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PERIODIC SOLUTIONS OF $x'' + cx' + g(x) = Ef(t)$ UNDER VARIATION OF CERTAIN PARAMETERS*†

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1. Introduction. This paper is a study of periodic solutions of the differential equation

$$x'' + cx' + g(x) = Ef(t) \qquad (' = d/dt) \tag{1.1}$$

where $f(t)$ is periodic of period T. The variables x and t are real, and it is always assumed that $g(x)$ and $f(t)$ have sufficient continuity properties so that a unique solution of (1.1) exists for any initial conditions, and that this solution can be continued for all t. We are particularly concerned with the existence, appearance, and disappearance of periodic solutions of (1.1) as the parameters c and E vary.

Equations of the type (1.1) appear very often in applications. Duffing's equation where $g(x) = \alpha x + \beta x^3$ is an important case which has been studied extensively (cf. [5] and [11]). Other important cases arising in the study of nonlinear electrical circuits containing a saturable-core inductance are described in [5]. In this last case the function $g(x)$ is the nonlinear characteristic of the inductance. If i is the current and φ is the resulting magnetic flux, then $i = g(\varphi)$.

In studying periodic solutions of (1.1) it is helpful to use the phase plane, with coordinates x and $y = x'$. The equation (1.1) then becomes the system

$$x' = y, \qquad y' = -cy - g(x) + Ef(t). \tag{1.2}$$

Consider the more general system

$$x' = F(t, x, y), \qquad y' = G(t, x, y), \tag{1.3}$$

where F and G have period T in t. Assume that F and G have sufficient continuity properties so that for any initial conditions there will exist a unique solution of (1.3) which can be continued for all t. Let $x(t, \xi, \eta)$, $y(t, \xi, \eta)$ be that solution of (1.3) for which $x(0, \xi, \eta) = \xi$, $y(0, \xi, \eta) = \eta$. Let $\xi' = x(T, \xi, \eta)$, $\eta' = y(T, \xi, \eta)$. The mapping

$$M: \qquad (\xi, \eta) \to (\xi', \eta') \tag{1.4}$$

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is then a one-to-one bicontinuous mapping of the x-y plane into itself. This follows from well-known properties of solutions of (1.3) (cf. [3], Ch. 1). Now if $x(t, \xi, \eta)$, $y(t, \xi, \eta)$ has period T, then $(\xi', \eta') = (\xi, \eta)$, so that the point (ξ, η) is a fixed point of the mapping M. If we define the iterates of the mapping by $M^2P = M(MP)$, $M^3P = M(M^2P)$, etc., then if $x(t, \xi, \eta)$, $y(t, \xi, \eta)$ is a subharmonic of order n, i.e., a solution of period nT but not of period T, the point (ξ, η) is a fixed point of the nth iterate, M^n, of the mapping M. In the study of the behavior of periodic solutions of (1.3) as parameters vary, it is helpful to visualize the fixed points of the mapping (1.4) or its iterates.

Suppose now that $x(t, \xi_0, \eta_0)$, $y(t, \xi_0, \eta_0)$ is a solution of the system (1.3) with period T. If a solution $x(t, \xi, \eta)$, $y(t, \xi, \eta)$ is such that

$$\lim_{t\to\infty} [x(t, \xi, \eta) - x(t, \xi_0, \eta_0)] = 0$$
$$\lim_{t\to\infty} [y(t, \xi, \eta) - y(t, \xi_0, \eta_0)] = 0,$$

the solution $x(t, \xi, \eta)$, $y(t, \xi, \eta)$ is said to *converge* to the periodic solution $x(t, \xi_0, \eta_0)$, $y(t, \xi_0, \eta_0)$. From the point of view of the mapping M, this means that the successive images MP, M^2P, M^3P, of the point $P(\xi, \eta)$ will approach the fixed point $P_0(\xi_0, \eta_0)$. It is known [2, 7, 10] that for c sufficiently large and positive in (1.1), and under mild restrictions on $g(x)$, there is a single periodic solution of (1.1) (and hence of the system (1.2)) to which all other solutions converge.

If the fixed point (ξ_0, η_0) has a neighborhood such that for any point (ξ, η) of this neighborhood, the solution $x(t, \xi, \eta)$, $y(t, \xi, \eta)$ converges to $x(t, \xi_0, \eta_0)$, $y(t, \xi_0, \eta_0)$, the periodic solution is called *asymptotically stable* ([3], Ch. 13).

The set of points (ξ, η) for which $x(t, \xi, \eta)$, $y(t, \xi, \eta)$ converges to an asymptotically stable periodic solution $x(t, \xi_0, \eta_0)$, $y(t, \xi_0, \eta_0)$ is called the *domain of attraction* of the fixed point (ξ_0, η_0). Some interesting experimental studies of domains of attraction are to be found in the work of Hayashi [5].

To study the stability of periodic solutions in general we consider the variation of solutions $x(t, \xi, \eta)$, $y(t, \xi, \eta)$ of the system (1.3) as the initial conditions (ξ, η) vary. In particular we are interested in

$$\frac{\partial x}{\partial \xi}(t, \xi_0, \eta_0), \qquad \frac{\partial x}{\partial \eta}(t, \xi_0, \eta_0),$$
$$\frac{\partial y}{\partial \xi}(t, \xi_0, \eta_0), \qquad \frac{\partial y}{\partial \eta}(t, \xi_0, \eta_0).$$

When the identities

$$x'(t, \xi, \eta) = F(t, x(t, \xi, \eta), y(t, \xi, \eta))$$
$$y'(t, \xi, \eta) = G(t, x(t, \xi, \eta), y(t, \xi, \eta))$$

are differentiated with respect to ξ and η, and after differentiation we set $\xi = \xi_0$, $\eta = \eta_0$, we obtain the linear equations

$$\left(\frac{\partial x}{\partial \xi}\right)'(t, \xi_0, \eta_0) = F_x(t, x(t, \xi_0, \eta_0), y(t, \xi_0, \eta_0))\frac{\partial x}{\partial \xi}(t, \xi_0, \eta_0) + F_y(t, x(t, \xi_0, \eta_0), y(t, \xi_0, \eta_0))\frac{\partial y}{\partial \xi}(t, \xi_0, \eta_0)$$

$$\left(\frac{\partial y}{\partial \xi}\right)'(t, \xi_0, \eta_0) = G_x(t, x(t, \xi_0, \eta_0), y(t, \xi_0, \eta_0))\frac{\partial x}{\partial \xi}(t, \xi_0, \eta_0) + G_y(t, x(t, \xi_0, \eta_0), y(t, \xi_0, \eta_0))\frac{\partial y}{\partial \xi}(t, \xi_0, \eta_0)$$

$$\left(\frac{\partial x}{\partial \eta}\right)'(t, \xi_0, \eta_0) = F_x(t, x(t, \xi_0, \eta_0), y(t, \xi_0, \eta_0))\frac{\partial x}{\partial \eta}(t, \xi_0, \eta_0) + F_y(t, x(t, \xi_0, \eta_0), y(t, \xi_0, \eta_0))\frac{\partial y}{\partial \eta}(t, \xi_0, \eta_0)$$

$$\left(\frac{\partial y}{\partial \eta}\right)'(t, \xi_0, \eta_0) = G_x(t, x(t, \xi_0, \eta_0), y(t, \xi_0, \eta_0))\frac{\partial x}{\partial \eta}(t, \xi_0, \eta_0) + G_y(t, x(t, \xi_0, \eta_0), y(t, \xi_0, \eta_0))\frac{\partial y}{\partial \eta}(t, \xi_0, \eta_0).$$

We also have the initial conditions, at $t = 0$, $\partial x/\partial \xi = \partial y/\partial \eta = 1$ and $\partial x/\partial \eta = \partial y/\partial \xi = 0$. Hence, $(\partial x/\partial \xi)(t, \xi_0, \eta_0)$, $(\partial y/\partial \xi)(t, \xi_0\ \eta_0)$ and $(\partial x/\partial \eta)(t, \xi_0, \eta_0)$, $(\partial y/\partial \eta)(t, \xi_0, \eta_0)$ are both solutions of the linear system

$$\begin{aligned} u' &= F_x(t, x(t, \xi_0, \eta_0), y(t, \xi_0, \eta_0))u + F_y(t, x(t, \xi_0, \eta_0), y(t, \xi_0, \eta_0))v \\ v' &= G_x(t, x(t, \xi_0, \eta_0), y(t, \xi_0, \eta_0))u + G_y(t, x(t, \xi_0, \eta_0), y(t, \xi_0, \eta_0))v, \end{aligned} \tag{1.5}$$

which is known as the variational system of (1.3) with respect to the periodic solution $x(t, \xi_0, \eta_0)$, $y(t, \xi_0, \eta_0)$. Now consider the matrix

$$\left\| \begin{matrix} \frac{\partial x}{\partial \xi}(t, \xi_0, \eta_0) & \frac{\partial x}{\partial \eta}(t, \xi_0, \eta_0) \\ \frac{\partial y}{\partial \xi}(t, \xi_0, \eta_0) & \frac{\partial y}{\partial \eta}(t, \xi_0, \eta_0) \end{matrix} \right\|. \tag{1.6}$$

