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TRANSFORMATION THEORY OF NON-LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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1. Introduction

Interest in the field of non-linear differential equations in bygone years centered around the dynamics of conservative systems, the problems arising mainly in celestial mechanics. Current interest is focused mainly on non-conservative systems. Nevertheless many of the advances made in the past are of the greatest relevance to the problems of current interest. Therefore we shall give an account here of certain of these past results relevant to current problems. We shall also adapt certain methods so as to get results for these current problems.

So far as the non-linear equations that we shall consider are concerned, the most challenging problem mathematically appears to be that of finding means of excluding the possibility of certain singular situations. We shall describe some of these singular possibilities. As yet there is no method available to indicate under what conditions they cannot arise.

Transformation theory as a method in differential equations is due to Poincaré. The type of transformation we shall be interested in here is that of the Euclidean plane into itself. Certain methods and results of Birkhoff will be shown to be of interest.

In connection with their work on second order non-linear equations, the non-linear terms of which have a small parameter as a coefficient, Kryloff and Bogoliuboff[1] make use of transformation theory, particularly the theory that has been developed in the study of curves on a torus by Poincaré and Denjoy.

The second order equation

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = e(t), \tag{1.0}$$

with $e(t)$ of period L, is of considerable interest in applied mathematics. By setting $\dot{x} = y$ this equation becomes

$$\dot{x} = y, \qquad \dot{y} = -f(x, y)y - g(x) + e(t). \tag{1.1}$$

The system (1.1) is a particular case of

$$\dot{x} = F(x, y, t), \qquad y = G(x, y, t), \tag{1.2}$$

where $F(x, y, t)$ and $G(x, y, t)$ are both periodic in t with period L. We shall assume that F and G are analytic in x, y, and t although much less would suffice for most of the discussion.

[1] N. M. KRYLOFF AND N. N. BOGOLIUBOFF, *Les Methodes de la Mécanique Non Linéaire Appliquées à la Théorie des Oscillations Stationnaires*, Monograph No. 8, Academy of Sciences of the Ukraines, 1934.

A system of the form (1.2) will be said to be of *class D, or to be a dissipative system for large displacements*, if there exists an R such that any solution $(x(t), y(t))$ of this system remains in the finite (x, y) plane as t increases and

$$\limsup_{t \to +\infty} x^2(t) + y^2(t) < R^2. \tag{1.3}$$

That is, any solution of a dissipative system of the form (1.2) eventually lies inside of a circle in the (x, y) plane with center at $(0, 0)$ and with radius depending only on the system. As might be expected from energy considerations, most systems which arise in practice are in class D.

This is a consequence of the following result: *Under rather general conditions*[2] *there exist simple closed curves in the (x, y) plane such that a solution $(x(t), y(t))$ of* (1.1) *can intersect any one of these curves only by crossing it from the domain exterior to the curve into the domain interior to the curve. Moreover through any point in the (x, y) plane sufficiently remote from the origin, there passes a curve with this property.*

Clearly if C_0 denote a simple closed curve of the type just described and if C_0 is sufficiently remote from the origin so that through every point exterior to it passes a curve of the type described above, then every solution of (1.1) starting in the domain exterior to C_0 must eventually cross into the interior of C_0 and stay there. Thus, under conditions generally met in practice, (1.1) is of class D.

It is well known that to a system such as (1.2) there corresponds a transformation of the (x, y) plane into itself. To see this let us consider the solution $(x(x_0, y_0, t), y(x_0, y_0, t))$ of (1.2) which when $t = t_0$ is at the point (x_0, y_0) of (x, y) plane. Let

$$x_n = x(y_0, t_0, t_0 + nL), \qquad y_n = y(x_0, y_0, t_0 + nL),$$

for any integer, n. Since F and G are of period L in t, it follows that

$$x_{n+m} = x(x_m, y_m, t_0 + nL), \qquad y_{n+m} = y(x_m, y_m, t_0 + nL). \tag{1.4}$$

If P_n denote the point (x_n, y_n), then we define a transformation T of the (x, y) plane into itself by $TP_0 = P_1$. By T^n is meant the transformation that takes P_0 into P_n. Clearly (1.4) is equivalent to $T^{n+m}P_0 = T^nP_m = T^nT^mP_0$. Moreover since solutions of (1.2) are continuous with respect to changes in initial values, T is continuous. If we now consider the solutions of (1.2) as curves in (x, y, t) space, it becomes clear that to study the behaviour of the solutions of (1.2) we have only to study the transformation T of the (x, y) plane into itself. In particular a fixed point under the transformation T^n corresponds to a periodic solution of (1.2) of period nL.

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[2] See N. Levinson, *On the Existence of Periodic Solutions for Second Order Differential Equations with a Forcing Term.* Journal of Math. and Physics, Vol. XXII (1943), p. 41, Theorem II.

