exp At &= S\exp(D+N)t\;\; S^{-1} = S\exp Dt\;\; \exp Nt\;\; S^{-1}\\ &= S\exp Dt\;\; S^{-1}S\exp Nt\;\; S^{-1}.\end{aligned}$$

If $x \in E_c$, then $y = S\exp Nt\;\; S^{-1}x \in E_c$. Thus, from the preceding we have

$$\|\exp At\;\; x\| = \|S\exp Dt\;\; S^{-1}y\| \geq \|S^{-1}\|^{-1}\|S^{-1}y\| = \|S^{-1}\|^{-1}\|\exp Nt\;\; Sx\|.$$

Now since $N$ is nilpotent,

$$\exp Nt = \sum_{k=0}^{p} \frac{t^k}{k!}N^k.$$

If $x \neq 0$, then $Sx \neq 0$ and there exists a largest integer $m$ between 0 and $p$ such that $N^m Sx \neq 0$. We have by the triangle inequality that

$$\|\exp Nt\;\; Sx\| = \|\sum_{k=0}^{m} \frac{t^k}{k!}N^k Sx\| \geq \frac{|t|^m}{m!}\|N^m Sx\| - \sum_{k<m}\frac{|t|^k}{k!}\|N^k Sx\|.$$

It follows that if $m > 0$, then $\liminf_{|t|\to\infty} \|\exp At\;\; x\|$ is equal to $+\infty$. When $m = 0$, it is bounded below by $\|S^{-1}\|^{-1}\|Sx\|$. □

The next result serves as a converse to Theorem 2.3.

**Theorem 2.4.** *Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. Let $x \in \mathbb{R}^n$.*
*If $\lim_{t\to\infty} \exp At\ x = 0$, then $x \in E_s$.*
*If $\lim_{t\to-\infty} \exp At\ x = 0$, then $x \in E_u$.*
*If $\lim_{|t|\to\infty} (1+|t|)^{-p} \exp At\ x = 0$, where $p$ is the size of the largest Jordan block of $A$, then $x \in E_c$.*

*Proof.* Again, we shall only prove the first statement. Let $x \in \mathbb{R}^n$. Then by Theorem 2.3, $\lim_{t\to\infty} \|\exp At\ P_s x\| = 0$. If $\lim_{t\to\infty} \exp At\ x = 0$, then we have that

$$0 = \lim_{t\to\infty} \exp At\ x = \lim_{t\to\infty} \exp At(P_s x + P_c x + P_u x) = \lim_{t\to\infty} \exp At(P_c x + P_u x). \tag{2.4}$$

By the triangle inequality, Theorem 2.3, and Corollary 2.2 we have

$$\begin{aligned}\|\exp At(P_u x + P_c x)\| &\geq \|\exp At\ P_u x\| - \|\exp At\ P_c x\| \\ &\geq C_1(1+t)^{-p} e^{\lambda_u t}\|P_u x\| - C_2(1+|t|)^p\|P_c x\|.\end{aligned}$$

The last expression grows exponentially when $P_u x \neq 0$, so (2.4) forces $P_u x = 0$. Therefore, we are left with $\lim_{t\to\infty} \|\exp At\ P_c x\| = 0$. By Lemma 2.3, we must also have $P_c x = 0$. Thus, $x = P_s x \in E_s$.

The proofs of the other two statements are similar. □

All of the results in this section hold for complex matrices $A$, except for the remarks concerning the projections on $\mathbb{R}^n$ and the ensuing definitions of real invariant subspaces. We shall only be concerned with real matrices, however.

## 2.6 Exercises

**Exercise 2.1.** Find $\exp At$ for the following matrices. (Suggestion: Use the definition of exp.)

(a) $A = \begin{bmatrix} \mu & \omega \\ \omega & \mu \end{bmatrix}$

(b) $A = \begin{bmatrix} \mu & \omega \\ -\omega & \mu \end{bmatrix}$

(c) $A = \begin{bmatrix} \mu & \omega & 1 & 0 \\ -\omega & \mu & 0 & 1 \\ 0 & 0 & \mu & \omega \\ 0 & 0 & -\omega & \mu \end{bmatrix}$

**Exercise 2.2.** (a) Write the second order equation

$$u'' + 2u' + u = 0 \tag{2.5}$$

as a first order system $x' = Ax$.

(b) Find $\exp At$ for the matrix $A$ in part (a).

(c) Use the answer from part (b) to find the solution of (2.5) with $u(0) = u_0$ and $u'(0) = u_1$.

**Exercise 2.3.** Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. Prove the following statements:

(a) If $A$ is symmetric, then $\exp At$ is symmetric.

(b) If $A$ is skew-symmetric, then $\exp At$ is orthogonal.

(c) If $\lambda \in \mathbb{R}$ is an eigenvalue of $A$ with corresponding eigenvector $x_0 \in \mathbb{R}^n$, then $u(t) = \mathrm{e}^{t\lambda} x_0$ is the solution of the initial value problem

$$u' = Au, \quad u(0) = x_0.$$

(d) If $A$ is block diagonal:

$$A = \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{bmatrix},$$

then

$$\exp At = \begin{bmatrix} \exp B_1 t & & & \\ & \exp B_2 t & & \\ & & \ddots & \\ & & & \exp B_k t \end{bmatrix}.$$

**Exercise 2.4.** Let $A$ be an $n \times n$ matrix over $\mathbb{R}$ with

$$p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

as its characteristic polynomial. The Cayley-Hamilton Theorem states that $p(A) = 0$. Use this fact to prove that the components $x_j(t)$ of the solution $x(t)$ to the linear system $x' = Ax$ satisfy the $n$th order equation

$$p(D)u = u^{(n)} + c_{n-1}u^{(n-1)} + \cdots + c_1 u' + c_0 u = 0.$$

**Exercise 2.5.** Define

$$A = \begin{bmatrix} 8 & 0 & 0 & 8 & 8 \\ 0 & 0 & 0 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

and

$$w_1 = \begin{bmatrix} 8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 0 \\ 8 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_4 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad w_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then

$$Aw_1 = 8w_1, \quad Aw_2 = 8w_2 + w_1, \quad Aw_3 = 0, \quad Aw_4 = w_3, \quad Aw_5 = 0.$$

(a) Find the Jordan normal form for $A$ in the basis $\{w_i\}_{i=1}^5$.
(b) Find the real stable, unstable, and center subspaces of $A$.
(c) Find the projections $P_s$, $P_u$, and $P_c$ in the standard basis.
(d) Find $\exp At$.

**Exercise 2.6.** Let $A$ be an $n \times n$ matrix over $\mathbb{R}$, and let $E_s$, $E_u$, and $E_c$ be its real stable, unstable, and center subspaces.

(a) Prove that $E_s$, $E_u$, and $E_c$ are invariant under $A$.
(b) Prove that $E_s$, $E_u$, and $E_c$ are invariant under $\exp At$.

**Exercise 2.7.** Find the stable, unstable, and center subspaces of the matrix

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

**Exercise 2.8.** Does there exist an invertible real $3 \times 3$ matrix whose stable, center, and unstable subspaces are all one dimensional? Explain.

**Exercise 2.9.** (a) Suppose that $A$ is an anti-symmetric $n \times n$ matrix over $\mathbb{R}$. Show that $\|\exp At\, x\| = \|x\|$, for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}$.
(b) Find the stable, center, and unstable subspaces of $A$.

# Chapter 3
# Existence Theory

## 3.1 The Initial Value Problem

Let $\Omega \subset \mathbb{R}^{1+n}$ be an open connected set. We will denote points in $\Omega$ by $(t, x)$ where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}^n$ be a continuous map. In this context, $f(t, x)$ is called a vector field on $\Omega$. Given any initial point $(t_0, x_0) \in \Omega$, we wish to construct a unique solution to the initial value problem

$$x'(t) = f(t, x(t)) \quad x(t_0) = x_0. \tag{3.1}$$

In order for this to make sense, $x(t)$ must be a $C^1$ function from some interval $I \subset \mathbb{R}$ containing the initial time $t_0$ into $\mathbb{R}^n$ such that the solution curve satisfies

$$\{(t, x(t)) : t \in I\} \subset \Omega.$$

Such a solution is referred to as a *local solution* when $I \neq \mathbb{R}$. When $I = \mathbb{R}$, the solution is called *global*.

## 3.2 The Cauchy-Peano Existence Theorem

**Theorem 3.1. (Cauchy-Peano).** *If $f : \Omega \to \mathbb{R}^n$ is continuous, then for every point $(t_0, x_0) \in \Omega$ the initial value problem (3.1) has local solution.*

The problem with this theorem is that it does not guarantee uniqueness. We will skip the proof, except to mention that it is uses a compactness argument based on the Arzela-Ascoli Theorem.

*Example 3.1.* Here is a simple example that demonstrates that uniqueness can indeed fail. Let $\Omega = \mathbb{R}^2$ and consider the autonomous vector field $f(t, x) = |x|^{1/2}$. When $(t_0, x_0) = (0, 0)$, the initial value problem has infinitely many solutions. In addition

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Atlantis Studies in Differential Equations 2, DOI: 10.2991/978-94-6239-021-8_3,

to the zero solution $x(t) = 0$, for any $\alpha, \beta \geq 0$, the following is a family of solutions.

$$x(t) = \begin{cases} -\frac{1}{4}(t+\alpha)^2, & t \leq -\alpha \\ 0, & -\alpha \leq t \leq \beta \\ \frac{1}{4}(t-\beta)^2, & \beta \leq t. \end{cases}$$

This can be verified by direct substitution.

## 3.3 The Picard Existence Theorem

The failure of uniqueness can be rectified by placing an additional restriction on the vector field. The next definitions introduce this key property.

**Definition 3.1.** *Let $S \subset \mathbb{R}^m$. Suppose $x \mapsto f(x)$ is a function from $S$ to $\mathbb{R}^n$.*

*The function $f$ is said to be Lipschitz continuous on $S$ if there exists a constant $C > 0$ such that*

$$\|f(x_1) - f(x_2)\|_{\mathbb{R}^n} \leq C\|x_1 - x_2\|_{\mathbb{R}^m},$$

*for all $x_1, x_2 \in S$.*

The function $f$ is said to be locally Lipschitz continuous on $S$ if for every compact subset $K \subset S$, there exists a constant $C_K > 0$ such that

$$\|f(x_1) - f(x_2)\|_{\mathbb{R}^n} \leq C_K\|x_1 - x_2\|_{\mathbb{R}^m},$$

for all $x_1, x_2 \in K$.

*Remark 3.1.* If a function is locally Lipschitz continuous on $S$, then it is continuous on $S$.

*Example 3.2.* The function $\|x\|^\alpha$ from $\mathbb{R}^m$ to $\mathbb{R}$ is Lipschitz continuous for $\alpha = 1$, locally Lipschitz continuous for $\alpha > 1$, and not Lipschitz continuous (on any neighborhood of 0) when $\alpha < 1$.

**Definition 3.2.** *Let $\Omega \subset \mathbb{R}^{1+n}$ be an open set. A continuous function $(t, x) \mapsto f(t, x)$ from $\Omega$ to $\mathbb{R}^n$ is said to be locally Lipschitz continuous in $x$ if for every compact set $K \subset \Omega$, there is a constant $C_K > 0$ such that*

$$\|f(t, x_1) - f(t, x_2)\| \leq C_K\|x_1 - x_2\|,$$

*for every $(t, x_1), (t, x_2) \in K$. If there is a constant for which the inequality holds for all $(t, x_1), (t, x_2) \in \Omega$, then $f$ is said to be Lipschitz continuous in $x$.*

**Lemma 3.1.** *If $f : \Omega \to \mathbb{R}^n$ is $C^1$, then it is locally Lipschitz continuous in $x$.*

**Theorem 3.2. (Picard)**. *Let $\Omega \subset \mathbb{R}^{1+n}$ be open. Assume that $f : \Omega \to \mathbb{R}^n$ is continuous and that $f(t,x)$ is locally Lipschitz continuous in x. Let $K \subset \Omega$ be any compact set. Then there is a $\delta > 0$ such that for every $(t_0, x_0) \in K$, the initial value problem (3.1) has a unique local solution defined on the interval $|t - t_0| < \delta$.*

Before proving this important theorem, it is convenient to have the following technical "Covering Lemma".

First, some notaion: Given a point $(t,x) \in \mathbb{R}^{1+n}$ and positive numbers $r$ and $a$, define the cylinder

$$\mathcal{C}(t,x) \equiv \{(t',x') \in \mathbb{R}^{1+n} : \|x - x'\| \le r,\ |t - t'| \le a\}.$$

**Lemma 3.2. (Covering Lemma)**. *Let $K \subset \Omega \subset \mathbb{R}^{1+n}$ with $\Omega$ an open set and $K$ a compact set. There exists a compact set $K'$ and positive numbers r and a such that $K \subset K' \subset \Omega$ and $\mathcal{C}(t,x) \subset K'$, for all $(t,x) \in K$.*

*Proof.* For every point $p = (t,x) \in K$, choose positive numbers $a(p)$ and $r(p)$ such that

$$D(p) = \{(t',x') \in \mathbb{R}^{1+n} : \|x - x'\| \le 2r(p),\ |t - t'| \le 2a(p)\} \subset \Omega.$$

This is possible because $\Omega$ is open.

Define the cylinders

$$C(p) = \{(t',x') \in \mathbb{R}^{1+n} : \|x - x'\| < r(p), |t - t'| < a(p)\}.$$

The collection of open sets $\{C(p) : p \in K\}$ forms an open cover of the compact set $K$. So there is a finite number of cylinders $C(p_1), \ldots, C(p_N)$ whose union contains $K$. The set

$$K' = \cup_{i=1}^N D(p_i)$$

is compact, and

$$K \subset \cup_{i=1}^N C(p_i) \subset \cup_{i=1}^N D(p_i) = K' \subset \Omega.$$

Define

$$a = \min\{a(p_i) : i = 1, \ldots, N\} \quad \text{and} \quad r = \min\{r(p_i) : i = 1, \ldots, N\}.$$

The claim is that, for this uniform choice of $a$ and $r$, $\mathcal{C}(t,x) \subset K'$, for all $(t,x) \in K$.

If $(t,x) \in K$, then $(t,x) \in \mathrm{C(p_i)}$ for some $i = 1, \ldots, N$. Let $(t',x') \in \mathcal{C}(t,x)$. Then

$$\|x' - x_i\| \le \|x' - x\| + \|x - x_i\| \le a + a(p_i) \le 2a(p_i)$$

and

$$|t' - t_i| \leq |t' - t| + |t - t_i| \leq r + r(p_i) \leq 2r(p_i).$$

This shows that $(t', x') \in D(p_i)$, from which follows the inclusion $\mathcal{C}(t, x) \subset D(p_i) \subset K'$. $\square$

*Remark 3.2.* This result is closely connected with the notion of a Lebesgue number in metric space.

*Proof of the Picard Theorem.* The first step of the proof is to reformulate the problem. If $x(t)$ is a $C^1$ solution of the initial value problem (3.1) for $|t - t_0| \leq \delta$, then by integration we find that

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \tag{3.2}$$

for $|t - t_0| \leq \delta$. Conversely, if $x(t)$ is a $C^0$ solution of the integral equation, then it is $C^1$ and it solves the initial value problem (3.1).

Given a compact subset $K \subset \Omega$, choose $a$, $r$, $K'$ as in the Lemma 3.2.

Choose $(t_0, x_0) \in K$. Let $\delta < a$ and set

$$I_\delta = \{|t - t_0| \leq \delta\}, \quad B_r = \{\|x - x_0\| \leq r\}, \quad X_\delta = C^0(I_\delta; B_r),$$

where $C^0(I_\delta; B_r)$ denotes the set of continuous functions from $I_\delta$ to $B_r$. Note that $X_\delta$ is a complete metric space with the sup norm metric.

By definition, if $x \in X_\delta$, then

$$(s, x(s)) \in \mathcal{C}(t_0, x_0) \subset K' \subset \Omega,$$

for $s \in I_\delta$. Thus, the operator

$$Tx(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

is well-defined on $X_\delta$ and the function $Tx(t)$ is continuous for $t \in I_\delta$.

Define $M_1 = \max_{K'} |f(t, x)|$. The claim is that if $\delta$ is chosen small enough so that $M_1\delta \leq r$, then $T : X_\delta \to X_\delta$. If $x \in X_\delta$, we have from (3.2)

$$\sup_{I_\delta} \|Tx(t) - x_0\| \leq M_1\delta \leq r,$$

for $t \in I_\delta$. Thus, $Tx \in X_\delta$.

Next, let $M_2$ be a Lipschitz constant for $f(t, x)$ on $K'$. If $\delta$ is further restricted so that $M_2\delta < 1/2$, then we claim that $T : X_\delta \to X_\delta$ is a contraction. Let $x_1, x_2 \in X_\delta$. Then from (3.2), we have

$$\sup_{I_\delta} \|Tx_1(t) - Tx_2(t)\| \leq M_2\delta \sup_{I_\delta} \|x_1(t) - x_2(t)\| \leq 1/2 \sup_{I_\delta} \|x_1(t) - x_2(t)\|.$$

So by the Contraction Mapping Principle, Theorem A.1, there exists a unique function $x \in X_\delta$ such that $Tx = x$. In other words, $x$ solves (3.2). □

*Remark 3.3.* Note that the final choice of $\delta$ is $\min\{a, r/M_1, 1/2M_2\}$ which depends only on the set $K$ and on $f$.

*Remark 3.4.* In the proof of the Picard Existence Theorem, we used the Contraction Mapping Principle, the proof of which is based on iteration: choose an arbitrary element $x_0 \in X_\delta$ of the metric space and define the sequence $x_k = Tx_{k-1}$, $k = 1, 2, \ldots$. Then the fixed point $x$ is obtained as a limit: $x = \lim_{k\to\infty} x_k$. In the context of the existence theorem, the sequence elements $x_k$ are known as the *Picard iterates*.

The alert reader will have noticed that the solution constructed above is unique within the metric space $X_\delta$, but it is not necessarily unique in $C^0(I_\delta, B_r)$. The next result fills in this gap.

**Theorem 3.3. (Uniqueness)**. *Suppose that $f : \Omega \to \mathbb{R}^n$ satisfes the hypotheses of the Picard Theorem. For $j = 1, 2$, let $x_j(t)$ be solutions of $x'(t) = f(t, x(t))$ on the interval $I_j$. If there is a point $t_0 \in I_1 \cap I_2$ such that $x_1(t_0) = x_2(t_0)$, then $x_1(t) = x_2(t)$ on the interval $I_1 \cap I_2$. Moreover, the function*

$$x(t) = \begin{cases} x_1(t), & t \in I_1 \\ x_2(t), & t \in I_2 \end{cases}$$

*defines a solution on the interval $I_1 \cup I_2$.*

*Proof.* Let $J \subset I_1 \cap I_2$ be any closed bounded interval with $t_0 \in J$. Let $M$ be a Lipschitz constant for $f(t, x)$ on the compact set

$$\{(t, x_1(t)) : t \in J\} \cup \{(t, x_2(t)) : t \in J\}.$$

The solutions $x_j(t)$, $j = 1, 2$, satisfy the integral equation (3.2) on the interval $J$. Thus, estimating as before

$$\|x_1(t) - x_2(t)\| \leq \left| \int_{t_0}^t M\|x_1(s) - x_2(s)\| ds \right|,$$

for $t \in J$. It follows from Gronwall's Lemma 3.3 (to follow) that

$$\|x_1(t) - x_2(t)\| = 0$$

for $t \in J$. Since $J \subset I_1 \cap I_2$ was any closed interval containing $t_0$, we have that $x_1 = x_2$ on $I_1 \cap I_2$.

From this it follows that $x(t)$ is well-defined, is $C^1$, and is a solution. $\square$

**Lemma 3.3.** **(Gronwall).** *Let $f(t)$, $\varphi(t)$ be nonnegative continuous functions on an open interval $J = (\alpha, \beta)$ containing the point $t_0$. Let $c_0 \geq 0$. If*

$$f(t) \leq c_0 + \left| \int_{t_0}^{t} \varphi(s) f(s) ds \right|,$$

*for all $t \in J$, then*

$$f(t) \leq c_0 \exp \left| \int_{t_0}^{t} \varphi(s) ds \right|,$$

*for $t \in J$.*

*Proof.* Suppose first that $t \in [t_0, \beta)$. Define

$$F(t) = c_0 + \int_{t_0}^{t} \varphi(s) f(s) ds.$$

Then $F$ is $C^1$ and

$$F'(t) = \varphi(t) f(t) \leq \varphi(t) F(t),$$

for $t \in [t_0, \beta)$, since $f(t) \leq F(t)$. This implies that

$$\frac{d}{dt} \left[ \exp \left( - \int_{t_0}^{t} \varphi(s) ds \right) \; F(t) \right] \leq 0,$$

for $t \in [t_0, \beta)$. Integrate this over the interval $[t_0, \tau)$ to get

$$f(\tau) \leq F(\tau) \leq c_0 \exp \int_{t_0}^{\tau} \varphi(s) ds,$$

for $\tau \in [t_0, \beta)$.

On the interval $(\alpha, t_0]$, perform the analogous argument to the function

$$G(t) = c_0 + \int_{t}^{t_0} \varphi(s) f(s) ds.$$

$\square$

## 3.4 Extension of Solutions

**Theorem 3.4.** *For every* $(t_0, x_0) \in \Omega$ *the solution to the initial value problem (3.1) extends to a maximal existence interval* $I = (\alpha, \beta)$. *Furthermore, if* $K \subset \Omega$ *is any compact set containing the point* $(t_0, x_0)$*, then there exist times* $\alpha(K)$ *and* $\beta(K)$ *such that* $\alpha < \alpha(K) < \beta(K) < \beta$ *and* $(t, x(t)) \in \Omega \setminus K$*, for* $t \in (\alpha, \beta) \setminus [(\alpha(K), \beta(K)]$.

*Proof.* Define $\mathcal{A}$ to be the collection of intervals $J$, containing the initial time $t_0$, on which there exists a solution $x_J$ of the initial value problem (3.1). The Existence Theorem 3.2 guarantees that $\mathcal{A}$ is nonempty, so we may define $I = \cup_{J \in \mathcal{A}} J$. Then $I$ is an interval containing $t_0$. Write $I = (\alpha, \beta)$. Note that $\alpha$ and/or $\beta$ could be infinite—that is ok.

Suppose $\bar{t} \in I$. Then $\bar{t} \in J$ for some $J \in \mathcal{A}$, and there is a solution $x_J$ of (3.1) defined on $J$. If $J' \in \mathcal{A}$ is any other interval containing the value $\bar{t}$ with corresponding solution $x_{J'}$, then by the Uniqueness Theorem 3.3, we have that $x_J = x_{J'}$ on $J \cap J'$. In particular, $x_J(\bar{t}) = x_{J'}(\bar{t})$. Therefore, the following function is well-defined on $I$:

$$x(t) = x_J(t), \quad t \in J, \quad J \in \mathcal{A}.$$

Moreover, it follows from this definition that $x(t)$ is a solution of (3.1) on $I$, and it is unique, again thanks to Theorem 3.3. Thus, $I \in \mathcal{A}$ is maximal.

Now let $K \subset \Omega$ be compact, with $(t_0, x_0) \in K$. Define

$$G(K) = \{t \in I : (t, x(t)) \in K\}.$$

By the Existence Theorem 3.2, we know $G(K)$ is nonempty. The set $G(K)$ is bounded since $K$ is compact, so $\beta(K) \equiv \sup G(K) < \infty$. We must show that $\beta(K) < \beta$.

If $t_1 \in G(K)$, then $(t_1, x_1) = (t_1, x(t_1)) \in K$. By the existence theorem, we can solve the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_1) = x_1$$

on the interval $\{|t - t_1| < \delta\}$. By the Uniqueness Theorem 3.3, $x(t) = y(t)$ on the interval $I \cap \{|t - t_1| < \delta\}$. Thus, $y(t)$ extends $x(t)$ to the interval $I \cup \{|t - t_1| < \delta\}$. But, by maximality, $I \cup \{|t - t_1| < \delta\} \subset I$. Therefore, $t_1 + \delta \leq \beta$, for all $t_1 \in G(K)$. Take the supremum over all $t_1 \in G(K)$. We conclude that $\beta(K) + \delta \leq \beta$. Thus, $\beta(K) < \beta$.

Similarly, $\alpha(K) = \inf G(K) > \alpha$. □

*Remark 3.5.* It is possible that $G(K) \neq [\alpha(K), \beta(K)]$. The solution curve $(t, x(t))$ may exit and reenter the set $K$ finitely many times on the interval $[\alpha(K), \beta(K)]$. See Fig. 3.1.

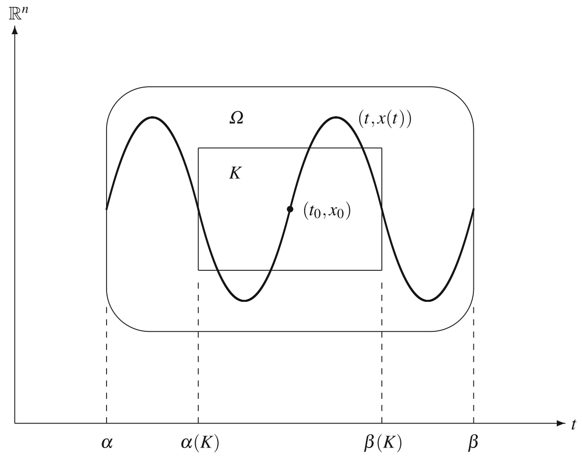


**Fig. 3.1** A solution curve $(t, x(t))$ in the domain $\Omega$ with maximal existence interval $(\alpha, \beta)$ and final exit times $\alpha(K), \beta(K)$ from the compact set $K$

*Example 3.3.* Consider the IVP

$$x' = x^2, \quad x(0) = x_0,$$

the solution of which is

$$x(t) = \frac{x_0}{1 - x_0\, t}.$$

We see that the maximal interval of existence depends on the initial value $x_0$:

$$I = (\alpha, \beta) = \begin{cases} (-\infty, \infty), & \text{if } x_0 = 0 \\ (-\infty, 1/x_0), & \text{if } x_0 > 0 \\ (1/x_0, \infty), & \text{if } x_0 < 0. \end{cases}$$

## 3.5 Continuous Dependence on Initial Conditions

**Definition 3.3.** *A function $g$ from $\mathbb{R}^m$ into $\mathbb{R} \cup \{\infty\}$ is lower semi-continuous at a point $y_0$ provided* $\liminf\limits_{y \to y_0} g(y) \geq g(y_0)$.

*Equivalently, a function $g$ into $\mathbb{R} \cup \{\infty\}$ is lower semi-continuous at a point $y_0$ provided for every $L < g(y_0)$ there is a neighborhood $V$ of $y_0$ such that $L \leq g(y)$ for $y \in V$.*

*Upper semi-continuity is defined in the obvious way.*

To study the dependence of the solution of the initial value problem upon its initial conditions, we now introduce some notation that will be used frequently. Let $\Omega \subset \mathbb{R}^{1+n}$ be an open set. Let $f : \Omega \to \mathbb{R}^n$ satisfy the hypotheses of the Picard

Theorem 3.2. Given $(t_0, x_0) \in \Omega$, let $x(t, t_0, x_0)$ denote the unique solution of the IVP

$$x' = f(t, x), \quad x(t_0) = x_0,$$

with maximal existence interval $I(t_0, x_0) = (\alpha(t_0, x_0), \beta(t_0, x_0))$.

**Theorem 3.5.** *The domain of existence of $x(t, t_0, x_0)$, namely*

$$D = \{(t, t_0, x_0) : (t_0, x_0) \in \Omega, \ t \in I(t_0, x_0)\},$$

*is an open set in $\mathbb{R}^{2+n}$.*

*The function $x(t, t_0, x_0)$ is continuous on $D$.*

*The function $\beta(t_0, x_0)$ is lower semi-continuous on $\Omega$, and the function $\alpha(t_0, x_0)$ is upper semi-continuous on $\Omega$.*

*Example 3.4.* Suppose that $\Omega = \mathbb{R}^{1+2} \setminus \{0\}$ and $f(t, x) = 0$ on $\Omega$. Then $x(t, t_0, x_0) = x_0$ where, because the solution curve must remain in $\Omega$, the maximal existence interval $I(t_0, x_0)$ has the right endpoint

$$\beta(t_0, x_0) = \begin{cases} 0, & \text{if } x_0 = 0, \ t_0 < 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus, we can not improve upon the lower semi-continuity of the function $\beta$.

The proof of this theorem will be based on the following lemma, see Fig. 3.2.

**Lemma 3.4.** *Choose any closed interval $J = [\bar{\alpha}, \bar{\beta}]$ such that $t_0 \in J \subset I(t_0, x_0)$. Given any $\varepsilon > 0$, there exists a neighborhood $V \subset \Omega$ containing the point $(t_0, x_0)$ such that for any $(t_1, x_1) \in V$, the solution $x(t, t_1, x_1)$ is defined for $t \in J$, and*

$$\|x(t, t_1, x_1) - x(t, t_0, x_0)\| < \varepsilon, \quad \textit{for} \quad t \in J.$$

Let us assume that Lemma 3.4 holds and use it to establish the theorem.

*Proof of Theorem 3.5.* To show that $x(t, t_0, x_0)$ is continuous, fix a point $(t', t_0, x_0) \in D$, with $(t_0, x_0) \in \Omega$, and let $\varepsilon > 0$ be given. Choose $J = [\bar{\alpha}, \bar{\beta}] \subset I(t_0, x_0)$ such that $\bar{\alpha} < t' < \bar{\beta}$. By Lemma 3.4 (using $\varepsilon/2$ for $\varepsilon$), there is a neighborhood $V \subset \Omega$ of the point $(t_0, x_0)$, such that for any $(t_1, x_1) \in V$, the solution $x(t, t_1, x_1)$ is defined for $t \in J$, and

$$\|x(t, t_1, x_1) - x(t, t_0, x_0)\| < \varepsilon/2, \quad \text{for} \quad t \in J.$$

Now $x(t, t_0, x_0)$ is continuous as a function of $t$ on $J \subset I(t_0, x_0)$, since it is a solution of the IVP. So there is a $\delta > 0$ with $\{|t - t'| < \delta\} \subset J$, such that

$$\|x(t, t_0, x_0) - x(t', t_0, x_0)\| < \varepsilon/2, \quad \text{provided} \quad |t - t'| < \delta.$$

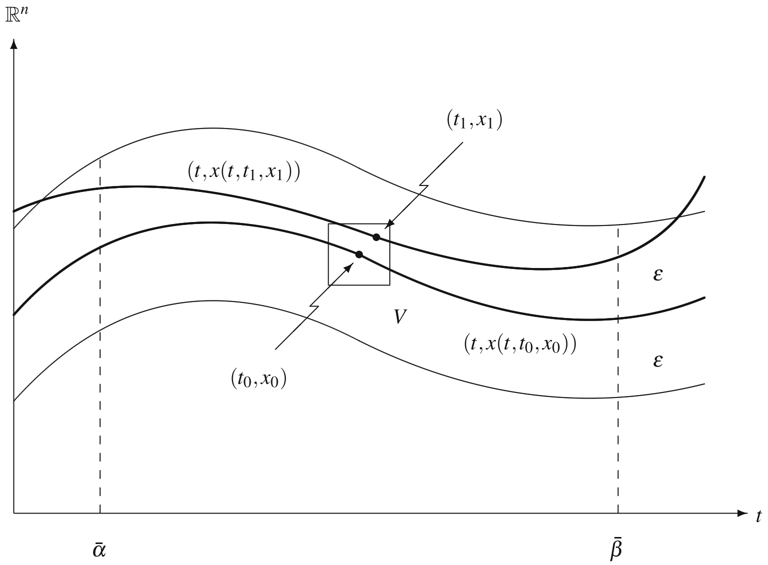


**Fig. 3.2** A tube of radius $\varepsilon$ about the solution curve $(t, x(t, t_0, x_0))$ over the time interval $J = (\bar{\alpha}, \bar{\beta})$, a neighborhood $V$ about the initial point $(t_0, x_0)$, and a nearby solution curve $(t, x(t, t_1, x_1))$

Thus, for any $(t, t_1, x_1)$ in the neighborhood $\{|t - t'| < \delta\} \times V$ of $(t', t_0, x_0)$, we have that

$$\begin{aligned}\|x(t, t_1, x_1) - x(t', t_0, x_0)\| &\leq \|x(t, t_1, x_1) - x(t, t_0, x_0)\| \\ &\quad + \|x(t, t_0, x_0) - x(t', t_0, x_0)\| \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

This proves continuity.

Let $(t', t_0, x_0) \in D$. In the preceding paragraph, we have constructed the set

$$\{|t - t'| < \delta\} \times V$$

which is a neighborhood of $(t', t_0, x_0)$ contained in $D$. This shows that $D$ is open.

Using Lemma 3.4 again, we have that for any $\bar{\beta} < \beta(t_0, x_0)$, there is a neighborhood $V \subset \Omega$ of $(t_0, x_0)$ such that $(t_1, x_1) \in V$ implies that $x(t, t_1, x_1)$ is defined for $t \in [t_1, \bar{\beta}]$. This says that $\beta(t_1, x_1) \geq \bar{\beta}$, in other words, the function $\beta$ is lower semi-continuous at $(t_0, x_0)$. The proof that $\alpha$ is upper semi-continuous is similar. $\square$

It remains to prove Lemma 3.4.

*Proof of Lemma 3.4.* Fix $(t_0, x_0) \in \Omega$. For notational convenience, we will use the abbreviation $x_0(t) = x(t, t_0, x_0)$. Let $J = [\bar{\alpha}, \bar{\beta}] \subset I(t_0, x_0)$.

Consider the compact set $K = \{(s, x_0(s)) : s \in J\}$. By Lemma 3.2, there exist a compact set $K'$ and numbers $a, r > 0$ such that $K \subset K' \subset \Omega$ and

$$\mathcal{C}(s, x_0(s)) = \{(s', x') : |s' - s| < a,\ \|x' - x_0(s)\| < r\} \subset K',$$

for all $s \in [\bar{\alpha}, \bar{\beta}]$. Define $M_1 = \max_{K'} \|f(t, x)\|$ and let $M_2$ be a Lipschitz constant for $f$ on $K'$.

Given $0 < \varepsilon < r$, we are going to produce a sufficiently small $\delta > 0$, such that the neighborhood $V = \{(t, x) : |t - t_0| < \delta,\ \|x - x_0\| < \delta\}$ satisfies the requirements of the lemma. In fact, choose $\delta$ small enough so that $\delta < a$, $\delta < r$, $\{|t - t_0| < \delta\} \subset [\bar{\alpha}, \bar{\beta}]$, and

$$\delta(M_1 + 1) \exp M_2(\bar{\beta} - \bar{\alpha}) < \varepsilon < r.$$

Notice that $V \subset \mathcal{C}(t_0, x_0) \subset K'$.

Choose $(t_1, x_1) \in V$. Let $x_1(t) = x(t, t_1, x_1)$. This solution is defined on the maximal interval $I(t_1, x_1) = (\alpha(t_1, x_1), \beta(t_1, x_1))$. Let $J^* = [\alpha^*, \beta^*]$ be the largest subinterval of $I(t_1, x_1)$ such that

$$\{(s, x_1(s)) : s \in J^*\} \subset K'.$$

Note that by the existence theorem, $J^* \neq \emptyset$ and that $t_1 \in J^*$. Since $|t_1 - t_0| < \delta$, we have that $t_1 \in J$. Thus, $J \cap J^* \neq \emptyset$. Since $(t, x_1(t))$ must eventually exit $K'$, we have that $J^*$ is properly contained in $I(t_1, x_1)$.

Recall that our solutions satisfy the integral equation

$$x_i(t) = x_i + \int_{t_i}^{t} f(s, x_i(s))ds, \quad i = 0, 1.$$

So we have for all $t \in J \cap J^*$,

$$\begin{aligned} x_1(t) - x_0(t) &= x_1 - x_0 + \int_{t_1}^{t} f(s, x_1(s))ds - \int_{t_0}^{t} f(s, x_0(s))ds \\ &= x_1 - x_0 + \int_{t_0}^{t_1} f(s, x_0(s))ds \\ &\quad + \int_{t_1}^{t} [f(s, x_1(s)) - f(s, x_0(s))]ds. \end{aligned}$$

Since the solution curves remain in $K'$ on the interval $J \cap J^*$, we now have the following estimate:

$$\|x_1(t) - x_0(t)\| \le \|x_1 - x_0\| + \left| \int_{t_0}^{t_1} \|f(s, x_0(s))\| ds \right|$$

$$
\begin{aligned}
&+\left|\int_{t_1}^{t} \|f(s,x_1(s)) - f(s,x_0(s))\| ds\right| \\
&\leq \delta + M_1|t_1 - t_0| + \left|\int_{t_1}^{t} M_2\|x_1(s) - x_0(s)\| ds\right| \\
&\leq \delta(1 + M_1) + \left|\int_{t_1}^{t} M_2\|x_1(s) - x_0(s)\| ds\right|.
\end{aligned}
$$

By Gronwall's inequality and our choice of $\delta$, we obtain

$$\|x_1(t) - x_0(t)\| \leq \delta(1 + M_1)\exp M_2|t - t_1| \leq \varepsilon < r,$$

for $t \in J \cap J^*$. This estimate shows that throughout the time interval $J \cap J^*$, $(t, x_1(t)) \in \mathcal{C}(t, x_0(t))$ and so $(t, x_1(t))$ is contained in the interior of $K'$. Thus, we have shown that $J \subset J^*$, and therefore $\bar{\beta} \leq \beta^* < \beta(t_1, x_1)$ and $x_1(t)$ remains within $\varepsilon$ of $x_0(t)$ on $J$. This completes the proof of the lemma. □

## 3.6 Flow of Nonautonomous Systems

Let $f : \Omega \to \mathbb{R}^n$ be a vector field which satisfies the hypotheses of the Picard Theorem 3.2. Given $(t_0, x_0) \in \Omega$, let $x(t, t_0, x_0)$ be the corresponding solution of the initial value problem defined on the maximal existence interval $I(t_0, x_0) = (\alpha(t_0, x_0), \beta(t_0, x_0))$. The domain of $x(t, t_0, x_0)$ is

$$D = \{(t, t_0, x_0) \in \mathbb{R}^{n+2} : (t_0, x_0) \in \Omega,\ t \in I(t_0, x_0)\}.$$

The domain $D$ is open, and $x(t, t_0, x_0)$ is continuous on $D$. The next result summarizes some key properties of the solution.

**Lemma 3.5.**

1. *If* $(s, t_0, x_0), (t, t_0, x_0) \in D$, *then* $(t, s, x(s, t_0, x_0)) \in D$, *and*

$$x(t, t_0, x_0) = x(t, s, x(s, t_0, x_0)) \quad \textit{for} \quad t \in I(t_0, x_0).$$

2. *If* $(s, t_0, x_0) \in D$, *then* $I(t_0, x_0) = I(s, x(s, t_0, x_0))$.
3. *If* $(t_0, t_0, x_0) \in D$, *then* $x(t_0, t_0, x_0) = x_0$.
4. *If* $(s, t_0, x_0) \in D$, *then* $(t_0, s, x(s, t_0, x_0)) \in D$ *and*

$$x(t_0, s, x(s, t_0, x_0)) = x_0.$$

*Proof.* The first two statements are a consequence of the Uniqueness Theorem 3.3. The two solutions $x(t, t_0, x_0)$ and $x(t, s, x(s, t_0, x_0))$ pass through the point

$(s, x(s, t_0, x_0))$, and so they share the same maximal existence interval and they are equal on that interval.

The third statement follows from the definition of $x$.

Finally, if $(s, t_0, x_0) \in D$, then by (2),

$$t_0 \in I(t_0, x_0) = I(s, x(s, t_0, x_0)).$$

Thus, $(t_0, s, x(s, t_0, x_0)) \in D$, and we may substitute $t_0$ for $t$ in (1) to get the result:

$$x_0 = x(t_0, t_0, x_0) = x(t_0, s, x(s, t_0, x_0)).$$

□

*Example 3.5.* Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. The solution of the initial value problem

$$x' = Ax, \quad x(t_0) = x_0$$

is $x(t, t_0, x_0) = \exp A(t - t_0)x_0$. Here, we have $\Omega = \mathbb{R}^{n+1}$, $I(t_0, x_0) = \mathbb{R}$, for every $(t_0, x_0) \in \mathbb{R}^{n+1}$, and $D = \mathbb{R}^{n+2}$. In this case, Lemma 3.5 says

1. $\exp A(t - t_0)\, x_0 = \exp A(t - s)\, \exp A(s - t_0)\, x_0$,
3. $\exp A(t_0 - t_0)\, x_0 = x_0$,
4. $\exp A(t_0 - s)\, \exp A(s - t_0)\, x_0 = x_0$,

which follow from Lemma 2.1.

**Definition 3.4.** *Let $t, s \in \mathbb{R}$. The flow of the vector field $f$ from time $s$ to time $t$ is the map $y \mapsto \Phi_{t,s}(y) \equiv x(t, s, y)$. The domain of the flow map is therefore the set*

$$U(t, s) \equiv \{y \in \mathbb{R}^n : (s, y) \in \Omega,\ t \in I(s, y)\} = \{y \in \mathbb{R}^n : (t, s, y) \in D\}.$$

Notice that $U(t, s) \subset \mathbb{R}^n$ is open because the domain $D$ is open. It is possible that $U(t, s)$ is empty for some pairs $t, s$.

*Example 3.6.* Continuing the previous example of linear systems, we have $\Phi_{t,t_0} = \exp A(t - t_0)$.

*Example 3.7.* Consider the scalar initial value problem

$$x' = x^2, \quad x(t_0) = x_0.$$

Then $\Phi_{t,t_0}(x_0) = x(t, t_0, x_0) = x_0/[1 - x_0(t - t_0)]$ on

$$D = \{(t, t_0, x_0) \in \mathbb{R}^3 : 1 - x_0(t - t_0) > 0\}.$$

If $t > t_0$, then $U(t, t_0) = (-\infty, (t - t_0)^{-1})$ and $U(t_0, t) = ((t_0 - t)^{-1}, \infty)$.

**Lemma 3.6.**

1. *If* $t_0 \le s \le t$ *or if* $t \le s \le t_0$*, then* $U(t, t_0) \subset U(s, t_0)$*.*
2. *Transitivity property:*

$$\Phi_{s,t_0} : U(t, t_0) \cap U(s, t_0) \to U(t, s)$$

   *and*

$$\Phi_{t,t_0} = \Phi_{t,s} \circ \Phi_{s,t_0}, \quad on \quad U(t, t_0) \cap U(s, t_0).$$

3. *If* $x_0 \in U(t_0, t_0)$*, then* $\Phi_{t_0,t_0}(x_0) = x_0$*.*
4. *Inverse property:*

$$\Phi_{t,t_0} : U(t, t_0) \to U(t_0, t)$$

   *and*

$$\Phi_{t_0,t} \circ \Phi_{t,t_0}(x_0) = x_0, \quad x_0 \in U(t, t_0).$$

5. $\Phi_{t,t_0}$ *is a homeomorphism from* $U(t, t_0)$ *onto* $U(t_0, t)$*.*

*Proof.* Suppose that $t_0 \le s \le t$. If $x_0 \in U(t, t_0)$, then $t \in I(t_0, x_0)$. Since $I(t_0, x_0)$ is an interval containing $t_0$, we must have $[t_0, t] \subset I(t_0, x_0)$, and so $[t_0, s] \subset [t_0, t] \subset I(t_0, x_0)$. Thus, $x_0 \in U(s, t_0)$. The other case is identical.

The remaining statements are simply a restatement of Lemma 3.5 using our new notation.

Lemma 3.5, (1), says that if $x_0 \in U(s, t_0) \cap U(t, t_0)$, then $\Phi_{s,t_0}(x_0) \in U(t, s)$ and $\Phi_{t,t_0}(x_0) = \Phi_{t,s} \circ \Phi_{s,t_0}(x_0)$, which is statement (2) above.

Statements (3) and (4) are equivalent to (3) and (4) of Lemma 3.5.

The continuity of $x(t, t_0, x_0)$ on $D$ implies that $\Phi_{t,t_0}$ is continuous on $U(t, t_0)$. It is one-to-one and onto by (4). □

*Remark 3.6.* If $\Phi_{t,t_0}$ is a map which satisfies the properties 2 and 3 in Lemma 3.6 and which is $C^1$ in the $t$-variable, then it is the flow of the vector field

$$f(t, x) = \frac{d}{ds} \Phi_{s,t}(x)\Big|_{s=t},$$

with domain $\Omega = \cup_t U(t, t)$.

## 3.7 Flow of Autonomous Systems

Suppose now that the vector field $f(t, x) = f(x)$ is autonomous. Then we may assume that its domain has the form $\Omega = \mathbb{R} \times \mathcal{O}$ for an open set $\mathcal{O} \subset \mathbb{R}^n$. As usual, given $x_0 \in \mathcal{O}$, $x(t, t_0, x_0)$ denotes the solution of the (autonomous) IVP

$$x' = f(x), \quad x(t_0) = x_0,$$

with maximal existence interval $I(t_0, x_0)$.

**Lemma 3.7.** **(Autonomous flow)**. *Let* $x_0 \in \mathcal{O}$, $t, \tau \in \mathbb{R}$. *Then*

$$t + \tau \in I(t_0, x_0) \quad \textit{if and only if} \quad t \in I(t_0 - \tau, x_0),$$

*and*

$$x(t + \tau, t_0, x_0) = x(t, t_0 - \tau, x_0), \quad \textit{for} \quad t \in I(t_0 - \tau, x_0).$$

*Proof.* Since $x(t, t_0, x_0)$ is a solution on $I(t_0, x_0)$, we have

$$x'(t, t_0, x_0) = f(x(t, t_0, x_0)), \quad \text{for all} \quad t \in I(t_0, x_0).$$

Fix $\tau \in \mathbb{R}$ and define $J = \{t : t + \tau \in I(t_0, x_0)\}$. Substituting $t + \tau$ for $t$, we see that

$$x'(t + \tau, t_0, x_0) = f(x(t + \tau, t_0, x_0)), \quad \text{for all} \quad t \in J. \tag{3.3}$$

Here is the key point. *Since the system is autonomous*, the translate

$$y(t) = x(t + \tau, t_0, x_0)$$

solves the equation on $J$. Using the chain rule and (3.3), we have

$$y'(t) = \frac{d}{dt}[x(t + \tau, t_0, x_0)] = x'(t + \tau, t_0, x_0) = f(x(t + \tau, t_0, x_0)) = f(y(t)),$$

on the interval $J$. Since $y(t_0 - \tau) = x_0$, it follows by the Uniqueness Theorem 3.3 that

$$x(t + \tau, t_0, x_0) = y(t) = x(t, t_0 - \tau, x_0),$$

and $I(t_0 - \tau, x_0) = J$. □

Using the flow notation, this can be restated as:

**Lemma 3.8.** **(Autonomous flow)**. *For* $t, t_0 \in \mathbb{R}$, *we have*

$$U(t + \tau, t_0) = U(t, t_0 - \tau) \quad \textit{and} \quad \Phi_{t+\tau, t_0} = \Phi_{t, t_0 - \tau}.$$

**Corollary 3.1.** **(Autonomous flow)**. *For* $t, s \in \mathbb{R}$, *we have that*

$$\Phi_{t+s,0} = \Phi_{t,0} \circ \Phi_{s,0}, \quad \textit{on the domain} \quad U(t + s, 0) \cap U(s, 0).$$

*Proof.* By the general result, Lemma 3.6, we have

$$\Phi_{t+s,0} = \Phi_{t+s,s} \circ \Phi_{s,0}, \quad \text{on the domain} \quad U(t+s,0) \cap U(s,0).$$

By Lemma 3.8, $\Phi_{t+s,s} = \Phi_{t,0}$. □

*Remark 3.7.* Because of Lemma 3.8, *valid only in the autonmous case*, there is no loss of generality in using $\Phi_{t,0}$ since $\Phi_{t-s,0} = \Phi_{t,s}$. As is commonly done, we shall use simply $\Phi_t$ to denote the flow $\Phi_{t,0}$.

*Example 3.8.* Suppose that $f(x) = Ax$ with $A$ an $n \times n$ matrix over $\mathbb{R}$. Then $\Phi_t = \Phi_{t,0} = \exp At$, and the corollary is the familiar property that $\exp A(t+s) = \exp At \ \exp As$.

*Example 3.9.* Suppose that $\Phi_{t,s}(x_0)$ is the flow associated to an $n$-dimensional nonautonomous initial value problem

$$x'(t) = f(t, x(t)), \quad x(s) = x_0.$$

We saw in Chap. 1 that a nonautonomous system can be reformulated as an autonomous system by treating the time variable as a dependent variable. If $y(t) = (u(t), x(t)) \in \mathbb{R}^{1+n}$, then the equivalent autonomous system is

$$y'(t) = g(y(t)), \quad y(s) = y_0,$$

with

$$g(y) = g(u,x) = (1, f(u,x)) \quad \text{and} \quad y_0 = (u_0, x_0).$$

By direct substitution, the flow of this system is

$$\Psi_{t,s}(y_0) = \Psi_{t,s}(u_0, x_0) = (t - s + u_0, \Phi_{t-s+u_0,u_0}(x_0)).$$

It is immediate that $\Psi_{t,s}$ satisfies the property of Lemma 3.8, namely

$$\Psi_{t+\tau,s}(y_0) = \Psi_{t,s-\tau}(y_0).$$

**Definition 3.5.** *Given $x_0 \in \mathcal{O}$, define the orbit of $x_0$ to be the curve*

$$\gamma(x_0) = \{x(t, 0, x_0) : t \in I(0, x_0)\}.$$

*The positive semi-orbit of $x_0$ is defined to be*

$$\gamma_+(x_0) = \{x(t, 0, x_0) : t \in I(0, x_0) \cap [0, \infty)\},$$

*and the negative semi-orbit $\gamma_-(x_0)$ is similarly defined.*

Notice that the orbit is a curve in the phase space $\mathcal{O} \subset \mathbb{R}^n$, as opposed to the solution trajectory $\{(t, x(t, t_0, x_0)) : t \in I(t_0, x_0)\}$ which is a curve in the space-time domain $\Omega = \mathbb{R} \times \mathcal{O} \subset \mathbb{R}^{1+n}$.

For autonomous flow, we have the following strengthening of the Uniqueness Theorem 3.3.

**Theorem 3.6.** *If $z \in \gamma(x_0)$, then $\gamma(x_0) = \gamma(z)$. Thus, if two orbits intersect, then they are identical.*

*Proof.* Suppose that $z \in \gamma(x_0)$. Then $z = x(\tau, 0, x_0)$, for some $\tau \in I(0, x_0)$. By Lemma 3.5, we have that $I(0, x_0) = I(\tau, z)$ and $x(t, 0, x_0) = x(t, \tau, z)$ on this interval. Thus, we may write

$$\begin{aligned}\gamma(x_0) &= \{x(t, 0, x_0) : t \in I(0, x_0)\}\\ &= \{x(t, \tau, z) : t \in I(\tau, z)\}\\ &= \{x(t + \tau, \tau, z) : t + \tau \in I(\tau, z)\}.\end{aligned}$$

On the other hand, Lemma 3.7, says that

$$t + \tau \in I(\tau, z) \quad \text{if and only if} \quad t \in I(0, z),$$

and

$$x(t + \tau, \tau, z) = x(t, 0, z), \quad \text{for} \quad t \in I(0, z).$$

Thus, we see that

$$\gamma(z) = \{x(t, 0, z) : t \in I(0, z)\} = \{x(t + \tau, \tau, z) : t \in I(\tau, z)\}.$$

This shows that $\gamma(x_0) = \gamma(z)$. □

From the existence and uniqueness theory for general systems, we have that the domain $\Omega$ is foliated by the solution trajectories

$$\{(t, x(t, t_0, x_0)) : t \in I(t_0, x_0)\}.$$

That is, every point $(t_0, x_0) \in \Omega$ has a unique trajectory passing through it. Theorem 3.6 says that, *for autonomous systems*, the phase space $\mathcal{O}$ is foliated by the orbits. Each point of the phase space $\mathcal{O}$ has a unique orbit passing through it. For this reason, phase diagrams are meaningful in the autonomous case.

Since $x' = f(x)$, the orbits are curves in $\mathcal{O}$ everywhere tangent to the vector field $f(x)$. They are sometimes also referred to as integral curves. They can be obtained by solving the system

$$\frac{dx_1}{f_1(x)} = \cdots = \frac{dx_n}{f_n(x)}.$$

*Example 3.10.* Consider the harmonic oscillator

$$x_1' = x_2, \quad x_2' = -x_1.$$

The system for the integral curves is

$$\frac{dx_1}{x_2} = \frac{dx_2}{-x_1}.$$

Solutions satisfy

$$x_1^2 + x_2^2 = c,$$

and so we confirm that the orbits are concentric circles centered at the origin.

## 3.8 Global Solutions

As usual, we assume that $f : \Omega \subset \mathbb{R}^{1+n} \to \mathbb{R}^n$ satisfies the hypotheses of the Picard Existence Theorem 3.2.

Recall that a global solution of the initial value problem is one whose maximal interval of existence is $\mathbb{R}$. For this to be possible, it is necessary for the domain $\Omega$ to be unbounded in the time direction. So let us assume that $\Omega = \mathbb{R} \times \mathcal{O}$, where $\mathcal{O} \subset \mathbb{R}^n$ is open.

**Theorem 3.7.** *Let $I = (\alpha, \beta)$ be the maximal interval of existence of some solution $x(t)$ of $x' = f(t, x)$. Then either $\beta = +\infty$ or for every compact set $K \subset \mathcal{O}$, there exists a time $\beta(K) < \beta$ such that $x(t) \in \mathcal{O} \setminus K$ for all $t \in (\beta(K), \beta)$.*

*Proof.* Assume that $\beta < +\infty$. Suppose that $K \subset \mathcal{O}$ is compact. Then $K' = [t_0, \beta] \times K \subset \Omega$ is compact. By Theorem 3.4, there exists a time $\beta(K') < \beta$ such that $(t, x(t)) \in \Omega \setminus K'$ for $t \in (\beta(K'), \beta)$. Since $t < \beta$, this implies that $x(t) \in \mathcal{O} \setminus K$ for $t \in (\beta(K'), \beta)$. □

Of course, the analogous statement holds for $\alpha$, the left endpoint of $I$.

**Theorem 3.8.** *If $\Omega = \mathbb{R}^{1+n}$, then either $\beta = +\infty$ or*

$$\lim_{t \to \beta^-} \|x(t)\| = +\infty.$$

*Proof.* If $\beta < +\infty$, then by the previous result, for every $R > 0$, there is a time $\beta(R)$ such that $x(t) \notin \{x : \|x\| \le R\}$ for $t \in (\beta(R), \beta)$. In other words, $\|x(t)\| > R$ for $t \in (\beta(R), \beta)$, i.e. the desired conclusion holds. □

**Corollary 3.2.** *Assume that $\Omega = \mathbb{R}^{1+n}$. If there exists a nonnegative continuous function $\psi(t)$ defined for all $t \in \mathbb{R}$ such that*

$$\|x(t)\| \leq \psi(t) \quad \textit{for all} \quad t \in [t_0, \beta),$$

*then* $\beta = +\infty$.

*Proof.* The assumed estimate rules out the second possibility in Theorem 3.8, so we must have that $\beta = +\infty$. □

**Theorem 3.9.** *Let* $\Omega = \mathbb{R}^{1+n}$. *Suppose that there exist continuous nonnegative functions* $c_1(t)$, $c_2(t)$ *defined on* $\mathbb{R}$ *such that the vector field* $f$ *satisfies*

$$\|f(t,x)\| \leq c_1(t)\|x\| + c_2(t), \quad \textit{for all} \quad (t,x) \in \mathbb{R}^{1+n}.$$

*Then the solution to the initial value problem is global, for every* $(t_0, x_0) \in \mathbb{R}^{1+n}$.

*Proof.* We know that a solution of the initial value problem also solves the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds,$$

for all $t \in (\alpha, \beta)$. Applying the estimate for $\|f(t,x)\|$, we find that

$$\begin{aligned}
\|x(t)\| &\leq \|x_0\| + \left| \int_{t_0}^{t} [c_1(s)\|x(s)\| + c_2(s)]ds \right| \\
&\leq \|x_0\| + \left| \int_{t_0}^{t} c_2(s)ds \right| + \left| \int_{t_0}^{t} c_1(s)\|x(s)\|ds \right| \\
&= C(t) + \left| \int_{t_0}^{t} c_1(s)\|x(s)\|ds \right|,
\end{aligned}$$

with

$$C(t) = \|x_0\| + \left| \int_{t_0}^{t} c_2(s)ds \right|.$$

We now adapt our version of Gronwall's inequality to this slightly more general situation. Notice that $C(t)$ is nondecreasing for $t \in [t_0, \beta)$. Fix $T \in [t_0, \beta)$. Then for $t \in [t_0, T)$ we have

$$\|x(t)\| \leq C(T) + \int_{t_0}^{t} c_1(s)\|x(s)\|ds.$$

Then Gronwall's Lemma 3.3 implies that

$$\|x(t)\| \leq C(T) \exp \int_{t_0}^{t} c_1(s)ds,$$

for $t \in [t_0, T]$. In particular, this holds for $t = T$. Thus, we obtain

$$\|x(T)\| \leq C(T) \exp \int_{t_0}^{T} c_1(s)ds,$$

for all $T \in [t_0, \beta)$. Therefore, by Corollary 3.2, we conclude that $\beta = +\infty$

In the same way, we have that

$$\|x(T)\| \leq C(T) \exp \int_{T}^{t_0} c_1(s)ds,$$

for all $T \in (\alpha, t_0]$, and hence $\alpha = -\infty$. □

**Corollary 3.3.** *Let $A(t)$ be an $n \times n$ matrix and let $F(t)$ be an $n$-vector which depends continuously on $t \in \mathbb{R}$. Then the linear initial value problem*

$$x'(t) = A(t)x(t) + F(t), \quad x(t_0) = x_0$$

*has a unique global solution for every $(t_0, x_0) \in \mathbb{R}^{1+n}$.*

*Proof.* This is an immediate corollary of the preceding result, since the vector field

$$f(t,x) = A(t)x + F(t)$$

satisfies

$$\|f(t,x)\| \leq \|A(t)\|\|x\| + \|F(t)\|.$$

□

## 3.9 Stability

Let $\Omega = \mathbb{R} \times \mathcal{O}$ for some open set $\mathcal{O} \subset \mathbb{R}^n$, and suppose that $f : \Omega \to \mathbb{R}^n$ satisfies the hypotheses of the Picard Theorem.

**Definition 3.6.** *A point $\bar{x} \in \mathcal{O}$ is called an equilibrium point (singular point, critical point) if $f(t, \bar{x}) = 0$, for all $t \in \mathbb{R}$.*

**Definition 3.7.** *An equilibrium point $\bar{x}$ is stable if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $\|x_0 - \bar{x}\| < \delta$, the solution of the initial value problem $x(t, 0, x_0)$ exists for all $t \geq 0$ and*

$$\|x(t,0,x_0) - \bar{x}\| < \varepsilon, \quad t \geq 0.$$

*An equilibrium point $\bar{x}$ is asymptotically stable if it is stable and there exists a $b > 0$ such that if $\|x_0 - \bar{x}\| < b$, then*

$$\lim_{t\to\infty} \|x(t, 0, x_0) - \bar{x}\| = 0.$$

*An equilibrium point $\bar{x}$ is unstable if it is not stable.*

*Remark 3.8.* If $\bar{x}$ is a stable equilibrium (or an asymptotically stable equilibrium), then the conclusion of Definition 3.7 holds for any initial time $t_0 \in \mathbb{R}$, by Lemma 3.4, see Exercise 3.13.

**Theorem 3.10.** *Let $A$ be an $n \times n$ matrix over $\mathbb{R}$, and define the linear vector field $f(x) = Ax$.*

*The equilibrium $\bar{x} = 0$ is asymptotically stable if and only if* $\operatorname{Re}\lambda < 0$ *for all eigenvalues of $A$.*

*The equilibrium $\bar{x} = 0$ is stable if and only if* $\operatorname{Re}\lambda \le 0$ *for all eigenvalues of $A$ and $A$ has no generalized eigenvectors corresponding to eigenvalues with* $\operatorname{Re}\lambda = 0$.

*Proof.* Recall that the solution of the initial value problem is $x(t, 0, x_0) = \exp At\ x_0$.

If $\operatorname{Re}\lambda < 0$ for all eigenvalues, then $E_s = \mathbb{R}^n$ and by Corollary 2.1, there are constants $C, \alpha > 0$ such that

$$\|x(t, 0, x_0)\| \le C e^{-\alpha t} \|x_0\|,$$

for all $t \ge 0$. Asymptotic stability follows from this estimate.

If $\bar{x} = 0$ is asymptotically stable, then by linearity,

$$\lim_{t\to\infty} x(t, 0, x_0) = \lim_{t\to\infty} \exp At\ x_0 = 0,$$

for all $x_0 \in \mathbb{R}^n$. By Theorem 2.4, we have that $E_s = \mathbb{R}^n$. Thus, $\operatorname{Re}\lambda < 0$ for all eigenvalues of $A$. This proves the first statement.

If $\operatorname{Re}\lambda \le 0$ for all eigenvalues, then $E_s \oplus E_c = \mathbb{R}^n$. By Corollary 2.1 and Remark 2.1,

$$\begin{aligned} \|x(t, 0, x_0)\| &= \|\exp At\ (P_s + P_c)x_0\| \\ &\le C e^{-\alpha t} \|P_s x_0\| + C(1 + (t - t_0)^p)\|P_c x_0\|, \end{aligned}$$

for all $t > t_0$, where $p + 1$ is the size of the largest Jordan block corresponding to eigenvalues with zero real part. So if $A$ has no generalized eigenvectors corresponding to eigenvalues with zero real part, then we may take $p = 0$ above, and thus, the right-hand side is bounded by $C\|x_0\|$. Stability follows from this estimate.

If $A$ has an eigenvalue with positive real part, then $E_u \ne \{0\}$. By Corollary 2.2,

$$\lim_{t\to\infty} \|x(t, 0, x_0)\| = \lim_{t\to\infty} \|\exp At\ x_0\| = +\infty,$$

for all $0 \ne x_0 \in E_u$. This implies that the origin is unstable.

Finally, if $A$ has an eigenvalue $\lambda$ with $\operatorname{Re}\lambda = 0$ and a generalized eigenvector, then there exists $0 \neq z_0 \in N(A - \lambda I)^2 \setminus N(A - \lambda I)$. Thus, $z_1 = (A - \lambda I)z_0 \neq 0$ and $(A - \lambda I)^k z_0 = 0$, $k = 2, 3, \ldots$. From the definition of the exponential, we have that

$$\begin{aligned} \exp At\ z_0 &= \exp \lambda t\ \exp(A - \lambda I)t\ z_0 \\ &= \exp \lambda t \cdot [I + t(A - \lambda I)]\ z_0 \\ &= \exp \lambda t\ (z_0 + t z_1). \end{aligned}$$

Since $\operatorname{Re}\lambda = 0$, this gives the estimate

$$\|\exp At\ z_0\| = \|z_0 + t z_1\| \geq t\|z_1\| - \|z_0\|, \quad t \geq 0.$$

Since $A$ is real, $x_0 = \operatorname{Re} z_0$ and $y_0 = \operatorname{Im} z_0$ belong to $E_c$. Writing $z_0 = x_0 + i y_0$, we have

$$\|\exp At\ z_0\| \leq \|\exp At\ x_0\| + \|\exp At\ y_0\|,$$

which implies that the origin is unstable. □

**Theorem 3.11.** *Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set, and let $f : \mathcal{O} \to \mathbb{R}^n$ be $C^1$. Suppose that $\bar{x} \in \mathcal{O}$ is an equilibrium point of $f$ and that the eigenvalues of $A = Df(\bar{x})$ all satisfy* $\operatorname{Re}\lambda < 0$. *Then $\bar{x}$ is asymptotically stable.*

*Proof.* Let $g(y) = f(\bar{x} + y) - Ay$ for $y \in \mathcal{O}' = \{y \in \mathbb{R}^n : x_0 + y \in \mathcal{O}\}$. Then $g \in C^1(\mathcal{O}')$, $g(0) = 0$, and $Dg(0) = 0$.