Its columns are linearly independent solutions of the system (1.5), so that the matrix is a fundamental matrix of the system (1.5), and is as a result nonsingular for all values of t. When $t = T$ the matrix (1.6) becomes

$$\left\| \begin{matrix} \frac{\partial x}{\partial \xi}(T, \xi_0, \eta_0) & \frac{\partial x}{\partial \eta}(T, \xi_0, \eta_0) \\ \frac{\partial y}{\partial \xi}(T, \xi_0, \eta_0) & \frac{\partial y}{\partial \eta}(T, \xi_0, \eta_0) \end{matrix} \right\| = \left\| \begin{matrix} \frac{\partial \xi'}{\partial \xi} & \frac{\partial \xi'}{\partial \eta} \\ \frac{\partial \eta'}{\partial \xi} & \frac{\partial \eta'}{\partial \eta} \end{matrix} \right\|_{\xi=\xi_0, \eta=\eta_0} \tag{1.7}$$

which is the Jacobian matrix of the mapping (1.4) at the fixed point (ξ_0, η_0). Consequently the eigenvalues of (1.7) will be significant in the study of the mapping and of the stability of the periodic solution. The eigenvalues of (1.7) are called the *characteristic multipliers* of the variational system (1.5). From the general theory of linear systems with periodic coefficients (cf. [3], Ch. 3), we have that if F, G, x, and y are all real, the characteristic multipliers are either both real, or else are conjugate complex. Moreover, their product is always positive. Using the terminology in Levinson's paper [8], we classify fixed points of the mapping (1.4) and the corresponding periodic solutions of (1.3) according to the characteristic multipliers, r_1 and r_2, of the corresponding variational system. A fixed point (ξ_0, η_0) is called:

Completely stable if $|r_1| < 1$ and $|r_2| < 1$,

Completely unstable if $|r_1| > 1$ and $|r_2| > 1$,

Directly unstable if $0 < r_1 < 1 < r_2$,

Inversely unstable if $r_1 < -1 < r_2 < 0$.

Since (1.7) is nonsingular, a characteristic multiplier is never zero. If a characteristic multiplier is equal to $+1$, there is a solution of (1.5) of period T. If a characteristic multiplier is equal to -1, there is a solution of (1.5) of period $2T$ which is odd-harmonic. (A function of period L is called odd-harmonic if $x(t + L/2) \equiv -x(t)$.) If r_1 and r_2 are not real and have absolute value unity, we call the fixed point quasi-stable. In this case no conclusion about the behavior of points in the neighborhood of the fixed point can be drawn from the variational equation.

If a fixed point is completely stable, then, since the eigenvalues of the Jacobian matrix of the mapping are less than one in absolute value, there is a neighborhood of the fixed point such that all points in this neighborhood tend to the fixed point under repeated applications of the mapping. Thus as $t \to +\infty$, the corresponding solutions tend to the periodic solution, so that the periodic solution is asymptotically stable. In the completely unstable case, we have the situation of points moving away from the fixed point under the mapping. In the directly unstable case, most but not all points will move away from the fixed point.

The purpose of this paper is to discuss periodic solutions of equation

(1.1) as the parameters E and c vary. We are particularly concerned with the movement in the phase plane of the corresponding fixed points of the mapping M. We are unable to give a complete rigorous mathematical treatment of the problem. In §2 we summarize the results of [9], which deals with the case of (1.1) when E is small, and c is small compared to E. In §3 we give a theorem on the movement of fixed points for a special case of (1.1). In §4 we characterize a part of the common boundary of the domains of attraction of two fixed points.

The remainder of the paper deals with an approximate technique for study of the problem, and with experimental results. In §5 we discuss an approximation procedure for location of fixed points. The procedure is exact for linear systems, and is a good approximation for systems that are nearly linear. We extend the predictions to systems that are not nearly linear. In §6 we then give the results of an experimental study, using a high-speed electronic computer, which shows the extent to which the approximate procedure gives results that agree with experiment. It is hoped that ultimately the rigorous theory can be extended to these cases.

2. Periodic solutions of (1.1) for small E. One of the present authors [9] has made a study of periodic solutions of (1.1) for small values of E. Certain of the results of [9] will be needed in this paper, and they will be summarized in the present section.

Let $g(x)$ have a continuous first derivative and a piecewise continuous second derivative. Let $xg(x)$ be always nonnegative (so that $g(x)$ acts like a restoring spring force), and let the set of values of x for which $g(x) = 0$ be either $x = 0$ alone or a closed interval about $x = 0$. It is then shown that the equation

$$x'' + g(x) = 0 \tag{2.1}$$

has all of its solutions periodic, with period varying with amplitude in general. In particular it is shown that if $g(x)$ is of softening characteristic, ($xg''(x) \leqq 0$), the period always increases with amplitude, while if $g(x)$ is of hardening characteristic, ($xg''(x) \geqq 0$), the period always decreases with amplitude. We shall exclude from consideration cases in which the derivative of the period with respect to amplitude is zero. In the following we denote the derivative of the period with respect to amplitude of the solutions of (2.1) by $-D$, so that D is positive for $g(x)$ of hardening characteristic and negative for $g(x)$ of softening characteristic.

Suppose now that $f(t)$ has period T, and that (2.1) has a solution of period L_0, where L_0 is a rational multiple of T. It is reasonable to expect that there should exist periodic solutions of

$$x'' + g(x) = Ef(t) \tag{2.2}$$

for small values of E, which reduce, as E approaches zero, to periodic solutions of (2.1). Let $x_0(t)$ be a solution of (2.1) of least period L_0, and let L be the least common multiple of T and L_0. Not only $x_0(t)$, but any translation, $x_0(t + \tau)$, of $x_0(t)$, is a solution of (2.1) of period L_0. It is shown that if

$$F_0(\tau) = \int_0^L x_0'(t)f(t - \tau)\,dt,$$

then for those values of τ for which $F_0(\tau) = 0$, while $F_0'(\tau) \neq 0$, there exist unique periodic solutions of (2.2) of period L of the form

$$x = x_0(t + \tau) + Ex_1(t) + o(E),$$

which reduce, as $E \to 0$, to the corresponding translations of $x_0(t)$. Here $x_1(t)$ is an approxpriately chosen periodic solution of

$$y'' + g'(x_0(t))y = f(t).$$

Moreover it is also shown that such a periodic function is quasi-stable if $EDF_0'(\tau)$ is negative, and is directly unstable if $EDF_0'(\tau)$ is positive.

If now a small positive damping term is added after E is fixed and sufficiently small, we obtain

$$x'' + cx' + g(x) = Ef(t). \tag{2.3}$$

In general c is small when E is small, and we understand that $c \to 0$ when $E \to 0$. It is shown in [9] that for the same values of τ there exist periodic solutions of (2.3) tending to the same translations of $x_0(t)$ as $E \to 0$. The stability result is the same except that for positive c the quasi-stable solutions become completely stable. The directly unstable solutions remain directly unstable.

Another case covered in [9] is that in which the periodic solution of (2.1) which is considered is the solution $x_0(t) \equiv 0$. Here it is shown that for sufficiently small E there exists a periodic solution of (2.2) or (2.3) of period T having the form

$$x(t) = Ex_1(t) + o(E),$$

provided that, in the case of (2.2), the linear equation

$$y'' + g'(0)y = 0$$

has no solution of period T. There is no requirement here that c be small. The function $x_1(t)$ is the unique periodic solution of

$$y'' + g'(0)y = f(t)$$

in the case of (2.2), and of

$$y'' + cy' + g'(0)y = f(t)$$

in the case of (2.3). The periodic solution is quasi-stable for (2.2) and is completely stable for (2.3) if c is positive.