2. Maximum finite invariant domain

If we consider the closed curve C_0, described above, it follows that under iterations of T all points exterior to C_0 are transformed into interior points. Moreover if $T^n C_0$ be denoted by C_n, it follows that C_1 lies in the domain interior to C_0. Thus C_2 lies in the interior of C_1, etc. Consider the closed domain interior to all C_n, $n \geqq 0$. Clearly this domain is finite and is invariant under T. Moreover it is obviously the maximum finite invariant domain. We shall now prove the existence of such a maximum finite invariant domain for every system, (1.2) of class D.

THEOREM I. *Every system* (1.2) *of class D, possesses a closed domain, I, invariant under T. The complement of I is an open simply connected domain if the point at infinity is adjoined to the* (x, y) *plane. Under iterations of T every point in the complement of I except the point at infinity tends to I. I is the maximum finite invariant domain under T.*

THEOREM II. *Every system* (1.2) *of class D has at least one fixed point under the transformation T. Thus* (1.2) *has at least one solution of period L.*

PROOF OF THEOREM I. By the property, (1.3), of systems of class D and by the continuity of solutions of (1.2) with respect to changes in initial values, there exists an $N > 0$ such that if R_0 represents the circle, $x^2 + y^2 < R^2$, then the open simply connected domain $T^N R_0 = R_N$ is interior to R_0 and remains interior to R_0 in further applications of T. From $R_N \subset R_0$ it follows at once that $T^{2N} R_0 = R_{2N}$ is interior to R_N, etc. Let the closed domain interior to all R_{kN}, where $k = 0, 1, 2, \cdots$, be denoted by I.

Let Q_0 be an open simply connected domain in the interior of which is R_0 and R_1. Then by the property of systems of class D there must exist an n_0 such that $T^{n_0 N} Q_0 \subset R_0$. Thus we have $T^{n_0 N} Q_0 \subset R_0 \subset Q_0$. Applying T^{Nk} we have

$$R_{Nk} \subset Q_{Nk} \subset R_{(-n_0+k)N} . \tag{2.0}$$

Since $R_{Nk} \rightarrow I$ as $k \rightarrow \infty$, it follows that as $k \rightarrow \infty$

$$Q_{Nk} \rightarrow I \tag{2.1}$$

From $R_1 \subset Q_0$, it follows that $R_0 \subset Q_{-1}$. Proceeding with Q_{-1} as we have above with Q_0 we find that analogous to (2.0) we have

$$R_{Nk} \subset Q_{Nk-1} \subset R_{(-n_1+k)N} .$$

Thus $Q_{Nk-1} \rightarrow I$ as $k \rightarrow \infty$, or applying T we find $Q_{Nk} \rightarrow TI$. Combining this with (2.1) we have $TI = I$. Thus I is invariant under T. The other statements of Theorem I follow immediately and thus we have demonstrated the existence of the maximum finite invariant domain under T.

PROOF OF THEOREM II. As in the proof of Theorem I let the open domain, $x^2 + y^2 < R^2$, be denoted by R_0. Clearly $I \subset R_0$ and $R_{N+k} \subset R_0$, $k \geqq 0$. Consider the open set $R_0 + R_1 + \cdots + R_{N-1}$. A point P. is said to be occluded[3]

[3] G. D. BIRKHOFF. *Surface Transformations and Their Dynamical Applications.* Acta Mathematica, Vol. 43, 1922, see p. 80.

by this open set if a simple closed curve can be drawn entirely within this set and enclosing P and I. Let K_0 be the set of points occluded by this open set. Clearly K_0 is simply connected. Let $TK_0 = K_1$. Then K_1 is the set occluded by $R_1 + R_2 + \cdots + R_N$. Since $R_0 \supset R_N$, it follows that $K_1 \subset K_0$. Since R_0 is a circle and T analytic, it follows that the boundary of K_0 is everywhere accessible. (In fact it is made up of analytic curve segments.) Thus by the Brouwer fixed point theorem, there exists a point in K_0 fixed under T. Clearly this fixed point must be contained in I.

3. Invariant closed curves and rotation numbers

As with Birkhoff[4] we shall mean by a *closed curve* the boundary of a simply connected open continuum in the finite plane. The plane is regarded as completed by the adjunction of the point at infinity. A closed curve invariant under T is called an invariant closed curve. Clearly the boundary of the complement of I, the maximum finite invariant domain, is an invariant closed curve. In addition there may be contained in I any number of invariant closed curves. In the simplest case where the right members of (1.2) actually do not involve t then any value of L is a period. In this case the invariant closed curves would be the limit cycles of Poincaré.

Let us denote an invariant closed curve by C. First let us consider the case where C is a simple closed curve of length λ. Consider all the solutions S_C of (1.2) starting on C when $t = t_0$. For $t_0 \leqq t \leqq t_0 + L$, these solutions, S_C, form a surface in (x, y, t) space which is bounded by C when $t = t_0$ and $t = t_0 + L$. Thus this surface can be mapped on the closed torus. The problem of the solutions, S_C, of (1.2) which emanate from C is reduced to the problem of the solution of a differential equation on a torus.

