INTRODUCTION TO
Nonlinear Analysis

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PREFACE

This book is intended to provide the engineer and scientist with information about some of the basic techniques for finding solutions for nonlinear differential equations having a single independent variable. Physical systems of many types and with very real practical interest tend increasingly to require the use of nonlinear equations in their mathematical description, in place of the much simpler linear equations which have often sufficed in the past. Unfortunately, nonlinear equations generally cannot be solved exactly in terms of known tabulated mathematical functions. The investigator usually must be satisfied with an approximate solution, and often several methods of approach must be used to gain adequate information. Methods which can be used for attacking a variety of problems are described here.

Most of the material presented in this book has been given for several years in a course offered to graduate students in electrical engineering at Yale University. Emphasis in this course has been on the use of mathematical techniques as a tool for solving engineering problems. There has been relatively little time devoted to the niceties of the mathematics as such. This viewpoint is evident in the present volume. Students enrolled in the course have had backgrounds in electrical circuits and mechanical vibrations and a degree of familiarity with linear differential equations. Their special field of interest has often been that of control systems. A knowledge of these general areas is assumed on the part of the reader of this volume.

Nonlinear problems can be approached in either of two very different ways. One approach is based on the use of a minimum of physical equipment, making use only of pencil, paper, a slide rule, and perhaps a desk calculator. The degree of complexity of problems which can be successfully handled in a reasonable time with such facilities is admittedly not very great. A second approach is based on the use of what may be exceedingly complicated computing machinery of either the analog or the digital variety. Such machinery allows much more information to be dealt with and makes feasible the consideration of problems of complexity far greater than can be handled in the first way. The efficient use of computing machinery is a topic too large to be considered in the present discussion. Only techniques useful with the modest facilities available to almost everyone are considered here.

A set of problems related directly to the subject matter of each chapter is added at the end of the text.

For a variety of reasons, it has seemed desirable to concentrate the bibliographic reference material in a special section also at the end of the text, rather than have it scattered throughout the book. In this way, a few comments can be offered about each reference and an effort made to guide the reader through what sometimes appears to be a maze of literature.

As is the case in most textbooks, by far the greater part of the material discussed here has developed through the continuing efforts of many workers, and but little is original with the present author. He would like, therefore, to acknowledge at this time his indebtedness to all those who have led the way through the thorny regions of nonlinearities.

W. J. Cunningham

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CHAPTER 1

INTRODUCTION

Whenever a quantitative study of a physical system is undertaken, it is necessary to describe the system in mathematical terms. This description requires that certain quantities be defined, in such a way and in such number that the physical situation existing in the system at any instant is determined completely if the numerical values of these quantities are known. In an electrical circuit, for example, the quantities may be the currents existing in the several meshes of the circuit. Alternatively, the quantities may be the voltages existing between some reference point and the several nodes of the circuit. Since the quantities, the currents and voltages of the example, change with time, time itself becomes a quantity needed to describe the system.

For a physical system, time increases continuously, independent of other occurrences in the system, and therefore is an independent quantity or an independent variable. If the system is made up of lumped elements, such as the impedance elements of the electrical circuit, time may be the only independent variable. If the system is distributed, as an electrical transmission line, additional independent variables such as the position along the line must be considered. The independent variables may change in any way without producing an effect upon one another.

Also in a physical system are certain parameters which must be specified but which do not change further or at least which change only in a specified way. Among these parameters are the resistance, inductance, and capacitance of an electrical circuit and the driving voltage applied to the circuit from a generator of known characteristics.

The remaining quantities that describe the system are dependent upon the values of the parameters and the independent variables. These remaining quantities are the dependent variables of the system. The system is described mathematically by writing equations in which the variables appear, either directly or as their derivatives or integrals. The constants in the equations are determined by the parameters of the system. A complicated system requires description by a number of simultaneous equations of this sort. The number of equations must be the same as the number of dependent variables which are the unknown quantities of the system. If a single independent variable is sufficient, only derivatives and integrals involving this one quantity appear. Furthermore, it is often possible to eliminate any integrals by suitable differentiation. The resulting equations having only derivatives with respect to one variable are called ordinary differential equations. Where there is more than one independent variable, partial differentials with respect to any or all such variables appear and the equations are partial differential equations. Only the case of ordinary differential equations is considered in this book.

The equations are said to be linear equations if the dependent variables or their derivatives appear only to the first power. If powers other than the first appear, or if the variables appear as products with one another or with their derivatives, the equations are nonlinear. Since a transcendental function, for example, can be expanded as a series with terms of progressively higher powers, an equation with a transcendental function of a dependent variable is nonlinear.

The following equation for a series electrical circuit is a linear equation:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = E\sin\omega t \tag{1.1}$$

In this equation, the time t is the independent variable and the current i is the dependent variable. The coefficients represent constant parameters. The equation describing approximately a circuit with an iron-core inductor which saturates magnetically is a nonlinear equation,

$$L(1 - ai^2)\frac{di}{dt} + Ri = E\sin\omega t \tag{1.2}$$

The term involving $i^2\,di/dt$ is a nonlinear term.

In the simplest case, the parameters of the physical system are constants and do not change under any condition that may exist in the system. For many cases, however, the parameters are not constants. If the parameters change in some fashion with a change in value of one or more of the dependent variables, the equation becomes nonlinear. This fact is evident since the parameter which is a function of the dependent variable appears as a coefficient of either that variable or another one. Thus, a term is introduced having a variable raised to a power other than unity, or appearing as a product with some other variable. Such a term appears in Eq. (1.2), where variable i is squared and its product with the derivative di/dt appears.

A second possibility is that the parameters change in some specified way as the independent variable changes. In this case, the equation remains a linear equation but has varying coefficients. The equation for the current in the electrical circuit containing a telephone transmitter subjected to a sinusoidal sound wave is linear but has a time-varying coefficient,

$$R_1(1 + a \sin \omega t)i + R_2 i = E \tag{1.3}$$

It is possible, of course, to have both nonlinear terms and terms with varying coefficients in the same equation. This would result for the example of a circuit having both a saturating inductor and a telephone transmitter. The equation would be a combination of Eqs. (1.2) and (1.3).

The solution for a set of differential equations is a set of functions of the independent variables in which there are no derivatives. When these functions are substituted into the original equations, identities result. It is generally quite difficult, and often impossible, to find a general solution for a differential equation. There are certain special kinds of equations, however, for which solutions are readily obtainable.

The most common type of differential equation for which solutions are easily found is a linear equation with constant coefficients. These equations can be solved by applying certain relatively simple rules. Operational methods are commonly employed for this purpose. If the number of dependent variables is large, or if the equations are otherwise complicated, application of the rules may be a fairly tedious process. In theory, at least, an exact solution is always obtainable.

A most important property of linear equations is that the principle of superposition applies to them. In essence, this principle allows a complicated solution for the equations to be built up as a linear sum of simpler solutions. For a homogeneous equation with no forcing function, the solution may be created from the several complementary functions, their number being determined by the order of the equation. An equation with a forcing function has as a solution the sum of complementary functions and a particular integral. A complicated forcing function may be broken down into simpler components and the particular integral found for each component. The complete particular integral is then the sum of all those found for the several components. This is the principle allowing a complicated periodic forcing function to be expressed as a Fourier series of simple harmonic components. Particular integrals for these components are easily found, and their sum represents the solution produced by the original forcing function. Superposition is not possible in a nonlinear system.

Much of the basic theory of the operation of physical systems rests on the assumption that they can be described adequately by linear equations with constant coefficients. This assumption is a valid one for many systems of physical importance. Most of the beautiful and complicated theory of electrical circuits, for example, is based on this assumption. It is probably no exaggeration to say, however, that all physical systems become nonlinear and require description by nonlinear equations under certain conditions of operation. Whenever currents or voltages in an electrical circuit become too large, nonlinear effects become evident. Iron cores saturate magnetically; the dielectric properties of insulating materials change; the temperature and resistance of conductors vary; rectification effects occur. Currents and voltages that are large enough to produce changes of this sort sometimes prove to be embarrassingly small. It is fortunate for the analyst of physical phenomena that his assumption of linearity with constant coefficients does apply to many systems of very real interest and importance. Nevertheless, he must remember that his assumption is likely to break down whenever the system is pushed toward its limits of performance.

The procedures for finding solutions for nonlinear equations, or equations with varying coefficients, generally are more difficult and less satisfying than are those for simpler equations. Only in a few cases can exact solutions in terms of known functions be found. Usually, only an approximate solution is possible, and this solution may apply with reasonable accuracy only within certain regions of operation. The analyst of a nonlinear system must use all the facilities at his disposal in order to predict the performance of the system. Often he must be guided by intuition based on a considerable insight into the physical operation of the system, and not be dependent upon a blind application of purely mathematical formulas. It is a truism to observe that it helps greatly to know the answer ahead of time. Experimental data concerning the operation of the system may be of great value in analyzing its performance.

If the amount of nonlinearity is not too large, or if the equations are of a few special types, analytical methods may be used to yield approximate solutions. Analytical methods give the solution in algebraic form without the necessity for inserting specific values for the numerical constants in the course of obtaining the solution. Once the solution is obtained, numerical values can be inserted and the effects of wide variations in these values explored rather easily.

If the amount of nonlinearity is larger, analytical methods may not be sufficient and solution may be possible only by numerical or graphical methods. These methods require that specific numerical values for the parameters of the equations and for the initial conditions of the variables be used in the course of obtaining the solution. Thus, any solution applies for only one particular set of conditions. Furthermore, the process of obtaining the solution is usually tedious, requiring considerable manipulation if good accuracy is to be obtained. Then, if a solution for some different set of values for the numerical parameters is required, the whole process of solution must be repeated. Because of the large amount of manipulation needed in solving equations by numerical methods, a suitable digital computer is almost a necessity if many equations of this sort are to be solved.

In summary, the situation facing the analyst can be described as follows: The equations describing the operation of many physical systems can be reduced to sets of simultaneous ordinary differential equations. When the equations are linear with constant coefficients, standard rules can be applied to yield a solution. If the number of dependent variables is large, or if the order of the differential equations is large, the application of the rules may be tedious and time-consuming. A solution is possible, at least in theory. When the differential equations are nonlinear, solution by analytical methods is possible only if the amount of nonlinearity is small and, even then, available solutions are usually only approximate. Equations with a considerable degree of nonlinearity can be solved only by numerical or graphical methods.

In many practical problems of interest, the situation is so complicated that an exact solution is not possible, or at least not possible in terms of available time and effort. In such a case, it becomes necessary to simplify the problem in some way by neglecting less important effects and concentrating upon the really significant features. Often suitable assumptions can be made which lead to simplified equations that can be solved. While these equations do not describe the situation exactly, they may describe important features of the phenomenon. The simplified equations can be studied by available techniques and certain information obtained. This process may be compared with setting up a simplified mathematical model analogous to the physical system of interest. The model may be studied in various ways so as to find how it behaves under a variety of conditions. While the information so obtained may not always be sufficient to allow a complete design of a physical device, the information may well be enough to provide useful criteria for the design. This kind of procedure has proved useful in many practical situations. An electronic oscillator, for example, is usually too complicated to be studied if all its details are taken into account. However, the classic van der Pol equation describes many of the operating features of the oscillator and is simple enough so that much information can be found about its solutions. A study of the van der Pol equation has led to much useful knowledge about the operation of self oscillators.

In the chapters which follow, methods are considered for the solution of ordinary differential equations that either are nonlinear or have varying coefficients. The first methods described are based on numerical or graphical techniques. While only relatively simple cases are considered here, these methods can be extended to apply to systems with considerable complexity. Usually, more information about the system can be obtained more quickly if an analytical solution is possible. Some of the simpler analytical procedures that can be used are considered in later chapters. Much of the work that has been done with nonlinear equations is limited to equations of first and second order. For this reason, the emphasis here is on equations of these orders. The methods of solution can be extended to equations of higher order, but complications increase rapidly when this is done.

It is assumed throughout the discussion that the reader is familiar with methods of solving the simpler kinds of differential equations and, in particular, differential equations that are linear with constant coefficients. The emphasis is on the use of mathematical techniques as a tool for studying physical systems, and no attempt has been made to provide elaborate justification for the techniques that are discussed. It is assumed that the reader has a reasonable degree of familiarity with analysis of electrical circuits and mechanical systems. A single method of attacking a nonlinear problem often gives only a partial solution, and other methods must be used to obtain still more information. For this reason, the same examples appear at a number of different places in the book, where different methods of analysis are applied to them. An attempt is made, however, to keep each chapter of the book as nearly self-contained as possible.

It is, perhaps, not out of place to remark that some of the procedures commonly used by mathematicians in studying a particular problem seem at first to be unessential and, in fact, confusing. In considering a single differential equation of high order, for example, it is common practice to replace it with a set of simultaneous first-order equations. This process is usually necessary when a numerical method of solution is being employed, since most numerical methods apply only to first-order equations. For other methods of analysis, separation into first-order equations is largely a convenience in that it allows better organization of the succeeding steps in the solution. In still other cases, it is actually necessary to recombine the equations into a single higher-order equation at certain points in the solution.

Another device commonly employed is that of changing the form of certain variables in the equation. Sometimes, a change of variable replaces an original equation which could not be solved with an alternate equation which can be solved. A few nonlinear equations can be made linear by appropriate changes of variable. At other times, a change of variable is made for purposes of convenience. Often, in problems concerning physical systems, simplification is possible by combining into single factors certain of the dimensional parameters that appear. By proper choice of factors, the resulting equation can be made dimensionless. The variables in the equation are then said to be normalized. A normalized equation and its solution often present information in a more compact and more easily analyzed form than does the equation before normalization. A normalized equation is free of extraneous constants, and the constants that do remain are essential to the analysis.

It should be recognized that this book, as well as most others of its type, represents the distillation of the work done by a great many investigators over a long period of time. The discussions presented here show the effects of continued revision and retain what are thought to be the most useful and essential principles. The examples that are chosen for discussion are admittedly ones which illustrate a particular principle and are chosen because a solution for them is possible. Changes of variable and definitions of new quantities are introduced at many points in the discussion. The fact that analysis is, after all, a sort of experimental science should not be overlooked. Often, the changes that are made are found only after a great many such modifications have been tried. The one which successfully simplifies the equation is the one, of course, which appears in the discussion. An analyst faced with a new and unknown problem may have to try many approaches before he finds one that gives the information he needs.

One of the fundamental problems in the analysis of any physical system is the question of its stability. For this reason, discussions relating to stability arise in many places in the following chapters. Because of the kinds of phenomena which occur in a nonlinear system, it is not possible to use a single definition for stability that is meaningful in all cases. Several definitions are necessary, and the appropriate one must be chosen in any given situation. Since the question of stability is fundamental to almost every problem, it is a topic which ought, perhaps, to be the first considered. On the other hand, the intricacies of the question can be appreciated only after background has been acquired. Therefore, points relating to stability appear at many places throughout the book. Then, in the last chapter, these points are brought together and presented in a more complete manner. A certain amount of repetition is necessarily involved in this kind of presentation, but it does provide a closing chapter in which many basic ideas are tied together.

CHAPTER 2

NUMERICAL METHODS

2.1. Introduction. Differential equations specify relations between the variables in question and their derivatives. The simplest differential equation is of the first order and contains only the first derivative. A first-order equation is

$$\frac{dx}{dt} = f(x,t) \tag{2.1}$$

where t is the independent variable, x is the dependent variable, and $f(x,t)$ is a specified function of these variables. In many physical problems of interest, the independent variable is time, and this is the reason for the choice of symbol t in the equation above. A solution for the equation is such a function

$$x = x(t,C) \tag{2.2}$$

that when it is substituted into the differential equation an identity results. The solution for a first-order equation contains a single arbitrary constant, the quantity C of Eq. (2.2). The value of this constant can be found only by specifying an initial or boundary condition which the solution must satisfy.

The ways of solving a differential equation fall into two general classes. One method may be termed analytical and is based on the process of integration. Only relatively simple equations can be solved analytically. For example, in the equation

$$\frac{dx}{dt} = ax \tag{2.3}$$

the variables can be separated and integrated to give as a solution

$$x = C \exp (ax) \tag{2.4}$$

In these equations, a is a parameter, and C is the arbitrary constant. Numerical values of a and C are not needed to obtain the general form of the solution. The solution here does involve the exponential function exp (ax). This is really a special function defined as the solution for an equation of the type of Eq. (2.3). Because it is a function of very general utility in mathematics, its properties have been studied in detail and numerical values for it are tabulated.

The second method of finding a solution for a differential equation may be termed numerical. It is a step-by-step process and gives a solution as a table of corresponding values of the independent and dependent variables. In theory, any equation can be solved numerically. Definite numerical values of all the parameters and initial conditions must be specified, and solution for just this special case is obtained. Sometimes, a numerical solution is carried out in part by a graphical construction, but the underlying principle is unchanged. A numerical solution for Eq. (2.3) with the value of parameter a specified as $a = 2$, and with the initial condition $x = 3$ at $t = 0$, may be tabulated as follows:

t	x
0	3.00
0.5	8.15
1.0	22.17
1.5	60.27
2.0	163.80

The tabulated values of x and t are accurate to two decimal places.

Since numerical methods may be used for solving any differential equation, they are of greatest generality. On the other hand, the numerical solution of a complicated equation requires an amount of manipulation that is formidable when it must be done by hand. For this reason, automatic digital computers become essential in the solution of complicated problems. Where the equations are simpler and only a modest amount of information about the solution is required, numerical computation by hand or with the aid of a desk calculator is not unreasonable.

Many methods have been developed for finding numerical solutions for a differential equation. These methods differ in the way in which the solution is accomplished, in what kind of information is used at each step, and in the feasibility of checking errors. It is not possible in a brief space to give an exhaustive treatment of these methods. Rather, in the present discussion a few of the simpler methods are considered, together with checking procedures, with the hope of conveying at least the spirit of numerical techniques.

The basic problem is the solution of the first-order equation

$$\frac{dx}{dt} = f(x,t) \tag{2.5}$$

subject to the initial conditions that, at $t = t_0$, $x = x_0$. It is assumed here that the function $f(x,t)$ is generally continuous and single-valued and has definite derivatives. There may, however, be certain singular points where $f(x,t)$ is undefined. Singularities may occur if this function has the form $f(x,t) = f_1(x,t)/f_2(x,t)$. A singularity exists if there is a combination of values of x and t for which $f(x,t)$ becomes indeterminate with the form zero over zero. The derivative dx/dt is then indeterminate. It is shown in Chap. 5 that the location and nature of singularities are most important in determining properties of solutions for the equation.

A numerical solution for Eq. (2.5) is built up as a table of the following form:

n	t	x
0	t_0	x_0
1	$t_0 + h$	x_1
2	$t_0 + 2h$	x_2
3	$t_0 + 3h$	x_3

The index n serves to identify the particular step in the solution and takes on successive integral values. The independent variable is assigned increments, and successive values of the dependent variable are found. Often, it is convenient to have a constant increment in the independent variable, as increment h in the table. Sometimes, however, the value of the increment must be changed in the course of a solution.

In order to carry out a solution, information available at any one step must be used to extend operations to the next step. Many procedures have been proposed for carrying out this process of extrapolation. These procedures differ in the complexity of the relations used and in the size of the increment which is allowed with good resulting accuracy.

2.2. Use of Taylor's Series. Perhaps the most obvious way of extrapolating from one step in a solution of a differential equation to the following step is to make use of Taylor's series. This infinite series has the form

$$x_{n+1} = x_n + m_n h + \tfrac{1}{2} m_n' h^2 + \tfrac{1}{6} m_n'' h^3 + \cdots \tag{2.6}$$

Here, $m_n = (dx/dt)_n$, $m_n' = (d^2x/dt^2)_n$, . . . , all evaluated at the point $t_n = t_0 + nh$, and increment $h = t_n - t_{n-1}$. The derivatives m_n, m_n', m_n'', . . . can be found by differentiating the original equation and putting in values of x_n and t_n. Such a series converges fast enough to be useful for purposes of calculation only if h is chosen sufficiently small.

The procedure for using Eq. (2.6) is as follows: At the nth step in the calculation, the various derivatives are all evaluated. Increment h is assigned and substituted into the equation. The value of x_{n+1} is calculated. It then may be possible to go ahead and substitute $2h$ in place of every h in Eq. (2.6) and thereby find x_{n+2}. This process cannot be repeated many times before a sum of few terms will fail to converge with sufficient accuracy. When this happens, it is possible to move ahead to the later step in calculation, redetermine all the derivatives, and extrapolate from this later step. In practical cases, this process usually requires more work than other ways of getting the same information and is rarely used in extending a solution. The Taylor's series is often used in starting a solution to be carried on by some other method.

Example 2.1

Find a solution for the equation

$$\frac{dx}{dt} = -x + t$$

using the Taylor's series with the initial condition $x_0 = 0$ at $t_0 = 0$. Choose as the constant increment in t the value $h = 0.1$. The derivatives are

$$\begin{aligned} m &= \frac{dx}{dt} = -x + t \\ m' &= \frac{d^2x}{dt^2} = -\frac{dx}{dt} + 1 = -m + 1 \\ m'' &= \frac{d^3x}{dt^3} = -m' \end{aligned}$$

At the initial point $n = 0$, the values of the derivatives are $m_0 = 0$, $m_0' = 1$, $m_0'' = 1$. The first three values of x as found from Eq. (2.6) are then

$$\begin{aligned} x_1 &= 0 + 0 + \tfrac{1}{2} \times 1 \times 0.1^2 + \tfrac{1}{6} \times (-1) \times 0.1^3 = 0.005 \\ x_2 &= 0 + 0 + \tfrac{1}{2} \times 1 \times 0.2^2 + \tfrac{1}{6} \times (-1) \times 0.2^3 = 0.019 \\ x_3 &= 0 + 0 + \tfrac{1}{2} \times 1 \times 0.3^2 + \tfrac{1}{6} \times (-1) \times 0.3^3 = 0.040 \end{aligned}$$

where calculations have been carried to three decimal places.

There is uncertainty in the third decimal place of x_3; so it is desirable to move ahead to x_2 and extrapolate from x_2 to x_3. At x_2, the derivatives are $m_2 = 0.181$, $m_2' = 0.819$, $m_2'' = -0.819$. The value of x_3 is now found as

$$x_3 = 0.019 + (0.181 \times 0.1) + \tfrac{1}{2} \times 0.819 \times 0.1^2 + \tfrac{1}{6} \times (-0.819) \times 0.1^3 = 0.041$$

In the foregoing example, too few significant figures are being carried to ensure that the results will be accurate to three decimal places. At least one figure beyond the number desired in the final result should be carried along. If three decimals are desired, at least four decimals should be carried in the computations.

The problem of rounding off continued decimal fractions appears in any numerical work, and the following is a good consistent rule for rounding: In rounding off a number to n digits, the number should be written with the decimal point following the nth digit. Rounding is accomplished thus:

1. If the digits following the decimal point are less than 0.500 . . . , they should be dropped.

2. If these digits are greater than 0.500 . . . , unity should be added to the left of the decimal.

3. If these digits are exactly 0.500 . . . , unity should be added to the left of the decimal if the digit there is odd or zero should be added if the digit is even. In either case, the final digit following the rounding is even.

This last criterion is based on the idea of making the rounded number too large half the time and too small half the time. Furthermore, an even number often leads to fewer remainders than an odd number in any computation.

Some examples of numbers rounded to three significant figures follow:

$$153.241 = 153 \qquad 781.500 = 782$$
$$24.681 = 24.7 \qquad 6.485 = 6.48$$

2.3. Modified Euler Method. If the increment h in the Taylor's series of Eq. (2.6) is made small enough, only the first two terms of the series are sufficient to provide fair accuracy. The series then becomes

$$x_{n+1} = x_n + m_n h \tag{2.7}$$

where again $m_n = (dx/dt)_n$. This equation is obviously inaccurate, and the first term that is dropped from the Taylor's series in writing it is of order h^2.

Accuracy can be improved without increasing the complexity of the equation by the following process: The value of x_{n-1} is given by the series of Eq. (2.6) by replacing each $+h$ where it appears by $-h$. When the resulting equation is subtracted from Eq. (2.6), the result is

$$x_{n+1} = x_{n-1} + 2m_n h \tag{2.8}$$

This equation is more accurate generally, since in writing it the first term that is dropped is of order h^3, and if h is small, h^3 is much smaller than h^2.

While a numerical solution can be carried out using Eq. (2.8) alone, it is possible to improve the accuracy and at the same time provide a check on the operation by using a second equation simultaneously. This second relation is the same as Eq. (2.7) except that the mean value of m is used in going from x_n to x_{n+1}, and the equation becomes

$$x_{n+1} = x_n + \tfrac{1}{2}(m_n + m_{n+1})h \tag{2.9}$$

where the two values of m are evaluated at t_n and t_{n+1}, respectively.

The two relations, Eqs. (2.8) and (2.9), can be used together in solving the original differential equation, Eq. (2.5). The procedure is to calculate a first value of x_{n+1} using Eq. (2.8). The corresponding value of m_{n+1} is then found from the differential equation. It is now used in Eq. (2.9) to find a second value for x_{n+1}. This second value can be shown to be more accurate than the first value. In fact, this second value can be used in the differential equation once more to give a new value of m_{n+1}, which in turn can be used in Eq. (2.9) to find a new and still more accurate value of x_{n+1}. This process is sometimes known under the name of the modified Euler method of solution.

It is evident from Eq. (2.8) that information about x_{n-1} and m_n is needed in order to find x_{n+1}. This information is not available when a calculation is first begun. It is necessary, therefore, to use the Taylor's series to find x_1 and m_1 from x_0 and t_0 and the various derivatives evaluated at t_0. Once this initial step has been taken, the calculation proceeds with Eqs. (2.8) and (2.9) alone.

Example 2.2

Find a solution for the equation of Example 2.1,

$$\frac{dx}{dt} = -x + t$$

with the initial condition $x_0 = 0$ at $t_0 = 0$. Choose as the constant increment in t the value $h = 0.1$. Use the modified Euler method, making use of the value for x_1 already found in Example 2.1 by the Taylor's series. This value is $x_1 = 0.005$.

Relations used in the tabulated calculations are the following:

$$
\begin{aligned}
m_n &= -x_n + t_n \\
x_{n+1}^{(1)} &= x_{n-1} + 2m_nh && (2.8)\\
x_{n+1}^{(2)} &= x_n + (m_n)_{av}h && (2.9)\\
(m_n)_{av} &= \tfrac{1}{2}(m_n + m_{n+1}) \\
c &= x_{n+1}^{(2)} - x_{n+1}^{(1)}
\end{aligned}
$$

n	t_n	x_n	m_n	$2m_nh$	$x_{n+1}^{(1)}$	m_{n+1}	$(m_n)_{av}$	$x_{n+1}^{(2)}$	c
0	0	0	0						
1	0.1	0.005	0.095	0.019	0.019	0.181	0.138	0.019	0.000
2	0.2	0.019	0.181	0.036	0.041	0.259	0.220	0.041	0.000
3	0.3	0.041	0.259	0.052	0.071	0.329	0.294	0.070	−0.001
4	0.4	0.070	0.330	0.066	0.107	0.393	0.362	0.106	−0.001
5	0.5	0.106	0.394	0.079	0.149	0.451	0.422	0.148	−0.001
6	0.6	0.148	0.452						

In this example, calculation proceeds across each row of the table in turn. The result of applying Eq. (2.8) is designated as $x_{n+1}^{(1)}$, and the result of Eq. (2.9) is designated as $x_{n+1}^{(2)}$. The difference between these two values is the correction, tabulated as c in the last column. For good accuracy, this correction should be small, in the order of 1 or 2 in the last decimal place that is being retained. If the correction is larger, or if it grows progressively through each step, the indication is that errors are accumulating and the results are inaccurate.

An exact solution for this differential equation can be found by analytical methods as

$$x = \exp(-t) + t - 1$$

Numerical values found from this solution, making use of tabulated values for the exponential function, agree with the numerical values in the table above except for the values of x_5 and x_6. The tabulated values are one unit too small in the third decimal place. An error of this magnitude is not unexpected.

In this type of calculation, as well as in other numerical solutions of differential equations, the choice of increment h is subject to compromise. If h is chosen quite small, the accuracy of each step is high. If a given range of values of t is to be covered, a small value of h leads to the requirement of a large number of steps in the calculation and the possibility of cumulative errors increases. If h is chosen larger, the accuracy of each step is reduced but fewer total steps are required. In a calculation such as that of this example, a correction c which grows or which is too large at the beginning indicates that the value of h being used is too large and should be reduced.

2.4. Adams Method. Points obtained from the numerical solution of a differential equation might be plotted in the form of a curve. In the Euler method of solution, extrapolation from one point to the succeeding point is carried out with the implicit assumption that the curve is a straight line over the interval between points. A more accurate method of extrapolation uses, instead of a straight line, a curve represented by an algebraic equation of degree higher than 1. This is the basis for what is known as the Adams method. It allows extrapolation over a larger interval with higher accuracy than is possible with the Euler method. The relations needed in the extrapolation are correspondingly more complex.

The curve next more complicated than a straight line is a parabolic curve represented by an algebraic equation of second degree. The use of the parabolic curve in extrapolation is described here, and extension to curves of still higher degree follows in similar manner. The equation for a parabola can be written as

$$x = x_n + a(t - t_n) + b(t - t_n)^2 \tag{2.10}$$

where a and b are constants and (x_n,t_n) are coordinates of a point through which the curve must pass. In order to use the parabola for extrapolation, constants a and b must be found in terms of quantities already known. These constants can be found by differentiating Eq. (2.10) to obtain

$$\frac{dx}{dt} = a + 2b(t - t_n) \tag{2.11}$$

Now, if $t = t_n$, $dx/dt = m_n$, in nomenclature already used, and thus $a = m_n$. Similarly, if $t = t_{n-1}$, $t_{n-1} - t_n = -h$,

$$\frac{dx}{dt} = m_{n-1} = m_n - 2bh$$

and thus $b = (m_n - m_{n-1})/2h$. A combination of all these equations gives as the relation for purposes of calculation

$$x_{n+1} = x_n + h[m_n + \tfrac{1}{2}(m_n - m_{n-1})] \tag{2.12}$$

Equation (2.12) is based on a parabolic curve of second degree for extrapolation. More accurate results can be obtained by using a curve of still higher degree. By a process similar to that just employed, a curve of fifth degree can be used, with the resulting equation for extrapolation

$$x_{n+1} = x_n + h(m_n + \tfrac{1}{2}\,\Delta m_{n-1} + \tfrac{5}{12}\,\Delta^2 m_{n-2} + \tfrac{3}{8}\,\Delta^3 m_{n-3} + \tfrac{251}{720}\,\Delta^4 m_{n-4}) \tag{2.13}$$

In this equation, the symbols are defined as

$$\begin{aligned}
\Delta m_{n-1} &= m_n - m_{n-1} \\
\Delta^2 m_{n-2} &= \Delta m_{n-1} - \Delta m_{n-2} \\
\Delta^3 m_{n-3} &= \Delta^2 m_{n-2} - \Delta^2 m_{n-3} \\
\Delta^4 m_{n-4} &= \Delta^3 m_{n-3} - \Delta^3 m_{n-4}
\end{aligned}$$

These are the successive differences in the quantity m, and similar successive differences occur in many types of tabulated numerical calculations.

It is worth noting that the terms in the parentheses of Eq. (2.13) represent corrections introduced by extrapolation curves of higher and higher degree. However, terms of higher order can be dropped from these parentheses at will without changing any terms of lower order. Thus, only the first term of the parentheses leads to the linear case of Eq. (2.7); just the first two terms lead to the parabolic case of Eq. (2.12); the first three terms correspond to extrapolation with a cubic curve.

An equation analogous to Eq. (2.13) can be set up in a similar manner, working backward rather than forward so as to give

$$x_{n-1} = x_n - h(m_n - \tfrac{1}{2}\,\Delta m_{n-1} - \tfrac{1}{12}\,\Delta^2 m_{n-2} - \tfrac{1}{24}\,\Delta^3 m_{n-3} - \tfrac{19}{720}\,\Delta^4 m_{n-4}) \tag{2.14}$$

In solving a differential equation using these last two relations, the procedure is to use first Eq. (2.13) to find a value for x_{n+1}, making use of thc value of x_n and the various values of m and its differences found at x_n. The value of x_{n+1} just found can then be checked by Eq. (2.14). This value of x_{n+1} and the various values of m and its differences found at x_{n+1} become the quantities identified as x_n, m_n, and so on, on the right side of Eq. (2.14). The value of x_{n-1} found from this equation should be the x_n from which the whole step of calculation began. If this is not the case, an error is indicated and the value of x_{n+1} found from Eq. (2.13) should be revised. The smaller coefficients appearing in Eq. (2.14) imply that it is the more accurate of the two equations.

Example 2.3

Find a solution for the equation of Example 2.1,

$$\frac{dx}{dt} = -x + t$$

with the initial condition $x_0 = 0$ at $t_0 = 0$. Choose as the constant increment in t the value $h = 0.1$. Use the Adams method with cubic extrapolation, making use of the first two values of x already found in Example 2.1 by the Taylor's series. These values are $x_1 = 0.005$ and $x_2 = 0.019$.

Relations used in the tabulated calculations are the following:

$$\begin{aligned} m_n &= -x_n + t_n \\ \Delta m_{n-1} &= m_n - m_{n-1} \\ \Delta^2 m_{n-2} &= \Delta m_{n-1} - \Delta m_{n-2} \\ A &= \tfrac{1}{2}\,\Delta m_{n-1} \\ B &= \tfrac{5}{12}\,\Delta^2 m_{n-2} \\ C &= \tfrac{1}{12}\,\Delta^2 m_{n-2} \\ x_{n+1} &= x_n + h(m_n + A + B) \qquad (2.13) \\ x_{n-1} &= x_n - h(m_n - A - C) \qquad (2.14) \end{aligned}$$

n	t_n	x_n	m_n	Δm_{n-1}	$\Delta^2 m_{n-2}$	A	B	x_{n+1}	C	x_{n-1}
0	0	0	0							
1	0.1	0.005	0.095	0.095						
2	0.2	0.019	0.181	0.086	−0.009	0.043	−0.004	0.041		
3	0.3	0.041	0.259	0.078	−0.008	0.039	....	.....	−0.001	0.019
3	0.3	0.041	0.259	0.078	−0.008	0.039	−0.003	0.071		
4	0.4	0.071	0.329	0.070	−0.008	0.035	....	.....	−0.001	0.041
4	0.4	0.071	0.329	0.070	−0.008	0.035	−0.003	0.107		
5	0.5	0.107	0.393	0.064	−0.006	0.032	....	.....	−0.001	0.071
5	0.5	0.107	0.393	0.064	−0.006	0.032	−0.003	0.149		
6	0.6	0.149	0.451	0.058	−0.006	0.029	....	.....	−0.001	0.107

In this example, as in Example 2.2, calculation proceeds across each row of the table in turn. The first application of Eq. (2.13) is in the line for $n = 2$, and the result is a value for x_3. This value is entered as x_n in the line for $n = 3$ just below the line for

$n = 2$. This value of x_n and its associated values of m_n and its differences are used in Eq. (2.14) to check backward on the value for x_2. This result for x_2 is tabulated as x_{n-1} at the extreme right of this first line for $n = 3$. Since this value for x_2 in the last column to the right is the same as the value for x_2 in the third column from the left and one row above, a check on the accuracy is provided. If the value of x_2 found from Eq. (2.14) had been different from that used initially with Eq. (2.13) to find x_3, the indication would be that the value of x_3 is in error and should be revised slightly.

Since a check has been obtained on the value for x_3, a new line for $n = 3$ is begun, using Eq. (2.13) to find x_4, after which, in a line marked $n = 4$, a check is made, using Eq. (2.14). Pairs of lines corresponding to successive applications of these equations are grouped together in the table. It is obvious that in this example considerable duplication of data occurs, and this could be eliminated if desired. In the example, a check was obtained immediately at each step, and no revision of the value of x_{n+1} first found was ever required.

The numerical values of x_n given in the table agree exactly with values found from an analytical solution for the differential equation, making use of tabulated values for the exponential function, except that x_4 is 0.001 too large.

In order not to obscure the principles involved in the application of the Euler method and the Adams method, the differential equation chosen for Examples 2.2 and 2.3 is an exceedingly simple linear equation. Its exact solution is readily found analytically. It should be evident that these methods are equally applicable to nonlinear equations of first order. The solution of a nonlinear equation would proceed in a manner analogous to that used in the examples, with the only difference appearing in the complexity of the algebraic manipulation required.

2.5. Checking Procedures. The numerical work needed to solve even a simple differential equation is quite extensive. Errors are likely to creep into the computations, and some means of checking for the appearance of errors becomes essential. A variety of ways of making such checks have been devised, and only a few are considered here. In both the Euler method and the Adams method, two alternate equations are suggested to be used in turn. If the result from using these two equations is not in proper agreement, a correction is indicated. Other possible checks are now described.

a. Checking by Plotting. If the differential equation meets the assumptions made initially, its solution will be a continuous single-valued function, well behaved mathematically. Numerical data from the solution will plot graphically as a smooth curve. This is the basis for what is perhaps the simplest checking procedure. As soon as data are obtained at each step of the solution, they are plotted as a curve. Any departure from a smooth curve indicates an error. Since the accuracy with which a curve of reasonable size can be read is quite limited, a plot of this sort will not detect small errors. Gross errors can be located, however, and it is often helpful to have a picture of how the solution is progressing while it is being obtained.

b. Checking by Successive Differences. A numerical solution leads to a table of corresponding values for the independent variable t and dependent variable x. In applying checks for accuracy, it is most convenient to have h, the increment in t, be of constant value, and this is assumed to be the case for the second means of checking. Perhaps the simplest check for arithmetic errors consists in finding the corresponding increments or differences in the successive tabulated values for x. The differences of these first differences are then found, and so on.

For example, the table representing values for the solution of a particular equation might be the columns of x and t given in the accompanying table, where values for t between 0 and 5 have been computed initially.

t	x	Δx	$\Delta^2 x$	$\Delta^3 x$	$\Delta^4 x$
0	1				
1	3	2			
2	15	12	10		
3	43	28	16	6	
4	93	50	22	6	0
5	171	78	28	6	0

In the third column, headed Δx, are given the differences between successive entries in the column headed x. In the fourth column, headed $\Delta^2 x$, are given the second differences, the differences between successive entries in the third column. The third differences $\Delta^3 x$ are constant in this example, and the fourth differences $\Delta^4 x$ are zero. This result, that the differences of a certain order are constant, is typical of tabulated numerical calculations of this kind. The reason for this constant difference is as follows:

The function representing the solution for the differential equation must be continuous and single-valued. It is possible to show that, if a function $x(t)$ is continuous and single-valued in an interval $t_1 \leq t \leq t_2$, a polynomial $P(t) = A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0$ can be found so that $|x(t) - P(t)| < \epsilon$, where ϵ is a quantity as small as desired. In other words, the function $x(t)$ can be approximated as closely as desired by means of a polynomial in t. If the error ϵ is required to be very small, the degree n of the polynomial must be large but the approximation requires only a finite number of terms. In Fig. 2.1 are shown a pair of curves representing the original function $x(t)$ and the polynomial $P(t)$ approximating it within the error ϵ. Because of the foregoing argument, it must be possible to express the solution of the differential equation as a polynomial, at least within an arbitrarily small error.

The differences of nth order for a polynomial of nth degree are constant.

The proof of this statement is straightforward. The polynomial of degree n is

$$P(t) = A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0 \tag{2.15}$$

If t is increased by an amount h, the increment in $P(t)$ is

$$\begin{aligned}
\Delta P(t) &= P(t + h) - P(t) \\
&= A_n[(t + h)^n - t^n] + A_{n-1}[(t + h)^{n-1} - t^{n-1}] + \cdots \\
&\quad + A_1[(t + h) - t] + A_0(1 - 1) \\
&= A_n[nt^{n-1}h + \tfrac{1}{2}n(n - 1)t^{n-2}h^2 + \cdots + h^n] \\
&\quad + A_{n-1}[(n - 1)t^{n-2}h + \tfrac{1}{2}(n - 1)(n - 2)t^{n-3}h^2 + \cdots + h^{n-1}] \\
&\quad + \cdots + A_1 h
\end{aligned}$$

This last expression is a polynomial of degree $n - 1$ in t. Thus, the first difference of a polynomial leads to a new polynomial having its

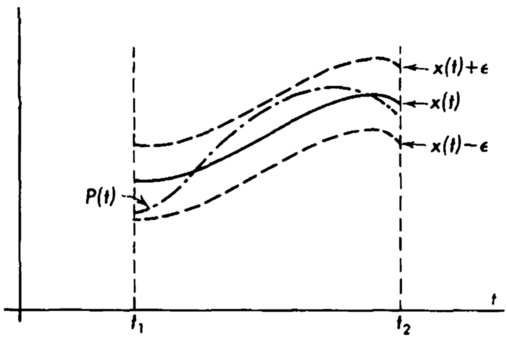


FIG. 2.1. Arbitrary function $x(t)$ and polynomial $P(t)$ approximating it within the error ϵ, over the interval $t_1 \leq t \leq t_2$.

degree reduced by one unit. It is evident that each successive differencing process reduces the degree of the polynomial one unit. If the nth difference is taken, the degree of the polynomial is reduced to zero, leaving just a constant quantity. In fact, if the process is carried out, it can be shown that the nth difference of the polynomial of nth degree is $\Delta^n P(t) = A_n n! h^n$. Differences of order higher than n are obviously zero.

The two steps of the discussion just given lead to the conclusion that the solution for the differential equation can be closely represented by a polynomial of definite degree and that therefore the tabular differences of definite order must be constant. The order of the difference which is constant depends upon the degree of the polynomial representing the solution of the equation within an accuracy determined by the number of significant figures carried in the computations.

In a typical numerical solution for a differential equation, the differences of any given order will not turn out to be exactly constant but will fluctuate around a constant value. Several effects contribute to this fluctuation. First, there are always small errors in any numerical computation, brought about by such things as the rounding off of numbers to an arbitrary number of significant figures. Further, the solution of a differential equation is only approximately represented by a polynomial, and any error here leads to a fluctuation in the differences. However, when this method of checking is applied, there should be some column of differences that are essentially constant within the number of figures carried in the computations. If an erratic fluctuation occurs, an error is indicated near the line in which this fluctuation takes place.

This method of checking does not verify that the column of values for x is, indeed, a solution for the original differential equation. It does verify, however, that the values of x represent a continuous, smoothly varying function of t, and this is an important condition that the solution must satisfy. Often, the kind of errors which arise in a computation are errors in arithmetic at some step in the process, and this kind of error is detected by the successive differences.

Example 2.4

Check the solution for Example 2.3 by the method of successive differences.

n	x_n	Δx_n	$\Delta^2 x_n$	$\Delta^3 x_n$
0	0			
1	0.005	0.005		
2	0.019	0.014	0.009	
3	0.041	0.022	0.008	−0.001
4	0.071	0.030	0.008	0
5	0.107	0.036	0.006	−0.002
6	0.149	0.042	0.006	0

The third differences here are not zero but fluctuate by about one unit in the last decimal place carried in the computation. This is about the order of constancy which can be expected.

The fact that differences of a certain order are constant can be put to a further use. Sometimes a table has been computed representing, say, the solution for a differential equation, and it is found later that additional values are necessary. If it is known that differences of a certain order have a definite constant value, this value can be used to work backward and find values of the function itself. If the differences are indeed constant, the function is found exactly. More often, the differences are only approximately constant, in which case the function is found only approximately, and extrapolation over too large a range should not be attempted.

Example 2.5

Extend the following table by making use of the fact that the third differences are constant:

	t	x	Δx	$\Delta^2 x$	$\Delta^3 x$
Values first computed...........	0	1			
	1	3	2		
	2	15	12	10	
	3	43	28	16	6
	4	93	50	22	6
	5	171	78	28	6
Values found from $\Delta^3 x = 6$.......	6	283	112	34	6
	7	435	152	40	6
	8	633	198	46	6

c. Checking by Numerical Integration. Still another method for checking the numerical solution for a differential equation requires integration. The differential equation is $dx/dt = m = f(x,t)$. If the value of x is $x = x_a$ at the point $t = t_a$, its value at the point $t = t_b$ is

$$x_b = x_a + \int_{t_a}^{t_b} m(t)\, dt \tag{2.16}$$

Values of t, x, and m are tabulated in the course of solving the differential equation. These values can be checked provided that the integral of Eq. (2.16) can be evaluated numerically. Several processes are available to carry out numerical integration.

The process of integration can be interpreted geometrically as the process of finding the area under a curve. Thus, any way of finding an area can be used for integration. In Fig. 2.2 is shown a curve representing the values of m as found in the numerical solution of the differential equation, with values of t running from t_0 to t_k. The integral $\int_{t_0}^{t_k} m(t)\, dt$ is the area under this curve. This area can be approximated by summing the areas of the trapezoids obtained by erecting vertical lines at each discrete value of t and connecting each adjacent value of m with straight lines. This is equivalent to replacing the actual curve of $m(t)$ with a series of straight-line segments. The area of each trapezoid is

$$A_1 = \frac{h}{2}(m_0 + m_1)$$

$$A_2 = \frac{h}{2}(m_1 + m_2)$$

$$\cdots\cdots\cdots\cdots$$

$$A_k = \frac{h}{2}(m_{k-1} + m_k)$$

The total area is

$$\int_{t_0}^{t_k} m(t)\,dt = h\left(\frac{m_0}{2} + m_1 + m_2 + \cdots + m_{k-1} + \frac{m_k}{2}\right) \qquad (2.17)$$

This equation is only approximately correct, since the straight-line segments only approximate the curve for $m(t)$. It can be shown that the principal error in the equation is $(h^2/12)[(dm/dt)_k - (dm/dt)_0]$, where the derivatives are evaluated at the two ends of the interval of integration. A positive error indicates that the result obtained from the equation is too large.

Equation (2.17) is known as the trapezoidal rule because it is based on the geometry of a trapezoid. It is less accurate than a similar rule for

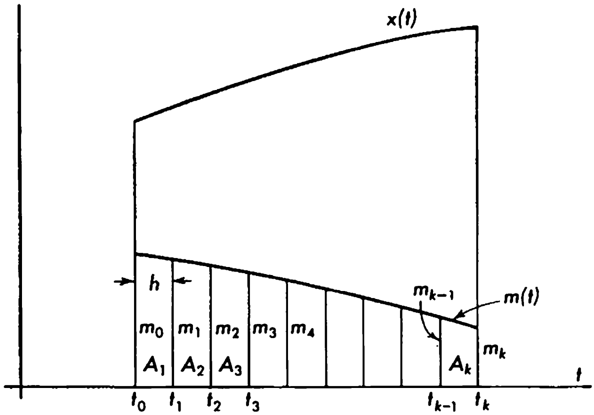


FIG. 2.2. Area under curve $m(t)$, broken into trapezoids of width h for numerical integration.

integration known as Simpson's one-third rule. This rule is obtained by connecting points on the curve of $m(t)$ with parabolic arcs, rather than straight-line segments. Provided the range of integration is divided into an even number of intervals (that is, k is even), the one-third rule is

$$\int_{t_0}^{t_k} m(t)\,dt = \frac{h}{3}[m_0 + 4(m_1 + m_3 + \cdots + m_{k-1}) + 2(m_2 + m_4 + \cdots + m_{k-2}) + m_k] \qquad (2.18)$$

This form of Simpson's rule is generally much more accurate than the trapezoidal rule and is only slightly more difficult to apply.

Either of these rules for numerical integration can be used to check values of x over every few steps of the computation used to solve a differential equation. The process verifies that the necessary integral relation does, indeed, exist between computed values of x and m.

Example 2.6

Check the solution for Example 2.3 by applying the trapezoidal rule.

$$\begin{aligned} x_6 &= x_0 + \int_0^{0.6} m(t)\,dt \\ &= 0 + 0.1\left(\frac{0}{2} + 0.095 + 0.181 + 0.259 + 0.329 + 0.393 + \frac{0.451}{2}\right) \\ &= 0.1483 \end{aligned}$$

The value of x_6 found in Example 2.3 is $x_6 = 0.149$. The difference of one unit in the last place carried in the computation is again what might be expected.

2.6. Equations of Order Higher than First. Several methods for solving first-order differential equations have been discussed. Equations of order higher than first can be solved in a similar manner by reducing them to a set of simultaneous first-order equations. This reduction requires that additional variables be defined.

For example, a third-order equation is

$$\frac{d^3x}{dt^3} + f_2(x,t)\frac{d^2x}{dt^2} + f_1(x,t)\frac{dx}{dt} + f_0(x,t) = 0 \tag{2.19}$$

where the functions f_2, f_1, and f_0 are single-valued, continuous, and differentiable. This equation can be reduced to three first-order equations by the substitutions $dx/dt = y$ and $d^2x/dt^2 = dy/dt = z$. The resulting equations are

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= z \\ \frac{dz}{dt} &= -f_2(x,t)z - f_1(x,t)y - f_0(x,t) \end{aligned} \tag{2.20}$$

These equations can be solved simultaneously making use of the methods already discussed for the first-order equation. Initial values of x, y, and z are needed to start the calculation, and solution for all three equations must be carried along simultaneously, one step at a time. The high-order equation is more tedious to solve than the first-order equation, but the principle is the same.

Example 2.7

Find a solution for the equation

$$\frac{d^3x}{dt^3} - \frac{dx}{dt} = t$$

with the initial conditions $x_0 = 1.000$, $(dx/dt)_0 = 0$, $(d^2x/dt^2)_0 = 1.000$ at $t_0 = 0$. Choose the constant increment in t as $h = 0.1$. Start the calculation with the Taylor's series, and then proceed with the Euler method.

It is first necessary to replace the original third-order equation with three first-order equations, defining new variables y and z as follows:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = z$$
$$\frac{dz}{dt} = y + t$$

Additional derivatives, together with their values at $t_0 = 0$, are

$$\frac{dz}{dt} = y + t \qquad \left(\frac{dz}{dt}\right)_0 = 0$$
$$\frac{d^2z}{dt^2} = z + 1 \qquad \left(\frac{d^2z}{dt^2}\right)_0 = 2.000$$
$$\frac{d^3z}{dt^3} = \frac{dz}{dt} \qquad \left(\frac{d^3z}{dt^3}\right)_0 = 0$$
$$\frac{d^4z}{dt^4} = \frac{d^2z}{dt^2} \qquad \left(\frac{d^4z}{dt^4}\right)_0 = 2.000$$

The values at $t_1 = h = 0.1$ are found from Taylor's series as

$$\begin{aligned}
x_1 &= x_0 + (y)_0h + \tfrac{1}{2}(z)_0h^2 + \tfrac{1}{6}\left(\frac{dz}{dt}\right)_0 h^3 \\
&= 1.000 + 0 + \tfrac{1}{2} \times 1 \times 0.1^2 + 0 = 1.005 \\
y_1 &= y_0 + (z)_0h + \tfrac{1}{2}\left(\frac{dz}{dt}\right)_0 h^2 + \tfrac{1}{6}\left(\frac{d^2z}{dt^2}\right)_0 h^3 \\
&= 0 + (1 \times 0.1) + 0 + \tfrac{1}{6} \times 2 \times 0.1^3 = 0.100 \\
z_1 &= z_0 + \left(\frac{dz}{dt}\right)_0 h + \tfrac{1}{2}\left(\frac{d^2z}{dt^2}\right)_0 h^2 + \tfrac{1}{6}\left(\frac{d^3z}{dt^3}\right)_0 h^3 \\
&= 1.000 + 0 + \tfrac{1}{2} \times 2 \times 0.1^2 + 0 = 1.010
\end{aligned}$$

Three sets of computations must be carried out at the same time, as given in the accompanying tables. Relations used with each table adjoin that particular table. The computations begin with the values of x_1, y_1, and z_1, just found, and proceed according to the following sequence: $x_{n+1}^{(1)}$, $y_{n+1}^{(1)}$, $x_{n+1}^{(2)}$, $z_{n+1}^{(1)}$, $y_{n+1}^{(2)}$, w_{n+1}, and $z_{n+1}^{(2)}$, after which the whole sequence is repeated.

$$\frac{dx}{dt} = y \qquad \begin{aligned}
x_{n+1}^{(1)} &= x_{n-1} + 2y_nh \\
x_{n+1}^{(2)} &= x_n + (y_n)_{av}h \\
(y_n)_{av} &= \tfrac{1}{2}(y_n + y_{n+1}^{(1)}) \\
c_x &= x_{n+1}^{(2)} - x_{n+1}^{(1)}
\end{aligned}$$

n	t_n	x_n	y_n	$2y_nh$	$x_{n+1}^{(1)}$	$y_{n+1}^{(1)}$	$(y_n)_{av}$	$x_{n+1}^{(2)}$	c_x
0	0.0	1.000	0						
1	0.1	1.005	0.100	0.020	1.020	0.202	0.015	1.020	0
2	0.2	1.020	0.203	0.041	1.046	0.308	0.026	1.046	0
3	0.3	1.046	0.310	0.062	1.082	0.421	0.037	1.083	0.001
4	0.4	1.083	0.423	0.085	1.131	0.543	0.048	1.131	0
5	0.5	1.131							

$$\frac{dy}{dt} = z \qquad \begin{aligned} y^{(1)}_{n+1} &= y_{n-1} + 2z_n h \\ y^{(2)}_{n+1} &= y_n + (z_n)_{av} h \\ (z_n)_{av} &= \tfrac{1}{2}(z_n + z^{(1)}_{n+1}) \\ c_y &= y^{(2)}_{n+1} - y^{(1)}_{n+1} \end{aligned}$$

n	t_n	y_n	z_n	$2z_n h$	$y^{(1)}_{n+1}$	$z^{(1)}_{n+1}$	$(z_n)_{av}$	$y^{(2)}_{n+1}$	c_y
0	0.0	0	1.000						
1	0.1	0.100	1.010	0.202	0.202	1.040	0.103	0.203	0.001
2	0.2	0.203	1.040	0.208	0.308	1.091	0.107	0.310	0.002
3	0.3	0.310	1.091	0.218	0.421	1.162	0.113	0.423	0.002
4	0.4	0.423	1.163	0.233	0.543	1.256	0.121	0.544	0.001
5	0.5	0.544							

$$\frac{dz}{dt} = w = y + t \qquad \begin{aligned} z^{(1)}_{n+1} &= z_{n-1} + 2w_n h \\ z^{(2)}_{n+1} &= z_n + (w_n)_{av} h \\ (w_n)_{av} &= \tfrac{1}{2}(w_n + w_{n+1}) \\ c_z &= z^{(2)}_{n+1} - z^{(1)}_{n+1} \end{aligned}$$

n	t_n	y_n	z_n	w_n	$2w_n h$	$z^{(1)}_{n+1}$	w_{n+1}	$(w_n)_{av}$	$z^{(2)}_{n+1}$	c_z
0	0.0	0	1.000	0						
1	0.1	0.100	1.010	0.200	0.040	1.040	0.403	0.030	1.040	0
2	0.2	0.203	1.040	0.403	0.081	1.091	0.610	0.051	1.091	0
3	0.3	0.310	1.091	0.610	0.122	1.162	0.823	0.072	1.163	0.001
4	0.4	0.423	1.163	0.823	0.165	1.256	1.044	0.093	1.256	0
5	0.5	0.544	1.256							

An exact solution for this differential equation can be found by analytical methods as

$$x = -1 + \exp(t) + \exp(-t) - \frac{t^2}{2}$$

Numerical values found from this solution differ by at most one unit in the third decimal place from the values in the tables.

Numerical solution for an equation of order higher than first is straightforward if the initial conditions are specified at a single point. Sometimes, however, boundary conditions are specified at more than one point. The specification of end points, for example, is not unusual. This may happen when the equation is of second order so that two conditions are required. These conditions may be the values of x at two particular values of t, the beginning and ending points of the solution curve for x as a function of t. While such conditions may be sufficient to determine a particular solution, they do not specify a value of dx/dt at a point where x is also known. The computation cannot be started with the assurance that it will be correct.

An example of this case is the prediction of the trajectory of a shell fired from a gun. The position of the gun and the target are known, and thus the end points of the trajectory are defined. The problem is essentially that of finding the elevation angle for the gun, which is the same as the slope of the trajectory at the beginning point. Usually, in a case of this sort it is necessary to assume a value for the slope in order to start a solution. The calculation is carried out, and it is determined how close the solution curve comes to the desired end point. It is likely that it will miss the end point; so the first estimate of the initial slope must be changed and the entire calculation repeated. It is evident that this process of successive approximations is likely to be long and tedious.

Because of the large amount of manipulation needed to solve an end-point problem by successive approximations, all possible information should be used at the beginning to make sure the first trial is as nearly correct as possible. An approximate graphical solution may be used initially to find something about the properties of the solution for the equation. If this kind of problem involves a linear differential equation, its solution can be simplified by recognizing that such an equation has as a solution the linear sum of complementary functions and a particular integral. The complementary function always includes arbitrary constants dependent upon boundary conditions. It may be possible to set up the procedure for calculation so as to find the parts of the complementary function directly, the arbitrary multiplying factors being adjusted at the end to satisfy the final end-point condition.

2.7. Summary. A great many methods have been devised for solving a differential equation numerically, and only a few methods of general utility have been considered here. These methods apply directly to first-order equations, but equations of higher order can be solved in a similar manner by reducing them to a set of simultaneous first-order equations. If all necessary conditions are specified at a single point, a numerical solution proceeds in straightforward fashion. If boundary conditions are specified at several points, a numerical solution may require tedious calculations involving successive approximations to the desired solution.

Special methods have been devised for finding solutions for certain specific forms of equations. For example, certain second-order equations can be solved directly without the requirement of breaking them into a pair of first-order equations. In particular, linear equations can often be handled by methods that are simpler than the more general methods applying to nonlinear equations.

In virtually all cases, a numerical solution requires a considerable amount of number work. One question that faces anyone setting up a numerical solution is that of deciding upon the size of increment h in the independent variable. If the solution must be known for small increments, then h is necessarily of small value. Sometimes the solution is required only at some end point. Then it is desirable to reduce the amount of manipulation so far as possible. A larger value of h reduces the number of steps in the computation but also tends to increase the error in each step. Here the analyst must choose between two alternatives. He may use a small increment, requiring many steps in the calculation but at the same time using relatively simple formulas. Alternatively, he may increase the size of the increment and reduce the number of steps by using more complicated formulas. The total amount of number work may not be too different for the two procedures.

In engineering work, numerical data are often known to only a limited degree of accuracy, so that solutions are needed with only limited accuracy. Then simple formulas and relatively large increments can be used to give a numerical solution that still meets the requirements.

When a problem can be solved by purely analytical methods, this kind of solution will usually give more information than a numerical solution requiring roughly the same amount of mathematical work. Occasionally a numerical solution may be shorter, even when an analytical solution is possible. This may be the case when the equation is reasonably complicated and its analytical solution involves complex combinations of special functions. If only a particular solution with one set of numerical parameters in the equation is desired, this one solution may be found more rapidly by purely numerical methods. An analytical solution, followed by the necessary substitutions of tabulated values of the functions involved, actually takes longer to carry out.

CHAPTER 3

GRAPHICAL METHODS

3.1. Introduction. When solutions of high accuracy are not required, it is often possible to use a graphical construction to solve a differential equation of low order. Most of the graphical methods are in principle similar to numerical methods. The difference is that some kind of geometrical construction is used to eliminate a part of the numerical computation. The accuracy is dependent upon the way the construction is performed and generally increases as the size of the figure is increased. A graphical method is usually relatively simple to utilize and may be especially useful as an exploratory tool when a nonlinear equation is first being attacked. Information about the operation of a physical system often is available only in the form of an experimentally determined curve, for which no mathematical relation is immediately known. A curve of this sort can often be incorporated directly into a graphical solution, and this may be a matter of considerable convenience. The graphical method may give much qualitative information rather quickly, and when this is known, other techniques can be used to gain additional and more exact information.

3.2. Isocline Method, First-order Equation. The basic graphical method is that known as the isocline method. The method applies directly to a first-order equation of the form

$$\frac{dx}{dt} = f(x,t) \tag{3.1}$$

It is assumed that function $f(x,t)$ is continuous and single-valued with the possible exception of certain singular points. At these singularities, $f(x,t)$ becomes indeterminate, with the form zero over zero. The isocline method of solution applies for all values of x and t, except those corresponding exactly to singularities.

The isocline method requires a graphical construction performed on axes of x and t and yields a solution as a curve, the so-called solution curve. Any parameters that appear in function $f(x,t)$ must be assigned numerical values before the construction can be started. For any given point on the xt plane, the numerical value of $f(x,t)$, and thus of dx/dt, can be calculated. Furthermore, the value of dx/dt at the point can be interpreted as the slope at the point which a curve must have if it represents a solution for the differential equation. Thus, at a given point in the plane a short line segment can be drawn with the calculated slope. This line represents a segment of the particular solution curve which passes through the point. A series of calculations can be made for many points in the plane and line segments of appropriate slope drawn through each point. Since the slope is indeterminate at a singularity, no line segment can be drawn there.

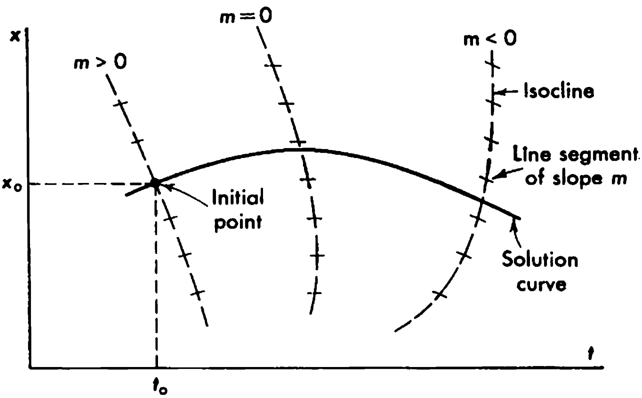


FIG. 3.1. Isoclines carrying line segments for three values of slope m. Solution curve sketched from initial point (x_0,t_0) following slope of line segments.

In the isocline method, this process is organized by assigning a specific numerical value, say, m, to the slope, thus giving the algebraic equation

$$f(x,t) = m \tag{3.2}$$

This equation fixes the locus of those values of x and t along which solution curves for the differential equation have the chosen slope m. Since a curve determined by Eq. (3.2) connects all points of constant slope, this curve is known as an isocline. If the isocline is plotted on the xt plane, line segments can be located all along it, having the chosen slope m. The numerical value of m can then be changed and a second isocline determined. In this way the entire xt plane can be filled with isoclines, each carrying directed line segments of appropriate slope. The result of such a construction is shown in Fig. 3.1. For clarity here, only three isoclines are shown, though many more are needed to give an accurate solution.

The particular curve representing the desired solution for the differential equation must be determined by a specified point on the curve.

Such a point is determined by specifying initial conditions, as $x = x_0$ at $t = t_0$. By starting at this point and sketching, always following the slope of the line segments, the solution curve can be found. If enough isoclines are set up in the beginning, and if care is used, the solution curve can be drawn with a reasonable degree of accuracy. The curve obviously may be extended in either direction from the initial point.

A different initial condition, in general, leads to a different solution curve corresponding to a different value for the one arbitrary constant which must appear in the solution for a first-order differential equation. Since it is required that the slope, given by $f(x,t)$, be single-valued at all points other than singularities, no two solution curves can cross one

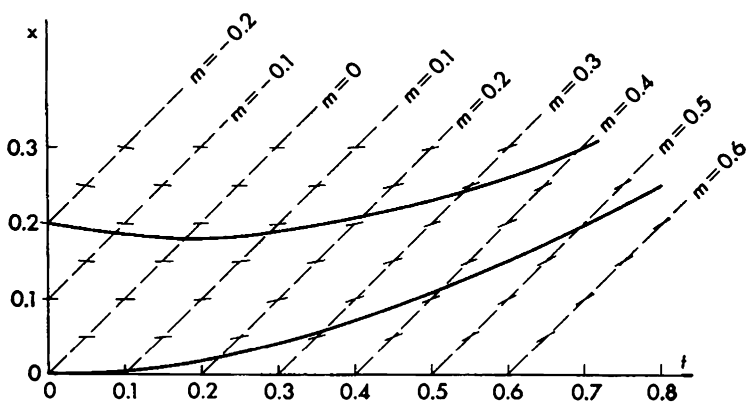


FIG. 3.2. Isocline construction for Example 3.1.

another. An infinity of solution curves may, however, approach or leave a singularity. A single initial condition is sufficient to define a unique solution curve.

Example 3.1

Find a solution for the equation used in Examples 2.2 and 2.3,

$$\frac{dx}{dt} = -x + t$$

using the isocline construction. Consider the initial conditions $x_0 = 0$ at $t_0 = 0$ and also $x_0 = 0.2$ at $t_0 = 0$.

For this linear equation, the isoclines are straight lines given by the relation $m = -x + t$. A family of isoclines, each carrying its line segments of proper slope, is shown in Fig. 3.2. Solution curves are sketched in the figure, starting at each of the specified initial points. There are no singularities for this simple equation.

It should be noted that $m = dx/dt$ is a quantity having dimensions and that the choice of numerical scales for the axes used in the construction is important. In Fig. 3.2, the scales along the x and t axes are the same, so that m can be interpreted directly as the slope of the solution curve in the geometrical sense of the word. If it were desired to expand the x scale, say, so as to display the shape of the solution curve to better advantage, this would not be so. In other words, with an expanded x scale, the particular value $m = f(x,t) = 1$ would not correspond to an angle of 45 degrees with respect to the horizontal axis. Rather, m would have to be interpreted as $m = \Delta x/\Delta t$, where Δx and Δt are increments measured along the scales of the two axes.

While the specification of an initial point is the most common way of determining a particular solution curve, other ways are also possible. For example, some sort of boundary condition such as a maximum value of x, say, x_m, might be specified. It may not be difficult to handle this kind of condition, since for a maximum $dx/dt = f(x,t) = 0$. The value of t, say, t_m, for $x = x_m$ is then determined from $f(x_m,t_m) = 0$. This resulting point (x_m,t_m) can be used as an initial point on the solution curve. Difficulty may arise if the condition $f(x_m,t_m) = 0$ leads to more than a single point. Further study of the equation is then necessary to decide which, if any, point is the desired one.

Example 3.2

Find a solution for the equation

$$\frac{dx}{dt} = x^2 + t^2 - 25$$

with the maximum value $x_m = 4$.

The value for t_m corresponding to x_m is then given by $f(x_m,t_m) = t_m^2 - 9 = 0$ or $t_m = +3$ or -3. The locus of $dx/dt = 0$ is the circle $x^2 + t^2 = 25$. Inside the circle, $dx/dt < 0$, while, outside, $dx/dt > 0$. It is evident that the point representing a local maximum of x must be $x_m = 4$ and $t_m = -3$. The solution curve must pass through this point.

Still a different kind of boundary condition is that for which the initial value of $x = x_0$, existing at $t = t_0$, is specified only as being some definite fraction of the maximum value x_m. Since values of x_0 and x_m are mutually dependent and neither is specified directly, no point is known on the solution curve. The only available procedure is to estimate a value for x_0, construct the solution curve, and find the resulting value of x_m. This value for x_m probably will not meet the required condition. It can be used to estimate a new and better value for x_0, and a second solution curve can be constructed. In this way, the desired solution curve usually can be obtained, provided, of course, that the specified condition is a possible one. The process may require several approximations.

The kind of boundary condition just described may be quite difficult to satisfy in a numerical solution for a differential equation. A preliminary exploration of the equation by the isocline method is often helpful in finding what kind of solutions may be expected and providing a guide for choosing initial numerical values.

3.3. Isocline Method, Second-order Equation. Phase-plane Diagram. In the discussion of numerical methods for solving differential equations, first-order equations are studied in the beginning. Then, equations of higher order are solved by replacing each equation of nth order by n equations of first order and solving these simultaneously. In like manner, the discussion of graphical methods has begun by considering the isocline method of solving a first-order equation. At first glance, it might appear that this method could be applied to higher-order equations in a way similar to that used with numerical methods. However, further consideration shows that this is not possible since there is not enough information available initially to construct isocline curves for the several first-order equations that would arise from a single higher-order equation.

Certain second-order equations can be solved by the isocline construction. These equations are those which can be reduced to a single first-order equation. An example is the equation

$$\frac{d^2x}{dt^2} + f\left(\frac{dx}{dt}, x\right) = 0 \tag{3.3}$$

This equation, in which independent variable t appears only in the derivatives, is called an autonomous equation. It is a particularly simple kind of equation to handle graphically. It may be reduced to a first-order equation by introducing the new variable $v = dx/dt$. Then,

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \left(\frac{dv}{dx}\right)\left(\frac{dx}{dt}\right) = v\frac{dv}{dx}$$

and Eq. (3.3) becomes

$$\frac{dv}{dx} = -\frac{f(v,x)}{v} \tag{3.4}$$

This is a first-order equation with x the independent variable, and it can be handled by the isocline method.

A specific case is the equation of simple-harmonic motion

$$\frac{d^2x}{dt^2} + \omega_0{}^2x = 0 \tag{3.5}$$

where $\omega_0{}^2$ is a positive real constant. A solution for this linear equation is well known to be

$$x = A\cos(\omega_0 t + \theta) \tag{3.6}$$

where A and θ are constants determined by initial conditions.

In a graphical solution, it is often desirable to reduce the equation to simplest possible form, and often the introduction of normalized dimensionless variables is convenient. In this equation, it is helpful to change to the new dimensionless time variable defined as $\tau = \omega_0 t$. At the same time, a further definition $\nu = dx/d\tau$ is made. The derivatives then become

$$\frac{dx}{dt} = \left(\frac{dx}{d\tau}\right)\left(\frac{d\tau}{dt}\right) = \omega_0 \nu$$

and
$$\frac{d^2x}{dt^2} = \left(\frac{d^2x}{d\tau^2}\right)\left(\frac{d\tau}{dt}\right)^2 + \left(\frac{dx}{d\tau}\right)\left(\frac{d^2\tau}{dt^2}\right) = \omega_0{}^2\frac{d\nu}{d\tau} = \omega_0{}^2\nu\frac{d\nu}{dx}$$

With these substitutions, Eq. (3.5) becomes

$$\frac{d\nu}{dx} + \frac{x}{\nu} = 0 \tag{3.7}$$

This is a first-order nonlinear equation having a variable coefficient, the independent variable being quantity x.

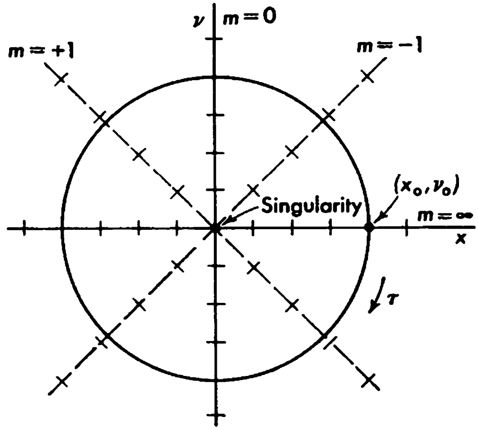


FIG. 3.3. Isocline construction for Eq. (3.7). This is a phase-plane diagram, plotting velocity as a function of displacement, with time increasing in a generally clockwise direction.

Equations of the type of Eq. (3.3) often arise in mechanical systems. Quantity x is usually the displacement of some point in the system, and $dx/dt = v$ is its velocity. Variable τ is a dimensionless measure of time t, and the derivative $dx/d\tau = \nu$ is the velocity measured in terms of τ. It is convenient to use the names displacement and velocity for quantities x and ν, even though these symbols may not refer directly to measured values in a mechanical system and a kind of normalization is involved in ν.

Isoclines can be constructed for Eq. (3.7) and evidently take the form of straight lines radiating from the origin, as shown in Fig. 3.3. All the isoclines run together at the origin, which is a singularity. The slope of a solution curve is not defined at the origin. The equation of the isocline connecting points where the solution curve has slope m is $\nu = -x/m$. The slope of the isocline is the negative reciprocal of the slope of the solution curve itself. Thus, a solution curve always crosses one of these isoclines perpendicularly.

A solution curve can be sketched through these isoclines provided an initial point, say, (x_0,ν_0), is specified. Such a curve is drawn in Fig. 3.3 and clearly is a circle. This can be shown easily by separating the variables in Eq. (3.7) and integrating to give

$$x^2 + \nu^2 = \text{constant} = x_0{}^2 + \nu_0{}^2 = R^2$$

which is the equation for a circle centered at the origin. The radius of the circle, R, is determined by the initial point (x_0,ν_0). The center of the circle is the singularity.

This figure displaying the velocity as a function of the displacement is often called a phase-plane diagram and is a figure of great utility in studying second-order systems. Instantaneous values of x and ν locate a particular point, or representative point, on a solution curve. As time progresses, the representative point moves along the solution curve, sometimes called the trajectory. In this example, the trajectory is a closed curve; so the representative point continually retraces the same path, and the solution is a periodic oscillation. The direction in which the representative point moves along the trajectory as time increases is determined by the fact that a positive velocity requires the displacement to become more positive with time. Thus, on the upper half of the phase plane where the velocity is positive, the representative point must move toward the right. Motion along the trajectory is in a generally clockwise direction. Zero velocity corresponds to a maximum or a minimum displacement; so the trajectory must cross the x axis perpendicularly.

The velocity $v = dx/dt$, or $\nu = dx/d\tau$, is the time rate of change of displacement in the system represented by the original differential equation. It must not be confused with the velocity with which the representative point moves along the trajectory. This latter velocity is called the phase velocity for the phase-plane diagram and is given by

$$\text{Phase velocity} = \frac{ds}{d\tau} = \left[\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{d\nu}{d\tau}\right)^2\right]^{1/2} \tag{3.8}$$

where s is the arc length measured along the trajectory. In terms of the original equation,

$$\text{Phase velocity} = \left\{\nu^2 + \left[\frac{f(\nu,x)}{\omega_0{}^2}\right]^2\right\}^{1/2} \tag{3.9}$$

In general, the phase velocity varies as the representative point moves along the trajectory. It is never zero except at a singularity, where both velocity $dx/d\tau$ and acceleration $[-f(\nu,x)]$ vanish simultaneously. The phase velocity for Fig. 3.3 is

$$\text{Phase velocity} = (\nu^2 + x^2)^{1/2} = R$$

and is constant along any single trajectory. The period, or interval required for a complete traverse of the trajectory, is $2\pi R/R = 2\pi$ measured in units of $\tau = \omega_0 t$. In real time, the period is $T = 2\pi/\omega_0$, as is well known.

The introduction of normalized time, $\tau = \omega_0 t$, is not essential in this kind of solution although it has several advantages. Since τ is dimensionless, the dimensions of x and of $\nu = dx/d\tau$ are the same, so that coordinates of the phase plane have the same dimensions. Often the numerical magnitude of x and ν in a given problem is similar, so that the same numerical scales occur along the two axes. In determining the slopes of solution curves, values of m from the equation are then directly the geometrical slopes to be plotted.

Example 3.3. van der Pol Equation

A classic nonlinear differential equation is the van der Pol equation, usually written

$$\frac{d^2x}{dt^2} - \epsilon(1 - x^2)\frac{dx}{dt} + x = 0 \tag{3.10}$$

This is the equation for an oscillatory system having variable damping. If the displacement x is small, the coefficient of dx/dt is negative and the damping is negative. If the displacement is large, damping becomes positive. The equation describes rather well the operation of certain types of electronic oscillators. The qualitative nature of the solution depends upon the value of parameter ϵ. If ϵ is small compared with unity, one type of solution results; if ϵ is large compared with unity, there is a different type of solution. Because of its importance, the van der Pol equation will be investigated by several different techniques, of which this is the first.

Find a phase-plane solution for the van der Pol equation with $\epsilon = 0.2$ and with $\epsilon = 5.0$ using the isocline method.

Since here the coefficient of x is unity, by analogy with Eq. (3.5) quantity ω_0^2 is unity and the normalized variables are simply $\tau = t$ and $\nu = v$. Substitution of $v = dx/dt$ leads to the equation

$$\frac{dv}{dx} = \frac{\epsilon(1 - x^2)v - x}{v}$$

The algebraic equation for an isocline curve is then

$$v = \frac{x}{\epsilon(1 - x^2) - m}$$

where m is a specified value of dv/dx, the slope of a solution curve. Isoclines found from this relation with indicated values of m are plotted in Figs. 3.4 and 3.5 for the two values of ϵ. Because the equation is nonlinear, the isoclines are curves and not simply straight lines. An infinity of isoclines come together at the origin, which is a singularity. For $\epsilon = 0.2$, the isoclines have a rather simple geometrical shape, reminiscent of Fig. 3.3. If ϵ is zero, Eq. (3.10) reduces to Eq. (3.5) with $\omega_0{}^2 = 1$, for which Fig. 3.3 applies. Line segments are located along the isoclines of the figures, drawn with the appropriate slope. For an accurate construction, more isoclines are needed than are actually shown here.

The most interesting property of the solution for the van der Pol equation is that there is a particular closed solution curve that is ultimately achieved, regardless of

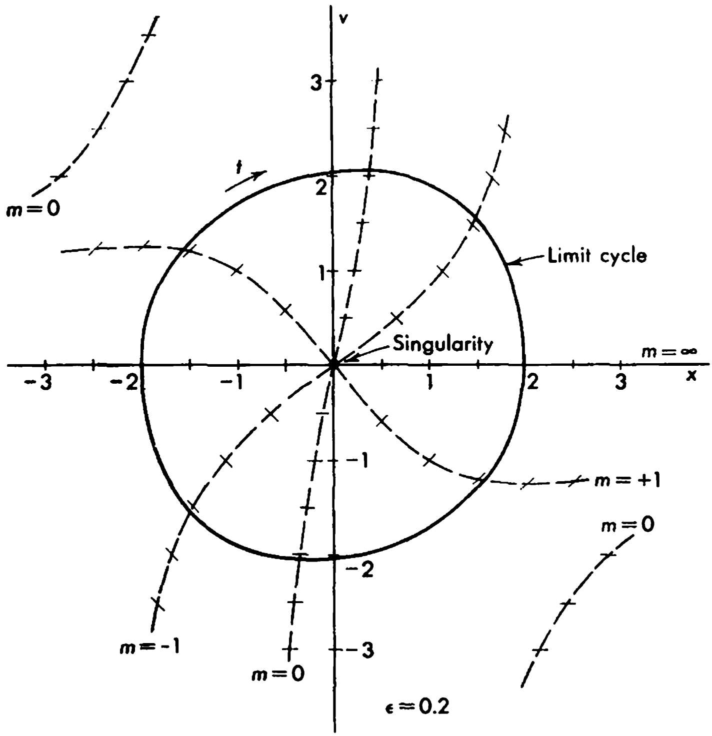


FIG. 3.4. Isocline construction and phase-plane diagram for van der Pol equation with $\epsilon = 0.2$. Only the limit cycle representing a steady-state oscillation is shown.

initial conditions. This closed curve represents a steady-state periodic oscillation, determined only by properties of the equation itself, and independent of the way the oscillation is started. It is a phenomenon characteristic of oscillators with nonlinear damping and cannot occur in a linear system. In Figs. 3.4 and 3.5, the only solution curve shown in each figure is the limit cycle. Further investigation shows that, if the initial point is inside the limit cycle, the ensuing solution curve spirals outward. If the initial point is outside the limit cycle, the solution curve spirals inward. In either case, ultimately the limit cycle is attained. This point is investigated more completely in Example 3.6. An important property of the limit cycle is that the maximum magnitude of x is always very close to two units, regardless of the value of ϵ.

3.4. Time Scale on the Phase Plane. The phase-plane curve involves time implicitly so that a time scale can be set up along a solution curve.

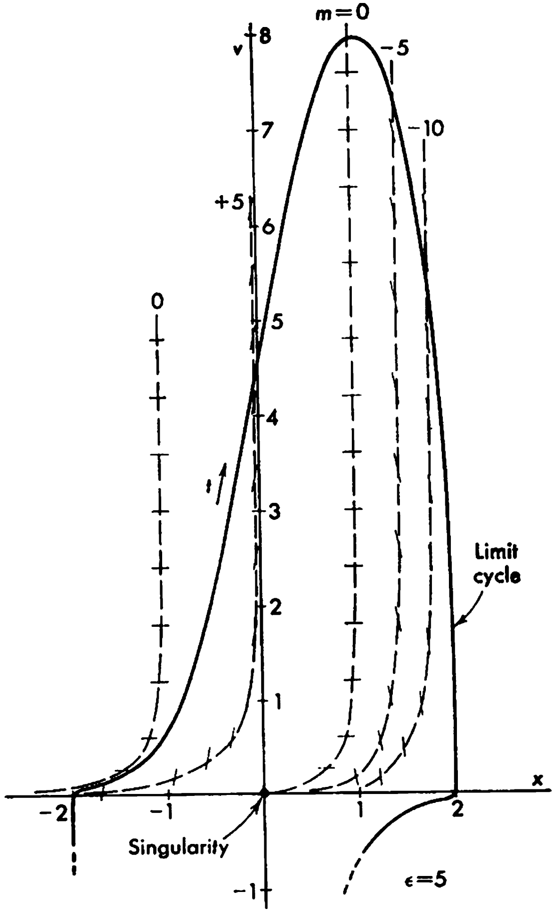


FIG. 3.5. Isocline construction and phase-plane diagram for van der Pol equation with $\epsilon = 5$. Only the upper half plane and the limit cycle are shown.

This process requires a step-by-step integration and can be performed in several ways. The most obvious way is based on the fact that, for small increments Δx and $\Delta\tau$, the average velocity is

$$\nu_{av} = \frac{\Delta x}{\Delta \tau}$$

A small increment Δx can be measured on the phase-plane curve and the corresponding ν_{av} determined. The increment in τ needed to traverse the increment Δx is then

$$\Delta\tau = \frac{\Delta x}{\nu_{av}} \tag{3.11}$$

In Fig. 3.6 is shown a portion of the phase-plane curve of Fig. 3.3. At $\tau = 0$, initial conditions are $x = x_0$ and $\nu = \nu_0 = 0$. Increments, negative here, in x are assigned as $\Delta x^{(1)}$, $\Delta x^{(2)}$, . . . , and corresponding average values of ν are found as $\nu_{av}^{(1)}$, $\nu_{av}^{(2)}$, . . . , also negative. The corresponding increments in τ are then determined from Eq. (3.11) and are positive, of course. The increments can be tabulated and summed to give the relation between x and τ. Individual points obtained this way are plotted in Fig. 3.7, where the expected cosine curve is shown. As in any numerical integration, the choice of the magnitude for Δx is a compromise. In general, Δx should be chosen small enough so that changes in x and ν during the increment are relatively small.

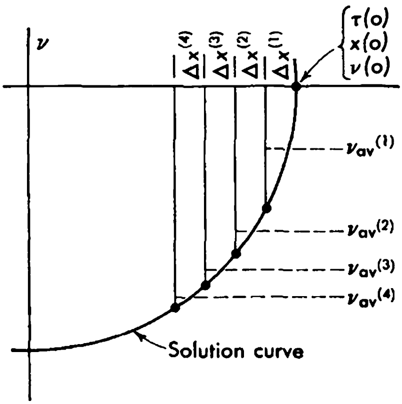


FIG. 3.6. Details of numerical integration for obtaining time scale from phase-plane diagram.

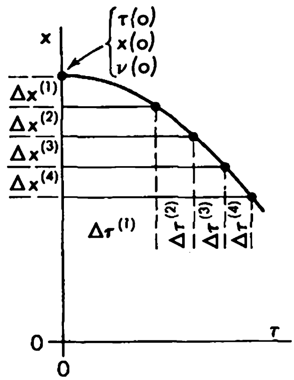


FIG. 3.7. Points obtained from numerical integration to give curve of displacement as a function of time.

A purely graphical construction can be used also for locating points equally spaced in time along a phase-plane curve. Equation (3.11) can be written

$$\Delta x = \nu_{av}\, \Delta\tau$$

If $\Delta\tau$ is small, it is nearly true that

$$\nu_{av} = \nu(0) + \frac{\Delta\nu}{2}$$

where $\nu(0)$ is the value of ν at the beginning of increment $\Delta\tau$ and $\Delta\nu$ is the change in ν during this increment. A combination of these relations gives

$$\Delta\nu = \left(\frac{2}{\Delta\tau}\right) \Delta x - 2\nu(0) \tag{3.12}$$

If a fixed value of $\Delta\tau$ is chosen, Eq. (3.12) represents a straight line of slope $2/\Delta\tau$ and $\Delta\nu$ intercept of $-2\nu(0)$, where $\Delta\nu$ and Δx are measured from $\nu(0)$ and $x(0)$, existing at the beginning of the increment. The intersection of this line with the solution curve locates the point satisfying simultaneously the original differential equation and also Eq. (3.12). Thus, the intersection is the point on the solution curve at the end of interval $\Delta\tau$.

This process can be mechanized by constructing a template in the form of a right triangle, one acute angle being $\tan^{-1}(2/\Delta\tau)$. This requires a choice for the value $\Delta\tau$. In going along the solution curve from the point $\tau = \tau(0)$ to $\tau = \tau(0) + \Delta\tau$, the point $[x(0),\nu(0)]$ at $\tau = \tau(0)$ is first located. A pin is then inserted in the paper at the point $[x(0), -\nu(0)]$. The template is placed against this pin, as shown in Fig. 3.8, with its right angle aligned with the coordinate axes. The intersection of the hypotenuse of the template with the solution curve locates the point $\tau = \tau(0) + \Delta\tau$. The process can be repeated, locating rather quickly a series of points equally spaced in time. Once more, the choice of increment $\Delta\tau$ needed to construct the template requires a compromise. Values of $\Delta\tau$ in the order of 0.2 to 0.4 are typical.

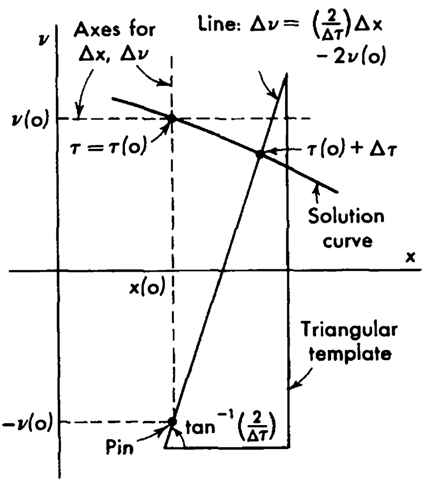


FIG. 3.8. Graphical construction for locating points spaced equal intervals $\Delta\tau$ along phase-plane curve. A triangular template is used as shown.

In solving a second-order equation such as Eq. (3.3), two integrations are necessary. The first integration is accomplished in the phase-plane diagram. The second integration requires either step-by-step numerical work or the graphical work of Fig. 3.8. Two initial conditions must be specified, the values of x and ν at some value of τ.

3.5. Delta Method, Second-order Equation. In starting a solution by the isocline method, the entire plane must be filled with line segments fixing the slope of a solution curve. If only a single solution curve is needed, only a few of these line segments are actually put to use. Considerable simplification would result if only information related directly to the desired solution curve were used. A technique of construction known as the delta (δ) method leads more directly to the desired solution.

The δ method applies to the solution of equations of the type

$$\frac{d^2x}{dt^2} + f\left(\frac{dx}{dt}, x, t\right) = 0 \tag{3.13}$$

where $f(dx/dt,x,t)$ must be continuous and single-valued but may be nonlinear and time-dependent. In applying the method, the equation is rewritten by adding and subtracting a term $\omega_0{}^2x$ to give

$$\frac{d^2x}{dt^2} + \omega_0{}^2x + f\left(\frac{dx}{dt}, x,t\right) - \omega_0{}^2x = 0 \tag{3.14}$$

The constant $\omega_0{}^2$ may be determined by the form of Eq. (3.13) itself or may have to be chosen from other information. Once again, it is convenient to introduce the definitions $\tau = \omega_0 t$ and $\nu = dx/d\tau$ so as to give

$$\omega_0{}^2\nu\frac{d\nu}{dx} + \omega_0{}^2x + [f(\nu,x,\tau) - \omega_0{}^2x] = 0$$

This can be written as

$$\frac{d\nu}{dx} = -\frac{x + \delta(\nu,x,\tau)}{\nu} \tag{3.15}$$

where

$$\delta(\nu,x,\tau) = \frac{1}{\omega_0{}^2} f(\nu,x,\tau) - x \tag{3.16}$$

Equation (3.15) is similar to Eq. (3.4) used in the discussion of the isocline method. Because of the way in which it is set up, the δ method is most immediately applicable to equations with oscillatory solutions, although it is not limited to this kind of equation.

Function δ of Eq. (3.16) depends upon all three variables ν, x, and τ, but for small changes in these variables it remains essentially constant. If δ can be assumed constant, the variables of Eq. (3.15) can be separated and integrated to give

$$\nu^2 + (x + \delta)^2 = \text{constant} = R^2 \tag{3.17}$$

This is the equation for a circle of radius R centered at the point ($\nu = 0$, $x = -\delta$). Thus, for a suitably small increment, the solution curve is the arc of a circle having these properties. The construction is shown in Fig. 3.9, where $[x(0),\nu(0)]$ represents a point on the solution curve at the time $\tau(0)$. These values used in Eq. (3.16) allow the calculation of quantity δ. This value of δ determines the center of the circular arc, located on the x axis. The radius R is automatically fixed. A short circular arc represents a portion of the solution curve. Actually, it is more

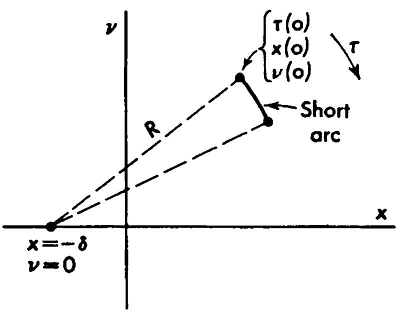


FIG. 3.9. Short arc of solution curve on phase plane as constructed by the δ method. The arc has radius R and center $(-\delta,0)$.

accurate to use the average values of ν, x, and τ existing during the increment in calculating the value of δ to be used for that increment. Again, the allowable length of arc is a compromise. In any case, it must be short enough so that the change in the variables is relatively small.

Example 3.4. Mass on Nonlinear Spring

The motion of a constant mass attached to a nonlinear spring which becomes relatively stiffer with increased deflection is described by an equation of the form

$$\frac{d^2x}{dt^2} + 25(1 + 0.1x^2)x = 0$$

Find a phase-plane solution curve for this equation by the δ method, with the initial condition that, at $t = 0$, $x = 3$ and $dx/dt = 0$.

In this equation, there is a term in x with a positive coefficient. It is simplest to take this coefficient as the value of $\omega_0{}^2$ in Eq. (3.14). Thus, for this example $\omega_0{}^2 = 25$, and $\delta = \delta(x) = 0.1x^3$, δ being dependent upon variable x alone. Axes for the phase-

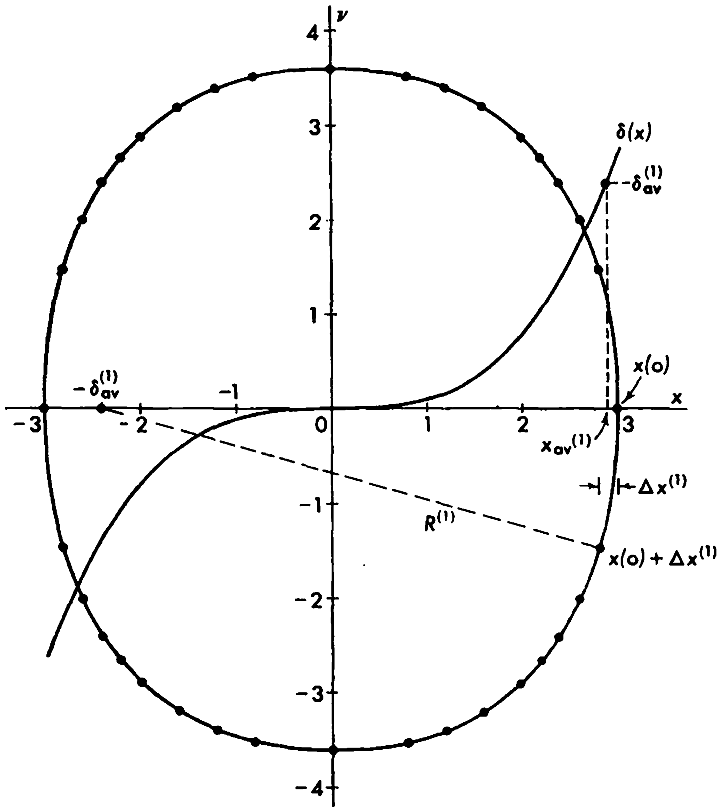


FIG. 3.10. Phase-plane diagram for Example 3.4 as constructed by δ method. The curve for nonlinear function $\delta(x)$ and the final solution curve are shown. Dots indicate junctions of successive arcs used in building up solution.

plane construction are shown in Fig. 3.10, where $\tau = \omega_0 t = 5t$ and $\nu = dx/d\tau = \frac{1}{5}\, dx/dt$. Since numerical values of δ are needed throughout the construction, it is desirable to have a curve of $\delta(x)$ at hand. Such a curve is shown in Fig. 3.10, plotted along the x axis, numerical values of δ being taken the same as numerical values of ν given along the vertical axis. For any assigned value of x, the corresponding value of δ can be read directly from this curve.

Construction proceeds as shown in the figure. The initial point is $x(0) = 3$ and $\nu(0) = 0$. As the first step, it is assumed that x decreases to the value 2.8, or $\Delta x^{(1)} = -0.2$, as shown. The average value of x during this interval is $x_{av}^{(1)} = 2.9$, for which the average value of δ can be read directly as $\delta_{av}^{(1)} = 2.4$, approximately. Thus, the center for the first circular arc is located at the point $x = -\delta_{av}^{(1)} = -2.4$ and $\nu = 0$. The radius is $R = 5.4$, and with this radius an arc is drawn to the point where $x = 2.8$, indicated by the small dot marked $x(0) + \Delta x^{(1)}$ in the figure.

By continuing in this manner, the entire solution curve can be built up as a sequence of circular arcs. Since the original equation involves no damping, the solution is a closed curve representing a periodic oscillation. Dots along the curve of Fig. 3.10 indicate the junction of successive arcs used in obtaining the complete solution curve.

The effect of the nonlinear term in the differential equation of this example is to make the phase-plane curve have a somewhat elliptical shape. A linear oscillator, described by Eq. (3.5), leads to the circular phase-plane curve of Fig. 3.3.

Example 3.5. Nonoscillatory Problem

While the δ method is most immediately applicable to equations with oscillatory solutions, it applies to other types of equations as well. The equation of this example has a solution which does not oscillate.

Find a phase-plane solution for the equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = 0$$

by the δ method with the initial conditions that, at $t = 0$, $x = 1$ and $dx/dt = 0$.

In this equation, there is no term in x with a positive coefficient. Thus, it is necessary to add and subtract such a term, the term $\omega_0^2 x$ in Eq. (3.14). The value of ω_0^2 is open to choice but is conveniently chosen of about the same magnitude as the coefficient of x in the original equation. Here a reasonable choice is $\omega_0^2 = 4$ so that $\omega_0 = 2$. With this choice, $\tau = 2t$ and $\nu = dx/d\tau = \frac{1}{2}\, dx/dt$. Function δ becomes $\delta = \nu/2 - 3x/2$ and is dependent upon both x and ν. This function can be written in two parts as $\delta = \delta_1(x) + \delta_2(\nu)$, where $\delta_1(x) = -3x/2$ and $\delta_2(\nu) = \nu/2$.

The phase plane is shown in Fig. 3.11. Plotted there is a curve of $\delta_1(x)$ using the numerical scale of ν for values of δ_1. Plotted similarly is a curve of $\delta_2(\nu)$, but for this curve the value for δ_2 is taken as the negative of x. This change of sign is made so that, if the figure were rotated 90 degrees clockwise to make the ν axis horizontal, the curve for δ_2 would fall in the first quadrant. This is the customary position for a curve of the sort $\delta_2 = \nu/2$. Values of δ_1 and δ_2 can be read from these curves. Since the original differential equation is linear, these curves are linear.

The initial point is $x(0) = 1$ and $\nu(0) = 0$. Since δ depends upon both x and ν, here, it is not possible to find an average value for δ during the first interval in quite so simple a fashion as was possible in the preceding example. Instead, a process of successive approximation is required. A reasonable procedure is to use values of x and ν existing at the beginning of the interval to determine a trial value of δ. Here the trial values are $\delta_1(x) = -1.5$ and $\delta_2(\nu) = 0$, so that $\delta = -1.5$. Thus, the trial arc is centered at $x = 1.5$ with radius $R = 0.5$. A short arc with these properties is drawn. Average values of x and ν for this arc are then used to find more accurate values for δ. By continuing in this way, a consistent set of values can soon be obtained. For this first arc of Fig. 3.11, $x_{av}^{(1)} = 1.1$, $[\delta_1]_{av}^{(1)} = 1.65$, $\nu_{av}^{(1)} = 0.2$, $[\delta_2]_{av}^{(1)} = 0.1$, so that $\delta_{av}^{(1)} = -1.55$. This value was used in drawing the first arc of the figure. By proceeding in this way, the solution curve can be extended as far as desired. Dots of Fig. 3.11 show the junctions of circular arcs. Once the trend of the curve is established, the successive approximation needed to find the average value of δ for an arc usually requires only two attempts.

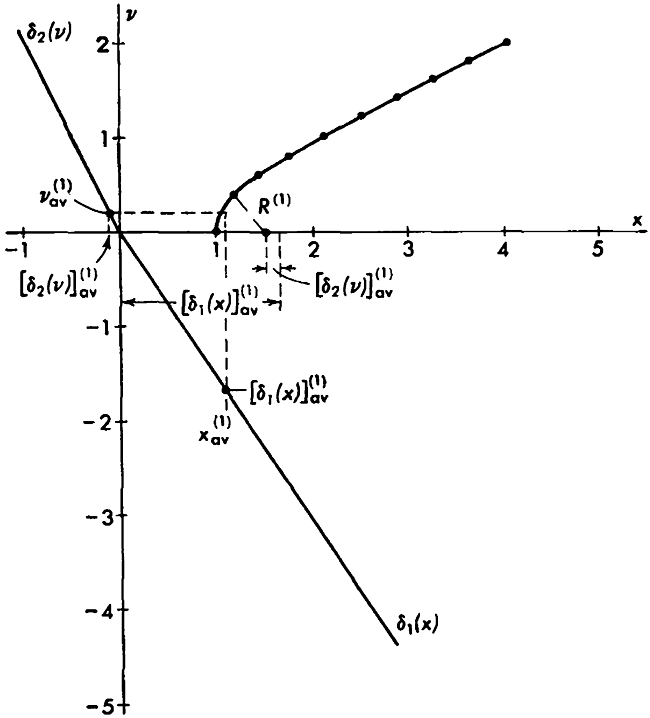


FIG. 3.11. Phase-plane diagram for Example 3.5 as constructed by δ method. Curves for functions $\delta_1(x)$ and $\delta_2(\nu)$ and the final solution curve are shown. Dots indicate junctions of successive arcs used in building up solution.

For the equation of this example, the values of x and ν continue to increase without limit, and there is no oscillation.

In Eq. (3.13), function $f(dx/dt,x,t)$, and ultimately δ also, has been allowed to depend upon the three variables dx/dt, x, and t. In Example 3.5, δ depends upon the two variables dx/dt and x. An equation with all three variables appearing in δ could be handled in much the same way. It would be necessary to keep track of t, or τ, as the solution was built up, so that the average value of this variable could be used to find the appropriate δ at each step. In general, a process of successive approximations, much like that of Example 3.5, would be necessary.

It should be noted that in the δ method it is essential to normalize the time scale to $\tau = \omega_0 t$ and to work with the corresponding normalized velocity $\nu = dx/d\tau$. It is only in this way, and with equal numerical scales along the x and ν axes, that the solution curve can be built up as arcs of circles in the manner described. Furthermore, with equal scales on the axes, the function δ typically takes on such numerical values that it may be plotted directly on the axes of the phase plane, as was done in Examples 3.4 and 3.5.

The second step in solving any of these second-order equations is to find x as a function of τ. This requires a second integration, which can be carried out in either of the ways described in Sec. 3.4. An alternate way of finding increments in τ is also possible in the δ method.

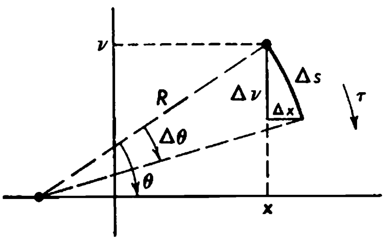


FIG. 3.12. Construction for δ method, showing equivalence of angle increment $\Delta\theta$ and time increment $\Delta\tau$.

In Fig. 3.12 is shown the construction for one short interval. Since time increases in a clockwise direction on the phase plane, the radius line R rotates in a clockwise direction. A positive increment Δs along the solution curve and a positive increment $\Delta\theta$ in angle of the radius line R are likewise taken in the clockwise direction of increasing τ. This positive direction for measuring θ and $\Delta\theta$ is, of course, opposite to the usual positive direction for measuring an angle.

The small right triangle in the figure having Δs as its hypotenuse is similar to the large right triangle with R as its hypotenuse. Thus, the relation can be written

$$\frac{\Delta x}{\Delta s} = \frac{\nu}{R}$$

Also, by definition of the angle $\Delta\theta$, $\Delta\theta = \Delta s/R$. Finally, the time increment is $\Delta\tau = \Delta x/\nu$. A combination of these relations leads to the result

$$\Delta\tau = \Delta\theta \tag{3.18}$$

valid for a very small increment, which is a requirement inherent throughout the method.

Actually, since $\Delta\tau$ must be small, $\Delta\theta$ will be a small angle and difficult to measure. Thus, Eq. (3.18) is of limited practical utility in giving information for x as a function of τ, and one of the methods of Sec. 3.4 is generally preferable. Equation (3.18) is of interest in a theoretical way in that it shows a direct correspondence between the time increment and a measurable geometrical quantity appearing in the construction.

3.6. Lienard Method. It is worth remarking that the δ method is an extension of an earlier method known as the Lienard method, which is of historical importance. The Lienard method is applicable to equations that can be reduced to the form

$$\frac{d\nu}{dx} = -\frac{x + \phi(\nu)}{\nu}$$

Here, function $\phi(\nu)$ is limited to dependence upon ν alone, while in the δ method the corresponding function called δ may involve ν, x, and τ. The Lienard method is included in the δ method, and the construction is identical.

3.7. Change of Variable. The solution of a differential equation can often be simplified by an appropriate change of variable. This comment is especially applicable to nonlinear equations. Unfortunately, the optimum change to be made is usually not obvious and often can be found only by trial and error.

One change of variable that is helpful where it can be used is to substitute for the dependent variable an integrated form of this variable. The reasoning here is that the process of integration tends to smooth out irregularities in a function. In general, the smoother a function is, the simpler it is to handle mathematically. Conversely, differentiation tends to accentuate irregularities in a function and make it more difficult to handle.

Example 3.6. Rayleigh Equation

Solve the van der Pol equation, already considered in Example 3.3, using the change of variable $y = \int x\,dt$. Use the δ method of construction, and consider the two cases of $\epsilon = 0.2$ and $\epsilon = 5$.

If $y = \int x\,dt$, then $x = dy/dt$ and the van der Pol equation [Eq. (3.10)] becomes

$$\frac{d^3y}{dt^3} - \epsilon\left[1 - \left(\frac{dy}{dt}\right)^2\right]\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$$

Each term can be integrated with respect to t and the constant of integration set arbitrarily to zero so as to give

$$\frac{d^2y}{dt^2} - \epsilon\left[\frac{dy}{dt} - \tfrac{1}{3}\left(\frac{dy}{dt}\right)^3\right] + y = 0 \tag{3.19}$$

If the constant of integration were allowed to be other than zero, it would merely introduce a constant term in the solution. Equation (3.19) is known as Rayleigh's equation. It is equivalent to the van der Pol equation but is simpler because the nonlinear term involves only dy/dt instead of both x and dx/dt. Since $x = dy/dt$, Eq. (3.19) can be rewritten for the δ method as

$$\frac{dx}{dy} = -\frac{y + \delta}{x}$$

where $\delta = -\epsilon(x - x^3/3)$. Since δ depends only upon x, the δ method is simple to apply.

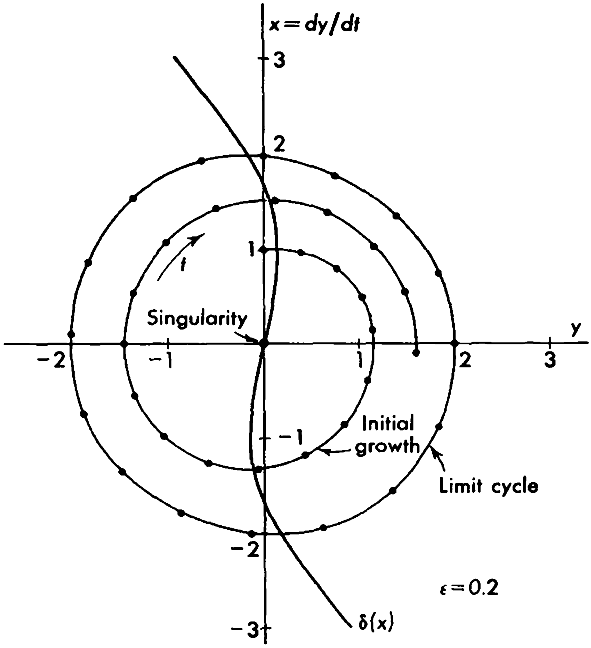


Fig. 3.13. Phase-plane diagram for Rayleigh equation with $\epsilon = 0.2$ obtained by δ method. Curves for nonlinear function $\delta(x)$ and solution curves for initial growth and the limit cycle are shown. Dots indicate points spaced equal increments $\Delta t = 0.4$ in time.

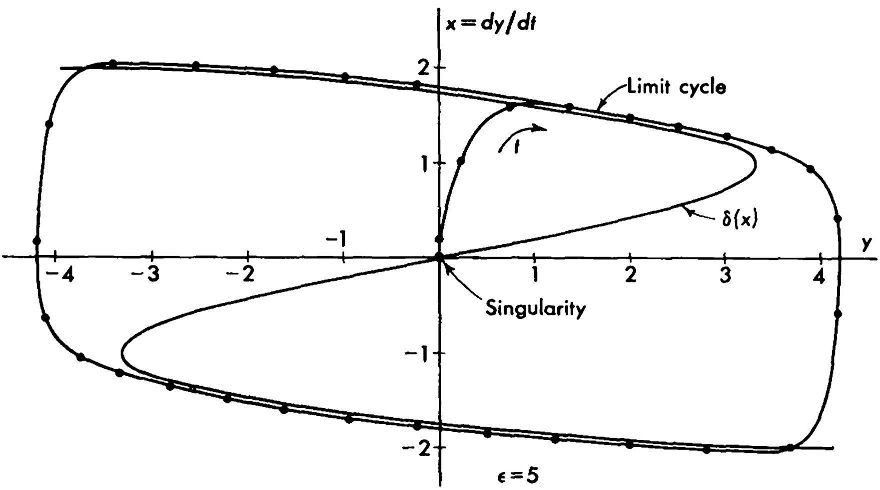


Fig. 3.14. Phase-plane diagram for Rayleigh equation with $\epsilon = 5$ obtained by δ method. Curves for nonlinear function $\delta(x)$ and solution curve for initial growth to limit cycle are shown. Dots indicate points spaced equal increments $\Delta t = 0.4$ in time.