Asymptotic stability of the equilibrium $x = \bar{x}$ for the system $x' = f(x)$ is equivalent to asymptotic stability of the equilibrium $y = 0$ for the system $y' = Ay + g(y)$.

Since $\operatorname{Re}\lambda < 0$ for all eigenvalues of $A$, know that from Corollary 2.1 that there exists $0 < \alpha < \lambda_s$ and a constant $C_1 > 0$ such that

$$\|\exp A(t - s)\| \leq C_1 \mathrm{e}^{-\alpha(t-s)}, \quad t \geq s.$$

Since $Dg(y)$ is continuous and $Dg(0) = 0$, given $\rho < \alpha/2C_1$, there is a $\mu > 0$ such that $\overline{B}_\mu(0) \subset \mathcal{O}'$ and

$$\|Dg(y)\| \leq \rho, \quad \text{for} \quad \|y\| \leq \mu.$$

Using the elementary formula

$$g(y) = \int_0^1 \frac{d}{ds}[g(sy)]ds = \int_0^1 Dg(sy)y ds,$$

we see that for $\|y\| \leq \mu$,

$$\|g(y)\| \le \int_0^1 \|Dg(sy)\|\|y\|ds \le \rho\|y\|.$$

Choose a $\delta < \mu$, and assume that $\|y_0\| < \delta$. Let $y(t) = y(t, 0, y_0)$ be the solution of the initial value problem

$$y'(t) = Ay(t) + g(y(t)), \quad y(0) = y_0,$$

defined for $t \in (\alpha, \beta)$. Define

$$T = \sup\{t \in [0, \beta) : \|y(s)\| < \mu, \ \text{ for } \ 0 \le s < t\}.$$

Then $T > 0$, by continuity, and of course, $T \le \beta$.

If we treat $g(y(t))$ as an inhomogeneous term, then by the Variation of Parameters formula, Corollary 4.1,

$$y(t) = \exp At \ y_0 + \int_0^t \exp A(t-s)g(y(s))ds.$$

Thus, since $\|y(t)\| < \mu$ on the interval $[0, T)$, we have

$$\|y(t)\| \le C_1 e^{-\alpha t}\|y_0\| + \int_0^t e^{-\alpha(t-s)} C_1\rho\|y(s)\|ds.$$

Set $z(t) = e^{\alpha t}\|y(t)\|$. Then

$$z(t) \le C_1\|y_0\| + \int_0^t C_1\rho z(s)ds.$$

By the Gronwall Lemma 3.3, we obtain

$$z(t) \le C_1\|y_0\|e^{C_1\rho t}, \quad 0 \le t < T.$$

In other words, we have

$$\|y(t)\| \le C_1\|y_0\|e^{(C_1\rho-\alpha)t} \le C_1\delta e^{-\alpha t/2}, \quad 0 \le t < T. \tag{3.4}$$

We may choose $\delta$ small enough so that in addition to $\delta < \mu$, as above, we also have $C_1\delta < \mu/2$. We have shown that

$$\|y(t)\| \le \mu/2, \quad 0 < t < T. \tag{3.5}$$

Now if $T < \infty$, then by Theorem 3.7, the curve $(t, x(t))$, $0 \le t < \beta$ would have to exit the compact set $K = [0, T] \times \overline{B}_\mu(0) \subset \Omega = \mathbb{R} \times \mathcal{O}'$. Our estimate (3.5) and the definition of $T$ prevents this from happening, so it must be the case that $T = \beta = \infty$. Now that $T = \infty$, stability and asymptotic stability follow from (3.4). □

*Remark 3.9.* Recall from Chap. 1 that the equation $y' = Ay$, with $A = Df(\bar{x})$, is the linearized equation for the nonlinear autonomous system $x' = f(x)$, near an equilibrium $\bar{x}$. The previous two results say that $\bar{x}$ is asymptotically stable for the nonlinear system if the origin is asymptotically stable for the linearized system.

*Example 3.11.* The scalar problem $x' = -x + x^2$ has equilibria at $x = 0, 1$. The origin is asymptotically stable for the linearized problem $y' = -y$, and so it is also asymptotically stable for the nonlinear problem. In this simple example, we can write down the explicit solution

$$x(t, 0, x_0) = \frac{x_0}{x_0 + (1 - x_0)e^t}.$$

Note that $x(t, 0, x_0) \to 0$ exponentially, as $t \to \infty$, provided $x_0 < 1$. This demonstrates both the asymptotic stability of $x = 0$ as well as the instability of $x = 1$. Of course, the easiest way to analyze stability in this example is to use the phase diagram in $\mathbb{R}$.

## 3.10 Liapunov Stability

Let $f(x)$ be a locally Lipschitz continuous vector field on an open set $\mathcal{O} \subset \mathbb{R}^n$. Assume that $f$ has an equilibrium point at $\bar{x} \in \mathcal{O}$.

**Definition 3.8.** *Let $U \subset \mathcal{O}$ be a neighborhood of $\bar{x}$. A Liapunov function for an equilibrium point $\bar{x}$ of a vector field $f$ is a function $E : U \to \mathbb{R}$ such that*

*(i)* $E \in C(U) \cap C^1(U \setminus \{\bar{x}\})$,
*(ii)* $E(x) > 0$ *for* $x \in U \setminus \{\bar{x}\}$ *and* $E(\bar{x}) = 0$,
*(iii)* $DE(x)\, f(x) \le 0$ *for* $x \in U \setminus \{\bar{x}\}$.

*If strict inequality holds in (iii), then $E$ is called a strict Liapunov function.*

**Theorem 3.12.** *If an equilibrium point $\bar{x}$ of $f$ has a Liapunov function, then it is stable.*

*If $\bar{x}$ has a strict Liapunov function, then it is asymptotically stable.*

*Proof.* Suppose that $E$ is a Liapunov function for $\bar{x}$.

Choose any $\varepsilon > 0$ such that $\overline{B}_\varepsilon(\bar{x}) \subset U$. Define

$$m = \min\{E(x) : \|x - \bar{x}\| = \varepsilon\} \quad \text{and} \quad U_\varepsilon = \{x \in U : E(x) < m\} \cap B_\varepsilon(\bar{x}).$$

Notice that $U_\varepsilon \subset U$ is a neighborhood of $\bar{x}$.

The claim is that for any $x_0 \in U_\varepsilon$, the solution $x(t) = x(t, 0, x_0)$ of the IVP $x' = f(x)$, $x(0) = x_0$ is defined for all $t \geq 0$ and remains in $U_\varepsilon$.

By the local existence theorem and continuity of $x(t)$, we have that $x(t) \in U_\varepsilon$ on some nonempty interval of the form $[0, \tau)$. Let $[0, T)$ be the maximal such interval. The claim amounts to showing that $T = \infty$.

On the interval $[0, T)$, we have that $x(t) \in U_\varepsilon \subset U$ and since $E$ is a Liapunov function,

$$\frac{d}{dt} E(x(t)) = DE(x(t)) \cdot x'(t) = DE(x(t)) \cdot f(x(t)) \leq 0.$$

From this it follows that

$$E(x(t)) \leq E(x(0)) = E(x_0) < m,$$

on $[0, T)$. So, if $T < \beta$, we would have $E(x(T)) \leq E(x(0)) < m$, and so, by definition of $m$, $x(T)$ cannot belong to the set $\|x - \bar{x}\| = \varepsilon$. Thus, we would have that $x(T) \in U_\varepsilon$. But this contradicts the maximality of the interval $[0, T)$. It follows that $T = \beta$. Since $x(t)$ remains in $U_\varepsilon$ on $[0, T) = [0, \beta)$, it remains bounded. So by Theorem 3.7, we have that $\beta = T = +\infty$.

We now use the claim to establish stability. Let $\varepsilon > 0$ be given. Without loss of generality, we may assume that $\overline{B}_\varepsilon(\bar{x}) \subset U$. Choose $\delta > 0$ so that $B_\delta(\bar{x}) \subset U_\varepsilon$. Then for every $x_0 \in B_\delta(\bar{x})$, we have that $x(t) \in U_\varepsilon \subset B_\varepsilon(\bar{x})$, for all $t > 0$.

Suppose now that $E$ is a strict Liapunov function, and let us prove asymptotic stability.

The equilibrium $\bar{x}$ is stable, so given $\varepsilon > 0$ with $\overline{B}_\varepsilon(\bar{x}) \subset U$, there is a $\delta > 0$ so that $x_0 \in B_\delta(\bar{x})$ implies $x(t) \in B_\varepsilon(\bar{x})$, for all $t > 0$.

Let $x_0 \in B_\delta(\bar{x})$. We must show that $x(t) = x(t, 0, x_0)$ satisfies $\lim\limits_{t \to \infty} x(t) = \bar{x}$. We may assume that $x_0 \neq \bar{x}$, so that, by uniqueness, $x(t) \neq \bar{x}$, on $[0, \infty)$.

Since $E$ is strict and $x(t) \neq \bar{x}$, we have that

$$\frac{d}{dt} E(x(t)) = DE(x(t)) \cdot x'(t) = DE(x(t)) \cdot f(x(t)) < 0.$$

Thus, $E(x(t))$ is a monotonically decreasing function bounded below by 0. Set $E^* = \inf\{E(x(t)) : t > 0\}$. Then $E(x(t)) \downarrow E^*$.

Since the solution $x(t)$ remains in the bounded set $U_\varepsilon$, it has a limit point. That is, there exist a point $z \in \overline{U_\varepsilon} \subset U$ and a sequence of times $t_k \to \infty$ such that $x(t_k) \to z$. We have, moreover, that $E^* = \lim_{k\to\infty} E(x(t_k)) = E(z)$.

Let $s > 0$. By the properties of autonomous flow, Lemma 3.7, we have that

$$\begin{aligned} x(s + t_k) &= x(s + t_k, 0, x_0) \\ &= x(s, 0, x(t_k, 0, x_0)) \\ &= x(s, 0, x(t_k)). \end{aligned}$$

By continuous dependence on initial conditions, Theorem 3.5, we have that

$$\lim_{k\to\infty} x(s+t_k) = \lim_{k\to\infty} x(s,0,x(t_k)) = x(s,0,z).$$

From this and the fact that $E(x(s,0,z))$ is nonincreasing, it follows that

$$E^* \le \lim_{k\to\infty} E(x(s+t_k)) = E(x(s,0,z)) \le E(x(0,0,z)) = E^*.$$

Thus, $x(s,0,z)$ is a solution along which $E$ is constant. But then

$$0 = \frac{d}{dt}E(x(t,0,z)) = DE(x(t,0,z))\ f(x(t,0,z)).$$

By assumption, this forces $x(t,0,z) = \bar{x}$ for all $t \ge 0$, and thus, $z = \bar{x}$. We have shown that the unique limit point of $x(t)$ is $\bar{x}$, which is equivalent to $\lim_{t\to\infty} x(t) = \bar{x}$. □

*Example 3.12.* (*Hamiltonian Systems*).
Let $H : \mathbb{R}^2 \to \mathbb{R}$ be a $C^2$ function. A system of the form

$$\begin{aligned}\dot{p} &= H_q(p,q)\\ \dot{q} &= -H_p(p,q)\end{aligned}$$

is referred to as Hamiltonian, the function $H$ being called the Hamiltonian. Suppose that $H(0,0) = 0$ and $H(p,q) > 0$, for $(p,q) \ne (0,0)$. Then since the origin is a minimum for $H$, we have that $H_p(0,0) = H_q(0,0) = 0$. In other words, the origin is an equilibrium for the system. Since

$$DH = (H_p, H_q)(H_q, -H_p)^T = 0,$$

we see that $H$ is a Liapunov function. Therefore, the origin is stable. Moreover, notice that for any solution $(p(t), q(t))$ we have

$$\frac{d}{dt}H(p(t),q(t)) = 0.$$

Thus, $H(p(t),q(t))$ is constant in time. This says that each orbit lies on a level set of the Hamiltonian.

More generally, we may take $H : \mathbb{R}^{2n} \to \mathbb{R}$ in $C^1$. Again with $H(0,0) = 0$ and $H(p,q) > 0$, for $(p,q) \ne (0,0)$. The origin is stable for the system

$$\begin{aligned}\dot{p} &= D_qH(p,q)\\ \dot{q} &= -D_pH(p,q).\end{aligned}$$

*Example 3.13.* (*Newton's equation*).

Let $G : \mathbb{R} \to \mathbb{R}$ be $C^2$, with $G(0) = 0$ and $G(u) > 0$, for $u \neq 0$. Consider the second order equation

$$\ddot{u} + G'(u) = 0.$$

With $x_1 = u$ and $x_2 = \dot{u}$, this is equivalent to the first order system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -G'(x_1).$$

Notice that the origin is an equilibrium, since $G'(0) = 0$. The system is, in fact, Hamiltonian with $H(x_1, x_2) = \frac{1}{2}x_2^2 + G(x_1)$. Since $H$ is positive away from the equilibrium, we have that the origin is stable.

The nonlinear pendulum arises when $G(u) = 1 - \cos u$.

*Example 3.14.* (*Lienard's equation*).

Let $\Phi : \mathbb{R} \to \mathbb{R}$ be $C^1$, with $\Phi(0) = 0$. Lienard's equation is

$$\ddot{u} + \Phi'(u)\dot{u} + u = 0.$$

It models certain electrical circuits, among other things. We rewrite this as a first order system in a nonstandard way. If we let

$$x_1 = u, \quad x_2 = \dot{u} + \Phi(u),$$

then

$$\dot{x}_1 = x_2 - \Phi(x_1), \quad \dot{x}_2 = -x_1.$$

Notice that the origin is an equilibrium.

Suppose that and $\Phi'(u) \geq 0$. The function $E(x) = \frac{1}{2}(x_1^2 + x_2^2)$ serves as a Liapunov function at the origin, since

$$DE(x)f(x) = -x_1\Phi(x_1) = -x_1 \int_0^{x_1} \Phi'(u)du \leq 0,$$

by our assumptions on $\Phi$. So the origin is a stable equilibrium. If $\Phi'(u) > 0$, for $u \neq 0$, then $E$ is a strict Liapunov function, and the origin is asymptotically stable.

## 3.11 Exercises

**Exercise 3.1.** Let $A$ be an $n \times n$ matrix over $\mathbb{R}$, and let $x_0 \in \mathbb{R}^n$. Calculate the Picard iterates for the initial value problem

$$x' = Ax, \quad x(0) = x_0,$$

and show that they converge uniformly to the exact solution on every closed, bounded interval of the form $[-T, T]$.

**Exercise 3.2.** Let $\Omega \subset \mathbb{R}^{1+n}$ be an open set, and suppose that $f : \Omega \to \mathbb{R}^n$ is continuous. Let $I$ be an interval containing the point $t_0$. Suppose that the function $t \mapsto (t, x(t))$ is continuous from $I$ into $\Omega$. Prove that $x(t)$ is a $C^1$ solution of the first order system

$$x'(t) = f(t, x(t)), \quad t \in I,$$

if and only if it is a $C^0$ solution of the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))ds, \quad t \in I.$$

**Exercise 3.3.** Let $I \subset \mathbb{R}$ be a closed, bounded interval, let $B \subset \mathbb{R}^n$ be a closed ball, and let $C^0(I, B)$ denote the set of continuous functions from $I$ to $B$. Define the sup norm

$$\|f\|_\infty = \sup_{t \in I} \|f(t)\|_{\mathbb{R}^n}.$$

Show that $C^0(I, B)$ is a complete metric space with the distance function

$$d(f, g) = \|f - g\|_\infty.$$

**Exercise 3.4.** Prove the following version of the Cauchy-Peano Existence Theorem: Suppose that $f$ is continuous on the rectangle

$$R = \{(t, x) \in \mathbb{R}^2 : |t - t_0| \le A, \ |x - x_0| \le B\}.$$

Let $M = \max_R |f(t, x)|$. Then the initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0,$$

has at least one solution defined for $|t - t_0| \le C = \min\{A, B/M\}$.

Suggestion: Define

$$x_n(t) = \begin{cases} x_0, & \text{if } |t - t_0| \le C/n \\ x_0 + \int_{t_0}^{t - \frac{C}{n}\operatorname{sgn}(t - t_0)} f(s, x_n(s))\, ds, & \text{if } C/n < |t - t_0| \le C. \end{cases}$$

Show that $x_n$ is well-defined, and apply the Arzela-Ascoli Theorem.

**Exercise 3.5.** Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that $f : \Omega \to \mathbb{R}^n$.

(a) Prove that if $f$ is locally Lipschitz continuous on $\Omega$, then it is continuous on $\Omega$.
(b) Prove that if $f$ is continuously differentiable on $\Omega$, then it is locally Lipschitz on $\Omega$.

**Exercise 3.6.** Show that the function $f(x) = \|x\|^\alpha, \ \alpha > 0$, from $\mathbb{R}^n$ to $\mathbb{R}$ is locally Lipschitz continuous if and only if $\alpha \geq 1$.

**Exercise 3.7.** Let $\Omega \subset \mathbb{R}^{1+n}$ be a nonempty, bounded, open set, with $\partial\Omega$ denoting its boundary. For $p \in \Omega$, define

$$\delta(p) = \mathrm{dist}(p, \partial\Omega) = \inf\{\|p - q\| : q \in \partial\Omega\}.$$

(a) Prove that $\delta$ is continuous on $\Omega$.
(b) Let $\varepsilon > 0$. Prove that the set $\{p \in \Omega : \delta(p) \geq \varepsilon\}$ is compact.
(c) Now suppose that $f : \Omega \to \mathbb{R}^n$ is a vector field which satisfies the hypotheses of the Picard existence theorem. Given a point $(t_0, x_0) \in \Omega$, let $x(t)$ be the solution of the IVP

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0,$$

defined on its maximal interval of existence $(\alpha, \beta)$. Prove that

$$\lim_{t \to \beta^-} \delta(t, x(t)) = 0.$$

**Exercise 3.8.** Suppose that $\Omega \subset \mathbb{R}^{1+n}$ is an open set and that $f : \Omega \to \mathbb{R}^n$ is a vector field which satisfies the hypotheses of the Theorems 3.2 and 3.3. Thus, for $(s, y) \in \Omega$, there exists a unique solution $x(t, s, y)$ of the initial value problem

$$x' = f(t, x), \quad x(s) = y,$$

defined on the maximal interval $I(s, y)$. Show that

$$\frac{d}{dt} x(t, s, y)\bigg|_{t=s} = f(s, y), \quad \text{for all} \quad (s, y) \in \Omega.$$

**Exercise 3.9.** Suppose that for all $t, s \in \mathbb{R}$, $\Phi_{t,s} : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the properties of a flow, namely:

- $(t, s, y) \mapsto \Phi_{t,s}(y)$ is continuous from $\mathbb{R}^{2+n}$ into $\mathbb{R}^n$.
- $t \mapsto \Phi_{t,s}(y)$ is $C^1$ from $\mathbb{R}$ into $\mathbb{R}^n$, for all $(s, y) \in \mathbb{R}^{n+1}$.
- $\Phi_{t,s} \circ \Phi_{s,\tau} = \Phi_{t,\tau}$, for all $t, s, \tau \in \mathbb{R}$.
- $\Phi_{t,t}$ is the identity map on $\mathbb{R}^n$.

Prove that $x(t, s, y) = \Phi_{t,s}(y)$ solves initial value problem

$$x' = f(t, x), \quad x(s) = y,$$

where the vector field $f$ is defined by

$$f(s, y) = \frac{d}{dt} \Phi_{t,s}(y)\bigg|_{t=s}, \quad \text{for all} \quad (s, y) \in \Omega.$$

**Exercise 3.10.** Using the notation from Definition 3.4, prove that if $s < t_1 < t_2$, then $U(t_2, s) \subset U(t_1, s)$.

**Exercise 3.11.** Consider the initial value problem

$$x' = (\cos t)\, x^2, \quad x(t_0) = x_0.$$

(a) Find the solution $x(t, t_0, x_0)$.
(b) Prove that if $|1/x_0 + \sin t_0| > 1$, then the maximal existence interval is $(-\infty, \infty)$.
(c) Prove that for every $t, s \in \mathbb{R}$,

$$\{y \in \mathbb{R} : |1/y + \sin s| > 1\} \subset U(t, s).$$

**Exercise 3.12.** Let $f : \mathbb{R}^{1+n} \to \mathbb{R}^n$ be the vector field

$$f(t, x) = \frac{(1 + t^2)\, x}{(1 + t^2 + \|x\|^2)^{1/2}}.$$

Prove that for every $(t_0, x_0) \in \mathbb{R}^{n+1}$, the initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0$$

has a unique solution defined on the interval $-\infty < t < \infty$.

**Exercise 3.13.** Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set, and set $\Omega = \mathbb{R} \times \mathcal{O}$. Suppose that $f : \Omega \to \mathbb{R}^n$ satisfies the hypotheses of Theorems 3.2 and 3.3. Suppose also that $\bar{x} \in \mathcal{O}$ is a stable equilibrium for $f$, according to Definition 3.7. Prove that for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}$, there exists a $\delta > 0$ such that if $\|x_0 - \bar{x}\| < \delta$, then the solution of the initial value problem $x(t, t_0, x_0)$ exists for all $t \geq t_0$ and

$$\|x(t, t_0, x_0) - \bar{x}\| < \varepsilon, \quad \text{for} \quad t \geq t_0.$$

(Thus, the notion of stability does not depend upon the choice of the initial time.)

**Exercise 3.14.** Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x) = Ax$, where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a nonzero matrix with real entries. Let

$$\operatorname{tr} A = a_{11} + a_{22} \quad \text{and} \quad \det A = a_{11}a_{22} - a_{12}a_{21}.$$

Prove the following statements.

(a) If $\operatorname{tr} A < 0$ and $\det A > 0$, then the origin is asymptotically stable for $f$.
(b) If $\operatorname{tr} A < 0$ and $\det A = 0$, then the origin is stable for $f$.
(c) If $\operatorname{tr} A = 0$ and $\det A > 0$, then the origin is stable for $f$.
(d) In all other cases, the origin is unstable for $f$.

**Exercise 3.15.** Let $A_k$ be $2 \times 2$ matrices over $\mathbb{R}$, and let $f_k(x) = A_k x$, $k = 1, 2$.

(a) Assume that the origin is asymptotically stable for $f_1$. Prove that there exists a $\delta > 0$ (depending on $A_1$) such that if $\|A_1 - A_2\| < \delta$, then the origin is asymptotically stable for $f_2$.
(b) Is the preceding statement true if "asymptotically stable" is replaced by "stable"? Explain.

**Exercise 3.16.** Let $\phi$ be a $C^1$ function on $\mathbb{R}$ with $\phi(0) = \phi'(0) = 0$. Let $a, b > 0$. Prove that the origin is asymptotically stable for the system

$$x' = y, \qquad y' = -ax - by + \phi(x).$$

# Chapter 4
# Nonautonomous Linear Systems

## 4.1 Fundamental Matrices

Let $t \mapsto A(t)$ be a continuous map from $\mathbb{R}$ into the set of $n \times n$ matrices over $\mathbb{R}$. Recall that, according to Corollary 3.3, the initial value problem

$$x'(t) = A(t)x(t), \quad x(t_0) = x_0,$$

has a unique global solution $x(t, t_0, x_0)$ for all $(t_0, x_0) \in \mathbb{R}^{n+1}$. Thus, the flow map $\Phi_{t,t_0}$is defined on $\mathbb{R}^n$ for every $t, t_0 \in \mathbb{R}$, i.e. $U(t, t_0) = \mathbb{R}^n$.

By linearity and the Uniqueness Theorem 3.3, it follows that

$$x(t, t_0, c_1x_1 + c_2x_2) = c_1x(t, t_0, x_1) + c_2x(t, t_0, x_2),$$

since both sides satisfy the same IVP. In other words, the flow map $\Phi_{t,t_0}(x_0)$ is linear in $x_0$. Thus, there is a continuous $n \times n$ matrix $X(t, t_0)$ such that

$$\Phi_{t,t_0}(x_0) = x(t, t_0, x_0) = X(t, t_0)x_0.$$

**Definition 4.1.** *The matrix $X(t, t_0)$ is called the state transition matrix for $A(t)$. The special case $X(t, 0)$ is called the fundamental matrix for $A(t)$, and we shall sometimes use the abbreviation $X(t)$ for $X(t, 0)$.*

By the general properties of the flow map, we have that

$$\Phi_{t,s} \circ \Phi_{s,t_0} = \Phi_{t,t_0}, \quad \Phi_{t,t} = id, \quad \Phi_{t,t_0}^{-1} = \Phi_{t_0,t}.$$

In the linear context, this implies that

$$X(t, s)X(s, t_0) = X(t, t_0), \quad X(t, t) = I, \quad X(t, t_0)^{-1} = X(t_0, t).$$

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Atlantis Studies in Differential Equations 2, DOI: 10.2991/978-94-6239-021-8_4,

Thus, for example, we may write

$$X(t,s) = X(t,0)X(0,s) = X(t,0)X(s,0)^{-1}.$$

Of course, if $A(t) = A$ is constant, then $X(t,t_0) = \exp A(t-t_0)$, and the preceding relations reflect the familiar properties of the exponential matrix for $A$.

Since

$$\frac{d}{dt}X(t,t_0)x_0 = \frac{d}{dt}x(t,t_0,x_0) = A(t)x(t,t_0,x_0) = A(t)X(t,t_0)x_0,$$

for every $x_0 \in \mathbb{R}^n$, we see that $X(t,t_0)$ is a matrix solution of

$$\frac{d}{dt}X(t,t_0) = A(t)X(t,t_0), \quad X(t_0,t_0) = I.$$

The $i$th column of $X(t,t_0)$, namely $y_i(t) = X(t,t_0)e_i$, satisfies

$$\frac{d}{dt}y_i(t) = A(t)y_i(t), \quad y_i(t_0) = e_i.$$

Since this problem has a unique solution for each $i = 1,\ldots,n$, it follows that $X(t,t_0)$ is unique.

From the formula $x(t,t_0,x_0) = X(t,t_0)x_0$, we have that every solution is a linear combination of the solutions $y_i(t)$. We say that the $y_i(t)$ span the solution space.

The next result generalizes the familiar Variation of Parameters formula from first order linear scalar ODES.

**Theorem 4.1. (Variation of Parameters)** *Let $t \mapsto A(t)$ be a continuous map from $\mathbb{R}$ into the set of $n \times n$ matrices over $\mathbb{R}$. Let $t \mapsto F(t)$ be a continuous map from $\mathbb{R}$ into $\mathbb{R}^n$. Then for every $(t_0,x_0) \in \mathbb{R}^n$, the initial value problem*

$$x'(t) = A(t)x(t) + F(t), \quad x(t_0) = x_0,$$

*has a unique global solution $x(t,t_0,x_0)$, given by the formula*

$$x(t,t_0,x_0) = X(t,t_0)x_0 + \int_{t_0}^{t} X(t,s)F(s)ds.$$

*Proof.* Global existence and uniqueness was shown in Corollary 3.3. So we need only verify that the solution $x(t) = x(t,t_0,x_0)$ satisfies the given formula.

Let $Z(t,t_0)$ be the state transition matrix for $-A(t)^T$. Set $Y(t,t_0) = Z(t,t_0)^T$. Then

$$\frac{d}{dt}Y(t,t_0) = \frac{d}{dt}Z(t,t_0)^T = [-A(t)Z(t,t_0)]^T = -Y(t,t_0)A(t),$$

and so

$$\begin{aligned}\frac{d}{dt}[Y(t,t_0)X(t,t_0)] &= [\frac{d}{dt}Y(t,t_0)]X(t,t_0) + Y(t,t_0)[\frac{d}{dt}X(t,t_0)] \\ &= -Y(t,t_0)A(t)X(t,t_0) + Y(t,t_0)A(t)X(t,t_0) \\ &= 0.\end{aligned}$$

Therefore, $Y(t,t_0)X(t,t_0) = Y(t,t_0)X(t,t_0)|_{t=t_0} = I$. In other words, we have that $Y(t,t_0) = X(t,t_0)^{-1} = X(t_0,t)$, from which we see that the state transition matrix is differentiable in both variables.

Now we mimic the solution method for the scalar case based on the integrating factor.

$$\begin{aligned}\frac{d}{dt}[Y(t,t_0)x(t)] &= [\frac{d}{dt}Y(t,t_0)]x(t) + Y(t,t_0)[\frac{d}{dt}x(t)] \\ &= -Y(t,t_0)A(t)x(t) + Y(t,t_0)[A(t)x(t) + F(t)] \\ &= Y(t,t_0)F(t).\end{aligned}$$

Upon integration, we find

$$\begin{aligned}x(t) &= Y(t,t_0)^{-1}x_0 + \int_{t_0}^{t} Y(t,t_0)^{-1}Y(s,t_0)F(s)ds \\ &= X(t,t_0)x_0 + \int_{t_0}^{t} X(t,t_0)X(s,t_0)^{-1}F(s)ds \\ &= X(t,t_0)x_0 + \int_{t_0}^{t} X(t,t_0)X(t_0,s)F(s)ds \\ &= X(t,t_0)x_0 + \int_{t_0}^{t} X(t,s)F(s)ds.\end{aligned}$$

□

**Corollary 4.1 (Variation of Parameters for Constant Coefficients).** *Let $A$ be a constant $n \times n$ matrix over $\mathbb{R}$, and let $F(t)$ be a continuous function from $\mathbb{R}$ into $\mathbb{R}^n$. For every $(t_0, x_0) \in \mathbb{R}^{n+1}$, the solution of the IVP*

$$x'(t) = Ax(t) + F(t), \quad x(t_0) = x_0,$$

*is given by*

$$x(t) = \exp At\ x_0 + \int_{t_0}^{t} \exp A(t-s)\ F(s)\ ds.$$

*Proof.* As noted above, $X(t,t_0) = \exp A(t-t_0)$. □

Although in general there is no formula for the state transition matrix, there is a formula for its determinant.

**Theorem 4.2.** *If $X(t, t_0)$ is the state transition matrix for $A(t)$, then*

$$\det X(t, t_0) = \exp \int_{t_0}^{t} \operatorname{tr} A(s) ds.$$

*Proof.* Regard the determinant of as a multi-linear function $\Delta$ of the rows $X_i(t, t_0)$ of $X(t, t_0)$. Then using multi-linearity, we have

$$\begin{aligned}
\frac{d}{dt} \det X(t, t_0) &= \frac{d}{dt} \Delta \begin{bmatrix} X_1(t, t_0) \\ \vdots \\ X_n(t, t_0) \end{bmatrix} \\
&= \Delta \begin{bmatrix} X_1'(t, t_0) \\ \vdots \\ X_n(t, t_0) \end{bmatrix} + \ldots + \Delta \begin{bmatrix} X_1(t, t_0) \\ \vdots \\ X_n'(t, t_0) \end{bmatrix}.
\end{aligned}$$

From the differential equation, $X'(t, t_0) = A(t)X(t, t_0)$, we get for each row

$$X_i'(t, t_0) = \sum_{j=1}^{n} A_{ij}(t) X_j(t, t_0).$$

Thus, using multi-linearity and the property that $\Delta = 0$ if two rows are equal, we obtain

$$\begin{aligned}
\Delta \begin{bmatrix} X_1(t, t_0) \\ \vdots \\ X_i'(t, t_0) \\ \vdots \\ X_n(t, t_0) \end{bmatrix} &= \Delta \begin{bmatrix} X_1(t, t_0) \\ \vdots \\ \sum_{j=1}^{n} A_{ij}(t) X_j(t, t_0) \\ \vdots \\ X_n(t, t_0) \end{bmatrix} = \sum_{j=1}^{n} A_{ij}(t) \Delta \begin{bmatrix} X_1(t, t_0) \\ \vdots \\ X_j(t, t_0) \\ \vdots \\ X_n(t, t_0) \end{bmatrix} \\
&= A_{ii}(t) \Delta \begin{bmatrix} X_1(t, t_0) \\ \vdots \\ X_i(t, t_0) \\ \vdots \\ X_n(t, t_0) \end{bmatrix} = A_{ii}(t) \det X(t, t_0).
\end{aligned}$$

The result now follows if we substitute this above. $\square$

## 4.2 Floquet Theory

Let $A(t)$ be a real $n \times n$ matrix, defined and continuous for $t \in \mathbb{R}$. Assume that $A(t)$ is $T$-periodic for some $T > 0$. Let $X(t) = X(t, 0)$ be the fundamental matrix for $A(t)$. We shall examine the form of $X(t)$.

**Theorem 4.3.** *There are $n \times n$ matrices $P(t)$, $L$ such that*

$$X(t) = P(t) \exp Lt,$$

*where $P(t)$ is $C^1$ and $T$-periodic and $L$ is constant.*

*Remarks 4.1.*

- Theorem 4.3 also holds for complex $A(t)$.
- Even if $A(t)$ is real, $P(t)$ and $L$ need not be real.
- Since $X(0) = I$, we have that $P(0) = I$. Hence,

  $$X(T) = P(T) \exp LT = \exp LT,$$

  for any $L$.
- $P(t)$ and $L$ are not unique. For example, let $S$ be an invertible matrix which transforms $L$ to Jordan normal form $J = \text{diag}[B_1, \ldots, B_p]$. For each Jordan block $B_j = \lambda_j I + N$, let $\tilde{B}_j = (\lambda_j + \frac{2\pi i k_j}{T})I + N$, for some $k_j \in \mathbb{Z}$. Then $B_j$ commutes with $\tilde{B}_j$ and

  $$\exp(B_j - \tilde{B}_j)T = \exp 2\pi i k_j I = I.$$

  Define $\tilde{J} = \text{diag}[\tilde{B}_1, \ldots, \tilde{B}_p]$ and $\tilde{L} = S\tilde{J}S^{-1}$. It follows that $L$ commutes with $\tilde{L}$ and

  $$\exp(L - \tilde{L})T = I.$$

  Now take

  $$\tilde{P}(t) = P(t) \exp(L - \tilde{L})t.$$

  Notice that $\tilde{P}(t)$ is $T$-periodic:

  $$\begin{aligned} \tilde{P}(t + T) &= P(t + T) \exp[(L - \tilde{L})(t + T)] \\ &= P(t) \exp(L - \tilde{L})t \exp(L - \tilde{L})T \\ &= \tilde{P}(t). \end{aligned}$$

  Since $L$ commutes with $\tilde{L}$ we have

$$\begin{aligned}\tilde{P}(t) \exp \tilde{L}t &= \tilde{P}(t) \exp(\tilde{L} - L)t \exp Lt \\ &= P(t) \exp Lt.\end{aligned}$$

**Definition 4.2.** *The eigenvalues of $X(T) = \exp LT$ are called the Floquet multipliers. The eigenvalues of L are called Floquet exponents.*

*Remarks 4.2.* The Floquet exponents are unique. Floquet multipliers are unique modulo $2\pi i/T$, see Lemma 4.2

**Theorem 4.4.** *If the Floquet multipliers of $X(t)$ all lie off the negative real axis, then the matrices $P(t)$ and L in Theorem 4.3 may be chosen to be real.*

**Theorem 4.5.** *The fundamental matrix $X(t)$ can be written in the form*

$$X(t) = P(t) \exp Lt,$$

*where $P(t)$ and L are real, $P(t+T) = P(t)R$, $R^2 = I$, and $RL = LR$.*

*Proof of Theorems 4.3, 4.4, and 4.5.* By periodicity of $A(t)$, we have that

$$X(t+T) = X(t)X(T), \quad t \in \mathbb{R},$$

since both sides satisfy the initial value problem $Z'(t) = A(t)Z(t)$, $Z(0) = X(T)$. The matrix $X(T)$ is nonsingular, so by Lemma 4.2 below, there exists a matrix $L$ such that

$$X(T) = \exp LT.$$

Set $P(t) = X(t)\exp(-Lt)$. Then to prove Theorem 4.3, we need only verify the periodicity of $P(t)$:

$$\begin{aligned}P(t+T) &= X(t+T) \exp[-L(t+T)] \\ &= X(t)X(T) \exp(-LT) \exp(-Lt) \\ &= X(t) I \exp(-Lt) \\ &= P(t),\end{aligned}$$

since by the choice of $L$, we have $X(T)\exp(-LT) = I$.

Theorem 4.4 is obtained by using Lemma 4.3 in place of Lemma 4.2.

Now we prove Theorem 4.5. By Lemma 4.4, there exist real matrices $L$ and $R$ such that

$$X(T) = R \exp LT, \quad RL = LR, \quad R^2 = I.$$

Define $P(t) = X(t)\exp(-Lt)$. Then exactly as before

$$P(t+T) = X(t+T) \exp[-L(t+T)]$$

$$
\begin{aligned}
&= X(t)X(T)\exp(-LT)\exp(-Lt) \\
&= X(t)R\exp(-Lt) \\
&= X(t)\exp(-Lt)R \\
&= P(t)R.
\end{aligned}
$$

Note we used the fact that $R$ and $L$ commute implies that $R$ and $\exp Lt$ commute. □

**Lemma 4.1.** *Let $N$ be an $n \times n$ nilpotent matrix over $\mathbb{C}$, with $N^{d+1} = 0$, $d \geq 1$. Then*

$$I - N = \exp L, \quad \textit{with} \quad L = -\sum_{k=1}^{d} \frac{1}{k} N^k.$$

*Proof.* Since $N^{d+1} = 0$, we have

$$I = I - (tN)^{d+1} = (I - tN) \sum_{k=0}^{d} (tN)^k.$$

Hence, $I - tN$ is invertible and $(I - tN)^{-1} = \sum_{k=0}^{d} (tN)^k$.

Set

$$L(t) = -\sum_{k=1}^{d} \frac{t^k}{k} N^k.$$

We will show that $\exp L(t) = I - tN$, i.e. $L(t) = \log(I - tN)$. We have, again since $N^{d+1} = 0$,

$$L'(t) = -\sum_{k=1}^{d} t^{k-1} N^k = -\sum_{k=0}^{d} t^k N^{k+1} = -(I - tN)^{-1} N.$$

Thus,

$$(I - tN)\, L'(t) = -N.$$

Another differentiation gives

$$-NL'(t) + (I - tN)\, L''(t) = 0.$$

This shows that

$$L''(t) = (I - tN)^{-1} (-N)\, L'(t) = -L'(t)^2.$$

Now we claim that

$$\frac{d}{dt} \exp L(t) = \exp L(t)\ L'(t).$$

Of course, this does not hold in general. But, because $L(t)$ is a polynomial in $tN$, we have that $L(t)$ and $L(s)$ commute for all $t, s \in \mathbb{R}$, and thus, the formula is easily verified using the definition of the derivative. Next, we also have

$$\frac{d^2}{dt^2} \exp L(t) = \exp L(t)[L''(t) + L'(t)^2] = 0.$$

Hence, $\exp L(t)$ is linear in $t$:

$$\exp L(t) = \exp L(0) + t \exp L(0)\ L'(0) = I - tN,$$

and as a consequence,

$$\exp L(1) = I - N.$$

This completes the proof. □

**Lemma 4.2.** *If $M$ is an $n \times n$ invertible matrix, then there exists an $n \times n$ matrix $L$ such that $M = \exp L$. If $\tilde{L}$ is another matrix such that $M = \exp \tilde{L}$, then the eigenvalues of $L$ and $\tilde{L}$ are equal modulo $2\pi i$.*

*Proof.* Choose an invertible matrix S which transforms M to Jordan normal form

$$S^{-1} M S = J.$$

J has the block structure

$$J = \mathrm{diag}[B_1, \ldots, B_p],$$

where the Jordan blocks have the form $B_j = \lambda_j I + N$ for some eigenvalue $\lambda_j \neq 0$, since M is invertible. Here, $N$ is the nilpotent matrix with 1 above the main diagonal.

Suppose that for each block we can find $L_j$ such that $B_j = \exp L_j$. Then if we set

$$L = \mathrm{diag}[L_1, \ldots, L_p],$$

we get

$$J = \exp L = \mathrm{diag}[\exp L_1, \ldots, \exp L_p].$$

Thus, we would have

$$M = SJS^{-1} = S \exp L S^{-1} = \exp SLS^{-1}.$$

We have therefore reduced the problem to that of finding the logarithm of an invertible Jordan block $B_j = \lambda_j I + N$. Since $-\lambda_j^{-1} N$ is nilpotent, by Lemma 4.1, there exists $\tilde{L}_j$ such that $I - (-\lambda_j^{-1} N) = \exp \tilde{L}_j$. Thus, we obtain

$$B_j = \exp(\log \lambda_j I + \tilde{L}),$$

where $\log \lambda_j = \log |\lambda_j| + i \arg \lambda_j$ is any complex logarithm of $\lambda_j$.

Suppose that $M = \exp L$ for some matrix $L$. Transform $L$ to Jordan canonical form $L = SJS^{-1} = S[D + N]S^{-1}$. We have that $J = D + N$ is an upper triangular matrix whose diagonal entries are the eigenvalues of $L$. Thus, $\exp J$ is also upper triangular with diagonal entries equal to the exponential of the eigenvalues of $L$. Now $M = \exp L = S \exp J\ S^{-1}$, has the same eigenvalues as $\exp J$, so it follows that the eigenvalues of $M$ are the exponentials of the eigenvalues of $L$. Therefore, if $\tilde{L}$ is another matrix such that $M = \exp \tilde{L}$, then $L$ and $\tilde{L}$ have the same eigenvalues, modulo $2\pi i$. □

**Lemma 4.3.** *If $M$ is a real $n \times n$ invertible matrix with no eigenvalues in $(-\infty, 0)$, then there exists a real $n \times n$ matrix $L$ such that $M = \exp L$.*

*Proof.* Suppose that $S$ reduces $M$ to Jordan canonical form. The columns of $S$ are a basis of generalized eigenvectors of $M$ in $\mathbb{C}^n$. Taking the real and imaginary parts of the elements of this basis, we obtain a basis for $\mathbb{R}^n$. Arranging these vectors appropriately in the columns of a matrix $T$, we obtain the real canonical form $M = TBT^{-1}$, where $B = \text{diag}[B_1, \ldots, B_d]$ is block diagonal. The blocks $B_j$ are real, and their form depends on whether the corresponding eigenvalue $\lambda_j$ is real or complex.

In the first case, when $\lambda_j$, $B_j$ is a standard Jordan block. We have seen in the proof of Lemma 4.2 how to to find a matrix $L_j$ such that $B_j = \exp L_j$, and moreover, when $\lambda_j > 0$, $L_j$ is real.

If $\lambda_j = \rho e^{i\theta}$, $0 < |\theta| < \pi$, is complex, $B_j$ has the block of the form

$$\begin{bmatrix} U_j & I & 0 & \ldots & 0 \\ 0 & U_j & I & \ldots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \ldots & 0 & U_j & I \\ 0 & \ldots & 0 & 0 & U_j \end{bmatrix}, \tag{4.1}$$

with

$$U_j = \begin{bmatrix} \rho \cos\theta & \rho \sin\theta \\ -\rho \sin\theta & \rho \cos\theta \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Notice that

$$B_j = U + N,$$

where $U = \text{diag}[U_j, \ldots, U_j]$, $N$ is nilpotent, and $UN = NU$. Now we have that

$$U_j = \exp W_j = \exp[\log \rho I + \theta V], \quad V = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Thus, $U$ is the exponential of the block diagonal matrix $X = \operatorname{diag}[W_j, \dots, W_j]$. We can use Lemma 4.1 to find a matrix $Y$ such that $I + U^{-1}N = \exp Y$. Since $Y$ is a polynomial in $U^{-1}N$ it is real and it commutes with $X$. We have, therefore, that

$$B_j = U(I + U^{-1}N) = \exp X \ \exp Y = \exp(X + Y).$$

Applying these procedures to each real block yields a real invertible matrix $T$ and a real block diagonal matrix $\mathcal{L}$ such that

$$M = T(\exp \mathcal{L})T^{-1} = \exp T\mathcal{L}T^{-1},$$

with $T\mathcal{L}T^{-1}$ real. □

**Lemma 4.4.** *If $M$ is a real $n \times n$ invertible matrix, then there exist real $n \times n$ matrices $L$ and $R$ such that $M = R \exp L$, $RL = LR$, and $R^2 = I$.*

*Proof.* We again begin by reducing $M$ to real canonical form

$$T^{-1}MT = J = \operatorname{diag}[B_1, \dots, B_d],$$

in which $B_j$ is either a Jordan block corresponding to a real eigenvalue of $M$ or $B_j$ has the form (4.1).

Define the block matrix

$$R = \operatorname{diag}[R_1, \dots, R_d],$$

where $R_j$ has the same size as $B_j$ and $R_j = -I$ if $B_j$ corresponds to an eigenvalue in $(-\infty, 0)$ and $R_j = I$, otherwise. Thus, $R^2 = I$.

Now the real matrix $RJ$ has no eigenvalues in $(-\infty, 0)$, and so by Lemma 2, there exists a real matrix $L$ such that $RJ = \exp L$. Clearly, $RL = LR$ since these two matrices have the same block structure and the blocks of $R$ are $\pm I$.

So now we have

$$M = TJT^{-1} = TR^2JT^{-1} = TR\exp LT^{-1} = \tilde{R}\exp \tilde{L},$$

in which $\tilde{R} = TRT^{-1}$ and $\tilde{L} = TLT^{-1}$. We have that $\tilde{R}^2 = I$ and $\tilde{R}\tilde{L} = \tilde{L}\tilde{R}$ as a consequence of the properties of $R$ and $L$. □

**Theorem 4.6.** *Let $A(t)$ be a continuous $T$-periodic $n \times n$ matrix over $\mathbb{R}$. Let $\{\mu_j\}_{j=1}^n$ be the Floquet multipliers and let $\{\lambda_j\}_{j=1}^n$ be a set of Floquet exponents. Then*

$$\prod_{j=1}^{n} \mu_j = \exp \int_0^T \operatorname{tr} A(t)dt,$$

*and*

$$\sum_{j=1}^{n} \lambda_j = \frac{1}{T}\int_0^T \operatorname{tr} A(t)dt, \qquad (\text{mod } 2\pi i/T).$$

*Proof.* Let $X(t)$ be the fundamental matrix for $A(t)$. By Theorem 4.2 we have that

$$\prod_{j=1}^{n} \mu_j = \det X(T) = \exp \int_0^T \operatorname{tr} A(t)dt.$$

By Theorem 4.3, we have $X(T) = \exp LT$. Since the Floquet exponents $\{\lambda_j\}_{j=1}^{n}$ are the eigenvalues of $L$, we have that the eigenvalues of $\exp LT$ are $\{e^{\lambda_j T}\}_{j=1}^{n}$, and hence

$$\det X(T) = \det \exp LT = \prod_{j=1}^{N} e^{\lambda_j T} = \exp\left(T\sum_{j=1}^{n} \lambda_j\right).$$

Thus,

$$T\sum_{j=1}^{n} \lambda_j = \log \det X(T) = \int_0^T \operatorname{tr} A(t)dt, \qquad (\text{mod } 2\pi i).$$

And so,

$$\sum_{j=1}^{n} \lambda_j = \frac{1}{T}\int_0^T \operatorname{tr} A(t)dt, \qquad (\text{mod } 2\pi i/T). \qquad \square$$

## 4.3 Stability of Linear Periodic Systems

**Theorem 4.7.** *Let $A(t)$ be a real $n \times n$ matrix which is continuous for $t \in \mathbb{R}$ and $T$-periodic for $T > 0$. By Theorem 4.3, the fundamental matrix $X(t)$ has the form*

$$X(t) = P(t) \exp Lt,$$

*where $P(t)$ is $T$-periodic.*

*The origin is stable for the system*

$$x'(t) = A(t)x(t) \tag{4.2}$$

*if and only if the Floquet multipliers $\mu$ satisfy $|\mu| \leq 1$ and there is a complete set of eigenvectors of $X(T)$ for any multipliers of modulus 1.*

*The origin is asymptotically stable if and only if $|\mu| < 1$ for all Floquet multipliers.*

*The stability of the origin for the system*

$$y'(t) = Ly(t) \tag{4.3}$$

*is the same as for* (4.2)

*Proof.* The solutions of the system (4.2) are given by

$$x(t) = P(t) \exp Lt\ x_0,$$

whereas the solutions of the system (4.3) are of the form

$$y(t) = \exp Lt\ x_0,$$

form some $x_0 \in \mathbb{R}^n$.

Now since $P(t)$ and $P(t)^{-1}$ are continuous and $T$-periodic, there exist positive constants $C_1, C_2$ such that

$$\sup_{t\in\mathbb{R}} \|P(t)\| = \sup_{0\leq t\leq T} \|P(t)\| = C_1$$

and

$$\sup_{t\in\mathbb{R}} \|P(t)^{-1}\| = \sup_{0\leq t\leq T} \|P(t)^{-1}\| = C_2.$$

This implies that

$$\|x(t)\| \leq C_1 \|\exp Lt\ x_0\| = C_1 \|y(t)\|, \quad t \in \mathbb{R}.$$

Thus, the stability or asymptotic stability of the origin for (4.3) implies the same for (4.2). Moreover, if the origin is unstable for (4.2), it also follows from this inequality that the origin is unstable for (4.3).

Similarly, we have that

$$\|y(t)\| \leq C_2 \|P(t) \exp Lt\ x_0\| = C_2 \|x(t)\|.$$

So the above statements hold with the roles of (4.2) and (4.3) reversed.

This shows that the stability of the origin for the two systems (4.2) and (4.3) is the same.

Let $\{\lambda_j\}_{j=1}^n$ be the eigenvalues of $L$. By Theorem 3.10, the origin is stable for (4.3) if and only if $\operatorname{Re}\lambda_j \leq 0$ for $j = 1, \ldots, n$ and there are no generalized eigenvectors corresponding to eigenvalues with $\operatorname{Re}\lambda_j = 0$. Now, the stability of the two systems is equivalent. The Floquet multipliers satisfy $\mu_j = \mathrm{e}^{\lambda_j T}$, and thus, the Floquet multipliers satisfy $|\mu_j| \leq 1$ if and only if $\operatorname{Re}\lambda_j \leq 0$. We see from the construction in Lemma 4.2 that $X(T) = \exp LT$ and $LT$ (and hence $L$) are reduced to Jordan normal form by the same matrix $S$. Thus, these two matrices have the same generalized eigenvectors, and so $L$ has no generalized eigenvectors with eigenvalue $\operatorname{Re}\lambda_j = 0$

if and only if $X(T)$ has no generalized eigenvectors with eigenvalue $|\mu_j| = 1$. We have shown that the origin is stable for (4.2) if and only if the Floquet multipliers satisfy $|\mu_j| \leq 1$ and there are no generalized eigenvectors for $X(T)$ corresponding to Floquet multipliers with norm 1.

The statement for asymptotic stability follows in the same way. □

*Example 4.1.* There is no simple relationship between $A(t)$ and the Floquet multipliers. Consider the $2\pi$-periodic coefficient matrix

$$A(t) = \begin{bmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\cos t \sin t \\ -1 - \frac{3}{2}\sin t \cos t & -1 + \frac{3}{2}\sin^2 t \end{bmatrix}.$$

By direct calculation it can be verified that

$$X(t) = \begin{bmatrix} \mathrm{e}^{t/2}\cos t & \mathrm{e}^{-t}\sin t \\ -\mathrm{e}^{t/2}\sin t & \mathrm{e}^{-t}\cos t \end{bmatrix}$$

is the fundamental matrix for $A(t)$. Since

$$X(2\pi) = \mathrm{diag}(\mathrm{e}^{\pi}\ \mathrm{e}^{-2\pi}),$$

we have that the Floquet multipliers are $\mathrm{e}^{\pi}$, $\mathrm{e}^{-2\pi}$, and so the origin is unstable. Indeed, $X(t)e_1$ is an unbounded solution.

On the other hand, the eigenvalues of $A(t)$ are

$$\lambda_1 = \frac{1}{4}[-1 + \sqrt{7}i], \quad \lambda_2 = \bar{\lambda}_1,$$

both of which have negative real parts. We see that the eigenvalues of $A(t)$ have no influence on stability.

Notice that

$$\mu_1\mu_2 = \exp \int_0^{2\pi} [\lambda_1 + \lambda_2] ds = \exp \int_0^{2\pi} \mathrm{tr}\, A(s) ds,$$

which confirms Theorem 4.6.

## 4.4 Parametric Resonance – The Mathieu Equation

The Mathieu equation is

$$u'' + (\omega^2 + \varepsilon \cos t)u = 0.$$

With $x(t) = [u(t)\ u'(t)]^T$ and

$$A(t;\omega,\varepsilon)=\begin{bmatrix}0 & 1\\ -(\omega^2+\varepsilon\cos t) & 0\end{bmatrix}$$

the equation can be written as a first order system

$$x'(t)=A(t;\omega,\varepsilon)x(t).$$

Notice that $A(t)$ is $2\pi$-periodic, and so Floquet theory applies. Let $X(t;\omega,\varepsilon)$ be the fundamental matrix of $A(t;\omega,\varepsilon)$. We shall use the fact, to be established in later in Theorem 6.2, that $X(t;\omega,\varepsilon)$ depends continuously on the parameters $\omega,\varepsilon$.

We ask the question: For which values of $\omega$ and $\varepsilon$ is the zero solution stable? Corollary 4.7 tells us to look at the Floquet multipliers in order to answer this question. In this case, since $T=2\pi$, the Floquet multipliers are the eigenvalues of $X(2\pi,\omega,\varepsilon)$.

Since $\operatorname{tr} A(t;\omega,\varepsilon)=0$ for every $t\in\mathbb{R}$, we have by Theorem 4.6 that the Floquet multipliers $\mu_1(\omega,\varepsilon)$, $\mu_2(\omega,\varepsilon)$ satisfy

$$\mu_1(\omega,\varepsilon)\mu_2(\omega,\varepsilon)=\det X(t,\omega,\varepsilon)=1,\quad t\in\mathbb{R}.$$

If $\mu_j(\omega,\varepsilon)\notin\mathbb{R}$, $j=1,2$, then $\mu_1(\omega,\varepsilon)=\bar{\mu}_2(\omega,\varepsilon)$. It follows that the Floquet multipliers are distinct points on the unit circle, and so there are no generalized eigenvectors for $X(2\pi;\omega,\varepsilon)$. Thus, in this case, we would have, by Corollary 4.7, that the origin is stable, but not asymptotically stable.

Notice that when $\varepsilon=0$, the system reduces to a harmonic oscillator. The fundamental matrix for this constant coefficient system is

$$X(t;\omega,0)=\begin{bmatrix}\cos\omega t & \omega^{-1}\sin\omega t\\ -\omega\sin\omega t & \cos\omega t\end{bmatrix}.$$

The Floquet multipliers are the eigenvalues of $X(2\pi,\omega,0)$. By direct calculation, we find

$$\mu_1(\omega,0)=\mathrm{e}^{i2\pi\omega},\quad \mu_2(\omega,0)=\mathrm{e}^{-i2\pi\omega}.$$

So $\mu_j(\omega,0)\in\mathbb{R}$ if and only if $2\omega\in\mathbb{Z}$.

Now by continuous dependence on the parameters $\omega,\varepsilon$, if the Floquet multipliers of $X(t;\omega,0)$ are not real, then the same is true for $X(t;\omega,\varepsilon)$ for $\varepsilon$ sufficiently small. Thus, for every $2\omega_0\notin\mathbb{Z}$, there is a small ball in the $(\omega,\varepsilon)$ plane with center $(\omega_0,0)$ where the origin is stable for Mathieu's equation.

It can also be shown, although we will not do so here, that there are regions of instability branching off of the points $(\omega_0,0)$ when $2\omega_0\in\mathbb{Z}$. This is the so-called parametric resonance.

## 4.5 Existence of Periodic Solutions

**Lemma 4.5.** *Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set and let $\Omega = \mathbb{R} \times \mathcal{O} \to \mathbb{R}^n$. Suppose that $f : \Omega \to \mathbb{R}^n$ satisfies the hypotheses of the Picard existence and uniqueness theorem and that there exists $T > 0$ such that*

$$f(t + T, x) = f(t, x) \quad \textit{for all} \quad (t, x) \in \Omega.$$

*If $x(t)$ is a solution of the system $x'(t) = f(t, x(t))$ defined on an interval containing $[0, T]$ and $x(0) = x(T)$, then $x(t)$ is a global $T$-periodic solution.*

*Proof.* Suppose that $x(t)$ is a solution defined on a maximal interval $I = (\alpha, \beta)$ containing $[0, T]$, such that $x(0) = x(T)$. Define

$$y(t) = x(t + T), \quad \text{for} \quad t \in I - T = \{t \in \mathbb{R} : t + T \in I\}.$$

Then

$$y'(t) = x'(t + T) = f(t + T, x(t + T)) = f(t, y(t)),$$

and

$$y(0) = x(T) = x(0).$$

Since $x$ and $y$ solve the same initial value problem, they are identical solutions on their common interval of definition $I \cap I - T$, by the Uniqueness Theorem 3.3. Thus, we have

$$x(t) = x(t + T) \quad \text{for} \quad t \in I \cap I - T. \tag{4.4}$$

Also, by Theorem 3.3, the solution $x(t)$ extends to $I \cup I - T$, but since $I$ is maximal, it follows that $\alpha = -\infty$. In the same way, by considering $x(t - T)$, we see that $\beta = \infty$.

Thus, $I = \mathbb{R}$, so that $x(t)$ is global. The solution $x(t)$ is $T$-periodic, by (4.4).

□

**Theorem 4.8.** *Let $A(t)$ be $T$-periodic and continuous. The system $x'(t) = A(t)x(t)$ has a nonzero $T$-periodic solution if and only if $A(t)$ has the Floquet multiplier $\mu = 1$.*

*Proof.* By Lemma 4.5, we have that a solution is $T$-periodic if and only if

$$x(T) = x(0). \tag{4.5}$$

Let $X(t)$ be the fundamental matrix for $A(t)$. Then every nonzero solution has the form $x(t) = X(t)x_0$ for some $x_0 \in \mathbb{R}^n$ with $x_0 \neq 0$. It follows that (4.5) holds if and only if $X(T)x_0 = x_0$. Thus, $x_0$ is an eigenvector for $X(T)$ with eigenvalue 1. But the eigenvalues of $X(T)$ are the Floquet multipliers.

□

**Theorem 4.9.** *Let $A(t)$ be an $n \times n$ matrix and let $F(t)$ be a vector in $\mathbb{R}^n$, both which are continuous functions of $t$. Assume that $A(t)$ and $F(t)$ are $T$-periodic. The equation*

$$x'(t) = A(t)x(t) + F(t) \tag{4.6}$$

*has a $T$-periodic solution if and only if*

$$\int_0^T y(t) \cdot F(t)dt = 0, \tag{4.7}$$

*for all $T$-periodic solutions $y(t)$ of the adjoint system*

$$y'(t) = -A(t)^T y(t). \tag{4.8}$$

*The system (4.6) has a unique $T$-periodic solution for every $T$-periodic function $F(t)$ if and only if the adjoint system has no nonzero $T$-periodic solutions.*

*Proof.* Let $X(t) = X(t, 0)$ be the fundamental matrix for $A(t)$. By Variation of Parameters, Theorem 4.1, we have that the solution $x(t) = x(t, 0, x_0)$ of (4.6) is given by

$$x(t) = X(t)x_0 + X(t)\int_0^t X(s)^{-1}F(s)ds.$$

By Lemma 4.5, $x(t)$ is $T$-periodic if and only if $x(T) = x(0) = x_0$. This is equivalent to

$$[I - X(T)]x_0 = X(T)\int_0^T X(s)^{-1}F(s)ds,$$

and so, multiplying both sides by $X(T)^{-1}$, we obtain an equivalent linear system of equations

$$Bx_0 = g, \tag{4.9}$$

in which

$$B = X(T)^{-1} - I \quad \text{and} \quad g = \int_0^T X(s)^{-1}F(s)ds.$$

Thus, $x(t, 0, x_0)$ is a $T$-periodic solution of (4.6) if and only if $x_0$ is a solution of (4.9).

By Lemma 4.6 below, the system (4.9) has a solution if and only if $g \cdot y_0 = 0$ for all $y_0 \in N(B^T)$.

We now characterize $N(B^T)$. Let $Y(t) = Y(t, 0)$ be the fundamental matrix for $-A(t)^T$. Then $Y(t) = [X(t)^{-1}]^T$, so

$$B^T = Y(T) - I.$$

Thus, $y_0 \in N(B^T)$ if and only if $y_0 = Y(T)y_0$. This, in turn, is equivalent to saying $y_0 \in N(B^T)$ if and only if $y(t) = Y(t)y_0$ is a $T$-periodic solution of the adjoint system (4.8).

Now we examine the orthogonality condition. We have

$$\begin{aligned} y_0 \cdot g &= \int_0^T y_0 \cdot X(s)^{-1} F(s)ds \\ &= \int_0^T y_0 \cdot Y(s)^T F(s)ds \\ &= \int_0^T Y(s)y_0 \cdot F(s)ds \\ &= \int_0^T y(s) \cdot F(s)ds. \end{aligned}$$

The result now follows from the following chain of equivalent statements:

- Equation (4.6) has a $T$-periodic solution.
- The system (4.9) has a solution.
- The orthogonality condition $y_0 \cdot g = 0$ holds for every $y_0 \in N(B^T)$.
- The orthogonality condition (4.7) holds for every $T$-periodic solution of (4.8).

$T$-periodic solutions are unique if and only if $B$ is invertible, i.e. $\det B = \det B^T \neq 0$. Thus, uniqueness of the $T$-periodic solution is equivalent to $N(B^T) = \{0\}$. In other words, $T$-periodic solutions are unique if and only if the adjoint system (4.8) has no nonzero $T$-periodic solutions. □

**Lemma 4.6. (Fredholm Alternative in Finite Dimensions).** *Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear operator. Define the range*

$$R(A) = \{f \in \mathbb{R}^m : f = Ax \text{ for some } x \in \mathbb{R}^n\}$$

*and the null space*

$$N(A) = \{x \in \mathbb{R}^n : Ax = 0\}.$$

*Then $R(A) = N(A^T)^\perp$.*

*Proof.* Let $f = Ax \in R(A)$. For any $y \in N(A^T)$, we have

$$f \cdot y = Ax \cdot y = x \cdot A^T y = 0.$$

Thus, $f \in N(A^T)^\perp$, which shows that $R(A) \subset N(A^T)^\perp$.

The inclusion $N(A^T)^\perp \subset R(A)$ is equivalent to $R(A)^\perp \subset N(A^T)$, since $S^{\perp\perp} = S$ for any subspace $S$ of a finite dimensional inner product space. Let $g \in R(A)^\perp$.

Then $g \cdot Ax = 0$, for all $x \in \mathbb{R}^n$, and so $A^T g \cdot x = 0$, for all $x \in \mathbb{R}^n$. Choose $x = A^T g$ to get $\|A^T g\| = 0$. Thus, $g \in N(A^T)$, proving $R(A)^\perp \subset N(A^T)$.

□

*Example 4.2.* Consider the periodically forced harmonic oscillator:

$$u'' + u = \cos \omega t, \quad \omega > 0.$$

This is equivalent to the first order system

$$x'(t) = Ax(t) + F(t) \tag{4.10}$$

with

$$x(t) = \begin{bmatrix} u(t) \\ u'(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad F(t) = \begin{bmatrix} 0 \\ \cos \omega t \end{bmatrix}.$$

Notice that $F(t)$ (and $A$) are $T$-periodic with $T = 2\pi/\omega$.

Since $-A^T = A$, the adjoint equation is

$$y' = Ay,$$

the solutions of which are

$$y(t) = \exp At \; y_0, \quad \text{with} \quad \exp At = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

All solutions of the adjoint equation are $2\pi$-periodic.

If the forcing frequency $\omega$ satisfies $\omega \neq 1$, then the adjoint equation has no nonzero $T$-periodic solutions. Thus, the system (4.10) has a unique $T$-periodic solution.

