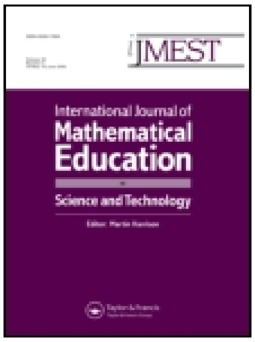

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Subharmonic solutions of order one-third

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A general approach for finding initial conditions for subharmonic solutions is discussed and an improved technique developed for finding approximate initial conditions that lead to subharmonic solutions of order 1/3 to differential equations of the form

$$\ddot{x} + g(x, \dot{x}) = F \cos \omega t$$

where $g(x, \dot{x})$ is generally a nonlinear function. In particular, this technique is illustrated by finding such subharmonic solutions for the forced spring Duffing type equation

$$\ddot{x} + x + \varepsilon x^3 = F \cos \omega t$$

Computer laboratory problems and student research problems arise naturally as the technique is general and can be applied to many of the classical nonlinear differential equations.

1. Introduction

Finding a periodic solution to a nonlinear ordinary differential equation is in general a difficult task. Only in a very few cases can direct methods be applied to an equation to find initial values leading to a solution of the corresponding initial value problem that is periodic. Oscillatory periodic solutions have such practical importance that great effort has gone into developing techniques to find such initial conditions. Recently a new technique was presented in this journal [1] for finding periodic solutions having the same period as the forcing part of the equation, the so called harmonic solution. In this article, we give a related technique for discovering initial values leading to solutions having period a multiple of the forcing, so called subharmonic solutions. We also give some problems suitable for undergraduate research that are designed to show the broad applicability of the technique.

We give an elementary solution technique to provide subharmonic solutions to differential equations of the form

$$\ddot{x} + g(x, \dot{x}) = F \cos \omega t \tag{1.1}$$

where $g(x, \dot{x})$ is a nonlinear restoring force term, $F \cos \omega t$ is called the forcing term, and the equation represents the displacement in some mechanical or electronic system. This form includes the undamped Duffing equation

$$\ddot{x} + x + \varepsilon x^3 = F \cos \omega t \tag{1.2}$$

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Since its introduction in 1918 [2], Duffing's equation has become one of the most studied nonlinear equations; it embodies all the bad behaviour one can expect from a nonlinear equation (sensitivity to initial conditions, unbounded solutions) as well as pleasing behaviour (bounded and periodic solutions).

A solution to (1.1) is called *harmonic* provided it is periodic with fundamental period $P = 2\pi/\omega$. A solution to (1.1) is called *subharmonic* provided it is periodic with fundamental period an integral multiple of the forcing period P. If the fundamental period of the solution is nP where n is a positive integer greater than 1, then the solution is called a *subharmonic solution of order* $1/n$.

Linear equations can have subharmonic solutions. For example, consider the equation

$$\ddot{x} + x = F\cos nt \tag{1.3}$$

where $n \neq 1$, which has the general solution

$$x(t) = c_1 \sin t + c_2 \cos t + \frac{F}{1-n^2}\cos nt \tag{1.4}$$

This solution has the form of a long-period oscillation of period 2π, with the shorter period of $2\pi/n$ of the forced oscillation superposed.

For the linear equation, any damping present causes the homogeneous part of the solution to become a transient and the subharmonic feature will have disappeared. For a nonlinear equation however, stable subharmonics may occur even in the presence of damping.

We shall concentrate on Duffing's equation (1.2) as representative, but the techniques developed can certainly be applied to other equations. Duffing's equation often has numerically stable subharmonic solutions of order 1/3 and such solutions are discussed at some length in [3] for the special case of the forcing frequency ω being close to 3. The development herein has no restriction on ω and the technique is applicable for other subharmonics. The search for subharmonic solutions generally is a complicated business and herein we only offer approximation techniques based on a second-order harmonic balance approach. Often subharmonics exist but the Fourier series coefficients of the solution converge so slowly to zero that the harmonic balance approach fails to be able to detect them; examples for this are given in [4] where subharmonic boundaries are discussed.

2. A harmonic balance approach

In this section, we imitate the harmonic balance approach that was successfully used to approximate harmonic solutions in [1]. The idea is to assume that the subharmonic solution of order $1/n$ will have a form very close to

$$x(t) = a(t)\cos\frac{\omega}{n}t + b(t)\sin\frac{\omega}{n}t + c(t)\cos\omega t + d(t)\sin\omega t \tag{2.1}$$

where $a(t), b(t), c(t)$ and $d(t)$ are slowly varying coefficients. In the text [3] motivated by the linear case, a simpler approach is used by assuming the solution has $d(t)$ identically zero and $c(t)$ constant equal to $F/(1-n^2)$. A comparison of the two approaches has shown (2.1) to be superior, as we will indicate below.