Clearly the equation of the surface formed by the solutions of (1.2) emanating from C can be written as $x = f(\theta, t)$ and $y = g(\theta, t)$ where f and g are of period 1 in θ and of period L in t. In fact f and g can be so chosen that, for any fixed $t = t_1$, θ is proportional to arc length on the curve, $x = f(\theta, t_1)$, $y = g(\theta, t_1)$. Differentiating x and y we find

$$\frac{dx}{dt} = \frac{\partial f}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial f}{\partial t}, \qquad \frac{dy}{dt} = \frac{\partial g}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial g}{\partial t}.$$

Using (1.2) and multiplying the first of the above equations by $\partial f/\partial\theta$ and the second by $\partial g/\partial\theta$ and adding we get

$$(3.0) \qquad \frac{d\theta}{dt} = \frac{1}{\left(\frac{\partial f}{\partial \theta}\right)^2 + \left(\frac{\partial g}{\partial \theta}\right)^2}\left[F(f, g, t)\frac{\partial f}{\partial \theta} + G(f, g, t)\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial t}\frac{\partial f}{\partial \theta} - \frac{\partial g}{\partial t}\frac{\partial g}{\partial \theta}\right].$$

Let λ_1 denote arc length on the curve $x = f(\theta, t_1)$, $y = g(\theta, t_1)$. Then since θ is proportional to arc length on each such closed curve we have

$$0 < \operatorname{Min} \lambda_1^2 \leqq \left(\frac{\partial f}{\partial \theta}\right)^2 + \left(\frac{\partial g}{\partial \theta}\right)^2 \leqq \operatorname{Max} \lambda_1^2 < \infty.$$

[4] Loc. cit. p. 79.

Thus the right member of (3.0) is bounded and is periodic of period 1 in θ and L in t. Its solutions as already stated are best represented geometrically on a torus.

Associated with such a differential equation, or with a family of curves on the torus, or with a transformation of a simple closed curve into itself is a rotation number, ρ.[5] This number is simply the average advance of θ, for an advance of $t = L$.

If ρ is rational and of the form p/q where p and q have no common factors, then (3.0) has solutions of period qL in t. Any non-periodic solution must tend toward such a periodic solution as $t \to \infty$. In this situation (1.2) has q^{th} subharmonics among its solutions.

If ρ is irrational there are two possibilities. One of these possibilities is termed the singular case. In case the right member of (3.0) is reasonably well behaved, for instance twice differentiable with respect to θ, the singular case is ruled out.[6] Under these latter circumstances the solutions of (3.0) are of the form

$$\theta(t) = \frac{t\rho}{L} + c + H\left(\frac{t}{L}, \frac{t\rho}{L} + c\right)$$

where[7] $H(u, v)$ is periodic in u and v of period 1 and c is an arbitrary constant. Clearly x and y are almost-periodic functions of t in this case. In fact they are of the form $h\left(\frac{t}{L}, \frac{t\rho}{L}\right)$ where $h(u, v)$ is of period 1 in u and v. In the case of certain systems which depart only slightly from linear Kryloff and Bogoliuboff[8] show that this is the case.

In the singular case there are solutions which trace out curves that together with their limit points intersect any meridian of the torus in a perfect, no-where-dense set of points.

There are cases of (1.2) of considerable practical interest where in effect, aside from a fixed point, there is only one invariant closed curve, C. If C were a reasonably well behaved curve the results just given would afford a complete qualitative solution of such cases. Even where there are several invariant closed curves the qualitative situation would be well in hand if these curves were known to be well behaved. Actually there is no indication that C need be a reasonable curve.

In fact Birkhoff[9] has given an example of an *analytic* transformation of the

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[5] POINCARÉ. *Collected Works.* Vol. I, p. 137, Chapter XV.

[6] A. DENJOY. *Sur les courbes definies par les equations differentielles a la surface du tore.* Journal d. Math. Vol. 11 (1932), p. 333.

[7] P. BOHL. *Über die Hinsichtlich der Unabhängigen und Abhängigen Variabeln Periodische Differential Gleichungen. Erster Ordnung.* Acta Math. Vol. 40 (1916), p. 321.

[8] Loc. cit.

[9] G. BIRKHOFF. *Sur quelques Courbes Fermées Remarquables.* Bull. de la Soc. Math. de France. Vol. 60, 1932, p. 1.

plane into itself which leaves a closed "curve," J, invariant. J divides the plane into two invariant open simply connected continua, S_i and S_e where to S_e is adjoined the point at infinity. Every point of J is a limit point of S_i or of S_e or of both. Those points of J accessible from S_e we denote by J_e, and from S_i by J_i. Associated with the transformation which carries J_e and J_i into themselves are rotation numbers ρ_e and ρ_i. What Birkhoff showed is that ρ_e need not equal ρ_i. This indicates how complicated J can be even though T is analytic. Unfortunately there is at present no basis for excluding curves of this type from the invariant closed curves of (1.2).

4. Fixed points

The solutions of (1.2) can be regarded geometrically as a family of curves in (x, y, t) space which, due to the periodicity of $F(x, y, t)$ and $G(x, y, t)$ in t need only be studied between the two planes $t = t_0$ and $t = t_0 + L$. Any point, (x_0, y_0), in the (x, y) plane when $t = t_0$ is carried into a point (x_1, y_1) in the (x, y) plane when $t = t_0 + L$ by the solution of (1.2) emanating from (x_0, y_0). This as we have seen defines the transformation T.