Constructions are shown in Figs. 3.13 and 3.14 for the two values of ϵ. Curves for $\delta(x)$ are plotted so that values of δ are read as the negative of numbers along the y axis. In Fig. 3.13 for $\epsilon = 0.2$, the limit cycle is shown, and also a portion of a solution curve starting at $x = 1$, $y = 0$. This curve spirals outward and ultimately would coalesce with the limit cycle. Similarly, an initial point outside the limit cycle would lead to a curve spiraling inward until it, too, would coalesce with the limit cycle. In Fig. 3.14 for $\epsilon = 5$, a curve starting at $x = 0.2$, $y = 0$ is shown. It very quickly leads to the

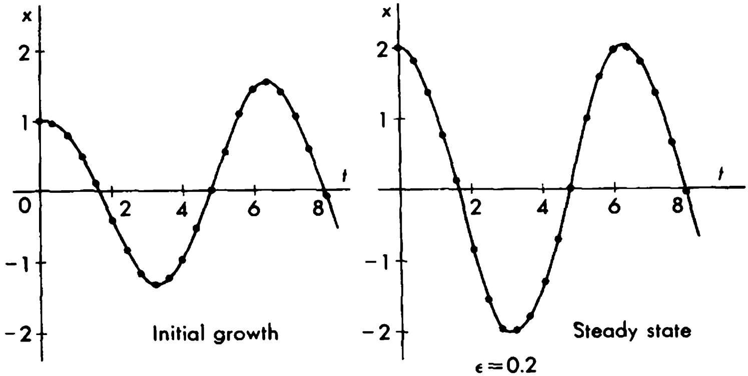


FIG. 3.15. Solution for van der Pol and Rayleigh equations with $\epsilon = 0.2$. Dots are points transferred directly from Fig. 3.13.

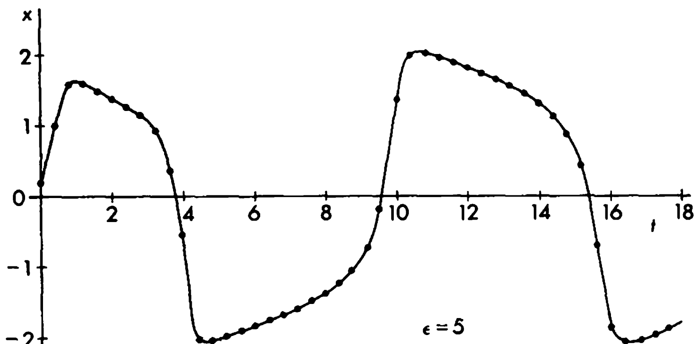


FIG. 3.16. Solution for van der Pol and Rayleigh equations with $\epsilon = 5$. Dots are points transferred directly from Fig. 3.14.

limit cycle. For ϵ small, a number of cycles of the oscillation are needed before the steady state, represented by the limit cycle, is achieved. For ϵ large, a steady state is reached almost at once. The relative simplicity of Figs. 3.13 and 3.14 for Rayleigh's equation is evident when they are compared with Figs. 3.4 and 3.5 for van der Pol's equation.

In both Figs. 3.13 and 3.14, a series of points equally spaced in time are shown constructed by the graphical method of Sec. 3.4 with the choice $\Delta t = 0.4$. Data from these points are transferred to the axes of x and t of Figs. 3.15 and 3.16. These curves show clearly the almost sinusoidal waveform for the case of $\epsilon \ll 1$ and the highly nonsinusoidal waveform for the case of $\epsilon \gg 1$. The latter waveform, characterized by skewed flattened peaks and abrupt transitions, is known as a relaxation oscillation. For ϵ small, the period in τ units is not far from 2π, its value for $\epsilon = 0$. For ϵ large, the period increases continuously as ϵ is increased.

It is worth noting that, if ϵ is very small, the limit cycles for both the Rayleigh equation and the van der Pol equation are almost circular in shape. If ϵ is made larger, the limit cycle for the Rayleigh equation has the simpler shape. If ϵ is made extremely large, the phase plane for the Rayleigh equation approaches the shape shown in Fig. 3.17. The Z-shaped curve for δ becomes broad and flat, with its points of infinite slope occurring for large magnitudes of y. The limit cycle stays quite close to this curve on the branches of negative slope and jumps abruptly from one branch to the other at the corners of infinite slope. This observation is of importance in estimating the period for large values of ϵ.

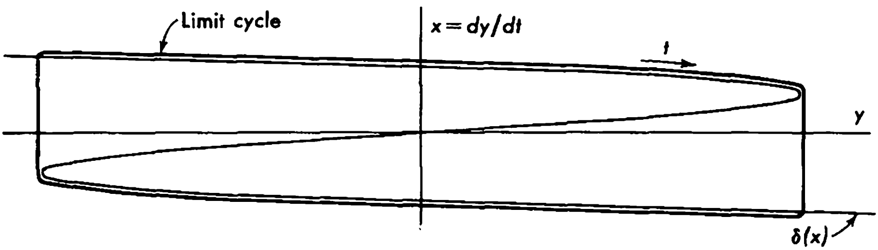


FIG. 3.17. Phase-plane diagram for Rayleigh equation with ϵ a large number. The solution curve closely follows the curve for $\delta(x)$, with abrupt transitions at points of infinite slope.

3.8. Graphical Integration.

The isocline method was described as being applicable to equations of the form $dx/dt = f(x,t)$. Two simpler forms of this equation may occur and can be solved by graphical integration.

The simplest form is

$$\frac{dx}{dt} = f(t) \tag{3.20}$$

which can be integrated directly to give

$$x = \int f(t)\,dt + C \tag{3.21}$$

where C is the constant of integration. Alternatively, the integral can be written

$$x_b = x_a + \int_{t_a}^{t_b} f(t)\,dt \tag{3.22}$$

where (x_a,t_a) and (x_b,t_b) are corresponding values. The integral can be found as the area under the curve of $f(t)$ as shown in Fig. 3.18. The curve $f(t)$ can be plotted and the area beneath it, shown shaded in the figure, can be found by such procedures as counting squares or using a planimeter. Alternatively, the area can be found by numerical integration, as by the trapezoidal rule. If a curve of x as a function of t is required, small increments in t must be taken so as to give a number of

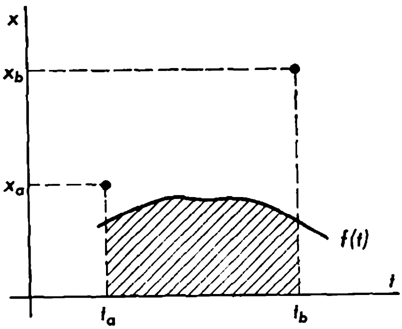


FIG. 3.18. Graphical integration for solving Eq. (3.20).

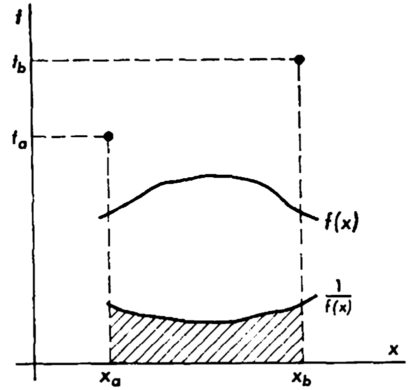


FIG. 3.19. Graphical integration for solving Eq. (3.24).

points along the curve for $x(t)$. The trapezoidal rule can then be simplified to become

$$x_b = x_a + \tfrac{1}{2}[f(t_a) + f(t_b)](t_b - t_a) \tag{3.23}$$

where $t_b - t_a$ is small.

A second simple equation is

$$\frac{dx}{dt} = f(x) \tag{3.24}$$

for which the solution is

$$t_b = t_a + \int_{x_a}^{x_b} \frac{dx}{f(x)} \tag{3.25}$$

Here integration must be performed with respect to x, whereas in Eq. (3.22) it was performed with respect to t. The corresponding construction is shown in Fig. 3.19. Again, if $(x_b - x_a)$ is small,

$$t_b = t_a + \frac{1}{2}\left[\frac{1}{f(x_a)} + \frac{1}{f(x_b)}\right](x_b - x_a) \tag{3.26}$$

Equations having the form of Eq. (3.24) often appear in electronic circuits where a nonlinear resistance, contributed by a vacuum tube perhaps, is used in a linear circuit with some type of reactor. An example of this kind is given in Example 3.7.

3.9. Preisman Method. The type of equation just described can be handled through a somewhat different approach closely related to graphical constructions often used for handling the nonlinear algebraic equations of electronic circuits. This method may be designated as the Preisman method[1] and it applies to equations of the type

[1] It should be noted that the procedure described here differs in slight detail from that given in reference 72 in the Bibliography at the end of the volume.

$$a\frac{dx}{dt} + bx + f(x) = c \tag{3.27}$$

where a, b, and c are constants and $f(x)$ is some generally nonlinear function of x. Equation (3.27) is the same as Eq. (3.24) except that the terms are written in slightly different form. This kind of equation often arises in connection with electrical circuits, where variable x is a voltage or current and $f(x)$ is a nonlinear relation, often known only as a curve found from experimental measurements.

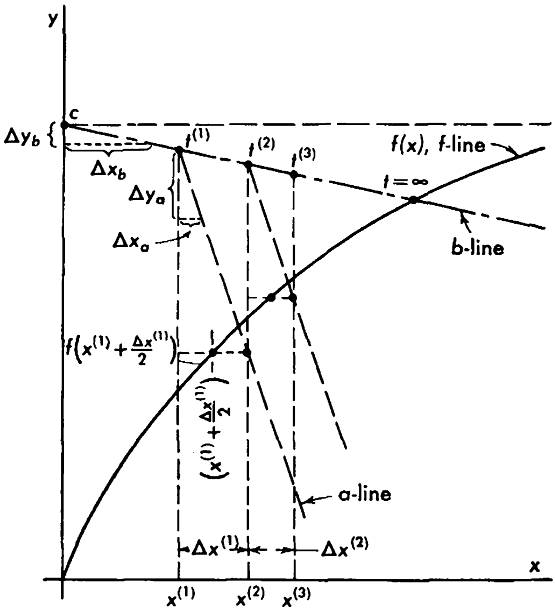


FIG. 3.20. Graphical construction of Preisman method of solving Eq. (3.27).

Solution for Eq. (3.27) is carried out on axes of y and x, where y is a quantity having the physical dimensions of $f(x)$. For example, in an electrical circuit x may be a voltage and y a current. A curve representing $f(x)$, identified as the f curve, is plotted on these axes as shown in Fig. 3.20. The point is located on the y axis, where $y = c$. At this point, a straight line, identified as the b line, is drawn with its slope determined by constant b so that $\Delta y_b/\Delta x_b = -b$, as shown in the figure. The intersection of this b line with the f curve determines the ultimate values of x and y after a very long time has elapsed. These values exist when there is no further change, so that $dx/dt = 0$ and a steady state exists. This construction is commonly used in studying electronic circuits.

The time-varying part of the solution is found in step-by-step fashion. At some point in the course of the solution, variable x has the value $x^{(1)}$, and the succeeding increment in x is $\Delta x^{(1)}$ occurring in time Δt. The equation relating these quantities is, from Eq. (3.27),

$$a\frac{\Delta x^{(1)}}{\Delta t} + b\left(x^{(1)} + \frac{\Delta x^{(1)}}{2}\right) + f\left(x^{(1)} + \frac{\Delta x^{(1)}}{2}\right) = c \tag{3.28}$$

Here the derivative is replaced by the ratio of finite increments, and the second and third terms are evaluated at the center of the increment in x. Equation (3.28) can be rearranged to give

$$\left(\frac{a}{\Delta t} + \frac{b}{2}\right)\Delta x^{(1)} = c - f\left(x^{(1)} + \frac{\Delta x^{(1)}}{2}\right) - bx^{(1)} \tag{3.29}$$

The terms on the right side of Eq. (3.29) can be found from the f curve and the b line already described. The term on the left side requires yet another line to be drawn on the figure. The slope of this line, identified as the a line, is determined by the relation $\Delta y_a/\Delta x_a = -(a/\Delta t + b/2)$. It is drawn from the point on the b line corresponding to the abscissa $x^{(1)}$ as shown in Fig. 3.20. A choice for the value of Δt must be made before the a line can be drawn. This value of Δt is subject to the usual compromises and should be such that a reasonable number of steps will occur in the solution. The increment $\Delta x^{(1)}$ is found, as shown in the figure, so that the f curve bisects the line representing $\Delta x^{(1)}$. The value of x for the beginning of the next step is $x^{(2)} = x^{(1)} + \Delta x^{(1)}$.

Construction proceeds in this way, building each step upon the result of the previous step. If a constant value of Δt is used, every a line has the same slope. The graphical work can be simplified by making a template in the form of a right triangle with the hypotenuse having the appropriate slope. This template can be used to draw each a line. Since the value of t is known at each step in the construction, it is a simple matter to transfer data directly to a curve giving x as a function of t.

Example 3.7. Diode Circuit

In the circuit of Fig. 3.21, the diode has a nonlinear relation between voltage e_b across it and current i through it. This relation is given only as the curve found experimentally and shown in Fig. 3.22. At the time $t = 0$, the switch in the circuit is closed, and the current, initially zero, starts to flow. Find the current as a function of time by graphical integration and by the Preisman method.

The equation for the circuit is

$$L\frac{di}{dt} + Ri + e_b(i) = E$$

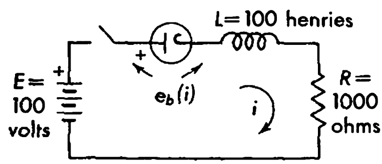


FIG. 3.21. Nonlinear diode circuit for Example 3.7.

This is of the form of Eq. (3.24), where variable i is analogous to x and $f(i) = [E - Ri - e_b(i)]/L = 1 - 10i - e_b(i)/100$. Graphical integration is immediately appli-

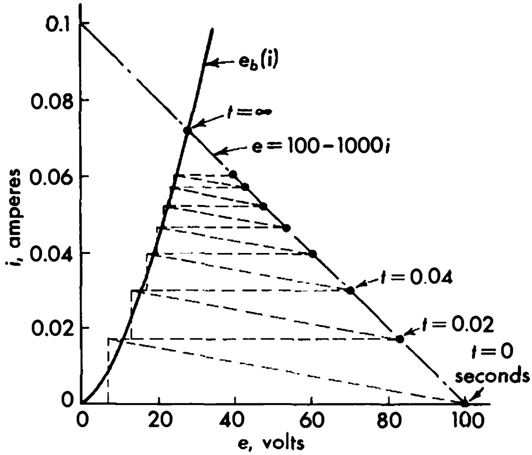


FIG. 3.22. Graphical construction for obtaining $f(i)$ for Example 3.7 and for Preisman method.

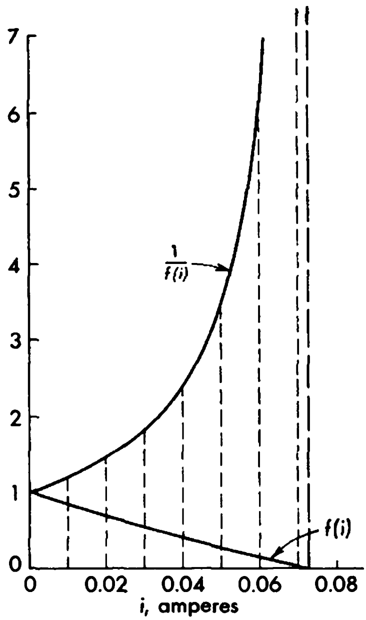


FIG. 3.23. Graphical integration of $1/f(i)$ for Example 3.7.

cable. The function $f(i)$ can be read directly from Fig. 3.22 by constructing the line $e = E - Ri = 100 - 1{,}000i$ on the axes of the figure. This line is the b line for the Preisman method. The horizontal distance between the curve for $e_b(i)$ and this b line, at any value of i, corresponds to the quantity $100 - 1{,}000i - e_b(i)$, which is evidently $100f(i)$. Values of $f(i)$ can be read in this way and are plotted in Fig. 3.23 as a function of i. Finally, the reciprocal $1/f(i)$ is also plotted in Fig. 3.23. The area under this last curve, taken from $i = 0$ to any chosen value of i, represents the time for the current change to occur.

Increments in current, $\Delta i = 0.01$ amp, are shown forming small regions beneath the curve for $1/f(i)$, and the area of each region is found by Eq. (3.26). The resulting time increments are used to plot the points indicated by crosses in Fig. 3.24, giving the variation of i with respect to t. Since the curve $1/f(i)$ goes to infinity for $i = 0.073$ amp, this current represents the ultimate steady state in the circuit, which is reached only after an infinite time has elapsed.

In applying the Preisman method, the equation for the circuit is rewritten as

$$\left(\frac{L}{\Delta t} + \frac{R}{2}\right) \Delta i = E - f\left(i + \frac{\Delta i}{2}\right) - Ri$$

Since vacuum-tube curves are commonly plotted with the axes oriented as in Fig. 3.22, the construction here is rotated 90 degrees with respect to that of Fig. 3.20. The

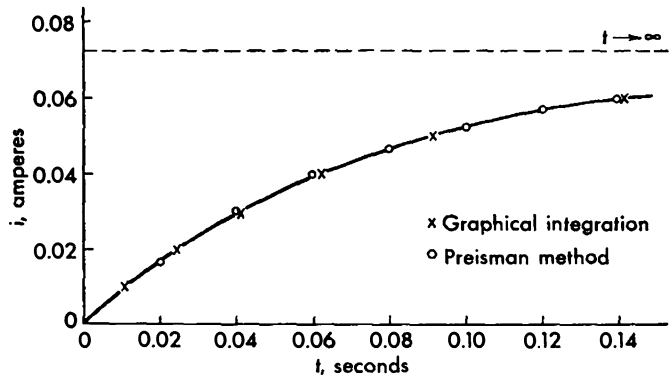


FIG. 3.24. Solution for Example 3.7 as obtained by graphical integration and by Preisman method.

b line has already been drawn as described for the first method, using the equation $e = 100 - 1{,}000i$. Its angle depends upon the ratio $\Delta e_b/\Delta i_b = -R = -1{,}000$ ohms. The slope of the a line depends upon the choice of increment Δt, and this is taken arbitrarily as $\Delta t = 0.02$ sec. With this choice, the angle of the a line is determined by the ratio $\Delta e_a/\Delta i_a = -(L/\Delta t + R/2) = -(100/0.02 + 1{,}000/2) = -5{,}500$ ohms. A triangular template is prepared with its angle giving these increments on the scales of Fig. 3.22. This template is used to draw the first a line, starting at the point on the b line corresponding to the initial current, $i = 0$. The current at the end of the first increment in t is located as described in Sec. 3.9. Successive values of current, equally spaced in time, are found by continuing in this way. Values so found are plotted as circles in Fig. 3.24. The two sets of points in this figure, found by the two methods of solution, are in good agreement.

3.10. Summary. The graphical methods described in this chapter apply directly to first-order equations, or to certain second-order equations that can be reduced to first-order equations. The methods can be extended to equations of higher order provided that these equations can be broken down into a sequence of equations of order no higher than the second. There have been many graphical methods devised for solving special equations, and the procedures described here are merely typical of those of rather general utility.

All the graphical methods are step-by-step processes. The number of steps needed to give a desired solution depends upon the size of the increments used. In general, the increments should be small enough so that only relatively small changes take place within an increment. If the increments are too small, however, construction is difficult to perform unless drawings of large dimensions are used. Whenever a solution is of such a nature that a great many increments are required, accuracy is likely to be poor. Small, unavoidable errors at each step tend to accumulate, and the latter portion of the solution is likely to become quite inaccurate.

Where the solution can be found without the necessity for too many increments, and where a high degree of accuracy is not essential, or even desired, a graphical process may lead to a solution relatively quickly and easily. It is often particularly useful in doing exploratory work with a new problem, since the qualitative features of the solution can often be determined fairly quickly. Those features of special interest can then be investigated by other more accurate, but more time-consuming, methods if this is desired.

CHAPTER 4

EQUATIONS WITH KNOWN EXACT SOLUTIONS

4.1. Introduction. In the two chapters preceding, several methods for solving differential equations by the use of numerical or graphical procedures have been considered. A single application of these methods gives only a single solution for one set of numerical parameters and initial conditions in the equation. A change of any of these quantities requires an entirely new solution. The process becomes tedious with even a simple equation if information concerning the solution with a wide range of parameters is desired. Counterbalancing the large amount of work is the fact that solutions may be obtained for equations that cannot be solved in any other way.

An analytical solution in algebraic form, even one that is only approximate, often allows information to be obtained relatively easily concerning effects of changes in parameters. For this reason, it may be worthwhile to spend considerable effort attempting to find an analytical solution. The approach needed to obtain such a solution may not be nearly so obvious as that for a numerical method, and considerable ingenuity may be required before a solution is achieved.

The present chapter contains a brief review of analytical techniques that can be used to find exact solutions for certain classes of differential equations. It includes a short discussion of systems which can be described by linear equations having known exact solutions, but where a different set of equations is required as certain variables of the system pass through critical values. Exact solutions can be found for such piecewise linear systems. The chapter closes with a discussion of equations which have solutions in terms of the specially defined elliptic functions.

4.2. Specific Types of Equations. Some of the standard forms for simple differential equations are listed here. In all cases, the various functions that appear are assumed to be single-valued, differentiable, and generally well behaved mathematically. Whether or not the indicated integrations can actually be carried out in a given equation depends, of course, upon the nature of the functions involved. If the functions are too complicated, integration in closed form may well prove to be impossible. In this section, the independent variable is given the symbol x and the dependent variable the symbol y, since these are the symbols commonly used in this kind of discussion.

a. Equation Directly Integrable

$$\frac{d^n y}{dx^n} = f(x) \tag{4.1}$$

This equation can be integrated directly to give

$$\frac{d^{n-1}y}{dx^{n-1}} = \int f(x)\,dx + C_1$$

thus reducing the order of the equation by 1. A total of n integrations is needed to remove the derivative, and n arbitrary constants, $C_1, \ldots, C_n$, appear.

Example

$$\begin{aligned}\frac{d^2y}{dx^2} &= ax^2 + \exp(bx) + c\\ \frac{dy}{dx} &= \frac{ax^3}{3} + \frac{1}{b}\exp(bx) + cx + C_1\\ y &= \frac{ax^4}{12} + \frac{1}{b^2}\exp(bx) + \frac{cx^2}{2} + C_1x + C_2\end{aligned}$$

o. First-order Linear Equation

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{4.2}$$

This is the standard linear equation of first order, although it may have a variable coefficient. All terms can be multiplied by the factor $\exp[\int P(x)\,dx]$ and integrated to give

$$y = C\exp[-\int P(x)\,dx] + \exp[-\int P(x)\,dx]\int Q(x)\exp[\int P(x)\,dx]\,dx$$

Again C is an arbitrary constant.

The process of introducing the multiplying factor is a common device for putting an equation into a form that can be integrated. Unfortunately, there is no general way of determining ahead of time just what the integrating factor should be. Ingenuity is called for here.

Example

$$\begin{aligned}\frac{dy}{dx} + axy &= bx\\ y &= C\exp\left(\frac{-ax^2}{2}\right) + \exp\left(\frac{-ax^2}{2}\right)\int bx\exp\left(\frac{ax^2}{2}\right)dx\\ &= C\exp\left(\frac{-ax^2}{2}\right) + \frac{b}{a}\end{aligned}$$

c. Higher-order Linear Equation with Constant Coefficients

$$f(D)y = Q(x) \tag{4.3}$$

Here the derivative operator is defined so that $D^n = d^n/dx^n$ and $f(D)$ is a polynomial in D of the type

$$f(D) = a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0$$

where the a's are all constants. This equation is the standard linear equation with constant coefficients. It is basic to the analysis of a large number of physical systems, and many techniques are available for its solution. Often operational methods that are quite straightforward in their application are used to obtain the solution.

In general, the solution consists of two parts, the complementary function and the particular integral. The complementary function is the so-called transient part of the solution. It is the complete solution if $Q(x) = 0$ and contains the same number of arbitrary constants as the order of the equation. Because the exponential function retains its form upon differentiation, the complementary function is built up from exponentials. The particular integral is the so-called steady-state solution and is determined completely by the equation itself, with the given $Q(x)$. For many physical systems, the transient solution ultimately disappears, leaving only the steady-state solution.

The principle of superposition applies to an equation of this kind. This principle states that, if the forcing function $Q(x)$ consists of several parts, each of these parts leads to one particular integral. The complete particular integral is merely the sum of the integrals for each component of $Q(x)$. Application of this principle allows the solution for a complicated $Q(x)$ function to be found as the superposition of relatively simpler functions. One important reason for the ease of solving linear equations in general is that this principle of superposition applies to them. The principle is not valid for nonlinear equations.

Linear equations with constant coefficients are considered further in Sec. 5.3.

Example

$$(D^2 + 5D + 6)y = \sin 4x$$
$$(D + 3)(D + 2)y = \sin 4x$$
$$y = C_1 \exp(-3x) + C_2 \exp(-2x) + \frac{1}{D^2 + 5D + 6} \sin 4x$$
$$= C_1 \exp(-3x) + C_2 \exp(-2x) - \tfrac{1}{25} \cos 4x - \tfrac{1}{50} \sin 4x$$

d. Equation with Variables Separable

$$\frac{dy}{dx} = \frac{f_1(x)}{f_2(y)} \tag{4.4}$$

Here the variables can be separated and integrated to give

$$\int f_1(x)\,dx = \int f_2(y)\,dy + C$$

where C is the usual constant of integration.

Example

$$\frac{dy}{dx} = \frac{\exp(ax)}{\sin by}$$

$$\frac{\exp(ax)}{a} = -\frac{(\cos by)}{b} + C$$

This form of solution does not give y explicitly.

e. Equation from an Exact Differential

$$\frac{dy}{dx} + \frac{f_1(x,y)}{f_2(x,y)} \tag{4.5}$$

This equation involves the exact differential of some function, $\phi(x,y) = C$, if the terms of the equation satisfy the Cauchy relation,

$$\frac{\partial f_1(x,y)}{\partial y} = \frac{\partial f_2(x,y)}{\partial x}$$

This relation is evident, since the differential of $\phi(x,y)$ is

$$d\phi = \left(\frac{\partial\phi}{\partial x}\right) dx + \left(\frac{\partial\phi}{\partial y}\right) dy = 0$$

This is equivalent to the original equation provided that the identification is made, $\partial\phi/\partial x = f_1(x,y)$ and $\partial\phi/\partial y = f_2(x,y)$. A further differentiation with respect to y and to x gives the identity $\partial^2\phi/(\partial y\,\partial x) = \partial^2\phi/(\partial x\,\partial y)$, which is the Cauchy relation. Thus, if the relation is satisfied, a function $\phi(x,y) = C$ can be found representing a solution for the equation. It may be necessary to group the terms of the equation suitably, and otherwise to use some skill, before the solution can be recognized.

Example

$$\frac{dy}{dx} = \frac{2ax + by + g\exp(gx)}{bx + 2hy}$$

Here, $\partial f_1/\partial y = b = \partial f_2/\partial x$, so that the Cauchy relation is satisfied. Furthermore, $\int f_1\,dx = ax^2 + bxy + \exp(gx)$ and $\int f_2\,dy = bxy + hy^2$. Thus, the solution in implicit form is

$$\phi = ax^2 + bxy + hy^2 + \exp(gx) = C$$

f. Equation with Only Second Derivative

$$\frac{d^2y}{dx^2} = f(y) \tag{4.6}$$

The terms of this equation can be multiplied by the factor

$$2\,(dy/dx)\,dx = 2\,dy$$

and integrated to give

$$\int \left(\frac{d^2y}{dx^2}\right) 2 \left(\frac{dy}{dx}\right) dx = \int 2f(y)\,dy + C$$
$$\left(\frac{dy}{dx}\right)^2 = 2 \int f(y)\,dy + C$$

The square root can be taken of each side of this equation and the variables separated and integrated as in the equation of Sec. 4.2*d*.

This kind of equation occurs in simple problems in mechanics, where variable x is replaced by t, the time, and variable y represents the displacement. The second derivative d^2y/dt^2 is the acceleration, the first derivative dy/dt is the velocity, and the velocity squared $(dy/dt)^2$ is proportional to the kinetic energy of the system.

Example

$$\frac{d^2y}{dx^2} = ay + by^2$$
$$\int \left(\frac{d^2y}{dx^2}\right) 2 \left(\frac{dy}{dx}\right) dx = \int 2(ay + by^2)\,dy + C$$
$$\left(\frac{dy}{dx}\right)^2 = 2\left(\frac{ay^2}{2} + \frac{by^3}{3}\right) + C$$

g. Second-order Equation without Dependent Variable Alone

$$f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, x\right) = 0 \tag{4.7}$$

In this second-order equation, the dependent variable y does not appear alone, outside the derivatives. The equation can be reduced to a first-order equation by the substitution $p = dy/dx$, giving $f(dp/dx, p, x) = 0$. This first-order equation may fall into one of the preceding classes for which solution is possible.

Example

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + ax = 0$$
$$\frac{dp}{dx} + xp + ax = 0 \qquad \text{where } p = \frac{dy}{dx}$$

This equation is that of Sec. 4.2*b*, for which the solution is

$$p = C_1 \exp\left(\frac{-x^2}{2}\right) - a = \frac{dy}{dx}$$

Thus,
$$y = -ax + C_1 \int \exp\left(\frac{-x^2}{2}\right) dx + C_2$$

This integral cannot be found in closed form, although the exponential function can be expanded in a series and integrated term by term to give

$$y = -ax + C_1\left(x - \frac{x^3}{6} + \frac{x^5}{40} - \cdots\right) + C_2$$

which is valid for x small.

h. Second-order Equation without Independent Variable Alone

$$f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y\right) = 0 \tag{4.8}$$

In this second-order equation, the independent variable x does not appear alone outside the derivatives. As in the preceding case, the equation can be reduced to first order by the substitution $p = dy/dx$ and $d^2y/dx^2 = dp/dx = (dp/dy)(dy/dx) = p\,dp/dy$, giving

$$f(p\,dp/dy, p, y) = 0$$

This kind of equation is generally more difficult to handle than that just preceding.

Example

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx}y = 0$$

$$p\frac{dp}{dy} - 2py = 0 \qquad \text{where } p = \frac{dy}{dx}$$

If $p \neq 0$, it can be divided out and the equation integrated,

$$\frac{dy}{dx} = p = y^2 + C_1^2$$

The variables can be separated, and integrated, to give

$$y = C_1 \tan C_1(x + C_2)$$

i. Bernoulli's Equation

$$\frac{dy}{dx} + f_1(x)y = f_2(x)y^n \tag{4.9}$$

This first-order nonlinear equation with variable coefficients is known as Bernoulli's equation. It is an example where a change of variable can be used to obtain a simpler form of the equation. The appropriate substitution here is $z = y^{1-n}$, a change in the dependent variable. Since $dz/dx = (1 - n)y^{-n}\,dy/dx$, this leads to the linear equation of Sec. 4.2*b*,

$$\frac{dz}{dx} + (1 - n)f_1(x)z = (1 - n)f_2(x)$$

Example

$$\frac{dy}{dx} + axy = \exp\left(\frac{ax^2}{2}\right) y^2$$

The substitution is $z = y^{-1}$, which leads to

$$\frac{dz}{dx} - axz = -\exp\left(\frac{ax^2}{2}\right)$$

$$z = (C - x)\exp\left(\frac{ax^2}{2}\right)$$

$$y = \left[(C - x)\exp\left(\frac{ax^2}{2}\right)\right]^{-1}$$

j. Riccati's Equation

$$\frac{dy}{dx} + ay^2 + f_1(x)y + f_2(x) = 0 \tag{4.10}$$

This first-order nonlinear equation with variable coefficients is a form of Riccati's equation. The change of variable is made $z = \exp\left(a\int_0^x y\,dx\right)$, so that $dz/dx = ayz$ and $d^2z/dx^2 = (a\,dy/dx + a^2y^2)z$. The original first-order equation then becomes a linear equation of second order,

$$\frac{d^2z}{dx^2} + f_1(x)\frac{dz}{dx} + af_2(x)z = 0$$

This equation is of the type considered in Sec. 4.2*l*, following.

Example

$$\frac{dy}{dx} + ay^2 + \left(\frac{1}{x}\right)y + \left(\frac{1}{a}\right) = 0$$

The substitution converts this equation to

$$\frac{d^2z}{dx^2} + \left(\frac{1}{x}\right)\frac{dz}{dx} + z = 0$$

which is Bessel's equation of order zero, with the solution

$$z = C_1J_0(x) + C_2Y_0(x)$$

where J_0 and Y_0 are the two kinds of Bessel functions of zero order. The original variable y is given by $y = (1/az)\,dz/dx$.

k. Euler-Cauchy Equation

$$(a_nx^nD^n + a_{n-1}x^{n-1}D^{n-1} + \cdots + a_1xD + a_0)y = Q(x) \tag{4.11}$$

This is the Euler-Cauchy equation, where coefficients $a_n, \ldots, a_0$ are constants and D is the derivative operator d/dx. It is a linear equation with variable coefficients, where the power of a given coefficient is the same as the order of the derivative it multiplies. The equation can be made linear with constant coefficients by the change of variable $x = \exp(z)$ or $z = \ln x$. This is a change of the independent variable. The derivatives become

$$\frac{dy}{dx} = \left(\frac{dy}{dz}\right)\left(\frac{dz}{dx}\right) = \left(\frac{1}{x}\right)\frac{dy}{dz}$$
$$\frac{d^2y}{dx^2} = \left(\frac{1}{x^2}\right)\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

and so on. Substitution of these quantities converts the original equation into an equation of the sort described in Sec. 4.2*c*.

Example

$$x^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 5y = 0$$

If $x = \exp(z)$, this becomes

$$\frac{d^2y}{dz^2} + 6\frac{dy}{dz} + 5y = 0$$
$$y = C_1 \exp(-5z) + C_2 \exp(-z)$$
$$y = C_1x^{-5} + C_2x^{-1}$$

l. Linear Equation with Varying Coefficients

$$\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_2(x)y = 0 \tag{4.12}$$

This is a linear equation with varying coefficients of arbitrary form. Several well-known equations are of this general type, among them being

Bessel's equation: $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

Legendre's equation:

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

Equations of this sort are usually attacked by the method of Frobenius, in which a solution is assumed in the form of a series, $y = \sum_{i=1}^{\infty} a_i x^{k+i}$, where the exponent k is not necessarily an integer. This series is substituted into the original equation and the coefficients a_i evaluated through recurrence relations. At best, this method gives a solution as a series of terms. The series may converge fast enough to be useful, or it may converge slowly, or perhaps it may not converge at all. In the last case, the solution is what is called a formal solution. The series satisfies the equation, in that upon substitution it reduces the equation to an identity, but the convergence is so poor that it is of no use for practical computation.

Example

Application of this method to Bessel's equation gives as one of the two independent solutions

$$y = J_n(x) = \sum_{i=0}^{\infty} \frac{(-1)^i (x/2)^{n+2i}}{i!\Gamma(n+i+1)}$$

where $\Gamma(n + i + 1)$ is the gamma function of the indicated argument.

4.3. Variation of Parameters. A rather powerful method that can sometimes be used for finding additional parts of the complete solution for a differential equation, once a portion of the solution has been found, is the method of variation of parameters. If the complementary function of a linear equation is known, the method can be used to find the particular integral. The method is based on the observation that the particular integral of a linear equation can be written as the product of one or more parts of the complementary function and one or more functions of the independent variable. A procedure based on this same idea may be applicable to a nonlinear equation also. If the nonlinear terms can be removed from the equation in such a way that the remaining linear equation can be solved, the method may be useful in finding a solution for the original nonlinear equation. Sometimes the required integrations cannot actually be performed, but the method may allow approximate solutions to be determined.

In applying the method of variation of parameters to an equation of high order, it is simplest to reduce the equation to a set of simultaneous first-order equations. The necessary manipulations are then more easily organized in straightforward fashion. As an example, a system of two equations of first order is considered, although the same process applies to systems of still higher order.

The two first-order equations are written as

$$\begin{aligned} \frac{dx}{dt} &= f_1(x,y,t) + \phi_1(x,y,t) \\ \frac{dy}{dt} &= f_2(x,y,t) + \phi_2(x,y,t) \end{aligned} \tag{4.13}$$

where the independent variable is now taken as t, while x and y are the two dependent variables. The way in which the functions on the right sides of these equations are separated depends upon the application of the method. In finding the particular integral for a linear equation, the functions ϕ represent the forcing function of the equation, or the $Q(x)$ of the equations of Secs. 4.2*b* and 4.2*c*. For a nonlinear equation, functions ϕ represent the nonlinear terms or other terms giving difficulty in the solution, and functions f represent the linear terms giving no difficulty.

Again, some ingenuity may be required in separating the terms to best advantage.

With the functions separated as described, a solution can be found for the simplified equations

$$\begin{aligned} \frac{dx}{dt} &= f_1(x,y,t) \\ \frac{dy}{dt} &= f_2(x,y,t) \end{aligned} \tag{4.14}$$

Usually, these equations must be combined into a single second-order equation in order to find this solution. The solutions are

$$\begin{aligned} x &= F_1(C_1,C_2,t) \\ y &= F_2(C_1,C_2,t) \end{aligned} \tag{4.15}$$

where C_1 and C_2 are arbitrary coustants that must be found from specified initial conditions. These solutions for the simplified equations are known as the generating solution.

In the method of variation of parameters, the quantities C_1 and C_2 are now allowed to become functions of the independent variable t, rather than remaining simple constants. Thus, C_1 becomes $C_1(t)$, and C_2 becomes $C_2(t)$. The generating solution, with the varying parameters, is substituted into the complete forms of the original equations so as to give

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial C_1}\frac{dC_1}{dt} + \frac{\partial F_1}{\partial C_2}\frac{dC_2}{dt} = f_1(F_1.F_2,t) + \phi_1(F_1,F_2,t) \\ \frac{dy}{dt} &= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial C_1}\frac{dC_1}{dt} + \frac{\partial F_2}{\partial C_2}\frac{dC_2}{dt} = f_2(F_1,F_2,t) + \phi_2(F_1,F_2,t) \end{aligned} \tag{4.16}$$

However, solutions F_1 and F_2 were determined so that

$$\begin{aligned} \frac{\partial F_1}{\partial t} &= f_1(F_1,F_2,t) \\ \frac{\partial F_2}{\partial t} &= f_2(F_1,F_2,t) \end{aligned}$$

and therefore Eqs. (4.16) reduce to

$$\begin{aligned} \frac{\partial F_1}{\partial C_1}\frac{dC_1}{dt} + \frac{\partial F_1}{\partial C_2}\frac{dC_2}{dt} &= \phi_1(F_1,F_2,t) \\ \frac{\partial F_2}{\partial C_1}\frac{dC_1}{dt} + \frac{\partial F_2}{dC_2}\frac{\partial C_2}{dt} &= \phi_2(F_1,F_2,t) \end{aligned} \tag{4.17}$$

The unknown quantities in these equations are the derivatives dC_1/dt and dC_2/dt. The derivatives can be found by solving these equations as a pair of simultaneous algebraic equations. The result is

$$\begin{aligned}\frac{dC_1}{dt} &= \frac{(\partial F_2/\partial C_2)\ \phi_1(F_1,F_2,t) - (\partial F_1/\partial C_2)\ \phi_2(F_1,F_2,t)}{(\partial F_1/\partial C_1)(\partial F_2/\partial C_2) - (\partial F_1/\partial C_2)(\partial F_2/\partial C_1)} \\ \frac{dC_2}{dt} &= \frac{(\partial F_1/\partial C_1)\ \phi_2(F_1,F_2,t) - (\partial F_2/\partial C_2)\ \phi_1(F_1,F_2,t)}{(\partial F_1/\partial C_1)(\partial F_2/\partial C_2) - (\partial F_1/\partial C_2)(\partial F_2/\partial C_1)}\end{aligned} \tag{4.18}$$

Once dC_1/dt and dC_2/dt are known, their integrals give $C_1(t)$ and $C_2(t)$, which are then inserted into the generating solution of Eq. (4.15) to give the final solution. In solving a particular equation by this method, it is usually preferable to go through the various steps with the equation itself, rather than to substitute blindly in Eq. (4.18)

Example 4.1

The equation

$$\frac{d^2x}{dt^2} - (\tan t)\frac{dx}{dt} - (\sec^2 t)x = \cos t$$

is known to have as its complementary function the solution

$$x = C_1 \sec t + C_2 \tan t$$

Find the particular integral by the method of variation of parameters.

This equation is linear but has varying coefficients. It can be written as two first-order equations by defining function y so that

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= (\tan t)y + (\sec^2 t)x + \cos t\end{aligned} \tag{4.19}$$

The generating solution is then

$$\begin{aligned}x &= C_1 \sec t + C_2 \tan t \\ y &= C_1 \sec t \tan t + C_2 \sec^2 t\end{aligned} \tag{4.20}$$

Parameters C_1 and C_2 are now allowed to vary, and, upon substitution into Eq. (4.19) and simplification, the result is

$$\begin{aligned}\frac{dC_1}{dt} \sec t + \frac{dC_2}{dt} \tan t &= 0 \\ \frac{dC_1}{dt} \sec t \tan t + \frac{dC_2}{dt} \sec^2 t &= \cos t\end{aligned}$$

Therefore, the derivatives are

$$\begin{aligned}\frac{dC_1}{dt} &= -\sin t \cos t \\ \frac{dC_2}{dt} &= \cos t\end{aligned}$$

which give upon integration

$$\begin{aligned}C_1 &= -\tfrac{1}{2} \sin^2 t \cos t + K_1 \\ C_2 &= \sin t + K_2\end{aligned} \tag{4.21}$$

where K_1 and K_2 are arbitrary constants of integration. Upon substitution of Eq. (4.21) into Eq. (4.20) and simplification, the final solution is found as

$$x = \tfrac{1}{2} \sin t \tan t + K_1 \sec t + K_2 \tan t$$

It is worth noting that the relation for dC_1/dt can be integrated differently to give $C_1 = \tfrac{1}{2} \cos^2 t + K_1$ and that this leads to an alternate form for the particular integral. Either form is, of course, correct.

Example 4.2

Find a solution for the equation

$$\frac{dy}{dx} + axy = \exp\left(\frac{ax^2}{2}\right) y^2$$

which is the nonlinear equation used as an example of the Bernoulli equation in Sec. 4.2*i*.

The linear terms are

$$\frac{dy}{dx} + axy = 0$$

which give the generating solution

$$y = C_1 \exp\left(\frac{-ax^2}{2}\right)$$

Parameter C_1 is now allowed to vary, and upon substitution into the original equation the result may be simplified to give

$$\frac{dC_1}{dx} = C_1^2$$

Integration gives

$$C_1 = -(x + K_1)^{-1}$$

and this used with the generating solution leads to the result

$$y = -(x + K_1)^{-1} \exp\left(\frac{-ax^2}{2}\right)$$

This is the same result found in Sec. 4.2*i*, although it is written in slightly different form.

A modification of the method of variation of parameters is considered further in Secs. 6.4 and 9.3*d*.

4.4. Equations Linear in Segments. Certain physical problems must be described by nonlinear equations when the entire range of the variables is considered. However, within certain limited ranges, the equations become linear. The problem can be set up in terms of several linear equations, with the transition from one equation to another taking place whenever certain of the variables take on known values. Such a problem is said to be linear in segments, or piecewise linear. It can be solved by fitting together solutions of strictly linear equations, which can be found with no difficulty.

Sometimes, a problem of this sort arises because of abrupt transitions in the nature of the physical phenomena involved. At other times, the phenomena actually vary smoothly and continuously, but nonlinearly. Only for the purposes of analysis are they replaced by relations linear within certain specified regions. Several examples of this kind of problem follow.

Example 4.3. Mechanical Oscillator with Dry Friction

To a rather crude approximation, the magnitude of the force required to cause one dry metal surface to slide along another is independent of the relative velocity of the two surfaces. The magnitude of the force depends upon the nature of the surfaces and the pressure one exerts upon the other. The direction of the force producing the motion is always the same as the direction of motion. If x is the displacement of one surface along the other, $\dot{x} = dx/dt$ is its velocity, and the force required to produce the motion can be written as a constant multiplying the ratio $\dot{x}/|\dot{x}|$, where the constant applies to a particular system. The ratio $\dot{x}/|\dot{x}|$ has the magnitude unity, but its sign is the sign of $\dot{x}$. It should be emphasized that the friction force for an actual physical system is generally found to be more complicated than that described by this simple relation. However, quite often the relation is used in analyzing a system with so-called Coulomb, or "dry," friction. In this case, the phenomenon can be assumed to be linear, but with an abrupt jump taking place when the velocity changes sign.

A constant mass may be attached to one end of a linear spring the other end of which is attached to a fixed point. The mass may be allowed to slide with dry friction over a surface parallel to the surface of the earth. If x is the deflection away from the rest position, the equation of motion for the system is

$$\ddot{x} + \frac{h\dot{x}}{|\dot{x}|} + \omega_0^2 x = 0 \tag{4.22}$$

where ω_0 is a constant dependent upon the mass and stiffness of the spring and h is a constant dependent upon the friction force and the mass. The equation may be rewritten

$$\ddot{x} + \omega_0^2 x = \frac{-h\dot{x}}{|\dot{x}|}$$

where the right side has the value $+h$ or $-h$, dependent upon the sign of $\dot{x}$.

Determine the motion of the system if at $t = 0$ the initial deflection is $x = A$ and velocity $\dot{x} = 0$.

The equation here involves the friction force, represented by the term in h, which jumps abruptly when the sign of the velocity changes. Considered as a single equation, it is nonlinear. It may be thought of as two linear equations, however.

A solution for the equation is

$$x = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t - \frac{(h/\omega_0^2)\dot{x}}{|\dot{x}|}$$
$$\dot{x} = -C_1\omega_0 \sin \omega_0 t + C_2\omega_0 \cos \omega_0 t$$

where C_1 and C_2 are constants to be determined. Since $\dot{x} = 0$ at $t = 0$, from the second relation it is necessary that $C_2 = 0$, and provided $C_1 > 0$, then $\dot{x} < 0$ for $0 < \omega_0 t < \pi$. The first interval, region 1, which can be described by a strictly linear equation, is then $0 < \omega_0 t < \pi$, and in this region the first relation gives

$$x = C_1 \cos \omega_0 t + \frac{h}{\omega_0^2}$$

and

$$C_1 = A - \frac{h}{\omega_0^2}$$

Constant C_1 is positive as assumed, if $A > h/\omega_0^2$. Thus, in region 1,

$$x = \left(A - \frac{h}{\omega_0^2}\right) \cos \omega_0 t + \frac{h}{\omega_0^2} \tag{4.23}$$

At the end of this region, $\omega_0 t = \pi$, and $x = -(A - 2h/\omega_0^2)$ with $\dot{x} = 0$.

These last values are used as initial conditions to redetermine constants C_1 and C_2 for the next region. Again from the second relation, $C_2 = 0$, and $\dot{x} > 0$ for $\pi < \omega_0 t < 2\pi$ provided $C_1 > 0$. Region 2 exists for $\pi < \omega_0 t < 2\pi$, and in this region the first

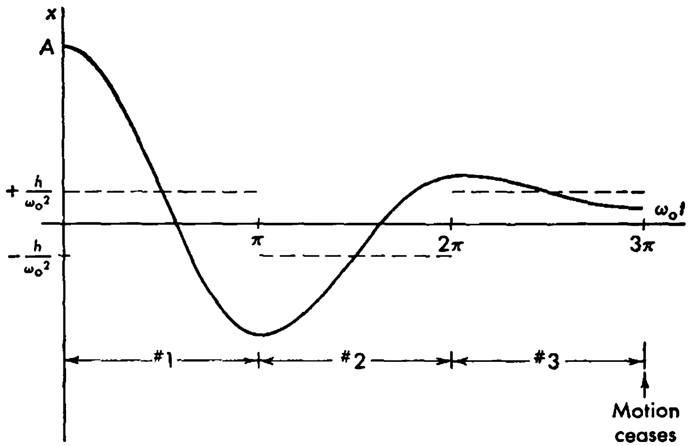


FIG. 4.1. Displacement of mechanical oscillator with dry friction of Example 4.3. Solution is composed of half cosine waves centered about dotted lines shown. Motion ceases after three half cycles.

relation gives $x = C_1 \cos \omega_0 t - h/\omega_0^2$ and $C_1 = (A - 3h/\omega_0^2)$. Constant $C_1 > 0$ if $A > 3h/\omega_0^2$. Thus, in region 2,

$$x = \left(A - \frac{3h}{\omega_0^2}\right) \cos \omega_0 t - \frac{h}{\omega_0^2} \tag{4.24}$$

At the end of this region, $\omega_0 t = 2\pi$, and $x = (A - 4h/\omega_0^2)$, with $\dot{x} = 0$.

In a similar manner, region 3 exists for $2\pi < \omega_0 t < 3\pi$, wherein $\dot{x} < 0$ if $C_1 > 0$, $C_1 = (A - 5h/\omega_0^2)$, and

$$x = \left(A - \frac{5h}{\omega_0^2}\right) \cos \omega_0 t + \frac{h}{\omega_0^2} \tag{4.25}$$

A complete solution can be built up in this manner from segments existing during each increment of π units in the variable $\omega_0 t$. At the boundary between adjoining regions, both x and $\dot{x}$ are continuous, but there is a discontinuous jump in $\ddot{x}$. Figure 4.1 shows a curve representing the motion of a system of this sort. It is plotted by making use of Eqs. (4.23) to (4.25) in their respective regions. There are two interesting features for this solution, different from the solution of a somewhat analogous system having viscous, or "wet," friction. There the friction force is directly proportional to velocity and is given by a constant times $\dot{x}$. With wet friction, the amplitude of successive cycles of the oscillation decays exponentially, so that motion ceases only after a very long time. With dry friction, the amplitude of successive cycles decreases by a fixed amount for each cycle. There is an abrupt ceasing of the motion when the deflection at the end of a region is smaller in magnitude than the quantity h/ω_0^2. In Fig. 4.1, motion ceases at the end of three half cycles. A larger initial amplitude A would cause motion to continue for a larger number of half cycles. The stopping of motion is accompanied by constant C_1 turning out to be negative when an attempt is made to evaluate it for the next region.

Example 4.4. Magnetic-amplifier Circuit

Special iron designed for use as core material of reactors for magnetic amplifiers and related control circuits is characterized by a magnetization curve of the kind shown in

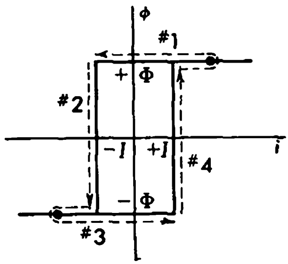


FIG. 4.2. Hysteresis loop for special iron used in magnetic amplifier. Time increases in the direction of the arrows.

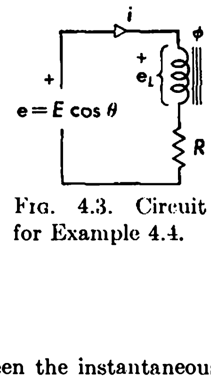


FIG. 4.3. Circuit for Example 4.4.

Fig. 4.2. This curve represents the relation between the instantaneous total magnetic flux ϕ in the core and the instantaneous current i in the coil wound around the core. The curve is not single-valued but is an open loop, known as the hysteresis loop. Iron, in general, displays a curved S-shaped hysteresis loop, but for this special iron the loop is made up of four essentially straight lines. Either the flux has one of the values $+\Phi$ or $-\Phi$, in which case the current is free to vary within certain limits, or else the current has one of the values $+I$ or $-I$, in which case the flux may vary within certain limits. As variation takes place, the instantaneous values of ϕ and i determine an operating point on the hysteresis loop. This point moves around the loop in a generally counterclockwise direction, as shown by the arrows of Fig. 4.2.

A reactor containing this special iron as its core material may be used in the circuit of Fig. 4.3 connected in series with a constant resistance R and an applied voltage $e = E \cos \theta$, where θ is merely the generalized phase angle ωt. The instantaneous voltage across the reactor is $e_L = N\, d\phi/dt$ where N is the number of turns of wire making up the coil surrounding the core. The equation for the operation of the circuit is

$$E \cos \theta = N \frac{d\phi}{dt} + Ri \tag{4.26}$$

where it is necessary to know the instantaneous operating conditions, in terms of the hysteresis loop, in order to be able to interpret $d\phi/dt$ and i correctly.

As a specific example, consider the case where voltage E is large enough to give a maximum current somewhat greater than I, and assume that the system has been in operation long enough for a steady state to exist. In general, there is always an initial transient period when voltage is first applied to the circuit, but this transient dies away, leaving a steady state because of the effect of resistance in the circuit.

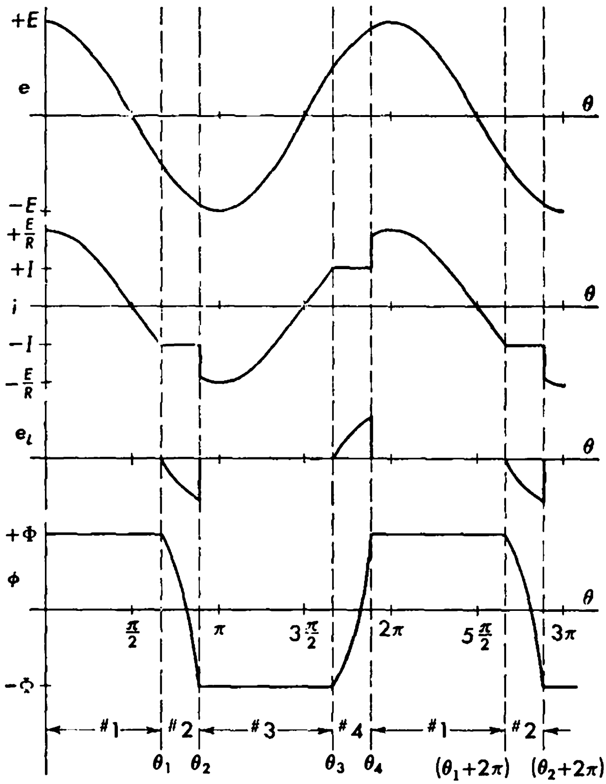


FIG. 4.4. Steady-state waveforms of instantaneous values of applied voltage e, current i, inductor voltage e_L, and flux ϕ, in circuit of Example 4.4.

In the steady state, operation during the positive half cycle is identical except for algebraic sign with that during the negative half cycle. The maximum positive current, indicated by the dot on the upper branch of the hysteresis loop of Fig. 4.2, is the same in magnitude as the maximum negative current, indicated by the dot on the lower branch of the loop.

In order to describe the solution, it is necessary to divide a complete cycle into four regions, determined by the portion of the hysteresis loop characterizing the operation. These four regions are indicated in Fig. 4.2. Operation in the steady state must be such as to pass through these four regions, around the hysteresis loop, and at the end of a complete cycle return to the original conditions.

A solution can be built up as shown in Fig. 4.4 and summarized in Table 4.1. In working out the solution, it is simplest to start at the time $t = 0$ when e has its maxi-

TABLE 4.1

Region	Circuit equation	i	ϕ	e_L	Region boundaries	Condition for upper boundary
1	$E\cos\theta = Ri$	$\frac{E}{R}\cos\theta$	$+\Phi$	0	$(\theta_4 - 2\pi),\ \theta_1$	$i = -I$
2	$E\cos\theta = N\frac{d\phi}{dt} - RI$	$-I$	$+\Phi + \frac{1}{N}\int_{\theta_1}^{\theta}(E\cos\theta + RI)\,d\theta$	$(E\cos\theta + RI)$	$\theta_1,\ \theta_2$	$\phi = -\Phi$
3	$E\cos\theta = Ri$	$\frac{E}{R}\cos\theta$	$-\Phi$	0	$\theta_2,\ \theta_3$	$i = +I$
4	$E\cos\theta = N\frac{d\phi}{dt} + RI$	$+I$	$-\Phi + \frac{1}{N}\int_{\theta_3}^{\theta}(E\cos\theta - RI)\,d\theta$	$(E\cos\theta - RI)$	$\theta_3,\ \theta_4$	$\phi = +\Phi$

mum positive value E. The iron core is saturated so that $\phi = \Phi$, voltage $e_L = 0$, and the current is at the point indicated by the dot of Fig. 4.2 and in value is $i = E/R$. Conditions during this first interval, region 1, appear in the first row of Table 4.1. In this region, the operating point representing instantaneous values of ϕ and i moves to the left along the upper branch of the hysteresis loop, maintaining constant flux, $\phi = \Phi$. Region 1 ends when this point reaches the left end of the upper branch, or when $i = -I$.

Region 2 exists while the flux is changing from $\phi = +\Phi$ to $\phi = -\Phi$, during which time the current is constant at $i = -I$. Voltage e_L across the reactor is not zero during this region since the flux is changing. Region 2 ends when the flux reaches its negative limit $\phi = -\Phi$.

Region 3 represents region 1, but with all signs changed. For the conditions illustrated in Fig. 4.4, region 1 actually begins slightly before $\theta = 0$, and region 3 begins before $\theta = \pi$. The transitions between regions depend upon circuit conditions and need not be exactly as shown. Region 4 repeats region 2, but with all signs reversed.

In each region, the current and flux are described by a purely linear equation which is easily solved. A complete solution requires the joining together of individual solutions of these linear equations, with appropriate adjustment of constants at each transition. This matching together of solutions is a tedious process.

As mentioned previously, the point in the cycle where transition from one region to another takes place depends upon the parameters of the circuit. It is possible to control these transition points by using an extra winding on the core and supplying an auxiliary control current to this winding. This is the basis for the use of this kind of reactor in a magnetic-amplifier system.

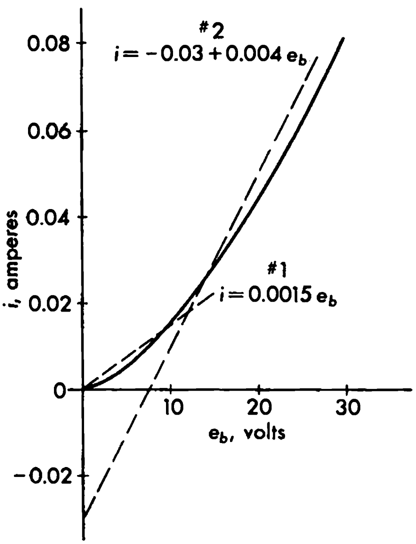


FIG. 4.5. Diode characteristic curve for Example 4.5 as approximated by two straight lines.

Example 4.5. Diode Circuit

In the two preceding examples, the physical phenomenon involved was described rather well by a relation made up of straight lines. A somewhat different situation is that where the phenomenon is described by a curve. This is the case for the electrical circuit already discussed in Example 3.7 and shown in Fig. 3.21. The diode characteristic is described by the nonlinear cu ve of Fig. 3.22. Sometimes it is convenient to replace a curve such as this by several straight lines. These lines can be chosen with such slope and in such number as to approximate the curve fairly well. Equations written for these lines will be linear and can be solved directly with no difficulty. Transition from one equation to another is necessary when transition from one straight line to another takes place.

Find the current in the circuit as a function of time with the initial condition that $i = 0$ at $t = 0$.

The diode curve of Fig. 3.22 is shown again in Fig. 4.5, where it is approximated by two straight lines. These lines are designated as region 1, for which $i = 0.0015e_b$, and region 2, for which $i = -0.03 + 0.004e_b$, where i is current in amperes and e_b is voltage across the diode. The lines are chosen, in part, so as to have simple numerical coefficients in their describing equations. The two lines intersect at $i = 0.018$ amp and $e_b = 12$ volts, which determines the boundary between regions.

The circuit equation in the first region is

$$E = L\frac{di}{dt} + Ri + e_b(i) \tag{4.27}$$

where $e_b(i)$ is the voltage for the diode in the first region. With numerical values inserted, this equation becomes

$$100 = 100\frac{di}{dt} + 1{,}000i + \frac{1}{0.0015}i$$

or

$$\frac{di}{dt} + 16.67i = 1$$

The solution is

$$i = C_1 \exp(-16.67t) + 0.060$$

Constant C_1 must be chosen to make $i = 0$ at $t = 0$, so that finally, in region 1

$$i = 0.060[1 - \exp(-16.67t)] \tag{4.28}$$

This solution is valid until i reaches the boundary for region 1 given by $i = 0.018$. Actually, some number work shows that, at $t = 0.02$ sec, $i = 0.017$ amp, and this is a convenient point to choose for transition to region 2.

In region 2, the circuit equation is

$$100 = 100\frac{di}{dt} + 1{,}000i + \frac{1}{0.004}(i + 0.03)$$

where the second relation is used for $e_b(i)$. This equation may be written

$$\frac{di}{dt} + 12.5i = 0.925$$

which has the solution

$$i = C_2 \exp(-12.5t) + 0.074$$

Constant C_2 must be chosen to make $i = 0.017$ at $t = 0.02$, the initial condition for region 2. The final solution for this region is

$$i = -0.073 \exp(-12.5t) + 0.074 \tag{4.29}$$

Numerical values obtained from Eqs. (4.28) and (4.29) are plotted in Fig. 4.6, where the circles indicate computed points. This figure should be compared with Fig. 3.24, which applies to the same problem as solved by two graphical methods.

Two features of this present solution should be observed. First, at the transition between regions, the characteristic of the diode is assumed to change abruptly. This causes the slope of the solution curve, di/dt, to change abruptly at the transition. The curve is continuous, but its slope is discontinuous. The discontinuity is small enough not to be readily apparent in Fig. 4.6. Second, the steady-state current, existing after a very long time, is given by Eq. (4.29) as $i = 0.074$ amp. This value is actually incorrect, because the linear approximation for the diode curve fits this curve rather poorly for values of current this large. A better approximation would have the straight line in region 2 intersect the actual curve at a value of current near the steady-state value. From the graphical solution of Example 3.7, it is known that the actual steady-state current is slightly less than $i = 0.073$ amp.

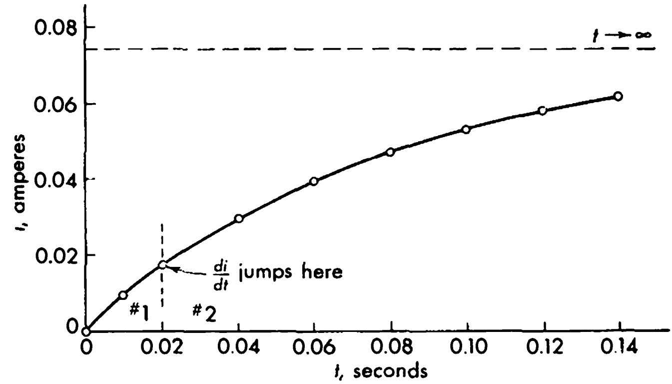


FIG. 4.6. Solution for diode circuit of Example 4.5 as obtained by method of linear segments.

In spite of these obvious errors, the curves of Figs. 3.24 and 4.6 are in generally good agreement. Any of the three methods leads to essentially the same results, and a choice among the three depends somewhat upon personal preference.

4.5. Equations Leading to Elliptic Functions. All the simpler mathematical functions can be considered to arise as the solutions for certain differential equations. The exponential function $x = \exp(at)$ comes from the equation $dx/dt = ax$. The circular trigonometric functions $x = \cos \omega t$ and $x = \sin \omega t$ arise from the equation $d^2x/dt^2 + \omega^2 x = 0$. The hyperbolic functions $x = \cosh \omega t$ and $x = \sinh \omega t$ originate in the equation $d^2x/dt^2 - \omega^2 x = 0$. All these equations are linear equations with constant coefficients. The functions they define are of great general utility in mathematics since they appear in the solution of all kinds of linear problems. For this reason, the functions have been studied in detail, and their properties are well known. Numerical values for the functions have been computed, and tables are available with values given to a high degree of accuracy.

Other more complicated differential equations can be considered to define additional functions. Bessel functions come from Bessel's equation, Legendre functions come from Legendre's equation, and so on. These additional functions are more complicated to employ than are the elementary functions, and more parameters are required in their tabulation. Among the more complicated functions, the elliptic function are of particular interest in the analysis of problems arising from oscillas tory physical systems.

A differential equation of the form

$$\left(\frac{dx}{dt}\right)^2 = \sum_{n=0}^{p} a_n x^n \tag{4.30}$$

arises in a number of problems of interest. For powers no higher than 2, so that $0 \leq p \leq 2$, it is possible to show that a solution always can be found for this equation in terms of the elementary functions. The form of the solution may be complicated, but it can be written with only elementary functions. For powers of 3 and 4, $3 \leq p \leq 4$, solution can be found in terms of a new kind of functions known as elliptic functions. For powers higher than 4, $5 \leq p \leq \infty$, still more complicated functions known as hyperelliptic functions are required.

A brief discussion of elliptic functions is given here. Properties of these functions have been studied, and numerical tables are available. These tables apply to a standard form for the differential equation and its solution. Tables are given for the so-called elliptic integral, of which there are three kinds. Only the first kind is considered here. The first elliptic integral is usually written in the standard form

$$u = F(k,\phi) = \int_0^\phi \frac{d\phi}{(1 - k^2 \sin^2 \phi)^{1/2}} \tag{4.31}$$

where k and ϕ are parameters determining the value of the integral u and k is limited to the range $0 \leq k \leq 1$. The name modulus of u is applied to k, and amplitude of u is applied to ϕ, so that, written symbolically, $k = \text{mod } u$ and $\phi = \text{am } u$. Tables of $u = F(k,\phi)$ are given with k and ϕ as parameters. Sometimes k is replaced by a parameter α, where $k = \sin \alpha$, thus making both parameters an angle. One of the difficulties in working with equations that have solutions as elliptic functions is that of modifying the original equation so as to put it into standard form and allow the use of existing tables.

Example 4.6. Mass on Nonlinear Spring

A specific physical example that leads to solution in elliptic functions is the following mechanical system: A constant mass is attached to the free end of a spring, the other end of the spring being fixed in position. It is assumed that there is no friction in the system, so that if the mass is pulled aside from its rest position and released, an oscillatory motion results. The simplest case which can exist is that in which there is a linear relation between the deflection of the spring and the force required to produce this deflection. The differential equation for the system is then a linear equation, and the motion is simple harmonic.

Most actual springs are not so simple; the deflection and force are rarely proportional to one another. Usually the force increases more rapidly than a linear function of the deflection. Such a spring is said to be a "hard" spring. It is also possible to construct a spring for which the force increases less rapidly than a linear function of the deflection. This kind of spring is called a "soft" spring. The equivalent stiffness of a hard spring increases with deflection, while that of a soft spring decreases. In either case, the differential equation of the system is no longer linear, and the motion is not simple harmonic.

In Fig. 4.7 is shown a plot of the force required to produce a given deflection for three kinds of spring. It is assumed here that the spring operates the same way for a deflection in either direction from its rest position. This assumption often does not

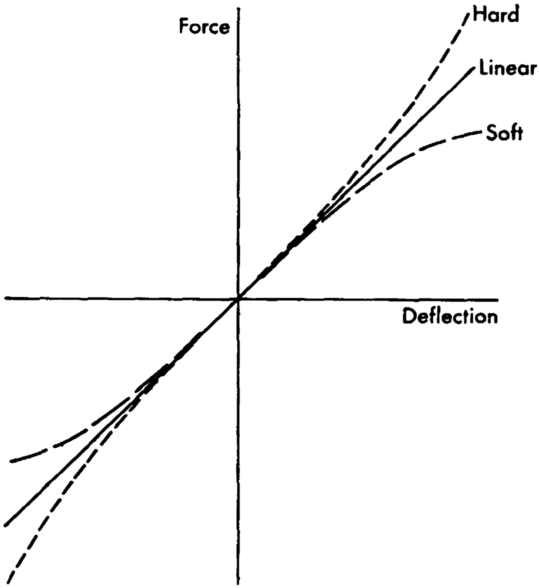


FIG. 4.7. Relation between force and deflection for three types of springs.

apply for an actual spring, of course. An equation describing the symmetrical hard spring is

$$\text{Force} = \kappa(1 + a^2x^2)x$$

where κ and a^2 are positive constants. If $a^2 = 0$, the spring is linear so that force $= \kappa x$ and κ is called the stiffness of the spring, constant in this case. If $a^2 > 0$, the spring is a hard spring, with its equivalent stiffness increasing with a deflection of either algebraic sign. A similar equation describing the symmetrical soft spring is

$$\text{Force} = \kappa(1 - b^2x^2)x$$

where κ and b^2 are positive constants. This equation may apply to an actual spring only so long as the deflection is not too large. It has the peculiarity that if x exceeds in magnitude $1/b$ the sign of the force reverses. The spring then tends to stretch itself further, rather than to return to its original length. While this behavior is unusual for a spring constructed by coiling a piece of wire, it may occur in such things as a nonuniform magnetic field that operates much like a spring. The equivalent spring for a simple pendulum is also of similar nature.

If a constant mass M is attached to the hard spring, the equation applying to the system is

$$M\frac{d^2x}{dt^2} + \kappa(1 + a^2x^2)x = 0 \tag{4.32}$$

This equation is of the type of Sec. 4.2*f* and can be integrated if each side is multiplied by $2\,(dx/dt)\,dt = 2\,dx$. At the same time, it is convenient to introduce the abbreviation $\omega_0{}^2 = \kappa/M$, so that

$$\int 2\left(\frac{dx}{dt}\right)\left(\frac{d^2x}{dt^2}\right)dt = -2\omega_0{}^2\int(1 + a^2x^2)x\,dx$$

and

$$\left(\frac{dx}{dt}\right)^2 = -\omega_0{}^2\left(x^2 + \frac{a^2x^4}{2} - C_1{}^2\right) \tag{4.33}$$

where $C_1{}^2$ is the constant of integration. This equation is of the form of Eq. (4.30). Therefore, a solution involving an elliptic function is indicated.

If the spring is linear, $a^2 = 0$ and the equation becomes

$$\left(\frac{dx}{dt}\right)^2 = -\omega_0{}^2\left(x^2 - C_1{}^2\right)$$

which can be solved in terms of elementary functions. In fact, if the square root is taken of each side of the equation and the variables separated, integration gives

$$\int\frac{dx}{\omega_0\,(C_1{}^2 - x^2)^{1/2}} = \int dt + C_2$$

or

$$x = C_1\sin\omega_0(t + C_2)$$

where C_2 is a second constant of integration. Of course, for the linear case, the solution can be written down at once without going through the intermediate steps.

In order to proceed further with the analysis of the hard spring, an initial condition must be specified so that the constant of integration can be determined. From experience with mass-spring systems having no friction, it is known that the system will oscillate with a periodic motion. Twice each cycle of oscillation the deflection is a maximum, and the velocity is zero. If the maximum deflection is identified as A, a necessary condition is that $x = A$ when $dx/dt = 0$. This condition used in Eq. (4.33) determines the constant as $C_1{}^2 = A^2 + a^2A^4/2$. Thus, Eq. (4.33) becomes

$$\left(\frac{dx}{dt}\right)^2 = \omega_0{}^2\left[(A^2 - x^2) + \frac{a^2}{2}(A^4 - x^4)\right]$$

The square root of each side of this equation, with the variables separated and integrated, gives

$$\int\omega_0\,dt = \pm\int\frac{dx}{[(A^2 - x^2) + (a^2/2)(A^4 - x^4)]^{1/2}} \tag{4.34}$$

A second initial condition is needed, either to specify limits on the integrals or to evaluate a constant of integration. Furthermore, it has already been predicted that an elliptic integral will result here, and it is desirable to have this integral in standard form. A little investigation shows that it is necessary to choose the time origin, $t = 0$, at the instant of maximum deflection, $x = A$, so as to get the standard elliptic integral. As time increases in the positive direction, the deflection decreases going in the negative direction. Therefore, the negative sign in front of the right-hand integral of Eq. (4.34) must be chosen. At any instant t, the corresponding deflection is x, and Eq. (4.34) becomes

$$\int_{t=0}^{t=t} \omega_0\, dt = \omega_0 t = -\int_{x=A}^{x=x} \frac{dx}{[(A^2 - x^2) + (a^2/2)(A^4 - x^4)]^{1/2}} \tag{4.35}$$

The standard elliptic integral is obtained after some factoring and a change of variable. The quantity in the integral can be written

$$(A^2 - x^2) + \frac{a^2}{2}(A^4 - x^4) = -\frac{a^2}{2}\left(x^2 + \frac{2}{a^2} + A^2\right)(x^2 - A^2) \tag{4.36}$$

For a linear oscillator with the initial conditions used here, the solution would be a cosine function of time. For the nonlinear system, the oscillation is somewhat like a cosine function, and a new variable is introduced as

$$x = A \cos \phi \tag{4.37}$$

This relation defines the angle ϕ, which is related to time in a manner to be determined. Upon substitution of Eqs. (4.36) and (4.37) into Eq. (4.35), the result is

$$\omega_0 t = \int_0^\phi d\phi \Bigg/ \left[\left(\frac{a^2}{2}\right)\left(A^2 + \frac{2}{a^2} + A^2 \cos^2 \phi\right)\right]^{1/2} \tag{4.38}$$

Finally, yet another quantity k is defined as

$$k^2 = \left[2\left(1 + \frac{1}{a^2 A^2}\right)\right]^{-1} \tag{4.39}$$

and this gives

$$\omega_0 t = (1 + a^2 A^2)^{-1/2} \int_0^\phi \frac{d\phi}{(1 - k^2 \sin^2 \phi)^{1/2}}$$

The integral here is the standard form of the elliptic integral of the first kind, so that the result may be written

$$\omega_0 t = (1 + a^2A^2)^{-1/2} F(k,\phi) = (1 + a^2A^2)^{-1/2} u \tag{4.40}$$

In the elliptic integral, the modulus k is determined from the maximum deflection by Eq. (4.39). The angle ϕ relates the instantaneous and maximum deflections, as given by Eq. (4.37). The integral gives the relation between instantaneous values of time and deflection. The instantaneous deflection can be written as

$$x = A \cos \phi = A \cos (\operatorname{am} u) = A \operatorname{cn} u \tag{4.41}$$

This defines the elliptic cosine function cn u of the integral u.

It is convenient to stop at this point in the discussion and consider briefly the analogous case of the constant mass attached to a frictionless soft spring. The equation is

$$M \frac{d^2x}{dt^2} + \kappa(1 - b^2x^2)x = 0$$

This equation can be analyzed in much the same way that the hard spring has been treated. Again, an elliptic integral results, and the solution should be adjusted to have this integral in standard form. For this purpose, initial conditions must be chosen as $x = A$ for $dx/dt = 0$ and $x = 0$ for $t = 0$, the latter condition differing from the choice made with the hard spring. In a linear system, these conditions would lead to a deflection varying as a sine function of time. Thus, the angle ϕ is defined for this case as

$$x = A \sin \phi \tag{4.42}$$

Similarly, the modulus k must be defined as

$$k^2 = \left(\frac{2}{b^2A^2} - 1\right)^{-1} \tag{4.43}$$

The final relation is

$$\omega_0 t = \left(1 - \frac{b^2A^2}{2}\right)^{-1/2} F(k,\phi) = \left(1 - \frac{b^2A^2}{2}\right)^{-1/2} u \tag{4.43a}$$

The instantaneous deflection is

$$x = A \sin \phi = A \sin (\text{am } u) = A \text{ sn } u \tag{4.44}$$

This defines the elliptic sine function sn u of the integral u.

4.6. Properties of Elliptic Functions. Certain general properties of the elliptic integral and of the functions related to it can be investigated. The numerical value of the integral depends upon the two parameters k and ϕ, both of which must be known. In order to compute the integral, it can be expanded as a binomial series and integrated term by term as

$$u = \int_0^\phi (1 + \tfrac{1}{2}k^2 \sin^2 \phi + \tfrac{3}{8}k^4 \sin^4 \phi + \tfrac{5}{16}k^6 \sin^6 \phi + \cdots) \, d\phi$$

Actually other series can be found that converge faster than this one and are more suitable for computation.

If ϕ has the value $\pi/2$, the integral becomes what is called the complete elliptic integral, given the symbol $K(k)$,

$$F\left(k, \frac{\pi}{2}\right) = K(k) = \frac{\pi}{2}\left(1 + \frac{k^2}{4} + \frac{9k^4}{64} + \cdots\right)$$

Numerical values have been calculated for these integrals and are available in tabular form. A curve plotted from these values is shown in Fig. 4.8, where the limiting values $k = 0$ and $k = 1$ are shown, together with an intermediate value for which $k^2 = 0.75$. Usually numerical tables are given only for the range $0 \leq \phi \leq \pi/2$. Extension can be made by using the following properties of the integral,

$$\begin{aligned} F(k,\phi) &= -F(k,-\phi) \\ F(k, \phi + n\pi) &= 2nK(k) + F(k,\phi) \end{aligned}$$

where $n = 0, 1, 2, \ldots$.

Once the elliptic integral is known, the elliptic cosine and sine functions, cn u and sn u, can be found from their defining relations, Eqs. (4.41) and (4.44). The details are shown in Fig. 4.9, where the choice of a particular value ϕ_1 is seen to lead to a particular value of the integral, u_1, and also

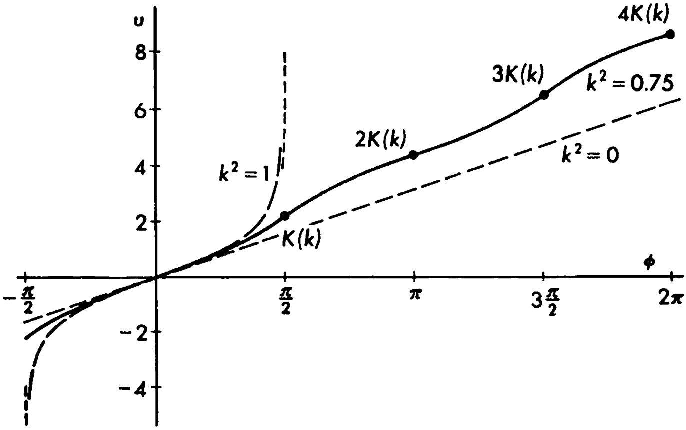


FIG. 4.8. Variation of the elliptic integral u of the first kind with ϕ, the amplitude of u. Three different values of modulus k are shown.

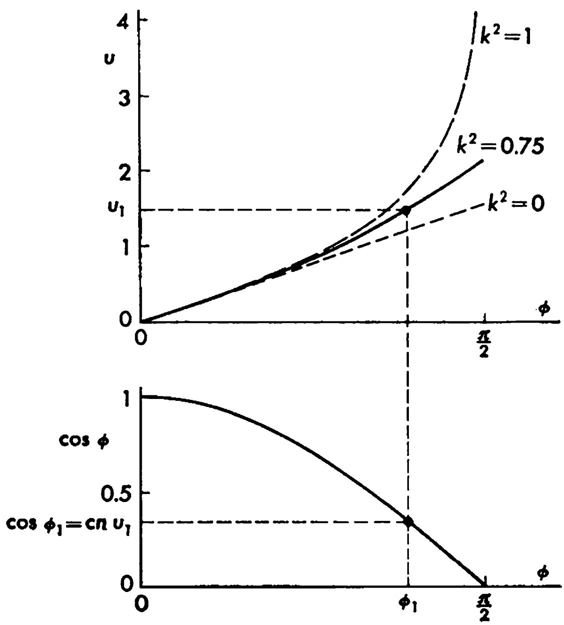


FIG. 4.9. Graphical construction for finding the elliptic cosine function. The upper curve is the elliptic integral and the lower curve is the circular cosine, both plotted as a function of angle ϕ.

to the elliptic cosine, cn $u_1 = \cos \phi_1$. Since the value u_1 depends upon the modulus k, the value of cn u_1 also depends upon k. If $k = 0$, the elliptic cosine function is the same as the circular cosine; as k is made greater than zero, the elliptic cosine departs further and further from the circular cosine, the effect being to accentuate the peak of the curve. In Fig. 4.10 are shown plots of cn u, obtained as indicated in Fig. 4.9. The elliptic sine function can be found in a similar manner, making use of the definition sn $u = \sin \phi$.

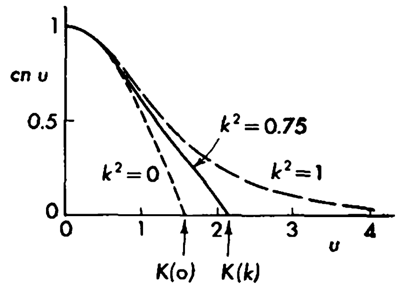


FIG. 4.10. One-quarter of a complete cycle of the elliptic cosine function. Three different values of modulus k are shown.

The elliptic functions have properties somewhat like the corresponding circular functions. Thus,

$$\text{cn } u = \cos \phi$$
$$\text{sn } u = \sin \phi$$

and therefore

$$\text{cn}^2 u + \text{sn}^2 u = \cos^2 \phi + \sin^2 \phi = 1$$

Also,
$$\text{cn }(u + 2K) = \cos(\phi + \pi) = -\cos \phi = -\text{cn } u$$
$$\text{sn }(u + 2K) = \sin(\phi + \pi) = -\sin \phi = -\text{sn } u$$

and
$$\text{cn }(u + 4K) = \cos(\phi + 2\pi) = \cos \phi = \text{cn } u$$
$$\text{sn }(u + 4K) = \sin(\phi + 2\pi) = \sin \phi = \text{sn } u$$

Both cn u and sn u are periodic functions with period $4K$. Again, $K(k)$ is dependent upon modulus k so that the period is dependent upon k. The curves of Fig. 4.10 are but the first quarter of a complete cycle.

Yet a third function dn u is sometimes useful. It is defined as

$$\text{dn } u = (1 - k^2 \sin^2 \phi)^{1/2} = (1 - k^2 \text{ sn}^2 u)^{1/2}$$

The elliptic integral is then $u = \int_0^\phi du/(\text{dn } u)$. The derivatives of the elliptic functions are

$$\frac{d(\text{cn } u)}{du} = -\text{sn } u \text{ dn } u$$
$$\frac{d(\text{sn } u)}{du} = \text{cn } u \text{ dn } u$$
$$\frac{d(\text{dn } u)}{du} = -k^2 \text{ sn } u \text{ cn } u$$

Curves displaying the shapes of all three of the elliptic functions are shown in Fig. 4.11. Only the first quarter of a complete cycle is given, and three values of k appear. The abscissa is given in normalized form as u/K so that a quarter cycle always takes place in the range $0 \leq u/K \leq 1$. If $k = 0$, the shapes are those of the circular functions. As k becomes large, the elliptic cosine approaches a sharply peaked wave, while the elliptic sine approaches a square wave. In each case, the period in u becomes longer and longer.

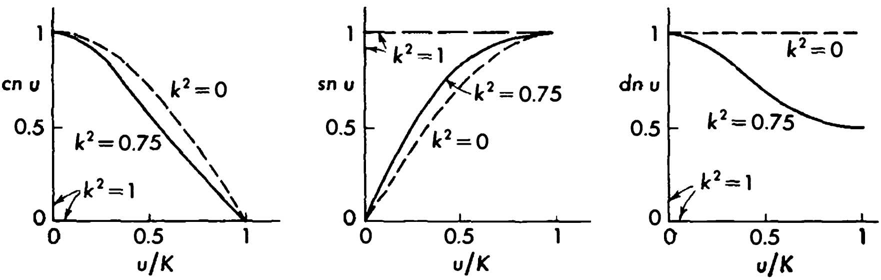


FIG. 4.11. One-quarter of a complete cycle of the elliptic functions cn u, sn u, and dn u plotted against the normalized abscissa u/K. Three values of modulus k are shown.

A further similarity between circular and elliptic sine functions may be noted. The inverse circular sine function may be written as the integral

$$v = \sin^{-1} y = \int_0^y \frac{dy}{(1 - y^2)^{1/2}} \qquad \text{where } 0 \leq y \leq 1$$

and

$$\sin v = y$$

For the elliptic sine function, the integral is

$$u = \int_0^\phi \frac{d\phi}{(1 - k^2 \sin^2 \phi)^{1/2}} \qquad \text{where } 0 \leq \phi \leq \frac{\pi}{2}$$

and

$$\text{sn } u = \sin \phi$$

The elliptic integral can be written in an alternate standard form using a new variable $z = \sin \phi$, so that

$$u = \int_0^z \frac{dz}{[(1 - z^2)(1 - k^2 z^2)]^{1/2}} \qquad \text{where } 0 \leq z \leq 1$$

The similarity with the integral for the inverse circular sine function is apparent.

Example 4.6 (continued). Period of Mass on Nonlinear Spring

The instantaneous deflection of the mass on the hard spring is shown in Eqs. (4.40) and (4.41) to be given by $x = A$ cn u, where $u = F(k,\phi) = (1 + a^2A^2)^{1/2}\omega_0 t$. The motion is periodic with period T in time, given by

$$T = \frac{4K(k)}{\omega_0(1 + a^2A^2)^{1/2}} \tag{4.45}$$

For a linear system $a^2 = 0$, $k^2 = 0$, $K(k) = \pi/2$, and the period is

$$T_0 = \frac{2\pi}{\omega_0}$$

Thus, the ratio of periods is

$$\frac{T}{T_0} = \frac{2K(k)}{\pi(1 + a^2A^2)^{1/2}} \tag{4.46}$$

The motion of the system is governed by the product a^2A^2. As this product is increased, modulus k is increased, although its maximum possible value in this example is $k^2 = \frac{1}{2}$. The increase in k also gives an increase in $K(k)$, but the extra factor a^2A^2 in the denominator of Eq. (4.45) leads to a decrease in period with an increase in a^2A^2. At the same time, the peaks of the waveform are accentuated.

For the system with soft spring, the instantaneous deflection is given by Eqs. (4.43a) and (4.44) as $x = A \text{ sn } u$, where $u = F(k,\phi) = (1 - b^2A^2/2)^{1/2}\omega_0 t$. The ratio of the actual period to the period of a linear system is

$$\frac{T}{T_0} = \frac{2K(k)}{\pi(1 - b^2A^2/2)^{1/2}} \tag{4.47}$$

As the product b^2A^2 is increased, the period is also increased and the peaks of the waveform are flattened. If $b^2A^2 = 1$, the value of k becomes unity, $K(k)$ becomes infinite, and the period becomes infinite. If the initial deflection is such that $b^2A^2 > 1$, the resulting motion is no longer periodic. A deflection this large has stretched the soft spring so far that it no longer tends to return to its original length but rather tends to stretch itself still further. A periodic motion could not be expected to exist. The solution as an elliptic function is not applicable to this nonperiodic motion.

This example involving the motion of a system with a nonlinear spring has, of course, involved a rather special and simple kind of nonlinearity. The nonlinear relation between force and deflection for the spring was assumed to be symmetrical and to have only a cubic term for the nonlinear function. Under this assumption, it was possible to find an exact solution for the motion in terms of elliptic functions which are available in tabulated form. A typical spring that might occur in an actual physical system probably would not have such a simple kind of nonlinearity. An exact analysis then would probably not be possible.

A classic problem somewhat similar to the preceding example is the motion of an ideal simple pendulum. A constant mass moving in a circular path about an axis of rotation normal to a constant gravitational field is governed by the nonlinear equation

$$\frac{d^2\theta}{dt^2} + \omega_0^2 \sin \theta = 0$$

where ω_0^2 is a constant dependent upon the radius of the circular path and the gravitational acceleration and θ is the angle of deflection measured from the vertical. Since the sine function can be expressed in the series form

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

it might well appear that the equation for the pendulum would lead to an equation of the form of Eq. (4.30) with powers going to infinity. In such a case, it would be expected that a solution would require hyperelliptic functions. It turns out, however, that the pendulum can be described in terms of simple elliptic functions.

The details of the solution are not given here, but, by adopting a procedure similar to that used with the nonlinear spring, the following relations can be shown to apply: The modulus and amplitude are given by $k = \sin(\theta_m/2)$ and $\sin \phi = (1/k) \sin(\theta/2)$, where θ_m is the maximum angular deflection. The time for an instantaneous deflection is then given by $\omega_0 t = u = F(k,\phi)$. The pendulum is somewhat similar to the system with the soft spring in that its period increases as the maximum deflection is increased.

One obvious difficulty in applying elliptic functions to the solution of a nonlinear differential equation is in recognizing which equations can be solved in this way and then transforming the original equation to the standard form in which the elliptic integral is tabulated. Considerable information on the details of these procedures is available in more complete works on elliptic functions, and no attempt is made to include them here.

4.7. Summary. Certain differential equations for which exact solutions are known in analytic form have been discussed in this chapter. Often a solution depends upon a change of variable or other modification of the original equation. Considerable ingenuity and experimentation may be necessary when a new and strange equation is first met before it can be modified in such a way as to allow solution. Furthermore, it should be recognized that only a very few equations that arise from actual physical systems are simple enough to allow exact solution in this way. Usually it is necessary to assume linearity for the system so as to allow exact solution for the equations, or to obtain only approximate solutions for the nonlinear equations. In either case, the resulting solution may apply only moderately well to the actual system.

A solution, even approximate, in algebraic analytic form is most convenient to use when the nature of solutions with a wide range of the parameters is being investigated. For this reason, it is usually highly desirable to find a solution in analytic form if this is at all possible.

CHAPTER 5

ANALYSIS OF SINGULAR POINTS

5.1. Introduction. The singular points of a differential equation are fundamental in determining properties of its solution. Considerable insight into the qualitative aspects of the solution, and some quantitative information as well, can be had through a study of singularities. A graphical representation of the location of singularities and the types of solution curves existing near them presents a picture in compact form of all the solutions that may exist. It shows what kinds of things may occur as the variables in the system take on all possible values, both large and small. A study of singularities is a combination of an analytical and graphical approach to the problem.

In order to investigate the singularities of a physical system, it is usually desirable to have appropriate equations relating all the variables of the system. This is not always necessary, however, and sometimes relations available only in graphical form can be used.

5.2. Type of Equation under Investigation. Because a graphical representation of solution curves on a plane surface with two dimensions is conveniently used in a study of singularities, such a study is usually limited to the case of two variables. A differential equation of the form

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} \tag{5.1}$$

can be investigated, where $P(x,y)$ and $Q(x,y)$ may be nonlinear functions of variables x and y. This equation is equivalent to the two equations

$$\begin{aligned} \frac{dx}{dt} &= P(x,y) \\ \frac{dy}{dt} &= Q(x,y) \end{aligned} \tag{5.2}$$

and is obtained merely by eliminating the independent variable t. Elimination of t makes Eq. (5.1) an autonomous equation and limits it to situations where any forcing function is either entirely absent or particularly simple. The two first-order equations may, in turn, have arisen from a single second-order equation. In analyzing a higher-order equation, it is commonly broken into simultaneous first-order equations.

Singularities of Eq. (5.1) are those values of x and y for which both P and Q simultaneously become zero. At the singular point, $x = x_s$, $y = y_s$, and both $P(x_s,y_s) = 0$ and $Q(x_s,y_s) = 0$, so that dy/dx becomes indeterminate. If P and Q are nonlinear functions, there may be a number of singularities. A singularity is always a point of equilibrium, since both dx/dt and dy/dt are zero. The resulting equilibrium may be stable or unstable.

The nature of solutions near a singularity may be explored by substituting $x = x_s + u$ and $y = y_s + v$, where u and v are small variations. With these substitutions, Eq. (5.1) becomes

$$\frac{dy}{dx} = \frac{dv}{du} = \frac{Q(x_s,y_s) + cu + dv + c_2u^2 + d_2v^2 + f_2uv + \cdots}{P(x_s,y_s) + au + bv + a_2u^2 + b_2v^2 + e_2uv + \cdots} \tag{5.3}$$

where a, b, c, d, . . . are real constants. The expansion of P and Q in power series may come directly or may require a Taylor's-series expression for these functions. The linear terms in u and v are most important in determining the nature of solutions near a singularity. The kind of singularity depends only upon the linear terms, provided these terms are present if nonlinear terms are present. In other words, if $c_2 \neq 0$ in Eq. (5.3), so that a term with u^2 appears in the numerator, the linear term cu must be present with $c \neq 0$ for the singularity to be simple. The same condition applies in both numerator and denominator for both u and v. Under this condition, with a simple singularity, the properties of the solution near a singularity depend upon the equation

$$\frac{dv}{du} = \frac{cu + dv}{au + bv} \tag{5.4}$$

with only linear terms. The coefficient d, multiplying v in the numerator, must not be confused with the derivative symbol on the left side of the equation.

If higher-power terms in u and v are present, but with the corresponding first-power terms missing, the singularity is not simple and cannot be studied from Eq. (5.4) alone. Only simple singularities are considered here.

5.3. Linear Transformations. Equation (5.4) is equivalent to the pair of simultaneous first-order equations

$$\begin{aligned} \frac{du}{dt} &= au + bv \\ \frac{dv}{dt} &= cu + dv \end{aligned} \tag{5.5}$$

This pair of equations is a simple case of the more general set of n simultaneous first-order equations, which may be written

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned} \tag{5.6}$$

where $x_1, \ldots, x_n$ are the n dependent variables, the dots signify differentiation with respect to independent variable t, and the a_{ij} coefficients are constants. This set of equations can be written more simply and more generally by using matrix notation as

$$\{\dot{x}\} = [A]\,\{x\} \tag{5.7}$$

where $\{x\}$ is the column matrix,

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{Bmatrix}$$

and $[A]$ is the square matrix of coefficients,

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The first subscript i on a_{ij} locates the row in which it falls, and the second subscript j locates the column.

Solutions for linear equations of the form of Eq. (5.6) are known to involve exponential functions which retain their form upon differentiation. The solution may be written $\{x\} = \{C_x\} \exp(\lambda t)$. Here λ is a constant determined by the coefficients in the differential equations, and, in general, there are as many values of λ as the number of first-order equations. Actually, a sum of exponentials, with each possible value of λ, is required to form a general solution. The constants $\{C_x\}$ cannot be found from the differential equations alone but require specification of initial conditions.

If the solution is differentiated and substituted into Eq. (5.7), the result is

$$\lambda\{x\} = [A]\{x\}$$

which can be written as

$$[[A] - \lambda[I]]\{x\} = \{0\}$$

where $[I]$ is the unit matrix,

$$[I] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

and $\{0\}$ is the zero matrix having every element zero. A property of this kind of equation is that it can be satisfied, except for the trivial case of $\{x\} = \{0\}$, only if the determinant vanishes,

$$|[A] - \lambda[I]| = 0 \tag{5.8}$$

With a few terms written out, this determinant is

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

Equation (5.8) is known as the characteristic equation, and values of λ satisfying it, designated as $\lambda_1, \lambda_2, \ldots, \lambda_n$, are characteristic roots. If coefficients a_{ij} are all real, the characteristic roots must be real or must occur as complex conjugate pairs.

The types of differential equations describing physical systems can often be written in a manner similar to Eq. (5.6), and usually the right sides of the equations involve several of the dependent variables. Coupling is then said to exist between the variables. A system in which there is no coupling would have on the right side of each equation only the single variable which appears on the left side. This kind of equation is said to be in normal form and is much simpler to study mathematically. A system with coupling between the variables can be converted to one having no coupling by the mathematical process of changing the variables through an appropriate linear transformation.

In a linear transformation, the original variables x of Eq. (5.7) are replaced by new variables y, where each x is made up of a linear combination of the variables y. In mathematical form,

$$\begin{aligned} x_1 &= p_{11}y_1 + p_{12}y_2 + \cdots + p_{1n}y_n \\ x_2 &= p_{21}y_1 + p_{22}y_2 + \cdots + p_{2n}y_n \\ &\cdots\cdots\cdots\cdots\cdots \\ x_n &= p_{n1}y_1 + p_{n2}y_2 + \cdots + p_{nn}y_n \end{aligned}$$

which in matrix form is

$$\{x\} = [P]\{y\} \tag{5.9}$$

The elements p_{ij} of $[P]$ are constant quantities. Equation (5.9) can be rewritten to give the y variables in terms of the x variables as

$$\{y\} = [P]^{-1}\{x\} \tag{5.10}$$

where matrix $[P]^{-1}$ is the inverse of matrix $[P]$. In order to be able to invert $[P]$, it is necessary that its determinant $|P|$ be different from zero.

A combination of Eqs. (5.7), (5.9), and (5.10) gives

$$[P]^{-1}\{\dot{x}\} = \{\dot{y}\} = [P]^{-1}[A]\{x\} = [P]^{-1}[A][P]\{y\}$$

and this can be written

$$\{\dot{y}\} = [B]\{y\} \tag{5.11}$$

where matrix $[B]$ is defined,

$$[B] = [P]^{-1}[A][P] \tag{5.12}$$

The linear transformation given by matrix $[P]$ has changed the original equations of Eq. (5.7) into the new form of Eq. (5.11). A solution for Eq. (5.11) has the form $\{y\} = \{C_y\} \exp(\lambda t)$, analogous to that of the solution for Eq. (5.7). The characteristic roots for Eq. (5.11) are given by the relation

$$|[B] - \lambda_B[I]| = 0 \tag{5.13}$$

where the λ_B are roots found from the matrix $[B]$. The matrices of Eq. (5.13) can be written

$$\begin{aligned} [B] - \lambda_B[I] &= [P]^{-1}[A][P] - \lambda_B[P]^{-1}[I][P] \\ &= [P]^{-1}[[A] - \lambda_B[I]][P] \end{aligned}$$

The determinant of each side of this equation must vanish to satisfy Eq. (5.13), and since both $|P| \neq 0$ and $|P^{-1}| \neq 0$, it is necessary that

$$|[A] - \lambda_B[I]| = 0$$

This relation is the same as Eq. (5.8), used to find characteristic roots from matrix $[A]$. Thus, the linear transformation $[P]$ changing $[A]$ to $[B]$ does not change the characteristic roots for the set of equations.

It is worth noting here that the trace of $[A]$, defined as the sum of terms down the main diagonal, $\text{tr}\,[A] = \sum_{i=1}^{n} a_{ii}$, is the same as the trace of $[B]$, $\text{tr}\,[B] = \sum_{i=1}^{n} b_{ii}$, a relation useful for checking purposes.

5.4. Equations in Normal Form. By means of a linear transformation employing matrix $[P]$, a set of equations described by matrix $[A]$ can be converted into an equivalent set described by matrix $[B]$, where

$$[B] = [P]^{-1}[A][P]$$

If the transformation matrix is chosen properly, matrix $[B]$ can be made a diagonal matrix, with all elements not on the main diagonal zero. The set of equations described by $[B]$ are then said to be in normal form. Since the characteristic roots of $[B]$ are the same as those of $[A]$, the elements on the main diagonal of $[B]$ must be the characteristic roots. Thus, the set of equations in normal form is

$$\left\{\begin{array}{l} \dot{y}_1 = \lambda_1 y_1 \\ \dot{y}_2 = \lambda_2 y_2 \\ \cdots\cdots \\ \dot{y}_n = \lambda_n y_n \end{array}\right. \qquad [B] = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{5.14}$$

If two roots happen to form a complex conjugate pair, as

$$\lambda_1 = \zeta + j\omega, \ \lambda_2 = \zeta - j\omega$$

these two elements of $[B]$ would be complex quantities. Often it is desirable that only real quantities appear. An equivalent form for the equations when λ_1 and λ_2 are complex conjugates is

$$\left\{\begin{array}{l} \dot{y}_1 = y_2 \\ \dot{y}_2 = -(\zeta^2 + \omega^2)y_1 + 2\zeta y_2 \\ \cdots\cdots\cdots\cdots\cdots \\ \dot{y}_n = \lambda_n y_n \end{array}\right. \qquad [B] = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ -(\zeta^2 + \omega^2) & 2\zeta & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{5.15}$$

Usually, when complex roots occur, this alternate form is preferable. The characteristic equation found from $[B]$ in the form of Eq. (5.15) is the same as that found from $[B]$ of Eq. (5.14).

The original set of equations $\{\dot{x}\} = [A]\{x\}$ has solutions

$$\{x\} = \{C_x\} \exp(\lambda t)$$

while the equations in normal form $\{\dot{y}\} = [B]\{y\}$ have solutions

$$\{y\} = \{C_y\} \exp(\lambda t)$$

The only feature of the solutions that can be determined from the differential equations alone is the characteristic roots, which are the same in each case. The coefficients $\{C_x\}$ and $\{C_y\}$ can be found only from initial conditions. Solution in terms of x and y can then be said to be equivalent, at least so far as qualitative properties are concerned.

For the sake of simplicity, the discussion now returns to a second-order system described by just two equations. The matrix $[P]$ needed to reduce this kind of system to normal form is easily found. In order to simplify notation, the following symbols are used:

$$
\begin{aligned}
\{x\} &= \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad [A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
\{y\} &= \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} \qquad [B] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \\
& \qquad [P] = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}
\end{aligned} \tag{5.16}
$$

Originally only $\{x\}$ and $[A]$ are known, and $[B]$, $[P]$, and $\{y\}$ must be found.

The characteristic equation is

$$|[A] - \lambda[I]| = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0$$

or

$$\lambda^2 - (a + d)\lambda - (bc - ad) = 0$$

The two characteristic roots are

$$(\lambda_1,\lambda_2) = \tfrac{1}{2}\{(a + d) \pm [(a + d)^2 + 4(bc - ad)]^{1/2}\} \tag{5.17}$$

where, in all that follows, λ_1 is the root found with the positive sign. If the roots are real, λ_1 is always the more positive root. These values of λ_1 and λ_2 serve to determine matrix $[B]$.

Matrix $[P]$ must satisfy Eq. (5.12), which can be written

$$[P][B] = [A][P]$$

When these matrix products are found, the result is

$$\begin{bmatrix} \alpha\lambda_1 & \beta\lambda_2 \\ \gamma\lambda_1 & \delta\lambda_2 \end{bmatrix} = \begin{bmatrix} \alpha a + \gamma b & \beta a + \delta b \\ \alpha c + \gamma d & \beta c + \delta d \end{bmatrix}$$

Since corresponding elements of equal matrices must be identical, the following simultaneous equations exist:

$$
\begin{aligned}
\alpha\lambda_1 &= \alpha a + \gamma b \\
\gamma\lambda_1 &= \alpha c + \gamma d
\end{aligned}
$$

The ratio γ/α defined as m_1 is

$$m_1 = \frac{\gamma}{\alpha} = \frac{\lambda_1 - a}{b} = \frac{c}{\lambda_1 - d} \tag{5.18}$$

In a similar manner, the ratio δ/β is defined as m_2 and is

$$m_2 = \frac{\delta}{\beta} = \frac{\lambda_2 - a}{b} = \frac{c}{\lambda_2 - d} \tag{5.19}$$

Both the forms given on the right sides of Eqs. (5.18) and (5.19) are necessary, since sometimes one form is indeterminate. These equations fix the ratios γ/α and δ/β and thereby fix the elements of matrix $[P]$ within constant factors.

The coordinate transformation is $\{x\} = [P]\{y\}$, or

$$\begin{aligned} x_1 &= \alpha y_1 + \beta y_2 \\ x_2 &= \gamma y_1 + \delta y_2 = m_1 \alpha y_1 + m_2 \beta y_2 \end{aligned} \tag{5.20}$$

The geometrical properties of this transformation can be explored by considering a straight line on the y_1y_2 plane, having the general equation $y_2 = my_1 + y_0$. This is a line of slope m and intercept y_0 on the y_2 axis.

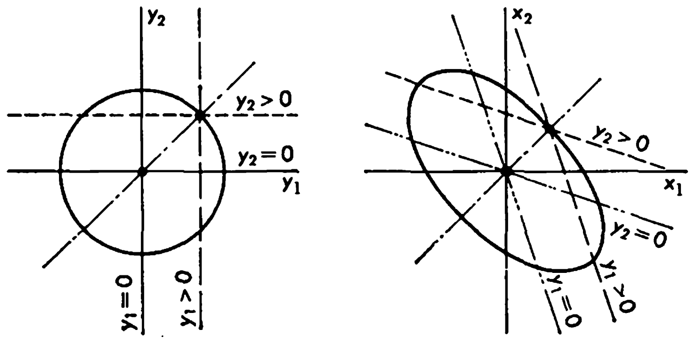


FIG. 5.1. Typical linear transformation from the y_1y_2 plane to the x_1x_2 plane. Corresponding lines and points are indicated.

The transformation, Eq. (5.20), gives the equivalent equation for the x_1x_2 plane as

$$x_2 = \frac{\gamma + \delta m}{\alpha + \beta m} x_1 + \frac{\alpha\delta - \beta\gamma}{\alpha + \beta m} y_0$$

This equation also represents a straight line. Furthermore, parallel lines on the y_1y_2 plane have the same slope m and lead to parallel lines with the same slope on the x_1x_2 plane.

The y_1 axis, for which $y_2 = 0$, corresponds to the line

$$x_2 = m_1 x_1 \tag{5.21}$$

and the y_2 axis, for which $y_1 = 0$, corresponds to the line

$$x_2 = m_2 x_1 \tag{5.22}$$

The quantities m_1 and m_2 are given by Eqs. (5.18) and (5.19).

A typical linear transformation is illustrated in Fig. 5.1. Several corresponding lines and curves are shown on the y_1y_2 and x_1x_2 planes. The figures on one plane are skewed upon being transferred to the other plane, but qualitative features are preserved. A circle in y_1y_2 becomes an ellipse on x_1x_2, but each figure is a closed curve.

5.5. Types of Singularities. The types of singularities of a second-order system can now be investigated and classified. The simplest cases are those in which the two characteristic roots are real. The equations for the system in normal form are

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$$

$$\frac{\dot{y}_2}{\dot{y}_1} = \frac{dy_2}{dy_1} = \frac{\lambda_2}{\lambda_1}\frac{y_2}{y_1}$$

The possibilities can be listed as follows, where always λ_1 designates the more positive root.

a. Both Roots Real and Positive

$$0 < \lambda_2 < \lambda_1$$

$$0 < \frac{\lambda_2}{\lambda_1} < +1$$

The equation for a curve representing a solution for the differential equation on the y_1y_2 plane can be found directly by integration as

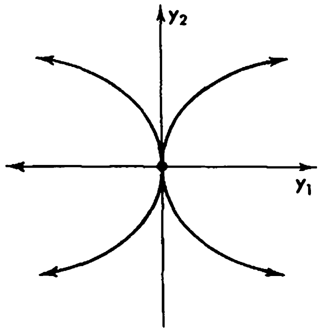
$$y_2 = Cy_1^{(\lambda_2/\lambda_1)}$$

where C is an arbitrary constant dependent upon initial conditions. These curves are generally parabolic in shape, with their exact shape determined by the ratio λ_2/λ_1 and constant C. The slope of the curves may be found from

$$\frac{dy_2}{dy_1} = C\frac{\lambda_2}{\lambda_1}y_1^{(\lambda_2/\lambda_1 - 1)}$$

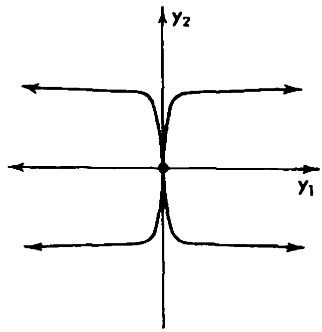
and, near the origin, $dy_2/dy_1 \to \infty$ as $y_1 \to 0$, since $\lambda_2/\lambda_1 < +1$. Thus, all solution curves have a definite direction near the origin, being parallel to the y_2 axis. A typical case is shown in Fig. 5.2, where values of λ_1 and λ_2 are not far different, $0 \ll \lambda_2/\lambda_1 < 1$. The curves represent the locus of points determined by corresponding values of y_1 and y_2. As independent variable t increases, the point relating instantaneous values of y_1 and y_2 moves along the curve in the direction of the arrowheads. Initial conditions determine the value of constant C and thus the quadrant within which a particular solution lies. Since the roots are positive, both y_1 and y_2 increase without bound as t increases and this type of singularity is said to be unstable. Conversely, if y_1 and y_2 were both ultimately to vanish as t increases, the singularity would be called stable. The question of the stability of a singularity is one of its important features. This particular type of singularity is given the name node, which refers to the fact that solution curves have a definite direction near the singularity, here located at the origin.