The foregoing ideas can be illustrated with Duffing's equation. The autonomous equation

$$x'' + \alpha x + \beta x^3 = 0, \tag{2.4}$$

is assumed to have $\alpha < 1$ and $\beta > 0$. Since $\alpha < 1$, the periods of solutions of (2.4) for very small amplitudes will be greater than 2π. Since $\beta > 0$, we have the hardening characteristic, so that the periods of solutions of (2.4) decrease as the amplitude increases. Hence there will be a periodic solution $x_0(t)$ of (2.4) of period 2π. Let us agree that at $t = 0$, $x_0(t)$ reaches its positive maximum. Then from the foregoing we know that

$$x'' + \alpha x + \beta x^3 = E \cos t \tag{2.5}$$

has periodic solutions of period 2π near to certain translations of $x_0(t)$ for small values of E. Because of symmetry in equation (2.4), $x_0(t)$ is an even function of time, and also odd-harmonic. Hence its Fourier series has the form

$$x_0(t) = A_1 \cos t + A_3 \cos 3t + \cdots . \tag{2.6}$$

Since $x_0(t)$ has its positive maximum at $t = 0$, A_1 is positive. Using $f(t) = \cos t$, we find that here

$$F_0(\tau) = \int_0^{2\pi} x_0'(t) \cos (t - \tau)\, dt = -\pi A_1 \sin \tau. \tag{2.7}$$

The expression determining stability is

$$EDF_0'(\tau) = -\pi EDA_1 \cos \tau. \tag{2.8}$$

It follows from (2.7) that there are two translations of $x_0(t)$, corresponding to $\tau = 0$ and $\tau = \pi$, near to which there are periodic solutions of (2.5) for small E. In (2.8) D and A_1 are both positive. Hence if E is positive, the solution near to $x_0(t)$ is quasi-stable, while that near to $x_0(t + \pi)$ is directly unstable. This result is in accord with well-known results for Duffing's equation (cf. [5] and [11]).

We may also deduce the existence, for small E, of a periodic solution near to $x = 0$, since for $\alpha < 1$, the linear equation

$$y'' + \alpha y = 0$$

has no solution of period 2π. We can draw similar conclusions for Duffing's equation with damping

$$x'' + cx' + \alpha x + \beta x^3 = E \cos t.$$

3. A theorem on movement of fixed points. The problem of a rigorous proof of the existence of periodic solutions of the equation (1.1) is in general quite difficult. If E is small, the results of [9] give considerable information. In this section we consider a special case of (1.1) without damping, in which it is possible to assert the existence of periodic solutions for all values of E, and to describe the movement in the phase plane of the corresponding fixed points of the mapping M. It would be desirable to have a general theory on movement of fixed points, but such does not seem to be available at present.

In the differential equation

$$x'' + g(x) = E \cos t \tag{3.1}$$

let $g(x)$ have the following properties:

1. $g(x) \in C^2$ for all x.
2. $g(x)$ is an odd function, and $xg(x) > 0$ for all $x \neq 0$.
3. $g(x)$ represents a softening spring force, i.e., $xg''(x) < 0$ for all $x \neq 0$.
4. $1 < g'(0) = \alpha^2 < 4$.
5. The limiting value $g'(\pm\infty)$ is less than 1.

The function $g(x) = \tanh 2x$, for example, satisfies all these conditions.

Using the results given in §2 we see that

$$x'' + g(x) = 0 \tag{3.3}$$

has all its solutions periodic. The periods vary with amplitude, and, because $xg''(x) < 0$, the period is an increasing function of amplitude. As the amplitude approaches zero, the period approaches $2\pi/\alpha$, while as the amplitude becomes infinite, the period approaches $2\pi/\sqrt{g'(\pm\infty)}$. Hence there is a unique amplitude corresponding to the period 2π. Let $x_0(t)$ be that solution of (3.3) having period 2π and such that at $t = 0$, $x_0(0) = A > 0$, and $x_0'(0) = 0$. Because of the symmetry in equation (3.3), $x_0(t)$ is even and odd-harmonic, so that $x_0(\pi/2) = x_0(3\pi/2) = 0$, and $x_0(\pi) = -A$.

Moreover, for small values of E, the results of §2 show that (3.1) has three solutions of period 2π, of the forms

$$x_0(t + \pi) + Ex_1(t) + o(E) \tag{3.4}$$

$$Ex_1(t) + o(E) \tag{3.5}$$

$$x_0(t) + Ex_1(t) + o(E) \tag{3.6}$$

where the functions $x_1(t)$ are appropriate solutions of period 2π of the linear equations

$$\begin{aligned} y'' + g'(x_0(t + \pi))y &= \cos t \\ y'' + g'(0)y &= \cos t \\ y'' + g'(x_0(t))y &= \cos t, \end{aligned}$$

respectively. The three solutions (3.4), (3.5), and (3.6) correspond for small E to fixed points of the mapping near to the points $(-A, 0)$, $(0, 0)$, and $(A, 0)$ respectively. These are points of the phase plane with coordinates x and $y = x'$, and are themselves the phase plane positions for $t = 0$ of the periodic solutions $x = x_0(t + \pi)$, $x = 0$, and $x = x_0(t)$ of equation (3.3). It also follows from the results of §2 that for E small and positive, (3.4) and (3.5) are quasi-stable, while (3.6) is directly unstable. Also, there are no other solutions of the above form tending as E approaches zero to $x_0(t)$ or to any of its translations.

THEOREM 1. *Under hypothesis* (3.2) *equation* (3.1) *has, for small positive E, three periodic solutions of period 2π. All three are odd-harmonic, with $x = 0$ for odd multiples of $\pi/2$ and $x' = 0$ for multiples of π. As E increases through positive values, the corresponding fixed points move along the x-axis as follows: The fixed point which is near $(-A, 0)$ for small E moves indefinitely far to the left. The fixed point near $(0, 0)$ for small E moves to the right. The fixed point near $(A, 0)$ for small E moves to the left. The latter two fixed points come together for a finite positive value of E, E_0. For $E > E_0$, the latter two fixed points no longer exist.*

Proof of Theorem 1. Let $x(t, \xi, E)$ be that solution of (3.1) for which $x(0, \xi, E) = \xi$, $x'(0, \xi, E) = 0$. Note that $x(t, -A, 0) = x_0(t + \pi)$, $x(t, 0, 0) = 0$, and $x(t, A, 0) = x_0(t)$. The solution $x(t, \xi, E)$ is an even function, and it will be periodic of period 2π if and only if $x'(\pi, \xi, E) = 0$. The cases just mentioned for $E = 0$ all have this property. Because of the assumed symmetry of $g(x)$ and because of the symmetry of the forcing term $E \cos t$, $x(t, \xi, E)$ will be periodic of period 2π if $x(\pi/2, \xi, E) = 0$, and in this case the solution is also odd-harmonic. Periodicity does not imply this last condition, so there is no assurance that for some values of E there will not be periodic solutions which are not odd-harmonic. We shall consider only periodic solutions for which $x(\pi/2, \xi, E) = 0$. These will be shown to exist for all E. It is known from [9] that at least for small E these are the only periodic solutions, so that the non-odd-harmonic periodic solutions are not present for small E.

Consider $x(\pi/2, \xi, E)$ as a function of ξ for fixed E. When $E = 0$, the graph of $x(\pi/2, \xi, 0)$ is as shown in Fig. 1. The graph crosses the ξ-axis at $(-A, 0)$, $(0, 0)$, and $(A, 0)$. The positive or negative character follows from the behavior of the periods of solutions of (3.3). We shall show that as E increases through positive values, the curve in Fig. 1 moves upward. This will prove the assertions of the theorem about the movement of the fixed points, since fixed points of the mapping are exactly the points $(\xi, 0)$ for which $x(t, \xi, E)$ is periodic, which correspond to the values of ξ for which $x(\pi/2, \xi, E) = 0$. The intersections of the curve in Fig. 1 with the ξ-axis as E varies move precisely as the fixed points of the mapping.

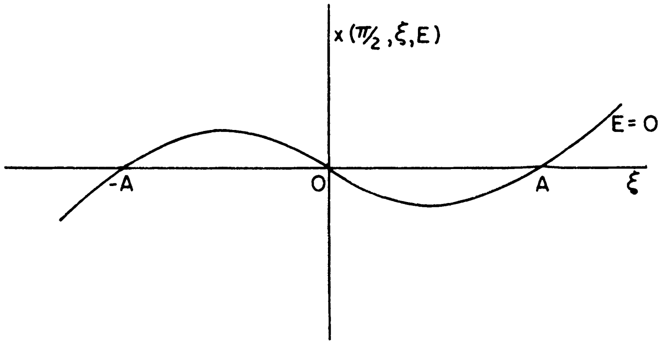

FIG. 1

In particular, the value E_0 is the value of E for which the minimum point on the curve is tangent to the ξ-axis.

The partial derivative $x_E(t, \xi, E)$ is that solution of the linear equation

$$y'' + g'(x(t, \xi, E))y = \cos t \tag{3.7}$$

for which $y = y' = 0$ at $t = 0$. It is known (cf. [3], Ch. 1) that the derivation of (3.7) from (3.1) by formal partial differentiation is legitimate. We write (3.7) in the form

$$y'' + \alpha^2 y = \cos t + [\alpha^2 - g'(x(t, \xi, E))]y, \tag{3.8}$$

and then transform to the integral equation

$$y(t) = \frac{\cos t - \cos \alpha t}{\alpha^2 - 1} + \int_0^t \frac{\sin \alpha(t - s)}{\alpha} [\alpha^2 - g'(x(s, \xi, E))]y(s)\, ds, \tag{3.9}$$

of which $x_E(t, \xi, E)$ is the unique solution. We shall show that $y(\pi/2)$ is positive for all values of E and ξ.