If $\omega = 1$, then $T = 2\pi$ and all solutions of the adjoint equation are $T$-periodic. The orthogonality condition can not be satisfied since

$$\begin{aligned}
\int_0^T y(s) \cdot F(s) ds &= \int_0^T \exp As \; y_0 \cdot F(s) ds \\
&= \int_0^T y_0 \cdot [\exp As]^T F(s) ds \\
&= \int_0^T y_0 \cdot [-\sin s \cos s \; e_1 + \cos^2 s \; e_2] ds \\
&= \pi \; y_0 \cdot e_2.
\end{aligned}$$

This is nonzero for $y_0 \cdot e_2 \neq 0$. Therefore, when $\omega \neq 1$, system (4.10) has no $2\pi$-periodic solutions. This is the case of resonance.

This overly simple example can be solved explicitly, since $A$ is constant. However, it illustrates the use of the Fredholm Alternative in such problems.

## 4.6 Exercises

**Exercise 4.1.** Let $a_0(t), \ldots, a_{n-1}(t), f(t)$ be continuous functions on some interval $I \subset \mathbb{R}$. Define the nth order linear differential operator

$$L = D_t^n + a_{n-1}(t)D_t^{n-1} + \cdots + a_1(t)D_t + a_0(t), \quad D_t = \frac{d}{dt}.$$

(a) Prove that for any $t_0 \in I$ and $(y_{n-1}, \ldots, y_0) \in \mathbb{R}^n$, the initial value problem

$$L[y](t) = f(t), \quad D_t^k y(t_0) = y_k; \quad k = 0, \ldots, n-1,$$

has a unique solution defined on $I$. *(Rewrite as a first order system.)*

(b) (Duhamel's Principle.) Show that the solution in (a) is represented by

$$y(t) = y_h(t) + \int_{t_0}^{t} Y(t,s) f(s) ds,$$

where $y_h(t)$ is the solution of the homogeneous equation $L[y_h] = 0$ with the initial data $D_t^k y_h(t_0) = y_k$, $k = 0, \ldots, n-1$, and $Y(t,s)$ is the solution of $L[Y] = 0$ with initial data

$$D_t^k Y(s,s) = 0, \; k = 0, \ldots, n-2, \; D_t^{n-1} Y(s,s) = 1.$$

**Exercise 4.2.** Let $A(t)$ be an $n \times n$ matrix over $\mathbb{R}$ and let $F(t) \in \mathbb{R}^n$, for every $t \in \mathbb{R}$. Suppose that $A$ and $F$ are continuous $T$-periodic functions of $t \in \mathbb{R}$. Prove that the solution, $x(t)$, of the initial value problem

$$x'(t) = A(t)x(t) + F(t), \quad x(0) = x_0,$$

is $T$-periodic if and only if $x(0) = x(T)$.

**Exercise 4.3.** Let

$$M = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a) Find a matrix $L$ such that $M = \exp L$.

(b) Find real matrices $L$ and $R$ such that

$$M = R \exp L, \quad R^2 = I, \quad RL = LR.$$

**Exercise 4.4.** Let $a : \mathbb{R} \to \mathbb{R}$ be continuous and $T$-periodic.

(a) Find the fundamental "matrix" $X(t) = X(t, 0)$ for $a(t)$.

(b) Find the Floquet multiplier.

(c) Find a continuous, $T$-periodic function $P : \mathbb{R} \to \mathbb{R}$ and an $L \in \mathbb{R}$ such that $X(t) = P(t) \exp Lt$. Why is it possible to have $P(t)$ and $L$ real here?

**Exercise 4.5.**

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $T$-periodic. Find all initial values $x_0 \in \mathbb{R}$ for which the initial value problem

$$x'(t) = x(t) + f(t), \quad x(0) = x_0$$

has a $T$-periodic solution. Reconcile your answer with Theorem 4.9.

(b) Let $a : \mathbb{R} \to \mathbb{R}$ be continuous and $T$-periodic. Find all initial values $x_0 \in \mathbb{R}$ for which the initial value problem

$$x'(t) = a(t)x(t) + 1, \quad x(0) = x_0$$

has a $T$-periodic solution. Reconcile your answer with Theorem 4.9.

# Chapter 5
# Results from Functional Analysis

## 5.1 Operators on Banach Space

**Definition 5.1** A Banach space is a complete normed vector space over $\mathbb{R}$ or $\mathbb{C}$.

Here are some examples of Banach spaces that will be relevant for us:

- Let $\mathcal{F} \subset \mathbb{R}^n$. $C^0(\mathcal{F}, \mathbb{R}^m)$ is set the of continuous functions from $\mathcal{F}$ into $\mathbb{R}^m$. Define the *sup*-norm
$$\|f\|_\infty = \sup_{x \in \mathcal{F}} \|f(x)\|.$$
Then
$$C_b^0(\mathcal{F}, \mathbb{R}^m) = \{f \in C^0(\mathcal{F}, \mathbb{R}^m) : \|f\|_\infty < \infty\},$$
is a Banach space. If $\mathcal{F}$ is compact, then $C_b^0(\mathcal{F}, \mathbb{R}^m) = C^0(\mathcal{F}, \mathbb{R}^m)$.
- $C^1(\mathcal{F}, \mathbb{R}^m)$ is set the of functions $f$ from $\mathcal{F}$ into $\mathbb{R}^m$ such that $Df(x)$ exists and is continuous. Define the norm
$$\|f\|_{C^1} = \|f\|_\infty + \|Df\|_\infty.$$
Then
$$C_b^1(\mathcal{F}, \mathbb{R}^m) = \{f \in C^1(\mathcal{F}, \mathbb{R}^m) : \|f\|_{C^1} < \infty\},$$
is a Banach space.
- $\mathrm{Lip}\,(\mathcal{F}, \mathbb{R}^m)$ is the set of Lipschitz continuous functions from $\mathcal{F}$ into $\mathbb{R}^m$ such that the norm
$$\|f\|_{\mathrm{Lip}} = \sup_{x \in \mathcal{F}} \|f(x)\| + \sup_{\substack{x,y \in \mathcal{F} \\ x \neq y}} \frac{\|f(x) - f(y)\|}{\|x - y\|}$$
is finite.

T. C. Sideris, *Ordinary Differential Equations and Dynamical Systems*,
Atlantis Studies in Differential Equations 2, DOI: 10.2991/978-94-6239-021-8_5,

Notice that Lip $(\mathcal{F}, \mathbb{R}^m) \subset C_b^0(\mathcal{F}, \mathbb{R}^m)$. If $\mathcal{F}$ is compact then $C_b^1(\mathcal{F}, \mathbb{R}^m) \subset$ Lip $(\mathcal{F}, \mathbb{R}^m)$.

**Definition 5.2** Let $X$ and $Y$ be Banach spaces. Let $A : X \to Y$ be a linear operator. $A$ is said to be bounded if

$$\sup_{\|x\|_X \neq 0} \frac{\|Ax\|_Y}{\|x\|_X} \equiv \|A\|_{X,Y} < \infty.$$

$\|A\|_{X,Y}$ is the operator norm.

*Remark 5.1* The set of all bounded linear operators from $X$ to $Y$ is denoted by $\mathcal{L}(X, Y)$. It is a Banach space with the operator norm.

*Remark 5.2* A linear operator from $X$ to $Y$ is bounded if and only if it is continuous.

*Remark 5.3* In the above definition, there appear three different norms. This is typical. For notational simplicity, we will often omit the subscripts on the different norms, since the norm being used can be inferred from the Banach space to which a given vector belongs.

**Definition 5.3** A bijective operator $A : X \to Y$ said to be an isomorphism if $A \in \mathcal{L}(X, Y)$ and $A^{-1} \in \mathcal{L}(Y, X)$.

We state without proof a fundamental result.

**Theorem 5.1 (Open Mapping Theorem).** *An operator in $\mathcal{L}(X, Y)$ is surjective if and only if it is an open mapping. In particular, a bijective mapping in $\mathcal{L}(X, Y)$ is an isomorphism.*

**Lemma 5.1** *Let $X$ be a Banach space, and let $A \in \mathcal{L}(X, X)$. If $\|A\| \leq \alpha$, for some $\alpha < 1$, then $I - A$ is an isomorphism, and*

$$\|(I - A)^{-1}\| \leq (1 - \alpha)^{-1}.$$

*If $B \in \mathcal{L}(X, X)$ is another operator such that $\|B\| \leq \alpha < 1$, then*

$$\|(I - A)^{-1} - (I - B)^{-1}\| \leq (1 - \alpha)^{-2}\|A - B\|.$$

*Proof.* If $A \in \mathcal{L}(X, X)$, then $A^j \in \mathcal{L}(X, X)$, for $j \in \mathbb{N}$, and $\|A^j\| \leq \|A\|^j$.

Define $M_k = \sum_{j=0}^{k} A^j$. If $\|A\| \leq \alpha < 1$, then $M_k$ is a Cauchy sequence in $\mathcal{L}(X, X)$. Therefore, $M_k \to M \in \mathcal{L}(X, X)$ and $\|M\| = \lim_k \|M_k\| \leq (1 - \alpha)^{-1}$.

Now $I - A$ is bijective and $M = (I - A)^{-1}$, since we have

$$M(I - A) = \lim_k M_k(I - A) = \lim_k (I - A^{k+1}) = I,$$

and likewise, $(I - A)M = I$.

Given $A, B \in \mathcal{L}(X, X)$ with $\|A\|, \|B\| \leq \alpha < 1$, we have

$$\begin{aligned}
\|(I-A)^{-1} - (I-B)^{-1}\| &= \|(I-A)^{-1}[(I-B)-(I-A)](I-B)^{-1}\| \\
&= \|(I-A)^{-1}(A-B)(I-B)^{-1}\| \\
&\leq \|(I-A)^{-1}\|\|A-B\|\|(I-B)^{-1}\| \\
&\leq (1-\alpha)^{-2}\|A-B\|. \qquad \square
\end{aligned}$$

## 5.2 Fréchet Differentiation

**Definition 5.4** Let $X$ and $Y$ be Banach spaces. Let $U \subset X$ be open. A map $f : U \to Y$ is Fréchet differentiable at a point $x_0 \in U$ if and only if there exists a bounded linear operator $Df(x_0) \in \mathcal{L}(X, Y)$ such that

$$f(x + x_0) - f(x_0) - Df(x_0)x \equiv Rf(x, x_0)$$

satisfies

$$\lim_{x \to 0} \frac{\|Rf(x, x_0)\|_Y}{\|x\|_X} = 0.$$

Equivalently, $f$ is Fréchet differentiable at $x_0$ if and only if given any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\|x\|_X < \delta \quad \text{implies} \quad \|Rf(x, x_0)\|_Y < \varepsilon\|x\|_X.$$

*Remark 5.4* $Df(x_0)$ is unique if it exists.

*Remark 5.5* If $f$ is Fréchet differentiable at $x_0$, then $Df(x_0)x$ can be computed as follows

$$Df(x_0)x = \lim_{\varepsilon \to 0} \varepsilon^{-1}[f(x_0 + \varepsilon x) - f(x_0)] = D_\varepsilon f(x_0 + \varepsilon x)|_{\varepsilon=0}.$$

**Definition 5.5** We say that $f : U \to Y$ is differentiable on $U$ if and only if $Df(x)$ exists for all $x \in U$.

$f$ is continuously differentiable on $U$ if and only if it is differentiable on $U$ and $Df(x)$ is a continuous map from $X$ into $\mathcal{L}(X, Y)$.

The set of continuously differentiable functions from $U$ to $Y$ is denoted by $C^1(U, Y)$.

*Remark 5.6* Since $\mathcal{L}(X, Y)$ is a Banach space, it is possible to define derivatives of arbitrary order. For example, we say that $f \in C^1(U, Y)$ is twice continuously differentiable on $U$ if $Df \in C^1(U, \mathcal{L}(X, Y))$. Thus, for each $x \in U$,

$$D^2 f(x) \in \mathcal{L}(X, \mathcal{L}(X, Y)).$$

This process can be continued inductively. The set of $k$ times continuously differentiable functions from $U$ to $Y$ is denoted by $C^k(U, Y)$.

The following results provide useful tools in establishing the differentiability of the functions that we will encounter in the sequel.

**Theorem 5.2** *Let $X$ and $Y$ be Banach spaces. Suppose that $A \in \mathcal{L}(X, Y)$. Then $f(x) = Ax$ is Fréchet differentiable on $X$ and $Df(x_0)x = Ax$, for all $x_0 \in X$.*

*Proof.* For any $x_0, x \in X$,

$$Rf(x, x_0) = f(x + x_0) - f(x_0) - Ax = 0. \qquad \square$$

**Lemma 5.2** *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be $C^1$. Define*

$$RF(y, y_0) = F(y) - F(y_0) - DF(y_0)y.$$

*For every $r > 0$, $\varepsilon > 0$, there is a $\delta > 0$ such that for all $\|y_0\| \le r$, we have that*

$$\|y\| < \delta \quad \textit{implies} \quad \|RF(y, y_0)\| < \varepsilon\|y\|.$$

*Remark 5.7* The point of this lemma is that $\delta$ can be chosen *uniformly* for $\|y_0\| \le r$, when $F$ is $C^1$.

*Proof.* Since $F$ is $C^1$, we may write

$$\begin{aligned} RF(y, y_0) &= \int_0^1 \frac{d}{d\sigma} F(\sigma y + y_0) d\sigma - DF(y_0)y \\ &= \int_0^1 [DF(\sigma y + y_0) - DF(y_0)]\, y\, d\sigma. \end{aligned}$$

The derivative $DF$ is continuous, so on the compact set $\overline{B}_{r+1}(0) \subset \mathbb{R}^n$, $DF$ is uniformly continuous. Given $\varepsilon > 0$, there is a $\delta > 0$ such that for all $y_1, y_2 \in \overline{B}_{r+1}(0)$,

$$\|y_1 - y_2\| < \delta \quad \text{implies} \quad \|DF(y_1) - DF(y_2)\| < \varepsilon.$$

So if $\|y_0\| \le r$, $\|y\| < \delta \le 1$, and $0 \le \sigma \le 1$, we have

$$\|DF(\sigma y + y_0) - DF(y_0)\| < \varepsilon,$$

and as a consequence, for all $\|y_0\| \le r$,

$$\|y\| < \delta \quad \text{implies} \quad \|RF(y, y_0)\| < \varepsilon\|y\|.$$

**Theorem 5.3** *Let $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Let $X = C_b^0(\mathcal{F}, \mathbb{R}^n)$ with the sup-norm. The map $\phi \mapsto f(\phi) = F \circ \phi$ is a $C^1$ map from $X$ into $X$, and $[Df(\phi_0)\phi](x) = DF(\phi_0(x))\phi(x)$.*

*Proof.* Define $RF(y, y_0)$ as in Lemma 5.2. Let $\phi_0 \in X$, and set $r = \|\phi_0\|_\infty$. Let $\varepsilon > 0$ be given, and choose $\delta > 0$ according to Lemma 5.2. Take any $\phi \in X$ with $\|\phi\|_\infty < \delta$. Notice that

$$\phi_0(x) \in \overline{B}_r(0) \quad \text{and} \quad \phi(x) \in B_\delta(0), \quad \text{for all} \quad x \in \mathcal{F}.$$

We have

$$\|Rf(\phi, \phi_0)\|_\infty = \sup_{x\in\mathcal{F}} \|RF(\phi(x), \phi_0(x))\| < \sup_{x\in\mathcal{F}} \varepsilon\|\phi(x)\| = \varepsilon\|\phi\|_\infty$$

This proves that $f$ is Fréchet differentiable at any $\phi_0 \in X$ together with the formula $[Df(\phi_0)\phi](x) = DF(\phi_0(x))\phi(x)$.

The map $\phi_0 \mapsto Df(\phi_0)$ is continuous from $X$ into $\mathcal{L}(X, X)$ because $DF$ is uniformly continuous on compact sets in $\mathbb{R}^n$. Thus, $f \in C^1(X, X)$.

□

**Theorem 5.4 (Chain Rule in Banach Space).** *Let $X$, $Y$, $Z$ be Banach spaces. Let $U \subset X$ and $V \subset Y$ be open. Suppose that $g : U \to V$ is Fréchet differentiable at $x_0 \in X$, and $f : V \to Z$ is Fréchet differentiable at $y_0 = f(x_0) \in V$. Then $f \circ g : U \to Z$ is Fréchet differentiable at $x_0$ and $D(f \circ g)(x_0) = Df(g(x_0))Dg(x_0)$.*

*If $f$ is continuously differentiable on $V$ and $g$ is continuously differentiable on $U$, then $f \circ g$ is continuously differentiable on $U$.*

*Proof.* Define

$$Rf(y, y_0) = f(y + y_0) - f(y_0) - Df(y_0)y$$
$$Rg(x, x_0) = g(x + x_0) - g(x_0) - Dg(x_0)x$$

and

$$R(f \circ g)(x, x_0) = f \circ g(x + x_0) - f \circ g(x_0) - Df(g(x_0))Dg(x_0)x.$$

Let $\varepsilon > 0$ be given. Choose $\mu > 0$ such that

$$\|Df(y_0)\|\mu < \varepsilon/2,$$

and then choose $\zeta > 0$ such that

$$\zeta(\|Dg(x_0)\| + \mu) < \varepsilon/2.$$

Since $f$ is differentiable at $y_0$ we may choose $\eta > 0$ such that

$$\|y\| < \eta \quad \text{implies} \quad \|Rf(y, y_0)\| < \eta\|y\|.$$

Finally, $g$ is differentiable at $x_0$, so we may choose $\delta > 0$ such that

$$\|x\| < \delta \quad \text{implies} \quad \|Rg(x, x_0)\| < \mu\|x\|, \quad (\|Dg(x_0)\| + \mu)\delta < \eta.$$

Notice that we may write

$$R(f \circ g)(x, x_0) = Rf(y, y_0) + Df(y_0)Rg(x, x_0),$$

with

$$y = Dg(x_0)x + Rg(x, x_0).$$

Now, $\|x\| < \delta$ implies that

$$\|y\| \le (\|Dg(x_0)\| + \mu)\|x\| < (\|Dg(x_0)\| + \mu)\delta < \eta.$$

Therefore, $\|x\| < \delta$ implies that

$$\begin{aligned}
\|R(f \circ g)(x, x_0)\| &\le \|Rf(y, y_0)\| + \|Df(y_0)Rg(x, x_0)\| \\
&\le \zeta\|y\| + \|Df(y_0)\| \, \|Rg(x, x_0)\| \\
&\le \zeta(\|Dg(x_0)\| + \mu) \, \|x\| + \|Df(y_0)\|\mu \, \|x\| \\
&< \varepsilon/2 \, \|x\| + \varepsilon/2 \, \|x\| \\
&= \varepsilon \, \|x\|.
\end{aligned}$$

This shows that $f \circ g$ is Fréchet differentiable at $x_0$. □

*Example 5.1* Here is a representative example that we will often encounter. Let $X = C_b^0([0, T], \mathbb{R}^n)$, with the sup norm. Let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Let $A$ be an $n \times n$ matrix. For $x \in X$, define a map $x \mapsto T(x)$ by the formula

$$T(x)(t) = \int_0^t \exp A(t - s) \; f(x(s))ds, \quad 0 \le t \le T.$$

Then $T : X \to X$. Notice that $T = F \circ G$, with

$$G(x) = f \circ x$$

and

$$F(y)(t) = \int_0^t \exp A(t - s)y(s)ds.$$

By Theorem 5.3, $G : X \to X$ is $C^1$ and $DG(x)(y)$ is the element $[Df \circ x]y \in X$. The map $F$ is a bounded linear transformation on $X$, so by Theorem 5.2, it is $C^1$ and for every $x, y \in X$,

$$DF(x)y = F(y).$$

Thus, by the chain rule, Theorem 5.4, $T : X \to X$ is $C^1$ and

$$DT(x)y = DF(G(x))DG(x)y = F(DG(x)y) = F(Df \circ x\ y).$$

Explicitly, we have

$$DT(x)y(t) = \int_0^t \exp A(t-s)\ Df(x(s))y(s)ds.$$

## 5.3 The Contraction Mapping Principle in Banach Space

Let $X$ be a Banach space. According to Definition 5.1, $X$ is a complete metric space with the distance function $d(x, y) = \|x - y\|_X$, $x, y \in X$. Any closed subset of $X$ is also a complete metric space. In this context, the Contraction Mapping Principle (Theorem A.1) says:

**Theorem 5.5 (Contraction Mapping Principle in Banach space).** *Let $V \subset X$ be a closed subset. Let $T : V \to V$ be a contraction mapping,* i.e. *there exists a constant $0 < \alpha < 1$ such that*

$$\|T(x) - T(y)\|_X \le \alpha\|x - y\|_X,$$

*for all $x, y \in V$. Then $T$ has a unique fixed point $x \in V$,* i.e. $T(x) = x$.

*Proof.* The set $V$ is closed, so $(V, d)$ is a complete metric space. Apply the standard Contraction Mapping Principle. □

We want to generalize this result to the situation where the mapping, and hence the corresponding fixed point, depends on parameters.

**Definition 5.6** Let $X$ and $Y$ be Banach spaces. Let $U \subset X$ and $V \subset Y$ be open sets. A mapping $T : U \times \overline{V} \to \overline{V}$ is called a uniform contraction if there is a constant $0 < \alpha < 1$ such that

$$\|T(x, y_1) - T(x, y_2)\|_Y \le \alpha\|y_1 - y_2\|_Y,$$

for all $x \in U$ and $y_1, y_2 \in \overline{V}$. Notice that the contraction number $\alpha$ is uniform for $x$ throughout $U$.

An application of the contraction mapping principle shows that if $T : U \times \overline{V} \to \overline{V}$ is a uniform contraction, then for every $x \in U$ there is a unique fixed point $g(x) \in \overline{V}$, i.e. a unique solution of the equation

$$T(x, g(x)) = g(x).$$

The next result shows that if the mapping $T$ is continuous or differentiable then the fixed point $g(x)$ depends continuously or differentiably on $x$. Following the approach of Chow and Hale [3], this will be the key step in proving the Implicit Function Theorem in the next section.

**Theorem 5.6 (Uniform Contraction Principle).** *Let $T$ be a uniform contraction, and let $g : U \to \overline{V}$ be the corresponding fixed point. If $T \in C^k(U \times \overline{V}, Y)$ for $k = 0$ or 1, then $g \in C^k(U, Y)$.*

*Proof.* The case $k = 0$ is easy. By the definition of $g$, the triangle inequality, and the uniform contraction hypothesis, we have

$$\begin{aligned}\|g(x+h) - g(x)\| &= \|T(x+h, g(x+h)) - T(x, g(x))\| \\ &\leq \|T(x+h, g(x+h)) - T(x+h, g(x))\| \\ &\quad + \|T(x+h, g(x)) - T(x, g(x))\| \\ &\leq \alpha\|g(x+h) - g(x)\| \\ &\quad + \|T(x+h, g(x)) - T(x, g(x))\|.\end{aligned}$$

Thus, since $\alpha < 1$, we get

$$\|g(x+h) - g(x)\| \leq \frac{1}{1-\alpha}\|T(x+h, g(x)) - T(x, g(x))\|.$$

But $T$ is assumed to be continuous, so

$$\lim_{h \to 0} \|g(x+h) - g(x)\| = 0,$$

i.e. $g$ is continuous at $x$.

Let's first look at the strategy for the case $k = 1$. Since $T(x, g(x)) = g(x)$, we would have by the Chain Rule, Theorem 5.4, *if $g$ were $C^1$* (remember that this is what we're trying to prove)

$$D_x T(x, g(x)) + D_y T(x, g(x)) Dg(x) = Dg(x).$$

Here, $D_{xT} = DT|_{X \times \{0\}}$ and $D_{yT} = DT|_{\{0\} \times Y}$. This inspires us to consider the operator equation

$$D_x T(x, g(x)) + D_y T(x, g(x)) M(x) = M(x). \tag{5.1}$$

We will first show that we can solve this for $M(x) \in \mathcal{L}(X, Y)$, and then we will show that $M(x) = Dg(x)$.

$T$ is assumed to be $C^1$, so for each $(x, y) \in U \times \overline{V}$, we have $D_y T(x, y) \in \mathcal{L}(Y, Y)$. Since $T$ is a uniform contraction, it can easily be shown that $\|D_y T(x, y)\| \leq \alpha < 1$. It follows from Lemma 5.1 that $I - D_y T(x, g(x))$ is invertible for all $x \in U$, its inverse depends continuously on $x$, and the inverse is bounded by $(1 - \alpha)^{-1}$. Thus, the solution of 5.1 is

$$M(x) = [I - D_y T(x, g(x))]^{-1} D_x T(x, g(x)) \in \mathcal{L}(X, Y), \tag{5.2}$$

and $M(x)$ depends continuously on $x \in U$.

Having constructed $M(x)$, it remains to show that $M(x) = Dg(x)$. Setting

$$Rg(x, h) \equiv g(x + h) - g(x) - M(x)h,$$

we are going to prove that for each fixed $x \in X$, given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\|h\| < \delta \quad \text{implies} \quad \|Rg(x, h)\| < \varepsilon \|h\|.$$

To simplify the notation in the remainder of the proof, define

$$\begin{aligned} \Delta g(x, h) &= g(x + h) - g(x) \\ RT((x, y), (\dot{x}, \dot{y})) &= T(x + \dot{x}, y + \dot{y}) - T(x, y) \\ &\quad - D_x T(x, y)\dot{x} - D_y T(x, y)\dot{y}. \end{aligned}$$

Then as above,

$$\begin{aligned} \Delta g(x, h) &= T(x + h, g(x + h)) - T(x, g(x)) \\ &= T(x + h, g(x) + \Delta g(x, h)) - T(x, g(x)) \\ &= D_x T(x, g(x))h + D_y T(x, g(x))\Delta g(x, h) \\ &\quad + RT((x, g(x)), (h, \Delta g(x, h))). \end{aligned} \tag{5.3}$$

Since $I - D_y T(x, g(x))$ is invertible, we have from (5.3) and (5.2)

$$\begin{aligned} \Delta g(x, h) &= [I - D_y T(x, g(x))]^{-1} D_x T(x, g(x))h \\ &\quad + [I - D_y T(x, g(x))]^{-1} RT((x, g(x)), (h, \Delta g(x, h))) \\ &= M(x)h + [I - D_y T(x, g(x))]^{-1} RT((x, g(x)), (h, \Delta g(x, h))). \end{aligned}$$

Therefore, we obtain

$$Rg(x, h) = [I - D_y T(x, g(x))]^{-1} RT((x, g(x)), (h, \Delta g(x, h))). \tag{5.4}$$

Since $T$ is $C^1$, for any $\eta > 0$ (to be selected below), there is a $\mu > 0$, such that

$$\|\dot{x}\| + \|\dot{y}\| < \mu \quad \text{implies} \quad \|RT((x, g(x))(\dot{x}, \dot{y}))\| < \eta(\|\dot{x}\| + \|\dot{y}\|).$$

Next, since $\Delta g(x, h)$ is continuous in $h$ and $\Delta g(x, 0) = 0$, we can find $\delta > 0$ such that $\delta < \mu/2$ and

$$\|h\| < \delta \quad \text{implies} \quad \|\Delta g(x, h)\| < \mu/2.$$

So for $\|h\| < \delta$, we have $\|h\| + \|\Delta g(x, h)\| < \mu/2 + \mu/2 = \mu$.

Combining the last two paragraphs, we see that if $\|h\| < \delta$, then

$$\|RT((x, g(x)), (h, \Delta g(x, h)))\| < \eta(\|h\| + \|\Delta g(x, h)\|). \tag{5.5}$$

So (5.4) and (5.5) imply that

$$\|Rg(x, h)\| \leq \frac{\eta}{1 - \alpha}[\|h\| + \|\Delta g(x, h)\|].$$

Now $\Delta g(x, h) = M(x)h + Rg(x, h)$, from which we obtain

$$\|Rg(x, h)\| \leq \frac{\eta}{1 - \alpha}[\|h\| + \|M(x)\|\|h\| + \|Rg(x, h)\|].$$

If we select $\eta$ so that $\eta/(1 - \alpha) < 1/2$, this implies that

$$\|Rg(x, h)\| \leq \frac{2\eta[1 + \|M(x)\|]}{1 - \alpha}\|h\|, \quad \text{for all} \quad \|h\| < \delta.$$

To conclude, we only need to further restrict $\eta$ so that

$$\frac{2\eta[1 + \|M(x)\|]}{1 - \alpha} < \varepsilon. \qquad \square$$

## 5.4 The Implicit Function Theorem in Banach Space

The next result generalizes the standard Implicit Function Theorem A.2 to the Banach space setting. It will be a fundamental tool in obtaining many of the results to follow.

**Theorem 5.7** *Suppose that $X$, $Y$, and $Z$ are Banach spaces. Let $F : \mathcal{O} \to Z$ be a $C^1$ mapping on an open set $\mathcal{O} \subset X \times Y$. Assume that there exists a point $(x_0, y_0) \in \mathcal{O}$ such that $F(x_0, y_0) = 0$ and $D_y F(x_0, y_0)$ has a bounded inverse in $\mathcal{L}(Z, Y)$. Then there exist a neighborhood $U \times V$ of $(x_0, y_0)$ contained in $\mathcal{O}$ and a $C^1$ mapping $g : U \to V$ such that*

$$y_0 = g(x_0) \quad \textit{and} \quad F(x, g(x)) = 0,$$

*for all* $x \in U$. *If* $F(x, y) = 0$ *for* $(x, y) \in U \times V$, *then* $y = g(x)$.

*Proof.* By performing a translation if necessary, we may assume without loss of generality that $(x_0, y_0) = (0, 0)$. We shall use the norm $\|(x, y)\|_{X \times Y} = \|x\|_X + \|y\|_Y$.

Let $L = D_y F(0, 0)^{-1}$, and define the $C^1$ map $G : \mathcal{O} \to Y$ by

$$G(x, y) = y - LF(x, y).$$

Notice that $G(x, y) = y$ if and only if $F(x, y) = 0$. We also have

$$G(0, 0) = 0 \quad \text{and} \quad D_y G(0, 0) = 0. \tag{5.6}$$

Let $A = D_x G(0, 0)$. Using the fact that $G$ is differentiable at $(0, 0)$ with (5.6), there exists $\delta_1 > 0$ such that

$$(x, y) \in B_{\delta_1}(0) \times B_{\delta_1}(0) \quad \text{implies} \quad \|G(x, y) - Ax\| < (1/2)\|(x, y)\|. \tag{5.7}$$

Since $D_y g$ is continuous, by (5.6) there exists $\delta_2 > 0$ such that

$$(x, y) \in B_{\delta_2}(0) \times B_{\delta_2}(0) \quad \text{implies} \quad \|D_y G(x, y)\| < 1/2. \tag{5.8}$$

Define

$$U = B_\mu(0) \quad \text{and} \quad V = B_\nu(0)$$

with

$$\nu = \min\{\delta_1, \delta_2\} \quad \text{and} \quad \mu = \min\{\nu, \frac{\nu}{2\|A\| + 1}\}.$$

By (5.7), we have

$$(x, y) \in U \times \overline{V} \quad \text{implies} \quad \|G(x, y)\| \le (\|A\| + 1/2)\|x\| + (1/2)\|y\| < \nu. \tag{5.9}$$

The fundamental theorem of calculus holds for Banach space-valued functions, so by the chain rule and (5.8), we obtain that

$$(x, y_1), (x, y_2) \in U \times \overline{V}$$

implies

$$\begin{aligned}
\|G(x, y_1) - G(x, y_2)\| &= \|\int_0^1 \frac{d}{d\sigma} G(x, \sigma y_1 + (1 - \sigma) y_2) d\sigma\| \\
&= \|\int_0^1 D_y G(x, \sigma y_1 + (1 - \sigma) y_2)(y_1 - y_2) d\sigma\|
\end{aligned}$$

$$\leq \sup_{U\times\overline{V}} \|D_y G(x,y)\| \, \|y_1 - y_2\|$$
$$\leq (1/2)\|y_1 - y_2\|.$$

We have thus shown that $G : U \times \overline{V} \to V \subset \overline{V}$ is a uniform contraction.

According to the Uniform Contraction Principle, Theorem 5.6, there is a $C^1$ map $g : U \to \overline{V}$ such that

$$G(x, g(x)) = g(x),$$

for all $x \in U$. Moreover, given $x \in U$, $y = g(x)$ is the unique point in $\overline{V}$ point such that $G(x, y) = y$. Note that strict inequality in (5.9) gives $g : U \to V$. □

**Corollary 5.1** *If, in addition to the hypotheses of Theorem 5.7, we have* $f \in C^k(\mathcal{O}, Z)$*, for some* $k > 1$*, then* $g \in C^k(U, V)$*.*

*Proof.* We continue with the notation used in the proof of the previous theorem.

The neighborhoods $U$, $V$ have the property that

$$(x, y) \in U \times V \quad \text{implies} \quad \|D_y G(x, y)\| < 1/2.$$

Since $D_y G(x, y) = I - L D_y F(x, y)$, Lemma 5.1 says that $L D_y F(x, y)$ is invertible for all $(x, y) \in U \times V$. Thus, we have that $D_y F(x, y)$ is also invertible for all $(x, y) \in U \times V$.

Since

$$F(x, g(x)) = 0 \quad \text{for all} \quad x \in U,$$

we obtain by the Chain Rule

$$D_x F(x, g(x)) + D_y F(x, g(x)) D_x g(x) = 0 \quad \text{for all} \quad x \in U.$$

Now by the invertibility of $D_y F(x, y)$ on $U \times V$, we get

$$D_x g(x) = -D_y F(x, g(x))^{-1} D_x F(x, g(x)) \quad \text{for all} \quad x \in U. \tag{5.10}$$

If $F \in C^k(\mathcal{O}, Z)$, then the mapping

$$(x, y) \mapsto D_y F(x, y)^{-1} D_x F(x, y)$$

belongs to $C^{k-1}(U \times V, \mathcal{L}(X, Y))$. So if $g \in C^1(U, V)$, then by the Chain Rule, the right hand side of (5.10) lies in $C^1(U, \mathcal{L}(X, Y))$. It follows that $g \in C^2(U, V)$. Now continue inductively. □

## 5.5 The Liapunov–Schmidt Method

Let $X$, $Y$, and $\Lambda$ be Banach spaces. Suppose that $A : X \to Y$ is a bounded linear map and that $N : X \times \Lambda \to Y$ is a $C^1$ map such that

$$N(0, 0) = 0, \quad D_x N(0, 0) = 0.$$

We are interested in finding nontrivial solutions $x \in X$ of the nonlinear equation

$$Ax = N(x, \lambda). \tag{5.11}$$

If $A \in \mathcal{L}(X, Y)$ is a bijection, then by the Open Mapping Theorem 5.1, it is an isomorphism, that is, $A^{-1} : Y \to X$ is bounded. The Implicit Function Theorem 5.7 ensures the existence of a $C^1$ function $\phi : U \to V$ from a neighborhood of the origin in $U \subset \Lambda$ to a neighborhood of the origin in $V \subset X$ such that

$$A\phi(\lambda) = N(\phi(\lambda), \lambda), \quad \lambda \in U.$$

The Liapunov–Schmidt technique deals with the situation where $A$ is *not* an isomorphism.

Let $K \subset X$ be the nullspace of $A$ and let $R \subset Y$ be the range. Assume that

(i) There exists a closed subspace $M \subset X$ such that $X = M \oplus K$, and
(ii) $R$ is closed.

It follows from the Open Mapping Theorem that the restriction of $A$ to $M$ is an isomorphism onto $R$. These assumptions are satisfied, for example, when $K$ is finite dimensional and $R$ has finite co-dimension. Such an operator is called *Fredholm*.

Suppose that $x$ is some solution of (5.11). Write $x = u + v \in M \oplus K$, $Y = R \oplus S$, and let $P_R$ be the projection of $Y$ onto $R$ along $S$. Then

$$\begin{aligned} Au = P_R N(u + v, \lambda) = F(u, v, \lambda) \\ (I - P_R) N(u, v, \lambda) = 0. \end{aligned} \tag{5.12}$$

Turn this around. By the Implicit Function Theorem 5.7, there exists a solution $u(v, \lambda) \in M$ of (5.12), for $(v, \lambda)$ in a neighborhood of the origin in $K \times \Lambda$. So now we want to solve

$$(I - P_R) N(u(v, \lambda) + v, \lambda) = 0. \tag{5.13}$$

This is called the *bifurcation equation*. We can then attempt to solve (5.13) for $\lambda$ in terms of $v$. If this is possible, then we get a family of solutions of (5.11) in the form

$$x(v) = u(v, \lambda(v)) + v.$$

A typical situation might be the case where $A$ is Fredholm and that the parameter space $\Lambda$ is finite dimensional. Then (5.13) is a system of finitely many equations in a finite number of unknowns. We can then attempt to solve it using the standard Implicit Function Theorem.

## 5.6 Exercises

**Exercise 5.1** Let $X$ be a Banach space. Prove that if $x_k \to x$ in $X$, then $\|x_k\| \to \|x\|$ in $\mathbb{R}$.

**Exercise 5.2** Let $X$ and $Y$ be Banach spaces. Prove that $\mathcal{L}(X, Y)$ is a Banach space with the operator norm.

**Exercise 5.3** Let $X$ be a Banach space. Prove that $\|AB\| \leq \|A\|\|B\|$, for every $A, B \in \mathcal{L}(X, X)$. (Therefore, $\mathcal{L}(X, X)$ is an algebra.)

**Exercise 5.4** Prove that $\mathrm{Lip}(\mathbb{R}^n, \mathbb{R}^m)$ is a Banach space.

**Exercise 5.5** Let $X$ be a Banach space. Let $A \in \mathcal{L}(X, X)$. Show that

$$\|A\| = \sup_{0 \neq \|x\| \leq \delta} \frac{\|Ax\|}{\|x\|}.$$

**Exercise 5.6** Let $X$ be a Banach space. Suppose that $f \in C^1(X, X)$ is Lipschitz continuous, i.e. there exists $0 < \alpha$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|,$$

for all $x, y \in X$. Prove that $\|Df(x)\| \leq \alpha$, for all $x \in X$.

**Exercise 5.7** Let $A$ be an $n \times n$ matrix over $\mathbb{R}$ all of whose eigenvalues satisfy $\operatorname{Re}\lambda < 0$. Suppose that $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with $f(0) = 0$ and $Df(0) = 0$. Use the Implicit Function Theorem 5.7 to prove that there exists an $r > 0$ such that the integral equation

$$y(t) = \exp At\ y_0 + \int_0^t \exp A(t - s)\ f(y(s))\, ds$$

has a solution $y \in C_b^0([0, \infty), \mathbb{R}^n)$, for all $y_0 \in \mathbb{R}^n$ with $\|y_0\| < r$. (Compare with Theorem 3.11.)

**Exercise 5.8** State and prove a version of the Inverse Function Theorem for Banach spaces. (Use the Implicit Function Theorem.)

**Exercise 5.9** Let $X$ and $Y$ be Banach spaces. Suppose that $L \in C^1(X, \mathcal{L}(Y, Y))$, and for some $x_0 \in X$, $L(x_0)$ has an inverse in $\mathcal{L}(Y, Y)$. Prove that there is a neighborhood $U \subset X$ of $x_0$ on which $L(x)^{-1}$ is defined and $L^{-1} \in C^1(U, \mathcal{L}(Y, Y))$.

# Chapter 6
# Dependence on Initial Conditions and Parameters

## 6.1 Smooth Dependence on Initial Conditions

We have seen in Theorem 3.5 that solutions of the initial value problem depend continuously on initial conditions. We will now show that this dependence is as smooth as the vector field.

**Theorem 6.1.** *Let $\Omega \subset \mathbb{R}^{1+n}$ be an open set, and suppose that $f : \Omega \to \mathbb{R}^n$ is $C^1$.*
*For $(s, p) \in \Omega$, the unique local solution $x(t, s, p)$ of the initial value problem*

$$\frac{d}{dt}x(t, s, p) = f(t, x(t, s, p)), \quad x(s, s, p) = p. \tag{6.1}$$

*is $C^1$ in its open domain of definition*

$$D = \{(t, s, p) \in \mathbb{R}^{2+n} : \alpha(s, p) < t < \beta(s, p), \ (s, p) \in \Omega\}.$$

*The derivative matrix $D_p x(t, s, p)$ satisfies the so-called linear variational equation*

$$\frac{d}{dt}D_p x(t, s, p) = D_x f(t, x(t, s, p))D_p x(t, s, p), \quad D_p x(s, s, p) = I. \tag{6.2}$$

*Also,*

$$\frac{\partial x}{\partial s}(t, s, p) = -D_p x(t, s, p) f(s, p). \tag{6.3}$$

*Proof.* Suppose, temporarily, that we have shown that $x(t, s, p) \in C^1(D)$. Then (6.2) follows immediately by taking the derivative of (6.1) with respect to $p$. Next, use the properties of the flow, Lemma 3.5, to write

$$x(t, s, p) = x(t, \tau, x(\tau, s, p)) \quad t, \tau \in I(s, p).$$

T. C. Sideris, *Ordinary Differential Equations and Dynamical Systems*,
Atlantis Studies in Differential Equations 2, DOI: 10.2991/978-94-6239-021-8_6,

Take the derivative of this with respect to $\tau$ to get

$$0 = \frac{\partial x}{\partial s}(t, \tau, x(\tau, s, p)) + D_p x(t, \tau, x(\tau, s, p)) \frac{\partial x}{\partial t}(\tau, s, p).$$

From this and the ODE (6.1), we obtain

$$\frac{\partial x}{\partial s}(t, \tau, x(\tau, s, p)) = -D_p x(t, \tau, x(\tau, s, p)) f(\tau, x(\tau, s, p)).$$

Equation (6.3) follows by letting $\tau = s$.

Since we already know that the solution is continuously differentiable in $t$, we must only establish continuous differentiability of $x(t, s, p)$ in $(s, p)$. We are going to do this by a uniqueness argument. The flow $x(t, s, p)$ satisfies the standard integral equation. We will show that the Implicit Theorem Function guarantees this equation has a unique $C^1$ solution and then invoke uniqueness. We now proceed to set this up precisely.

For an arbitrary point $(s_0, p_0) \in \Omega$, let $x_0(t) = x(t, s_0, p_0)$ be the corresponding solution to the initial value problem (6.1), defined on the maximal interval $I(s_0, p_0) = (\alpha(s_0, p_0), \beta(s_0, p_0))$. Choose an arbitrary closed subinterval $J = [a, b]$ with

$$\alpha(s_0, p_0) < a < s_0 < b < \beta(s_0, p_0).$$

Define the compact set

$$K = \{(t, x(t, s_0, p_0)) \in \Omega : t \in J\}.$$

By the Covering Lemma 3.2, there exist numbers $\delta, \rho > 0$ and a compact set $K' \subset \Omega$ such that for every $(s, p) \in K$, the cylinder

$$\mathcal{C}(s, p) = \{(s', p') \in \mathbb{R}^{1+n} : |s' - s| \le \delta,\ \|p' - p\| \le \rho\}$$

satisfies

$$\mathcal{C}(s, p) \subset K'.$$

Define the Banach spaces $X = \mathbb{R}^{1+n}$ and $Y = Z = C(J, \mathbb{R}^n)$ with the sup norm. Let $U = (a, b) \times \mathbb{R}^n$ and

$$V = \{x \in Y : \|x - x_0\| = \sup_J \|x(t) - x_0(t)\| < \rho\} = B_\rho(x_0) \subset Y.$$

We have that $U \subset X$ and $V \subset Y$ are open.

Suppose that $x \in V$. Then for any $\sigma \in J$, we have that

$$\|x(\sigma) - x_0(\sigma)\| < \rho,$$

and so, $(\sigma, x(\sigma)) \in \mathcal{C}(\sigma, x_0(\sigma)) \subset K' \subset \Omega$, for any $\sigma \in J$. Therefore, the operator

$$T : U \times V \to Z$$

given by

$$(s, p, x) \mapsto T(s, p, x)(t) = x(t) - p - \int_s^t f(\sigma, x(\sigma))d\sigma, \quad t \in J,$$

is well-defined. It follows from Theorems 5.2, 5.3, and 5.4 that $T$ is $C^1$. In particular, we have that $D_x T(s, p, x)$ is a bounded linear map from $Y$ to $Z(= Y)$ which takes $y \in Y$ to the function $D_x T(s, p, x)[y] \in Z$ whose value at a point $t \in J$ is

$$D_x T(s, p, x)[y](t) = y(t) - \int_s^t A(\sigma)y(\sigma)d\sigma \quad \text{with} \quad A(\sigma) = D_x f(\sigma, x(\sigma)).$$

(It is here that we are using the assumption $f \in C^1(\Omega)$.)

Since $x_0(t) = x(t, s_0, p_0)$ solves (6.1) we have that

$$T(s_0, p_0, x_0) = 0.$$

Now we claim that $D_x T(s_0, p_0, x_0)$ is invertible as a linear map from $Y$ to $Z$. Let $g \in Z$. The equation $D_x T(s_0, p_0, x_0)[y] = g$ can be written explictly as

$$y(t) - \int_{s_0}^t A(\sigma)y(\sigma)d\sigma = g(t). \tag{6.4}$$

Letting $u(t) = y(t) - g(t)$, this is equivalent to

$$u(t) = \int_{s_0}^t A(\sigma)[u(\sigma) + g(\sigma)]d\sigma.$$

Notice that the right-hand side is $C^1$ in $t$. So this is equivalent to

$$u'(t) = A(t)[u(t) + g(t)], \quad t \in J, \quad u(s_0) = 0.$$

We can represent the unique solution of this initial value problem using the Variation of Parameters formula, Theorem 4.1. Let $W(t, s)$ be the state transition matrix for $A(t)$. Then

$$u(t) = \int_{s_0}^t W(t, \sigma)A(\sigma)g(\sigma)d\sigma.$$

Since $u = y - g$, this is equivalent to

$$y(t) = g(t) + \int_{s_0}^t W(t, \sigma)A(\sigma)g(\sigma)d\sigma. \tag{6.5}$$

We have shown that for every $g \in Z$, the Eq. (6.4) has a unique solution $y \in Y$ given by (6.5). This proves th $D_x T(s_0, p_0, x_0)$ is a bijection. Finally, from the formula (6.5) (or by the Open Mapping Theorem 5.1), we see that the inverse is a bounded map from $Z$ to $Y$.

By the Implicit Function Theorem 5.7, there are a neighborhood $U_0 \subset U$ of $(s_0, p_0)$, a neighborhood $V_0 \subset V$ of $x_0$, and a $C^1$ map $\phi : U_0 \to V_0$ such that $\phi(s_0, p_0) = x_0$ and $T(s, p, \phi(s, p)) = 0$ for all $(s, p) \in U_0$. It follows that $\phi(s, p)(t)$ solves (6.1) on the interval $J$. By the Uniqueness Theorem 3.3, we have that

$$\phi(s, p)(t) = x(t, s, p), \quad t \in J.$$

Therefore, $x(t, s, p)$ is $C^1$ as a function of $(s, p)$ on $J \times U_0$. Since, $(s_0, p_0)$ is an arbitrary point in $\Omega$, and $J$ is an arbitrary subinterval of $I(s_0, p_0)$, it follows that $x(t, s, p)$ is $C^1$ on all of $D$. □

**Corollary 6.1.** *If $f(t, x)$ is continuous on $\Omega \subset \mathbb{R}^{1+n}$ and $C^k$ in $x$, then $x(t, s, p)$ is $C^k$ in $(s, p)$ on its open domain of definition. (If $f(t, x)$ is in $C^k(\Omega)$, then $x(t, s, p)$ is $C^k$ in $(t, s, p)$ on its domain of definition.)*

*Proof.* If $f(t, x)$ is $C^k$ in $x$, then the map $T(s, p, x)$ defined in the proof of Theorem 6.1 belongs to $C^k(U \times V, Z)$. It follows by Corollary 5.1 that the function $\phi$ constructed there lies in $C^k(U_0, V_0)$. Since we have identified $\phi(s, p)(\cdot)$ with $x(\cdot, s, p)$, we conclude that the solution $x(\cdot, s, p)$ is $C^k$ in $(s, p)$. □

**Corollary 6.2.** *The derivative matrix satisfies*

$$\det D_p x(t, s, p) = \exp \int_s^t \sum_{j=1}^n \frac{\partial f_j}{\partial x_j}(\tau, x(\tau, s, p)) d\tau.$$

*Proof.* This follows by applying Theorem 4.2 to the variational equation:

$$\frac{d}{dt} D_p x(t, s, p) = D_x f(t, x(t, s, p)) D_p x(t, s, p),$$

and noting that

$$\operatorname{tr} D_x f = \sum_{j=1}^n \frac{\partial f_j}{\partial x_j}.$$

□

## 6.2 Continuous Dependence on Parameters

**Theorem 6.2.** *Let $f(t, x, \lambda)$ be continuous on an open set $\Omega \subset \mathbb{R}^{1+n+m}$ with values in $\mathbb{R}^n$. Assume that $f(t, x, \lambda)$ is locally Lipschitz with respect to $(x, \lambda)$. Then given $(t_0, x_0, \lambda) \in \Omega$, the initial value problem*

$$x' = f(t, x, \lambda), \quad x(t_0) = x_0$$

*has a unique local solution* $x(t, t_0, x_0, \lambda)$ *on a maximal interval of definition*

$$I(t_0, x_0, \lambda) = (\alpha(t_0, x_0, \lambda), \beta(t_0, x_0, \lambda))$$

*where*

*(i)* $\alpha(t_0, x_0, \lambda)$ *is upper semi-continuous.*
*(ii)* $\beta(t_0, x_0, \lambda)$ *is lower semi-continuous.*
*(iii)* $x(t, t_0, x_0, \lambda)$ *is continuous on its open domain of definition*

$$\{(t, t_0, x_0, \lambda) : t \in I(t_0, x_0, \lambda);\ (t_0, x_0, \lambda) \in \Omega\}.$$

*Proof.* Here's the trick: turn the parameter $\lambda$ into a dependent variable and use the old continuous dependence result, Theorem 3.5. Define a new vector

$$y = \begin{bmatrix} x \\ \lambda \end{bmatrix} \in \mathbb{R}^{n+m},$$

and a new vector field

$$F(t, y) = \begin{bmatrix} f(t, x, \lambda) \\ 0 \end{bmatrix},$$

on $\Omega$. This vector field satisfies the hypothesis of the Picard existence and uniqueness theorem. Apply Theorem 6.1 to the so-called *suspended system*

$$y' = F(t, y), \quad y(t_0) = y_0 = \begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix}.$$

Since the vector field $F$ is 0 in its last $m$ components, the last $m$ components of $y$ are constant and, hence, equal to $\lambda_0$. Extraction of the first $n$ components yields the desired result for $x(t, t_0, x_0, \lambda_0)$. □

*Remark 6.1.* This result is still true even if $f(t, x, \lambda)$ is not locally Lipschitz continuous in $\lambda$, however the easy proof no longer works.

**Corollary 6.3.** *If* $f(t, x, \lambda)$ *is in* $C^k(\Omega)$*, then* $x(t, t_0, x_0, \lambda)$ *is in* $C^k$ *on its open domain of definition.*

*Proof.* Use the trick above, and then apply Corollary 6.1. □

## 6.3 Exercises

**Exercise 6.1.** Find an expression for the solution $x(t, s, x_0, p, q)$ of the initial value problem

$$x' = (1 + t^2)^{-q/2}|x|^p, \quad x(s) = x_0, \quad p > 1, \quad q > 0$$

and its maximal existence time $\beta(s, x_0, p, q)$.

**Exercise 6.2.** Show that the fundamental matrix $X(t; \omega, \varepsilon)$ of the Mathieu equation

$$x'(t) = \begin{bmatrix} 0 & 1 \\ -(\omega^2 + \varepsilon \cos t) & 0 \end{bmatrix} x(t)$$

belongs to $C^\infty(\mathbb{R}^3)$.

# Chapter 7
# Linearization and Invariant Manifolds

## 7.1 Autonomous Flow at Regular Points

**Definition 7.1.** For $j = 1, 2$, let $\mathcal{O}_j \subset \mathbb{R}^n$ be open sets and let

$$f_j : \mathcal{O}_j \to \mathbb{R}^n,$$

be $C^1$ autonomous vector fields. Let $\Phi_t^{(j)}$, $j = 1, 2$, be the associated flows. We say that the two flows $\Phi_t^{(1)}$ and $\Phi_t^{(2)}$ are topologically conjugate if there exists a homeomorphism $\eta : \mathcal{O}_1 \to \mathcal{O}_2$ such that $\eta \circ \Phi_t^{(1)} = \Phi_t^{(2)} \circ \eta$.

If the map $\eta$ can be chosen to be a diffeomorphism, then we will say that the flows are diffeomorphically conjugate.

If $\Phi_t^{(1)}$ and $\Phi_t^{(2)}$ are conjugate, then the maximal interval of existence of $\Phi_t^{(1)}(p)$ is the same as for $\Phi_t^{(2)}(\eta(p))$, for every $p \in \mathcal{O}_1$.

Conjugacy is an equivalence relation.

**Definition 7.2.** A point $x \in \mathcal{O}$ is a regular point for an autonomous vector field $f$ if $f(x) \neq 0$.

**Theorem 7.1.** *Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set, and suppose that $f : \mathcal{O} \to \mathbb{R}^n$ is a $C^1$ autonomous vector field with a regular point at $x_0 \in \mathcal{O}$. Then there is a neighborhood $V \subset \mathcal{O}$ of $x_0$ on which $f$ is diffeomorphically conjugate to the flow of the constant vector field $e_n = (0, \dots, 0, 1) \in \mathbb{R}^n$ near the origin.*

*Proof.* Let $v_n = f(x_0)$, and choose $v_1, \dots, v_{n-1} \in \mathbb{R}^n$ such that the set $\{v_j\}_{j=1}^n$ is linearly independent. Let $M$ be the $n \times n$ matrix whose columns are $v_1, \dots, v_n$. Then $M$ is nonsingular. Also, let $\widetilde{M}$ be the $n \times n - 1$ matrix with columns $v_1, \dots, v_{n-1}$.

T. C. Sideris, *Ordinary Differential Equations and Dynamical Systems*,
Atlantis Studies in Differential Equations 2, DOI: 10.2991/978-94-6239-021-8_7,

Let $\Phi_t(q)$ be the flow of $f$. Given $p \in \mathbb{R}^n$, write $p = (\tilde{p}, p_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. For $p$ near the origin we may define the map

$$\eta(p) = \eta((\tilde{p}, p_n)) = \Phi_{p_n}(x_0 + \widetilde{M}\tilde{p}).$$

Then $\eta(0) = x_0$ and $D\eta(0) = M$ is nonsingular. By the Inverse Function Theorem A.3, there are neighborhoods $0 \in U \subset \mathbb{R}^n$ and $x_0 \in V \subset \mathbb{R}^n$ such that $\eta : U \to V$ is a diffeomorphism. Without loss of generality, we may assume that

$$U = \widetilde{U} \times (-\delta, \delta) \subset \mathbb{R}^{n-1} \times \mathbb{R}.$$

Note that the flow of the constant vector field $e_n$ is $\Psi_t(p) = p + te_n = (\tilde{p}, p_n + t)$. Therefore, for $p \in U$ and $-\delta < p_n + t < \delta$, we have

$$\begin{aligned}\eta \circ \Psi_t(p) &= \Phi_{p_n+t}(x_0 + \widetilde{M}\tilde{p}) \\ &= \Phi_t \circ \Phi_{p_n}(x_0 + \widetilde{M}\tilde{p}) \\ &= \Phi_t \circ \eta(p).\end{aligned}$$

□

*Remarks 7.1.* The neighborhood $U = \widetilde{U} \times (-\delta, \delta)$ is called a *flow box*.

## 7.2 The Hartman-Grobman Theorem

**Definition 7.3.** An $n \times n$ matrix is said to be hyperbolic if its eigenvalues have nonzero real parts.

Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set. Let $F : \mathcal{O} \to \mathbb{R}^n$ be a $C^1$ autonomous vector field. We say an equilibrium point $x_0 \in \mathcal{O}$ is hyperbolic if $DF(x_0)$ is hyperbolic.

Suppose that $F$ has an equilibrium point at $x_0 = 0$. Set $A = DF(0)$. Writing

$$F(x) = Ax + [F(x) - Ax] = Ax + f(x),$$

we have that $f(x)$ is $C^1$, $f(0) = 0$, and $D_x f(0) = 0$.

We are going to consider the flow of the vector field $Ax + f(x)$ in relation to the flow of the linear vector field $Ax$. The Hartman-Grobman theorem says that the two flows are topologically conjugate, if $A$ is hyperbolic. Before coming to a precise statement of this theorem, we need to define some spaces and set down some notation.

Define the Banach spaces

$$X = \{g \in C_b^0(\mathbb{R}^n, \mathbb{R}^n) : g(0) = 0\}$$

and

$$Y = \{f \in \text{Lip}\,(\mathbb{R}^n, \mathbb{R}^n) : f(0) = 0\}$$

with the usual norms:

$$\|g\|_X = \sup_x \|g(x)\| = \|g\|_\infty,$$

and

$$\|f\|_Y = \|f\|_\infty + \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|} = \|f\|_{\text{Lip}}.$$

Notice that $Y \subset X$ and $\|f\|_\infty \leq \|f\|_{\text{Lip}}$.

We assume that $f(x)$ lies in $Y$. This would appear to be a strong restriction. However, since ultimately we are only interested in the flow near the origin, the behavior of the vector field away from the origin is unimportant. We will return to this point in Theorem 7.5. One advantage of having $f \in Y$ is that the flow $\psi_t(p) = x(t, p)$ of $Ax + f(x)$ is then globally defined by Theorem 3.9, since $f$ is Lipschitz and bounded. Of course, the flow of the vector field $Ax$ is just $\phi_t(p) = \exp At\ p$.

**Theorem 7.2.** *(**Hartman-Grobman**). Let $A$ be hyperbolic, and suppose that $f \in Y$. Let $\phi_t$ be the flow of $Ax$, and let $\psi_t$ be the flow of $Ax + f(x)$.*

*There is a $\delta > 0$ such that if $\|f\|_Y < \delta$ then there exists a unique homeomorphism $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Lambda - I \in X$ and*

$$\Lambda \circ \phi_t = \psi_t \circ \Lambda.$$

*Moreover, the map $f \to \Lambda - I$ is continuous from $Y$ into $X$.*

*Remarks 7.2.* Here and below $I$ denotes the identity map on $\mathbb{R}^n$. Thus, $\Lambda - I$ is the function whose value at a point $x$ is $\Lambda(x) - x$.

*Remarks 7.3.* We will be using four different norms: the Euclidean norm of vectors in $\mathbb{R}^n$, the operator norm of $n \times n$ matrices, and the norms in the spaces $X$ and $Y$. The first two will be written $\|\cdot\|$ as usual, and the last two will be denoted $\|\cdot\|_\infty$ and $\|\cdot\|_{\text{Lip}}$, respectively.

*Remarks 7.4.* The following proof is due to Michael Crandall.

*Proof.* Let $P_s$, $P_u$ be the projections onto the stable and unstable subspaces of $A$. Given $(f, g) \in Y \times X$, define the function

$$\begin{aligned} T(f, g)(p) = &\int_{-\infty}^0 P_s\phi_{-\tau} \circ f \circ (I + g) \circ \phi_\tau(p) d\tau \\ &- \int_0^\infty P_u\phi_{-\tau} \circ f \circ (I + g) \circ \phi_\tau(p) d\tau. \end{aligned}$$

Recall that by Corollary 2.1, there are positive constants $C_0$, $\lambda$ such that

$$\|P_s\phi_t\| = \|P_s \exp At\| \leq C_0 \mathrm{e}^{-\lambda t}, \quad t \geq 0$$

and

$$\|P_u\phi_t\| = \|P_u \exp At\| \le C_0 e^{\lambda t}, \quad t \le 0.$$

Thus, we have that

$$\begin{aligned}\|T(f,g)(p)\| &\le \int_{-\infty}^{0} C_0 e^{\lambda\tau}\|f\|_\infty \, d\tau + \int_0^\infty C_0 e^{-\lambda\tau}\|f\|_\infty \, d\tau \\ &= (2C_0/\lambda)\|f\|_\infty \equiv C_1\|f\|_\infty,\end{aligned}$$

i.e. $T(f,g)$ is bounded. From this estimate, it also follows by the dominated convergence theorem that, for every $p \in \mathbb{R}^n$,

$$\lim_{y\to p} T(f,g)(y) = T(f,g)(p),$$

i.e. $T(f,g)$ is continuous. Note that $T(f,g)(0) = 0$, as a consequence of the fact that $f(0) = g(0) = 0$. This proves that $T(f,g) \in X$ and $\|T(f,g)\|_\infty \le C_1\|f\|_\infty$, for every $(f,g) \in Y \times X$.

In the same way, since $T$ is linear in $f$, we have for $(f_1,g), (f_2,g) \in Y \times X$

$$\|T(f_1,g) - T(f_2,g)\|_\infty = \|T(f_1 - f_2, g)\|_\infty \le C_1\|f_1 - f_2\|_\infty.$$

Finally, given $(f,g_1), (f,g_2) \in Y \times X$, we have the estimate

$$\|T(f,g_1) - T(f,g_2)\|_\infty \le C_1\|f\|_{\mathrm{Lip}}\|g_1 - g_2\|_\infty.$$

Therefore, we have that

$$T : Y \times X \to X$$

is continuous. Moreover, if $C_1 r < 1$, then

$$T : B_r(0) \times X \to X$$

is a uniform contraction.

By the Uniform Contraction Principle, Theorem 5.6, there is a continuous function

$$\Phi : B_r(0) \subset Y \to X$$

such that for every $f \in B_r(0)$, $\Phi(f)$ is the fixed point solution in $X$ of

$$\Phi(f) = T(f, \Phi(f)). \tag{7.1}$$

To simplify the notation, we now fix $f \in B_r(0)$, and set $g = \Phi(f)$, i.e. $g = T(f,g)$.

By (7.1) and the properties of the linear flow $\phi_t$ we have

$$\begin{aligned}
g \circ \phi_t(p) =& T(f,g) \circ \phi_t(p) \\
&= \int_{-\infty}^{0} P_s \phi_{-\tau} \circ f \circ (I+g) \circ \phi_{t+\tau}(p) d\tau \\
&\quad - \int_{0}^{\infty} P_u \phi_{-\tau} \circ f \circ (I+g) \circ \phi_{t+\tau}(p) d\tau \\
&= \int_{-\infty}^{t} P_s \phi_{t-\sigma} \circ f \circ (I+g) \circ \phi_{\sigma}(p) d\sigma \\
&\quad - \int_{t}^{\infty} P_u \phi_{t-\sigma} \circ f \circ (I+g) \circ \phi_{\sigma}(p) d\sigma \\
&= \int_{-\infty}^{0} P_s \phi_{t-\sigma} \circ f \circ (I+g) \circ \phi_{\sigma}(p) d\sigma \\
&\quad - \int_{0}^{\infty} P_u \phi_{t-\sigma} \circ f \circ (I+g) \circ \phi_{\sigma}(p) d\sigma \\
&\quad + \int_{0}^{t} \phi_{t-\sigma} \circ f \circ (I+g) \circ \phi_{\sigma}(p) d\sigma \\
&= \phi_t \circ g(p) + \int_{0}^{t} \phi_{t-\sigma} \circ f \circ (I+g) \circ \phi_{\sigma}(p) d\sigma.
\end{aligned}$$

Adding $\phi_t(p)$ to both sides, we obtain finally

$$(I+g) \circ \phi_t(p) = \phi_t \circ (I+g)(p) + \int_0^t \phi_{t-\sigma} \circ f \circ (I+g) \circ \phi_\sigma(p) d\tau.$$

Notice that this is precisely the integral equation satisfied by the solution of the nonlinear initial value problem with initial data $(I+g)(p)$, namely $\psi_t \circ (I+g)(p)$. By uniqueness, we conclude that

$$\psi_t \circ (I+g) = (I+g) \circ \phi_t, \tag{7.2}$$

for all $t \in \mathbb{R}$.

It remains to show that $\Lambda = I + g = I + \Phi(f)$ is a homeomorphism from $\mathbb{R}^n$ onto $\mathbb{R}^n$.