It is possible for initial conditions to be such that one of the variables y_1 or y_2 is exactly zero. In such a case, this variable remains zero, and the solution curve is either the y_2 or the y_1 axis, respectively. The axes then represent special cases of solution curves, corresponding to special initial conditions.



$0 \ll \lambda_2/\lambda_1 < 1$

FIG. 5.2. Typical solution curves for equations in normal form where singularity is an unstable node and roots are of approximately the same magnitude. Arrows indicate direction of increasing t.



$0 < \lambda_2/\lambda_1 \ll 1$

FIG. 5.3. Typical solution curves for equations in normal form where singularity is an unstable node and roots are of considerably different magnitude.

If the values of λ_1 and λ_2 differ considerably, so that $0 < \lambda_2/\lambda_1 \ll 1$, the solution curves take the shape of Fig. 5.3. These curves are qualitatively similar to those of Fig. 5.2 but have a much sharper break as they change direction. If $t \to +\infty$, solution curves are almost straight and parallel to the y_1 axis. If $t \to -\infty$, solution curves are almost straight and parallel to the y_2 axis. These observations are supported by considering the solutions, which have the form

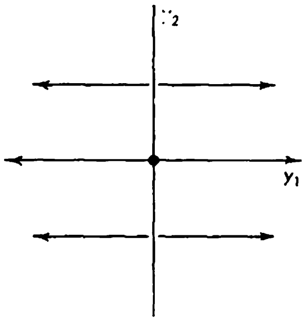
$$y_1 = C_1 \exp(\lambda_1 t)$$
$$y_2 = C_2 \exp(\lambda_2 t)$$

so that

$$\frac{dy_2}{dy_1} = \frac{\lambda_2 C_2 \exp(\lambda_2 t)}{\lambda_1 C_1 \exp(\lambda_1 t)}$$

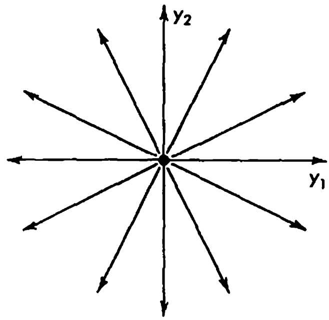
where the relation exists $0 < \lambda_2 \ll \lambda_1$. If t is large and positive, $\exp(\lambda_1 t) \gg \exp(\lambda_2 t)$ and dy_2/dy_1 is essentially zero. If t is large and negative, $\exp(\lambda_1 t) \ll \exp(\lambda_2 t)$ and dy_2/dy_1 is essentially infinite. Thus, if λ_1 and λ_2 differ sufficiently in magnitude, transition between these two conditions occurs suddenly and solution curves are essentially two straight lines with a sharp break joining them.

Two limiting cases are illustrated in Figs. 5.4 and 5.5, which represent the extremes for the ratio $0 \leq \lambda_2/\lambda_1 \leq 1$. For Fig. 5.4, one root is zero, while for Fig. 5.5 the two roots are equal.



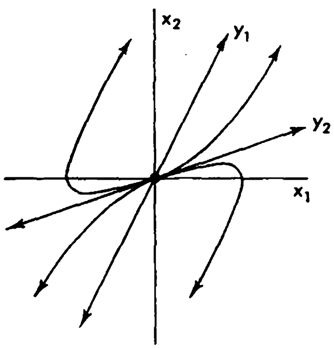
$\lambda_1 > 0,\ \lambda_2 = 0,\ \lambda_2/\lambda_1 = 0$

FIG. 5.4. Typical solution curves for equations in normal form where singularity is an unstable node and one root is zero.



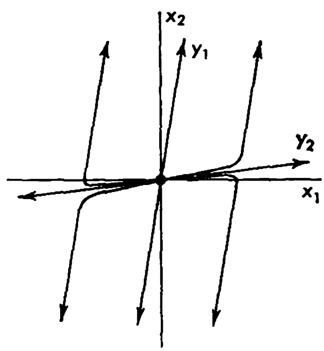
$0 < \lambda_1 = \lambda_2,\ \lambda_2/\lambda_1 = 1$

FIG. 5.5. Typical solution curves for equations in normal form where singularity is an unstable node and both roots are equal.



$0 \ll \lambda_2/\lambda_1 < 1$

FIG. 5.6. Typical solution curves for equations in general form where singularity is an unstable node and roots are of approximately the same magnitude. This figure may be compared with Fig. 5.2.



$0 < \lambda_2/\lambda_1 \ll 1$

FIG. 5.7. Typical solution curves for equations in general form where singularity is an unstable node and roots are of considerably different magnitude. This figure may be compared with Fig. 5.3.

The discussion thus far and the figures shown have applied to the case where the differential equations are in normal form. In the more general case, the equations are not in normal form, but a linear transformation relates the two sets of equations. For this case of positive real characteristic roots, leading to an unstable node, the general forms of the solution curves on the x_1x_2 plane are shown in Figs. 5.6 and 5.7.

Straight lines corresponding to the y_1 and y_2 axes are shown here, skewed in accordance with the transformation. Figure 5.6 is entirely analogous to Fig. 5.2 and Fig. 5.7 to Fig. 5.3, the only difference being the skewing that has occurred. The location of lines on the x_1x_2 plane corresponding

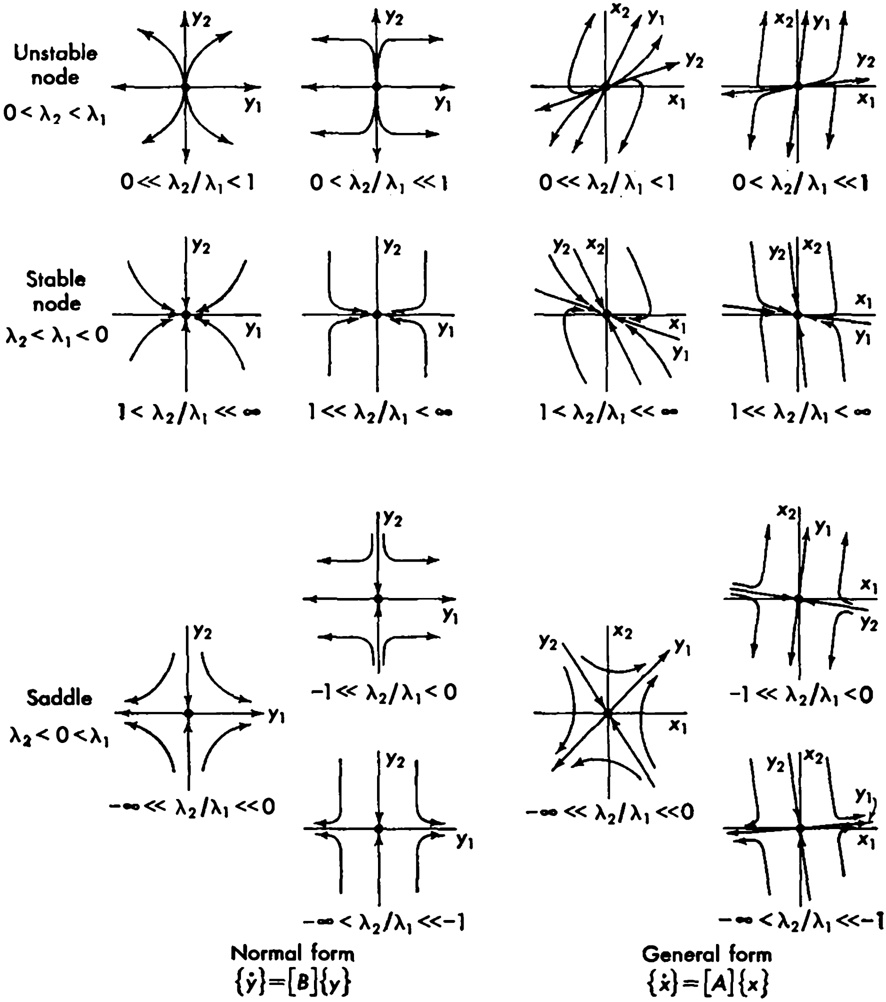


FIG. 5.8. Typical solution curves for equations both in normal form and in general form where singularity is an unstable node, a stable node, or a saddle. Cases are shown where the roots are of approximately the same magnitude and of considerably different magnitude. Arrows indicate direction of increasing t.

to the y_1 and y_2 axes can be found from Eqs. (5.21) and (5.22). If roots λ_1 and λ_2 differ sufficiently in magnitude, solution curves are essentially formed of two straight lines parallel to these axes. The exact location of these lines depends, of course, upon initial conditions.

The case of the unstable node is one of those listed in the collection of Fig. 5.8. Other cases considered in the following sections are also listed here.

b. Both Roots Real and Negative

$$\lambda_2 < \lambda_1 < 0$$
$$\frac{\lambda_2}{\lambda_1} > +1$$

Solution curves for the normal form of the equations are once more parabolic, but near the origin $dy_2/dy_1 = 0$, and the curves are parallel to the y_1 axis. This singularity is a node, but since negative characteristic roots lead to the ultimate disappearance of both y_1 and y_2 with increasing t, the node is stable. As t increases, the point representing corresponding values of y_1 and y_2 moves along the curve approaching the singularity at the origin but reaches it only at an infinite value of t. Again, the y_1 and y_2 axes are special solution curves. Typical solution curves are shown in Fig. 5.8 for both the normal and general forms of the differential equations.

c. Both Roots Real and Opposite in Sign

$$\lambda_2 < 0 < \lambda_1$$
$$\frac{\lambda_2}{\lambda_1} < 0$$

In this case, $dy_2/dy_1 = -|\lambda_2/\lambda_1|(y_2/y_1)$, and $y_2 y_1^{|\lambda_2/\lambda_1|} = C$. Solution curves in the y_1y_2 plane are hyperbolic in shape and generally pass by the singularity at the origin. Since root λ_1 is positive, y_1 ultimately increases without bound and the solution is unstable, even though y_2 ultimately vanishes. The singularity is called a saddle, since a three-dimensional surface, with height above the y_1y_2 plane proportional to the value of constant C, has a shape somewhat like a saddle. Peaks in the first and third quadrants are separated by valleys in the second and fourth quadrants. Once more, the y_1 and y_2 axes are special solution curves. Typical solution curves are shown in Fig. 5.8.

The second general possibility for the characteristic roots is that they form a complex conjugate pair. In this case, the normal form for the equations can be written

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -(\delta^2 + \omega^2) & 2\zeta \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$$

where the notation is slightly different from that following Eq. (5.14), and

$$\frac{\dot{y}_2}{\dot{y}_1} = \frac{dy_2}{dy_1} = \frac{-(\delta^2 + \omega^2)y_1 + 2\,\delta y_2}{y_2}$$

d. Roots Pure Imaginaries

$$\delta = 0 \qquad \begin{cases} \lambda_1 = +j\omega \\ \lambda_2 = -j\omega \end{cases}$$

The equation for a solution curve can be found from $dy_2/dy_1 = -\omega^2 y_1/y_2$ as $\omega^2 y_1^2 + y_2^2 = C$. This is the equation of an ellipse about the singularity at the origin. A typical case is shown in Fig. 5.9. These closed curves completely encircling the singularity represent a vortex. The solution is a periodic oscillation in time, with no change in amplitude. Since there is neither growth nor decay, the solution has neutral stability. The amplitude, and the size of the ellipse, are determined by the initial conditions.

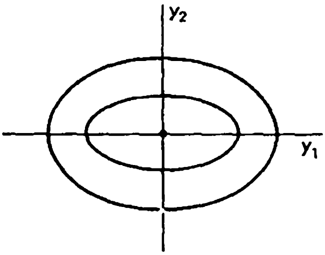


FIG. 5.9. Typical solution curves for equations in normal form where singularity is a vortex.

Linear transformation from the y_1y_2 plane to the x_1x_2 plane merely skews the elliptical curves without otherwise changing their form.

e. Roots Complex Conjugates

$$\lambda_1 = \delta + j\omega$$
$$\lambda_2 = \delta - j\omega$$

The solution curves depend upon the equation

$$\frac{dy_2}{dy_1} = \frac{-(\delta^2 + \omega^2)\, y_1 + 2\,\delta y_2}{y_2}$$

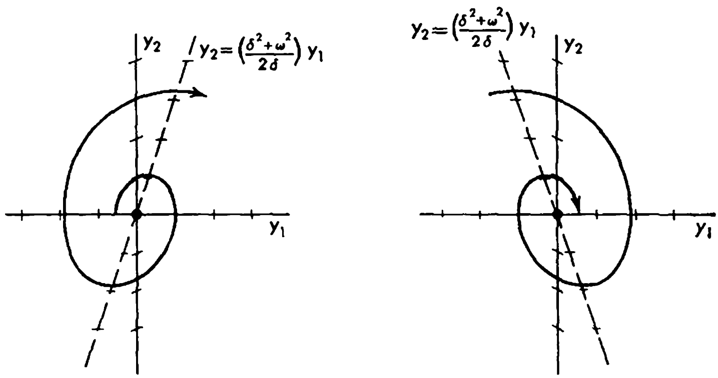
This equation cannot be integrated directly, although integration is possible following an appropriate change of variable. The qualitative nature of its solution curves can be found more easily from an isocline construction. The following isoclines are evident:

$$\frac{dy_2}{dy_1} = 0 \qquad \text{along} \qquad y_2 = \frac{\delta^2 + \omega^2}{2\delta}\, y_1$$
$$\frac{dy_2}{dy_1} = \infty \qquad \text{along} \qquad y_2 = 0$$
$$\frac{dy_2}{dy_1} = 2\delta \qquad \text{along} \qquad y_1 = 0$$

These isoclines carrying directed line segments of appropriate slope are shown in Fig. 5.10, and typical solution curves are sketched properly. The curves form spirals about the singularity at the origin, which is designated a focus. If $\delta > 0$, the solution ultimately grows without bound and is unstable. If $\delta < 0$, the solution ultimately vanishes and is stable.

Again, a linear transformation to the x_1x_2 plane merely skews the spirals without otherwise changing their form.

For simple singularities of two first-order linear equations, there are only four possibilities: node, saddle, vortex, and focus. The node and focus may be either stable or unstable. The saddle is always unstable. The vortex is neutrally stable. Only the types of solution curves associated with these four singularities can exist.



$\delta > 0$ $\delta < 0$

FIG. 5.10. Typical solution curves for equations in normal form where singularity is a focus. Both the unstable case ($\delta > 0$) and the stable case ($\delta < 0$) are shown. Arrows indicate direction of increasing t.

5.6. Sketching of Solution Curves. Information about the singularity can be used to sketch a family of solution curves for a pair of first-order equations. In general form the equations are

$$\left.\begin{aligned} \frac{dx}{dt} &= P(x,y) \\ \frac{dy}{dt} &= Q(x,y) \end{aligned}\right\} \quad \text{or} \quad \frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} \tag{5.23}$$

Among the information that is readily available is the following:

1. Location of singularities, (x_s,y_s), determined by the simultaneous requirement that both $P(x_s,y_s) = 0$ and $Q(x_s,y_s) = 0$.

2. Type of each singularity, provided it is simple, found by the change $x = x_s + u$ and $y = y_s + v$ which gives

$$\left.\begin{aligned} \frac{du}{dt} &= au + bv \\ \frac{dv}{dt} &= cu + dv \end{aligned}\right\} \quad \text{or} \quad \frac{dv}{du} = \frac{cu + dv}{au + bv} \tag{5.24}$$

In writing these equations, care must be taken not to change coefficients a, b, c, d in going from the two first-order equations to the single equation found as their ratio. For example, it is not permissable to divide out a common factor in the coefficients in the second form of equation since this changes the values of the coefficients in the two first equations. Any change in these coefficients is equivalent to a change of time scale. For example, if all the coefficients were reversed in sign, the value of dv/du would not be changed at all. However, a change in sign of all coefficients reverses the signs of the characteristic roots. The conditions for stability are thereby reversed. In summary, the coefficients must be carefully retained in their original form.

Characteristic roots are

$$(\lambda_1, \lambda_2) = \tfrac{1}{2}\{(a + d) \pm [(a + d)^2 + 4(bc - ad)]^{1/2}\} \tag{5.25}$$

The types of singularities are as follows:

a. Node if both roots real and of same sign
b. Saddle if both roots real and of opposite sign
c. Vortex if both roots pure imaginary
d. Focus if roots complex conjugates

If there is a real part of a characteristic root that is positive, the solution is unstable. Negative real parts give stable solutions. The vortex, where the real parts are zero, has neutral stability. In setting up these criteria, a definition of stability for essentially linear systems has been made. The relations are summarized in Table 5.1.

TABLE 5.1. RELATIONS AMONG COEFFICIENTS FOR EACH TYPE OF SINGULARITY

Saddle: $4(bc - ad) > 0$
Node: $-(a + d)^2 < 4(bc - ad) < 0$
Focus: $4(bc - ad) < -(a + d)^2$
  For node or focus:
    Stable: $a + d < 0$
    Unstable: $a + d > 0$
Vortex: Both $a + d = 0$
    and $4(bc - ad) < 0$

3. Direction of solution curves near each node or saddle, as $t \to +\infty$, found from

$$v = m_1 u = \frac{(\lambda_1 - a)u}{b} = \frac{cu}{\lambda_1 - d} \tag{5.26}$$

and as $t \to -\infty$, found from

$$v = m_2 u = \frac{(\lambda_2 - a)u}{b} = \frac{cu}{\lambda_2 - d} \tag{5.27}$$

If the two roots are of considerably different magnitude, solution curves are essentially two straight lines having these directions.

The following relations, easily shown to be true, are sometimes useful for purposes of checking computations:

$$\lambda_1\lambda_2 = ad - bc$$
$$\lambda_1 + \lambda_2 = a + d$$
$$m_1m_2 = -\frac{c}{b}$$
$$m_1 + m_2 = \frac{d - a}{b}$$

4. Isoclines for solution curves near each singularity, for zero slope $dv/du = 0$ found from

$$v = -\left(\frac{c}{d}\right)u \tag{5.28}$$

and for infinite slope $dv/du = \infty$ found from

$$v = -\left(\frac{a}{b}\right)u \tag{5.29}$$

With this information, solution curves near each singularity can be sketched. Since dy/dx for Eq. (5.23) is assumed to be uniquely defined, except at singularities, solution curves can never cross. Thus, solution curves coming from the neighborhood of one singularity can be joined smoothly to those near another singularity to give a complete solution curve. By making use of all the available information, solution curves can be sketched with a fair degree of accuracy.

These solution curves show the relation between x and y with t as a parameter. As t increases, the representative point relating corresponding instantaneous values of x and y moves along the curve. In order to find the speed with which it moves, and thus the relation between x and t or y and t, it is necessary to return to the differential equation. During a short interval Δt in t, changes Δx and Δy occur in x and y. The average values during this interval are x_{av} and y_{av}. The original differential equations can then be written for this interval as

$$\Delta t = \frac{\Delta x}{P(x_{av},y_{av})} = \frac{\Delta y}{Q(x_{av},y_{av})} \tag{5.30}$$

A numerical calculation beginning at the initial values of x, y, and t and working ahead in small increments is necessary to find relations between x and y and t.

Example 5.1. Volterra's Competition Equations

The following example of the sketching of solution curves is based on what are known as Volterra's equations describing competition between two species of animals.

In these equations, N_1 and N_2 are the populations of the two species existing in some prescribed region. The two species are in competition with one another, perhaps because they consume the same food supply. For actual colonies of animals, quantities N_1 and N_2 would be limited to positive integers, but for this cxample they are assumed to be continuous variables which may become negative. Under certain conditions, Volterra argued that the populations change with time according to the equations

$$\begin{aligned} \frac{dN_1}{dt} &= k_1N_1 - k_3N_1N_2 \\ \frac{dN_2}{dt} &= k_2N_2 - k_4N_1N_2 \end{aligned} \tag{5.31}$$

where k_1, k_2, k_3, and k_4 are positive real constants. For this example, it is assumed arbitrarily that $k_2 > k_1$. In these equations, the left side represents the growth rate at any instant. The first term on the right side represents the normal increase in growth rate which would occur if there were no limiting influences. The second term represents a reduction in growth rate which comes about because of some kind of competition between the two species.

The two equations can be combined to give

$$\frac{dN_2}{dN_1} = \frac{k_2N_2 - k_4N_1N_2}{k_1N_1 - k_3N_1N_2}$$

Singular points for this equation are located at $N_{1s} = 0$, $N_{2s} = 0$ and at $N_{1s} = k_2/k_4$, $N_{2s} = k_1/k_3$. Near the first singularity, N_1 and N_2 themselves are both very small, and the equation is essentially $dN_2/dN_1 = k_2N_2/k_1N_1$, which is of the gcneral form of Eq. (5.24). Here the coefficients may be identified as $a = k_1$, $b = 0$, $c = 0$, and $d = k_2$, so that the characteristic roots become $\lambda_1 = k_2$ and $\lambda_2 = k_1$, since it was assumed that $k_2 > k_1 > 0$. Both roots are real and positive, and the first singularity is an unstable node. The slope of solution curves as $t \to +\infty$ is $m_1 = (\lambda_1 - a)/b = (k_2 - k_1)/0 = \infty$, while, for $t \to -\infty$, $m_2 = c/(\lambda_2 - d) = 0/(k_1 - k_2) = 0$. Furthermore, $dN_2/dN_1 = 0$ along $N_2 = 0$, and $dN_2/dN_1 = \infty$ along $N_1 = 0$. Solution curves near the first singularity at the origin are sketched in Fig. 5.11 in the area designated as region 1.

Near the second singularity, $N_1 = (k_2/k_4) + n_1$, and $N_2 = (k_1/k_3) + n_2$, where n_1 and n_2 are small. This change is equivalent to setting up a new coordinate system with coordinates n_1 and n_2 centered at the second singularity. The linearizcd equation near this singularity is

$$\frac{dn_2}{dn_1} = \frac{-(k_1k_4/k_3)n_1}{-(k_2k_3/k_4)n_2}$$

which is of the general form of Eq. (5.24). Here the coefficients are $a = 0$, $b = -(k_2k_3/k_4)$, $c = -(k_1k_4/k_3)$, and $d = 0$, where care has been taken not to modify the original form of these coefficients. In other words, the negative signs of both b and c are carefully retained. The characteristic roots are $\lambda_1 = +(k_1k_2)^{1/2}$ and $\lambda_2 = -(k_1k_2)^{1/2}$. Both roots are real but of opposite sign; so this singularity is a saddle. The slopes of solution curves as $t \to +\infty$ is $m_1 = (\lambda_1 - a)/b = -(k_4/k_3)(k_1/k_2)^{1/2}$, while for $t \to -\infty$ $m_2 = (\lambda_2 - a)/b = +(k_4/k_3)(k_1/k_2)^{1/2}$. Furthermore, $dn_2/dn_1 = 0$ along $n_1 = 0$, or $N_1 = k_2/k_4$, while $dn_2/dn_1 = \infty$ along $n_2 = 0$, or $N_2 = k_1/k_3$. Solution curves near this second singularity, designated as region 2, are sketched in Fig. 5.11. Continuous curves must fill the entire N_1N_2 plane, running from region 1 to region 2. Several such curves are sketched in the figure.

A study of Fig. 5.11 gives considerable insight into the kinds of solutions which may exist for the competition equations. It is evident that, under all conditions, either N_1 or N_2 ultimately vanishes and the other increases without limit. Particular initial conditions determine which of the two will ultimately survive. It is not necessarily the one initially the smaller which finally disappears. A separatrix curve can be found, separating those curves for which N_1 ultimately vanishes from those for which N_2 ultimately vanishes. This separatrix is shown in the figure and is the one curve beginning and ending at the singularities. Since curves cannot cross the N_1 or N_2 axis, the algebraic signs of these quantities can never reverse.

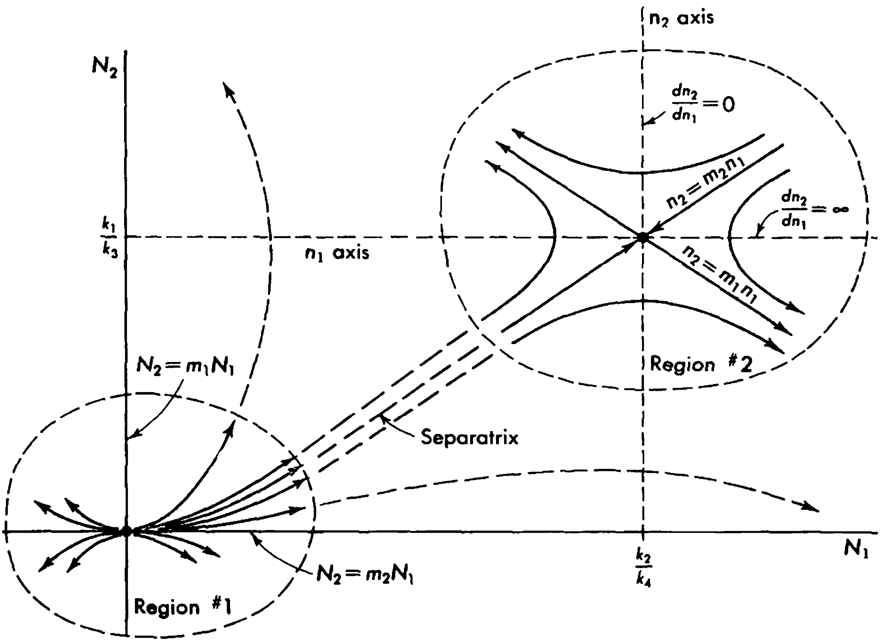


FIG. 5.11. Solution curves sketched for Example. 5.1. Region 1 contains an unstable node, and region 2 contains a saddle. Curves of proper shape are sketched near each singularity, and smooth connecting curves fill the intervening region. The separatrix divides curves ultimately tending one way from those tending the other way.

5.7. Forcing Functions as Steps or Ramps.

The simple analysis in terms of singularities is applicable to systems which can be described by Eq. (5.1), in which independent variable t does not appear. This requirement implies either that the system has no forcing function applied to it or that the forcing function is particularly simple. The equation for a second-order system might be

$$\frac{d^2x}{dt^2} + f\left(\frac{dx}{dt}, x\right)\frac{dx}{dt} + x = F(t) \tag{5.32}$$

A forcing function in the form of a step, $F(t) = F_0$ for $t \geq 0$, can be handled, as sometimes can be a ramp, $F(t) = Gt$ for $t \geq 0$. If the forcing function is a step, time does not enter and there is no difficulty. If the forcing function is a ramp, a change of dependent variable can be used to eliminate time.

In the general case where both step and ramp are included,

$$F(t) = F_0 + Gt$$

the change of variable required is $z = x - F(t) = x - F_0 - Gt$. The substitution for Eq. (5.32) is $x = z + F_0 + Gt$, $dx/dt = dz/dt + G$, $d^2x/dt^2 = d^2z/dt^2$, and the equation becomes

$$\frac{d^2z}{dt^2} + f\left(\frac{dz}{dt} + G, z + F_0 + Gt\right)\left(\frac{dz}{dt} + G\right) + z = 0$$

This equation will be autonomous, with t only in the derivatives, either if $G = 0$, which is the step function alone, or if $f(dx/dt,x) \equiv f(dx/dt)$ so that x does not appear in the multiplier of the dx/dt term. In either of these cases, the equation can be studied easily in terms of its singularities.

Example 5.2. Damped Oscillatory System with Forcing Function

The equation describing a linear damped oscillatory system with a single degree of freedom, and first with no forcing function, is

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^2x = 0 \tag{5.33}$$

where α and ω_0 are constants. It is convenient to use a normalized time variable in studying this equation, $\tau = \omega_0 t$, in terms of which

$$\frac{dx}{dt} = \omega_0\frac{dx}{d\tau} = \omega_0\nu$$
$$\frac{d^2x}{dt^2} = \omega_0^2\frac{d^2x}{d\tau^2} = \omega_0^2\nu\frac{d\nu}{dx}$$

These are the same changes of variable introduced in Sec. 3.3. The equation can then be written

$$\frac{d\nu}{dx} = \frac{-(2\alpha/\omega_0)\nu - x}{\nu}$$

There is one singularity at $x_s = 0$, $\nu_s = 0$. The coefficients are $a = 0$, $b = 1$, $c = -1$, $d = -2\alpha/\omega_0$, so that $(\lambda_1,\lambda_2) = -(\alpha/\omega_0) \pm [(\alpha/\omega_0)^2 - 1]^{1/2}$. The product $\lambda_1\lambda_2$ is here always $\lambda_1\lambda_2 = +1$, so that the two roots are of the same sign if they are real, and a saddle is impossible.

One case of some interest is that in which $\alpha/\omega_0 \gg 1$, corresponding to a high degree of damping. For this case, approximately $\lambda_1 \doteq -1/(2\alpha/\omega_0)$, and $\lambda_2 \doteq -(2\alpha/\omega_0)$. Furthermore, $m_1 = (\lambda_1 - a)/b = \lambda_1$, and $m_2 = (\lambda_2 - a)/b = \lambda_2$. The value of $d\nu/dx$ is infinite along $\nu = 0$ and zero along $\nu = x/(-2\alpha/\omega_0)$. Thus, $d\nu/dx = 0$ along the line very nearly the same as $\nu = m_1x$.

The relation between ν and x is sketched in Fig. 5.12. This is a phase-plane diagram plotting the velocity, normalized in this case, as a function of displacement. For this example with $\alpha/\omega_0 \gg 1$, damping is quite high. Initial conditions determine the initial location of the representative point on a solution curve. As time progresses, the point moves generally toward the singularity at the origin, as indicated by the arrowheads. The figure can be divided into regions in which there is overshoot or no overshoot in the solution following the initial conditions. If the point is initially in a region giving overshoot, the magnitude of the displacement x either increases beyond its initial value or goes through zero and increases before ultimately returning to zero. In a region giving no overshoot, the point moves steadily to the origin, with the magnitude of x decreasing monotonically. The solution curve for which x can start with a given initial value and decay toward zero most rapidly is the curve given by $\nu = m_2x$. The slowest decay is along the curve $\nu = m_1x$.

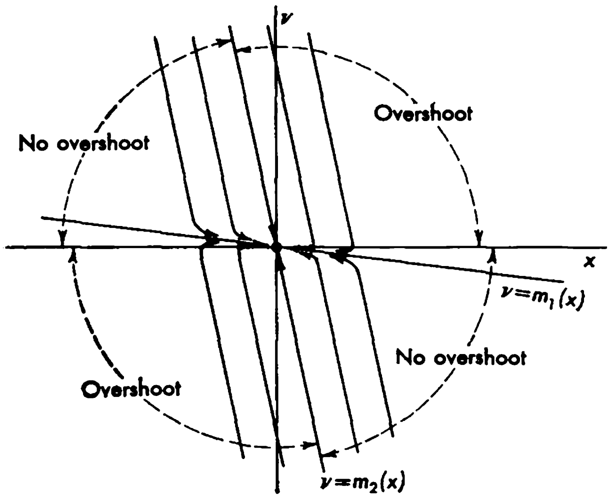


FIG. 5.12. Solution curves for Example 5.2. The phase plane is divided into regions where the solution may overshoot or may not overshoot.

A forcing function consisting of both a step and a ramp applied to this same oscillatory system leads to the equation

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^2x = F_0 + Gt \tag{5.34}$$

replacing Eq. (5.33). In order to make this equation autonomous, the substitution $z = x - F_0/\omega_0^2 - Gt/\omega_0^2$ is necessary, where the terms of the forcing function must be divided by the coefficient of x in Eq. (5.34). Further changes of variable with $\tau = \omega_0 t$ and $\nu = dx/d\tau$ as before, together with the new definition $\zeta = dz/d\tau$, lead to the relations $\zeta = \nu - G/\omega_0^3$ and $d^2z/d\tau^2 = \zeta\, d\zeta/dz = d^2x/d\tau^2 = \nu\, d\nu/dx$. Therefore, Eq. (5.34) becomes

$$\zeta\frac{d\zeta}{dz} + \frac{2\alpha}{\omega_0}\left(\zeta + \frac{G}{\omega_0^3}\right) + z = 0$$

or

$$\frac{d\zeta}{dz} = \frac{-(2\alpha/\omega_0)(\zeta + G/\omega_0^3) - z}{\zeta}$$

This equation is obviously similar to that found with no forcing function except that its singularities are located at $\zeta_s = 0$ and $z_s = -2\alpha G/\omega_0^4$. In terms of the original variables, the singularity is at

$$x_s = z_s + \frac{F_0}{\omega_0^2} + \frac{Gt}{\omega_0^2} = \frac{F_0}{\omega_0^2} - \frac{2\alpha G}{\omega_0^4} + \frac{Gt}{\omega_0^2}$$
$$\nu_s = \frac{1}{\omega_0}\left(\frac{dx}{dt}\right)_s = \zeta_s + \frac{G}{\omega_0^3} = \frac{G}{\omega_0^3}$$

Thus, the singularity, or equilibrium condition, for the original system involves a steady-state velocity, $dx/dt = G/\omega_0^2$, dependent upon the ramp portion of the forcing function. The equilibrium value for x depends upon both the step and the ramp. The nature of the singularity is determined by the coefficients of the equation, which are the same as with no forcing function. If the singularity is a node, as considered in the first part of the example, the steady-state velocity will be approached with no oscillation taking place about it.

5.8. Analysis Combining Singularities and Linear Segments.

A system involving a nonlinear element can often be studied profitably in terms of its singular points, by considering the nonlinear element as made up of several linear elements. This is much the same process as described in Sec. 4.4. An example of this kind is an electronic circuit employing a nonlinear negative resistance.

Example 5.3 Negative-resistance Trigger Circuit and Oscillator

By any of several means, it is possible to obtain a two-terminal resistance in which instantaneous current i_r and voltage e_r are related by a curve similar to that shown in Fig. 5.13. The curve is described by the relation $i_r = f(e_r)$, where function $f(e_r)$ is defined by the curve, and may be found, perhaps, only as the result of experiment. The variational resistance at any point on the curve is $r = 1/(di_r/de_r)$, and since the curve has a negative slope within the central region, the variational resistance here is negative. Since a negative resistance implies a source of power, and since any physically realizable source can supply only a finite amount of power, a realizable resistance can be negative only within a finite region. Outside this region, the slope of the curve becomes positive, and the variational resistance is positive. A curve such as that of Fig. 5.13 applies to a voltage-controlled nonlinear negative resistance since the current is a unique function of voltage.

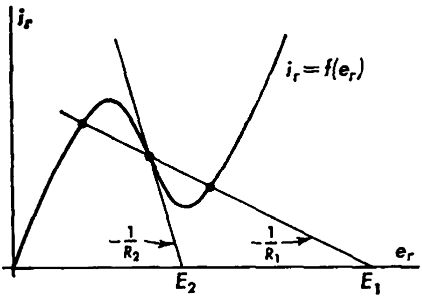


FIG. 5.13. Typical characteristic curve for a negative-resistance element. Construction is shown for load lines corresponding to two cases of load resistance and applied voltage. A single intersection occurs in one case, while three occur in the other case.

This kind of curve is typical of a number of electronic circuits where there is some kind of voltage-controlled feedback.

An element having a characteristic similar to that of Fig. 5.13 can be used in the circuit of Fig. 5.14. The box contains the negative resistance, and the other components are assumed to be simple linear elements of constant value. A constant voltage E is applied in the circuit.

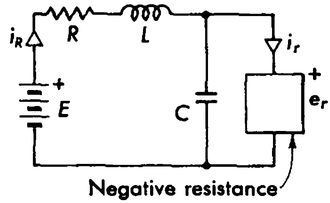


FIG. 5.14. Negative-resistance circuit for Example 5.3.

By proper choice of variables, operation of the circuit can be described in terms of two simultaneous first-order equations,

$$\begin{aligned} \frac{di_R}{dt} &= \frac{1}{L}(E - Ri_R - e_r) \\ \frac{de_r}{dt} &= \frac{1}{C}[i_R - f(e_r)] \end{aligned} \tag{5.35}$$

where i_R is the current through the linear resistor. Singular, or equilibrium, points exist where simultaneously

$$\begin{aligned} E &= Ri_{Rs} + e_{rs} = Ri_{rs} + e_{rs} \\ i_{Rs} &= f(e_{rs}) = i_{rs} \end{aligned}$$

The second of these conditions indicates that at a singularity $i_{Rs} = i_{rs}$ and the current into the capacitor is zero. The singularities can be found by the graphical construction commonly used in studying electronic circuits and shown in Fig. 5.13. A load line is erected on the $i_r e_r$ characteristic. It is drawn from the point on the e_r axis corresponding to steady voltage E, with its slope determined by resistance R. The slope of the load line is $-1/R$, where consistent units must be used for R, e_r, and i_r in drawing the load line. The intersection of this load line with the curve, $i_r = f(e_r)$, determines the singular points. Cases of special interest have an intersection in the region of negative resistance. Two possibilities are shown in Fig. 5.13. In one case, E and R are both large enough, (E_1,R_1), so that there are three intersection points. In the second case, E and R are smaller, (E_2,R_2), and a single intersection occurs. Operation turns out to be basically different in the two cases.

Near a singularity, the substitutions can be made, $i_R = i_{Rs} + i$ and $e_r = e_{rs} + e$, where i and e are small changes. Then Eq. (5.35) becomes

$$\begin{aligned} \frac{di}{dt} &= \frac{1}{L}(-Ri - e) \\ \frac{de}{dt} &= \frac{1}{C}\left(i - \frac{1}{r}e\right) \end{aligned} \tag{5.36}$$

In the second of these equations, the Taylor's-series expansion has been used,

$$f(e_r) = f(e_{rs}) + \left.\frac{df(e_r)}{de_r}\right]_{e_{rs}} e + \cdots$$

with only the first two terms being retained. The definition has been made, $df(e_r)/de_r\Big]_{e_{rs}} = 1/r$, where r is the variational resistance as defined previously and evaluated at the singularity.

The ratio of Eq. (5.36) is

$$\frac{di}{de} = \frac{(1/L)(-e - Ri)}{(1/C)[-(1/r)e + i]} \tag{5.37}$$

The coefficients are $a = -1/rC$, $b = 1/C$, $c = -1/L$, and $d = -R/L$, and the characteristic roots are

$$(\lambda_1,\lambda_2) = \frac{1}{2}\left\{ -\left(\frac{1}{rC} + \frac{R}{L}\right) \pm \left[\left(\frac{1}{rC} + \frac{R}{L}\right)^2 + \frac{4}{LC}\left(-1 - \frac{R}{r}\right)\right]^{1/2} \right\} \tag{5.38}$$

In order to carry the analysis further, it is convenient to replace the actual curved characteristic of Fig. 5.13 with the characteristic shown in Fig. 5.15 and composed of three straight lines, meeting at sharp corners. This straight-line figure approximates the true characteristic. The figure can be divided into the three regions shown, with a linear relation between i_r and e_r existing within each region. In this first case, it is assumed that E and R are large enough and of such value that three intersections occur, one within each region of the figure.

In region 1, the negative-resistance element has a variational resistance $r = -r_1$, where r_1 is a positive quantity, the magnitude of the resistance. The slope of the load line, $-1/R$, is less in magnitude than the slope of the negative-resistance characteristic, which in this region is $-1/r_1$. Thus, for this condition to exist, $R > r_1$. Both characteristic roots for the singularity in region 1 are real and of opposite sign; so this singularity is a saddle and is unstable.

In regions 2 and 3, the variational resistance is $r = +r_2$ and $r = +r_3$, respectively, both positive quantities. The singularities in these regions are both stable and are either nodes or foci, depending upon the relations among the values of all the circuit elements.

Solution curves for the small variations in current and voltage, i and e, may now be sketched near each singularity. This requires knowledge of the types of singularities, already found; of the fact that $di/de = 0$ along $i = -e/R$, which is the resistance load line, and $di/de = \infty$ along $i = e/r$, which is the characteristic of the negative-resistance element; and of the slopes m_1 and m_2 near each singularity. Several solution curves are sketched in Fig. 5.15, where stable nodes are assumed to exist in both regions 2 and 3. These curves represent the way small variations i in the current i_R and e in the voltage e_r change with time, following some initial disturbance in the system. Since the system of the figure is linear within any one region, i and e are not necessarily limited to small values. The stable equilibrium points are those in regions 2 and 3, and the system always comes to rest at one or the other of these points. It does not necessarily come to rest at the equilibrium

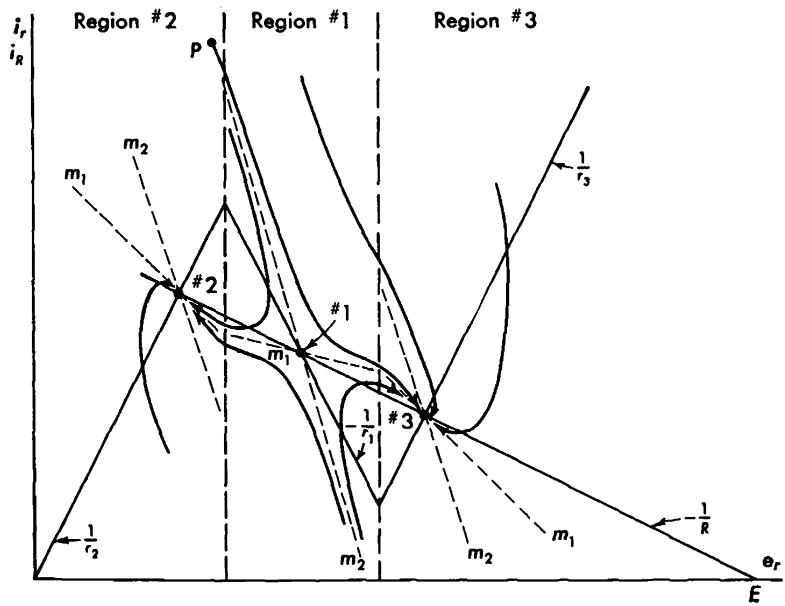


FIG. 5.15. Solution curves sketched for Example 5.3. The curved characteristic of Fig. 5.13 is replaced by three straight lines, and the load line is chosen to intersect the characteristic at three points. A saddle occurs in region 1, and stable nodes occur in regions 2 and 3. Arrows indicate direction of increasing time.

point nearest the initial point. For example, if the initial point is point P in region 2, the representative point moves across region 1 to the singularity in region 3.

A circuit of this sort is the basis for certain types of triggering circuits, characterized as having two stable states separated by an unstable state. The circuit can be caused to trigger from one stable state to the other by suitable influences applied from outside it. The circuit of Fig. 5.14 can be made to trigger by changing steady voltage E momentarily. As a specific case, it will be assumed that initially the circuit is in the stable state characterized by the singularity in region 2. Voltage E is then raised by an amount ΔE sufficiently to cause the load line to shift to the position shown in Fig. 5.16. For the conditions of these figures, ΔE is actually a relatively large change. The singularity in region 3 has merely moved upward and to the right a short distance, to the position marked 3′. Singularities associated with regions 1 and 2 have moved outside these regions, so that they no longer actually exist. However, by extending the straight-line characteristics of the negative-resistance element, what might be called virtual singularities can be located as shown in Fig. 5.16. Virtual singularity 1′ is associated

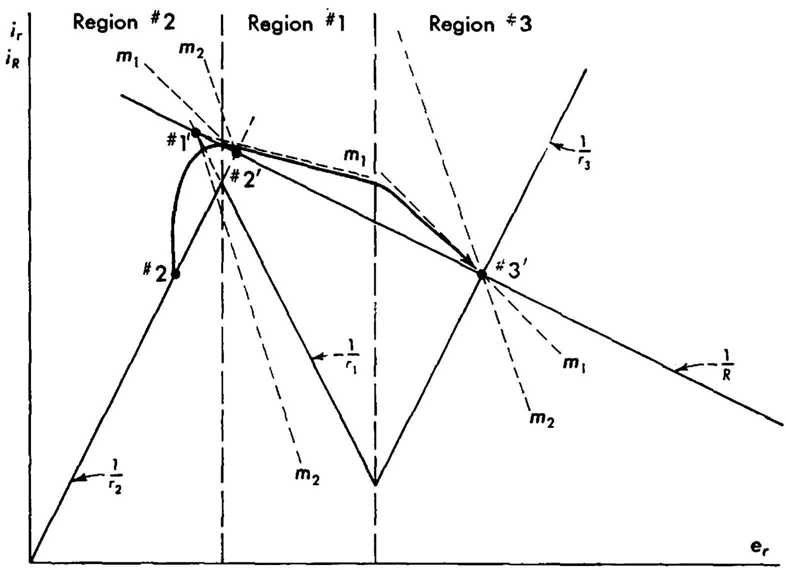


FIG. 5.16. Solution curves sketched for Example 5.3. The load line of Fig. 5.15 has been shifted upward and to the right. A real singularity still exists in region 3, while virtual singularities exist in regions 1 and 2. The path is sketched from original singularity 2 to final singularity 3′.

with region 1 even though it is actually located in region 2. Similarly, 2′ is associated with region 2 even though it is actually in region 1. The singularities are of the same type as before the change in E, and solution curves can be sketched just as before.

The only singularity actually existing is 3′, and it is stable just as before the change in E. Thus, if just previous to the change in E the system was at singularity 2 and the change in E takes place very suddenly, the system follows the solution curve shown in Fig. 5.16 as it triggers to singularity 3′. If now E is lowered to its original value, the system remains in region 3. If, further, E is suddenly and momentarily reduced sufficiently, the system will trigger back to region 2.

The solution curve represents the path followed by a point relating instantaneous values of i_R and e_r. The point relating instantaneous values of i_r and e_r must, of course, follow the characteristic of the negative-resistance element, composed of the straight lines. The difference between these two paths represents the small voltages and currents associated with the reactive elements of the system during the intervals of change. The time required for triggering to take place can be found by numerical integration following the particular solution curve.

In designing an actual trigger circuit, usually it is desirable that the time required for the shift from one stable state to the other be as small as possible. This time is dependent, in part, upon the magnitude of the reactive elements, L and C, in the circuit. A smaller transition time results if these elements are reduced in size. Usually, in an actual circuit, they consist only of parasitic inductance and capacitance always present in electronic components and associated wiring.

An obvious question is whether the reactive elements need be considered at all in an analysis of the circuit. A consideration of Fig. 5.16 indicates that at least one reactive element is essential for an explanation of what takes place during the interval of triggering. During this interval, the path followed by the point relating instantaneous values of i_r and e_r must be the zigzag characteristic of the negative-resistance element. If there were no reactances, the point relating instantaneous values of i_R and e_r would have to move along the straight-line path of the resistance load line. Without reactance, i_R must equal i_r and points corresponding to these two currents must coincide. This could not be the case except at singularities, and triggering between singularities could not occur. A difference between paths can be allowed only through a component of voltage drop across series inductance L or current into parallel capacitance C. One or the other of these elements must be present.

It would appear that perhaps only one of the reactive elements is essential, in which case the system can be described by a single first-order differential equation. If the series inductance is missing, so that $L = 0$, the equation describing the circuit is

$$C \frac{de}{dt} + \frac{r + R}{rR} e = 0$$

where r is $-r_1$, $+r_2$, or $+r_3$, depending upon the region of operation. The single characteristic root is $\lambda = -(1/C)[(r + R)/rR]$. In region 1, $r < 0$, and $R > |r|$, so that $\lambda > 0$ and the system is unstable. In regions 2 and 3, $r > 0$ so that $\lambda < 0$ and the system is stable. These results agree qualitatively with those already found with both L and C present.

In the alternative case with the series inductance present, but the parallel capacitance missing so that $C = 0$, the equation for the circuit is

$$L\frac{di}{dt} + (R + r)i = 0$$

The single characteristic root is $\lambda = -(R + r)/L$. In regions 2 and 3, $r > 0$ so that $\lambda < 0$ and the system is stable. In region 1, $r < 0$, and $R > |r|$, so that again $\lambda < 0$ and the system is predicted as being stable. This result is just the opposite to what has been found in the two analyses already given. It is opposite to what is found experimentally, and the obvious conclusion is that it is incorrect. In other words, the capacitance in parallel with the voltage-controlled negative resistance is essential in obtaining the operating characteristics of the circuit.

This analysis brings out a fundamental difficulty in considering certain systems of this type. There are actually present reactive elements of small magnitude. At first glance, it would appear that one, or all, of these reactive elements could be ignored. A simple examination of currents and voltages in the system is convincing that at least one of the elements must be considered. Different conclusions are reached depending upon which one element is considered and which is disregarded. There appears to be no clear-cut way of deciding in advance just which parasitic elements must be considered and which can safely be neglected in an analysis of such a system.

In the first analysis of the circuit of Fig. 5.14, the assumption is made that the load line intersects the negative-resistance characteristic at three points. This requires that $R > r_1$, where r_1 is the magnitude of the variational resistance in region 1. A second, basically different type of operation occurs if there is but a single intersection, in which case $R < r_1$. This case is shown in Fig. 5.17.

There is but a single intersection between the load line and the negative-resistance characteristic. This intersection determines the location of the singularity for region 1. However, if the portions of the negative-resistance characteristic with positive slope are extended, two additional intersections occur, as shown in the figure. These are virtual intersections, associated with regions 2 and 3, but actually located in region 1.

The most interesting case is that for which the elements are chosen to make $r_1 < L/RC$, so that the magnitude of the negative resistance, in the region of negative slope, is less than the impedance of the parallel LCR circuit at its resonance frequency. With this condition, the two characteristic roots given by Eq. (5.38) are certain to have a positive real part in region 1, and the circuit is unstable. Further, with $R < r_1$, the singularity cannot be a saddle, and so it must be either a node or a focus. If $(4/LC)(1 + R/r_1) > (1/r_1C + R/L)^2$, the singularity is a focus, implying the production of oscillations. For the virtual singularities, 2′ and 3′, $r > 0$, and the characteristic roots must have negative real parts, representing stable operation. These singularities may again be either nodes or foci, this time stable.

A possible set of solution curves is sketched in Fig. 5.17. Here it is assumed that an unstable focus exists with singularity 1, while stable foci exist for both 2′ and 3′. The solution curves relating instantaneous values of i_R and e_r are spiral curves, spiraling outward from singularity

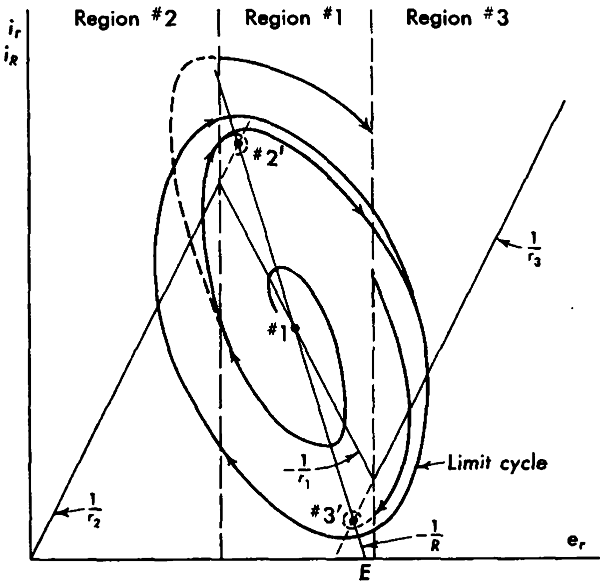


FIG. 5.17. Solution curves sketched for Example 5.3. The load line is chosen to intersect the negative-resistance characteristic at only one real point, singularity 1, which is an unstable focus. There are two virtual singularities, 2′ and 3′, both stable foci. The solution curve is ultimately a limit cycle.

1. When the curves cross into regions 2 or 3, they become spiral curves going inward toward virtual singularities 2′ and 3′. An interesting and most important result is that eventually a closed solution curve is reached, indicated by the elliptical path of Fig. 5.17. This closed curve is called a limit cycle and appears only in systems having nonlinear negative resistance. It is the same kind of limit cycle found in the graphical analysis of the van der Pol equation of Example 3.3. The van der Pol equation is a description of a circuit such as that of Fig. 5.14. It is evident from Fig. 5.17 that the limit cycle is made up of portions of expanding curves associated with one singularity and contracting curves associated with other singularities. It represents a periodic, and

generally nonsinusoidal, oscillation. The amplitude and waveform of this oscillation depend only upon the properties of the system and are independent of initial conditions starting the oscillation. Under some conditions, the waveform is exceedingly nonsinusoidal, with abrupt transitions, and represents what is known as a relaxation oscillation. This type of oscillation is described in Examples 3.3 and 3.6.

The presence of a limit cycle can be justified in the following way: If initially the amplitude is very small, the solution curve is entirely in region 1 and the amplitude must grow. Alternatively, if initially an extremely large amplitude were to exist, the solution curve would be mostly in regions 2 and 3, where the amplitude decays. Such a small portion of the curve would be in region 1, with increasing amplitude, that the net effect would be decay. Thus, initial small amplitude grows; initial large amplitude decays. There must be some intermediate amplitude which neither grows nor decays. One conceivable solution, nonperiodic and different from a limit cycle, would be that representing an oscillation of fixed amplitude but with a mean value that fluctuates in some way. This would be represented by a spiral curve of some sort but would require that successive turns of the spiral cross over one another. Such crossing of solution curves is not allowed, since their slope at any point other than a singularity is uniquely defined. Thus, the conclusion is reached that a system such as this must lead to a steady-state oscillation of constant amplitude and constant mean value. This is the limit cycle of Fig. 5.17.

It should be noted that the closed curve of the limit cycle looks somewhat like the closed curve associated with a vortex point. The two are fundamentally different, however. A vortex may occur in a system that is completely linear. A closed curve around a vortex may have any amplitude, as determined by initial conditions. A limit cycle is one particular closed curve, occurring only in a nonlinear system. It represents a combination of solution curves associated with several singularities. Some curves grow, while others decay, and the combination leads to a definite amplitude.

5.9. Potential Energy of Conservative Systems. Solution curves on the phase plane for conservative systems having no dissipation can be sketched through a consideration of the potential energy for the system. The particular example of a mass on a nonlinear spring is considered here.

A soft spring of simple type is described by the relation

$$F = kx - gx^3$$

where F is the force needed to deflect the spring a distance x from its rest position and k and g are positive constants. A plot of this relation is shown in Fig. 5.18*a*. When the magnitude of x becomes larger than $x_1 = (k/g)^{1/2}$, the force reverses sign, so that the spring delivers force rather than requires force to produce additional deflection. This case is that of the symmetrical soft spring already discussed in Example 4.6. This kind of relation between force and deflection may exist for combinations of linear springs together with other nonlinear mechanisms (e.g., magnetic) for obtaining forces. The restoring force of a pendulum approximates this kind of relation.

The potential energy stored in the deflected spring is

$$V = \int_0^x F\,dx = \frac{kx^2}{2} - \frac{gx^4}{4} \qquad (5.39)$$

This is plotted in Fig. 5.18*b*. This curve has zero slope at those values of x making $F = 0$. The potential energy becomes negative for sufficiently large deflection, which indicates that a soft spring delivers work in this region as deflection increases.

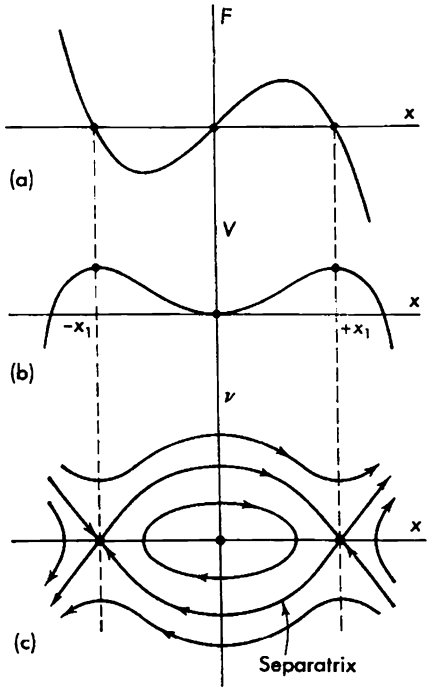


FIG. 5.18. Curves for system consisting of constant mass and nonlinear soft spring, showing spring force F, potential energy V, and normalized velocity ν, as a function of deflection x. Three singularities exist, a vortex and two saddles.

If the soft spring is combined with a constant mass M, giving a nonlinear conservative oscillatory system, the differential equation describing the system is

$$M\frac{d^2x}{dt^2} + kx - gx^3 = 0$$

By making the definitions $\omega_0^2 = k/M$, $\alpha = g/M\omega_0^2 = g/k$, $\tau = \omega_0 t$, $\nu = dx/d\tau$, $d^2x/dt^2 = \omega_0^2\, d^2x/d\tau^2 = \omega_0^2 \nu\, d\nu/dx$, the equation becomes

$$\frac{d\nu}{dx} = \frac{-x + \alpha x^3}{\nu} \qquad (5.40)$$

There are three singularities, all with $\nu_s = 0$ and with $x_s = 0$ or

$$x_s = \pm x_1 = \pm\left(\frac{k}{g}\right)^{1/2}$$

An investigation shows that the singularity at the origin is a vortex, and the singularities at $\pm x_1$ are saddles. Solution curves can be sketched as shown in Fig. 5.18c. Near the origin, these curves are closed, representing periodic oscillations. Time increases as shown by the arrowheads. Well away from the origin, the solution curves are not closed, and there is no oscillation. There is one particular curve running between the two outer singularities which is a separatrix curve. It divides the phase plane into regions of qualitatively different types of solutions. The three basically different types of solutions are shown in Fig. 5.19, where the deflection is shown as a function of time. The type of solution existing in any given case is determined by the initial conditions.

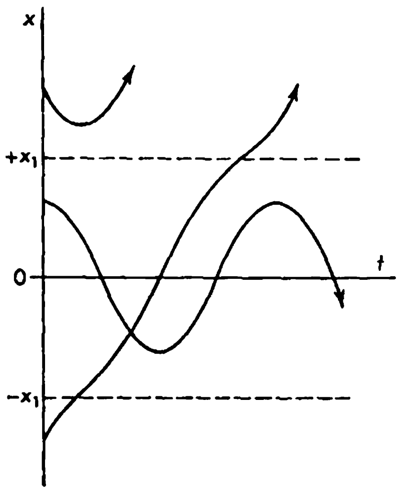


FIG. 5.19. Three possible variations of deflection with time, for mass on soft spring of Fig. 5.18.

An important observation can be made from Fig. 5.18. Singularities are associated with values of x where $dV/dx = 0$. If the point is a local minimum of potential energy, the associated singularity is a vortex and is neutrally stable. If the point is a local maximum of potential energy, the singularity is a saddle and is unstable. These conclusions are generally valid for any conservative system with a constant mass.

Furthermore, for a conservative system, the sum of kinetic and potential energies is always constant. In algebraic form, this statement is

$$\frac{M}{2}\left(\frac{dx}{dt}\right)^2 + V(x) = \frac{M}{2}\left(\frac{dx}{dt}\right)_0^2 + V_0(x)$$

where $V_0(x)$ is the potential energy under the condition $dx/dt = (dx/dt)_0$ or $\nu = \nu_0$. The sum of terms on either side of the equation is the total energy of the system. The value of the normalized velocity ν at any instant can be found from

$$\nu^2 = \nu_0{}^2 + \frac{2}{k}(V_0 - V) \tag{5.41}$$

and is independent of the value of mass M. If $V(x)$ is known, the value of $\nu(x)$ can readily be found. Solution curves as shown in Fig. 5.18c are symmetrical about the x axis and always cross the x axis perpendicularly.

In general, for a conservative system with a single degree of freedom, where potential energy varies only with x and kinetic energy varies only with dx/dt, the following statements are true: Equilibrium, or singular, points are located where dV/dx vanishes. Also, solution curves have zero slope, $d\nu/dx = 0$, at values of x corresponding to singular points. If the point is a local minimum of potential energy, the singularity is a vortex; if the point is a local maximum, the singularity is a saddle. A further example of this sort is shown in Fig. 5.20. Here the potential

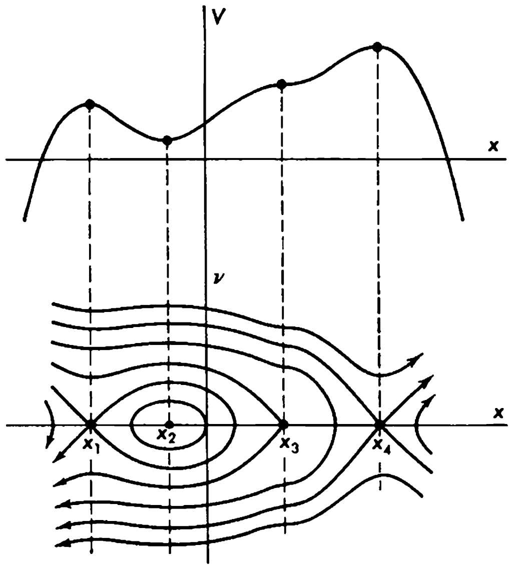


FIG. 5.20. Relation between potential energy of conservative system with constant mass, and corresponding phase-plane diagram. Four singularities are shown. Two are saddles; one is a vortex; the third is neither a saddle nor a vortex but has properties of both.

energy is assumed to vary in a complicated way with x, and there are four points, x_1, x_2, x_3, x_4, where $dV/dx = 0$. Points x_1 and x_4 are potential maxima and are saddles. Point x_2 is a potential minimum and is a vortex. Point x_3 is neither a potential maximum nor minimum, although it is a point where $dV/dx = 0$. It is not a simple singularity in the sense that the equation for $d\nu/dx$ cannot be reduced to a fraction with only linear terms in ν and x remaining in numerator and denominator. Just to the left of x_3, V varies as near a maximum, so that here ν varies as near a saddle. Just to the right of x_3, V varies as near a minimum, so that here ν varies as near a vortex. A singularity of this kind is generally described as being unstable and therefore rather more like a saddle than like a vortex.

Example 5.4. Nonsimple Singularity

An example of a singularity of the foregoing type that is not simple is shown in Fig. 5.21. This is the case of a constant mass mounted on a nonlinear spring for which the force varies with deflection as shown in Fig. 5.21*a*. The important feature here is that there is a certain deflection, $x = x_1$, for which the force returns to zero and at the same point its derivative is zero. In other words, at $x = x_1$, $F = 0$, and $dF/dx = 0$. The potential energy associated with this kind of force relation is shown in

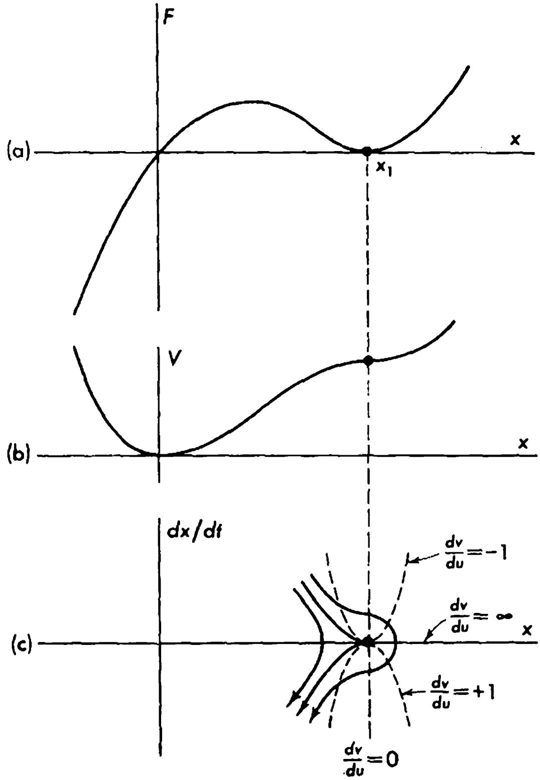


FIG. 5.21. Force, potential energy, and phase-plane diagram for Example 5.4.

Fig. 5.21*b* and displays a minimum at $x = 0$ and a point where $dV/dx = 0$ at $x = x_1$. This latter point is a singularity that is not simple.

This kind of force relation can be described by the equation

$$F = kx - gx^2 + hx^3$$

where k, g, and h are all positive constants. The derivative is

$$\frac{dF}{dx} = k - 2gx + 3hx^2$$

At $x = x_1$, it is required that

$$F = 0 = (k - gx_1 + hx_1^2)x_1 \tag{5.42}$$

and

$$\frac{dF}{dx} = 0 = k - 2gx_1 + 3hx_1^2 \tag{5.43}$$

It is evident that, in order to satisfy these two conditions, two specified relations must exist among the four parameters k, g, h, and x_1. In terms of k and x_1, these conditions are easily shown to be $g = 2k/x_1$ and $h = k/x_1^2$.

With a constant mass, the equation for the system is

$$M\frac{d^2x}{dt^2} + kx - gx^2 + hx^3 = 0 \tag{5.44}$$

If variable y is defined as $y = dx/dt$, this equation becomes

$$\frac{dy}{dx} = \frac{(1/M)(-kx + gx^2 - hx^3)}{y} \tag{5.45}$$

The singularity of interest is that at $x = x_1$, $y = 0$, and near this singularity $x = x_1 + u$, $y = 0 + v$. If these substitutions are made in Eq. (5.45) and the requirements of Eqs. (5.42) and (5.43) are recognized, the result is

$$\frac{dv}{du} = \frac{(1/M)[(g - 3hx_1)u^2 - hu^3]}{v} \tag{5.46}$$

The linear terms in u have disappeared, while nonlinear terms remain, and the singularity is not simple.

Because of the requirements on g and h, the coefficient of u^2 becomes

$$g - 3hx_1 = \frac{2k}{x_1} - \frac{3k}{x_1} = -\frac{k}{x_1}$$

which is a negative quantity. Sufficiently close to the singularity, the u^3 term is small compared with the u^2 term, and the equation is essentially

$$\frac{dv}{du} = -\left(\frac{k}{Mx_1}\right)\frac{u^2}{v} \tag{5.47}$$

This can be integrated to give

$$v^2 + Au^3 = C_1 \tag{5.48}$$

where $A = 2k/3Mx_1$ and C_1 is a constant determined by initial conditions.

Equation (5.48) is not one of the simple geometrical curves. It can be sketched on the phase plane by making use of several isoclines. The slopes of the solution curves can be found from Eq. (5.47), and several values are

$$\frac{dv}{du} = 0 \qquad \text{along} \qquad u = 0$$

$$\frac{dv}{du} = \infty \qquad \text{along} \qquad v = 0$$

$$\frac{dv}{du} = \pm 1 \qquad \text{along} \qquad v = \mp \tfrac{3}{2}Au^2$$

Some typical solution curves are sketched in Fig. 5.21c making use of these facts. The singularity is evidently not simple but is somewhat like a saddle for $x < x_1$ and somewhat like a vortex for $x > x_1$.

5.10. Summary.

A system described by a pair of simultaneous first-order differential equations can be studied in some detail by considering its singularities. These singularities are points of equilibrium for the system and may be either stable or unstable. Simple singularities are those which can be described by an equation of the form

$$\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{cx + dy}{ax + by}$$

The singularity for this equation is located at $x = 0$, $y = 0$. Only four types of simple singularities exist, and the particular type is determined by the coefficients a, b, c, and d. Once the location and type of singularity are known, it is not difficult to sketch solution curves showing the variation of y with respect to x. These sketches can be made easily and with some accuracy where characteristic roots are both real and well separated in value. The resulting figure is a compact representation of considerable information about solutions for the equation. Determination of the way in which x and y vary with t generally requires a numerical integration.

In studying a nonlinear system, it is often helpful to consider it as several linear systems. Operation of the nonlinear system is built up by considering the several regions of operation, each of which in itself is entirely linear. The accuracy obtained in such an analysis depends upon how closely the nonlinear properties are approximated by the several linear regions.

An important phenomenon arising in certain oscillatory systems with nonlinear negative resistance is the appearance of a limit cycle. This is a periodic oscillation with its amplitude and waveform determined entirely by properties of the system itself. Initial conditions serve only to start the oscillation but do not affect the ultimate steady state.

Conservative systems with a single degree of freedom, where potential energy V varies only with x and kinetic energy varies only with dx/dt, can be studied in terms of the variation of V with x. Equilibrium points are those where dV/dx vanishes. The only equilibrium points in such a system that are not unstable are vortex points, associated with minima of potential energy.

CHAPTER 6

ANALYTICAL METHODS

6.1. Introduction. There is usually considerable advantage in finding an analytical solution for a differential equation when this is possible. The analytical solution is obtained in algebraic form without the necessity of introducing numerical values for parameters or initial conditions during the process. Once the solution is obtained, any desired numerical values can be inserted and the entire possible range of solutions explored. Because of this flexibility, it is often worth expending considerable effort to find a solution in analytical form.

In order to be able to obtain an analytical solution, all relations needed to describe the physical system must be expressed in mathematical terms. Appropriate equations, either algebraic or differential, must be found to relate all variables of the system. Any empirical relation resulting from experimental measurements plotted as a curve between certain quantities must be converted into the form of an equation. It is obviously desirable to keep all relations as simple mathematically as possible, and suitable approximations must be made. The resulting equations for the system are finally analyzed to obtain a solution in the form of an equation.

Some of the methods for attacking linear differential equations can be extended to nonlinear equations as well. A powerful method often used with linear equations is based on finding a solution as a power series in the independent variable. For example, an equation in the form $f(x,dx/dt,d^2x/dt^2,t) = 0$ is assumed to have a solution as the series $x = a_0 + a_1t + a_2t^2 + \cdots$. The series is substituted into the equation, and coefficients $a_0, a_1, a_2, \ldots$ are adjusted to satisfy the equation. Certain linear equations lead to particular series of sufficient general importance for a name to be given to the series. The equation

$$dx/dt - kx = 0$$

leads to the series

$$x = a_0\left[1 + (kt) + \frac{(kt)^2}{2!} + \frac{(kt)^3}{3!} + \cdots\right]$$

and the series in the brackets is defined as the exponential function and written exp (kt). Because of its importance in many problems, numerical values for the exponential function have been worked out and are tabulated in detail. Similarly, other equations lead to the trigonometric functions, both circular and hyperbolic, and these are tabulated. Still other mathematical functions arise in a similar way.

The method of finding a solution as a power series in the independent variable can sometimes be extended to nonlinear equations. Usually the series so obtained cannot be recognized as any well-known tabulated functions. Furthermore, the series may converge only slowly or not at all. The series then provides a formal solution for the equation in the sense that it satisfies the equation when substituted into it. On the other hand, such a formal solution may be of no practical utility in providing quantitative information about the equation.

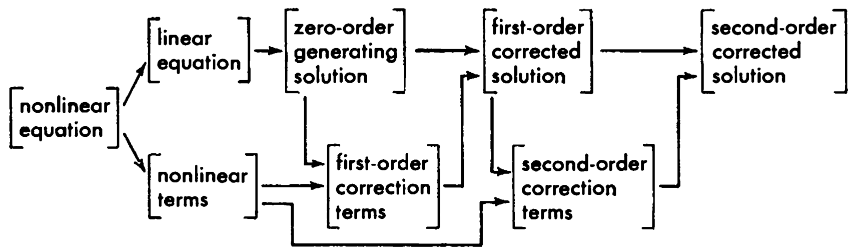


FIG. 6.1. Typical outline of steps followed in finding an approximate solution for a nonlinear equation.

Many of the analytical methods of practical importance are based on attempting to find a solution as a combination of well-known tabulated mathematical functions. It is recognized that an exact solution probably cannot be found, but an approximate solution of sufficient accuracy may be possible. While details of the various methods differ, most of them are rather similar and follow the steps diagramed in Fig. 6.1. The original nonlinear equation is first separated into two parts. One part is a linear equation that is simple enough to allow exact solution. The other part contains any terms that are difficult to handle and will usually involve the nonlinear terms of the equation, and perhaps other terms as well. The linear equation is solved so as to give the zero-order, or generating, solution. This generating solution is then employed in some way with the nonlinear terms of the original equation to produce first-order correction terms. These correction terms are then combined with the generating solution to yield a first-order corrected solution, which is an approximate solution of the original equation. The exact form of the correction terms depends upon the particular details of the method being employed. In some cases, the correction terms are merely added to the generating solution. In other cases, the correction terms produce changes in amplitude or phase of the generating solution.

If the degree of nonlinearity is sufficiently small, a single application of this process, yielding a first-order corrected solution, may give sufficient accuracy. If the degree of nonlinearity is greater, it is sometimes possible to obtain better accuracy by applying the method a second time, so as to yield a second-order correction. Further repeated applications of the method are possible in theory, but in practice usually the mathematical complications become so great as to outweigh any small increase in accuracy. Actually, an important uncertainty inherent in methods of this sort is the error in the solution which they yield. It is not always a simple matter to evaluate the error.

6.2. Perturbation Method. One of the analytical methods of widest general utility is the perturbation method. This method is applicable to equations where a small parameter is associated with the nonlinear terms. An approximate solution is found as a power series, with terms of the series involving the small parameter raised to successively higher powers. These powers of the parameter are associated with functions of the independent variable. If the magnitude of the parameter is small enough, only the first few terms of the series are sufficient to give a solution of fair accuracy.

The application of the perturbation method to a pair of first-order differential equations proceeds as follows, extension to a larger number of first-order equations being carried out in much the same way: The two equations may be written

$$\begin{aligned} \frac{dx}{dt} &= \psi_1(x,y,t) \\ \frac{dy}{dt} &= \psi_2(x,y,t) \end{aligned} \tag{6.1}$$

with initial conditions that, at $t = t_0$,

$$\begin{aligned} x(t_0) &= a \\ y(t_0) &= b \end{aligned}$$

These two equations may have come directly from the analysis of some system or may have come from what was originally a single second-order equation. Functions ψ_1 and ψ_2 are generally nonlinear. These functions are first broken into two parts,

$$\begin{aligned} \frac{dx}{dt} &= f_1(x,y,t) + \mu\phi_1(x,y,t) \\ \frac{dy}{dt} &= f_2(x,y,t) + \mu\phi_2(x,y,t) \end{aligned} \tag{6.2}$$

Terms which give no difficulty in the equation are written in functions f_1 and f_2. Nonlinear terms, and others which do give difficulty, are written in functions ϕ_1 and ϕ_2. Associated with ϕ_1 and ϕ_2 is parameter μ. This parameter ideally should be a dimensionless number, and its magnitude should be small. Solution is found as a power series in μ, and the smaller the magnitude of μ, the more rapidly will the series converge. In practice, μ is often allowed to have physical dimensions. Sometimes a convenient parameter does not appear in the original equations, and it is necessary to introduce the parameter artificially.

Under the foregoing conditions, solution for Eq. (6.1) can often be found in series form as

$$\begin{aligned} x &= x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \cdots \\ y &= y_0(t) + \mu y_1(t) + \mu^2 y_2(t) + \cdots \end{aligned} \tag{6.3}$$

If μ is sufficiently small, these series converge fast enough so that only the first two or three terms give good accuracy. If μ is not sufficiently small, the series may not converge at all and the solutions are of no practical usefulness. Solutions, Eq. (6.3), are usually adjusted so that, if $\mu \to 0$, the generating solution

$$\begin{aligned} x &= x_0(t) \\ y &= y_0(t) \end{aligned}$$

is the exact solution for the linear equations

$$\begin{aligned} \frac{dx}{dt} &= f_1(x,y,t) \\ \frac{dy}{dt} &= f_2(x,y,t) \end{aligned}$$

with the initial conditions

$$\begin{aligned} x(t_0) &= x_0(t_0) = a \\ y(t_0) &= y_0(t_0) = b \end{aligned}$$

The various functions of t in Eq. (6.3), $x_0, x_1, x_2, \ldots, y_0, y_1, y_2, \ldots,$ can all be determined from the solution of purely linear equations. Initial conditions are used to evaluate arbitrary constants that arise at each step of the solution. The method of finding the various functions appearing in the solution is most easily shown by considering a particular example.

Example 6.1. Capacitor Discharge through Diode

The circuit of Fig. 6.2 represents a linear capacitor of value C connected in series with a diode. The relation between instantaneous values of current i and voltage e of the diode can be written

$$i = ae + be^2$$

where a and b are positive constants. This equation may hold fairly well for currents between zero and some maximum value. The equation is plotted in Fig. 6.3. At $t = 0$, the voltage across the capacitor is $e = E$. Find the ensuing variation in voltage with time t.

The equation for the circuit is

$$C\frac{de}{dt} + ae + be^2 = 0$$

or

$$\frac{de}{dt} = -Ae - Be^2 \qquad (6.4)$$

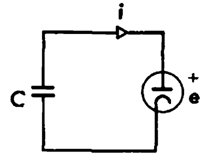


FIG. 6.2. Circuit for Example 6.1, consisting of a capacitor discharging through a nonlinear diode.

where $A = a/C$ and $B = b/C$. Equation (6.4) is a first-order nonlinear equation of the form of Eq. (6.1). It is convenient to introduce a dimensionless parameter μ before the nonlinear term so as to give

$$\frac{de}{dt} = -Ae - \mu Be^2 \qquad (6.5)$$

The value of μ must bc unity, of course, to give Eq. (6.4). A solution is sought in the form of Eq. (6.3) as

$$e = e_0(t) + \mu e_1(t) + \mu^2 e_2(t) + \cdots \qquad (6.6)$$

This solution, introduced into Eq. (6.5), gives

$$\dot{e}_0 + \mu\dot{e}_1 + \mu^2\dot{e}_2 = -Ae_0 - \mu Ae_1 - \mu^2 Ae_2 - \mu Be_0^2 - \mu^2 2Be_0e_1 \qquad (6.7)$$

where dots indicate differentiation with respect to t. Only terms in μ raised to powers of 0, 1, and 2 are retained. This arbitrary choice leads to a solution containing second-order correction terms.

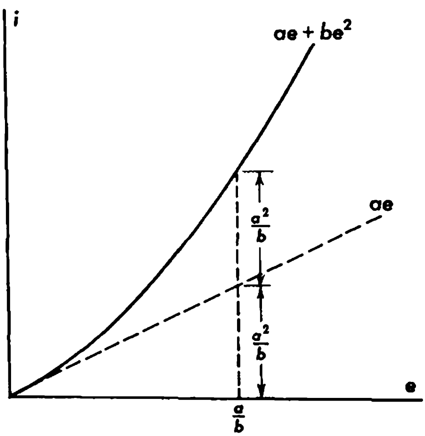


FIG. 6.3. Characteristic of diode for circuit of Example 6.1.

It is now argued that the solution is required to be exact if $\mu = 0$ and is to remain as nearly exact as possible as μ is allowed to increase from zero. Under this condition, terms in any given power of μ, chosen from the equation, must themselves form an equality. It is thus possible to set up several simultaneous equations by picking terms in the successive powers of μ from Eq. (6.7).

The generating solution is found from the terms of Eq. (6.7) having μ raised to the zero power,

$$\mu^0: \qquad \dot{e}_0 = -Ae_0$$

which is merely Eq. (6.5) with the nonlinear term dropped. Its solution is

$$e_0 = k_0 \exp(-At)$$

Subject to the initial condition $e_0 = E$ at $t = 0$, constant k_0 is found as $k_0 = E$. Thus, the generating solution is

$$\text{Generating solution:} \qquad e_0(t) = E\exp(-At) \qquad (6.8)$$

The first-order correction terms are found from those terms of Eq. (6.7) having μ raised to the first power,

$$\mu^1: \qquad \dot{e}_1 = -Ae_1 - Be_0^2 = -Ae_1 - BE^2 \exp(-2At)$$

where the value of e_0, already found, has been inserted. Here, both a complementary function and a particular integral appear in the solution for e_1, which is

$$e_1 = k_1 \exp(-At) + \frac{BE^2}{A} \exp(-2At)$$

Since the initial condition for the complete solution, $e = E$ at $t = 0$, has already been used in evaluating constant k_0 appearing in the generating solution, the initial condition for this first-order correction is merely $e_1 = 0$ at $t = 0$. With this condition, $k_1 = -BE^2/A$, and

First-order correction:

$$e_1 = \frac{BE^2}{A} [\exp(-2At) - exp(-At)] \tag{6.9}$$

Terms from Eq. (6.7) leading to the second-order correction are

$$\mu^2: \qquad \dot{e}_2 = -Ae_2 - 2Be_0e_1 = -Ae_2 - \frac{2B^2E^3}{A} [\exp(-3At) - \exp(-2At)]$$

where values of e_0 and e_1, already found, have been used. The solution here, subject to the initial condition $e_2 = 0$ at $t = 0$, is

Second-order correction:

$$e_2 = \frac{B^2E^3}{A^2} [\exp(-3At) - 2\exp(-2At) + \exp(-At)] \tag{6.10}$$

The complete solution to the second order of approximation is the sum of Eqs. (6.8) to (6.10), written in the form of Eq. (6.6). After some factoring to put it into better algebraic form, the complete solution can be written as

$$e = E \exp(-At) \left\{1 + \frac{\mu BE}{A} [\exp(-At) - 1] + \left(\frac{\mu BE}{A}\right)^2 [\exp(-At) - 1]^2\right\} \tag{6.11}$$

Parameter μ was introduced rather arbitrarily in going from Eq. (6.4) to Eq. (6.5). It must have the value unity to agree with the circuit equation [Eq. (6.4)]. Thus, in the solution [Eq. (6.11)], $\mu = 1$ to apply to the circuit equation. The quantity $BE/A = bE/a$ in the solution is dimensionless and appears raised to successive powers. Evidently the magnitude of this quantity must be small, less than unity, if the series is to converge fast enough to allow only a few terms of the series solution to be used. Thus, the solution is useful provided only that the initial voltage E is not too large when compared with the ratio a/b. If the voltage across the diode has the value a/b, the situation is illustrated in Fig. 6.3. Half the total current of $2a^2/b$ is contributed by the linear term ae and half by the square term be^2.

It is worth noting that coefficient B appears in both the original differential equation and the final solution in exactly the same way as does parameter μ. For this reason, often the series solution is written initially in powers of B directly instead of in powers of μ as was done in Eq. (6.6). Since B has dimensions, however, there may be some difficulty in deciding just what is meant by requiring that B be small for the series to converge rapidly. Here, of course, it turns out that the requirement is for B to be small compared with the ratio A/E.

Equation (6.4) for this circuit is one of the few nonlinear equations simple enough to allow an exact solution to be found. It can be solved as a Bernoulli equation (Sec. 4.2*i*) or by variation of parameters (Sec. 4.3) and the exact solution written

$$e = \frac{E \exp(-At)}{1 - (BE/A)[\exp(-At) - 1]} \tag{6.12}$$

If the denominator of this relation is expanded in the binomial series, the first three terms are found to be exactly the terms in the braces of Eq. (6.11). In this example, then, the perturbation method leads to a solution which represents the first terms of a series solution which would be exact if carried far enough.

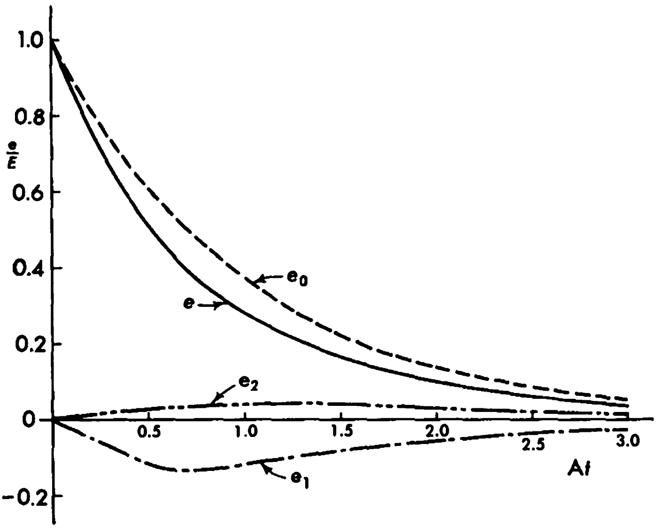


FIG. 6.4. Exact solution for circuit of Example 6.1, together with first three components of approximate solution, for case of $BE/A = \frac{1}{2}$.

A consideration of either form of the solution shows that the voltage across the capacitor in this circuit decays from its initial value of E to zero as time increases. The decay is not exponential, however. The decay for large voltages, not much smaller than E, takes place relatively more rapidly than as a simple exponential. The decay for small voltages, not much greater than zero, is almost a simple exponential. The nonlinear diode can be compared with a resistance whose value is smaller for large voltages than for small voltages.

In Fig. 6.4 are plotted the three components e_0, e_1, and e_2 of Eq. (6.11) and their sum for the case of $BE/A = \frac{1}{2}$. Within the accuracy of plotting this figure, the approximate solution agrees with the exact solution.

Example 6.2. Low-level Bombing with Air Resistance

The friction force produced by air resistance acting upon a projectile moving with moderate velocity is approximately proportional to the square of this velocity and is directed to oppose the velocity. Thus, the equations of motion for a projectile of mass M, moving in the coordinate system of Fig. 6.5, may be written by considering forces acting along the trajectory and normal to the trajectory as

Forces along trajectory: $Mg \sin \theta - kv^2 = M\dot{v}$
Forces normal to trajectory: $Mg \cos \theta = Mv\dot{\theta}$

With the equations reversed in order and simplified, they become

$$\begin{aligned} \dot{\theta} &= \frac{g}{v} \cos \theta \\ \dot{v} &= g \sin \theta - hv^2 \end{aligned} \tag{6.13}$$

where k is the constant determining the effect of air resistance and $h = k/M$. The velocity along the trajectory is v, while θ is the angle in which the velocity is directed, measured below the horizontal. These are nonlinear simultaneous first-order equations. If the projectile falls for a very long time, ultimately $\theta \rightarrow \pi/2$ radians, $\dot{v} \rightarrow 0$, and the final velocity is $v_f = (g/h)^{1/2} = (gM/k)^{1/2}$. If angle θ is limited to sufficiently small values, an approximate solution can be found by the perturbation method.

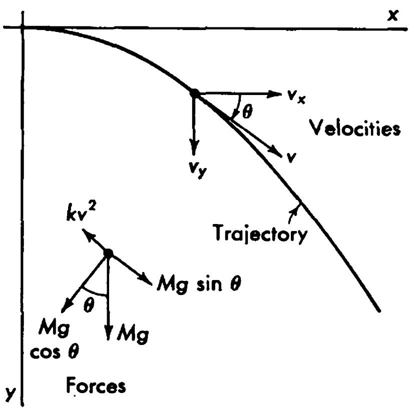


FIG. 6.5. Coordinates for trajectory of Example 6.2, together with diagram of forces acting on bomb.

Consider the trajectory of a bomb dropped at time $t = 0$ from a low-flying airplane moving horizontally with a velocity $v = V$ and angle $\theta = 0$.

If the airplane is flying close to the surface of the earth, the time elapsed during the fall of the bomb is small and angle θ will remain small over the entire trajectory. Thus, the perturbation method is applicable to this case of low-level bombing. It would not be immediately applicable to bombing from a high altitude.

If θ is small, approximate forms for the trigonometric functions may be used as

$$\cos \theta = 1 - \frac{\theta^2}{2}$$
$$\sin \theta = \theta - \frac{\theta^3}{6}$$

and Eq. (6.13) written

$$\begin{aligned} \dot{\theta} &= \mu \left(\frac{g}{v} - \frac{g\theta^2}{2v} \right) \\ \dot{v} &= g\theta - \mu \left(\frac{g\theta^3}{6} + hv^2 \right) \end{aligned} \tag{6.14}$$

where the numerical parameter μ has been introduced before each nonlinear term. An attempt is now made to find a solution as a power series in μ of the form

$$\begin{aligned} \theta &= \theta_0(t) + \mu\theta_1(t) \\ v &= v_0(t) + \mu v_1(t) \end{aligned} \tag{6.15}$$

where only first-order corrections are considered. Substitution into Eq. (6.14) gives

$$
\begin{aligned}
\dot{\theta}_0 + \mu\dot{\theta}_1 &= \mu\left(\frac{g}{v_0} - \frac{g\theta_0^2}{2v_0}\right) \\
\dot{v}_0 + \mu\dot{v}_1 &= g\theta_0 + \mu\left(g\theta_1 - \frac{g\theta_0^3}{6} - hv_0^2\right)
\end{aligned}
\tag{6.16}
$$

where only terms in μ to the powers 0 and 1 are retained.

The generating solution is found from those terms in μ to the zero power. The equation for θ_0 is merely $\dot{\theta}_0 = 0$, or $\theta_0 = C_1$. The initial condition is $\theta_0 = 0$ at $t = 0$, so that $C_1 = 0$, and thus the generating solution is $\theta_0 = 0$. The equation for v_0 is $\dot{v}_0 = g\theta_0 = 0$, since θ_0 has just been found to be zero. Its solution is $v_0 = C_2$. The initial condition is $v_0 = V$ at $t = 0$, so that $C_2 = V$, and the generating solution is $v_0 = V$. In summary, the complete generating solution is

$$
\begin{aligned}
\theta_0 &= 0 \\
v_0 &= V
\end{aligned}
\tag{6.17}
$$

The first-order correction is found from those terms in μ raised to the first power. The equation for θ_1 is $\dot{\theta}_1 = g/v_0 - g\theta_0^2/2v_0 = g/V$. Its solution is $\theta_1 = gt/V + C_3$ and since $\theta_1 = 0$ at $t = 0$, constant $C_3 = 0$. Similarly, the equation for v_1 is $\dot{v}_1 = g\theta_1 - g\theta_0^3/6 - hv_0^2 = g^2t/V - hV^2$. Its solution is $v_1 = g^2t^2/2V - hV^2t + C_4$, and since $v_1 = 0$ at $t = 0$, constant $C_4 = 0$. In summary, the first-order corrections are

$$
\begin{aligned}
\theta_1 &= \frac{gt}{V} \\
v_1 &= \frac{g^2t^2}{2V} - hV^2t
\end{aligned}
\tag{6.18}
$$

The first-order corrected solution is the combination of Eqs. (6.17) and (6.18) and is

$$
\begin{aligned}
\theta &= \mu\frac{gt}{V} \\
v &= V + \mu\left(\frac{g^2t^2}{2V} - hV^2t\right)
\end{aligned}
\tag{6.19}
$$

where μ must be unity to fit the original equations.

These results can be checked in part by comparing them with the known exact solutions with no air resistance, so that $k = 0$ and $h = 0$. In that case, the horizontal and vertical components of velocity are, respectively, $v_x = V$ and $v_y = gt$ for the initial conditions assumed here. The velocity along the trajectory is $v = (v_x^2 + v_y^2)^{1/2} = [V^2 + (gt)^2]^{1/2}$. If t is small enough so that $gt \ll V$, this is approximately $v = V[1 + \frac{1}{2}(gt/V)^2]$, which is the value of v from Eq. (6.19) if $h = 0$. Similarly, the angle is given by $\tan\theta = v_y/v_x = gt/V$, and if again $gt \ll V$, the angle is small and $\tan\theta \doteq \theta$ and the result is the value of θ from Eq. (6.19).

It should be recognized, of course, that the solution, Eq. (6.19), is only approximate and becomes more inaccurate as t and θ increase. The effect of air resistance is to reduce the velocity and thereby make the bomb strike the earth after having traveled a shorter horizontal distance from the release point.

A complication arising in certain applications of the perturbation method, but not immediately apparent, is the appearance of what are called secular terms in the analysis of an equation having an oscillatory solution. Secular terms are oscillatory terms having an amplitude growing indefinitely with time. Since a physical system with finite available energy can never have amplitudes increasing without limit, the analysis must be adjusted to eliminate secular terms as they arise.

Secular terms appear, for example, in the analysis of an oscillatory system where the period of oscillation is a function of the amplitude. This is the case for the mechanical system consisting of a constant mass mounted on a nonlinear spring discussed in Example 4.6, where the relation between period and amplitude is given in Eqs. (4.46) and (4.47). The generating solution for such a system has the form $\cos \omega_0 t$, where ω_0 is the angular frequency for the system with only linear terms present. The effect of nonlinear terms is to change the angular frequency to a new value $\omega_0 + \omega_1$, so that the corrected solution should have the form $\cos (\omega_0 + \omega_1)t$. In the perturbation method as described thus far, correction terms are found as additional terms in a power series. Thus, the perturbation method would give for the corrected solution the alternate series form

$$\begin{aligned}\cos (\omega_0 + \omega_1)t &= \cos \omega_1 t \cos \omega_0 t - \sin \omega_1 t \sin \omega_0 t \\ &= \left(1 - \frac{\omega_1^2 t^2}{2} + \frac{\omega_1^4 t^4}{24} - \cdots\right) \cos \omega_0 t \\ &\qquad - \left(\omega_1 t - \frac{\omega_1^3 t^3}{6} + \frac{\omega_1^5 t^5}{120} - \cdots\right) \sin \omega_0 t\end{aligned}$$

In general, only a few correction terms will be obtained, and the summation of these terms to give the closed form $\cos (\omega_0 + \omega_1)t$ will not be recognized. The individual terms, other than the single one, (1) $\cos \omega_0 t$, all have as coefficients time t raised to some positive power and represent growing oscillations. These are secular terms, and they must be eliminated from the solution.

Secular terms may be avoided by recognizing at the beginning the possibility of a change in frequency with amplitude. In addition to the series substitution for x and y, as given in Eq. (6.3), a series is needed for ω as

$$\omega = \omega_0 + \mu b_1(A) + \mu^2 b_2(A) + \cdots \tag{6.20}$$

where ω is the actual fundamental frequency of the nonlinear system, ω_0 is the angular frequency for the linearized system with $\mu = 0$, and $b_1(A)$, $b_2(A)$, . . . are functions of the amplitude A which must be determined so as to remove secular terms as they arise.

Example 6.3. Mass on Nonlinear Spring

Find an approximate solution for the motion of a constant mass mounted on a nonlinear spring of the sort considered in Example 4.6, with the initial conditions that $x = A_0$ and $\dot{x} = 0$ at $t = 0$.

The equation for this system can be written

$$\ddot{x} + \omega_0^2 x + hx^3 = 0 \tag{6.21}$$

where ω_0 is the natural angular frequency if nonlinear terms are missing and h is a constant associated with the degree of nonlinearity. Coefficient h is positive for a hard spring and negative for a soft spring.

If h is small enough, an approximate solution can be found by the perturbation method. Actually, of course, h has dimensions, and in starting the solution it is not apparent what is the quantity with respect to which h must be small. In previous examples, a numerical parameter μ has been introduced artificially to multiply the nonlinear terms of the original equation, and solution has been found as a power series in μ. It is just as convenient here to use coefficient h as the parameter for the series solution, and this will be done. Furthermore, Eq. (6.21) is a second-order equation, and it might be replaced by a pair of simultaneous first-order equations. Again, it is just as convenient to work with the second-order equation, and this also will be done.

A solution for Eq. (6.21) is sought in the form

$$x = x_0(t) + hx_1(t) + h^2x_2(t) \tag{6.22}$$

h being the parameter in the series, and powers of h no greater than 2 are retained. Furthermore, since the solution is known to be oscillatory, it is reasonable to expect that the fundamental frequency of oscillation will change with amplitude for this nonlinear system. Thus, to be able to remove secular terms as they arise, it is necessary to assume

$$\omega = \omega_0 + hb_1(A) + h^2b_2(A) \tag{6.23}$$

where A is the amplitude and ω is the actual fundamental frequency of oscillation. Actually, since only ω_0^2, rather than ω_0, appears in Eq. (6.21), it is more convenient to write

$$\omega^2 = \omega_0^2 + hb_1(A) + h^2b_2(A) \tag{6.24}$$

Functions b_1 and b_2 are, of course, different for Eqs. (6.23) and (6.24).

Equation (6.24) can be rewritten as

$$\omega_0^2 = \omega^2 - hb_1(A) - h^2b_2(A)$$

and, along with Eq. (6.22), substituted into Eq. (6.21) to give

$$\ddot{x}_0 + h\ddot{x}_1 + h^2\ddot{x}_2 + \omega^2x_0 + h\omega^2x_1 + h^2\omega^2x_2 \\ - hb_1x_0 - h^2b_1x_1 - h^2b_2x_0 + hx_0^3 + h^2 3x_0^2x_1 = 0$$

Only terms leading to a second-order approximation are retained.

The generating solution is found from

h^0: $$\ddot{x}_0 + \omega^2x_0 = 0$$

Its solution is

$$\begin{aligned} x_0 &= P_0 \cos \omega t + Q_0 \sin \omega t \\ \dot{x}_0 &= -\omega P_0 \sin \omega t + \omega Q_0 \cos \omega t \end{aligned}$$

Subject to the initial condition $x_0 = A_0$, $\dot{x}_0 = 0$ at $t = 0$, the constants are $P_0 = A_0$ and $Q_0 = 0$, so that the generating solution becomes

$$\begin{aligned} x_0 &= A_0 \cos \omega t \\ \omega &= \omega_0 \end{aligned} \tag{6.25}$$

The generating solution is valid if $h = 0$, corresponding to the linear case, and thus the actual frequency here is simply the linear frequency.

The first-order correction terms are found from

$$h^1: \quad \ddot{x}_1 + \omega^2 x_1 = b_1 x_0 - x_0^3$$
$$= \left(b_1 A_0 - \frac{3A_0^3}{4}\right) \cos \omega t - \frac{A_0^3}{4} \cos 3\omega t$$

where the value for x_0 has been inserted and the identity

$$\cos^3 \theta = \tfrac{3}{4} \cos \theta + \tfrac{1}{4} \cos 3\theta$$

has been used. The solution for x_1 is

$$x_1 = P_1 \cos \omega t + Q_1 \sin \omega t + \frac{\omega t}{2}\left(b_1 A_0 - \frac{3A_0^3}{4}\right) \sin \omega t + \frac{A_0^3}{32\omega^2} \cos 3\omega t$$

The first two terms are the complementary function. The third term is the particular integral arising from the forcing function having the same frequency as the natural frequency of the equation. This corresponds to a condition of resonance in a dissipationless system and thus leads to the term having t in the coefficient. This is, of course, the secular term, which must be removed. The fourth term in x_1 is the particular integral arising from the forcing term at the angular frequency 3ω.

The secular term can be removed by requiring that its coefficient vanish, so that $b_1 A_0 - 3A_0^3/4 = 0$. This requirement is satisfied if either $A_0 = 0$ or $b_1 = 3A_0^2/4$. The first of these conditions is trivial, and the condition of interest is the second one, which fixes function $b_1(A)$ in Eq. (6.24). Initial conditions for x_1 are that $x_1 = 0$, $\dot{x}_1 = 0$ at $t = 0$. The constants are thus found to be $P_1 = -A_0^3/32\omega^2$ and $Q_1 = 0$, so that the first-order correction is

$$x_1 = -\left(\frac{A_0^3}{32\omega^2}\right)(\cos \omega t - \cos 3\omega t)$$
$$b_1 = \frac{3A_0^2}{4} \tag{6.26}$$

The second-order correction terms are found from

$$h^2: \quad \ddot{x}_2 + \omega^2 x_2 = b_1 x_1 + b_2 x_0 - 3x_0^2 x_1$$
$$= \frac{3A_0^2}{4}\left(\frac{-A_0^3}{32\omega^2}\right)(\cos \omega t - \cos 3\omega t) + b_2 A_0 \cos \omega t$$
$$-3(A_0^2 \cos^2 \omega t)\left(\frac{-A_0^3}{32\omega^2}\right)(\cos \omega t - \cos 3\omega t)$$

The right side can be simplified through the identities

$$\cos^3 \theta = \tfrac{3}{4} \cos \theta + \tfrac{1}{4} \cos 3\theta$$
$$\cos^2 \theta \cos 3\theta = \tfrac{1}{4} \cos \theta + \tfrac{1}{2} \cos 3\theta + \tfrac{1}{4} \cos 5\theta$$

Function b_2 is found in the course of removing a secular term as $b_2 = -3A_0^4/128\omega^2$. Then, with the initial condition $x_2 = 0$, $\dot{x}_2 = 0$ at $t = 0$, the second-order correction is found to be

$$x_2 = -\left(\frac{A_0^5}{1{,}024\omega^4}\right)(\cos \omega t - \cos 5\omega t)$$
$$b_2 = \frac{-3A_0^4}{128\,\omega^2} \tag{6.27}$$

The solution to the second-order correction is therefore

$$
\begin{aligned}
x &= A_0 \cos \omega t - \frac{hA_0^3}{32\omega^2}(\cos \omega t - \cos 3\omega t) - \frac{h^2A_0^5}{1{,}024\omega^4}(\cos \omega t - \cos 5\omega t) \\
\omega^2 &= \omega_0^2 + \frac{h3A_0^2}{4} - \frac{h^2 3A_0^4}{128\omega^2}
\end{aligned}
\tag{6.28}
$$

The effect of the nonlinearity is to distort the oscillation by introducing odd-order harmonics and to change the fundamental frequency of oscillation.

In the correction terms for x, the dimensionless quantity $hA_0^2/32\omega^2$ appears to the first power in x_1 and to the second power in x_2. Evidently, if the solution is to hold with some degree of accuracy, this quantity must be small compared with unity. Thus, in writing the original solution in the form of Eq. (6.22), the requirement is that $h \ll 32\omega^2/A_0^2$.

To the first order of approximation, the actual fundamental angular frequency is given by $\omega^2 = \omega_0^2 + h3A_0^2/4 = \omega_0^2(1 + 3hA_0^2/4\omega_0^2)$, or approximately $\omega = \omega_0(1 + 3hA_0^2/8\omega_0^2)$. If the actual period is $T = 2\pi/\omega$ and the period if linear is $T_0 = 2\pi/\omega_0$, the ratio of periods is

$$\frac{T}{T_0} = \frac{1}{1 + \dfrac{3hA_0^2}{8\omega_0^2}}$$

This ratio can be compared with the corresponding ratios found in the exact analysis of Example 4.6, given in Eqs. (4.46) and (4.47). After some algebra, it can be shown that, to this order of approximation, the value of the ratio found from the perturbation method agrees with the exact values. For a hard spring, parameter h is positive, and the period is reduced as A_0 increases. This occurs because the equivalent stiffness of the spring increases with increasing deflection. Just the opposite effect takes place with the soft spring, for which h is negative.

An important observation concerning the perturbation method is that in this method the effect of nonlinear terms is accounted for by adding correction terms to the generating solution. If the nature of the problem is such that these additive terms are easily interpreted and easily describe the phenomenon, then the perturbation method is a suitable approach. On the other hand, the perturbation method is much less useful for certain other problems. For example, an oscillating system in which the amplitude changes with time is not easily analyzed by this method. The change in amplitude is not easily described by additional correction terms, and another method of approach is desirable.

6.3. Reversion Method. In applying the perturbation method to the solution of a nonlinear equation, a solution is first assumed in the form of a power series in some small parameter, and this series is inserted into the equation to determine all the unknown functions involved. Sometimes it is convenient to have the various steps of this process reduced to a set of formulas. Such formulas are provided by the reversion method. These two methods, perturbation and reversion, are essentially the same except for the details.

In the reversion method, the original nonlinear differential equation is written as

$$f_1(D)x + f_2(D)x^2 + \cdots + f_n(D)x^n = \mu F(t) \tag{6.29}$$

where $f_1(D), \ldots, f_n(D)$ are functions of the derivative operator $D = d/dt$, $F(t)$ is a forcing function, and μ is a numerical parameter introduced to facilitate solution. Ultimately μ is allowed to become unity. The function $f_1(D)$ operating on the linear terms of x must be present, so that it is required that $f_1(D) \neq 0$. A solution for Eq. (6.29) is sought in the form

$$x = \mu(x_0 + \mu x_1 + \mu^2 x_2 + \cdots) \tag{6.30}$$

where $x_0, x_1, x_2, \ldots$ are functions of t to be determined. The powers of x are found as

$$\begin{aligned} x^2 &= \mu^2(x_0^2 + \mu 2x_0x_1 + \cdots) \\ x^3 &= \mu^3(x_0^3 + \cdots) \end{aligned}$$

and so on. Upon substitution into Eq. (6.29), the result is

$$\begin{aligned} &f_1(D)[\mu(x_0 + \mu x_1 + \mu^2 x_2 + \cdots)] \\ &\qquad + f_2(D)[\mu^2(x_0^2 + \mu 2x_0x_1 + \cdots)] \\ &\qquad\qquad + f_3(D)[\mu^3(x_0^3 + \cdots)] + \cdots = \mu F(t) \end{aligned}$$

Just as in the perturbation method, terms of successive powers of μ are picked from this equation to form a new set of simultaneous equations. This set is the following:

$$\begin{aligned} &\mu^1: & f_1(D)x_0 &= F(t) \\ &\mu^2: & f_1(D)x_1 &= -f_2(D)x_0^2 \\ &\mu^3: & f_1(D)x_2 &= -f_3(D)x_0^3 - 2f_2(D)x_0x_1 \end{aligned} \tag{6.31}$$

These equations are applied in turn, appropriate initial conditions being used at each step so as to evaluate arbitrary constants as they arise. The first equation yields x_0, which can be used in the second equation to find x_1. These two functions are then used in the third equation to find x_2. While only corrections to the second order have been considered here, it is obviously possible to obtain in the same manner corrections of any desired order.

It is worth noting that the reversion method as described does not allow for the removal of secular terms should they arise. Therefore, this method cannot be applied directly to the solution of problems in nonlinear oscillatory systems. For nonoscillatory systems, however, it reduces the problem to the application of a set of formulas.

Example 6.4. Capacitor Discharge through Diode

The circuit of Example 6.1 can be considered also by the reversion method. The equation for the circuit is

$$\dot{e} + Ae + Be^2 = 0 \tag{6.32}$$

with the initial condition $e = E$ at $t = 0$.

In the notation of Eq. (6.29), the functions are

$$\begin{aligned} f_1(D) &= D + A \qquad F(t) = 0 \\ f_2(D) &= B \\ f_3(D) &= 0 \end{aligned}$$

A solution is sought as

$$e = \mu(e_0 + \mu e_1 + \mu^2 e_2)$$

where ultimately μ is allowed to be unity. From the first of Eq. (6.31), $(D + A)e_0 = 0$, and, with the initial condition $e_0 = E$ at $t = 0$, the result is $e_0 = E \exp(-At)$. From the second of Eq. (6.31), $(D + A)e_1 = -BE^2 \exp(-2At)$, and, with the initial condition $e_1 = 0$ at $t = 0$, the result is $e_1 = (BE^2/A)[\exp(-2At) - \exp(-At)]$. Finally, from the third of Eq. (6.31), $(D + A)e_2 = -2B[E \exp(-At)](BE^2/A)[\exp(-2At) - \exp(-At)]$, and, with the initial condition $e_2 = 0$ at $t = 0$, the result is $e_2 = (B^2E^3/A^2)[\exp(-3At) - 2\exp(-2At) + \exp(-At)]$. The sum $e_0 + e_1 + e_2$, with $\mu = 1$, is the approximate solution for the original nonlinear equation. It is, of course, the same as Eq. (6.11), as found in Example 6.1.

6.4. Variation of Parameters. The perturbation method is primarily useful where the effect of nonlinearity in a differential equation is easily accounted for by additional correction terms in the solution. This method is relatively less useful where additive terms do not easily correct for nonlinearity. This is the case of nonlinear oscillatory systems where changes in amplitude or phase result. For such systems, approximate solutions following the method of variation of parameters, already considered in Sec. 4.3, are useful. The procedure is as follows:

A second-order equation for a nonlinear oscillatory system, with no driving function, can be written

$$\ddot{x} + \omega_0^2 x + \mu\phi(x,\dot{x},t) = 0 \tag{6.33}$$

where function $\phi(x,\dot{x},t)$ contains nonlinear terms and other terms causing difficulty in the solution and μ is a small parameter. It is simplest to reduce the original second-order equation to a pair of first-order equations by defining a new variable $y = \dot{x}$. The resulting equations are

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega_0^2 x - \mu\phi(x,y,t) \end{aligned} \tag{6.34}$$

A generating solution is found by ignoring function ϕ and considering only the linear terms. This is equivalent to considering the equation

$\ddot{x} + \omega_0^2 x = 0$, which has the solution

$$\begin{aligned} x &= A \cos (\omega_0 t + \theta) \\ y &= -A\omega_0 \sin (\omega_0 t + \theta) \end{aligned} \tag{6.35}$$

where A and θ are constants dependent upon initial conditions. It is convenient to define a single symbol ψ to represent the total angle of the trigonometric functions, so that $\psi = \omega_0 t + \theta$.