Since for $t = 0$, $y = y' = 0$, it is clear from (3.7) that $y''(0) = +1$, so that y is positive for small positive t. Since $1 < \alpha < 2$,

$$\frac{\cos t - \cos \alpha t}{\alpha^2 - 1}$$

is positive for $0 < t \leqq \pi/2$, and at $\pi/2$ its value is positive and independent of ξ and E. Also both of

$$\frac{\sin \alpha(t - s)}{\alpha}, \qquad \alpha^2 - g'(x(s, \xi, E))$$

are nonnegative for $0 \leqq s \leqq t \leqq \pi/2$. As a result, so long as y remains positive, the integral is a nondecreasing function of t. But this implies that $y(t)$ surely does remain positive until at least $\pi/2$, and that its value there has a positive lower bound independent of ξ and E. Therefore, for all ξ and E, the curve in Fig. 1 does indeed move upward as E increases, so that the assertions of the theorem hold. This completes the proof of Theorem 1.

REMARK. For negative values of E, the stability of the outer two fixed points is reversed, and as E decreases through negative values it is the left two fixed points that come together. This is implied by the fact that $x_E(\pi/2, \ \xi, \ E) > 0$.

4. A theorem on domains of attraction. In this section we characterize limited portions of the domains of attraction of two completely stable fixed points for a case similar to that discussed in the preceding section. The general question of the appearance of a domain of attraction is quite difficult. Experimental studies [5] have shown that even for very simple equations, domains of attraction can be very complicated.

Suppose that in a region R of the x-y plane there are exactly three fixed points P_1, P_2, and P_3 of the mapping (1.3) discussed in §1. Suppose further that P_1 and P_2 are completely stable, while P_3 is directly unstable (see Fig. 2). This is a case discussed in [8] as a possibility for a system which is dissipative for large amplitudes. It is known that through P_3 there are two curves, C_1 and C_2, which are invariant under the mapping. Points on C_1 approach P_3 under iterations of the mapping, while points on C_2 approach P_3 under iterations of the inverse mapping.

Consider the images MR, M^2R, M^3R, $\cdots$ of the region R. For systems that are dissipative for large amplitudes (which is the case for (1.1) if c is positive), there exists a maximum finite invariant domain, $I = \bigcap_{n=0}^{\infty} M^nR$. For the case of (1.1) with positive c it is known that I has zero area (8). We shall assume that for the case under discussion I consists of the three fixed points and a simple arc connecting them. P_1 and P_2 will be on the ends, with P_3 between them. The arc is the curve C_2 mentioned earlier.

Now let Q and S be points on C_1, one on each side of P_3, and let Q' and S' be their images under the mapping. Because P_3 is directly unstable, Q' and S' will be located as shown in Fig. 3. Finally, we suppose that we

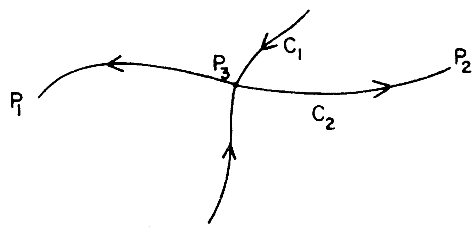

FIG. 2

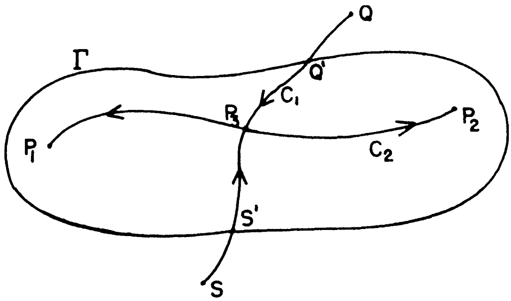

FIG. 3

can construct a simple closed curve, Γ, lying in R, and containing I in its interior, such that Γ intersects C_1 in the points Q' and S', and such that the image under the mapping of every point of Γ is interior to Γ. Then C_1 divides the interior of Γ into two parts, D_1 and D_2, with $P_1 \in D_1$ and $P_2 \in D_2$, each of D_1 and D_2 being open and connected.

THEOREM 2. *Every point of the domain D_1 is in the domain of attraction of P_1, while every point of the domain D_2 is in the domain of attraction of P_2, so that the curve C_1 forms a part of the boundary between the two domains of attraction.*

Proof of Theorem 2. It is to be shown that under iterations of the mapping every point of D_1 tends to P_1. It will then follow that similarly every point of D_2 tends to P_2 which will establish the theorem.

Since every point of D_1 is interior to Γ, the successive images of D_1 are likewise interior to Γ, and since D_1 is a subset of R, the limiting points of the images of points of D_1 must lie in I. Now since the points of I other than P_3 are such that their images tend to either P_1 or P_2, it follows by continuity of the mapping that the later images of any point of D_1 will also tend to P_1 or to P_2 depending on which part of C_2 they approach.

It is impossible for the images of any point of D_1 to approach P_3, since only the points of C_1 can tend to P_3, and the preimage of any point of C_1 interior to Γ lies on the arc QS of C_1 and never in D_1. Finally it is not possible for the image of a point in D_1 to lie in D_2. For suppose that $P \in D_1$ while $MP \in D_2$. Connect P to P_1 by an arc C lying wholly in D_1. Since the image of C lies interior to Γ, and connects $P_1 \in D_1$ with $MP \in D_2$, the image of C must intersect C_1. But this is impossible since the preimage of a point of C_1 interior to Γ can not lie in D_1. This shows that points of D_1 are mapped into D_1. Hence the successive images of a point in D_1 tend to the arc P_3P_1 of C_2, and so ultimately tend to P_1. This completes the proof of Theorem 2.

REMARK. The assumptions made about Γ sidestep some real topological problems. This was done deliberately, since the point intended was merely the fact that C_1 separates the domains of attraction.

5. An approximate technique for locating fixed points. In this section we consider a method introduced by van der Pol and used extensively by Krylov and Bogolyubov for the study of periodic solutions of systems differing slightly from harmonic oscillators (see [6] and [1]). For a proof of the validity of the method, due to Lefschetz, see [1], p. 341.

We use the method to obtain formulas which predict the locations of fixed points of the type considered in the earlier sections of this paper. The formulas are exact for linear systems, and are good approximations for systems differing but slightly from linear. We shall write the formulas for systems differing more than slightly from linear. Then in the following section there will be given experimental results showing to what extent these formulas are reliable.

Consider the system

$$\begin{aligned} x' &= F(x, y) + A \cos t + B \sin t \\ y' &= G(x, y) + C \cos t + D \sin t \end{aligned} \tag{5.1}$$

where

$$F(0, 0) = G(0, 0) = 0.$$

If (5.1) is linear, the system becomes

$$\begin{aligned} x' &= F_1 x + F_2 y + A \cos t + B \sin t \\ y' &= G_1 x + G_2 y + C \cos t + D \sin t. \end{aligned} \tag{5.2}$$

We consider (5.1) and (5.2) as special cases of (1.2), and seek solutions of period 2π and the corresponding fixed points. The linear system (5.2) will have a unique solution of period 2π unless the homogeneous system has a nontrivial solution of least period 2π. We do not consider the exceptional situation. If F and G are sufficiently close to linear, the solutions of (5.1) will follow the solutions of (5.2) arbitrarily closely for any preassigned interval of time. Therefore we expect the solution of (5.1) which reduces to the periodic solution of (5.2) as (5.1) is made linear to have the form:

$$\begin{aligned} x &= a(t) \cos t + b(t) \sin t \\ y &= c(t) \cos t + d(t) \sin t \end{aligned} \tag{5.3}$$

where $a(t)$, $b(t)$, $c(t)$, and $d(t)$ are slowly varying functions of time. If (5.3) is substituted into (5.1), we obtain

$$-a \sin t + b \cos t + a' \cos t + b' \sin t$$
$$= F(a \cos t + b \sin t, c \cos t + d \sin t) + A \cos t + B \sin t$$
$$- c \sin t + d \cos t + c' \cos t + d' \sin t$$
$$= G(a \cos t + b \sin t, c \cos t + d \sin t) + C \cos t + D \sin t.$$

The approximate method consists in ignoring the variation of a, b, c, and d in F and G, and expanding in Fourier series. Then in the resulting series neglect higher harmonics, and equate coefficients of $\sin t$ and $\cos t$. This will give

$$-a + b' - B = \frac{1}{\pi} \int_0^{2\pi} F \sin t \, dt$$
$$b + a' - A = \frac{1}{\pi} \int_0^{2\pi} F \cos t \, dt$$
$$-c + d' - D = \frac{1}{\pi} \int_0^{2\pi} G \sin t \, dt$$
$$d + c' - C = \frac{1}{\pi} \int_0^{2\pi} G \cos t \, dt$$

where we have not written the arguments of F and G. Solving for a', b', c', and d', we have

$$\begin{aligned} a' &= -b + A + \frac{1}{\pi} \int_0^{2\pi} F \cos t \, dt \\ b' &= a + B + \frac{1}{\pi} \int_0^{2\pi} F \sin t \, dt \\ c' &= -d + C + \frac{1}{\pi} \int_0^{2\pi} G \cos t \, dt \\ d' &= c + D + \frac{1}{\pi} \int_0^{2\pi} G \sin t \, dt. \end{aligned} \tag{5.4}$$

This is a system of autonomous differential equations for a, b, c, and d. Since we seek solutions for which a, b, c, and d are slowly varying, we take those values of a, b, c, and d for which the right members of (5.4) are zero. Each such set of values gives us

$$x = a \cos t + b \sin t, \qquad y = c \cos t + d \sin t \tag{5.5}$$

as an approximate solution of period 2π of (5.1). When $t = 0$, we have $x = a$, $y = c$, so that the point (a, c) is an approximation to the corresponding fixed point.