Let us consider the change in area in the (x, y) plane under the transformation T. Let $(x(x_0, y_0, t))$, $(y(x_0, y_0, t))$ denote the solution of (1.2) at (x_0, y_0) when $t = t_0$. Let the Jacobian of $(x(x_0, y_0, t), y(x_0, y_0, t))$ with respect to (x_0, y_0) be $\Delta(t)$. That is

$$\Delta(t) = \begin{vmatrix} \dfrac{\partial x}{\partial x_0} & \dfrac{\partial x}{\partial y_0} \\ \dfrac{\partial y}{\partial y_0} & \dfrac{\partial y}{\partial y_0} \end{vmatrix}. \tag{4.0}$$

Differentiating (4.0) and using (1.2) we find

$$\frac{d\Delta}{dt} = \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\right)\Delta.$$

Integrating and using $\Delta(t_0) = 1$, we find that $\Delta(t_0 + L)$, the Jacobian of the transformation T for the point (x_0, y_0) is given by

$$J\left(\frac{x_1, y_1}{x_0, y_0}\right) = \begin{vmatrix} \dfrac{\partial x_1}{\partial x_0} & \dfrac{\partial x_1}{\partial x_0} \\ \dfrac{\partial y_1}{\partial x_0} & \dfrac{\partial y_1}{\partial y_0} \end{vmatrix} = \exp\left[\int_{t_0}^{t_0+L} \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\right) dt\right]. \tag{4.1}$$

Thus the element of area $dx_0\, dy_0$ is carried by T into $J((x_1, y_1)/x_0, y_0))\, dx_0\, dy_0$.

Now let us study the transformation T in the neighborhood of a fixed point of the transformation. This will be facilitated by making a transformation of coordinates which takes the fixed point into the origin. That is let $(\bar{x}(t), \bar{y}(t))$ be a solution of (1.2) of period L. Then the point $(\bar{x}(t_0), \bar{y}(t_0))$ is a fixed point under T. Let P_0 be the point $(\bar{x}(t_0) + u_0, \bar{y}(t_0) + v_0)$ in the (x, y) plane. Denote TP_0 by P_1 with coordinates $(\bar{x}(t_0) + u_1, \bar{y}(t_0) + v_1)$. It is well known that the solution $(x(t), y(t))$ starting at P_0 when $t = t_0$ can be represented by a power series in u_0 and v_0 with coefficients functions of t. Thus

$$x(t) = \bar{x}(t) + c_1(t)u_0 + c_2(t)v_0 + c_3(t)u_0^2 + c_4(t)u_0v_0 + \cdots$$

and similarly for $y(t)$. In particular setting $t = t_0 + L$ we get

$$u_1 = au_0 + bv_0 + \cdots \qquad (4.2)$$
$$v_1 = cu_0 + dv_0 + \cdots$$

where the terms not explicitly given in the right members are of degree two or higher in u_0 and v_0.

If we denote $(\bar{x}(t_0) + u_0, \bar{y}(t_0) + v_0)$ by (x_0, y_0) and $(\bar{x}(t_0) + u_1, \bar{y}(t_0) + v_1)$ by (x_1, y_1) then clearly

$$J\left(\frac{x_1, y_1}{x_0, y_0}\right) = J\left(\frac{u_1, v_1}{u_0, v_0}\right).$$

Using (4.1) and (4.2) we have

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \exp\left[\int_{t_0}^{t_0+L}\left(\frac{\partial F(\bar{x}, \bar{y}, t)}{\partial x} + \frac{\partial G(\bar{x}, \bar{y}, t)}{\partial y}\right)dt\right]. \qquad (4.3)$$

Thus $ad - bc > 0$.

The transformation (4.2) has been very much studied. For small values of u_0 and v_0, its character is determined by its linear terms. That is, the transformation can be characterized by the roots of the equation

$$(a - \rho)(d - \rho) - bc = 0. \qquad (4.4)$$

If one or both roots of (4.4) is 1, this means that $(\bar{x}(t_0), \bar{y}(t_0))$ is a multiple fixed point of T. A slight change in the parameters of the equation (1.2) will ordinarily separate these points and thus the situation requires no special treatment but can be studied as a limiting case of simple fixed points. We shall therefore assume the roots of (4.4) to be different from 1 in the discussion that follows. We shall also assume the roots different from -1, which indicates multiple fixed points under T^2.

Let us denote the roots of (4.4) by ρ_1 and ρ_2. Since $ad - bc > 0$, it follows from (4.4) that $\rho_1\rho_2 > 0$. We now characterize the periodic solution, $(\bar{x}(t), \bar{y}(t))$, or what is the same, the fixed point, $(\bar{x}(t_0), \bar{y}(t_0))$, as *completely stable* if $|\rho_1| < 1$, $|\rho_2| < 1$. In this case the point (u_1, v_1) is nearer to $(0, 0)$ than (u_0, v_0). Iterations of T bring (u_0, v_0) nearer and nearer to $(0, 0)$. That is, all solutions of (1.2) near $(\bar{x}(t), \bar{y}(t))$ move closer and closer to $(\bar{x}(t), \bar{y}(t))$ at $t \to \infty$.