In applying the method of variation of parameters, quantities A and θ are allowed to be functions of time t. Derivatives of the generating solution, Eq. (6.35), are taken, and substituted into the nonlinear equation, Eq. (6.34). After certain terms have been canceled, the result is

$$\begin{aligned} \dot{A} \cos \psi - \dot{\theta} A \sin \psi &= 0 \\ -\dot{A}\omega_0 \sin \psi - \dot{\theta}\omega_0 A \cos \psi &= -\mu\phi(A \cos \psi, -\omega_0 A \sin \psi, t) \end{aligned}$$

where $\dot{A} = dA/dt$ and $\dot{\theta} = d\theta/dt$. These equations may be solved for $\dot{A}$ and $\dot{\theta}$, with the result

$$\begin{aligned} \dot{A} &= \mu \left(\frac{1}{\omega_0}\right) \sin \psi \; \phi(A \cos \psi, -\omega_0 A \sin \psi, t) \\ \dot{\theta} &= \mu \left(\frac{1}{\omega_0 A}\right) \cos \psi \; \phi(A \cos \psi, -\omega_0 A \sin \psi, t) \end{aligned} \tag{6.36}$$

These simultaneous differential equations in A and θ are usually sufficiently complicated so that exact solutions for them cannot be found. However, it should be noted that, if parameter μ in the original equation is zero, both $\dot{A}$ and $\dot{\theta}$ are zero and neither A nor θ changes with time. It is argued, therefore, that, if μ is sufficiently small, changes in A and θ occur relatively slowly. In particular, the relative change in either A or θ during one cycle of the oscillation is assumed small. Under this condition, the average value of $\dot{A}$ and $\dot{\theta}$ over a cycle can be considered rather than the instantaneous value at every instant during the cycle.

The averages taken over a cycle are found in the usual way,

$$\begin{aligned} [\dot{A}]_{\text{av}} &= \frac{\mu}{2\pi\omega_0} \int_0^{2\pi} \sin \psi \; \phi(A \cos \psi, -\omega_0 A \sin \psi, t) \, d\psi \\ [\dot{\theta}]_{\text{av}} &= \frac{\mu}{2\pi\omega_0 A} \int_0^{2\pi} \cos \psi \; \phi(A \cos \psi, -\omega_0 A \sin \psi, t) \, d\psi \end{aligned} \tag{6.37}$$

In evaluating these averages, it is assumed that both A and θ remain essentially constant over the duration of a cycle of the oscillation. In mathematical terminology, the assumption is that μ is small enough so that both

$$\left|\frac{\dot{A}}{A}\right|\frac{2\pi}{\omega_0} \ll 1$$

$$\frac{|\dot{\theta}|}{\omega_0} \ll 1$$

Under these conditions, the averages of $\dot{A}$ and $\dot{\theta}$ are taken to be the quantities of interest.

The integral of $\dot{A}$ gives the variation of A with time. The total phase angle is $\psi = \omega_0 t + \theta$, and its derivative $d\psi/dt$ may be defined as the instantaneous angular frequency ω, so that $\omega = d\psi/dt = \omega_0 + \dot{\theta}$. Thus, the instantaneous angular frequency is found directly from $\dot{\theta}$.

In the analysis just presented, the generating solution is taken as $x = A \cos(\omega_0 t + \theta)$, where A and θ are the parameters that are allowed to vary. These are the magnitude and phase angle, and this process may be considered to be performed in a polar coordinate system. An alternate way of applying this same technique may be considered as performed in a rectangular coordinate system. In this alternate approach, the generating solution for Eq. (6.34) is written as

$$\begin{aligned} x &= P \cos \omega_0 t + Q \sin \omega_0 t \\ y &= -P\omega_0 \sin \omega_0 t + Q\omega_0 \cos \omega_0 t \end{aligned} \tag{6.38}$$

where the parameters are P and Q, the amplitudes of a pair of components in quadrature. By allowing P and Q to be functions of time and proceeding just as before, equations for $\dot{P}$ and $\dot{Q}$ are found as

$$\begin{aligned} \dot{P} &= \mu \frac{1}{\omega_0} \sin \omega_0 t \; \phi(x,y,t) \\ \dot{Q} &= -\mu \frac{1}{\omega_0} \cos \omega_0 t \; \phi(x,y,t) \end{aligned} \tag{6.39}$$

Once again the assumption can be made that μ is small enough so that

$$\left|\frac{\dot{P}}{P}\right|\frac{2\pi}{\omega_0} \ll 1 \qquad \text{and} \qquad \left|\frac{\dot{Q}}{Q}\right|\frac{2\pi}{\omega_0} \ll 1$$

and the averages $[\dot{P}]_{av}$ and $[\dot{Q}]_{av}$ found as before.

In most problems where variation of parameters is applicable, it is more convenient to have the result in terms of amplitude and phase, so that operations in polar coordinates are usually preferable. Occasionally, however, the alternate procedure in rectangular coordinates is more useful.

Example 6.5. Mass on Nonlinear Spring

The equation for the mechanical system consisting of a constant mass mounted on a nonlinear spring, already considered in Examples 4.6 and 6.3, is

$$\ddot{x} + \omega_0^2 x + hx^3 = 0 \tag{6.40}$$

Find a solution for this equation by the method of variation of parameters, with the initial conditions $x = A_0$, $\dot{x} = 0$ at $t = 0$.

Equation (6.40) can be written

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega_0^2 x - hx^3 \end{aligned} \tag{6.41}$$

which has the generating solution

$$\begin{aligned} x &= A \cos (\omega_0 t + \theta) = A \cos \psi \\ y &= -A\omega_0 \sin \psi \end{aligned} \tag{6.42}$$

With A and θ allowed to vary, the equations become

$$\begin{aligned} \dot{A} \cos \psi - \dot{\theta} A \sin \psi &= 0 \\ -\dot{A}\omega_0 \sin \psi - \dot{\theta}\omega_0 A \cos \psi &= -hA^3 \cos^3 \psi \end{aligned}$$

The values for $\dot{A}$ and $\dot{\theta}$ are

$$\begin{aligned} \dot{A} &= \frac{hA^3}{\omega_0} \cos^3 \psi \sin \psi \\ \dot{\theta} &= \frac{hA^2}{\omega_0} \cos^4 \psi \end{aligned}$$

The averages are readily found by making use of the identities

$$\begin{aligned} \cos^3 \psi \sin \psi &= \tfrac{1}{8} \sin 4\psi + \tfrac{1}{4} \sin 2\psi \\ \cos^4 \psi &= \tfrac{1}{8} \cos 4\psi + \tfrac{1}{2} \cos 2\psi + \tfrac{3}{8} \end{aligned}$$

Since the average over a cycle of either a cosine or a sine function is zero, the average value of these identities is merely the constant term appearing in them. Therefore, the averages are

$$\begin{aligned} [\dot{A}]_{\text{av}} &= 0 \\ [\dot{\theta}]_{\text{av}} &= \frac{3hA^2}{8\omega_0} \end{aligned} \tag{6.43}$$

Their integrals with respect to time are

$$\begin{aligned} A &= C_1 \\ \theta &= \frac{3hA^2 t}{8\omega_0} + C_2 \end{aligned}$$

so that the generating solution, Eq. (6.42), is modified as

$$x = C_1 \cos \left(\omega_0 t + \frac{3hA^2 t}{8\omega_0} + C_2 \right)$$

Subject to the initial condition that $x = A_0$, $\dot{x} = 0$ at $t = 0$, the constants are found as $C_1 = A_0$, $C_2 = 0$, and the final solution is

$$x = A_0 \cos \left(\omega_0 + \frac{3hA_0^2}{8\omega_0} \right) t \tag{6.44}$$

which has the instantaneous angular frequency

$$\omega = \omega_0 + \dot{\theta} = \omega_0 + \frac{3hA_0^2}{\omega_0} \tag{6.45}$$

To this first order of approximation, the nonlinearity in the spring of the mechanical system has produced no change in amplitude but has introduced a change in frequency of the oscillation. The modification in frequency found in Example 6.3 for this same system, but using the perturbation method, is

$$\omega^2 = \omega_0^2 + \frac{3hA_0^2}{4}$$

and if h is small, this becomes

$$\omega = \omega_0 + \frac{3hA_0^2}{8\omega_0}$$

just as found here. The requirement for applying the method of variation of parameters is that $|\dot{\theta}|/\omega_0 \ll 1$. In this case, $|\dot{\theta}|/\omega_0 = 3hA_0^2/8\omega_0^2$, and h must be small enough so that this quantity is much less than unity, or $h \ll 8\omega_0^2/3A_0^2$.

Example 6.6. Damped Linear Oscillator

The equation for a linear oscillator with damping is

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0 \tag{6.46}$$

where α and ω_0 are constants. This is the equation used in Example 5.2, where a case of large damping is considered. If small damping exists, an approximate solution can be found by variation of parameters.

Find a solution for Eq. (6.46) with the initial condition that $x = A_0$, $\dot{x} = 0$ at $t = 0$.

The generating solution here is simply Eq. (6.35), and when A and θ are allowed to vary so as to account for the term $2\alpha\dot{x}$ in Eq. (6.46), the result is

$$\begin{aligned} \dot{A} &= -2\alpha A \sin^2 \psi \\ \dot{\theta} &= -2\alpha \sin \psi \cos \psi \end{aligned}$$

which have the average values

$$\begin{aligned} [\dot{A}]_{av} &= -\alpha A \\ [\dot{\theta}]_{av} &= 0 \end{aligned} \tag{6.47}$$

The integrals are

$$\begin{aligned} A &= C_1 \exp(-\alpha t) \\ \theta &= C_2 \end{aligned}$$

so that the generating solution becomes

$$\begin{aligned} x &= C_1 \exp(-\alpha t) \cos(\omega_0 t + C_2) \\ y &= C_1 \exp(-\alpha t)[-\alpha \cos(\omega_0 t + C_2) - \omega_0 \sin(\omega_0 t + C_2)] \end{aligned}$$

Subject to the initial conditions that $x = A_0$, $\dot{x} = 0$ at $t = 0$, and with the further assumption that $\alpha/\omega_0 \ll 1$, which is necessary for this method of solution to be valid, the result is

$$x = A_0 \exp(-\alpha t) \cos\left(\omega_0 t - \frac{\alpha}{\omega_0}\right) \tag{6.48}$$

To this first order of approximation, the effect of dissipation is to produce a change in amplitude but no change in the apparent frequency of oscillation. The requirement for applying this method of solution is that $|\dot{A}/A|(2\pi/\omega_0) \ll 1$, and here $|\dot{A}/A| = \alpha$ so that it is necessary for $\alpha \ll \omega_0/2\pi$.

It is known that the exact solution for the linear equation, Eq. (6.46), is

$$x = A_0\omega_0(\omega_0^2 - \alpha^2)^{-1/2} \exp(-\alpha t) \cos\left[(\omega_0^2 - \alpha^2)^{1/2}t - \tan^{-1}\frac{\alpha}{\omega_0}\right]$$

Subject to the condition that $\alpha/\omega_0 \ll 1$, this is the same as Eq. (6.48).

6.5. Use of Several Methods. In finding as much information as possible about a typical nonlinear system, it is usually necessary to apply several different methods of analysis. One method yields certain information, while another method yields different information.

Example 6.7. Negative-resistance Oscillator

a. Preliminary Discussion of Circuit. The analytical methods just described can be applied to the analysis of the operation of a nonlinear negative-resistance oscillator. Essentially, this same system has been considered in Examples 3.3 and 3.6, where a purely graphical method is used, and in Example 5.3, where a piecewise linear analysis considering singular points is used. A still different approach is the perturbation method and the method of variation of parameters.

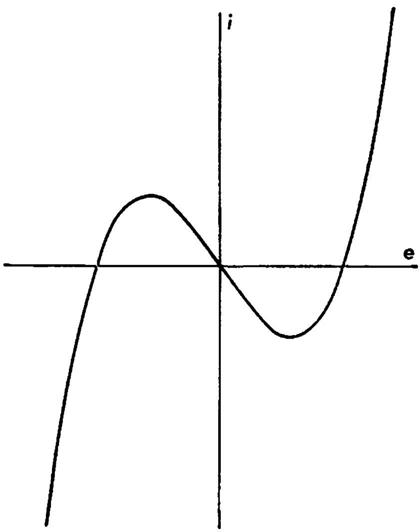


FIG. 6.6. Characteristic curve for nonlinear negative-resistance element of Example 6.7.

A nonlinear voltage-controlled negative resistance has its instantaneous current i and voltage e related by a curve similar to that shown in Fig. 5.13. For an analytical study, an algebraic equation for this curve is necessary. The simplest equation describing a curve with odd-order symmetry of this sort is

$$i = -ae + be^3 \tag{6.49}$$

where a and b are positive constants. A curve is plotted from this equation in Fig. 6.6. The origin is at the center of the region of negative slope and is shifted from the position of Fig. 5.13. This shift in origin merely represents a change in coordinates. For small currents and voltages, the variational resistance is negative; for large currents and voltages, it is positive.

A simple self-oscillatory circuit can be made by using the negative resistance in the circuit of Fig. 6.7, where L, R, and C are constant linear elements. This is essentially the same as Fig. 5.14 with details of any steady-voltage supply omitted.

The equation for the circuit of Fig. 6.7 is easily found from the condition

$$i_C + i + i_L = 0$$

where $i_C = C\dot{e}$, $i = -ae + be^3$, and $L\, di_L/dt + Ri_L = e$. After substitution and simplification, the result in terms of voltage e is

$$\ddot{e} - \frac{a}{C}\left(1 - \frac{3be^2}{a}\right)\dot{e} + \frac{e}{LC} + \frac{R}{L}\left(\dot{e} - \frac{ae}{C} + \frac{be^3}{C}\right) = 0$$

Under many conditions of interest for an oscillatory circuit, the resistance R is small enough so that the last term with the parentheses can be neglected. The equation is considerably simplified by omitting this term in R, and the assumption is made here that it can be neglected. Actually, it is possible later in the analysis to make certain observations about the effect of R. With R neglected, the equation can be written

$$\ddot{e} - \alpha(1 - \beta e^2)\omega_0\dot{e} + \omega_0^2 e = 0 \qquad (6.50)$$

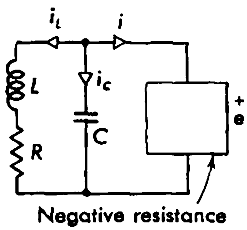


FIG. 6.7. Circuit of negative-resistance oscillator of Example 6.7.

where $\alpha = a/C\omega_0$, $\beta = 3b/a$, and $\omega_0^2 = 1/LC$. This is essentially van der Pol's equation, Eq. (3.10), except for the presence of the extra parameters β and ω_0. Parameter α is entirely equivalent to parameter ϵ of the van der Pol equation.

Equation (6.50) has a nonlinear coefficient for the damping term in $\dot{e}$. Evidently, if voltage e is small, damping is negative, while if e is large, damping is positive. Negative damping represents the introduction of energy to the system; positive damping represents the removal of energy.

Under some conditions, voltage e varies almost as a simple-harmonic function of time with angular frequency ω_0. This is certainly the case if $\alpha = 0$. It has been found to be so in Example 3.6 and is verified by experiments with a circuit such as Fig. 6.7. A useful parameter for describing an oscillatory circuit with simple-harmonic oscillations is its circuit Q. One definition for the Q of this kind of circuit is

$$Q = \pi\,\frac{\text{maximum stored energy}}{\text{energy lost per half cycle}}$$

If voltage e is $e = E \sin \omega_0 t$, the maximum energy stored in the capacitor during a cycle of oscillation is

$$\text{Maximum stored energy} = \frac{CE^2}{2}$$

This is the energy circulating in the LC circuit. Any change in circulating energy is produced by the effects of dissipation, now assumed to exist only in the negative-resistance element. If voltage e is small, the variational resistance is merely $1/(di/de) = -1/a$, where $ae \gg be^3$. The magnitude of the change in energy per half cycle is then

$$\text{Energy change per half cycle} = \frac{E^2a}{2}\frac{\pi}{\omega_0}$$

The circuit Q under these conditions is

$$Q = \pi \frac{CE^2/2}{\pi a E^2/2\omega_0} = \frac{C\omega_0}{a} = \frac{1}{\alpha} = \frac{1}{a}\left(\frac{C}{L}\right)^{1/2}$$

Thus, the circuit Q is merely the reciprocal of the parameter α.

If $Q \gg 1$ or $\alpha \ll 1$, any change in energy during a cycle is but a small fraction of the circulating energy and the change in amplitude per cycle is relatively small. If $Q \ll 1$, or $\alpha \gg 1$ the change in energy per cycle is large compared with the circulating energy and the amplitude may change abruptly. In general, only relatively slow changes can be considered by analytical methods, and these methods are accordingly limited to the case of $\alpha \ll 1$.

b. Perturbation Method, Steady State. In applying the perturbation method, a solution is sought in the form

$$e = e_0(t) + \alpha e_1(t) + \alpha^2 e_2(t) + \cdots \tag{6.51}$$

Because this system is oscillatory, a possible change in frequency with amplitude A must be allowed for, so that a second part of the solution is

$$\omega = \omega_0 + \alpha b_1(A) + \alpha^2 b_2(A) + \cdots$$

or

$$\omega_0 = \omega - \alpha b_1(A) - \alpha^2 b_2(A) - \cdots \tag{6.52}$$

It is worth noting that here correction is applied to ω_0 raised to the first power, since the first power appears in Eq. (6.50). In Eq. (6.24) of Example 6.3 the correction was applied to ω_0^2, since only the square appeared in Eq. (6.21). Substitution of Eqs. (6.51) and (6.52) into Eq. (6.50) gives

$$\begin{aligned}\ddot{e}_0 &+ \alpha\ddot{e}_1 + \alpha^2\ddot{e}_2 - \alpha\omega\dot{e}_0 - \alpha^2\omega\dot{e}_1 + \alpha^2 b_1\dot{e}_0 + \alpha\beta\omega e_0^2\dot{e}_0 \\ &+ 2\alpha^2\beta\omega e_0 e_1\dot{e}_0 - \alpha^2\beta b_1 e_0^2\dot{e}_0 + \alpha^2\beta\omega e_0^2\dot{e}_1 + \omega^2 e_0 \\ &+ \alpha\omega^2 e_1 + \alpha^2\omega^2 e_2 - 2\alpha b_1\omega e_0 - 2\alpha^2 b_1\omega e_1 \\ &+ \alpha^2 b_1^2 e_0 - 2\alpha^2 b_2\omega e_0 = 0\end{aligned} \tag{6.53}$$

Initial conditions are taken that $e = E$, $\dot{e} = 0$ at $t = 0$. It turns out in the course of the solution that the initial voltage E cannot be chosen at will but must have a particular value.

The generating solution is found from

α^0: $$\ddot{e}_0 + \omega^2 e_0 = 0$$

and is $e_0 = P_0 \cos \omega t + Q_0 \sin \omega t$. Subject to the initial condition $e_0 = E$, $\dot{e}_0 = 0$ at $t = 0$, the generating solution is

$$\begin{aligned} e_0 &= E \cos \omega t \\ \omega &= \omega_0 \end{aligned} \tag{6.54}$$

The first-order correction is found from

$$\begin{aligned} \alpha^1: \quad \ddot{e}_1 + \omega^2 e_1 &= \omega \dot{e}_0 - \beta \omega e_0^2 \dot{e}_0 + 2b_1 \omega e_0 \\ &= \left(-\omega^2 E + \frac{\beta \omega^2 E^3}{4}\right) \sin \omega t + 2b_1 \omega E \cos \omega t \\ &\qquad + \frac{\beta \omega^2 E^3}{4} \sin 3\omega t \end{aligned}$$

where the identity

$$\cos^2 \theta \sin \theta = \tfrac{1}{4}(\sin \theta + \sin 3\theta)$$

has been used. In order to avoid secular terms, it is necessary that both $\omega^2 E(1 - \beta E^2/4) = 0$ and $2b_1 \omega E = 0$. The possibility of $E = 0$ is trivial. The only interesting possibility is that both

$$E^2 = \frac{4}{\beta} \qquad \text{or} \qquad E = \pm \frac{2}{\beta^{1/2}} = \pm 2 \left(\frac{a}{3b}\right)^{1/2}$$

and $\quad b_1 = 0$

To this first-order of approximation, there is no change in frequency of oscillation, and a definite amplitude is required. The first-order correction is

$$e_1 = P_1 \cos \omega t + Q_1 \sin \omega t - \frac{\beta E^3}{32} \sin 3\omega t$$

Subject to the initial condition $e_1 = 0$, $\dot{e}_1 = 0$ at $t = 0$ and to the further requirement $\beta E^2 = 4$, this correction is

$$\begin{aligned} e_1 &= \frac{E}{8} (3 \sin \omega t - \sin 3\omega t) \\ b_1 &= 0 \end{aligned} \tag{6.55}$$

The second-order correction is found from

$$\begin{aligned} \alpha^2: \quad \ddot{e}_2 + \omega^2 e_2 &= \omega \dot{e}_1 - b_1 \dot{e}_0 - 2\beta \omega e_0 e_1 \dot{e}_0 + \beta b_1 e_0^2 \dot{e}_0 \\ &\qquad - \beta \omega e_0^2 \dot{e}_1 + 2b_1 \omega e_1 - b_1^2 e_0 + 2b_2 \omega e_0 \\ &= \omega^2 E \left(\tfrac{1}{8} + \frac{2b_2}{\omega}\right) \cos \omega t \\ &\qquad + \omega^2 E \left(-\tfrac{3}{4}\right) \cos 3\omega t + \omega^2 E \left(\tfrac{5}{8}\right) \cos 5\omega t \end{aligned}$$

where simplification has made use of several trigonometric identities. In order to avoid a secular term, it is necessary that $\frac{1}{8} + 2b_2/\omega = 0$ or $b_2 = -\omega/16$. The second-order correction is

$$e_2 = P_2 \cos \omega t + Q_2 \sin \omega t + \tfrac{3}{32}E \cos 3\omega t - \tfrac{5}{192}E \cos 5\omega t$$

Subject to the initial condition $e_2 = 0$, $\dot{e}_2 = 0$ at $t = 0$, this becomes

$$\begin{aligned} e_2 &= -\frac{E}{192}(13 \cos \omega t - 18 \cos 3\omega t + 5 \cos 5\omega t) \\ b_2 &= -\frac{\omega}{16} \end{aligned} \tag{6.56}$$

Thus, to a second order of approximation, a solution is

$$\begin{aligned} e &= E \cos \omega t + \alpha \frac{E}{8}(3 \sin \omega t - \sin 3\omega t) \\ &\qquad - \alpha^2 \frac{E}{192}(13 \cos \omega t - 18 \cos 3\omega t + 5 \cos 5\omega t) \qquad (6.57) \\ \omega &= \omega_0 + \alpha(0) + \alpha^2\left(-\frac{\omega}{16}\right) \end{aligned}$$

where $E = \pm 2/\beta^{1/2} = \pm 2(a/3b)^{1/2}$.

A number of observations can be made about this solution for the oscillator described by Eq. (6.50). This solution found by the perturbation method is first of all a steady-state solution. It applies only after a steady state has been reached, following any initial transient interval. There is a definite amplitude of oscillation, regardless of initial conditions, a situation which corresponds to a limit cycle. The waveform is not simple-harmonic but is distorted by odd-order harmonics, with third and fifth harmonics appearing in this approximation. The relative magnitude of the harmonics increases as parameter α increases. The entire solution is valid only so long as α is small compared with unity.

The steady-state voltage amplitude $E = 2(a/3b)^{1/2}$ is determined entirely by coefficients a and b for the negative-resistance element. This steady-state amplitude is easily related to the geometry of the negative-resistance characteristic. The relation between current and voltage is $i = -ae + be^3$. The slope of the characteristic is zero,

$$\frac{di}{de} = 0,$$

at $e = \pm(a/3b)^{1/2} = \pm 1/\beta^{1/2}$. At these points, the current is

$$i = \pm \frac{2a/3}{\beta^{1/2}}$$

Lines can be drawn with horizontal slope tangent to the characteristic as shown in Fig. 6.8. These lines intersect the portion of the characteristic with positive slope at the points $e = \pm 2/\beta^{1/2} = \pm E$. Thus, the path of operation relating instantaneous values of i and e is that portion of the characteristic indicated by the heavy curve of Fig. 6.8.

If parameter α is very small, in the limit $\alpha = 0$, the voltage waveform is merely a cosine function of time. By projecting this cosine through the operating path, the waveform of the current as a function of time can be found, as shown in Fig. 6.8. Under this impractical assumption that the voltage is a pure cosine of the fundamental frequency, the current is

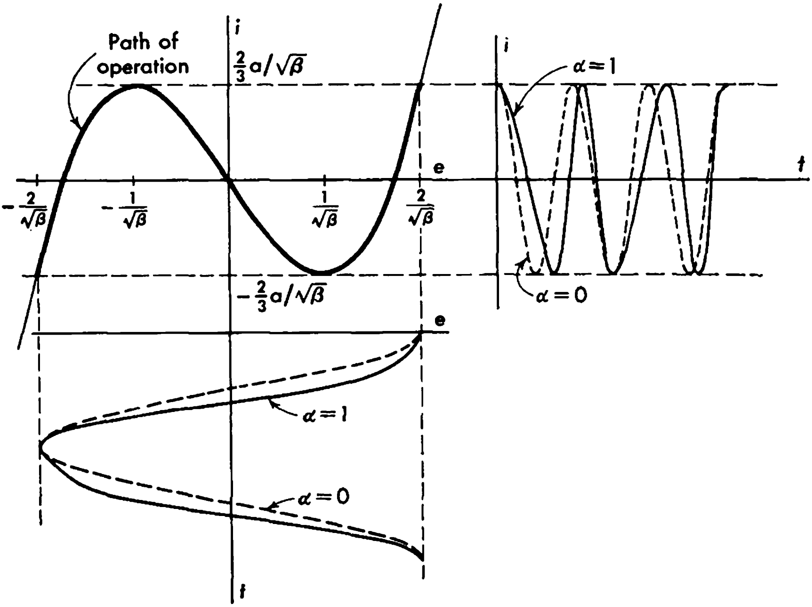


FIG. 6.8. Path of operation on negative-resistance characteristic of Example 6.7. Corresponding waveforms of voltage e and current i are shown for two values of parameter α.

a pure cosine of the third-harmonic frequency. In a more practical case, $\alpha > 0$, and the voltage is not a pure cosine. If $\alpha = 1$, a value too large for the foregoing analysis to be accurate, Eq. (6.57) gives the steady-state waveform also plotted in Fig. 6.8. The current wave corresponding to this voltage wave evidently contains a component at the fundamental frequency as well as higher-order components.

To the first order of approximation, the fundamental angular frequency of oscillation is merely ω_0, the natural frequency of the linear system. To the second order of approximation, the fundamental angular frequency becomes

$$\omega = \frac{\omega_0}{1 + \alpha^2/16}$$

which is slightly less than ω_0. The reason for this shift is as follows: With $\alpha > 0$, the voltage is not simple-harmonic. Acting with the nonlinear resistance, it produces a current which is far from simple-harmonic. One component of this current is at the fundamental frequency. The frequency must be such that the fundamental current multiplied by the impedance of the LC circuit at this frequency gives the fundamental voltage. This condition requires that the fundamental frequency of oscillation be slightly lower than the resonance frequency of the LC circuit. The frequency shift is, however, a second-order effect.

In the foregoing analysis, the resistance R appearing in series with the inductor in the circuit of Fig. 6.7 has been assumed negligibly small. Its effect can be considered for this case of small α in the following way: It has been shown that oscillation occurs very nearly at the resonance frequency of the tuned circuit. At its resonance frequency, an LRC circuit of this sort has the impedance of a pure resistance $R' = L/RC$. This positive resistance R' appears in parallel with the negative-resistance element and has the effect of adding a positive conductance $1/R'$ to the negative conductance $-a$. The effect of resistance R is primarily, therefore, to modify the equation for the negative-resistance element so that it becomes

$$i = -\left(a - \frac{1}{R'}\right)e + be^3 = -a'e + be^3$$

The modified coefficient $a' = a - 1/R'$ takes the place of the original coefficient a throughout the analysis. It is evident, for example, that the amplitude of oscillation $E = 2(a'/3b)^{1/2}$ will become zero and imaginary if a' becomes zero and negative. Thus, a condition for oscillation is that $1/R' = RC/L$ be less in magnitude than coefficient a. This sets an upper limit on the allowable resistance R in the tuned circuit if self-oscillation is to occur.

c. Variation of Parameters, Amplitude Growth. Insight into the way oscillation in the system builds up or decays can be found by applying the method of variation of parameters. The differential equation for the circuit is still Eq. (6.50),

$$\ddot{e} - \alpha(1 - \beta e^2)\omega_0\dot{e} + \omega_0^2 e = 0 \tag{6.50}$$

For this method, a generating solution is first found by neglecting the term with coefficient α, as

$$e = A\cos(\omega_0 t + \theta) = A\cos\psi \tag{6.58}$$

When A and θ are allowed to become functions of time, the simultaneous relations are

$$\dot{A}\cos\psi - \dot{\theta}A\sin\psi = 0$$
$$-\dot{A}\omega_0\sin\psi - \dot{\theta}A\omega_0\cos\psi = \alpha(1 - \beta A^2\cos^2\psi)\omega_0(-A\omega_0\sin\psi)$$

Solution for $\dot{A}$ and $\dot{\theta}$ gives

$$\dot{A} = \alpha\omega_0 A \sin^2 \psi(1 - \beta A^2 \cos^2 \psi)$$
$$\dot{\theta} = \alpha\omega_0 \sin \psi \cos \psi(1 - \beta A^2 \cos^2 \psi)$$

The average values taken over a cycle are

$$[\dot{A}]_{av} = \alpha\omega_0 A \left(\tfrac{1}{2} - \frac{\beta A^2}{8}\right)$$
$$[\dot{\theta}]_{av} = 0$$

To this first-order approximation, the value of $\dot{\theta}$ is zero, so that the actual frequency of oscillation remains at the linear value, and $\omega = \omega_0$. This result agrees with that of the perturbation analysis, where the change in frequency has been found as only a second-order effect.

The equation for $\dot{A}$ can be written as

$$\dot{A} = \frac{\alpha\omega_0 A}{2}\left(1 - \frac{\beta A^2}{4}\right) \tag{6.59}$$

It is conveniently studied by plotting $\dot{A}$ as a function of A, to give a kind of phase-plane diagram. Since this is a first-order equation, only a single curve appears on the phase plane instead of the family of curves usually associated with a second-order equation. The curve is shown in Fig. 6.9. There are three points where $\dot{A} = 0$, these points being $A = 0$ and $A = \pm 2/\beta^{1/2}$. These are equilibrium points. The maximum value of $\dot{A}$ occurs where $d\dot{A}/dA = 0$. For this condition, $A = \pm 2/(3\beta)^{1/2}$ and $\dot{A} = \pm(2\alpha\omega_0/3)/(3\beta)^{1/2}$.

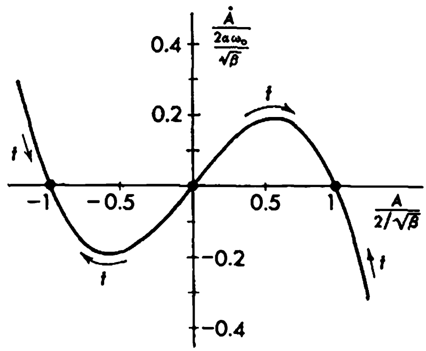


FIG. 6.9. Phase-plane diagram for amplitude A of oscillation for circuit of Example 6.7. Arrows indicate direction of increasing time.

As in all phase-plane diagrams, time is a parameter. Time progresses generally to the right in the upper half plane and to the left in the lower half plane. Thus, time progresses in the direction of the arrows of Fig. 6.9. The equilibrium point at the origin is therefore unstable, since a point on the curve tends to move away from the origin as time progresses. Similarly, the equilibrium points at $A = \pm 2/\beta^{1/2}$ are stable, since a point on the curve tends toward these points. The stable value of A is the same as the steady-state value for E found by the perturbation analysis. It is worth noting that equilibrium points are those where $\dot{A} = 0$. Stable equilibrium is associated with points where $d\dot{A}/dA < 0$; unstable equilibrium is associated with those where $d\dot{A}/dA > 0$.

Equation (6.59) can be solved exactly as a Bernoulli equation, Sec. 4.2*i*, or by variation of parameters. Its solution is

$$A = \frac{2/\beta^{1/2}}{[1 - (1 - 4/\beta A_0^2) \exp(-\alpha\omega_0 t)]^{1/2}} \tag{6.60}$$

where the initial condition is $A = A_0$ at $t = 0$. This solution may be plotted as in Fig. 6.10, where several different values for A_0 are chosen. If $A_0 = 0$, amplitude A remains zero in theory, although it is unstable. If $A_0 \neq 0$, the amplitude ultimately assumes the value $A = 2/\beta^{1/2}$,

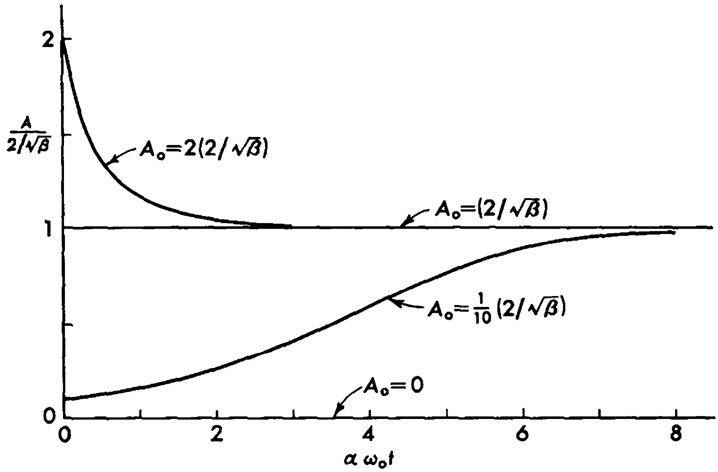


FIG. 6.10. Possible variations of amplitude with time for circuit of Example 6.7. Four different initial conditions are shown.

growing if A_0 is initially small and decaying if A_0 is initially large. In all cases, A approaches its ultimate value monotonically with no overshoot.

This kind of oscillation, which is self-starting so long as there is any arbitrarily small initial disturbance, is known as a "soft" oscillator. Other types of negative-resistance elements lead to a "hard" oscillator, which requires an initial disturbance exceeding some minimum value for oscillation to continue.

For this method of variation of parameters to be valid, it is necessary that always $|\dot{A}/A|(2\pi/\omega_0) \ll 1$. If A grows from an initial small value, the maximum value of $\dot{A}$ is $\dot{A}_{max} = (2\alpha\omega_0/3)/(3\beta)^{1/2}$ existing when $A = 2/(3\beta)^{1/2}$. Thus, $\dot{A}_{max}/A = \alpha\omega_0/3$, and, for this method of analysis to be valid, it is required that $\alpha \ll 3/2\pi$, which is about one-half. With this requirement, a number of cycles of oscillation will exist during growth from some initial small amplitude.

The solution for the instantaneous voltage to this order of approximation, as found by the method of variation of parameters, is

$$e = A(t) \cos (\omega_0 t + \theta_0)$$

where $A(t)$ is given by Eq. (6.60) and θ_0 is a constant dependent upon initial conditions.

d. Large Values of α. Both the perturbation method and variation of parameters require that parameter α in the equation of the self-oscillator be small compared with unity. If $\alpha \ll 1$, the resulting waveform of voltage is nearly simple-harmonic, the frequency is near the natural frequency of the tuned circuit, and any changes in amplitude take place

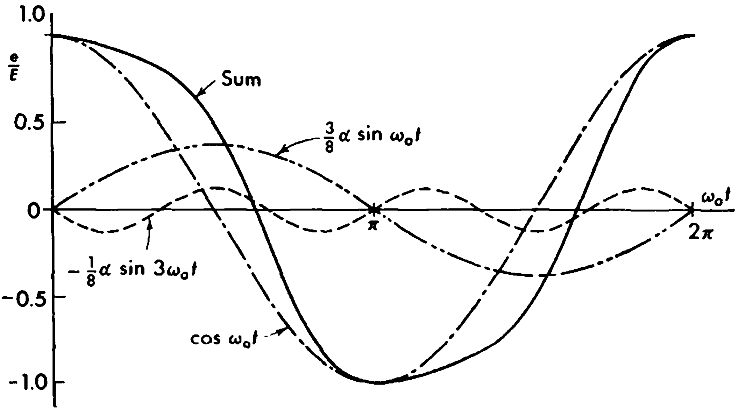


FIG. 6.11. Waveform of voltage in circuit of Example 6.7, showing fundamental- and third-harmonic components to first-order approximation, with $\alpha = 1$.

relatively slowly. A basically different kind of operation takes place if parameter α is large. Some information about this case, at least for the van der Pol equation, has been found in Examples 3.3 and 3.6, where graphical methods are used. Additional information can be found analytically.

The first-order steady-state solution, found by the perturbation method, is

$$e = E \cos \omega_0 t + \alpha \frac{E}{8} (3 \sin \omega_0 t - \sin 3\omega_0 t) \tag{6.61}$$

When plotted for $\alpha = 1$, this equation gives the curve of Fig. 6.11. The use of this approximate solution for so large a value of α is not actually justified, but the result is qualitatively correct. The effect of the first-order correction terms is to produce a somewhat flattened, though skewed, peak to the wave, with more abrupt transitions between positive and negative values. The larger the value of α, the greater the effect of the additional terms. It is not unreasonable to expect that values of α much larger than unity would lead to peaks that are quite flat, though sloping, and almost instantaneous transitions. Sharp corners would occur in the wave wherever the transition begins and ends. This observation agrees with that made in Example 3.6, where the case of ϵ very large in van der Pol's equation is considered. Furthermore, it is evident from Fig. 3.17 of Example 3.6 that relatively slow changes in x take place as x decreases from its maximum value to half its maximum. A very rapid change then occurs as x jumps to the maximum of the opposite sign. This information is used in the analysis following.

If the waveform of e is composed of essentially straight lines, joined with sharp corners, the second derivative $\ddot{e}$ in Eq. (6.50) can be neglected everywhere except at the corners. Then, Eq. (6.50) becomes

$$-\alpha(1 - \beta e^2)\omega_0\left(\frac{de}{dt}\right) + \omega_0^2 e = 0 \tag{6.62}$$

Variables can be separated and integrated to give

$$\int_0^t dt = \int_{2/\beta^{1/2}}^{e} \frac{\alpha}{\omega_0}\left(\frac{1}{e} - \beta e\right) de \tag{6.63}$$

where it is assumed that $e = 2/\beta^{1/2}$, its maximum value, at $t = 0$, and instantaneous voltage e exists at time t. Integration gives the result

$$t = \frac{\alpha}{\omega_0}\left(\ln\frac{\beta^{1/2}e}{2} - \frac{\beta e^2}{2} + 2\right) \tag{6.64}$$

When the voltage has decreased to the value $e = 1/\beta^{1/2}$, the slope de/dt becomes infinite and transition to negative values of e takes place. Because of symmetry in the negative-resistance characteristic, the negative half cycle is identical with the positive half cycle except for sign. Thus, the interval during which e changes from $2/\beta^{1/2}$ to $1/\beta^{1/2}$ is half a complete period T, and Eqs. (6.63) and (6.64) hold only for the interval $0 \leq t \leq T/2$. This statement is based on the assumption that transition is instantaneous, requiring no time at all.

The period can be found by putting the condition that $t = T/2$ at $e = 1/\beta^{1/2}$ into Eq. (6.64), which gives

$$\frac{T}{2} = \frac{\alpha}{\omega_0}(\ln \tfrac{1}{2} + \tfrac{3}{2}) \doteq 0.81\frac{\alpha}{\omega_0}$$

The period for $\alpha \gg 1$ is approximately

$$T = 1.62\frac{\alpha}{\omega_0} \tag{6.65}$$

The waveform can be found by plotting Eq. (6.64) as in Fig. 6.12, where positive and negative half cycles have the same shape. This waveform, applying approximately when α is very large, may be compared with Fig. 3.16 obtained graphically for $\epsilon = 5$. The two curves are qualitatively similar, the primary difference being that the curve for $\epsilon = 5$ shows corners that are not sharp and transitions that are not instantaneous. It would be expected therefore that a waveform such as Fig. 6.12 would require an exceedingly large value of α. Similarly, Eq. (6.65) for the period gives a result which is actually too small unless α is exceedingly large.

The kind of operation associated with large values of α is known as relaxation oscillation. It is characterized by waveforms showing abrupt transitions from one state, during which relatively slow changes take

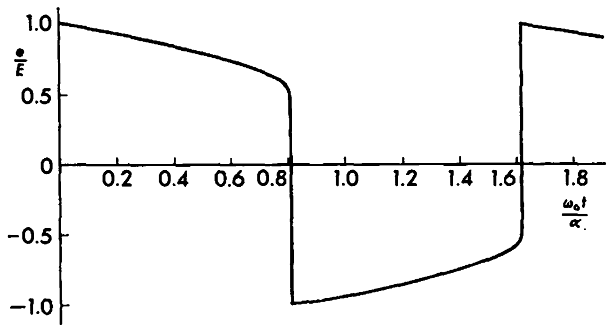


FIG. 6.12. Approximate waveform of voltage in circuit of Example 6.7, with α exceedingly large.

place, to a second such state. The period of the relaxation oscillation, given by Eq. (6.65), depends upon the quantity $\alpha/\omega_0 = a/C\omega_0^2 = aL$. This quantity involves the value of the negative conductance coefficient a and the single reactive element L. By contrast, the period of the sinusoidal oscillation existing for $\alpha \ll 1$ is $T = 2\pi/\omega_0 = 2\pi(LC)^{1/2}$. It depends upon the two reactive elements L and C and is independent of the resistance. The peak values for the instantaneous voltage e, interestingly enough, remain essentially the same, $E = \pm 2/\beta^{1/2}$, for both sinusoidal and relaxation oscillations.

More exact analyses are available[1] giving the variations of voltage amplitude E and period T with α. The results of these analyses are plotted in Fig. 6.13, together with results found in the simpler treatments given here.

6.6. Averaging Methods Based on Residuals. *a. Galerkin's Method.* A somewhat different approach to the question of finding an approximate solution for a nonlinear differential equation is based on the consideration of a kind of error, called the residual, associated with this solution. The

[1] See Fisher, reference 22 in the List of References in the Bibliography at the end of this volume.

nonlinear equation may have the form

$$f(D,x,t) = 0 \tag{6.66}$$

where $f(D,x,t)$ is some generally nonlinear function of the derivative operator $D = d/dt$ and of dependent variable x and independent variable t. An approximate solution for Eq. (6.66) is assumed on the basis of information available about it. This assumed solution must contain enough constants to allow fitting of initial conditions and, in addition, certain parameters which are to be adjusted to make the solution optimum in an appropriate sense.

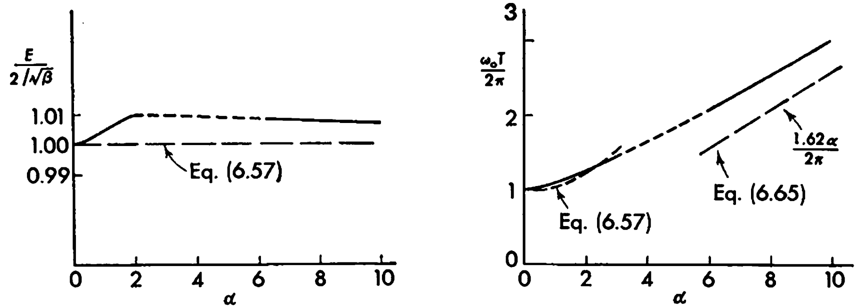


FIG. 6.13. Variation of maximum amplitude E and period T for circuit of Example 6.7, with parameter α. Approximate relations found here, together with relations found by more accurate analysis from Fisher, are shown.

Exact details may vary somewhat, but a reasonable procedure is to assume the solution as a linear combination of suitably chosen functions. In mathematical terminology, $x(t)$ is approximated by $X(t)$, as

$$x(t) \doteq X(t) = \phi_0(t) + C_1\phi_1(t) + \cdots + C_m\phi_m(t) \tag{6.67}$$

where $\phi_0, \phi_1, \ldots, \phi_m$ are appropriate linearly independent functions, chosen to be of such type and in such number to give a solution of desired accuracy. The first function, ϕ_0, is chosen to approximate the solution as well as possible and to satisfy the n initial conditions which must be imposed in an nth-order equation. This ϕ_0 is analogous to the generating solution, which was found as the first step in the perturbation method and then adjusted to take care of initial conditions. The remaining functions, $\phi_1, \ldots, \phi_m$, represent corrections applied to ϕ_0. Initial conditions for these functions are all zero, since function ϕ_0 includes any nonzero initial conditions for the entire solution. The constants $C_1, \ldots, C_m$ are to be adjusted to optimize the solution.

It is recognized that the assumed solution may be fairly accurate only within a limited range of the independent variable. It is therefore necessary to specify a range, say, $a \leq t \leq b$, within which the solution is to apply. If the solution is periodic, the range would probably be chosen as one period. If the solution is not periodic, the choice of range is more arbitrary.

The selection of the functions appearing in Eq. (6.67) is a crucial part of this method of solution. These functions should be of such a nature that the sum of only a few of them approximates the solution reasonably well. If the differential equation is complicated, probably a relatively complicated sort of function is required to represent its solution. On the other hand, since mathematical operations have to be performed on the sum of functions of Eq. (6.67), it is most desirable that these functions be as simple as possible. Obviously, some knowledge of the properties of the solution is necessary to allow an intelligent choice of the functions.

The approximate solution of Eq. (6.67) is substituted into Eq. (6.66), and since the solution is not exact, it will not satisfy the equation identically. Thus, a residual $\epsilon(t)$ may be defined as

$$\epsilon(t) = f[D,X(t),t] \tag{6.68}$$

This residual is a measure of the unbalance in the differential equation with the assumed approximate solution substituted. The residual is not, of course, directly the difference between the approximate solution and an exact solution. An exact solution for the equation, in general, remains unknown.

A measure of the accuracy of the approximate solution is how closely the residual ϵ approaches and remains near zero over the range of interest, $a \leq t \leq b$. Since ϵ may be of either algebraic sign and in either case represents an error, its square ϵ^2 is a convenient and simple measure of accuracy. Furthermore, the integral

$$J = \int_a^b \epsilon^2(t)\,dt \tag{6.69}$$

is a measure of the error over the whole interval involved. A criterion of accuracy with some validity is the requirement that this integral J be a minimum. This criterion follows naturally from the principle of least squares, often used in assigning values to a few constants in an empirical solution so as to obtain optimum approximation to a larger number of known conditions.

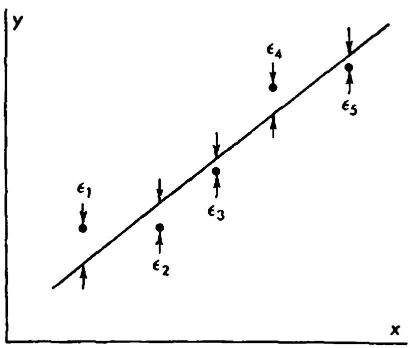


FIG. 6.14. Least-square fitting of straight-line relation to experimental data.

The principle of least squares is applicable, for example, to a situation of the sort shown in Fig. 6.14. This figure represents five experimental values of some quantity y, which is known from theory to be a linear function of another quantity x. It is desired to choose optimum values for constants C_1 and C_2 in the linear equation $y = C_1x + C_2$. This is done by defining the five residuals $\epsilon_1, \ldots, \epsilon_5$ as the differences between the experimental values of y and the corresponding values given by the optimum linear relation. It is then required that the sum $\sum_{i=1}^{5} \epsilon_i^2$ be a minimum. Where a continuous function, rather than discrete points, is involved, the sum becomes an integral instead.

The application of this principle to the solution of differential equations is known as Galerkin's method.

In the approximate solution, Eq. (6.67), the constants $C_1, \ldots, C_m$ are to be determined. These constants appear in the residual $\epsilon(t)$ and in the integral J. In order to minimize J, it is evidently necessary that

$$\begin{aligned} \frac{\partial J}{\partial C_1} &= \int_a^b 2\epsilon(t) \frac{\partial \epsilon(t)}{\partial C_1}\, dt = 0 \\ &\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial J}{\partial C_m} &= \int_a^b 2\epsilon(t) \frac{\partial \epsilon(t)}{\partial C_m}\, dt = 0 \end{aligned} \tag{6.70}$$

It is possible, of course, that these conditions will lead to a maximum for J, rather than a minimum. Usually it is not difficult to verify whether a maximum or a minimum is involved. An ultimate test for a minimum is that the second derivative of J be positive where the first derivative vanishes. The constants $C_1, \ldots, C_m$ are chosen to satisfy Eq. (6.70). The mathematical details involved here may be relatively simple in the case of linear differential equations. Nonlinear equations, however, typically lead to considerable mathematical complexity.

Example 6.8. Capacitor Discharge through Diode

The circuit consisting of a capacitor discharging through a nonlinear diode has already been considered in Examples 6.1 and 6.4. The equation is

$$\dot{e} + Ae + Be^2 = 0 \tag{6.71}$$

with $e = E$ at $t = 0$. Find an approximate solution by the Galerkin method.

If the nonlinear term were missing, $B = 0$, the exact solution would be $e = E \exp(-At)$, where the initial condition has already been used. This is a reasonable function to use as the first term, ϕ_0, in an approximate solution for the nonlinear equation. A suitable form for the correction terms must now be chosen. Because of mathematical difficulties, only a single correction term, ϕ_1, is considered here. This term must be chosen with such a form as to have the value zero when time is either zero or infinity, so that initial and final values of e will be correct. A possible correction term having this property is $t \exp(-At)$. It must be recognized that this is but one of many possible forms for the correction term. In particular, the choice of exponent

At is a purely arbitrary one. At any rate, a possible form for an approximate solution for Eq. (6.71) is

$$e = E \exp(-At) + kt \exp(-At) \tag{6.72}$$

where k is the multiplying constant which is to be chosen by the Galerkin procedure. Substitution of Eq. (6.72) into Eq. (6.71) gives for the residual

$$\epsilon = k \exp(-At) + B(E + kt)^2 \exp(-2At) \tag{6.73}$$

Its partial derivative is

$$\frac{\partial \epsilon}{\partial k} = \exp(-At) + 2Bt(E + kt) \exp(-2At) \tag{6.74}$$

The requirement is that k be chosen to satisfy the condition

$$\int_0^\infty \epsilon \left(\frac{\partial \epsilon}{\partial k}\right) dt = 0 \tag{6.75}$$

where the range of integration has been taken arbitrarily to include all positive values of time.

Equations (6.73) and (6.74) can be substituted into Eq. (6.75) and the integration carried out. As is typical of the application of this method to even relatively simple nonlinear equations, such as Eq. (6.71), the details of the integration become quite tedious and a large number of terms arise. A rather complicated cubic algebraic equation for k results. This equation cannot readily be solved in general terms, though if specific numerical values were given, a solution would be possible.

This same problem is considered by a similar, but simpler, method in Example 6.11, where it is carried to completion.

Example 6.9. Mass on Nonlinear Spring

The motion of a constant mass mounted on a nonlinear spring has already been considered in Examples 4.6, 6.3, and 6.5. The equation is

$$\ddot{x} + \omega_0^2 x + hx^3 = 0 \tag{6.76}$$

with initial conditions $x = A$, $\dot{x} = 0$ at $t = 0$. Find an approximate solution by Galerkin's method.

It is known that the solution for Eq. (6.76) is periodic oscillatory and is nearly simple-harmonic if the nonlinear term is not too large. Thus, an approximate solution can be assumed as

$$x = \phi_0(t) = A \cos \omega t \tag{6.77}$$

This is analogous to Eq. (6.67) with only a single function used in the approximation. This function satisfies the initial conditions but contains the unknown angular frequency ω. This quantity can be determined by the Galerkin method even though it does not take the form of an additive correction term. The procedure here is similar to that followed in the perturbation method when secular terms arise in an oscillatory system.

The residual for Eq. (6.76) when Eq. (6.77) is substituted becomes

$$\epsilon = \left(-\omega^2 A + \omega_0^2 A + \frac{3hA^3}{4}\right) \cos \omega t + \frac{hA^3}{4} \cos 3\omega t \tag{6.78}$$

In order to minimize integral J, it is necessary that

$$\int_0^{2\pi} \epsilon \left(\frac{\partial \epsilon}{\partial A}\right) d(\omega t) = 0 \tag{6.79}$$

Since the solution is periodic, it is natural to fit the approximate solution over a complete period and the range of integration is chosen as $0 \leq \omega t \leq 2\pi$. The relation between ω and A of the assumed solution is to be determined, and this can be found through differentiation of ϵ with respect to A. The result is

$$\int_0^{2\pi} \left[\left(-\omega^2 A + \omega_0{}^2 A + \frac{3hA^3}{4}\right) \cos \omega t + \frac{hA^3}{4} \cos 3\omega t\right] \left[\left(-\omega^2 + \omega_0{}^2 + \frac{9hA^2}{4}\right) \cos \omega t + \frac{3hA^2}{4} \cos 3\omega t\right] d(\omega t) = 0$$

In order to satisfy this requirement, it is necessary that

$$\left(-\omega^2 A + \omega_0{}^2 A + \frac{3hA^3}{4}\right)\left(-\omega^2 + \omega_0{}^2 + \frac{9hA^2}{4}\right) + \left(\frac{hA^3}{4}\right)\left(\frac{3hA^2}{4}\right) = 0$$

Then, either $A = 0$, which is trivial, or

$$(\omega_0{}^2 - \omega^2)^2 + 3hA^2(\omega_0{}^2 - \omega^2) + \frac{15h^2A^4}{8} = 0$$

Solution of this quadratic equation allows the result to be written

$$\omega^2 = \omega_0{}^2 + khA^2$$

where constant k may have either of the values 0.89 or 2.11.

The determination of which value of k minimizes integral J can be made as follows: An alternate way of writing the result is $(\omega^2 - \omega_0{}^2)/kh = A^2$. In addition to the two values of k found from the quadratic, the possible zero value for A may be considered to correspond to an infinite value of k. The value of integral J can be expected to vary continuously if k is allowed to change, with maxima and minima occurring alternately. It is possible, of course, but unlikely, that J might have a point where the derivative is zero but which is neither a maximum nor a minimum, in which case the assumed alternation of maxima and minima would not occur. Here, if k is infinite and A is zero, ϵ is zero and integral J is clearly a minimum. If k is reduced from infinity to 2.11, the next larger value of interest, J should be a maximum. Finally, if k is further reduced to 0.89, J should be a minimum. This latter value is evidently the one applying in this problem. Therefore, the parameters of approximate solution, Eq. (6.77), should be related as

$$\omega^2 = \omega_0{}^2 + 0.89hA^2 \tag{6.80}$$

This result may be compared with the results found in Examples 6.3 and 6.5, where the equivalent of constant k turned out to have the value 0.75.

A more accurate solution could be obtained by this method if two components, a fundamental and a third-harmonic cosine, were used in place of the single component of Eq. (6.77). The analysis is greatly complicated by the additional component, however.

Example 6.10. Mass on Square-law Spring

The nonlinear spring of the preceding example, first discussed in Example 4.6, has its force of deflection related to the deflection through an equation with a linear term and a cubic term. A kind of spring can be imagined in which the force is proportional only to the square of the deflection, with no linear term, but with the force always in such a direction as to tend to reduce the deflection. A constant mass mounted on a spring of this kind is described by the equation

$$\ddot{x} + a^2 x|x| = 0 \tag{6.81}$$

where a^2 is a positive constant dependent upon the physical properties of the mass and spring. Since there is no linear term in x, solution by variation of parameters as described in Sec. 6.4, for example, cannot be carried out. If initial conditions are $x = A$, $\dot{x} = 0$ at $t = 0$, find an approximate solution.

The system can be expected to oscillate so that an approximate solution can be assumed in the form

$$x = \phi_0(t) = A \cos \omega t \tag{6.82}$$

When substituted into Eq. (6.81) the residual is

$$\epsilon = -\omega^2 A \cos \omega t \pm a^2 A^2 \cos^2 \omega t$$

where the plus sign applies for $-\pi/2 \leq \omega t \leq +\pi/2$ and the minus sign for $\pi/2 \leq \omega t \leq 3\pi/2$. The condition to be met is

$$\int_{-\pi/2}^{\pi/2} \epsilon \left(\frac{\partial \epsilon}{\partial A}\right) d(\omega t) = 0$$

where integration over only half a complete period is needed, since the two half periods are symmetrical. The integral becomes

$$\int_{-\pi/2}^{\pi/2} (-\omega^2 A \cos \omega t + a^2 A^2 \cos^2 \omega t)(-\omega^2 \cos \omega t + 2a^2 A \cos^2 \omega t)\, d(\omega t) = 0$$

This may be satisfied either by the trivial case of $A = 0$ or by the condition

$$\omega^4 - \frac{8}{\pi} a^2 A \omega^2 + \tfrac{3}{2} a^4 A^2 = 0$$

The result may be written $\omega^2 = ka^2 A$, where k may have either of the values 1.63 or 0.92. An argument similar to that used in the preceding example shows that the minimum value for integral J corresponds to the case

$$\omega^2 = 0.92 a^2 A \tag{6.83}$$

b. Ritz Method. Under certain conditions, Eq. (6.70), which is the requirement to minimize integral J of Eq. (6.69), can be shown to be equivalent to the simpler equations

$$\begin{aligned} &\int_a^b \epsilon(t)\phi_1(t)\, dt = 0 \\ &\cdots\cdots\cdots\cdots \\ &\int_a^b \epsilon(t)\phi_m(t)\, dt = 0 \end{aligned} \tag{6.84}$$

where $\phi_1, \ldots, \phi_m$ are the functions in the assumed solution of Eq. (6.67). The equivalence of Eqs. (6.70) and (6.84) exists if the original differential equation, Eq. (6.66), is linear and if functions $\phi_0, \phi_1, \ldots, \phi_m$ are orthogonal over the range $a \leq t \leq b$. Orthogonal functions have the property that

$$\int_a^b \phi_i(t)\phi_j(t)\, dt \begin{cases} = 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j \end{cases}$$

Usually Eq. (6.84) is much simpler to apply than Eq. (6.70), and thus these second conditions are often used with nonlinear equations. The solution thus obtained is not quite the same as that found through minimizing integral J.

This second, somewhat simpler, process is known as the Ritz method. It is identical with Galerkin's method for linear equations but differs somewhat for nonlinear equations.

For an oscillatory system, the use of Eq. (6.84) can be justified by a kind of physical argument. In an equation of the form $M\ddot{x} + f(\dot{x},x) = F(t)$ applying to a mechanical system, variable x might be interpreted as a displacement, while the several terms of the equation are forces. If $F(t)$ is a periodic driving force and damping is present, or if $F(t)$ is zero and there is no damping, a steady-state periodic oscillation ultimately exists in the system. The displacement can then be represented approximately as a finite sum of terms of the form

$$x \doteq X(t) = C_1 \cos \omega_1 t + C_2 \cos 2\omega_1 t + \cdots + C_m \cos m\omega_1 t$$

If this approximate solution is substituted into the differential equation, the residual ϵ may be found. This residual is dimensionally a force and can be thought of as being the excess force existing at any instant in the cycle because of error inherent in the approximate solution. The product of this force ϵ and the assumed displacement x has the dimensions of energy and can be interpreted as a kind of error in the work done by the system. A reasonable requirement might be that the integral of this work over a complete cycle should be zero. In mathematical terminology, the requirement is

$$\int_0^{2\pi} \epsilon(t)X(t)\, d(\omega_1 t) = 0$$

Since $X(t)$ is written as a sum of harmonically related cosine terms, which are orthogonal over the range of integration, this one equation can be split into the several equations of Eq. (6.84).

Another way of looking at the Ritz method is to observe that it minimizes an integral of the alternate form $\int_a^b F(x,\dot{x},t)\, dt$, in place of integral J of Eq. (6.69). Function $F(x,\dot{x},t)$ of this alternate integral is that function which makes the Euler equation

$$\frac{d}{dt}\left[\frac{\partial F}{\partial \dot{x}}\right] - \frac{\partial F}{\partial x} = 0$$

correspond to the nonlinear equation in question. This process of minimizing the alternate integral is analogous to an application of Hamilton's principle from mechanics. The series of cosine terms in the approximate solution found through this method can be said to be chosen in the Hamilton sense. The cosine terms found through the minimization of integral J are chosen to fit the Fourier conditions. The two procedures give the same results for linear equations, or where an infinite number of terms are used in writing the approximate solution.

Finally, another interpretation of the Ritz method is to consider the functions $\phi_1, \ldots, \phi_m$ in Eq. (6.84) as weighting functions. The first factor ϵ in the integrand measures the error. Multiplication by functions $\phi_1, \ldots, \phi_m$ before integration applies a certain weight to the error at each instant in the solution. This process serves to make the effect of the error, as measured by ϵ, more important when the values of functions $\phi_1, \ldots, \phi_m$ are greatest.

Example 6.11. Capacitor Discharge through Diode

The circuit consisting of a capacitor discharging through a nonlinear diode has been considered in Examples 6.1, 6.4, and 6.8. The equation is

$$\dot{e} + Ae + Be^2 = 0$$

with $e = E$ at $t = 0$. Find an approximate solution by the Ritz method.

The solution assumed in Example 6.8 is

$$e = \phi_0 + k\phi_1 = E \exp(-At) + kt \exp(-At)$$

for which the residual is

$$\epsilon = k \exp(-At) + B(E + kt)^2 \exp(-2At)$$

The integral for the Ritz method is

$$\int_0^\infty \epsilon\phi_1 \, dt = \int_0^\infty [kt \exp(-2At) + Bt(E + kt)^2 \exp(-3At)] \, dt = 0$$

This integral can be evaluated without too much difficulty and leads to the quadratic equation in k,

$$\frac{2B}{27A^2} k^2 + \left(\frac{1}{4} + \frac{4BE}{27A}\right) k + \frac{BE^2}{9} = 0 \tag{6.85}$$

A particular case examined numerically in Example 6.1 is that where $BE/A = \frac{1}{2}$. If this value is used here and at the same time the dimensionless quantity K is defined as $K = k/AE$, Eq. (6.85) becomes

$$4K^2 + 35K + 6 = 0$$

The roots of this equation are $K = -8.60$ and $K = -0.172$. The first root can be dismissed as much too large in magnitude for the present problem, and so the appropriate value of k must be $k = -0.172AE$. The approximate solution can then be written

$$e = E(1 - 0.172At) \exp(-At) \tag{6.86}$$

This equation is plotted in Fig. 6.15, together with the exact solution for this problem as found in Example 6.1. Agreement is seen to be relatively good, although not so good as for the approximate solution found to a second-order approximation by the

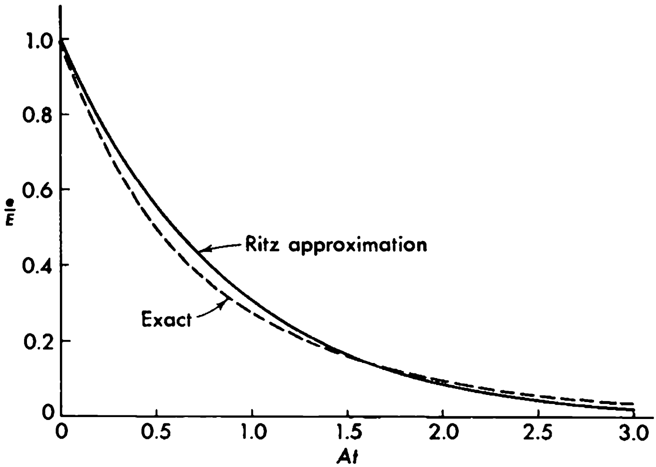


FIG. 6.15. Solution for Example 6.11 found by Ritz method and compared with exact solution.

perturbation method and plotted in Fig. 6.4. Equation (6.86) is, however, much simpler than Eq. (6.11).

Example 6.12. Mass on Nonlinear Spring

The motion of a mass on a nonlinear spring has been considered in Examples 4.6, 6.3, 6.5, and 6.9. The equation is

$$\ddot{x} + \omega_0^2 x + hx^3 = 0$$

with initial conditions $x = A$, $\dot{x} = 0$ at $t = 0$. Find an approximate solution by the Ritz method.

The solution is known to be periodic and not to differ much from simple harmonic if the nonlinear term is small. Thus, the first term in an approximate solution might be

$$x = \phi_0 = A \cos \omega t$$

which fits the initial conditions. The effect of the nonlinearity is to distort the solution from simple harmonic. However, the nature of the differential equation is such that its solution must show odd-order symmetry about both $t = 0$ and $x = 0$. Thus, the first important correction term might be expected to be a cosine term of the third-harmonic frequency. If such a term is added, the amplitude of the fundamental term must be changed so as to maintain the assigned initial condition. This change in fundamental can be considered as yet another correction term, so that an approximate solution with third-harmonic correction might be

$$\begin{aligned} x &= \phi_0 + C_1\phi_1 + C_2\phi_2 \\ &= A \cos \omega t + C_1 \cos \omega t + C_2 \cos 3\omega t \end{aligned}$$

where C_1 and C_2 are constants. Clearly, in order to meet the initial conditions,

$C_1 = -C_2$, and if C_2 is called A_3 to keep a consistent notation, the approximate solution becomes

$$x = A_1 \cos \omega t + A_3 \cos 3\omega t \tag{6.87}$$

where
$$A_1 = A - A_3$$

The residual can be found by substituting the assumed solution, Eq. (6.87), into the original equation. Since there are two correcting functions in the assumed solution, two integrals are involved in the Ritz method. These integrals are

$$\int_0^{2\pi} \epsilon\phi_1\, d(\omega t) = \int_0^{2\pi} \left[\left(-\omega^2 A_1 + \omega_0{}^2 A_1 + \frac{3hA_1{}^3}{4} + \frac{3hA_1{}^2 A_3}{4} + \frac{3hA_1 A_3{}^2}{2} \right) \cos \omega t + \cdots \right] \cos \omega t\, d(\omega t) = 0$$

$$\int_0^{2\pi} \epsilon\phi_2\, d(\omega t) = \int_0^{2\pi} \left[\left(-9\omega^2 A_3 + \omega_0{}^2 A_3 + \frac{hA_1{}^3}{4} + \frac{3hA_1{}^2 A_3}{2} + \frac{3hA_3{}^3}{4} \right) \cos 3\omega t + \cdots \right] \cos 3\omega t\, d(\omega t) = 0$$

In each pair of brackets, only those terms are written which lead to a value of the integral different from zero. Thus, the simultaneous algebraic equations that must be solved for a relation among the quantities A_1, A_3, and ω are found by setting to zero the collection of terms in the two pairs of parentheses of the integrands above, together with the relation $A_1 + A_3 = A$, where A is the initial value of x.

These equations involve cubes of both A_1 and A_3, and a general algebraic solution is not possible. Particular numerical solutions could, of course, be obtained. The relative algebraic complexity of this result is typical of that encountered when several terms are used in the approximate solution.

A simple solution can be found if the third-harmonic term is neglected, in which case $A_3 = 0$ and $A_1 = A$. Then only the first integral need be considered, and it leads to the result

$$\omega^2 = \omega_0{}^2 + \tfrac{3}{4} hA^2$$

which is the same as that already found in Examples 6.3 and 6.5.

Example 6.13. Spring with Discontinuous Stiffness

A mechanical oscillating system somewhat different from those previously considered combines a constant mass with a spring for which the stiffness varies in a discontinuous manner. If the spring becomes stiffer symmetrically for a critical deflection either side of the rest position, it may be described by the curve of Fig. 6.16. The relations applying to such a spring are

$x \leq -x_1$:	$F = k_2(x + x_2)$
$-x_1 \leq x \leq +x_1$:	$F = k_1 x$
$x \geq +x_1$:	$F = k_2(x - x_2)$

where F is the force to deflect the spring, x is the deflection, and $k_2 > k_1$ and $x_1 > x_2$. If a constant mass M is attached to the spring, the equations of motion are

$x \leq -x_1$:	$\ddot{x} + \omega_2{}^2(x + x_2) = 0$
$-x_1 \leq x \leq +x_1$:	$\ddot{x} + \omega_1{}^2 x = 0$
$x \geq +x_1$:	$\ddot{x} + \omega_2{}^2(x - x_2) = 0$

where $\omega_1{}^2 = k_1/M$ and $\omega_2{}^2 = k_2/M$. If the initial conditions are $x = A$, $\dot{x} = 0$ at $t = 0$, find an approximate solution.

An exact solution could be built up for these piecewise linear equations by using the process described in Sec. 4.4. On the other hand, an approximate solution can be obtained with the Ritz method. Since an oscillatory solution is expected, a first approximation might have the form

$$x = A \cos \omega t$$

This, of course, fits the initial conditions.

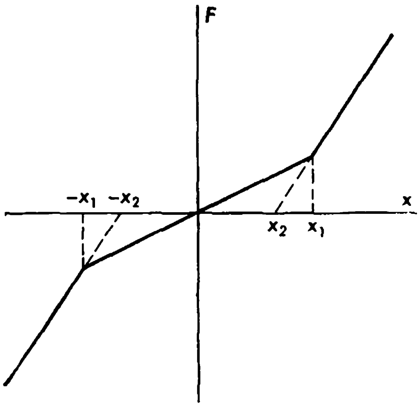


FIG. 6.16. Relation between force and deflection for spring of Example 6.13.

The integrals required in the minimizing process must be written in three parts as

$$\int_0^{\omega t_1} [(-\omega^2 A + \omega_2^2 A) \cos \omega t - \omega_2^2 x_2] \cos \omega t \, d(\omega t)$$
$$+ \int_{\omega t_1}^{\omega t_2} [(-\omega^2 A + \omega_1^2 A) \cos \omega t] \cos \omega t \, d(\omega t)$$
$$+ \int_{\omega t_2}^{\pi} [(-\omega^2 A + \omega_2^2 A) \cos \omega t + \omega_2^2 x_2] \cos \omega t \, d(\omega t) = 0$$

The limits of the integrals depend upon the conditions

$$\begin{array}{lll} x = A & \text{at} & t = 0 \\ x = +x_1 & \text{at} & t = t_1 \\ x = -x_1 & \text{at} & t = t_2 \\ x = -A & \text{at} & t = \dfrac{T}{2} \end{array}$$

where T is the period of oscillation. Because of symmetry, only half a period need be considered. The values of ωt_1 and ωt_2 can be found from the relations $+x_1 = A \cos \omega t_1$ and $-x_1 = A \cos \omega t_2$.

Integration here once again leads to rather complicated algebraic expressions that cannot be solved in general terms. However, if specific numerical values for the parameters of the equations are available, a particular solution could be carried out. A cubic equation in the unknown frequency of oscillation is involved.

Example 6.14. Vertical Fall with Air Resistance

In Example 6.2, motion of a projectile moving with moderate velocity through air in a gravitational field has been considered. Friction force produced by the air is assumed to vary with the square of the velocity. If motion takes place in only the vertical direction, the equation is

$$M\dot{v} = Mg - kv^2$$

where M is the mass of the moving body, v is its downward velocity, g is the gravitational acceleration, and k is the constant relating retarding force to v^2. This equation is analogous to Eq. (6.13). The equation may be rewritten as

$$\dot{v} = g - hv^2 \tag{6.88}$$

where $h = k/M$. If the body falls from rest, so that $v = 0$ at $t = 0$, find an approximate solution for its motion.

Even though Eq. (6.88) is nonlinear, it can be solved exactly, but the resulting solution is rather complicated, and for some purposes an approximate solution of simpler mathematical form might be preferable. Such a solution can be found by the Ritz method.

The velocity of the body increases from its initial value of zero, but its acceleration becomes progressively less and less. After a long time, the velocity is constant. The final velocity, attained when $\dot{v} = 0$, is $v_f = (g/h)^{1/2} = (Mg/k)^{1/2}$. An approximate solution for Eq. (6.88) might be assumed of the form

$$v = v_f[1 - \exp(-Ct)] \tag{6.89}$$

where C is a constant to be determined. The time function, $\phi(t) = 1 - \exp(-Ct)$, is only an estimate and is assumed merely because it has the proper initial and final values. Choice of constant C is made to give an optimum solution in the sense of the Ritz method.

The residual can be found as

$$\epsilon = (v_f C - 2g)\exp(-Ct) + g\exp(-2Ct)$$

The Ritz criterion is

$$\int_0^\infty \epsilon\phi\, dt = 0$$

The range of integration, or the range of the optimized solution, is taken to cover all positive time. Upon carrying out the evaluation of the integral, constant C is found as $C = \frac{5}{3}(kg/M)^{1/2}$, so that the approximate solution becomes

$$v = v_f\left\{1 - \exp\left[-\frac{5}{3}\left(\frac{kg}{M}\right)^{1/2} t\right]\right\} \tag{6.90}$$

This solution gives correct values for the velocity at both zero and infinite values of time. The acceleration is correct at infinite time but is incorrect at zero time. Since the function $\phi(t)$ appears as a kind of weighting function in the integral and is largest for large values of time, the approximate solution could be expected to be more nearly correct as time increases.

An exact solution for Eq. (6.88) can be found by separating the variables, Sec. 4.2d, and is

$$\left(\frac{1 - v/v_f}{1 + v/v_f}\right)^{1/2} = \exp\left[-\left(\frac{kg}{M}\right)^{1/2} t\right] \tag{6.91}$$

It is obviously not a simple matter to visualize the properties of this rather complicated relation. A plot of the approximate solution, Eq. (6.90), and the exact solution, Eq. (6.91), is shown in Fig. 6.17. As expected, the agreement becomes better as time increases. If particular interest is centered on small velocities near zero time, a more accurate solution could be found by reducing the range of integration.

The result of an alternate approach is also shown in Fig. 6.17. A solution is assumed in the form of Eq. (6.89), and constant C is adjusted to make the starting slope of the solution curve correct. In other words, constant C is made equal to $(kg/M)^{1/2}$. This mode of approximation is seen to be relatively poor.

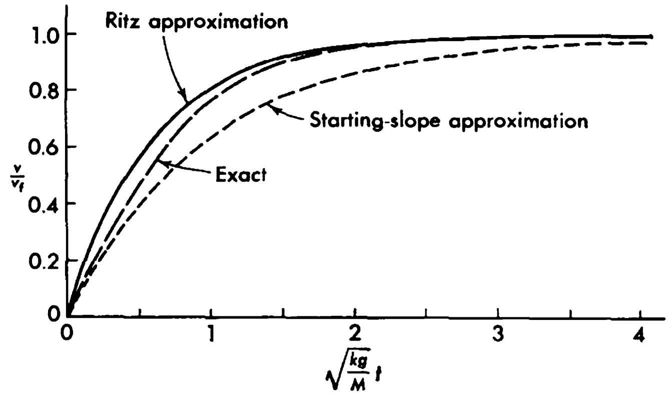


FIG. 6.17. Solution for Example 6.14 found by Ritz method and compared with exact solution. Another inferior approximation is also shown.

6.7. Principle of Harmonic Balance. Systems having oscillatory solutions are often analyzed by any of a variety of methods differing but slightly and based on what is sometimes called the principle of harmonic balance. The idea here is that oscillations in a nonlinear system are rarely simple-harmonic functions of time but often are periodic, or nearly so. A periodic oscillation can be expressed as a Fourier series of sine and cosine components. In many cases, only the component of fundamental frequency, and perhaps one or two additional components, is of significant amplitude. According to the principle of harmonic balance, an approximate solution of first-order accuracy is obtained if only the fundamental component is considered and the solution is adjusted to satisfy all terms of fundamental frequency in the equation. Better approximations may be obtained if, in addition to the fundamental component, higher-order components are also accounted for and are adjusted to satisfy all terms at their respective frequencies.

The principle of harmonic balance arises naturally from a consideration of the Ritz method as applied to an oscillatory system. The assumed oscillatory solution might be of the form

$$x = C_1 \cos \omega t + C_m \cos (n\omega t + \theta_m)$$

with only the two angular frequencies ω and $n\omega$ considered. With a nonlinear equation, the residual ϵ will contain components of angular frequencies ω and $n\omega$, together with other components as well. Because of the orthogonality of the circular trigonometric functions, the integrals of the Ritz method

$$\int_0^{2\pi} \epsilon(t) \cos \omega t \, d(\omega t) = 0 \qquad \text{and} \qquad \int_0^{2\pi} \epsilon(t) \cos n\omega t \, d(\omega t) = 0$$

result in the requirement that those coefficients of terms in $\cos \omega t$ and $\cos n\omega t$ appearing in the residual must individually sum to zero. This is precisely the requirement of the principle of harmonic balance.

A somewhat different way of showing the operation of the principle is by studying the second-order equation used in discussing the method of variation of parameters, Sec. 6.4. This equation is

$$\ddot{x} + \omega_0^2 x + \mu\phi(x,\dot{x},t) = 0 \tag{6.92}$$

where ω_0^2 is a positive constant, function $\phi(x,\dot{x},t)$ contains terms causing difficulty in solution, and μ is a small parameter. A solution for this equation is sought in the form

$$x = A \cos (\omega_0 t + \theta) = A \cos \psi \tag{6.93}$$

where both A and θ may be functions of time. The derivatives of x are

$$\begin{aligned} \dot{x} &= \dot{A} \cos \psi - (\omega_0 + \dot{\theta})A \sin \psi \\ \ddot{x} &= [\ddot{A} - (\omega_0 + \dot{\theta})^2 A] \cos \psi - 2(\omega_0 + \dot{\theta})\dot{A} \sin \psi \end{aligned}$$

The same kind of assumptions on the magnitudes of $\dot{A}$ and $\dot{\theta}$ are made here as were made in the discussion of variation of parameters. The assumption is that parameter μ is small enough for the relative change in amplitude or phase during one cycle to be small, so that $|\dot{A}/A|(2\pi/\omega_0) \ll 1$ and $|\dot{\theta}|/\omega_0 \ll 1$.

Substitution of the derivatives into Eq. (6.92) gives

$$\begin{aligned} [\ddot{A} - (\omega_0 + \dot{\theta})^2 A] \cos \psi - 2(\omega_0 + \dot{\theta})\dot{A} \sin \psi + \omega_0^2 A \cos \psi \\ = -\mu\phi(A \cos \psi, -\omega_0 A \sin \psi, t) \end{aligned} \tag{6.94}$$

Since function ϕ on the right side of this equation is multiplied by the small parameter μ, when $\dot{x}$ is used in ϕ only the largest term of $\dot{x}$ is significant and thus only $-\omega_0 A \sin \psi$ is retained.

On the left side of Eq. (6.94) are sine and cosine terms of fundamental frequency. Coefficients of these terms may vary but are required to do so relatively slowly, and so the product remains very nearly a sine or cosine function. In general, function ϕ on the right side will also contain similar sine and cosine terms of fundamental frequency. For simplicity, function ϕ can be broken into parts as

$$\phi = \Phi_c \cos \psi + \Phi_s \sin \psi + \phi_r \tag{6.95}$$

where Φ_c is the coefficient of all terms with $\cos \psi$, Φ_s is the coefficient of all terms with $\sin \psi$, and ϕ_r is a function containing all the remaining terms. According to the principle of harmonic balance, terms on the left side of the equation with $\cos \psi$ must equal those on the right side in $\cos \psi$. A similar equality must exist between terms in $\sin \psi$. The equating of these terms gives

$$\begin{aligned} \cos \psi: \quad & \ddot{A} - (\omega_0 + \dot{\theta})^2 A + \omega_0^2 A = -\mu\Phi_c \\ \sin \psi: \quad & -2(\omega_0 + \dot{\theta})\dot{A} = -\mu\Phi_s \end{aligned} \tag{6.96}$$

The assumptions on the size of $\dot{A}$ and $\dot{\theta}$ can be written

$$|\ddot{A}| \ll |\dot{A}| \frac{\omega_0}{2\pi} \ll |A| \left(\frac{\omega_0}{2\pi}\right)^2$$
$$|\dot{\theta}| \ll \omega_0$$

The use of these assumptions allows Eq. (6.96) to become

$$\begin{aligned} \dot{\theta} &= \frac{\mu\Phi_c}{2\omega_0 A} \\ \dot{A} &= \frac{\mu\Phi_s}{2\omega_0} \end{aligned} \tag{6.97}$$

Coefficients Φ_c and Φ_s can be found from function ϕ by the usual process of Fourier analysis. If both sides of Eq. (6.95) are multiplied by $\cos \psi$ and integrated with respect to ψ over a complete period, the result is

$$\int_0^{2\pi} \phi \cos \psi \, d\psi = \int_0^{2\pi} \Phi_c \cos^2 \psi \, d\psi + \int_0^{2\pi} \Phi_s \sin \psi \cos \psi \, d\psi + \int_0^{2\pi} \phi_r \cos \psi \, d\psi$$

Only the first of the three integrals on the right differs from zero, and it has the value $\pi\Phi_c$. Thus, the coefficient is found as

$$\Phi_c = \frac{1}{\pi} \int_0^{2\pi} \phi \cos \psi \, d\psi$$

In a similar manner, Φ_s is found as

$$\Phi_s = \frac{1}{\pi} \int_0^{2\pi} \phi \sin \psi \, d\psi$$

Actually, of course, if $\dot{A}$ and $\dot{\theta}$ are not actually zero, these integrals are not quite correct. If $\dot{A}$ and $\dot{\theta}$ have the assumed smallness, however, the error is small. These coefficients used in Eq. (6.97) give the result

$$\dot{A} = \frac{\mu}{2\pi\omega_0}\int_0^{2\pi} \phi(A\cos\psi, -\omega_0 A\sin\psi, t)\sin\psi\, d\psi$$
$$\dot{\theta} = \frac{\mu}{2\pi\omega_0 A}\int_0^{2\pi} \phi(A\cos\psi, -\omega_0 A\sin\psi, t)\cos\psi\, d\psi \tag{6.98}$$

These relations are the same as Eq. (6.37) found by variation of parameters.

The conclusion is that for Eq. (6.92), at least, the two quite different methods of variation of parameters and that based on harmonic balance lead to the same result to this first order of approximation.

Sometimes this process is used to replace the original nonlinear equation, Eq. (6.92), with a linear equation which has the same solution to the first-order approximation. This procedure is given the name of equivalent linearization. The price paid for obtaining a linear equation is that the coefficients of this equation are not constants but are variables, changing with amplitude or time.

The assumed form for the linear equation is

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0 \tag{6.99}$$

Its coefficients γ and ω^2 are to be adjusted to give the same first-order solution as Eq. (6.92). The solution is of the form of Eq. (6.93). When this solution and its derivatives are substituted into Eq. (6.99) the result is

$$[\ddot{A} - (\omega_0 + \dot{\theta})^2 A]\cos\psi - 2(\omega_0 + \dot{\theta})\dot{A}\sin\psi + 2\gamma\dot{A}\cos\psi$$
$$-2\gamma(\omega_0 + \dot{\theta})A\sin\psi + \omega^2 A\cos\psi = 0$$

Again, cosine and sine terms are collected as

$$\cos\psi: \qquad \ddot{A} - (\omega_0 + \dot{\theta})^2 A + 2\gamma\dot{A} + \omega^2 A = 0$$
$$\sin\psi: \qquad -2(\omega_0 + \dot{\theta})\dot{A} - 2\gamma(\omega_0 + \dot{\theta})A = 0$$

The second of these gives the value for γ as

$$\gamma = -\frac{\dot{A}}{A}$$

Subject to the assumption $|\dot{A}| \ll |A|(\omega_0/2\pi)$, there is then the further requirement $|\gamma| \ll (\omega_0/2\pi)$. This condition used in the first equation above gives the result

$$\omega^2 = (\omega_0 + \dot{\theta})^2$$

Thus, Eq. (6.99) may be rewritten as

$$\ddot{x} - \frac{2\dot{A}}{A}\dot{x} + (\omega_0 + \dot{\theta})^2 x = 0 \tag{6.100}$$

where the coefficients are variables and must be found from Eq. (6.98).

While Eq. (6.100) is a linear equation, it is not at all unlikely that the variable coefficients may prevent a solution being found for it in any simple manner.

Example 6.15. Nonlinear Oscillator with Damping

A nonlinear oscillator with damping may be described by the equation

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x + hx^3 = 0 \tag{6.101}$$

This is a combination of Eq. (6.40) for Example 6.5 and Eq. (6.46) for Example 6.6. Replace this equation with an equivalent linear equation.

The equivalent linear equation is of the form of Eq. (6.100), with a solution of the form of Eq. (6.93). From Eq. (6.98), $\dot{A}$ and $\dot{\theta}$ are found as

$$\dot{A} = \frac{1}{2\pi\omega_0}\int_0^{2\pi} (hA^3\cos^3\psi - 2\alpha\omega_0 A\sin\psi)\sin\psi\, d\psi = -\alpha A$$

$$\dot{\theta} = \frac{1}{2\pi\omega_0 A}\int_0^{2\pi} (hA^3\cos^3\psi - 2\alpha\omega_0 A\sin\psi)\cos\psi\, d\psi = \frac{3hA^2}{8\omega_0}$$

The equivalent linear form for Eq. (6.101) is therefore

$$\ddot{x} + 2\alpha\dot{x} + \left(\omega_0 + \frac{3hA^2}{8\omega_0}\right)^2 x = 0 \tag{6.102}$$

Evidently this equation represents an oscillation with both amplitude and frequency varying with respect to time. The amplitude A decays and ultimately disappears as time progresses. The value of the frequency similarly decreases as the amplitude decays with increasing time, with ω approaching the limiting value ω_0.

Example 6.16. van der Pol Equation

The van der Pol equation of Examples 3.3 and 3.6 is

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0 \tag{6.103}$$

Replace this equation with an equivalent linear form.

Values of $\dot{A}$ and $\dot{\theta}$ are found from Eq. (6.98) as

$$\begin{aligned}\dot{A} &= -\frac{\epsilon}{2\pi}\int_0^{2\pi} (1 - A^2\cos^2\psi)(-A\sin\psi)\sin\psi\, d\psi \\ &= \epsilon\frac{A}{2}\left(1 - \frac{A^2}{4}\right)\end{aligned}$$

$$\dot{\theta} = -\frac{\epsilon}{2\pi A}\int_0^{2\pi} (1 - A^2\cos^2\psi)(-A\sin\psi)\cos\psi\, d\psi = 0$$

Thus, the equivalent linear equation becomes

$$\ddot{x} - \epsilon\left(1 - \frac{A^2}{4}\right)\dot{x} + x = 0 \tag{6.104}$$

Evidently a steady-state oscillation may exist if $A = \pm 2$, which is the amplitude of the limit cycle found in Example 3.3.

6.8. Summary. Several procedures have been described for finding approximate solutions for a nonlinear differential equation by an analytical method. All these methods are limited in their application to those equations in which the degree of nonlinearity is not too large. The solutions obtained in this way are always only approximate, and it is often difficult to evaluate the error in any given solution.

The perturbation method and its modification, the reversion method, account for the effect of nonlinearity by additive correction terms. This method is relatively straightforward, at least where linear terms as well as nonlinear terms appear in the differential equation. It is quite useful where solution in series form readily expresses the nonlinear effects. Where the solution is oscillatory, however, secular terms with indefinitely increasing amplitude may arise, and the procedure must be modified to eliminate such terms. An oscillation with amplitude or phase changing with time is not easily handled by the perturbation method. The method may be applied successively in order to obtain additional terms in the series solution and thereby to achieve better accuracy. Each successive application, however, generally becomes significantly more tedious.

When the solution of the nonlinear equation is an oscillation having its amplitude or phase changing with time, but doing so relatively slowly, the method of variation of parameters is useful. Slow variation is required because usually the details of the solution can be worked out only by using values of certain of the quantities as averaged over a cycle of the oscillation.

The Galerkin and Ritz methods require an initial assumption of the form of an approximate solution. The residual is found by substituting the assumed solution into the differential equation, and parameters in the solution are adjusted to minimize some property of this residual. Some knowledge of the equation and its solution is obviously required in order to make an appropriate choice for the form of the solution. On the other hand, a rather simple form may sometimes be used, in place of the much more complicated form of a more exact solution. The mathematical operations of the Ritz method are the simpler to apply and often yield results of adequate accuracy. The methods can be used with equations having either oscillatory or nonoscillatory solutions. They are applicable to certain equations in which a linear term is lacking and to which other methods of solution cannot be applied. The approximate solution is fitted only within a certain chosen range of the independent variable, and if a wide range is required in the final solution, it may have to be built up of several subranges. In theory, enough terms can be used in the assumed solution to give a high degree of accuracy, but in practice this may not be feasible because of mathematical complexities.

Finally, oscillatory solutions can be obtained by methods based on the principle of harmonic balance. This principle asserts that the most important criterion for an approximate oscillatory solution is that the fundamental component be adjusted to fit the conditions of the equation as closely as possible. The application of this principle to a second-order equation is easily shown to give the same result as the method of variation of parameters and to follow directly from the Ritz method. Sometimes the principle is used to replace a nonlinear equation with a linear equation, approximately equivalent, but having variable coefficients.

In the following chapter, this principle is used in several ways in the analysis of nonlinear systems driven by oscillatory driving forces.

As is true for virtually all methods of attacking nonlinear problems, no one of these methods gives all the information that may be desired about the problem. Often it is necessary to use several of the methods, each method yielding certain information characteristic of that particular method. In Example 6.7, for instance, the perturbation method gives information about the steady-state operation of the negative-resistance oscillator but is of little use in determining how the oscillation builds up to the steady state. The method of variation of parameters allows the growth period to be studied but gives steady-state information only to the first order of approximation.

CHAPTER 7

FORCED OSCILLATING SYSTEMS

7.1. Introduction. In the preceding chapter on analytical methods, several such methods are described and applications to examples given. While most of the methods are applicable to systems with or without driving forces, all the examples of that chapter are concerned with free systems having no externally applied driving forces. In the present chapter, nonlinear systems driven with periodic oscillating forces are considered. Usually either the perturbation method or some method based essentially on the principle of harmonic balance is most useful in considering forced oscillatory systems. In applying these methods, the system must be described by appropriate mathematical relations, and solutions are found in the form of equations. Furthermore, in this chapter the discussion is confined entirely to steady-state conditions. Transient conditions in a driven nonlinear system may be of considerable interest but are also usually quite difficult to determine.

In contrast with the preceding chapters, which have been concerned primarily with techniques for analyzing nonlinear problems, the present chapter deals more with the application of techniques already discussed to certain types of problems with interesting solutions. A number of interesting phenomena occur in systems of this sort, phenomena which are peculiar to nonlinear systems and which cannot occur in strictly linear systems. These phenomena include such things as appearance of discontinuous jumps in amplitude, generation of harmonics of the driving frequency, generation of subharmonics, and the entrainment of frequency of a self-oscillator. Phenomena of these kinds occur in relatively complex systems, where a detailed analysis is virtually impossible. In this chapter, only simple systems are considered. Analysis of these simple systems provides the basis for insight into the operation of much more complicated physical systems.

It is characteristic of the analysis of forced nonlinear systems that even relatively simple nonlinear differential equations lead to algebraic equations of considerable complexity. It is not at all easy to obtain information from these algebraic equations in a form easy to visualize.

Often it is necessary to resort to numerical or graphical methods in order to obtain any information whatever. The examples in the following sections are intended to be relatively simple, but even so the algebraic complexities are quite apparent.

7.2. Principle of Harmonic Balance. The application to a driven system of methods based on the principle of harmonic balance is similar to their application to a free system as described in Sec. 6.7. Essentially, only the fundamental component of a nonsinusoidal variation is considered, and parameters are adjusted so that this fundamental component fits the nonlinear equation as well as possible.

The type of equation under consideration is the following

$$\ddot{x} + \omega_0^2 x + \mu\phi(x,\dot{x}) = G(t) \tag{7.1}$$

where ω_0^2 is a positive constant, $\phi(x,\dot{x})$ is some kind of nonlinear function, and $G(t)$ represents a forcing function. Usually $G(t)$ is taken as a simple harmonic variation, $G(t) = G \cos \omega_1 t$, where angular frequency ω_1 is generally different from ω_0. If Eq. (7.1) applies to a mechanical system where x is a displacement, the dimensions of $G(t)$ are actually not directly force, but rather force per unit mass. Small parameter μ is associated with the nonlinear function.

Any practical system always contains dissipation, although a linear dissipative term in $\dot{x}$ is not specifically written in Eq. (7.1). A solution for the equation would therefore be expected, in general, to involve something analogous to a transient part, which eventually disappears, and a steady-state part, which remains. The solution for a nonlinear equation cannot, of course, be separated into two parts so easily as this, because the principle of superposition is not valid. The steady-state part of the solution is the thing of primary interest in all of the present chapter.

If a steady-state solution for Eq. (7.1) is desired, it is common practice to assume a solution in the form

$$x = A_1 \cos (\omega_1 t + \theta_1) \tag{7.2}$$

where A_1 and θ_1 are constants to be determined. These constants are found by substituting the assumed solution into the equation and adjusting the constants to satisfy the equation so far as terms of fundamental frequency ω_1 are concerned. Terms of frequency other than ω_1 are certain to be present also and are ignored to this order of approximation. This assumed solution is at the frequency of the driving force. In some cases, important solutions of some other frequency occur and are desired, and this possibility must be recognized. Often it is more convenient to allow for a possible phase difference between $G(t)$ and $x(t)$

by putting a phase angle into $G(t)$, writing it as

$$\begin{aligned} G(t) &= G \cos (\omega_1 t + \theta_0) \\ &= G_c \cos \omega_1 t + G_s \sin \omega_1 t \end{aligned} \tag{7.3}$$

and allowing $x(t)$ to be merely $x = A_1 \cos \omega_1 t$. This procedure simplifies $x(t)$ when it must be differentiated and substituted into the differential equation.

As in all nonlinear problems, some knowledge of the kind of solution desired is almost essential. Even so simple-appearing an equation as Eq. (7.1) has a variety of types of solutions, and no one approach gives them all. It is necessary to adapt the details of the solution somewhat to fit the kind of solution being sought.

Example 7.1. Duffing's Equation, Ferroresonance, Jumps, Harmonics

a. Preliminary Discussion of Circuit. The mechanical system consisting of a mass on a nonlinear spring has already been considered in Examples 4.6, 6.3, 6.5, 6.9, and 6.12, where free oscillation is involved. This same system can be driven by applying to the moving mass a force which varies as a simple-harmonic function of time. The resulting equation becomes

$$\ddot{x} + \omega_0^2 x + hx^3 = G \cos \omega_1 t \tag{7.4}$$

where the driving function is $G \cos \omega_1 t$. Coefficient G is the ratio between the amplitude of the force and the mass of the moving system. Coefficients ω_0^2 and h depend upon the parameters of the system, with h being positive for a hard spring and negative for a soft spring.

Equation (7.4) is a classic equation of the theory of nonlinear systems and is known as Duffing's equation. Besides describing the mechanical system of a mass on a nonlinear spring, it also describes an electrical system consisting of the series combination of a capacitor and inductor, one of which is nonlinear. This circuit is driven by a sinusoidal voltage. Since phenomena associated with the saturation of an iron-core inductor occur frequently in electrical systems, further discussion of this kind of system is in order.

The magnetization curve for an iron core surrounded by a coil of wire carrying a current is shown in Fig. 7.1. This curve relates the total instantaneous flux ϕ in the core to the instantaneous current i in the coil. If the current varies sinusoidally, a double-valued relation is found, characterized by the hysteresis loop of the figure. Time increases around the loop in the direction of the arrows. If the amplitude of current is changed, the tips of the hysteresis loops trace the magnetization curve. This curve typically flattens, and the iron is said to saturate, for large values of current. There is usually a region of reverse curvature near the origin. The shape of the curve of Fig. 7.1 is typical of many kinds of iron used in rotating electrical machinery and transformers. It is, of course, quite different from special types of iron developed for magnetic amplifiers and described in Example 4.4.

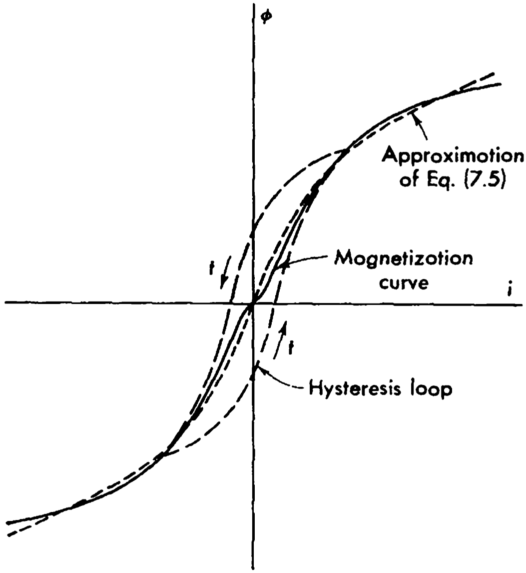


FIG. 7.1. Magnetization curve and hysteresis loop for iron core of Example 7.1. An approximation to the magnetization curve is shown. Arrows indicate direction of increasing time.

The magnetization curve of Fig. 7.1 can be described approximately by the equation

$$i = \frac{N\phi}{L_0} + a\phi^3 \tag{7.5}$$

where N is the number of turns on the coil carrying current i and L_0 and a are positive constants. Constant L_0 is the self-inductance which would exist if saturation effects were absent, so that $a = 0$. An equation with only a single nonlinear term, as Eq. (7.5), cannot fit the magnetization curve of Fig. 7.1 with its multiple curvature. The best that can be done is to achieve an approximate fit as exemplified by the dotted curve of the figure. It is evident, therefore, that Eq. (7.5) is far too simple to describe the phenomena associated with magnetic saturation of an iron core. This equation cannot account for saturation and at the same time fit the curve well near the origin. Furthermore, it yields a single-valued curve, and hysteresis is ignored completely. Hysteresis can be shown to represent a loss of energy in the iron-core inductor, and thus, if hysteresis is ignored, a mechanism of dissipation is not accounted for. Even though Eq. (7.5) describes only certain features of the operation of an iron-core inductor, it does allow prediction of phenomena observed in circuits employing this kind of inductor.

If an iron-core inductor is used in the circuit of Fig. 7.2, the circuit equation can be written

$$\frac{q}{C} + e_L = E \sin \omega_1 t$$

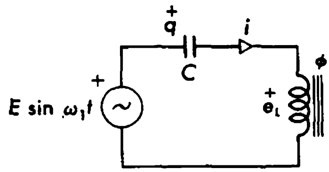


FIG. 7.2. Circuit with iron-core inductor for Example 7.1.

where q is the instantaneous charge on the capacitor, C is the capacitance, e_L is the instantaneous voltage across the inductor, and $E \sin \omega_1 t$ is the driving voltage. The rate of change of charge must be the current i in the circuit, and this is related to the instantaneous flux ϕ in the core by Eq. (7.5), so that

$$\dot{q} = i = \frac{N\phi}{L_0} + a\phi^3$$

Further, the instantaneous voltage across the inductor is proportional to the rate of change of flux, $e_L = N\, d\phi/dt$. If the circuit equation is differentiated with respect to time and these relations substituted, the result is

$$\ddot{\psi} + \omega_0^2\psi + h\psi^3 = E\omega_1 \cos \omega_1 t \tag{7.6}$$

where $\psi = N\phi$, $\omega_0^2 = 1/L_0C$, and $h = a/N^3C$. Equation (7.6) is of the same form as Eq. (7.4), Duffing's equation.

A somewhat similar saturation phenomenon occurs in certain types of capacitors. The charge shows a saturation effect as voltage across the capacitor is increased, and the two are related by an equation analogous to Eq. (7.5). A circuit with such a nonlinear capacitor and a linear inductor would also be governed by Duffing's equation.

b. Discontinuous Jumps in Amplitude as Frequency Varies. One of the kinds of phenomena appearing in systems governed by Duffing's equation is that of discontinuous jumps in amplitude as either the frequency or the magnitude of the driving function is varied smoothly and continuously. At the same time, harmonics of the driving frequency are generated.

A solution for Eq. (7.4) is sought in the form of Eq. (7.2) as $x = A \cos \omega_1 t$. In Eq. (7.4), no term in $\dot{x}$, representing dissipation, is included, and thus it is expected that x will be exactly in, or out of, phase with the driving force. Therefore, no phase angle is included. The assumed solution is differentiated and substituted into Eq. (7.4), giving

$$\left(-\omega_1^2 A + \omega_0^2 A + \frac{3hA^3}{4}\right) \cos \omega_1 t + \frac{hA^3}{4} \cos 3\omega_1 t = G \cos \omega_1 t$$

where the identity for $\cos^3 \omega_1 t$ has been used. It is now required that terms of fundamental frequency satisfy the equation, and thus all terms in $\cos \omega_1 t$ are collected. The one term in $\cos 3\omega_1 t$ is ignored until later. The coefficients are

$$\cos \omega_1 t: \qquad -\omega_1^2 A + \omega_0^2 A + \frac{3hA^3}{4} = G \tag{7.7}$$

This equation is a cubic equation in the unknown amplitude A. Evidently it will always yield one real value of A, and there may be as many

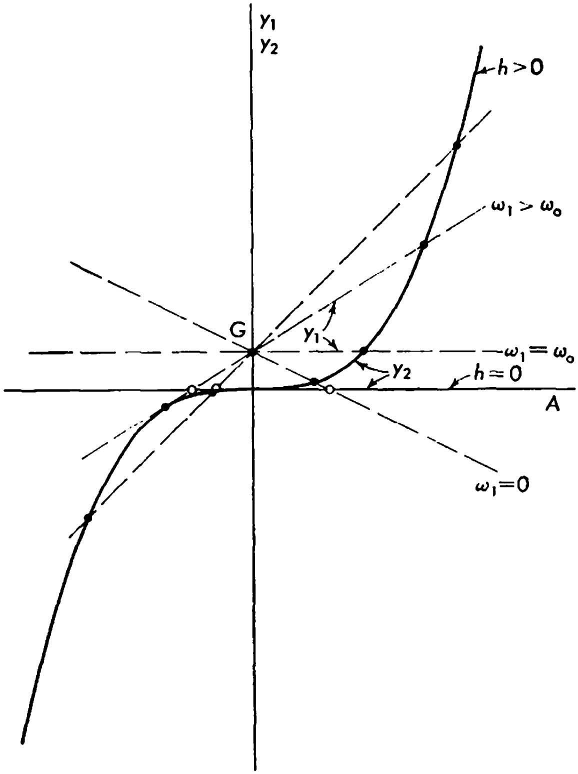


FIG. 7.3. Construction for determining the amplitude A from the cubic equation for it in Example 7.1.

as three real values. It is thus possible to obtain the multiple-valued solution needed to allow jumps in amplitude.

Equation (7.7) is most easily investigated through a graphical construction based upon separating it into the two parts,

$$y_1 = (\omega_1^2 - \omega_0^2)A + G$$

$$y_2 = \frac{3hA^3}{4}$$

and requiring that $y_1 = y_2$ to determine a solution for A. Curves can be plotted for y_1 and y_2 as functions of A, the intersection of these curves locating points where $y_1 = y_2$. Curves of this sort are shown in Fig. 7.3. A particular value of G has been chosen here, but ω_1 is allowed to take on different values. Several particular curves for y_1 are shown as straight lines passing through the point $A = 0$, $y_1 = G$ with varying slopes. Similarly, two curves are shown for y_2, the case of $h = 0$ for a linear system and that for $h > 0$ for a system with a hard spring or a saturating inductor.

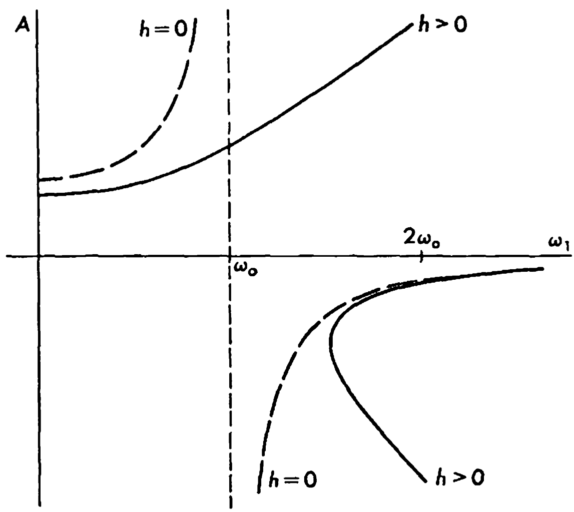


FIG. 7.4. Amplitude as a function of driving frequency for constant amplitude of driving function in Example 7.1. Both the nonlinear and linear cases are shown.

Intersections between curves for y_1 and y_2 locate possible values for A, and values found this way are shown in Fig. 7.4. There are two branches for the curves of Fig. 7.4, one with positive and one with negative values of A. The difference in sign merely represents a reversal in phase and indicates that the solution is either in, or out of, phase with the driving function. Curves of Fig. 7.4 show that A is a single-valued function of ω_1 if the system is linear and $h = 0$ but that A may be triple-valued if $h > 0$. The three values correspond to three real roots of Eq. (7.7). Only a single real value of A results if there are two complex roots.

The curves of Fig. 7.4 are more easily interpreted in terms of a familiar phenomenon if only the magnitude of A is plotted, as in Fig. 7.5. The curve for $h = 0$ is then easily recognized as being the familiar resonance curve for a linear system without dissipation. Resonance occurs when $\omega_1 = \omega_0$, and with no dissipation the amplitude A at resonance becomes infinite. For frequencies either side of resonance, the amplitude is reduced.

The curve of Fig. 7.5 for the nonlinear system with $h > 0$ can now be interpreted as being also a kind of resonance curve. Instead of having its two branches located about a vertical spine, its spine is shifted off to the right. Qualitatively, what happens is the following: For a system having a hard spring, with $h > 0$, the effective stiffness becomes greater with increasing deflection. Thus, as the driving frequency increases from a value much below ω_0, resonance is approached and the amplitude increases. The increased amplitude leads to an increased apparent stiffness, which pushes the resonance frequency higher. This process continues indefinitely in the dissipationless system of Fig. 7.5 and results in the skewed resonance curve.

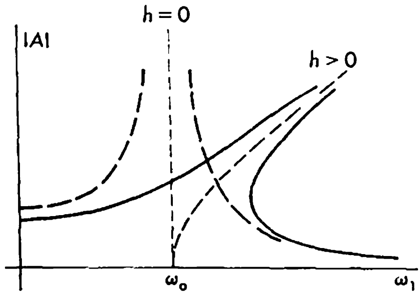


FIG. 7.5. Resonance curves for Example 7.1 with constant amplitude of driving function. Both the nonlinear and linear cases are shown.

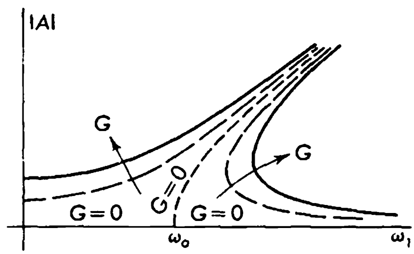


FIG. 7.6. Family of nonlinear resonance curves for Example 7.1 showing effect of changing amplitude of driving function.

The curves of Figs. 7.3 to 7.5 are drawn for a particular value of forcing function G. Similar curves may be obtained if G is changed, and Fig. 7.6 illustrates a family of curves with G allowed to vary. These curves are all located about the same spine, given by Eq. (7.7) with $G = 0$, which can be written

$$\omega_1^2 = \omega_0^2 + \frac{3hA^2}{4} \tag{7.8}$$

This agrees with Eq. (6.28) found in Example 6.3 for this system in free oscillation.