We begin by illustrating this technique with the Duffing problem

$$\ddot{x} + x + \varepsilon x^3 = F\cos\omega t \tag{2.2}$$

2.1. *Seeking subharmonic solutions of order 1/3*

We assume that the subharmonic solution of order 1/3 to equation (2.2) has the form

$$x(t) = a(t)\cos\frac{\omega}{3}t + b(t)\sin\frac{\omega}{3}t + c(t)\cos\omega t + d(t)\sin\omega t \tag{2.3}$$

Substituting into equation (2.2), simplifying, neglecting the harmonics of order different from $\omega/3$ and ω, and balancing coefficients, we have the coefficients of $\cos(\omega/3)t$, $\sin(\omega/3)t$, $\cos\omega t$, and $\sin\omega t$ giving rise to the equations respectively:

$$a\left(1-\frac{\omega^2}{9}\right) + \frac{3\varepsilon}{4}a(a^2+b^2) + \frac{3\varepsilon}{4}c(a^2-b^2) + \frac{3\varepsilon}{2}a(c^2+bd+d^2) + \frac{2\omega}{3}\dot{b} + \ddot{a} = 0 \tag{2.4}$$

$$b\left(1-\frac{\omega^2}{9}\right) + \frac{3\varepsilon}{4}b(a^2+b^2) + \frac{3\varepsilon}{4}d(a^2-b^2) + \frac{3\varepsilon}{2}b(c^2-ac+d^2) - \frac{2\omega}{3}\dot{a} + \ddot{b} = 0 \tag{2.5}$$

$$c(1-\omega^2) + \frac{\varepsilon}{4}a(a^2-3b^2) + \frac{3\varepsilon}{2}c(a^2+b^2) + \frac{3\varepsilon}{4}c(c^2+d^2) + 2\omega\dot{d} + \ddot{c} = F \tag{2.6}$$

$$d(1-\omega^2) + \frac{\varepsilon}{4}b(3a^2-b^2) + \frac{3\varepsilon}{2}d(a^2+b^2) + \frac{3\varepsilon}{4}d(c^2+d^2) - 2\omega\dot{c} + \ddot{d} = 0 \tag{2.7}$$

By assumption, the derivatives are all small and by neglecting them, we have the system of equations

$$a\left(1-\frac{\omega^2}{9}\right) + \frac{3\varepsilon}{4}a(a^2+b^2) + \frac{3\varepsilon}{4}c(a^2-b^2) + \frac{3\varepsilon}{2}a(c^2+bd+d^2) = 0 \tag{2.8}$$

$$b\left(1-\frac{\omega^2}{9}\right) + \frac{3\varepsilon}{4}b(a^2+b^2) + \frac{3\varepsilon}{4}d(a^2-b^2) + \frac{3\varepsilon}{2}b(c^2-ac+d^2) = 0 \tag{2.9}$$

$$c(1-\omega^2) + \frac{\varepsilon}{4}a(a^2-3b^2) + \frac{3\varepsilon}{2}c(a^2+b^2) + \frac{3\varepsilon}{4}c(c^2+d^2) = F \tag{2.10}$$

$$d(1-\omega^2) + \frac{\varepsilon}{4}b(3a^2-b^2) + \frac{3\varepsilon}{2}d(a^2+b^2) + \frac{3\varepsilon}{4}d(c^2+d^2) = 0 \tag{2.11}$$

The simultaneous solutions of these equations we call *critical values* (they are critical values of a first-order differential system). These equations are easy to solve numerically using a computer algebra system, we use *Mathematica*'s **NSolve** routine (for details on this routine see [5]).