For the system (5.2) this process gives the exact values of a, b, c, and d for which (5.5) is a periodic solution.

We now apply this technique to the equation

$$x'' + cx' + g(x) = E\cos t \tag{5.6}$$

which we write as a system

$$x' = y, \qquad y' = -g(x) - cy + E\cos t. \tag{5.7}$$

Setting $x = a\cos t + b\sin t$, $y = m\cos t + n\sin t$, in (5.7) we find analogous to (5.4)

$$\begin{aligned} a' &= -b + m \\ b' &= a + n \\ m' &= -n - cm + E - \frac{1}{\pi}\int_0^{2\pi} g(a\cos t + b\sin t)\cos t\,dt \\ n' &= m - cn - \frac{1}{\pi}\int_0^{2\pi} g(a\cos t + b\sin t)\sin t\,dt. \end{aligned} \tag{5.8}$$

Setting the right members in (5.8) equal to zero we find $n = -a$, $m = b$. Using these in the second two equations we find

$$\begin{aligned} -a + cb - E &= -\frac{1}{\pi}\int_0^{2\pi} g(a\cos t + b\sin t)\cos t\,dt \\ -b - ca &= -\frac{1}{\pi}\int_0^{2\pi} g(a\cos t + b\sin t)\sin t\,dt. \end{aligned} \tag{5.9}$$

The system (5.9) is then solved for a and b as functions of c and E. Since the approximate fixed point is $(a, m) = (a, b)$, we are thus furnished with a predicted fixed point for the system (5.7).

To solve for a and b it is convenient to use polar coordinates, so we set

$$a = r\cos\theta, \qquad b = r\sin\theta.$$

(5.9) becomes

$$\begin{aligned} r(\cos\theta - c\sin\theta) + E &= \frac{1}{\pi}\int_0^{2\pi} g(r\cos(t-\theta))\cos t\,dt \\ r(\sin\theta + c\cos\theta) &= \frac{1}{\pi}\int_0^{2\pi} g(r\cos(t-\theta))\sin t\,dt. \end{aligned}$$

Now in the integrals set $t = u + \theta$, and use the periodicity to obtain

$$\begin{aligned} r(\cos\theta - c\sin\theta) + E &= \frac{1}{\pi}\int_0^{2\pi} g(r\cos u)\cos(u+\theta)\,du \\ r(\sin\theta + c\cos\theta) &= \frac{1}{\pi}\int_0^{2\pi} g(r\cos u)\sin(u+\theta)\,du. \end{aligned} \tag{5.10}$$

If we now set

$$\phi(r) = \frac{1}{\pi}\int_0^{2\pi} g(r\cos u)\cos u\,du = \frac{2}{\pi}\int_{-1}^{1} g(ry)\frac{y}{\sqrt{1-y^2}}\,dy$$

$$\psi(r) = \frac{1}{\pi}\int_0^{2\pi} g(r\cos u)\sin u\,du$$

the system (5.10) can be written[3]

$$\begin{aligned} r - \phi(r) &= -E\cos\theta \\ cr - \psi(r) &= E\sin\theta, \end{aligned} \tag{5.11}$$

which we use to predict the location of fixed points. As was said earlier, these equations are exact when $g(x)$ is linear. In that case, if $g(x) = kx$, we have

$$\begin{aligned} r - kr &= -E\cos\theta \\ cr &= E\sin\theta, \end{aligned} \tag{5.12}$$

which give:

$$\tan\theta = \frac{c}{k-1}, \qquad r = \frac{E}{\sqrt{c^2 + (k-1)^2}}.$$

We now consider the predicted loci of fixed points as the parameters c and E vary. If E is fixed and c varies, we obtain on eliminating c from (5.11) the equation

$$\phi(r) - r = E\cos\theta, \tag{5.13}$$

which we shall call a constant-E curve. If c is fixed, and E varies, we obtain on eliminating E from (5.11)

$$\frac{cr - \psi(r)}{\phi(r) - r} = \tan\theta \tag{5.14}$$

which we shall call a constant-c curve. If several constant-E curves and constant-c curves are plotted for a given differential equation, a picture of the predicted movement of fixed points is obtained. One feature of these curves is apparent from (5.13) and (5.14). If r_0 is a root of the equation $\phi(r) = r$, every curve of both families will pass through the point $r = r_0$, $\theta = \pi/2$.

We now discuss more completely four cases of equation (5.1). We shall say that $g(x)$ represents a hardening spring force in case $xg''(x)$ is positive for nonzero x, and a softening spring force in case $xg''(x)$ is negative for

[3] Note added in proof: It is always the case that $\psi(r) \equiv 0$.

all nonzero x. In either case we assume that $xg(x)$ is positive for all nonzero x. In each of these two cases the relation of $g'(0)$ to unity is important. In the hardening case if $g'(0) < 1$, and in the softening case if $g'(0) > 1$, the undamped unforced equation

$$x'' + g(x) = 0$$

will have periodic solutions of period 2π for some unique amplitude. We then expect that for small values of E there will be at least three periodic solutions. However, in the hardening case if $g'(0) > 1$, and in the softening case if $g'(0) < 1$, the undamped unforced equation will have no solution of least period 2π, so we expect only one periodic solution for small E.

We shall assume in addition that $g(x)$ is an odd function. It will follow that $\psi(r) \equiv 0$. Moreover, since $\phi''(r)$ is given by

$$\phi''(r) = \frac{2}{\pi} \int_{-1}^{1} g''(ry) \frac{y^3}{\sqrt{1 - y^2}} \, dy, \tag{5.15}$$

it follows that for positive r, $\phi''(r)$ is positive in the hardening case, and negative in the softening case.

We rewrite equations (5.13) and (5.14) for the cases to be discussed as

$$\phi(r) - r = E \cos \theta \tag{5.13}$$

$$\frac{\phi(r)}{r} = 1 + c \cot \theta. \tag{5.14a}$$

Case I. Hardening system, $g'(0) < 1$. In this case there is exactly one positive value of r, r_0, for which $\phi(r) = r$, with $\phi(r) < r$ for $r < r_0$ and $\phi(r) > r$ for $r > r_0$. Hence the possible intersections of constant-E curves and constant-c curves can lie in the second quadrant if $r < r_0$, and in the

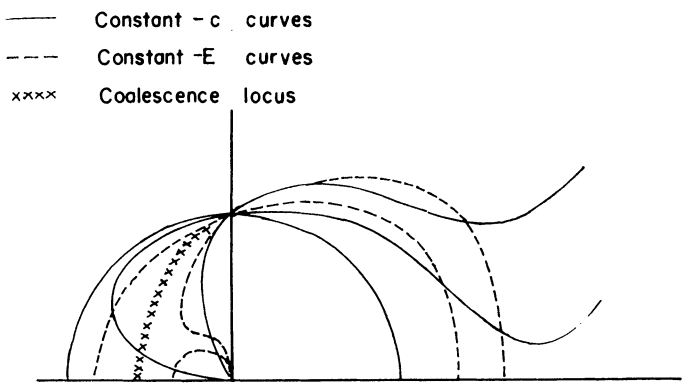

FIG. 4

first quadrant if $r > r_0$. A few typical curves of each family are sketched in Fig. 4. Note that all curves of both families pass through $r = r_0$, $\theta = \pi/2$.

Case II. Hardening system, $g'(0) > 1$. In this case $\phi(r) > r$ for all values of r. Hence possible intersections of constant-c curves and constant-E curves all lie in the first quadrant (see Fig. 5).

Case III. Softening system, $g'(0) < 1$. In this case $\phi(r) < r$ for all values of r. Hence possible intersections of constant-c curves and constant-E curves all lie in the second quadrant (see Fig. 6).