In any physical system, dissipation is always present. The original equation should, therefore, contain a dissipation term to be realistic. The inclusion of a term in $\dot{x}$ markedly complicates the analysis, however. Furthermore, in an electrical circuit such as Fig. 7.2, dissipation is present not only in the ohmic resistance of the coil but also because of eddy-current and hysteresis losses in the iron core. Accurate formulation of these effects is quite difficult. Thus, only a qualitative discussion of the system with dissipation is presented at first.

The effect of dissipation in a linear resonant circuit is well known. If the amount of dissipation is relatively small, its effect is important only near the resonance frequency. There it prevents an infinite amplitude of the solution at resonance and provides a rounded peak for the resonance curve near resonance. The height of the peak of the curve is reduced by additional dissipation. It would be expected that qualitatively much the same effects would be produced by small dissipation in a nonlinear system. Thus, the curve of Fig. 7.5 with no dissipation is modified to that of Fig. 7.7 by small dissipation. A rounded peak, whose location depends upon the relative amount of dissipation, appears on the skewed resonance curve.

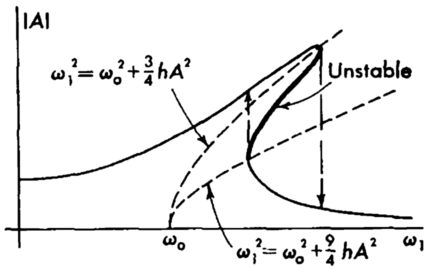


FIG. 7.7. Nonlinear resonance curve for Example 7.1, with constant amplitude of driving function and including effects of small dissipation. Conditions for discontinuous jumps in amplitude are shown.

The curve of Fig. 7.7 explains the possible appearance of discontinuous jumps in amplitude. If a given value of G is used and ω_1 is chosen initially well below ω_0, the resulting amplitude is determined by a point on the curve well to the left. If ω_1 is now increased, the value of $|A|$ increases as the resonance curve is traversed. When the peak of the curve is reached and ω_1 is further increased, $|A|$ must jump discontinuously to a smaller value as shown by the dashed line. Further increase in ω_1 leads to further slow reduction in $|A|$. If now ω_1 is reduced, $|A|$ rises slowly as it follows the lower branch of the curve. It remains on the lower branch until the curve takes on an infinite slope. Further reduction in ω_1 requires that $|A|$ jump discontinuously to the upper branch, as indicated by the second dashed line, after which it remains on the upper branch. Thus, there is a range of frequencies, located between the dashed lines of the figure, for which either of two different values of $|A|$ may exist. Which value does exist depends upon the past history of the system.

Between the dashed lines of Fig. 7.7 the curve actually shows three possible values of $|A|$, while only two have been discussed. The third value, given by the heavy center line of the figure, is unstable and is never found in experiments. A discussion of the exact reason for this instability must be deferred until Sec. 10.5, but a qualitative argument can be based on Fig. 7.6. It is evident from an examination of this figure in the region asserted to be unstable that an increase in driving function G leads to a reduction in amplitude $|A|$. Just the opposite kind of change is characteristic of stable systems, where an increase in driving function leads to an increased response. Thus, it might be expected that instability will occur as described.

The condition for the downward jump of $|A|$ in Fig. 7.7 is determined by the position of the peak of the skewed resonance curve, and this depends upon the relative dissipation in the system. About all that can be said about the location of this point from the present discussion is that, if the dissipation is relatively small, the point will be located nearly on the spine of the curve, given by Eq. (7.8). If dissipation is increased, the height of the resonance peak is reduced. If dissipation becomes too great, the triple-valued part of the curve disappears and no jump phenomenon is possible.

The point on the curve of Fig. 7.7 where the upward jump of $|A|$ occurs is the point where the resonance curve has infinite slope. This point is well away from the peak of the curve and is but little influenced by dissipation if the relative dissipation is small. Thus, the point of upward jump can be located approximately by differentiating Eq. (7.7). The condition is that $dA/d\omega_1 = \infty$, or, more simply, $d\omega_1^2/dA = 0$. This latter derivative is merely

$$\frac{d\omega_1^2}{dA} = \frac{G}{A^2} + \frac{3hA}{2} = 0$$

This may be written $G = -3hA^3/2$ and used in Eq. (7.7) to give

$$\omega_1^2 = \omega_0^2 + \frac{9hA^2}{4} \tag{7.9}$$

Equations (7.8) and (7.9) are plotted in Fig. 7.7 and show the approximate loci of jumps downward or upward as driving frequency ω is changed, for small dissipation. The downward jump, occurring at the higher frequency, is always the more spectacular.

It should be remarked that all the foregoing discussion of Duffing's equation, as well as that which follows, has been concerned with the case of parameter h being positive. This case corresponds to that observed in most physical systems. If the sign of h is allowed to reverse, as does sometimes occur, much the same kind of phenomena exist. The primary difference is that now the nonlinear resonance curve is skewed toward lower, rather than higher, frequencies.

c. Analysis with Small Dissipation. If an analysis of the system with a small linear dissipation term is attempted, it would proceed as follows: The modified equation for the system is

$$\begin{aligned} \ddot{x} + 2\alpha\dot{x} + \omega_0^2 x + hx^3 &= G\cos(\omega_1 t + \theta) \\ &= G_c \cos\omega_1 t + G_s \sin\omega_1 t \end{aligned} \tag{7.10}$$

where ω_0^2 and h are the same coefficients as in Eq. (7.4), α is the dissipation coefficient, and G_c and G_s are the amplitudes of the cosine and sine components of the forcing function. A solution is assumed of the form

$$x = A \cos \omega_1 t$$

This simple form is easy to substitute into the differential equation, and the use of cosine and sine components in $G(t)$ allows for a phase difference between the solution and the forcing function. Upon substitution into Eq. (7.10) and collection of terms in $\cos \omega_1 t$ and $\sin \omega_1 t$, the resulting pair of algebraic equations is

$$\cos \omega_1 t: \qquad -\omega_1^2 A + \omega_0^2 A + \frac{3hA^3}{4} = G_c$$

$$\sin \omega_1 t: \qquad -2\alpha\omega_1 A = G_s$$

The square of the amplitude of the forcing function is $G^2 = G_c^2 + G_s^2$ and thus

$$\left[(\omega_0^2 - \omega_1^2)A + \frac{3hA^3}{4}\right]^2 + [2\alpha\omega_1 A]^2 = G^2 \tag{7.11}$$

This equation determines the amplitude of the solution in terms of parameters of the original equation. If there is no dissipation, $\alpha = 0$ and it reduces to Eq. (7.7) as it should. This is a cubic equation in A^2, with all terms present, and therefore it cannot be studied by the graphical process used with Eq. (7.7). Some pertinent information can be found about it, however.

The condition for a discontinuous jump in amplitude is $dA/d\omega_1 = \infty$, or, more simply, $d\omega_1^2/dA = 0$, as before. After differentiation and simplification, this condition for jump is found to be

$$A\left\{\left[(\omega_0^2 - \omega_1^2) + \frac{3hA^2}{4}\right]\left[(\omega_0^2 - \omega_1^2) + \frac{9hA^2}{4}\right] + [2\alpha\omega_1]^2\right\} = 0 \tag{7.12}$$

Since $A = 0$ is a trivial case, the quantity in the braces must be zero. If $\alpha = 0$, so that there is no dissipation, this requirement gives the two conditions of Eqs. (7.8) and (7.9) already found, and plotted in Fig. 7.7 and again in Fig. 7.8. If $\alpha > 0$, so that dissipation is present, Eq. (7.12) gives a single curve also plotted in Fig. 7.8. This curve falls within the region between the two curves for $\alpha = 0$. The minimum value for ω_1 on this new curve is somewhat greater than ω_0 and can be found from the condition that, once more, $d\omega_1^2/dA = 0$. Upon evaluating this derivative and carrying out the necessary substitutions, the condition is found to be

$$\omega_1^2 = \omega_0^2 + \frac{9hA^2}{8} \tag{7.13}$$

reminiscent of Eqs. (7.8) and (7.9). Solution for A from Eqs. (7.12) and (7.13) gives, following considerable algebra,

$$hA^2 = \frac{16\alpha^2}{3}\left[1 \pm \left(1 + \frac{\omega_0^2}{3\alpha^2}\right)^{\frac{1}{2}}\right] \tag{7.14}$$

If dissipation is sufficiently small so that $\omega_0^2/3\alpha^2 \gg 1$, this equation is approximately

$$hA^2 = \frac{16\alpha\omega_0}{3(3)^{\frac{1}{2}}} \tag{7.15}$$

From Eqs. (7.15) and (7.13), it is evident that the value of A for the upward jump, and the minimum value of ω_1 associated with it, increases as the dissipation, determined by α, increases.

The whole situation is summarized in Fig. 7.8, which represents the response of a system with small relative dissipation. Curves plotted from Eqs. (7.8), (7.9), and (7.11) to (7.13) are shown, with three values of the amplitude of the driving function, G, being considered. If G is large enough, $G = G_1$, the response is multivalued and jumps in amplitude of the solution may occur. If G has the critical value, $G = G_2$, the curve has a vertical slope at one point but no jump occurs. If G is less than the critical value, as $G = G_3$, there is no point with infinite slope. The accurate plotting of a family of curves such as Fig. 7.8 obviously requires a considerable amount of arithmetic with the various quantities describing the system.

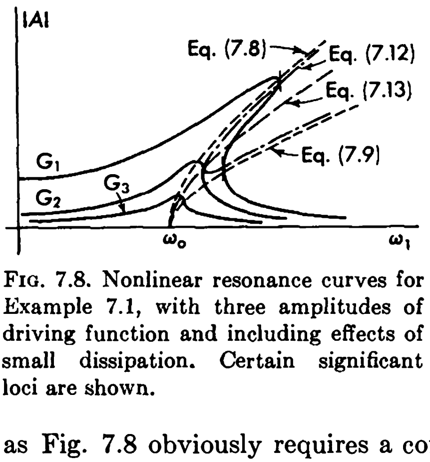


FIG. 7.8. Nonlinear resonance curves for Example 7.1, with three amplitudes of driving function and including effects of small dissipation. Certain significant loci are shown.

A somewhat different approach to this same problem is described in Example 7.6.

d. Generation of Third Harmonic. The foregoing solutions, both with and without dissipation, have been based upon just the fundamental component of what actually is a nonsinusoidal solution for the original nonlinear differential equation. In fact, a third-harmonic term, arising when the assumed solution was substituted into the equation, has been completely ignored. The solution is therefore only approximately correct.

An obvious way of attempting to improve the accuracy of the solution is to assume the form

$$x = A_1 \cos \omega_1 t + A_3 \cos 3\omega_1 t \tag{7.16}$$

having both fundamental- and third-harmonic terms, in place of Eq. (7.2) with just the fundamental. Coefficients A_1 and A_3 must be determined. If this assumed solution, Eq. (7.16), is substituted into Eq. (7.4), applying to the system with no dissipation, and terms in $\cos \omega_1 t$ and $\cos 3\omega_1 t$ are collected, the result is a pair of simultaneous algebraic equations for the coefficients,

$$\cos \omega_1 t: \quad (\omega_0^2 - \omega_1^2)A_1 + \frac{3hA_1^3}{4} + \frac{3hA_1A_3}{4}(A_1 + 2A_3) = G$$

$$\cos 3\omega_1 t: \quad (\omega_0^2 - 9\omega_1^2)A_3 + \frac{3hA_3^3}{4} + \frac{hA_1^2}{4}(A_1 + 6A_3) = 0$$

While, in theory, these equations can be solved for A_1 and A_3, their solution is certainly not simple. Numerical solution is possible in a

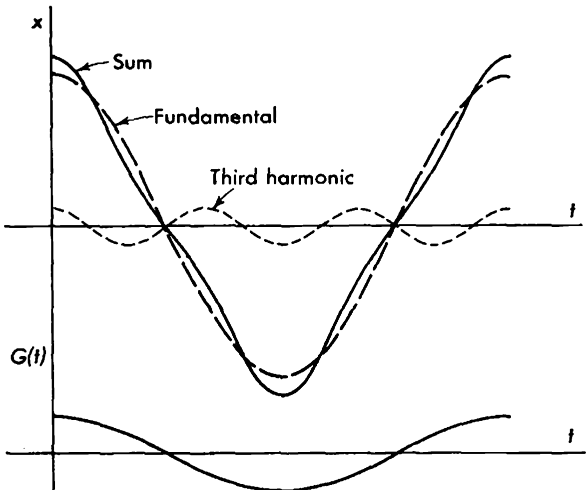


Fig. 7.9. Approximate steady-state solution for Example 7.1, showing presence of third-harmonic component and relation with driving function, for $\omega_1 > \omega_0/3$.

particular case, but a general representation of the solution cannot be obtained.

If conditions are assumed to be such that the third harmonic is relatively small, $A_3/A_1 \ll 1$, the second equation can be rewritten as

$$\frac{A_3}{A_1} = \left(\frac{-hA_1^2}{4}\right) \Big/ (\omega_0^2 - 9\omega_1^2) \tag{7.17}$$

and the first equation becomes the same as Eq. (7.7). Thus, A_1 might be found from Eq. (7.7) as before and A_3 found from Eq. (7.17).

If $\omega_1 > \omega_0/3$, the algebraic signs for A_1 and A_3 are the same, so that the third-harmonic component is of such phase as to accentuate the peaks of the wave, as shown in Fig. 7.9. If conditions are such that A_1

is given by the upper branch of Fig. 7.7, the variation of x is exactly in phase with the driving force. If A_1 is given by the lower branch of Fig. 7.7, x is out of phase with the driving force. If $\omega_1 < \omega_0/3$, A_1 and A_3 are of opposite signs and the peaks of the resulting wave are flattened.

It is worth remarking that if the physical system corresponding to the Duffing equation is a mechanical mass-spring system, variable x represents the deflection. Under typical conditions, the wave representing the deflection is more peaked because of the third-harmonic components. If the physical system is an iron-core inductor and capacitor, x represents the flux in the core. The quantity more easily measured is the voltage across the inductor, which is proportional to the time derivative of the flux. If the flux wave has its peak accentuated, the voltage wave has its peak flattened.

e. Discontinuous Jumps in Amplitude as Force Varies. A further modification of the phenomenon of discontinuous jumps occurs if the driving frequency is held constant but the driving force is varied, so that G varies while ω_1 is constant. By extending the analysis of the preceding sections, it is not difficult to show that, if ω_1 is chosen properly, $|A|$ is related to G by a curve similar to Fig. 7.10, applying for relatively small dissipation. Once again, there is a range of values of G for which three values of $|A|$ are possible. The upper and lower of these values are stable, while the middle value is unstable and is not observed in experiments. Discontinuous jumps in $|A|$ occur if G is varied smoothly and continuously. This phenomenon is considered in more detail in Example 7.6.

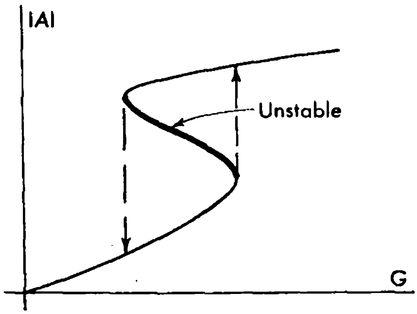


FIG. 7.10. Variation of amplitude with amplitude of driving function of Example 7.1. Driving frequency is constant, and small dissipation is included. Discontinuous jumps are indicated.

An iron-core circuit, such as that of Fig. 7.2, displays discontinuous jumps of the kind described in Figs. 7.8 and 7.10. Since this is a resonance phenomenon associated with a ferromagnetic circuit element, this jump phenomenon is sometimes given the name ferroresonance. This same name is sometimes applied, by extension, to essentially the same phenomenon appearing in any physical system described by Duffing's equation, even though the physical system has no immediate connection with an iron-core electrical circuit.

The discussion of the circuit given here deals only with steady-state conditions which may possibly occur. A theoretical analysis is available in the literature[1] giving information about transient conditions also. This analysis is based on an approach through the methods of harmonic balance and variation of parameters, and the results are presented graphically as a diagram with singular points. The diagram applying to a system with a particular set of numerical parameters is shown in Fig. 7.11. The solution is assumed to be simple-harmonic at the frequency of the driving function. The axes of the diagram represent

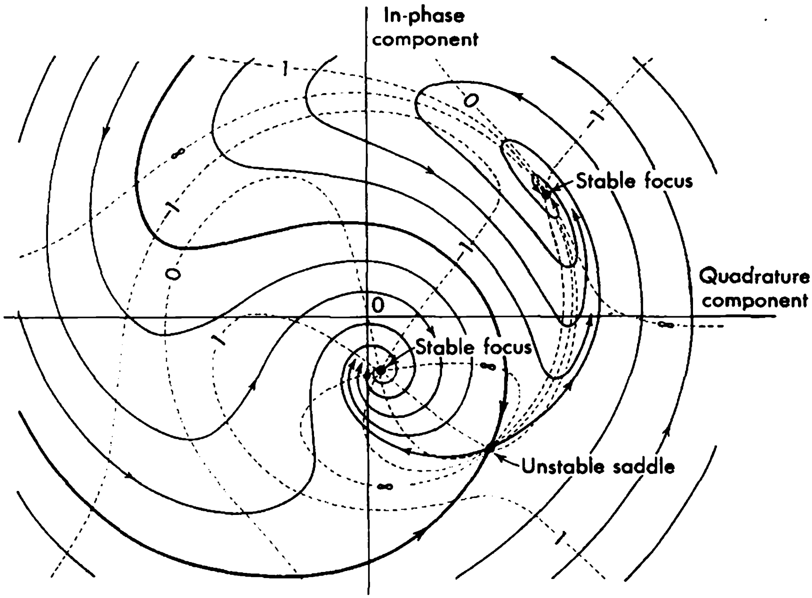


FIG. 7.11. Diagram from Hayashi, showing in-phase and quadrature components of the solution for a system similar to that of Example 7.1, but including the effects of dissipation. Solid curves are solution curves, while dotted curves are isoclines corresponding to the slopes indicated. Time increases in the direction of the arrows. There are two stable conditions having different amplitudes, separated by an unstable condition of intermediate amplitude.

components of the solution which are in phase and in quadrature with the driving function. A line drawn from the origin to any point in the diagram, representing some particular solution, has its length proportional to the amplitude of that solution, and its angle is the phase angle of the solution. The two stable steady-state conditions in the figure correspond to stable focal points, although with other numerical parameters these may be stable nodal points. The unstable condition corresponds to a saddle point. The stable condition of smaller amplitude represents a solution almost exactly out of phase with the driving function; the

[1] See Hayashi, references 28 to 32 in the List of References in the Bibliography at the end of the volume.

solution of larger amplitude lags the driving function by about 50 degrees because dissipation is assumed to be present. Solution curves in the regions between the singularities show what sort of changes in amplitude and phase must take place during jumps in amplitude. Because of inherent limitations in the method of analysis, changes in amplitude must be assumed to take place relatively slowly, with at least several cycles of the oscillation occurring during the change from one state to the other.

Separatrix curves can be located in the diagram, separating initial conditions which lead to one stable state from those which lead to the other. Experimental tests with an electric circuit including an iron-core inductor arranged to allow controllable initial conditions show generally good agreement with the theoretical analysis.

Example 7.2. Temperature Fluctuations in a Lamp Bulb

An incandescent lamp bulb consists of a fine tungsten wire supported in a vacuum and connected in a circuit so that it carries an electric current. If the current is alternating, the temperature of the tungsten filament fluctuates slightly. The equations governing this fluctuation are highly nonlinear.

The resistance of tungsten is nearly proportional to its absolute temperature θ. Thus, the relation between instantaneous values of voltage e across the filament and current i through it is

$$e = c\theta i$$

where c is a constant dependent upon the dimensions of the filament and the properties of tungsten. The instantaneous rate at which electrical energy is supplied to the filament is the product ei, and the rate at which heat energy is supplied is proportional to θi^2, where voltage e has been replaced by product θi. A long filament in a vacuum loses most of its heat energy by radiation, and thus the rate of heat loss is proportional to $\theta^4 - \theta_a{}^4$, where θ_a is the ambient absolute temperature. A filament is usually operated hot enough so that $\theta^4 \gg \theta_a{}^4$, and this condition is assumed. The difference between heat energy supplied and that lost determines the rate of change of heat in the filament itself. This is proportional to its rate of change of temperature, $d\theta/dt$. Thus, the equation for the temperature is

$$\frac{d\theta}{dt} = \dot{\theta} = a\theta i^2 - b\theta^4 \tag{7.18}$$

where a and b are constants dependent upon the geometry and material of the filament.

If a direct current I_s flows in the filament, $\dot{\theta} = 0$ and θ takes on the steady value

$$\theta_s = \left(\frac{aI_s{}^2}{b}\right)^{1/3}$$

If the filament carries an alternating current, its heating is dependent primarily upon the rms value I_r of the current. If the frequency of the current is not too low, the temperature of the filament fluctuates only a small amount about a mean value almost the same as θ_s, where instead of I_s the value I_r is used. The fluctuation in temperature can be found by the principle of harmonic balance.

It is convenient to write the instantaneous alternating current as

$$i = 2^{1/2} I_r \sin \omega t \tag{7.19}$$

where I_r is the rms value of the wave. Each time the current passes through zero, the temperature drops slightly, so that its fluctuation is at twice the frequency of the current. Furthermore, the filament cannot heat and cool instantaneously, and so its temperature fluctuation must lag slightly behind the alternating current. Thus, a reasonable assumption for the instantaneous temperature in the steady state is

$$\theta = \theta_0 - \theta_1 \sin (2\omega t + \phi) \tag{7.20}$$

This form can be expected to hold only if $\theta_1/\theta_0 \ll 1$. Otherwise, the fluctuation will not be essentially sinusoidal, and additional harmonic terms are required in an expression for θ. The choice of negative sign before θ_1 is arbitrary at this point but ultimately proves to be advantageous.

Substitution of Eqs. (7.19) and (7.20) into Eq. (7.18) gives the relation

$$2\omega\theta_1 \cos (2\omega t + \phi) = a[\theta_0 + \theta_1 \sin (2\omega t + \phi)] I_r^2 (1 - \cos 2\omega t) \\ - b\theta_0^4 - 4b\theta_0^3\theta_1 \sin (2\omega t + \phi)$$

where terms in $\theta_0^2\theta_1^2$, $\theta_0\theta_1^3$, and θ_1^4 have been dropped as being negligibly small in comparison with θ_0^4 and $\theta_0^3\theta_1$. This equation can be expanded by the use of appropriate trigonometric identities, and among the terms that result are certain constant terms and certain terms varying as sin $2\omega t$ and cos $2\omega t$. These terms are collected in three separate equations and the additional terms dropped. At the same time it is convenient to introduce the definitions $A = 2\omega/aI_r^2$ and $B = b\theta_0^3/aI_r^2$. On the basis of the observation that the value of θ_0 must be about the same as θ_s, the value of B is almost unity.

The collection of terms leads to the three equations

$$\begin{array}{ll} \text{Constant:} & (B - 1)\theta_0 = \dfrac{\theta_1}{2} \sin \phi \\ \cos 2\omega t: & A\theta_1 \cos \phi + 3\theta_1 \sin \phi = \theta_0 \\ \sin 2\omega t: & -A\theta_1 \sin \phi + 3\theta_1 \cos \phi = 0 \end{array}$$

In the last two of these equations, the condition $B = 1$ has already been inserted since here it is a good approximation. This condition evidently cannot be used in the first equation for it would then require that $\theta_1 = 0$.

The last equation leads immediately to the condition $\tan \phi = 3/A$, and when this value for ϕ is used in the second equation, ratio θ_1/θ_0 can be found as

$$\frac{\theta_1}{\theta_0} = (A^2 + 9)^{-1/2}$$

Finally, from the first equation a better value for B is found,

$$B = 1 + \frac{\theta_1}{2\theta_0} \sin \phi = 1 + \frac{3/2}{A^2 + 9}$$

The mean temperature is $\theta_0 = (aI_r^2B/b)^{1/3}$, or approximately

$$\theta_0 = \left(\frac{aI_r^2}{b}\right)^{1/3} \left[1 + \frac{1}{2(A^2 + 9)}\right]$$

provided $A^2 + 9 \gg 1$.

In Fig. 7.12 is shown the variation of angle ϕ, ratio θ_1/θ_0, and parameter B as parameter A is varied. If A is made smaller, either by increasing current I_r or decreasing frequency ω, all three quantities ϕ, θ_1/θ_0, and B increase in magnitude.

For a 115-volt 6-watt incandescent bulb, the value of parameter a is about 9,000 amp^{-2} sec^{-1}. Thus, when operated normally at a frequency of 60 cycles per second, parameter A is about 30. Ratio θ_1/θ_0 is then about $\frac{1}{30}$, angle ϕ is less than 6 degrees, and parameter B is less than 1.002.

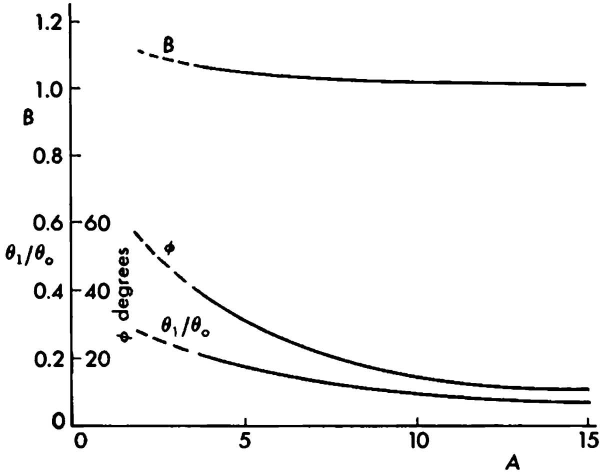


FIG. 7.12. Variation of significant quantities in the solution for Example 7.2.

7.3. Iteration Procedure.

A method not unlike the perturbation method is based on the process of iteration. It allows a more accurate solution to be found, once a first-order solution has been obtained. Iteration may be performed in a number of ways, but the basic principle is first to solve the equation with certain terms neglected. The resulting solution is then used with the terms first neglected and a second solution obtained. Under many conditions, this process leads to a solution of improved accuracy.

A forced nonlinear second-order system might be described by the equation

$$\ddot{x} + f(x) = G \cos \omega_1 t \tag{7.21}$$

where $f(x)$ is a nonlinear function of x. If an approximate solution, say, $x = x_0(t)$, has been found, it is desired to obtain a more accurate solution. Iteration may be applied by solving Eq. (7.21) for $\ddot{x}$ and substituting $x_0(t)$ into the function $f(x)$. The result is

$$\ddot{x} = -f(x_0) + G \cos \omega_1 t \tag{7.22}$$

Since x_0 has been found so that it is an approximate solution for Eq. (7.21), the right side of Eq. (7.22) is merely $\ddot{x}_0$ plus certain correction terms.

When integrated twice, the result is

$$x = x_0 + (\text{integral of correction terms})$$

and if the correction terms are small in magnitude, the new solution is a more accurate one.

Example 7.3. Duffing's Equation, Harmonics

Duffing's equation for Example 7.1 is

$$\ddot{x} + \omega_0^2 x + hx^3 = G \cos \omega_1 t \tag{7.23}$$

and an approximate solution has been found as

$$x_0 = A_1 \cos \omega_1 t \tag{7.24}$$

where a necessary condition is

$$(\omega_0^2 - \omega_1^2)A_1 + \frac{3hA_1^3}{4} = G \tag{7.25}$$

The iteration process can be applied as follows:

Equation (7.23) is solved for $\ddot{x}$ and x_0 from Eq. (7.24) inserted in the right side to give

$$\ddot{x} = -\omega_0^2 A_1 \cos \omega_1 t - hA_1^3(\tfrac{3}{4} \cos \omega_1 t + \tfrac{1}{4} \cos 3\omega_1 t) + G \cos \omega_1 t$$

But, from Eq. (7.25), this latter equation becomes

$$\ddot{x} = -\omega_1^2 A_1 \cos \omega_1 t - \frac{hA_1^3}{4} \cos 3\omega_1 t$$

Upon integration, a more accurate solution is found as

$$x = A_1 \cos \omega_1 t + \frac{hA_1^3}{36\omega_1^2} \cos 3\omega_1 t \tag{7.26}$$

Constants of integration are set to zero since a periodic solution is expected, with zero mean value.

This result shows the appearance of a third-harmonic component. The relative amplitude of this component is essentially the same as that found in Eq. (7.17) provided $\omega_0^2 \ll 9\omega_1^2$.

7.4. Perturbation Method. The perturbation method can be applied to driven systems much as it was applied to free systems as described in Sec. 6.2. Again, the details of its application depend somewhat upon what kind of solution is sought, and some knowledge of the desired solution is essential.

The same kind of driven system is considered, as in Sec. 7.2. The system is described by the equation

$$\ddot{x} + \omega_0^2 x + \mu\phi(x,\dot{x}) = G(t) \tag{7.27}$$

Small parameter μ is associated with the nonlinear function $\phi(x,\dot{x})$.

The perturbation method yields a solution which differs but slightly from some generating solution obtained for the linearized equation. Accordingly, it is necessary to choose the generating solution, and thus the details of the analysis, so as to fit the kind of phenomenon being sought. Practically, this means that, in applying the perturbation method to Eq. (7.27), it may, or may not, be desirable to associate parameter μ with the forcing function $G(t)$ as well as with the nonlinear function $\phi(x,\dot{x})$. If the desired solution is near the free oscillation of the linear system, μ should be associated with $G(t)$. If the desired solution is near a forced oscillation, μ should not be associated with G.

A solution for Eq. (7.27) is sought in the form

$$x = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \cdots \tag{7.28}$$

just as in applications of the method to a free system. Functions x_0, x_1, x_2, . . . are to be determined. Similarly, if free oscillations are involved in the solution, it may become necessary to eliminate secular terms by assuming

$$\omega = \omega_0 + \mu b_1(A) + \mu^2 b_2(A) + \cdots \tag{7.29}$$

where ω is the actual frequency of oscillation and functions b_1, b_2, . . . depend upon amplitude A. If $G(t) = G \cos \omega_1 t$, usually ω must be the same as ω_1, or related to it in some predetermined way. In certain problems, it may be apparent that the solution involves only the equivalent of a particular integral and must oscillate at the driving frequency ω_1. In such cases, the relation of Eq. (7.29) need not be used.

The relations of Eq. (7.28) and (7.29) are substituted into Eq. (7.27) and terms of each power of μ collected, just as described in Sec. 6.2. This process leads to several simultaneous equations, which can be solved in sequence. The result is a solution in the form of a series of successive correction terms applied to the zero-order, or generating, solution.

Example 7.4. Duffing's Equation, Jumps, Harmonics

The Duffing equation of Example 7.1 can be studied by the perturbation method. Again, the phenomena of discontinuous jumps and harmonic generation are of first interest.

Since it is known that the jump phenomena occur near the natural frequency of oscillation of the free system, the generating solution for the perturbation method is taken as the free oscillation usually associated with a transient. The equation for the system, analogous to Eq. (7.4), is accordingly written

$$\ddot{x} + \omega_0^2 x + \mu h x^3 = \mu G \cos \omega_1 t \tag{7.30}$$

where small parameter μ appears before both the nonlinear term and the forcing function. This implies that solution is sought near the free oscillation.

The form of the solution to the first order of approximation is

$$\begin{aligned} x &= x_0(t) + \mu x_1(t) \\ \omega_1^2 &= \omega_0^2 + \mu b_1(A) \end{aligned} \tag{7.31}$$

Only a steady-state solution is desired, and it is required that the actual frequency of oscillation be the same as the driving frequency. Evidently, the second of Eq. (7.31) requires that ω_1 and ω_0 be not too far different, since μ is assumed small. It is further assumed that the solution must fit the initial conditions $x = A$, $\dot{x} = 0$ at $t = 0$.

Substitution of Eq. (7.31) into Eq. (7.30) and collection of like powers of μ gives a set of simultaneous equations. For the generating solution, the equation is

$$\mu^0: \qquad \ddot{x}_0 + \omega_1^2 x_0 = 0$$

The generating solution, subject to the conditions $x_0 = A$, $\dot{x}_0 = 0$ at $t = 0$, is

$$x_0 = A \cos \omega_1 t \tag{7.32}$$

The equation for the first-order correction is

$$\mu^1: \qquad \ddot{x}_1 + \omega_1^2 x_1 = \left(b_1 A - \frac{3hA^3}{4} + G\right) \cos \omega_1 t - \frac{hA^3}{4} \cos 3\omega_1 t$$

where the value of x_0 and the identity for $\cos^3 \omega_1 t$ have been used on the right side. The secular term is removed by the requirement $b_1 = 3hA^2/4 - G/A$ and used in Eq. (7.31), and the result is

$$(\omega_1^2 - \omega_0^2)A + G = \frac{3hA^3}{4} \tag{7.33}$$

which is the same as Eq. (7.7). The first-order correction subject to the initial conditions $x_1 = 0$, $\dot{x}_1 = 0$ at $t = 0$ is

$$x_1 = -\left(\frac{hA^3}{32\omega_1^2}\right)(\cos \omega_1 t - \cos 3\omega_1 t)$$

The first-order approximate solution is therefore

$$x = A \cos \omega_1 t - \frac{hA^3}{32\omega_1^2}(\cos \omega_1 t - \cos 3\omega_1 t) \tag{7.34}$$

subject to the relation Eq. (7.33).

This result is essentially the same as Eqs. (7.2), (7.7), and (7.17) found in Example 7.1 and Eq. (7.26) found in Example 7.3. There are minor differences in the amplitudes of Eq. (7.34), but the properties of the solutions are all essentially the same. All the solutions hold only so long as the distortion, represented by the third-harmonic term, is not too large in magnitude.

Only a first-order correction is considered here. Higher-order correction terms would introduce components at 3, 5, 7, . . . times the frequency of the driving function.

Example 7.5. Duffing's Equations, Subharmonics

All the systems governed by Duffing's equation display a phenomenon basically different from those found in Examples 7.1, 7.3, and 7.4. In these examples, the appearance of a harmonic of the driving frequency has been observed. This harmonic is a component of the solution having its frequency an integral multiple of the driving frequency. Solutions to the accuracy of the examples here showed only a third harmonic. Solutions to a higher order would exhibit fifth, seventh, and other odd-order harmonics as well.

Under suitable conditions, solutions for Duffing's equation also exist as subharmonics. These are components having their frequency an integral submultiple of the driving frequency. The subharmonic most easily obtained experimentally in a system described by Duffing's equation is at one-third the driving frequency. Other orders may be found, however, and are predicted in the analysis.

It is found from experiment that starting conditions are quite important if a subharmonic oscillation is to exist in a physical system. The amplitude and frequency of the driving force must fall within certain definite limits. Appropriate initial conditions must exist within the system itself. Because of this strong dependence upon initial conditions, it is apparent that a subharmonic oscillation is associated with what would basically be called a transient oscillation in a linear oscillatory system. An oscillating system, either linear or nonlinear, can be set into oscillation by an initial shock of some kind. Because any realizable physical system always involves dissipation, any oscillation following a shock would be expected to die away. In a nonlinear system, the oscillation is nonsinusoidal and contains harmonics or integral multiples of the fundamental frequency. It turns out to be possible to maintain the oscillation in a steady state, under some conditions, by supplying energy to the system at one of these harmonic frequencies. Thus, the driving force is then at an integral multiple of the fundamental frequency of oscillation, or the fundamental oscillation frequency is an integral submultiple of the driving frequency. This is the condition for subharmonic generation.

The possible appearance of subharmonics in the solution for Duffing's equation,

$$\ddot{x} + \omega_0^2 x + \mu h x^3 = G \cos \omega_1 t \tag{7.35}$$

can be predicted by the perturbation method. Here, both the free and forced oscillation are desired in the generating solution, and small parameter μ is associated with the nonlinear term only, in contrast to Eq. (7.30). From the discussion of the preceding paragraph, it is expected that the fundamental frequency of the subharmonic oscillation will be somewhere near the natural frequency of the system. Accordingly, the driving frequency will be at an integral multiple of the natural frequency. A solution for Eq. (7.35) is sought in the form

$$\begin{aligned} x &= x_0(t) + \mu x_1(t) \\ \omega^2 &= \omega_0^2 + \mu b_1(A) \end{aligned} \tag{7.36}$$

where only first-order corrections are used. The actual fundamental frequency of the solution is ω, and this is to be an integral submultiple of the driving frequency ω_1, say, $\omega = \omega_1/n$, with n an integer.

Upon substitution of Eq. (7.36) into Eq. (7.35), the result is

$$\ddot{x}_0 + \mu\ddot{x}_1 + \omega^2 x_0 + \mu\omega^2 x_1 - \mu b_1 x_0 + \mu h x_0^3 = G \cos \omega_1 t$$

It is expected that the solution of interest here will include components at the subharmonic frequency and at the driving frequency, as well as other components at multiples of these frequencies. It is convenient in interpreting the result physically to obtain the amplitudes of these components separately. For this reason, it turns out, as is apparent shortly, that the only initial condition readily specified is that $\dot{x} = 0$ at $t = 0$.

The generating solution is found from the relation

$$\mu_0: \qquad \ddot{x}_0 + \omega^2 x_0 = G \cos \omega_1 t$$

and is

$$x_0 = A \cos \omega t + C \cos \omega_1 t \tag{7.37}$$

where A is the amplitude of the subharmonic component,

$$C = \frac{G}{(\omega^2 - \omega_1^2)}$$

is the amplitude of the component of driving frequency, and $\omega^2 \neq \omega_1^2$. As written, this solution satisfies the initial condition $\dot{x}_0 = 0$ at $t = 0$, while amplitude A remains to be determined. Furthermore, the sum of the two cosine functions with their frequencies integrally related is a function of even symmetry, as must be the case for a solution for Eq. (7.35), since this equation is unchanged if $+t$ in it is replaced by $-t$.

The generating solution applies exactly to a linear dissipationless system governed by Eq. (7.35) with $\mu = 0$. The first term represents the free oscillation, and the second represents the forced oscillation, both of constant amplitude. In a linear system, there is no necessary relation involving the frequencies and amplitudes of these two components. The free oscillation is at the natural frequency of the system, and its amplitude depends upon initial conditions. The forced oscillation is at the frequency of the driving function, and its amplitude depends upon both amplitude and frequency of the driving function. There is no necessary relation connecting the two frequencies. In the nonlinear system, it turns out that definite relations between the two frequencies and their amplitudes are required.

The first-order correction terms are found from the relation

$$
\begin{aligned}
\mu^1\colon\quad \ddot{x}_1 + \omega^2 x_1 &= b_1 x_0 - h x_0^3 \\
&= b_1 A \cos \omega t + b_1 C \cos \omega_1 t \\
&\quad - hA^3(\tfrac{3}{4} \cos \omega t + \tfrac{1}{4} \cos 3\omega t) \\
&\quad - 3hA^2C[\tfrac{1}{2} \cos \omega_1 t + \tfrac{1}{4} \cos (2\omega + \omega_1)t \\
&\qquad\qquad + \tfrac{1}{4} \cos (2\omega - \omega_1)t] \\
&\quad - 3hAC^2[\tfrac{1}{2} \cos \omega t + \tfrac{1}{4} \cos (2\omega_1 + \omega)t \\
&\qquad\qquad + \tfrac{1}{4} \cos (2\omega_1 - \omega)t] \\
&\quad - hC^3(\tfrac{3}{4} \cos \omega_1 t + \tfrac{1}{4} \cos 3\omega_1 t)
\end{aligned} \tag{7.38}
$$

where simplifying identities have been used.

Because of the way the angular frequencies ω and ω_1 enter terms on the right side of this equation, it is now necessary to choose the relation between these frequencies. The subharmonic most commonly observed in this kind of system is of one-third order, in which case $n = 3$ and $\omega_1 = 3\omega$. This case is examined first. For this case,

$$\cos (2\omega - \omega_1)t = \cos \omega t$$

and the condition needed to avoid a secular term is

$$b_1 A - \frac{3hA^3}{4} - \frac{3hA^2C}{4} - \frac{3hAC^2}{2} = 0$$

This condition can be satisfied if either $A = 0$ or

$$b_1 = \frac{3h}{4} (A^2 + AC + 2C^2)$$

The first possibility is trivial, since it removes the subharmonic, and the second is the one of interest. This value of b_1 used in the second of Eq. (7.36) gives as a necessary relation

$$\omega_0^2 = \omega^2 - \frac{3h}{4} (A^2 + AC + 2C^2)$$

where μ has been set equal to unity. Moreover, it is already known that $C = G/(\omega^2 - \omega_1^2)$, or here $C = -G/8\omega^2$. Upon substituting this value for C and carrying out algebraic simplification, the relation can be written

$$\omega^6 - \omega^4\omega_0^2 = \frac{3hG^2}{128} \left(1 - \frac{4A\omega^2}{G} + \frac{32A^2\omega^4}{G^2}\right) \tag{7.39}$$

This algebraic equation provides a necessary relation between the amplitude A and the angular frequency ω of the subharmonic oscillation. While the equation is rather complicated, certain information about it can be obtained. It is, first of all, a cubic equation in ω^2 and a quadratic in A. A real value for A exists only so long as ω^2 exceeds a certain minimum. This minimum can be found in several ways, but the simplest is to find the condition for $dA/d\omega^2 = \infty$, which is $A = G/16\omega^2$. For this value of A, the minimum ω^2 is given by

$$\omega^6 - \omega^4\omega_0^2 = \frac{21hG^2}{1{,}024}$$

If A has the value zero, the value of ω^2 is given by

$$\omega^6 - \omega^4\omega_0^2 = \frac{3hG^2}{128}$$

For both these latter relations, with h a small quantity, the only positive real value of ω is just greater than ω_0. If ω/ω_0 is written as $\omega/\omega_0 = 1 + \epsilon$ with $\epsilon \ll 1$, the minimum value of the ratio is approximately determined by

$$(1 + \epsilon)^6 - (1 + \epsilon)^4 = 2\epsilon = \frac{21hG^2}{1{,}024\omega_0^6}$$

or

$$\frac{\omega_{\min}}{\omega_0} = 1 + \frac{21hG^2}{2{,}048\omega_0^6}$$

The minimum value for ω is accordingly

$$\omega_{\min} = \omega_0 + \frac{21hG^2}{2{,}048\omega_0^5}$$

This result indicates that the minimum allowable fundamental frequency of the subharmonic oscillation is just slightly greater than the natural frequency of the linearized system.

The relation between the amplitude and frequency of the subharmonic oscillation, given in Eq. (7.39), can be sketched as shown in Fig. 7.13. The curve relating A and ω is a kind of parabola, one portion having positive slope and one portion having negative slope. The portion of negative slope turns out to be unstable and is therefore never observed experimentally. A demonstration of this point must be deferred to Sec. 10.5. The branch of positive slope is stable and may be observed. There is a minimum value for the amplitude of the stable subharmonic, given by $A_{\min} = G/16\omega^2$, ω being its frequency.

Also plotted in Fig. 7.13 is the amplitude of the component of the solution at the driving frequency 3ω, three times the subharmonic frequency. This amplitude is always negative, indicating a difference in phase between the two components. The phase relations lead to the waveform of Fig. 7.14, showing a flattened top. Only the two components of the driving frequency and the subharmonic frequency are considered in plotting Fig. 7.14. Actually, if details of finding the first-order correction from Eq. (7.38) are carried out, components arise having frequencies of 1, 3, 5, 7, and 9 times the frequency of the subharmonic. These are $\frac{1}{3}$, 1, $\frac{5}{3}$, $\frac{7}{3}$, and 3, respectively, times the driving frequency. Thus, a more exact solution could be expected to have an even more

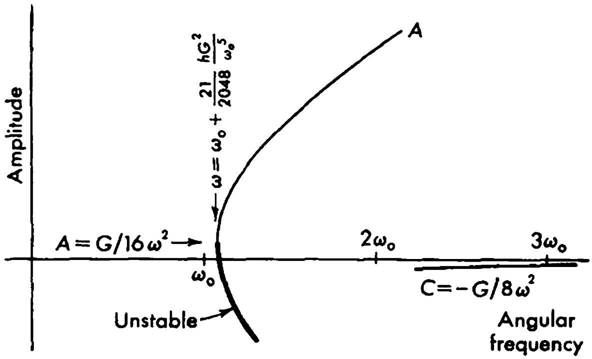


FIG. 7.13. Variation with frequency of amplitudes of fundamental component and one-third-order subharmonic component for Example 7.5. Amplitude of the driving function is constant while its frequency is changed.

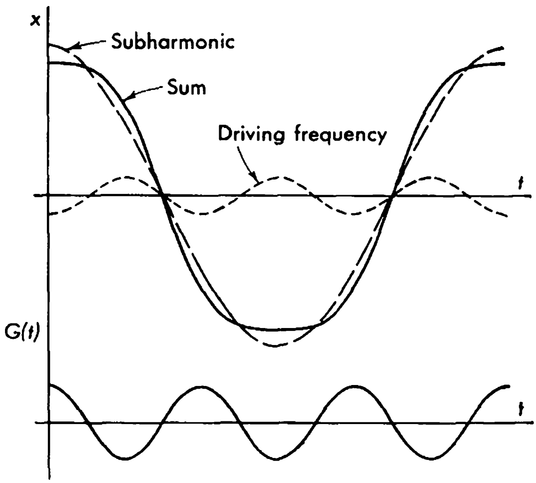


FIG. 7.14. Approximate steady-state solution for Example 7.5, showing components at the driving frequency and at the one-third-order subharmonic frequency.

complicated shape than that of Fig. 7.14 because of the additional components.

Any physically realizable system will, of course, include dissipation, and this should be considered in the analysis. Such inclusion will complicate the analysis still further. The qualitative effects of dissipation could be expected to be somewhat similar to its effect in the jump phenomenon of Example 7.1. Primarily, this effect is to prevent the oscillation in question, here the subharmonic, from continuing if the driving frequency is raised indefinitely. In other words, the curve for the subharmonic in Fig. 7.13 could not be expected to extend indefinitely far upward and to the right but would terminate at some definite point. The greater the relative dissipation in the system, the smaller the maximum allowable amplitude for the subharmonic. If the dissipation becomes too large, the subharmonic cannot exist at all.

In any subharmonic oscillation, starting conditions are most important, and if these are incorrect, the subharmonic is not established. Details of these starting conditions have not been considered in the present discussion.

The foregoing analysis applies to the case of a subharmonic at one-third the driving frequency. Much the same kind of analysis can be used to consider subharmonics of other orders as well. Even though a cubic, odd-order, nonlinear term appears in the original differential equation, Eq. (7.35), it is possible for even-order subharmonics (and harmonics, also) to appear in the solution. The most likely even-order subharmonic is at one-half the driving frequency, and since this is a case of some interest, it is considered briefly.

The analysis for a subharmonic of order ½ is similar to that already given for order ⅓. A solution for Eq. (7.35) is sought in the form of Eq. (7.36), and this leads to the generating solution of Eq. (7.37). The first-order correction terms are found from Eq. (7.38), where now $n = 2$ and $\omega_1 = 2\omega$. The condition to eliminate a secular term is

$$b_1 A - \frac{3hA^3}{4} - \frac{3hAC^2}{2} = 0.$$

Since $A = 0$ is trivial, the value for b_1 must be

$$b_1 = \frac{3h}{4}(A^2 + 2C^2)$$

and thus

$$\omega_0^2 = \omega^2 - \frac{3h}{4}\left(A^2 + \frac{2G^2}{9\omega^4}\right) \tag{7.40}$$

since here $C = G/(\omega^2 - \omega_1^2) = -G/3\omega^2$. This equation is analogous to Eq. (7.39) and fixes the necessary relation between amplitude and frequency for the subharmonic oscillation. Again, allowed values of ω are slightly greater than ω_0, and both positive and negative values of A are possible. In this case, it would be expected that there should be no difference in the stability of the solution with either sign of A, since for a one-half-order subharmonic a change in sign of A is equivalent merely to a shift in time of one period of the driving force.

The point of some interest in connection with this even-order subharmonic is that, when first-order correction terms are evaluated, a constant term of zero frequency is found and appears added to the solution. This term represents a shift in the mean value of the solution. When expressed in the form of a polynomial about this new mean value of x as an origin, the simple cubic nonlinearity with x^3 in Eq. (7.35) acquires a quadratic nonlinear term with x^2 as well. The presence of this quadratic term easily explains the possibility of an even-order subharmonic oscillation, based on the trigonometric identity

$$\cos^2 \omega t = \tfrac{1}{2}(1 + \cos 2\omega t)$$

By extrapolation, it would be expected that, any time a harmonic or subharmonic solution of even order is found for Duffing's equation, a shift away from zero for the mean value will also occur.

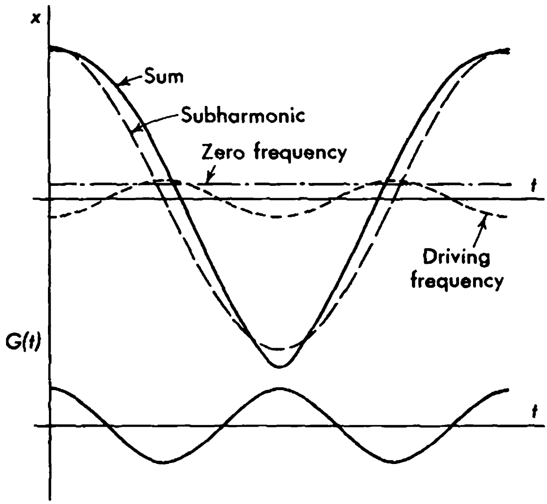


FIG. 7.15. Approximate steady-state solution for Example 7.5, showing components at the driving frequency, at the one-half-order subharmonic frequency, and at zero frequency.

The waveform of the second-order subharmonic solution is shown in Fig. 7.15, where only the three components at the subharmonic frequency, the driving frequency, and zero frequency are used. In addition to these terms, the first-order correction terms include components of 0, 1, 2, 3, 4, 5, and 6 times the fundamental or subharmonic frequency, or 0, $\tfrac{1}{2}$, 1, $\tfrac{3}{2}$, 2, $\tfrac{5}{2}$, and 3 times the driving frequency. The waveform is even more complicated than shown in Fig. 7.15.

The subharmonic oscillation of order $\tfrac{1}{2}$ can be found in physical systems governed by Duffing's equation. Typically, it is somewhat more difficult to excite than the one-third-order subharmonic and exists over a smaller range of the system parameters. As for all subharmonic oscillations, starting conditions are highly important.

It should be evident from the preceding analyses of Duffing's equation that a strong response is possible at a frequency other than the driving frequency. A component of considerable magnitude will occur at a frequency near the linear resonance frequency ω_0. The driving frequency ω_1 may be either greater or less than ω_0. Harmonic or subharmonic generation of any order may be studied in much the same way that one-third- and one-half-order subharmonics have been investigated.

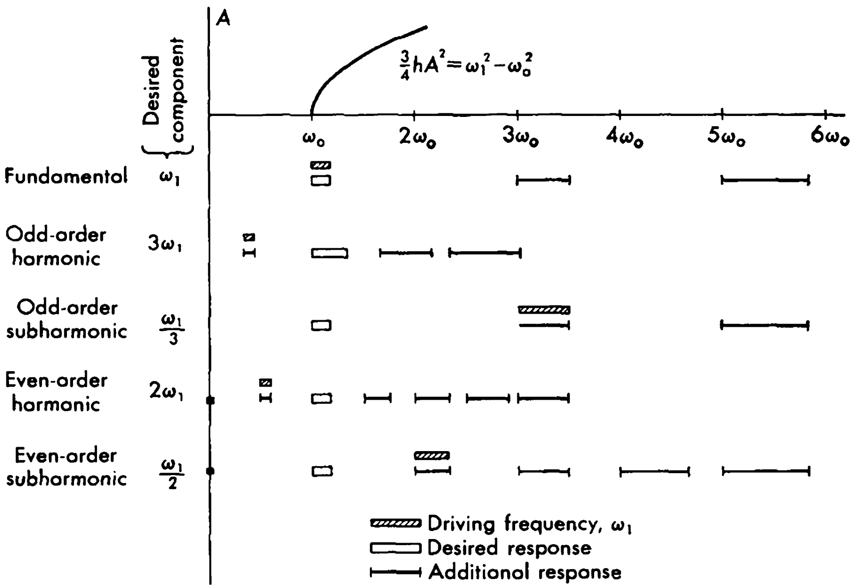


FIG. 7.16. Summary of different types of steady-state responses for a system governed by Duffing's equation. In each case, the desired response occurs near the frequency ω_0. The necessary location of the driving frequency to achieve the desired condition is shown. Additional responses occur at multiples of the lowest frequency present. Responses of even order also show a response at zero frequency.

The results of such analyses are summarized in Fig. 7.16, which presents several typical cases. The component of special interest has frequency ω, which is always slightly higher than frequency ω_0 if parameter h is positive, as assumed here. The driving frequency ω_1 may bear any of various relations with ω. If $\omega = \omega_1$, the interesting response is at the fundamental frequency, although odd-order harmonics occur also. If $\omega = m\omega_1$ with m an integer, the interesting response is at a harmonic frequency. If $\omega = \omega_1/n$ with n an integer, the interesting response is at a subharmonic frequency. Additional components always exist in the solution at integral multiples of the lowest frequency present. Only odd multiples occur for odd-order-harmonic or -subharmonic generation. Both odd and even multiples, as well as a zero-frequency component, occur for even-order-harmonic or -subharmonic generation. When a zero-frequency component is not present, the nonlinear terms in the basic equation are of odd power and these lead to only odd-order harmonics. When a zero-frequency component is present, the effect is to shift the point about which operation occurs in such a way as to give the equivalent of both even- and odd-power terms in the equation. These terms give both even- and odd-order harmonics.

7.5. Extension of Concepts from Linear Systems. In the analysis of linear systems, particularly electrical networks, several concepts have been developed which have been found to be of fundamental importance. Among these are the concepts of the various impedances and transfer functions of the system. These quantities are defined for linear systems to which a simple-harmonic driving force is applied and for which steady-state conditions exist. Any steady-state response of a physical system, which is certain to have dissipation, must take place at the frequency of the driving force. Thus, all time variations in the system are at the same frequency. The properties of the system are independent of the amplitudes involved.

Because of general familiarity with concepts such as these, and because of their proved utility in linear systems, it seems natural to extend them to nonlinear systems. This sort of extension can be justified only so long as the nonlinearity is not too great. The steady-state response of a nonlinear system to a simple-harmonic driving force is not simple-harmonic. Instead of the response having only a component at the driving frequency, it has additional components at multiples, or even submultiples, of this frequency. Extension of the linear concepts to the nonlinear system is justifiable only so long as the additional components are relatively small in comparison with the component at the driving frequency. Fortunately, this condition is often valid in systems of practical importance. There are a number of practical systems in which the nonlinear effects are inherently relatively small but which nevertheless must be accounted for in any realistic analysis. There are other systems in which some kind of low-pass filtering action takes place, thereby minimizing any harmonic frequencies introduced by the nonlinearity. In systems such as these, analysis based on linear concepts may lead to results of useful accuracy.

a. The Impedance Concept. In linear electrical circuits, the self-impedance of a two-terminal element is defined when sinusoidal voltages and currents exist. If a current $I \sin \omega t$ exists in the element, producing a voltage $E \sin (\omega t + \theta)$ across it, the impedance is the complex ratio of voltage to current, where these are expressed as complex quantities. The magnitude of the impedance is $|Z| = E/I$, and the angle of the impedance is $\angle Z = \theta$. Depending upon the type of analysis concerned, the impedance may be written in polar form as magnitude and angle or in rectangular form as real and imaginary parts. The real part is called the resistance $R = (E/I) \cos \theta$, and the imaginary part is called the reactance $X = (E/I) \sin \theta$. For general linear impedances, both components of a complex impedance are functions of frequency but are independent of the amplitudes of current and voltage. Simple rules are well known from which the impedance of even complicated networks can be written down at once. For example, the impedance of a circuit having resistance R, inductance L, and capacitance C in series is easily found as

$$Z = R + jX = R + j\left(L\omega - \frac{1}{C\omega}\right) \tag{7.41}$$

where j is the imaginary unit and ω is the angular frequency.

A kind of self-impedance for a nonlinear element can be found in a similar manner by considering only components of current and voltage at the fundamental frequency. For example, a saturating capacitor has instantaneous values of its charge q and voltage e related approximately according to the equation

$$e = \frac{q}{C_0}(1 + aq^2) \tag{7.42}$$

where C_0 would be the capacitance for a linear unit and a is a coefficient describing the nonlinearity. This equation indicates that the charge increases less and less rapidly as the voltage is increased. If the charge is assumed to vary as a simple-harmonic function of time, it could be given by $q = Q \cos \omega t$, where Q is the amplitude and ω is the angular frequency. The instantaneous current is the time derivative of the charge and is $i = dq/dt = -Q\omega \sin \omega t = -I \sin \omega t$, where $I = Q\omega$. The instantaneous voltage across the capacitor is found by using the instantaneous charge variation in Eq. (7.42) and is

$$\begin{aligned} e &= \frac{Q}{C_0} \cos \omega t + \frac{aQ^3}{C_0} \cos^3 \omega t \\ &= E \cos \omega t + \frac{aQ^3}{4C_0} \cos 3\omega t \end{aligned}$$

where $E = (I/C_0\omega)(1 + 3aI^2/4\omega^2)$. If coefficient a were zero, the capacitor would be linear and its impedance would be a pure reactance given by $X_C = E/(-I) = -1/C_0\omega$. By analogy, an equivalent reactance for the nonlinear capacitor is found as

$$X_C = \frac{E}{(-I)} = \frac{-1}{C_0\omega}\left(1 + \frac{3aI^2}{4\omega^2}\right) \tag{7.43}$$

It should be noted that in this equation I is the current amplitude, so that I^2 is twice the square of the rms current. The equivalent reactance is found by neglecting entirely that component of voltage at the third-harmonic frequency. The reactance depends upon both frequency and current.

In a similar way, an equivalent reactance can be found for a saturating inductor. The saturation effects of an iron-core inductor have been considered in Example 7.1, where the relation between instantaneous values of flux ϕ and current i is given as Eq. (7.5). The flux increases less and less rapidly as current is increased. Unfortunately, the form of Eq. (7.5) does not lend itself to a simple expression for the equivalent reactance. Rather, a relation must be written,

$$N\phi = L_0 i(1 - bi^2) \tag{7.44}$$

where N is the number of turns on the winding of the inductor, L_0 would be the inductance for a linear unit, and b is a coefficient related to the nonlinearity. It should be noted that this relation is not even qualitatively correct for large currents, for if i^2 exceeds $1/3b$, the relation indicates a decrease in flux, rather than an increase, with increasing current. If $i^2 = 1/b$, the flux is reduced to zero, obviously incorrect. If a simple harmonic current $i = I \cos \omega t$ is used in Eq. (7.44), the instantaneous voltage is found as

$$\begin{aligned} e = N\frac{d\phi}{dt} &= \left(\frac{d}{dt}\right)(L_0 I \cos \omega t - bL_0 I^3 \cos^3 \omega t) \\ &= -E \cos \omega t + \frac{3bL_0\omega I^3}{4} \sin 3\omega t \end{aligned}$$

where $E = IL_0\omega(1 - 3bI^2/4)$. The equivalent reactance of the saturating inductor is

$$X_L = \frac{E}{I} = L_0\omega\left(1 - \frac{3bI^2}{4}\right) \tag{7.45}$$

where again I is the current amplitude and I^2 is twice the square of the rms current.

Example 7.6. Nonlinear Resonant Circuit

A nonlinear resonant circuit may consist of a saturating inductor in series with a linear resistor and capacitor and a generator of sinusoidal voltage. This is the circuit of Fig. 7.2 with resistance added, and it has been considered in Example 7.1. The magnitude for the impedance of this circuit is

$$\begin{aligned} Z &= [R^2 + (X_L + X_C)^2]^{1/2} \\ &= \left\{R^2 + \left[L_0\omega\left(1 - \frac{3bI_r^2}{2}\right) - \frac{1}{C\omega}\right]^2\right\}^{1/2} \end{aligned} \tag{7.46}$$

where R is the resistance, C is the capacitance, and ω is the angular frequency of the generator. The reactance of the inductor is taken to be given by Eq. (7.45), but in Eq. (7.46) the rms value I_r of the current is used. The numerical coefficient in Eq. (7.46) becomes $\frac{3}{2}$ rather than $\frac{3}{4}$ as in Eq. (7.45). The rms value of current is advantageous in a practical problem, first because many meters read proportional to rms values, and second because the rms value is but little affected by the presence of relatively small components at other than the fundamental frequency. The impedance of Eq. (7.46) is, of course, the magnitude $Z = E_r/I_r$, where E_r is the rms voltage applied to the circuit.

This example is essentially an analysis of the Duffing equation of Example 7.1. The procedure followed here differs in a number of respects, however.

In the analysis of any problem of this kind, it is often desirable to normalize the equations, replacing the actual variables of the system with dimensionless variables.

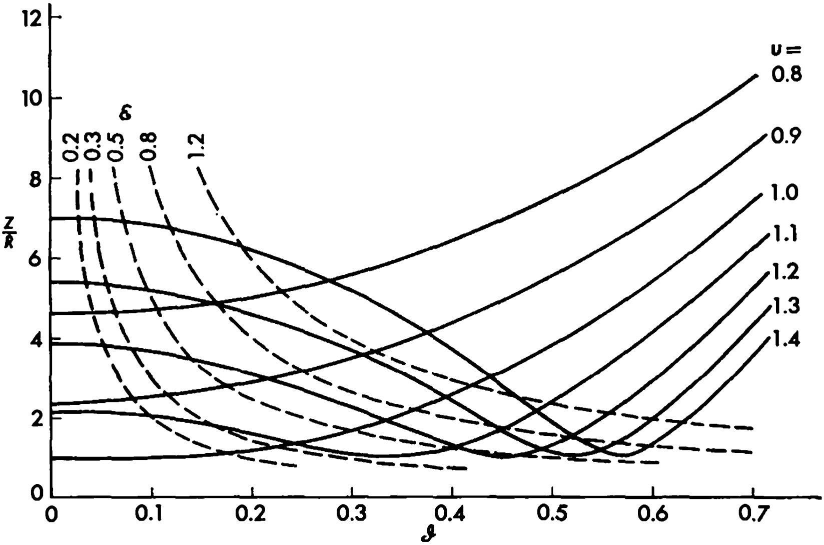


FIG. 7.17. Curves for nonlinear resonant circuit of Example 7.6, showing normalized values of impedance Z/R as a function of current $\mathcal{J}$, with parameters voltage $\mathcal{E}$ and frequency u. The circuit Q is 10.

In this example, suitable dimensionless variables can be set up from the angular frequency of resonance, if linear, $\omega_0 = 1/(L_0C)^{1/2}$, and the circuit Q at resonance, if linear, $Q = L_0\omega_0/R$. The new variables become $u = \omega/\omega_0$, $\mathcal{J} = b^{1/2}I_r$, and $\mathcal{E} = b^{1/2}E_r/R$, all of which are dimensionless. The relation then becomes

$$\frac{Z}{R} = \frac{\mathcal{E}}{\mathcal{J}} = \left\{1 + Q^2\left[u\left(1 - \frac{3\mathcal{J}^2}{2}\right) - \frac{1}{u}\right]^2\right\}^{1/2} \tag{7.47}$$

The impedance of the system, given by Eq. (7.47), depends upon both frequency u and current $\mathcal{J}$. If the current were very small, essentially zero, saturation effects would be absent and a resonance frequency would exist for which the total reactance is zero. At this resonance frequency, $u = 1$, and the impedance has its minimum value, which is simply R. If the current is greater than zero, the impedance is equal to R at some frequency for which $u > 1$. Several curves showing the variation of Z with $\mathcal{J}$ are shown in Fig. 7.17 with u as the parameter. For this figure, Q has arbitrarily been given the value $Q = 10$, which is reasonable for a typical inductor. A considerable amount of numerical work is needed to give data for plotting these curves.

Impedance is defined so that its magnitude is the ratio of the amplitudes of voltage and current. Thus $\mathcal{E} = Z\mathcal{I}$, and if an alternating voltage of chosen amplitude were applied to the circuit described by Fig. 7.17, the corresponding values of impedance Z and current $\mathcal{I}$ must be related as shown by the dashed hyperbolic curves of the figure. Several such curves are shown for different values of $\mathcal{E}$.

If the generator voltage and frequency are specified, both $\mathcal{E}$ and u are known. These values of $\mathcal{E}$ and u select one of each of the families of curves of Fig. 7.16. The

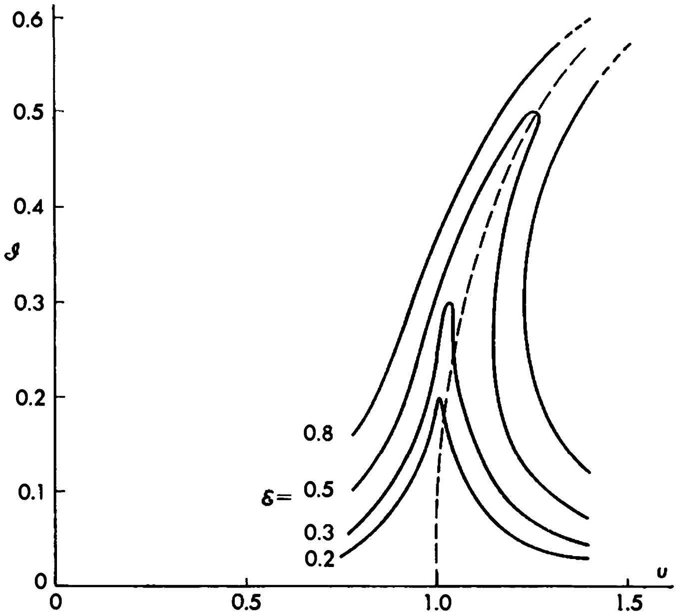


FIG. 7.18. Nonlinear resonance curves for Example 7.6. Discontinuous jumps may occur if $\mathcal{E}$ exceeds about 0.3.

resulting current in the circuit is determined by the intersection of the curves corresponding to the specified $\mathcal{E}$ and u. In Fig. 7.18 are shown several curves of current as a function of frequency for particular values of driving voltage. These curves are plotted by using data from Fig. 7.17 as taken along a particular dashed curve. For small values of $\mathcal{E}$, a simple resonance peak exists. If $\mathcal{E}$ exceeds about 0.3, the peak becomes skewed toward higher frequencies and the possibility of jumps in amplitude is indicated. These curves are analogous to those of Fig. 7.8.

In Fig. 7.19 are shown curves of current as a function of applied voltage for particular values of frequency. These curves are also plotted with data from Fig. 7.17 as taken along a particular solid curve. For driving frequencies less than that of linear resonance where $u = 1$, current increases smoothly as voltage increases, although saturation effects are apparent. If the driving frequency is such that u exceeds about 1.1, the curve becomes multiple-valued and discontinuous jumps may occur. These curves are analogous to those of Fig. 7.10.

It should be evident that the underlying principle of considering only the fundamental component is the same for both this present example and Example 7.1. The methods predict the same phenomena, and both are subject to increasing errors as the waveforms become increasingly nonsinusoidal.

b. The Transfer-function Concept. In linear electrical systems, and by extension in other types of linear systems as well, the concept of a transfer function is used with four-terminal networks. One pair of terminals is defined as the input terminals to which some driving force is applied. The second pair of terminals is defined as the output terminals at which

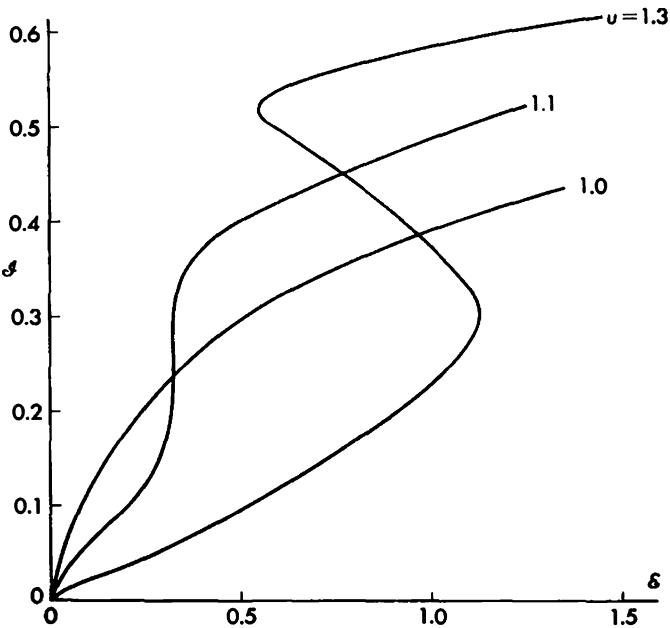


FIG. 7.19. Current as a function of applied voltage for Example 7.6. Discontinuous jumps may occur if u exceeds about 1.1.

a response is observed. The transfer function is defined when simple-harmonic variations with respect to time exist. It is the complex ratio of the resulting response to the applied driving force, each taken as a complex quantity. Depending upon the particular situation, the driving force and the response may be either currents or voltages, in any combination. The transfer function may therefore have dimensions of impedance, admittance, or pure numeric and consists of real and imaginary parts, or magnitude and angle. Both parts are generally functions of frequency but not of amplitude for a linear system. Through the application of well-known rules of circuit theory, it is usually not difficult to determine transfer functions for networks. For example, the transfer function, taken as the ratio of output voltage to input voltage, for the network of Fig. 7.20 is easily found as

$$H = \frac{E_2}{E_1} = \frac{R}{R + j(L\omega - 1/C\omega)}$$

A kind of transfer function for a system with a nonlinear element can be found in a similar manner by considering only components at the fundamental driving frequency. In the course of analyzing a nonlinear system, it is often desirable to separate the linear and nonlinear elements so far as possible. The transfer function for the linear elements is found in the usual way and generally is a function of frequency but not of amplitude. A kind of equivalent transfer function for the nonlinear elements is found, considering fundamental components only. This equivalent function is called the describing function for the system. Typically, it is a function dependent upon amplitude, which may or may not also depend upon frequency. It is meaningful and leads to results of useful accuracy only so long as the nonlinearity is such that the response to a simple-harmonic driving force does not itself differ too much from being simple-harmonic.

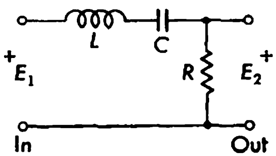


FIG. 7.20. Simple circuit having transfer function $H = E_2/E_1$.

Example 7.7. Transmission Element Saturating Abruptly

A number of physical transmission elements have the property of saturation. The output quantity is related linearly to the input quantity so long as the input magnitude is less than some critical value. If the input magnitude exceeds this critical value, the output quantity remains constant. The transition between these two relations takes place more or less abruptly, depending upon the particular phenomenon involved. The magnetic saturation of Example 7.1, approximated by a cubic equation, is an example of rather gradual saturation. The magnetic saturation of Example 4.4, on the other hand, takes place suddenly. If saturation is assumed to take place abruptly, the input-output relation can be described simply in terms of several linear algebraic equations. If the instantaneous value of the input quantity is x and of the output quantity is y, they may be related as shown in Fig. 7.21. The relations are

$$\begin{aligned} -x_c \le x \le +x_c&: \qquad y = kx \\ x \ge +x_c&: \qquad y = +kx_c \\ x \le -x_c&: \qquad y = -kx_c \end{aligned}$$

where k is a positive constant. Electronic amplifiers often saturate abruptly in this fashion. A mechanical overrunning clutch may have this property.

The describing function H can be found for this kind of element by assuming that the input quantity varies simple harmonically as $x = X \cos \omega t$, where X is the amplitude and ω is the angular frequency. The output quantity will also vary simple harmonically if $X \le x_c$, in which case $y = kX \cos \omega t$. If $X > x_c$, the output quantity is not simple-harmonic but must be expressed as a Fourier series. The component

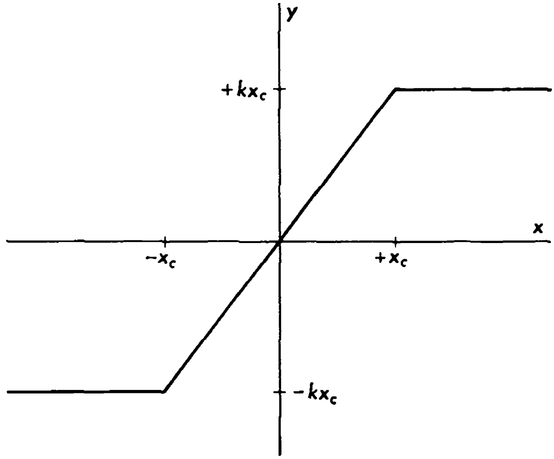


Fig. 7.21. Transmission characteristics of element saturating abruptly for Example 7.7.

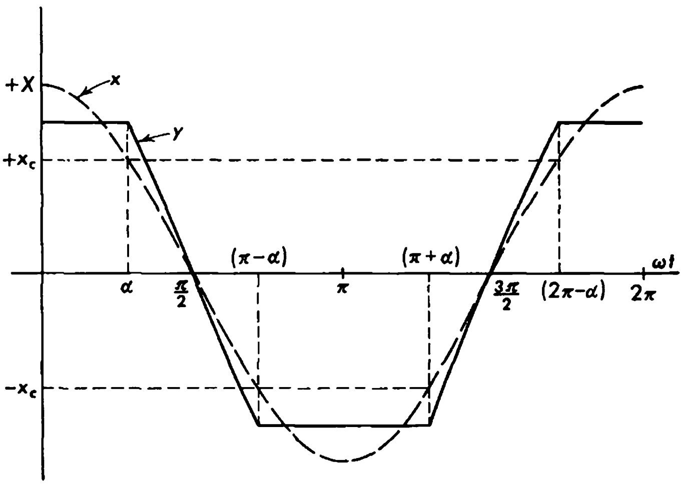


Fig. 7.22. Input waveform x and output waveform y for saturating element of Example 7.7.

of fundamental frequency is the one of interest, and it is found in the usual manner. The input and output quantities are plotted in Fig. 7.22 as functions of time. For the first half cycle, the relations are

$$
\begin{array}{rcccl}
0 \leq \omega t \leq \alpha & \quad & x \geq +x_c & \quad & y = +kx_c \\
\alpha \leq \omega t \leq \pi - \alpha & & -x_c \leq x \leq +x_c & & y = kX \cos \omega t \\
\pi - \alpha \leq \omega t \leq \pi & & x \leq -x_c & & y = -kx_c
\end{array}
$$

where α is the angle for which $\cos \alpha = x_c/X$ and $X > x_c$. The second half cycle is symmetrical with the first and need not be considered in the Fourier analysis. Because of the symmetry, only cosine components appear in the output wave. The amplitude of the fundamental cosine component is

$$A_1 = \frac{2}{\pi}\left[\int_0^{\alpha} kx_c \cos \omega t \, d(\omega t) + \int_{\alpha}^{(\pi - \alpha)} kX \cos^2 \omega t \, d(\omega t) + \int_{(\pi - \alpha)}^{\pi} (-kx_c) \cos \omega t \, d(\omega t)\right].$$

When the integration is carried out, A_1 can be expressed as

$$A_1 = kX\left[1 - \frac{2}{\pi}(\alpha - \sin \alpha \cos \alpha)\right]$$

If $X = x_c$, then $\alpha = 0$ and $A_1 = kx_c$. If $X \gg x_c$, then $\alpha \to \pi/2$ and $A_1 \to 4kx_c/\pi$.

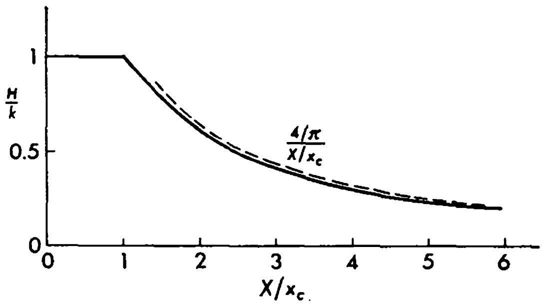


FIG. 7.23. Describing function for the saturating element of Example 7.7. For large values of X/x_c, the curve is asymptotic to the dotted curve shown.

The describing function is the ratio $H = A_1/X$ and is

$$\begin{aligned} X &\leq x_c: \quad & H &= k \\ X &\geq x_c: \quad & H &= k\left[1 - \frac{2}{\pi}(\alpha - \sin \alpha \cos \alpha)\right] \end{aligned} \tag{7.48}$$

where $\cos \alpha = x_c/X$. The describing function depends upon the input amplitude but not upon frequency. It is a real number having magnitude but zero angle. It varies as a function of the ratio X/x_c as shown in Fig. 7.23.

Example 7.8. Mass Sliding with Dry Friction

A somewhat more complicated situation exists when the nonlinear transmission element is of such a nature that it requires description by a differential equation. Here the describing function will depend upon frequency as well as upon amplitude. An example of this kind is a mechanical system involving a mass sliding across a surface with dry friction. A driving force is applied to the mass, and the describing function is to relate the resulting velocity to the force. A practical example of this sort might involve the application of a torque to a shaft which carries a load having appreciable moment of inertia. The angular velocity of the load is of interest.

If the force varies simple harmonically, the system is described by the equation

$$M\dot{v} + \frac{hv}{|v|} = F \cos \omega t$$

where M is the mass, v is its velocity, and $F \cos \omega t$ is the applied force having amplitude F and angular frequency ω. The force needed to overcome friction is in magnitude h, and its algebraic sign is the same as the sign of the velocity.

The exact steady-state motion of the mass can be determined by a piecewise linear solution of two equations, as in Sec. 4.4, with the applicable equation depending upon the sign of v. The equations with their regions of applicability are

$$v \geq 0: \qquad \beta \leq \omega t \leq (\pi + \beta)$$
$$\dot{v} = \frac{1}{M}(F \cos \omega t - h)$$
$$v = \frac{1}{M\omega}\left[F \sin \omega t - h\left(\omega t - \frac{\pi}{2} - \beta\right)\right]$$
$$v \leq 0: \qquad (\pi + \beta) \leq \omega t \leq (2\pi + \beta)$$
$$\dot{v} = \frac{1}{M}(F \cos \omega t + h)$$
$$v = \frac{1}{M\omega}\left[F \sin \omega t + h\left(\omega t - \frac{3\pi}{2} - \beta\right)\right]$$

where $\sin \beta = -\pi h/2F$. A plot of the variation of force and velocity is shown in Fig. 7.24, where the arbitrary choice has been made, $h/F = 1/\pi$, so that $\beta = -30$

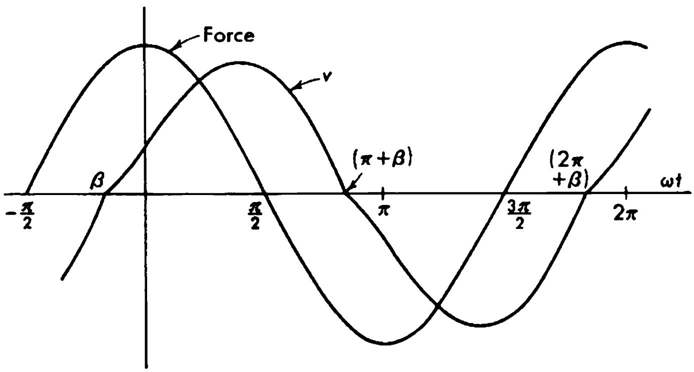


FIG. 7.24. Waveforms of force and velocity in a system of mass and dry friction of Example 7.8. This is an exact steady-state solution for the case of $h/F = 1/\pi$, or $\beta = -30$ degrees.

degrees. The velocity is zero at instants determined by angle β. This angle is negative and reduces to zero only if h is negligibly small in comparison with F. The defining relation for β leads to an imaginary angle if $h/F > 2/\pi$. If h/F exceeds this value, the velocity does not vary smoothly through zero, but so-called dead bands with no motion exist. In fact, if $h/F > 1$, no steady-state motion is possible at all.

The fundamental component of the velocity can be found by the usual Fourier analysis. Both cosine and sine components exist, but because of symmetry only half a complete cycle need be considered in the analysis. The fundamental cosine component has the amplitude

$$\begin{aligned} A_1 &= \frac{2}{\pi}\int_{\beta}^{\pi+\beta} \frac{1}{M\omega}\left[F \sin \omega t - h\left(\omega t - \frac{\pi}{2} - \beta\right)\right] \cos \omega t \, d(\omega t) \\ &= \frac{2}{\pi M \omega} 2h \cos \beta \end{aligned}$$

The fundamental sine component has the amplitude

$$B_1 = \frac{2}{\pi} \int_{\beta}^{\pi+\beta} \frac{1}{M\omega} \left[F \sin \omega t - h \left(\omega t - \frac{\pi}{2} - \beta \right) \right] \sin \omega t \, d(\omega t)$$
$$= \frac{2}{\pi M \omega} \left(\frac{\pi F}{2} + 2h \sin \beta \right)$$

The amplitude of the fundamental component of velocity is

$$V = (A_1^2 + B_1^2)^{1/2}$$

The describing function is the ratio $H = V/F$, which becomes in magnitude

$$|H| = \frac{1}{M\omega} \left[1 - \left(\frac{\pi^2}{4} - 1 \right) \left(\frac{4h}{\pi F} \right)^2 \right]^{1/2} \tag{7.49}$$

and in angle

$$\angle H = -\tan^{-1} \frac{B_1}{A_1}$$
$$= -\tan^{-1} \left(\frac{\pi F}{4h \cos \beta} + \tan \beta \right) \tag{7.50}$$

These relations are valid so long as $h/F \leq 2/\pi$. In general, however, a describing function is meaningful only so long as the variations do not differ too much from simple-harmonic functions of time. This requirement inherently limits h/F to small

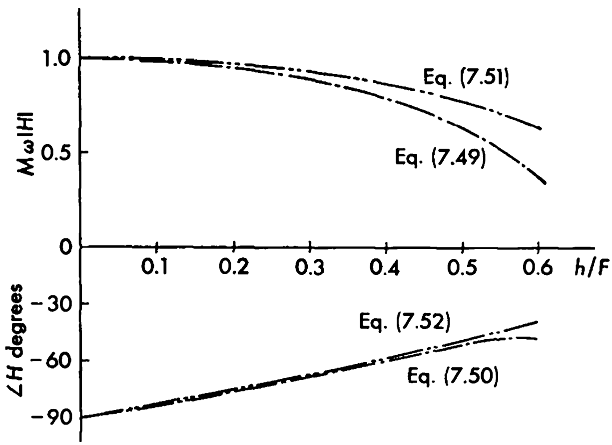


FIG. 7.25. Magnitude and angle of describing function for the mass-friction system of Example 7.8 as found by alternate methods of analysis.

values. The components of H as given by Eqs. (7.49) and (7.50) are plotted in Fig. 7.25.

A somewhat different approach to the question of determining a describing function in a case of this kind is based on the Ritz method. An approximate solution for the velocity is assumed as

$$v = V \cos (\omega t + \theta)$$

having the fundamental frequency of the driving force, and amplitude V and phase angle θ. The relation between instantaneous values of force and velocity is as shown in Fig. 7.26, where, as in Fig. 7.24, the arbitrary choice has been made, $h/F = 1/\pi$. For a system with both mass and friction, it can be recognized at once that velocity will lag the applied force, and phase angle θ is negative. The instant of zero velocity in Fig. 7.26 cannot be expected to be exactly the instant of zero velocity of Fig. 7.24, however.

Upon substitution of the assumed velocity into the differential equation for the system, the residual is found as

$$\epsilon = -M\omega V \sin(\omega t + \theta) \pm h - F \cos \omega t$$

where the sign preceding h must be the same as the sign of the velocity. In a problem such as this, the resulting velocity is not in phase with the driving force, and two

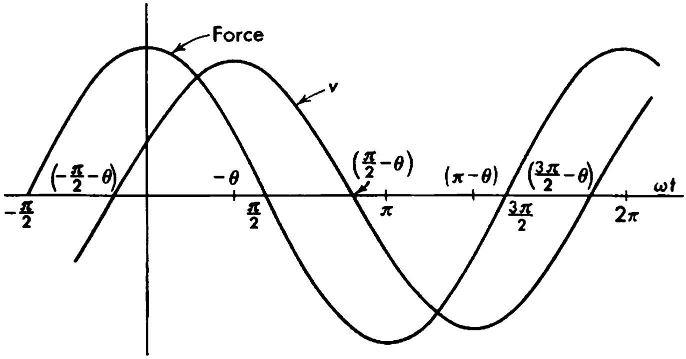


FIG. 7.26. Waveforms of force and velocity for the mass-friction system of Example 7.8. Here the velocity is assumed to vary sinusoidally, and the Ritz method of approximation is used. Again, $h/F = 1/\pi$.

parameters of the solution, V and θ, must be found. Two integrals of the Ritz method which must be satisfied are

$$\int_0^{2\pi} \epsilon \cos \omega t \, d(\omega t) = 0$$

$$\int_0^{2\pi} \epsilon \sin \omega t \, d(\omega t) = 0$$

The weighting functions, $\cos \omega t$ and $\sin \omega t$, are, respectively, in and out of phase with the driving force. Accordingly, the two integrals deal with components of the velocity in phase and out of phase with the driving force.

The first of the integrals leads to the requirement

$$-\frac{\pi M\omega V}{2} \sin \theta + 2h \cos \theta - \frac{\pi F}{2} = 0$$

and the second to

$$-\frac{\pi M\omega V}{2} \cos \theta - 2h \sin \theta = 0$$

Solution for θ gives $\cos \theta = 4h/\pi F$, where the negative value of θ must be taken. Solution for V gives

$$V = \frac{F}{M\omega}\left[1 - \left(\frac{4h}{\pi F}\right)^2\right]^{1/2}$$

The describing function is in magnitude

$$|H| = \frac{V}{F} = \frac{1}{M\omega}\left[1 - \left(\frac{4h}{\pi F}\right)^2\right]^{1/2} \tag{7.51}$$

and in angle

$$\angle H = \theta = \cos^{-1} \frac{4h}{\pi F} \tag{7.52}$$

where the negative sign must be taken for θ. As in the preceding analysis, this result is obviously meaningful only if the ratio h/F is not too large. Here the equations are completely invalid if $h/F > \pi/4$.

Components of H as given by Eqs. (7.51) and (7.52) are plotted in Fig. 7.25. Good agreement is seen to exist between the results of the two ways of finding the describing function, so long as h/F is less than, say, 0.5. If h/F is this large, however, the waveform of the velocity is beginning to depart considerably from a sinusoid and the describing function no longer has real significance.

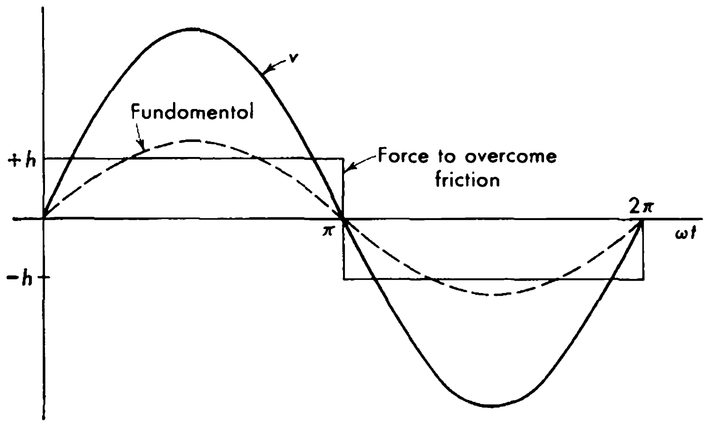


FIG. 7.27. Sinusoidal velocity in the mass-friction system of Example 7.8, together with the force needed to overcome friction and the fundamental component of this force.

Finally, a third way of finding the describing function for this same system is based on considering an equivalent resistance for the friction effect. If just the friction element is considered and it is assumed to be forced into a sinusoidal velocity $v = V \sin \omega t$, the instantaneous force and velocity are as shown in Fig. 7.27. The force needed to overcome friction is a square wave of amplitude h. The fundamental component of this square wave has the amplitude $4h/\pi$. A mechanical-resistance coefficient R_m describing the friction effect may be defined as

$$R_m = \frac{\text{fundamental force}}{\text{fundamental velocity}} = \frac{4h/\pi}{V} = \frac{4h}{\pi V}$$

This equivalent resistance is not constant in value but becomes infinitely large as the velocity is made smaller.

If now both the mass and friction are considered, the equivalent mechanical impedance becomes

$$Z_m = R_m + jM\omega = \frac{F}{V} = (R_m^2 + M^2\omega^2)^{1/2} \angle \tan^{-1} \frac{M\omega}{R_m}$$

where F is the driving force. The describing function has been defined as $H = V/F$, and thus it is in magnitude

$$|H| = \frac{1}{M\omega} \left[1 - \left(\frac{4h}{\pi F} \right)^2 \right]^{1/2} \tag{7.53}$$

and in angle

$$\angle H = \tan^{-1}\frac{M\omega}{R_m} = \cos^{-1}\frac{4h}{\pi F} \tag{7.54}$$

where the negative sign must be taken. These two equations are identical with those found just previously as Eqs. (7.51) and (7.52).

In summary, three methods have been presented for obtaining a describing function for this system. All of them are meaningful only so long as the ratio h/F is not too large. The last two processes, based on the Ritz method and upon obtaining an equivalent resistance for the friction effect, lead to the same results. The first method, involving first a piecewise linear solution of two equations and then a Fourier analysis, is a longer process and leads to slightly different results. So long as ratio h/F is small, however, the three results are essentially the same.

7.6. Forced Oscillation in Self-oscillatory System. The type of system discussed thus far in the present chapter is such that the system is quiescent unless an external driving force is applied. When a driving force is present, the nature of the resulting response depends upon properties of both the system and the force. There are, however, self-oscillating systems in which oscillations occur with no external driving force. Such a system has been discussed in some detail in Example 6.7. A simple-harmonic driving force can also be applied to this kind of system and the resulting response investigated. Analysis shows that several new phenomena occur here, including that of frequency entrainment, in which the driving force takes over control of the oscillation frequency.

Example 7.9. Forced Negative-resistance Oscillator

In Example 6.7, the operation of a negative-resistance oscillator is considered. Analytical methods are used there to study the self-oscillation of the system under conditions such that the waveform is essentially sinusoidal. Under this condition, any change which may take place in the amplitude can occur only relatively slowly. Ultimately, a steady state is established, corresponding to the appearance of a limit cycle with the amplitude of oscillation dependent solely upon circuit parameters. This self-oscillation can be thought of as a kind of sustained free oscillation in the system.

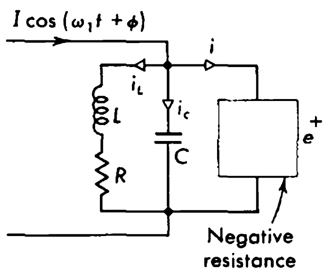


FIG. 7.28. Nonlinear negative-resistance oscillator driven by an injected simple-harmonic current.

A somewhat different situation exists if the same oscillator is driven from an external source. A sinusoidal current can be supplied to the circuit as shown in Fig. 7.28. This is the same as the circuit of Fig. 6.7, with any resistance in the self-inductance neglected, and with a current supplied from an external generator. For this condition, forced oscillations as well as free oscillations may occur.

If the negative resistance is described by Eq. (6.49) as before, the equation for the circuit is

$$C\dot{e} - ae + be^3 + \frac{1}{L}\int e\,dt = I\cos(\omega_1 t + \phi)$$

where I is the amplitude of the driving current, its angular frequency is ω_1, and angle ϕ is a phase angle inserted in the expression for the current as matter of convenience. Other coefficients of the equation are the same as those of Example 6.7. Upon differentiation to remove the integral, the equation becomes

$$\ddot{e} - \alpha(1 - \beta e^2)\omega_0\dot{e} + \omega_0^2 e = -\left(\frac{I\omega_1}{C}\right)\sin(\omega_1 t + \phi) \tag{7.55}$$

where $\alpha = a/C\omega_0$, $\beta = 3b/a$, and $\omega_0^2 = 1/LC$ as before. This is Eq. (6.50) with the addition of the forcing term on the right side.

In general, it might be expected that a solution for Eq. (7.55) would involve both a free oscillation, analogous to that already found in Example 6.7, and a forced oscillation produced by the driving current. It turns out that under certain conditions only one or the other of these components exists, while under other conditions both exist simultaneously. If parameter α is small compared with unity, the free oscillation has been found to be essentially sinusoidal. It is not unreasonable to expect that the forced oscillation also will be essentially sinusoidal for this condition. Thus, an approximate solution for Eq. (7.55) might be expected to have the form

$$e = E\cos\omega t + E_1\cos\omega_1 t \tag{7.56}$$

The first term here represents the free oscillation at a frequency ω determined by circuit parameters. The second term represents the forced oscillation at the driving frequency ω_1. A case of particular interest is that where ω and ω_1 are not far different. If this is so, there is no reason for the ratio ω/ω_1 to be a rational fraction and e, given by the sum in Eq. (7.56), is not generally periodic. For this reason, there is no necessity for including a phase angle in either term of the sum, and simple cosine functions are sufficient. The single phase angle ϕ associated with the current in Eq. (7.55) is all that is required.

The four unknown quantities in the solution are the two amplitudes E and E_1, the frequency ω of the free oscillation, and the phase angle ϕ. Four simultaneous equations for determining these unknowns can be set up by substituting the assumed solution of Eq. (7.56) into the circuit equation, Eq. (7.55). Quantities needed in this substitution are the following.

$$
\begin{aligned}
\dot{e} &= -\omega E \sin \omega t - \omega_1 E_1 \sin \omega_1 t \\
\ddot{e} &= -\omega^2 E \cos \omega t - \omega_1{}^2 E_1 \cos \omega_1 t \\
e^2\dot{e} &= -\left(\frac{\omega E^3}{4}\right)(\sin \omega t + \sin 3\omega t) \\
&\quad - \left(\frac{\omega E^2 E_1}{2}\right)[\sin (2\omega + \omega_1)t + \sin (2\omega - \omega_1)t] \\
&\quad - \left(\frac{\omega E E_1{}^2}{4}\right)[2 \sin \omega t + \sin (\omega + 2\omega_1)t + \sin (\omega - 2\omega_1)t] \\
&\quad - \left(\frac{\omega_1 E_1 E^2}{4}\right)[2 \sin \omega_1 t + \sin (\omega_1 + 2\omega)t + \sin (\omega_1 - 2\omega)t] \\
&\quad - \left(\frac{\omega_1 E E_1{}^2}{2}\right)[\sin (2\omega_1 + \omega)t + \sin (2\omega_1 - \omega)t] \\
&\quad - \left(\frac{\omega_1 E_1{}^3}{4}\right)(\sin \omega_1 t + \sin 3\omega_1 t)
\end{aligned}
$$

Trigonometric identities have been used in writing this last relation. Upon substitution into Eq. (7.55) and collection of corresponding terms, the following four relations result.

$$\cos \omega t: \qquad E(\omega_0{}^2 - \omega^2) = 0 \tag{7.57}$$

$$\sin \omega t: \qquad \alpha\omega\omega_0 E\left[1 - \frac{\beta}{4}(E^2 + 2E_1{}^2)\right] = 0 \tag{7.58}$$

$$\cos \omega_1 t: \qquad E_1(\omega_0{}^2 - \omega_1{}^2) = -\left(\frac{I\omega_1}{C}\right)\sin \phi \tag{7.59}$$

$$\sin \omega_1 t: \qquad \alpha\omega_1\omega_0 E_1\left[1 - \frac{\beta}{4}(E_1{}^2 + 2E^2)\right] = -\left(\frac{I\omega_1}{C}\right)\cos \phi \tag{7.60}$$

In accordance with the principle of harmonic balance, only those terms having the two frequencies ω and ω_1, the fundamental frequencies of the free and forced oscillations, are considered. Terms of other frequencies are neglected. Further, there is the implicit assumption that the two frequencies are different, $\omega \neq \omega_1$. If the two were the same, $\omega = \omega_1$, additional terms would need to be considered, since then, for example, $\sin (2\omega - \omega_1)t = \sin (2\omega - \omega)t = \sin \omega t$.

From Eq. (7.57), it is evident that either $E = 0$, in which case there is no free oscillation, or else $\omega^2 = \omega_0{}^2$ and the frequency of free oscillation is the same as ω_0. This is the result found in Example 6.7.

From Eq. (7.58), either $E = 0$ or $1 - (\beta/4)(E^2 + 2E_1{}^2) = 0$. This result can be rewritten $E^2 = 4/\beta - 2E_1{}^2$. If there is no forced oscillation, $E_1 = 0$ and $E^2 = 4/\beta$, which is the result found in Example 6.7.

If a forced oscillation is present and in magnitude sufficient so that $E_1^2 \geq 2/\beta$, amplitude E becomes imaginary and there is no free oscillation. Free oscillation can exist simultaneously with a forced oscillation only if the amplitude of the forced oscillation is within the limits set by $0 \leq E_1^2 \leq 2/\beta$.

Evidently, if a forced oscillation exists of amplitude large enough so that $E_1^2 \geq 2/\beta$, there is no free oscillation and $E = 0$. This is a case of some interest. If $E = 0$, Eq. (7.60) becomes

$$\alpha\omega_1\omega_0 E_1\left(1 - \frac{\beta E_1^2}{4}\right) = -\left(\frac{I\omega_1}{C}\right)\cos\phi$$

Both sides of this relation and of Eq. (7.59) can be squared and the two added to give

$$E_1^2\left[(\omega_0^2 - \omega_1^2)^2 + (\alpha\omega_1\omega_0)^2\left(1 - \frac{\beta E_1^2}{4}\right)^2\right] = \left(\frac{I\omega_1}{C}\right)^2 \tag{7.61}$$

This equation can be put into simpler form by dividing each term by the quantity $(\alpha\omega_1\omega_0E_1)^2$ and introducing the dimensionless parameters $x = (\omega_1^2 - \omega_0^2)/\alpha\omega_0\omega_1$, $y = E_1^2/(4/\beta)$, and $A^2 = \beta(I/2\alpha\omega_0C)^2$. If ω_1 and ω_0 are not far different, approximately $\omega_1 + \omega_0 = 2\omega_1$ and

$$x = \frac{2(\omega_1 - \omega_0)}{\alpha\omega_0}$$

Thus, very nearly parameter x is proportional to the difference $\omega_1 - \omega_0$ and measures the difference between the driving frequency and the frequency of free oscillation with no externally applied driving current. Parameter y is proportional to E_1^2, the square of the amplitude of the forced oscillation, the only oscillation existing under the assumptions now being considered. Finally, parameter A is proportional to the injected current I, the amplitude of the forcing function.

With these definitions, Eq. (7.61) becomes

$$[x^2 + (1 - y)^2]y = A^2 \tag{7.62}$$

valid for $E = 0$, which requires $y \geq \frac{1}{2}$. If the free oscillation is present, so that $E \neq 0$, an exactly similar procedure can be followed with the addition of the use of Eq. (7.58) to remove the term in E. The result is

$$[x^2 + (1 - 3y)^2]y = A^2 \tag{7.63}$$

This equation is valid for $E \neq 0$, which in turn requires $0 \leq E_1^2 \leq 2/\beta$ or $0 \leq y \leq \frac{1}{2}$.

A plot of Eq. (7.62), applying to the forced oscillation alone, $E = 0$, is shown in Fig. 7.29. Since the equation involves only x^2, it is evident that curves must be symmetrical about $x = 0$. The curves are somewhat similar to response curves for a driven resonant circuit. The parameter for the family of curves is A^2, which depends upon the square of the amplitude of the injected current. If $A^2 = 0$, either $y = 0$ or else $y = 1$ and $x = 0$. If A^2 is just greater than zero, there are two branches to the curve. One branch is near the x axis, while the other is a closed curve about the point $y = 1$, $x = 0$. If A^2 has the value $A^2 = 4/27$, these two branches join together at the point $y = 1/3$, $x = 0$. A still

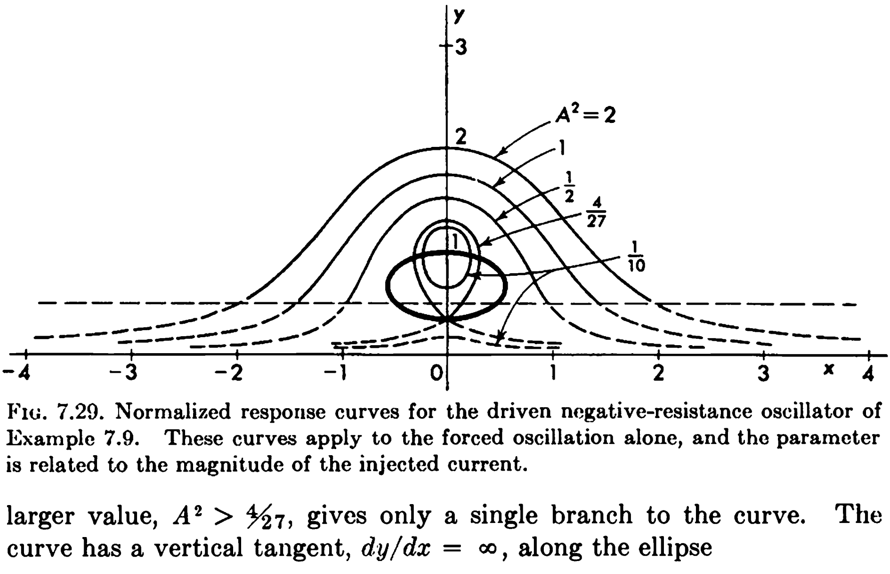


FIG. 7.29. Normalized response curves for the driven negative-resistance oscillator of Example 7.9. These curves apply to the forced oscillation alone, and the parameter is related to the magnitude of the injected current.

larger value, $A^2 > 4/27$, gives only a single branch to the curve. The curve has a vertical tangent, $dy/dx = \infty$, along the ellipse

$$3y^2 - 4y + x^2 + 1 = 0$$

shown in Fig. 7.29. Any curve passing within this ellipse has certain values of x for which three values of y exist. If $A^2 = 8/27$, the curve is just tangent to the ellipse, and if $A^2 > 8/27$, the curve remains outside the ellipse and y is always a single-valued function of x. It should be recognized, of course, that Eq. (7.62) is valid only for $y \geq 1/2$, and the dashed portions of Fig. 7.29 for $y < 1/2$ do not actually apply to the system.

If $y \leq 1/2$, both free and forced oscillations exist and the solution consists of their sum, as in Eq. (7.56). It has already been assumed that the frequency ratio ω/ω_1 is irrational, and thus there is no periodicity in the resulting solution. In experimental practice, where components at several frequencies exist simultaneously, it is often convenient to describe the amplitude of the complex wave in terms of its rms value. Many laboratory instruments are constructed so that they measure the rms value directly. Therefore, in this case, where both free and forced oscillations exist, it is desirable to describe the sum in terms of its rms value. Actually, dimensionless variable y is defined as the square of the voltage amplitude, $y = E_1^2/(4/\beta)$. Thus, where both free and forced oscillations are present, the analogous quantity is $\bar{y} = (E_1^2 + E^2)/(4/\beta)$. In either case, y or $\bar{y}$ is proportional to the square of the indication of a voltmeter reading the rms value of voltage e.

It has already been shown from Eq. (7.58) that, with both free and forced oscillation present, it is necessary that $1 - (\beta/4)(E + 2E_1^2) = 0$, and thus $\bar{y}$ can be written

$$\bar{y} = 1 - y \tag{7.64}$$

In summary, then, Eq. (7.62) applies under the requirement that $y > \frac{1}{2}$ so that only the forced oscillation exists and $E = 0$. If $y < \frac{1}{2}$, both

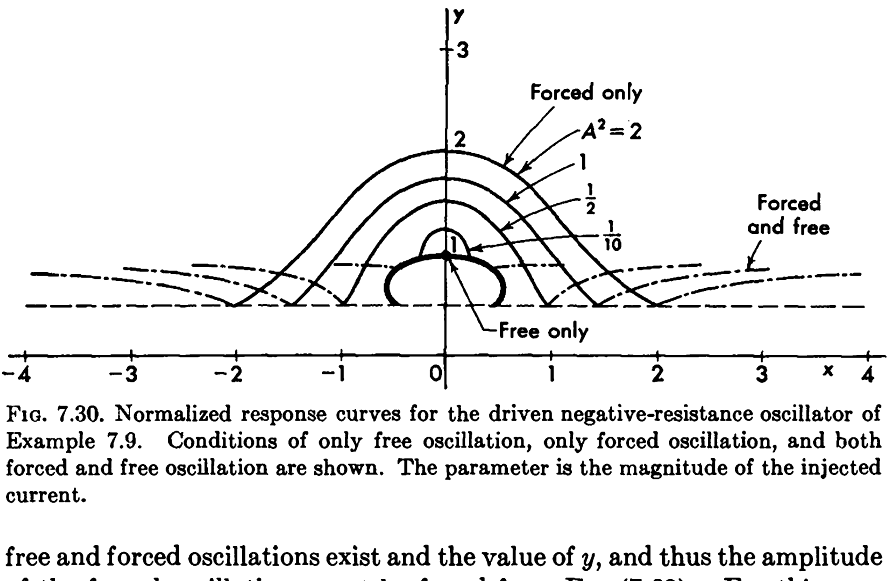


FIG. 7.30. Normalized response curves for the driven negative-resistance oscillator of Example 7.9. Conditions of only free oscillation, only forced oscillation, and both forced and free oscillation are shown. The parameter is the magnitude of the injected current.

free and forced oscillations exist and the value of y, and thus the amplitude of the forced oscillation, must be found from Eq. (7.63). For this condition, the mean-square amplitude of the complex wave can be found from Eq. (7.64) for $\bar{y}$.

In Fig. 7.30 is plotted a family of curves showing y or $\bar{y}$, proportional to the mean-square amplitude, as a function of x, where the appropriate equation is used in each region of the figure. In general, the ellipse along which $dy/dx = \infty$ for Eq. (7.62) can be shown to represent a transition from one mode of operation to another. Similarly, the line $y = \frac{1}{2}$ outside the ellipse represents a transition. A detailed analysis in the neighborhood of the ellipse indicates that jump phenomena and hysteresis effects may appear.

If the definitions for y, x, and A^2, together with the value for A^2/y as found from Eq. (7.62), are used in Eq. (7.59), the phase angle ϕ may be found as

$$\tan \phi = \frac{x}{1 - y} \tag{7.65}$$

This is the phase angle between the driving current and the resulting forced oscillation. If $x > 0$, or driving frequency ω_1 exceeds frequency ω_0, phase angle ϕ is positive and the driving current leads the oscillating voltage. Similarly, if $x < 0$, $\omega_1 < \omega_0$, and the current lags the voltage. Contours of constant phase angle ϕ are plotted in Fig. 7.31. This figure, of course, applies to the case where only the forced oscillation is present.

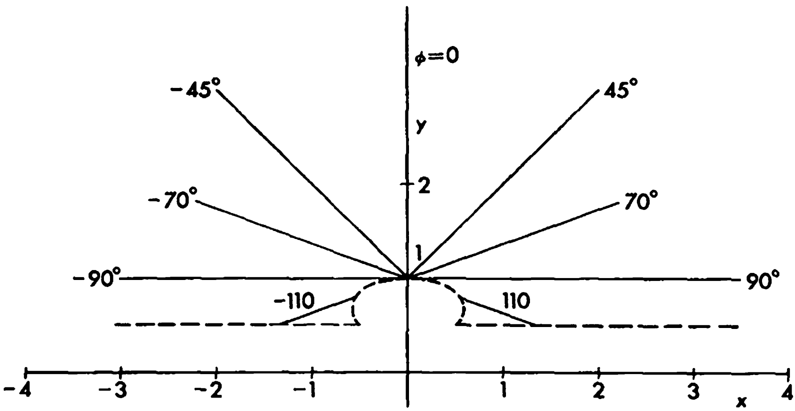


FIG. 7.31. Phase angle for the forced oscillation in the driven negative-resistance oscillator of Example 7.9.