Suppose that $(I+g)(p) = (I+g)(p')$. Then by (7.2),

$$\begin{aligned}
\phi_t(p-p') &= \phi_t(p) - \phi_t(p') \\
&= -g \circ \phi_t(p) + g \circ \phi_t(p') + \psi_t \circ (I+g)(p) - \psi_t \circ (I+g)(p') \\
&= -g \circ \phi_t(p) + g \circ \phi_t(p').
\end{aligned}$$

The right-hand side is bounded for all $t \in \mathbb{R}$ because $g \in X$. By Corollary 2.2 and the fact that $A$ is hyperbolic, we have that $p - p' = 0$, i.e. $I + g$ is injective.

By Lemma 7.1 below, we have that $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ is surjective.

Let $K \subset \mathbb{R}^n$ be a closed set. Let $y_i = \Lambda(p_i) \in \Lambda(K)$ be a sequence in $\Lambda(K)$ converging to $y$. Since $g \in X$, we see that $p_i = y_i - g(p_i)$ is a bounded sequence. Hence, it has a subsequence $p_{i_j}$ converging to a point $p$, which belongs to $K$ since $K$ is closed. By continuity of $\Lambda$ we have $\Lambda(p) = \lim_j \Lambda(p_{i_j}) = \lim_j y_{i_j} = y$. This shows that $y \in \Lambda(K)$, and hence, $\Lambda(K)$ is closed. Thus, $\Lambda^{-1}$ is continuous.

Altogether, we conclude that $\Lambda$ is a homeomorphism. □

**Lemma 7.1.** *Suppose that $g : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and bounded. Define $\Lambda(x) = x + g(x)$. Then $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ is surjective.*

*Proof.* A point $y \in \mathbb{R}^n$ belongs to the range of $\Lambda$ if the function $q(x) = y - g(x)$ has a fixed point.[1] Since $\|q\|_\infty \leq \|y\| + \|g\|_\infty \equiv r$, it follows that $q : \mathbb{R}^n \to \overline{B}_r(0)$. In particular, $q$ maps the convex compact set $\overline{B}_r(0)$ into itself. By the Brouwer Fixed Point Theorem,[2] $q$ has a fixed point. □

*Remarks 7.5.* Notice that this theorem only guarantees the existence of a *homeomorphism* $\Lambda$ which conjugates the linear and nonlinear flows. In general, $\Lambda$ will not be smooth, unless a certain nonresonance condition is satisfied. This is the content of Sternberg's Theorem, the proof of which is substantially more difficult. There is an example due to Hartman that illustrates the limitation on the smoothness of the linearizing map $\Lambda$, see Exercise 7.5.

The proof of the next result follows Chow and Hale [3].

**Theorem 7.3.** *Let $A$ and $B$ be hyperbolic $n \times n$ real matrices. There exists a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f \circ \exp At = \exp Bt \circ f$ for all $t \in \mathbb{R}$ if and only if the stable subspaces of $A$ and $B$ have the same dimension.*

*Proof.* Let $E_s$, $E_u$ be the stable and unstable subspaces of $A$, and let $P_s$, $P_u$ be the respective projections. Define the linear transformation $L = -P_s + P_u$.

The first part of the proof will be to find a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$h \circ \exp At = \exp Lt \circ h = \mathrm{e}^{-t} P_s \circ h + \mathrm{e}^{t} P_u \circ h.$$

This will done by constructing homeomorphisms

$$h_s : E_s \to E_s \quad \text{and} \quad h_u : E_u \to E_u$$

such that

$$h_s \circ P_s \circ \exp At = \mathrm{e}^{-t} h_s \circ P_s \quad \text{and} \quad h_u \circ P_u \circ \exp At = \mathrm{e}^{t} h_u \circ P_u,$$

for then, if we set $h = h_s \circ P_s + h_u \circ P_u$, $h$ is a homeomorphism and

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1 We cannot invoke the Contraction Mapping Principle because $g \in X$ is not Lipschitz continuous.

2 See Milnor [7] for a short proof.

$$\begin{aligned}
h \circ \exp At &= h_s \circ P_s \circ \exp At + h_u \circ P_u \circ \exp At \\
&= \mathrm{e}^{-t} h_s \circ P_s + \mathrm{e}^{t} h_u \circ P_u \\
&= \exp Lt \circ h.
\end{aligned}$$

*Construction of* $h_s$. Let $\{e_i\}_{i=1}^p$ be a basis of generalized eigenvectors in $E_s^{\mathbb{C}}$. Then

$$Ae_i = \lambda_i e_i + \delta_i e_{i-1}, \quad i = 1, \ldots, p$$

where $\mathrm{Re}\lambda_i < 0$ and $\delta_i = 0$ or 1. Given $\mu > 0$, set $f_i = \mu^{-i} e_i$. Then

$$Af_i = \lambda_i f_i + \frac{\delta_i}{\mu} f_{i-1}.$$

If $T$ is the $n \times p$ matrix whose columns are the $f_i$, then

$$AT = T(D + N),$$

with $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ and $N$ nilpotent. The $p \times p$ nilpotent matrix $N$ has zero entries except possibly $1/\mu$ may occur above the main diagonal. Let $\{g_i\}_{i=1}^p$ be the dual basis, i.e. $\langle f_i, g_j \rangle_{\mathbb{C}^n} = \delta_{ij}$. Let $S$ be the $p \times n$ matrix whose rows are the $g_i$. Then $ST = I$ $(p \times p)$ and

$$SAT = (D + N).$$

Thus, Re $SAT$ is negative definite for $\mu$ large enough.

For $x \in E_s$, define

$$\phi(x) = \frac{1}{2}\|Sx\|^2$$

and

$$\Sigma = \{x \in E_s : \phi(x) = 1\}.$$

Then $D_x\phi(x)y = \mathrm{Re}\langle Sx, Sy \rangle_{\mathbb{C}^n}$. So since $TS = P_s$, we have that

$$\frac{d}{dt}\phi(\exp At\ x) = \mathrm{Re}\langle S \exp Atx, SA \exp Atx \rangle_{\mathbb{C}^n} = \mathrm{Re}\langle y, SATy \rangle_{\mathbb{C}^n},$$

in which $y = S \exp Atx$. Thus, we can find constants $0 < c_1 < c_2$ such that

$$-c_2\phi(\exp At\ x) \le \frac{d}{dt}\phi(\exp At\ x) \le -c_1\phi(\exp At\ x).$$

We see that $\phi(\exp Atx)$ is strictly decreasing as a function of $t$, and

$$\frac{1}{2}\|Sx\|^2 \mathrm{e}^{-c_2 t} = \phi(x)\mathrm{e}^{-c_2 t} \le \phi(\exp At\ x) \le \phi(x)\mathrm{e}^{-c_1 t} = \frac{1}{2}\|Sx\|^2 \mathrm{e}^{-c_1 t}. \quad (7.3)$$

Given $0 \neq x \in \mathbb{R}^n$, it follows that there exists a unique $t(x)$ such that

$$\phi(\exp At(x)x) = 1.$$

The function $t(x)$ is continuous for $x \neq 0$. (Actually, it is $C^1$ by the implicit function theorem.) Now define

$$h_s(x) = \begin{cases} \mathrm{e}^{t(x)} \exp At(x)\ x, & 0 \neq x \in E_s \\ 0, & x = 0. \end{cases}$$

Since $\phi(\exp At(x)x) = 1$, it follows from (7.3) that $\lim_{x \to 0} \|h_s(x)\| = 0$. Thus, $h_s$ is continuous on all of $\mathbb{R}^n$.

Given $0 \neq y \in E_s$, there is a unique time $t_0 \in \mathbb{R}$ such that $\mathrm{e}^{-t_0} y \in \Sigma$, namely $\frac{1}{2}\mathrm{e}^{-2t_0}\|Sy\|^2 = 1$. Setting $x = \mathrm{e}^{-t_0} \exp(-At_0)\ y$, we have $t(x) = t_0$ and $h_s(x) = y$. We have shown that $h_s : E_s \to E_s$ is one-to-one and onto. Continuity of $h_s^{-1}$ follows from its explicit formula. Therefore, $h_s$ is a homeomorphism.

For any $0 \neq x \in \mathbb{R}^n$, set $x_s = P_s x$ and $y = \exp At\ x_s$. Note that $t(y) = t(x_s) - t$. From the definition of $h_s$, we have

$$\begin{aligned} h_s \circ P_s \circ \exp At(x) &= h_s(\exp At\ x_s) \\ &= \mathrm{e}^{t(y)} \exp At(y)\ y \\ &= \mathrm{e}^{t(x_s)-t} \exp A[t(x) - t] \exp At\ x_s \\ &= e^{-t} h_s \circ P_s(x), \end{aligned}$$

so $h_s$ has all of the desired properties.

In a similar fashion, we can construct $h_u$, and as explained above we get a homeomorphism $h \circ \exp At = \exp Lt \circ h$.

Let $\widetilde{E}_s$, $\widetilde{E}_u$ be the stable and unstable subspaces for $B$, with their projections $\widetilde{P}_s$, $\widetilde{P}_u$. Set $\widetilde{L} = -\widetilde{P}_s + \widetilde{P}_u$. Let $g$ be a homeomorphism such that $g \circ \exp Bt = \exp \widetilde{L}t \circ g$.

Since $E_s$ and $\widetilde{E}_s$ have the same dimension, there is an isomorphism $M : \mathbb{R}^n \to \mathbb{R}^n$ such that $ME_s = \widetilde{E}_s$ and $ME_u = \widetilde{E}_u$. Define $f = g^{-1} \circ M \circ h$. Then

$$\begin{aligned} f \circ \exp At &= g^{-1} \circ M \circ h \circ \exp At \\ &= g^{-1} \circ M \circ \exp Lt \circ h \\ &= g^{-1} \circ (M \circ \exp Lt \circ M^{-1}) \circ M \circ h \\ &= g^{-1} \circ \exp \widetilde{L}t \circ M \circ h \\ &= \exp Bt \circ g^{-1} \circ M \circ h \\ &= \exp Bt \circ f. \end{aligned}$$

Conversely, suppose that $f$ is a homeomorphism which conjugates the flows, $f \circ \exp At = \exp Bt \circ f$. If $x_s \in E_s$ then $\lim_{t \to \infty} \|\exp Bt f(x_s)\| = \lim_{t \to \infty} \|$

$f(\exp Atx_s)\| = 0$. Thus, $f : E_s \to \widetilde{E}_s$. By symmetry, $f^{-1} : \widetilde{E}_s \to E_s$. Thus, $E_s$ and $\widetilde{E}_s$ are homeomorphic. By the Invariance of Domain Theorem, $E_s$ and $\widetilde{E}_s$ have the same dimension. □

Combining Theorems 7.2 and 7.3 yields

**Theorem 7.4.** *Let $A$ and $B$ be hyperbolic $n \times n$ matrices whose stable subspaces have the same dimension. Let $f, g \in Y$ with $\|f\|_{\text{Lip}}, \|g\|_{\text{Lip}} \leq \mu$. If $\mu$ is sufficiently small, then the two flows generated by the vector fields $Ax + f(x)$, $Bx + g(x)$ are topologically conjugate.*

Next we remove the boundedness restriction on the nonlinearity in Theorem 7.3.

**Theorem 7.5.** *Let $A$ be hyperbolic, and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be $C^1$ with $f(0) = 0$ and $Df(0) = 0$. Let $\phi_t(p)$ and $\psi_t(p)$ be the flows generated by $Ax$ and $Ax + f(x)$, respectively. There is a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ and a neighborhood $0 \in U \subset \mathbb{R}^n$ such that for all $p \in U$*

$$h \circ \psi_t(p) = \phi_t \circ h(p),$$

*as long as $\psi_t(p) \in U$.*

*Proof.* Since $f(0) = 0$ and $Df(0) = 0$, we may write

$$f(x) - f(y) = \int_0^1 \frac{d}{d\sigma} f(\sigma x + (1-\sigma)y)d\sigma = \int_0^1 Df(\sigma x + (1-\sigma)y)d\sigma \ (x-y).$$

Choose $0 < r < 1$ so that $\|Df(x)\| \leq \delta/2$ for $\|x\| \leq r$. If $\|x\|, \|y\| \leq r$, then

$$\|f(x) - f(y)\| \leq (\delta/2)\|x - y\| \quad \text{and} \quad \|f(x)\| \leq (\delta/2)\|x\| \leq \delta/2.$$

Thus, $\|f\|_{\text{Lip}(B_r(0))} \leq \delta$.

Now extend $f$ outside $B_r(0)$ to a bounded continuous function

$$\tilde{f}(x) = \begin{cases} f(x), & \|x\| \leq r \\ f(rx/\|x\|), & \|x\| > r. \end{cases}$$

Then $\|\tilde{f}\|_{\text{Lip}(\mathbb{R}^n)} = \|f\|_{\text{Lip}}(B_r(0)) \leq \delta$.

The Hartman-Grobman Theorem says that the flow $\tilde{\psi}_t$ of $Ax + \tilde{f}(x)$ and the flow $\phi_t$ are topologically conjugate under some homeomorphism $h$.

Since $f = \tilde{f}$ in $B_r(0)$, the Uniqueness Theorem 3.3 implies that for all $p \in B_r(0)$

$$\psi_t(p) = \tilde{\psi}_t(p),$$

as long as $\psi_t(p) \in B_r(0)$. In this case, we have

$$h \circ \psi_t(p) = h \circ \tilde{\psi}_t(p) = \phi_t \circ h(p). \qquad \square$$

*Remarks 7.6.* Note that if $q \in E_s$, the stable subspace of A, then

$$\lim_{t \to \infty} \phi_t(q) = 0.$$

Thus, if $\|q\| \leq \varepsilon$ for $\varepsilon$ small enough, then $\phi_t(q) \in h(U) \cap E_s$ for all $t > 0$. By Theorem 7.5, if $q = h(p)$, then $\psi_t(p)$ is defined for all $t > 0$, $\psi_t(p) \in U \cap h^{-1}(E_s)$, and $\lim_{t\to\infty} \psi_t(p) = 0$.

Similarly, if $h(p) \in E_u$ with $\|h(p)\| < \varepsilon'$, then $\psi_t(p)$ is defined and remains in $U \cap h^{-1}(E_u)$, for all $t < 0$.

The sets $U \cap h^{-1}(E_s)$ and $U \cap h^{-1}(E_u)$ are called the local stable and unstable manifolds relative to $U$, respectively.

## 7.3 Invariant Manifolds

**Definition 7.4.** Let $F \in C^1(\mathcal{O}, \mathbb{R}^n)$ be an autonomous vector field. A set $\mathcal{A} \subset \mathcal{O}$ is called an invariant set if $\mathcal{A}$ contains the orbit through each of its points; i.e. $x(t, x_0) \in \mathcal{A}$ for all $\alpha(x_0) < t < \beta(x_0)$. A set $\mathcal{A}$ is positively (negatively) invariant if $x(t, x_0) \in \mathcal{A}$ for all $0 < t < \beta(x_0)$ ($\alpha(x_0) < t < 0$).

Throughout this section, we assume that the vector field

$$F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$$

has a hyperbolic equilibrium at $x = 0$.

Set $A = DF(0)$, and write $F(x) = Ax + f(x)$. Then $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $f(0) = 0$, and $D_x f(0) = 0$. As usual, $x(t, x_0)$ will denote the (possibly local) solution of the initial value problem

$$x' = Ax + f(x), \quad x(0) = x_0.$$

Let $E_s$ and $E_u$ be the stable and unstable subspaces of the hyperbolic matrix $A$ with their projections $P_s$ and $P_u$. Since $A$ is hyperbolic, we have that $\mathbb{R}^n = E_s \oplus E_u$.

The stable and unstable subspaces are invariant under the linear flow. We shall be interested in what happens to these sets under nonlinear perturbations. In fact, we have already seen in Remark 7.6 that the Hartman-Grobman Theorem implies that there is a neighborhood of the origin in $E_s$ which is homeomorphic to positively invariant set under the nonlinear flow. Similarly, $E_u$ corresponds to a negatively invariant set for the nonlinear flow. We will show that these invariant sets are $C^1$ manifolds which are tangent to the corresponding subspaces at the origin.

Recall that by Corollary 2.3, there exists constants $C_0 > 0$ and $\lambda > 0$ such that $\lambda_s, \lambda_u > \lambda$ and

$$\|\exp At \; P_s x\| \leq C_0 e^{-\lambda t} \|P_s x\|, \quad t \geq 0, \tag{7.4}$$

and

$$\| \exp At \; P_u \, x \| \leq C_0 e^{\lambda t} \| P_u x \|, \quad t \leq 0, \tag{7.5}$$

for all $x \in \mathbb{R}^n$.

**Definition 7.5.** The stable manifold (of the equilibrium at the origin) is the set

$$W_s(0) = \{x_0 \in \mathbb{R}^n : x(t, x_0) \text{ exists for all } t \geq 0 \text{ and } \lim_{t \to \infty} x(t, x_0) = 0\}.$$

Of course, it remains to be shown that $W_s(0)$ is indeed a manifold.

**Definition 7.6.** Let $\mathcal{U}$ be a neighborhood of the equilibrium at the origin. A set $W_s^{loc}(0)$ is called a local stable manifold relative to $\mathcal{U}$ if

$$W_s^{loc}(0) = \{x_0 \in W_s(0) : x(t, x_0) \in \mathcal{U} \text{ for all } t \geq 0\}.$$

The notions of an unstable manifold and a local unstable manifold are similarly defined.

*Remarks 7.7.*

- If $F(x) = Ax$ is linear, then $E_s = W_s(0)$, by Theorems 2.3 and 2.4.
- The stable and local stable manifolds are positively invariant under the flow.

**Theorem 7.6.** *(***Local Stable Manifold Theorem***) Let $x(t, x_0)$ be the flow of the vector field $A + f(x)$, where $A$ is hyperbolic and $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ satisfies $f(0) = 0$ and $Df(0) = 0$.*

*There exists a function $\eta$ defined and $C^1$ on a neighborhood of the origin $U \subset E_s$ with the following properties:*

*(i)* $\eta : U \to E_u$, $\eta(0) = 0$, *and* $D\eta(0) = 0$.
*(ii)* *Define* $\mathcal{A} = \{x_0 \in \mathbb{R}^n : P_s x_0 \in U, \;\; P_u x_0 = \eta(P_s x_0)\}$. *If* $x_0 \in \mathcal{A}$, *then* $x(t, x_0)$ *is defined for all* $t \geq 0$, *and*

$$\|x(t, x_0)\| \leq C \|P_s x_0\| \exp(-\lambda t/2).$$

*(iii)* *$\mathcal{A}$ is a local stable manifold relative to $B_r(0)$, for a certain $r > 0$.*
*(iv)* *If $x_0 \in \mathcal{A}$, then $z(t) = P_s x(t, x_0)$ solves the initial value problem*

$$z'(t) = Az(t) + P_s f(z(t) + \eta(z(t))), \quad z(0) = P_s x_0.$$

*Remarks 7.8.*

- If $f \in C^k(\mathbb{R}^n)$ then $\eta \in C^k(U)$.
- There is an obvious version of this result for local unstable manifolds. This can be proven simply by reversing time (i.e. $t \to -t$) in the equation and then applying the Theorem 7.6.

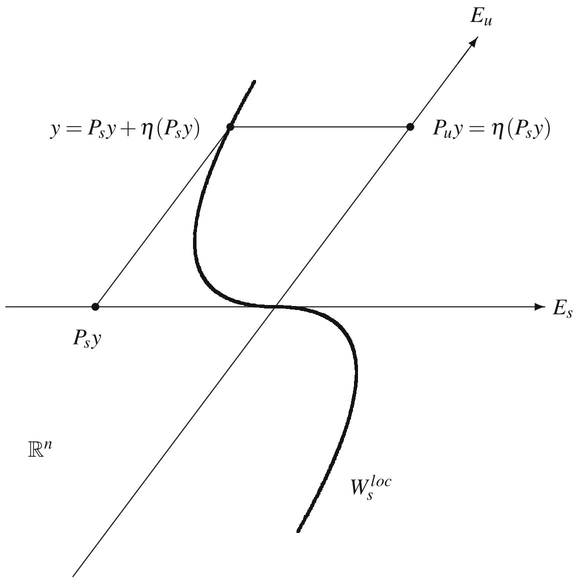


**Fig. 7.1** The local stable manifold $W_s^{loc}$ as a graph over $E_s$

- The initial value problem *(iv)* governs the flow on the local stable manifold.

Throughout this section, we are going to use the alternate norm

$$\|x\|_* = \|P_s x\|_{\mathbb{R}^n} + \|P_u x\|_{\mathbb{R}^n}.$$

The norm $\|\cdot\|_*$ is equivalent to the Euclidean norm $\|\cdot\|_{\mathbb{R}^n}$. Its advantage is that the operator norms of the projections $P_s$ and $P_u$ with respect to $\|\cdot\|_*$ are equal to 1. To keep the notation simple, we shall write $\|\cdot\|$ for the norm $\|\cdot\|_*$ without further comment.

We shall also use the notation

$$X = C_b^0([0,\infty), \mathbb{R}^n) = \{x \in C^0([0,\infty), \mathbb{R}^n) : \|x\|_\infty = \sup_{t \geq 0} \|x(t)\| < \infty\}.$$

We break the rather long proof into a series of Lemmas.

**Lemma 7.2.** *For $g \in X$, define*

$$I_s g(t) = \int_0^t \exp A(t-\tau)\ P_s g(\tau) d\tau,$$

*and*

$$I_u g(t) = \int_t^\infty \exp A(t-\tau)\ P_u g(\tau) d\tau.$$

*Then $I_s, I_u \in \mathcal{L}(X, X)$ and $\|I_s\|, \|I_u\| \le C_0/\lambda$.*

*Proof.* Using (7.4), we have

$$\|I_s g(t)\| \le \int_0^t C_0 \mathrm{e}^{-\lambda(t-\tau)} \|g(\tau)\| d\tau \le \int_0^t C_0 \mathrm{e}^{-\lambda(t-\tau)} d\tau\ \|g\|_\infty \le (C_0/\lambda)\|g\|_\infty.$$

Similarly, by (7.5), we have

$$\|I_u g(t)\| \le \int_t^\infty C_0 \mathrm{e}^{\lambda(t-\tau)} \|g(\tau)\| d\tau \le \int_t^\infty C_0 \mathrm{e}^{\lambda(t-\tau)} d\tau\ \|g\|_\infty \le (C_0/\lambda)\|g\|_\infty.$$

Thus, $I_s g$ and $I_u g$ are bounded functions. They are continuous ($C^1$, in fact). Hence, $I_s$ and $I_u$ map $X$ into itself. These maps are linear, and they are bounded as a consequence of the estimates given above. □

**Lemma 7.3.** *Suppose that $x_0 \in \mathbb{R}^n$ is such that $x(t, x_0)$ is defined for all $t \ge 0$ and belongs to $X$. Then*

$$P_u x_0 + \int_0^\infty \exp(-A\tau) P_u f(x(\tau, x_0)) d\tau = 0,$$

*and $y(t) = x(t, x_0)$ solves the integral equation*

$$y(t) = \exp At\ y_0 + \int_0^t \exp A(t-\tau) P_s f(y(\tau)) d\tau - \int_t^\infty \exp A(t-\tau) P_u f(y(\tau)) d\tau, \tag{7.6}$$

*for all $t \ge 0$, where $y_0 = P_s x_0$.*

*Conversely, if $y(t, y_0)$ is a solution of (7.6) in $X$, then $P_u y_0 = 0$ and $y(t, y_0) = x(t, x_0)$ with $x_0 = y(0)$.*

*Proof.* Suppose that $x_0 \in \mathbb{R}^n$ is such that $x(t, x_0)$ is defined for all $t \ge 0$ and $\sup_{t \ge 0} \|x(t, x_0)\| < \infty$. Using Variation of Parameters, Corollary 4.1, we can write, for all $t \ge 0$,

$$x(t, x_0) = \exp At\ x_0 + \int_0^t \exp A(t-\tau)\ f(x(\tau, x_0)) d\tau. \tag{7.7}$$

Since $x(t, x_0)$ is bounded, so too is $f(x(t, x_0))$, and hence by Lemma 7.2, we may write

$$\begin{aligned}\int_0^t &\exp A(t-\tau)\ P_u f(x(\tau, x_0))d\tau \\ &= \int_0^\infty \exp A(t-\tau)\ P_u f(x(\tau, x_0))d\tau \\ &\quad - \int_t^\infty \exp A(t-\tau)\ P_u f(x(\tau, x_0))d\tau \\ &= \exp At \int_0^\infty \exp(-A\tau)\ P_u f(x(\tau, x_0))d\tau \\ &\quad - \int_t^\infty \exp A(t-\tau)\ P_u f(x(\tau, x_0))d\tau.\end{aligned}$$

Combining this with (7.7), we get

$$\begin{aligned}x(t, x_0) &= \exp At\ x_0 + \int_0^t \exp A(t-\tau)\ P_s f(x(\tau, x_0))d\tau \\ &\quad - \int_t^\infty \exp A(t-\tau)\ P_u f(x(\tau, x_0))d\tau \\ &\quad + \exp At \int_0^\infty \exp(-A\tau)\ P_u f(x(\tau, x_0))d\tau \\ &= \exp At\ P_s x_0 + \int_0^t \exp A(t-\tau)\ P_s f(x(\tau, x_0))d\tau \\ &\quad - \int_t^\infty \exp A(t-\tau)\ P_u f(x(\tau, x_0))d\tau \\ &\quad + \exp At \left[ P_u x_0 + \int_0^\infty \exp(-A\tau)\ P_u f(x(\tau, x_0))d\tau \right].\end{aligned}$$

Applying $P_u$ to both sides , we find that

$$\begin{aligned}&\exp At \left[ P_u x_0 + \int_0^\infty \exp(-A\tau)\ P_u f(x(\tau, x_0))d\tau \right] \\ &= P_u x(t, x_0) + \int_t^\infty \exp A(t-\tau)\ P_u f(x(\tau, x_0))d\tau.\end{aligned}$$

By Lemma 7.2 and our assumption on $x(t, x_0)$, the right-hand side is bounded.

Let

$$z = P_u x_0 + \int_0^\infty \exp(-A\tau)\ P_u f(x(\tau, x_0))d\tau.$$

Then we have shown that $\| \exp At \ z\|$ is uniformly bounded for all $t \geq 0$. By Corollary 2.3, we have $z = P_u z = 0$.

As a consequence of this fact, if $x(t, x_0)$ is a bounded solution of (7.7), then it satisfies

$$x(t, x_0) = \exp At \ P_s x_0 + \int_0^t \exp A(t-\tau) \ P_s f(x(\tau, x_0)) d\tau$$
$$- \int_t^\infty \exp A(t-\tau) \ P_u f(x(\tau, x_0)) d\tau.$$

Thus $y(t) = x(t, x_0)$ satisfies (7.6) with $y_0 = P_s x_0$.

Suppose now that $y(t, y_0)$ is a solution of (7.6) in $X$. Applying $P_u$ to (7.6), we obtain

$$\exp At \ P_u y_0 = P_u y(t, y_0) + \int_t^\infty \exp A(t-\tau) \ P_u f(y(\tau, y_0)) d\tau.$$

By Lemma 7.2, the right-hand side is bounded. Using Corollary 2.3 again, we conclude that $P_u y_0 = 0$.

Finally, as above, we may rearrange the terms in (7.6) to get

$$y(t, y_0) = \exp At \left[ y_0 - \int_0^\infty \exp A(t-\tau) \ P_u f(y(\tau, y_0)) d\tau \right]$$
$$+ \int_0^t \exp A(t-\tau) f(y(\tau, y_0)) d\tau$$
$$= \exp At \ y(0, y_0) + \int_0^t \exp A(t-\tau) f(y(\tau, y_0)) d\tau.$$

Thus, by uniqueness of solutions to the initial value problem, we have that $y(t, y_0) = x(t, x_0)$ with $x_0 = y(0, y_0)$. □

Lemma 7.3 gives a big hint for how to find the local stable manifold.

**Lemma 7.4.** *There exists an $r > 0$ such that if $y(\cdot, y_0) \in X$ is a solution of* (7.6) *with $\|y(\cdot, y_0)\|_\infty < r$, then*

$$\|y(t, y_0))\| \leq 2C_0 \exp(-\lambda t/2) \|y_0\|, \quad \textit{for all} \quad t \geq 0.$$

*Proof.* Since $f$ is $C^1$ and $f(0) = 0$, we have if $\|x\| \leq r$

$$\|f(x)\| = \| \int_0^1 Df(\sigma x) x d\sigma \| \leq \max_{\|p\| \leq r} \|Df(p)\| \|x\| = C_r \|x\|.$$

Note that $C_r \to 0$ as $r \to 0$, because $Df(0) = 0$. Using this, we have for $y \in X$

$$\|y\|_\infty \leq r \quad \text{implies} \quad \|f(y(\tau))\| \leq C_r \|y(\tau)\|.$$

Suppose that $y(t, y_0)$ is a solution of (7.6) with $\|y(\cdot, y_0)\|_\infty < r$. Then by Lemma 7.3, $y_0 \in E_s$. By (7.4) and Lemma 7.2, we have the estimate

$$\begin{aligned}\|y(t, y_0)\| &\leq C_0 \mathrm{e}^{-\lambda t}\|y_0\| + C_0 C_r \int_0^t \mathrm{e}^{-\lambda(t-\tau)}\|y(\tau, y_0)\| d\tau \\ &\quad + C_0 C_r \int_t^\infty \mathrm{e}^{\lambda(t-\tau)}\|y(\tau, y_0)\| d\tau. \qquad (7.8)\end{aligned}$$

Choose $r$ sufficiently small so that $4C_0C_r/ < \lambda$. The result is a consequence of the next lemma. □

**Lemma 7.5.** *Let $A, B, \lambda$ be positive constants. Suppose that $u(t)$ is a nonnegative, bounded continuous function such that*

$$u(t) \leq Ae^{-\lambda t} + B\int_0^t e^{-\lambda(t-\tau)}u(\tau)d\tau + B\int_t^\infty e^{\lambda(t-\tau)}u(\tau)d\tau,$$

*for all $t \geq 0$. If $4B < \lambda$, then $u(t) \leq 2A\exp(-\lambda t/2)$.*

*Proof.* Since $u(t)$ is bounded, we may define $\rho(t) = \sup_{\tau\geq t} u(\tau)$. The function $\rho(t)$ is nonincreasing, nonnegative, and continuous, as can easily be verified. Fix $t \geq 0$. For any $\varepsilon > 0$, there is a $t_1 \geq t$ such that $\rho(t) < u(t_1) + \varepsilon$. Then, since $u(t) \leq \rho(t)$, we have

$$\begin{aligned}\rho(t) - \varepsilon < u(t_1) &\leq A\mathrm{e}^{-\lambda t_1} + B\int_0^{t_1} \mathrm{e}^{-\lambda(t_1-\tau)}u(\tau)d\tau \\ &\quad + B\int_{t_1}^\infty \mathrm{e}^{\lambda(t_1-\tau)}u(\tau)d\tau \\ &\leq A\mathrm{e}^{-\lambda t_1} + B\int_0^{t_1} \mathrm{e}^{-\lambda(t_1-\tau)}\rho(\tau)d\tau \\ &\quad + B\int_{t_1}^\infty \mathrm{e}^{\lambda(t_1-\tau)}\rho(\tau)d\tau \\ &\leq A\mathrm{e}^{-\lambda t_1} + B\int_0^{t} \mathrm{e}^{-\lambda(t_1-\tau)}\rho(\tau)d\tau \\ &\quad + B\int_t^{t_1} \mathrm{e}^{-\lambda(t_1-\tau)}\rho(\tau)d\tau + B\int_{t_1}^\infty \mathrm{e}^{\lambda(t_1-\tau)}\rho(\tau)d\tau \\ &\leq A\mathrm{e}^{-\lambda t} + B\int_0^{t} \mathrm{e}^{-\lambda(t-\tau)}\rho(\tau)d\tau \\ &\quad + B\rho(t)\int_t^{t_1} \mathrm{e}^{-\lambda(t_1-\tau)}d\tau + B\rho(t)\int_{t_1}^\infty \mathrm{e}^{\lambda(t_1-\tau)}d\tau\end{aligned}$$

$$\leq A\mathrm{e}^{-\lambda t} + B\mathrm{e}^{-\lambda t} \int_0^t \mathrm{e}^{\lambda\tau} \rho(\tau) d\tau + (2B/\lambda)\rho(t).$$

This holds for every $\varepsilon > 0$. So if we send $\varepsilon \to 0$ and set $z(t) = \mathrm{e}^{\lambda t}\rho(t)$, we get

$$(1 - 2B/\lambda)z(t) \leq A + B \int_0^t z(\tau) d\tau.$$

Our assumption $\lambda > 4B$ implies that $(1-2B/\lambda)^{-1} < 2$ and $B(1-2B/\lambda)^{-1} < \lambda/2$. Thus,

$$z(t) \leq 2A + (\lambda/2) \int_0^t z(\tau) d\tau.$$

Gronwall's Lemma 3.3 implies that $z(t) \leq 2A \exp(\lambda t/2)$, so that $u(t) \leq \rho(t) \leq 2A \exp(-\lambda t/2)$. □

**Lemma 7.6.** *There are neighborhoods* $0 \in U_1 \subset E_s$, $0 \in V_1 \subset X$, *and a* $C^1$ *mapping* $\phi : U_1 \to V_1$ *such that* $\phi(0) = 0$ *and* $\phi(y_0)$ *solves* (7.6) *for all* $y_0 \in U_1$.

*Moreover, if* $y_0 \in U_1$, $y \in V_1$ *solve* (7.6)*, then* $y = \phi(y_0)$.

*Proof.* Given $y_0 \in E_s$, $y \in X$, define the mapping

$$\begin{aligned} T(y_0, y)(t) =& y(t) - \exp At \; y_0 - \int_0^t \exp A(t-\tau) \; P_s f(y(\tau)) d\tau \\ &+ \int_t^\infty \exp A(t-\tau) \; P_u f(y(\tau)) d\tau \\ =& y(t) - \exp At \; y_0 - I_s f \circ y(t) + I_u f \circ y(y). \end{aligned} \tag{7.9}$$

By Theorems 5.2, 5.3, 5.4 and Lemma 7.2, we have that

$$T : E_s \times X \to X$$

is a $C^1$ mapping.

Now we proceed to check that $T$ fulfills the hypotheses of the Implicit Function Theorem 5.7 at (0, 0). From the Definition (7.9) we see that $T(0,0) = 0$ and that

$$\begin{aligned} D_y T(y_0, y) z(t) = z(t) &- \int_0^t \exp A(t-\tau) \; P_s Df(y(\tau)) z(\tau) d\tau \\ &+ \int_t^\infty \exp A(t-\tau) \; P_u Df(y(\tau)) z(\tau) d\tau. \end{aligned}$$

Since $Df(0) = 0$, we get $D_y T(0,0) = I$. The result follows by the Implicit Function Theorem 5.7. □

*Proof of Theorem 7.6.* Using Lemma 7.6, for $y_0 \in U_1$, set $y(t, y_0) = \phi(y_0)(t)$. Then $y(t, y_0)$ solves (7.6) in $X$. Letting $t = 0$, we have

$$y(0, y_0) = P_s y(0, y_0) + P_u y(0, y_0) = y_0 - \int_0^\infty \mathrm{e}^{-A\tau} P_u f(y(\tau, y_0))d\tau. \quad (7.10)$$

For $y_0 \in U_1$, define

$$\eta(y_0) = P_u y(0, y_0) = -\int_0^\infty \mathrm{e}^{-A\tau} P_u f(y(\tau, y_0))d\tau. \quad (7.11)$$

From this definition, we see that $\eta(0) = 0$, since $\phi(0)(t) = y(t, 0) = 0$, and it also follows that $\eta : U_1 \to E_u$. Since $y_0 \mapsto \phi(y_0)$ is $C^1$ from $U_1$ into $X$ and since $g \mapsto g(0)$ is $C^1$ from $X$ into $\mathbb{R}^n$, we have that $y_0 \mapsto \eta(y_0) = \phi(y_0)(0)$ is $C^1$ from $U_1$ into $\mathbb{R}^n$, by the chain rule. So we can compute

$$D\eta(y_0) = -\int_0^\infty \exp(-A\tau)\ P_u Df(y(\tau, y_0))D_{y_0} y(\tau, y_0)d\tau.$$

Again since $\phi(0) = y(\cdot, 0) = 0$ and $Df(0) = 0$, we have $D\eta(0) = 0$. Thus, the function $\eta$ satisfies the properties of *(i)*.

In addition to the restriction $4C_0C_r < \lambda$ placed on $r$ by Lemma 7.4, choose $r > 0$ small enough such that

$$\{y_0 \in E_s : \|y_0\| < r\} \subset U_1, \quad \text{and} \quad \{y \in X : \|y\|_\infty < r\} \subset V_1.$$

Define

$$V = \{y \in X : \|y\|_\infty < r\} \subset V_1$$

and

$$U = \{y_0 \in U_1 : \|\phi(y_0)\|_\infty < r\} = \phi^{-1}(V).$$

Note that $y_0 \in U$ implies

$$\|y_0\| = \|P_s y(0, y_0)\| \le \|y(0, y_0)\| \le \|\phi(y_0)\|_\infty < r.$$

(Here we use the fact $\|P_s\| = 1$ in the special norm we have chosen.) Thus, $U \subset B_r(0) \subset U_1 \subset E_s$.

Note that $\phi : U \to V$ is $C^1$, since $U \subset U_1$, and so we may pass to the smaller pair of neighborhoods $U$ and $V$ while maintaining the validity of the properties in *(i)*. The reason for these choices will become clear below.

Define the following sets

$$\mathcal{A} = \{x_0 \in \mathbb{R}^n : P_s x_0 \in U,\ P_u x_0 = \eta(P_s x_0)\}$$
$$\mathcal{B} = \{x_0 \in \mathbb{R}^n : P_s x_0 \in U,\ x(t, x_0) = y(t, P_s x_0),\ t \geq 0\}$$
$$\mathcal{C} = \{x_0 \in \mathbb{R}^n : P_s x_0 \in U,\ \|x(t, x_0)\|_\infty < r,\ t \geq 0\}.$$

We will now show that these three sets are equal.

Suppose that $x_0 \in \mathcal{A}$. Then $y_0 = P_s x_0 \in U$, and $\phi(y_0)(t) = y(t, y_0)$ is a solution of (7.6) in $X$. By Lemma 7.3 and (7.10), (7.11), we have that $y(t, y_0) = x(t, p)$ with $p = y(0, y_0) = P_s y(0, y_0) + P_u y(0, y_0) = y_0 + \eta(y_0) = P_s x_0 + P_u x_0 = x_0$. Since $P_s x_0 \in U$, we have shown that $x_0 \in \mathcal{B}$; i.e. $\mathcal{A} \subset \mathcal{B}$.

Let $x_0 \in \mathcal{B}$. Then $P_u x_0 = P_u x(0, x_0) = P_u y(0, P_s x_0) = \eta(P_s x_0)$, by (7.10), (7.11). Thus, $x_0 \in \mathcal{A}$, i.e. $\mathcal{B} \subset \mathcal{A}$.

Again take $x_0 \in \mathcal{B}$. Then $y_0 = P_s x_0 \in U$ implies that $\phi(y_0) \in V = B_r(0) \subset X$. Thus, we have $\|x(\cdot, x_0)\|_\infty = \|y(\cdot, y_0)\|_\infty = \|\phi(y_0)\|_\infty < r$. So $x_0 \in \mathcal{C}$; i.e. $\mathcal{B} \subset \mathcal{C}$.

Finally, suppose that $x_0 \in \mathcal{C}$. Then $x(t, x_0)$ is a solution of the initial value problem in $X$, so by Lemma 7.3, we have that it is also a solution of (7.6) with $y_0 = P_s x_0$. Thus, $(y_0, x(\cdot, x_0)) \in U \times V \subset U_1 \times V_1$ satisfies $T(y_0, x(\cdot, x_0)) = 0$. By the uniqueness of solutions to this equation in $U_1 \times V_1$, we have $x(\cdot, x_0) = \phi(y_0) = y(\cdot, y_0) = y(\cdot, P_s x_0)$. Thus, $x_0 \in \mathcal{B}$; i.e. $\mathcal{C} \subset \mathcal{B}$.

This completes the proof that $\mathcal{A} = \mathcal{B} = \mathcal{C}$.

We now establish the inequality of *(ii)*. If $x_0 \in \mathcal{A} = \mathcal{C}$, then $x(t, x_0)$ is defined for all $t \geq 0$, and $\sup_{t \geq 0} \|x(t, x_0)\| < r$. But $x_0 \in \mathcal{A} = \mathcal{B}$ implies that $x(t, x_0) = y(t, y_0)$, with $y_0 = P_s x_0$. Since $\|y(\cdot, y_0)\|_\infty = \|x(\cdot, x_0)\|_\infty < r$, part *(ii)* of the Theorem follows from Lemma 7.4.

This also shows that $\mathcal{A} \subset W_s(0)$.

Define
$$W_s^{loc}(0) = \{x_0 \in W_s(0) : x(t, x_0) \in B_r(0),\ t \geq 0\}.$$

That is, $W_s^{loc}(0)$ is the local stable manifold relative to $B_r(0) \subset \mathbb{R}^n$. We claim that $\mathcal{A} = W_s^{loc}(0)$.

First let $x_0 \in \mathcal{A}$. Then by part *(ii)*, we have that $x_0 \in W_s(0)$. Since $x_0 \in \mathcal{C}$, we also have that $\|x(t, x_0)\| < r$ for all $t \geq 0$. Thus, $x_0 \in W_s^{loc}(0)$.

Now take $x_0 \in W_s^{loc}(0)$. Then $\|x(t, x_0)\| < r$ implies that $x(\cdot, x_0) \in V_1$. Set $y_0 = P_s x_0$. We have $\|y_0\| = \|P_s x_0\| \leq \|x_0\| = \|x(0, x_0)\| < r$. (Using our special norm again.) Thus $y_0 \in B_r(0) \subset U_1$. By Lemma 7.3, $x(\cdot, x_0) \in V_1$ is a solution of (7.6) with $y_0 = P_s x_0 \in U_1$. By the uniqueness portion of Lemma 7.6, we have that $x(\cdot, x_0) = \phi(y_0)$. Thus, we have that $\|\phi(y_0)\|_\infty < r$. In other words, $y_0 = P_s x_0 \in \phi^{-1}(V) = U$. This proves that $x_0 \in \mathcal{C} = \mathcal{A}$.

Finally, having shown that $\mathcal{A} = W_s^{loc}(0)$, we see that $\mathcal{A}$ is positively invariant because $W_s^{loc}(0)$ is. Thus, if $x_0 \in \mathcal{A} = W_s^{loc}(0)$, then $x(t, x_0)$ is defined for all $t \geq 0$ and $P_u x(t, x_0) = \eta(P_s x(t, x_0))$. Setting $z(t) = P_s x(t, x_0)$, we have that

$$x(t, x_0) = z(t) + \eta(z(t)).$$

So *(iv)* follows by applying $P_s$ to both sides of the differential equation satisfied by $x(t, x_0)$:

$$x'(t) = Ax(t) + f(x(t)), \quad x(0) = x_0.$$

□

*Remarks 7.9.* We remark that the local stable manifold relative to some neighborhood $\mathcal{U}$ is unique because Definition 7.6 is intrinsic.

**Definition 7.7.** A $C^1$ manifold of dimension $k$ is a set $M$ together with a family of injective mappings $\psi_\alpha : U_\alpha \to M$, with each $U_\alpha$ an open set in $\mathbb{R}^k$, satisfying

1. $M = \cup_\alpha \psi_\alpha(U_\alpha)$, and
2. if $W = \psi_\alpha(U_\alpha) \cap \psi_\beta(U_\beta) \neq \emptyset$, then the sets $\psi_\alpha^{-1}(W)$ and $\psi_\beta^{-1}(W)$ are open and $\psi_\beta^{-1} \circ \psi_\alpha$ is a $C^1$ mapping between them.

Each pair $(\psi_\alpha, U_\alpha)$ is called a local coordinate chart on $M$.

**Theorem 7.7.** *(***The manifold structure of** $W_s^{loc}(0)$ **and** $W_s(0)$*). The stable and local stable manifolds of a hyperbolic equilibrium are $C^1$ manifolds of the same dimension as $E_s$ tangent to $E_s$ at the origin.*

*Proof.* The local stable manifold $W_s^{loc}(0) = \mathcal{A}$ constructed in Theorem 7.6 is a $C^1$ manifold of the same dimension as $E_s$ with a single coordinate chart $(\psi, U)$, $\psi(y) = y + \eta(y)$. Strictly speaking, $U$ is an open set in $E_s$, but $E_s$ is isomorphic to $\mathbb{R}^k$, for some $k \leq n$. The map $\psi$ is injective since $\psi(y) = \psi(y')$ implies $y = P_s\psi(y) = P_s\psi(y') = y'$. In other words, $\psi^{-1} = P_s$.

The stable manifold $W_s(0)$ is also a $C^1$ manifold of the same dimension as $E_s$. Let $\Phi_t$ be the flow of the vector field $Ax + f(x)$. Then $W_s(0) = \cup_{t\leq 0}\Phi_t(W_s^{loc}(0))$, where

$$\Phi_t(S) = \{\Phi_t(x) : x \in S \cap \text{domain}(\Phi_t)\}.$$

Thus, the family $\{(\Phi_t \circ \psi, U) : t \leq 0\}$ comprises a system of local coordinate charts for $W_s(0)$.

Finally, we note that $W_s^{loc}(0)$ (and hence also $W_s(0)$) is tangent to $E_s$ at the origin. Let $\gamma(t)$ be any $C^1$ curve on $W_s^{loc}(0)$ with $\gamma(0) = 0$. Then

$$\gamma(t) = P_s\gamma(t) + \eta(P_s\gamma(t)),$$

so since $\eta$ is $C^1$ and $D\eta(0) = 0$, we have that

$$\gamma'(0) = \gamma'(t)|_{t=0} = P_s\gamma'(t) + D\eta(P_s\gamma(t))P_s\gamma'(t)|_{t=0} = P_s\gamma'(0).$$

Thus, the tangent vector to $\gamma$ at the origin lies in $E_s$. □

Now let's consider the problem of approximating the function $\eta$ which defines the local stable manifold. Let $x_0 \in W_s^{loc}(0)$. Then $x(t, x_0)$ is defined for all $t \geq 0$.

Set $x(t) = x(t, x_0)$ and $y(t) = P_s x(t, x_0)$. Then we have

$$P_u x(t) = \eta(y(t)).$$

If we differentiate:

$$P_u x'(t) = D\eta(y(t))y'(t),$$

use the equations:

$$P_u[Ax(t) + f(x(t))] = D\eta(y(t))[Ay(t) + f(y(t) + \eta(y(t))],$$

and set $t = 0$, then

$$A\eta(P_s x_0) + P_u f(P_s x_0 + \eta(P_s x_0)) = D\eta(P_s x_0)[AP_s x_0 + P_s f(P_s x_0 + \eta(P_s x_0))],$$

for all $x_0 \in W_s^{loc}(0)$. This is useful in approximating the function $\eta$.

**Definition 7.8.** A function $f$ defined in a neighborhood $U$ of the origin in any normed space $X$ into another normed space $Y$ is said to be $O(\|x\|^k)$ as $x \to 0$ for some $k \geq 0$ if there exists a constant $C > 0$ such that $\|f(x)\| \leq C\|x\|^k$, for all $x \in U$.

**Theorem 7.8.** *(**Approximation**) Let $U \subset E_s$ be a neighborhood of the origin. Suppose that $h : U \to E_u$ is a $C^1$ mapping such that $h(0) = 0$ and $Dh(0) = 0$. If*

$$Ah(x) + P_u f(x + h(x)) - Dh(x)[Ax + P_s f(x + h(x))] = O(\|x\|^k),$$

*then*

$$\eta(x) - h(x) = O(\|x\|^k),$$

*for $x \in U$ and $\|x\| \to 0$.*

For a proof of this result, see the book of Carr [2]. The way in which this is used is to plug in a finite Taylor expansion for $h$ into the equation for $\eta$ and grind out the coefficients. The theorem says that this procedure is correct.

*Example 7.1.* Consider the nonlinear system

$$x' = x + y^3, \quad y' = -y + x^2.$$

The origin is a hyperbolic equilibrium, and the stable and unstable subspaces for the linearized problem are

$$E_s = \{(0, y) \in \mathbb{R}^2\} \quad \text{and} \quad E_u = \{(x, 0) \in \mathbb{R}^2\}.$$

The local stable manifold has the description

$$W_s^{loc}(0) = \{(x, y) \in \mathbb{R}^2 : x = f(y)\},$$

for some function $f$ such that $f(0) = f'(0) = 0$. Following the procedure described above, we find upon substitution,

$$f(y) + y^3 = f'(y)[-y + f(y)^2].$$

If we use the approximation $f(y) \approx Ay^2 + By^3$, we obtain

$$A = 0 \quad \text{and} \quad B = -1/4.$$

Thus, by Theorem 7.8, we have $f(y) \approx (-1/4)y^3$.
In the same fashion, we find

$$W_u^{loc}(0) = \{(x, y) \in \mathbb{R}^2 : y = g(x)\}, \quad \text{with} \quad g(x) \approx (1/3)x^2.$$

## 7.4 Exercises

**Exercise 7.1.** Illustrate Theorem 7.3 by comparing the flows of the one-dimensional equations $x' = x$ and $x' = \alpha x$, $\alpha \neq 0$.

(a) Find a homeomorphism which conjugates the two flows when $\alpha > 0$.
(b) Show that that the homeomorphism above can fail to be differentiable.
(c) Show that Theorem 7.3 fails when $\alpha < 0$.

**Exercise 7.2.**

(a) Find a homeomorphism defined in a neighborhood of the equilibrium $x = 0$ which conjugates the flow of $x' = x + x^2$ with the flow of the linearized equation at $x = 0$.
(b) Do the same for the equilibrium at $x = -1$.

**Exercise 7.3.** Let $f_i : \mathcal{O}_i \to \mathbb{R}^n$, $i = 1, 2$, be a pair of $C^1$ autonomous vector fields, and let $\psi_t^{(i)}$, $i = 1, 2$, be their flows. Suppose that $h$ is a diffeomorphism from $\mathcal{O}_1$ onto $\mathcal{O}_2$ such that $h \circ \psi_t^{(1)} = \psi_t^{(2)} \circ h$. Prove that $Dh(p)\ f^{(1)}(p) = f^{(2)} \circ h(p)$, for all $p \in \mathcal{O}_1$.

**Exercise 7.4.** Prove Theorem 7.4.

**Exercise 7.5.** Show that there does not exist a $C^1$ diffeomorphism from a neighborhood $0 \in U \subset \mathbb{R}^3$ into $\mathbb{R}^3$ which conjugates the flow of the linear vector field $Ax$ with the flow of the nonlinear vector field $Ax + f(x)$, where $A = \text{diag}[\alpha, \alpha - \gamma, -\gamma]$, $\alpha > \gamma > 0$, and

$$f(x) = \begin{bmatrix} 0 \\ \varepsilon x_1 x_3 \\ 0 \end{bmatrix}, \quad \varepsilon \neq 0.$$

(Exercise 7.1, Hartman [6])

**Exercise 7.6.** Let $A$ be a hyperbolic $n \times n$ matrix over $\mathbb{R}$. Let $P_s$, $P_u$ denote the projections onto $E_s$, $E_u$.

(a) Prove that if $E_s \neq \{0\}$, then $\|P_s\| \geq 1$, where $\|P_s\|$ is the usual operator norm of $P_s$.
(b) Prove that $\|P_s\| = 1$ if and only if $E_s \neq \{0\}$ and $E_s \perp E_u$.
(c) For $x \in \mathbb{R}^n$, define $\|x\|_* = \|P_s x\| + \|P_u x\|$. Prove that $\|\cdot\|_*$ is a norm on $\mathbb{R}^n$.
(d) Prove that if $E_s \neq \{0\}$, then $\|P_s\|_* = 1$, where $\|P_s\|_*$ is the operator norm of $P_s$ in the norm $\|\cdot\|_*$, i.e.

$$\|P_s\|_* = \sup_{x \neq 0} \frac{\|P_s x\|_*}{\|x\|_*}.$$

**Exercise 7.7.** Prove that the stable manifold and the local stable manifold of a hyperbolic equilibrium are positively invariant.

**Exercise 7.8.** Let $W_s(0)$ be the stable manifold of a hyperbolic equilibrium at 0 and suppose that $W_s^{loc}(0)$ is a local stable manifold relative to some neighborhood $U$ of 0. Prove that

$$W_s(0) = \{x_0 \in \mathbb{R}^n : x(t, x_0) \in W_s^{loc}(0) \text{ for some } t \in \mathbb{R}\}.$$

**Exercise 7.9.** Formulate an *unstable manifold theorem* and prove it using the stable manifold theorem.

**Exercise 7.10.** The system

$$\begin{aligned} x_1' &= x_1 + x_2 + x_2 x_3 \\ x_2' &= x_2 + x_3^2 \\ x_3' &= -x_3 + x_1 x_2 \end{aligned}$$

has a hyperbolic equilibrium at the origin. Approximate the local stable and unstable manifolds at the origin to within an error of order $O(\|x\|^3)$.

**Exercise 7.11.** Find $W_s(0)$ and $W_u(0)$ for the system

$$\begin{aligned} x_1' &= -x_1 \\ x_2' &= x_2 + x_1^2. \end{aligned}$$

**Exercise 7.12.** Let $W_s^{loc}(0)$ and $W_u^{loc}(0)$ be local stable and unstable manifolds relative to $B_r(0)$ for some $C^1$ vector field with a hyperbolic equilibrium at the origin. Prove that if $r > 0$ is sufficiently small, then $W_s^{loc}(0) \cap W_u^{loc}(0) = \{0\}$.

**Exercise 7.13.** [3]Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. Define $\lambda_u$ as in Theorem 2.3. Choose $0 \leq r < \lambda_u$ and define

$$X_r = \{x \in C^0([0,\infty);\mathbb{R}^n) : \sup_{t \geq 0} \mathrm{e}^{-rt} \|P_u x(t)\| < \infty\}.$$

Suppose that $h \in X_r$. (a) A solution of

$$x'(t) = Ax(t) + h(t), \quad x(0) = x_0 \tag{7.12}$$

belongs to $X_r$ if and only if

$$P_u x_0 + \int_0^\infty \exp(-A\tau) P_u h(\tau) d\tau = 0.$$

(b) A solution $x(t)$ of (7.12) in $X_r$ has the representation

$$x(t) = (I - P_u)\left[\exp At\ x_0 + \int_0^t \exp A(t-\tau) h(\tau) d\tau\right] - \int_t^\infty \exp A(t-\tau) P_u h(\tau) d\tau.$$

(c) Set

$$y(t) = \exp At\ y_0 + \int_0^t \exp A(t-\tau)(I - P_u) h(\tau) d\tau - \int_t^\infty \exp A(t-\tau) P_u h(\tau) d\tau.$$

If $y \in X_r$, then $y$ solves (7.12) with $x_0 = y_0 - \int_0^\infty \exp(-A\tau) P_u h(\tau) d\tau$, and hence $P_u y_0 = 0$.

[3] Compare with Lemma 7.3.

# Chapter 8
# Periodic Solutions

## 8.1 Existence of Periodic Solutions in $\mathbb{R}^n$: Noncritical Case

We are going to consider time $T$-periodic perturbations of an autonomous vector field with an equilibrium. If the linearized problem has no nonzero $T$-periodic solution, then we shall demonstrate the existence of a unique $T$-periodic solution of the perturbed equation near the equilibrium. This is the so-called *noncritical* case.

The linear version of this result corresponds to Theorem 4.9. If $p(t)$ is a $T$-periodic function for some $T > 0$, then the linear system $x' = Ax + \varepsilon p(t)$ has a unique $T$-periodic solution for all $\varepsilon$ if and only if the adjoint system $x' = -A^T x$ has no $T$-periodic solutions. This condition is equivalent to assuming that $-A^T$, and hence also $A$, has no eigenvalues in the set $\{2\pi i k/T : k \in \mathbb{Z}\}$. We now formulate a nonlinear version of this result.

**Theorem 8.1.** *Let $f(t, x, \varepsilon)$ be a $C^1$ map from $\mathbb{R} \times \mathbb{R}^n \times (-\varepsilon_0, \varepsilon_0)$ into $\mathbb{R}^n$ such that*

*(i) There is a $T > 0$, such that*

$$f(t + T, x, \varepsilon) = f(t, x, \varepsilon) \quad \textit{for all} \quad (t, x, \varepsilon) \in \mathbb{R} \times \mathbb{R}^n \times (-\varepsilon_0, \varepsilon_0).$$

*(ii) $f(t, x, 0) = f_0(x)$ is autonomous and $f_0(0) = 0$.*

*Set $A = Df_0(0)$. If $T\lambda \notin 2\pi i\mathbb{Z}$ for each eigenvalue $\lambda$ of $A$, then there exists a neighborhood of $U$ of $0 \in \mathbb{R}^n$ and an $0 < \varepsilon_1 < \varepsilon_0$ such that for every $|\varepsilon| < \varepsilon_1$ there is a unique initial point $p = p(\varepsilon) \in U$ for which the solution, $x(t, p, \varepsilon)$, of the initial value problem*

$$x' = f(t, x, \varepsilon), \quad x(0, p, \varepsilon) = p$$

*is $T$-periodic.*

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Atlantis Studies in Differential Equations 2, DOI: 10.2991/978-94-6239-021-8_8,

*Remarks 8.1.*

- The assumption on the eigenvalues of $A$ means that the linear equation $x' = Ax$ (and its adjoint system) has no nonzero $T$-periodic solutions.
- Under the assumptions of the theorem, we can write

$$f(t, x, \varepsilon) = f_0(x) + \varepsilon \tilde{f}(t, x, \varepsilon), \quad \text{where} \quad \tilde{f}(t, x, \varepsilon) = \int_0^1 \frac{\partial f}{\partial \varepsilon}(t, x, \sigma\varepsilon) d\sigma.$$

Notice that $f_0 \in C^1$ and if $\frac{\partial f}{\partial \varepsilon} \in C^1$, then $\tilde{f} \in C^1$, and

$$\tilde{f}(t + T, x, \varepsilon) = \tilde{f}(t, x, \varepsilon).$$

Thus, we have generalized the linear case $Ax + \varepsilon p(t)$, where $p(t)$ is $T$-periodic.

*Proof.* Let $I(p, \varepsilon) = (\alpha(p, \varepsilon), \beta(p, \varepsilon))$ be the maximal existence interval for the solution $x(t, p, \varepsilon)$. Since $f$ is $C^1$, $x$ is $C^1$ in $(t, p, \varepsilon)$, by Corollary 6.3. Moreover, since $x(t, 0, 0) \equiv 0$, we have that $I(0, 0) = (-\infty, \infty)$. Therefore, by continuous dependence (Theorem 6.2), we know that $[0, T] \subset I(p, \varepsilon)$ for all $(p, \varepsilon)$ sufficiently small, say in a neighborhood $U' \times (-\varepsilon', \varepsilon')$ of $(p, \varepsilon) = (0, 0) \in \mathbb{R}^n \times \mathbb{R}$.

Let $Y(t) = D_p x(t, 0, 0)$. Then since (by Theorem 6.1) $D_p x(t, p, \varepsilon)$ solves the linear variational equation

$$\frac{d}{dt} D_p x(t, p, \varepsilon) = D_x f(t, x(t, p, \varepsilon), \varepsilon) D_p x(t, p, \varepsilon), \quad D_p x(0, p, \varepsilon) = I,$$

we have that $Y(t)$ solves

$$Y'(t) = D_x f(t, 0, 0) Y(t) = D_x f_0(0) Y(t) = AY(t), \quad Y(0) = I.$$

Thus, we see that $Y(t) = \exp At$ is the fundamental matrix of $A$.

For $(p, \varepsilon) \in U' \times (-\varepsilon', \varepsilon')$, the "time $T$" or Poincaré map

$$\Pi(p, \varepsilon) = x(T, p, \varepsilon)$$

is well-defined. Note that $\Pi(0, 0) = 0$, and by smooth dependence,

$$\Pi : U' \times (-\varepsilon', \varepsilon') \to \mathbb{R}^n \quad \text{is} \quad C^1.$$

By our definitions, we have

$$D_p \Pi(0, 0) = D_p x(T, p, \varepsilon)\Big|_{(p,\varepsilon)=(0,0)} = Y(T) = \exp AT.$$

Now set $Q(p, \varepsilon) = \Pi(p, \varepsilon) - p$. If we transfer the properties of $\Pi$ to $Q$, we see that $Q : U' \times (-\varepsilon', \varepsilon') \to \mathbb{R}^n$ is $C^1$, $Q(0, 0) = 0$, and $D_p Q(0, 0) = e^{AT} - I$.

Our assumptions on $A$ ensure that this matrix is invertible. By the Implicit Function Theorem 5.7, there are neighborhoods $(-\varepsilon_1, \varepsilon_1) \subset (-\varepsilon', \varepsilon')$, $0 \in U \subset U'$, and a $C^1$ function $p : (-\varepsilon_1, \varepsilon_1) \to U$ such that $Q(p(\varepsilon), \varepsilon) = 0$, for all $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$. Moreover, if $(p, \varepsilon) \in U \times (-\varepsilon_1, \varepsilon_1)$ satisfies $Q(p, \varepsilon) = 0$, then $p = p(\varepsilon)$.

This is the same as saying that

$$x(T, p(\varepsilon), \varepsilon) = \Pi(p(\varepsilon), \varepsilon) = p(\varepsilon).$$

Thanks to the periodicity of the vector field $f(t, x, \varepsilon)$ in $t$, Lemma 4.5 implies that $x(t, p(\varepsilon), \varepsilon)$ is a global $T$-periodic solution. □

*Remark 8.2.* If the matrix $A$ is hyperbolic, then there are no eigenvalues on the imaginary axis and the condition of Theorem 8.1 is met.

*Remark 8.3.* A typical situation where this Theorem might be applied is $f(t, x, \varepsilon) = f_0(x) + \varepsilon p(t)$, with $f_0(0) = 0$ and $p(t + T) = p(t)$.

*Example 8.1.* Consider Newton's equation $u'' + g(u) = \varepsilon p(t)$, in which $g(0) = 0$, $g'(0) \neq 0$, and $p(t + T) = p(t)$ for all $t \in \mathbb{R}$.

If $g'(0) = -\gamma^2$, then the matrix $A$ is hyperbolic. There exists a small $T$-periodic solutions for small enough $\varepsilon$.

If $g'(0) = \gamma^2$, then there is a $T$-periodic solution for small $\varepsilon$ provided that $\gamma T/2\pi$ is not an integer.

Consider the linear equation,

$$u'' + u = \varepsilon \cos(\omega t).$$

Here we have $g'(0) = 1$ and $T = 2\pi/\omega$. According to Theorem 8.1, for small $\varepsilon$ there is a $T$-periodic solution if $1/\omega$ is not an integer. This is consistent with (but weaker than) the familiar nonresonance condition which says that if $\omega^2 \neq 1$, then

$$u_\varepsilon(t) = \frac{\varepsilon}{1 - \omega^2} \cos(\omega t)$$

is the unique $T$-periodic solution for every $\varepsilon$.

*Example 8.2.* Consider the Duffing equation

$$u'' + \alpha u' + \gamma^2 u - \varepsilon u^3 = B \cos \omega t, \quad \alpha > 0.$$

Notice that the periodic forcing term does not have small amplitude. The nonlinear term is "small", however, which allows us to rewrite the equation in a form to which the Theorem applies. Let $v = \varepsilon^{1/2} u$. If we multiply the equation by $\delta = \varepsilon^{1/2}$, then there results

$$v'' + \alpha v' + \gamma^2 v - v^3 = B\delta \cos \omega t.$$

When this equation is written as a first order system, it has the form

$$x' = f(x) + \delta p(t), \quad p(t + 2\pi/\omega) = p(t)$$

with $Df(0)$ hyperbolic, so that Theorem 8.1 ensures the existence of a $2\pi/\omega$-periodic solution.