Similarly if $|\rho_1| > 1$, $|\rho_2| > 1$, then we shall term the solution $(\bar{x}(t), \bar{y}(t))$ *completely unstable.*

If $\rho_1 > 1 > \rho_2 > 0$, then we shall call the solution $(\bar{x}(t), \bar{y}(t))$, *directly unstable.*

If $\rho_2 < -1 < \rho_1 < 0$, then the solution $(\bar{x}(t), \bar{y}(t))$, will be called *inversely unstable.*

The only other possibility is the case $| \rho_1 | = | \rho_2 | = 1$. In this case $\rho_2 = \bar{\rho}_1$. This is a very important case in conservative systems. In this case stability can be investigated only by considering higher powers of u_0 and v_0 in (4.2). Since in this case stability is not determined by the linear part of the transformation we shall call it the *undetermined* case.

In the case of directly or inversely unstable points there is a curve invariant under T passing through the fixed point[10]. Points on this invariant curve move farther and farther away from the fixed point under iterations of T. There is another invariant curve passing through the fixed point, points of which move toward the fixed point under iterations of T. Aside from the case where ρ_1 or ρ_2 are ± 1, the above classification of stability is exhaustive.

5. The Fixed Point Equation

A fruitful tool in the investigation of the transformation, T, involves the use of the vector field in the (x, y) plane where from each point, P_0, there emanates a vector directed toward and terminating in $TP_0 = P_1$. The application we shall make here is a special case of results of Birkhoff.[11]

Clearly the number of revolutions made by P_0P_1 as P_0 traces out a closed curve in the (x, y) plane must be an integer since P_0 returns to its starting position. This integer is called the index of the curve. The curve, we assume does not pass through any fixed points of T.

We shall first show that the closed curve which is the boundary of K_0, the continuum introduced in the proof of Theorem II, has index 1. Clearly this curve is free of fixed points. Since $TK_0 \subset K_0$, the index would certainly be 1 if K_0 were the interior of a circle. For in this case the vector $\overrightarrow{P_0P_1}$ must make an angle of less than 90° with the radius from the center of the circle to P_0, and the radius makes one revolution when P_0 does.

We now consider the general case of K_0 and gradually deform it into the interior of a circle. It follows that since the index of the boundary changes continuously with the deformation and since it is an integer, it must be 1.

(We assume in what follows that there is no curve in the (x, y) plane each point of which is invariant under T.) We enclose each fixed point by a small circle. We join the circles together by curves so that if the curves be regarded as cuts, the curves and circles together form a single closed curve. Shrink the boundary of K_0 down to this closed curve. Since the index will change continuously, it must remain 1 for the single closed curve made up of the circles and several curves joining them. In determining the index of the single closed curve, each curve segment, or cut, joining a pair of circles is traversed first in one direction and then the other. Thus the net effect of these segments on the index is zero.

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[10] J. Hadamard, *Sur l'iteration et les solutions asymptotiques des equations différentielles.* Bull. de la Soc. math. de France, Vol. 26, 1901. Actually there is an invariant curve if ρ_1 and ρ_2 are real and unequal.

[11] *Dynamical systems with two degrees of freedom.* Trans. of the Am. Math. Soc., vol. 18 (1917), p. 287. The method is due to Poincaré, Collected Works, Vol. 1, p. 25.

Therefore the sum of the indices of the several circles enclosing the fixed points of T must be 1.

Let P_0 be a point on the circumference of such a small circle. Let ϕ be the angle P_0P_1 makes with the x axis, or what is the same thing, the u axis. Then

$$\phi = \tan^{-1}\frac{y_1 - y_0}{x_1 - x_0} = \tan^{-1}\frac{v_1 - v_0}{v_1 - v_0}$$

or using (4.2)

$$\phi = \tan^{-1}\frac{cu_0 + (d-1)v_0 + \cdots}{(a-1)u_0 + bv_0 + \cdots}. \tag{5.0}$$

If the radius of the small circle is r and if the angle the radius to P_0 makes with the u axis is θ, then using (5.0) we get

$$\phi = \tan^{-1}\frac{c\cos\theta + (d-1)\sin\theta + r[\cdots]}{(a-1)\cos\theta + b\sin\theta + r[\cdots]}. \tag{5.1}$$

From (5.1)

$$\frac{d\phi}{d\theta} = \frac{(a-1)(d-1) - bc + r[\cdots]}{[(a-1)\cos\theta + b\sin\theta + \cdots]^2 + [\cos\theta + (d-1)\sin\theta + \cdots]^2}. \tag{5.2}$$