Case IV. Softening system, $g'(0) > 1$. In this case there is exactly one positive value of r, r_0, for which $\phi(r) = r$. If $r < r_0$, $\phi(r) > r$, while if $r > r_0$, $\phi(r) < r$. Hence the possible intersections of constant-c curves and constant-E curves lie in the first quadrant if $r < r_0$, and in the second quadrant if $r > r_0$. All curves of both families pass through $r = r_0$, $\theta = \pi/2$ (see Fig. 7).

When $g(x)$ is linear, $\phi(r) \equiv g'(0)r$, and the constant-E curves are semicircles, while the constant-c curves are radial straight lines. All are located in the second quadrant if $g'(0) < 1$, and in the first quadrant if $g'(0) > 1$. In the linear case these curves are the precise loci of fixed points, and each constant-E curve intersects each constant-c curve exactly once. If $g(x) = x$, the constant-E curves and the constant-c curves all coincide with the line $\theta = \pi/2$.

Cases II and III are similar to the linear case. The constant-c loci and constant-E loci resemble radial straight lines and semicircles, and again

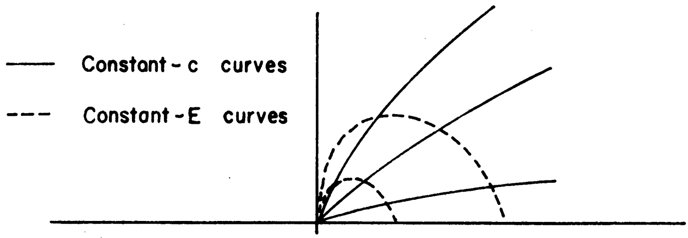

FIG. 5

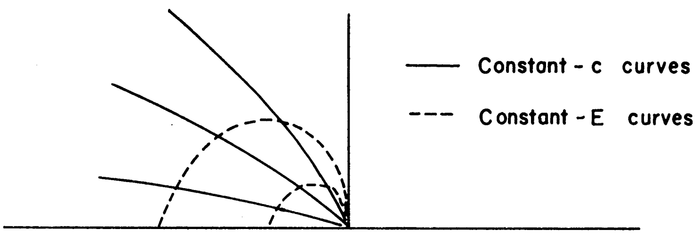

FIG. 6

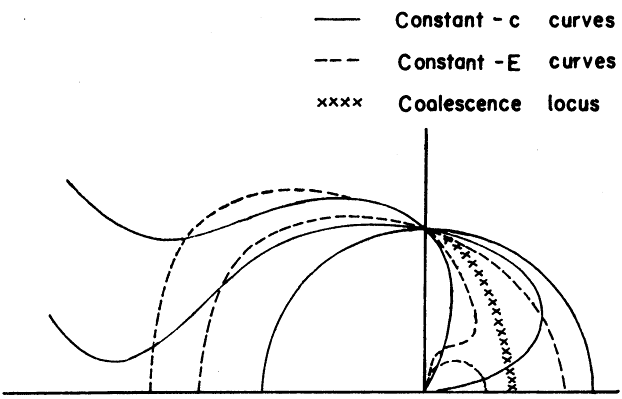

FIG. 7

for each value of E and c there is exactly one intersection of the corresponding curves, and exactly one predicted fixed point.

Cases I and IV are more complicated. A constant-c curve and a constant-E curve may have three intersections, one intersection, or they may be tangent. (We do not count the point $(r_0, \pi/2)$.) Before giving a discussion of these cases, we investigate the occurrence of tangency of constant-E curves and constant-c curves. If the differential equations for the families (5.13) and (5.14a) are determined, it is found that the locus of tangencies is

$$\frac{\phi'(r) - 1}{\phi'(r) - \phi(r)/r} = \sin^2\theta. \tag{5.15}$$

This locus is real on the range $r_1 \leqq r \leqq r_0$, where r_1 is the root of $\phi'(r) = 1$, and r_0 is the root of $\phi(r) = r$. This locus lies in the first quadrant in Case IV and in the second quadrant in Case I. It is the locus of predicted coalescences of fixed points. It is shown in Figs. 4 and 7.

If the coordinate r in (5.15) is regarded as a parameter, the corresponding values of E and c for points of the locus (5.15) can be determined. It is found that

$$\begin{aligned} E &= \sqrt{(\phi(r) - r)(\phi(r) - r\phi'(r))} \\ c &= \sqrt{(\phi(r)/r - 1)(1 - \phi'(r))}. \end{aligned} \tag{5.16}$$

If the equations (5.16) are sketched in a c-E plane, we obtain Fig. 8. The point for $r = r_1$ corresponds to $c = 0$, $E = E_0$ (cf. Theorem 1). Let the cusp occur for $c = c_1$, $E = E_1$. In Fig. 8, points inside the curve correspond to three predicted fixed points in Cases I and IV, while points outside the curve correspond to one predicted fixed point.

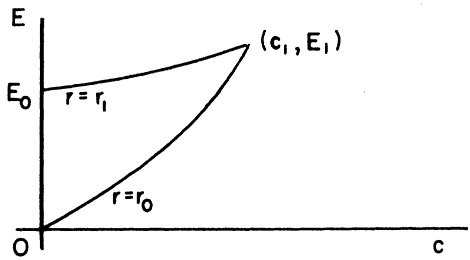

FIG. 8

We can now discuss the intersections of constant-E curves and constant-c curves in Figs. 4 and 7. Consider first a constant-c curve with $c < c_1$. For a certain range of E, between the two branches of the curve in Fig. 8, this constant curve will have three intersections with the corresponding constant-E curve. As E decreases toward zero, the two fixed points more remote from the origin move together and coalesce for that value of E on the lower branch of the curve in Fig. 8. For very small values of E there is but one intersection with the constant-E curve. As E increases, the two fixed points nearer to the origin on the constant-c curve move together and coalesce for that value of E on the upper branch of the curve in Fig. 8. For large E there is only one intersection with the constant-E curve. Now consider a constant-c curve for $c > c_1$. For this there will be but one intersection with any constant-E curve.

For a constant-E curve with $E < E_0$, there are three intersections with constant-c curves for small c. As c increases, the two intersections more remote from the origin on the constant-E curve come together, and coalesce for that value of c on the lower branch of the curve in Fig. 8. For larger c there is but one intersection with the constant-c curve. For constant-E curves with $E_0 < E < E_1$ there is a range of c for which such a constant-E curve has three intersections with a constant-c curve. As c decreases, the two intersections nearer to the origin on the constant-E curve come together for that value of c on the upper branch of the curve in Fig. 8. For smaller c there is but one intersection. As c increases, the two intersections more remote from the origin on the constant-E curve come together for that value of c on the lower branch of the curve in Fig. 8. For larger c there is but one intersection. For a constant-E curve with $E > E_1$, there is but one intersection with any constant-c curve.

The preceding discussion is a prediction based on a technique that is known to be a good approximation only for cases in which the differential equation is nearly linear. However, we can mention the following agreements with known results.

1. For large c there is known to be but one periodic solution, and this is predicted by the above.

2. For small E the results in [9] as mentioned in §3 are in agreement with the above predictions, at least qualitatively.

3. For $c = 0$, the movement of fixed points in Case IV as predicted agrees with the result of Theorem 1.

We may also make a prediction of stability of the various fixed points predicted in the above discussion. When there is but one predicted fixed point, one would expect it to be completely stable for positive c. When there are three fixed points, at least one must be directly unstable. Extrapolating from the known results for small E in [9], one would predict that on any constant-c curve or constant-E curve, the intersections nearest to and farthest from the origin are completely stable fixed points, while the third is directly unstable. In Figs. 4 and 7, the predicted directly unstable fixed points all lie on the opposite side from the origin of the predicted locus of coalescences.

In the following section, results from a study of one example from each of Cases I and IV are given. Those fixed points that were experimentally located show qualitative agreement with the predicted number, location, and stability of fixed points.

The example from [5] cited in the introduction can be used to illustrate appearance and disappearance of periodic solutions (cf. [5], pp. 23–27). In the circuit there described, the dependent variable x is proportional to the flux in the saturable-core inductance, and E is proportional to the amplitude of the forcing voltage. The governing differential equation is

$$x'' + cx' + g(x) = E \cos t.$$

It is observed for fixed c, not too large, and for a range of E, that there are two stable steady states of oscillation of the circuit, having different amplitudes. These correspond to stable periodic solutions of the differential equation. There must also be a third, unstable, periodic solution of the differential equation, but this will not be observed experimentally. In Fig. 8, the situation corresponds to the region between the two branches of the curve. We must have $c < c_1$, and E between the lower and upper branches of the curve.