Example 2.1. Find subharmonic solutions of order 1/3 using the harmonic balance method for the equation

$$\ddot{x} + x - \frac{1}{6}x^3 = \frac{1}{3}\cos\frac{3}{5}t \tag{2.12}$$

Solving the equations (2.8–2.11) simultaneously, there are 21 critical values all having $d = 0$. Three are of the form $(0, 0, c, 0)$.

$$(0, 0, 0.5540519, 0) \quad (0, 0, -2.488308, 0) \quad (0, 0, 1.934256, 0)$$

The one of these having $c = 0.5540519$ yields very nearly the harmonic solution (which arises from $x(0) = 0.5507900342$ and $\dot{x}(0) = 0$); the other two of this form lead to unbounded solutions, see [1] as these critical values arise in second-order van der Pol plane analysis when seeking harmonic solutions. The remaining 18 critical values arise in the form of six *reflection triples* (these values have been rounded to 6 significant digits).

$$\begin{array}{ll} (2.95691, 0, -0.860894, 0) & (-2.93512, 0, 0.467376, 0) \\ (-1.47845, \pm 2.56076, -0.860894, 0) & (1.46756, \pm 2.54189, 0.467376, 0) \end{array}$$

$$\begin{array}{ll} (-1.51156, 0, -1.30746, 0) & (1.04552, 0, -2.09489, 0) \\ (0.75578, \pm 1.30905, -1.30746, 0) & (-0.522759, \pm 0.905446, -2.09489, 0) \end{array}$$

$$\begin{array}{ll} (0.295852, 0, 1.87583, 0) & (0.148402, 0, 1.92003, 0) \\ (-0.147926, \pm 0.256216, 1.87583, 0) & (-0.0742012, \pm 0.12852, 1.92003, 0) \end{array}$$

From the first two reflection triples, notice that $2.95691 \div 2 \approx 1.47845$ and $2.93512 \div 2 \approx 1.46756$ and that $\sqrt{3}(1.47845) \approx 2.56076$ and $\sqrt{3}(1.46756) \approx 2.54189$. This follows in part from the equation (2.11); for if we know $d = 0$, then, assuming $b \neq 0$, this equation reduces to $b^2 = 3a^2$. More specifically, if a critical value has $a \neq 0$, and $b = 0$, then the solution has the form

$$x(t) = a\cos\frac{\omega}{3}t + c\cos\omega t$$

But since the equation (2.2) is invariant under the time translation $\tau = t \pm 2\pi/\omega$, it follows that there are corresponding solutions

$$\begin{aligned} \tilde{x}\left(t \pm \frac{2\pi}{\omega}\right) &= a\cos\frac{\omega}{3}\left(t \pm \frac{2\pi}{\omega}\right) + c\cos\omega\left(t \pm \frac{2\pi}{\omega}\right) \\ &= a\left\{\cos\frac{\omega}{3}t\cos\frac{2\pi}{3} \mp \sin\frac{\omega}{3}t\sin\frac{2\pi}{3}\right\} + c\cos\omega t \\ &= -\frac{a}{2}\cos\frac{\omega}{3}t \mp \sqrt{3}\frac{a}{2}\sin\frac{\omega}{3}t + c\cos\omega t \end{aligned}$$

which gives rise to the reflection triples. Thus each reflection triple solution arises from one solution to the differential equation which is given as time translation according to which set of initial values determines the solution.

To find the subharmonic solutions, we solve the initial value problems

$$\ddot{x} + x - \frac{1}{6}x^3 = \frac{1}{3}\cos\frac{3}{5}t \quad x(0) = a + c, \quad \dot{x}(0) = \frac{\omega}{3}b$$

The first reflection triple leads to only unbounded solutions. The critical value $(2.95852, 0, -0.860894, 0)$ leads to an unbounded solution with $x(0) = a + c = 2.09601$, $\dot{x}(0) = 0$. The other two critical values yield unbounded solutions and no bounded solutions exist for nearby initial values. The second, third and fourth reflection triples behave similarly.

For the fifth and sixth reflection triples, both first critical values of the form $(a, 0, c, 0)$ lead to unbounded solutions for $x(0) = a + c$ and $\dot{x}(0) = 0$. However, each of the remaining critical values of the form $(a, \pm b, c, 0)$ lead to, by tweaking, the subharmonic of order 1/3 determined by $x(0) = 1.816$ and $\dot{x}(0) = 0$. The trajectory of the subharmonic is shown in figure 1.