## 8.2 Stability of Periodic Solutions to Nonautonomous Periodic Systems

Having established the existence of periodic solutions to certain periodic nonautonomous systems, we now examine their stability.

We shall explicitly label our assumptions for this section and the next.

A1. Let $f : \mathbb{R}^{1+n} \to \mathbb{R}^n$ be a $C^1$ nonautonomous $T$-periodic vector field: $f(t+T, x) = f(t, x)$ for all $(t, x) \in \mathbb{R}^{1+n}$.

A2. Given $(\tau, x_0) \in \mathbb{R}^{1+n}$, let $x(t, \tau, x_0)$ represent the solution of the initial value problem

$$x' = f(t, x), \quad x(\tau) = x_0.$$

A3. Assume that $\varphi(t)$ is a $T$-periodic solution of $x' = f(t, x)$, defined for all $t \in \mathbb{R}$.

A4. Let $Y(t)$ be the fundamental matrix for the $T$-periodic matrix $A(t) = Df(\varphi(t))$. That is,

$$Y'(t) = A(t)Y(t), \quad Y(0) = I.$$

This is the situation to which Floquet theory applies. By Theorem 4.5, there exist real matrices $P(t)$, $B$, and $R$ such that

$$Y(t) = P(t) \exp Bt, \quad P(t+T) = P(t)R, \quad R^2 = I, \quad RB = BR.$$

If $\exp BT$ has no eigenvalues in the interval $(-\infty, 0)$, then by Theorem 4.4, $R = I$ and $P(t)$ is $T$-periodic. Otherwise, $P(t)$ is $2T$-periodic.

**Definition 8.1.** *The eigenvalues of* $\exp BT$ *are called the Floquet multipliers of* $\varphi$*. The eigenvalues of* $B$ *are called the Floquet exponents of* $\varphi$*.*

*Remark 8.4.* The Floquet multipliers of $\varphi$ are unique. The Floquet exponents of $\varphi$ are unique modulo $2\pi i/T$, by Lemma 4.2.

**Definition 8.2.** *A periodic solution* $\varphi(t)$ *is stable if for every* $\varepsilon > 0$ *and every* $\tau \in \mathbb{R}$ *there is a* $\delta > 0$ *such that*

$$\|x_0 - \varphi(\tau)\| < \delta \quad \text{implies that} \quad \|x(t, \tau, x_0) - \varphi(t)\| < \varepsilon,$$

*for all* $t \geq \tau$*.*

*If in addition,* $\delta$ *may be chosen so that*

$$\|x_0 - \varphi(\tau)\| < \delta \quad \textit{implies that} \quad \lim_{t\to\infty} \|x(t,\tau,x_0) - \varphi(t)\| = 0,$$

*then $\varphi(t)$ is said to be asymptotically stable.*

**Theorem 8.2.** *Suppose that A1-A4 hold. If the Floquet exponents of $\varphi$ satisfy* $\operatorname{Re}\lambda < -\lambda_0 < 0$ *(equivalently, if the Floquet multipliers satisfy $|\mu| < e^{-\lambda_0} < 1$), then $\varphi$ is asymptotically stable. There exist $C > 0$, $\delta > 0$ such that if $\|x_0 - \varphi(\tau)\| < \delta$ for some $\tau \in \mathbb{R}$, then*

$$\|x(t,\tau,x_0) - \varphi(t)\| \le Ce^{-\lambda_0(t-\tau)}\|x_0 - \varphi(\tau)\|,$$

*for all $t \ge \tau$.*

A5. Define

$$H(t,z) = P(t)^{-1}[f(t,\varphi(t) + P(t)z) - f(t,\varphi(t)) - A(t)P(t)z].$$

Note that $H \in C^1(\mathbb{R}^{1+n},\mathbb{R}^n)$, $H(t,0) = 0$, $D_z H(t,0) = 0$, and

$$H(t+T,z) = RH(t,Rz).$$

Let $z(t,\tau,z_0)$ be the solution of the initial value problem

$$z' = Bz + H(t,z), \quad z(\tau) = z_0. \tag{8.1}$$

The proof of Theorem 8.2 is based on a reduction to the system (8.1) given by the next lemma. This reduction will be used again in the following section.

**Lemma 8.1.** *Suppose that A1-A5 hold. For $(t,z) \in \mathbb{R}^{1+n}$, define the map*

$$\xi(t,z) = \varphi(t) + P(t)z.$$

*If $x_0 = \xi(\tau,z_0)$, then*

$$x(t,\tau,x_0) = \xi(t,z(t,\tau,z_0)).$$

*Proof.* Since $P(t) = Y(t)\exp(-Bt)$, we have

$$P'(t) = A(t)Y(t)\exp(-Bt) - Y(t)\exp(-Bt)B = A(t)P(t) - P(t)B.$$

Given $(\tau,x_0) \in \mathbb{R}^{1+n}$, let $x(t) = x(t,\tau,x_0)$ and define $z(t)$ by means of

$$x(t) = \xi(t,z(t)) = \varphi(t) + P(t)z(t).$$

Then we have

$$\begin{aligned}
f(t,\varphi(t)+P(t)z(t)) &= f(t,x(t))\\
&= x'(t)\\
&= \varphi'(t)+P'(t)z(t)+P(t)z'(t)\\
&= f(t,\varphi(t))+[A(t)P(t)-P(t)B]z(t)+P(t)z'(t)\\
&= f(t,\varphi(t))+A(t)P(t)z(t)+P(t)[-Bz(t)+z'(t)].
\end{aligned}$$

A simple rearrangement gives

$$z'(t) = Bz(t) + H(t, z(t)).$$

Letting $z_0 = z(\tau)$, we have $x_0 = \xi(\tau, z_0)$. Therefore, we have shown that $x(t,\tau,x_0) = z(t,\tau,z_0)$, by the Uniqueness Theorem 3.3. □

*Proof of Theorem 8.2.* Recall Theorem 3.11 which said that if $F(x)$ is a $C^1$ autonomous vector field on $\mathbb{R}^n$ with $F(0) = 0$, $DF(0) = 0$, and $A$ is an $n \times n$ matrix whose eigenvalues all have negative real part, then the origin is asymptotically stable for $x' = Ax + F(x)$. The present situation differs in that the nonlinear portion of the vector field $H(t,z)$ is nonautonomous. However, the fact that $H(t+2T, z) = H(t,z)$ gives uniformity in the $t$ variable which permits us to use the same argument as in the autonomous case. The outcome is the following statement:

There exist $C_0 > 0$, $\delta_0 > 0$ such that if $\|z_0\| < \delta_0$, then $z(t,\tau,z_0)$ is defined for all $t \geq \tau$, and

$$\|z(t,\tau,z_0)\| < C_0 e^{-\lambda_0(t-\tau)}\|z_0\|, \tag{8.2}$$

for every $t \geq \tau$.

Given $(\tau, x_0)$, set $z_0 = P(\tau)^{-1}[x_0 - \varphi(\tau)]$. Thus, by Lemma 8.1, we have

$$x(t,\tau,x_0) - \varphi(t) = P(t)z(t,\tau,z_0).$$

The constant

$$M = \max_{0\leq t\leq 2T}[\|P(t)\| + \|P(t)^{-1}\|].$$

bounds $\|P(t)\| + \|P(t)^{-1}\|$ for all $t \in \mathbb{R}$, since $P$ and $P^{-1}$ are $2T$-periodic. Put $\delta = \delta_0/M$. If $\|x_0 - \varphi(\tau)\| < \delta$, then $\|z_0\| < \delta_0$ and so by (8.2)

$$\|x(t,\tau,x_0) - \varphi(\tau)\| \leq M\|z(t,\tau,z_0)\| \leq C_0 M^2 e^{-\lambda_0(t-\tau)}\|x_0 - \varphi(\tau)\|. \quad \square$$

*Remark 8.5.* The defect of this theorem is that it is necessary to at least estimate the Floquet multipliers. We have already seen that this can be difficult. However, suppose that we are in the situation of Sect. 8.1, namely the vector field has the form

$$f(t,x,\varepsilon) = f_0(x) + \varepsilon\tilde{f}(t,x,\varepsilon),$$

with $f_0(0) = 0$ and $\tilde{f}(t + T, x, \varepsilon) = \tilde{f}(t, x, \varepsilon)$. Let $\varphi_\varepsilon$ be the unique $T$-periodic orbit of $f(t, x, \varepsilon)$ near the origin. Then by continuous dependence

$$A_\varepsilon(t) = D_x f(t, \varphi_\varepsilon(t), \varepsilon) \approx D_x f_0(0) = A.$$

Again by continuous dependence, the fundamental matrix $Y_\varepsilon(t)$ of $A_\varepsilon(t)$ is close to $\exp At$. So for $\varepsilon$ sufficiently small the Floquet multipliers of $\varphi_\varepsilon$ are close to the eigenvalues of $\exp AT$. If the eigenvalues of $A$ all have $\operatorname{Re}\lambda < 0$, then $\varphi_\varepsilon$ is asymptotically stable, for $\varepsilon$ small.

## 8.3 Stable Manifold Theorem for Nonautonomous Periodic Systems

We continue the notation and assumptions of the previous section.

**Theorem 8.3.** *Suppose that A1-A5 hold. Assume that $B$ is hyperbolic, and let $P_s$, $P_u$ be the projections onto the stable and unstable subspaces $E_s$, $E_u$ of $B$.*

*There is a neighborhood of the origin $U \subset E_s$ and a $C^1$ function $\eta : \mathbb{R} \times U \to E_u$ such that*

*(i)* $RU = U$, $\eta(\tau, 0) = 0$ *and* $D_y\eta(\tau, 0) = 0$ *for all* $\tau \in \mathbb{R}$, *and*

$$\eta(\tau + T, y) = R\eta(\tau, Ry), \text{ for all } (\tau, y) \in \mathbb{R} \times U.$$

*(ii) Define*

$$\mathcal{A} \equiv \{(\tau, z_0) \in \mathbb{R}^{1+n} : P_s z_0 \in U,\ P_u z_0 = \eta(\tau, P_s z_0)\}.$$

*There are positive constants $C_0$, $\lambda$ such that for all $(\tau, z_0) \in \mathcal{A}$, $z(t, \tau, z_0)$ is defined for $t \geq \tau$ and*

$$\|z(t, \tau, z_0)\| \leq C_0 e^{-[\lambda(t-\tau)/2]} \|P_s z_0\|. \tag{8.3}$$

*(iii) Define*

$$\mathcal{B} = \{(\tau, z_0) \in \mathbb{R}^{1+n} : \|z(t, \tau, z_0)\| < \rho, \text{ for all } t \geq \tau\}.$$

*If $\rho > 0$ is sufficiently small, then $\mathcal{B} \subset \mathcal{A}$ and $\mathcal{B}$ is a positively invariant manifold of dimension* $\dim E_s + 1$ *and tangent to $\mathbb{R} \times E_s$ along $\mathbb{R} \times \{0\}$.*

*(iv) If $\lim_{t\to\infty} z(t, \tau, z_0) = 0$, then there exists $t_0 \geq \tau$ such that $(t, z(t, \tau, z_0)) \in \mathcal{A}$, for all $t \geq t_0$.*

*Proof.* We will merely provide a sketch of the proof since the details are similar to the autonomous case, Theorem 7.6.

By Corollary 2.1, there exist constants $C_0, \lambda > 0$ such that

$$\|\exp Bt\ \ P_s z\| \le C_0 \mathrm{e}^{-\lambda t} \|P_s z\|,\ \ t \ge 0 \tag{8.4}$$

and

$$\|\exp Bt\ \ P_u z\| \le C_0 \mathrm{e}^{\lambda t} \|P_u z\|,\ \ t \le 0. \tag{8.5}$$

If $z \in E_s$, then since $R$ and $B$ commute,

$$\exp Bt\ \ Rz = R \exp Bt\ z \to 0, \quad \text{as} \quad t \to \infty,$$

by (8.4). Thus, by Theorem 2.4, $Rz \in E_s$, and we have that $R : E_s \to E_s$. Similarly, $R : E_u \to E_u$. It follows that $R$ commutes with $P_s$ and $P_u$.

Throughout the proof we will use a special norm on $\mathbb{R}^n$:

$$\|z\| = \|P_s z\|_{\mathbb{R}^n} + \|P_u z\|_{\mathbb{R}^n} + \|P_s R z\|_{\mathbb{R}^n} + \|P_u R z\|_{\mathbb{R}^n},$$

where $\|\cdot\|_{\mathbb{R}^n}$ is the standard Euclidean norm. Since $R$ commutes with $P_s$ and $P_u$, we have in our chosen norm

$$\|P_s\| = \|P_u\| = \|R\| = 1.$$

Since $H(t + 2T, z) = H(t, z)$, for any $r > 0$,

$$\begin{aligned} C_r &= \sup\{\|D_z H(t, z)\| : (t, z) \in \mathbb{R} \times B_r(0)\} \\ &= \max\{\|D_z H(t, z)\| : (t, z) \in [0, 2T] \times B_r(0)\}. \end{aligned}$$

Since $D_z H(t, 0) = 0$,

$$\lim_{r \to 0} C_r = 0$$

and

$$\|H(t, z_1) - H(t, z_2)\| \le C_r \|z_1 - z_2\|, \quad \text{for all} \quad z_1, z_2 \in B_r(0). \tag{8.6}$$

Define the Banach space

$$Y = C_b^0(\{(t, \tau) \in \mathbb{R}^2 : t \ge \tau\}; \mathbb{R}^n)$$

with the sup norm.

The operators

$$\begin{aligned} I_s(g)(t, \tau) &= \int_\tau^t \exp B(t - \sigma)\ \ P_s g(\sigma, \tau)\, d\sigma \\ I_u(g)(t, \tau) &= \int_t^\infty \exp B(t - \sigma)\ \ P_u g(\sigma, \tau)\, d\sigma \end{aligned}$$

belong to $\mathcal{L}(Y, Y)$ and by (8.4), (8.5)

$$\|I_s\|, \|I_u\| \leq C_0/\lambda.$$

If $z(t, \tau, z_0)$ is bounded for all $t \geq \tau$, then $z(t, \tau, z_0)$ solves

$$\begin{aligned} y(t) = & \exp B(t-\tau)\ y_0 + \int_\tau^t \exp B(t-\sigma)\ P_s H(\sigma, y(\sigma))\, d\sigma \\ & - \int_t^\infty \exp B(t-\sigma)\ P_u H(\sigma, y(\sigma))\, d\sigma, \end{aligned} \tag{8.7}$$

with $y_0 = P_s z_0$.

If $y(t)$ is bounded for $t \geq \tau$ and solves (8.7) for some $(\tau, y_0) \in \mathbb{R}^{1+n}$, then $y_0 \in E_s$ and

$$y(t) = z(t, \tau, z_0), \quad \text{with} \quad z_0 = y(\tau).$$

Suppose that $y_i(t)$ are solutions of (8.7) with data $(\tau, y_{i0})$ such that $\|y_i(t)\| < r$, $t \geq \tau$, $i = 1, 2$. It follows from (8.4–8.6), (8.7) that

$$\begin{aligned} \|y_1(t) - y_2(t)\| \leq\ & C_0 e^{-\lambda(t-\tau)} \|y_{10} - y_{20}\| \\ & + C_0 C_r \int_\tau^t e^{-\lambda(t-\sigma)} \|y_1(\sigma) - y_2(\sigma)\| d\sigma \\ & + C_0 C_r \int_t^\infty e^{\lambda(t-\sigma)} \|y_1(\sigma) - y_2(\sigma)\| d\sigma. \end{aligned}$$

If the condition

$$4C_0 C_r/\lambda < 1 \tag{8.8}$$

holds, then by Lemma 7.5, we obtain the estimate

$$\|y_1(t) - y_2(t)\| \leq C_0 e^{-\lambda(t-\tau)/2} \|y_{10} - y_{20}\|. \tag{8.9}$$

It follows that solutions of (8.7) are unique among functions with sup norm smaller than $r$.

Define the mapping $F : E_s \times Y \to Y$ by

$$\begin{aligned} F(y_0, y)(t, \tau) = & y(t, \tau) - \exp B(t-\tau) y_0 \\ & - \int_\tau^t \exp B(t-\sigma) P_s H(\sigma, y(\sigma, \tau)) d\sigma \\ & + \int_t^\infty \exp B(t-\sigma) P_u H(\sigma, y(\sigma, \tau)) d\sigma. \end{aligned}$$

$F$ is a $C^1$ mapping, $F(0, 0) = 0$, and $D_y(0, 0) = I$ is invertible.

By the Implicit Function Theorem 5.7, there exist neighborhoods of the origin $U \subset E_s$ and $V \subset Y$ and a $C^1$ map $\phi : U \to V$ such that $\phi(0) = 0$ and $F(y_0, \phi(y_0)) = 0$, for all $y_0 \in U$. In addition, if $(y_0, y) \in U \times V$ and $F(y_0, y) = 0$, then $y = \phi(y_0)$. Write $y(t, \tau, y_0) = \phi(y_0)(t, \tau)$.

Without loss of generality, we may assume that $U$ and $V$ are balls, in which case $RU = U$ and $RV = V$, thanks to our choice of norm. We choose $V = B_r(0)$ with $r > 0$ so that (8.8) holds.

Define $S_T$ by

$$S_T y(t, \tau) = y(t - T, \tau - T).$$

Then $S_T \in \mathcal{L}(Y, Y)$. In fact, we have $\|S_T g\| = \|g\|$, for $g \in Y$. A direct calculation yields that

$$F(y_0, y) = RS_T F(Ry_0, RS_{-T} y).$$

Since $V$ is a ball in $Y$ we have $S_T V = V$. It follows that if $y_0 \in U$, then $Ry_0 \in U$ and $RS_{-T}\phi(y_0) \in V$, hence

$$\phi(Ry_0) = RS_{-T}\phi(y_0). \tag{8.10}$$

Define $\eta(\tau, y_0) = P_u\phi(y_0)(\tau, \tau)$. Then

$$\eta \in C^1(\mathbb{R} \times U; E_u), \quad \eta(\tau, 0) = 0, \quad D_y\eta(\tau, 0) = 0.$$

Moreover, from (8.10) we have that

$$\eta(\tau, Ry_0) = R\eta(\tau + T, y_0), \quad \text{for all} \quad (\tau, y_0) \in \mathbb{R} \times U. \tag{8.11}$$

So $\eta$ has the desired properties of part (i).

Let $\mathcal{A}$ be defined as in (ii). If $(\tau, z_0) \in \mathcal{A}$, then $y_0 = P_s z_0 \in U$, so $\phi(y_0) \in V$. Thus, we have that $y(t, \tau, y_0) = \phi(y_0)(t, \tau)$ is a solution of (8.7) with $\|y(\cdot, \tau, y_0)\| < r$. From (8.9) with $y_2 = 0$ we obtain exponential decay for $y(t, \tau, y_0)$. Now $y(\tau, \tau, y_0) = y_0 + \eta(\tau, y_0) = z_0$, and so $y(t, \tau, y_0)$ solves the IVP with initial data $(\tau, z_0)$, i.e. $y(t, \tau, y_0) = z(t, \tau, z_0)$. This is part (ii).

Let $\mathcal{B}$ be as in (iii), with $0 < \rho < r$ chosen small enough so that $B_\rho(0) \subset U$. If $(\tau, z_0) \in \mathcal{B}$, then $z(t, \tau, z_0)$ solves (8.7) with data $(\tau, y_0) = (\tau, P_s z_0)$. We have that $\|y_0\| = \|P_s z_0\| = \|P_s z(\tau, \tau, z_0)\| < \rho$, so $y_0 \in U$. Thus, $\phi(y_0) \in V$ and consequently, $y(t, \tau, y_0)$ is also a solution of (8.7) with sup norm smaller than $r$. By uniqueness, $z(t, \tau, z_0) = y(t, \tau, y_0)$, and $z_0 = y(\tau, \tau, y_0) = y_0 + \eta(\tau, y_0)$. This shows that $(\tau, z_0) \in \mathcal{A}$.

On the other hand, suppose that $\rho' < \min\{\rho, r/(2C_0)\}$. Then $B_{\rho'}(0) \subset U$. By part (ii), (8.3), we have that

$$\mathcal{A}' = \{(t, z_0) : P_s z_0 \in B_{\rho'}(0), \; P_u z_0 = \eta(\tau, P_s z_0)\} \subset \mathcal{B} \subset \mathcal{A}.$$

This implies that $\mathcal{B}$ is a nontrivial $C^1$ manifold. The properties of $\eta$ imply that $\mathcal{B}$ is tangent to $\mathbb{R} \times E_s$ along $\mathbb{R} \times \{0\} \subset \mathcal{B}$.

The set $\mathcal{B}$ is positively invariant in the sense that if $(\tau, z_0) \in \mathcal{B}$, then $(\sigma, z(\sigma, \tau, z_0)) \in \mathcal{B}$ for all $\sigma \geq \tau$. This follows by the flow property

$$z(t, \sigma, z(\sigma, \tau, z_0)) = z(t, \tau, z_0), \quad t \geq \sigma \geq \tau. \tag{8.12}$$

This proves (iii).

Finally, if $\lim_{t\to\infty} z(t, \tau, z_0) = 0$, then there exists $t_0 \geq \tau$ such that

$$\|z(t, \tau, z_0)\| < \rho, \quad t \geq t_0.$$

It follows from (8.12) that $\sigma, z(\sigma, \tau, z_0)) \in \mathcal{B}$, for $\sigma \geq t_0$. This proves (iv). □

**Definition 8.3.** *The stable manifold of a periodic solution $\varphi$ is the set*

$$W_s(\varphi) = \{(\tau, x_0) \in \mathbb{R}^{1+n} : x(t, \tau, x_0) \text{ exists for all } t \geq \tau, \\ \text{and } \lim_{t\to\infty} [x(t, \tau, x_0) - \varphi(t)] = 0\}.$$

*A set $W_s^{loc}(\varphi)$ is called a local stable manifold of $\varphi$ if there is a neighborhood $\mathcal{V}$ of the curve $\{(t, \varphi(t)) : t \in \mathbb{R}\}$ in $\mathbb{R}^{1+n}$ such that*

$$W_s^{loc}(\varphi) = W_s(\varphi) \cap \mathcal{V}.$$

The unstable and local unstable manifolds of a periodic solution are defined in the analogous manner.

**Theorem 8.4.** *Let A1-A5 hold. Assume that $B$ is hyperbolic. Define $\mathcal{A}(\varphi)$, $\mathcal{B}(\varphi)$ to be the images under the mapping*

$$(t, z) \mapsto (t, \xi(t, z)) = (t, \varphi(t) + P(t)z)$$

*of the sets $\mathcal{A}$, $\mathcal{B}$ constructed in Theorem 8.3 Then $\mathcal{B}(\varphi) \subset \mathcal{A}(\varphi)$ is a positively invariant local stable manifold. The global stable manifold is a $C^1$ manifold of dimension equal to* $\dim E_s + 1$. *The set $\mathcal{A}(\varphi)$ is $T$-periodic.*

*Proof.* Except for the final statement, this is an immediate consequence of Theorem 8.3 since

$$W_s(\varphi) = \{(\tau, x_0) \in \mathbb{R}^{1+n} : (t, x(t, \tau, x_0)) \in \mathcal{B}(\varphi) \text{ for some } t \in \mathbb{R}\}.$$

We now show that $\mathcal{A}(\varphi)$ is $T$-periodic. By the property (8.11) that

$$\eta(t + T, P_s z) = R\eta(t, P_s Rz), \quad z \in U,$$

we have that

$$(t, z) \in \mathcal{A} \quad \text{if and only if} \quad (t + T, Rz) \in \mathcal{A}.$$

By the periodicity of $\varphi$ and the property $P(t + T) = P(t)R$, we also have

$$\xi(t + T, z) = \varphi(t + T) + P(t + T)z = \varphi(t) + P(t)Rz = \xi(t, Rz).$$

These facts now yield

$$\begin{aligned}
\mathcal{A}(\varphi) &= \{(t, x) : x = \xi(t, z),\ (t, z) \in \mathcal{A}\} \\
&= \{(t + T, x) : x = \xi(t + T, z),\ (t + T, z) \in \mathcal{A}\} \\
&= \{(t + T, x) : x = \xi(t, Rz),\ (t + T, z) \in \mathcal{A}\} \\
&= \{(t + T, x) : x = \xi(t, z),\ (t + T, Rz) \in \mathcal{A}\} \\
&= \{(t + T, x) : x = \xi(t, z),\ (t, z) \in \mathcal{A}\}.
\end{aligned}$$

□

*Remarks 8.6.*

- By reversing the sense of time in the system, this theorem also immediately ensures the existence of a negatively invariant local unstable manifold $W_u^{loc}(\varphi)$.
- The periodic solution is the intersection of the local stable and unstable manifolds. We say that the periodic solution $\varphi$ is hyperbolic.
- A tangent vector $v$ to $W_s(\varphi)$ at $(\tau, \phi(\tau))$ which is perpendicular to $(t, \varphi'(\tau))$ moves by parallel transport over one period to the tangent vector $Rv$ at $(\tau + T, \varphi(\tau))$. If there are Floquet multipliers on the negative real axis, then $R \neq I$. The local stable manifold makes an odd number of twists around $\varphi$ over one period, and so it is a generalized Möbius band.

*Example 8.3.* (Newton's Equation with Small Periodic Forcing).
Recall that Newton's equation is the second order equation,

$$u'' + g(u) = 0,$$

which is equivalent to the first order system

$$x' = f(x), \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u \\ u' \end{bmatrix}, \quad f(x) = \begin{bmatrix} x_2 \\ -g(x_1) \end{bmatrix}.$$

Here we assume that $g : \mathbb{R} \to \mathbb{R}$ is a $C^3$ function with $g(0) = 0$ and $g'(0) = \gamma^2 > 0$, so that the origin is a non-hyperbolic equilibrium point for this system. In fact, if $G'(u) = g(u)$, then solutions satisfy $x_2^2(t)/2 + G(x_1(t)) = \text{Const.}$, which shows that all orbits near the origin are closed, and hence periodic.

Let $p(t)$ be a continuous $T$-periodic function. Consider the forced equation

$$u''(t) + g(u(t)) = \varepsilon p(t),$$

or equivalently,

$$x' = f(t, x, \varepsilon), \quad f(t, x, \varepsilon) = \begin{bmatrix} x_2 \\ -g(x_1) + \varepsilon p(t) \end{bmatrix}.$$

In Example 8.1, we saw that if

$$\gamma T \neq 2j\pi, \quad j \in \mathbb{Z}, \tag{8.13}$$

then, for small $\varepsilon$, there is a unique periodic solution $x_\varepsilon(t)$ of the forced equation near the origin. Throughout this example, we shall rely heavily on the fact that $x_\varepsilon$ depends smoothly on the parameter $\varepsilon$, by Theorem 6.3.

In spite of the fact that the origin is not a hyperbolic equilibrium point when $\varepsilon = 0$, we will show that the periodic solution $x_\varepsilon(t)$ of the forced equation is hyperbolic, under certain restrictions on $g$ and $p$, see (8.16) and (8.23). This will follow from Theorem 8.4 if we show that the Floquet multipliers of $x_\varepsilon$ lie off of the unit circle.

Let $u_\varepsilon(t)$ denote the first component of $x_\varepsilon(t)$. Then $u_\varepsilon(t)$ solves

$$u_\varepsilon''(t) + g(u_\varepsilon(t)) = \varepsilon p(t). \tag{8.14}$$

Set

$$A(t, \varepsilon) = D_x f(t, x_\varepsilon(t), 0) = \begin{bmatrix} 0 & 1 \\ -g'(u_\varepsilon(t)) & 0 \end{bmatrix},$$

and let $Y(t, \varepsilon)$ be its fundamental matrix. That is,

$$Y'(t, \varepsilon) = A(t, \varepsilon)Y(t, \varepsilon) \quad Y(0, \varepsilon) = I. \tag{8.15}$$

The Floquet multipliers of $x_\varepsilon$ are the eigenvalues of $Y(T, \varepsilon)$. We shall denote them by $\mu_1(\varepsilon)$ and $\mu_2(\varepsilon)$, and without loss of generality we assume that $\operatorname{Re}\mu_1(\varepsilon) \geq \operatorname{Re}\mu_2(\varepsilon)$.

Observe that by Theorem 4.6, $\mu_1(\varepsilon)$ and $\mu_2(\varepsilon)$ satisfy

$$\mu_1(\varepsilon)\mu_2(\varepsilon) = \det Y(T, \varepsilon) = \exp\left(\operatorname{tr}\int_0^T A(t, \varepsilon)dt\right) = 1.$$

This means that one of the following hold:

I. $\mu_1(\varepsilon) = \overline{\mu_2(\varepsilon)}$, $|\mu_1(\varepsilon)| = |\mu_2(\varepsilon)| = 1$
II. $\mu_1(\varepsilon) = \mu_2(\varepsilon)^{-1}$

   A. $\mu_1(\varepsilon) > 1 > \mu_2(\varepsilon) > 0$
   B. $\mu_2(\varepsilon) < -1 < \mu_1(\varepsilon) < 0$

Note that only case II implies hyperbolicity. Since the Floquet multipliers are roots of the characteristic polynomial,

$$\mu^2 - \tau(\varepsilon)\mu + 1 = 0,$$

with $\tau(\varepsilon) = \mu_1(\varepsilon) + \mu_2(\varepsilon) = \operatorname{tr} Y(T, \varepsilon)$, these cases can be distinguished as follows: I. $|\tau(\varepsilon)| \leq 2$, II A. $\tau(\varepsilon) > 2$, and II B. $\tau(\varepsilon) < -2$.

Our strategy will be to obtain an expansion for the trace of the form

$$\tau(\varepsilon) = \tau(0) + \varepsilon\tau'(0) + \frac{\varepsilon^2}{2}\tau''(0) + O(\varepsilon^3).$$

Since $\tau^{(k)}(0) = \operatorname{tr}\left[\frac{d^k Y}{d\varepsilon^k}(T, 0)\right]$, this can be accomplished by first finding an expansion for the fundamental matrix $Y(t, \varepsilon)$:

$$\begin{aligned} Y(t, \varepsilon) &= Y(t, 0) + \varepsilon\frac{dY}{d\varepsilon}(t, 0) + \frac{\varepsilon^2}{2}\frac{d^2 Y}{d\varepsilon^2}(t, 0) + O(\varepsilon^3) \\ &= Y_0(t) + \varepsilon Y_1(t) + \frac{\varepsilon^2}{2} Y_2(t) + O(\varepsilon^3). \end{aligned}$$

The existence of this expansion is implied by the smooth dependence of $Y(t, \varepsilon)$ on the parameter $\varepsilon$.

The terms $Y_k(t)$ will be found by successive differentiation of the variational equation (8.15) with respect to $\varepsilon$. Thus, for $k = 0$ we have

$$Y_0'(t) = A_0 Y_0(t), \quad Y_0(0) = I,$$

in which

$$A_0 \equiv A(t, 0) = D_x f(t, 0, 0) = \begin{bmatrix} 0 & 1 \\ -\gamma^2 & 0 \end{bmatrix}.$$

This has the solution

$$Y_0(t) = \exp A_0 t = \begin{bmatrix} \cos\gamma t & \gamma^{-1}\sin\gamma t \\ -\gamma\sin\gamma t & \cos\gamma t \end{bmatrix}.$$

Thus, $\tau(0) = \operatorname{tr} Y_0(T) = 2\cos\gamma T$, so that $|\tau(0)| \leq 2$. The assumption (8.13) which guarantees the existence of a periodic solution, rules out $\tau(0) = 2$. So we have $-2 \leq \tau(0) < 2$. If $-2 < \tau(0) < 2$, then by continuous dependence on parameters $-2 < \tau(\varepsilon) < 2$, for small $\varepsilon$. This corresponds to the non-hyperbolic case I. We are led to the condition $\tau(0) = -2$, and therefore, we now assume that

$$\gamma T = (2j + 1)\pi, \quad j \in \mathbb{Z}. \tag{8.16}$$

Our expansion now reads

$$\tau(\varepsilon) = -2 + \tau'(0)\varepsilon + \frac{1}{2}\tau''(0)\varepsilon^2 + O(\varepsilon^3). \tag{8.17}$$

We shall now show that case II B, $\tau(\varepsilon) < -2$ holds, under appropriate assumptions.

Next let's look at $\tau'(0)$. For $k = 1$, we get

$$Y_1'(t) = A_0 Y_1(t) + A_1(t) Y_0(t), \quad Y_1(0) = 0,$$

with

$$A_1(t) = \frac{dA}{d\varepsilon}(t, 0) = \frac{d}{d\varepsilon}\begin{bmatrix} 0 & 0 \\ -g'(u_\varepsilon(t)) & 0 \end{bmatrix}\Bigg|_{\varepsilon=0} = \begin{bmatrix} 0 & 0 \\ -g''(0)q(t) & 0 \end{bmatrix}, \tag{8.18}$$

and

$$q(t) = \frac{d}{d\varepsilon} u_\varepsilon(t)\Big|_{\varepsilon=0}. \tag{8.19}$$

Using the Variation of Parameters formula, Corollary 4.1, together with the previous computation, we obtain the representation

$$\begin{aligned} Y_1(t) &= \int_0^t \exp A_0(t-s) A_1(s) Y_0(s) ds \\ &= \exp A_0 t \int_0^t \exp(-A_0 s) A_1(s) Y_0(s) ds \\ &= Y_0(t) \int_0^t B(s) ds \end{aligned} \tag{8.20}$$

with

$$B(s) = Y_0(-s) A_1(s) Y_0(s). \tag{8.21}$$

If we set $t = T$ and use (8.16), we see that $Y_0(T) = -I$, so

$$Y_1(T) = -\int_0^T B(s) ds.$$

Hence, $\tau'(0) = \operatorname{tr} Y_1(T) = -\int_0^T \operatorname{tr} B(s) ds$. Now $B(s)$ is similar to $A_1(s)$. Similar matrices have identical traces, and $A_1(s)$ has zero trace. Therefore, $\tau'(0) = 0$. We get no information from this term!

So we move on to the coefficient of the second order term $\tau''(0)$. If $k = 2$, then

$$Y_2'(t) = A_0 Y_2(t) + 2A_1(t) Y_1(t) + A_2(t) Y_0(t), \quad Y_2(0) = 0,$$

in which

$$A_2(t) = \frac{d^2 A}{d\varepsilon^2}(t, 0).$$

As before, we have

$$\begin{aligned}
Y_2(T) &= \int_0^T \exp A_0(T-s)[2A_1(s)Y_1(s) + A_2(s)Y_0(s)]ds \\
&= -2\int_0^T Y_0(-s)A_1(s)Y_1(s)ds - \int_0^T Y_0(-s)A_2(s)Y_0(s)ds.
\end{aligned}$$

The second term has zero trace, by the similarity argument above. Therefore, we have by (8.20)

$$\begin{aligned}
\tau''(0) &= \operatorname{tr} Y_2(T) \\
&= -2 \ \operatorname{tr} \int_0^T Y_0(-s)A_1(s)Y_1(s)ds \\
&= -2 \ \operatorname{tr} \int_0^T Y_0(-s)A_1(s)Y_0(s)\left(\int_0^s B(\sigma)d\sigma\right)ds \\
&= -2 \ \operatorname{tr} \int_0^T B(s)\left(\int_0^s B(\sigma)d\sigma\right)ds \\
&= -2 \int_0^T \int_0^s \operatorname{tr} B(s)B(\sigma)d\sigma ds \\
&= -2 \int_0^T \int_\sigma^T \operatorname{tr} B(s)B(\sigma)ds d\sigma \\
&= -2 \int_0^T \int_s^T \operatorname{tr} B(\sigma)B(s)d\sigma ds,
\end{aligned}$$

where we have interchanged the order of integration in the next to last step.

Now, for any two square matrices $L$ and $M$, there holds $\operatorname{tr} LM = \operatorname{tr} ML$. Therefore, we conclude that

$$\begin{aligned}
\tau''(0) &= -\operatorname{tr} \int_0^T \int_0^s B(s)B(\sigma)d\sigma ds - \operatorname{tr} \int_0^T \int_s^T B(s)B(\sigma)d\sigma ds \\
&= -\operatorname{tr} \int_0^T \int_0^T B(s)B(\sigma)d\sigma ds \\
&= -\operatorname{tr}\left(\int_0^T B(s)ds\right)^2.
\end{aligned}$$

Using the definitions (8.18) (8.21), we find

$$B(s) = -g''(0)q(s)\begin{bmatrix} -\gamma^{-1}\cos\gamma s\sin\gamma s & -\gamma^{-2}\sin^2\gamma s \\ \cos^2\gamma s & \gamma^{-1}\cos\gamma s\sin\gamma s \end{bmatrix}. \tag{8.22}$$

If $B(s)$ is such that

$$\tau''(0) < 0, \tag{8.23}$$

then by (8.17), $\tau''(\varepsilon) < -2$ for small $\varepsilon$, since $\tau'(0) = 0$.

Let's see what this condition means. We obtain from (8.22)

$$\begin{aligned}\tau''(0) = -\left(\frac{g''(0)}{\gamma^2}\right)^2 \Bigg[ &\left(\int_0^T q(s)\cos\gamma s\sin\gamma s ds\right)^2 \\ &- \left(\int_0^T q(s)\cos^2\gamma s ds\right)\left(\int_0^T q(s)\sin^2\gamma s ds\right)\Bigg].\end{aligned}$$

So the first thing we note is that $g''(0) \neq 0$ must hold, which means the function $g$ must be nonlinear. Taking the derivative of (8.14), we see that the function $q(t)$ defined in (8.19) is a $T$-periodic solution of the linear equation

$$q''(t) + \gamma^2 q(t) = p(t).$$

Since $\gamma T \neq 2\pi$, this solution is unique. If, for example, we take the periodic forcing term in (8.14) to be

$$p(t) = \sin 2\gamma t,$$

so that $\gamma T = \pi$, then

$$q(t) = -(1/3\gamma^2)\sin 2\gamma t.$$

The last two integrals vanish, while the first is nonzero. We therefore see that $\tau''(0) < 0$, and so $\tau(\varepsilon) < -2$. It follows that the periodic solution $x_\varepsilon$ is hyperbolic, for small $\varepsilon \neq 0$. Moreover, the Floquet multipliers lie on the negative real axis, and so from (the proof of) Lemma 4.4 we see that $R = -I$. This implies that the stable manifold is a Möbius band.

## 8.4 Stability of Periodic Solutions to Autonomous Systems

Our aim in this section is to investigate the orbital stability of periodic solutions to autonomous systems. Recall that in Sect. 8.2, Theorem 8.2, we established the stability of periodic solutions to periodic *nonautonomous* systems. In that case, we saw that if the Floquet multipliers of a periodic solution are all inside the unit disk, then the periodic solution is asymptotically stable. For autonomous flow, this hypothesis can never hold because, as we will see in the next lemma, at least one Floquet multiplier is always equal to 1.

**Lemma 8.2.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ autonomous vector field, and suppose that $\varphi(t)$ is a nonconstant $T$-periodic solution of the system*

$$x' = f(x). \tag{8.24}$$

*At least one Floquet multiplier of $\varphi$ is equal to unity.*

*Proof.* If we differentiate the equation $\varphi'(t) = f(\varphi(t))$ with respect to $t$, we get

$$\varphi''(t) = DF(\varphi(t))\varphi'(t) = A(t)\varphi'(t).$$

This says that $\varphi'(t)$ is a solution of the linear variational equation which is given by

$$\varphi'(t) = Y(t)\varphi'(0).$$

The $T$-periodicity of $\varphi$ is inherited by $\varphi'$, so

$$\varphi'(0) = \varphi'(T) = Y(T)\varphi'(0).$$

Note that $\varphi'(0) = f(\varphi(0)) \neq 0$, for otherwise, $\varphi$ would be an equilibrium point, contrary to assumption. So we have just shown that $\varphi'(0)$ is a eigenvector of $Y(T)$ with eigenvalue 1. Thus, one of the Floquet multipliers is unity, and it is referred to as the trivial Floquet multiplier. □

We now need some definitions.

**Definition 8.4.** *A $T$-periodic orbit $\gamma = \{\varphi(t) : 0 \le t \le T\}$ is said to be* orbitally stable *if for every $\varepsilon > 0$ there is a $\delta > 0$ such that* $\operatorname{dist}(x_0, \gamma) < \delta$ *implies that $x(t, x_0)$ is defined for all $t \ge 0$ and* $\operatorname{dist}(x(t, x_0), \gamma) < \varepsilon$*, for all $t \ge 0$. If in addition,* $\lim_{t\to\infty} \operatorname{dist}(x(t, x_0), \gamma) = 0$*, for all $x_0$ sufficiently close to $\gamma$, then the orbit is said to be* asymptotically orbitally stable.

This is weaker than the definition of stability given in Definition 8.2, where solutions are compared at the same time values.

**Definition 8.5.** *A periodic orbit $\gamma = \{\varphi(t) : 0 \le t \le T\}$ is asymptotically orbitally stable with asymptotic phase if it is asymptotically orbitally stable and there exists a $\tau \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} \|x(t, x_0) - \varphi(t + \tau)\| = 0.$$

**Theorem 8.5.** *Suppose that $\varphi(t)$ is a $T$-periodic solution of the autonomous equation (8.24) whose nontrivial Floquet multipliers satisfy $|\mu_i| \le \bar{\mu} < 1$, $i = 1, \ldots, n-1$. Then $\varphi(t)$ is asymptotically orbitally stable with asymptotic phase.*

*Proof.* Let $Y(t)$ be the fundamental matrix of the $T$-periodic matrix $A(t) = Df(\varphi(t))$. According to Theorem 4.5, there are real matrices $P(t)$, $B$, and $R$ such that

$$Y(t) = P(t) \exp Bt,$$

with $P(t+T) = P(t)R$, $R^2 = I$, and $BR = RB$. The Floquet multipliers are the eigenvalues $\{\mu_i\}_{i=1}^n$ of the matrix $Y(T) = R \exp BT$.

To start out with, we recall some estimates for the fundamental matrix $\exp Bt$. Our assumption is equivalent to saying that the Floquet exponents, i.e. the eigenvalues of $BT$, have strictly negative real parts except for a simple zero eigenvalue. Thus, if $\{\lambda_i\}_{i=1}^n$ denote the eigenvalues of $B$, then there is a $\lambda > 0$ such that

$$\operatorname{Re} \lambda_i \leq -\lambda < 0, \ i = 1, \ldots, n-1; \quad \lambda_n = 0.$$

So $B$ has an $(n-1)$-dimensional stable subspace $E_s$, and a 1-dimensional center subspace $E_c$, spanned by the vector $v_n = \varphi'(0)$. Let $P_s$ and $P_c$ be the corresponding projections. By Theorem 2.3 and Corollary 2.1, we have the estimates

$$\begin{aligned} &\| \exp Bt P_s x \| \leq C_0 \mathrm{e}^{-\lambda t} \| P_s x \|, \ t \geq 0 \\ &\| \exp Bt P_c x \| \leq C_0 \| P_c x \|, \ t \in \mathbb{R}, \end{aligned} \tag{8.25}$$

for all $x \in \mathbb{R}^n$. Note that $\exp Bt$ is bounded on $E_c$, since $v_n$ is a simple eigenvalue.

With these preliminaries, we now discuss the strategy of the proof. Let $x_0 \in \mathbb{R}^n$ be a given point close to $\varphi(0)$. We want to show that there is a small phase shift $\tau \in \mathbb{R}$ such that

$$x(t, x_0) - \varphi(t + \tau) \to 0, \quad \text{as} \quad t \to \infty.$$

Similar to what we did in the nonautonomous case in Sect. 8.2, define a function $z(t, \tau)$ by

$$P(t+\tau) z(t, \tau) = x(t, x_0) - \varphi(t + \tau).$$

Note that $P'(t+\tau) = A(t+\tau)P(t+\tau) - P(t+\tau)B$. Since $f$ is autonomous, $\varphi(t+\tau)$ is a solution of (8.24) for all $\tau \in \mathbb{R}$. As in Lemma 8.1, we see that $z(t, \tau)$ solves

$$\begin{aligned} z'(t, \tau) &= Bz(t, \tau) + P(t+\tau)^{-1} [f(\varphi(t+\tau) + P(t+\tau)z(t, \tau)) \\ &\qquad - f(\varphi(t+\tau)) - A(t+\tau)P(t+\tau)z(t, \tau)] \\ &\equiv Bz(t, \tau) + H(t+\tau, z(t, \tau)), \end{aligned} \tag{8.26}$$

with $H(\sigma, 0) = 0$, $D_z H(\sigma, 0) = 0$, and $H(\sigma + T, z) = RH(\sigma, Rz)$, for all $\sigma \in \mathbb{R}$ and $z \in \mathbb{R}^n$.

Conversely, if $z(t, \tau)$ is a solution of (8.26), then $x(t) = \varphi(t+\tau) + P(t+\tau)z(t, \tau)$ solves (8.24) with initial data $z(0, \tau) + P(\tau)\varphi(\tau)$. Our strategy will be to construct exponentially decaying solutions of (8.26) and then to adjust the initial data appropriately.

By Variation of Parameters, Corollary 4.1, a solution of (8.26) satisfies

$$z(t,\tau) = \exp Bt\ z(0,\tau) + \int_0^t \exp B(t-\sigma)H(\sigma+\tau, z(\sigma,\tau))d\sigma.$$

Moreover, if $z(t,\tau)$ is exponentially decaying, then using (8.25), this is equivalent to

$$z(t,\tau) = \exp Bt\ P_s z(0,\tau) + \int_0^t \exp B(t-\sigma)P_s H(\sigma+\tau, z(\sigma,\tau))d\sigma$$
$$- \int_t^\infty \exp B(t-\sigma)P_c H(\sigma+\tau, z(\sigma,\tau))d\sigma.$$

Now we come to the main set-up. Take a $0 < \beta < \lambda$. Define the set of functions

$$Z = \{z \in C(\mathbb{R}^+, \mathbb{R}^n) : \|z\|_\beta \equiv \sup_{t\geq 0} \mathrm{e}^{\beta t}\|z(t)\| < \infty\}.$$

$Z$ is a Banach space with the indicated norm. For $\tau \in \mathbb{R}$, $z_0 \in E_s$, and $z \in Z$, define the mapping

$$G(\tau, z_0, z)(t) = z(t) - \exp Bt z_0 - \int_0^t \exp B(t-\sigma)P_s H(\sigma+\tau, z(\sigma))d\sigma$$
$$+ \int_t^\infty \exp B(t-\sigma)P_c H(\sigma+\tau, z(\sigma))d\sigma.$$

Now $G : \mathbb{R}\times E_s \times Z \to Z$ is a well-defined $C^1$ mapping such that $G(0,0,0) = 0$ and $D_z G(0,0,0) = I$. By the Implicit Function Theorem 5.7, there is a neighborhood $U$ of the origin in $\mathbb{R}\times E_s$ and a $C^1$ mapping $\psi : U \to Z$ such that

$$\psi(0,0) = 0 \quad \text{and} \quad G(\tau, z_0, \psi(\tau, z_0)) = 0, \quad \text{for all} \quad (\tau, z_0) \in U.$$

It follows that $z(t,\tau,z_0) = \psi(\tau,z_0)(t)$ is a solution of (8.26), and since $z(t,\tau,z_0) \in Z$, we have $\|z(t,\tau,z_0)\| \leq C\mathrm{e}^{-\beta t}$. Thus, $x(t) = \varphi(t+\tau) + P(t+\tau)z(t,\tau,z_0)$ is a solution of (8.24) for all $(\tau, z_0) \in U$, with data $x(0) = \varphi(\tau) + P(\tau)z(0,\tau,z_0)$.

Define a map $F : E_s \times \mathbb{R} \to \mathbb{R}^n$ by

$$F(\tau, z_0) = \varphi(\tau) + P(\tau)z(0,\tau,z_0).$$

Then $F$ is $C^1$ and $F(0,0) = \varphi(0)$. We are going to show, using the Inverse Function Theorem A.3, that $F$ is an invertible map from a neighborhood $U$ of $(0,0) \in E_s \times \mathbb{R}$ to a neighborhood $V$ of $\varphi(0)$ in $\mathbb{R}^n$.

We have set $z(t,\tau,z_0) = \psi(\tau,z_0)(t)$, so $z(t,0,0) = \psi(0,0)(t) = 0$. Since $H(\tau,0) = 0$ for all $\tau$, we have that $D_\tau H(\tau,0) = 0$. It follows that $D_\tau G(0,0,0) = 0$. We also know that $D_z H(\tau,0) = 0$, so $D_z G(0,0,0) = I$. Now, by definition of $z(t,\tau,z_0)$, we have

$$G(\tau, z_0, z(t, \tau, z_0))(t) = 0. \tag{8.27}$$

Upon differentiating in $\tau$, we obtain by the chain rule

$$D_\tau G(\tau, z_0, z(t, \tau, z_0))(t) + D_z G(\tau, z_0, z(t, \tau, z_0)) D_\tau z(t, \tau, z_0)(t) = 0.$$

Letting $(\tau, z_0) = (0, 0)$, we find

$$D_\tau z(t, 0, 0) = 0.$$

From this it follows that

$$D_\tau F(0, 0) = \varphi'(\tau).$$

Since $G(\tau, z_0, z)$ is linear in $z_0$, we have $D_{z_0} G(\tau, z_0, z)(t) = -\exp Bt$. Taking now the derivative of (8.27) in $z_0$ we get

$$D_{z_0} G(\tau, z_0, z(t, \tau, z_0))(t) + D_z G(\tau, z_0, z(t, \tau, z_0)) D_{z_0} z(t, \tau, z_0)(t) = 0.$$

and then setting $(t, \tau, z_0) = (0, 0, 0)$ yields

$$D_{z_0} z(t, 0, 0) = I.$$

Therefore, we have shown that

$$DF(0, 0)(\bar{\tau}, \bar{z}_0) = D_\tau F(0, 0)\bar{\tau} + D_{z_0} F(0, 0)\bar{z}_0 = \varphi'(0)\bar{\tau} + \bar{z}_0.$$

We claim that this map is invertible from $\mathbb{R} \times E_s$ into $\mathbb{R}^n$. Given $x \in \mathbb{R}^n$, there is a unique decomposition $x = P_s x + P_c x$. Since $E_c$ is spanned by $\varphi'(0)$, we have $P_c x = \varphi'(0)\nu$, for a unique $\nu \in \mathbb{R}$, and $DF(0, 0)^{-1}(x) = (\nu, P_s x)$.

So by the Inverse Function Theorem, there is a neighborhood $V \subset \mathbb{R}^n$ containing $\varphi(0)$ and a neighborhood $U \subset \mathbb{R} \times E_s$ containing $(0, 0)$ such that $F : U \to V$ is a diffeomorphism.

In other words, for every $x_0 \in V$, there is a $(\tau, z_0) \in U$ such that

$$x_0 = \varphi(\tau) + P(\tau) z(0, \tau, z_0).$$

From our earlier discussion, we have

$$x(t, x_0) = \varphi(t + \tau) + P(t + \tau) z(t, \tau, z_0),$$

since both sides satisfy (8.24) with the same initial data. Thus, since $\|P(t + \tau)\|$ is uniformly bounded and $\|z(t, \tau, z_0)\| \le C\mathrm{e}^{-\beta t}$, given $x_0 \in V$, we have found a phase $\tau \in \mathbb{R}$ with

$$\|x(t, x_0) - \varphi(t + \tau)\| \le C\mathrm{e}^{-\beta t}, \quad t \ge 0.$$

Since the argument works for any point along the periodic orbit (not only $\varphi(0)$), we have established asymptotic orbital stability with asymptotic phase. □

*Remarks 8.7.*

- The proof shows that the convergence to the periodic orbit occurs at an exponential rate.
- The Theorem implies that if the nontrivial Floquet multipliers of a periodic orbit lie within the unit disk, then the orbit is a local attractor.
- There are other proofs of Theorem 8.5. See Hartman [6] (Theorem 11.1), and Hale [5] (Chap. 6, Theorem 2.1). The one presented above emphasizes the similarities with the Center Manifold Theorem 9.1.
- If the nontrivial Floquet multipliers all lie off the unit circle (but not necessarily inside), then a version of the stable/unstable manifold theorem can be formulated. More or less routine (by now) modifications of the preceding argument could be used to prove it.

## 8.5 Existence of Periodic Solutions in $\mathbb{R}^n$: Critical Case

We now return to the question of existence of periodic solutions to systems which are periodic perturbations of an autonomous system with a critical point. Although this topic has been studied in Sect. 8.1 when the perturbation is noncritical, here we will be concerned with the case of a critical perturbation.

Once again we consider a one parameter family of nonautonomous vector fields $f(t, x, \varepsilon)$. Assume that

(i) $f : \mathbb{R}^{1+n} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ is $C^\infty$.
(ii) There is a $T > 0$, such that

$$f(t + T, x, \varepsilon) = f(t, x, \varepsilon) \quad \text{for all} \quad (t, x, \varepsilon) \in \mathbb{R}^{1+n} \times (-\varepsilon_0, \varepsilon_0).$$

(iii) $f(t, x, 0) = f_0(x)$ is autonomous and $f_0(0) = 0$.
(iv) *All* solutions of $x' = Ax$ are $T$-periodic, where $A = Df(0)$.

In order to avoid counting derivatives, we assume that the vector field is infinitely differentiable in *(i)*.

Of interest here is the last condition which is what is meant by a critical perturbation. Recall that in Sect. 8.1 we assumed that the eigenvalues, $\lambda$, of $A$ satisfy $\lambda T \notin 2\pi i\mathbb{Z}$, so that no solution of $x' = Ax$ is $T$-periodic, see Theorem 8.1. Admittedly, some ground is left uncovered between these two extremes. Although we will not discuss what happens when some, but not all, solutions of $x' = Ax$ are $T$-periodic, a combination of the two approaches would yield a result here, as well.

There are various statements that are equivalent to assumption *(iv)*. One is that $\exp A(t + T) = \exp At$ for all $t \in \mathbb{R}$. It is also equivalent to saying that the set of eigenvalues of $A$ are contained in $(2\pi i/T)\mathbb{Z}$, the set of integer multiples of $2\pi i/T$, and that $A$ has a basis of eigenvectors in $\mathbb{C}^n$.

In the following paragraphs we transform the equation

$$x' = f(t, x, \varepsilon) \tag{8.28}$$

to the so-called Lagrange standard form, with which most books begin.

As in Sect. 8.1, we can write

$$f(t, x, \varepsilon) = f_0(x) + \varepsilon \widetilde{f}(t, x, \varepsilon), \quad \text{where} \quad \widetilde{f}(t, x, \varepsilon) = \int_0^1 \frac{\partial f}{\partial \varepsilon}(t, x, \sigma\varepsilon) d\sigma.$$

Notice that $\widetilde{f}$ is $C^\infty$ and $\widetilde{f}(t+T, x, \varepsilon) = \widetilde{f}(t, x, \varepsilon)$. Moreover, let's write

$$f_0(x) = Ax + [f_0(x) - Ax] \equiv Ax + f_1(x).$$

We see that $f_1 \in C^\infty$, $f_1(0) = 0$, and $Df_1(0) = 0$. Using integration by parts, we obtain another expression for the function $f_1$:

$$\begin{aligned} f_1(x) &= \int_0^1 \frac{d}{d\sigma}[f_1(\sigma x)] d\sigma \\ &= \sum_{i=1}^n \int_0^1 \frac{\partial f_1}{\partial x_i}(\sigma x) x_i d\sigma \\ &= \sum_{i=1}^n \left( \int_0^1 \frac{d}{d\sigma} \left[ \frac{\partial f_1}{\partial x_i}(\sigma x) \right] (1-\sigma) d\sigma \right) x_i \\ &= \sum_{i,j=1}^n \left( \int_0^1 \frac{\partial^2 f_1}{\partial x_i \partial x_j}(\sigma x)\ (1-\sigma) d\sigma \right) x_i x_j. \end{aligned}$$

Notice that the expressions enclosed in parentheses above are in $C^\infty$, since $f_1 \in C^\infty$.

Now suppose that $x(t)$ is a solution of (8.28) and perform the rescaling

$$x(t) = \sqrt{\varepsilon} y(t).$$

Then

$$\begin{aligned} \sqrt{\varepsilon} y' &= x' \\ &= f(t, x, \varepsilon) \\ &= Ax + f_1(x) + \varepsilon \widetilde{f}(t, x, \varepsilon) \\ &= \sqrt{\varepsilon} Ay + f_1(\sqrt{\varepsilon} y) + \varepsilon \widetilde{f}(t, \sqrt{\varepsilon} y, \varepsilon) \\ &\equiv \sqrt{\varepsilon} Ay + \varepsilon g_1(y, \sqrt{\varepsilon}) + \varepsilon \widetilde{g}(t, y, \sqrt{\varepsilon}), \end{aligned}$$

in which

$$g_1(y,\mu) = \sum_{i,j=1}^{n} \left( \int_0^1 \frac{\partial^2 f_1}{\partial x_i \partial x_j}(\sigma\mu y)\ (1-\sigma) d\sigma \right) y_i y_j,$$

and

$$\widetilde{g}(t, y, \mu) = \widetilde{f}(t, \mu y, \mu^2).$$

Both $g_1$ and $\widetilde{g}$ are $C^\infty$ functions of their arguments. To summarize, we have transformed the original problem to

$$y' = Ay + \sqrt{\varepsilon} g(t, y, \sqrt{\varepsilon}),$$

with $g$ in $C^\infty$ and $T$-periodic in the "$t$" variable.

As a final reduction, set $\mu = \sqrt{\varepsilon}$ and $y(t) = \mathrm{e}^{At} z(t)$. Then

$$z' = \mu h(t, z, \mu),$$

with $h$ in $C^\infty$, and since $\exp A(t+T) = \exp At$, we have $h(t+T, z, \mu) = h(t, z, \mu)$ for all $(t, z, \mu)$, with $|\mu| < \sqrt{\varepsilon_0}$. This is the standard form we were after.

Having reduced (8.28) to the standard form, let's go back to the original variables. We have

$$x' = \varepsilon f(t, x, \varepsilon) \tag{8.29}$$

with $f$ in $C^\infty$ and $f(t+T, x, \varepsilon) = f(t, x, \varepsilon)$ for all $(t, x, \varepsilon)$, with $|\varepsilon| < \sqrt{\varepsilon_0}$.

We will use the so-called *method of averaging*, one of the central techniques used in the study of periodic systems, to compare solutions of (8.29) with those of

$$y' = \varepsilon \bar{f}(y), \tag{8.30}$$

where $\bar{f}$ denotes the *averaged vector field*

$$\bar{f}(x) = \frac{1}{T} \int_0^T f(t, x, 0) dt. \tag{8.31}$$

(Recall that having changed coordinates, $f(t, x, 0)$ is no longer autonomous.) We will show

**Theorem 8.6.** *Suppose that the averaged vector field $\bar{f}$ has a critical point at $p_0 \in \mathbb{R}^n$. Let $B = D\bar{f}(p_0)$, and assume that $B$ is invertible. Then there is an $\varepsilon_0 > 0$, such that for all $|\varepsilon| < \varepsilon_0$ the Eq. (8.29) has a unique $T$-periodic solution $\varphi_\varepsilon(t)$ near $p_0$.*

*If the eigenvalues of $B$ lie in the negative half plane (so that $p_0$ is asymptotically stable for (8.30)), then the Floquet multipliers of $\varphi_\varepsilon(t)$ lie inside the unit disk (i.e. the periodic solution $\varphi_\varepsilon(t)$ is asymptotically stable for (8.29)).*

*If the eigenvalues of $B$ lie off the imaginary axis (so that $p_0$ is hyperbolic for (8.30)), then the Floquet multipliers of $\varphi_\varepsilon(t)$ lie off the unit circle (i.e. the periodic solution $\varphi_\varepsilon(t)$ is hyperbolic for (8.29)).*

Before starting the proof of the theorem, we prepare the following lemma which lies behind a crucial change of coordinates.

**Lemma 8.3.** *Suppose that $w : \mathbb{R}^{1+n} \to \mathbb{R}^n$ is $C^k$ and that for some $T > 0$,*

$$w(t+T, y) = w(t, y) \quad \textit{for all} \quad (t, y) \in \mathbb{R}^{1+n}.$$

*For any $y_0 \in \mathbb{R}^n$ and $R > 1$, there exists an $\varepsilon_0 > 0$ such that for all $|\varepsilon| < \varepsilon_0$ the following hold:*

*(i) The mapping*
$$y \mapsto y + \varepsilon w(t, y)$$

*is a $T$-periodic family of $C^k$ diffeomorphisms on $B_R(y_0)$, whose range contains the ball $B_{R-1}(y_0)$.*

*(ii) The Jacobian matrix*
$$I + \varepsilon D_y w(t, y)$$

*is nonsingular for all $t \in \mathbb{R}$, and $y \in B_R(y_0)$. Its inverse is $C^{k-1}$ in $(y, t, \varepsilon)$, is $T$-periodic in $t$, and satisfies the estimates*

$$\|(I + \varepsilon D_y w(t, y))^{-1}\| \leq \frac{1}{1 - |\varepsilon/\varepsilon_0|} \tag{8.32}$$

*and*

$$\|(I + \varepsilon D_y w(t, y))^{-1} - I\| \leq \frac{|\varepsilon/\varepsilon_0|}{1 - |\varepsilon/\varepsilon_0|}. \tag{8.33}$$

*Proof.* Set

$$M = \max\{\|w(t, y)\| + \|D_y w(t, y)\| : (t, y) \in [0, T] \times \overline{B}_R(y_0)\},$$

and define $\varepsilon_0 = M^{-1}$. For all $y_1, y_2 \in B_R(y_0)$, we have

$$\begin{aligned} w(t, y_1) - w(t, y_2) &= \int_0^1 \frac{d}{d\sigma} w(t, \sigma y_1 + (1-\sigma) y_2) d\sigma \\ &= \int_0^1 D_y w(t, \sigma y_1 + (1-\sigma) y_2) d\sigma \, (y_1 - y_2). \end{aligned}$$

Note that $\sigma y_1 + (1-\sigma) y_2 \in B_R(y_0)$ for all $0 \leq \sigma \leq 1$. Hence, we have the estimate

$$\|w(t, y_1) - w(t, y_2)\| \leq M \|y_1 - y_2\|.$$

So if $y_1, y_2 \in B_R(y_0)$ and $|\varepsilon| < \varepsilon_0$, we have

$$\begin{aligned}\|(y_1 + w(t, y_1)) - (y_2 + w(t, y_2))\| \\ &\geq \|y_1 - y_2\| - |\varepsilon|\|w(t, y_1) - w(t, y_2)\| \\ &\geq (1 - |\varepsilon|M)\|y_1 - y_2\| \\ &= (1 - |\varepsilon/\varepsilon_0|)\|y_1 - y_2\|.\end{aligned}$$

This implies that the mapping $y \mapsto y + \varepsilon w(t, y)$ is one-to-one on $B_R(y_0)$ for any $|\varepsilon| < \varepsilon_0$ and $t \in \mathbb{R}$, and is therefore invertible.

If $x \in B_{R-1}(y_0)$, then for each $|\varepsilon| < \varepsilon_0$, $t \in \mathbb{R}$, the map $y \mapsto -\varepsilon w(t, y) + x$ is a contraction of $B_R(y_0)$ into itself. The Contraction Mapping Principle A.1 implies that there exists a point $y \in B_R(y_0)$ such that

$$y = -\varepsilon w(t, y) + x$$

which in turn implies that the function $y \mapsto y + \varepsilon w(t, y)$ maps $B_R(y_0)$ onto $B_{R-1}(y_0)$.

Next, for any $|\varepsilon| < \varepsilon_0$, $t \in \mathbb{R}$, and $y \in B_R(y_0)$, consider the Jacobian

$$I + \varepsilon D_y w(t, y).$$

Since $|\varepsilon|M < \varepsilon_0 M = 1$, we have that the Jacobian is nonsingular.

By the Inverse Function Theorem A.3, the mapping $y + \varepsilon w(t, y)$ is a local diffeomorphism in a neighborhood of any point $y \in B_R(y_0)$. But since we have shown that this map is one-to-one on all of $B_R(y_0)$, is a diffeomorphism on all of $B_R(y_0)$.