If the circuit is oscillating in the steady state with lower amplitude and E is increased, the system jumps to the higher amplitude state when E increases beyond the upper limit of the allowable range, i.e., when E goes beyond the upper branch in Fig. 8. This jump phenomenon corresponds to the disappearance of the low amplitude steady state, and to the disappearance (by coalescence) of two of the three periodic solutions of the differential equation. Similarly if the circuit is in the steady state with higher amplitude, and E is decreased, the system jumps to the lower amplitude state when E decreases beyond the lower limit of the allowable range, i.e., when E goes below the lower branch in Fig. 8. This again illustrates disappearance of periodic solutions. Appearance of periodic solutions would correspond to E entering the allowable range instead of leaving it. There is no jump phenomenon accompanying the appearance of a second steady state as there is with its disappearance.

6. Experimental results. Two specific equations were studied using the approximate methods of §5 and numerical integration. The numerical integration used the Gill [4] adaptation of the Runge-Kutta process, and was carried out on a high-speed digital computer, the Univac Scientific Computer. The equations studied had the form $x'' + cx' + g(x) = E \cos t$, where for the hardening case

$$g(x) = \begin{cases} (9/4)\,(x - 1) + \frac{1}{4} & (x > 1) \\ (1/4)\,x & (|\,x\,| \leqq 1) \\ (9/4)\,(x + 1) - \frac{1}{4} & (x < -1) \end{cases}$$

and for the softening case

$$g(x) = \begin{cases} (9/4)\,x & (|\,x\,| \leqq 1) \\ (9/4) \operatorname{sgn} x & (|\,x\,| > 1)\,. \end{cases}$$

Using the approximate methods of §5, the loci of fixed points for several values of the damping parameter c and the forcing parameter E were determined. These curves are shown in Figs. 9 and 10. Where a constant-c curve and a constant-E curve intersect there should be a fixed point for

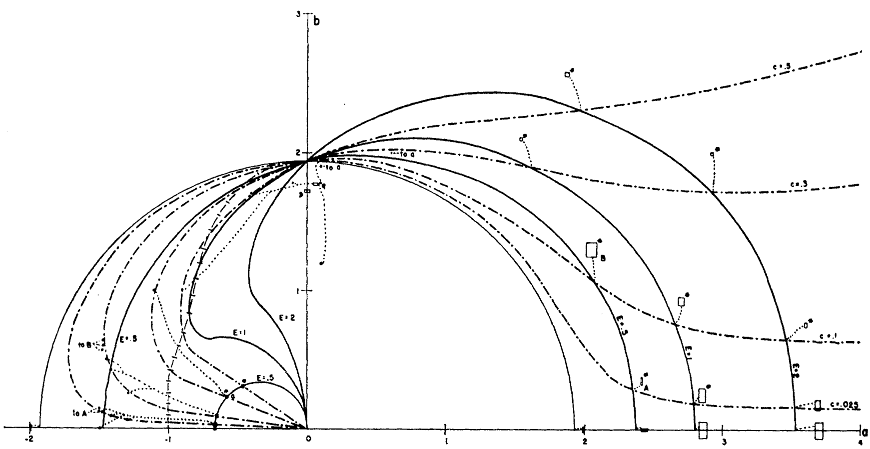

FIGURE 9. PREDICTED AND EXPERIMENTALLY DETERMINED DATA FOR THE HARDENING SYSTEM. THE PREDICTED CURVES ARE REPRESENTED AS FOLLOWS: CONSTANT –E CURVES BY ——, CONSTANT –c CURVES BY —·—·—, AND COALESCENCE LINE BY —I—I—. RECTANGLES DENOTE EXPERIMENTAL FIXED POINTS. A (•) DENOTES A POINT CONTAINING THE POINT OF COMMON INTERSECTION IN ITS ZONE OF ATTRACTION.

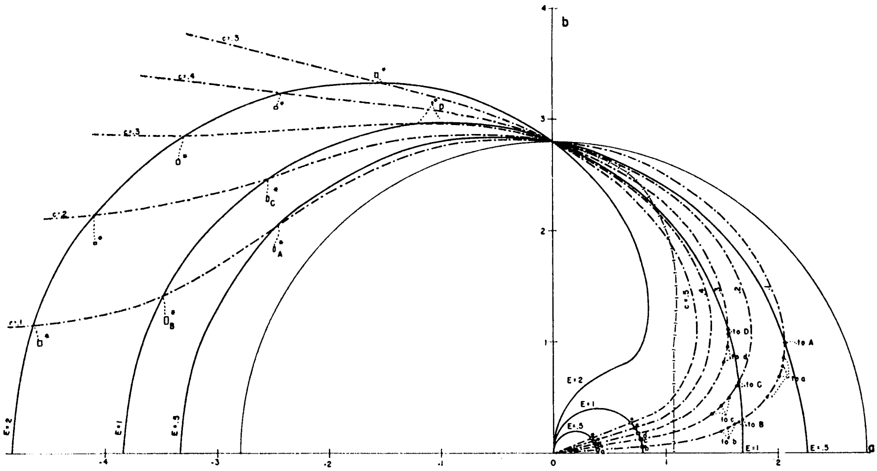

FIGURE 10. PREDICTED AND EXPERIMENTALLY DETERMINED DATA FOR THE SOFTENING SYSTEM. THE PREDICTED CURVES ARE REPRESENTED AS FOLLOWS: CONSTANT -E CURVES BY——, CONSTANT -c CURVES BY—·—·—, AND COALESCENCE LINE BY—I—I—. RECTANGLES DENOTE EXPERIMENTAL FIXED POINTS. A (*) DENOTES A POINT CONTAINING THE POINT OF COMMON INTERSECTION IN ITS ZONE OF ATTRACTION.

TABLE 1

Experimental data for the fixed points of the hardening system

c	E	a	b	a	b
		Stable Fixed Points			
0	0	1.98946 ± .00005	0.00 ± .01	−1.98946 ± .00005	0.00 ± .01
0	0	1.72 ± .01	0.00 ± .01		
0	.5	2.44 ± .03	0.00 ± .01	−.6667 ± .0001	0.00 ± .01
.025	.5	2.415 ± .005	0.35 ± .03	−.66 ± .01	0.03 ± .01
.1	.5	2.05 ± .04	1.30 ± .05	−.65 ± .01	0.09 ± .01
.3	.5			−.58 ± .01	0.23 ± .01
.5	.5			−.46 ± .01	0.30 ± .01
0	1	2.86 ± .03	0.00 ± .06		
.025	1	2.85 ± .03	0.25 ± .05		
.1	1	2.70 ± .02	0.93 ± .03		
.3	1	1.55 ± .01	2.10 ± .01		
.5	1	0.06 ± .02	1.77 ± .01		
0	2	3.70 ± .03	0.00 ± .06		
.025	2	3.69 ± .02	0.20 ± .03		
.1	2	3.60 ± .01	0.76 ± .02		
.3	2	2.92 ± .01	2.00 ± .01		
.5	2	1.87 ± .02	2.57 ± .01		
		Unstable Fixed Points			
.025	.5	−1.50 ± .02	0.09 ± .04		
.1	.5	−1.5 ± .1	0.5 ± .1		

that c and E. Note that every c-curve and E-curve passes through a common point of the b-axis, which point shall be called the $*$-point. The approximate methods were also used to find the line of coalescences. Consider the fixed points in the quadrant containing this line. The fixed points between the origin and the coalescence line should be stable, whereas points on the other side of the coalescence line should be unstable.

These approximate results were checked by integrating the equations numerically. The coordinates of the experimentally determined fixed points are given in Tables 1 and 2 and are indicated by rectangles in Figs. 9 and 10. A fixed point of the predicted stability was found at or near each point of intersection except the point of common intersection, the $*$-point. The predicted fixed points in the linear region of $g(x)$, i.e., for $|x| \leqq 1$, appeared exact within the computational error. The locations of the unstable fixed points were established from two properties. First, the un-

TABLE 2

Experimental data for the fixed points of the softening system

c	E	a	b	a	b
		Stable Fixed Points			
.1	.5	$-2.48 \pm .01$	$1.84 \pm .02$	$0.395 \pm .005$	$.030 \pm .005$
.2	.5			$0.390 \pm .005$	$.06 \pm .01$
.3	.5			$0.380 \pm .005$	$.09 \pm .01$
.4	.5			$0.36 \pm .01$	$.11 \pm .01$
.5	.5			$0.345 \pm .005$	$.140 \pm .005$
.1	1	$-3.44 \pm .02$	$1.21 \pm .02$	$0.80 \pm .01$	$.065 \pm .005$
.2	1	$-2.54 \pm .01$	$2.29 \pm .02$	$0.780 \pm .005$	$.13 \pm .01$
.3	1	$-1.06 \pm .01$	$3.15 \pm .01$	$0.755 \pm .005$	$.185 \pm .005$
.4	1			$0.725 \pm .005$	$.235 \pm .005$
.5	1			$0.69 \pm .01$	$.28 \pm .01$
.1	1.309305	$-3.82 \pm .03$	$1.10 \pm .03$	$1.08 \pm .01$	$.08 \pm .01$
.1	1.318279	$-3.83 \pm .015$	$1.10 \pm .03$		
.1	2	$-4.58 \pm .02$	$0.99 \pm .02$		
.2	2	$-4.08 \pm .01$	$1.89 \pm .01$		
.3	2	$-3.33 \pm .02$	$2.62 \pm .02$		
.4	2	$-2.46 \pm .02$	$3.11 \pm .01$		
.5	2	$-1.57 \pm .02$	$3.40 \pm .02$		
		Unstable Fixed Points			
.1	.5	$2.10 \pm .05$	$0.8 \pm .1$		
.1	1	$1.65 \pm .03$	$0.25 \pm .05$		
.2	1	$1.82 \pm .03$	$0.48 \pm .06$		
.3	1	$1.59 \pm .03$	$0.9 \pm .1$		
.1	1.309305	$1.18 \pm .04$	$0.10 \pm .04$		

stable fixed point is on the boundary between the zones of attraction of the stable fixed points. Second, the distance between the initial point and its first image under the mapping (1.3) becomes small as the initial point approaches the unstable fixed point.