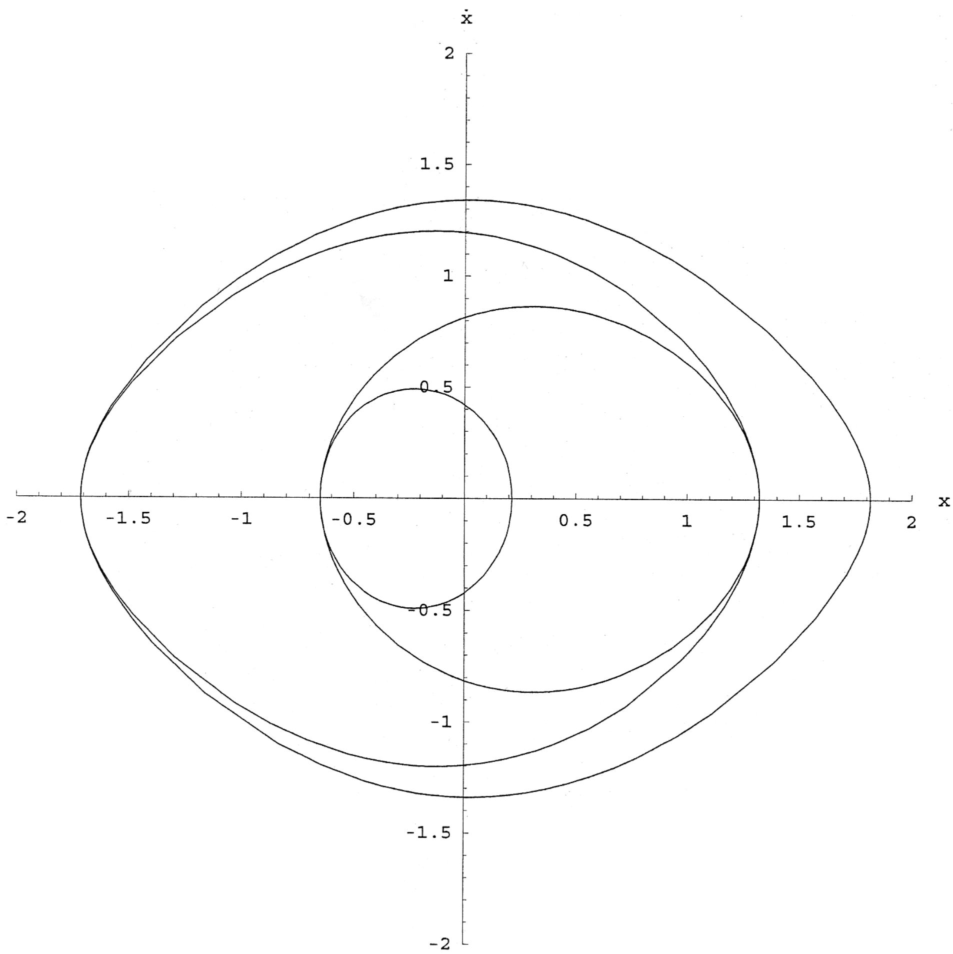

Figure 1. The trajectory of the subharmonic of order 1/3 for Examble 2.1 $x(0) = 1.816$, $\dot{x}(0) = 0$.

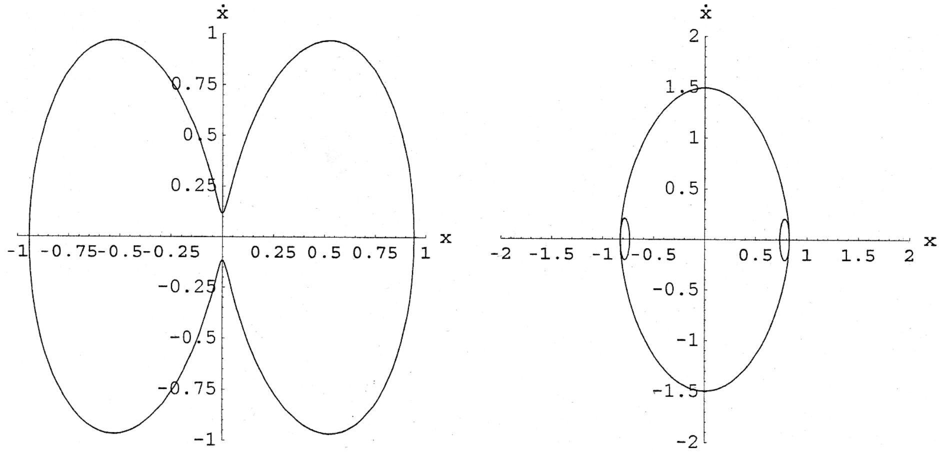

Figure 2. The unstable and subharmonic trajectories for Example 2.2.