Since we have excluded the case where (4.4) has roots equal to 1,

$$(a-1)(d-1) - bc \neq 0.$$

Thus it is clear from (5.2) that $d\phi/d\theta$ is of fixed sign which it takes from $(a-1)(d-1) - bc$. Also the denominator of the right member of (5.1) vanishes twice as θ goes from 0 to 2π. Since ϕ is either monotonically increasing or decreasing, this means that ϕ changes by 2π or -2π according to the sign of $(a-1)(d-1) - bc$ when P_0 traverses the circle. In other words each circle enclosing a fixed point is of index equal to $[(a-1)(d-1) - bc]$. A consideration of the relationship of this sign to the location of the roots of $(a-\rho)(d-\rho) - bc$ reveals immediately that in the cases completely stable, completely unstable, inversely unstable, and undetermined, $(a-1)(d-1) - bc > 0$. In the case directly unstable, $(a-1)(d-1) - bc < 0$. Since the sum of the indices is 1, we have

THEOREM III. *If C denote the number of completely stable, completely unstable, and undetermined points of T, if I denote the number of inversely unstable points, and D the number of directly unstable points we have*

$$C + I = 1 + D. \tag{5.3}$$

Although (5.3) was proved for T, the proof would obviously apply to the transformation T^n where n is any integer. Thus (5.3) applies to the solutions of (1.2) of period nL.

6. A Theorem on Subharmonics

A solution of (1.2) is called subharmonic if its least period is equal to qL where $q > 1$ is an integer. We shall now show, for any q, that not all the subharmonic solutions of least period of qL can be stable. To be more specific we shall show

THEOREM IV. *If (1.2) is of class D and possesses N subharmonic solutions of least period qL, then $N = 2kq$, where k is an integer. Moreover kq, that is one half, of the solutions are directly unstable.*

Let $A(m)$ represent the number of completely stable, completely unstable, inversely unstable, and undetermined solutions of (1.2) of *least* period mL, and let $D(m)$ represent the number of directly unstable solutions of *least* period mL.

Clearly if m_1 is a factor of m, then the solutions of (1.2) of period m_1L appear among those of period mL. In other words among the fixed points of T^m are those of T^{m_1}. Let

$$m = p_1^{n_1} p_2^{n_2} \cdots p_n^{n_n}$$

where p_j, $j = 1, 2, \cdots, n$, are prime numbers. Then the total number of solutions of (1.2) of period mL is given by

$$\sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \cdots \sum_{j_n=0}^{n_n} [A(p_1^{j_1} p_2^{j_2} \cdots p_n^{j_n}) + D(p_1^{j_1} p_2^{j_2} \cdots p_n^{j_n})].$$

Applying (5.3) to the transformation T^m we find

$$\sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \cdots \sum_{j_n=0}^{n_n} [A(p_1^{j_1} p_2^{j_2} \cdots p_n^{j_n}) - D(p_1^{j_1} p_2^{j_2} \cdots p_n^{j_n})] = 1. \tag{6.0}$$

If we now apply (6.0) to the case where $m = p_1^{n_1-1} p_2^{n_2} \cdots p_n^{n_n}$ and subtract the equation so obtained from (6.0), we get

$$\sum_{j_2=0}^{n_2} \cdots \sum_{j_n=0}^{n_n} [A(p_1^{n_1} p_2^{j_2} \cdots p_n^{j_n}) - D(p_1^{n_1} p_2^{j_2} \cdots p_n^{j_n})] = 0. \tag{6.1}$$

If we consider (6.1) for $m = p_1^{n_1} p_2^{n_2-1} p_3^{n_3} \cdots p_n^{n_n}$ and subtract the result so obtained from (6.1) we get

$$\sum_{j_3=0}^{n_3} \cdots \sum_{j_n=0}^{n_n} [A(p_1^{n_1} p_2^{n_2} p_3^{j_3} \cdots p_n^{j_n}) - D(p_1^{n_1} p_2^{n_2} p_3^{j_3} \cdots p_n^{j_n})] = 0. \tag{6.2}$$

Proceeding in this way we finally get

$$A(p_1^{n_1} p_2^{n_2} \cdots p_n^{n_n}) = D(p_1^{n_1} p_2^{n_2} \cdots p_n^{n_n}).$$

In other words

$$A(m) = D(m). \tag{6.3}$$

Let $D(q)$ be the number of directly unstable fixed points under T^q but not fixed for T^n where $n > 0$ is less than q. Consider such a fixed point P. Then P, TP, T^2P, $\cdots$ $T^{q-1}P$, are all distinct. Thus the fixed points under consideration fall into mutually exclusive sets of q points. That is $D(q) = kq$ where k is an integer. Now from (6.3) it follows that $A(q) = D(q)$. Thus the total number of points fixed under T^q and not T^n, $0 < n < q$, is $D(q) + A(q) = 2kq$. This completes the proof of the theorem.

7. Some Examples

The simplest possible maximum finite invariant domain is that consisting of a single point. This is actually the case for the following two equations[12], (7.0) and (7.1):

$$\ddot{x} + f(x)\dot{x} + x = e(t) \tag{7.0}$$

where $e(t)$ is periodic, $xf(x) > 0$ for $x \neq 0$, and $\int^{\infty} f(x)\,dx = \infty$;

$$\ddot{x} + F(\dot{x}) + x = e(t) \tag{7.1}$$

where $F(y) > 0$ for $y \neq 0$ and $F(y) \to \infty$ as $y \to \infty$.