We now consider in more detail the results for the hardening case. For $E = 1$ and $c = .5$, the intersection of the constant-c and the constant-E curves is so near the $*$-point that they are indistinguishable. In this case a fixed point was found fairly near the $*$-point. It was found that the $*$-point, (0, 1.9433), was in the zone of attraction of each of the experimentally located fixed points in the first quadrant. If there was no fixed point for a certain pair of values of c and E, the $*$-point was in the zone of attraction of the stable fixed point for that c and E in the second quadrant. The fixed points that contain the $*$-point in their zones of attraction are designated by an affixed $*$ in Fig. 9. The fixed points for the $*$-point were not determined for any pair of parameters where $c = 0$ or for the pair $c = .025$, $E = 2$ because in these cases the damping parameter is so much smaller than the forcing parameter that the convergence of a solution to a periodic solution is extremely slow.

The results for the softening system show many details similar to those of the hardening system. In the softening system the $*$-point, (0, 2.8027), is in the zone of attraction of each of the fixed points in the second quadrant and also in the zone of attraction of each of the stable fixed points in the first quadrant which correspond to pairs of values of E and c not possessed by any fixed point in the second quadrant.

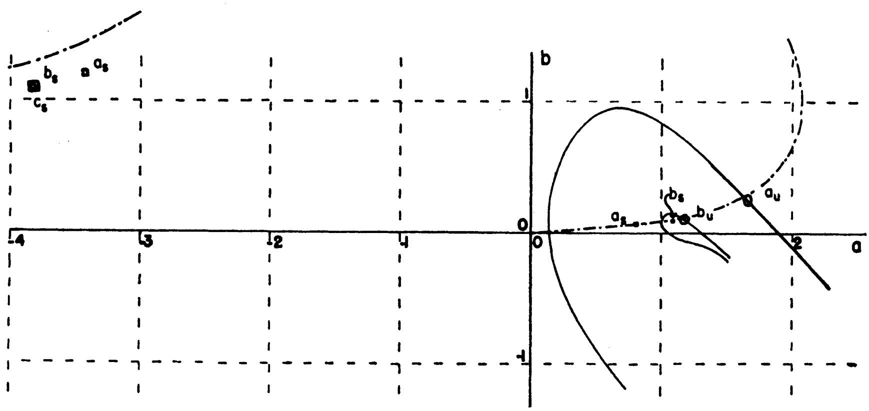

FIG. 11. *Motion of the fixed points of the softening system as E increases and* $c = .1$. The broken line is the predicted constant-c curve. The solid lines are the boundaries of the zones of attraction of the stable fixed points. The points designated by a, b, and c correspond to $E = 1$, 1.309305, and 1.318279, respectively. The subscripts indicate the stability.

For the softening system with $c = .1$ more information was obtained using additional values of E. The approximate calculations predict that as E increases (for constant c) the stable fixed point in the second quadrant should move out along the constant-c curve toward more negative a. The other stable fixed point and the unstable fixed point should approach each other along the constant-c curve as E increases until they meet and annihilate. Experimentally this qualitative behavior was observed as shown in Fig. 11. As E changed from 1 to 1.309305 to 1.318279 the second quadrant stable fixed point moved out to more negative a along a line a little below the predicted line for $c = .1$. As E took the values 1 and 1.309305 the first quadrant stable and unstable fixed points approached one another approximately along the $c = .1$ curve. For $E = 1.318279$ no fixed point was found in the first quadrant implying that when E has reached this value the stable and unstable fixed points have met and annihilated. The

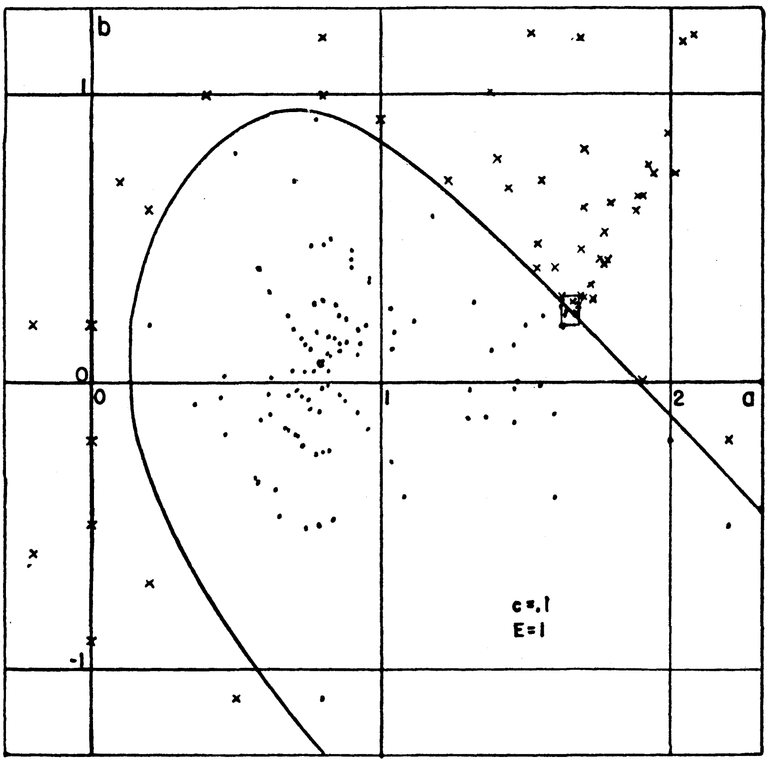

FIG. 12. *Boundary of the zones of attraction of the stable fixed points of the softening system for* $c = .1$, $E = 1$. The fixed points represented by • converge to (.80 ± .01, .065 ± .005), and those represented by × converge to (−3.44 ± .02, 1.21 ± .02). There is an unstable fixed point at (1.65 ± .03, .25 ± .05) on the boundary.

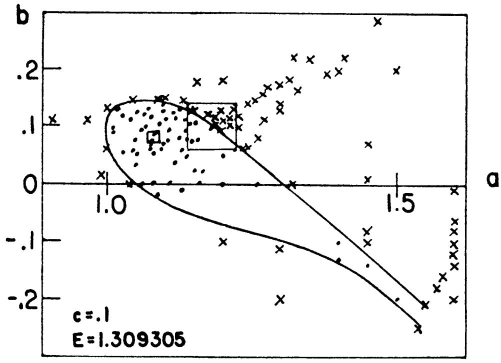

FIG. 13. *Boundary of the zones of attraction of the stable fixed points of the softening system for* $c = .1, E = 1.309305$. The fixed points represented by • converge to $(1.08 \pm .01, .08 \pm .01)$, and those represented by × converge to $(-3.82 \pm .03, 1.10 \pm .03)$. There is an unstable fixed point at $(1.18 \pm .04, .10 \pm .04)$ on the boundary.

predicted value of E for the coalescence was $E = 1.305$. The predicted point (a, b) for the coalescence was $(1.08, .09)$ and the observed point was $(1.12, .08)$.

The zones of attraction of the stable fixed points for $c = .1$ and $E = 1$ and $E = 1.309305$ are shown in Figs. 11, 12, and 13. The zone of attraction of the first quadrant stable point is much smaller for $E = 1.309305$ than for $E = 1$. The zones for these points have narrowing "tails" which must extend to infinity.

The treatment of §4 shows that the boundary between the zones of attraction of the two stable points is the invariant curve c_1, which is the locus of the image points that approach the unstable fixed point from both sides. These curves could be found by starting just on either side of the unstable fixed point and integrating the equation for decreasing time, i.e., by using negative time in the equation. However this method was found to be impractical because the locations of the unstable fixed points were not known accurately enough. The inaccuracy of the coordinates of the unstable point caused the image points to be far from the desired boundary after a very few cycles. Hence the boundaries were found by experimentally determining to which zone of attraction points near the estimated boundary belong. The points near the boundary and their successive images are shown in Figs. 12 and 13.

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