Example 2.2. Find subharmonic solutions of order 1/3 using the harmonic balance method for the equation

$$\ddot{x} + x - \frac{1}{6}x^3 = \frac{3}{2}\cos 2.85t \tag{2.13}$$

This example was used in [3] to illustrate subharmonics of order 1/3. Applying the technique, we obtain 21 critical values, 14 of which are complex and discarded; all have $d = 0$. To find the subharmonic solutions, we solve the initial value problems having $x(0) = a + c$, and $\dot{x}(0) = (\omega/3)b$.

Of the seven real critical values $(a, b, c, 0)$, one critical value $(0, 0, 0.210436, 0)$ leads (by tweaking) to the harmonic solution. The other six critical values occur in two reflection triples. The first reflection triple is

$$(-0.7388768,\ 0,\ -0.204181, 0)$$

$$(0.3693841,\ \pm 0.639792, 0.204181, 0)$$

Tweaking these initial conditions slightly so that $x(0) = -0.9422, \dot{x}(0) = 0,$ produces the subharmonic trajectory shown in figure 2. This subharmonic is numerically unstable as to generate the solution accurately over long time intervals one has to increase the working precision to maintain accurate computations. The second reflection triple

$$(0.943773,\ 0,\ -0.208828, 0)$$

$$(-0.471636,\ \pm 0.816898,\ -0.208828, 0)$$

leads, upon tweaking the initial conditions, to a very nice stable subharmonic of order 1/3 with $x(0) = 0.734444$ and $\dot{x}(0) = 0$. This solution stays accurate and well behaved over the time interval $[0, 100]$; its trajectory over the interval $[0, 5\pi]$ is shown in figure 2. For more details on working precision and solving these types of equations numerically see [5] and [1].

3. Simplifying this approach

Since $d = 0$ for all the calculations in the previous section, it is natural to simplify the method by supposing the solution has the form

$$x(t) = a(t)\cos\frac{\omega}{n}t + b(t)\sin\frac{\omega}{n}t + c(t)\cos\omega t \tag{3.1}$$

Substituting into equation (2.2), simplifying, neglecting the harmonics of order different from $\omega/3$ and ω, and balancing coefficients, we have the coefficients of $\cos(\omega/3)t$, $\sin(\omega/3)t$, $\cos\omega t$, and $\sin\omega t$ giving rise to the equations respectively:

$$a\left(1 - \frac{\omega^2}{9}\right) + \frac{3\varepsilon}{4}a(a^2 + b^2) + \frac{3\varepsilon}{4}c(a^2 - b^2) + \frac{3\varepsilon}{2}ac^2 + \frac{2\omega}{3}\dot{b} + \ddot{a} = 0 \tag{3.2}$$

$$b\left(1 - \frac{\omega^2}{9}\right) + \frac{3\varepsilon}{4}b(a^2 + b^2) + \frac{3\varepsilon}{2}b(c^2 - ac) - \frac{2\omega}{3}\dot{a} + \ddot{b} = 0 \tag{3.3}$$

$$c(1 - \omega^2) + \frac{\varepsilon}{4}a(a^2 - 3b^2) + \frac{3\varepsilon}{2}c(a^2 + b^2) + \frac{3\varepsilon}{4}c^3 + \ddot{c} = F \tag{3.4}$$

$$\frac{\varepsilon}{4}b(3a^2 - b^2) - 2\omega\dot{c} = 0 \tag{3.5}$$

By assumption, the derivatives are all small and by neglecting them, we have the system of equations

$$a\left(1 - \frac{\omega^2}{9}\right) + \frac{3\varepsilon}{4}a(a^2 + b^2) + \frac{3\varepsilon}{4}c(a^2 - b^2) + \frac{3\varepsilon}{2}ac^2 = 0 \tag{3.6}$$

$$b\left(1 - \frac{\omega^2}{9}\right) + \frac{3\varepsilon}{4}b(a^2 + b^2) + \frac{3\varepsilon}{2}b(c^2 - ac) = 0 \tag{3.7}$$

$$c(1 - \omega^2) + \frac{\varepsilon}{4}a(a^2 - 3b^2) + \frac{3\varepsilon}{2}c(a^2 + b^2) + \frac{3\varepsilon}{4}c^3 = F \tag{3.8}$$

$$\frac{\varepsilon}{4}b(3a^2 - b^2) = 0 \tag{3.9}$$

These equations are precisely those that are obtained when substituting $d = 0$ into the equations (2.8) – (2.11).