The operation of the oscillator with a driving current injected can be summarized as follows, with reference to Figs. 7.30 and 7.31: If the amplitude of the injected current is zero, only the free oscillation exists, with its frequency ω the same as ω_0 and $\bar{y} = 1$. If the injected current has an amplitude greater than zero and its frequency ω_1 is near ω_0, only the forced oscillation exists, with its frequency ω the same as ω_1. If ω_1 is changed, remaining always sufficiently near ω_0, the frequency of oscillation ω always follows ω_1. This phenomenon, in which the oscillator is locked, so to speak, with the frequency of the injected current, is known as frequency entrainment. It is peculiar to nonlinear systems. The phase of the injected current leads that of the oscillation voltage if ω_1 exceeds ω_0 and lags if ω_1 is less than ω_0. If ω_1 becomes sufficiently different from ω_0, free oscillation begins and both free oscillation at ω_0 and forced oscillation at ω_1 exist simultaneously. The more ω_1 differs from ω_0, the smaller are both the amplitude of the forced oscillation and its effect upon the free oscillation.

This analysis has been concerned with the case where the frequency ω_1 of the injected current is not far different from the frequency ω_0 of the free oscillation. Under certain conditions, frequency entrainment has been found to occur, with the free oscillation being controlled entirely by the injected current. A similar phenomenon may occur also if the frequency of the injected current is near an integral multiple of the free frequency, $\omega_1 = n\omega_0$, where n is an integer. In this case, frequency demultiplication is said to exist, with the frequency of oscillation being $\omega = \omega_1/n$. For a system with a cubic nonlinearity, such as considered here, odd values for integer n are most easily found experimentally, although even values may exist also.

7.7. Summary. A number of examples have been given in the present chapter for nonlinear systems driven by a simple-harmonic forcing function. The process of solution used in each case is essentially an obvious extension of one or another of the analytical methods developed in the preceding chapter. Because of the variety of solutions which can be obtained for even a relatively simple-appearing nonlinear equation, and because no single approach generally gives all these solutions, it is important to know what kind of solution is being sought and to arrange details so as to lead to this kind of solution.

In a great many cases, about the only approach to a driven system is that based on the principle of harmonic balance. The solution is obtained as just the fundamental component of what is recognized to be a complex wave made up of many components in addition to the fundamental. In theory, additional components can be considered in the analysis, but in practice the algebra involved usually becomes too unwieldy to lead to useful results. Often some kind of perturbation or iteration process can be used, once the fundamental component has been found, so as to give an indication of the importance of additional components. If the additional components are relatively small, this process may lead to solutions of acceptable accuracy.

Certain concepts based on linear theory have been extended to nonlinear systems. These include the concepts of equivalent impedance and of describing functions. They have been applied with some success, particularly in the study of the stability of nonlinear systems. This topic is considered further in Sec. 10.6.

A number of phenomena occur in driven nonlinear systems which cannot exist in purely linear systems. Among these phenomena are the appearance of discontinuous jumps in amplitude, generation of harmonics and subharmonics, rectification, and frequency entrainment.

CHAPTER 8

SYSTEMS DESCRIBED BY DIFFERENTIAL-DIFFERENCE EQUATIONS

8.1. Introduction. The discussion in all the preceding chapters has centered around pure differential equations and types of physical systems that can be described by this sort of equations. In a pure differential equation with time the independent variable, each term in the equation is evaluated at the same instant in time. A great many physical systems can be described accurately by equations of this kind.

There are, however, certain types of systems in which fixed time delays exist. The most obvious situation leading to a definite delay is that where the quantity in the system represented by the dependent variable undergoes transmission over a fixed distance in space with a finite velocity. An obvious case of this sort is a processing machine of some kind in which the material being processed moves from one position to another. The ultimate performance of the machine depends upon what has gone on at each position, with definite time delays associated with transportation from one position to the next. An equation describing the over-all operation of this kind of system must account for the time differences between successive steps.

An analogous delay exists in any kind of signal-transmission system in which an electrical or acoustical transmission line is involved. A transmission line has the property of requiring a fixed time interval for a signal to propagate from one end of the line to the other. Any system incorporating a transmission line requires consideration of the associated time delay.

Mathematical description of most physical systems requires the use of time derivatives of the dependent variables, thus leading to differential equations. If, in addition, constant time delays appear in the system, the description requires what are known as differential-difference equations. These equations include terms with derivatives, and the various terms of the equation are not all evaluated at the same instant in time. Just as in the case of pure differential equations, a realistic description of a typical physical system requires the use of nonlinear terms. Nonlinear differential-difference equations therefore arise. These equations are characteristically more difficult to solve than are qualitatively similar pure differential equations. Their solutions are comparably more complicated.

In the present chapter is given a brief discussion of certain differential-difference equations, and several examples of physical systems are worked out. Techniques of solution used are modifications of those already discussed for pure differential equations. A phenomenon often introduced into a physical system because of delays is the appearance of oscillations in what would otherwise be a nonoscillatory system. For this reason, the following discussion deals primarily with oscillatory solutions.

8.2. Linear Difference Equations. A linear pure difference equation may have the general form

$$a_n x_{(t-n\tau)} + a_{(n-1)} x_{[t-(n-1)\tau]} + \cdots + a_0 x_t = F(t) \tag{8.1}$$

where $a_n, a_{n-1}, \ldots, a_0$ are constant coefficients, x is the dependent variable, t is the independent variable, τ is a fixed time delay, and $F(t)$ is a forcing function. The several terms on the left side of the equation represent x as evaluated at integral multiples, $n, n-1, \ldots, 0$, of τ previous to the instant t. This equation must hold as time t varies continuously.

It can be shown that an equation of this kind has an exact solution which is similar in many respects to the solution for a linear pure differential equation with constant coefficients. The order of the equation is n, the number multiplying the basic delay interval τ so as to give the maximum delay in the equation. The solution consists of two parts, a complementary function and a particular integral. The complementary function satisfies the equation if $F(t) = 0$ and contains as many arbitrary constants as the order of the equation. Initial or boundary conditions are needed to evaluate these constants. The particular integral satisfies the equation if $F(t) \neq 0$, and is completely determined by the equation with $F(t)$ included.

While an exact solution for Eq. (8.1) can be found in a relatively simple manner, the general possibility of highly complicated solutions for equations with fixed delays can be recognized at once. The difference equation may be converted into a pure differential equation, with all terms evaluated at the same instant in time, by using the Taylor's-series forms

$$\begin{aligned} x_{(t-\tau)} &= x_t - \dot{x}\tau_t + \frac{\tau^2}{2}\ddot{x}_t - \frac{\tau^3}{6}\dddot{x}_t + \cdots \\ x_{(t-n\tau)} &= x_t - n\tau\dot{x}_t + \frac{n^2\tau^2}{2}\ddot{x}_t - \frac{n^3\tau^3}{6}\dddot{x}_t + \cdots \end{aligned} \tag{8.2}$$

These are infinite series. When they are substituted into Eq. (8.1), the result is a pure differential equation with every term evaluated at the same instant. Because derivatives of all orders are present, the differential equation is of infinite order. It can be expected to have an infinity of components in its solution. The delay time τ may, of course, be small enough for reasonable accuracy to be obtained with only a few terms in the series of Eq. (8.2). If this is the case, the approximately equivalent differential equation is of lower order and its solution is less complicated.

For a pure difference equation, such as Eq. (8.1), terms of the Taylor's series combine in such a way that the solution is actually quite simple. If derivatives appear in addition to the terms of Eq. (8.1), giving a differential-difference equation, highly complicated solutions can be expected.

The complementary function for Eq. (8.1) can be found in much the same way as this solution is obtained for linear differential equations with constant coefficients. The process is described in Sec. 5.3 and is based on properties of the exponential function. The exponential function retains its form upon successive differentiation. Such a function can be substituted into a differential equation, the common exponential appearing in each term can be factored out, and the remaining algebraic equation solved for allowed values of the exponent. Much the same operation can be used with Eq. (8.1).

If a solution is assumed as

$$x_t = C \exp(\lambda t) \tag{8.3}$$

and substituted into Eq. (8.1) with $F(t) = 0$, the result is

$$C \exp(\lambda t)\{a_n \exp(-n\lambda\tau) + a_{(n-1)} \exp[-(n-1)\lambda\tau] + \cdots + a_0\} = 0 \tag{8.4}$$

where C is an arbitrary constant. Evidently a nontrivial solution requires the quantity in the brackets to be zero. This gives the so-called characteristic equation. Terms of this equation involve $\exp(\lambda)$ raised to certain powers. It is convenient to write $k = \exp(\lambda)$, which gives the alternate form for the characteristic equation

$$a_n k^{-n\tau} + a_{(n-1)} k^{-(n-1)\tau} + \cdots + a_0 = 0 \tag{8.5}$$

This is an algebraic polynomial of degree n in the quantity $k^{-\tau}$. In general, there are n roots, from which n values of k can be found. If coefficients $a_n, \ldots, a_0$ are real, values of k must be real, or occur in complex conjugate pairs. If k is known, exponent λ can be found from the defining relation. Because λ may be complex, more than one value for λ may arise from a single value of k. If the characteristic equation is written in terms of exp (λ), rather than k, it is typically a transcendental equation with an infinity of roots.

The complementary function for Eq. (8.1) may have either of the alternate forms

$$\begin{aligned} x &= C_1 k_1{}^t + C_2 k_2{}^t + \cdots + C_n k_n{}^t \\ x &= C_1 \exp(\lambda_1 t) + C_2 \exp(\lambda_2 t) + \cdots + C_n \exp(\lambda_n t) \end{aligned} \tag{8.6}$$

where $k_1, \ldots, k_n$ and $\lambda_1, \ldots, \lambda_n$ are found from the roots of the characteristic equation. If a root k_1 is repeated, occuring twice, say, the form of the solution must be modified as $x = (C_1 + C_2 t)k_1{}^t + \cdots$, just as in the case of differential equations. Arbitrary constants $C_1, \ldots, C_n$ must be found from initial conditions.

If $F(t)$ in Eq. (8.1) is not zero, a complete solution requires a particular integral in addition to the complementary function. The particular integral, also, can be found by procedures similar to those used with differential equations.

It is worth noting that, if the assumed exponential solution of Eq. (8.3) is used in the Taylor's series of Eq. (8.2), the result for $x_{(t-n\tau)}$, for example, is

$$\begin{aligned} x_{(t-n\tau)} &= C \exp(\lambda t) \left[1 - n\tau\lambda + \frac{(n\tau\lambda)^2}{2} - \frac{(n\tau\lambda)^3}{6} + \cdots \right] \\ &= C \exp(\lambda t) \exp(-n\tau\lambda) \end{aligned}$$

The terms of the series can be combined in closed form as the exponential term appearing in the braces of Eq. (8.4). Thus, the pure difference equation does not have the infinity of components in its solution which might at first be expected from the infinite number of derivatives in the Taylor's series.

Example 8.1. Linear Oscillator with Delay

A simple system with a fixed delay is shown in Fig. 8.1. It represents an electronic amplifier with amplification G combined with some kind of transmission line giving a fixed delay τ. Instantaneous voltages at certain points are indicated as e_1, e_2, and e_3. Because of the circuit arrangement, $e_1 = e_3$. In the simplest case, $G = e_3/e_2$ is a real number, although G may be either positive or negative. Similarly, the delay time is such that $e_2(t) = e_1(t - \tau)$, with no change in magnitude or sign. Actual construction of a system meeting these conditions is difficult, since any physical amplifier always shows low-pass properties and G ultimately drops in magnitude and acquires a phase angle as frequency increases. Similarly any physical delay system usually shows changes in delay time and losses in transmission as frequency increases.

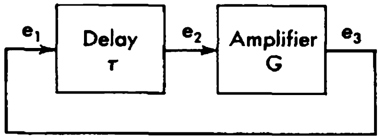


FIG. 8.1. Circuit for linear oscillator with delay of Example 8.1.

Operation of the system of Fig. 8.1 can be predicted from a simple physical consideration. Oscillation may occur at any frequency for which the phase shift of transmission, $\omega\tau$, through the loop made up of the amplifier and delay circuit is an integral multiple of 2π radians. If G is positive, evidently the angular frequency of oscillation must be such that the phase shift $\omega\tau$ is an integral multiple of 2π. If G is negative, equivalent to an inversion of waveform by the amplifier, $\omega\tau$ must be an odd multiple of π. If the magnitude of G is exactly unity, the signal is of constant amplitude as it circulates around the system. If $|G| > 1$, the signal grows progressively, while if $|G| < 1$, the signal decays. This system has the interesting property of being a completely linear system capable of oscillating simultaneously at an infinite number of harmonically related frequencies.

The equation describing the system is

$$x_{(t)} - Gx_{(t-\tau)} = 0 \tag{8.7}$$

where $x_{(t)}$ represents voltage e_1 at the input of the delay circuit. This is a linear difference equation of the form of Eq. (8.1). A solution for it may be assumed in the form of Eq. (8.3). Upon substitution of this form, the characteristic equation is found as

$$1 - G \exp(-\lambda\tau) = 1 - Gk^{-\tau} = 0 \tag{8.8}$$

where $k = \exp(\lambda)$. The first-order difference equation leads to a single possible value of k, which is $k = G^{1/\tau}$. In terms of λ, however, complications arise. Since oscillations are expccted, the root λ must be allowed to be complex, as $\lambda = \delta \pm j\omega$, and the characteristic equation can be split into real and imaginary parts as

$$\begin{aligned} &\text{Real:} && \exp(-\delta\tau)\cos\omega\tau = \frac{1}{G} \\ &\text{Imaginary:} && \exp(-\delta\tau)\sin\omega\tau = 0 \end{aligned} \tag{8.9}$$

Since $\exp(-\delta\tau)$ is not zero in cases of interest, the second of these relations requires $\sin \omega\tau = 0$, or $\omega\tau$ must be an integral number of π radians. Then, from the first relation, the possibilities can be tabulated as:

$\omega\tau$	. . .	-3π	-2π	$-\pi$	0	$+\pi$	$+2\pi$	$+3\pi$	. . .
$\cos\omega\tau$	. . .	-1	$+1$	-1	$+1$	-1	$+1$	-1	. . .
$\exp(-\delta\tau)$. . .	. . .	$-1/G$	$+1/G$	$-1/G$	$+1/G$	$-1/G$	$+1/G$	$-1/G$	. . .
Sign of G. . .	. . .	$-$	$+$	$-$	$+$	$-$	$+$	$-$	. . .

Since $\exp(-\delta\tau)$ must be positive, the algebraic sign of G determines which series of allowed values of $\omega\tau$ can exist. If $G > 0$, $\omega\tau$ may be any integral multiple of 2π; if $G < 0$, $\omega\tau$ must be an odd multiple of π. Since ω is the angular frequency of oscillation, an infinite number of discrete frequencies is allowed, with their values dependent upon the sign of G. Exponent δ determines whether the oscillation grows or decays with time, positive δ leading to growth. Always $\delta\tau = \ln |G|$. In summary, a solution for Eq. (8.7) may have the general form

$$x_{(t)} = \exp(\delta t) \sum_{i=1}^{\infty} A_i \cos(\omega_i t + \theta_i) \tag{8.10}$$

where δ and ω_i meet the requirements which have already been stated and which are plotted in Fig. 8.2. Constants A_i and θ_i depend upon initial conditions. If $G > 0$, a possible value for ω is zero and a nonoscillatory solution may exist. It is worth noting again that the quite complicated solution, Eq. (8.10), arises from the deceptively simple equation, Eq. (8.7).

Because of the infinity of arbitrary constants in the solution, an infinity of initial conditions is required. The value of $x_{(t)}$ must be specified for the entire interval $-\tau \leq t \leq 0$. What happens physically is that a voltage varying in some prescribed manner is applied at the input of the delay circuit, but the output voltage from the delay circuit remains zero for an interval τ. Only after this interval has passed does a voltage actually reach the amplifier. Thus, a particular kind of voltage variation of duration τ is effectively stored within the delay circuit at any instant. Initial conditions are obviously rather different from those of a system described by a pure

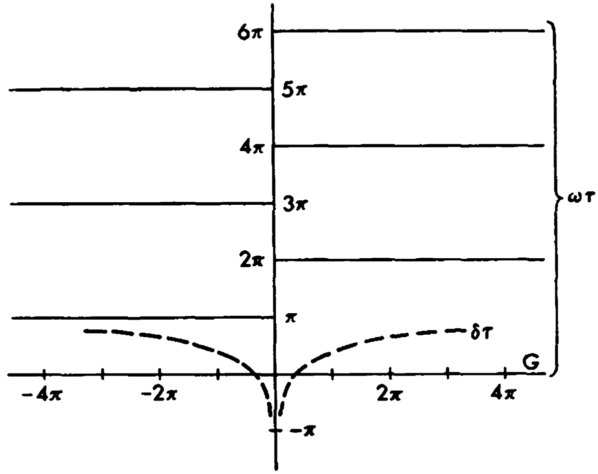


FIG. 8.2. Relations for the characteristic roots for the oscillator of Example 8.1.

differential equation. Furthermore, the waveform generated by this kind of oscillator depends upon how the system is put into oscillation.

8.3. Linear Differential-difference Equation.

If in addition to the terms with x evaluated at different instants in time, as written in Eq. (8.1), derivatives of x also appear, the equation becomes a differential-difference equation. In simplest form, only first powers of x and its derivatives are present, and the equation is linear. The complementary function for a linear differential-difference equation may be found in much the same manner as though only one of the two types of terms, derivatives and shifted terms, appeared. The characteristic equation for a differential-difference equation is usually a transcendental algebraic equation and may have an infinity of roots, both real and complex.

Example 8.2. Linear Oscillator with Delay and Low-pass Amplifier

A physically realizable electronic amplifier is a low-pass device because of the effects of shunt capacitances and series inductances that are always present. The simplest system which displays the effects of these low-pass properties is illustrated in Fig. 8.3. The first box, containing the delay circuit, and the third box, containing the amplifier, are assumed to be the same as the analogous boxes of Fig. 8.1. The middle box of Fig. 8.3 contains a four-terminal combination of resistance R and capacitance C which produces a low-pass action. Instantaneous voltages e_1, e_2, e_3, and e_4 exist at the points indicated. Relations applying to each of the three boxes are

$$\begin{aligned} e_2(t) &= e_1(t - \tau) \\ (RCD + 1)e_3(t) &= e_2(t) \\ e_4(t) &= Ge_3(t) \\ e_1(t) &= e_4(t) \end{aligned}$$

where $D = d/dt$. When combined, these relations lead to the single equation

$$\dot{x}_{(t)} + ax_{(t)} - bx_{(t-\tau)} = 0 \tag{8.11}$$

where $x = e_1$ is the voltage at the input of the delay circuit, $a = 1/RC$, and $b = G/RC$.

Equation (8.11) is a linear differential difference equation.

The physical effect of the RC circuit in the middle box of Fig. 8.3 is to provide a reduction in transmission and to introduce a phase lag as frequency increases. Thus, while again oscillation may occur if the total phase shift around the loop is an integral multiple of 2π radians, the possible frequencies are not related simply in the ratio of integers.

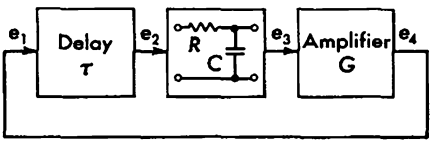


Fig. 8.3. Circuit for linear oscillator with delay and low-pass amplifier of Example 8.2.

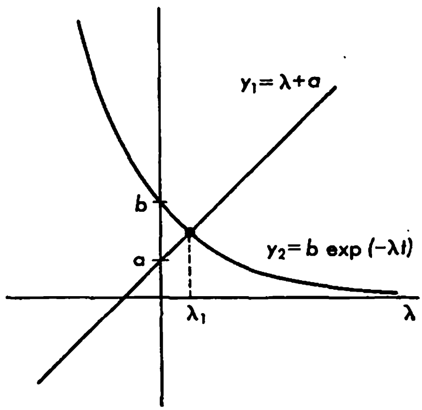


Fig. 8.4. Construction for locating real characteristic root for oscillator of Example 8.2.

If a solution is assumed in the form of Eq. (8.3), $x = C_0 \exp(\lambda t)$, the resulting characteristic equation is

$$\lambda + a - b \exp(-\lambda\tau) = 0 \tag{8.12}$$

which involves root λ in the transcendental function $\exp(-\lambda\tau)$. Since λ may be complex, it is conveniently written $\lambda = \delta \pm j\omega$ and substituted into Eq. (8.12) so as to give

$$\begin{aligned} &\text{Real:} & \delta + a - b \exp(-\delta\tau) \cos \omega\tau &= 0 \\ &\text{Imaginary:} & \omega - b \exp(-\delta\tau) \sin \omega\tau &= 0 \end{aligned} \tag{8.13}$$

An infinite number of characteristic roots can be found from Eqs. (8.12) and (8.13).

The simplest case is that where the roots are real. This case is governed by Eq. (8.12) with λ a real quantity. It is most easily studied graphically, by writing Eq. (8.12) in two parts as

$$\begin{aligned} y_1 &= \lambda + a \\ y_2 &= b \exp(-\lambda\tau) \end{aligned}$$

and plotting curves of y_1 and y_2 as functions of λ. Intersections of these curves give allowed values of λ. In Fig. 8.4 is shown the construction for the case of $b > a > 0$.

For the circuit of Fig. 8.3 with positive elements, coefficient a will be positive, while the sign of coefficient b is the same as the sign on G. However, if Eq. (8.11) is considered in a purely mathematical sense, without interpreting it in physical terms, the algebraic signs of a and b might be either positive or negative. Depending upon conditions, there may be one or two real values of λ as found from a construction analogous to Fig. 8.4. The situation is summarized in Fig. 8.5, where dimensionless products $a\tau$ and $b\tau$ are the coordinates and λ_1 and λ_2 represent allowed values of λ

In general, if $b\tau > 0$, a single value of λ exists, although it may be either positive or negative. If $b\tau < -\exp(-1 - a\tau)$, no real value exists. If $-\exp(-1 - a\tau) < b\tau < 0$, two real values exist. Only a small region of Fig. 8.5 has combinations of a

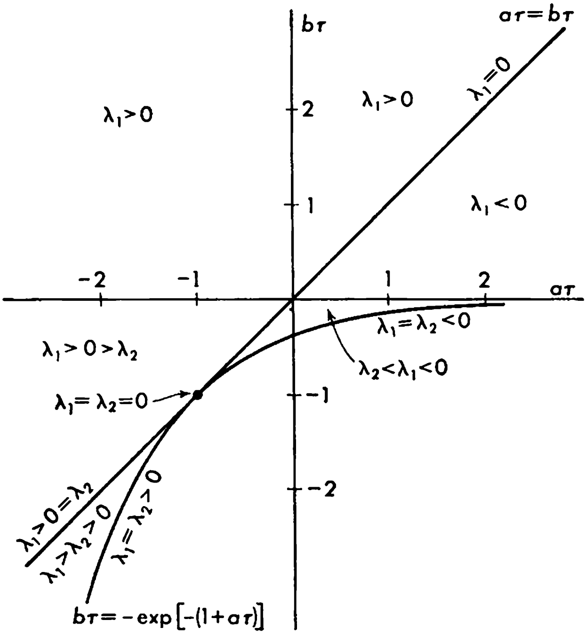


FIG. 8.5. Relation between real characteristic roots and parameters of oscillator of Example 8.2.

and b leading to stable nonoscillatory solutions, for which λ is real and negative. The conditions for this kind of solution are $a\tau > b\tau > -1$ and simultaneously $b\tau > -\exp(-1 - a\tau)$. Other conditions give no real values, or give real values that are positive and lead to unstable solutions.

If oscillatory solutions exist, λ becomes complex and Eq. (8 13) must be used. The simplest possibility is that the solution is a steady-state oscillation with $\delta = 0$ and λ pure imaginary. In this case, Eq. (8.13) becomes

$$\begin{aligned} a &= -\frac{\omega}{\tan \omega\tau} \\ b &= -\frac{\omega}{\sin \omega\tau} = \frac{a}{\cos \omega\tau} \end{aligned} \tag{8.14}$$

Because of the periodicity of the trigonometric functions, an infinity of values for ω is possible. Contours are plotted in Fig. 8.6 for the combinations of $a\tau$ and $b\tau$ leading to pure imaginary values of λ. Values of $\omega\tau$ at certain points on these contours are shown. Considerable numerical work is necessary to obtain data for plotting Fig. 8.6.

In general, if magnitude $|b\tau|$ is increased beyond a value corresponding to a contour of Fig. 8.6, keeping $a\tau$ constant, root λ acquires a positive real part in addition to its imaginary part. This represents a solution which is a growing oscillation. Thus,

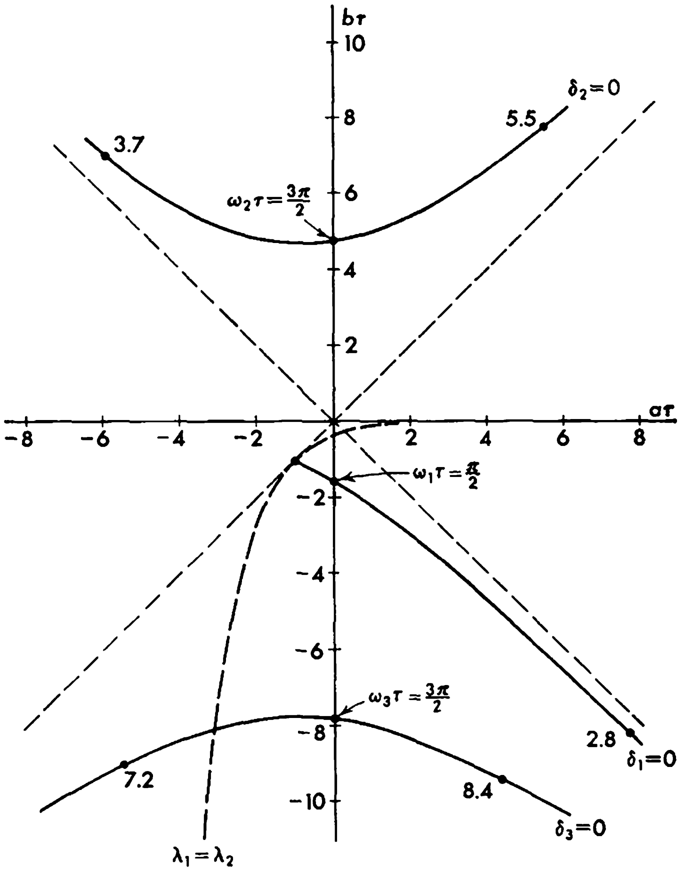


FIG. 8.6. Relation between complex characteristic roots and parameters of oscillator of Example 8.2.

unstable growing oscillations exist above the contours for $\delta = 0$ in the first and second quadrants or below these contours in the third and fourth quadrants.

A general solution is again of the form

$$x_{(t)} = B_1 \exp(\lambda_1 t) + B_2 \exp(\lambda_2 t) + \sum_{i=1}^{\infty} A_i \exp(\delta_i t) \cos(\omega_i t + \theta_i) \quad (8.15)$$

where λ_1 and λ_2 are real roots found from Eq. (8.12), $\delta_i \pm j\omega_i$ are complex roots found from Eq. (8.13), and B_1, B_2, A_i, and θ_i are constants found from initial conditions. Once more the initial conditions essentially require specification of $x_{(t)}$ for the entire interval $-\tau \leq t \leq 0$.

It is worth noting that Eq. (8.11) for the system can be put into somewhat simpler form by the change of variable $y = x \exp(at)$. The new equation is $\dot{y}_{(t)} - b \exp(a\tau) y_{(t-\tau)} = 0$, which has only two terms in place of three. The two parameters a and b now appear in the single coefficient $b \exp(a\tau)$, rather than in two separate terms. Depending upon the type of analysis being considered, the revised form may or may not be preferable to the original equation.

8.4. Nonlinear Difference Equation.

If nonlinear terms appear in addition to the linear terms of Eq. (8.1), a nonlinear difference equation results. Just as methods for solving a linear differential equation can be extended to linear difference equations, so also can methods developed for attacking nonlinear differential equations be extended to nonlinear difference equations. Details of applying the methods may have to be changed somewhat, but much the same general approach can be used.

Example 8.3. Nonlinear Oscillator with Delay

The system of Example 8.1 has been shown to have a growing amplitude of oscillation if the gain of the amplifier is sufficient so that $|G| > 1$. In such a case, the amplitude would theoretically increase indefinitely. Actually, no physical amplifier is capable of transmitting amplitudes that are indefinitely large. Rather, some kind of nonlinear limiting action always takes place, reducing the effective gain of the amplifier as the amplitude increases. Ultimately, a steady state is achieved where the effective amplification is just sufficient to maintain oscillation.

A system which illustrates the operation of an amplifier with limiting action is shown in Fig. 8.7.

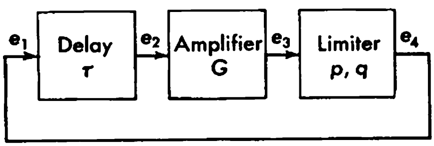


FIG. 8.7. Circuit for nonlinear oscillator with delay of Example 8.3.

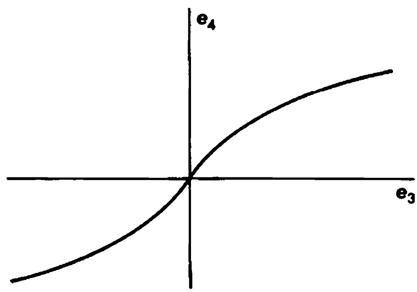


FIG. 8.8. Characteristics of limiter for oscillator of Example 8.3.

The delay circuit and amplifier of the first and second boxes here are the same as those of Fig. 8.1. The third box contains a limiter which transmits a smaller and smaller fraction of voltage applied to it as the magnitude of the voltage increases. Instantaneous voltages at particular points are e_1, e_2, e_3, and e_4.

The simplest mathematical description for a limiter is

$$e_3 = pe_4 + qe_4^3 \tag{8.16}$$

where e_3 and e_4 are instantaneous input and output voltages, while p and q are positive constants. This relation corresponds to the curve of Fig. 8.8. A typical nonlinear amplifier saturates more abruptly than illustrated in the figure, but Eq. (8.16) has the advantage of simplicity for analysis.

The relations for the system are

$$\begin{aligned}
e_2(t) &= e_1(t - \tau) \\
e_3(t) &= Ge_2(t) \\
e_3(t) &= pe_4(t) + q[e_4(t)]^3 \\
e_1(t) &= e_4(t)
\end{aligned}$$

When combined, these relations lead to the single equation

$$x_{(t)} - Cx_{(t-T)} + gx^3_{(t)} = 0 \tag{8.17}$$

where $x = e_1$, $C = G/p$, and $g = q/p$. If the limiter were missing, $q = 0$ and $g = 0$ and Eq. (8.17) becomes the same as Eq. (8.7). Because of the cubic term, Eq. (8.17) is a nonlinear difference equation.

A case of some practical interest is that where the oscillating system is self-starting but ultimately assumes a steady state. This requires that the amplifier have enough gain to cause growth for small amplitudes. The limiter then provides a throttling action as the amplitude increases. From the discussion of Example 8.1, it is evident that the growth from small amplitudes will occur if $|C| > 1$, with more rapid growth taking place if $|C|$ is larger.

A physical discussion can be used to explain the operation qualitatively. Possible frequencies of oscillation are the same as those for the linear system of Example 8.1 and are in the ratio of small integers. If initially only a single frequency is present, the nonlinear action of the limiter will generate harmonics and sum and difference frequencies, so that other frequencies are soon present. Ultimately, a steady state is reached. In the steady state, the signal at the amplifier input undergoes changes as it passes through the amplifier and limiter. These changes must be such that the resulting signal from the limiter is identical with that going into the amplifier. The limiter is nonlinear and is the only distorting element. Thus, if the waveform of the signal is to be preserved, it must have a fixed magnitude. The voltage at the amplifier input must satisfy the condition $|G|E = pE + qE^3$, where E is this voltage. Either $E = 0$, or

$$E^2 = \frac{|G| - p}{q} = \frac{|C| - 1}{g} \tag{8.18}$$

The first possibility, representing no signal, must be unstable if the system is adjusted for growth. The second possibility is a stable magnitude.

The sign of E from Eq. (8.18) may be either positive or negative. Thus, the waveform may be either a positive or negative constant or a rectangular wave jumping discontinuously between a positive and negative constant. If coefficient C of Eq. (8.17) is negative, the voltage of the system cannot remain constant and only odd harmonics of the fundamental frequency can exist, as shown in Fig. 8.2. Thus, if $C < 0$, the only steady-state solution is a square wave with equal duration for positive and negative values and with amplitude given by Eq. (8.18). If coefficient C is positive, the voltage may be constant or may have all harmonics of a fundamental frequency. Thus, this case may lead to a constant voltage or to some kind of rectangular wave with discontinuous jumps between positive and negative voltages of magnitude given by Eq. (8.18).

A method used in studying growth of oscillation in the negative-resistance oscillator of Example 6.7 is that of variation of parameters. The same process with minor changes can be used here. This method is applicable only provided that amplitude changes take place slowly, requiring many cycles to occur. If the system of Fig. 8.7 is to be self-starting and at the same time is to have only relatively slow growth, it is necessary that coefficient C have a magnitude just greater than unity. It is therefore convenient to write $C = C_0(1 + h)$, where $C_0 = \pm 1$ and h is a small positive quantity $0 < h \ll 1$. Equation (8.17) then becomes

$$x_{(t)} - C_0x_{(t-T)} - hC_0x_{(t-T)} + gx^3_{(t)} = 0 \tag{8.19}$$

In the qualitative discussion of the system, it was pointed out that any steady-state operation of the system must involve many frequencies simultaneously because of the nonlinear action of the limiter. For the sake of simplicity, however, the following discussion is confined to the case of a single frequency in the system, although it is recognized that this is not a realistic situation. Two frequencies are considered in Example 10.10.

In the method of variation of parameters, a generating solution is found first by neglecting any terms in the equation which cause difficulty. The terms of Eq. (8.19) having h and g in their coefficients can be omitted and a generating solution written as

$$x = A \cos (\omega t + \theta) = A \cos \psi \tag{8.20}$$

The frequency ω must be of such value that $\omega\tau$ satisfies the conditions illustrated in Fig. 8.2.

In order to correct for the effect of the terms first neglected, amplitude A and phase angle θ are allowed to become functions of time. An approximate value for $x_{(t-\tau)}$ is then found as

$$\begin{aligned} x_{(t-\tau)} &= (A - \dot{A}\tau) \cos [\omega(t - \tau) + \theta - \dot{\theta}\tau] \\ &= (A - \dot{A}\tau) \cos (\psi - \omega\tau) + \dot{\theta}\tau A \sin (\psi - \omega\tau) \end{aligned} \tag{8.21}$$

where $\dot{A} = dA/dt$ and $\dot{\theta} = d\theta/dt$, both values being averaged over a cycle of the oscillation. It is necessary to use average values at this point in the analysis, rather than further on as can be done with a differential equation. It is assumed that only slow changes take place, so that $\dot{A}\tau/A \ll 1$, $\dot{\theta} \ll \omega$, and $\dot{\theta}\tau \ll 1$, and only first-order variations are retained. At the same time, approximately,

$$hC_0x_{(t-\tau)} = hC_0A \cos (\psi - \omega\tau) \tag{8.22}$$

and

$$gx^3_{(t)} = gA^3 \cos^3 \psi = \frac{gA^3}{4} (3 \cos \psi + \cos 3\psi) \tag{8.23}$$

with only first-order variations retained.

Upon substitution of Eqs. (8.20) to (8.23) into Eq. (8.19), the result is

$$\begin{aligned} A \cos \psi - C_0(A - \dot{A}\tau) \cos (\psi - \omega\tau) - C_0\dot{\theta}\tau A \sin (\psi - \omega\tau) \\ - hC_0A \cos (\psi - \omega\tau) + \frac{gA^3}{4} (3 \cos \psi + \cos 3\psi) = 0 \end{aligned} \tag{8.24}$$

The generating solution satisfies the relation

$$A \cos \psi - C_0A \cos (\psi - \omega\tau) = 0$$

which can be used to remove the first two terms of Eq. (8.24). The remaining terms can be expanded by using trigonometric identities for $\cos (\psi - \omega\tau)$ and $\sin (\psi - \omega\tau)$ to give terms in $\cos \psi$ and $\sin \psi$ alone. If the resulting equation is to be an identity, valid at all instants in time, these terms must individually add to zero. The term in $\cos 3\psi$ is neglected here, as has been done in application of the principle of harmonic balance. The resulting equations are

$$\begin{aligned} \cos \psi: &\quad \dot{A}\tau C_0 \cos \omega\tau + \dot{\theta}\tau AC_0 \sin \omega\tau = hC_0A \cos \omega\tau - \frac{3gA^3}{4} \\ \sin \psi: &\quad \dot{A}\tau C_0 \sin \omega\tau - \dot{\theta}\tau AC_0 \cos \omega\tau = hC_0A \sin \omega\tau \end{aligned} \tag{8.25}$$

From the discussion following Eq. (8.9) of Example 8.1, it is evident that here $\sin \omega\tau = 0$ and $C_0 \cos \omega\tau = +1$. Thus, Eq. (8.25) can be simplified to give

$$\begin{aligned} \dot{A} &= \frac{1}{\tau}\left(hA - \frac{3gA^3}{4}\right) \\ \dot{\theta} &= 0 \end{aligned} \tag{8.26}$$

Since $\dot{\theta} = 0$, the frequency of oscillation remains at ω and is not affected by the limiting action. The equation for $\dot{A}$ is reminiscent of Eq. (6.59) for the negative-resistance oscillator of Example 6.7. It may be solved easily as an example of a Bernoulli equation, Sec. 4.2*i*, with the result

$$A(t) = \frac{U}{[1 - (1 - U^2/A_0^2) \exp(-2ht/\tau)]^{1/2}} \tag{8.27}$$

where $U = (4h/3g)^{1/2}$ is the ultimate steady-state amplitude and A_0 is the amplitude at $t = 0$. Clearly, the situation is quite analogous to that illustrated in Figs. 6.9 and 6.10. Equilibrium values of A exist at $A = 0$ and $A = \pm U$. The first of these is unstable, while the second and third are stable. Any initial disturbance in the system will cause the steady-state oscillation to build up. The final approximate solution is $x = A(t) \cos(\omega t + \theta_0)$, where $A(t)$ is given by Eq. (8.27) and θ_0 depends upon initial conditions.

The steady-state amplitude produced by this recognizably incomplete analysis, based on a single frequency, is $A = U = (4h/3g)^{1/2} = 1.15E$, where E is the amplitude of the rectangular wave predicted in the qualitative discussion leading to Eq. (8.18), where $E = [(|C| - 1)/g]^{1/2} = (h/g)^{1/2}$. The fundamental component of a square wave of amplitude E is known to have an amplitude $4E/\pi = 1.27E$. Thus, the amplitude found from this analysis with the single frequency is not far from the value of the fundamental component of the correct square wave.

A more exact analysis might be based on the assumption of a generating solution having more than a single frequency present. Complications multiply quickly in such a case, however, and an analysis of this kind is not given here. One case is given, however, in Example 10.10.

8.5. Nonlinear Differential-difference Equations.

If a combination of delayed terms, derivatives, and nonlinear terms appear in a single equation, the result is a nonlinear differential-difference equation. Again it may be possible to extend methods developed for attacking nonlinear differential equations to this somewhat more complicated kind of equation.

Example 8.4. Nonlinear Oscillator with Delay and Low-pass Amplifier

The combination of the systems of Figs. 8.3 and 8.7 leads to a fairly realistic system such as might be created physically. It involves both the low-pass and limiting

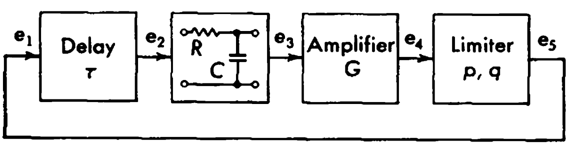


FIG. 8.9. Circuit for nonlinear oscillator with delay and low-pass amplifier of Example 8.4.

effects of a typical amplifier and may be represented as in Fig. 8.9. The elements in the boxes of this figure are assumed to be the same as those used in Examples 8.2 and 8.3. The equation governing the system is accordingly

$$\dot{x}_{(t)} + ax_{(t)} - aCx_{(t-\tau)} + g(ax^3_{(t)} + 3x^2_{(t)}\dot{x}_{(t)}) = 0 \tag{8.28}$$

where x is the voltage going into the delay circuit, $a = 1/RC$, $C = G/p$, and $g = q/p$, as before. This is a nonlinear differential-difference equation.

Probably the only case of practical interest here is that where just the lowest possible frequency of oscillation, ω_1 corresponding to the contour of $\delta_1 = 0$ of Fig. 8.6, exists. Coefficient b of this figure corresponds to the product aC of Eq. (8.28). For steady-state oscillation in a linear system, aC must have the particular value aC_0 given for b in Fig. 8.6, where a has already been specified. For a self-starting oscillation, aC must be slightly larger, say, $aC = aC_0(1 + h)$, with $0 < h \ll 1$. If the amplifier gain, and thus the value of C, is chosen to meet this condition, oscillation at any higher-order frequencies will decay.

An approximate solution can be found by variation of parameters, just as in Example 8.3. The generating solution is $x = A \cos(\omega t + \theta) = A \cos \psi$. Upon allowing A and θ to vary with time and carrying out the details of the solution, it is found that the average value of $\dot{A}$ is given by

$$\dot{A} = \left(h - \frac{3gA^2}{4}\right)\mu A \tag{8.29}$$

where

$$\mu = \frac{a(1 + a\tau) + \omega^2\tau}{(1 + a\tau)^2 + (\omega\tau)^2}$$

Again, $A = 0$ is an unstable equilibrium amplitude, while $A = \pm U = \pm(4h/3g)^{1/2}$ are stable equilibrium amplitudes. Also, the value of $\dot{\theta}$ is $\dot{\theta} = 0$ when $A = \pm U$.

Example 8.5. Population Growth

Under certain idealistic conditions, a colony of animals of a single species grows so that its population varies according to the relation

$$\frac{dx}{dt} = \dot{x} = rx$$

where x is the population at any instant, a positive number, r is the reproduction rate, and t is time. Solutions for this equation are, of course, increasing exponential functions if r is a positive number. Population ultimately grows without bounds.

If the colony is confined to a finite living space or is limited in its food supply, there is a definite maximum population which can be supported. In this case, a relation describing the population may be

$$\dot{x} = rx\left(1 - \frac{x}{x_s}\right)$$

where x_s is the steady-state population ultimately attained. The extra factor here introduces a kind of saturation effect and limits the maximum population. It makes the equation nonlinear, but an exact solution can be found as a Bernoulli equation, Sec. 4.2i. The exact solution is

$$x = \frac{x_s}{1 - (1 - x_s/x_0)\exp(-rt)}$$

where $x = x_0$ at $t = 0$. This solution can be represented as a curve rising initially with time and finally approaching x_s monotonically.

Experimental studies with animal populations almost invariably show a variation with time much like this. However, instead of approaching the ultimate value monotonically, there are usually overshoots and decaying oscillations about this value. Sometimes, even, violent oscillations about this value are observed. The growth equation can be modified further to allow for oscillatory solutions of this type in the following way:

$$\dot{x}_{(t)} = rx_{(t)}\left(1 - \frac{x_{(t-\tau)}}{x_s}\right) \tag{8.30}$$

where $\dot{x}_{(t)}$ and $x_{(t)}$ are evaluated at the time t, while $x_{(t-\tau)}$ is evaluated at the earlier time $t - \tau$. Equation (8.30) is thus a nonlinear differential-difference equation. It carries the implication that the population does not react immediately to its increasing number, but rather that a delay time τ is involved. Just what is the mechanism of this delay is not always clear, although it is probably a combination of several factors. There are certain obvious fixed times in animal growth, such as gestation time, the time for death to occur after all food is removed, and the like, all of which may well be involved.

The nature of solutions for Eq. (8.30) can be explored by converting it into a pure differential equation through the Taylor's series of Eq. (8.2). With only the first three terms the series is

$$x_{(t-\tau)} = x_t - \tau\dot{x}_t + \frac{\tau^2}{2}\ddot{x}_t \tag{8.31}$$

The use of three terms in place of the infinite series is only approximately correct but is more accurate the smaller the value of τ. When Eq. (8.31) is used in Eq. (8.30), the result is

$$\frac{\tau^2}{2}\ddot{x} - x_s\left(\frac{\tau}{x_s} - \frac{1}{rx}\right)\dot{x} + x = x_s \tag{8.32}$$

This is a nonlinear pure differential equation with all terms evaluated at the same instant. It is approximately equivalent to Eq. (8.30) provided τ is small enough.

An equilibrium point for Eq. (8.32), existing if $\ddot{x} = 0$ and $\dot{x} = 0$ simultaneously, is $x = x_s$. Near this equilibrium value, x may be written as $x = x_s + u$, where u is a small variation. If this form for x is substituted into Eq. (8.32), the result is the linear variational equation

$$\left(\frac{\tau^2}{2}\right)\ddot{u} - \left(\tau - \frac{1}{r}\right)\dot{u} + u = 0 \tag{8.33}$$

The characteristic roots may be written

$$(\lambda_1\tau, \lambda_2\tau) = \frac{1}{r\tau}\{(r\tau - 1) \pm [(r\tau - 1)^2 - 2(r\tau)^2]^{1/2}\}$$

Both these roots are real and negative, representing a decaying nonoscillatory solution for u, if $r\tau < 2^{1/2} - 1$. This is a singularity of the kind designated as a stable node in Sec. 5.5. If $2^{1/2} - 1 < r\tau < 1$, the solution is a decaying oscillation, or a stable focus. If $r\tau > 1$, the solution is a growing oscillation, or an unstable focus. If $r\tau = 1$, the oscillation is steady state and has the angular frequency $\omega = 2^{1/2}/\tau$ and the period $T = 2^{1/2}\pi\tau \doteq 4.4\tau$.

There is a second equilibrium value for the original equation, Eq. (8.30), where $x = 0$, making the saturation factor essentially unity. Near this equilibrium value, the solution grows, and this point corresponds to an unstable node.

By combining these results, the solution for Eq. (8.30) might be expected to be qualitatively as shown in Fig. 8.10.

This simple analysis indicates a growing oscillation about x_s if $r\tau > 1$. However, Eq. (8.30) shows that, if $x = 0$, $\dot{x} = 0$, and therefore x can never change its sign. Thus, if conditions are such that the solution is a growing oscillation, its amplitude must be limited in some way by the fact that x must remain positive. Actually, because Eq. (8.30) is nonlinear, a growing oscillation ultimately assumes a steady state corresponding to the appearance of a limit cycle. The value of x becomes small but never becomes negative.

This analysis is only approximately correct for Eq. (8.30) because of the few terms used in the series of Eq. (8.31). The first term neglected here is $-(\tau^3/6)\dddot{x}$, while the last term retained is $(\tau^2/2)\ddot{x}$. If the three terms are to be a good approximation to the infinite series, the ratio of these two terms must be small, $(\tau/3)|\dddot{x}/\ddot{x}| \ll 1$. With $r\tau = 1$ and a steady-state oscillation for the solution, the variation about x_s has the form $u = x - x_s = A \cos (2^{1/2}t/\tau)$, so that $\ddot{x} = -(2/\tau^2)A \cos (2^{1/2}t/\tau)$ and $\dddot{x} = (2^{3/2}/\tau^3)A \sin (2^{1/2}t/\tau)$. The test ratio is $(\tau/3)|\dddot{x}/\ddot{x}| = 2^{1/2}/3 \doteq 0.47$. Since this ratio is not small compared with unity, it is apparent that the solution can be only approximately correct. On the other hand, if $r\tau$ is small, say, $r\tau = 0.1$, the characteristic roots for variations near x_s are real and have the approximate values $\lambda_1 = -0.1/\tau$ and $\lambda_2 = -18/\tau$. The second of these represents a solution decaying relatively rapidly, and accordingly it quickly disappears. The first has a much slower decay

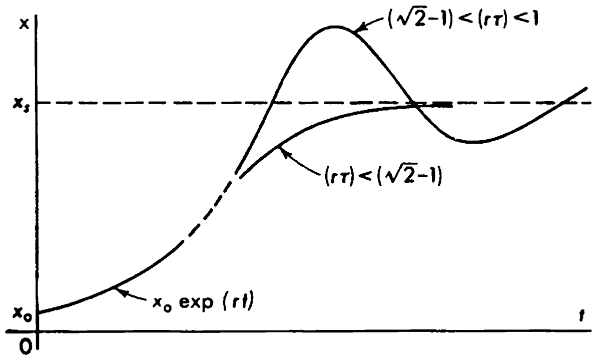


FIG. 8.10. Solution curves sketched for Example 8.5 following the approximate analysis based on Taylor's series.

and corresponds to the part of the solution which will be most apparent. The important variation near x_s now has the form $u = A \exp (-0.1t/\tau)$ so that $\ddot{x} = (0.1/\tau)^2u$ and $\dddot{x} = -(0.1/\tau)^3u$. The test ratio is $(\tau/3)|\dddot{x}/\ddot{x}| = 0.1/3 \doteq 0.033$. This ratio is indeed small compared with unity, and therefore this solution should be rather accurate.

These observations apply to any analysis of a differential-difference equation in which just a few terms of a Taylor's series are used to reduce the equation to a pure differential equation. The process yields accurate solutions only so long as the delay time is relatively small. The error becomes progressively larger as the delay is increased. Furthermore, with only three terms in the Taylor's series, the resulting differential equation is of second order. It can have only a single frequency of oscillation. Analyses, such as that of Example 8.1, show that difference equations with oscillatory solutions typically have an infinity of possible frequencies. Only the lowest of these is obtained from the simple differential equation. Higher frequencies are completely ignored in this method.

Since it has been predicted that if $r\tau$ is large enough a steady-state oscillation will exist about a fixed mean value, additional information can be obtained from the original differential-difference equation, Eq. (8.30). A solution may be assumed as

$$x = C + A \cos \omega t \tag{8.34}$$

where C, A, and ω are quantities to be determined. The process is essentially the principle of harmonic balance of Sec. 6.7.

If Eq. (8.34) is substituted into Eq. (8.30), the result may be written

$$\frac{-\omega A}{r}\sin \omega t = C + A\cos \omega t - \frac{1}{x_s}\left[C^2 + AC\cos \omega t + AC\cos \omega(t-\tau) + A^2\cos \omega t\cos \omega(t-\tau)\right] \quad (8.35)$$

Trigonometric identities can be used to give terms in $\cos \omega t$ and $\sin \omega t$. These terms can then be collected as

Constant: $$0 = C - \frac{C^2}{x_s} - \left(\frac{A^2}{4x_s}\right)\cos \omega\tau$$

Cos ωt: $$0 = A - \frac{AC}{x_s} - \frac{AC}{2x_s}\cos \omega\tau$$

sin ωt: $$\frac{-\omega A}{r} = \frac{-AC}{2x_s}\sin \omega\tau$$

where terms in $\cos 2\omega t$ and $\sin 2\omega t$ are neglected. The first two of these equations can be combined to show that always $A = 2^{1/2}C$. This result cannot be correct, for it requires that the amplitude of the oscillatory term exceed the constant term, which would necessitate x becoming negative for a portion of each cycle. This is impossible since x cannot change sign. This poor result arises because the assumed form of solution is not accurate but should include terms of higher frequency.

Since the solution is obviously not very accurate, it is reasonable to assume at once that $C = x_s$, as was found in the first consideration of the problem. If this is so, the first two equations are satisfied if $\cos \omega\tau = 0$, or $\omega\tau = \pi/2$. With this result, $\sin \omega\tau = 1$, and the last equation requires $r\tau = \pi$. Thus, an approximate solution is

$$x = x_s + 2^{1/2}x_s \cos \frac{\pi t}{2\tau} \quad (8.36)$$

with the requirement that $r\tau = \pi$. This result is to be compared with that found from the approximate differential equation of Eq. (8.32), for which a steady-state oscillation is predicted if $r\tau = 1$ with the angular frequency such that $\omega\tau = 2^{1/2}$. The required values of $r\tau$ differ considerably as found from these two types of analyses, although the resulting angular frequencies are quite similar.

A more complete study of the three equations gives additional results for the constants. The minimum value for $r\tau$ to give a steady-state oscillation is $r\tau = 3$, for which $\omega\tau = 0$, $C/x_s = 2/3$, and always $A = 2^{1/2}C$. If $r\tau$ is increased, $\omega\tau$ and C/x_s both increase, and if $r\tau \to \infty$, $\omega\tau \to \pi$ and $C/x_s \to 2$. Higher-order frequencies of oscillation become possible if $r\tau$ is increased sufficiently.

A better result might be expected if in place of Eq. (8.34) a solution is assumed as

$$x = x_1 + x_2 = x_s + A\cos \omega t + A_2\cos(2\omega t + \theta) \quad (8.37)$$

where x_1 is the approximate solution already found in Eq. (8.36). A second-harmonic term, x_2, is added to account in part for terms of this frequency dropped in Eq. (8.35) Quantities A and ω of Eq. (8.34), which forms x_1, have been chosen as in Eq. (8.36) to give best agreement with fundamental terms in Eq. (8.35). Quantities A_2 and θ of x_2 are now to be chosen to fit the second-harmonic terms.

If Eq. (8.37) is substituted into Eq. (8.30), terms of fundamental frequency will disappear, since x_1 has been adjusted to fit the equation. The most important remaining terms are

$$\frac{\dot{x}_2}{r} = x_2 - \frac{A^2}{4x_s}\cos \omega\tau \cos 2\omega t - \frac{A^2}{4x_s}\sin \omega\tau \sin 2\omega t$$

where the last two terms are those neglected in Eq. (8.35). If x_2 is inserted here and trigonometric identities are used, terms in cos $2\omega t$ and sin $2\omega t$ can be obtained and collected as

$$\cos 2\omega t: \qquad -A_2 \sin \theta = A_2 \cos \theta$$

$$\sin 2\omega t: \qquad -A_2 \cos \theta = -A_2 \sin \theta - \frac{x_s}{2}$$

where use has been made of the conditions $A^2 = 2x_s^2$, $\omega\tau = \pi/2$, $\sin \omega\tau = 1$, $\cos \omega\tau = 0$, and $r\tau = \pi$. The first of these relations gives $\tan \theta = -1$ so that $\theta = -\pi/4$ radians, and the second then gives $A_2 = 2^{1/2}x_s/4$. Thus, the approximate solution of Eq. (8.37) becomes

$$x = x_s + 2^{1/2}x_s \cos \frac{\pi t}{2\tau} + \frac{2^{1/2}x_s}{4} \cos \left(\frac{\pi t}{\tau} - \frac{\pi}{4}\right) \tag{8.38}$$

with the requirement that $r\tau = \pi$.

This approximate solution is plotted in Fig. 8.11. The wave still becomes negative for a portion of the cycle, which is incorrect. The second-harmonic term does reduce

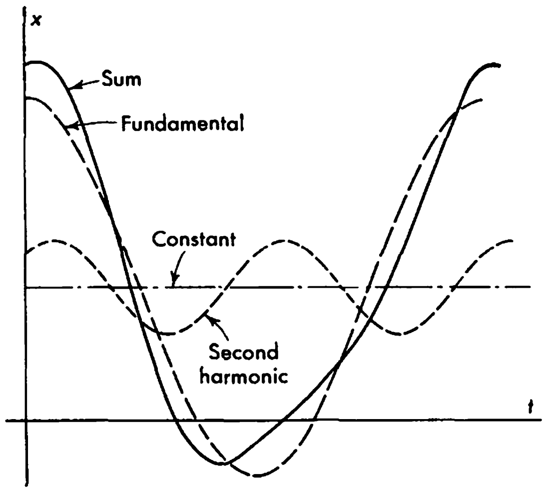


FIG. 8.11. Approximate steady-state solution for Example 8.5 based on harmonic balance, with $r\tau = \pi$. Three components are considered.

the swing into the negative region, however, and the indication is that if still more components were included the wave would remain positive, as it should. The second harmonic tends to accentuate the peak of the wave, flatten the trough, decrease the rate of growth, and increase the rate of decay. A further solution, found in Example 8.6, shows that these effects are qualitatively correct.

8.6. Graphical Solutions. Just as graphical methods of solution are often useful with certain differential equations, so also are they useful with certain differential-difference equations. In Sec. 3.2, the isocline method is described as it applies to a first-order differential equation of the form $dx/dt = \dot{x} = f(x,t)$. The value of $\dot{x}$ is interpreted as the slope of a solution curve on the xt plane. The differential equation allows this slope to be determined at all points in the plane, and the solution curves can be sketched in following these slopes.

In Sec. 3.3, the isocline method is extended to second-order equations of the form $d^2x/dt^2 = \ddot{x} = f(\dot{x},x)$. The procedure is to reduce this equation to first order by the substitution $\ddot{x} = \dot{x}\, d\dot{x}/dx$, giving the new form $d\dot{x}/dx = f(\dot{x},x)/\dot{x}$. This equation may be solved by the isocline method, working in the $\dot{x}x$ plane, known as the phase plane. The isocline construction is equivalent to a first integration. A second integration is needed to find x as a function of t, and this can be carried out by methods described in Sec. 3.4.

A somewhat similar graphical construction can be used with differential-difference equations of the form

$$\dot{x}_{(t)} = f(x_{(t)},x_{(t-\tau)}) \tag{8.39}$$

where $\dot{x}_{(t)}$ and $x_{(t)}$ are evaluated at the time t, while $x_{(t-\tau)}$ is evaluated at the earlier time $t - \tau$, and function $f(x_{(t)},x_{(t-\tau)})$ is a single-valued, generally nonlinear function. Because this function, and thus $\dot{x}_{(t)}$, depends upon $x_{(t-\tau)}$ as well as $x_{(t)}$, the simple isocline method cannot be used. Furthermore, instead of a single initial condition being sufficient to define a unique solution, it is necessary to specify $x_{(t)}$ for the entire interval $-\tau \leq t \leq 0$.

The construction is begun by taking the assigned value of $x_{(t)}$ for the initial interval $-\tau \leq t \leq 0$. With appropriate differentiation, $\dot{x}_{(t)}$ is found for this interval. A curve is then drawn on the phase plane with $\dot{x}_{(t)}$ as a function of $x_{(t)}$ for this interval, as shown in Fig. 8.12. It is now necessary to choose an increment Δt for the succeeding steps of the construction. The size of Δt is subject to the usual compromises. Too small an increment requires many steps in construction, and the cumulative error is likely to be large. Too large an increment involves more error at each step, but fewer total steps are needed. Also, it is convenient to have Δt be an integral submultiple of τ. Once Δt is chosen, points corresponding to $-\tau$, $\Delta t - \tau$, $2\Delta t - \tau$, . . . , 0 should be located along the initial curve of $\dot{x}$ versus x. This is shown in Fig. 8.12, where Δt has been chosen as $\tau/4$. At $t = 0$, the values are $\dot{x}(0)$ and $x(0)$.

The next step in extending the solution curve is to find values $\dot{x}(\Delta t)$ and $x(\Delta t)$ existing at $t = \Delta t$. These values can be found by satisfying simultaneously two conditions. The first condition is given by the differential-difference equation, Eq. (8.39), itself. The most convenient way of using this equation is to construct curves on the $\dot{x}_{(t)}x_{(t)}$ plane, the parameter being $x_{(t-\tau)}$. A sequence of values can be assigned to $x_{(t-\tau)}$ and corresponding values of $\dot{x}_{(t)}$ and $x_{(t)}$ calculated from the equation. Contours can be plotted from these data as in Fig. 8.12. The value of $\dot{x}_{(t)}$ and $x_{(t)}$ at any instant t must lie on the contour for the value of $x_{(t-\tau)}$ existing at the previous instant $t - \tau$. This value is known from the way the solution is built up.

The second condition which must be satisfied is the integration over one increment, given by the relation $\Delta x = \dot{x}_{av}\, \Delta t$, where Δx is the increment in x taking place in increment Δt and $\dot{x}_{av}$ is the average value of $\dot{x}$ over this increment. This condition can most simply be used by the graphical construction of Sec. 3.4, illustrated in Fig. 8.12. A template is constructed in the form of a right triangle having as one angle $\tan^{-1}(2/\Delta t)$, where Δt is the chosen time increment. Since Δt is a quantity with

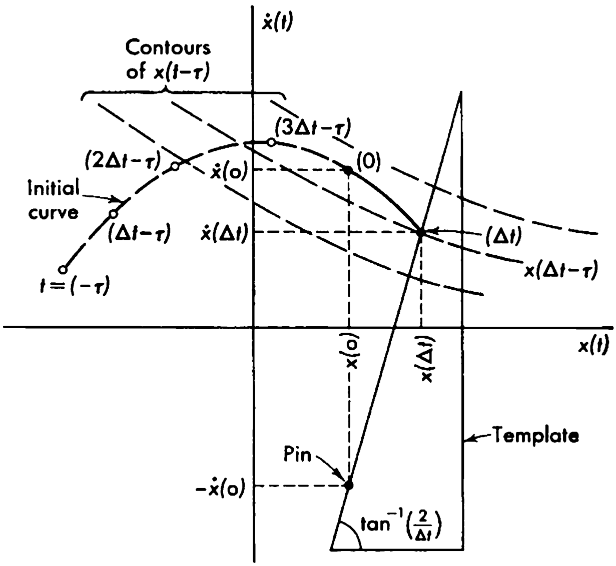


FIG. 8.12. Details of graphical construction for solving the equation $\dot{x}_{(t)} = f(x_{(t)}, x_{(t-\tau)})$. The construction involves an initial curve for $x_{(t)}$ with $-\tau \le t \le 0$, contours of $x_{(t-\tau)}$, and a triangular template dependent upon interval Δt.

physical dimensions, the angle for the template must be found by considering the scales for the $\dot{x}$ and x axes. The template must be constructed so that the ratio of the number of $\dot{x}$ units along the vertical side of the triangle to the number of x units along the horizontal side is equal to the fraction $2/\Delta t$. The template is used at the point $[\dot{x}(0),x(0)]$ as shown in Fig. 8.12. A pin is placed on the figure at the point $[-\dot{x}(0),x(0)]$. The hypotenuse of the template is placed against this pin and the sides aligned with the $\dot{x}x$ axes. The point on the solution curve at $t = \Delta t$ must lie on the hypotenuse of the template.

Since the point at $t = \Delta t$ must lie on both the contour for $x_{(\Delta t-\tau)}$ and on the hypotenuse of the template, the intersection of these two lines locates the point $[\dot{x}(\Delta t),x(\Delta t)]$ at $t = \Delta t$. This is the next point of the solution curve. In a similar way, construction proceeds on to the point where $t = 2\Delta t$; it must lie on the contour for $x_{(2\Delta t-\tau)}$. Successive points of the solution curve can be located in this manner, and a smooth curve can be sketched through the points. Since the points are separated by the constant increment Δt in time, information is immediately available for plotting a curve of x versus t.

The most laborious part of this graphical process is the calculation and plotting of the family of contours of $x_{(t-\tau)}$. This is somewhat analogous to plotting the family of directed line segments in the isocline construction. Once the family of contours is obtained, the remainder of the construction is quite straightforward.

One feature of the resulting phase-plane curves is somewhat different from curves for a pure differential equation. For this latter type of equation, the slope $d\dot{x}/dx$ of the curve at any point on the phase plane depends only upon values of $\dot{x}$ and x existing at that point. Therefore, only a single solution curve can pass through a given point, and two solution curves never intersect. This is true except for certain singular points where the slope is undefined. These singular points have been shown in Sec. 5.4 to play an important role in determining properties of the solution.

For the differential-difference equation, the slope $(d\dot{x}/dt)_{(t)}$ of the phase-plane curve depends upon $x_{(t-\tau)}$ as well as upon $\dot{x}_{(t)}$ and $x_{(t)}$. The slope does not have a unique value at any point in the phase plane, since the point specifies only $\dot{x}_{(t)}$ and $x_{(t)}$ but not $x_{(t-\tau)}$. It is possible, therefore, for several solution curves for the differential-difference equation to intersect or for a single curve to return and cross over itself.

Another feature worth noting is in regard to specification of the initial curve. Presumably, this curve can be specified arbitrarily, with no restrictions. On the other hand, an arbitrary initial curve may have a shape radically different from that representing the solution for the equation, and there may be a short interval of abrupt transition just following $t = 0$. An abrupt break of this sort appears in the curve in Fig. 8.13 for $r\tau = 1$.

Example 8.6. Population Growth

Under certain conditions, the population of a colony composed of a single species of animals is given by the equation developed in Example 8.5 as

$$\dot{x}_{(t)} = rx_{(t)}\left(1 - \frac{x_{(t-\tau)}}{x_s}\right) \tag{8.40}$$

This equation is of the form of Eq. (8.39), and a graphical solution can be obtained. If x is much less than x_s, the solution is approximately $x = x_0 \exp[r(t - t_0)]$, where $x = x_0$ at $t = t_0$.

Find a solution by graphical construction where the numerical values are $r = 1$, $x_s = 20$, $\Delta t = 0.5$, a consistent set of units being employed. The value of τ remains to be chosen. Assume, at the time $t = t_0 = -2$, $x = x_0 = 2$, so that, for the initial interval $-2 \le t \le 0$, approximately $x = 2 \exp (t + 2)$ and $\dot{x} = 2 \exp (t + 2)$.

Construction proceeds as shown in Fig. 8.13. The initial curve is plotted for the interval $-2 \le t \le 0$, with circles indicating points separated $\Delta t = 0.5$. At $t = 0$, $\dot{x}(0) = 14.8$, and $x(0) = 14.8$. Contours of $x_{(t-\tau)}$ can be plotted from the relation $\dot{x}_{(t)}/x_{(t)} = 1 - x_{(t-\tau)}/20$. These contours are straight lines radiating out from the origin. Their slope is positive if $0 \le x < x_s$ and negative if $x > x_s$. Only a

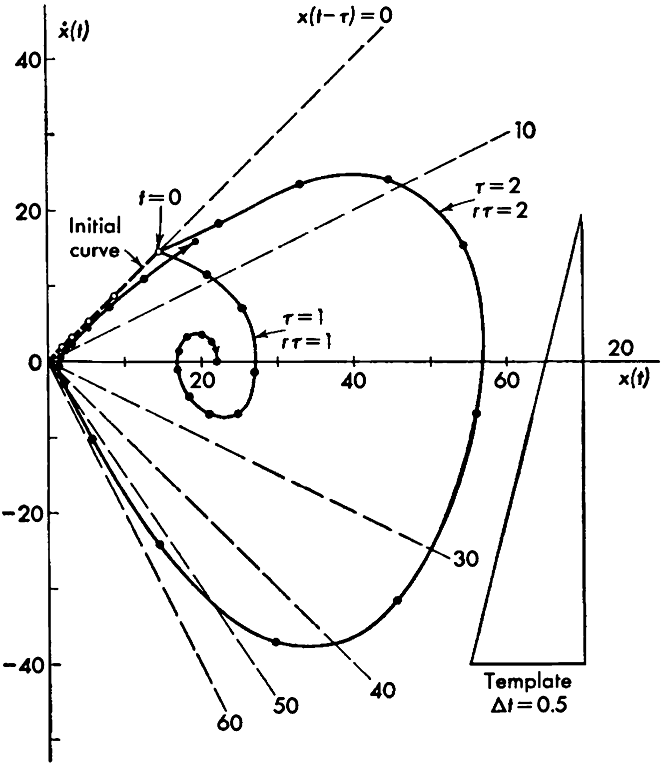


FIG. 8.13. Graphical solution for Example 8.6. Two choices for $r\tau$ are shown, but the same initial curve is used in each case. Dots indicate points separated $\Delta t = 0.5$.

few contours are shown in the figure, and more should be drawn for an accurate construction.

A triangular template is constructed with its angle determined by $\Delta t = 0.5$, as shown in Fig. 8.13.

A value for τ must now be chosen. A case of some interest is that where $\tau = 2$, for which $\tau = 4\,\Delta t$ and $r\tau = 2$. The first point in the construction for this case, located at $t = \Delta t = 0.5$, must lie on the contour for $x_{(\Delta t-\tau)} = x_{(-1.5)} = 3.30$. The position on this contour is found with the template against a pin located at the point $\dot{x} = -\dot{x}(0) = -14.8$ and $x = x(0) = 14.8$. Thus, at $t = 0.5$, values are approximately $\dot{x}(0.5) = 18.0$ and $x(0.5) = 22.5$. Successive points separated by $\Delta t = 0.5$ are located by continuing this process and are shown as the dots on Fig. 8.13. A curve drawn through these points is the solution curve for the original equation. After just fewer than 18 Δt intervals, the curve closes upon itself, and a limit cycle representing a steady-state periodic oscillation results.

A second value of τ of some interest is $\tau = 1$, for which $\tau = 2\,\Delta t$ and $r\tau = 1$. The construction for this case is also shown in Fig. 8.13. This time the solution curve spirals inward about the point $\dot{x} = 0$ and $x = x_s = 20$. The solution is accordingly a decaying oscillation about this point.

Values of x taken from the points of the two curves of Fig. 8.13 are plotted as a function of t in Fig. 8.14. In Example 8.5, the analysis based on the Taylor's-series approximation led to the prediction that if $r\tau = 1$ a steady-state oscillation of period $T = 4.4\tau$ would exist about $x = x_s$. The analysis based on harmonic balance led to the prediction that if $r\tau = \pi = 3.14$ a similar steady-state oscillation would exist with $\omega\tau = \pi/2$, or period $T = 4\tau$. The curves of Fig. 8.14 indicate that the actual solutions have properties somewhere between these predictions from the two types

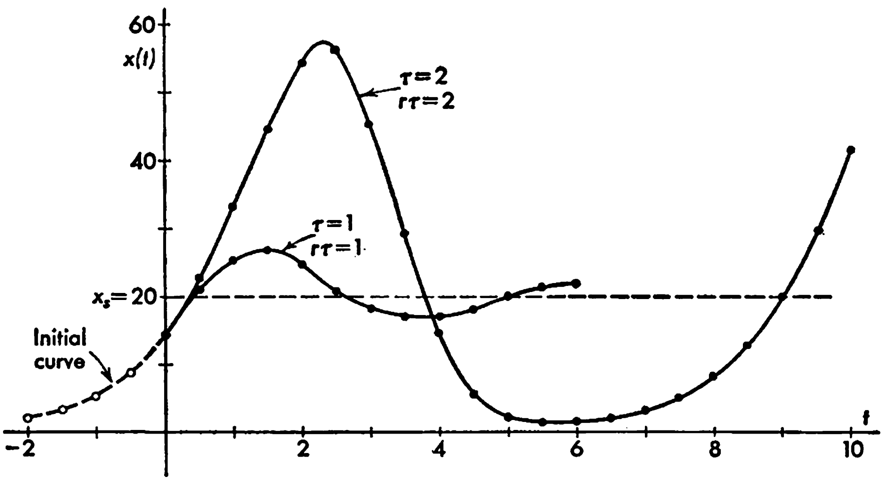


FIG. 8.14. Solution for Example 8.6. Dots are points separated $\Delta t = 0.5$ taken from Fig. 8.13.

of analysis. If $r\tau = 1$, the curve is a damped oscillation with the apparent period about $T = 4.5 = 4.5\tau$. If $r\tau = 2$, the curve is a limit cycle, indicating a growing oscillation about $x = x_s$ until the nonlinear throttling action takes place. The period in this case is about $T = 8.7 = 4.35\tau$. The waveform for the case of $r\tau = 2$ is qualitatively similar to that of Fig. 8.11, with the accentuated peak, flattened trough, and the rise occurring more slowly than the drop. The important difference is that the curve of Fig. 8.14 goes to small positive values of x but never crosses the x axis into the region of negative values. It remains in the region of small positive values for a large fraction of the period of the oscillation.

8.7. Summary. Several methods have been discussed for finding information about solutions for nonlinear differential-difference equations. These methods are similar to those developed for pure differential equations.

While the two kinds of equations have many similarities, there are important differences that must be kept in mind. The differential-difference equation is formally like a pure differential equation of infinite order. For this reason, solutions for the differential-difference equation can be expected to be considerably more complicated than solutions for a differential equation that looks superficially similar. An infinity of possible frequencies of oscillation can occur, for example, in a differential-difference equation containing a single first derivative. A pure differential equation with just a first derivative can have no oscillatory solutions at all. In addition to the complexity of the solution, a more complicated kind of initial condition must be specified for a differential-difference equation. Instead of being able to specify a few values at certain chosen points, it is necessary to specify all values over a finite range of the independent variable. A complete initial curve is required, rather than just a few initial points.

Perhaps the simplest method of attempting a solution for a differential-difference equation is to convert it into a pure differential equation. This can be done by using a Taylor's series for those terms which are evaluated at different times. If the time differences involved are small enough, only a few terms of the Taylor's series need be used and the resulting equation can be of low order and still apply fairly accurately. If the time differences are not small, many terms of the series are necessary and the resulting equation must be of such high order as to be unwieldy. In any case, if the solution is oscillatory, the number of possible frequencies of oscillation obtained in this procedure can never exceed half the order of the differential equation, while the differential-difference equation may have an infinity of frequencies.

The differential-difference equation itself may be attacked by methods used with pure differential equations. Procedures are much the same, although typically extra complications arise in the course of the analysis.

CHAPTER 9

LINEAR DIFFERENTIAL EQUATIONS WITH VARYING COEFFICIENTS

9.1. Introduction. Most of the preceding chapters have been concerned with nonlinear equations and physical systems to which they apply. An outstanding characteristic of nonlinear equations is that the principle of superposition is not valid. There is a class of differential equations to which superposition does apply but which is more complicated than the class of linear equations with constant coefficients. This class includes linear equations with coefficients which are functions of the independent variable. While exact solutions can, in theory, be found for equations of this type, in most cases details of finding the solution become exceedingly complicated. For this reason, approximate solutions found by some method similar to those used with nonlinear equations may be more useful.

Equations with variable coefficients may arise from physical systems in which some parameter is caused to vary with time because of the action of a process outside the system itself. This occurs, for example, when modulation takes place in a communication system. The resistance of a carbon microphone or the capacitance of a condenser microphone varies in response to a sound wave striking the microphone. The varying resistance or capacitance leads to an electrical circuit with a varying parameter. Certain ways of modulating the frequency of a carrier wave involve the equivalent of a varying capacitance in a resonant circuit. The effective negative resistance in the circuit of a superregenerative receiver is made to vary with time by the action of a subsidiary oscillator. There are mechanical systems in which the effective mass or stiffness of a component varies with time because of some external means. Finally, in testing the stability of certain nonlinear oscillating systems, equations with varying coefficients arise.

In this chapter is given a brief discussion of this kind of equation, considerable space being devoted to the Mathieu equation. This is a standard equation which applies to certain systems of interest and for which a considerable theory is available.

9.2. First-order Equation. The standard form for a linear first-order equation with a varying coefficient has been given in Sec. 4.2b as

$$\frac{dx}{dt} + P(t)x = Q(t) \tag{9.1}$$

where t is the independent variable and x is the dependent variable. The coefficient $P(t)$ of the term in x is a function of t, and a forcing function $Q(t)$ exists on the right side of the equation. The standard form for the solution has been given as

$$x = C \exp\left[-\int P(t)\,dt\right] + \exp\left[-\int P(t)\,dt\right] \int Q(t) \exp\left[\int P(t)\,dt\right] dt \tag{9.2}$$

where C is an arbitrary constant determined by initial conditions. Provided the necessary integrations can be carried out, an exact solution for the equation can be obtained.

Example 9.1. Capacitor Discharge

In Examples 6.1 and 6.4, the discharge of a capacitor through a nonlinear resistor has been considered. In those examples, the value of the resistance depends in nonlinear fashion upon the voltage across the resistance, and the circuit is governed by a nonlinear differential equation. A circuit somewhat similar is that in which a capacitor discharges through a resistance for which the value is a controlled function of time.

A specific example involves a capacitance C discharging through a resistance having the value

$$r(t) = R \exp\left(\frac{t}{T_1}\right)$$

where the resistance is $r = R$ at $t = 0$ and r approaches infinity exponentially as t increases, as shown in Fig. 9.1. The rate of increase is controlled by constant T_1. A resistor changing in this manner might be achieved in the plate circuit of a triode having appropriate biasing voltages and a resistor-capacitor discharge circuit connected to its grid.

The equation for the complete circuit is

$$C\frac{de}{dt} + \frac{e}{r(t)} = 0$$

or

$$\frac{de}{dt} + \frac{1}{T_2}\exp\left(\frac{-t}{T_1}\right) e = 0 \tag{9.3}$$

where $T_2 = RC$ and e is the instantaneous voltage across the capacitor. Equation (9.3) is of the form of Eq. (9.1) where $P(t) = (1/T_2)\exp(-t/T_1)$ and $Q(t) = 0$. The integral needed in the solution is $\int P(t)\,dt = -(T_1/T_2)\exp(-t/T_1)$. When this result is used in Eq. (9.2), the solution becomes

$$e = C_0 \exp\left[\frac{T_1}{T_2}\exp\left(\frac{-t}{T_1}\right)\right]$$

where C_0 is the arbitrary constant. Since $e = E$ at $t = 0$, the constant must be $C_0 = E\exp(-T_1/T_2)$ and the final solution is

$$e = E \exp\left\{\frac{-T_1}{T_2}\left[1 - \exp\left(\frac{-t}{T_1}\right)\right]\right\} \tag{9.4}$$

This result, involving the exponential function having an exponent which is itself an exponential function, is typical of the general complexity of solutions for problems of this type. A somewhat different form, perhaps more easily interpreted, can be obtained by using the series for the first exponential,

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

giving

$$\begin{aligned} e &= E \exp\left(\frac{-T_1}{T_2}\right)\left[\exp\left(\frac{T_1}{T_2}\right)\exp\left(\frac{-t}{T_1}\right)\right] \\ &= E \exp\left(\frac{-T_1}{T_2}\right)\left[1 + \frac{T_1}{T_2}\exp\left(\frac{-t}{T_1}\right)\right. \\ &\quad + \frac{1}{2}\left(\frac{T_1}{T_2}\right)^2 \exp\left(\frac{-2t}{T_1}\right) \\ &\quad \left. + \frac{1}{6}\left(\frac{T_1}{T_2}\right)^3 \exp\left(\frac{-3t}{T_1}\right) + \cdots\right] \end{aligned} \tag{9.5}$$

Here the solution is given as the sum of a constant term and exponentials decaying at different rates.

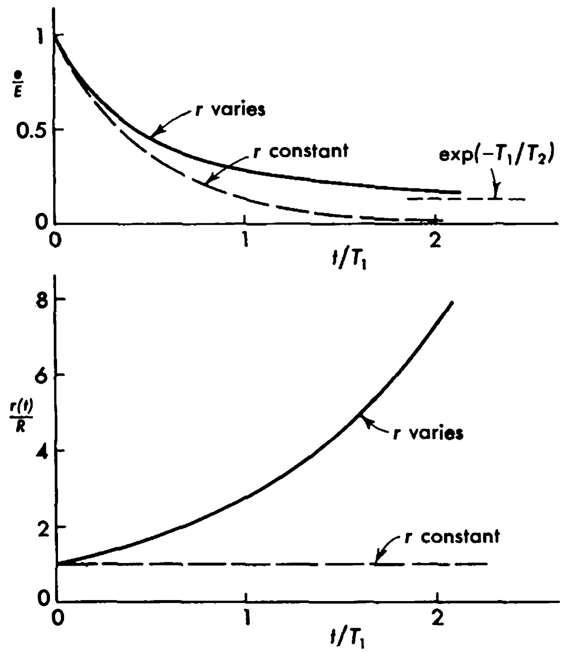


FIG. 9.1. Variation of resistance and of capacitor voltage for circuit of Example 9.1.

A plot of the solution for the particular case of $T_1/T_2 = 2$ is shown in Fig. 9.1, together with the case for which the resistance is constant at the value R and does not vary. If the resistance is constant, the capacitor ultimately discharges completely and voltage e approaches zero. If the resistance increases as assumed, even though it becomes infinite only after infinite time, the capacitor never completely discharges. Instead, voltage e approaches a definite nonzero value determined by ratio T_1/T_2. A larger value of this ratio corresponds to a relatively slower change in resistance value and leads to a more complete discharge of the capacitor.

Example 9.2. Carbon-microphone Circuit

A carbon microphone has a small container of carbon granules coupled to a diaphragm exposed to the air. A pair of electrodes is arranged so that an electric current can flow through the granules If a sound wave strikes the diaphragm, motion occurs so as to change the way the granules are packed together. This produces a change in the resistance between the electrodes. Thus, the resistance between terminals of the microphone is varied by the action of a sound wave striking the microphone.

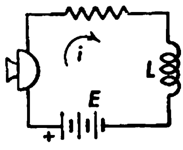


FIG. 9.2. Carbon-microphone circuit for Example 9.2.

The microphone may be used in a circuit such as shown in Fig. 9.2. A battery of voltage E causes an instantaneous current i to flow in the circuit consisting of the microphone and the primary of a transformer having a self-inductance L. The total resistance in the circuit includes the resistance of the microphone itself as well as any additional resistance which may be present. In the simplest case, the resistance of the microphone may be assumed to vary sinusoidally with time, so that the total resistance is $R(1 + m \sin \omega t)$, where m is the fractional variation in the total resistance, $0 < m < 1$, and ω is the angular frequency of the variation. The equation for the circuit is, accordingly,

$$L \frac{di}{dt} + R(1 + m \sin \omega t)i = E$$

This may be simplified slightly to give

$$\frac{di}{dt} + a(1 + m \sin \omega t)i = \frac{E}{L} \tag{9.6}$$

where $a = R/L$.

Equation (9.6) is of the form of Eq. (9.2), where now $P(t) = a(1 + m \sin \omega t)$ and $Q(t) = E/L$. The integral needed in the solution is $\int P(t)\,dt = a[t - (m/\omega) \cos \omega t]$. When this integral is used in Eq. (9.2), terms appear with the reasonably complicated form $\exp\,[(am/\omega) \cos \omega t]$, again typical of this kind of problem. This form can be expanded into a series of Bessel functions as

$$\exp\left(\frac{am}{\omega} \cos \omega t\right) = I_0\left(\frac{am}{\omega}\right) + 2 \sum_{n=1}^{\infty} I_n\left(\frac{am}{\omega}\right) \cos n\omega t$$

where $I_0, \ldots, I_n$ are modified Bessel functions of the first kind and order n.

When this series is used in the integral of the second term of Eq. (9.2), the result is

$$\int \frac{E}{L} \exp\,(at) \left[I_0\left(\frac{-am}{\omega}\right) + 2 \sum_{n=1}^{\infty} I_n\left(\frac{-am}{\omega}\right) \cos n\omega t \right] dt$$

A form needed to evaluate this integral is

$$\int \exp\,(at) \cos n\omega t\,dt = A_n \exp\,(at) \cos\,(n\omega t - \phi_n)$$

where $A_n = (n^2\omega^2 + a^2)^{-1/2}$ and $\tan \phi_n = n\omega/a$. Upon combining these various relations, a solution may be written as

$$i = \left[I_0\left(\frac{am}{\omega}\right) + 2\sum_{p=1}^{\infty} I_p\left(\frac{am}{\omega}\right)\cos p\omega t\right]$$

$$\left\{C\exp(-at) + \frac{E}{L}\left[\frac{1}{a}I_0\left(\frac{am}{\omega}\right) + 2\sum_{n=1}^{\infty} I_n\left(\frac{-am}{\omega}\right)A_n\cos(n\omega t - \phi_n)\right]\right\} \quad (9.7)$$

Different indices must appear in the two summations here. Use has been made of the even symmetry of the I_0 function, $I_0(-x) = I_0(+x)$. Coefficient C depends upon initial conditions.

The solution evidently has a transient part with an infinity of component frequencies, all of which decay as time progresses, and a steady-state part including both a constant component and an infinity of frequencies. The steady-state components include the fundamental frequency and all its integral multiples. Their amplitudes depend upon the parameter m, which is the relative resistance change produced by the microphone. Thus, even though the resistance of the microphone changes sinusoidally with time, the variation in current may be far from sinusoidal. The larger the value of m, the greater is the distortion in the current.

A considerably simpler, though only approximate, solution may be obtained by applying the perturbation method of Sec. 6.2 to Eq. (9.6). This method will yield results of good accuracy only if parameter m is small, and thus this parameter itself may be used in writing the assumed solution as

$$i = i_0 + mi_1 \quad (9.8)$$

Only a first-order approximation is used here.

The generating solution is found from the relation

$$\frac{di_0}{dt} + ai_0 = \frac{E}{L}$$

and is

$$i_0 = C_0\exp(-at) + \frac{E}{aL}$$

If the initial current is assumed to be zero, $i_0 = 0$ at $t = 0$ and $C_0 = -E/aL$. The generating solution is

$$i_0 = \frac{E}{aL}[1 - \exp(-at)] \quad (9.9)$$

The first-order correction is found from

$$\frac{di_1}{dt} + ai_1 = -a(\sin\omega t)i_0 = -\frac{E}{L}\sin\omega t[1 - \exp(-at)]$$

and is

$$i_1 = C_1\exp(-at) + \frac{E}{L}(\omega^2 + a^2)^{-1/2}\cos(\omega t + \theta) - \frac{E}{L\omega}\exp(-at)\cos\omega t \quad (9.10)$$

where $\tan\theta = a/\omega$. Constant C_1 must be chosen to make $i_1 = 0$ at $t = 0$ and is $C_1 = (E/L)[(1/\omega) - (\omega^2 + a^2)^{-1/2}\cos\theta]$.

The solution to the first-order approximation is

$$i = \frac{E}{R}\exp(-at)\left[-1 + \frac{am}{\omega}\left(\frac{a^2}{\omega^2 + a^2} - \cos\omega t\right)\right]$$

$$+ \frac{E}{R}[1 + am(\omega^2 + a^2)^{-1/2}\cos(\omega t + \theta)] \quad (9.11)$$

where terms have been grouped as transient and steady-state components and $\tan \theta = a/\omega$.

To this first order of approximation, the steady-state solution includes only a constant component and a component at the fundamental frequency. If the process is extended to second- or higher-order approximations, distortion terms appear.

9.3. Second-order Equation. While a standard form is available for the solution of a first-order linear equation with a variable coefficient, this is not the case for equations of higher order. Indeed, as the order of the equation increases, the difficulty of finding a solution increases very rapidly. Certain second-order equations with varying coefficients have been studied in detail, and considerable information about their solutions is available. There is a great deal of information on Bessel's equation and Legendre's equation, mentioned in Sec. 4.2*l*. These are second-order equations with certain types of varying coefficients. Solutions for these equations are usually described as being obtained in series form by the method of Frobenius. While this is a suitable method where an intensive study of a single equation is contemplated, it is rather impractical as a general tool for attacking a variety of problems.

A number of physical systems are described by a second-order equation in which one coefficient varies periodically with time. This is the case for certain modulation systems, already mentioned. The equations here are of the general type of Hill's equation or, in simpler form, Mathieu's equation, for which considerable information is available. The process of finding exact solutions for these equations is a formidable task, and certain approximate methods of solution are often much more convenient and lead to solutions of useful accuracy.

a. Removal of First-derivative Term. The standard form of certain well-known second-order equations of the type under consideration here usually is written with only a second derivative and no first derivative. Physical problems, however, often lead to this general type of equation, but with a first-derivative term present. It is necessary, therefore, to be able to remove a first-derivative term if one appears. This is always possible in a linear equation by a simple change of variable.

A general linear second-order equation might have the form

$$\ddot{x} + 2P(t)\dot{x} + R^2(t)x = Q(t) \tag{9.12}$$

where x is the dependent variable, t is the independent variable, $P(t)$ and $R^2(t)$ are variable coefficients, and $Q(t)$ is a forcing function. It is desired to remove the term in $\dot{x}$. This may be accomplished by changing the dependent variable from x to y through the definition

$$y = x \exp\left[\int P(t)\,dt\right] \qquad \text{or} \qquad x = y \exp\left[-\int P(t)\,dt\right] \tag{9.13}$$

If this change of variable is introduced into Eq. (9.12), it becomes

$$\ddot{y} + [R^2(t) - P^2(t) - \dot{P}(t)]y = Q(t) \exp [\int P(t)\, dt] \tag{9.14}$$

where $\dot{P}(t) = d[P(t)]/dt$. This equation does not have the term in the first derivative.

Example 9.3

Find a solution for the equation

$$\ddot{x} + 2a\dot{x} + \omega^2 x = E \cos \omega_1 t$$

where a, ω^2, E, and ω_1 are constants, by removing the first-derivative term.

Here the coefficients are all constants, and the necessary transformation is $x = y \exp (-at)$. The equation in y becomes

$$\ddot{y} + (\omega^2 - a^2)y = E \exp (at) \cos \omega_1 t$$

and the solution for y is

$$y = A \cos (\omega^2 - a^2)^{1/2} t + B \sin (\omega^2 - a^2)^{1/2} t + \frac{E \exp (at)}{(\omega^2 - \omega_1^2)^2 + (2a\omega_1)^2} [(\omega^2 - \omega_1^2) \cos \omega_1 t + 2a\omega_1 \sin \omega_1 t]$$

The solution for x is finally

$$x = \exp (-at) [A \cos (\omega^2 - a^2)^{1/2} t + B \sin (\omega^2 - a^2)^{1/2} t] + \frac{E}{(\omega^2 - \omega_1^2)^2 + (2a\omega_1)^2} [(\omega^2 - \omega_1^2) \cos \omega_1 t + 2a\omega_1 \sin \omega_1 t]$$

where A and B are arbitrary constants.

This simple equation can, of course, be solved with no difficulty in the form it first appears, without the change of variable.

b. Determination of Particular Integral. The standard form of certain well-known second-order equations is often written with no forcing function, so that the right side of the equation is zero. A solution for an equation with no forcing function is merely the complementary function. If the right side is not zero, a particular integral is needed to complete the solution. In other words, the general equation

$$\ddot{x} + 2P(t)\dot{x} + R^2(t)x = Q(t) \tag{9.15}$$

might be solved initially to give only the complementary function. This function might be designated as

$$x_c = Ax_1(t) + Bx_2(t) \tag{9.16}$$

where $x_1(t)$ and $x_2(t)$ are the two independent parts required for a second-order equation and A and B are the associated arbitrary constants.

One way of finding the particular integral needed to complete the solution for Eq. (9.15) is by using the method of variation of parameters described in Sec. 4.3. In this process, quantities A and B of Eq. (9.16) are allowed to become functions of t. It is simplest to reduce Eq. (9.15) to a pair of simultaneous first-order equations through the substitution $z = \dot{x}$, so as to give

$$\begin{aligned} \dot{x} &= z \\ \dot{z} &= -2P(t)z - R^2(t)x + Q(t) \end{aligned} \tag{9.17}$$

The complementary function for Eq. (9.16) is

$$\begin{aligned} x &= Ax_1 + Bx_2 \\ z &= A\dot{x}_1 + B\dot{x}_2 \end{aligned}$$

Upon substitution of these values into Eq. (9.17), with A and B allowed to vary with t, and after some simplification, the result is

$$\begin{aligned} \dot{A}x_1 + \dot{B}x_2 &= 0 \\ \dot{A}\dot{x}_1 + \dot{B}\dot{x}_2 &= Q(t) \end{aligned}$$

These equations may be solved as simultaneous algebraic equations for $\dot{A}$ and $\dot{B}$ so as to give

$$\begin{aligned} \dot{A} &= \frac{-Qx_2}{x_1\dot{x}_2 - \dot{x}_1x_2} \\ \dot{B} &= \frac{+Qx_1}{x_1\dot{x}_2 - \dot{x}_1x_2} \end{aligned}$$

Finally, these relations may be integrated to give values for $A(t)$ and $B(t)$ and the results used in Eq. (9.16) as

$$x = \left(-\int \frac{Qx_2}{x_1\dot{x}_2 - \dot{x}_1x_2}\,dt + C_1\right)x_1 + \left(\int \frac{Qx_1}{x_1\dot{x}_2 - \dot{x}_1x_2}\,dt + C_2\right)x_2 \tag{9.18}$$

where C_1 and C_2 are the necessary arbitrary constants. It is evident that the integrals here may well turn out to be exceedingly difficult to evaluate in a practical problem.

Example 9.4

Find the particular integral for the equation

$$\ddot{x} + 2a\dot{x} + \omega^2 x = E \cos \omega_1 t$$

if the complementary function is known to be

$$x_c = \exp(-at)(A \cos \omega_2 t + B \sin \omega_2 t)$$

where $\omega_2^2 = \omega^2 - a^2$. This is the equation already solved in Example 9.3.

The two parts of the complementary function are

$$\begin{aligned} x_1 &= \exp(-at) \cos \omega_2 t \\ x_2 &= \exp(-at) \sin \omega_2 t \end{aligned}$$

and their derivatives are

$$\begin{aligned} \dot{x}_1 &= \exp(-at)(-a \cos \omega_2 t - \omega_2 \sin \omega_2 t) \\ \dot{x}_2 &= \exp(-at)(-a \sin \omega_2 t + \omega_2 \cos \omega_2 t) \end{aligned}$$

Thus, the denominator of the integrals of Eq. (9.18) is

$$x_1\dot{x}_2 - \dot{x}_1 x_2 = \exp(-2at)\omega_2$$

The integrals themselves are

$$\int \frac{Qx_2\,dt}{x_1\dot{x}_2 - \dot{x}_1x_2} = \frac{E}{\omega_2}\int \exp(at)\cos\omega_1 t \sin\omega_2 t\,dt$$
$$\int \frac{Qx_1\,dt}{x_1\dot{x}_2 - \dot{x}_1x_2} = \frac{E}{\omega_2}\int \exp(at)\cos\omega_1 t \cos\omega_2 t\,dt$$

When integration is carried out and the results substituted into Eq. (9.18), a complicated collection of terms appears. By appropriate simplification, however, it is possible to obtain the same form of the particular integral as found in Example 9.3.

c. The WKBJ Approximation. There are a number of physical problems in which the varying coefficient executes only relatively small changes about a large mean value. If the system can be described by a second-order equation, this equation can be put into the form

$$\frac{d^2x}{dt^2} + G^2(t)x = 0 \tag{9.19}$$

where $G^2(t)$ includes the varying coefficient. Function $G^2(t)$ has a relatively large mean value about which small variations take place. The right side of Eq. (9.19) is zero, implying that no forcing function exists, and the solution contains only a complementary function. If there is a forcing function, the resulting particular integral might be found by the method of the preceding section.

An approximate solution for Eq. (9.19) is known in the literature by some combination of the names Wentzel, Kramers, Brillouin, and Jeffreys, or as the WKBJ approximation. This approximation has the form

$$x = [G(t)]^{-\frac{1}{2}}\{C_1 \exp[j\phi(t)] + C_2 \exp[-j\phi(t)]\} \tag{9.20}$$

where $\phi(t) = \int G(t)\,dt$ and j is the imaginary unit. If Eq. (9.20) is differentiated twice, it can be shown to satisfy exactly the equation

$$\frac{d^2x}{dt^2} + \left[G^2 + \frac{\ddot{G}}{2G} - \frac{3}{4}\left(\frac{\dot{G}}{G}\right)^2\right]x = 0 \tag{9.21}$$

where $\dot{G} = dG/dt$ and $\ddot{G} = d^2G/dt^2$. Evidently, if $G^2(t)$ is such a function that it meets the requirement

$$|G^2| \gg \left|\frac{\ddot{G}}{2G} - \frac{3}{4}\left(\frac{\dot{G}}{G}\right)^2\right| \tag{9.22}$$

Eq. (9.21) is essentially the same as Eq. (9.19) and Eq. (9.20) represents an approximate solution for Eq. (9.19). If $G^2(t)$ has a large mean value about which only small fluctuations occur, the inequality of Eq. (9.22) is usually satisfied, although abrupt changes in $G^2(t)$ may give values of $\ddot{G}$ so large that the inequality is not justified. Finally, if $G^2(t)$ is a positive function of the sort described, so that $G(t)$ and $\phi(t)$ are both real, Eq. (9.20) may be written in the convenient form

$$x = [G(t)]^{-\frac{1}{2}}[A \cos \phi(t) + B \sin \phi(t)] \tag{9.23}$$

where A and B are arbitrary constants and $\phi(t) = \int G(t)\, dt$. This relation is the WKBJ approximation to the solution for Eq. (9.19).

Example 9.5. Resonant Circuit with Varying Capacitance

In some types of frequency modulation, the frequency of an electronic oscillator is controlled by a resonant circuit consisting of inductance and capacitance. The capacitance is made to vary as a function of time by the modulating signal. This capacitance variation, in turn, causes the frequency of the oscillation to vary. The resonant circuit is as shown in Fig. 9.3.

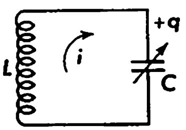


FIG. 9.3. Resonant circuit with varying capacitance for Example 9.5.

The equation for the circuit is

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0$$

where L is the inductance, C is the capacitance, and q is the instantaneous charge on the capacitor. In a typical frequency-modulation system, the change in capacitance is but a small fraction of the average value. If a simple-harmonic variation is assumed, the instantaneous capacitance is

$$C = C_0(1 + m \cos \omega_1 t)$$

where C_0 is the mean value, m is the relative variation, and ω_1 is the angular frequency of the variation. It is assumed that $0 < m \ll 1$. The equation for the system accordingly becomes

$$\frac{d^2q}{dt^2} + \frac{\omega_0^2}{1 + m \cos \omega_1 t} q = 0$$

where $\omega_0^2 = 1/LC_0$. If m is as small as assumed, this is nearly equivalent to

$$\frac{d^2q}{dt^2} + \omega_0^2(1 - m \cos \omega_1 t)q = 0 \tag{9.24}$$

Under the assumed conditions, Eq. (9.24) is of the type of Eq. (9.19) and the WKBJ approximation of Eq. (9.23) can be applied. Function $G^2(t)$ of Eq. (9.19) is

$$G^2(t) = \omega_0^2(1 - m \cos \omega_1 t)$$

so that

$$G(t) = \omega_0\left(1 - \frac{m}{2} \cos \omega_1 t\right)$$

with the assumed smallness of m. Thus, the integral becomes

$$\phi(t) = \int G(t)\, dt = \omega_0\left(t - \frac{m}{2\omega_1} \sin \omega_1 t\right)$$

The WKBJ approximate solution is

$$q = \omega_0^{-1/2}\left(1 + \frac{m}{4}\cos\omega_1 t\right)$$
$$\left[A\cos\omega_0\left(t - \frac{m}{2\omega_1}\sin\omega_1 t\right) + B\sin\omega_0\left(t - \frac{m}{2\omega_1}\sin\omega_1 t\right)\right] \quad (9.25)$$

where again the fact has been used that $m \ll 1$ in writing $G^{1/2}$.

Equation (9.25) shows that the frequency at which the instantaneous charge varies is a function of the modulation of the capacitance, as expected. Furthermore, the amplitude of the oscillating charge also varies slightly in accordance with the modulation.

The process of frequency modulation produces a whole series of side frequencies, and these can be found by expanding the sine and cosine terms of Eq. (9.25) in a series of Bessel functions. The identities involved are

$$\cos(a\sin b) = J_0(a) + 2J_2(a)\cos 2b + 2J_4(a)\cos 4b + \cdots$$
$$\sin(a\sin b) = 2J_1(a)\sin b + 2J_3(a)\sin 3b + \cdots$$

Therefore, the terms in the brackets of Eq. (9.25) become

$$\cos\omega_0\left(t - \frac{m}{2\omega_1}\sin\omega_1 t\right)$$
$$= \cos\omega_0 t[J_0(k) + 2J_2(k)\cos 2\omega_1 t + 2J_4(k)\cos 4\omega_1 t + \cdots]$$
$$+ \sin\omega_0 t[2J_1(k)\sin\omega_1 t + 2J_3(k)\sin 3\omega_1 t + \cdots]$$
$$\sin\omega_0\left(t - \frac{m}{2\omega_1}\sin\omega_1 t\right)$$
$$= \sin\omega_0 t[J_0(k) + 2J_2(k)\cos 2\omega_1 t + 2J_4(k)\cos 4\omega_1 t + \cdots]$$
$$- \cos\omega_0 t[2J_1(k)\sin\omega_1 t + 2J_3(k)\sin 3\omega_1 t + \cdots]$$

where $k = m\omega_0/2\omega_1$. The terms with products of sine and cosine functions may be rewritten with functions of sum and difference frequencies and substituted into Eq. (9.25) to give

$$q = \omega_0^{-1/2}\left(1 + \frac{m}{4}\cos\omega_1 t\right)$$
$$\begin{bmatrix} A\{J_0(k)\cos\omega_0 t + J_2(k)[\cos(\omega_0 + 2\omega_1)t + \cos(\omega_0 - 2\omega_1)t] \\ \quad + J_4(k)[\cos(\omega_0 + 4\omega_1)t + \cos(\omega_0 - 4\omega_1)t] + \cdots \\ \quad - J_1(k)[\cos(\omega_0 + \omega_1)t - \cos(\omega_0 - \omega_1)t] \\ \quad - J_3(k)[\cos(\omega_0 + 3\omega_1)t - \cos(\omega_0 - 3\omega_1)t] - \cdots\} \\ + B\{J_0(k)\sin\omega_0 t \\ \quad + J_2(k)[\sin(\omega_0 + 2\omega_1)t + \sin(\omega_0 - 2\omega_1)t] \\ \quad + J_4(k)[\sin(\omega_0 + 4\omega_1)t + \sin(\omega_0 - 4\omega_1)t] + \cdots \\ \quad - J_1(k)[\sin(\omega_0 + \omega_1)t - \sin(\omega_0 - \omega_1)t] \\ \quad - J_3(k)[\sin(\omega_0 + 3\omega_1)t - \sin(\omega_0 - 3\omega_1)t] - \cdots\} \end{bmatrix} \quad (9.26)$$

This result shows that the instantaneous charge has components at the natural frequency ω_0 for the unmodulated circuit and at sums and differences of ω_0 and integral multiples of ω_1, the modulating frequency. The amplitudes of these additional side-frequency components depend upon Bessel functions of the quantity $k = m\omega_0/2\omega_1$. A larger relative fluctuation m in capacitance leads to a larger value of k and to generally larger amplitudes of the side components. In addition, the amplitudes of the side-frequency components vary slightly with the modulating frequency because of the coefficient $1 + (m/4)\cos\omega_1 t$.

Example 9.6. Driven Resonant Circuit with Varying Resistance

In certain types of systems, as, for example, the superregenerative receiver, the effective resistance of the circuit is caused to vary periodically by means of an external oscillator. The equation for a series resonant circuit in which the resistance is made to change with time might be

$$L\frac{d^2q}{dt^2} + r(t)\frac{dq}{dt} + \frac{q}{C} = E\cos\omega_2 t \tag{9.27}$$

where L is the self-inductance, C is the capacitance, q is the instantaneous charge on the capacitor, $E\cos\omega_2 t$ is a driving voltage of amplitude E and angular frequency ω_2, and $r(t)$ is the instantaneous resistance in the circuit. If the resistance varies periodically about a mean value R_0, it might be given by the relation

$$r(t) = R_0(1 + m\cos\omega_1 t)$$

where m is the relative variation and ω_1 is the angular frequency. It is assumed that $0 < m \ll 1$.

The equation for the circuit then becomes

$$\frac{d^2q}{dt^2} + 2P(t)\frac{dq}{dt} + \omega_0^2 q = \frac{E}{L}\cos\omega_2 t$$

where $2P(t) = (\omega_0/Q_0)(1 + m\cos\omega_1 t)$ and $Q_0 = L\omega_0/R_0$ is the mean value of the circuit Q. This equation is of the type of Eq. (9.12) with the first-derivative term present. This term must be removed in order to be able to apply the WKBJ approximation. The necessary change of variable is $y = q\exp[\int P(t)\,dt]$, which gives the new equation

$$\ddot{y} + \left[\omega_0^2 - \left(\frac{\omega_0}{2Q_0}\right)^2(1 + m\cos\omega_1 t)^2 + \frac{m\omega_0\omega_1}{2Q_0}\sin\omega_1 t\right]y = \frac{E}{L}\cos\omega_2 t\exp\left[\frac{\omega_0}{2Q_0}\left(t + \frac{m}{\omega_1}\sin\omega_1 t\right)\right]$$

If the circuit Q is large, $(\omega_0/2Q_0)^2 \ll \omega_0^2$ and, further, parameter m has already been assumed small, so that the quantity in the first brackets becomes, approximately,

$$G^2(t) = \left[\omega_0^2 - \frac{m}{2}\left(\frac{\omega_0}{Q_0}\right)^2\cos\omega_1 t + \frac{m\omega_0\omega_1}{2Q_0}\sin\omega_1 t\right]$$

If Q_0 is large enough and ω_1 is not too small, the cosine term can be neglected, leaving

$$G^2(t) = \omega_0^2\left(1 + \frac{m\omega_1}{2Q_0\omega_0}\sin\omega_1 t\right)$$

The WKBJ approximation can now be used to find the complementary function. Functions needed here are

$$G(t) = \omega_0\left(1 + \frac{m\omega_1}{4Q_0\omega_0}\sin\omega_1 t\right)$$

and

$$\phi(t) = \int G(t)\,dt = \omega_0\left(t - \frac{m}{4Q_0\omega_0}\cos\omega_1 t\right)$$

where the assumed smallness of m has been considered. The WKBJ approximation for the complementary function is

$$y_c = \omega_0^{-1/2}\left(1 - \frac{m\omega_1}{8Q_0\omega_0}\sin\omega_1 t\right)$$
$$\left[A\cos\left(\omega_0 t - \frac{m}{4Q_0}\cos\omega_1 t\right) + B\sin\left(\omega_0 t - \frac{m}{4Q_0}\cos\omega_1 t\right)\right]$$

The particular integral might be found by the method of Sec. 9.3*b*. Finally, the charge must be found by converting back through the relation

$$q = y\exp\left[-\int P(t)\,dt\right] = y\exp\left[-\frac{\omega_0}{2Q_0}\left(t + \frac{m}{\omega_1}\sin\omega_1 t\right)\right]$$

The complexity of the situation needs no comment.

d. Approximation through Variation of Parameters. A second-order equation arising in a number of physical systems of interest has the form

$$\frac{d^2x}{dt^2} + \omega_0^2[1 + m\phi(t)]x = 0 \tag{9.28}$$

where ω_0^2 and m are constants and $\phi(t)$ is some specified function of t. If coefficient m is zero, Eq. (9.28) has as a solution a simple-harmonic oscillation of angular frequency ω_0. If the varying coefficient $m\phi(t)$ changes sufficiently slowly, so that the relative change per cycle of the oscillation is small, an approximate solution can be found by the method of variation of parameters of Sec. 4.3. The requirement is that $0 < m \ll 1$ and that $|(d\phi/dt)/\phi|(\omega_0/2\pi) \ll 1$.

With these conditions applying to $m\phi(t)$, a generating solution for Eq. (9.28) can be found by first ignoring the term with coefficient m, giving

$$x = A\cos(\omega_0 t + \theta) = A\cos\psi$$

where $\psi = \omega_0 t + \theta$ and A and θ are parameters to be varied. If now the term with m is considered, A and θ must vary as

$$\frac{dA}{dt} = m\omega_0 A\phi(t)\sin\psi\cos\psi \tag{9.29}$$

$$\frac{d\theta}{dt} = m\omega_0\phi(t)\cos^2\psi \tag{9.30}$$

so as to modify the generating solution to allow for the varying coefficient.