The differential equation

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = e(t),$$

under the conditions enumerated in the paper referred to in connection with this equation in §1, and with the further condition

$$f(x, y) + \frac{\partial f}{\partial y} y > 0, \qquad (y \neq 0, \quad x \neq 0), \tag{7.2}$$

has its maximum finite invariant domain of zero area. To see this we note that the left member of (7.2) is $-(\partial F/\partial x + \partial G/\partial y)$ in our general notation. Thus by (4.1) the Jacobian of T under these conditions is always less than 1. Using the notation in the proof of Theorem II we note that I is the inner limiting set of the domains enclosed by the curves K_n. Denoting by A_0 the area bounded by K_0,

$$\text{area of } I = \lim_{n\to\infty} \iint_{A_0} dx_0\, dy_0 \exp\left[-\int_{t_0}^{t_0+nL} \left(f(x, \dot{x}) + \dot{x}\frac{\partial f}{\partial x}\right) dt\right]. \tag{7.3}$$

But by (7.2), for all (x_0, y_0) in K_0 there exists a $\delta > 0$ such that

$$\int_{t_0}^{t_0+L} \left(f + x\frac{\partial f}{\partial x}\right) dt \geqq \delta.$$

Thus (7.3) becomes

$$\text{area of } I \leqq \lim_{n\to\infty} \iint_{A_0} dx_0\, dy_0\, e^{-n\delta} = 0.$$

[12] N. Levinson. *On a Non-Linear Differential Equation of the Second Order.* Journal of Math. and Physics, Vol. 22 (1943), p. 181.

The statement that I is of zero area clearly implies that every point of I is a limit point of the open continuum exterior to I. This case includes the important case of

$$\ddot{x} + \epsilon\dot{x} + g(x) = e(t), \qquad \epsilon > 0.$$

Being of zero areas, I is of a comparatively simple character and a number of special results can be given. For the present we point out a few simple possible types of I of zero area. The case where I is a single point, which arises for instance with equations (7.0) or (7.1), signifies that there exists one periodic solution of period L toward which all other solutions tend.

The case where I is a segment of a curve with three fixed points is of considerable interest. Let I consist of the curve segment ABC where A and C are the end points of the segment and A, B, and C are all fixed under T. A and C are completely stable fixed points, and B is directly unstable. ABC is shown in Fig. 1.

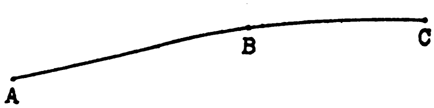

FIG. 1

In case parameters in the differential equation are varied, ABC will change its shape, but we shall suppose ABC remains topologically equivalent to a line segment. This turns out to be the case in many situations of practical importance. What does happen though, very often, is that as the parameters in the differential equation change B will move up gradually to the point C, and will actually come into coincidence with C. Then as the parameters are varied further, I will consist merely of the point A.

If we consider the periodic solution corresponding to the fixed point C, what we find is that this solution remains stable until B comes into coincidence with C, whereupon the only stable solution is the one corresponding to A. This situation is well known in practice as the "jump" phenomenon whereby a non-linear system may with a small change in parameter jump from one steady state to another.

Another I of interest is shown in Fig. 2. Here 0 is the only fixed point under T and is completely stable. We have $TA = B$, $TB = C$, $TC = A$ and similarly $T^3D = T^2E = TF = D$. That is A, B, C, D, E, and F are fixed under T^3. Moreover under T^3, A, B, and C are completely stable and D, E, and F directly unstable. In this case the rotation number of I is $2\pi/3$. By Theorem IV we see that this case corresponds to the simplest I for which we can have a subharmonic of period $3L$.

Another I of interest is shown in Fig. 3. Here the portion of I from O through D continues on by wrapping itself an infinite number of times around the curve segment containing A. The same is true of the two other branches of I going through E and F. As before $TA = B$, $TB = C$, $TC = A$ and similarly for D, E, and F. We also have $TG = H$, $TH = I$, $TI = J$, $TJ = K$, $TK = L$, and $TL = G$. That is, these points are fixed under T^6. Thus in this case A, B, and C are inversely unstable under T^3 and D, E, and F are directly unstable under T^3. Here although the rotation number is $2\pi/3$, we have no stable third