Let $A = \varepsilon D_y w(t, y)$. Then

$$\|A\| \leq |\varepsilon|M \leq |\varepsilon/\varepsilon_0| < 1.$$

By Lemma 5.1, $I + A$ is invertible, and we obtain the estimates (8.32) and (8.33).

Finally, the smoothness of $Z(t, y, \varepsilon) = (I + \varepsilon D_y w(t, y))^{-1}$ in $(y, t, \varepsilon)$ follows from the Implicit Function Theorem, since it solves the equation

$$(I + \varepsilon D_y w(t, y))Z - I = 0. \qquad \square$$

*Proof of Theorem 8.6.* Having defined the averaged vector field $\bar{f}$ in (8.31), set

$$\widetilde{f}(t, x, \varepsilon) = f(t, x, \varepsilon) - \bar{f}(x).$$

Then $\widetilde{f}$ is $C^\infty$, $T$-periodic, and

$$\int_0^T \widetilde{f}(t, x, 0)dt = 0. \tag{8.34}$$

Define

$$w(t, y) = \int_0^t \widetilde{f}(s, y, 0)ds. \tag{8.35}$$

Then $w \in C^\infty$, and thanks to (8.34), $w(t+T, y) = w(t, y)$. This function will be used to change coordinates.

Suppose that $p_0$ is a critical point for $\bar{f}$. Choose $R > 1$ and set

$$\varepsilon_0^{-1} = M = \max\{\|w(t, y)\| + \|D_y w(t, y)\| : (t, y) \in [0, T] \times B_R(p_0)\}.$$

By Lemma 8.3, we know that for all $t \in \mathbb{R}$ and $|\varepsilon| < \varepsilon_0$, the mapping

$$F(y, t, \varepsilon) = y + \varepsilon w(t, y)$$

is a diffeomorphism on $B_R(p_0)$ whose range contains $B_{R-1}(p_0)$.

Let $x(t)$ be any solution of (8.29) in $B_{R-1}(p_0)$. Then the formula

$$x(t) = y(t) + \varepsilon w(t, y(t))$$

defines a smooth curve $y(t)$ in $B_R(p_0)$. We need to calculate the differential equation satisfied by $y(t)$. Using (8.29), we have

$$\begin{aligned}
y'(t) + &\varepsilon D_y w(t, y(t)) y'(t) + \varepsilon \frac{\partial w}{\partial t}(t, y(t)) \\
&= x'(t) \\
&= \varepsilon f(t, x(t), \varepsilon) \\
&= \varepsilon \bar{f}(x(t)) + \varepsilon \widetilde{f}(t, x(t), \varepsilon) \\
&= \varepsilon \bar{f}(y(t) + \varepsilon w(t, y(t))) + \varepsilon \widetilde{f}(t, y(t) + \varepsilon w(t, y(t)), \varepsilon).
\end{aligned}$$

Notice that by (8.35), $\dfrac{\partial w}{\partial t}(t, y(t)) = \widetilde{f}(t, y(t), 0)$, so we obtain

$$\begin{aligned}
[I + \varepsilon D_y w(t, y(t))] y'(t) = \varepsilon &\bar{f}(y(t) + \varepsilon w(t, y(t))) \\
&+ \varepsilon[\widetilde{f}(t, y(t) + \varepsilon w(t, y(t)), \varepsilon) - \widetilde{f}(t, y(t), 0)].
\end{aligned}$$

By Lemma 8.3, the matrix $I + \varepsilon D_y w(t, y(t))$ is invertible, so this may be rewritten as

$$\begin{aligned}
y' = \varepsilon &\bar{f}(y) + \varepsilon([I + \varepsilon D_y w(t, y)]^{-1} - I)\bar{f}(y) \\
&+ \varepsilon[I + \varepsilon D_y w(t, y)]^{-1}[\bar{f}(y + \varepsilon w(t, y)) - \bar{f}(y)] \\
&+ \varepsilon[I + \varepsilon D_y w(t, y)]^{-1}[\widetilde{f}(t, y + \varepsilon w(t, y), \varepsilon) - \widetilde{f}(t, y, 0)].
\end{aligned}$$

Now using the estimates (8.32), (8.33) and Taylor expansion, the last three terms on the right can be grouped and written in the form

$$\varepsilon^2 \hat{f}(t, y, \varepsilon),$$

with $\hat{f}$ in $C^\infty$ and $T$-periodic. We have shown that $y(t)$ solves

$$y' = \varepsilon \bar{f}(y) + \varepsilon^2 \hat{f}(t, y, \varepsilon), \quad y(0) = y_0. \tag{8.36}$$

Conversely, a solution of (8.36) in $B_R(p_0)$ gives rise to a solution $x(t)$ to the original problem (8.29) via the mapping $x = y + \varepsilon w(t, y)$. If $y(t)$ is $T$-periodic, then the same holds for $x(t)$.

So now let's consider the initial value problem (8.36) and let $y(t, y_0, \varepsilon)$ denote its local solution. (By smooth dependence, it is $C^\infty$ in its arguments.) As in Sect. 8.1, we are going to look for periodic solutions as fixed points of the period $T$ map, however the argument is a bit trickier than in Sect. 8.1 because the perturbation is critical.

Note that $y(t, p, 0) = p$ is a global solution (the right hand side vanishes), so for $|\varepsilon|$ sufficiently small, $y(t, p, \varepsilon)$ is defined for $0 \le t \le T$ for all $p \in B_R(p_0)$, by continuous dependence. For such $p$ and $\varepsilon$, define

$$Q(p, \varepsilon) = \varepsilon^{-1}[y(T, p, \varepsilon) - p].$$

We are going to construct a curve of zeros $p(\varepsilon)$ of $Q(p, \varepsilon)$ near $p_0$ with $p(0) = p_0$ using the Implicit Function Theorem. Each zero corresponds to a $T$-periodic solution of (8.36), by Lemma 4.5.

As noted above, $y(t, p, 0) = p$. Thus, we have

$$Q(p, \varepsilon) = \varepsilon^{-1} \int_0^1 \frac{d}{d\sigma} y(T, p, \sigma\varepsilon) d\sigma = \int_0^1 \frac{\partial y}{\partial \varepsilon}(T, p, \sigma\varepsilon) d\sigma,$$

which shows that $Q(p, \varepsilon)$ is $C^\infty$ and also that

$$Q(p, 0) = \frac{\partial y}{\partial \varepsilon}(T, p, 0). \tag{8.37}$$

To further evaluate this expression, differentiate equation (8.36) with respect to the parameter $\varepsilon$:

$$\begin{aligned} \frac{d}{dt} \frac{\partial y}{\partial \varepsilon}(t, p, 0) &= \frac{\partial}{\partial \varepsilon} [\varepsilon \bar{f}(y(t, p, \varepsilon)) + \varepsilon^2 \hat{f}(y(t, p, \varepsilon))] \Big|_{\varepsilon=0} \\ &= \bar{f}(y(t, p, 0)) = \bar{f}(p), \end{aligned}$$

and

$$\frac{\partial y}{\partial \varepsilon}(0, p, 0) = \frac{\partial}{\partial \varepsilon} p \Big|_{\varepsilon=0} = 0.$$

This, of course, is easily solved to produce

$$\frac{\partial y}{\partial \varepsilon}(T, p, 0) = T\bar{f}(p),$$

which when combined with (8.37) gives us

$$Q(p, 0) = T\bar{f}(p).$$

and hence, in particular,

$$Q(p_0, 0) = T\bar{f}(p_0) = 0.$$

From this we see that

$$D_p Q(p_0, 0) = D_p Q(p, 0)|_{p=p_0} = T D_p \bar{f}(p_0) = TB,$$

is nonsingular, by assumption on $B$.

So by the Implicit Function Theorem, there is a $C^1$ curve $p(\varepsilon)$ defined near $\varepsilon = 0$ such that

$$p(0) = 0 \quad \text{and} \quad Q(p(\varepsilon), \varepsilon) = 0.$$

Moreover, there is a neighborhood $N$ of $(p_0, 0)$ such that if $(q, \varepsilon) \in N$ and $Q(q, \varepsilon) = 0$, then $q = p(\varepsilon)$.

Thus, we have constructed a unique family $y_\varepsilon(t) = y(t, p(\varepsilon), \varepsilon)$ of $T$-periodic solutions of (8.36) in a neighborhood $N$ of $(p_0, 0)$. By continuous dependence, the Floquet exponents of $y_\varepsilon$ are close to the eigenvalues of $\varepsilon TB$, for $\varepsilon$ small. This gives the statements on stability. Finally, the analysis is carried over to the original equation (8.29) using our change of variables. □

*Example 8.4. (Duffing's Equation).* We can illustrate the ideas with the example of Duffing's equation

$$u'' + u + \varepsilon\beta u + \varepsilon\gamma u^3 = \varepsilon F \cos t,$$

which models the nonlinear oscillations of a spring. Here $\beta$, $\gamma$, and $F \neq 0$ are fixed constants, and $\varepsilon$ is a small parameter. Notice that when $\varepsilon = 0$, the unperturbed equation is

$$u'' + u = 0$$

solutions of which all have period $2\pi$, the same period as the forcing term, so we are in a critical case.

In first order form, the system looks like

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - \varepsilon\beta x_1 - \varepsilon\gamma x_1^3 + \varepsilon F \cos t \end{bmatrix}.$$

The vector field is smooth and $2\pi$-periodic in $t$. When $\varepsilon = 0$, the system is linear with a critical point at $x = 0$:

$$x' = Ax, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

All solutions are $2\pi$-periodic.

We can express the system as

$$x' = Ax + \varepsilon f(t, x),$$

with

$$f(t, x) = \begin{bmatrix} 0 \\ -\beta x_1 - \gamma x_1^3 + F \cos t \end{bmatrix}.$$

Since the unperturbed system is linear, reduction to standard form is easily achieved without the rescaling step. Set

$$x = \exp At\ z.$$

Then

$$z' = \varepsilon \exp(-At) f(t, \exp At\ z).$$

Of interest are the critical points of the averaged vector field

$$\bar{f}(z) = \frac{1}{2\pi} \int_0^{2\pi} \exp(-At) f(t, \exp At\ z) dt.$$

Evaluation of this formula is a rather messy, but straightforward calculation. We summarize the main steps, skipping over all of the algebra and integration. First, since

$$\exp At = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix},$$

we have

$$\begin{aligned} &\exp(-At) f(t, \exp At\ z) \\ &\quad = \left\{ -\beta(z_1 \cos t + z_2 \sin t) - \gamma(z_1 \cos t + z_2 \sin t)^3 + F \cos t \right\} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}. \end{aligned}$$

Expanding this and averaging in $t$, we get

$$\bar{f}(z) = \frac{1}{2} \begin{bmatrix} (\beta + \frac{3}{4}\gamma r^2) z_2 \\ -(\beta + \frac{3}{4}\gamma r^2) z_1 + F \end{bmatrix}, \quad r^2 = z_1^2 + z_2^2.$$

From this, we see that there are critical points when

$$\frac{3}{4}\gamma z_1^3 + \beta z_1 - F = 0, \quad \text{and} \quad z_2 = 0.$$

The first equation is called the frequency response curve. Suppose that $\gamma > 0$, this is the case of the so-called *hard* spring. Also let $F > 0$. Then there is always one positive root of the frequency response function for all $\beta$.

If we set $\beta_0 = -(3/2)^{4/3}F^{2/3}\gamma^{1/3}$, then for $\beta < \beta_0$ there are an additional pair of negative roots of the frequency response function. This holds for all $F, \gamma > 0$.

In order to apply Theorem 8.6, we need to verify that $D\bar{f}$ is nonsingular at the equilibria determined by the frequency response function. Since $z_2 = 0$, it follows that at the critical points of $\bar{f}$ we have

$$D\bar{f} = \begin{bmatrix} 0 & (\beta + \frac{3}{4}\gamma z_1^2) \\ -(\beta + \frac{3}{4}\gamma z_1^2) - \frac{3}{2}\gamma z_1^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & F/z_1 \\ (2\beta z_1 - 3F)/z_1 & 0 \end{bmatrix}.$$

This matrix is nonsingular provided that $2\beta z_1 - 3F \neq 0$ at the zeros of the frequency response equation. This is indeed the case when $\beta \neq \beta_0$ because then $z_1 = 3F/2\beta$ is not a zero of the frequency response equation.

It follows that there are either one or three periodic orbits of the original equation for $\varepsilon$ small depending on whether $\beta > \beta_0$ or $\beta < \beta_0$. These periodic orbits are located close to the value of the corresponding critical point of $\bar{f}$.

If $2\beta z_1 - 3F > 0$ at any root $z_1$ of the frequency response equation, then the corresponding periodic orbit will be hyperbolic with one dimensional stable and unstable manifolds. If $2\beta z_1 - 3F < 0$ at a root, then the stability of its periodic orbit can not be determined by Theorem 8.6.

Consider the positive root. If $\beta \leq 0$, then clearly $2\beta z_1 - 3F < 0$. If $\beta > 0$, then the positive root is less than $3F/2\beta$. So we see that for all $\beta$ the stability of the periodic orbit corresponding to the positive root can not be determined by Theorem 8.6.

When $\beta < \beta_0$, there are two negative roots of the frequency response function. One of them is larger than $3F/2\beta$, and its corresponding periodic solution is hyperbolic. The other is less than $3F/2\beta$, and the stability of its periodic orbit can not be determined from Theorem 8.6.

## 8.6 The Poincaré-Bendixson Theorem

**Definition 8.6.** *A Jordan curve is a continuous map $\gamma[t_1, t_2] \to \mathbb{R}^2$ such that $\gamma(t) = \gamma(s)$ for $t < s$ if and only if $t = t_1$ and $s = t_2$.*

All of the results in this section hold only in the plane because of the next result, known as the Jordan curve theorem.

**Theorem 8.7. (Jordan Curve Theorem).** *If $\gamma$ is a Jordan curve, then $\mathbb{R}^2 \backslash \gamma = I \cup E$ in which $I$ and $E$ are disjoint connected open sets, $I$ is bounded, and $E$ is unbounded.*

*The curve $\gamma$ is the boundary of $I$ and $E$. The sets $I$ and $E$ are called the interior and exterior of $\gamma$, respectively.*

**Definition 8.7.** *Let $x(t, x_0)$ be the flow of an autonomous vector field. Assume that for some $x_0$, $x(t, x_0)$ is defined for all $t \geq 0$. Define the omega limit set of $x_0$, denoted $\omega(x_0)$, to be the set of points $p$ such that there exist an increasing sequence of times $t_k$ with*

$$t_k \to \infty, \quad x(t_k, x_0) \to p, \quad as \quad k \to \infty.$$

*If $x(t, x_0)$ is exists for all $t \leq 0$, the alpha limit set, $\alpha(x_0)$, is defined as the set of points $p$ such that there exist a decreasing sequence of times $t_k$ with*

$$t_k \to -\infty, \quad x(t_k, x_0) \to p, \quad as \quad k \to \infty.$$

*Remark 8.6.* Following the notation of Definition 3.5, $\gamma_+(x_0)$ denotes the positive semi-orbit of a point $x_0$.

**Lemma 8.4.** *Suppose that $f : \mathcal{O} \subset \mathbb{R}^2 \to \mathbb{R}^2$ is a $C^1$ autonomous vector field. If $\gamma_+(x_0)$ is contained in a compact subset of $\mathcal{O}$, then $\omega(x_0)$ is compact, connected, positively invariant, and*

$$\omega(x_0) = \bigcap\nolimits_{\tau>0} \overline{\{x(t, x_0) : t \geq \tau\}}.$$

The next theorem is the famous Poincaré-Bendixson theorem, a general result on the existence of periodic solutions for planar autonomous flow. Its proof will be divided into a series of short steps some of which are of interest in their own right.

**Theorem 8.8. (Poincaré-Bendixson Theorem).** *Let $f : \mathcal{O} \to \mathbb{R}^2$ be a $C^1$ autonomous vector field defined on an open set $\mathcal{O} \subset \mathbb{R}^2$. Let $x(t, x_0)$ be its flow. If for some $x_0 \in \mathcal{O}$, $\gamma_+(x_0)$ is contained in a compact subset of $\mathcal{O}$ and $\omega(x_0)$ contains no critical points of $f(x)$, then $\omega(x_0)$ coincides with the orbit of a periodic solution $x(t, y_0)$. If $t_0$ is the smallest period of $x(t, y_0)$, then $\{x(t, y_0) : 0 \leq t \leq t_0\}$ is a Jordan curve.*

*Proof.* §1. Assume that $\gamma_+(x_0)$ is contained in a compact set $K \subset \mathcal{O}$. Then by Theorem 3.7 $x(t, x_0)$ is defined for all $t \geq 0$, and it follows that $\emptyset \neq \omega(x_0) \subset K$. If $\gamma_+(x_0)$ is a periodic orbit, then $\gamma_+(x_0) = \omega(x_0)$, and we are done. So we may suppose that $x(t, x_0)$ is not periodic.

§2. If $p$ is any regular point, by Theorem 7.1, the flow near $p$ is topologically conjugate to the flow of a constant vector field $e_2 = (0, 1) \in \mathbb{R}^2$. There exist a neighborhood $V$ of $p$, a box $U = (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$, and a diffeomorphism $\psi : U \to V$ such that $\psi(0) = p$ and

$$x(t, \psi(q)) = \psi(q + te_2),$$

for all $q \in U$ and $t \in (-\varepsilon, \varepsilon)$ such that $q + te_2 \in U$.

Define a so-called *transversal*

$$\mathcal{T} = \{\psi(re_1) : r \in (-\varepsilon, \varepsilon)\}$$

and the domains

$$V_{\pm} = \{\psi(re_1 + se_2) : (r, \pm s) \in (-\varepsilon, \varepsilon) \times (0, \varepsilon)\}.$$

Then $V = V_- \cup \mathcal{T} \cup V_+$ is a partition. If an orbit $\gamma_+(x_0)$ intersects $V$, then it follows that each connected component of $\gamma_+(x_0) \cap V$ has the form

$$\{x(t, x_0) : \tau - \varepsilon < t < \tau + \varepsilon\}, \quad \text{for some} \quad \tau,$$

and

$$x(t, x_0) \in \begin{cases} V_-, \ \tau - \varepsilon < t < \tau \\ \mathcal{T}, \quad t = \tau \\ V_+, \ \tau < t < \tau + \varepsilon. \end{cases}$$

§3. Suppose that $\omega(x_0)$ contains no critical points. Since $\omega(x_0) \neq \emptyset$, choose $p \in \omega(x_0)$. Then $\gamma_+(p) \subset \omega(x_0)$, because $\omega(x_0)$ is positively invariant. We are going to show that $\gamma_+(p)$ is a periodic orbit and $\gamma_+(p) = \omega(x_0)$.

Since $p \in \omega(x_0)$, there exists an increasing sequence of times $t_k \to \infty$ such that $x(t_k, x_0) \to p$, as $k \to \infty$. Since the point $p$ is regular, we can set up the conjugacy discussed in §2. There is a $k_0$ such that for all $k \geq k_0$, $x(t_k, x_0) \in V$. For $k \geq k_0$, there exists $\tau_k$ such that $|t_k - \tau_k| < \varepsilon$ and $x(\tau_k, x_0) \in \mathcal{T}$. There also exists $q_k = r_k e_1 + s_k e_2 \in U$ such that $\psi(q_k) = x(t_k, x_0)$. Since $x(t_k, x_0) \to p$, we have $q_k \to 0$. Hence, $r_k \to 0$, and so $x(\tau_k, x_0) = \psi(r_k e_1) \to p$. We have constructed an increasing sequence of times $\tau_k \to \infty$ such that $x(\tau_k, x_0) \to p$ and $x(\tau_k, x_0) \in \mathcal{T}$, $k \geq k_0$.

§4. Now consider the set of all crossings of $\mathcal{T}$:

$$\mathcal{C} = \{\sigma > 0 : x(\sigma, x_0) \in \mathcal{T}\}.$$

Since by §2, $x(t, x_0)$ can cross $\mathcal{T}$ at most once in a time interval of length $2\varepsilon$, the set $\mathcal{C}$ is countable. Moreover, $\mathcal{C}$ contains the sequence $\{\tau_k\}$ of §3, so we have that $\mathcal{C} = \{\sigma_k\}$ is an increasing sequence with $\sigma_k \to \infty$ (Fig. 8.1).

For each $k$, write $x(\sigma_k, x_0) = \psi(\rho_k e_1)$, $\rho_k \in (-\varepsilon, \varepsilon)$. Since in §1 we have assumed that $x(t, x_0)$ is not periodic, we have that $x(\sigma_j, x_0) \neq x(\sigma_k, x_0)$, for $j \neq k$. Thus, $\rho_j \neq \rho_k$, for $j \neq k$. We are going to show that $\{\rho_k\}$ is a monotone sequence.

Choose any $k$, and put $\rho'_j = \rho_{k+j}$, $x(\sigma'_j, x_0) = \psi(\rho'_j)$, $j = 0, 1, 2$. Assume that $\rho'_0 < \rho'_1$. Define the curves

$$\xi_1 = \{x(t, x_0) : \sigma'_0 \leq t \leq \sigma'_1\}, \quad \xi_2 = \{\psi(\rho e_1) \in \mathcal{T} : \rho'_0 < \rho < \rho'_1\}.$$

Then $\xi = \xi_1 \cup \xi_2$ is a Jordan curve. Let $E$ denote its exterior and $I$ its interior.

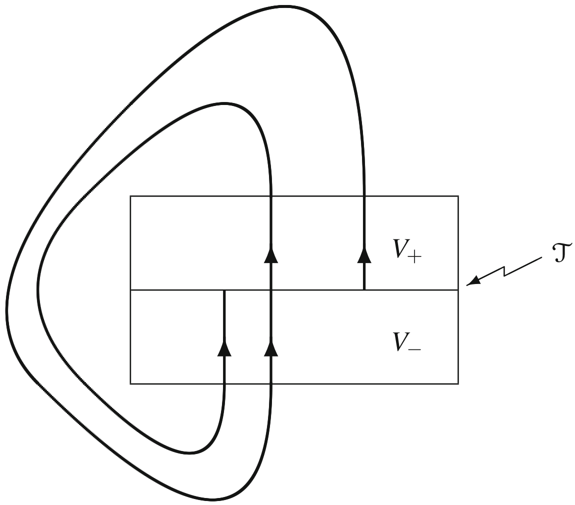



**Fig. 8.1** The flow near a transversal

The curve $\{x(t, x_0) : \sigma_1' < t < \sigma_2'\}$ does not intersect $\xi_1$ because the solution is not periodic, so it is contained in $E$ or $I$. In the first case, we have

$$\{\psi(\rho e_1 + se_2) : -\varepsilon < \rho < \sigma_1', \ -\varepsilon < s < 0\} \subset I,$$

and

$$\{\psi(\rho e_1 + se_2) : \sigma_1' < \rho < \varepsilon, \ -\varepsilon < s < 0\} \subset E.$$

It follows that $\rho_2' > \rho_1'$. The conclusion is the same if the roles of $E$ and $I$ are reversed. Similar arguments apply when $\rho_0' > \rho_1'$. We therefore see that $\{\rho_k\}$ is monotone.

As a consequence of monotonicity, the sequence $\{\rho_k\}$ has at most one limit point. From §3, we know that $\rho = 0$ is a limit point for $\{\rho_k\}$, so it is unique.

§5. Suppose that $p' \in \mathcal{T} \cap \omega(x_0)$. Then arguing as in §3, we can find an increasing sequence of times $\tau_k'$ such that $\tau_k' \to \infty$, $x(\tau_k', x_0) \in \mathcal{T}$, and $x(\tau_k', x_0) \to p'$, as $k \to \infty$. Writing $p' = \psi(\rho' e_1)$, we see that $\rho'$ is a limit point of $\{\rho_k\}$. Thus, by §4, we see that $\rho' = 0$ and $p' = p$. This shows that $\omega(x_0) \cap \mathcal{T} = \{p\}$.

§6. Let $y_0 \in \omega(x_0)$. The fact that $\omega(x_0) \subset K$ implies that $x(t, y_0)$ is defined for all $t \geq 0$ and $\gamma_+(y_0) \subset \omega(x_0)$. Thus, $\omega(y_0) \subset \omega(x_0)$. Take $p \in \omega(y_0)$. Construct the conjugacy of §2 at $p$. By §3, there exists an increasing sequence of times $\tau_k$ such that $\tau_k \to \infty$, $x(\tau_k, y_0) \in \mathcal{T}$, and $x(\tau_k, y_0) \to p$, as $k \to \infty$. Each point $x(\tau_k, y_0)$ belongs to $\mathcal{T} \cap \omega(x_0)$. By uniqueness, §5, $x(\tau_k, y_0) = p$. This proves that $\gamma_+(y_0)$ is a periodic orbit in $\omega(x_0)$, for any $y_0 \in \omega(x_0)$.

§7. It remains to show that $\gamma_+(y_0) = \omega(x_0)$, for any $y_0 \in \omega(x_0)$. The orbit $\gamma_+(y_0)$ is closed, and thus relatively closed in $\omega(x_0)$. Let $p \in \gamma_+(y_0)$. By §2 and §5, we have that $\gamma_+(y_0) \cap V = \omega(x_0) \cap V$, where $V$ is the conjugacy neighborhood centered at

$p$. So we see that $p \in \gamma_+(y_0) \cap V = \omega(x_0) \cap V$, Thus, $\gamma_+(y_0)$ is relatively open in $\omega(x_0)$. The set $\omega(x_0)$ is connected. Thus, $\gamma_+(y_0) = \omega(x_0)$.

**Corollary 8.1.** *A compact set in $\mathbb{R}^2$ which is positively invariant under the flow of a $C^1$ autonomous vector field and which contains no critical points must contain a nontrivial periodic orbit.*

## 8.7 Exercises

**Exercise 8.1.** Supply the details for the proof of Theorem 8.2.

**Exercise 8.2.** Let $h : [0, \infty) \to (0, \infty)$ be $C^1$. For $x = (x_1, x_2) \in \mathbb{R}^2$, let $r = \|x\|$. For $p \in \mathbb{R}^2$, let $x(t, p)$ denote the solution of the IVP

$$x_1' = h(r^2)x_2, \quad x_2' = -h(r^2)x_1 \quad x(0, p) = p.$$

(a) Show that for every $p \in \mathbb{R}^2$ and $t \in \mathbb{R}$, $\|x(t, p)\| = \|p\|$ .
(b) Show that $x(t, p)$ is periodic with period $2\pi/h(\|p\|^2)$, for $p \neq 0$.
(c) If $h$ is strictly increasing, show that for each $p$, $x(t, p)$ is orbitally stable, but not stable.

**Exercise 8.3.** Consider the system

$$x_1' = x_2 + \varepsilon(x_1^2 \sin t - \sin t), \quad x_2' = -x_1.$$

(a) Verify that this is a critical perturbation.
(b) Compute a Lagrange standard form for this system.
(c) Find the corresponding averaged equations.
(d) What does the averaged system say about the existence and stability of periodic solutions?

**Exercise 8.4.** Prove Lemma 8.4.

**Exercise 8.5.** Consider the system

$$x_1' = x_1 + x_2^2 - \sin t, \quad x_2' = -x_2.$$

(a) Find the solution to the initial value problem $x(t, p)$.
(b) Show that there is a unique $p \in \mathbb{R}^2$ such that $x(t, p)$ is periodic.
(c) Find the Floquet multipliers of this periodic solution.
(d) Show that the periodic solution is hyperbolic and find the local stable and unstable manifolds.

**Exercise 8.6.** Let $r = \|x\|$. Construct a $C^1$ planar autonomous vector field $f(x)$ with an unstable equilibrium at the origin, an asymptotically orbitally stable periodic orbit at $r = 1$, and an orbitally unstable periodic orbit at $r = 2$. *Suggestion: Polar coordinates.*

**Exercise 8.7.** Show that Mathieu's equation

$$u'' + (\omega^2 + \varepsilon \cos t)u = 0$$

has no nonzero $2\pi$-periodic solutions for $\omega \neq 1, 2, \ldots$, when $\varepsilon$ is small enough.

# Chapter 9
# Center Manifolds and Bifurcation Theory

## 9.1 The Center Manifold Theorem

**Definition 9.1.** *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ vector field with $F(0) = 0$. A center manifold for $F$ at $0$ is an invariant manifold containing $0$ which is tangent to and of the same dimension as the center subspace of $DF(0)$.*

Assume $F : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^1$ vector field with $F(0) = 0$. Set $A = DF(0)$, and let $E_s$, $E_u$, and $E_c$ be its stable, unstable, and center subspaces with their corresponding projections $P_s$, $P_u$, and $P_c$. Assume that $E_c \neq 0$. By Theorem 2.3 there exist constants $C_0, \lambda > 0, d \geq 0$ such that

$$\|\exp At\ P_s x\| \leq C_0 e^{-\lambda t} \|P_s x\|, \qquad t \geq 0 \tag{9.1}$$

$$\|\exp At\ P_u x\| \leq C_0 e^{\lambda t} \|P_u x\|, \qquad t \leq 0 \tag{9.2}$$

$$\|\exp At\ P_c x\| \leq C_0 (1 + |t|^d) \|P_c x\|, \qquad t \in \mathbb{R}. \tag{9.3}$$

Write $F(x) = Ax + f(x)$. Then $f : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$, $f(0) = 0$, and $Df(0) = 0$. Moreover, we temporarily assume that

$$\|f\|_{C^1} = \sup_{x \in \mathbb{R}^n} (\|f(x)\| + \|Df(x)\|) \leq M. \tag{9.4}$$

This restriction will be removed later, at the expense of somewhat weakening the conclusions of the next result.

As usual, we denote by $x(t, x_0)$ the solution of the initial value problem

$$x' = Ax + f(x), \quad x(0) = x_0.$$

Thanks to the strong bound assumed for the nonlinear portion of the vector field, the flow is globally defined for all initial points $x_0 \in \mathbb{R}^n$.

T. C. Sideris, *Ordinary Differential Equations and Dynamical Systems*, Atlantis Studies in Differential Equations 2, DOI: 10.2991/978-94-6239-021-8_9,

**Theorem 9.1.** (Center Manifold Theorem) *Let the constant $M$ in (9.4) be sufficiently small. There exists a function $\eta$ with the following properties:*

(i) $\eta : E_c \to E_s + E_u$ *is* $C^1$, $\eta(0) = 0$, *and* $D\eta(0) = 0$.
(ii) *The set*
$$W_c(0) = \{x_0 \in \mathbb{R}^n : P_s x_0 + P_u x_0 = \eta(P_c x_0)\}$$
*is invariant under the flow.*
(iii) *If $x_0$ has the property that there exist* $0 < \alpha < \lambda$ *and* $C > 0$ *such that*
$$\|P_s x(t, x_0)\| \le C\mathrm{e}^{-\alpha t}, \quad for\ all\ t < 0,$$
*and*
$$\|P_u x(t, x_0)\| \le C\mathrm{e}^{\alpha t}, \quad for\ all\ t > 0,$$
*then* $x_0 \in W_c(0)$.
(iv) *If* $x_0 \in W_c(0)$, *then* $w(t) = P_c x(t, x_0)$ *solves*
$$w' = Aw + P_c f(w + \eta(w)), \quad w(0) = P_c x_0.$$

*Remarks 9.1.*

- It follows from (i),(ii) that $W_c(0)$ is a center manifold.
- Statement (iii) is the nonlinear version of Corollary 2.3.

*Proof.* Fix $0 < \varepsilon < \lambda$. Let
$$X_\varepsilon = \{y \in C(\mathbb{R}, \mathbb{R}^n) : \|y\|_\varepsilon \equiv \sup_{t\in\mathbb{R}} \mathrm{e}^{-\varepsilon|t|}\|y(t)\| < \infty\}.$$

$X_\varepsilon$ is a Banach space with the norm $\|\cdot\|_\varepsilon$. Define a mapping $T : E_c \times X_\varepsilon \to X_\varepsilon$ by
$$\begin{aligned} T(y_0, y)(t) = \exp At\ y_0 &+ \int_0^t \exp A(t-\tau)\ P_c f(y(\tau))d\tau \\ &+ \int_{-\infty}^t \exp A(t-\tau)\ P_s f(y(\tau))d\tau \\ &- \int_t^\infty \exp A(t-\tau)\ P_u f(y(\tau))d\tau. \end{aligned}$$

The following estimate shows that $T(y_0, y)$ is a well-defined function in $X_\varepsilon$. Let $y_0 \in E_c$ and $y \in X_\varepsilon$. Then by (9.1)–(9.3), we have

$$\begin{aligned}
\|T(y_0, y)(t)\| &\le C_0(1+|t|^d)\|y_0\| + C_0 M \left| \int_0^t (1+|t-\tau|^d) d\tau \right| \\
&\quad + C_0 M \int_{-\infty}^t e^{-\lambda(t-\tau)} d\tau + C_0 M \int_t^\infty e^{\lambda(t-\tau)} d\tau \\
&\le CM\, e^{\varepsilon|t|}.
\end{aligned}$$

If $M$ is small enough, then $T$ is a uniform contraction on $X_\varepsilon$. Given $y_0 \in E_c$ and $y, z \in X_\varepsilon$, we have

$$\begin{aligned}
\|T(y_0, y)(t) - T(y_0, z)(t)\| &\le C_0 M \left| \int_0^t (1+|t-\tau|^d)\|y(\tau)-z(\tau)\| d\tau \right| \\
&\quad + C_0 M \int_{-\infty}^t e^{-\lambda(t-\tau)} \|y(\tau)-z(\tau)\| d\tau \\
&\quad + C_0 M \int_t^\infty e^{\lambda(t-\tau)} \|y(\tau)-z(\tau)\| d\tau \\
&\le CM e^{\varepsilon|t|} \|y-z\|_\varepsilon.
\end{aligned}$$

Thus,

$$\|T(y_0, y) - T(y_0, z)\|_\varepsilon \le CM\|y-z\|_\varepsilon \le (1/2)\|y-z\|_\varepsilon,$$

for $M$ small.

It follows from the Uniform Contraction Principle,[1] Theorem 5.6, that for every $y_0 \in E_c$ there is a unique fixed point $\psi(y_0) \in X_\varepsilon$:

$$T(y_0, \psi(y_0)) = \psi(y_0).$$

The assumptions on $f(x)$ also imply that $T$ is $C^1$, and so $\psi : E_c \to X_\varepsilon$ is a $C^1$ mapping. For notational convenience we will write $y(t, y_0) = \psi(y_0)(t)$. Note that $y(t, y_0)$ is $C^1$ in $y_0$. Since $y(t, y_0)$ is a fixed point, we have explicitly

$$\begin{aligned}
y(t, y_0) &= \exp At\; y_0 + \int_0^t \exp A(t-\tau)\; P_c f(y(\tau, y_0)) d\tau \\
&\quad + \int_{-\infty}^t \exp A(t-\tau)\; P_s f(y(\tau, y_0)) d\tau \\
&\quad - \int_t^\infty \exp A(t-\tau)\; P_u f(y(\tau, y_0)) d\tau.
\end{aligned} \tag{9.5}$$

Now define

[1] We use the Contraction Principle rather than the Implicit Function Theorem because this gives a solution for every $y_0 \in E_c$, thanks to the fact that $M = \|f\|_{C^1}$ is small.

$$\begin{aligned}\eta(y_0) =&(I - P_c)y(0, y_0)\\ =&\int_{-\infty}^{0} \exp(-A\tau)\ P_s f(y(\tau, y_0))d\tau\\ &- \int_{0}^{\infty} \exp A\tau\ P_u f(y(\tau, y_0))d\tau.\end{aligned} \tag{9.6}$$

$\eta : E_c \to E_s + E_u$ is $C^1$, and since $y(t, 0) = 0$ we have that $\eta(0) = 0$ and $D\eta(0) = 0$. Thus, the function $\eta$ fulfills the requirements of (i).

Define $W_c(0) = \{x_0 \in \mathbb{R}^n : (P_s + P_u)x_0 = \eta(P_c x_0)\}$.

As a step towards proving invariance, we verify the property

$$y(t + s, y_0) = y(t, P_c y(s, y_0)). \tag{9.7}$$

Fix $s \in \mathbb{R}$ and set $z(t) = y(t + s, y_0)$. Then from (9.5), we have

$$\begin{aligned}z(t) &= y(t + s, y_0)\\ &= \exp A(t + s)\ y_0 + \int_{0}^{t+s} \exp A(t + s - \tau)\ P_c f(y(\tau, y_0))d\tau\\ &\quad + \int_{-\infty}^{t+s} \exp A(t + s - \tau)\ P_s f(y(\tau, y_0))d\tau\\ &\quad - \int_{t+s}^{\infty} \exp A(t + s - \tau)\ P_u f(y(\tau, y_0))d\tau\\ &= \exp A(t + s)\ y_0\\ &\quad + \int_{0}^{s} \exp A(t + s - \tau)\ P_c f(y(\tau, y_0))d\tau\\ &\quad + \int_{s}^{t+s} \exp A(t + s - \tau)\ P_c f(y(\tau, y_0))d\tau\\ &\quad + \int_{-\infty}^{t+s} \exp A(t + s - \tau)\ P_s f(y(\tau, y_0))d\tau\\ &\quad - \int_{t+s}^{\infty} \exp A(t + s - \tau)\ P_u f(y(\tau, y_0))d\tau.\end{aligned}$$

Now factor $\exp At$ out of the first two terms, and make the change of variables $\sigma = \tau - s$ in the last three integrals. This results in

$$\begin{aligned}z(t) &= \exp At \left[\exp As\ P_c y_0 + \int_{0}^{s} \exp A(s - \tau)\ P_c f(y(\tau, y_0))d\tau\right]\\ &\quad + \int_{0}^{t} \exp A(t - \sigma)\ P_c f(y(\sigma + s, y_0))d\sigma\end{aligned}$$

$$
\begin{aligned}
&+\int_{-\infty}^{t} \exp A(t-\sigma)\ P_s f(y(\sigma+s, y_0))d\sigma \\
&-\int_{t}^{\infty} \exp A(t-\sigma)\ P_u f(y(\sigma+s, y_0))d\sigma \\
=\ & \exp At\ P_c y(s, y_0) + \int_0^t \exp A(t-\sigma)\ P_c f(z(\sigma))d\sigma \\
&+\int_{-\infty}^{t} \exp A(t-\sigma)\ P_s f(z(\sigma))d\sigma \\
&-\int_{t}^{\infty} \exp A(t-\sigma)\ P_u f(z(\sigma))d\sigma \\
=\ & T(P_c y(s, y_0), z)(t).
\end{aligned}
$$

By uniqueness of fixed points, (9.7) follows.

Notice that from (9.5) and (9.6), we have that

$$
y(t, y_0) = \exp At[y_0 + \eta(y_0)]\ +\ \int_0^t \exp A(t-\tau)\ f(y(\tau, y_0))d\tau = x(t, y_0 + \eta(y_0)).
$$

Thus, $x_0 \in W_c(0)$ if and only if $y(t, P_c x_0) = x(t, x_0)$.

Let $x_0 \in W_c(0)$. Using (9.7) and then (9.6), we obtain

$$
\begin{aligned}
(P_s + P_u)x(t, x_0) &= (P_s + P_u)y(t, P_c x_0) \\
&= (P_s + P_u)y(0, P_c y(t, P_c x_0)) \\
&= (I - P_c)y(0, P_c y(t, P_c x_0)) \\
&= \eta(P_c y(t, P_c x_0)) \\
&= \eta(P_c x(t, x_0)).
\end{aligned}
$$

This proves invariance of the center manifold (ii).

Let $x_0 \in \mathbb{R}^n$ be a point such that

$$
\begin{aligned}
\|P_s x(t, x_0)\| &\leq C\mathrm{e}^{-\alpha t},\ \text{for all } t < 0, \\
\|P_u x(t, x_0)\| &\leq C\mathrm{e}^{\alpha t},\quad \text{for all } t > 0,
\end{aligned}
\tag{9.8}
$$

for some $\alpha < \lambda$ and $C > 0$. From the linear estimates (9.1), (9.2) and (9.8), we get

$$
\|\exp(-At)\ P_s x(t, x_0)\| \leq C_0 \mathrm{e}^{\lambda t}\|P_s x(t, x_0)\| \leq C\mathrm{e}^{(\lambda-\alpha)t}, t \leq 0
$$

and

$$
\|\exp(-At)\ P_u x(t, x_0)\| \leq C_0 \mathrm{e}^{-\lambda t}|P_u x(t, x_0)\| \leq C\mathrm{e}^{(\alpha-\lambda)t}, t \geq 0.
$$

Next, from the Variation of Parameters formula,

$$x(t, x_0) = \exp At\ x_0 + \int_0^t \exp A(t-\tau)\ f(x(\tau, x_0))d\tau,$$

it follows that

$$P_s x(t, x_0) = \exp At \left[ P_s x_0 + \int_0^t \exp(-A\tau)\ P_s f(x(\tau, x_0))d\tau \right],$$

and

$$P_u x(t, x_0) = \exp At \left[ P_u x_0 + \int_0^t \exp(-A\tau)\ P_u f(x(\tau, x_0))d\tau \right].$$

Combining these formulas with the previous estimates, we get

$$\left\| P_s x_0 + \int_0^t \exp(-A\tau)\ P_s f(x(\tau, x_0))d\tau \right\| \leq C\mathrm{e}^{(\lambda-\alpha)t}, \quad \text{for all} \quad t < 0,$$

and

$$\left\| P_u x_0 + \int_0^t \exp(-A\tau)\ P_u f(x(\tau, x_0))d\tau \right\| \leq C\mathrm{e}^{(\alpha-\lambda)t}, \quad \text{for all} \quad t > 0.$$

Hence, if we send $t \to -\infty$ in the first inequality and $t \to \infty$ in the second, we obtain

$$P_s x_0 + \int_0^{-\infty} \exp(-A\tau)\ P_s f(x(\tau, x_0))d\tau = 0,$$

and

$$P_u x_0 + \int_0^{\infty} \exp(-A\tau)\ P_u f(x(\tau, x_0))d\tau = 0.$$

It follows that $x(t, x_0)$ solves the integral equation (9.5) with $y_0 = P_c x_0$. By uniqueness of fixed points, we have $x(t, x_0) = y(t, P_c x_0)$. This proves that $x_0 \in W_c(0)$, which is (iii).

Finally, let $x_0 \in W_c(0)$ and set $w(t) = P_c x(t, x_0)$. Since the center manifold is invariant under the flow, we have that

$$x(t, x_0) = w(t) + \eta(w(t)).$$

Multiply the differential equation by $P_c$ to get

$$\begin{aligned} w'(t) = P_c x'(t, x_0) &= P_c[Ax(t, x_0) + f(x(t, x_0))] \\ &= Aw(t) + P_c f(w(t) + \eta(w(t))), \end{aligned}$$

with initial condition

$$w(0) = P_c x(t, x_0) = P_c x_0.$$

This is *(iv)*. □

Without the smallness restriction (9.4), we obtain the following weaker result:

**Corollary 9.1.** (Local Center Manifold Theorem) *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$ with $f(0) = 0$ and $Df(0) = 0$. There exists a $C^1$ function $\eta$ and a small neighborhood $U = B_r(0) \subset \mathbb{R}^n$ with the following properties:*

(i) $\eta : E_c \to E_s + E_u$, $\eta(0) = 0$, *and* $D\eta(0) = 0$.
(ii) *The set* $W_c^{loc}(0) = \{x_0 \in U : P_s x_0 + P_u x_0 = \eta(P_c x_0)\}$ *is invariant under the flow in the sense that if* $x_0 \in W_c^{loc}(0)$, *then* $x(t, x_0) \in W_c^{loc}(0)$ *as long as* $x(t, x_0) \in U$.
(iii) *If* $x_0 \in W_c(0)$, *then* $w(t) = P_c x(t, x_0)$ *solves*

$$w' = Aw + P_c f(w + \eta(w)), \quad w(0) = P_c x_0,$$

*as long as* $x(t, x_0) \in U$.

**Definition 9.2.** *A set* $W_c^{loc}(0)$ *which satisfies (i) and (ii) is called a local center manifold.*

*Proof of Corollary 9.1.* By choosing $r > 0$ sufficiently small, we can find $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(x) = \tilde{f}(x)$ for all $x \in B_r(0) = U$ and (9.4) holds for $\tilde{f}$, with $M$ as small as we please. For example, choose $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ such that

$$\varphi(s) = \begin{cases} 1, & s \le 1 \\ 1/s, & s \ge 2. \end{cases}$$

Then we have

$$|s\varphi(s)| + |s\varphi'(s)| \le C \quad \text{for all} \quad s \in \mathbb{R}.$$

Define

$$\tilde{f}(x) = f(\varphi(\|x\|/r)\, x),$$

for $r$ sufficiently small.

Let $x(t, x_0)$, $\tilde{x}(t, x_0)$ be the flows of $Ax + f(x)$, $Ax + \tilde{f}(x)$, respectively. By uniqueness, we have that $x(t, x_0) = \tilde{x}(t, x_0)$, as long as $x(t, x_0) \in U$.

Fix the matrix $A$. Choose $r$ (and hence $M$) sufficiently small so that the Center Manifold Theorem applies for $\tilde{x}(t, x_0)$. The conclusions of the Corollary follow immediately for $x(t, x_0)$. □

*Example 9.1. (Nonuniqueness of the Center Manifold)* The smallness condition (9.4) leads to the intrinsic characterization of a center manifold given by (iii) in the Center Manifold Theorem 9.1. However, the following example illustrates that there may exist other center manifolds.

Consider the $2 \times 2$ system

$$x_1' = -x_1^3, \quad x_2' = -x_2; \qquad x_1(0) = \alpha, \ x_2(0) = \beta$$

which has the form

$$x' = Ax + f(x),$$

with

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad f(x) = \begin{bmatrix} -x_1^3 \\ 0 \end{bmatrix}.$$

The eigenvalues of $A$ are $\{0, -1\}$, and $A$ has one-dimensional center and stable subspaces spanned by the standard unit vectors $e_1$ and $e_2$, respectively. The projections are given by

$$P_c x = x_1 e_1 \quad \text{and} \quad P_s x = x_2 e_2.$$

Note that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is $C^\infty$ with $f(0) = 0$ and $Df(0) = 0$.

This system is easily solved, of course, since it is uncoupled. We have

$$x_1(t, \alpha, \beta) = \frac{\alpha}{\sqrt{1 + 2\alpha^2 t}},$$
$$x_2(t, \alpha, \beta) = \beta \mathrm{e}^{-t}.$$

Notice that the orbits are given by

$$x_2 \exp\left(\frac{1}{2x_1^2}\right) = \beta \exp\left(\frac{1}{2\alpha^2}\right).$$

Let $c_1, c_2 \in \mathbb{R}$ be arbitrary. Define the $C^\infty$ function

$$\xi(s) = \begin{cases} c_1 \exp\left(-\frac{1}{2s^2}\right), & s < 0 \\ 0, & s = 0 \\ c_2 \exp\left(-\frac{1}{2s^2}\right), & s > 0. \end{cases}$$

Use the function $\xi$ to obtain a mapping $\eta : E_c \to E_s$ by

$$\eta(x_1 e_1) = \xi(x_1) e_2.$$

Note that $\eta$ is $C^1$, $\eta(0) = 0$, and $D\eta(0) = 0$. Let

$$W_\xi = \{x \in \mathbb{R}^2 : P_s x = \eta(P_c x)\} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = \xi(x_1)\}.$$

The set $W_\xi$ is tangent to $E_c$ at the origin, and it is invariant because it is the union of orbits. Therefore it is a center manifold for every $\xi$.

**Theorem 9.2.** (Approximation of the Center Manifold) *Let $U \subset E_c$ be a neighborhood of the origin. Let $h : U \to E_s + E_u$ be a $C^1$ mapping with $h(0) = 0$ and $Dh(0) = 0$. If for $x \in U$,*

$$Ah(x) + (P_s + P_u)f(x + h(x)) - Dh(x)[Ax + P_c f(x + h(x))] = O(\|x\|^k),$$

*as $\|x\| \to 0$, then there is a a $C^1$ mapping $\eta : E_c \to E_s + E_u$ with $\eta(0) = 0$ and $D\eta(0) = 0$ such that*

$$\eta(x) - h(x) = O(\||x\|^k),$$

*as $\|x\| \to 0$, and*

$$\{x + \eta(x) : x \in U\}$$

*is a local center manifold.*

A proof of this result can be found in Carr [1]. The motivation for the result is the same as in the case of the stable manifold. If $\eta$ defines a local center manifold, then

$$A\eta(x) + (P_s + P_u)f(x + \eta(x)) - D\eta(x)[Ax + P_c f(x + \eta(x))] = 0 \quad x \in U.$$

*Remarks 9.2.* In the case where the local center manifold has a Taylor expansion near the origin, this formula can be used to compute coefficients. See Guckenheimer and Holmes [2] for an example in which this does not work.

## 9.2 The Center Manifold as an Attractor

**Definition 9.3.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $F : \Omega \to \mathbb{R}^n$ be an autonomous vector field with flow $x(t, x_0)$. A compact, connected, and positively invariant set $\mathcal{A} \subset \Omega$ is a local attractor for the flow if there exists an open set $U$ with $\mathcal{A} \subset U \subset \Omega$ such that for all $x_0 \in U$, the flow $x(t, x_0)$ is defined for all $t \geq 0$ and dist$(x(t, x_0), \mathcal{A}) \to 0$ as $t \to \infty$. If the neighborhood $U = \Omega$, then $\mathcal{A}$ is called a global attractor.*

We assume that:

- $F : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^1$ autonomous vector field with $F(0) = 0$.
- $F(x) = Ax + f(x)$, with $A = DF(0)$.
- The eigenvalues of $A$ satisfy $\mathrm{Re}\,\lambda \leq 0$.
- $x(t, x_0)$ is the (local) solution of

$$x' = Ax + f(x) \tag{9.9}$$

with initial data $x(0, x_0) = x_0$.

- According to Corollary 9.1, $\eta : E_c \to E_s$ defines a local center manifold

$$W_c^{loc}(0) = \{x \in B_\varepsilon(0) : P_c x \in B_\varepsilon(0) \cap E_c,\ \ P_s x = \eta(P_c x)\}$$

for (9.9).
- $w(t, w_0)$ is the (local) solution of the reduced flow on $W_c^{loc}(0)$

$$w' = Aw + P_c f(w + \eta(w)), \tag{9.10}$$

with $w(0, w_0) = w_0 \in B_\varepsilon(0) \cap E_c$.

**Theorem 9.3.** *Assume that $w = 0$ is a stable fixed point for* (9.10)*. There is a neighborhood of the origin $V \subset B_\varepsilon(0) \subset \mathbb{R}^n$ with the property that for every $x_0 \in V$, $x(t, x_0)$ is defined and remains in $B_\varepsilon(0)$, for all $t \geq 0$, and there corresponds a unique $w_0 \in V \cap E_c$ such that*

$$\|x(t, x_0) - w(t, w_0) - \eta(w(t, w_0))\| \leq C\mathrm{e}^{-\beta t},$$

*for all $t \geq 0$, where $0 < \beta < \lambda_s$.*

*Remarks 9.3.*

- The Theorem implies, in particular, that $x = 0$ is a stable fixed point for (9.9) and that $W_c^{loc}(0)$ is a local attractor for (9.9).
- The Theorem holds with the word "stable" replaced by "asymptotically stable". Corresponding "unstable" versions are also true as $t \to -\infty$.

*Proof.* The proof works more smoothly if instead of the usual Euclidean norm on $\mathbb{R}^n$, we use the norm $\|x\| = \|P_s x\| + \|P_c x\|$ so that $\|P_s\|, \|P_c\| \leq 1$. Alternatively, we could change coordinates so that $E_s \perp E_c$.

Thanks to the stability assumption, there is a $\delta_1 < \varepsilon$ such that if $w_0 \in B_{\delta_1}(0) \cap E_c$, then $w(t, w_0) \in B_{\varepsilon/4}(0)$, for all $t \geq 0$. By continuity of $\eta$, we may assume, moreover, that $\delta_1$ has been chosen small enough so that $\eta(w(t, w_0)) \in B_{\varepsilon/4}(0)$, for all $t \geq 0$.

Let $M/2 = \max\{\|Df(x)\| : x \in B_\varepsilon(0)\}$. By changing $f$ outside of $B_\varepsilon(0)$, we may assume that $\|Df(x)\| \leq M$, for all $x \in \mathbb{R}^n$. By uniqueness of solutions to the initial value problem, the modified flow is same as the original flow in $B_\varepsilon(0)$. Since $W_c^{loc}(0) \subset B_\varepsilon(0)$, $W_c^{loc}(0)$ remains a local center manifold for the modified flow.

Define the $C^1$ function $g : B_\varepsilon(0) \cap E_c \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$g(w, z) = f(z + w + \eta(w)) - f(w + \eta(w)).$$

Then for all $w \in B_\varepsilon(0) \cap E_c$ and $z \in \mathbb{R}^n$,

$$g(w, 0) = 0, \quad \|D_z g(w, z)\| \leq M, \quad \|g(w, z)\| \leq M\|z\|. \tag{9.11}$$

Given $x_0 \in \mathbb{R}^n$ and $w_0 \in B_{\delta_1}(0) \cap E_c$, set

$$z(t) = x(t, x_0) - w(t, w_0) - \eta(w(t, w_0)).$$

Since both $x(t, x_0)$ and $w(t, w_0) + \eta(w(t, w_0))$ solve the differential equation (9.9), $z(t)$ solves

$$z'(t) = Az(t) + g(w(t, w_0), z(t)), \quad w_0 \in B_{\delta_1}(0) \cap E_c \tag{9.12}$$

with data

$$z(0) = x_0 - w_0 - \eta(w_0).$$

Conversely, if $z(t)$ is a solution of (9.12) in $B_{\varepsilon/2}(0)$, then

$$x(t) = z(t) + w(t, w_0) + \eta(w(t, w_0))$$

is a solution of (9.9) in $B_\varepsilon(0)$ with data $x(0) = z(0) + w_0 + \eta(w_0)$.

The strategy will be to construct exponentially decaying solutions of (9.12) and then to adjust the initial data appropriately.

Given $0 < \beta < \lambda_s$, define the set of functions

$$X = \{z \in C(\mathbb{R}^+, \mathbb{R}^n) : \|z\|_\beta \equiv \sup_{t \geq 0} \mathrm{e}^{\beta t} \|z(t)\| < \infty\}.$$

$X$ is a Banach space with norm $\|\cdot\|_\beta$.

Now if $z_0 \in B_{\delta_1}(0)$, then $w_0 = P_c z_0 \in B_{\delta_1}(0) \cap E_c$. Thus, $w(t, w_0) \in B_{\varepsilon/4}(0)$, $\eta(w(t, w_0))$ is defined, and also remains in $B_{\varepsilon/4}(0)$. Next, if $z \in X$, then by the properties (9.11), we have that $\|g(w(\tau, w_0), z(\tau))\| \leq M\|z\|_\beta \, \mathrm{e}^{-\beta\tau}$. It follows by (9.1) and (9.3) that

$$\begin{aligned} \left\| \int_0^t \exp A(t-\tau) \; P_s g(w(\tau, w_0), z(\tau)) d\tau \right\| &\leq C\|z\|_\beta \int_0^t \mathrm{e}^{-\lambda_s(t-\tau)} \mathrm{e}^{-\beta\tau} d\tau \\ &\leq C\mathrm{e}^{-\beta t} \|z\|_\beta, \end{aligned}$$

and

$$\begin{aligned} \left\| \int_t^\infty \exp A(t-\tau) \; P_c g(w(\tau, w_0), z(\tau)) d\tau \right\| &\leq C\|z\|_\beta \int_t^\infty (1+\tau-t)^d \mathrm{e}^{-\beta\tau} d\tau \\ &= C\|z\|_\beta \int_0^\infty (1+\sigma)^d \mathrm{e}^{-\beta(\sigma+t)} d\sigma \\ &= C\mathrm{e}^{-\beta t} \|z\|_\beta. \end{aligned}$$

From this we see that the mapping

$$T(z_0, z)(t) = z(t) - \exp At\ P_s z_0 - \int_0^t \exp A(t-\tau)\ P_s g(w(\tau, P_c z_0), z(\tau))d\tau$$
$$+ \int_t^\infty \exp A(t-\tau)\ P_c g(w(\tau, P_c z_0), z(\tau))d\tau.$$

is well-defined and that $T : B_{\delta_1}(0) \times X \to X$. $T$ is a $C^1$ mapping, $T(0,0) = 0$, and $D_z T(0,0) = I$. The Implicit Function Theorem 5.7 says that there exists $\delta_2 < \delta_1$ and a $C^1$ map $\phi : B_{\delta_2}(0) \to X$ such that

$$\phi(0) = 0, \quad \text{and} \quad T(z_0, \phi(z_0)) = 0, \quad \text{for all} \quad z_0 \in B_{\delta_2}(0).$$

Set $z(t, z_0) = \phi(z_0)(t)$. By continuity, we may assume that $\delta_2$ is so small that $\|z(t, z_0)\|_\beta < \varepsilon/2$, for all $z_0 \in B_{\delta_2}(0)$.

The function $z(t, z_0)$ is an exponentially decaying solution of (9.12) which remains in $B_{\varepsilon/2}(0)$. Thus, we have that

$$x(t) = z(t, z_0) + w(t, w_0) + \eta(w(t, w_0))$$

is a solution of (9.9) in $B_\varepsilon(0)$ with initial data

$$F(z_0) = z(0, z_0) + P_c z_0 + \eta(P_c z_0).$$

It remains to show that given any $x_0 \in \mathbb{R}^n$ close to the origin, we can find a $z_0 \in B_{\delta_2}(0)$ such that $x_0 = F(z_0)$.

Notice that $F : B_{\delta_2}(0) \to B_\varepsilon(0)$ is $C^1$, $F(0) = 0$, and $DF(0) = I$. So by the Inverse Function Theorem A.3, there are neighborhoods of the origin $V \subset B_{\delta_2}(0)$ and $W \subset B_\varepsilon(0)$ such that $F : V \to W$ is a diffeomorphism. Therefore, given any $x_0 \in W$, there is a $z_0 = F^{-1}(x_0) \in V$. Then $w_0 = P_c z_0 \in B_{\delta_2}(0) \cap E_c$, and

$$\|x(t) - w(t, w_0) - \eta(w(t, w_0))\| = \|z(t, z_0)\| < (\varepsilon/2)\mathrm{e}^{-\beta t}. \qquad \square$$

*Example 9.2. (The Lorenz equation)* Let us take a look at the following system, known as the Lorentz equation,

$$\begin{aligned} x_1' &= -\sigma x_1 + \sigma x_2 \\ x_2' &= x_1 - x_2 - x_1 x_3 \qquad \beta, \sigma > 0 \\ x_3' &= -\beta x_3 + x_1 x_2. \end{aligned}$$

This can be expressed in the general form

$$x' = Ax + f(x)$$

with

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}, \quad \text{and} \quad f(x) = \begin{bmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{bmatrix}.$$

The eigenvalues of $A$ are $\{0, -(\sigma+1), -\beta\}$, and so $A$ has a one-dimensional center subspace and a two-dimensional stable subspace. Since $f(x)$ is a $C^1$ map with $f(0) = 0$ and $Df(0) = 0$, the center subspace goes over to a one-dimensional local center manifold $W_c^{loc}(0)$ for the nonlinear equation. We are going to approximate the flow on $W_c^{loc}(0)$. We will show that the origin is an asymptotically stable critical point for the flow on $W_c^{loc}(0)$. It follows from Theorem 9.3 that $W_c^{loc}(0)$ is a local attractor.

To simplify the analysis, we use the eigenvectors of $A$ to change coordinates. Corresponding to the eigenvalues

$$\lambda_1 = 0, \ \lambda_2 = -(\sigma+1), \ \lambda_3 = -\beta,$$

$A$ has the eigenvectors

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -\sigma \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If $S$ is the matrix whose columns are $u_1$, $u_2$, and $u_3$, then $AS = SB$ with $B = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$.

Make the change of variables $x = Sy$. Then

$$Sy' = x' = Ax + f(x) = ASy + f(Sy),$$

so that

$$y' = S^{-1}ASy + S^{-1}f(Sy) = By + g(y),$$

with

$$g(y) = S^{-1}f(Sy) = \begin{bmatrix} -\frac{\sigma}{1+\sigma}(y_1 - y_2)y_3 \\ -\frac{1}{1+\sigma}(y_1 - \sigma y_2)y_3 \\ (y_1 - \sigma y_2)(y_1 + y_2) \end{bmatrix}.$$

Of course we still have $g(0) = 0$ and $Dg(0) = 0$. Since the linear equations are now diagonal, we have

$$\begin{array}{ll} E_c = \text{span}\{e_1\} & E_s = \text{span}\{e_2, e_3\} \\ P_cy = y_1e_1 & P_sy = y_2e_2 + y_3e_3. \end{array}$$

This means that a local center manifold

$$W_c^{loc}(0) = \{y \in \mathbb{R}^n : P_cy \in U, \ P_sy = \eta(P_cy)\},$$

can be determined by a function $\eta : E_c \to E_s$ which has the form

$$\eta(y) = \eta(y_1 e_1) = \eta_2(y_1)e_2 + \eta_3(y_1)e_3, \quad y = y_1 e_1 \in E_c.$$

To approximate $\eta$, use the equation

$$B\eta(y) + P_s g(y + \eta(y)) - D\eta(y)[By + P_c g(y + \eta(y))] = 0, \quad y \in E_c.$$

This is equivalent to the system

$$\begin{aligned} -(\sigma+1)\eta_2(y_1) + g_2(y + \eta(y)) &= \eta_2'(y_1)g_1(y + \eta(y)) \\ -\beta\eta_3(y_1) + g_3(y + \eta(y)) &= \eta_3'(y_1)g_1(y + \eta(y)). \end{aligned} \tag{9.13}$$

Since $\eta(0) = 0$ and $D\eta(0) = 0$, the first term in the approximation of $\eta$ near 0 is of the form

$$\eta_2(y_1) = \alpha_2 y_1^2 + \cdots, \quad \eta_3(y_1) = \alpha_3 y_1^2 + \cdots.$$

Note that the right-hand side of (9.13) is of third order or higher. Therefore, using the form of $\eta$ and $g$, we get

$$\begin{aligned} -(\sigma+1)\eta_2(y_1) - \frac{1}{\sigma+1}(y_1 - \sigma\eta_2(y_1))\eta_3(y_1) &= O(|y_1|^3), \\ -\beta\eta_3(y_1) + (y_1 - \sigma\eta_2(y_1))(y_1 + \eta_2(y_1)) &= O(|y_1|^3). \end{aligned}$$

There can be no quadratic terms on the left, so

$$\begin{aligned} -(\sigma+1)\eta_2(y_1) &= 0, \\ -\beta\eta_3(y_1) + y_1^2 &= 0. \end{aligned}$$

This forces us to take $\alpha_2 = 0$ and $\alpha_3 = 1/\beta$. In other words, we have

$$W_c^{loc}(0) = \{y \in \mathbb{R}^3 : y_1 \in U,\ y_2 = O(y_1^3),\ y_3 = y_1^2/\beta + O(y_1^3)\}.$$

The flow on $W_c^{loc}(0)$ is governed by the equation

$$w' = Bw + P_c g(w + \eta(w)), \quad w(0) = w_0 \in E_c.$$

Since $w = P_c w = w_1 e_1$, this reduces to

$$\begin{aligned} w_1' &= -\frac{\sigma}{\sigma+1}(w_1 - \eta_2(w_1))\eta_3(w_1) \\ &\approx -\frac{\sigma}{\beta(\sigma+1)} w_1^3. \end{aligned}$$

Therefore, since $\beta, \sigma > 0$, the origin is an asymptotically stable critical point for the reduced flow on $W_c^{loc}(0)$. A simple calculation shows that

$$w_1(t) = \frac{w_1(0)}{\left[1 + \left(\frac{2\sigma w_1^2(0)}{\beta(\sigma+1)}\right) t\right]^{1/2}} \left[1 + O\left(\frac{1}{1+t}\right)\right].$$

Now $W_c^{loc}(0)$ is exponentially attracting, so

$$y(t) = w_1(t)e_1 + O(\mathrm{e}^{-Ct}).$$

It follows that all solutions in a neighborhood of the origin decay to 0 at a rate of $t^{-1/2}$, as $t \to \infty$, which is a purely nonlinear phenomenon.