Equation (9.29) can be solved for A to give

$$A = B\exp\left[\frac{m\omega_0}{2}\int\phi(t)\sin 2\psi\,dt\right]$$

where B is an arbitrary constant. Under the assumed conditions, angle θ changes only slowly during a cycle of the oscillation and can be assumed essentially constant for the integration required here. Furthermore, since $|d\phi/dt| \ll |\phi|(\omega_0/2\pi)$, the integral is nearly

$$\int \phi(t) \sin 2\psi \, dt = -\frac{\phi(t)}{2\omega_0} \cos 2\psi$$

and therefore, approximately,

$$A = B \exp\left[-\frac{m}{4} \phi(t) \cos 2\psi\right]$$

This relation implies a fluctuation in amplitude A at the high-frequency rate determined by the angle $2\psi = 2(\omega_0 t + \theta)$. Actually, under the assumed condition on $m\phi(t)$, the amplitude can vary only slowly, with a small change during any one cycle of oscillation. The values of A when x is of greatest magnitude are accordingly the things of interest. This process of considering values of A only at a certain instant in each cycle is sometimes known as the stroboscopic method. The generating solution shows that the magnitude of x is a maximum when $\cos \psi = \pm 1$, or $\psi = 0, \pi, 2\pi, \ldots$. At these instants, $\cos 2\psi = +1$, and the equation for A becomes

$$A = B \exp\left[-\frac{m}{4} \phi(t)\right] = B\left[1 - \frac{m}{4} \phi(t)\right] \tag{9.31}$$

where the second form applies because m is small.

In Eq. (9.30), for the variation of θ it is sufficient to take the average value over a cycle of the oscillation, which is

$$\left(\frac{d\theta}{dt}\right)_{\text{av}} = \frac{m\omega_0}{2} \phi(t) \tag{9.32}$$

so that $\theta = (m\omega_0/2)\int \phi(t) \, dt + \theta_0$, where θ_0 is an arbitrary constant.

Finally, then, an approximate solution for Eq. (9.28), subject to the conditions on $m\phi(t)$, becomes

$$x = B\left[1 - \frac{m}{4} \phi(t)\right] \cos\left[\omega_0 t + \frac{m\omega_0}{2} \int \phi(t) \, dt + \theta_0\right] \tag{9.33}$$

where B and θ_0 are constants dependent upon initial conditions.

A somewhat different application of Eqs. (9.29) and (9.30) is made in finding an approximate solution for the Mathieu equation in Sec. 9.4*b*.

Example 9.7. Resonant Circuit with Varying Capacitance

An electrical circuit with varying capacitance has been studied in Example 9.5. The instantaneous capacitance varies as $C = C_0(1 + m \cos \omega_1 t)$, and, when used with a constant inductance L, the instantaneous charge q is given by the equation

$$\frac{d^2q}{dt^2} + \omega_0^2(1 - m \cos \omega_1 t)q = 0$$

which is Eq. (9.24). It is of the same form as Eq. (9.28) with $x = q$ and $\phi(t) = -\cos \omega_1 t$. An approximate solution is given by Eq. (9.33), provided $m \ll 1$ and $\omega_1 \ll \omega_0$, so that $|\dot{\phi}| \ll |\phi|(\omega_0/2\pi)$.

The integral is

$$\int \phi(t)\, dt = -\frac{1}{\omega_1} \sin \omega_1 t$$

and the approximate solution becomes

$$q = B\left(1 + \frac{m}{4} \cos \omega_1 t\right) \cos \left(\omega_0 t - \frac{m\omega_0}{2\omega_1} \sin \omega_1 t + \theta_0\right)$$

This is the same as Eq. (9.25) found by the WKBJ method.

9.4. Mathieu Equation. A standard form for a second-order equation with a periodic coefficient is

$$\frac{d^2y}{dz^2} + [a - 2q\phi(z)]y = 0 \tag{9.34}$$

where y is the dependent variable, z is the independent variable, coefficients a and q are constants, and $\phi(z)$ is a periodic function. The period of $\phi(z)$ is Z, so that $\phi(z) = \phi(z + Z)$, and the maximum magnitude of $\phi(z)$ is unity. An equation of the general type of Eq. (9.34) is known as Hill's equation.

The simplest form of Hill's equation results if $\phi(z)$ is merely a cosine function, in which case the equation may be written

$$\frac{d^2y}{dz^2} + (a - 2q \cos 2z)y = 0 \tag{9.35}$$

where the period of the varying coefficient is π in variable z. This is a standard form for the Mathieu equation, about which there is considerable literature. Unfortunately, different authors use various combinations of numerical parameters in writing the equation, with the result that some care must be used in comparing results taken from different places in the literature.

It should be evident from Examples 9.5 and 9.6 that solutions for this general type of equation are likely to be extremely complicated. Under some conditions, the periodic coefficient leads to solutions which are themselves periodic. Usually, however, the solutions are not strictly periodic.

One important feature of solutions, not considered before, is their stability. With some combinations of coefficients a and q, the solution for y of Eq. (9.35) grows without bound as z increases, and the solution is said to be unstable. With other combinations of a and q, the solutions remain bounded and are said to be stable. Combinations of a and q corresponding to stable and unstable solutions are of considerable importance in applications of the equation and can be found without too much difficulty.

It can be observed at once that if q is zero, so that the coefficients are constant, and a is positive, the equation is that of simple-harmonic motion. Its solution is a sum of circular cosine and sine functions which oscillate periodically but remain bounded and are stable. If, now, q is zero but is small enough in magnitude so that $|2q| < a$, the coefficient of y in Eq. (9.35) fluctuates but remains positive. In this case, it might be expected that solutions would be somewhat like the circular functions. In Fig. 9.4 are shown axes of q and a. Combinations of q and a lying between the lines for $+2q = a > 0$ and $-2q = a > 0$ might be expected

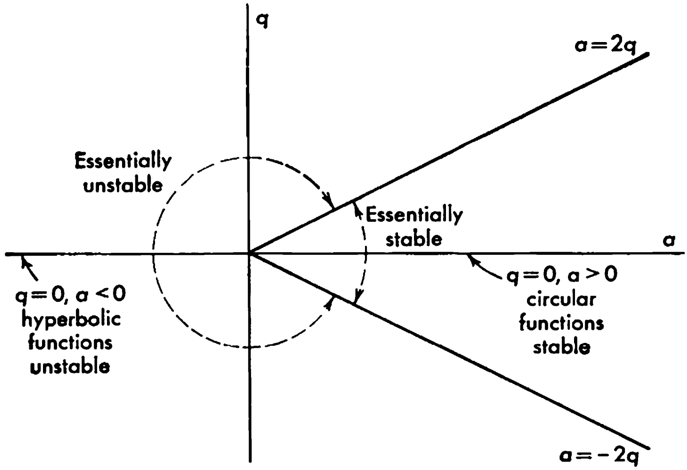


FIG. 9.4. Figure showing the general trend of stability of solutions for the Mathieu equation. This figure should be compared with Fig. 9.11.

to lead to essentially stable solutions. It is evident from the fact that q is multiplied by the oscillating function $\cos 2z$ in the Mathieu equation that a change in sign of q corresponds merely to a shift along the z scale. Thus, a stability diagram, such as Fig. 9.4, must be symmetrical about the a axis.

Similarly, if q is zero and a is negative, the solution is a sum of hyperbolic cosine and sine functions. These functions grow without bound and are unstable. Thus, if a is negative, or if q is large enough so that $|2q| > a$, the coefficient of y becomes negative for at least part of its cycle of fluctuation. It might be expected that then solutions would be somewhat like the unstable hyperbolic functions. Thus, a large region of Fig. 9.4 can be expected to lead to essentially unstable solutions.

It is shown in the analysis to follow that actually the boundaries between stable and unstable regions are highly complicated in shape but that they do have the general trend shown in Fig. 9.4.

a. Basic Theory. In the general form of the Mathieu equation, Eq. (9.35), independent variable z and coefficients a and q are assumed to be real quantities for the following discussion. Solutions for dependent variable y may be complex quantities, in general. For most physical problems, however, y must be real, but it may be made up of the sum of complex components with imaginary parts that cancel.

The writing of Eq. (9.35) can be condensed by the notation

$$\frac{d^2y}{dz^2} + f(z)y = 0 \tag{9.36}$$

where function $f(z)$ is $f(z) = a - 2q \cos 2z$. This function varies periodically with period π in z. A theory, known as the Floquet theory, exists for linear differential equations with periodic coefficients, of which Eq. (9.36) is an example. This theory gives certain basic information concerning solutions for the equation, although determination of exact solutions may be a difficult process. A brief discussion of the Floquet theory as it applies to the Mathieu equation follows.

Since Eq. (9.35) is a linear second-order equation, two independent nonzero solutions can be found as $y_1(z)$ and $y_2(z)$. These solutions can be combined linearly to give any other solution, say, $y(z)$, in this way,

$$y(z) = C_1y_1(z) + C_2y_2(z) \tag{9.37}$$

where C_1 and C_2 are constants chosen to fit the solutions. In order that $y_1(z)$ and $y_2(z)$ be independent, they must satisfy the Wronskian determinant

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

Primes indicate differentiation with respect to variable z.

In Eq. (9.35), independent variable $+z$ can be replaced with $-z$ with no change in the equation. Thus, if $y(+z)$ is a certain solution, then $y(-z)$ also is a solution for the equation. If $y(z)$ has no symmetry about the point $z = 0$, it can be split into components having even and odd symmetry by the usual process,

$$\text{Even:} \qquad \tfrac{1}{2}[y(+z) + y(-z)] = C_1y_1(z)$$

$$\text{Odd:} \qquad \tfrac{1}{2}[y(+z) - y(-z)] = C_2y_2(z)$$

where the even component of $y(z)$ has been designated as $y_1(z)$ and the odd component has been designated as $y_2(z)$. The symmetry requirements are, of course, that

$$y_1(+z) = y_1(-z)$$

$$y_2(+z) = -y_2(-z)$$

Thus, the members of the pair of independent solutions for Eq. (9.35) can be chosen to have even and odd symmetry about $z = 0$.

Furthermore, in Eq. (9.35) z can be replaced by $z + \pi$ with no change. If $y_1(z)$ and $y_2(z)$ are solutions, then $y_1(z + \pi)$ and $y_2(z + \pi)$ also are solutions. This statement does not imply that $y(z + \pi)$ is the same as $y(z)$ and that $y(z)$ is necessarily periodic. Rather, it means that the form of Eq. (9.37) applies and that the following equations hold:

$$\begin{aligned} y_1(z + \pi) &= k_{11}y_1(z) + k_{12}y_2(z) \\ y_2(z + \pi) &= k_{21}y_1(z) + k_{22}y_2(z) \end{aligned} \tag{9.38}$$

It is convenient to assume that the y_1 and y_2 functions are real and thus that the k_{ij} coefficients are real. In Eq. (9.37), variable z can be replaced by $z + \pi$ and use made of Eq. (9.38) to give

$$\begin{aligned} y(z + \pi) &= C_1y_1(z + \pi) + C_2y_2(z + \pi) \\ &= (C_1k_{11} + C_2k_{21})y_1(z) + (C_1k_{12} + C_2k_{22})y_2(z) \end{aligned} \tag{9.39}$$

Coefficients k_{ij} are specified by Eq. (9.38), but the determination of constants C_1 and C_2 depends upon the choice of $y(z)$. If $y(z)$ is a suitably chosen solution, the following equations apply,

$$\begin{aligned} C_1k_{11} + C_2k_{21} &= \sigma C_1 \\ C_1k_{12} + C_2k_{22} &= \sigma C_2 \end{aligned} \tag{9.40}$$

where σ is a constant that cannot be chosen independently. A comparison of Eqs. (9.37), (9.39), and (9.40) shows that

$$y(z + \pi) = \sigma[C_1y_1(z) + C_2y_2(z)] = \sigma y(z) \tag{9.41}$$

Thus, for certain solutions $y(z)$, a constant factor σ relates the value of the solution at any z to the value at $z + \pi$. A solution that has the property indicated in Eq. (9.41) is called a normal solution. It has been assumed that the y_1 and y_2 functions and the k_{ij} coefficients are real. Factor σ may be complex, so that normal solutions may be complex.

If constants C_1 and C_2 are not to be zero in the simultaneous homogeneous equations, Eq. (9.40), the determinant of their coefficients must be zero. Expansion of this determinant gives the relation for factor σ as

$$\sigma^2 - (k_{11} + k_{22})\sigma + (k_{11}k_{22} - k_{12}k_{21}) = 0 \tag{9.42}$$

The two roots of Eq. (9.42) give two possible values for factor σ, dependent upon the values of coefficients k_{ij}. Because the k_{ij} are assumed real, the two values of σ must be either real or conjugate complex quantities.

The two independent solutions, $y_1(z)$ and $y_2(z)$, can be chosen to meet the following conditions at $z = 0$, with no loss in generality:

$$\begin{aligned} y_1(0) &= 1 \qquad & y_2(0) &= 0 \\ y_1'(0) &= 0 \qquad & y_2'(0) &= 1 \end{aligned} \tag{9.43}$$

Normalization may be necessary to give the specified values to $y_1(0)$ and $y_2'(0)$. A typical pair of such solutions is plotted in Fig. 9.5 with the kinds of symmetry assumed. The conditions of Eq. (9.43) give the value unity to the Wronskian at $z = 0$, and the solutions are independent. If the conditions are used in Eq. (9.38), and the derivatives of these equations, the conditions at $z = \pi$ are found

$$\begin{aligned} y_1(\pi) &= k_{11} \qquad & y_2(\pi) &= k_{21} \\ y_1'(\pi) &= k_{12} \qquad & y_2'(\pi) &= k_{22} \end{aligned} \tag{9.44}$$

Since both y_1 and y_2 are solutions of Eq. (9.36), the relation holds

$$\frac{y_1''}{y_1} = \frac{y_2''}{y_2} = -f(z)$$

This equation can be rewritten as

$$y_2''y_1 - y_1''y_2 = 0$$

and integrated by parts to give

$$y_1y_2' - y_2y_1' = \text{constant} = 1$$

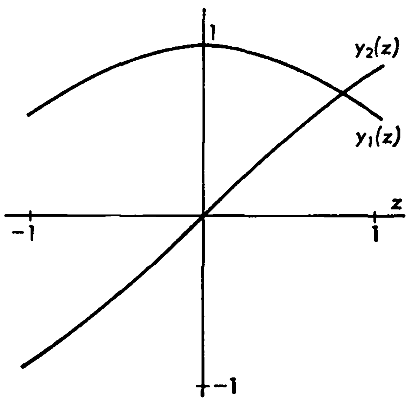


FIG. 9.5. Two independent functions meeting the following requirements: y_1 has even symmetry, $y_1(0) = 1$, $y_1'(0) = 0$; y_2 has odd symmetry, $y_2(0) = 0$, $y_2'(0) = 1$.

The terms on the left side form the Wronskian, which has been shown to have the value unity at $z = 0$. The constant of integration must also have the value unity, and the Wronskian is constant for all values of z. A relation among the coefficients k_{ij} can be found from Eq. (9.44), at $z = \pi$, as

$$W(y_1,y_2) = k_{11}k_{22} - k_{12}k_{21} = 1$$

Thus, Eq. (9.42) becomes

$$\sigma^2 - 2m\sigma + 1 = 0 \tag{9.45}$$

where $2m = k_{11} + k_{22}$. The roots of Eq. (9.45) are

$$\sigma_1, \sigma_2 = m \pm j(1 - m^2)^{1/2}$$

The roots are either real or complex, depending upon the value of m. Loci of σ_1 and σ_2 are plotted in Fig. 9.6 as m varies.

For any value of m, the product of σ_1 and σ_2 is $\sigma_1\sigma_2 = 1$. For the range $-1 < m < +1$, σ is a complex quantity of magnitude unity and may be written as $(\sigma_1,\sigma_2) = \exp(\pm j\beta\pi)$, with β in the range $0 < \beta < 1$. For the range $|m| > 1$, σ is a real quantity and may be written as

$$(\sigma_1,\sigma_2) = \pm \exp(\pm\alpha\pi)$$

with α in the range $0 < \alpha < \infty$. For $m = \pm 1$, the two values of σ are the same $\sigma_1 = \sigma_2 = \pm 1$. In both the last two cases where σ is real, the algebraic signs of m and σ are the same.

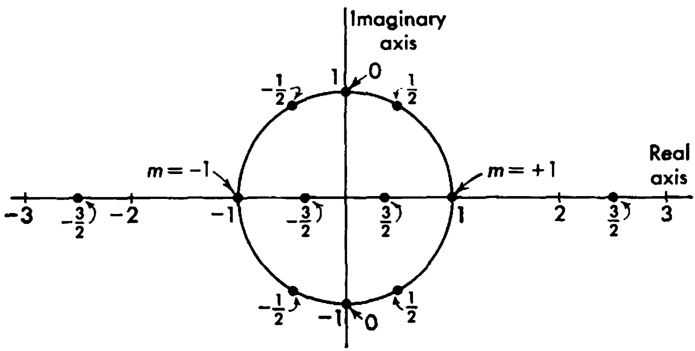


FIG. 9.6. Locus of factor σ as quantity m varies.

When σ has the value either $+1$ or -1, the normal solution $y(z)$ is periodic. From Eq. (9.41) with $\sigma = +1$, it is evident that $y(z + \pi) = y(z)$, and the period of the solution is π. With $\sigma = -1$, then

$$y(z + 2\pi) = -y(z + \pi) = y(z)$$

and the period is 2π. If the magnitude of σ is greater than unity, as will occur for $|m| > 1$, solution $y(z)$ increases without bound as z increases. Such a solution is termed unstable. Similarly, a solution that remains bounded as z increases is termed stable. A stable solution exists if the magnitude of σ is unity or less. The boundaries between stable and unstable solutions occur for $|\sigma| = 1$, corresponding to $|m| = 1$.

As has been indicated above, a general form for complex factor σ is

$$\sigma = \exp(\alpha + j\beta)\pi = \exp(\mu\pi)$$

where one of the quantities α or β is zero, as determined by the value of m. A new function $P(z)$ can be defined so that

$$y(z) = \exp(\mu z)P(z) \tag{9.46}$$

Then, in accordance with Eq. (9.41),

$$\begin{aligned} y(z+\pi) &= \exp\,[\mu(z+\pi)]P(z+\pi) \\ &= \exp\,(\mu\pi)y(z) \\ &= \exp\,(\mu\pi)\exp\,(\mu z)P(z) \end{aligned}$$

A comparison of terms shows that $P(z+\pi) = P(z)$ and $P(z)$ is a periodic function of period π in z.

Since the Mathieu equation is unchanged if the algebraic sign of z is reversed, the sign of z can be changed in the solution, Eq. (9.46). So long as σ has a value other than $+1$ or -1, this sign reversal gives an independent solution. Thus, a complete solution for Eq. (9.35) is

$$y = A\exp\,(\mu z)P(z) + B\exp\,(-\mu z)P(-z) \tag{9.47}$$

where A and B are arbitrary constants. This solution holds so long as σ is different from either 0 or j. It is evidently unstable if σ is other than a pure imaginary number.

In the special case of $\sigma = +1$ or $\mu = 0$, the two parts of Eq. (9.47) are not independent and the solution can be shown to take the form

$$y = AP(z) + BzQ(z) \tag{9.48}$$

The first term here is periodic with period π in z. Similarly, in the special case of $\sigma = -1$ or $\mu = \pm j$, the solution must take the form

$$y = A\exp\,(jz)P(z) + Bz\exp\,(jz)Q(z) \tag{9.49}$$

The first term is again periodic, but of period 2π in z. In each case, $Q(z)$ is a periodic function of period π in z. Because the multiplying factor z appears in each of the second terms, this term grows without bound and accordingly is unstable.

An important result of all this is that the transition between stable and unstable solutions occurs for $\sigma = +1$ and $\sigma = -1$. In each case, one part of the solution is periodic with periods of π or 2π, respectively. The other part of the solution is not stable and grows without bound. Any stable solution, other than a transitional solution, which happens to be periodic has a period greater than 2π. Furthermore, the solutions may have either even or odd symmetry.

b. Approximate Solutions by Variation of Parameters. The method of variation of parameters of Sec. 9.3*d* can be used to find an approximate solution for the Mathieu equation for combinations of coefficients a and q near $a = 1$ and $q = 0$. While this is a limited region, it is nevertheless a region of considerable interest for practical problems.

The Mathieu equation is

$$\frac{d^2y}{dz^2} + (a - 2q \cos 2z)y = 0$$

which is analogous to Eq. (9.28). If the term with coefficient q is first neglected, the generating solution is $y = A \cos (a^{1/2}z + \theta)$. If now q is assumed small, but different from zero, the necessary variations in A and θ are given by

$$\frac{dA}{dz} = -\frac{qA}{a^{1/2}} \cos 2z \sin 2(a^{1/2}z + \theta) \tag{9.50}$$

$$\frac{d\theta}{dz} = -\frac{q}{a^{1/2}} \cos 2z[1 + \cos 2(a^{1/2}z + \theta)] \tag{9.51}$$

which are analogous to Eqs. (9.29) and (9.30)

A particular case of interest is that where $a = 1$ and q is small. For this case, Eq. (9.51) becomes

$$\frac{d\theta}{dz} = -q[\cos 2z + \tfrac{1}{2} \cos (4z + 2\theta) + \tfrac{1}{2} \cos 2\theta]$$

The first two terms in the brackets are oscillatory with respect to z and have average values that are zero. The last term is a constant. Thus, the average value of the derivative is

$$\left(\frac{d\theta}{dz}\right)_{\text{av}} = -\frac{q}{2} \cos 2\theta$$

This equation can be plotted in a kind of phase-plane diagram as shown in Fig. 9.7, where q is assumed to be positive. Equilibrium points for θ occur when θ has values that are odd multiples of $\pi/4$ radians. Variable z increases as shown by the arrows of Fig. 9.7, and evidently stable values exist for $\theta = \ldots, -5\pi/4, -\pi/4, 3\pi/4, \ldots$, while unstable values are $\theta = \ldots, -3\pi/4, \pi/4, 5\pi/4, \ldots$. Thus, as z increases, θ will change from whatever its initial value may be to one of the stable equilibrium values. The simplest stable equilibrium value is $\theta = -\pi/4$ radians, for q a positive number.

With $a = 1$ and $\theta = -\pi/4$, Eq. (9.50) becomes

$$\frac{dA}{dz} = -qA \cos 2z \sin 2\left(z - \frac{\pi}{4}\right) = qA \cos^2 2z$$

The average value is

$$\left(\frac{dA}{dz}\right)_{\text{av}} = \frac{qA}{2}$$

and amplitude A must vary as

$$A = A_0 \exp\left(\frac{qz}{2}\right)$$

where $A = A_0$ at $z = 0$. Thus, an approximate solution with $a = 1$ and $0 < q \ll 1$ is

$$y = A_0 \exp\left(\frac{qz}{2}\right) \cos\left(z - \frac{\pi}{4}\right) \tag{9.52}$$

obtained by using the equilibrium value for θ and the varying value for A in the original generating solution. The solution in Eq. (9.52) is an

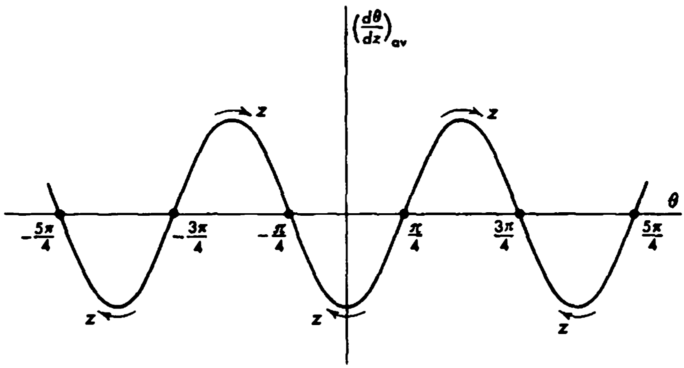


FIG. 9.7. Phase-plane diagram for angle θ in approximate solution of Mathieu equation, with $a = 1$ and $0 < q \ll 1$.

oscillation growing in amplitude without bound as z increases and is accordingly unstable. This solution is plotted in Fig. 9.8, where q has been given the arbitrary value, $q = 0.02$.

If q is negative and small, the equilibrium value for θ is $\theta = +\pi/4$ and the solution is again unstable.

A second case of interest is that where coefficient a differs from unity by a small amount, say, $a = 1 + \epsilon$, where $0 < |\epsilon| \ll 1$, and q is small. Equation (9.50) may be written

$$\frac{dA}{dz} = -\frac{qA}{2a^{1/2}}\left\{\sin 2\,[(a^{1/2} + 1)z + \theta] + \sin 2[(a^{1/2} - 1)z + \theta]\right\}$$

and integrated to give

$$A = C \exp\left(\frac{q}{2a^{1/2}}\left\{\frac{\cos 2[(a^{1/2} + 1)z + \theta]}{2(a^{1/2} + 1)} + \frac{\cos 2[(a^{1/2} - 1)z + \theta]}{2(a^{1/2} - 1)}\right\}\right)$$

where C is a constant of integration. In this integration, it is assumed that θ is a constant. Actually, this is incorrect, but changes in θ are small if q is small, as is necessary for this method of solution to be useful. If, now, $a = 1 + \epsilon$ so that very nearly $a^{1/2} = 1 + \epsilon/2$, the second exponent in the brace of the equation for A is much larger than the first,

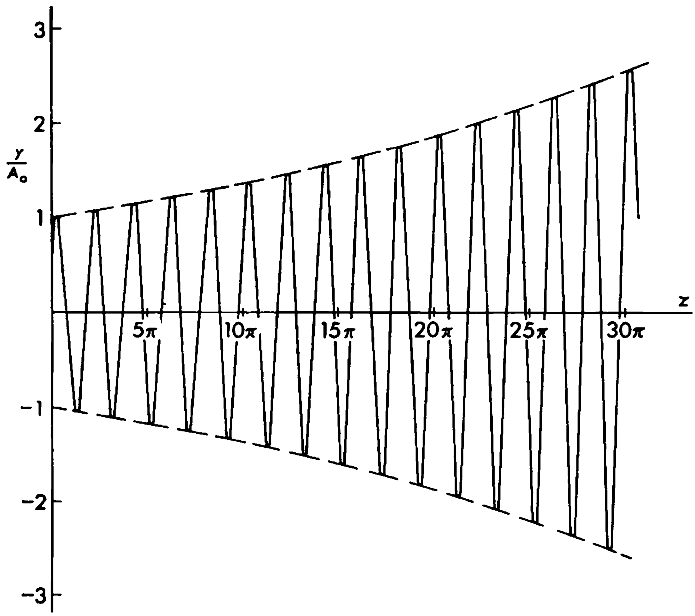


FIG. 9.8. Approximate solution for Mathieu equation, with $a = 1$, $q = 0.02$, and $\theta = -\pi/4$.

because its denominator is $2(a^{1/2} - 1) = \epsilon$, which is very small. Thus, very nearly,

$$A = C \exp\left[\frac{q}{2\epsilon} \cos\,(\epsilon z + 2\theta)\right]$$

and if $q/2\epsilon \ll 1$, approximately

$$A = C\left[1 + \frac{q}{2\epsilon} \cos\,(\epsilon z + 2\theta)\right]$$

Thus, the amplitude of the solution undergoes a kind of modulation.

At the same time, Eq. (9.51) becomes

$$\begin{aligned}\frac{d\theta}{dz} &= -q\{\cos 2z + \tfrac{1}{2} \cos 2[(a^{1/2} + 1)z + \theta] + \tfrac{1}{2} \cos 2[(a^{1/2} - 1)z + \theta]\} \\ &= -q\{\cos 2z + \tfrac{1}{2} \cos\,[(4 + \epsilon)z + 2\theta] + \tfrac{1}{2} \cos\,(\epsilon z + 2\theta)\}\end{aligned}$$

where $a^{1/2} = 1 + \epsilon/2$. There are three terms here, but the last, which varies most slowly with respect to z, is the one which makes the most apparent difference in the form of the solution. This term is

$$\frac{d\theta}{dz} = -\frac{q}{2}\cos(\epsilon z + 2\theta)$$

If q is very small, θ changes but slightly from a mean value, say, θ_0, so that very nearly

$$\theta = -\frac{q}{2\epsilon}\sin(\epsilon z + 2\theta_0) + \theta_0$$

An approximate solution for $a = 1 + \epsilon$, $0 < |\epsilon| \ll 1$, and $0 < |q/2\epsilon| \ll 1$ becomes

$$y = C\left[1 + \frac{q}{2\epsilon}\cos(\epsilon z + 2\theta_0)\right] \cos\left[\left(1 + \frac{\epsilon}{2}\right)z - \frac{q}{2\epsilon}\sin(\epsilon z + 2\theta_0) + \theta_0\right] \quad (9.53)$$

where constants C and θ_0 depend upon initial conditions. This solution represents an oscillation with z in which modulation of both amplitude

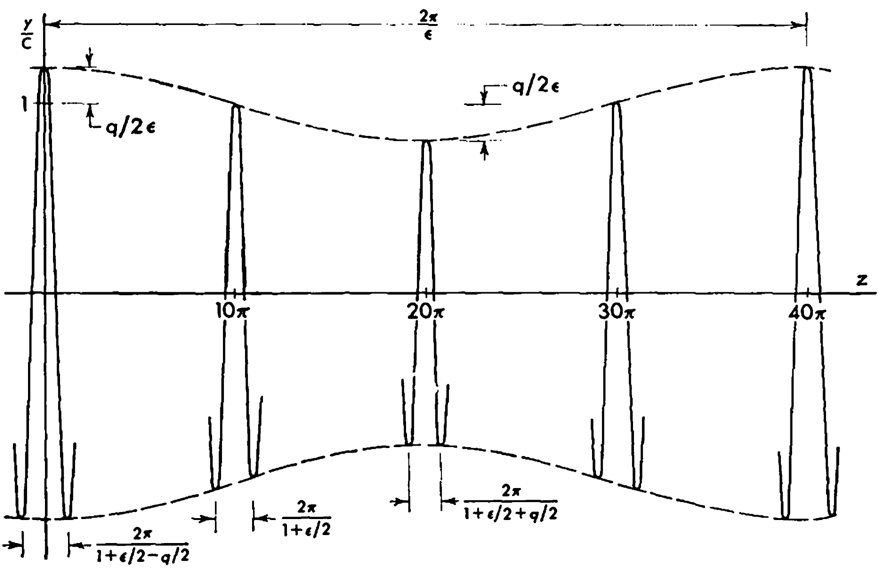


FIG. 9.9. Approximate solution for Mathieu equation, with $a = 1.05$, $q = 0.02$, and $\theta_0 = 0$. Only a few cycles of a continuing oscillation are shown, and differences in apparent period are exaggerated.

and frequency takes place. A plot of Eq. (9.53) is shown in Fig. 9.9, where the arbitrary choice has been made, $q = 0.02$, $\epsilon = 0.05$, and $\theta_0 = 0$. Changes in apparent period are greatly exaggerated in this figure for the sake of clarity. As amplitude and instantaneous frequency fluctuate, maximum amplitude and maximum apparent period of one oscillation of the solution occur together, as do minimum values for these quantities. The period for the amplitude variation is $2\pi/\epsilon$. The solution itself is generally not strictly periodic.

This method of variation of parameters leads to just a first-order approximation and is generally useful only near the combination of coefficients $a = 1$ and $q = 0$. The case of $a = 1$ and $0 < |q| \ll 1$ has been shown in Eq. (9.52) to lead to an unstable solution. Other unstable solutions exist near $a = 2^2, 3^2, \ldots$, but these cannot be found through this method of approach.

c. Location of Stability Boundaries by Perturbation Method. A second way of finding information about approximate solutions for the Mathieu equation is by the perturbation method of Sec. 6.2. This method can be used provided coefficient q is small, $|q| \ll 1$.

The equation itself is

$$\frac{d^2y}{dz^2} + (a - 2q \cos 2z)y = 0 \tag{9.54}$$

Since q is small, approximate solutions can be found in the series form

$$\begin{aligned} y(z) &= y_0(z) + qy_1(z) + q^2y_2(z) + \cdots \\ a(q) &= a_0(q) + qa_1(q) + q^2a_2(q) + \cdots \end{aligned} \tag{9.55}$$

The second relation involving a and q is necessary to eliminate secular terms as they arise. If Eq. (9.55) is substituted into Eq. (9.54) and terms in like powers of q collected, the result to the second order is

$$q^0: \qquad y_0'' + a_0y_0 = 0 \tag{9.56}$$

$$q^1: \qquad y_1'' + a_0y_1 = -a_1y_0 + 2y_0 \cos 2z \tag{9.57}$$

$$q^2: \qquad y_2'' + a_0y_2 = -a_2y_0 - a_1y_1 + 2y_1 \cos 2z \tag{9.58}$$

A feature of particular interest is the combinations of values of coefficients a and q that represent transitions between stable and unstable solutions. In the discussion of Sec. 9.4*a*, it is shown that the transitional solutions have two parts, of which one is unstable and the other is oscillatory with period either π or 2π in z. Thus, transitional combinations of a and q can be investigated by considering those combinations which lead to periodic solutions of period π or 2π.

The generating solution is found from Eq. (9.56) and is

$$y_0 = A_0 \cos a_0^{1/2}z + B_0 \sin a_0^{1/2}z$$

If a_0 has the value unity, $a_0 = 1$, these solutions will be periodic with period 2π. In this case, Eq. (9.57) becomes, after some trigonometric simplification,

$$y_1'' + y_1 = -(a_1 - 1)A_0 \cos z - (a_1 + 1)B_0 \sin z + A_0 \cos 3z + B_0 \sin 3z$$

Two secular terms arise here, either but not both of which can be removed by appropriate choice of a_1. If the choice is $a_1 = +1$, y_1 becomes

$$y_1 = A_1 \cos z + B_1 \sin z + B_0 z \cos z + \tfrac{1}{8}(A_0 \cos 3z + B_0 \sin 3z)$$

The first two and last two terms here, combined, have a period 2π and are stable, while the third term grows without bound and is unstable. This is typical of a transitional solution. Evidently the boundary for this transitional solution is located for $a = a_0 + qa_1 = 1 + q$. If the choice is made $a_1 = -1$, y_1 is the same form as obtained previously, except that now the third term is $A_0 z \sin z$. The same remarks can be made about this solution, so that a second boundary for transition is $a = a_0 + qa_1 = 1 - q$.

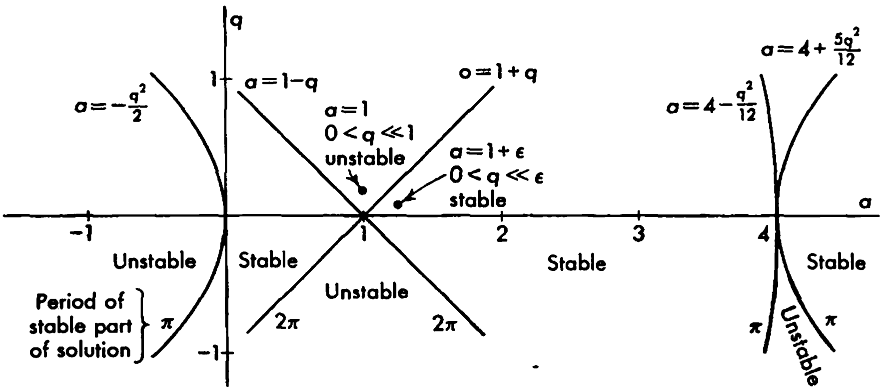


FIG. 9.10. Approximate location of boundaries between stable and unstable solutions for Mathieu equation, valid for q small.

These two boundaries for transition between stable and unstable solutions are plotted in Fig. 9.10. They are applicable, of course, only so long as q is small. Also plotted in the figure are two points corresponding to the two cases investigated by variation of parameters in the preceding section. These points determine the nature of stability in the regions where they are located. It should be noted further that a point exactly on a transitional boundary leads to a solution having one part periodic and stable and the other part growing and unstable. The boundary, therefore, is unstable in general.

In a similar manner, if $a_0 = 4$ in Eq. (9.56), the generating solution is

$$y_0 = A_0 \cos 2z + B_0 \sin 2z$$

and is periodic with period π. In this case, Eq. (9.57) becomes, after some simplification,

$$y_1'' + 4y_1 = -a_1(A_0 \cos 2z + B_0 \sin 2z) + A_0 + A_0 \cos 4z + B_0 \sin 4z$$

A secular term can be avoided by requiring $a_1 = 0$, in which case

$$y_1 = A_1 \cos 2z + B_1 \sin 2z + \frac{A_0}{4} - \tfrac{1}{12}(A_0 \cos 4z + B_0 \sin 4z)$$

The use of y_0 and y_1 in Eq. (9.58) then gives, after simplification,

$$\begin{aligned} y_2'' + 4y_2 = {} & (-a_2 + \tfrac{5}{12})A_0 \cos 2z - (a_2 + \tfrac{1}{12})B_0 \sin 2z \\ & + A_1 + A_1 \cos 4z + B_1 \sin 4z - \frac{A_0}{12} \cos 6z - \frac{B_0}{12} \sin 6z \end{aligned}$$

Here, again, secular terms arise from both the first two terms on the right side of the equation. Either of these may be eliminated by appropriate choice of a_2. If the choice is $a_2 = \frac{5}{12}$, the coefficient of the first term is zero, while the second term remains to give a secular term in the solution. This choice gives the boundary

$$a = a_0 + qa_1 + q^2a_2 = 4 + \frac{5q^2}{12}$$

If alternatively the choice is $a_2 = -\frac{1}{12}$, the second term is removed while the first leads to a secular term in the solution. This choice gives the boundary $a = 4 - q^2/12$. These boundaries are plotted in Fig. 9.10.

Yet a third case is that of $a_0 = 0$ in Eq. (9.56) so that the generating solution is

$$y_0 = A_0 + B_0 z$$

This solution is not periodic, while it has been asserted that transitional solutions must have one periodic component. The periodic component in this case arises from the succeeding correction terms. Since the second term here grows with z, it can be ignored at once by requiring $B_0 = 0$. After substitution and simplification, Eq. (9.57) becomes

$$y_1'' = -a_1A_0 + 2A_0 \cos 2z$$

To avoid a growing term, it is necessary that $a_1 = 0$, so that

$$y_1 = -\left(\frac{A_0}{2}\right) \cos 2z$$

which is periodic with period π. Further, Eq. (9.58) becomes

$$y_2'' = -(a_2 + \tfrac{1}{2})A_0 - \frac{A_0}{2} \cos 4z$$

A growing term is avoided if $a_2 = -\frac{1}{2}$. The boundary between stable and unstable regions is then $a = a_0 + qa_1 + q^2a_2 = -q^2/2$. This boundary is plotted in Fig. 9.10.

The perturbation method, and Fig. 9.10 based on this method, is of good accuracy only so long as $|q| \ll 1$.

By other methods of analysis, it is possible to find exact values of those combinations of coefficients a and q that represent boundaries between stable and unstable solutions. A diagram based on exact values is shown in Fig. 9.11, which may be compared with Fig. 9.10. Both these figures are symmetrical about the a axis. It is evident that solutions are essentially stable if $a > 0$ and $|2q| < a$ and essentially unstable otherwise, as predicted in Fig. 9.4. However, there are narrow regions in which this general observation is incorrect. In particular, there are

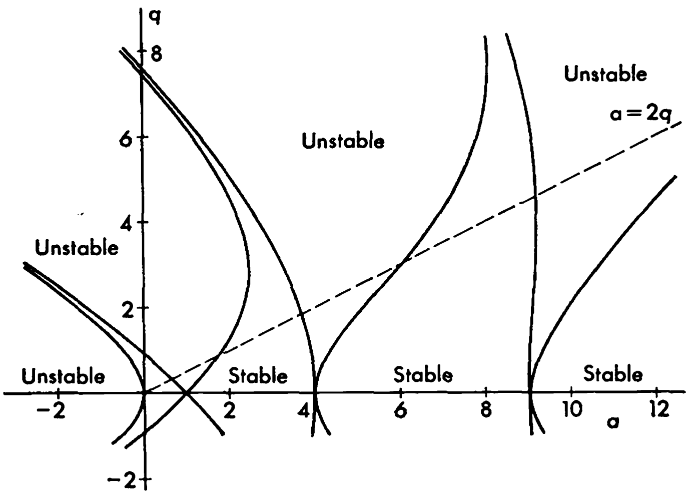


FIG. 9.11. Exact location of boundaries between stable and unstable solutions for Mathieu equation.

unstable regions associated with small values of q near $a = 1^2, 2^2, 3^2, \ldots$. There are certain narrow stable regions for a negative.

The form of typical solutions for the Mathieu equation may be exceedingly complicated. Several examples as obtained with an analog computer are shown in Fig. 9.12. Each of these solutions corresponds to initial conditions that $y = C$ and $dy/dz = 0$ at $z = 0$, with C a constant. The curves corresponding to cases studied in Sec. 9.4*b* agree well with the conclusions reached there.

d. Applications. The Mathieu equation arises in several different types of physical problems. It was studied first historically in connection with boundary-value problems in which one of the boundaries is elliptical. Investigations of the motion of an elliptical membrane or of waves in a waveguide of elliptical cross section lead to this equation. It arises in the analysis of oscillating systems with a parameter that is periodic in either time or space. Finally, a test for the stability of certain nonlinear oscillating systems leads to the equation.

Boundary-value problems are beyond the scope of the present discussion. Examples are given here, however, of several systems with varying parameters. In the following chapter, the stability of several nonlinear oscillating systems is explored, and examples of the use of the equation appear.

Example 9.8. Pendulum with Oscillating Support

A simple pendulum is governed by the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

where θ is the angle of deflection away from the stable rest position with the pendulum hanging vertically downward, g is the gravitational acceleration, and l is the length

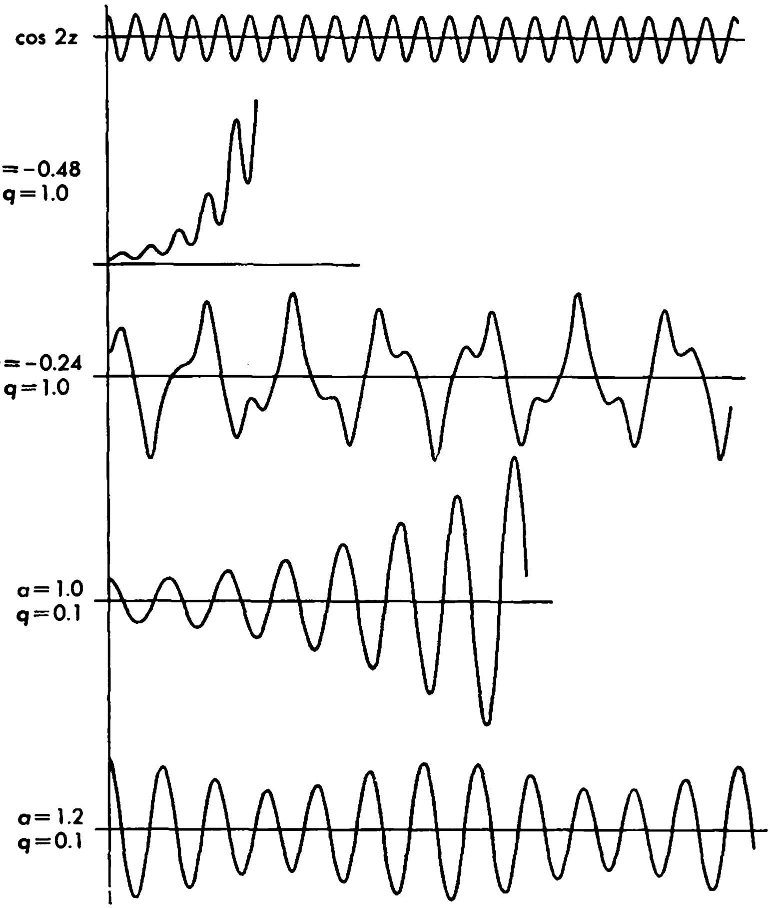


FIG. 9.12. Solutions for Mathieu equation as found with analog computer. Values of $z = 0$. These curves should be compared

of the pendulum. If the angle of deflection is always small, so that $\theta \ll 1$, the approximation can be made, $\sin\theta = \theta$, so that the equation becomes linear.

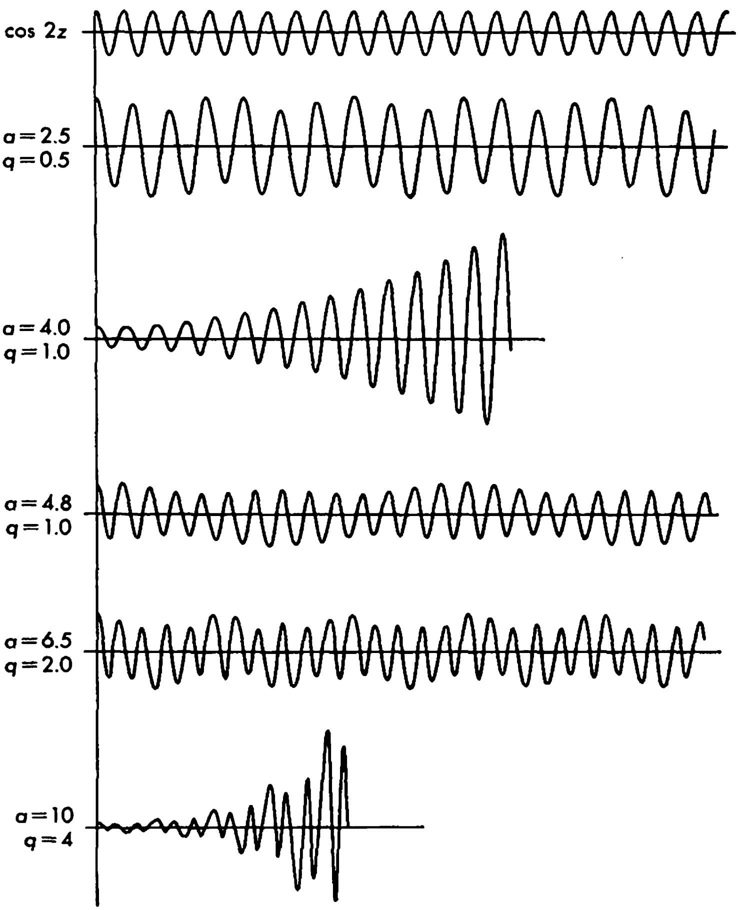
A simple pendulum can be arranged so that its pivot point is caused to oscillate in a vertical direction, as $u = U \cos \omega_1 t$, where u is positive upward, U is the amplitude, and ω_1 is the angular frequency of the oscillation. Such motion accelerates the entire pendulum, so that the gravitational acceleration is modified to have the effective value

$$g_{eff} = g + \ddot{u} = g - A \cos \omega_1 t$$

where $A = \omega_1{}^2 U$ is the maximum acceleration of the pivot. The equation for the pendulum, with small deflections and vertical motion of the pivot, is

$$\frac{d^2\theta}{dt^2} + \left(\frac{g}{l} - \frac{A}{l}\cos \omega_1 t\right)\theta = 0 \tag{9.59}$$

This is a form of the Mathieu equation.



coefficients a and q are indicated. In each case, y = constant and $dy/dz = 0$ at with Figs. 9.8, 9.9, and 9.11.

In order to put the pendulum equation into the standard form of the Mathieu equation, Eq. (9.35), the arguments of the cosine functions must be made identical. This leads to the equivalences

$$\omega_1 t = 2z \qquad \theta = y$$

$$\frac{d\theta}{dt} = \left(\frac{dy}{dz}\right)\left(\frac{dz}{dt}\right) = \frac{\omega_1}{2}\left(\frac{dy}{dz}\right)$$

$$\frac{d^2\theta}{dt^2} = \left(\frac{d^2y}{dz^2}\right)\left(\frac{dz}{dt}\right)^2 + \left(\frac{dy}{dz}\right)\left(\frac{d^2z}{dt^2}\right)$$

$$= \left(\frac{\omega_1}{2}\right)^2\left(\frac{d^2y}{dz^2}\right)$$

With these substitutions, Eq. (9.59) becomes

$$\frac{d^2y}{dz^2} + \left(\frac{2}{\omega_1}\right)^2\left(\frac{g}{l} - \frac{A}{l}\cos 2z\right) y = 0 \tag{9.60}$$

Identification of the coefficients here with those of Eq. (9.35) gives

$$a = \frac{4g}{\omega_1{}^2 l}$$

and

$$q = \frac{2A}{\omega_1{}^2 l}$$

or for a given pendulum of constant length

$$q = \frac{A}{2g}\,a$$

This linear relation between q and a can be plotted on a stability diagram for the Mathieu equation, such as Fig. 9.11, as shown in Fig. 9.13. Both a and q are positive here, so that the line lies in the first quadrant. Its slope depends upon A, the maximum acceleration of the pivot. The point on the line corresponding to a chosen frequency ω_1 of pivot oscillation moves toward the origin as ω_1 increases.

Evidently near $a = 1$ the combination of a and q leads to an unstable solution. The greater the ratio A/g, the larger the slope of the line and the wider the unstable region. If $a = 1$, $\omega_1{}^2 = 4g/l = (2\omega_0)^2$, where $\omega_0 = (g/l)^{1/2}$ is the natural angular frequency of the pendulum with no motion of the pivot. Thus, if the frequency of pivot oscillation is close to twice the natural frequency, the amplitude of motion of the pendulum itself tends to increase. What happens physically is that the mass of the pendulum is raised twice each cycle of its oscillation, and, each time, work is done against the centrifugal force produced by its motion. Energy is thus supplied to the pendulum. The pendulum is lowered when its angular velocity and centrifugal force are small, and little energy is removed. The net result is an increase in energy, and this appears as increased amplitude of swing. This is the process used by a child in propelling himself on a playground swing.

Near $a = 4, 9, 16, \ldots$, there are other unstable conditions which might cause the pendulum to be unstable. Usually friction effects, disregarded here but present in any practical pendulum, are sufficient for these other unstable conditions not to be observed experimentally. If combinations of a and q fall in a stable region, the average amplitude of the pendulum swing is unchanged by motion of its pivot.

The unstable case in which energy is supplied by causing the pivot to oscillate at twice the frequency at which the pendulum swings represents a kind of subharmonic generation of order $\frac{1}{2}$. The mechanism here is entirely different from that of subharmonic generation in a system governed by Duffing's equation, as discussed in Example 7.5. Duffing's equation is nonlinear with constant coefficients, while Mathieu's equation is linear with a periodic coefficient.

A second case of some interest is that where the pendulum is inverted, with the equilibrium position taken vertically above its pivot. This position is, of course,

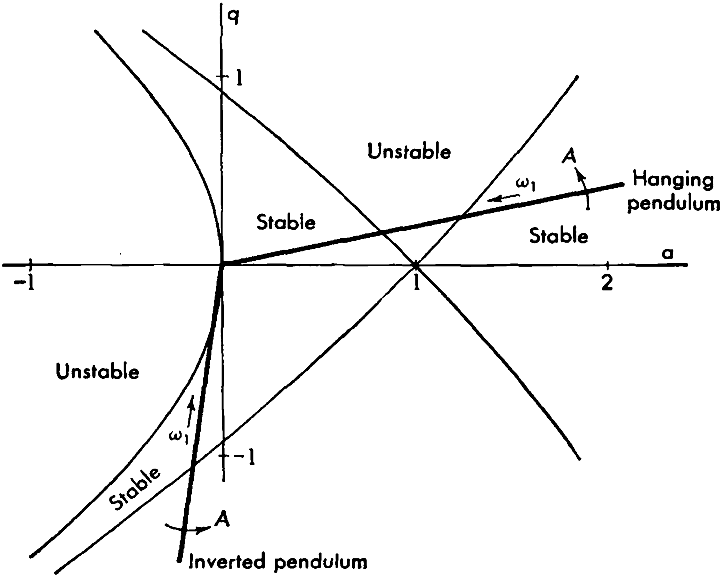


FIG. 9.13. Stability diagram for Mathieu equation, applying to hanging pendulum and inverted pendulum, each having an oscillating support as in Example 9.8.

unstable for the simple pendulum, and it falls away from this position when released. The equation for the inverted pendulum is

$$\frac{d^2\phi}{dt^2} - \frac{g}{l}\sin\phi = 0$$

where ϕ is the angle measured from the unstable equilibrium point. If the pivot of the inverted pendulum is caused to oscillate as before, the equation becomes, in place of Eq. (9.59),

$$\frac{d^2\phi}{dt^2} + \left(\frac{-g}{l} + \frac{A}{l}\cos\omega_1 t\right)\phi = 0 \tag{9.61}$$

where angle ϕ is assumed to be small. The only difference here is the change in algebraic signs of the coefficients in the parentheses. The coefficients of the corresponding Mathieu equation are accordingly $a = -4g/\omega_1^2 l$ and $q = -2A/\omega_1^2 l$, both negative, so that again $q = (A/2g)a$. This relation is plotted in the third quadrant of Fig. 9.13.

The interesting feature here is that, if ω_1 is chosen large enough, coefficients of a and q fall in a stable region, so that the pendulum is stable and will remain in the inverted position. In a practical model illustrating this phenomenon, the amplitude U of motion of the pivot must be somewhat less than the length l of the pendulum, so that $q = -2U/l$ will be a number certainly no greater than unity in magnitude.

Examination of Fig. 9.13 shows that the stable region with q in the order of unity requires a to be somewhat smaller than q. Since, for the pendulum, $q = (A/2g)a$, it is required that $q/a = A/2g > 1$. The conclusion is reached that, if the inverted pendulum is to be stable, the maximum acceleration of its pivot point must exceed somewhat the gravitational acceleration.

In summary, vertical oscillation of the pivot point of a simple pendulum leads to two phenomena not present in the pendulum alone. If the pendulum is hanging downward in what is normally a stable position, it may become unstable. If the pendulum is inverted into a normally unstable position, it may be made stable. In each case, parameters of the vertical oscillation must be properly chosen.

Example 9.9. Resonant Circuit with Varying Capacitance

In Example 9.5, the case is investigated of a dissipationless electrical circuit having a constant inductance and a capacitance varying periodically with time. It is evident that this circuit is described by an equation of the Mathieu type. It has been shown that, under some conditions, solutions for this equation become unstable. It is therefore pertinent to determine whether the electrical circuit can become unstable in the sense that voltages and currents in the circuit might increase without bound. Conditions leading to such instability should be known.

The same circuit is considered as shown in Fig. 9.3, except that now a constant resistance is included. The equation for the circuit is

$$L\frac{d^2x}{dt^2} + R\frac{dx}{dt} + \frac{x}{C} = 0$$

where t is time, x is the instantaneous charge on the capacitor, R is the resistance, and C is the instantaneous capacitance. The symbol x is used here for charge to avoid confusion with coefficient q of the Mathieu equation. The capacitance is assumed to vary as

$$C = C_0(1 + m\cos\omega_1 t)$$

where m is the relative variation about the mean capacitance C_0 and ω_1 is the angular frequency of the variation. It is assumed that m is small so that $0 < m \ll 1$. The equation for the circuit accordingly becomes

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^2(1 - m\cos\omega_1 t)x = 0 \tag{9.62}$$

where $2\alpha = R/L$, $\omega_0^2 = 1/LC_0$, and use has been made of the assumed smallness of m.

Equation (9.62) is similar to the Mathieu equation, except that it has a term including the first derivative. This term can be removed by the change of variable described in Sec. 9.3a, which is here $y = x\exp(\alpha t)$ or $x = y\exp(-\alpha t)$. When this change is made, Eq. (9.62) becomes

$$\frac{d^2y}{dt^2} + (\omega_0^2 - \alpha^2 - m\omega_0^2\cos\omega_1 t)y = 0 \tag{9.63}$$

In order to put this into the standard form of the Mathieu equation, Eq. (9.35), the arguments of the cosine functions must be identical. The equivalences are

$$\begin{aligned}\omega_1 t &= 2z\\ \frac{dy}{dt} &= \frac{\omega_1}{2}\left(\frac{dy}{dz}\right)\\ \frac{d^2y}{dt^2} &= \left(\frac{\omega_1}{2}\right)^2\left(\frac{d^2y}{dz^2}\right)\end{aligned}$$

When these changes are used in Eq. (9.63), it becomes

$$\frac{d^2y}{dz^2} + \frac{4}{\omega_1^2}(\omega_0^2 - \alpha^2 - m\omega_0^2 \cos 2z)y = 0$$

This is the standard form for the Mathieu equation, and the coefficients can be identified as

$$a = \frac{4}{\omega_1^2}(\omega_0^2 - \alpha^2)$$

$$q = \frac{2m\omega_0^2}{\omega_1^2}$$

or

$$q = \frac{(m\omega_0^2/2)}{(\omega_0^2 - \alpha^2)} a \tag{9.64}$$

In Eq. (9.64), all quantities in the parentheses except parameter m are determined by the circuit elements and are fixed for a particular circuit. Parameter m depends upon the relative change in capacitance. Equation (9.64) can be plotted on a

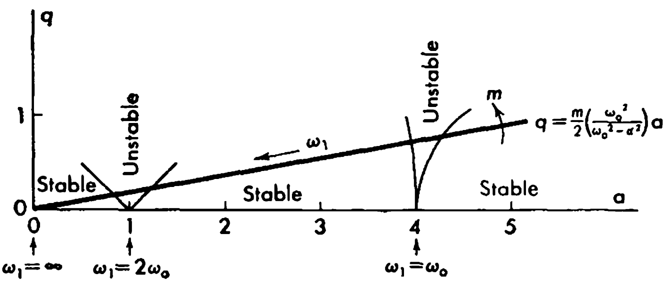


FIG. 9.14. Stability diagram for Mathieu equation, applying to resonant circuit with varying capacitance of Example 9.9.

stability diagram as shown in Fig. 9.14. The relation between q and a is a straight line having a slope that is increased if m is increased. The point on this line corresponding to a particular condition of operation depends upon the frequency ω_1 of capacitance variation. As ω_1 is increased, the point moves toward the origin. Evidently certain values of ω_1 cause the point to fall in an unstable region, and unstable solutions for variable y occur.

The instantaneous charge x on the capacitor is related to variable y through the equation $x = y \exp(-\alpha t) = y \exp(-2\alpha z/\omega_1)$ so that, even if variable y is unstable and grows with z, x may or may not grow with t. If the solution for y is unstable, it will contain a factor of the form $\exp(\mu z)$, where exponent μ is positive and depends upon coefficients a and q. The charge x will then have the factor $\exp[(\mu - 2\alpha/\omega_1)z]$, and the requirement for x to be unstable is that $\mu > 2\alpha/\omega_1$.

Coefficient a can be expressed in terms of the circuit Q as

$$a = \frac{4}{\omega_1^2}(\omega_0^2 - \alpha^2) = \left(\frac{2\omega_0}{\omega_1}\right)^2\left(1 - \frac{1}{4Q_0^2}\right)$$

where $Q_0 = L\omega_0/R$ is the circuit Q at the resonant frequency ω_0. In many practical cases, Q_0 is large enough so that $Q_0^2 \gg 1$, and approximately $a = (2\omega_0/\omega_1)^2$. If this is so, very nearly $q = ma/2$.

The simplest case of instability is that for $a = 1$, investigated in Sec. 9.4*b*. For this case of $a = 1$ and $Q_0 \gg 1$, the frequency of capacitance variation must be $\omega_1 = 2\omega_0$, or twice the resonant frequency of the circuit itself. It has been shown that with these conditions and q small, the amplitude grows as exp $(qz/2)$ so that the exponent is $\mu = q/2 = m/4$. Thus, if the actual charge in the circuit is to be unstable for $\omega_1 = 2\omega_0$, the requirement is $\mu > 2\alpha/\omega_1$, or $m/4 > R/L\omega_1 = R/2L\omega_0 = 1/2Q_0$, or, finally, $m > 2/Q_0$.

In summary, then, a circuit with small dissipation or high Q_0 will become unstable if the capacitance is made to vary at twice the resonant frequency of the circuit, with the relative variation large enough so that $m > 2/Q_0$. Under these conditions, the charge on the capacitor, and thus the voltage across it, will increase without bound. Energy used to produce the change in capacitance, as by rotating a variable capacitor against electrostatic forces of attraction, is converted into electrical energy. Ultimately, of course, some part of any practical circuit becomes nonlinear and breaks down, and an upper limit is set on the developed voltage.

Figure 9.14 indicates possible instability for other conditions as well, the next being near $a = 4$, or $\omega_1 = \omega_0$. An investigation here shows that exponent μ, relating to the rate of increase of y, is much smaller than for the case of $a = 1$. The chance of an actual circuit becoming unstable is accordingly much less. Similarly, other unstable regions, near $a = 9, 16, \ldots$, are unlikely to produce instability in an actual circuit. In a typical frequency-modulation system, for example, $\omega_1 \ll \omega_0$ so that coefficient a is a very large number and instability is highly unlikely to occur.

In the stable regions for coefficients a and q, the changing capacitance produces the multiplicity of side-frequency components found in Example 9.5.

9.5. Summary. Linear equations with varying coefficients arise in the description of linear physical systems having parameters which are made to vary by some external agency. Since these equations are linear, the principle of superposition applies and a complete solution can be built up as a linear sum of several components. Techniques generally applicable to linear equations can be used. The details of finding a solution are usually exceedingly complicated.

A standard form is known for the solution of a first-order equation with a varying coefficient. When this form is applied to a typical case, it leads to an integral which is usually difficult to evaluate.

Equations of order higher than the first and having varying coefficients are even more difficult to solve. Several methods of finding approximate solutions for second-order equations of a particular form have been presented.

One second-order equation with a varying coefficient about which there is much information is the Mathieu equation. This equation has a coefficient varying as a simple-harmonic function. Under some conditions, its solutions are stable, while under other conditions they are unstable. The stability is known in terms of the combinations of coefficients of the equation. Exact solutions for the Mathieu equation are generally difficult to find, although for certain combinations of the coefficients the solutions are relatively simple.

CHAPTER 10

STABILITY OF NONLINEAR SYSTEMS

10.1. Introduction. In many places in the preceding chapters, there are discussions regarding the stability of a particular physical system and of solutions for the equations describing it. In a general way, the question of stability is concerned with the determination of conditions of equilibrium and with what happens if the system is disturbed slightly near an equilibrium condition. Again in general terms, any disturbance near an unstable equilibrium condition leads to a larger and larger departure from this condition. Near a stable equilibrium condition, the opposite is the case. It is usually not difficult to define exactly what is meant by stability in a linear system. Because of new types of phenomena which may arise in a nonlinear system, it is not possible to use a single definition for stability which is meaningful in every case. For this reason, the present chapter is concerned with the problem of defining stability for a nonlinear system and of applying these definitions to several typical systems.

10.2. Structural Stability. A concept known as structural stability is sometimes introduced in discussions of physical systems and is used in a sense somewhat different from the more general concept that might be called dynamical stability. The observation is made that the coefficients in equations describing physical systems are never known to a high degree of accuracy. These coefficients must be found empirically as the result of experimental measurements which are always subject to error. Particularly in the case of nonlinear systems, where the coefficients are functions of the operating conditions, it is difficult to determine values of the coefficients with high accuracy. Furthermore, the physical parameters are often subject to change with such ambient and uncontrolled conditions as time, temperature, humidity, and the like. Changes of this sort are probably not included in equations describing the system. As a result, coefficients in the equations are invariably subject to considerable uncertainty.

The mathematical solutions for nonlinear equations can usually be found only approximately. Depending upon the nature of the nonlinearity and the method employed, the solution may or may not be of a high degree of accuracy, but some uncertainty is almost sure to be present. Because of this uncertainty about solving the equations themselves, in addition to the uncertainty about whether or not the numerical values of coefficients are actually valid for the physical system being studied, there is reason to question whether or not the solution finally obtained actually applies to the physical system under study.

Certain types of solutions depend critically upon relative values of parameters in the equation. This is the case, for example, of the equilibrium point described in Sec. 5.5 as a vortex point. A vortex point in a linear system is associated with a continuing steady-state oscillation which neither grows nor decays but maintains a constant amplitude determined by initial conditions. A vortex point requires the net dissipation of the system to be exactly zero. Any physical system always involves positive dissipation, and its solution can correspond to a vortex point only if negative dissipation is also present. Negative and positive dissipation must exactly cancel one another. This exact cancellation requires a precise adjustment of coefficients in the physical system. Any small change from the proper adjustment will lead to either a growing or a decaying oscillation, types of solution which are basically different from each other and from the steady-state oscillation. A system of this kind is said to lack structural stability.

Structural stability, then, is the property of a physical system such that the qualitative nature of its operation remains unchanged if parameters of the system are subject to small variations. The properties of the mathematical solutions for equations describing the system are unchanged if small variations occur in coefficients of the equations. Because of the inherent uncertainty in relating mathematical solutions to physical systems, it is often well to require that the system be arranged in such a way that it possesses the property of structural stability.

10.3. Dynamical Stability. A physical system may be described by a set of simultaneous differential equations of the form

$$
\begin{aligned}
\frac{dx_1}{dt} &= \dot{x}_1 = f_1(x_1,x_2, \ . \ . \ . \ ,x_n) \\
\frac{dx_2}{dt} &= \dot{x}_2 = f_2(x_1,x_2, \ . \ . \ . \ ,x_n) \\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\tag{10.1}
$$

where t is the independent variable, here considered to be time, x_1, x_2, . . . , x_n are the n dependent variables, and the functions f_1, f_2, . . . , f_n are generally nonlinear functions of the dependent variables. Most of the preceding chapters have been concerned with ways of finding solutions for equations of this sort where only one or two first-order equations appear. A complicated physical system may obviously lead to more than two first-order equations.

The simplest equilibrium, or singular, points are those points where all the derivatives, $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n$ are simultaneously zero. The system is accordingly at rest, since all the dependent variables are constant and unvarying with time.

A linear system is characterized by only linear functions $f_1, f_2, \ldots, f_n$ in Eq. (10.1). When the derivatives are set to zero, these linear functions give the conditions for equilibrium as

$$\begin{aligned} 0 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ 0 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{10.2}$$

where the a_{ij} coefficients are constants arising from the parameters of the physical system. In general, the determinant of the coefficients does not vanish,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

In this case, the condition for the variables $x_1, x_2, \ldots, x_n$ satisfying Eq. (10.2) is that they all be zero. A linear system described by Eq. (10.1) has only a single equilibrium point at which all the dependent variables vanish.

A nonlinear system requires nonlinear functions $f_1, f_2, \ldots, f_n$ in Eq. (10.1) and leads to nonlinear algebraic equations corresponding to Eq. (10.2). These nonlinear equations may be satisfied by values for variables $x_1, x_2, \ldots, x_n$ which are not zero, and more than a single set of values may exist. Nonlinear systems, therefore, may have many equilibrium points.

In investigating the stability of a system near a chosen equilibrium point, essentially what is done is to disturb the system slightly by changing all the x from their equilibrium values. If, as t increases indefinitely, the x return to their original equilibrium values, the system is said to be asymptotically stable. On the other hand, if the x depart further from their original equilibrium values with increasing t, the system is said to be dynamically unstable. In a few special cases, the x may neither return to their original values nor depart further from them. A system with this property is said to be neutrally, or temporally, stable. Certain solutions for the Mathieu equations of Sec. 9.4 are said to be stable in this sense. For a nonlinear system, it is necessary to require that the initial disturbances of the x be small enough to keep them in the region controlled by the equilibrium point in question. If the initial disturbances are too large, the x may then be located in a region controlled by some other singular point.

A linear physical system can be described as stable or unstable in accordance with the preceding discussion. It is usually required that a linear system of practical importance be stable in this sense. If it is unstable, the implication is that, as time increases, variables describing the system may increase without bound. In a physical system, this leads to mechanical elements that strike against stops or break, or electrical elements that saturate or burn out. In any case, the system no longer operates as intended and is, of course, no longer linear.

In addition to the possible appearance of many equilibrium points, the operation of a nonlinear system may be complicated by new phenomena associated with the appearance of limit cycles. Limit cycles are steady-state periodic oscillations with their properties determined entirely by parameters of the system. A somewhat similar kind of steady-state oscillation may occur in a hypothetical dissipationless system or in a system driven by an oscillating forcing function. Because the variables in the system are undergoing continuous periodic change in these cases, a different definition of stability may be required.

The most rigid definition for the stability of an oscillating system is similar to the definition of asymptotic stability. According to this test for stability, the x of the system are disturbed slightly from their steady-state motion. If the differences between the x of the ensuing motion and the original undisturbed motion ultimately return to zero, the system may be said to be asymptotically stable. If the differences neither vanish nor increase, the system is neutrally stable.

A conservative nonlinear oscillating system is usually unstable according to this definition. In a nonlinear system of this sort, as, for example, the mass and nonlinear spring of Example 4.6, the period of the oscillation is dependent upon the amplitude. If this kind of system is in motion, it can be represented by a closed curve on the phase plane, as shown in Fig. 10.1. When the system is tested for stability as described, the representative point on the curve is displaced slightly, after which it follows a different closed curve as shown in the figure. Since the new condition corresponds to a period different from the old condition, the representative points of the two trajectories travel around the curves at different rates. As a result, even though the points are initially separated by only a small amount, the distance of separation grows as time increases. This is the condition corresponding to what has been called asymptotically unstable.

In many applications, a change of period, accompanying only a small and nongrowing change in amplitude, does not seem to fit the usual connotations of instability. For this reason, yet another definition, that of orbital stability, is made. The solution for a system having a steady-state oscillatory motion may be represented as a closed curve in the phase plane. If a small disturbance applied to the system results in a curve which ultimately returns to the first curve, the system is said to have orbital stability. If the small disturbance results in a curve which leaves the first curve, the system is orbitally unstable. The definition here is related only to the amplitude and not to the period of the oscillation. A conservative nonlinear oscillating system is not orbitally unstable, although it is asymptotically unstable.

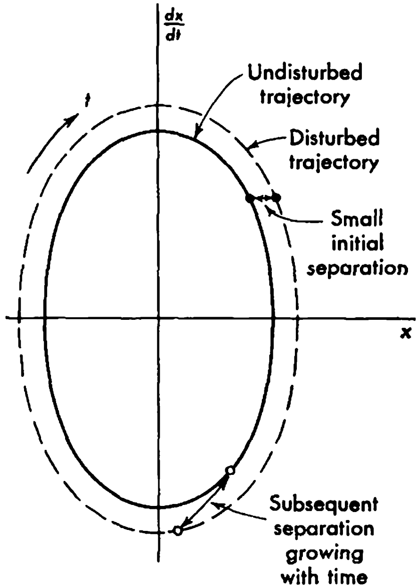


FIG. 10.1. Phase-plane diagram for conservative nonlinear oscillating system, showing asymptotic instability because of change of period with amplitude.

All these definitions of stability are concerned only with what happens as time increases indefinitely. A system in which disturbances increase, even after a very long time, is classified as unstable. Under some conditions, operation of a system takes place over only short time intervals. Any instability which requires a long time to lead to disturbances of serious magnitude may actually be unimportant in short-time operation. Under these conditions, the instability must be expressed in some kind of quantitative terms to be meaningful.

10.4. Test for Stability of System with Nonoscillatory Steady State. The procedure for testing the stability of nonoscillating systems is relatively straightforward, although details of the steps may be difficult to carry out. The process here is essentially that described in Chap. 5, where only second-order systems are considered, although the principles can readily be extended to systems of higher order.

The first step is to represent the system by a set of simultaneous differential equations such as Eq. (10.3), which are the same as Eq. (10.1)

$$\begin{aligned}
\frac{dx_1}{dt} &= \dot{x}_1 = f_1(x_1,x_2, \ldots ,x_n) \\
\frac{dx_2}{dt} &= \dot{x}_2 = f_2(x_1,x_2, \ldots ,x_n) \\
&\cdots\cdots\cdots\cdots\cdots \\
\frac{dx_n}{dt} &= \dot{x}_n = f_n(x_1,x_2, \ldots ,x_n)
\end{aligned} \tag{10.3}$$

These equations will generally be nonlinear and may involve constant differences in time as well as time derivatives.

The second step is to determine the equilibrium conditions, or singular points, for the system. This is done by setting all derivatives and time differences to zero and solving the resulting set of static algebraic equations, as in Eq. (10.4),

$$\begin{aligned} f_1(x_1,x_2, \ldots ,x_n) &= 0 \\ \cdots\cdots\cdots &\cdots \\ f_n(x_1,x_2, \ldots ,x_n) &= 0 \end{aligned} \tag{10.4}$$

Nonlinear systems may lead to more than one equilibrium condition. The singular values of the x can be designated as $x_{1s}, x_{2s}, \ldots, x_{ns}$, where a different set applies to each singularity.

The third step is to investigate the effect of small changes in the x near each singularity. Each x_i in Eq. (10.4) is replaced by $x_{is} + u_i$, where u_i is a small change away from the singularity. This leads to a set of linear variational equations,

$$\begin{aligned} \frac{du_1}{dt} &= \dot{u}_1 = \phi_1(u_1,u_2, \ldots ,u_n,x_{1s},x_{2s}, \ldots ,x_{ns}) \\ \cdots\cdots &\cdots\cdots\cdots\cdots \\ \frac{du_n}{dt} &= \dot{u}_n = \phi_n(u_1,u_2, \ldots ,u_n,x_{1s},x_{2s}, \ldots ,x_{ns}) \end{aligned} \tag{10.5}$$

where the ϕ functions are new functions, linear in the variations u_i.

If the original f functions of Eq. (10.3) are nonlinear, they must be expanded in Taylor's series or some equivalent form about the equilibrium values of the x. Usually it is sufficient to retain only the first powers of the u in these expansions, in which case the singularity is said to be simple. If the first powers of the u are absent in the expansions but higher powers are present, the singularity is not simple and higher-power terms must be retained. In this case, the ϕ functions of Eq. (10.5) will not be linear.

A simple singularity leads to Eq. (10.5) in the form

$$\begin{aligned} \dot{u}_1 &= c_{11}u_1 + c_{12}u_2 + \cdots + c_{1n}u_n \\ \dot{u}_2 &= c_{21}u_1 + c_{22}u_2 + \cdots + c_{2n}u_n \\ \cdots\cdots &\cdots\cdots\cdots\cdots \\ u_n &= c_{n1}u_1 + c_{n2}u_2 + \cdots + c_{nn}u_n \end{aligned} \tag{10.6}$$

where the c_{ij} coefficients are all constants with values which may depend upon the x_{is} values.

The fourth step is to find the characteristic algebraic equation for the set of variational equations of Eq. (10.6). A solution is assumed in the form $u_i = U_i \exp(\lambda t)$, where U_i is a constant determined by initial conditions and λ is an exponent to be found. When substituted into Eq. (10.6), the assumed solution gives the relations

$$\lambda U_1 = c_{11}U_1 + c_{12}U_2 + \cdots + c_{1n}U_n$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\lambda U_n = c_{n1}U_1 + c_{n2}U_2 + \cdots + c_{nn}U_n$$

If the U are not to be identically zero, it is necessary that the determinant vanish,

$$\begin{vmatrix} c_{11} - \lambda & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} - \lambda & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} - \lambda \end{vmatrix} = 0 \tag{10.7}$$

When expanded, this nth-order determinant leads to the characteristic equation, which has the general form

$$f(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0 \tag{10.8}$$

For a set of pure differential equations, as just described, or a set of pure difference equations, the characteristic equation is an algebraic polynomial of degree n with constant coefficients. For a set of mixed differential-difference equations, the characteristic equation is transcendental, as described in Chap. 8, and is accordingly considerably more difficult to solve.

The fifth step is to determine the algebraic signs of the real parts of the roots of the characteristic equation. If any root has a real part that is positive, the system is asymptotically unstable. Any disturbance in the system will grow with time. If all roots have only negative real parts, the system is asymptotically stable. If any root has a real part that is zero, neutral stability is indicated. A practical system with a root of zero real part indicates a precise balance among certain coefficients of the system, and it is likely that the system does not possess the structural stability discussed in Sec. 10.2.

The determination of signs of the real parts of the roots may be carried out conveniently through either the Routh-Hurwitz analytical method or the Nyquist numerical method. Both these methods merely indicate the presence or absence of roots with positive real parts, and which of the methods is preferable depends upon the data available. In some cases, determination of actual numerical values for all the roots is desirable. This requires solution for all the roots of the characteristic equation, which may be a formidable task if the equation is of high order.

The Routh-Hurwitz method is based on consideration of a set of n determinants set up from the coefficients of the nth degree characteristic equation, Eq. (10.8). These determinants are formed as follows:

$$\Delta_1 = |a_1|$$
$$\Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}$$
$$\Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} \tag{10.9}$$
$$\Delta_n = \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ a_5 & a_4 & a_3 & a_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2n-1} & a_{2n-2} & \cdots & \cdots & a_n \end{vmatrix}$$

All coefficients a_r, for which $r > n$ or $r < 0$, are replaced by zeros. The condition that all roots of the characteristic equation have only negative real parts is that all these determinants be positive, $\Delta_1 > 0$, $\Delta_2 > 0$, $\Delta_3 > 0$, . . . , $\Delta_n > 0$.

Since the bottom row of determinant Δ_n is composed all of zeros, except for the one term a_n, its value is $\Delta_n = a_n\Delta_{n-1}$. Thus, for stability it is required that both $a_n > 0$ and $\Delta_{n-1} > 0$, and Δ_n need not actually be evaluated.

Furthermore, it is necessary that all coefficients, a_r, be positive for stability, and the presence of a negative coefficient immediately indicates instability. This is not a sufficient condition, however, since a characteristic equation with all positive coefficients may have roots with positive real parts if the values of the coefficients have certain relationships.

The Routh-Hurwitz criterion for stability requires that the characteristic equation and its coefficients be known. The test is applied to the coefficients and can be carried out in analytical form without the necessity of substituting numerical values during the algebraic work. Proof of the validity of this process is beyond the scope of the present discussion.

The Nyquist method is based on conformal mapping of complex quantities. The value of λ in the characteristic equation, Eq. (10.8), is generally a complex number and can be represented on the complex λ plane, as in Fig. 10.2, where λ is taken as $\lambda = \delta + j\omega$. Any value of λ that is a root of Eq. (10.8) and leads to an unstable solution has a positive real part and would be located on the right half of the complex λ plane. This region is shaded in Fig. 10.2. Its boundaries can be traced out by starting at the lower end of the $j\omega$ axis, where $\lambda = j\omega \to -j\infty$, moving up this axis until $\lambda = j\omega \to +j\infty$, turning clockwise through a right angle, and returning to the starting point along a semicircle of very large radius. At the starting point, a second clockwise right-angled turn is needed to begin retracing the original path. The shaded region is always to the right of this boundary as it is traced in the direction indicated.

The characteristic equation, Eq. (10.8), involves the algebraic function $f(\lambda)$. This function can be plotted on the complex f plane, also shown in Fig. 10.2, as λ traces out the boundary of the shaded area just described. For this figure, the simple case is taken of $f(\lambda) = \lambda^2 + g\lambda + h$, with g and h constants. Again λ is allowed to be a pure imaginary, $\lambda = j\omega$,

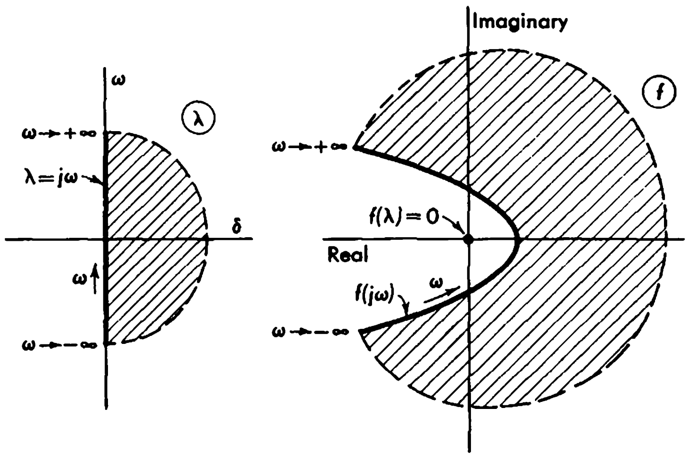


FIG. 10.2. $j\omega$ axis of complex λ plane and corresponding contour $f(j\omega)$ on complex f plane. Values of λ with positive real part are in shaded area of λ plane. Values of $f(\lambda)$ corresponding to a value of λ with positive real part are in shaded area of f plane. Since origin of f plane is not in shaded area, system is indicated to be stable.

and to take on successive values between $-j\infty$ and $+j\infty$. Corresponding values of $f(j\omega)$ are calculated and plotted. This leads to the heavy curves shown on the two planes of the figure. The dotted curve enclosing the shaded region of the λ plane is found by turning clockwise through a right angle at the point $\omega \to +\infty$, and following a semicircle of large radius to the starting point, $\omega \to -\infty$. Similarly, the dotted curve of the f plane is found by tracing the solid curve to the point $\omega \to +\infty$, turning clockwise through a right angle, and returning to the point $\omega \to -\infty$ along a circular path. As the boundaries in either the λ plane or the f plane are traced in the directions indicated, the shaded areas correspond and are located to the right of the path.

Because the polynomial $f(\lambda)$ is of degree n in λ, each point on the f plane has corresponding to it n points, generally all different, in the λ plane. On the other hand, a given point on the λ plane has only one point corresponding to it on the f plane. Any point in the shaded region of the λ plane must have corresponding to it a point in the shaded region of the f plane. These points represent values of λ which would lead to unstable solutions.

The characteristic equation is $f(\lambda) = 0$, which is represented by the origin of the f plane. Thus, the roots of the characteristic equation, which determine properties of the solution for the original differential equation, are represented by those points in the λ plane which correspond to the origin of the f plane. If the origin of the f plane is in the shaded area, at least one of the roots has a positive real part and instability is indicated. If the origin of the f plane is not in the shaded area, roots have only negative real parts and there are only stable solutions. Since the shaded area of the f plane of Fig. 10.2 does not include the origin, a stable solution is indicated.

If the degree of the characteristic equation is large, the curve in the f plane corresponding to the $j\omega$ axis in the λ plane may be exceedingly complicated in shape. Care must be taken to keep track of just where the shaded area of the f plane is located. Basically, however, instability is indicated whenever the origin of the f plane is located to the right of the contour corresponding to the $j\omega$ axis of the λ plane, as this contour is traced from $\omega \to -\infty$ to $\omega \to +\infty$.

If the characteristic equation is of the form of Eq. (10.8), $f(\lambda) = 0$, the general form of the geometrical figures is as shown in Fig. 10.2. The Nyquist method is widely used in the analysis of feedback systems, where the equation being studied often involves fractions of the form $f(\lambda) = 1 + P_1(\lambda)/P_2(\lambda) = 0$, where $P_1(\lambda)$ and $P_2(\lambda)$ are polynomials in λ. Here, when an attempt is made to carry out the mapping process as described, it may be that certain values of $\lambda = j\omega$ will make

$$P_2(\lambda) = P_2(j\omega) = 0$$

so that $f(\lambda)$ becomes infinite. These are so-called poles of $f(\lambda)$. In applying the mapping process, it is necessary to avoid poles of $f(\lambda)$, in tracing along the $j\omega$ axis of the λ plane, by following a small semicircle around them. For example, if $f(\lambda) = 1 + k/\lambda(\lambda + l)$, where k and l are constants, a pole is located at $\lambda = 0$. This case is entirely analogous to that already used with Fig. 10.2. The path to be followed is shown in Fig. 10.3, where a small detour is needed around the origin of the λ plane. The corresponding path in the f plane is also shown, where a large semicircle appears corresponding to the small semicircle about the origin of the λ plane. In each case as ω increases from negative to positive values, a clockwise right-angled turn is made as the semicircular path is entered. The example of Fig. 10.3 is again indicated to be a stable system.

The Nyquist method typically involves considerable numerical calculation. On the other hand, it can be applied to systems where certain information is available only as empirically determined curves and where

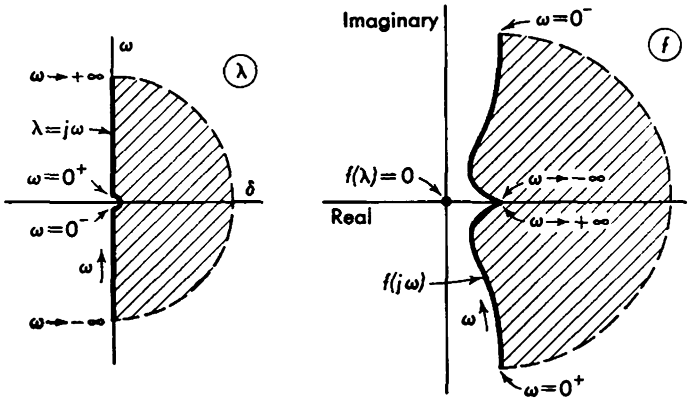


FIG. 10.3. $j\omega$ axis of complex λ plane and corresponding contour $f(j\omega)$ on complex f plane for the case of a system with a pole at the origin of the λ plane. This system is indicated to be stable.

coefficients of differential equations for the system are not readily obtainable.

Example 10.1. Nonlinear Mechanical System

A mechanical system is shown in Fig. 10.4. It consists of a simple pendulum with mass M and length l. The pendulum rod is fastened to a small, light pulley of radius r. Wound around this pulley is a thin, flexible string which is attached to a spring of stiffness k. The other end of the spring is attached to a second string, which passes over a second light pulley and carries a mass M_0. This mass, in turn, is attached to a dashpot giving a mechanical resistance R. The deflection of the pendulum away from the vertical is angle θ. The downward displacement of mass M_0 is x, measured from a position corresponding to $\theta = 0$ and no deflection of the spring. This is a system with two degrees of freedom, nonlinear because of the pendulum.

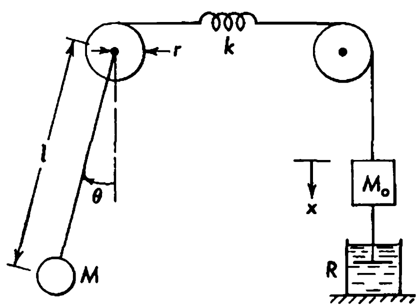


FIG. 10.4. Mechanical system for Example 10.1.

The system is governed by the differential equations

$$\begin{aligned} Ml^2\ddot{\theta} + Mgl\sin\theta + kr^2\theta - krx &= 0 \\ M_0\ddot{x} + R\dot{x} + kx - kr\theta &= M_0 g \end{aligned} \tag{10.10}$$

where g is the gravitational acceleration. A complete solution here involves solving these two simultaneous second-order equations and ultimately requires finding complex roots of a fourth-degree characteristic equation.

Equilibrium conditions for the system are found by setting all derivatives to zero and are

$$\dot{\theta}_s = 0 \qquad \dot{x}_s = 0$$
$$\sin\theta_s = \frac{M_0 r}{Ml} \qquad x_s = \frac{M_0 g}{k} + r\theta_s$$

Evidently, if $0 < M_0 r/Ml < 1$, there are two different values for θ_s, the first located so that $0 < \theta_s < \pi/2$ and the second so that $\pi/2 < \theta_s < \pi$. If $M_0 r/Ml > 1$, there is no equilibrium condition.

The stability of the system can be tested near each equilibrium point by making the substitutions $x = x_s + u$ and $\theta = \theta_s + v$, where u and v are small changes. These substitutions give the linear variational equations

$$Ml^2\ddot{v} + (Mgl\cos\theta_s + kr^2)v - kru = 0$$
$$M_0\ddot{u} + R\dot{u} + ku - krv = 0$$

where $\sin(\theta_s + v) = \sin\theta_s\cos v + \cos\theta_s \sin v \doteq \sin\theta_s + (\cos\theta_s)v$ if v is small. Solutions may be assumed in the form $u = U\exp(\lambda t)$ and $v = V\exp(\lambda t)$, where U and V are arbitrary constants. The characteristic equation is then found as

$$\begin{vmatrix} Ml^2\lambda^2 + Mgl\cos\theta_s + kr^2 & -kr \\ -kr & M_0\lambda^2 + R\lambda + k \end{vmatrix} = 0$$

When expanded, this equation can be written

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

where

$$a_0 = 1$$
$$a_1 = \frac{R}{M_0}$$
$$a_2 = \frac{k}{M_0} + \frac{kr^2}{Ml^2} + \frac{g}{l}\cos\theta_s$$
$$a_3 = \frac{R}{M_0}\left(\frac{kr^2}{Ml^2} + \frac{g}{l}\cos\theta_s\right)$$
$$a_4 = \frac{kg}{M_0 l}\cos\theta_s$$

Since the algebraic sign of $\cos\theta_s$ may be either positive or negative, the sign of a_4 may be negative and instability is certainly possible. If a_4 is positive, the system may be stable but further investigation is necessary.

The determinants of the Routh-Hurwitz method are

$$\Delta_1 = \frac{R}{M_0}$$
$$\Delta_2 = \frac{Rk}{M_0^2}$$
$$\Delta_3 = \frac{R^2k^2r^2}{MM_0^3l^2}$$
$$\Delta_4 = \Delta_3\frac{kg}{M_0 l}\cos\theta_s$$

The first three of these determinants are always positive for a system with positive coefficients. The sign of the fourth determinant is positive if $0 < \theta_s < \pi/2$, and the system is accordingly stable for this case. The sign of the fourth determinant is negative if $\pi/2 < \theta_s < \pi$, and this is an unstable case. Thus, of the two equilibrium conditions for the pendulum, that corresponding to the larger angle of steady deflection is unstable, and the pendulum tends to move away from this position. The smaller angle of deflection is stable, and the pendulum tends to move toward this position, perhaps with a damped oscillation.

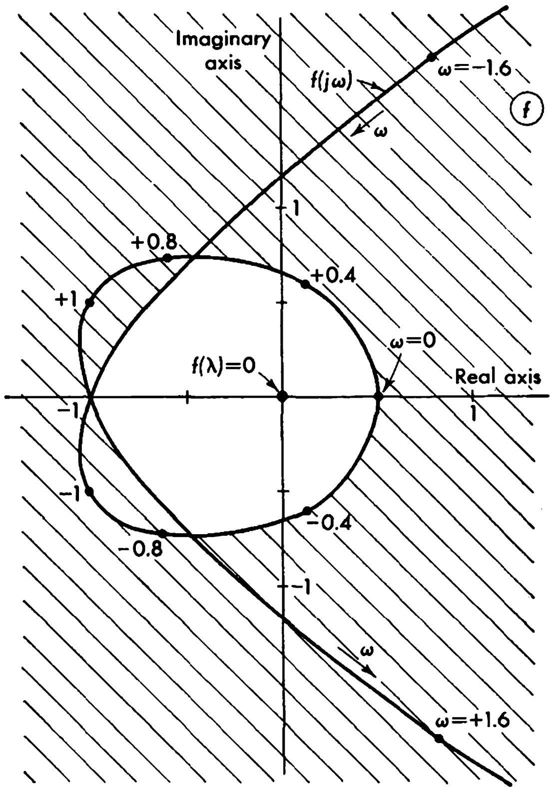


FIG. 10.5. Nyquist stability diagram for system of Example 10.1 near the equilibrium value, $\theta_s = 60$ degrees. This condition is stable.

In order to apply the Nyquist method, numerical values are necessary. A specific case is that in which the coefficients of the system are such that the following hold,

$$\frac{R}{M_0} = 1 \qquad \frac{g}{l} = 1$$
$$\frac{k}{M_0} = 1 \qquad \cos\theta_s = \pm 0.5$$
$$\frac{k^2r^2}{Ml^2} = 1$$

where a consistent system of units is used. The algebraic sign of $\cos\theta_s$ depends upon whether $\theta_s = 60$ degrees or $\theta_s = 120$ degrees, respectively. The coefficients of the characteristic equation are accordingly

$$a_0 = 1 \qquad a_3 = 1 + \cos\theta_s$$
$$a_1 = 1 \qquad a_4 = \cos\theta_s$$
$$a_2 = 2 + \cos\theta_s$$

If $\theta_e = 60$ degrees, so that $\cos \theta_e = +0.5$, the characteristic equation is

$$f(\lambda) = \lambda^4 + \lambda^3 + 2.5\lambda^2 + 1.5\lambda + 0.5 = 0$$

With $\lambda = j\omega$, the equation to be plotted becomes

$$f(j\omega) = (\omega^4 - 2.5\omega^2 + 0.5) + j(-\omega^3 + 1.5\omega)$$

The curve on the complex f plane appears in Fig. 10.5, with certain values of ω indicated along the curve. As ω varies from $-\infty$ to $+\infty$, the shaded area lying to the

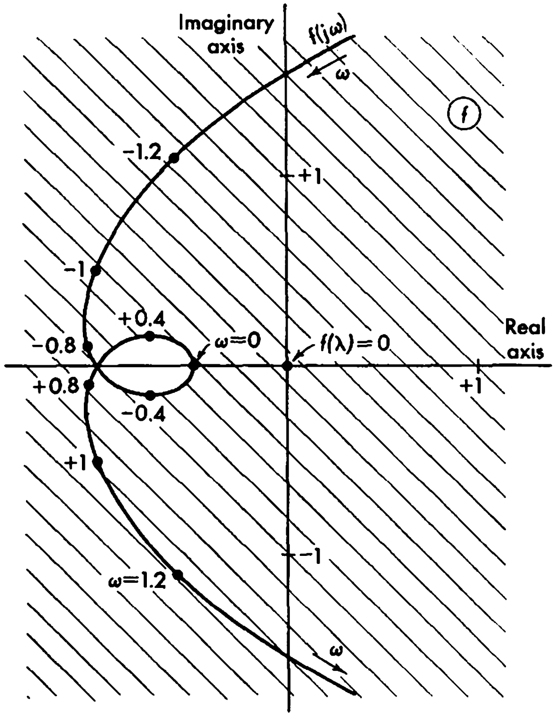


FIG. 10.6. Nyquist stability diagram for system of Example 10.1 near the equilibrium value, $\theta_e = 120$ degrees. This condition is unstable.

right of at least some part of the path fills all the plane except for the interior of the central loop of the curve. This small unshaded area is always to the left of the path. Since it contains the origin of the f plane, this case is indicated to be stable.

If $\theta_e = 120$ degrees, so that $\cos \theta_e = -0.5$, the equation becomes

$$f(\lambda) = \lambda^4 + \lambda^3 + 1.5\lambda^2 + 0.5\lambda - 0.5$$

or

$$f(j\omega) = (\omega^4 - 1.5\omega^2 - 0.5) + j(-\omega^3 + 0.5\omega)$$

The curve for this case appears in Fig. 10.6. As ω varies from $-\infty$ to $+\infty$, the shaded area to the right of some part of the path again includes all the f plane except the interior of the small loop. Since the origin of the f plane is not inside this loop, this case is indicated to be unstable.

Both the Routh-Hurwitz method and the Nyquist method lead to the same conclusions regarding stability and instability. The first method is the simpler to apply in this particular example. Neither method gives actual values for the roots of the characteristic equation. These might be desired to determine, for example, whether the system is oscillatory or whether it is overdamped and will not oscillate. If the actual roots are desired, the fourth-degree characteristic equation must be solved.

10.5. Test for Stability of Oscillating Systems. If a nonlinear system has a steady-state solution which is oscillatory, the determination of its stability is often difficult and must be interpreted with some care. Essentially the procedure of the preceding section is followed, but various modifications may be necessary. A second-order nonlinear equation might be of the form

$$\ddot{x} + \omega_0^2 x + \mu\phi(x,\dot{x}) = G(t) \tag{10.11}$$

which is the same as Eq. (7.1). Here ω_0^2 is a constant, $\phi(x,\dot{x})$ is a nonlinear function with a small multiplier μ, and $G(t)$ is a forcing function. Often $G(t)$ is a simple-harmonic function. In Chap. 7, methods are discussed for finding approximate steady-state solutions for this kind of equation. Typically, the solutions are of the form

$$x = x_s = A \cos \omega t \tag{10.12}$$

where only a component of fundamental frequency is considered. If this solution is substituted into Eq. (10.11), it becomes approximately an identity, so that

$$\ddot{x}_s + \omega_0^2 x_s + \mu\phi(x_s,\dot{x}_s) - G(t) = 0$$

A test for asymptotic stability might be attempted as described in the preceding section, by assigning a small variation u to x, so that $x = x_s + u$. The resulting variational equation is then of the form

$$\ddot{u} + \omega_0^2 u + \mu\phi(x_s + u,\ \dot{x}_s + \dot{u}) = 0$$

Because $x_s = A \cos \omega t$ appears in the function $\phi(x_s + u,\ \dot{x}_s + \dot{u})$, this function is not only nonlinear but also involves a time-varying coefficient. If the variation u is small enough, only linear terms need be retained in the variational equation but the equation has a periodic coefficient. The nature of solutions for such an equation must be known in order to predict stability.

In simple cases, the variational equation may be put into the standard form for the Mathieu equation. Since the stability or instability for solutions of this equation is known, the stability for the nonlinear oscillating system can be predicted.

Example 10.2. Free Motion of Mass on Nonlinear Spring

In Example 6.3 is considered the free motion of a mass on a nonlinear spring governed by the equation

$$\ddot{x} + \omega_0^2 x + hx^3 = 0 \tag{10.13}$$

which is Eq. (6.21), where ω_0^2 and h are constants dependent upon properties of the mass and spring. An approximate solution is found as

$$x = x_s = A \cos \omega t$$
$$\omega^2 = \omega_0^2 + \frac{3hA^2}{4}$$

which is Eq. (6.28), where A is the amplitude of oscillation and it is necessary that $3hA^2/4 \ll \omega_0^2$.

If a small variation u is assigned to the steady-state oscillation, x_s, Eq. (10.13) becomes

$$(\ddot{x}_s + \omega_0^2 x_s + hx_s^3) + (\ddot{u} + \omega_0^2 u + 3hx_s^2 u) = 0$$

where only linear terms in u are retained. Since x_s is an approximate solution for Eq. (10.13), the quantity in the first parentheses is essentially identically zero. With the value for x_s inserted, that in the second parentheses becomes

$$\ddot{u} + \omega_0^2 u + 3hA^2(\cos^2 \omega t)u = 0 \tag{10.14}$$

Use of the identity for $\cos^2 \omega t$ allows this equation to be written

$$\frac{d^2u}{dt^2} + \left[\left(\omega_0^2 + \frac{3hA^2}{2}\right) + \frac{3hA^2}{2} \cos 2\omega t\right] u = 0$$

This may be put into the standard form of the Mathieu equation of Sec. 9.4

$$\frac{d^2y}{dz^2} + (a - 2q \cos 2z)y = 0$$

where

$$y = u \qquad a = \frac{\omega_0^2 + 3hA^2/2}{\omega^2}$$
$$z = \omega t \qquad q = \frac{-3hA^2/4}{\omega^2}$$

Coefficient q will be small because of limitations on the approximate solution. Since, for the approximate solution, $\omega^2 = \omega_0^2 + 3hA^2/4$, the relations for a and q can be combined as

$$a = 1 - q$$

This is the relation between coefficients of the Mathieu equation applying to the stability test for the oscillation system.

In Fig. 9.10, the line $a = 1 - q$ is shown to be the boundary between stable and unstable solutions for the Mathieu equation, where q is a small number. Furthermore, of the two independent solutions arising with these coefficients, one is periodic, while the other grows indefinitely and is unstable. Thus, the boundary represents an unstable condition, and the indication is that the approximate solution for Eq. (10.13) is unstable.

This instability comes about because the test applied here is for what has been termed asymptotic stability. Small variation u, away from the steady-state oscillation $x_s = A \cos \omega t$, has been found to increase indefinitely with time. This increase is produced by the change in period associated with a change in amplitude for this sort of nonlinear oscillator. The situation is that shown in Fig. 10.1, where a small initial change on the phase plane increases progressively because of the changed period. The amplitude of the oscillation, of course, does not grow indefinitely.

Example 10.3. Stability of Fundamental Solution for Duffing's Equation

In Example 7.1, an approximate solution of fundamental frequency has been found for Duffing's equation,

$$\ddot{x} + \omega_0^2 x + hx^3 = G \cos \omega_1 t \tag{10.15}$$

which is Eq. (7.4), where ω_0^2, h, G, and ω_1 are constants. The approximate solution is

$$x = x_s = A \cos \omega_1 t$$

where amplitude A satisfies the cubic equation

$$-\omega_1^2 A + \omega_0^2 A + \frac{3hA^3}{4} = G \tag{10.16}$$

which is Eq. (7.7). For some combinations of G and ω_1, three possible values for A exist. The largest and smallest of these have been asserted to be stable, while the intermediate value has been asserted to be unstable.

A small variation u applied to the approximate steady-state solution leads to the variational equation

$$\ddot{u} + \omega_0^2 u + 3hA^2(\cos^2 \omega_1 t)u = 0 \tag{10.17}$$

This is the same as Eq. (10.14) of the preceding example, except that here the frequency of the varying coefficient is fixed by the driving frequency ω_1. In the preceding example, it is determined by the frequency ω of free oscillation, which changes with amplitude. Upon converting the form of the variational equation to that of the standard Mathieu equation, the coefficients are found as

$$a = \frac{\omega_0^2 + 3hA^2/2}{\omega_1^2}$$
$$q = \frac{-3hA^2/4}{\omega_1^2}$$

Stability of solutions associated with these coefficients may be investigated either numerically for specific cases or analytically. Certain approximations are necessary in the analytical treatment, but the result is more generally informative. An analytical discussion is given here.

If the value for ω_1^2 from Eq. (10.16) is used in the denominators of a and q, the result is

$$a = \frac{\omega_0^2 + 3hA^2/2}{\omega_0^2 + 3hA^2/4 - G/A}$$
$$\doteq \left(1 + \frac{3hA^2}{2\omega_0^2}\right)\left(1 - \frac{3hA^2}{4\omega_0^2} + \frac{G}{A\omega_0^2}\right)$$

or approximately

$$a = 1 + \frac{3hA^2}{4\omega_0^2} + \frac{G}{A\omega_0^2} \tag{10.18}$$

and

$$q = \frac{-3hA^2/4}{\omega_0^2 + 3hA^2/4 - G/A}$$

or approximately

$$q = \frac{-3hA^2}{4\omega_0^2} \tag{10.19}$$

The approximate forms for a and q here are based on the assumptions that $3hA^2/4\omega_0^2 \ll 1$ and $G/A\omega_0^2 \ll 1$, which often is the case. The sum of Eqs. (10.18) and (10.19) is

$$a + q = 1 + \frac{G}{A\omega_0^2} \tag{10.20}$$

The analysis of a system governed by Duffing's equation in Example 7.1 has shown that for chosen values of G and ω_1, with ω_1 somewhat greater than ω_0, three values for A may exist, as indicated in Fig. 10.7, analogous to Fig. 7.4. Each of these values must be tested for stability.

The first case is that of A_1 in Fig. 10.7, for which amplitude A is positive. Corresponding values of coefficients a and q of the Mathieu equation are found from Eqs. (10.18) to (10.20) and can be represented on the diagram of Fig. 10.8. If $A > 0$,

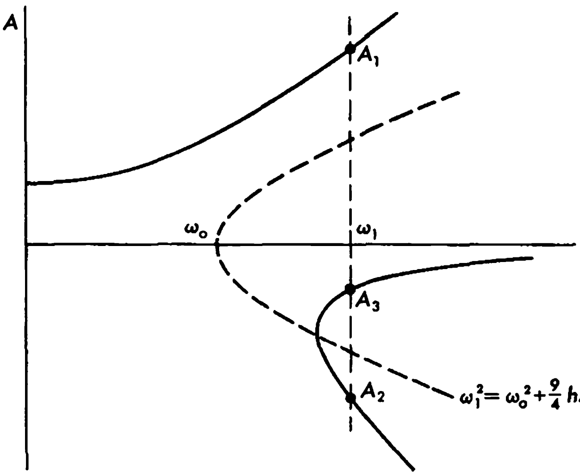


FIG. 10.7. Amplitudes of three possible fundamental solutions for Duffing's equation of Example 10.3.

from Eq. (10.18) $a > 1$ and the corresponding point must be located in the region identified by symbol ①. From Eq. (10.19), $q < 0$, which requires the point to be in region ②. Finally, from Eq. (10.20), $a + q > 1$, which requires the point to be in region ③. The only area common to all three of these regions has the three kinds of crosshatching in Fig. 10.8. A comparison with Fig. 9.10 shows this to correspond to a stable combination of coefficients a and q. Thus, the solution for the Duffing equation having amplitude A_1, the positive and largest amplitude, is stable.

For both amplitudes A_2 and A_3 of Fig. 10.7, A is negative. If $A < 0$, from Eq. (10.19) $q < 0$, which puts the point in region ② of Fig. 10.9. From Eq. (10.20), $a + q < 1$, which requires the point to be in region ③. The difference between Eqs. (10.18) and (10.19) is

$$a - q = 1 + \frac{3hA^2}{2\omega_0^2} + \frac{G}{A\omega_0^2} \tag{10.21}$$

If $a - q < 1$, the point in Fig. 10.9 is in region ① and the area common to all three regions is stable. If $a - q > 1$, the point must be on the side of the line $a - q = 1$ opposite from region ①, where the area is not crosshatched. The area common to all

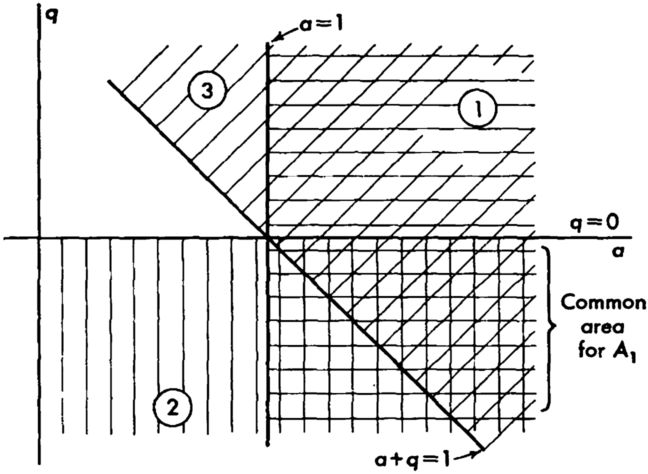


FIG. 10.8. Stability diagram for Mathieu equation, applying near amplitude A_1 of Fig. 10.7.

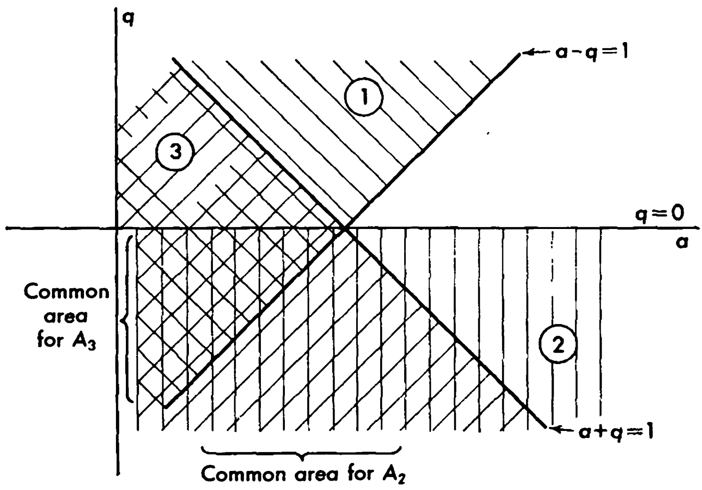


FIG. 10.9. Stability diagram for Mathieu equation, applying near amplitudes A_2 and A_3 of Fig. 10.7.

three regions is then unstable. Thus, the test for stability of amplitudes A_2 and A_3 is the sign of $a - q - 1$, a positive sign corresponding to instability.

In Example 7.1, it has been shown that the condition separating the two negative amplitudes A_2 and A_3 is

$$\omega_1^2 = \omega_0^2 + \frac{9hA^2}{4} \tag{10.22}$$

which is Eq. (7.9). From Fig. 10.7, it is apparent that amplitude A_2 is larger in magnitude than necessary to satisfy this relation, while A_3 is smaller in magnitude.

This relation used in Eq. (10.16) gives

$$\frac{-3hA^3}{2} = G$$

and if this is used in Eq. (10.21), the result is

$$a - q - 1 = 0$$

Thus, if the amplitude is A_2, larger in magnitude than given by Eq. (10.22), $a - q - 1 > 0$ and this case is unstable. If the amplitude is A_3, smaller in magnitude than given by Eq. (10.22), $a - q - 1 < 0$ and this case is stable.