Example 3.1. Find subharmonic solutions of order 1/3 using the simplified harmonic balance method for the equation

$$\ddot{x} + x - \frac{1}{5}x^3 = \frac{1}{5}\cos\frac{3}{7}t \tag{3.10}$$

We obtain 21 real critical values of the form (a, b, c). Three of these are of the form $(0, 0, c)$.

$$(0, 0, -2.44685) \quad (0, 0, 2.19906) \quad (0, 0, 0.247796)$$

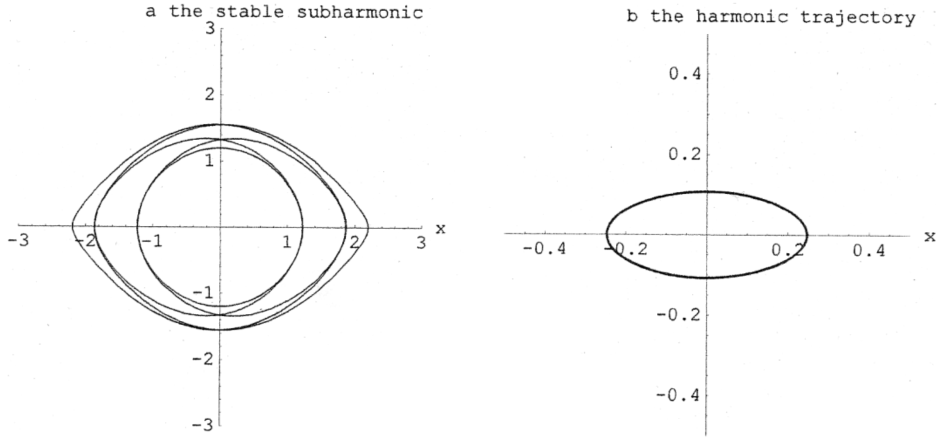

Figure 3. Subharmonic and harmonic trajectories for Examble 3.1.

Setting $x(0) = c$ and $\dot{x}(0) = 0$ and solving: the first one of these leads to an unbounded solution; the second leads to a very nice stable subharmonic of order 1/3 whose trajectory is shown in figure 3(a); the third leads to the harmonic solution whose trajectory is shown in figure 3(b).

The remaining critical values occur in six reflection triples. For each of these critical values, setting $x(0) = a + c$ and $\dot{x}(0) = (\omega/3)b$, leads to either a bounded solution obviously not a subharmonic of order 1/3 or to an unbounded solution.

Student problem 3.2. Since all the reflection triples are determined by a critical value having $b = 0$, determine if things simplify by applying the harmonic balance approach using the equation

$$x(t) = a(t)\cos\frac{\omega}{3}t + c(t)\cos\omega t$$

For example, apply this approach to the equation

$$\ddot{x} + x - \frac{1}{6}x^3 = \frac{1}{3}\cos\frac{3}{5}t$$

Repeat this for the other examples above. How does this approach compare with the others above?

For the sake of simplicity, we have considered only the undamped case, but damping in the form $k\dot{x}$ is not any more difficult.

Student problem 3.3. Investigate the existence of subharmonics of order 1/3 for the damped Duffing equation

$$\ddot{x} + k\dot{x} + x + \varepsilon x^3 = F\cos\omega t$$

For example, investigate

$$\ddot{x} + k\dot{x} + x - x^3/6 = \frac{1}{3}\cos\frac{3}{5}t$$

for various values of k; to begin with let $k = 0.2$ and $k = 0.002$.

Student problem 3.4. Investigate the existence subharmonic solutions of order 1/3 to the forced van der Pol equation

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = F\cos\omega t$$

4. The classical approach

Viewing the differential equation (2.2) as being 'weakly nonlinear', and since the particular solution of

$$\ddot{x} + \ x = F\cos\omega t$$

is

$$x(t) = \frac{F}{1-\omega^2}\cos\omega t$$

one can argue (see [3, Chapter 7]) that the harmonic term of the solution should have very nearly the form

$$-\frac{F}{8}\cos\omega t$$

when seeking a subharmonic of order 1/3 (there is other supporting evidence from the perturbation method that this approach is good when ω is close to 3). Thus it is reasonable, in this case, to try the approach of setting

$$x(t) = a(t)\cos\frac{\omega}{3}t + b(t)\sin\frac{\omega}{3}t - \frac{F}{8}\cos\omega t \tag{4.1}$$

Student problem 4.1. Compare the classical approach with the approaches used in Sections 2 and 3 by investigating the examples in those sections. Show that if ω is not close to 3, then the classical approach fails to produce subharmonics of order 1/3 while if ω is close to 3, then the approaches work equally well.

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