## 9.3 Co-Dimension One Bifurcations

Consider the system $x' = f(x, \mu)$ in which $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a $C^\infty$ autonomous vector field depending on the parameter $\mu \in \mathbb{R}$. Suppose that $f(0, 0) = 0$; i.e. $x = 0$ is a critical point when $\mu = 0$. We wish to understand if and how the character of the flow near the origin changes when we vary the parameter $\mu$.

**Definition 9.4.** *The value $\mu = 0$ is called a bifurcation value if, for $\mu \neq 0$, the flow of $f(x, \mu)$ near the origin is not topologically conjugate to the flow of $f(x, \mu)$ for $\mu = 0$.*

Let $A(\mu) = D_x f(0, \mu)$. If $A(0)$ hyperbolic, then by the Implicit Function Theorem, there is a smooth curve of equilibria $x(\mu)$ such that $x(0) = 0$ and $f(x(\mu), \mu) = 0$, for $\mu$ near 0. Moreover, $x(\mu)$ is the only critical point of $f(x, \mu)$ in a neighborhood of the origin. Again for $\mu$ small, we see that by continuity $A(\mu)$ is hyperbolic, and it has stable and unstable subspaces of the same dimension as $A(0)$. By the Theorems 7.3 and 7.5 the flow of $x' = f(x, \mu)$ is topologically equivalent for all $\mu$ near 0. This argument shows that in order for a bifurcation to occur, $A(0)$ can not be hyperbolic.

The simplest case is when $A(0)$ has a single eigenvalue on the imaginary axis. Since $A(0)$ is real, that eigenvalue must be zero and all other eigenvalues must have non-zero real part. This situation is called the *co-dimension one* bifurcation.

Bifurcation problems are conveniently studied by means of the so-called *suspended system*

$$x' = f(x, \mu) \quad \mu' = 0. \tag{9.14}$$

This is obviously equivalent to the original problem, except that now $\mu$ is viewed as a dependent variable instead of a parameter. The orbits of the suspended system lie on planes $\mu = Const.$ in $\mathbb{R}^{n+1}$. The suspended system will have a center manifold of one dimension greater than the unperturbed equation $x' = f(x, 0)$. So in the co-dimension one case, it will be two-dimensional. We now proceed to reduce the flow of the suspended system to the center manifold.

Let $A = A(0)$ have invariant subspaces $E^A_c$, $E^A_s$, and $E^A_u$, with the projections $P^A_c$, $P^A_s$, and $P^A_u$. We are assuming that $E^A_c$ is one-dimensional, so let $E^A_c$ be spanned by the vector $v_1 \in \mathbb{R}^n$. Let $v_2, \ldots, v_n \in \mathbb{R}^n$ span $E^A_s \oplus E^A_u$, so that the set $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$.

Now returning to the suspended system, if we set

$$y = \begin{bmatrix} x \\ \mu \end{bmatrix}, \quad \text{and} \quad g(y) = \begin{bmatrix} f(x, \mu) \\ 0 \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R},$$

then (9.14) can be written as

$$y' = g(y).$$

Note that $g(0) = 0$ and

$$D_y g(0) = \begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix} \equiv B,$$

in which $C = \frac{\partial f}{\partial \mu}(0, 0) \in \mathbb{R}^n$. Write $C = C_1 + C_2$, with $C_1 = P^A_c C = \sigma v_1$ and $C_2 = (P^A_s + P^A_u)C$. Now $A$ is an isomorphism on $E^A_s \oplus E^A_u$, so there is a vector $v_0 \in E^A_s \oplus E^A_u$ such that $Av_0 = -C_2$.

The vectors

$$u_0 = \begin{bmatrix} v_0 \\ 1 \end{bmatrix}, \quad u_1 = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}, \quad \ldots, \quad u_n = \begin{bmatrix} v_n \\ 0 \end{bmatrix},$$

form a basis for $\mathbb{R}^{n+1}$. Let $E^B_s$, $E^B_c$, $E^B_u$ be the stable, center, and unstable subspaces for $B$ in $\mathbb{R}^{n+1}$. Note that

$$Bu_1 = 0 \quad \text{and} \quad Bu_0 = \begin{bmatrix} Av_0 + C \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma v_1 \\ 0 \end{bmatrix} = \sigma u_1;$$

i.e. $u_0$ is an eigenvector and $u_1$ is a generalized eigenvector for the eigenvalue 0. So $E^B_c$ is (at least) two-dimensional. Since the restriction of $B$ to the subspace spanned by the vectors $\{u_2, \ldots, u_n\}$ is the same as $A$ on $E^A_s \oplus E^A_u$, it follows that $\{u_2, \ldots, u_n\}$ spans $E^B_s \oplus E^B_u$. So it now also follows that $E^B_c$ is spanned by $\{u_0, u_1\}$.

We see that there is a two-dimensional local center manifold for the suspended system described by a $C^1$ function $\eta : E^B_c \to E^B_s \oplus E^B_u$, with $\eta(0) = 0$, $D\eta(0) = 0$. The reduced flow is then

$$w' = P^B_c g(w + \eta(w)), \quad w \in E^B_c.$$

Since

$$f(x,\mu) = f(y) = \sum_{j=1}^{n} f_j(y)v_j,$$

we have

$$g(y) = \begin{bmatrix} f(y) \\ 0 \end{bmatrix} = \sum_{j=1}^{n} f_j(y)u_j.$$

Therefore, given that $E_c^B$ is spanned by $u_0, u_1$, we can write

$$w = w_0u_0 + w_1u_1, \quad \text{and} \quad P_c^B g(y) = f_1(y)u_1.$$

In more explicit form, the reduced equation is

$$w_0'u_0 + w_1'u_1 = f_1(w_0u_0 + w_1u_1 + \eta(w_0u_0 + w_1u_1))u_1.$$

Define the scalar function

$$h(w_0, w_1) = f_1(w_0u_0 + w_1u_1 + \eta(w_0u_0 + w_1u_1)).$$

Compare the coefficients of the two components to get the system

$$w_0' = 0, \quad w_1' = h(w_0, w_1).$$

Looking at the definition of $h$ and $f_1$, we see that $h(0,0) = 0$ and $D_{w_1}h(0,0) = 0$, since

$$\begin{aligned}
&D_{w_1}h(0,0)u_1 \\
&\qquad = D_{w_1}[f_1(w_0u_0 + w_1u_1 + \eta(w_0u_0 + w_1u_1))]u_1|_{(w_0,w_1)=(0,0)} \\
&\qquad = D_{w_1}[P_c^B g(w_0u_0 + w_1u_1 + \eta(w_0u_0 + w_1u_1))]|_{(w_0,w_1)=(0,0)} \\
&\qquad = P_c^B Dg(0)[u_1 + D\eta(0)u_1] \\
&\qquad = P_c^B Bu_1 \\
&\qquad = 0.
\end{aligned}$$

Therefore, we have shown that, after relabelling $\mu = w_0$ and $x = w_1$, the co-dimension one bifurcation problem reduces to the study of a scalar equation

$$x' = h(\mu, x)$$

with $h : \mathbb{R}^2 \to \mathbb{R}$ smooth and $h(0,0) = 0$, $D_x h(0,0) = 0$.

In the following, we consider the three basic ways in which this type of bifurcation occurs.

## *Saddle-Node Bifurcation*

The generic example is illustrated by the vector field

$$f(x,\mu) = \varepsilon_1\mu - \varepsilon_2 x^2, \quad \varepsilon_i = \pm 1,\ i = 1, 2.$$

If, for example, $\varepsilon_1 = \varepsilon_2 = 1$, then $f(x,\mu)$ has no critical points when $\mu < 0$, one critical point when $\mu = 0$, and two critical points when $\mu > 0$. Thus, we have three distinct phase portraits, as shown in Fig. 9.1.

The situation can be summarized in a single bifurcation diagram as in Fig. 9.2. The solid line indicates a branch of stable critical points, and the dashed line indicates unstable critical points. The picture is reflected in the $\mu$-axis when the sign of $\varepsilon_1$ is negative and in the $x$-axis when $\varepsilon_2$ changes sign.

The general saddle-node bifurcation occurs when $f(\mu, x)$ satisfies

$$f(0,0) = D_x f(0,0) = 0, \quad \text{and} \quad D_\mu f(0,0) \neq 0, \quad D_x^2 f(0,0) \neq 0.$$

By the Implicit Function Theorem, the equation

$$f(x,\mu) = 0$$

can be solved for $\mu$ in terms of $x$. There is a smooth function $\mu(x)$ defined in a neighborhood of $x = 0$ such that

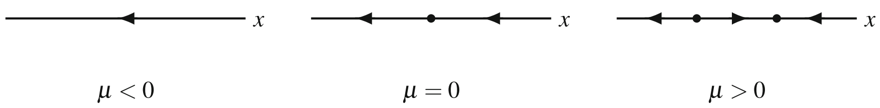
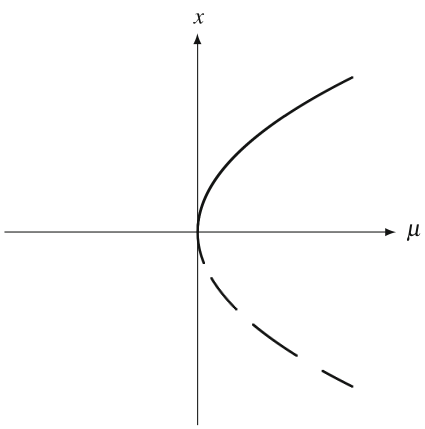



**Fig. 9.1** Phase diagrams for the saddle-node bifurcation

**Fig. 9.2** Saddle-node bifurcation diagram



$$\mu(0) = 0, \quad f(x, \mu(x)) = 0,$$

and all zeros of $f$ near $(0, 0)$ lie on this curve. If we differentiate the equation $f(x, \mu(x)) = 0$ and use the facts that $D_x f(0, 0) = 0$ and $D_\mu f(0, 0) \neq 0$ we find

$$\mu'(0) = 0 \quad \text{and} \quad \mu''(0) = -\frac{D_x^2 f(0, 0)}{D_\mu f(0, 0)} \neq 0.$$

So we can write

$$f(x, \mu) = g(x, \mu)(\mu - \mu(x)), \quad \text{with} \quad g(x, \mu) = \int_0^1 D_\mu f(x, \sigma\mu + (1 - \sigma)\mu(x))d\sigma.$$

Since $g(0, 0) \neq 0$, we obtain bifurcation diagrams analogous to the model case.

## *Transcritical Bifurcation*

The basic example is

$$f(x, \mu) = \alpha x^2 + 2\beta\mu x + \gamma\mu^2.$$

In order to have interesting dynamics for $\mu \neq 0$, there should be some critical points besides $x = 0$, $\mu = 0$. So we assume that $\alpha \neq 0$ and $\beta^2 - \alpha\gamma > 0$. This yields two lines of critical points

$$x_\pm(\mu) = \mu r_\pm, \quad \text{with} \quad r_\pm = [-\beta \pm \sqrt{\beta^2 - \alpha\gamma}]/\alpha,$$

and we can write

$$f(x, \mu) = \alpha(x - x_+(\mu))(x - x_-(\mu)).$$

For $\mu \neq 0$, there are a pair of critical points, one stable, the other unstable, and their stability is exchanged at $\mu = 0$.

Note that stability of both fixed points can be determined by the sign of $\alpha$, since $x_-(\mu) < -\beta\mu/\alpha < x_+(\mu)$ and $f(-\beta\mu/\alpha, \mu) = -[\beta^2 - \alpha\gamma]\mu^2/\alpha$.

In general, a transcritical bifurcation occurs when

$$f(0, 0) = D_x f(0, 0) = D_\mu f(0, 0) = 0, \quad \text{and} \quad D_x^2 f(0, 0) \neq 0.$$

In order to have critical points near $(0, 0)$, $f$ cannot have a relative extremum at $(0, 0)$. This is ruled out by assuming that the Hessian is negative. Let

$$\alpha = D_x^2 f(0, 0) \quad \beta = D_x D_\mu f(0, 0) \quad \gamma = D_\mu^2(0, 0).$$

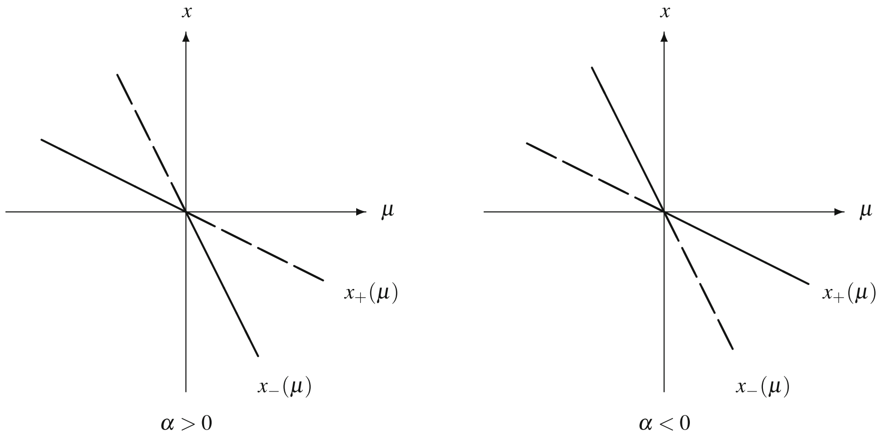



**Fig. 9.3** Transcritical bifurcation diagrams

Then $(0, 0)$ is not a local extremum for $f$ provided that

$$H = \det \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} = \alpha\gamma - \beta^2 < 0.$$

Suppose that $x(\mu)$ is a smooth curve of equilibria with $x(0) = 0$. From $f(x(\mu), \mu) = 0$ and implicit differentiation, we get

$$D_x f(x(\mu), \mu)x'(\mu) + D_\mu f(x(\mu), \mu) = 0,$$

and

$$\begin{aligned} &D_x^2 f(x(\mu), \mu)[x'(\mu)]^2 + 2D_x D_\mu f(x(\mu), \mu)x'(\mu) \\ &\quad + D_\mu^2 f(x(\mu), \mu) + D_x f(x(\mu), \mu)x''(\mu) = 0. \end{aligned}$$

If we set $\mu = 0$, we see that $x'(0)$ is a root of

$$\alpha\xi^2 + 2\beta\xi + \gamma = 0.$$

The solvability condition is precisely $H < 0$. From this we can expect to find two curves

$$x_\pm(\mu) = \mu r_\pm + \ldots \quad \text{with} \quad r_\pm = -[\beta \pm \sqrt{H}]/\alpha.$$

Now we proceed to construct the curves $x_\pm(\mu)$. By Taylor's theorem, we may write

$$f(x, \mu) = \frac{1}{2}\alpha x^2 + \beta\mu x + \frac{1}{2}\gamma\mu^2 + \delta(x, \mu),$$

in which the remainder has the form

$$\delta(x,\mu) = A(x,\mu)x^3 + B(x,\mu)x^2\mu + C(x,\mu)x\mu^2 + D(x,\mu)\mu^3,$$

with $A$, $B$, $C$, $D$ smooth functions of $(x,\mu)$.

Next consider the function

$$g(y,\mu) = \mu^{-2} f(\mu(y+r_+),\mu).$$

Thanks to the above expansion, we have that

$$g(y,\mu) = \frac{1}{2}\alpha(y+r_+)^2 + \beta(y+r_+) + \frac{1}{2}\gamma + \mu F(y,\mu),$$

with $F(y,\mu)$ a smooth function. It follows that $g(y,\mu)$ is smooth, $g(0,0) = 0$, and $D_y g(0,0) = \sqrt{-H} > 0$. If we apply the Implicit Function Theorem, we get a smooth function $y_+(\mu)$ such that

$$y_+(0) = 0, \quad g(y_+(\mu),\mu) = 0.$$

Set $x_+(\mu) = \mu(y_+(\mu) + r_+)$. Then

$$x_+(0) = 0, \quad x'_+(0) = r_+, \quad f(x_+(\mu),\mu) = 0.$$

In the same way we get a second curve $x_-(\mu)$ with slope $r_-$ at 0.

Since $x_- < (x_+ + x_-)/2 < x_+$, the stability can be determined by the sign of $f((x_+ + x_-)/2,\mu)$ which a simple calculation shows is approximated by $-H\mu^2/\alpha$, so we get diagrams similar to the one above.

## *Pitchfork Bifurcation*

The fundamental example is

$$f(x,\mu) = \varepsilon_1\mu x - \varepsilon_2 x^3, \quad \varepsilon_i = \pm 1,\ i = 1,2.$$

For example, if $\varepsilon_1 = \varepsilon_2 = 1$, then for $\mu < 0$ there is one stable critical point at $x = 0$ and for $\mu > 0$ two new critical points at $x = \pm\sqrt{\mu}$ are created. The stability of $x = 0$ for $\mu < 0$ is passed to the newly created pair for $\mu > 0$. The one-dimensional phase portraits are shown in Fig. 9.4. The bifurcation diagram is shown in Fig. 9.5. The other cases are similar.

The bifurcation is said to be *supercritical* when $\varepsilon_1\varepsilon_2 > 0$ since the new branches appear when $\mu$ exceeds the bifurcation value $\mu = 0$. When $\varepsilon_1\varepsilon_2 < 0$, the bifurcation is *subcritical*.

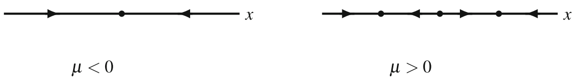



**Fig. 9.4** Phase diagrams for the pitchfork bifurcation

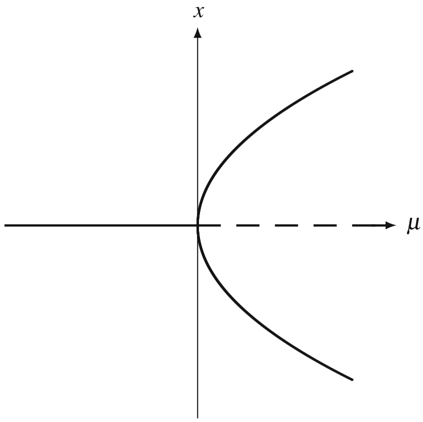



**Fig. 9.5** Supercritical pitchfork bifurcation diagram

The general case is identified by the following conditions

$$f(0,0) = D_x f(0,0) = D_\mu f(0,0) = D_x^2 f(0,0) = 0,$$
$$D_\mu D_x f(0,0) \neq 0, \quad D_x^3 f(0,0) \neq 0.$$

From the above analysis, we expect to find a pair of curves $x(\mu) \approx c_1\mu$ and $\mu(x) \approx \pm c_2 x^2$ to describing the two branches of equilibria. If such curves exist, then implicit differentiation shows that a necessary condition is

$$x(0) = 0 \quad x'(0) = \sigma \equiv -D_\mu^2 f(0,0)/D_\mu D_x f(0,0)$$

and

$$\mu(0) = \mu'(0) = 0 \quad \mu''(0) = \rho = -D_x^3 f(0,0)/3 D_x D_\mu f(0,0) \neq 0.$$

Therefore, the first curve can be found by applying the implicit function theorem to

$$g(y,\mu) = \mu^{-2} f(\mu(\sigma + y), \mu) = 0$$

to get $y(\mu)$ satisfying

$$y(0) = 0, \qquad g(y(\mu), \mu) = 0.$$

Then the curve is obtained by setting $x(\mu) = \mu(\sigma + y(\mu))$.

The second branch comes by considering the equation

$$h(x, \lambda) = x^{-2} f(x, x^2(\rho/2 + \lambda)) = 0.$$

The implicit function theorem yields a function $\lambda(x)$ such that

$$\lambda(0) = 0, \qquad g(x, \lambda(x)) = 0.$$

Finally, let $\mu(x) = x^2(\rho/2 + \lambda(x))$. The bifurcation is supercritical when $\rho > 0$ and subcritical when $\rho < 0$.

*Example 9.3. (The Lorenz Equation, II)* Let us have another look at the Lorenz equations considered in Sect. 9.2. This time, however, we add a small bifurcation parameter:

$$\begin{aligned} x_1' &= -\sigma x_1 + \sigma x_2 \\ x_2' &= (1 + \mu)x_1 - x_2 - x_1 x_3 \\ x_3' &= -\beta x_3 + x_1 x_2. \end{aligned}$$

Here $\beta, \sigma > 0$ are regarded as being fixed, and $\mu \in \mathbb{R}$ is small. The system has the form $x' = f(x, \mu)$ with $f(0, 0) = 0$. Moreover, we have already seen that the matrix

$$D_x f(0, 0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix},$$

has a one-dimensional center subspace, so that for $\mu \neq 0$ a bifurcation could occur. To study possible bifurcations near $(x, \mu) = (0, 0)$, we will consider the suspended system and reduce to the flow on the local center manifold. The suspended system is:

$$\begin{aligned} x_1' &= -\sigma x_1 + \sigma x_2 \\ x_2' &= x_1 - x_2 + \mu x_1 - x_1 x_3 \\ x_3' &= -\beta x_3 + x_1 x_2 \\ \mu' &= 0. \end{aligned}$$

Notice that because $\mu$ is being regarded as a dependent variable, $\mu x_1$ is considered to be a nonlinear term. So we are going to approximate the flow on the two-dimensional center manifold of the equation

$$z' = Az + g(z), \tag{9.15}$$

where

$$z = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \mu \end{bmatrix}, \quad A = \begin{bmatrix} -\sigma & \sigma & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad g(z) = \begin{bmatrix} 0 \\ x_1(\mu - x_3) \\ x_1 x_2 \\ 0 \end{bmatrix}. \tag{9.16}$$

By our earlier calculations, we have the following eigenvalues:

$$\lambda = 0, -(\sigma+1), -\beta, 0$$

and corresponding eigenvectors:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -\sigma \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

for the matrix $A$. Thus, the matrix

$$S = \begin{bmatrix} 1 & -\sigma & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

can be used to diagonalize A:

$$AS = SB, \quad \text{with} \quad B = \operatorname{diag}(0, -(\sigma+1), -\beta, 0).$$

If we make the change of coordinates $z = Sy$, then (9.15) and (9.16) become

$$y' = By + h(y),$$

in which $h(y) = S^{-1}g(Sy)$. More explicitly, we have

$$S^{-1} = \begin{bmatrix} \frac{1}{\sigma+1} & \frac{\sigma}{\sigma+1} & 0 & 0 \\ -\frac{1}{\sigma+1} & \frac{1}{\sigma+1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and hence

$$h(y) = \begin{bmatrix} \frac{\sigma}{\sigma+1}(y_1 - \sigma y_2)(y_4 - y_3) \\ \frac{1}{\sigma+1}(y_1 - \sigma y_2)(y_4 - y_3) \\ (y_1 - \sigma y_2)(y_1 + y_2) \\ 0 \end{bmatrix}.$$

Tracing back through the substitutions, we have $y_4 = z_4 = \mu$.

The local center manifold is described as

$$W_c^{loc}(0) = \{y \in \mathbb{R}^4 : P_c y \in U,\ P_s y = \eta(P_c y)\},$$

with $\eta : U \subset E_c \to E_s$ a $C^1$ map such that $\eta(0) = 0$, $D\eta(0) = 0$. Since $B$ is diagonal, it follows that $E_c = \text{span}\{e_1, e_4\}$ and $E_s = \text{span}\{e_2, e_3\}$. Given the form of $E_c$ and $E_s$, we can write

$$W_c^{loc}(0) = \{y \in \mathbb{R}^4 : (y_1, y_4) \in U,\ y_2 = \eta_2(y_1, y_4),\ y_3 = \eta_3(y_1, y_4)\}.$$

Using Theorem 9.2, we can approximate $\eta$ to third order. We find that $\eta$ satisfies

$$\begin{aligned} -(\sigma+1)\ \eta_2(y_1, y_4) + \frac{1}{\sigma+1}\ y_1 y_4 &= O(\|y\|^3) \\ -\beta\ \eta_3(y_1, y_4) + y_1^2 &= O(\|y\|^3). \end{aligned}$$

This shows that

$$\eta_2(y_1, y_4) = (\sigma+1)^{-2} y_1 y_4 + O(\|y\|^3), \quad \eta_3(y_1, y_4) = \beta^{-1} y_1^2 + O(\|y\|^3).$$

By part (iii) of Corollary 9.1, the flow on $W_c^{loc}(0)$ is governed by

$$w' = P_c h(w + \eta(w)), \quad w = P_c y,$$

which, in this case, is equivalent to

$$\begin{aligned} y_1' &= \frac{\sigma}{\sigma+1}(y_1 - \sigma\eta_2(y_1, y_4))(y_4 - \eta_3(y_1, y_4)) \\ y_4' &= 0, \end{aligned}$$

with initial conditions $y_1(0)$ and $y_4(0) = \mu$. Since $y_4 = \text{Const.} = \mu$, we end up with a scalar equation for $v = y_1$ of the form

$$v' = F(v, \mu),$$

with

$$\begin{aligned} F(v, \mu) &= \frac{\sigma}{\sigma+1}(v - \sigma\eta_2(v, \mu))(\mu - \eta_3(v, \mu)) \\ &\approx \frac{\sigma}{\sigma+1}\left[v - \frac{\sigma}{(\sigma+1)^2}\mu v\right]\left[\mu - \frac{1}{\beta}v^2\right] + \cdots \\ &= \frac{\sigma}{\sigma+1}\left[\mu v - \frac{1}{\beta}v^3 - \frac{\sigma}{(\sigma+1)^2}\mu^2 v\right] + \cdots \end{aligned}$$

Notice that

$$F(0,0) = D_v F(0,0) = D_\mu F(0,0) = D_v^2 F(0,0) = 0,$$

and

$$D_v D_\mu F(0,0) = \frac{\sigma}{\sigma+1} > 0, \quad D_v^3 F(0,0) = \frac{6\sigma}{\beta(\sigma+1)} > 0.$$

Thus we have a supercritical pitchfork bifurcation. For $\mu \le 0$ there is a single asymptotically stable fixed point at $x = 0$. If $\mu < 0$, then all solutions decay towards the origin exponentially fast. For $\mu > 0$ there is an unstable fixed point at $x = 0$. It has a two-dimensional stable manifold and a one-dimensional unstable manifold. Additionally, there are a pair of asymptotically stable fixed points. These three critical points lie on the invariant curve obtained by intersecting the center manifold of the suspended equation with the plane $\mu = Const$. This curve contains the unstable manifold of $x = 0$.

## 9.4 Poincaré Normal Forms

The analysis of the reduced flow on the center manifold can be a difficult task in more than one dimension. In one dimension, we have seen how the isolation of certain terms of the Taylor expansion of the vector field is essential in understanding the nature of a bifurcation. The *reduction to normal form* is a systematic procedure for eliminating all inessential terms. The presentation in this action largely follows Arnol'd [3].

Consider the flow of

$$x' = Ax + f(x) \tag{9.17}$$

where $A$ is an $n \times n$ real matrix and $f : \mathbb{R}^n \to \mathbb{R}^n$ is $C^\infty$ with $f(0) = 0$, $Df(0) = 0$. Suppose we make a change of variables

$$x = y + h(y) = \Phi(y), \tag{9.18}$$

with $h : \mathbb{R}^n \to \mathbb{R}^n$ in $C^\infty$ and $h(0) = 0$, $Dh(0) = 0$. The Inverse Function Theorem ensures us that $\Phi$ is a local diffeomorphism near the origin. That is, there exist neighborhoods of the origin $U, V \subset \mathbb{R}^n$ such that $\Phi : U \to V$ is a diffeomorphism.

Let us see what happens to the system under this change of coordinates. Suppose that $x(t)$ is a solution of (9.17) in $V$ and define $y(t)$ through (9.18). By the chain rule, we have

$$x' = y' + Dh(y)y' = [I + Dh(y)]y'.$$

Then using the fact that $x$ is a solution and then the invertibility of $D\Phi$, we get

$$\begin{aligned}
y' &= [I + Dh(y)]^{-1}x' \\
&= [I + Dh(y)]^{-1}[Ax + f(x)] \\
&= [I + Dh(y)]^{-1}[A(y + h(y)) + f(y + h(y))].
\end{aligned} \tag{9.19}$$

This is simpler than the proof of Theorem 8.6 where we studied the nonautonomous analog of this computation.

Suppose that the terms of the Taylor expansion of $f(x)$ at $x = 0$ all have degree $r \geq 2$. Write

$$f(x) = f_r(x) + O(\|x\|^{r+1}),$$

where $f_r(x)$ is a vector field all of whose components are homogeneous polynomials of degree $r$ and $O(\|x\|^{r+1})$ stands for a smooth function whose Taylor expansion at $x = 0$ starts with terms of degree at least $r + 1$. Take the function $h(y)$ in the coordinate change (9.18) to have the same form as $f_r(x)$, that is, suppose that $h(y)$ is a vector-valued homogeneous polynomial of degree $r$. Then since

$$[I + Dh(y)]^{-1} = I - Dh(y) + Dh(y)^2 - \ldots = I - Dh(y) + O(\|y\|^{2(r-1)}),$$

we have when we go back to (9.19)

$$y' = Ay + Ah(y) - Dh(y)Ay + f_r(y) + O(\|y\|^{r+1}).$$

Then expression

$$Ah(y) - Dh(y)Ay + f_r(y)$$

is homogeneous of degree $r$. We can attempt to kill these terms by choosing the function $h(y)$ so that

$$L_A h(y) \equiv Dh(y)Ay - Ah(y) = f_r(y). \tag{9.20}$$

The expression $L_A$ is known as the *Poisson* or *Lie bracket* of the vector fields $Ax$ and $h(x)$.

Equation (9.20) is really just a linear algebra problem. The set of all homogeneous vector-valued polynomials of degree $r$ in $\mathbb{R}^n$ is a finite-dimensional vector space, call it $\mathcal{H}_r^n$, and $L_A\mathcal{H}_r^n \to \mathcal{H}_r^n$ is a linear map.[2] So we are looking for a solution of the linear system $L_A h = f_r$ in $\mathcal{H}_r^n$.

When does (9.20) have solutions? In order to more easily examine our linear system in the vector space $\mathcal{H}_r^n$, we need to introduce a bit of notation. An $n$-tuple of nonnegative integers

$$m = (m_1, m_2, \ldots, m_n)$$

is called a multi-index. Its order or degree is defined to be

[2] The dimension of $\mathcal{H}_r^n$ is $n\begin{pmatrix} r+n-1 \\ r \end{pmatrix}$.

$$|m| = m_1 + m_2 + \ldots + m_n.$$

Given an $n$-tuple $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and a multi-index $m$ of degree $r$, monomials of degree $r$ are conveniently notated as

$$y^m = y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n}.$$

Now if $\{v_k\}_{k=1}^n$ is a basis for $\mathbb{R}^n$, then

$$\{y^m v_k : k = 1, \ldots, n;\ |m| = r\}$$

is a basis for $\mathcal{H}_r^n$.

**Lemma 9.1.** *Let $A$ be an $n \times n$ matrix, and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be its $n$-tuple of eigenvalues. Then the eigenvalues of the linear transformation $L_A$ on $\mathcal{H}_r^n$ are given by*

$$\lambda \cdot m - \lambda_j$$

*where $|m| = r$ is an multi-index of degree $r$ and $j = 1, \ldots, n$.*

*Proof.* We shall only sketch the proof in the case where $A$ is diagonalizable. Let $S$ be the matrix whose columns are a basis of eigenvectors $v_1, \ldots, v_n$, and let $\Lambda$ be the diagonal matrix of the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Then $A = S\Lambda S^{-1}$. The set of homogeneous polynomials

$$f_{m,j} : \mathbb{R}^n \to \mathbb{C}^n \quad \text{with} \quad f_{j,m}(x) = (S^{-1}x)^m v_j, \quad j = 1, \ldots, n, \quad |m| = r$$

forms a basis for (complex) $\mathcal{H}_r^n$ as well as being a set of eigenvectors for $L_A$. A direct calculation shows that $L_A \ f_{m,j}(x) = (\lambda \cdot m - \lambda_j) f_{m,j}(x)$. This confirms the statement about the eigenvalues, and it also shows that, in the case that $A$ is diagonalizable, the range of $L_A$ is spanned by $f_{m,j}(x)$ for $m$ and $j$ such that $\lambda \cdot m - \lambda_j \neq 0$. A real basis for the range of $L_A$ is obtained by taking real and imaginary parts, and this is summarized in the next corollary. If $A$ is not diagonalizable, the argument is a bit more complicated. □

**Corollary 9.2.** *Let $A$ be a diagonalizable $n \times n$ matrix with eigenvalues $\{\lambda_j\}$. Define the polynomials $f_{j,m}(x)$, $j = 1, \ldots, n$, $|m| = r$, as above. Then*

$$\mathcal{H}_r^n = R_r(L_A) \oplus N_r(L_A),$$

*where*

$$R_r(L_A) = \text{span}\{\text{Re} f_{j,m}(x),\ \text{Im}\, f_{j,m}(x) : \lambda \cdot m - \lambda_k \neq 0\}$$

*is the range of $L_A$, and*

$$N_r(L_A) = \text{span}\{\text{Re} f_{j,m}(x),\ \text{Im}\, f_{j,m}(x) : \lambda \cdot m - \lambda_k = 0\}$$

*is the null space.*

**Definition 9.5.** *A multi-index $m$ of degree $r$ is resonant for $A$ if $\lambda \cdot m - \lambda_k = 0$, for some $k$.*

Thus, if $A$ has no resonant multi-indices of degree $r$, then $L_A$ is invertible on $\mathcal{H}_r^n$ and Eq. (9.20) has a unique solution $h \in \mathcal{H}_r^n$ for every $f_r \in \mathcal{H}_r^n$.

*Example 9.4.* Let us compute the resonances of degree two and three for the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues of $A$ are $\lambda = (i, -i)$, and $A$ is diagonalizable. First, we list $\lambda \cdot m$ for the three multi-indices $m$ of degree two:

$$(2,0) \cdot (i,-i) = 2i, \quad (1,1) \cdot (i,-i) = 0, \quad (0,2) \cdot (i,-i) = -2i.$$

None of these numbers is an eigenvalue for $A$, so there are no resonances of order 2. Now we do the same thing for $r = 3$. There are 4 multi-indices of degree 3:

$$\begin{gathered} (3,0) \cdot (i,-i) = 3i, \quad (2,1) \cdot (i,-i) = i = \lambda_1, \\ (1,2) \cdot (i,-i) = -i = \lambda_2, \quad (0,3) \cdot (i,-i) = -3i. \end{gathered}$$

We find two resonant multi-indices of degree 3, namely (2, 1) and (1, 2). Applying the formula of Corollary 9.2, we obtain

$$N_3(L_A) = \text{span}\left\{ \|x\|^2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \|x\|^2 \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \right\}.$$

**Definition 9.6.** *The* convex hull *of a set $\Omega \subset \mathbb{R}^n$ is the smallest convex set which contains $\Omega$. The convex hull is denoted by co $\Omega$.*

**Lemma 9.2.** *Let $A$ be an $n \times n$ matrix, and let $\sigma(A)$ be the set containing its eigenvalues. If* $0 \notin \text{co}\,\sigma(\text{A})$*, then $A$ has at most finitely many resonances.*

*Proof.* Since $\sigma(A)$ is finite, co $\sigma(A)$ is closed. Therefore, if $0 \notin$ co $\sigma(A)$, then there is a $\delta > 0$ such that $|z| \geq \delta$ for all $z \in$ co $\sigma(A)$.

Let $m$ be a multi-index with $|m| = r$. Then

$$\lambda \cdot m/r = \sum_{j=1}^{n} \frac{m_j}{r} \lambda_j \in \text{co}\,\sigma(A),$$

so $|\lambda \cdot m/r| \geq \delta$.

Choose $M > 0$ so that $|\lambda_k| < M$, for all $\lambda_k \in \sigma(A)$. If $r \geq M/\delta$, then $|\lambda \cdot m| \geq r\delta \geq M > \lambda_k$, for all $\lambda_k \in \sigma(A)$. Thus, $x^m$ is nonresonant if $|m| \geq M/\delta$. □

*Remark 9.4.* If $A$ is real, then since complex eigenvalues come in conjugate pairs, $0 \notin \mathrm{co}\, \sigma(A)$ if and only if $\sigma(A)$ lies on one side of the imaginary axis.

**Theorem 9.4.** *Let $Ax + f(x)$ be a smooth vector field with $f(0) = 0$ and $Df(0) = 0$. Consider the Taylor expansion of $f(x)$ near $x = 0$:*

$$f(x) = f_2(x) + f_3(x) + \cdots + f_p(x) + Rf_{p+1}(x),$$

*in which $f_r(x) \in \mathcal{H}_r^n$, $r = 2, \ldots, p$ and the remainder $Rf_{p+1}$ is smooth and $O(\|x\|^{p+1})$.*

*For each $r = 1, \ldots, p$ decompose $\mathcal{H}_r^n$ as $\mathcal{H}_r^n = R_r(L_A) \oplus G_r(L_A)$, where $R_r(L_A)$ is the range of $L_A$ on $\mathcal{H}_r^n$ and $G_r(L_A)$ is a complementary subspace. There exists a polynomial change of coordinates, $x = \Phi(y)$, which is invertible in a neighborhood of the origin, which transforms the equation*

$$x' = Ax + f(x)$$

*into Poincaré normal form*

$$y' = Ay + g_2(y) + \ldots + g_p(y) + Rg_{p+1}(y),$$

*with $g_r \in G_r$ and the remainder $Rg_{p+1}$ smooth and $O(\|y\|^{p+1})$ near $y = 0$.*

*Proof.* We will proceed inductively, treating terms of order $r = 2, \ldots, p$ in succession. First, write

$$f_2(x) = f_2(x) - g_2(x) + g_2(x),$$

with $f_2(x) - g_2(x) \in R(L_A)$ and $g_2(x) \in G_2$. Let $h_2 \in \mathcal{H}_2^n$ be a solution of

$$L_A h_2 = f_2 - g_2.$$

Then $x = y + h_2(y)$ transforms $x' = Ax + f(x)$ into

$$y' = Ay + g_2(y) + \tilde{f}_3(y) + \cdots + \tilde{f}_p(y) + O(\|y\|^{p+1}).$$

If we continue in the same manner to eliminate terms of order 3, then we will have a coordinate change of the form $y = z + h_3(z)$ with $h_3 \in \mathcal{H}_3^n$. The point is that this change of coordinates will not affect terms of lower order. Thus, we can proceed to cancel terms of successively higher order. □

*Example 9.5.* Consider the system $x' = Ax + f(x)$ with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and $f(0) = 0$, $Df(0) = 0$. Since $A$ has the double eigenvalue 0, all multi-indices are resonant! Nevertheless, $L_A$ is nonzero, so we can remove some terms. Let's study $L_A$ on $\mathcal{H}_2^2$ and determine the normal form of degree 2.

First, for an arbitrary function $H(x) = \begin{bmatrix} H_1(x) \\ H_2(x) \end{bmatrix}$, we have

$$L_A H(x) = DH(x)Ax - AH(x) = \begin{bmatrix} x_2 \partial_1 H_1(x) - H_2(x) \\ x_2 \partial_1 H_2(x) \end{bmatrix}.$$

Choose a basis for $\mathcal{H}_2^2$:

$$\begin{array}{ll} h_1(x) = \frac{1}{2} x_1^2 e_1 & h_4(x) = \frac{1}{2} x_1^2 e_2 \\ h_2(x) = x_1 x_2 e_1 & h_5(x) = x_1 x_2 e_2 \\ h_3(x) = \frac{1}{2} x_2^2 e_1 & h_6(x) = \frac{1}{2} x_2^2 e_2 \end{array}$$

We have used the standard basis vectors $e_1, e_2$ here because they are generalized eigenvectors for $A$. After a little computation, we find that

$$\begin{array}{ll} L_A h_1 = h_2 & L_A h_2 = 2h_3 \\ L_A h_3 = 0 & L_A h_4 = -h_1 + h_5 \\ L_A h_5 = -h_2 + 2h_6 & L_A h_6 = -h_3 \end{array}$$

Thus, the range of $L_A$ is given by

$$R(L_A) = \text{span } \{h_2, h_3, -h_1 + h_5, h_6\}.$$

We can choose the complementary subspace

$$G_2 = \text{span } \{h_1, h_4\}.$$

Now using Taylor's theorem, expand $f(x)$ to second order near $x = 0$:

$$\begin{aligned} f(x) &= f_2(x) + O(\|x\|^3) \\ &= \sum_{i=1}^{6} \alpha_i h_i(x) + O(\|x\|^3) \\ &= [\alpha_2 h_2(x) + \alpha_3 h_3(x) + \alpha_5(-h_1(x) + h_5(x)) + \alpha_6 h_6(x)] \\ &\quad + [(\alpha_1 + \alpha_5) h_1(x) + \alpha_4 h_4(x)] + O(\|x\|^3) \\ &\equiv \bar{f}_2(x) + g_2(x) + O(\|x\|^3), \end{aligned}$$

with $f_2 \in R_2(L_A)$ and $g_2 \in G_2(L_A)$. Set

$$H(x) = \alpha_2 h_1(x) + \frac{1}{2}\alpha_3 h_2(x) + \alpha_5 h_4(x) + \frac{1}{2}\alpha_6(h_1(x) - h_5(x)).$$

Then

$$L_A H(x) = \bar{f}_2(x),$$

and using the transformation $x = y + H(y)$, we can achieve the normal form

$$y' = Ay + g_2(y) + O(\|y\|^3).$$

To complete this section we state a general theorem about reduction to normal form.

**Theorem 9.5.** (Sternberg's Theorem) *Let $A$ be an $n \times n$ matrix with at most a finite number of resonances. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be $C^\infty$ with $f(0) = 0$, $Df(0) = 0$. For any positive integer $k$, there exists a $C^k$ function $h : \mathbb{R}^n \to \mathbb{R}^n$ with $h(0) = 0$, $Dh(0) = 0$ such that the change of variables $x = y + h(y)$ transforms $x' = Ax + f(x)$ in a neighborhood of the origin into*

$$y' = Ay + g_2(y) + \cdots + g_p(y),$$

*where $g_r \in \mathcal{H}_r^n \ominus R(L_A)$, $r = 2, \ldots, p$, and where $p$ is the maximum order of any resonance.*

*Remarks 9.5.*

- The difficult proof can be found in the book of Hartman [4] (see Theorem 12.3).
- The result is the smooth analogue of the Hartman-Grobman theorem 7.2. It says, in particular, that if $A$ has no resonances, then the flow is $C^k$ conjugate to the linear flow.
- The hypothesis that $A$ has finitely many resonances implies that $A$ is hyperbolic.
- In general, the size of the neighborhood in which the transformation is invertible shrinks as $k$ increases. Nothing is being said here about the existence of a $C^\infty$ change of coordinates. Theorems of that sort have been established in the analytic case, see Arnol'd [3].

## 9.5 The Hopf Bifurcation

The Hopf bifurcation occurs when a pair of distinct complex conjugate eigenvalues of an equilibrium point cross the imaginary axis as the bifucation parameter is varied. At the critical bifurcation value, there are two (nonzero) eigenvalues on the imaginary axis. So this is an example of a co-dimension two bifurcation. As the bifurcation parameter crosses the critical value, a periodic solution is created.

The general case will be reduced to the following simple planar autonomous system, with higher order perturbative terms. For now, the illustrative paradigm is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (x_1^2 + x_2^2)\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{9.21}$$

Here, as usual, $\mu$ denotes the bifurcation parameter. The remaining constants $a$, $b$, $\omega$ are fixed with $a \neq 0$, $\omega > 0$. Notice that $x = 0$ is a fixed point, for all $\mu \in \mathbb{R}$.

The corresponding linear problem

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

has eigenvalues $\mu \pm i\omega$. Thus, when $\mu < 0$, the critical point $x = 0$ is asymptotically stable for (9.21), and when $\mu > 0$, it is unstable. When $\mu = 0$, the eigenvalues lie on the imaginary axis.

In order to find the periodic solution of (9.21), we change to polar coordinates. Let

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta.$$

The equations transform into

$$\begin{aligned} r' &= \mu r + a r^3 \\ \theta' &= \omega + b r^2. \end{aligned}$$

The interesting feature here is that the first equation is independent of $\theta$. In fact, we recognize the equation for $r$ as the basic example of a pitchfork bifurcation. If $a < 0$ the bifurcation is supercritical, and if $a > 0$ it is subcritical.

Suppose that $a < 0$. Then the asymptotically stable critical point which appears at $r = (-\mu/a)^{1/2}$ when $\mu > 0$ corresponds to an asymptotically orbitally stable periodic orbit for (9.21). If we set $\alpha = 1/\sqrt{-a}$ and $\beta = b/(-a)$, then this periodic solution is explicitly represented by

$$\begin{aligned} x_1(t) &= \alpha\sqrt{\mu}\cos[(\omega + \beta\mu)t], \\ x_2(t) &= \alpha\sqrt{\mu}\sin[(\omega + \beta\mu)t]. \end{aligned}$$

The amplitude is $\alpha\sqrt{\mu}$ and the period is $2\pi/(\omega + \beta\mu)$.

Simple as this example is, surprisingly it contains the essential information necessary for understanding the general case. However, in order to see this it will be necessary to make several natural changes of variables.

We begin with a planar autonomous equation depending on a parameter

$$x' = f(x, \mu), \tag{9.22}$$

with a critical point at the origin when $\mu = 0$, i.e. $f(0, 0) = 0$. Think of this equation as the result of reducing a higher dimensional system with a critical point at the origin when $\mu = 0$ to its two-dimensional center manifold. If $D_x f(0, 0)$ is invertible, then by the Implicit Function Theorem there exists a smooth curve of equilibria $x(\mu)$ for $\mu$ near 0. If we set $g(x, \mu) = f(x + x(\mu), \mu)$, then $g(0, \mu) = 0$. Therefore, we will consider vector fields in this form. Altogether we will assume that

A1. $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is $C^\infty$,
A2. $f(0, \mu) = 0$,
A3. $D_x f(0, 0)$ has eigenvalues $\pm i\omega$, with $\omega > 0$, and
A4. $\tau(\mu) = \operatorname{tr} D_x f(0, \mu)$ satisfies $\tau'(0) \neq 0$.

Condition A3 implies that $D_x f(0, \mu)$ has eigenvalues of the form $\xi(\mu) \pm i\eta(\mu)$ with $\xi(0) = 0$ and $\eta(0) = \omega$. Assumption A4 says that $\xi'(0) \neq 0$, since $\tau(\mu) = 2\xi(\mu)$. It means that as the bifurcation parameter passes through the origin, the eigenvalues cross the imaginary axis with nonzero speed.

The first step will be to show:

**Lemma 9.3.** *Let $f(x, \mu)$ be a vector field which satisfies the assumptions A1–A4 above. For $|\mu|$ small, there is a smooth change of coordinates $(t, x, \mu) \mapsto (s, y, \lambda)$ which transforms solutions $x(t, \mu)$ of (9.22) to solutions $y(s, \lambda)$ of*

$$y' = A_\lambda y + g(y, \lambda), \tag{9.23}$$

*in which $g(y, \lambda)$ is smooth and*

$$A_\lambda = \begin{bmatrix} \lambda & -\omega \\ \omega & \lambda \end{bmatrix}, \quad g(0, \lambda) = 0, \quad D_y g(0, \lambda) = 0.$$

*Proof.* Let $\xi(\mu) \pm i\eta(\mu)$ denote the eigenvalues of $A_\mu = D_x f(0, \mu)$. Choose a basis which reduces $A_\mu$ to canonical form:

$$S_\mu^{-1} A_\mu S_\mu = \begin{bmatrix} \xi(\mu) & -\eta(\mu) \\ \eta(\mu) & \xi(\mu) \end{bmatrix}.$$

Then the transformation $x(t, \mu) = S_\mu y(t, \mu)$ transforms (9.22) into $y' = g(y, \mu)$ in which $g(y, \mu) = S_\mu^{-1} f(S_\mu y, \mu)$ still satisfies A1–A4 and in addition

$$D_y g(0, \mu) = \begin{bmatrix} \xi(\mu) & -\eta(\mu) \\ \eta(\mu) & \xi(\mu) \end{bmatrix}.$$

Next we rescale time through the change of variables $t = (\omega/\eta(\mu))s$. Then $y((\omega/\eta(\mu))s, \mu)$ solves

$$\frac{d}{ds}y = h(y, \mu),$$

with

$$h(y, \mu) = (\omega/\eta(\mu))g(y, \mu).$$

Again $h(y, \mu)$ satisfies A1–A4, and also

$$D_y h(0, \mu) = \begin{bmatrix} \omega \dfrac{\xi(\mu)}{\eta(\mu)} & -\omega \\ \omega & \omega \dfrac{\xi(\mu)}{\eta(\mu)} \end{bmatrix}.$$

Finally, because of our assumptions, the function $\psi(\mu) = \omega\xi(\mu)/\eta(\mu)$ has $\psi'(0) \neq 0$ and is therefore locally invertible. Set $\lambda = \psi(\mu)$. Then $y(s, \lambda)$ is a solution of $y' = h(y, \lambda)$, and

$$D_y h(0, \lambda) = \begin{bmatrix} \lambda & -\omega \\ \omega & \lambda \end{bmatrix}.$$

This completes the proof of the lemma. □

We continue from (9.23) with the variables renamed. We are now going to apply the normal form technique to the suspended system

$$x' = A_\mu x + g(x, \mu), \quad \mu' = 0.$$

From this point of view, we should write this as

$$x' = A_0 x + \mu x + g(x, \mu), \quad \mu' = 0,$$

since $\mu x$ is a quadratic term. The eigenvalues for the linearized problem

$$x' = A_0 x, \quad \mu' = 0$$

are $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (i\omega, -i\omega, 0)$.

The resonances of order $|m| = 2$ occur when

$$m = (0, 1, 1),\ (1, 0, 1),\ (1, 1, 0),\ (0, 0, 2).$$

Following the procedure of Corollary 9.2, we have that the corresponding resonant monomials are

$$N_2 = \text{span}\left\{ \mu \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix},\ \mu \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix},\ \|x\|^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \mu^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The last two of these monomials do not occur because the third component of the vector field is trivial. By the properties of $g(x,\mu)$, the quadratic terms of its Taylor expansion only involve the variables $x$ and not $\mu$. So by a quadratic change of variables of the form $x = y + h(y)$ on $\mathbb{R}^2$, we can eliminate all quadratic terms from $g$ and (after relabeling $y$ with $x$) transform the system to

$$x' = A_0 x + \mu x + \tilde{g}(x,\mu), \quad \mu' = 0,$$

in which $\tilde{g}$ is at least cubic in $x, \mu$ and still satisfies $\tilde{g}(0,\mu) = 0$ and $D_x\tilde{g}(0,\mu) = 0$.

The resonances of order $|m| = 3$ are given by

$$m = (2,1,0),\ (1,2,0),\ (1,1,1),\ (0,1,2),\ (1,0,2),\ (0,0,3).$$

Using Corollary 9.2 again, we find the corresponding resonant monomials

$$N_3 = \left\{ \|x\|^2 \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix},\ \|x\|^2 \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix},\ \mu^2 \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix},\ \mu^2 \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix},\ \mu\|x\|^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \mu^3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The last four of these do not occur since $\tilde{g}$ is at least quadratic in $x$ and the third equation is still trivial. Thus, using a cubic change of variables of the form $x = y + h(y,\mu)$, with $h(0,\mu) = 0$, $D_y h(0,\mu) = 0$, we achieve the normal form

$$x' = A_0 x + \mu x + \|x\|^2 Bx + \tilde{\tilde{g}}(x,\mu), \quad \mu' = 0, \tag{9.24}$$

with

$$B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

and with $\tilde{\tilde{g}}$ of fourth order in $x, \mu$, satisfying $\tilde{\tilde{g}}(0,\mu) = 0$, $D_x\tilde{\tilde{g}}(0,\mu) = 0$. We will make use of this fact in the next section.

Recognize that our normal form (9.24) is, of course, just a perturbation of the model case (9.21). Does the periodic solution survive the perturbation?

We know that the sign of the constant $a$ is important. Guckenheimer and Holmes [2] give a formula for $a$ in terms of $\tilde{g}$ and its derivatives to third order at $(x,\mu) = (0,0)$.

We will look at the supercritical case when $a < 0$. In the subcritical case, an analysis similar to what follows can be performed. When $a = 0$, there is a so-called generalized Hopf bifurcation to a periodic solution, but its properties depend on higher order terms.

Suppose that $-a \equiv 1/\alpha > 0$. We will show that there is a $\mu_0 > 0$ such that for all $0 < \mu < \mu_0$, the annulus

$$\mathcal{A} = \{x \in \mathbb{R}^2 : (\alpha\mu)^{1/2} - \mu^{3/4} \le \|x\| \le (\alpha\mu)^{1/2} + \mu^{3/4}\} \tag{9.25}$$

is positively invariant under the flow of (9.24). Since $\mathcal{A}$ contains no critical points, it follows by the Poincaré-Bendixson Theorem 8.8 that $\mathcal{A}$ contains a periodic orbit.

Let's check the invariance. We will show that the flow of (9.24) goes into $\mathcal{A}$. Let $x(t)$ be a solution of (9.24), let $\alpha = 1/(-a)^{1/2}$, and set $r(t) = \|x(t)\|$. Multiply (9.24) by $x/r$:

$$r' = \mu r - r^3/\alpha + \frac{x}{r} \cdot \tilde{\tilde{g}}(x, \mu).$$

Now $\tilde{\tilde{g}}$ is fourth order near the origin, so we have that

$$|\frac{x}{r} \cdot \tilde{\tilde{g}}(x, \mu)| \leq \|\tilde{\tilde{g}}(x, \mu)\| \leq C(r^4 + \mu^4),$$

and so for $x \in \mathcal{A}$ we have $\frac{x}{r} \cdot \tilde{\tilde{g}}(x, \mu) = O(\mu^2)$. Now consider $r = (\alpha\mu)^{1/2} + \mu^{3/4}$ on the outer boundary of $\mathcal{A}$. Then

$$r' = \mu[(\alpha\mu)^{1/2} + \mu^{3/4}] - [(\alpha\mu)^{1/2} + \mu^{3/4}]^3/\alpha + O(\mu^2) = -2\mu^{7/4} + O(\mu^2) < 0,$$

for $0 < \mu \leq \mu_0$, sufficiently small. Similarly, if $r = (\alpha\mu)^{1/2} - \mu^{3/4}$, then

$$r' = 2\mu^{7/4} + O(\mu^2) > 0,$$

for $0 < \mu \leq \mu_0$. This proves invariance.

Within the region $\mathcal{A}$, it is easy to show that the trace of the Jacobian of the vector field is everywhere negative. This means that all periodic solutions inside $\mathcal{A}$ have their nontrivial Floquet multiplier inside the unit circle and are therefore orbitally stable. It follows that there can be only one periodic solution within $\mathcal{A}$, and it encircles the origin since there are no critical points in $\mathcal{A}$.

Define the ball

$$B = \{x \in \mathbb{R}^2 : \|x\| < |\alpha\mu|^{1/2} + |\mu|^{3/4}\}. \tag{9.26}$$

It is immediate to check that $r' > 0$ in $B \setminus \mathcal{A}$, so that the origin is asymptotically unstable, and there are no periodic orbits in $B \setminus \mathcal{A}$.

On the other hand, if $a < 0$, $\mu < 0$, then it is straightforward to show that $r = (x_1^2 + x_2^2)^{1/2}$ is a strict Liapunov function for (9.24) in the ball $B$. Thus, the origin is asymptotically stable, and there are no periodic orbits In $B$.

Similar arguments can be made when $a > 0$.

The results can be summarized in the following:

**Theorem 9.6.** *Suppose that $a \neq 0$ and that $\mu_0 > 0$ is sufficiently small. Let $\mathcal{A}$ and $B$ be defined as in (9.25), (9.26).*

*Supercritical case, $a < 0$: If $0 < \mu < \mu_0$, then (9.24) has an asymptotically orbitally stable periodic orbit inside $\mathcal{A}$. It is the unique periodic orbit in $B$. The* *origin is asymptotically unstable. If* $-\mu_0 < \mu < 0$*, then the origin is asymptotically stable for (9.24), and there are no periodic orbits in* $B$*.*

*Subcritical case,* $a > 0$*: If* $-\mu_0 < \mu < 0$*, then (9.24) has an asymptotically orbitally unstable periodic orbit inside* $\mathcal{A}$*. It is the unique periodic orbit in* $B$*. The origin is asymptotically stable. If* $0 < \mu < \mu_0$*, then the origin is asymptotically unstable for (9.24), and there are no periodic orbits in* $B$*.*

There are a few questions that remain open. Does the periodic solution depend continuously on $\mu$? How does the period depend on $\mu$? In the next section, we use the Liapunov–Schmidt method to study these questions.

## 9.6 Hopf Bifurcation via Liapunov–Schmidt

Let's return to the reduced problem (9.24) given in Sect. 9.5 after the normal form transformation:

$$x' = A_0 x + \mu x + \|x\|^2 Bx + g(x, \mu), \tag{9.27}$$

with

$$A_0 = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \quad B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \tag{9.28}$$

and $g(x, \mu) \in C^\infty(\mathbb{R}^{2+1}, \mathbb{R}^2)$ such that

$$g(0, \mu) = 0, \quad D_x g(0, \mu) = 0, \quad g(x, \mu) = O[\|x\|^2(\|x\|^2 + \mu^2)]. \tag{9.29}$$

For notational convenience, we have dropped the double tilde from $\tilde{\tilde{g}}$. We will now show that the family of periodic solutions constructed in Sect. 9.5 is a smooth curve in $Z = C_b^1(\mathbb{R}, \mathbb{R}^2)$.

**Theorem 9.7.** *Let* $I = \{0 \leq s < s_0\}$*. For* $s_0$ *sufficiently small, there exist* $\lambda, \mu \in C^2(I, \mathbb{R})$ *and* $\psi \in C^3(I, Z)$ *such that*

$$\mu(s) \approx -as^2 + \dots, \quad \lambda(s) \approx -bs^2 + \dots,$$

$\psi(0) = 0$*, and* $x(t, s) = \psi(s)(t)$ *is a periodic solution of* (9.27)–(9.29) *with* $\mu = \mu(s)$ *and with period* $2\pi(1 + \lambda(s))/\omega$*.*

*Remarks 9.6.*

- Since $s \mapsto \psi(s)$ is $C^1$ and $\psi(0) = 0$, we have that $\|x(\cdot, s)\|_{C^1} \leq C|s|$. The fact that $|s| \approx |\mu/a|^{1/2}$ implies that the solution $x(t, s)$ belongs to the ball $B$ defined in (9.26), and thus it coincides with the solution of Theorem 9.6.
- This theorem allows $a = 0$, however in this case we know less about the nature of periodic solutions since Theorem 9.6 is not available.
- When $a \neq 0$, Theorems 9.6 and 9.7 show that the Hopf bifurcation for the perturbed case is close to the unperturbed case.

*Proof.* Let $x(t, \mu)$ be a small solution of (9.27)–(9.29). We will account for variations in the period by introducing a second parameter. Set

$$y(t, \lambda, \mu) = x((1+\lambda)t, \mu), \quad \lambda \approx 0.$$

Then $y$ is $T_0$-periodic if and only if $x$ is $T_0(1+\lambda)$-periodic, and $y$ satisfies

$$y' = (1+\lambda)[A_0 y + \mu y + \|y\|^2 B y + g(y, \mu)].$$

Viewed as a function of $(y, \lambda, \mu)$, the right-hand side is quadratic (or higher) except for the term $A_0 y$. To be able to apply the Liapunov-Schmidt technique, we will eliminate this term by introducing rotating coordinates (cf. Sects. 8.2 and 8.4). Let

$$U(t) = \exp A_0 t = \begin{bmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix},$$

and set

$$y(t) = U(t) z(t).$$

A routine calculation shows that

$$z' = \lambda A_0 z + (1+\lambda)\Big[\mu z + \|z\|^2 U(-t) B(t) z + U(-t) g(U(t) z, \mu)\Big]. \tag{9.30}$$

We are going to construct $T_0$-periodic solutions of this equation with $T_0 = 2\pi/\omega$.

Here comes the set-up for Liapunov–Schmidt. Let

$$X = \{z \in C_b^1(\mathbb{R}, \mathbb{R}^2) : z(t + T_0) = z(t)\},$$

with the norm

$$\|z\|_X = \max_{0 \leq t \leq T_0} \|z(t)\| + \max_{0 \leq t \leq T_0} \|z'(t)\|.$$

Also, let

$$Y = \{h \in C_b^0(\mathbb{R}, \mathbb{R}^2) : h(t + T_0) = h(t)\},$$

with the norm

$$\|h\|_Y = \max_{0 \le t \le T_0} \|h(t)\|.$$

If we define $L = \dfrac{d}{dt}$, then $L : X \to Y$ is a bounded linear map. It is clear that the null space of $L$ is

$$K = \{z \in X : z(t) = v = \text{Const.}\},$$

and that the range of $L$ is

$$R = \{h \in Y : \int_0^{T_0} h(t)dt = 0\}.$$

So we have

$$X = M \oplus K, \quad \text{and} \quad Y = R \oplus S,$$

with

$$M = \{z \in X : \int_0^{T_0} z(t)dt = 0\},$$

and

$$S = \{h \in Y : h(t) = \text{Const.}\}.$$

$M$ is a closed subspace of $X$, $R$ is a closed subspace of $Y$, and $L : M \to R$ is an isomorphism, since

$$L^{-1}h(t) = \int_0^t h(\tau)d\tau$$

is bounded from $R$ to $M$. Note that $L$ is Fredholm, since $K$ and $S$ are finite (two) dimensional. The projection of $Y$ onto $R$ along $S$ is given by

$$P_R h(t) = h(t) - \frac{1}{T_0}\int_0^{T_0} h(\tau)d\tau.$$

Now our problem (9.30) can be encoded as

$$Lz = N(z, \lambda, \mu),$$

in which

$$N(z, \lambda, \mu) = \lambda A_0 z + (1+\lambda)\Big[\mu z + \|z\|^2 U(-t)BU(t)z + U(-t)g(U(t)z, \mu)\Big].$$

Note that $N \in C^\infty(X \times \mathbb{R}^2, Y)$ and

$$N(0, \lambda, \mu) = 0, \quad D_z N(0, 0, 0) = 0. \tag{9.31}$$

All of the conditions for Liapunov–Schmidt are fulfilled, so we proceed to analyze the problem

$$\begin{aligned} Lu &= P_R N(u+v,\lambda,\mu), \\ (I-P_R)N(u+v,\lambda,\mu) &= 0, \end{aligned}$$

with $u \in M$ and $v \in K = \mathbb{R}^2$.

By the Implicit Function Theorem, Corollary 5.1, we get a $C^3$ mapping $\phi(v,\lambda,\mu)$ from a neighborhood of the origin in $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ into $M$ such that

$$\phi(0,0,0) = 0 \quad \text{and} \quad L\phi(v,\lambda,\mu) = P_R N(\phi(v,\lambda,\mu)+v,\lambda,\mu).$$

(This is the minimum regularity which permits our subsequent computations). By the uniqueness part of the Implicit Function Theorem, we get that $\phi(0,\lambda,\mu) = 0$. Next, from

$$LD_v\phi(v,\lambda,\mu) = P_R[D_z N(\phi(v,\lambda,\mu)+v,\lambda,\mu)(D_v\phi(v,\lambda,\mu)+I)],$$

we see using (9.31) that

$$LD_v\phi(0,\lambda,\mu) = P_R[D_z N(0,\lambda,\mu)(D_v\phi(0,\lambda,\mu)+I)] = 0.$$

It follows that $D_v\phi(0,\lambda,\mu) = 0$, since $L$ is an isomorphism.