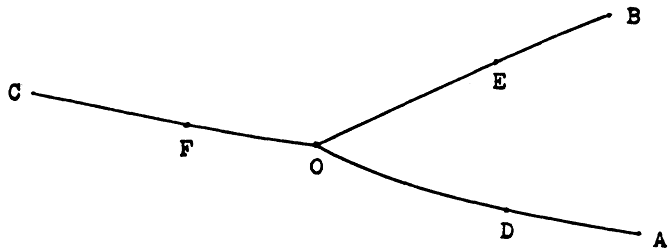

FIG. 2

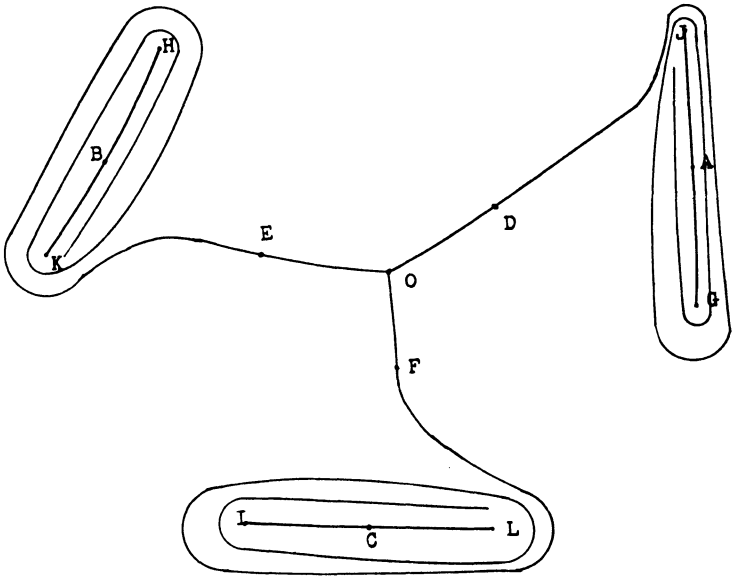

FIG. 3

order subharmonics. However corresponding to G we will have a stable subharmonic of order 6.

Another case, a singular case which we cannot rule out as a possibility, has an I which has only one fixed point, 0, under T or any iterate of T. Topologically I is equivalent to certain radii of a circle with 0 as its center. These radii emanate from 0 and terminate in a perfect no-where dense set of points on the circumference of this circle. A solution corresponding to a point of this perfect set is of the type Birkhoff calls discontinuous recurrent. It would be very desirable to have a criterion for ruling out this type of I.

As a final example let us take the van der Pol equation with a forcing term

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = e(t),$$

with μ large. In this case I is not of zero area. If $e(t)$ is not too large, and if the period of $e(t)$ is likewise not too large, there is only one fixed point, 0, under T. Moreover 0 is completely unstable. Furthermore if we consider a circle, S_0, of small radius about 0, then the open simply connected domain, $S_0 + TS_0 + T^2S_0 + \cdots$, is invariant under T. I minus this open domain is a "curve" every point of which is a limit point of the open domain exterior to the curve or the open domain interiot to the curve. As we have already stated, we can unfortunately not say anything about the analytic character of such a curve. This curve could conceivably be as bad as the curve, J, in the example of Birkhoff already cited.

8. Completely stable invariant closed curve

A "curve," J, is an invariant completely stable closed curve under T if (a) it divides the plane into two open simply connected invariant continua, S_i and S_e where to S_e is adjoined the point at infinity; (b) every point of J is a limit point of S_i or of S_e or of both; and (c) for any small ϵ there exists an open continuum, $0(\epsilon)$ containing J and with distance from J less than ϵ and such that

$$\lim_{n\to\infty} T^n(0(\epsilon)) = J.$$

A curve is said to be an invariant completely unstable closed curve if it is an invariant completely stable closed curve under T^{-1}. Using occluded sets much as in the proof of Theorem II, it is easy to show that an invariant completely stable closed curve is contained in an open set O_0, arbitrarily close to J, such that $O_0 \supset TO_0 \supset TO_0^2$ etc. Moreover O_0 can be chosen so that its exterior and interior boundaries are simple closed curves.

The curve, J, considered in the last example of the preceding section, is clearly an invariant completely stable closed curve.

We shall now prove

THEOREM V. *J is an invariant completely stable (or unstable) closed curve. If J contains N fixed points under T^q, $q \geqq 1$, but not fixed under T^j, $0 < j < q$, then $N = 2kq$ where k is an integer. Moreover kq of the fixed points are directly unstable.*

First consider the open simply connected domain consisting of O_0, described above, and all points in S_i, the interior of J. Let $A_0(q)$ be the number of completely stable, completely unstable, inversely unstable and undetermined points invariant under T^q, and no smaller power of T, and contained in the open simply connected domain under consideration. Similarly $D_0(q)$ is the number of directly unstable points. Then using exactly the same procedure as in the proofs of Theorem III and IV, we have

$$\begin{aligned} A_0(1) &= 1 + D_0(1) \\ A_0(q) &= D_0(q), \qquad q > 1. \end{aligned} \tag{8.0}$$

Consider next the open simply connected domain bounded by the interior boundary of O_0. Call the invariant points for this domain $A_1(q)$ and $D_1(q)$, all these points being fixed under T^q but not under any smaller power of T. Now applying T^{-q} we get as before

$$\begin{aligned} A_1(1) &= 1 + D_1(1) \\ A_1(q) &= D_1(q), \qquad q > 1. \end{aligned} \tag{8.1}$$

Subtracting (8.1) from (8.0) we have for any q

$$A(q) = D(q)$$

where $A(q)$ and $D(q)$ are the invariant points of T^q, but not T_k, $0 < k < q$, contained in J. This result, as with (5.3) in the proof of Theorem IV, leads to statement of Theorem V.

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