Set $u(t,v,\lambda,\mu) = \phi(v,\lambda,\mu)(t)$. Then

$$u(t,0,\lambda,\mu) = 0, \quad \text{and} \quad D_v u(t,0,\lambda,\mu) = 0.$$

We can therefore write

$$\begin{aligned} u(t,v,\lambda,\mu) &= \int_0^1 \frac{d}{d\sigma} u(t,\sigma v,\lambda,\mu)d\sigma \\ &= \int_0^1 D_v u(t,\sigma v,\lambda,\mu)v d\sigma \\ &= J(t,v,\lambda,\mu)v, \end{aligned}$$

with

$$J(t,0,0,0) = 0, \quad \text{and} \quad (v,\lambda,\mu) \mapsto J(\cdot,v,\lambda,\mu) \quad \text{in } C^2, \tag{9.32}$$

as a map of a neighborhood of the origin in $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ into $M$.

We now need to study the bifurcation function

$$
\begin{aligned}
\mathcal{B}(v,\lambda,\mu) &= (I-P_R)N(\phi(v,\lambda,\mu)+v,\lambda,\mu)\\
&= \frac{1}{T_0}\int_0^{T_0} N(u(t,v,\lambda,\mu)+v,\lambda,\mu)dt\\
&= [\lambda A_0 + (1+\lambda)\mu]v\\
&\quad + \frac{1+\lambda}{T_0}\int_0^{T_0} \|u(t,v,\lambda,\mu)+v\|^2 U(-t)BU(t)[u(t,v,\lambda,\mu)+v]dt\\
&\quad + \frac{1+\lambda}{T_0}\int_0^{T_0} U(-t)g(U(t)[u(t,v,\lambda,\mu)+v],\mu)dt.
\end{aligned}
$$

Consider $v$ in the form $v=(s,0)=se_1$. Define $\mathcal{B}_0(s,\lambda,\mu)=\frac{1}{s}\mathcal{B}(se_1,\lambda,\mu)$. Since $u(t,se_1,\lambda,\mu)+se_1 = s[J(t,se_1,\lambda,\mu)+I]e_1$, we have

$$
\mathcal{B}_0(s,\lambda,\mu) = [\lambda A_0 + (1+\lambda)\mu I]e_1 + s^2\mathcal{B}_1(s,\lambda,\mu) + \mathcal{B}_2(s,\lambda,\mu),
$$

with

$$
\begin{aligned}
\mathcal{B}_1(s,\lambda,\mu) = \frac{1+\lambda}{T_0}\int_0^{T_0} &\|[J(t,se_1,\lambda,\mu)+I]e_1\|^2 U(-t)BU(t)\\
&\times [J(t,se_1,\lambda,\mu)+I]e_1 dt
\end{aligned}
$$

and

$$
\mathcal{B}_2(s,\lambda,\mu) = \frac{1+\lambda}{sT_0}\int_0^{T_0} U(-t)g(U(t)s[J(t,se_1,\lambda,\mu)e_1+e_1],\mu)dt.
$$

By (9.32), we have

$$
\mathcal{B}_1(0,0,0) = \frac{1}{T_0}\int_0^{T_0} U(-t)BU(t)e_1 dt = \begin{bmatrix} a\\ b\end{bmatrix}.
$$

By (9.29) and (9.32), we obtain

$$
\mathcal{B}_2(0,0,0) = \frac{\partial}{\partial s}\mathcal{B}_2(0,0,0) = \frac{\partial^2}{\partial s^2}\mathcal{B}_2(0,0,0) = 0.
$$

Now we solve the bifurcation equation. The function $\mathcal{B}_0 : \mathbb{R}^2\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2$ is $C^2$, $\mathcal{B}_0(0,0,0)=0$, and

$$
D_{\lambda,\mu}\mathcal{B}_0(0,0,0) = \begin{bmatrix} 0 & 1\\ \omega & 0\end{bmatrix},
$$

is invertible. By the Implicit Function Theorem, there are $C^2$ curves $\lambda(s)$, $\mu(s)$ such that $\lambda(0)=0$, $\mu(0)=0$, and $\mathcal{B}_0(s,\lambda(s),\mu(s))=0$.

This construction yields a unique (up to a phase shift) family of periodic solutions

$$x(t,s) = U(t/(1+\lambda(s)))\,[u(t/(1+\lambda(s)), se_1, \lambda(s), \mu(s)) + se_1]\,.$$

The map $s \mapsto x(\cdot, s)$ is $C^3$ into $Z$. The period is $T_0(1+\lambda(s))$.

Differentiation of the bifurcation equation with respect to $s$ gives $\lambda'(0) = \mu'(0) = 0$, and

$$\begin{bmatrix} 0 & 1 \\ \omega & 0 \end{bmatrix} \begin{bmatrix} \lambda''(0) \\ \mu''(0) \end{bmatrix} = -2 \begin{bmatrix} a \\ b \end{bmatrix}. \qquad \square$$

## 9.7 Exercises

**Exercise 9.1.** Consider the system

$$\begin{aligned} x' &= xy + ax^3 + bxy^2 \\ y' &= -y + cx^2 + dx^2 y. \end{aligned}$$

Prove that:

(a) If $a + c < 0$, then the origin is asymptotically stable.
(b) If $a + c > 0$, then the origin is unstable.
(c) If $a + c = 0$ and

  (i) if $cd + bc^2 < 0$, then the origin is asymptotically stable.
  (ii) if $cd + bc^2 > 0$, then the origin is unstable.

**Exercise 9.2.** Analyze the bifurcation that occurs when $\mu = 0$ for the following system:

$$x_1' = x_2, \quad x_2' = -x_2 - \mu + x_1^2.$$

**Exercise 9.3.**

(a) Given that $A = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$, calculate $L_A h(x)$, where $h(x) = x^m e_j$, $|m| = 2$, and $j = 1$ or $2$.
(b) Find a change of coordinates of the form $x = y + h(y)$ which transforms the system

$$\begin{aligned} x_1' &= 3x_2 - x_1^2 + 7x_1x_2 + 3x_2^2 \\ x_2' &= 2x_1 + 4x_1x_2 + x_2^2 \end{aligned}$$

into the form

$$\begin{aligned} y_1' &= 3y_2 + O(\|y\|^3) \\ y_2' &= 2y_1 + O(\|y\|^3). \end{aligned}$$

**Exercise 9.4.** Prove that both the Rayleigh equation

$$\ddot{x} + \dot{x}^3 - \mu\dot{x} + x = 0$$

and the Van der Pol equation

$$\ddot{x} + \dot{x}(\mu - x^2) + x = 0,$$

undergo Hopf bifurcations at $\mu = 0$. In both cases, sketch the phase portraits for $\mu < 0$, $\mu = 0$, and $\mu > 0$. State what type of Hopf bifurcation takes place.

# Chapter 10
# The Birkhoff Smale Homoclinic Theorem

## 10.1 Homoclinic Solutions of Newton's Equation

The starting point is Newton's equation

$$\ddot{\varphi} + g(\varphi) = 0. \tag{10.1}$$

The nonlinearity $g$ will be assumed to satisfy:

$$g \in C^\infty(\mathbb{R}, \mathbb{R}), \tag{N1}$$

$$g(0) = 0, \quad g'(0) = -\alpha^2 < 0, \tag{N2}$$

$$g(\varphi) = -g(-\varphi), \quad \varphi \in \mathbb{R}, \tag{N3}$$

$$g''(\varphi) > 0, \quad \varphi > 0, \tag{N4}$$

$$g(\varphi) \to \infty, \quad \text{as} \quad \varphi \to \infty. \tag{N5}$$

A simple example is $g(\varphi) = -\varphi + \varphi^3$.

Define

$$G(\varphi) = \int_0^\varphi g(s)ds.$$

By (N3), $G$ is even, and by (N5), $G(\varphi) \to \infty$, as $|\varphi| \to \infty$. There are unique numbers $0 < \varphi_0 < \zeta_0$ such that

$$g(\varphi_0) = 0 \quad \text{and} \quad G(\zeta_0) = 0.$$

T. C. Sideris, *Ordinary Differential Equations and Dynamical Systems*,
Atlantis Studies in Differential Equations 2, DOI: 10.2991/978-94-6239-021-8_10,

Using Taylor's theorem, we may write

$$g(\varphi) = g'(0)\varphi + g_1(\varphi)\varphi^2, \quad g_1(\varphi) = \int_0^1 (1-s)g''(\varphi s)ds, \tag{10.2}$$

and

$$G(\varphi) = \frac{1}{2}g'(0)\varphi^2 + g_2(\varphi)\varphi^3, \quad g_2(\varphi) = \frac{1}{2}\int_0^1 (1-s)^2 g''(\varphi s)ds. \tag{10.3}$$

Note that by (N4), we have $g_k(\varphi) \geq 0$, $\varphi > 0$, $k = 1, 2$.

We know that the Hamiltonian

$$E(\varphi, \dot\varphi) = \tfrac{1}{2}\dot\varphi^2 + G(\varphi)$$

is conserved along solutions of (10.1). Since $G(\varphi) \to \infty$, as $|\varphi| \to \infty$, we see that the level curves of $E$ in $\mathbb{R}^2$ are bounded. It follows that all solutions of the system are defined for all $t \in \mathbb{R}$, and by (N1), all solutions lie in $C^\infty(\mathbb{R}, \mathbb{R})$.

**Lemma 10.1.** *Let $p(t)$ be the solution of* (10.1) *with $p(0) = \zeta_0$ and $\dot p(0) = 0$. Set $\alpha = \sqrt{-g'(0)}$. Then $p(t)$ has the following properties:*

$$E(p(t), \dot p(t)) = 0, \tag{10.4}$$

$$p(t) = p(-t), \tag{10.5}$$

$$0 < p(t) \leq \zeta_0 e^{-\alpha|t|}, \quad t \in \mathbb{R}, \tag{10.6}$$

$$|\dot p(t)| \leq \alpha\zeta_0 e^{-\alpha|t|}, \quad t \in \mathbb{R} \tag{10.7}$$

$$|\ddot p(t)| \leq Ce^{-\alpha|t|}, \quad t \in \mathbb{R}. \tag{10.8}$$

*Proof.* By conservation of energy, we have $E(p(t), \dot p(t)) = E(\zeta_0, 0) = 0$.

The symmetry (10.5) is a consequence of (N3) and uniqueness.

The solution $p(t)$ never vanishes because the level curve $E = 0$ intersects $\{p = 0\}$ only at the origin which is an equilibrium.

The solution $p$ has a local maximum at $t = 0$, since the ODE tells us that $p''(0) = -g(\zeta_0) < 0$. The solution $p$ has no other extrema, and it decreases monotonically to 0 on the interval $t > 0$.

Thus, for $t > 0$ we see from (10.4) and the expansion (10.3) that

$$\dot p(t)^2 = -G(p(t)) = \alpha^2 p(t)^2[1 - 2p(t)g_2(p(t))/\alpha].$$

Since $-G(p) \geq 0$ for $0 < p \leq \zeta_0$, we can say that for $t > 0$

$$\dot p(t) = -\alpha p(t)q(t), \quad 0 \leq q \leq 1. \tag{10.9}$$

It follows that

$$\frac{d}{dt}[e^{\alpha t}p(t)] \leq 0, \quad t \geq 0,$$

and so

$$p(t) \leq \zeta_0 e^{-\alpha t}, \quad t \geq 0.$$

The bound for $t < 0$ can be obtained using (10.5).

The bound (10.7) for $\dot{p}$ comes from (10.9) and (10.6).

Finally, the bound (10.8) for $\ddot{p}$ is a consequence of (10.2) and then (10.6):

$$|\ddot{p}(t)| = |g(p(t)| \leq |p(t)| \left[ |\alpha^2 + \max_{|\varphi| \leq \zeta_0} |\varphi g_1(\varphi)| \right] \leq C|p(t)|.$$

□

We can also express the problem (10.1) as a first order system

$$x' = f(x) \quad \text{with} \quad f(x) = \begin{bmatrix} x_2 \\ -g(x_1) \end{bmatrix}. \tag{10.10}$$

Note that $\bar{x}(t) = (p(t), \dot{p}(t))$ is a solution of (10.10) which, by (10.6) and (10.7), satisfies $\lim_{t \to \pm\infty} \bar{x}(t) = 0$.

**Definition 10.1.** *A solution of a first order autonomous system whose $\alpha$- and $\omega$-limits sets consist of a single equilibrium is called a homoclinic orbit.*

By (N2), the system (10.10) has a hyperbolic equilibrium at the origin, since the eigenvalues of $Df(0)$ are $\pm\alpha$. The stable and unstable manifolds are given by

$$W_s(0) = W_u(0) = \{(x_1, x_2) \in \mathbb{R}^2 : E(x_1, x_2) = 0\}.$$

This set is the union of the equilibrium solution and the two homoclinic orbits $\pm\bar{x}(t)$ (Fig. 10.1).

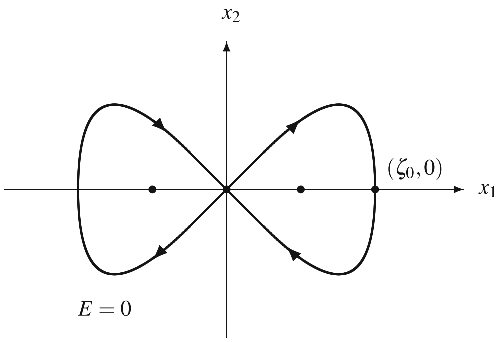



**Fig. 10.1** The homoclinic orbits as the zero level set of the energy function

## 10.2 The Linearized Operator

For any interval $I \subset \mathbb{R}$, define

$$C_b^k(I) = \{z \in C^k(I, \mathbb{R}) : \|z\|_k = \sum_{j=0}^{k} \sup_I \|z^{(j)}\| < \infty\}.$$

The interval $I$ will be either $\mathbb{R}$, $\mathbb{R}_+ = [\,0, \infty)$, or $\mathbb{R}_- = (-\infty, 0\,]$. Note that in this chapter our notation for the sup norm will be $\|\cdot\|_0$ which deviates from our custom.

**Lemma 10.2.** *Define the linearized operator*

$$L = \frac{d^2}{dt^2} + g'(p(t)).$$

*Let $h \in C_b^0(\mathbb{R}_\pm)$, and suppose that $z \in C^2(\mathbb{R}_\pm, \mathbb{R})$ solves $Lz = h$. Then $z \in C_b^2(\mathbb{R}_\pm)$, i.e. $z$ is bounded on $\mathbb{R}_\pm$, if and only if*

$$z(0) = -\frac{1}{g(\zeta_0)} \int_0^{\pm\infty} \dot{p}(s)h(s)ds. \tag{10.11}$$

*Proof.* By (10.5), we have that $L[z(-t)] = Lz(-t)$, so it is enough to prove the result for $h \in C_b^0(\mathbb{R}_+)$.

Given the homoclinic orbit $\bar{x}(t)$, observe that

$$y(t) \equiv \bar{x}'(t) = \begin{bmatrix} \dot{p}(t) \\ \ddot{p}(t) \end{bmatrix}$$

satisfies the linearized system

$$y'(t) = Df(\bar{x}(t))y(t), \tag{10.12}$$

with initial data

$$y(0) = \begin{bmatrix} 0 \\ -g(\zeta_0) \end{bmatrix}.$$

Let $X(t)$ be the fundamental matrix for $Df(\bar{x}(t))$. Write

$$X(t) = \begin{bmatrix} X_{11}(t) & X_{12}(t) \\ X_{21}(t) & X_{22}(t) \end{bmatrix},$$

and also denote the columns of $X(t)$ as

$$X_1(t) = \begin{bmatrix} X_{11}(t) \\ X_{21}(t) \end{bmatrix}, \quad X_2(t) = \begin{bmatrix} X_{12}(t) \\ X_{22}(t) \end{bmatrix}.$$

Thus, we have that

$$y(t) = X(t)y(0) = -g(\zeta_0)X_2(t),$$

and in particular,

$$X_{12}(t) = -\frac{1}{g(\zeta_0)}\dot{p}(t).$$

We see from this, (10.7) and (10.8) that

$$\|X_2(t)\| \le C\mathrm{e}^{-\alpha t}, \quad t \ge 0. \tag{10.13}$$

Let $h \in C_b^0(\mathbb{R}_+)$, and write $H(t) = \begin{bmatrix} 0 \\ h(t) \end{bmatrix}$. By Variation of Parameters, Theorem 4.1, we have that a solution of

$$w'(t) = Df(\bar{x}(t))w(t) + H(t), \tag{10.14}$$

satisfies

$$w(t) = X(t)\left[w(0) + \int_0^t X(s)^{-1}H(s)ds\right], \tag{10.15}$$

as well as

$$X(t)^{-1}w(t) = w(0) + \int_0^t X(s)^{-1}H(s)ds. \tag{10.16}$$

Since tr $Df(\bar{x}(t)) = 0$, we have by Theorem 4.2 that det $X(t) = 1$, and so

$$X(t)^{-1} = \begin{bmatrix} X_{22}(t) & -X_{12}(t) \\ -X_{21}(t) & X_{11}(t) \end{bmatrix}.$$

With this, the first row of (10.16) says

$$X_{22}(t)w_1(t) - X_{12}(t)w_2(t) = w_1(0) - \int_0^t X_{12}(s)h(s)ds.$$

If $\|w(t)\|$ is bounded, then by (10.13) the left-hand side tends to zero, as $t \to \infty$. It follows that

$$w_1(0) = \int_0^\infty X_{12}(s)h(s)ds = -\frac{1}{g(\zeta_0)} \int_0^\infty \dot{p}(s)h(s)ds,$$

so that (10.11) holds.

Conversely, suppose that (10.11) holds. Going back to (10.15), the formula for $w$ can be rewritten as

$$\begin{aligned} w(t) =& X_1(t)\left[w_1(0) - \int_0^t X_{12}(s)h(s)ds\right] \\ &+ X_2(t)\left[w_2(0) + \int_0^t X_{11}(s)h(s)ds\right] \\ =& X_1(t)\left[\int_t^\infty X_{12}(s)h(s)ds\right] \\ &+ X_2(t)\left[w_2(0) + \int_0^t X_{11}(s)h(s)ds\right]. \end{aligned} \tag{10.17}$$

Let us temporarily accept the claim that

$$\|X_1(t)\| \leq C\mathrm{e}^{\alpha t}, \quad t \geq 0. \tag{10.18}$$

By (10.13) and (10.18), we then have that

$$\left\| X_1(t)\left[\int_t^\infty X_{12}(s)h(s)ds\right]\right\| \leq C\mathrm{e}^{\alpha t} \cdot \mathrm{e}^{-\alpha t}\|h\|_0,$$

and also that

$$\left\| X_2(t)\left[w_2(0) + \int_0^t X_{11}(s)h(s)ds\right]\right\| \leq C\mathrm{e}^{-\alpha t}\big[|w_2(0)| + \mathrm{e}^{\alpha t}\|h\|_0\big].$$

So from (10.17), we see that (10.14) has a bounded solution $w(t)$, provided (10.11) holds.

It remains to verify the claim (10.18). Define

$$u(t) = \Lambda X_1(t), \quad \Lambda = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}.$$

Then since $X_1(t)$ solves (10.12), we find

$$u'(t) = \Lambda Df(\bar{x}(t))\Lambda^{-1}u(t).$$

Upon integration, it follows that

$$u(t) = u(0) + \int_0^t \Lambda Df(\bar{x}(s))\Lambda^{-1}u(s)ds,$$

and so by Gronwall's inequality, Lemma 3.3, we have

$$\|u(t)\| \le \|u(0)\| \exp \int_0^t \|\Lambda Df(\bar{x}(s))\Lambda^{-1}\|ds, \quad t \ge 0. \tag{10.19}$$

A simple calculation gives

$$\|\Lambda Df(\bar{x}(s))\Lambda^{-1}\| = \max\{\alpha, |g'(p(s))|/\alpha\}.$$

By (N4), we have that $g'(p)$ is monotonically increasing for $p > 0$. Since $g'(0) = -\alpha^2 < 0$, there is a $p_0 > 0$ such that $g'(p_0) = 0$, and

$$-\alpha^2 = g'(0) \le g'(p) \le 0, \quad 0 \le p \le p_0.$$

Since $p(s) \searrow 0$ as $s \to \infty$, there is a $T > 0$ such that $0 < p(s) \le p_0$, for $s > T$. Thus, we have

$$|g'(p(s))| \le \alpha^2, \quad s > T,$$

which implies that

$$\|\Lambda Df(\bar{x}(s))\Lambda^{-1}\| = \alpha, \quad s > T.$$

The claim (10.18) now follows from (10.19):

$$\begin{aligned}
\|X_1(t)\| &\le \|\Lambda^{-1}\|\|u(t)\| \\
&\le \|\Lambda^{-1}\|\|u(0)\| \left(\exp \int_0^T \|\Lambda Df(\bar{x}(s))\Lambda^{-1}\|ds + \int_T^t \alpha\, ds\right) \\
&\le Ce^{\alpha t},
\end{aligned}$$

for all $t \ge T$. □

Define

$$D_\pm(L) = \{z \in C_b^2(\mathbb{R}_\pm) : \dot{z}(0) = 0\}.$$

**Theorem 10.1.** *The linear operator $L$ is bounded from $C_b^2(\mathbb{R}_\pm)$ to $C_b^0(\mathbb{R}_\pm)$, and*

$$L : D_\pm(L) \to C_b^0(\mathbb{R}_\pm)$$

*is an isomorphism.*

*Proof.* The boundedness of $L$ is a consequence of the fact that $p$, and hence $g \circ p$, belong to $C_b^0(\mathbb{R}_\pm)$.

According to Lemma 10.2, given $h \in C_b^0(\mathbb{R}_\pm)$, there exists a unique solution of $Lz = h$ in $D_\pm(L)$, namely the one with initial data

$$z(0) = -\frac{1}{g(\zeta_0)} \int_0^{\pm\infty} \dot{p}(s)h(s)ds \quad \text{and} \quad \dot{z}(0) = 0.$$

Thus, $L : D_\pm(L) \to C_b^0(\mathbb{R}_\pm)$ is a bijection.

The domains $D_\pm(L)$ are closed subspaces of $C_b^2(\mathbb{R}_\pm)$, and so each is a Banach space. The boundedness of $L^{-1} : C_b^0(\mathbb{R}_\pm) \to D_\pm(L)$ follows from the Open Mapping Theorem 5.1. $\square$

Set

$$\begin{aligned} K(L) &= \{z \in C_b^2(\mathbb{R}) : Lz = 0\} \\ R(L) &= \{h \in C_b^0(\mathbb{R}) : Lz = h, \text{ for some } z \in C_b^2\} \\ D(L) &= \{z \in C_b^2(\mathbb{R}) : \dot{z}(0) = 0\}. \end{aligned}$$

**Theorem 10.2.** *The linear operator $L$ is bounded from $C_b^2(\mathbb{R})$ to $C_b^0(\mathbb{R})$. The null space and range of $L$ are characterized by*

$$K(L) \;=\; \text{span}\,\{\dot{p}\} = \{c\dot{p} : c \in \mathbb{R}\}, \tag{10.20}$$

*and*

$$R(L) = \left\{ h \in C_b^0(\mathbb{R}) : \int_{\mathbb{R}} \dot{p}(t)h(t)dt = 0 \right\}. \tag{10.21}$$

*Finally, $L : D(L) \to R(L)$ is an isomorphism.*

*Proof.* The boundedness of $L$ again follows from $g \circ p \in C_b^0(\mathbb{R})$.

First note that $\dot{p}$ is in $C^\infty(\mathbb{R}, \mathbb{R})$. Moreover, since $L\dot{p} = 0$, the bounds (10.7) and (10.8) imply that span$\{\dot{p}\} \subset K(L)$.

Conversely, if $z \in C_b^2(\mathbb{R})$ and $Lz = 0$ then by Lemma 10.2 we have $z(0) = 0$. Thus,

$$z(t) = X_{12}(t)z'(0) = -[z'(0)/g(\zeta_0)]\dot{p}(t) \in \text{span}\{\dot{p}\},$$

so that $K(L) \subset \text{span}\{\dot{p}\}$. This proves (10.20).

If $h \in R(L)$, then there exists $z \in C_b^2(\mathbb{R})$ such that $Lz = h$. By Lemma 10.2, there holds

$$z(0) = -\frac{1}{g(\zeta_0)} \int_{\mathbb{R}_+} \dot{p}(s)h(s)ds = \frac{1}{g(\zeta_0)} \int_{\mathbb{R}_-} \dot{p}(s)h(s)ds,$$

and thus,

$$\int_{\mathbb{R}} \dot{p}(s)h(s)ds = 0. \tag{10.22}$$

Conversely, assume that $h \in C_b^0(\mathbb{R})$ satisfies (10.22). Applying Theorem 10.1, we can find unique functions $z_\pm \in D_\pm(L) \subset C_b^2(\mathbb{R}_\pm)$ such that $Lz_\pm = h$ on $\mathbb{R}_\pm$, and

$$z_\pm(0) = -\frac{1}{g(\zeta_0)} \int_{\mathbb{R}_\pm} \dot{p}(s)h(s)ds \quad \text{and} \quad \dot{z}_\pm(0) = 0.$$

Our assumption (10.22) implies that $z_-(0) = z_+(0)$. Therefore, the function

$$z(t) = \begin{cases} z_+(t), & t \geq 0 \\ z_-(t), & t \leq 0 \end{cases}$$

lies in $C_b^1(\mathbb{R})$ and solves $Lz = h$ in $\mathbb{R}\backslash\{0\}$. It follows that $z'' = -g'(p)z + h \in C_b^0(\mathbb{R})$, and so $z$ a solution of $Lz = h$ in $D(L) \subset C_b^2(\mathbb{R})$. This proves the statement (10.21).

The argument in the preceding paragraph also shows that the map $L : D(L) \to R(L)$ is surjective. Thus, since $D(L) \cap K(L) = \emptyset$, by (10.20), this map is a bijection. By the Open Mapping Theorem 5.1, $L^{-1} : R(L) \to D(L)$ is bounded. □

## 10.3 Periodic Perturbations of Newton's Equation

Suppose that $q \in C^\infty(\mathbb{R}, \mathbb{R})$ is T-periodic. For the remainder of the chapter we will be interested in Newton's equation with a periodic perturbation

$$\ddot{\varphi} + g(\varphi) = \varepsilon q(t), \tag{10.23}$$

or its first order equivalent

$$\dot{x}(t) = f(x) + \varepsilon Q(t) = Ax + h(x) + \varepsilon Q(t), \tag{10.24}$$

where

$$f(x) = \begin{bmatrix} x_2 \\ -g(x_1) \end{bmatrix}, \; A = Df(0), \; h(0) = 0, \; Dh(0) = 0, \text{ and } Q(t) = \begin{bmatrix} 0 \\ q(t) \end{bmatrix}.$$

**Lemma 10.3.** *Solutions of* (10.24) *are globally defined, for all initial data.*

*There exist positive constants $C_0$, $\varepsilon_0$, and a neighborhood of the origin $U_0 \subset \mathbb{R}^2$ such that for all $|\varepsilon| < \varepsilon_0$, the system* (10.24) *has a $T$-periodic solution $x^\varepsilon(t)$ with $|x^\varepsilon(t)| \le C_0\varepsilon$, for all $t \in \mathbb{R}$. It is the unique $T$-periodic solution with initial data in $U_0$.*

*The Floquet multipliers of $x^\varepsilon(t)$ satisfy*

$$0 < \mu_1^\varepsilon < 1 < \mu_2^\varepsilon, \quad \mu_1^\varepsilon \mu_2^\varepsilon = 1, \quad \mu_1^0 = \exp(-\alpha T). \tag{10.25}$$

*Remark 10.1.* The corresponding periodic solution of (10.23) shall be written as $\varphi^\varepsilon(t) = x_1^\varepsilon(t)$.

*Proof.* From the properties (N3) and (N5) the antiderivative $G(\varphi)$ is an even function and $G(\varphi) \to \infty$, as $\varphi \to \infty$. Therefore, $G(\varphi)$ has an absolute minimum value $-m$, which is negative by (N2). Note that $-m$ is also the absolute minimum for $E(x_1, x_2)$.

For any solution $x(t)$ of (10.24), we have

$$\begin{aligned}
\left| \frac{d}{dt}[E(x_1(t), x_2(t)) + m] \right| &= |\varepsilon q(t) x_2(t)| \\
&\le \frac{1}{2} x_2(t)^2 + \frac{1}{2}\varepsilon^2 q(t)^2 \\
&= E(x_1(t), x_2(t)) - G(x_1(t)) + \frac{1}{2}\varepsilon^2 q(t)^2 \\
&\le E(x_1(t), x_2(t)) + m + \frac{1}{2}\varepsilon^2 q(t)^2 \\
&\le E(x_1(t), x_2(t)) + m + C.
\end{aligned}$$

This implies that

$$E(x_1(t), x_2(t)) + m \le [E(x_1(0), x_2(0)) + m]e^{|t|} + C[e^{|t|} - 1], \quad t \in \mathbb{R},$$

which prevents $E(x_1(t), x_2(t))$ from going to $+\infty$ in finite time. It follows that $x(t)$ is a global solution.

The existence of a unique periodic solution for small $\varepsilon$ follows from Theorem 8.1 as already discussed in Example 8.1. The estimate for its size is a consequence of its $C^1$ dependence on the parameter $\varepsilon$ and the fact that $x^0 = 0$.

To find the Floquet multipliers of $x^\varepsilon$, we examine the linearized equation

$$\dot{y}(t) = Df(x^\varepsilon(t))y(t) = A^\varepsilon(t)y(t).$$

Let $Y^\varepsilon(t)$ be the fundamental matrix for $A^\varepsilon(t)$. Since $\operatorname{tr} Df(x) = 0$, we have by Theorem 4.2 that $\det A^\varepsilon(t) = 1$. Recall that the Floquet multipliers of $x^\varepsilon(t)$ are the eigenvalues $\mu_1^\varepsilon, \mu_2^\varepsilon$ of $Y^\varepsilon(T)$. Thus, we have $\mu_1^\varepsilon \mu_2^\varepsilon = 1$. Since $A^0(t) = A$, we have by continuous dependence that $Y^\varepsilon(T) \approx \exp AT$. The eigenvalues of $\exp AT$ are $\mu_1 = \exp(-\alpha T)$ and $\mu_2 = \exp(\alpha T)$ which satisfy

$$0 < \mu_1 < 1 < \mu_2.$$

This rules out a pair of complex conjugate eigenvalues for $Y^\varepsilon(T)$, and so the Floquet multipliers must be real and satisfy the inequalities (10.25). □

Having established the existence of a unique hyperbolic periodic solution of the periodically perturbed Newton equation near the origin, we shall discuss the properties of its stable and unstable manifolds which were introduced in Sect. 8.3. Let $\Phi^\varepsilon_{t,\tau}$ denote the globally defined flow of (10.24). Define

$$W_s^\varepsilon = \{(\tau, y) \in \mathbb{R} \times \mathbb{R}^2 : \lim_{t\to\infty} \|\Phi^\varepsilon_{t,\tau}(y) - x^\varepsilon(t)\| = 0\},$$
$$W_u^\varepsilon = \{(\tau, y) \in \mathbb{R} \times \mathbb{R}^2 : \lim_{t\to-\infty} \|\Phi^\varepsilon_{t,\tau}(y) - x^\varepsilon(t)\| = 0\}.$$

These are smooth manifolds which are invariant under the flow and $T$-periodic, by Theorem 8.4.

**Lemma 10.4.** *There exist small constants $\varepsilon_0, \eta_0 > 0$ and a constant $C_0 > 0$ such that*

$$|\varepsilon| < \varepsilon_0 \quad \textit{and} \quad \|\Phi^\varepsilon_{t,\tau}(y) - x^\varepsilon(t)\| < \eta_0, \quad t \ge \tau$$

*imply that*

$$\|\Phi^\varepsilon_{t,\tau}(y) - x^\varepsilon(t)\| < C_0\eta_0 \exp(-\alpha t/4), \quad t \ge \tau,$$

*where, as above, $\alpha = \sqrt{-g'(0)}$.*

*It follows that, for $|\varepsilon| < \varepsilon_0$, the stable and unstable manifolds can be characterized as*

$$W_s^\varepsilon = \{(\tau, y) \in \mathbb{R} \times \mathbb{R}^2 : \limsup_{t\to\infty} \|\Phi^\varepsilon_{t,\tau}(y) - x^\varepsilon(t)\| < \eta_0\},$$
$$W_u^\varepsilon = \{(\tau, y) \in \mathbb{R} \times \mathbb{R}^2 : \limsup_{t\to-\infty} \|\Phi^\varepsilon_{t,\tau}(y) - x^\varepsilon(t)\| < \eta_0\}.$$

*Proof.* This is a consequence Theorem 8.4. The uniformity in $\varepsilon$ follows from the smooth dependence of the fundamental matrix $Y^\varepsilon(t)$ upon $\varepsilon$ and periodicity. □

Next, we define the time slices

$$\mathcal{W}_+^\varepsilon(\tau) = \{y \in \mathbb{R}^2 : (\tau, y) \in W_s^\varepsilon\},$$
$$\mathcal{W}_-^\varepsilon(\tau) = \{y \in \mathbb{R}^2 : (\tau, y) \in W_u^\varepsilon\}.$$

For each $\tau \in \mathbb{R}$, these are smooth one-dimensional manifolds. The $T$-periodicity of the stable and unstable manifolds gives

$$\mathcal{W}_{\pm}^{\varepsilon}(\tau + T) = \mathcal{W}_{\pm}^{\varepsilon}(\tau).$$

Since the stable and unstable manifolds are invariant under the flow, we have

$$\Phi_{t,\tau}^{\varepsilon}(\mathcal{W}_{\pm}^{\varepsilon}(\tau)) = \mathcal{W}_{\pm}^{\varepsilon}(t).$$

In particular, if we define the associated Poincaré map

$$\Pi_{\tau}^{\varepsilon} = \Phi_{\tau+T,\tau}^{\varepsilon} ,$$

then

$$\Pi_{\tau}^{\varepsilon}(\mathcal{W}_{\pm}^{\varepsilon}(\tau)) = \mathcal{W}_{\pm}^{\varepsilon}(\tau).$$

Of course, we have that

$$\Pi_{\tau}^{\varepsilon}(x^{\varepsilon}(\tau)) = x^{\varepsilon}(\tau)$$

and

$$x^{\varepsilon}(\tau) \in \mathcal{W}_{+}^{\varepsilon}(\tau) \cap \mathcal{W}_{-}^{\varepsilon}(\tau),$$

for all $\tau \in \mathbb{R}$. In the next section, we will construct another solution with this last property by perturbing from the homoclinic orbit $\bar{x}(t)$.

## 10.4 Existence of a Transverse Homoclinic Point

**Definition 10.2.** *A homoclinic point for the Poincaré map $\Pi_{\tau}^{\varepsilon}$ is a point with the property that*

$$\Pi_{\tau}^{\varepsilon}(y) \in \mathcal{W}_{+}^{\varepsilon}(\tau) \cap \mathcal{W}_{-}^{\varepsilon}(\tau), \quad \textit{and} \quad \Pi_{\tau}^{\varepsilon}(y) \neq y.$$

*A homoclinic point $y$ for $\Pi_{\tau}^{\varepsilon}$ is transverse if $\mathcal{W}_{+}^{\varepsilon}(\tau)$ and $\mathcal{W}_{-}^{\varepsilon}(\tau)$ intersect transversely at $y$.*

If $y$ is a homoclinic point, then $(\Pi_{\tau}^{\varepsilon})^{k}(y)$, $k \in \mathbb{Z}$, is a doubly infinite sequence of distinct points in $\mathcal{W}_{+}^{\varepsilon}(\tau) \cap \mathcal{W}_{-}^{\varepsilon}(\tau)$ converging to $x^{\varepsilon}(\tau)$ as $k$ tends to $\pm\infty$.

We are going to construct solutions of the periodically perturbed Newton equation

$$\ddot{\varphi} + g(\varphi) = \varepsilon q(t), \quad \varphi(\tau) = \zeta_0 + z_0, \quad \dot{\varphi}(\tau) = \lambda, \tag{10.26}$$

close to the homoclinic solution $p(t)$, whose behavior has been detailed in Lemma 10.1. We remind the reader that the nonlinear function $g$ satisfies the assumptions (N1)–(N5) and that $q(t)$ is $C^{\infty}$ and $T$-periodic, for some $T > 0$. Solutions $\varphi$

will have the perturbative form

$$\varphi(t) = p(t-\tau) + \lambda u(t-\tau) + z(t-\tau), \tag{10.27}$$

where

$$u = -\alpha^{-2}\dot{p}.$$

We have that

$$p(0) = \zeta_0, \quad \dot{p}(0) = u(0) = 0, \quad \dot{u}(0) = 1, \quad \text{and} \quad Lu = 0.$$

Therefore, $\varphi$ in (10.27) solves the IVP (10.26) if and only if $z$ solves the IVP

$$Lz = N(z, \lambda, \tau, \varepsilon), \quad z(0) = z_0, \quad \dot{z}(0) = 0, \tag{10.28}$$

in which

$$N(z, \lambda, \tau, \varepsilon)(t) = R(t, z(t) + \lambda u(t)) + \varepsilon q(t+\tau) \tag{10.29}$$

with

$$\begin{aligned} R(t, w) &= -\, g(p(t)+w) + g(p(t)) + g'(p(t))w \\ &= -\, w^2 \int_0^1 (1-\sigma) g''(p(t)+\sigma w)\, d\sigma \\ &\equiv -\, w^2 \bar{R}(t, w). \end{aligned} \tag{10.30}$$

**Lemma 10.5.** *The mapping* $(z, \lambda, \tau, \varepsilon) \mapsto N(z, \lambda, \tau, \varepsilon)$ *defined in* (10.29) *and* (10.30) *satisfies*

(i) $N \in C^2(C_b^2(\mathbb{R}) \times \mathbb{R}^3; C_b^0(\mathbb{R}))$,
(ii) $N(0, 0, \tau, 0) = 0$, *and*
(iii) $D_z N(0, 0, \tau, 0) = 0$.

*Remark 10.2.* We also have $N \in C^2(C_b^2(\mathbb{R}_\pm) \times \mathbb{R}^3; C_b^0(\mathbb{R}_\pm))$.

**Lemma 10.6.** *For each* $\tau_0 \in [0, T)$, *there exist a neighborhood* $U$ *of* $(0, \tau_0, 0) \in \mathbb{R}^3$, *a neighborhood* $V_\pm$ *of* $0 \in C_b^2(\mathbb{R}_\pm)$, *and a* $C^1$ *mapping*

$$(\lambda, \tau, \varepsilon) \mapsto z_\pm(\cdot, \lambda, \tau, \varepsilon)$$

*from* $U$ *into* $V_\pm$ *such that* $z_\pm(\cdot, 0, \tau, 0) = 0$ *and* $z_\pm(t, \lambda, \tau, \varepsilon)$ *is the unique solution of* (10.28) *in* $V_\pm$.

*Proof.* Consider that mapping

$$(z, \lambda, \tau, \varepsilon) \mapsto F(z, \lambda, \tau, \varepsilon) = Lz - N(z, \lambda, \tau, \varepsilon).$$

Then $F$ is a $C^1$ map from $D_\pm(L) \times \mathbb{R}^3$ into $C^0_b(\mathbb{R}_\pm)$, where

$$D_\pm(L) = \{z \in C^2_b(\mathbb{R}_\pm) : \dot{z}(0) = 0\}.$$

And moreover, for each $\tau_0 \in [0, T)$, we have

$$F(0, 0, \tau_0, 0) = 0, \quad D_z F(0, 0, \tau_0, 0) = L.$$

By Theorem 10.1, $L : D_\pm(L) \to C^0_b(\mathbb{R}_\pm)$ is an isomorphism. The result follows by the Implicit Function Theorem.

**Lemma 10.7.** *There exist* $\lambda_0 > 0$*,* $\varepsilon_0 > 0$*, such that for all* $|\varepsilon| < \varepsilon_0$*, the curves*

$$\lambda \mapsto y_\pm(\lambda, \tau, \varepsilon) = (\zeta_0 + z_\pm(0, \lambda, \tau, \varepsilon), \lambda), \quad |\lambda| < \lambda_0, \tag{10.31}$$

*are local parameterizations of* $\mathcal{W}^\varepsilon_\pm(\tau)$ *near* $(\zeta_0, 0)$.

*Proof.* Since the map $(\tau, \lambda, \varepsilon) \mapsto z_\pm(\cdot, \lambda, \tau, \varepsilon)$ is $C^1$ from $U$ into $C^2_b(\mathbb{R}_\pm)$ and $z_\pm(\cdot, 0, \tau, 0) = 0$, we have that $\|z_\pm(\cdot, \lambda, \tau, \varepsilon)\|_2 \le C(|\lambda| + |\varepsilon|)$. Through the definition (10.26), the solution $z_\pm(t, \lambda, \tau, \varepsilon)$ gives rise to a solution $\Phi^\varepsilon_{t,\tau}(y_\pm(\lambda, \tau, \varepsilon))$ of (10.24) which, since $p$ decays exponentially in $C^2$ by Lemma 10.1, satisfies

$$\limsup_{t \to \pm\infty} \|\Phi^\varepsilon_{t,\tau}(y_\pm(\lambda, \tau, \varepsilon))\| \le C(|\lambda| + |\varepsilon|).$$

The periodic solution $x^\varepsilon$ also satisfies $\|x^\varepsilon(t)\| \le C\varepsilon$. Thus, if $\lambda_0$ and $\varepsilon_0$ are sufficiently small, we have that

$$\limsup_{t \to \pm\infty} \|\Phi^\varepsilon_{t,\tau}(y_\pm(\lambda, \tau, \varepsilon)) - x^\varepsilon(t)\| \le C(|\lambda| + |\varepsilon|) < \eta_0.$$

This shows that $y_\pm(\lambda, \tau, \varepsilon)$ belongs to $\mathcal{W}^\varepsilon_\pm(\tau)$, for $|\lambda| < \lambda_0$, by Lemma 10.4. $\square$

**Definition 10.3.** *The Melnikov function is defined as*

$$\mu(\tau, \varepsilon) = z_+(0, 0, \tau, \varepsilon) - z_-(0, 0, \tau, \varepsilon).$$

The points $\zeta_0 + z_\pm(0, 0, \tau, \varepsilon)$ are the locations where the stable and unstable manifolds cross the horizontal axis near $(\zeta_0, 0)$ in the phase plane. The Melnikov function will be used to measure the separation of these manifolds near a homoclinic point.

**Lemma 10.8.** *The Melnikov function* $\mu(\tau, \varepsilon)$ *is* $C^1$ *in a neighborhood of* $(0, 0)$*, and if*

$$h(\tau) = \int_{-\infty}^{\infty} u(t) q(t + \tau) dt, \tag{10.32}$$

*then*

$$\|\mu(\tau,\varepsilon)-\varepsilon h(\tau)\|+\|D_\tau[\mu(\tau,\varepsilon)-\varepsilon h(\tau)]\|\le C\varepsilon^2.$$

*Proof.* The map which takes $(\lambda,\tau,\varepsilon)$ to $z_\pm(\cdot,\lambda,\tau,\varepsilon)$ in $C_b^2(\mathbb{R}_\pm)$ is $C^1$. So the Melnikov function is $C^1$ by its definition. Since $z_\pm$ is a bounded solution of (10.28), we have by Lemma 10.2 that

$$z_\pm(0,\lambda,\tau,\varepsilon)=\int_0^{\pm\infty} u(t)[R(t,z_\pm(t,\lambda,\tau,\varepsilon)+\varepsilon q(t+\tau)]dt.$$

Therefore,

$$\mu(\tau,\varepsilon)-\varepsilon h(\tau)=\int_0^{\infty} u(t)R(t,z_+(t,0,\tau,\varepsilon))dt-\int_0^{-\infty} u(t)R(t,z_-(t,0,\tau,\varepsilon))dt.$$

Since $\|z_\pm(\cdot,0,\tau,\varepsilon)\|_2+\|D_\tau z_\pm(\cdot,0,\tau,\varepsilon)\|_2\le C|\varepsilon|$, we see from (10.30) that the desired estimate holds. $\square$

**Lemma 10.9.** *Suppose that for $|\varepsilon|<\varepsilon_0$, there exists a small number $\tau(\varepsilon)$ such that $y^\varepsilon\equiv y_+(0,\tau(\varepsilon),\varepsilon)=y_-(0,\tau(\varepsilon),\varepsilon)$ is a homoclinic point. Then*

$$D_\tau\mu(\tau(\varepsilon),\varepsilon)\ne 0,\quad 0<|\varepsilon|<\varepsilon_0 \tag{10.33}$$

*implies that $\mathcal{W}_+^\varepsilon(\tau(\varepsilon))$ and $\mathcal{W}_-^\varepsilon(\tau(\varepsilon))$ are transverse at $y^\varepsilon$.*

*Proof.* By Lemma 10.7, the tangents to $\mathcal{W}_\pm^\varepsilon(\tau(\varepsilon))$ at the homoclinic point $y^\varepsilon$ are

$$D_\lambda y_\pm(0,\tau(\varepsilon),\varepsilon)=(D_\lambda z_\pm(0,0,\tau(\varepsilon),\varepsilon),1).$$

Thus, transversality is equivalent to

$$D_\lambda z_+(0,0,\tau(\varepsilon),\varepsilon)-D_\lambda z_-(0,0,\tau(\varepsilon),\varepsilon)\ne 0. \tag{10.34}$$

Consider the points $y_\pm(0,\tau,\varepsilon)=(\zeta_0+z_\pm(0,0,\tau,\varepsilon),0)$ where $\mathcal{W}_\pm^\varepsilon(\tau)$ cross the horizontal axis in the phase plane. Since $\Phi^\varepsilon_{\tau,\tau(\varepsilon)}$ maps $\mathcal{W}_\pm^\varepsilon(\tau(\varepsilon))$ diffeomorphically to $\mathcal{W}_\pm^\varepsilon(\tau)$, we can find values $\lambda_\pm(\tau)$ such that

$$y_\pm(0,\tau,\varepsilon)=\Phi^\varepsilon_{\tau,\tau(\varepsilon)}(y_\pm(\lambda_\pm(\tau),\tau(\varepsilon),\varepsilon)).$$

For $\tau=\tau(\varepsilon)$, we have $\lambda_\pm(\tau(\varepsilon))=0$ and

$$y_\pm(\lambda_\pm(\tau(\varepsilon)),\tau(\varepsilon),\varepsilon)=y^\varepsilon.$$

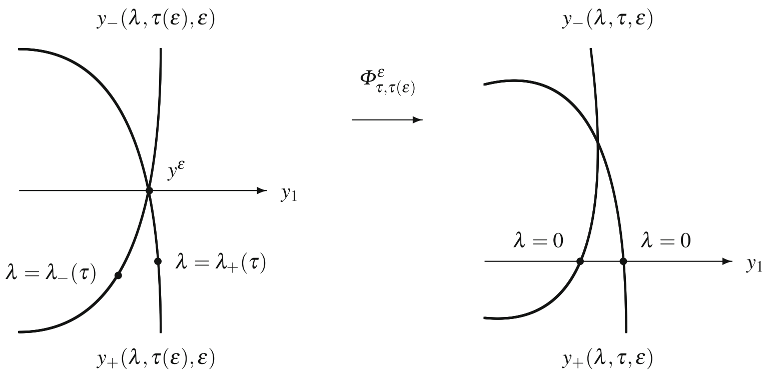



**Fig. 10.2** Slices of the invariant manifolds at time $\tau(\varepsilon)$ and their images under the flow $\Phi^\varepsilon_{\tau,\tau(\varepsilon)}$ for a time $\tau \sim \tau(\epsilon)$

See Fig. 10.2. Using the chain rule and the fact that $D_y\Phi^\varepsilon_{\tau,\tau}(y) = I$, we have

$$\begin{aligned}
&D_\tau[y_+(0,\tau,\varepsilon) - y_-(0,\tau,\varepsilon)]|_{\tau=\tau(\varepsilon)} \\
&\quad = D_\tau[\Phi^\varepsilon_{\tau,\tau(\varepsilon)}(y_+(\lambda_+(\tau),\tau(\varepsilon),\varepsilon)) - \Phi^\varepsilon_{\tau,\tau(\varepsilon)}(y_+(\lambda_+(\tau),\tau(\varepsilon),\varepsilon))]|_{\tau=\tau(\varepsilon)} \\
&\quad = \dot{\Phi}^\varepsilon_{\tau,\tau(\varepsilon)}(y_+(\lambda_+(\tau),\tau(\varepsilon),\varepsilon))|_{\tau=\tau(\varepsilon)} \\
&\qquad - \dot{\Phi}^\varepsilon_{\tau,\tau(\varepsilon)}(y_-(\lambda_-(\tau),\tau(\varepsilon),\varepsilon))]|_{\tau=\tau(\varepsilon)} \\
&\qquad + D_y\Phi^\varepsilon_{\tau,\tau(\varepsilon)}(y_+(\lambda_+(\tau),\tau(\varepsilon),\varepsilon))\ D_\lambda y_+(\lambda_+(\tau),\tau(\varepsilon),\varepsilon))\ D_\tau\lambda_+(\tau)|_{\tau=\tau(\varepsilon)} \\
&\qquad - D_y\Phi^\varepsilon_{\tau,\tau(\varepsilon)}(y_-(\lambda_-(\tau),\tau(\varepsilon),\varepsilon))\ D_\lambda y_-(\lambda_-(\tau),\tau(\varepsilon),\varepsilon))\ D_\tau\lambda_-(\tau)|_{\tau=\tau(\varepsilon)} \\
&\quad = D_\lambda y_+(0,\tau(\varepsilon),\varepsilon))\ D_\tau\lambda_+(\tau(\varepsilon)) - D_\lambda y_-(0,\tau(\varepsilon),\varepsilon))\ D_\tau\lambda_-(\tau(\varepsilon)).
\end{aligned}$$

Given the definition (10.31) of the curves $y_\pm$, expressing the above calculation in coordinates yields the system

$$\begin{aligned}
\begin{bmatrix} D_\tau\mu(\tau(\varepsilon),\varepsilon) \\ 0 \end{bmatrix} = &\begin{bmatrix} D_\lambda z_+(0,0,\tau(\varepsilon),\varepsilon) \\ 1 \end{bmatrix} D_\tau\lambda_+(\tau(\varepsilon)) \\
&- \begin{bmatrix} D_\lambda z_-(0,0,\tau(\varepsilon),\varepsilon) \\ 1 \end{bmatrix} D_\tau\lambda_-(\tau(\varepsilon))
\end{aligned}$$

Thus, (10.33) implies (10.34). □

**Theorem 10.3.** *Suppose that the function $h(\tau)$ definied in (10.32) satisfies*

$$h(0) = 0 \quad \textit{and} \quad h'(0) \neq 0. \tag{10.35}$$

*Then there is a function $\tau(\varepsilon)$ defined for $|\varepsilon| < \varepsilon_0$ such that $|\tau(\varepsilon)| < A|\varepsilon|$, for some $A > 0$, and $y^\varepsilon = y_\pm(0,\tau(\varepsilon),\varepsilon)$ is a transverse homoclinic point for $\Pi^\varepsilon_{\tau(\varepsilon)}$, for $0 < |\varepsilon| < \varepsilon_0$.*

*Proof.* We shall construct a solution of (10.28) in $C_b^2(\mathbb{R})$, using the Liapunov–Schmidt method, see Sect. 5.5. Since we seek a homoclinic point on the axis, we can take $\lambda = 0$. If $z(t, \tau, \varepsilon) \in C_b^2(\mathbb{R})$ solves (10.28) with $\lambda = 0$, then by Theorem 10.2, we must have $N(z, 0, \tau, \varepsilon) \in R(L)$.

Define the projection $P : C_b^0(\mathbb{R}) \to R(L)$ by

$$Pv(t) = v(t) - \frac{\int_{\mathbb{R}} u(s)v(s)ds}{\int_{\mathbb{R}} u(s)^2 ds} u(t).$$

Consider the problem of solving

$$Lz = PN(z, 0, \tau, \varepsilon), \quad z \in D(L).$$

To this end, define

$$F(z, \tau, \varepsilon) = Lz - PN(z, 0, \tau, \varepsilon).$$

Then $F \in C^2(D(L), R(L))$, $F(0, \tau, 0) = 0$, and $D_z F(0, \tau, 0) = L$ is an isomorphism from $D(L)$ to $R(L)$, by Theorem 10.2. By the Implicit Function Theorem, there exists a $C^2$ map

$$(\tau, \varepsilon) \mapsto z(\cdot, \tau, \varepsilon)$$

defined for $(\tau, \varepsilon)$ in a neighborhood of the origin in $\mathbb{R}^2$ into a neighborhood of the origin in $D(L)$ such that

$$z(\cdot, \tau, 0) = 0, \quad \text{and} \quad F(z(\cdot, \tau, \varepsilon), \tau, \varepsilon) = 0.$$

Next, following the Liapunov–Schmidt method, we show that for each $|\varepsilon| < \varepsilon_0$ we can find $\tau(\varepsilon)$ such that

$$PN(z(\cdot, \tau, \varepsilon), \tau, \varepsilon)|_{\tau=\tau(\varepsilon)} = N(z(\cdot, \tau, \varepsilon), \tau, \varepsilon)|_{\tau=\tau(\varepsilon)}.$$

Thus, the bifurcation equation is

$$b(\tau, \varepsilon) = \frac{1}{c_0^2} \int_{\mathbb{R}} u(t)[R(t, z(t, \tau, \varepsilon)) + \varepsilon q(t + \tau)]dt = 0, \quad c_0^2 = \int_{\mathbb{R}} u(t)^2 dt.$$

Now $(\tau, \varepsilon) \mapsto z(\cdot, \tau, \varepsilon) \in C_b^2(\mathbb{R})$ is $C^2$ and $z(\cdot, \tau, 0) = 0$. Thus, we may write

$$z(\cdot, \tau, \varepsilon) = \varepsilon \int_0^1 D_\varepsilon z(\cdot, \tau, \sigma\varepsilon) d\sigma = \varepsilon \bar{z}(\cdot, \tau, \varepsilon),$$

where $(\tau, \varepsilon) \mapsto \bar{z}(\cdot, \tau, \varepsilon)$ is a $C^1$ map into $C_b^2(\mathbb{R})$.

By (10.30), we have that

$$R(\cdot, z(\cdot, \tau, \varepsilon)) = \varepsilon^2 \bar{z}(\cdot, \tau, \varepsilon)^2 \bar{R}(\cdot, z(\cdot, \tau, \varepsilon)),$$

is a $C^1$ map of $(\tau, \varepsilon)$ into $C_b^2(\mathbb{R})$. And so, it follows that

$$\int_{\mathbb{R}} R(t, z(t, \tau, \varepsilon))u(t)dt = \varepsilon^2 \rho(\tau, \varepsilon),$$

where $\rho$ is a $C^1$ function.

We can obtain solutions of the bifurcation equation by solving

$$\mathcal{B}(\tau, \varepsilon) = h(\tau) + \varepsilon\rho(\tau, \varepsilon) = 0.$$

Since $\mathcal{B}(0, 0) = 0$ and $D_\tau\mathcal{B}(0, 0) \neq 0$, the Implicit Function Theorem yields a $C^1$ curve $\tau(\varepsilon)$ such that

$$\tau(0) = 0 \quad \text{and} \quad \mathcal{B}(\tau(\varepsilon), \varepsilon) = 0.$$

We conclude that $z(\cdot, \tau(\varepsilon), \varepsilon)$ solves (10.28) with $\lambda = 0$ in $C_b^2(\mathbb{R})$. Since $\|z(\cdot, \tau, \varepsilon)\|_2 < C|\varepsilon|$, for small $|\varepsilon|$, we may assert that

$$z(\cdot, \tau(\varepsilon), \varepsilon)|_{\mathbb{R}_\pm} = z_\pm(\cdot, 0, \tau(\varepsilon), \varepsilon),$$

by the uniqueness portion of Lemma 10.6. Therefore,

$$y^\varepsilon = (\zeta_0 + z(0, \tau(\varepsilon), \varepsilon), 0) = (\zeta_0 + z_\pm(0, 0, \tau(\varepsilon), \varepsilon), 0) = y_\pm(0, \tau(\varepsilon), \varepsilon)$$

is a homoclinic point for $\Pi^\varepsilon_{\tau(\varepsilon)}$. By (10.35) and Lemma 10.8, we have that the Melnikov function satisfies

$$D_\tau \mu(\tau, \varepsilon)|_{\tau=\tau(\varepsilon)} \neq 0, \quad 0 < |\varepsilon| < \varepsilon_0.$$

By Lemma 10.9, the homoclinic point $y^\varepsilon$ is transverse. □

## 10.5 Chaotic Dynamics

In this final section, we summarize the connection between the existence of a transverse homoclinic point and chaotic dynamics.

Fix $0 < |\varepsilon| < \varepsilon_0$, and let $y = y^\varepsilon$ be a transverse homoclinic point for $\Pi = \Pi^\varepsilon_{\tau(\varepsilon)}$, as constructed in the previous section. Then the sequence $\{\Pi^k(y) : k \in \mathbb{Z}\}$ belongs to $\mathcal{W}^\varepsilon_+(\tau(\varepsilon)) \cap \mathcal{W}^\varepsilon_-(\tau(\varepsilon))$, and $\lim_{k\to\pm\infty} \Pi^k(y) = x = x^\varepsilon(\tau(\varepsilon))$. Thus, there are an infinite number of transverse crossings of the stable and unstable manifolds near the fixed point $x$, and the two manifolds create a so-called homoclinic tangle.

For any positive integer $\ell$ and any neighborhood $S$ of $x$, the set $\Lambda_\ell = \cap_{k\in\mathbb{Z}}(\Pi^\ell)^k(S)$ is nonempty, and $\Pi^\ell : \Lambda_\ell \to \Lambda_\ell$ is a bijection.

Let $\Sigma = \{a : \mathbb{Z} \to \{0, 1\}\}$, the set of doubly infinite sequences in 0 and 1. The set $\Sigma$ becomes a metric space with the distance function

$$d(a, b) = \sum_{j\in\mathbb{Z}} 2^{-|j|}|a_j - b_j|.$$

We define the shift $\sigma : \Sigma \to \Sigma$ by

$$\sigma(a)_j = a_{j-1}, \quad j \in \mathbb{Z}.$$

**Theorem 10.4.** (Birkhoff-Smale) *There exist a neighborhood $S$ of $x$, an $\ell$ sufficently large, and a homeomorphism $h : \Lambda_\ell \to \Sigma$ such that $h \circ \Pi^\ell = \sigma \circ h$.*

We say that $\Pi^\ell|\Lambda_\ell$ is topologically conjugate to a shift on two symbols.

We shall not attempt to prove this result. The crux of the matter is that $\Pi^\ell|_S$ can be modeled by the Smale horseshoe map, which in turn can be linked to the shift on two symbols. This is discussed in many places, see for example the books by Guckenheimer and Holmes [1], Moser [2], or Perko [3].

**Corollary 10.1.** *The map $\Pi^\ell : \Lambda_\ell \to \Lambda_\ell$ is bijective.*

- *$\Lambda_\ell$ contains a countable set of periodic orbits of arbitrarily long period.*
- *$\Lambda_\ell$ contains an uncountable set of aperiodic orbits.*
- *$\Lambda_\ell$ contains a dense orbit.*

## 10.6 Exercises

**Exercise 10.1.**

(a) *Verify that $p(t) = \sqrt{2}\,\mathrm{sech}\, t$ solves $\ddot{\varphi} - \varphi + \varphi^3 = 0$.*

(b) *Show that $x = (p, \dot{p})$ is a homoclinic orbit for the system*

$$x_1' = x_2, \quad x_2' = x_1 - x_1^3.$$

(c) *Show that the Poincaré map for perturbed system*

$$x_1' = x_2, \quad x_2' = x_1 - x_1^3 + \varepsilon \cos t$$

*has a transverse homoclinic point for $\varepsilon$ small.*

# Appendix A
# Results from Real Analysis

In this brief appendix we collect a few standard facts from analysis that will be used repeatedly. A good reference is the book by Rudin [10]. These results are generalized in Chap. 5.

**Definition A.1** Let $(X, d)$ be a metric space. A map $T : X \to X$ is called a contraction if there exists a constant $\alpha > 0$ such that

$$d(Tx, Ty) \leq \alpha\, d(x, y), \quad \text{for all} \quad x, y \in X.$$

**Definition A.2** Let $T : X \to X$. A point $x \in X$ is called a fixed point of $T$ if $Tx = x$.

**Theorem A.1** *(Contraction Mapping Principle). A contraction mapping on a complete metric space $(X, d)$ has a unique fixed point.*

*Proof* Choose an arbitrary point $x_0 \in X$, and define an iterative sequence

$$x_k = Tx_{k-1}, \quad k = 1, 2, \ldots.$$

Then we have

$$d(Tx_{k+1}, Tx_k) = d(T^k x_1, T^k x_0) \leq \alpha^k d(x_1, x_0),$$

and so, for any $m > n$ we obtain by the triangle inequality

$$d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \leq \sum_{k=n}^{m-1} \alpha^k d(x_1, x_0) \leq \frac{\alpha^n}{1-\alpha} d(x_1, x_0).$$

It follows that the sequence $\{x_k\}_{k=0}^{\infty}$ is Cauchy. Its limit $x \in X$ is a fixed point of $T$ since $T$ is continuous and

T. C. Sideris, *Ordinary Differential Equations and Dynamical Systems*,
Atlantis Studies in Differential Equations 2, DOI: 10.2991/978-94-6239-021-8,

$$x = \lim_k x_k = \lim_k T x_k = T x.$$

To prove uniqueness, notice that if $x, x' \in X$ are fixed points of $T$, then

$$d(x, x') = d(Tx, Tx') \leq \alpha\, d(x, x')$$

implies that $d(x, x') = 0$.

**Theorem A.2** *(Implicit Function Theorem in Finite Dimensions). Let $F : \mathcal{O} \to \mathbb{R}^n$ be a $C^1$ mapping on an open set $\mathcal{O} \subset \mathbb{R}^m \times \mathbb{R}^n$. If $(x_0, y_0) \in \mathcal{O}$ is a point such that $F(x_0, y_0) = 0$ and $D_y F(x_0, y_0)$ is invertible, then there exist a neighborhood $U \times V \subset \mathcal{O}$ of $(x_0, y_0)$ and a $C^1$ map $g : U \to V$ such that*

$$y_0 = g(x_0) \quad \textit{and} \quad F(x, g(x)) = 0, \quad \textit{for all} \quad x \in U.$$

*If $(x, y) \in U \times V$ and $F(x, y) = 0$, then $y = g(x)$.*

A more general version of this result is proven in Chap. 5, see Theorem 5.7.

**Theorem A.3** *(Inverse Function Theorem). Let $f : \mathcal{O} \to \mathbb{R}^n$ be a $C^1$ mapping on an open set $\mathcal{O} \subset \mathbb{R}^n$. If $Df(x_0)$ is invertible for some $x_0 \in \mathcal{O}$, then there exist neighborhoods $x_0 \in U \subset \mathcal{O}$ and $f(x_0) \in V \subset \mathbb{R}^n$ such that $f : U \to V$ is a bijection. The mapping $f^{-1} : V \to U$ is $C^1$, and*

$$D(f^{-1})(f(x)) = (Df)^{-1}(x), \quad \textit{for all} \quad x \in U.$$

*Proof* Apply the Implicit Function Theorem to the mapping $F(v, \boldsymbol{w}) = v - f(\boldsymbol{w})$ near the point $(v_0, \boldsymbol{w}_0) = (f(x_0), x_0)$ in the open set $\mathbb{R}^n \times \mathcal{O}$. There exist neighborhoods $v_0 \in V \subset \mathbb{R}^n$, $\boldsymbol{w}_0 \in W \subset \mathcal{O}$ and a $C^1$ mapping $g : V \to W$ such that

$$v = f(g(v)), \quad \text{for all} \quad v \in V.$$

Set $U = g(V)$. Then $f : U \to V$ is a bijection, so $U = f^{-1}(V)$ is open, and $g = f^{-1}$. □

**Definition A.3** A $C^1$ bijection whose inverse is also $C^1$ is called a diffeomorphism.

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