In summary, then, of the three possible amplitudes of the approximate steady-state solution shown in Fig. 10.7, those of largest and smallest magnitude, A_1 and A_3, are stable, while that of intermediate magnitude, A_2, is unstable. Since the frequency of the steady-state oscillation here is fixed by the driving function, $G(t) = G \cos \omega_1 t$, instability in the solution must be associated with an actual increasing change in the amplitude of oscillation. Thus, any small disturbance in the system will cause it to move away from amplitude A_2 toward either of the amplitudes A_1 or A_3.

Example 10.4. Stability of One-third-order Subharmonic Solution for Duffing's Equation

It has been shown in Example 7.5 that Duffing's equation

$$\ddot{x} + \omega_0^2 x + hx^3 = G \cos \omega_1 t$$

which is Eq. (7.35), has an approximate solution of the form

$$x = A \cos \omega t + C \cos \omega_1 t$$

which is Eq. (7.37). The second term is of the driving frequency, while the first term is of the subharmonic frequency, here taken to be of order $\frac{1}{3}$, so that $\omega = \omega_1/3$. Amplitude C is very nearly $C = -G/8\omega^2$, while amplitude A must fit the complicated relation

$$\omega_0^2 = \omega^2 - \frac{3h}{4}(A^2 + AC + 2C^2)$$

It is shown in Fig. 7.13 that the subharmonic may exist provided $\omega = \omega_1/3$ is slightly greater than ω_0. Furthermore, the curve relating amplitude A to frequency ω is roughly parabolic in shape, and it has been asserted that the value of A corresponding to the branch with negative slope is unstable. The value of A where the slope changes sign is $A = G/16\omega^2 = -C/2$. The stability of the subharmonic solution can be tested just as in the preceding example.

The variational equation in this case is

$$\ddot{u} + \omega_0^2 u + \frac{3h}{2}[(A^2 + C^2) + (A^2 + 2AC) \cos 2\omega t]u = 0$$

where only linear terms in u and the varying coefficient of lowest frequency are retained. When converted to the standard Mathieu equation, the coefficients are approximately

$$a = 1 + \left(\frac{3h}{4\omega_0^2}\right)(A^2 - AC)$$

$$q = \left(\frac{-3h}{4\omega_0^2}\right)(A^2 + 2AC)$$

The sum and difference are

$$a + q = 1 - \frac{3h}{4\omega_0^2} 3AC$$
$$a - q = 1 + \frac{3h}{4\omega_0^2} (2A^2 + AC)$$

In any case, $C < 0$, while A may be of either sign. The first possibility is that $A > 0$ and $C < 0$, which makes $a > 1$ and $a + q > 1$. Investigation of the stability diagram for the Mathieu equation, Fig. 9.10, shows that a stable solution requires, in addition, $a - q > 1$, or $A > -C/2$. If A is positive and exceeds $C/2$ in magnitude, it must be located on the branch of Fig. 7.13 having positive slope. Thus, this branch is stable. If A is positive and smaller in magnitude than $C/2$, it is on the branch of negative slope and is unstable.

The second possibility is that $A < 0$ and $C < 0$, which makes $q < 0$ and $a + q < 1$ and $a - q > 1$. Investigation shows that these conditions give a point always in an unstable region of the stability diagram. Thus, this possibility, which also corresponds to points on the branch of the parabolic curve of Fig. 7.13 having negative slope, is unstable. Any point on this branch of negative slope is unstable, regardless of the sign of A.

10.6. Stability of Feedback Systems. Feedback systems are commonly described with reference to a block diagram such as Fig. 10.10. The two boxes represent certain combinations of transmission elements appearing in the complete system. Transmission is assumed to take place only toward the right in the upper box and toward the left in the lower box. Instantaneous values of the dependent variables describing the system are indicated as x_0, x_1, x_2, and x_3, all of which are functions of the independent variable t. Quantities x_1 and x_2, at the input and output terminals, respectively, of the upper box, are related through function H_1 as $x_2 = H_1(D)x_1$, where H_1 is generally a function of the derivative operator $D = d/dt$. Similarly, x_2 and x_3 for the lower box are related as $x_3 = H_2(D)x_2$. The input quantity x_1 for the upper box is obtained by subtracting, from the input x_0 to the entire system, the output x_3 of the lower box. The subtracting device is indicated by the circle of the figure.

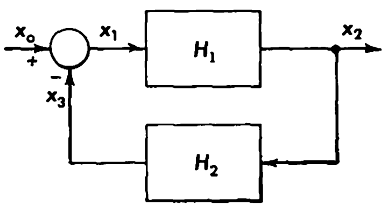


FIG. 10.10. Block diagram of typical feedback system.

Because of these relations, the entire system can be described by the equations

$$x_1 = x_0 - x_3 = \frac{x_2}{H_1(D)} = x_0 - H_2(D)x_2$$

or

$$x_2 = \frac{H_1(D)}{1 + H_1(D)H_2(D)} x_0 \tag{10.23}$$

Asymptotic stability of the system is determined by what happens in the absence of any input signal to the system, so that $x_0 = 0$. If $x_0 = 0$, Eq. (10.23) becomes

$$[1 + H_1(D)H_2(D)]x_2 = 0 \tag{10.24}$$

A solution for this differential equation may be assumed as $x_2 = X \exp(\lambda t)$, where X is an arbitrary constant and λ is the characteristic exponent. If the solution is substituted into Eq. (10.24), the result is the algebraic characteristic equation

$$f(\lambda) = 1 + H_1(\lambda)H_2(\lambda) = 0 \tag{10.25}$$

which is the same as Eq. (10.8). Thus, the stability of the system of Fig. 10.10 can be tested by either of the methods described in Sec. 10.4. Because of the kind of information often available concerning actual physical systems, the Nyquist method is commonly used.

In applying the Nyquist method, the curve representing $f(\lambda)$, with $\lambda = j\omega$ so as to give $f(j\omega)$, is plotted on the complex f plane. If, as ω varies from $-\infty$ to $+\infty$, the origin of the f plane is always to the left of the curve, the system is asymptotically stable. Otherwise, a characteristic root with a positive real part leading to instability is indicated. Because of the form of Eq. (10.25), it is slightly simpler to plot, not $f(j\omega)$, but rather the curve representing just the product $H_1(j\omega)H_2(j\omega)$. Stability is governed by the relation of this curve to the point at $-1 + j0$, in exactly the same way it is governed by the relation of the curve for $f(j\omega)$ with the origin. Furthermore, any physical system is a low-pass device so that product $H_1(j\omega)H_2(j\omega)$ approaches zero as ω approaches infinity. Thus, the curve representing the product closes on itself. The system is stable if the curve for $H_1(j\omega)H_2(j\omega)$ does not enclose the point $-1 + j0$. The system is unstable if the point is enclosed.

In applying the Nyquist criterion as described, the functions H_1 and H_2 for the parts of the system become merely the transfer functions defined with simple-harmonic variations. If some part of the system is slightly nonlinear, its transfer function becomes the describing function discussed in Sec. 7.5*b*. Thus, the Nyquist criterion may be applied to testing the stability of a nonlinear system. Provided that the nonlinearity is not too great and that the waveforms in the system are essentially sinusoidal in shape, predictions of this kind may be essentially correct. If the system is such that the waveforms depart considerably from a sinusoidal shape, testing the stability in this way is open to question and may lead to erroneous conclusions.

Example 10.5. Phase-shift Oscillator

A circuit sometimes used as an oscillator is shown in Fig. 10.11. It consists of an electronic amplifier with a phase-shifting network connected between its output and input terminals. Any practical amplifier saturates if the magnitude of the signal voltage applied to it becomes too large, and is accordingly nonlinear. In the first investigation of this circuit, however, the amplifier is assumed to be linear without saturation effects. The limiter of Fig. 10.11 is first not considered, and voltages e_2 and e_3 are taken as identical. The limiter is considered in a second investigation.

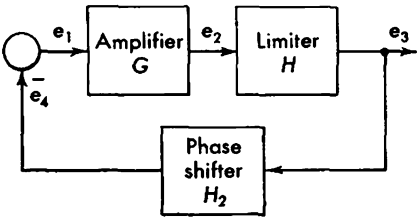


FIG. 10.11. Phase-shift oscillator for Example 10.5.

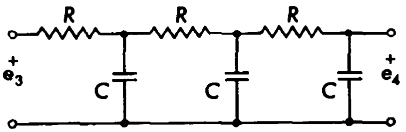


FIG. 10.12. Phase-shifting network for Example 10.5.

A variety of phase-shifting networks may be employed with oscillators of this kind. One sometimes used is shown in Fig. 10.12. It consists of a ladder composed of three equal resistances R and three equal capacitances C. The transfer function for this network is

$$H_2(D) = \frac{e_4}{e_3} = \frac{1}{(RCD)^3 + 5(RCD)^2 + 6(RCD) + 1} \tag{10.26}$$

A sinusoidal voltage applied at the input terminals of this network is attenuated and shifted in phase when it reaches the output terminals. Attenuation and phase shift both increase as frequency increases.

As a specific example, it is assumed that the amplifier is a single-stage vacuum-tube circuit, with voltage amplification $G = e_2/e_1 = 35$. An amplifier of this sort typically reverses the polarity of the signal. This is equivalent to the subtraction which is indicated to take place at the input of the upper box of Figs. 10.10 and 10.11, and thus the algebraic signs of the equations are correct. With a linear amplifier, the stability of the system is governed by the equation

$$1 + GH_2(\lambda) = 0 \tag{10.27}$$

which is merely Eq. (10.25). In Eq. (10.27) for this example, $G = 35$ and $H_2(\lambda)$ is given by Eq. (10.26) with D replaced by λ.

In order to test the stability, the curve is plotted for the product

$$GH_2(j\omega) = \frac{35}{[-5(RC\omega)^2 + 1] + j[-(RC\omega)^3 + 6(RC\omega)]}$$

as ω varies from $-\infty$ to $+\infty$. This curve is shown in Fig. 10.13. As ω increases in the positive direction, the point $-1 + j0$ is always to the right of the curve and is completely encircled by the curve. Thus, the system is asymptotically unstable, and oscillation will occur with increasing amplitude.

A practical amplifier always saturates if the magnitude of the signal becomes too large. The saturation effect can be considered by including a limiter following the ideal linear amplifier, as in Fig. 10.11. A simple limiter is analyzed in Example 7.7, and the result of that analysis is used here. The limiter is assumed to saturate abruptly when the magnitude of the input voltage e_2 exceeds a critical value e_c so that the following relations apply:

$-e_c \leq e_2 \leq +e_c$:	$e_3 = e_2$
$e_2 \geq +e_c$:	$e_3 = +e_c$
$e_2 \leq -e_c$:	$e_3 = -e_c$

These are the relations of Example 7.7 with parameter k equal to unity. With a sinusoidal input voltage of amplitude E_2 applied to the limiter, the describing function for it has been shown to be

$$E_2 \leq e_c: \qquad H = 1$$

$$E_2 \geq e_c: \qquad H = \left[1 - \frac{2}{\pi}(\alpha - \sin\alpha\cos\alpha)\right]$$

where $\cos\alpha = e_c/E_2$ and $k = 1$. These are Eq. (7.48), plotted in Fig. 7.23.

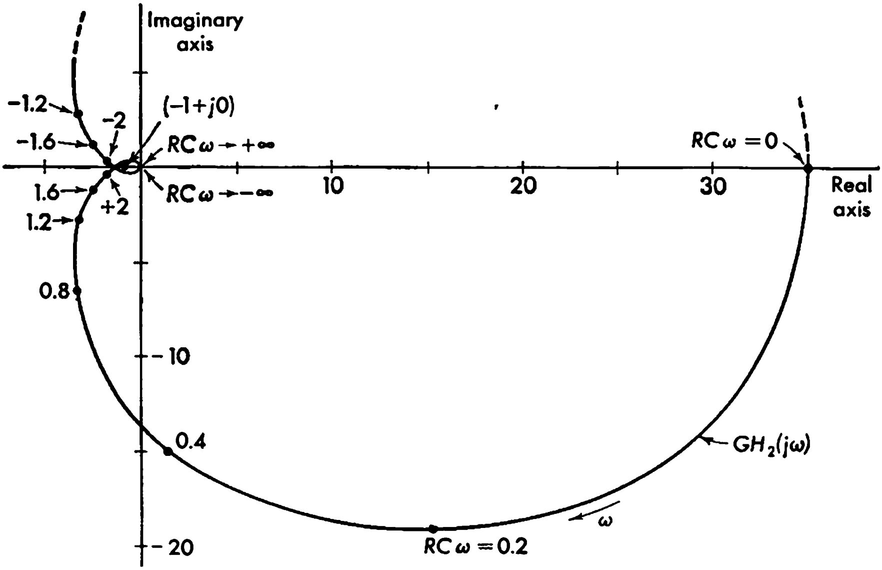


FIG. 10.13. Nyquist stability diagram for oscillator of Example 10.5, with only linear amplifier. Values of quantity $RC\omega$ are indicated along the contour. The system is asymptotically unstable.

The stability of the system including the limiter is accordingly governed by the relation

$$1 + GH(E_2)H_2(\lambda) = 0$$

This is merely Eq. (10.25) with $H_1 = GH(E_2)$, where $H(E_2)$ is the describing function for the limiter. The equation can be put into more convenient form for study by writing it

$$\frac{1}{GH_2(\lambda)} = -H(E)_2 \qquad (10.28)$$

which is the condition for steady-state oscillation. For this example, $G = 35$, $H_2(\lambda)$ is given by Eq. (10.26) with D replaced by λ, and $H(E_2)$ is given by Fig. 7.23 with

$k = 1$ and $X = E_2$. In Fig. 10.14 are plotted two curves representing the two sides of Eq. (10.28) with λ replaced by $j\omega$. The left side of the equation, $1/[GH_2(j\omega)]$, is a curve with certain values of the quantity $RC\omega$ indicated along it. The right side of the equation, $-H(E_2)$, is merely that portion of the negative real axis lying between the point $-1 + j0$ and the origin. Certain values of the ratio E_2/e_c are indicated along this line. Equation (10.28) is satisfied when the two curves intersect, and conditions determined by the intersection point correspond to steady-state conditions in the nonlinear circuit. What happens is that, as the amplitude of oscillation increases from an initial small value because of the asymptotic instability, the limiter reduces the effective amplification until at steady state the effective amplification of amplifier plus limiter is just sufficient to overcome attenuation in the phase-shift network.

The intersection of the curves in Fig. 10.14 indicates that, in the circuit analyzed, steady-state oscillation should occur with a frequency such that $RC\omega = 2.45$ and an

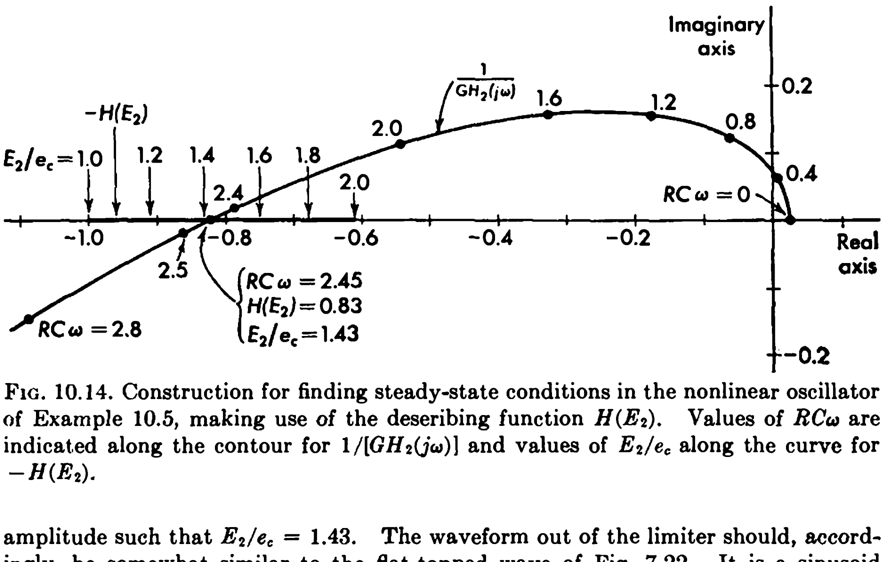


FIG. 10.14. Construction for finding steady-state conditions in the nonlinear oscillator of Example 10.5, making use of the describing function $H(E_2)$. Values of $RC\omega$ are indicated along the contour for $1/[GH_2(j\omega)]$ and values of E_2/e_c along the curve for $-H(E_2)$.

amplitude such that $E_2/e_c = 1.43$. The waveform out of the limiter should, accordingly, be somewhat similar to the flat-topped wave of Fig. 7.22. It is a sinusoid with a peak 1.43 times the limiting level. Because the phase-shifting network of Fig. 10.12 is a low-pass filter, the waveform at the input of the amplifier should be nearly sinusoidal. Since this is the case, the prediction based on the describing function for the nonlinear element should be fairly accurate. A larger quantitative error could be expected if amplification G were increased. The waveform out of the limiter would then approach more nearly a square wave and depart further from a sinusoid. The describing function would then be less applicable.

In analyzing a nonlinear system, such as this, it is always most convenient to write the relation for testing the stability as is done in Eq. (10.28). Those terms depending upon frequency appear on one side of the equation, and those depending upon amplitude appear on the opposite side. This separation allows the curves, such as Fig. 10.14, to be plotted and interpreted in simplest fashion. Separation of this sort can be achieved where operation of the nonlinear element depends only upon amplitude of the signal, while that of the linear element depends only upon frequency. A more complicated situation arises if the nonlinear effeet is also frequency dependent.

Example 10.6. Servo with Dry Friction

A somewhat more complicated nonlinear feedback system is a positioning servomechanism in which the mechanical load includes both mass and dry friction. This kind of load has been discussed in Example 7.8, where translational motion is considered. For consistency, the same symbols are used in the present example, although here the motion is rotational.

A simple system of this kind is shown in Fig. 10.15. An electronic amplifier has zero-frequency voltage amplification G and, because of its low-pass properties, has an associated time constant T. The amplifier provides voltage e_2 across the armature of a separately excited shunt motor, with armature resistance R. The torque f, generated by the motor, is related to the armature current i by constant k_2, so that $f = k_2 i$. The counter electromotive force (emf) due to rotation of the armature is related to the angular velocity v by constant k_1, so that counter emf $= k_1 v$. The mechanical load driven by the motor has moment of inertia M and dry-friction effects described by coefficient h. This quantity h is the constant value of the torque

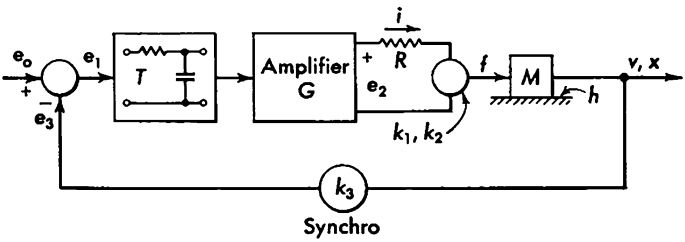


FIG. 10.15. Servo system with mechanical load, including mass and dry friction for Example 10.6.

necessary to overcome the dry-friction effect, regardless of the velocity. The angular velocity of the load is v, the same as the motor. Since the servo is to control the angular position x, rather than velocity, position of the load is sensed by the synchro. It has the property that $e_3 = k_3 \int v\, dt$ where k_3 is a constant.

The combination of mass and dry friction is a nonlinear element. It has been analyzed in Example 7.8 and its describing function found. The describing function applies to the case of a simple-harmonic driving force of amplitude F and gives the relation with the component of the resulting velocity having fundamental frequency. The describing function is the complex ratio $H = V/F$, where V is the amplitude of the fundamental component of velocity. One method of analysis gives the results

$$
\begin{aligned}
|H| &= \frac{1}{M\omega}\left[1 - \left(\frac{4h}{\pi F}\right)^2\right]^{1/2} \\
\angle H &= \cos^{-1}\frac{4h}{\pi F}, \text{ a negative angle}
\end{aligned}
\tag{10.29}
$$

which are Eqs. (7.51) and (7.52), plotted in Fig. 7.25. For this element, H depends upon both frequency ω and amplitude F.

The equations for the various elements of the system are

$$
\begin{aligned}
e_2 &= \left(\frac{G}{TD + 1}\right) e_1 \\
e_2 &= Ri + k_1 v \\
f &= k_2 i \\
e_3 &= k_3 \frac{v}{D}
\end{aligned}
$$

where all the variables are instantaneous values and $D = d/dt$. If only simple-harmonic variations of angular frequency ω are considered, D can be replaced by $j\omega$, and the describing function H, for the mass-friction element, can be used. The transfer through the upper path of Fig. 10.15 becomes

$$H_1 = \frac{V}{E_1} = \frac{G}{j\omega T + 1} \, \frac{1/k_1}{R/k_1k_2H + 1}$$

where, in the denominator of the last factor, $H = H(\omega,F)$ is the describing function for the mass-friction element. Transfer function H_1 must also depend upon both ω and F. The transfer through the lower path of Fig. 10.15 is

$$H_2 = \frac{E_3}{V} = \frac{k_3}{j\omega}$$

Asymptotic stability is governed by the relation

$$1 + H_1H_2 = 0$$

which is Eq. (10.25).

If the system is asymptotically unstable, oscillation will build up to what is ultimately a steady state and the stability relation becomes an equality. For a steady state, therefore,

$$\frac{G}{j\omega T + 1} \, \frac{1/k_1}{R/k_1k_2H + 1} \, \frac{k_3}{j\omega} = -1$$

Describing function H depends upon both frequency ω and amplitude of torque F from the motor, neither of which is known. Solution for this equation therefore requires a kind of trial-and-error process. This is done most simply by first putting H on one side of the equation by itself. After some algebraic manipulation, the equation can be written

$$H(\omega,F) = \frac{-R/k_1k_2}{1 + \dfrac{Gk_3/k_1}{j\omega(j\omega T + 1)}} = P(j\omega) \tag{10.30}$$

where $P(j\omega)$ represents the complicated combination of terms in the fraction.

For a specific numerical example, the following values are assumed,

$$\frac{R}{k_1k_2} = 1 \qquad T = 1$$
$$\frac{Gk_3}{k_1} = 10 \qquad M = 0.2$$

where a consistent system of units is used. With these values, the right side of Eq. (10.30) can be written

$$P(j\omega) = \frac{(9\omega^2 - \omega^4) - j10\omega}{(10 - \omega^2)^2 + \omega^2}$$

A curve of $P(j\omega)$ plotted on the complex plane is shown in Fig. 10.16, with points representing certain values of ω indicated along it.

The left side of Eq. (10.30), $H(\omega,F)$, is given by Eq. (10.29) and depends upon both ω and F. If parameter $4h/\pi F$ is given a constant value, the angle of H is constant. Plotted in Fig. 10.16 are several straight lines radiating from the origin and representing loci for $H(\omega,F)$ as $4h/\pi F$ has the constant values indicated. Also plotted in the figure are loci for $H(\omega,F)$ if ω is held constant. These are roughly circular curves cutting the first family of radial lines.

If Eq. (10.30) is to be satisfied, an intersection must occur between the curve for $P(j\omega)$ and the contours for $H(\omega,F)$, with the point of intersection corresponding to some constant value of ω for both curves. Some investigation of Fig. 10.16 shows that if $\omega = 2.55$, approximately, such an intersection exists. At this intersection, $4h/\pi F = 0.53$ and $H(\omega,F) = P(j\omega) = 0.88 - j1.41$. Thus, the system is predicted to be asymptotically unstable, with steady-state oscillation ultimately occurring at the angular frequency $\omega = 2.55$. The amplitude of the torque developed by the motor is then $F = (4h/\pi)/0.53 = 2.4h$. The amplitude of the resulting velocity is $V = |H|F = 1.66(2.4h) = 4h$, and the corresponding deflection is $X = V/\omega = 4h/2.55 = 1.56h$, approximately.

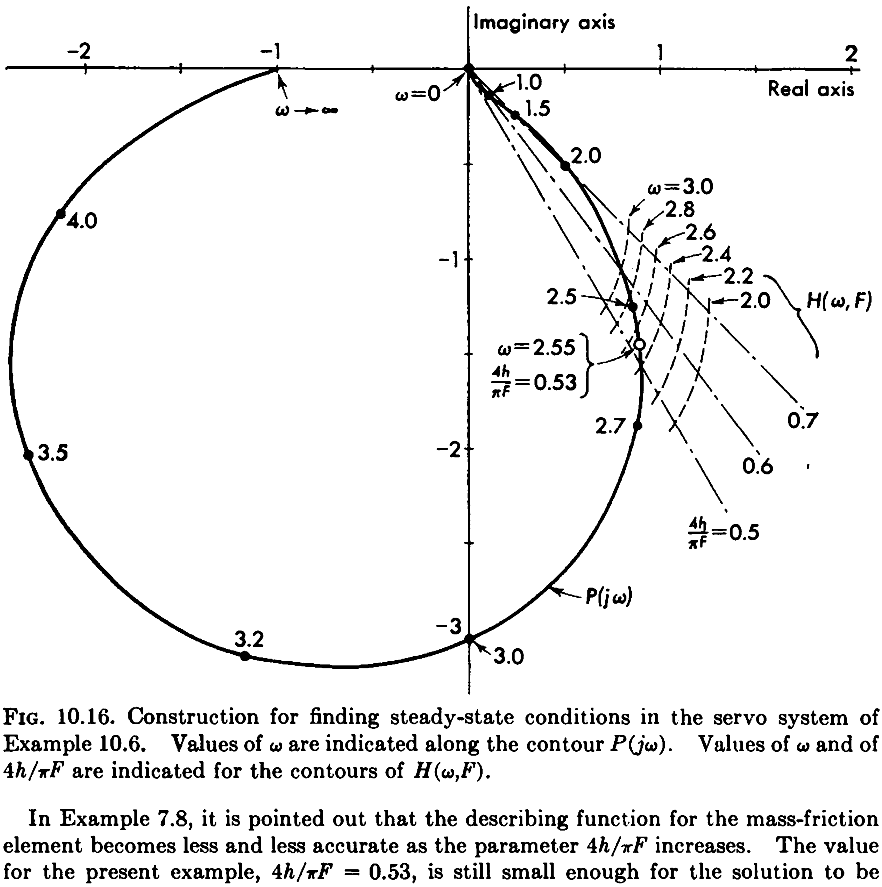


FIG. 10.16. Construction for finding steady-state conditions in the servo system of Example 10.6. Values of ω are indicated along the contour $P(j\omega)$. Values of ω and of $4h/\pi F$ are indicated for the contours of $H(\omega,F)$.

In Example 7.8, it is pointed out that the describing function for the mass-friction element becomes less and less accurate as the parameter $4h/\pi F$ increases. The value for the present example, $4h/\pi F = 0.53$, is still small enough for the solution to be reasonably accurate.

Any system which has a steady-state oscillation requires some initial disturbance to start the oscillation. In this particular system, with its dry friction, an initial disturbance of considerable magnitude is necessary. No motion of the mechanical elements can occur until the torque from the motor exceeds the friction torque. Thus, any initial disturbance which leads ultimately to the oscillation must be great enough to overcome this friction. The system is therefore asymptotically stable for small initial disturbances but is unstable for large initial disturbances.

It should be evident that the analysis of a nonlinear system of this kind, where the describing function for the nonlinear element involves both frequency and amplitude, requires a considerable amount of calculation. Much less numerical work is needed where the describing function involves only amplitude, as in Example 10.5. Actually, in this present example, it is possible to arrange the analysis so that only amplitude appears in the describing function. What must be done is to combine the linear mass element directly with the motor. The nonlinear friction element is considered by itself and a describing function found for it. This function will involve only amplitude. The resulting analysis is somewhat simpler than that presented here.

10.7. Orbital Stability of Self-oscillating Systems. In Example 10.2, a conservative nonlinear oscillating system is shown to be asymptotically unstable. This instability comes about because the period of the oscillation depends upon amplitude. Any small initial disturbance produces a change in amplitude, and the accompanying change in period causes the disturbance to increase in magnitude when considered from the viewpoint of a phase-plane diagram. Often a system operating in this way is more reasonably thought of as being stable or, at least, not unstable. For this reason, the concept of orbital stability is used in connection with a self-oscillating system.

Orbital stability is defined in relation to just the amplitude of the oscillation, without regard to the period. An oscillating system has orbital stability if any small change in amplitude disappears as time progresses. It is orbitally unstable if a small initial change in amplitude leads to an increasing change.

An oscillating system has an approximate solution of the form

$$x = A \cos \omega t$$

where A is the amplitude of oscillation. It is assumed here that the oscillation is essentially simple-harmonic, so that only a term of fundamental frequency is important. In order to test for orbital stability, it is necessary to obtain a relation between the time rate of change of amplitude, $\dot{A}$, and the amplitude A itself. If $\dot{A}$ is plotted as a function of A, the result is a kind of phase-plane curve, as shown in Fig. 10.17. This curve typically shows odd-order symmetry, since a change of algebraic sign of A is merely equivalent to a shift in phase of the solution and should not change the nature of the stability. Since time increases along a phase-plane curve generally toward the right in the upper half plane, where $\dot{A} > 0$, and toward the left in the lower half plane, where $\dot{A} < 0$, the direction of increasing time is indicated by the arrows of the figure. Those points on the horizontal axis, where $\dot{A} = 0$, are equilibrium points for amplitude A. In Fig. 10.17, points $-A_2$, A_0, and $+A_2$ are orbitally stable equilibrium values, since time always progresses toward them. Points $-A_1$ and $+A_1$ are unstable values, since time always progresses away from them. This figure is typical of a "hard" oscillator, which requires an initial disturbance of some considerable magnitude to cause it to continue oscillating in the steady state. Here, the initial disturbance must exceed A_1.

Examination of Fig. 10.17 shows that, near a point of stable amplitude, the slope of the curve of $\dot{A}$ versus A is negative, while near an unstable amplitude the slope is positive. Thus, the criterion for orbital stability is that the derivative $d\dot{A}/dA$ be negative near the equilibrium amplitude. If $d\dot{A}/dA$ is positive near the equilibrium amplitude, the system is orbitally unstable.

For a conservative system, $\dot{A}$ is identically zero, and $d\dot{A}/dA = 0$. This condition corresponds to neither stability nor instability, and thus

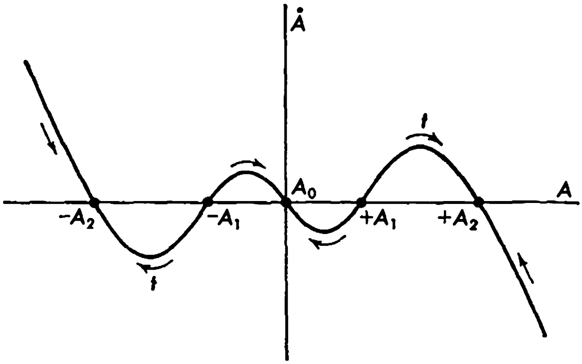


FIG. 10.17. Phase-plane diagram for determining orbital stability of a self-oscillating system. Arrows indicate direction of increasing time.

the system might be said to have neutral stability. Thus, the mass-spring system of Example 10.2 is neutrally stable in the sense of orbital stability.

Example 10.7. Negative-resistance Oscillator

A negative-resistance oscillator has been investigated in Example 6.7. The amplitude of oscillation is given by the relation

$$\dot{A} = \frac{\alpha\omega_0 A}{2}\left(1 - \frac{\beta A^2}{4}\right)$$

which is Eq. (6.59), where α, β, and ω_0 are constant parameters for the system. Equilibrium values of the amplitude occur for $\dot{A} = 0$ and are $A = 0$ and $A = \pm 2/\beta^{1/2}$. The derivative is

$$\frac{d\dot{A}}{dA} = \left(\frac{\alpha\omega_0}{2}\right)\left(1 - \frac{3\beta A^2}{4}\right)$$

At $A = 0$, $d\dot{A}/dA = \alpha\omega_0/2$ and is positive; so this equilibrium point is orbitally unstable. At $A = \pm 2/\beta^{1/2}$, $d\dot{A}/dA = -\alpha\omega_0$ and is negative; so this equilibrium amplitude is stable. These are, of course, the conclusions reached in Example 6.7.

Example 10.8. Phase-shift Oscillator

A phase-shift oscillator employing a saturating amplifier and a feedback system is investigated in Example 10.5. The analysis there is based on the Nyquist criterion for stability of a feedback system. A relation for the amplitude of oscillation is not immediately available, so that the derivative $d\dot{A}/dA$ cannot be found readily. Orbital stability can be investigated in qualitative fashion, however.

The Nyquist diagram for the circuit of Fig. 10.11, excluding the limiter so that it is linear, is shown in Fig. 10.13. A portion of this figure, considerably enlarged, is shown

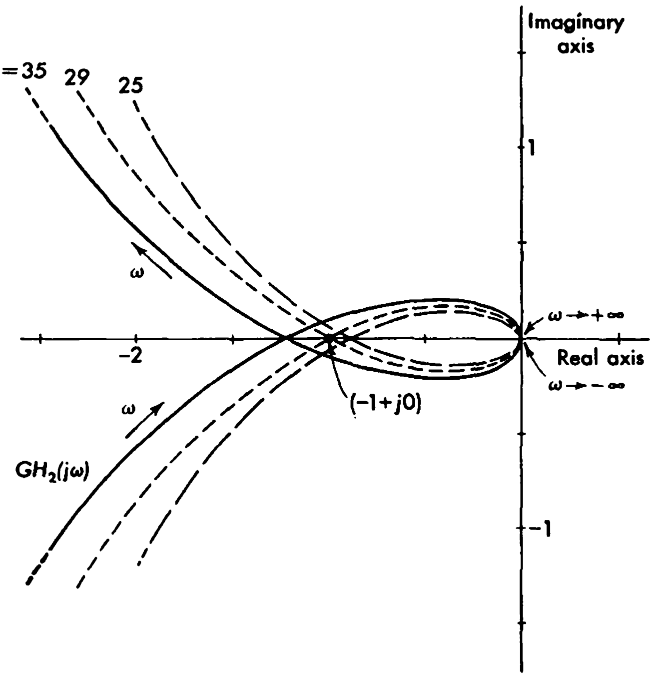


Fig. 10.18. Nyquist stability diagram for oscillator of Example 10.5, with only linear amplifier. Three values of amplification G are shown. The system is asymptotically unstable if G exceeds 29.

in Fig. 10.18. Asymptotic stability of this linear system is determined by the relation between the contour for product $GH_2(j\omega)$ and the point $-1 + j0$. If the contour encircles the point, a characteristic root with a positive real part is indicated in the solution. This implies a growing solution, here known to be oscillatory. If the contour does not encircle the point, all real parts of the characteristic roots are negative, indicating only decaying solutions.

In Example 10.5, the amplification of the amplifier is assumed as $G = 35$, which leads to the contour enclosing the point $-1 + j0$, as shown in Fig. 10.18. Thus, with $G = 35$, the amplitude A of oscillation grows, and $\dot{A} > 0$. If the amplification is chosen smaller, as $G = 25$, the contour does not encircle the point, and the amplitude of oscillation must decay, so that $\dot{A} < 0$. For the special value $G = 29$, the contour passes through the point, and $\dot{A} = 0$. These cases are also shown in Fig. 10.18.

The effect of the limiter added to the linear amplifier of Fig. 10.11 is to cause the effective amplification of the combination to become smaller as the signal amplitude grows. The effect is illustrated in Fig. 7.23 of Example 7.7.

In Example 10.5, amplification G is assigned as $G = 35$, and this applies for small signal amplitudes. Thus, if the amplitude A is small (or E_2 is small in the notation of Example 10.5), $\dot{A}$ is positive. If amplitude A becomes large, the effective amplification is reduced and ultimately $\dot{A}$ becomes negative. In Fig. 10.14 is given a construction for finding the condition for equilibrium amplitude, where $\dot{A} = 0$. Since $\dot{A}$ changes sign from positive to negative as A passes through this equilibrium value, the derivative $d\dot{A}/dA$ is negative and the system is orbitally stable.

10.8. Orbital Stability with Two Components.

Some types of self-oscillators have the property of oscillating simultaneously at more than a single frequency. This is the case for a negative-resistance oscillator, such as that of Example 6.7, with more than one resonant circuit, or for oscillators with constant time delay, as in Example 8.1. If the solution can be represented by components at only two frequencies, the orbital stability of the system can be investigated through a consideration of singular points as discussed in Chap. 5. The limitation to only two components comes about because of the necessity of plotting geometrically on a two-dimensional plane.

The approximate solution for the system may have the form

$$x = A_1 \cos (\omega_1 t + \theta_1) + A_2 \cos (\omega_2 t + \theta_2)$$

where A_1 and A_2 are the amplitudes, θ_1 and θ_2 are the phase angles, and ω_1 and ω_2 are the angular frequencies of the two components. In general, all six of these quantities may be unknown, and an analysis is then extremely complicated. For some systems, however, certain of the quantities are known initially, or it may be possible to separate them so that all six need not be considered simultaneously. Ingenuity is called for here. In investigating orbital stability, what is desired is a plot of solution curves for A_2 as a function of A_1, made on axes of A_1 and A_2. It is necessary to find relations for $\dot{A}_1$ and $\dot{A}_2$ and from these to locate equilibrium conditions where simultaneously $\dot{A}_1 = 0$ and $\dot{A}_2 = 0$. These equilibrium conditions can be considered as singular points in the A_1A_2 plane and the nature of each singularity investigated. Those singularities which are stable represent conditions of orbital stability for the system.

Example 10.9. Negative-resistance Oscillator with Two Modes

The negative-resistance oscillator of Example 6.7 may be used with two resonant circuits instead of just one. In this case, the circuit of Fig. 6.7 is modified to that of Fig. 10.19. The negative-resistance element is assumed to be described by the same equation as before,

$$i = -ae + be^3$$

which is Eq. (6.49), where a and b are positive constants. The two resonant circuits have elements L_1, C_1 and L_2, C_2, and both are assumed free of dissipation. Instantaneous voltages across certain parts of the circuit are indicated as e, e_1, and e_2.

Because of the two resonant circuits, a fourth-order differential equation is necessary to describe the system of Fig. 10.19. It is much more convenient to work instead with two separate second-order equations. This can be done by replacing the nonlinear negative resistance with an equivalent linear conductance in parallel with each resonant circuit. This is the concept of Sec. 7.5a.

The equation for a linear circuit consisting of inductance L, capacitance C, and conductance G in parallel is

$$\ddot{e} + \frac{G}{C}\dot{e} + \frac{1}{LC}e = 0$$

where e is the instantaneous voltage across all the elements. A solution for this equation is well known as

$$e = A_0 \exp\left(\frac{-Gt}{2C}\right) \cos\left\{\left[\frac{1}{LC} - \left(\frac{G}{2C}\right)^2\right]^{1/2} t + \theta\right\}$$

where A_0 and θ are arbitrary constants. If $G/2C$ is not too large, the solution is oscillatory with an amplitude A which varies with time as $A = A_0 \exp(-Gt/2C)$. Thus, the variation of amplitude is

$$\dot{A} = \frac{-G}{2C}A \qquad (10.31)$$

and is dependent upon the parallel conductance of the circuit.

In Fig. 10.19, the instantaneous voltage across the negative-resistance element is

$$e = e_1 + e_2$$

and if simple-harmonic oscillations were to exist simultaneously across both resonant circuits, this would become

$$e = E_1 \cos\psi_1 + E_2 \cos\psi_2$$

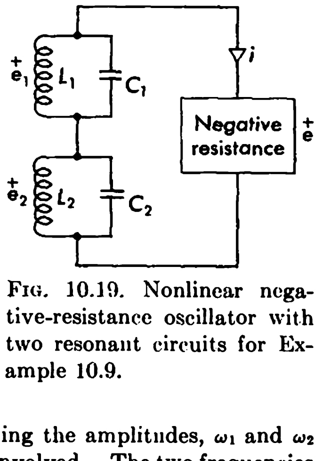


FIG. 10.19. Nonlinear negative-resistance oscillator with two resonant circuits for Example 10.9.

where $\psi_1 = \omega_1 t + \theta_1$ and $\psi_2 = \omega_2 t + \theta_2$, E_1 and E_2 being the amplitudes, ω_1 and ω_2 the angular frequencies, and θ_1 and θ_2 the phase angles involved. The two frequencies of oscillation may be expected to be very close to the resonant frequencies of the two circuits and are determined by products L_1C_1 and L_2C_2. The instantaneous current in the negative resistance is accordingly

$$\begin{aligned} i &= -a(E_1\cos\psi_1 + E_2\cos\psi_2) + b(E_1\cos\psi_1 + E_2\cos\psi_2)^3 \\ &= \left(-aE_1 + \frac{3bE_1^3}{4} + \frac{3bE_1E_2^2}{2}\right)\cos\psi_1 + \left(-aE_2 + \frac{3bE_2^3}{4} + \frac{3bE_1^2E_2}{2}\right)\cos\psi_2 \\ &\qquad + \text{other terms} \end{aligned}$$

where trigonometric identities have been used. The additional terms not written here involve cosine functions of the angles $3\psi_1$, $3\psi_2$, $\psi_1 \pm 2\psi_2$, and $\psi_2 \pm 2\psi_1$. If the two resonant circuits are adjusted so that frequencies ω_1 and ω_2 are incommensurate, none of these additional terms are of the frequencies of ψ_1 and ψ_2. They are neglected in the following analysis. Furthermore, with this condition phase angles θ_1 and θ_2 have no significance and need not be considered.

The omitted terms may have to be considered if ω_1 and ω_2 are related as small integers. If $\omega_2 = 3\omega_1$, for example, a cosine term with angle $3\psi_1$ is of the same frequency as that with ψ_2 and similarly a term with angle $\psi_2 - 2\psi_1$ is of the same frequency as that with ψ_1. In this case, certain of the terms omitted here would have to be considered, and phase angles would be involved.

For the simplest case, where ω_1 and ω_2 are incommensurate, the equivalent conductance presented by the negative-resistance element at frequency ω_1 is

$$G_1 = \frac{I_1}{E_1} = \left(-a + \frac{3bE_1^2}{4} + \frac{3bE_2^2}{2}\right)$$

where I_1 is the amplitude of the component of current at ω_1. Similarly, the equivalent conductance at ω_2 is

$$G_2 = \frac{I_2}{E_2} = \left(-a + \frac{3bE_2^2}{4} + \frac{3bE_1^2}{2}\right)$$

The rates of change of amplitude for the two components can be found from Eq. (10.31) and their ratio written as

$$\frac{\dot{E}_2}{\dot{E}_1} = \frac{dE_2}{dE_1} = \frac{-(1/2C_2)(-a + 3bE_2^2/4 + 3bE_1^2/2)E_2}{-(1/2C_1)(-a + 3bE_1^2/4 + 3bE_2^2/2)E_1} \tag{10.32}$$

It is worth noting here that if only a single component is present, so that, say, $E_2 = 0$, the equation for the amplitude of the remaining component is the same as that found previously in Example 6.7, as Eq. (6.59).

Equilibrium points, or singularities, for Eq. (10.32) are located where numerator and denominator vanish simultaneously. A total of nine singularities exist, although they fall into only four fundamentally different groups. The nature of each singularity may be determined by the methods of Sec. 5.6. Singular values of E_1 and E_2 from Eq. (10.32) are E_{1s} and E_{2s}. Near each singularity, the changes are made, $E_1 = E_{1s} + u_1$ and $E_2 = E_{2s} + u_2$, where u_1 and u_2 are small variations. The nature of solutions for the resulting variational equations are determined from the coefficients involved. The results of the investigation are summarized in Table 10.1.

TABLE 10.1*

Designation of singularity	E_{1s}	E_{2s}	Type
I	0	0	Unstable node
II	$\pm 2/\beta^{1/2}$	0	Stable node
III	0	$\pm 2/\beta^{1/2}$	Stable node
IV	$\pm 2/(3\beta)^{1/2}$	$\pm 2/(3\beta)^{1/2}$	Saddle

* $\beta = 3b/a$, as in Example 6.7.

A plot of the singularities on the E_1E_2 plane is shown in Fig. 10.20. Isoclines for $dE_2/dE_1 = 0$, or $\dot{E}_2 = 0$, and for $dE_2/dE_1 = \infty$, or $\dot{E}_1 = 0$, are shown dotted. Several solution curves are sketched in the figure, showing possible variations of E_2 with E_1 as t increases. It is evident that singularities of designation II and III are stable and represent orbitally stable solutions for the oscillator, while all the others are unstable. These two stable kinds of singularities correspond to the existence of only a single component of one frequency in the circuit. Oscillation of one frequency building up in the circuit serves to suppress oscillation at the other frequency. Simultaneous oscillation at two incommensurate frequencies is orbitally unstable. Stable amplitudes for a single frequency are the same as those found in Example 6.7.

As pointed out previously, if the two resonant circuits are adjusted so that the two frequencies of possible oscillation are related as small integers, the foregoing analysis must be modified. Extra terms appear in the equations for the equivalent conductances. It turns out that simultaneous oscillation may sometimes be orbitally

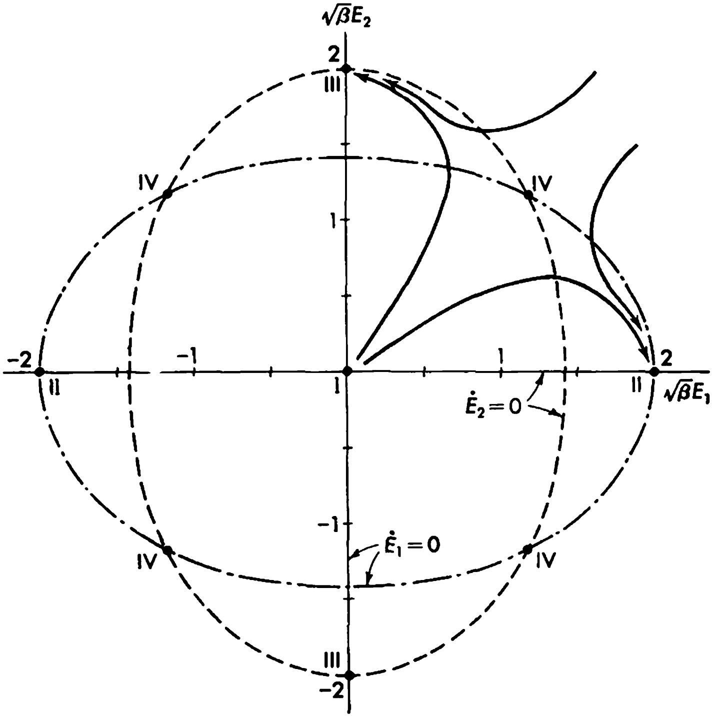


FIG. 10.20. Diagram illustrating the orbital stability of the two components generated by the oscillator of Example 10.9, where the frequencies of these components are incommensurate. Arrows indicate direction of increasing time.

stable when the two frequencies bear special relations. The situation is somewhat like the forced negative-resistance oscillator of Example 7.9.

Example 10.10. Nonlinear Oscillator with Delay

A somewhat more complicated example is the delay oscillator of Example 8.3, which has the circuit arrangement of Fig. 8.7. It consists of a linear amplifier and nonlinear limiter combined with a delay circuit producing a constant time delay. The system is shown to be governed by the equation

$$x_{(t)} - C_0 x_{(t-\tau)} - hC_0 x_{(t-\tau)} + g x^3_{(t)} = 0 \tag{10.33}$$

which is Eq. (8.19). In the previous analysis, coefficient C_0 is assigned the value $C_0 = \pm 1$. The net amplification around the loop of the system is $C = C_0(1 + h)$, where $0 < h \ll 1$. Coefficient g is related to the nonlinear properties of the limiter. Under these conditions, the oscillator is self-starting from any small initial disturbance and builds up relatively slowly to a steady state. It has been shown that, if $C_0 = -1$, the steady state must consist of some kind of rectangular wave composed only of odd harmonics of a fundamental frequency. The amplitude of the rectangular wave is given by $(h/g)^{1/2}$.

In Example 8.3, an approximate solution is found as $x = A \cos \omega t$, having only a single frequency. It is recognized that this is not a realistic solution, since a single frequency cannot be maintained in the system because of the nonlinear action of the limiter. Additional harmonic frequencies are sure to be generated. The steady-state amplitude of this single frequency has been found in Eq. (8.27) to be

$$U = \left(\frac{4h}{3g}\right)^{1/2}$$

A more realistic solution for Eq. (10.33) has two components,

$$\begin{aligned} x_{(t)} &= A_1 \cos (\omega_1 t + \theta_1) + A_2 \cos (\omega_2 t + \theta_2) \\ &= A_1 \cos \psi_1 + A_2 \cos \psi_2 \end{aligned} \tag{10.34}$$

where A_1, A_2, θ_1, and θ_2 are to be determined. It is known from previous work with this system that the two frequencies must be such that both $\omega_1\tau$ and $\omega_2\tau$ are odd multiples of π radians, if $C_0 = -1$, as assumed here. The simplest case is that where $\omega_1\tau = \omega_2\tau/3 = \pi$.

Solution proceeds with the two components here much as with the single component in Example 8.3. All four of the unknown quantities are allowed to vary slowly with time. An approximate value for $x_{(t-\tau)}$ is

$$\begin{aligned} x_{(t-\tau)} = (A_1 - \dot{A}_1\tau) \cos (\psi_1 - \omega_1\tau) + \dot{\theta}_1\tau A_1 \sin (\psi_1 - \omega_1\tau) \\ + (A_2 - \dot{A}_2\tau) \cos (\psi_2 - \omega_2\tau) + \dot{\theta}_2\tau A_2 \sin (\psi_2 - \omega_2\tau) \end{aligned} \tag{10.35}$$

and also, approximately,

$$hC_0x_{(t-\tau)} = hC_0A_1 \cos (\psi_1 - \omega_1\tau) + hC_0A_2 \cos (\psi_2 - \omega_2\tau) \tag{10.36}$$

The cubic term, $x^3_{(t)}$, becomes quite complicated and can be put into the following form:

$$\begin{aligned} x^3_{(t)} = \frac{3A_1^3}{4} \cos \psi_1 + \frac{A_1^3}{4} \cos (\psi_2 + 3\theta_1 - \theta_2) \\ + \frac{3A_1^2A_2}{2} \cos \psi_2 + \frac{3A_1^2A_2}{4} \cos (-\psi_1 + 3\theta_1 - \theta_2) \\ + \frac{3A_1A_2^2}{2} \cos \psi_1 + \frac{3A_2^3}{4} \cos \psi_2 \\ + \text{other terms} \end{aligned} \tag{10.37}$$

The terms not explicitly written here have frequencies different from ω_1 and ω_2 and are not used in the following work. In writing the relation for $x^3_{(t)}$, the following identities have been used:

$$\begin{aligned} \cos^3 \alpha &= \tfrac{3}{4} \cos \alpha + \tfrac{1}{4} \cos 3\alpha \\ \cos^2 \alpha \cos \beta &= \tfrac{1}{2} \cos \beta + \tfrac{1}{4} \cos (2\alpha + \beta) + \tfrac{1}{4} \cos (2\alpha - \beta) \\ 3\psi_1 &= \psi_2 + 3\theta_1 - \theta_2 \\ 2\psi_1 - \psi_2 &= -\psi_1 + 3\theta_1 - \theta_2 \end{aligned}$$

When Eqs. (10.34) to (10.37) are substituted into Eq. (10.33), with $C_0 = -1$, a great many terms appear. Those terms in $\cos\psi_1$, $\sin\psi_1$, $\cos\psi_2$, and $\sin\psi_2$ can individually be set to zero, giving four simultaneous equations. Further, from the conditions on $\omega_1\tau$ and $\omega_2\tau$, it is required that

$$\sin\omega_1\tau = \sin\omega_2\tau = 0$$

and

$$\cos\omega_1\tau = \cos\omega_2\tau = -1$$

The relations become

$$\cos\psi_1:\quad \dot{A}_1\tau = hA_1 - \frac{3gA_1^3}{4} - \frac{3gA_1^2A_2}{4}\cos(3\theta_1 - \theta_2) - \frac{3gA_1A_2^2}{2} \tag{10.38}$$

$$\sin\psi_1:\quad -\dot{\theta}_1A_1\tau = -\frac{3gA_1^2A_2}{4}\sin(3\theta_1 - \theta_2) \tag{10.39}$$

$$\cos\psi_2:\quad \dot{A}_2\tau = hA_2 - \frac{gA_1^3}{4}\cos(3\theta_1 - \theta_2) - \frac{3gA_1^2A_2}{2} - \frac{3gA_2^3}{4} \tag{10.40}$$

$$\sin\psi_2:\quad -\dot{\theta}_2A_2\tau = \frac{gA_1^3}{4}\sin(3\theta_1 - \theta_2) \tag{10.41}$$

Although the four quantities A_1, A_2, θ_1, and θ_2 are unknown, it is possible to determine θ_1 and θ_2 first and then to find A_1 and A_2.

The ratio of Eqs. (10.39) and (10.41) is

$$\frac{\dot{\theta}_2}{\dot{\theta}_1} = \frac{d\theta_2}{d\theta_1} = \frac{-(g/4\tau)(A_1^3/A_2)\sin(3\theta_1 - \theta_2)}{(3g/4\tau)A_1A_2\sin(3\theta_1 - \theta_2)} \tag{10.42}$$

Evidently either of two equilibrium conditions may exist for which both numerator and denominator of Eq. (10.42) vanish simultaneously. The first possibility is that $3\theta_{1s} - \theta_{2s} = 0$, where θ_{1s} and θ_{2s} are the singular values. Near these values, the substitutions can be made $\theta_1 = \theta_{1s} + \beta_1$, $\theta_2 = \theta_{2s} + \beta_2$, where β_1 and β_2 are small variations. Then, approximately, $\sin(3\theta_1 - \theta_2) = 3\beta_1 - \beta_2$, and

$$\frac{d\beta_2}{d\beta_1} = \frac{-(g/4\tau)(A_1^3/A_2)(3\beta_1 - \beta_2)}{(3g/4\tau)A_1A_2(3\beta_1 - \beta_2)}$$

If A_1 and A_2 are of the same algebraic sign, $A_1A_2 > 0$, the equilibrium point can be shown to be an unstable node, following the procedure of Sec. 5.6. The second possibility is that $3\theta_{1s} - \theta_{2s} = \pi$, in which case, approximately, $\sin(3\theta_{1s} - \theta_{2s}) = -(3\beta_1 - \beta_2)$, and the equilibrium point is a stable node. Thus, if $A_1A_2 > 0$, the phase angles tend to adjust themselves to the stable condition $3\theta_{1s} - \theta_{2s} = \pi$ and at equilibrium $\cos(3\theta_{1s} - \theta_{2s}) = -1$ in Eqs. (10.38) and (10.40). If A_1 and A_2 are of opposite signs, $A_1A_2 < 0$, these conditions are just reversed.

The equation describing changes in amplitude is the ratio of Eqs. (10.38) and (10.40) and can be written

$$\frac{\dot{a}_2}{\dot{a}_1} = \frac{da_2}{da_1} = \frac{(h/\tau)[a_2 + \frac{1}{3}(\pm a_1^3 - 6a_1^2a_2 - 3a_2^3)]}{(h/\tau)[a_1 - (a_1^3 \mp a_1^2a_2 + 2a_1a_2^2)]}$$

where $a_1 = A_1/U$, $a_2 = A_2/U$, and $U^2 = 4h/3g$. The upper algebraic signs apply if $A_1A_2 > 0$ and the lower signs if $A_1A_2 < 0$.

A total of 11 singularities occur here, where both numerator and denominator vanish simultaneously. The nature of these singularities can be investigated, again following Sec. 5.6, and the results tabulated as in Table 10.2. Singular values are identified as a_{1s} and a_{2s}.

TABLE 10.2

Designation of singularity	a_{1s}	a_{2s}	Type
I	0	0	Unstable node
II	0	± 1	Stable node
III	± 1.07	± 0.32	Stable node
IV	± 0.55	± 0.75	Saddle

A plot of these singularities on the a_1a_2 plane is shown in Fig. 10.21. Out of the 11 singularities, there are only 4 fundamentally different kinds. The others merely correspond to reversal of algebraic signs. There are 1 singularity of designation I, 2 of designation II, and 4 each of designations III and IV. Isoclines for $da_2/da_1 = 0$, or $\dot{a}_2 = 0$, and for $da_2/da_1 = \infty$, or $\dot{a}_1 = 0$, are shown dotted in the figure. Several solution curves showing possible variations of a_2 with a_1 as t increases are sketched.

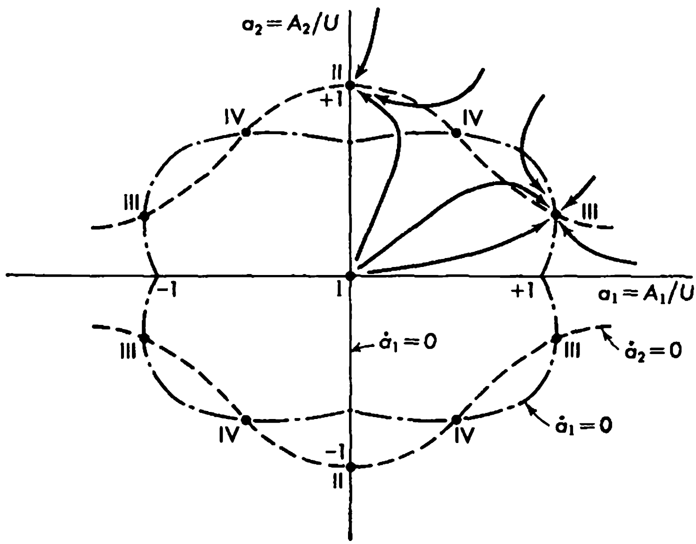


FIG. 10.21. Diagram illustrating the orbital stability of the two components of the approximate solution assumed for the oscillator of Example 10.10. The two frequencies are in the ratio of 1:3. Arrows indicate direction of increasing time. A change in relative phase is involved in crossing any axis of this figure.

It should be noted that the four quadrants of the figure are not strictly consistent, since angles θ_1 and θ_2 have stable values in quadrants 1 and 3 where $A_1A_2 > 0$, which differ from their values in quadrants 2 and 4, where $A_1A_2 < 0$. In other words, a change in phase angles is necessary in crossing any axis of Fig. 10.21.

It appears that singularities of designation II are stable, indicating an orbitally stable solution. This case represents a solution with just the third-harmonic component present. It is unrealistic because the nonlinear action of the limiter is certain to generate additional frequencies. The present analysis indicates that this is a stable solution only because of the erroneous original assumption of no frequencies higher than the third harmonic in the approximate solution.

The second case with orbital stability is that of designation III. An approximate steady-state solution for the oscillator is here

$$\frac{x}{U} = 1.07 \cos (\omega t + \theta_1) + 0.32 \cos (3\omega t + \theta_2)$$

with the requirement from Example 8.1 that $\omega\tau$ be an odd multiple of π radians and that $3\theta_1 - \theta_2 = \pi$ radians. The corresponding waveform is plotted in Fig. 10.22, where the arbitrary choice has been made, $\theta_1 = 0$. Also plotted is the square wave of amplitude $(h/g)^{1/2} = (3/4)^{1/2}U$, known to be an exact solution. The wave with only two components is seen to agree quite well with the square wave.

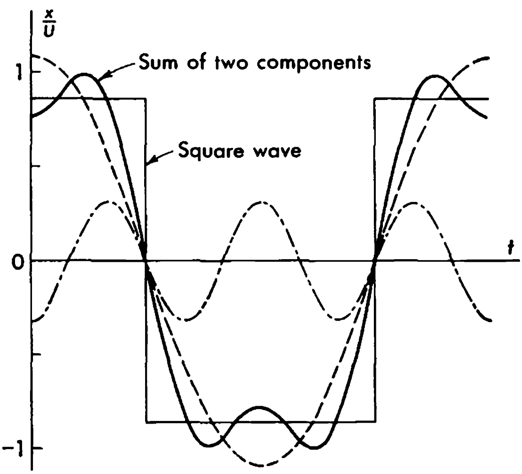


FIG. 10.22. Approximate orbitally stable solution with two components, and exact solution as square wave, for oscillator of Example 10.10.

10.9. Summary.

Definitions and tests for the stability of a physical system differ, depending upon whether the equilibrium condition is stationary or oscillatory. The simplest case is that of a linear system having a single equilibrium point with a stationary solution, at which point time derivatives of all the dependent variables are zero. Stability of this point is tested by applying a small disturbance to all the variables and finding whether or not these disturbances ultimately are reduced to zero. This is the case if the real parts of all the characteristic roots for the system are negative, and the system is then said to be asymptotically stable. Both the Routh-Hurwitz method, based on analytical work with coefficients of the differential equation, and the Nyquist method, based on numerical work with transfer functions, can be used to test whether or not positive real parts of characteristic roots exist.

If the system is nonlinear, more than a single equilibrium condition may appear. The simplest case is again that where all equilibrium points represent stationary solutions. Asymptotic stability can be tested in essentially the same way as for linear systems, except that the nonlinear equations may be replaced by linear variational equations near each equilibrium point.

If the system is a conservative oscillating system, or if it is a dissipative system driven by an oscillatory driving force, or if it is a nonlinear dissipative system having a solution as a limit cycle, an equilibrium condition involves a steady-state oscillation. A test for asymptotic stability may be applied in these cases also. A nonlinear equation leads to a variational equation with a periodic coefficient. Solution for this variational equation determines the asymptotic stability of the original system.

In many cases, orbital stability may be a more meaningful concept for an oscillatory system than is asymptotic stability. If a small change in amplitude of an oscillatory solution near equilibrium ultimately disappears, the system has orbital stability, regardless of any associated change in period. Orbital stability may be tested provided a relation is known between the amplitude of oscillation and the time rate of change of this amplitude.

PROBLEMS

Chapter 2

2.1. Solve, by the modified Euler method, $dy/dx = 4 - y^2$, with $y = 0.2$ at $x = 0.1$ and with $h = 0.05$. An exact solution for this equation can be found analytically.

2.2. Solve, by the Adams method, $dy/dx = y - y^2$, with $y = 0.2$ at $x = 0$ and with $h = 0.2$. An exact solution for this equation can be found analytically.

2.3. Check your results for Probs. 2.1 and 2.2 by the method of successive differences and by integration over an interval.

2.4. Tabulate values of $\sin \theta$ to four decimal places for increments $\Delta\theta = 5$ degrees over the range $0 \leq \theta \leq 30$ degrees, making use of standard tables of the sine function. Show that successive differences of third order are essentially constant. Extend the table to $\theta = 60$ degrees, making use of this constant third-order difference. Compare your table with accepted values of the sine function.

2.5. Solve, by the modified Euler method, $d^2y/dx^2 - 2(dy/dx)y = 0$, with $y = 0$, $dy/dx = 1$ at $x = 0$, and with $h = 0.1$. An exact solution for this equation can be found analytically.

2.6. Find a solution for the simultaneous equations

$$\frac{dy}{dx} = y^2 + xz$$

$$\frac{dz}{dx} = x^2 + yz$$

with $y = 0$, $z = 0$ at $x = 0$, and with $h = 0.1$.

2.7. The diode of the circuit shown in Fig. P2.7 has its instantaneous current i in amperes related to the instantaneous voltage e across it, in volts, by the equation $i = 0.002e^{3/2}$ for $e > 0$. Other elements are $C = 1{,}000$ microfarads, $R = 200$ ohms, and $E = 100$ volts. The switch is closed at time $t = 0$ with $e_R = 0$. Find a numerical solution for e_R as a function of t.

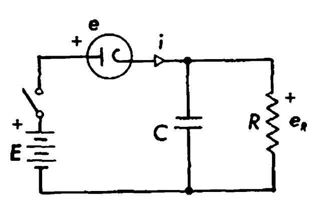


FIG. P2.7

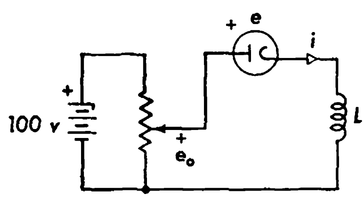


FIG. P2.8

2.8. The instantaneous voltage e in volts across the diode of the circuit shown in Fig. P2.8 is related to the instantaneous current i in amperes by the equation $e = 50i^{2/3}$ for $e > 0$. The self-inductance has the value $L = 10$ henrys. The slider on the potentiometer is made to move with time t so that voltage e_0 has the value

$$e_0 = 100(1 - 5t) \qquad \text{for } 0 \leq t \leq 0.2 \text{ sec}$$
$$e_0 = 0 \qquad \text{for } t \geq 0.2 \text{ sec}$$

Find a numerical solution for i as a function of t, with the initial condition that $i = 0$ at $t = 0$.

Chapter 3

3.1. Obtain a graphical solution for Prob. 2.7 by means of an isocline construction.

3.2. Obtain a graphical solution for Prob. 2.8 by means of an isocline construction. Consider the three initial conditions that, at $t = 0$, the current has each of the values $i = 0$, 2.5 amp, and $i_{max}/2$. In the last case, the initial current is to be half the maximum current ever attained.

3.3. A mechanical system consisting of a mass mounted on a linear spring, and sliding over a smooth surface with friction, is governed approximately by the equation

$$\frac{d^2x}{dt^2} + h\frac{dx/dt}{|dx/dt|} + \omega_0{}^2x = 0$$

discussed in Example 4.3. Coefficients h and $\omega_0{}^2$ are positive constants.

Construct a phase-plane diagram for this system by the δ method, with the initial conditions that $dx/dt = 0$ and $x = 20h/\omega_0{}^2$ at $t = 0$. Sketch a curve for x as a function of $\tau = \omega_0 t$ until motion ceases.

3.4. Using the isocline method, construct a phase-plane diagram for the system consisting of a mass on a hard spring, governed by the equation discussed in Example 4.6,

$$\frac{d^2x}{dt^2} + \omega_0{}^2(1 + a^2x^2)x = 0$$

with the constant chosen as $a^2 = \frac{1}{2}$. Consider the two initial conditions that $dx/dt = 0$ and both $x = 2$ and $x = 1$ at $t = 0$.

By means of numerical integration, obtain values of x as a function of $\tau = \omega_0 t$. Plot the ratio x/x_{max} against τ for the two initial conditions of this problem and also for the case of a linear oscillator for which $a^2 = 0$.

3.5. Using the δ method, construct a phase-plane diagram for the system consisting of a mass on a soft spring, governed by the equation discussed in Example 4.6,

$$\frac{d^2x}{dt^2} + \omega_0{}^2(1 - b^2x^2)x = 0$$

with the constant chosen as $b^2 = \frac{1}{4}$. Consider the two initial conditions that, at $\tau = 0$, both $dx/d\tau = 0$, $x = 1.5$, and also $dx/d\tau = 1.5$, $x = 0$, where $\tau = \omega_0 t$.

By means of a graphical construction using a triangular template based on $\Delta\tau = 0.4$, locate points equally spaced in time along the phase-plane curves. Plot x against τ for the two initial conditions.

3.6. An oscillatory system with a constant mass is governed by the equation

$$\frac{d^2x}{dt^2} + \frac{1}{4}\left(\frac{dx}{dt}\right)\left|\frac{dx}{dt}\right| + 2x|x| = 0$$

The spring carrying the mass has a restoring force which varies as the square of the deflection and is always directed opposite to the deflection. There is a friction force which varies as the square of the velocity and is always directed to oppose the motion.

Using the δ method, construct a phase-plane diagram for this system with the initial condition that $dx/dt = 0$ and $x = 2$ at $t = 0$. Since there is no linear term in x in the equation, it is necessary to add and subtract a term $\omega_0^2 x$. An appropriate choice is $\omega_0^2 = 4$.

By means of a graphical construction using a triangular template based on $\Delta\tau = 0.4$, locate points equally spaced in time along the phase-plane curve. Plot x against both τ and t.

3.7. Using the δ method, construct a phase-plane diagram for the Rayleigh equation,

$$\frac{d^2y}{dt^2} - \epsilon\left[1 - \tfrac{1}{3}\left(\frac{dy}{dt}\right)^2\right]\frac{dy}{dt} + y = 0$$

with $\epsilon = 1$. The initial conditions are $dy/dt = 0.5$ and $y = 0$ at $t = 0$.

3.8. Using the δ method, construct a phase-plane diagram for the van der Pol equation,

$$\frac{d^2x}{dt^2} - \epsilon(1 - x^2)\frac{dx}{dt} + x = 0$$

with $\epsilon = 1$. The initial conditions are $dx/dt = 0$ and $x = 0.5$ at $t = 0$.

3.9. By means of the Preisman construction, solve Prob. 2.7, and obtain a curve for e_R as a function of t.

Chapter 4

Solve the following equations for $y(x)$, making use of the specified initial conditions to evaluate all arbitrary constants that appear in the solution:

4.1. $\dfrac{d^2y}{dx^2} = ax^2 + bx,\ x_0 = 0,\ y_0 = 0,\ \left(\dfrac{dy}{dx}\right)_0 = 0.$

4.2. $\dfrac{dy}{dx} + y(\sin x) = \exp(\cos x),\ x_0 = 0,\ y_0 = 1.$

4.3. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 4y = \exp(2x),\ x_0 = 0,\ y_0 = 1,\ \left(\dfrac{dy}{dx}\right)_0 = 0.$

4.4. $\dfrac{dy}{dx} = \dfrac{1 + y}{1 - x^2},\ x_0 = 0,\ y_0 = 0.$

4.5. $\dfrac{dy}{dx} = \dfrac{2x - 3y}{3x + 8y},\ x_0 = 0,\ y_0 = 0.$

4.6. $\dfrac{d^2y}{dx^2} = 2y^3,\ x_0 = 1,\ y_0 = 1,\ \left(\dfrac{dy}{dx}\right)_0 = -1.$

4.7. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} - x^2 = 0,\ x_0 = 1,\ y_0 = 0,\ \left(\dfrac{dy}{dx}\right)_0 = 0.$

4.8. $\dfrac{d^2y}{dx^2} + y\left(\dfrac{dy}{dx}\right)^2 - \exp(2y - y^2) = 0,\ x_0 = 0,\ y_0 = 0,\ \left(\dfrac{dy}{dx}\right)_0 = 1.$

A solution in closed form for this equation can be found for dy/dx, but not for y. A series for $x = x(y)$ can obtained, and this should be done.

4.9. $\dfrac{dy}{dx} + (y - 2y^3)\sin x = 0,\ x_0 = 0,\ y_0 = 1.$

Solve this equation as an example of Bernoulli's equation and also by variation of parameters.

4.10. $2x^2\dfrac{dy}{dx} + 4x^2y^2 + 14xy + 5 = 0,\ x_0 = 1,\ y_0 = 0.$

4.11. $x^2\dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} - 5y = x^2,\ x_0 = 1,\ y_0 = 0,\ \left(\dfrac{dy}{dx}\right)_0 = 1$

4.12. Replace the diode characteristic curve of Prob. 2.8 with two linear segments for $e > 0$. Solve for and plot the current in the circuit as a function of time with the initial condition that $i = 0$ at $t = 0$.

4.13. Consider the integral $\int dt/[T(t)]^{1/2}$, where the function in the denominator is

$$T(t) = (t - a)(t - b)(t - c) = t^3 + At^2 + Bt + C$$

with constants a, b, and c all real and ordered as $a > b > c$. Show that the substitution $t = c + (b - c)\sin^2\phi$, for $c \leq t \leq b$, transforms the integral into the standard elliptic integral of the first kind

$$\frac{2}{(a - c)^{1/2}} \int \frac{d\phi}{(1 - k^2 \sin^2\phi)^{1/2}}$$

where $k^2 = (b - c)/(a - c)$.

4.14. The equation for the motion of a simple pendulum of length l is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

where θ is the angle of deflection away from the vertical and g is the gravitational acceleration. Show that, if the maximum deflection is θ_m, the substitutions $k = \sin(\theta_m/2)$ and $k\sin\phi = \sin(\theta/2)$ lead to the relation $(g/l)^{1/2}t = u = F(k,\phi)$, where $F(k,\phi)$ is the elliptic integral of the first kind.

4.15. Calculate and plot the ratio x/x_{max} for the mass and hard spring of Prob. 3.4 for the three conditions considered there, making use of tabulated values of the elliptic integral.

4.16. A system consisting of a mass and a soft spring is governed by the equation

$$\frac{d^2x}{dt^2} + \omega_0^2(1 - b^2x^2)x = 0$$

The system is operated with the maximum value of x equal to one unit and with the modulus k for the elliptic integral in the solution having the value $k^2 = 0.9924$.

Using tabulated values for the elliptic integral, calculate and plot x and $dx/d\tau$ as a function of $\tau = \omega_0 t$. Plot also $dx/d\tau$ against x to obtain a phase-plane diagram.

4.17. Using tabulated values for the elliptic integral, find a solution for Prob. 3.5 for the one initial condition which can be treated in this manner.

Chapter 5

5.1. A pair of simultaneous differential equations is

$$\frac{dy_1}{dt} = 5y_1$$

$$\frac{dy_2}{dt} = 4y_2$$

where the matrix of coefficients is $[B]$. Plot a solution curve for these equations on the y_1y_2 plane, passing through the points (0,0) and (1,1). Transform the equations by the matrix

$$[P] = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$$

used as $[B] = [P]^{-1}[A][P]$ so as to obtain a new set of equations in x_1 and x_2. Plot curves on the x_1x_2 plane corresponding to the axes of the y_1y_2 plane and to the curve first plotted there.

5.2. Write the simultaneous differential equations for the instantaneous values of voltages e_1, e_2, and e_3 at the nodes of the circuit of Fig. P5.2, with $R = 5 \times 10^5$ ohms and $C = 1$ microfarad. Because of coupling in the circuit, more than a single voltage appears in each equation. Show that the matrix

$$[P] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

when used with the matrix of coefficients of the original system of equations as $[B] = [P]^{-1}[A][P]$ transforms these equations to a system in normal form with no coupling. Calculate the characteristic roots from both matrices $[A]$ and $[B]$. Show a circuit which is described by the equations in normal form. Write the algebraic equations which relate the variables in the two forms of the circuit.

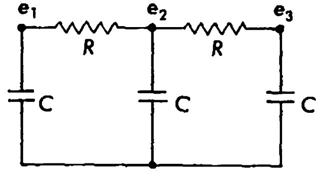


FIG. P5.2

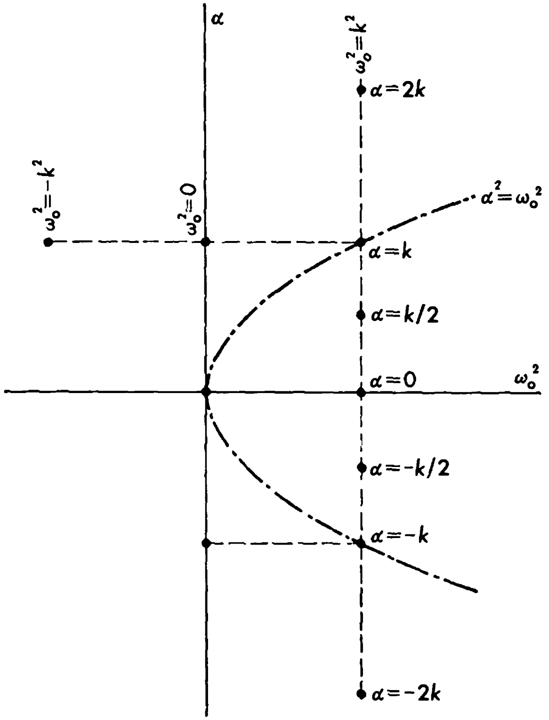


FIG. P5.3

5.3. The differential equation for a linear oscillator with damping can be written

$$\frac{d^2x}{dt^2} + 2\alpha \frac{dx}{dt} + \omega_0^2 x = 0$$

which has characteristic roots $(\lambda_1,\lambda_2) = \alpha \pm j\omega$ with $\omega = (\omega_0^2 - \alpha^2)^{1/2}$.

Eleven different combinations of values for coefficients α and ω_0 are shown in Fig. P5.3, where $k^2 > 0$, $k > 0$ is a specified constant. For each of these 11 combinations, sketch roughly the following:

(a) Location of λ_1 and λ_2 on the complex plane
(b) Typical solution curves on the phase plane, identifying the type of singularity
(c) Typical solution curves on the xt plane

5.4. Locate all the singularities for each of the following equations, and determine the nature of solution curves near each singularity:

van der Pol equation: $$\frac{d^2x}{dt^2} - \epsilon(1 - x^2)\frac{dx}{dt} + x = 0$$

Mass on hard spring: $$\frac{d^2x}{dt^2} + \omega_0^2(1 + a^2x^2)x = 0$$

Mass on soft spring: $$\frac{d^2x}{dt^2} + \omega_0^2(1 - b^2x^2)x = 0$$

Simple pendulum: $$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

5.5. In Volterra's competition equations, Eq. (5.31), of Example 5.1 positive values of the coefficients correspond to the case of two species of somewhat similar animals competing for the same food supply or living space. Consider the case in which $k_1 = 2$, $k_2 = 8$, $k_3 = 1$, and $k_4 = 2$ with consistent units. Locate singularities and sketch solution curves on the N_1N_2 plane.

If the algebraic signs of both coefficients in the equation for N_2 are reversed, the situation corresponds to the case of the second species preying upon the first. The second species then starves if the first is absent but increases in population more rapidly as the population of the first increases. These equations have been used to explain periodic changes in population of soles and sharks observed by commercial fishermen. Consider the case in which $k_1 = 2$, $k_2 = -8$, $k_3 = 1$, and $k_4 = -2$. Locate singularities and sketch solution curves on the N_1N_2 plane.

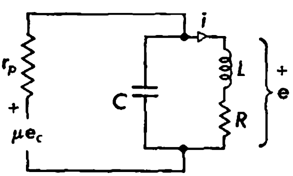


FIG. P5.6

5.6. The circuit of Fig. P5.6 represents an electron tube and a parallel resonant circuit. Values of the components are $C = 1$ microfarad, $L = 1$ henry, $R = 2{,}500$ ohms, and $r_p = 10{,}000$ ohms. The tube is initially cut off, but at $t = 0$ the equivalent voltage μe_c jumps suddenly to the value $\mu e_c = 100$ volts.

Set up equations for de/dt and di/dt, where e is the voltage across the resonant circuit and i is the current in the inductor. Locate singular points in the ie plane. Sketch solution curves for the initial conditions of $i = 0$, $e = 0$ at $t = 0$ and for $i = 0$, $e = 20$ volts at $t = 0$.

5.7. A mechanical oscillator is arranged to have considerable damping for small deflections but negligible damping for large deflections. It is governed by the equation

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + 9x = 0$$

where for

$-2 \leq x \leq +2$: $\quad \alpha = 10\frac{1}{2}$
$x > +2$: $\quad \alpha = 0$
$x < -2$: $\quad \alpha = 0$

At $t = 0$, the system is given an initial deflection x_0 and released with zero initial velocity, $dx/dt = 0$. Sketch a family of phase-plane curves for the system. Estimate maximum values of x_0 for which the ensuing motion:

(a) Does not overshoot
(b) Overshoots no greater than two units
(c) Has a single overshoot

5.8. A negative-resistance element has the type of characteristic curve shown in Fig. 5.13 and is used in the circuit of Fig. P5.8. Values of the components measured in a consistent system of units are $L = 4$, $C = 1$, $R = 1$, and $R_0 = 3$. Steady voltage E is chosen so as to locate an equilibrium condition in the center of the region of negative slope of the characteristic for the negative-resistance element. If the slope in this negative-resistance region is $di/de = -1/r$, determine the nature of the equilibrium point if r is allowed to take on values between zero and infinity. Determine values of r for transitions between different types of solutions.

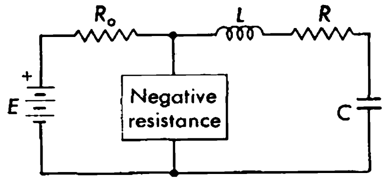


FIG. P5.8

5.9. Numerical values in the trigger circuit of Fig. 5.14 are $R = 10^4$ ohms, $L = 10^{-4}$ henry, and $C = 10^{-10}$ farad. The negative-resistance characteristic is composed of three straight lines as shown in Fig. P5.9. Initially, $E = -70$ volts, while at $t = 0$ it jumps suddenly to $E = +70$ volts.

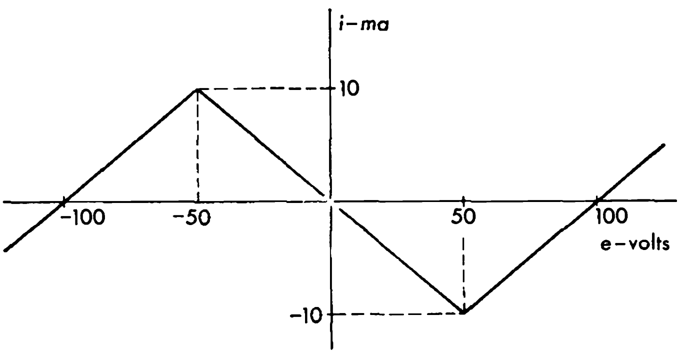


FIG. P5.9

Locate the singularities, and sketch a figure analogous to Fig. 5.16, showing the path followed as the circuit triggers from one state to the other. Calculate the time required for the circuit to come within 1 volt of the final steady state.

5.10. Two identical series d-c generators are connected in parallel. The field winding of each generator has a resistance r and an inductance L. The armature resistance is negligible. The two generators in parallel supply current to a load of resistance R. The relation between the generated voltage e and the current i of each machine is $e = ai - bi^3$, where a and b are positive constants and $a > r$.

Equilibrium values of the two currents, i_1 and i_2, may exist for $i_1 = i_2$ and also for $i_1 = -i_2$. Determine the values of these equilibrium currents. Find the nature of solutions near each of the singularities. Is this mode of operation a useful one?

5.11. A parallel connection for two d-c generators, different from that of Prob. 5.10, has the field winding of either generator connected in series with the armature of the other generator. With this connection $e_1 = ai_2 - bi_2^3$, and vice versa, where the quantities are the same as those of Prob. 5.10. The parallel combination of generators supplies current to the load of resistance R.

Determine equilibrium conditions for this mode of operation, and find the nature of solutions near each singularity.

5.12. The angular deflection of an electrodynamometer ammeter is governed by the equation

$$\frac{d^2\theta}{dt^2} + a\theta - bI^2 \cos \theta = 0$$

where θ is the angle of deflection, a and b are positive constants, and I is the steady current in the meter.

Find the equilibrium deflection for a specified current. Under what conditions may there be more than a single equilibrium deflection? If there are three equilibrium deflections, show a phase-plane diagram for the system.

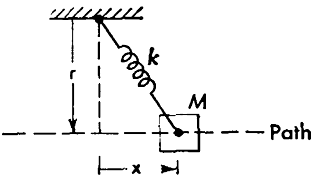


FIG. P5.13

5.13. A mass M is constrained to move only along the straight-line path of Fig. P5.13. The perpendicular distance between this path and a fixed point is r. A linear spring of stiffness k and unstretched length l, somewhat greater than r, is connected between the fixed point and the mass. Coordinate x determines the position of the mass, with $x = 0$ when the spring is compressed to length r. As x varies, the spring is alternately compressed and stretched.

Show that the system is governed by an equation of the general type $d^2x/dt^2 - \omega_0^2 x + hx^3 = 0$, where ω_0^2 and h are positive constants. Because of the geometry of this system, the actual equation is more complicated. Determine equilibrium values of x for this simpler equation and the nature of solutions near each singularity. Sketch a phase-plane diagram for the system.

5.14. A dissipationless, spring-loaded pendulum has been adjusted so that the restoring torques are

Gravitational:	$Mgl \sin \theta$
Spring:	$Mgl \dfrac{\theta}{\pi^2}$

where θ is the angle of deflection from the vertical, M is the mass, l is the length, and g is the gravitational acceleration.

Sketch a curve for potential energy as a function of θ. Determine equilibrium values of θ and the nature of solutions near each singularity. Sketch a phase-plane diagram for the system. Describe the various types of motion which may occur.

5.15. A conservative mechanical system consists of a permanent magnet suspended by a linear spring located over a fixed iron plate, as shown in Fig. P5.15. Displacement is measured downward from the end of the unstretched spring. In a consistent system of units, the upward forces on the magnet are

Gravitational:	-4
Spring:	$2x$
Magnetic:	$\dfrac{-32}{12 - x}$

The deflection cannot exceed 12 units. The mass is $M = 8$.

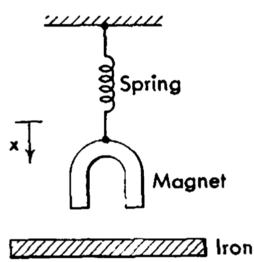


FIG. P5.15

Plot the total potential energy of the system as a function of the deflection. Locate singularities, determine their types, and sketch a phase-plane diagram for the system. Locate the separatrix curve, and sketch solution curves on either side of it. Sketch typical curves for deflection as a function of time.

Chapter 6

6.1. The voltage e across a saturating capacitor is related to the charge q as $e = q/C_0 + hq^3$, where C_0 and h are positive constants. A resistance R is connected across the terminals of the capacitor. If $q = Q_0$ at $t = 0$, determine the charge as a function of time to the second-order approximation, using the perturbation method. Find also an exact solution for the nonlinear differential equation.

6.2. Determine and plot the trajectory of a bomb released from an airplane flying close and parallel to the surface of the earth. At $t = 0$, the velocity is $V = 320$ ft per sec, and the angle is $\theta = 0$. The properties of the bomb are such that, if it were allowed to fall freely for a very long distance through air, its terminal velocity would be $v_f = 320$ ft per sec. Plot also the trajectory the bomb would follow if there were no air resistance.

6.3. Show that to a first order of approximation the ratio of periods T/T_0 found for the mass and nonlinear spring in Example 6.3 is the same as this ratio found in Eqs. (4.46) and (4.47). Consider both the hard and the soft spring. The series expansion for the elliptic integral $K(k)$ must be used here.

6.4. Calculate and plot the ratio x/x_{max} for the mass and hard spring of Prob. 3.4 for the three conditions considered there, making use of the perturbation analysis of this system. Compare these results with those of Prob. 4.15.

6.5. An asymmetrical spring becomes stiffer for deflections in one direction but less stiff for deflections in the opposite direction. When combined with a constant mass, the system is governed by the equation

$$\frac{d^2x}{dt^2} + \omega_0^2 x + hx^2 = 0$$

where ω_0^2 and h are positive constants. Initial conditions are that $x = A$, $dx/dt = 0$ at $t = 0$. Determine a solution for the motion to a second-order of approximation.

6.6. Bessel's equation of order zero may be written

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$$

If x is large, the coefficient of dy/dx is small and changes relatively slowly. Under these conditions, find an approximate solution for the equation by the method of variation of parameters. The two arbitrary constants of the solution cannot, of course, be evaluated. Compare your result with the usual asymptotic form of Bessel functions of order zero and large argument.

6.7. The equation for a simple pendulum is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

where θ is the angle of deflection and g and l are constants. Find an approximate value for the fundamental frequency of oscillation by the method of variation of parameters.

6.8. A mechanical system with dry friction and an asymmetrical spring is governed by the equation

$$\frac{d^2x}{dt^2} + h\frac{dx/dt}{|dx/dt|} + \omega_0^2 x + gx^2 = 0$$

where ω_0^2, g, and h are positive constants. Initial conditions are that $x = 20h/\omega_0^2$, $dx/dt = 0$ at $t = 0$. Determine an approximate solution by the method of variation of parameters, assuming that g and h are relatively small. Sketch a curve for x as a function of t, and compare with the results of Prob. 3.3.

6.9. A negative-resistance element is described by the equation $i = ae - be^3 + ce^5$ where a, b, and c are positive constants. Find a necessary relation among these constants in order that there be five real values of e at which $i = 0$.

This negative resistance is used with an inductance L and a capacitance C to form a self-oscillator as in Fig. 6.7. Determine the equilibrium conditions for oscillation. Find a relation among constants a, b, c so that there will be five equilibrium amplitudes. Determine which amplitudes are stable and which are unstable.

This kind of system is known as a "hard" oscillator.

6.10. An oscillatory system with a special type of damping is governed by the equation

$$\frac{d^2x}{dt^2} + \alpha x^2 \frac{dx}{dt} + \omega_0^2 x = 0$$

where α and ω_0^2 are positive constants.

If initial conditions are $x = A$, $dx/dt = 0$ at $t = 0$, find an approximate solution. It is known that many cycles of oscillation occur before motion ceases.

6.11. A tuning fork driven by a magnetic system supplied from an external generator is governed by the equation

$$\frac{d^2x}{dt^2} - 2\alpha \frac{dx}{dt} + \beta \left(\frac{dx}{dt}\right)^3 + \omega_0^2 x = 0$$

where α, β, and ω_0^2 are positive constants. Find an approximate solution, and determine steady-state values of amplitude.

6.12. A projectile of mass M is shot vertically upward from the surface of the earth with initial velocity V. It is subject to the gravitational attraction of the earth and to a retarding force kv^2, where v is velocity with respect to the earth and k is a constant. Find relations for the velocity and for height above the earth as functions of time.

6.13. Complete the details of Example 6.8, and find an optimum value of k with the condition $BE/A = \frac{1}{2}$. Compare your result with that of Example 6.11.

6.14. The rest mass of a particle is m_0, while at high velocities $\dot{x}$ the relativistic mass becomes $m = m_0/(1 - \dot{x}^2/c^2)^{1/2}$, where c is the velocity of light. If such a particle could be arranged to move in a force field having an equivalent constant stiffness k, the equation of motion would be[1]

$$\frac{d}{dt} \frac{m_0 \dot{x}}{(1 - \dot{x}^2/c^2)^{1/2}} + kx = 0$$

An alternate form of the equation is

$$\frac{d}{dt} \frac{\dot{y}}{(1 - \dot{y}^2)^{1/2}} + \omega_0^2 y = 0$$

[1] L. A. MacColl, Theory of the Relativistic Oscillator, *Am. J. Phys*, **25**:535–538 (1957).

where $y = x/c$ and $\omega_0^2 = k/m_0$. If the velocity of the particle is not too great, $\dot{y} \ll 1$. Find an approximate solution for this condition. Show how the period of the resulting oscillation varies with amplitude.

Chapter 7

7.1. Duffing's equation with damping has been written as Eq. (7.10)

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^2 x + hx^3 = G\cos(\omega_1 t + \theta)$$

with an approximate steady-state solution $x = A\cos\omega_1 t$. If ω_1 is fixed and G is varied, the amplitude of the solution may jump discontinuously at points where $dA^2/dG = \infty$. If α is less than a critical value α_c, there are two points where $dA^2/dG = \infty$. If α exceeds α_c, there is no such point.

Determine the value of α_c in terms of ω_0 and ω_1. Determine values of G and A^2 where $dA^2/dG = \infty$ if $\alpha = 0$ and if $\alpha = \alpha_c$.

7.2. It has been shown in Example 7.5 that a possible steady-state solution for Duffing's equation is a subharmonic oscillation of order $\frac{1}{3}$. Sketch a figure analogous to Fig. 7.13, showing the relation between the amplitude and the driving frequency for this subharmonic. Carry out the perturbation analysis for this subharmonic, finding a solution to the first-order approximation.

7.3. Carry out the perturbation analysis for the one-third-order subharmonic response of the Duffing equation to the first-order approximation.

7.4. Carry out the perturbation analysis for the second-order harmonic response of the Duffing equation to the first-order approximation.

7.5. Consider Duffing's equation, Eq. (7.4), with no dissipation but with nonlinear coefficient h a negative constant. This corresponds to a mechanical system with a soft spring and requires a rather special physical arrangement. Only positive values of driving frequency are meaningful.

Sketch a figure for this system analogous to Fig. 7.6. Show that if driving force G is small the jump phenomena may occur, while if G is large jumps are impossible. Determine the maximum value of G for which jumps may take place.

7.6. The mechanical system of Prob. 5.13 is driven by a simple-harmonic driving force, so that the governing equation is

$$\frac{d^2x}{dt^2} - \omega_0^2 x + hx^3 = G\cos\omega_1 t$$

where ω_0, h, G, and ω_1 are all positive constants. Among possible solutions are those periodic with period the same as the driving force. One solution of this kind is periodic about a mean value of x that is zero. For this solution, sketch a figure analogous to Fig. 7.6, extending the figure to include negative as well as positive values of ω_1.

A second possible solution is periodic about a mean value of x that is not zero. Sketch a figure for this case, showing both the mean value and the amplitude of the periodic component. Over what range of G and ω_1 may this solution exist?

7.7. A pair of springs is arranged so that the relation between force and deflection is as shown in Fig. P7.7. The combination is driven so that the deflection varies as a simple-harmonic function of time with amplitude $2x_1$. Note there are intervals when the spring force is zero. Find an equivalent stiffness for the spring.

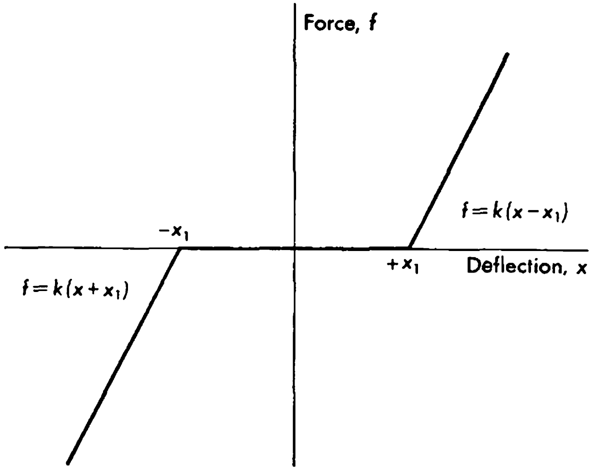


FIG. P7.7

7.8. Consider the springs of Prob. 7.7, now driven so that the force varies as a simple harmonic function of time. The amplitude of the variation in force is chosen so that the fundamental component of the variation in deflection is $2x_1$. Note that the deflection jumps discontinuously at certain points. Find an equivalent stiffness for the spring. Compare with the result of Prob. 7.7.

7.9. The instantaneous output signal y of a nonlinear transmission system is related to the instantaneous input signal x as $y = y(x) = a_1x + a_2x^2 + a_3x^3$, where a_1, a_2, and a_3 are real coefficients. If x varies as $x = A \cos \omega t$, find the fundamental component of y as $y_1 = B \cos \omega t$.

Reverse the polynomial for $y = y(x)$ by assuming that $x = x(y) = b_1y + b_2y^2 + b_3y^3$, where b_1, b_2, and b_3 are coefficients to be determined. Substitute the polynomial for $x(y)$ into that for $y(x)$, require that the result be an identity, and solve for b_1, b_2, b_3 in terms of a_1, a_2, a_3. If y varies as $y = B \cos \omega t$, where B is the amplitude of y_1 found above, find the fundamental component of x as $x_1 = A' \cos \omega t$. Show that, if $a_2A/a_1 < \frac{1}{3}$ and $a_3A^2/a_1 < \frac{1}{3}$, amplitudes A and A' are essentially the same.

Note that this result is applicable to the calculation of a describing function, or an equivalent impedance, for an element having a moderate degree of nonlinearity. It implies that essentially the same result is obtained by requiring either the input signal or the output signal to be simple-harmonic and finding the fundamental component of the other.

Chapter 8

8.1. The thickness of sheet metal coming from a rolling mill is measured at a point some distance from the rolls. If the thickness x differs from that desired, x_0, a control system corrects the spacing between the rolls. Because of the separation of rolls and measurement point, combined with constant linear velocity of the metal sheet, a fixed time delay τ is involved. A simplified equation describing the system might be $\dot{x}_{(t)} = -k[x_{(t-\tau)} - x_0]$, where $\dot{x}_{(t)}$ is evaluated at time t, while $x_{t-\tau}$ is evaluated at the earlier time $t - \tau$. Coefficient k is a positive constant, determined by the control system.

A solution for this equation has the form $x = x_0 + \sum_i A_i \exp(m_i t)$, where A_i is a coefficient determined by any unwanted departure from proper thickness x_0 and m_i is an exponent determined by properties of the system. Exponent m_i may have an infinity of values and may be real or complex. What is the largest value of product $k\tau$ for which real values of m may exist? What is the largest value of $k\tau$ for which the system is stable, in the sense that any real part of m is never positive?

If $k\tau$ is small enough, complex values of m have large negative real parts and these values contribute little to the solution. Calculate and plot x as a function of t/τ if $x = 15$ at $t = -\tau$ and $x = 5$ at $t = 0$, for the two cases of $k\tau = 0.3$ and $k\tau = \exp(-1)$. Note that actually x must be completely defined for $-\tau \le t \le 0$, while here only two points are considered.

8.2. Carry out the details of the solution for Example 8.4, leading to the results given in Eq. (8.29).

8.3. An active mechanism investigated as a means for minimizing the rolling of a ship has been shown to be described by the equation[1]

$$\ddot{\theta}_{(t)} + a\dot{\theta}_{(t)} + b\dot{\theta}_{(t-\tau)} + \omega_0^2\theta_{(t)} = 0$$

where θ is the angle of roll, a and ω_0^2 are positive constants determined by properties of the ship, and b is a positive constant determined by the stabilizing equipment. All terms are evaluated at time t, except the third term, which is evaluated at time $t - \tau$ because of a fixed time delay τ in the operation of the stabilizer. If time τ were zero, the stabilizer would act to increase damping of the rolling action. Because τ is not zero, it is found that for some adjustments of coefficient b the net damping becomes zero and a steady-state rolling occurs at the angular frequency ω.

Determine conditions for this steady-state rolling. In particular, find the range of values within which product $\omega\tau$ must lie. Find the transcendental equations from which ω and b could be determined if numerical values of a, ω_0^2, and τ were known.

Note that $\omega\tau$ is sufficiently large here so that an approximation to $\dot{\theta}_{(t-\tau)}$ using just a few terms of a Taylor series is not adequate.

8.4. If the rolling of the ship of Prob. 8.3 becomes large, a nonlinear term in θ^3 must be introduced into the equation. Consider the case in which coefficient b is made slightly different from the value b_0 needed to give a steady-state oscillation in the linear case, so that $b = b_0(1 + h)$ with $|h| \ll 1$. The equation is

$$\ddot{\theta}_{(t)} + a\dot{\theta}_{(t)} + b_0(1 + h)\dot{\theta}_{(t-\tau)} + \omega_0^2\theta_{(t)} + g\theta^3{}_{(t)} = 0$$

where coefficients h and g are relatively small. Investigate the possible appearance of a steady-state oscillation in this case, and find its equilibrium amplitude.

8.5. Construct a graphical solution for Eq. (8.28) of Example 8.4 with the parameters chosen as $a = 0$, $aC = aC_0(1 + h)$ with $aC_0 = -\pi/4$ and $h = 0.2$, $g = 10^{-4}$, $\tau = 2$, and $\Delta t = 0.5$, all quantities being measured in a consistent system of units. Assume that x has the initial value $x = 10$ for the entire interval $-\tau \le t \le 0$. Compare the resulting equilibrium amplitude with that found from Eq. (8.29).

8.6. Change the variable of Eq. (8.28) to $y(t)$, where the definition is made, $x(t) = \dot{y}(t)$. Repeat the graphical solution of Prob. 8.5 with the resulting simplified equation in y.

[1] N. Minorsky, Self-excited Mechanical Oscillations, *J. Appl. Phys.*, **19**:332–338 (1948).

8.7. Repeat the construction of Prob. 8.6 with all the parameters the same, except now with $h = 5$. This large value of h cannot be used in the analytical solution of Eq. (8.29).

Chapter 9

9.1. A capacitor microphone produces at its terminals a capacitance of value dependent upon the sound wave striking the microphone. With a simple-harmonic wave, the instantaneous capacitance C is $C = C_0(1 + m \sin \omega t)$, where C_0 is the mean value and $m \ll 1$. The microphone is used in a series circuit consisting of a resistance R and a battery of constant voltage E.

Find the instantaneous voltage across the resistance as a function of time to an order of approximation high enough to show the generation of distortion. Show what criteria are important in minimizing this distortion and in keeping the instantaneous voltage as nearly as possible in phase with the instantaneous capacitance.

9.2. An a-c generator provides a voltage $e_0 = E \sin \omega_0 t$ to a load of resistance R. It is proposed to increase the power available in the load by connecting a controllable inductance in series with the generator and load. The inductance is to be made to vary with time as $L = L_1[1 + m \sin (\omega_1 t + \theta)]$, where L_1 is the mean inductance and m is the relative variation. Determine angular frequency ω_1 and phase angle θ for the varying inductance so that maximum power is supplied to the load resistance.

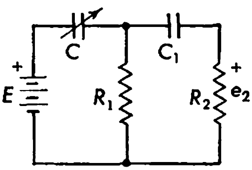


FIG. P9.3

9.3. A capacitor microphone is used in the circuit of Fig. P9.3. The capacitance of the microphone varies as $C = C_0(1 + m \sin \omega t)$, where $m \ll 1$. The circuit elements are chosen so that $C_1 \gg C_0$, while R_1 and R_2 are of similar values. Find the instantaneous voltage e_2 across resistance R_2.

9.4. A circuit consists of a constant inductance L and a capacitance which is caused to vary with time as $C = C_0 \exp(-t/T)$, where C_0 is the initial value and time constant T is a measure of the rapidity of variation. Set up the differential equation for the instantaneous charge on the capacitor, and find an approximate solution by the WKBJ method. Interpret your result physically. What is the limitation on time constant T in order that this method of solution be valid?

9.5. A mechanical system consists of a bucket suspended on a spring of stiffness k. Water flows into the bucket at a constant rate, so that the total mass on the spring increases linearly with time as $M = M_0(1 + t/T)$, where time constant T is a measure of the rate of increase. Set up the differential equation for the system, and find an approximate solution by the WKBJ method. What is the limitation on time constant T in order that this method of solution be valid?

9.6. Frequency modulation is applied to the negative-resistance oscillator of Example 6.7 by causing the capacitance to vary as a function of time[1] so that its instantaneous value is $C = C_0[1 + m\phi(t)]$. The time variation is given by function $\phi(t)$, with the requirements that both $\phi(t) = 0$ for $t \leq 0$ and the relative change in $\phi(t)$ for $t > 0$ is small during a cycle of the oscillation. At $t = 0$, $C = C_0$. The equation for the modulated oscillator, analogous to Eq. (6.50), becomes

$$C\ddot{e} + 2\dot{C}\dot{e} + \ddot{C}e - a\dot{e} + 3be^2\dot{e} + \frac{e}{L} = 0$$

[1] W. J. Cunningham, Amplitude Variations in a Frequency-modulated oscillator, *J. Franklin Inst.*, **256**:311–323 (1953).

Find an approximate solution for this equation by the method of variation of parameters. Assume that the oscillator has been operating long enough prior to time $t = 0$ so that a steady state has been attained. Modulation of both frequency and amplitude is of interest here.

9.7. An inverted pendulum of the sort described in Example 9.8 is to be constructed. The length of the pendulum is chosen to be 10 cm. Its pivot point is to be made to oscillate by driving it from a crank attached to the shaft of a motor running at 1,800 rpm. How large must be the amplitude of oscillation of the pivot in order that the inverted pendulum be stable?

9.8. The motion of an ideal stretched string is governed by the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where coordinate x measures position along the string, t is time, and u is deflection from the equilibrium position. The velocity of propagation of a disturbance is c, where $c^2 = F/m$, F being the tension in the string and m the mass per unit length. The string can be loaded so that its mass varies periodically with position as $m = m_0(1 + \epsilon \cos 2\pi x/x_0)$, where m_0 is the mean value, ϵ is the relative variation assumed small so that $\epsilon \ll 1$, and x_0 is the length for one complete variation of mass.

Assume the deflection u may be written as the product of two functions, $u(x,t) = X(x)T(t)$, separate the variables, and show that function $X(x)$ is given by a Mathieu equation.

Chapter 10

10.1. A negative-resistance element is described by the equation $i = -ae + be^3$, where i and e are its instantaneous current and voltage, respectively, and a and b are positive constants. The element is used in the circuit of Fig. P10.1. The parameters of the circuit are $a = 20$, $b = 2$, $L = 0.1$, $C_1 = 10$, $C_2 = 10$, and $G = 12$, where a consistent system of units is used and G is the conductance across which voltage e_2 appears.

Set up the differential equation for voltage e_1. Note that this equation need be only third-order. Locate equilibrium values of e_1, and determine which are stable.

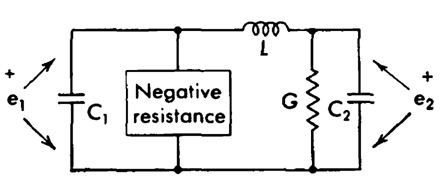


FIG. P10.1

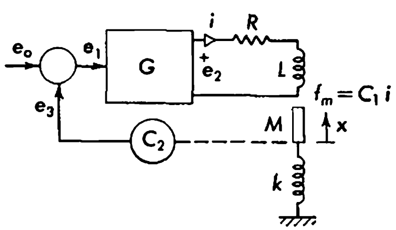


FIG. P10.2

10.2. A simple positioning servo system is shown in Fig. P10.2. It consists of a solenoid, of resistance R and self-inductance L, which produces a force f_m on a plunger of mass M mounted on a spring of stiffness k. Force f_m is given by $f_m = C_1 i$, where i is the current in the solenoid and C_1 is a constant. The resulting deflection of the plunger is x, and it produces a feedback voltage $e_3 = C_2 x$, where C_2 is a constant. Voltage e_2 applied to the solenoid is $e_2 = Ge_1$, where G is the amplification of the amplifier and $e_1 = e_0 - e_3$, e_0 being the input control voltage. It is desired that deflection x be proportional to voltage e_0.

Parameters of the system, measured in a consistent system of units, are $R = 10$, $L = 0.1$, $M = 50$, $k = 5{,}000$, $C_1 = 1{,}000$, $C_2 = 10$, and $G = 50$.

Investigate the system, and determine whether or not it will be stable. Will a change in G modify the stability?

10.3. The system of Prob. 10.2 is modified by the addition of a dashpot to the mechanical components. The dashpot provides a retarding force proportional to the velocity $\dot{x}$ and effectively adds mechanical resistance. This resistance is adjusted so that the mass-spring–dashpot combination is critically damped.

What is the stability of the resulting system? What value of amplification G marks the transition between stability and instability?

10.4. Determine the steady-state conditions for the servo system of Example 10.6 by the method outlined in its closing paragraph. In this alternate approach, consider the total torque developed by the motor as the sum of two components. One component accelerates the linear mass; the other overcomes the nonlinear friction. A describing function can be used for just the friction and thus will involve only amplitude and not frequency. Show that this alternate procedure leads to essentially the same results found in Example 10.6.

10.5. The servo system of Example 10.6 has been shown to be asymptotically unstable, with a steady-state oscillation developing, if the quantity $Gk_3/k_1 = 10$. Determine whether or not the system is still unstable if the gain of the amplifier is reduced so that $Gk_3/k_1 = 2$.

10.6. Investigate the stability of the two types of solution for the driven nonlinear mechanical system of Prob. 7.6.

10.7. In Example 7.5 and Prob. 7.2, the subharmonic response of order $\frac{1}{2}$ is considered for Duffing's equation. Investigate the stability of this subharmonic.

10.8. Consider the conditions for orbital stability of the oscillator of Example 10.10, described by Eq. (10.33), with coefficient C_0 chosen as $C_0 = +1$. A possible form of solution in this case, with only two components, is $x = A_0 + A_1 \cos \omega t$. Find possible equilibrium values of amplitudes A_0 and A_1, and determine which are orbitally stable.

10.9. Consider the negative-resistance oscillator of Example 10.9 with the resonant circuits adjusted so that the two possible frequencies of oscillation are related as $\omega_2 = 3\omega_1$. If it is assumed that any oscillation which occurs does so exactly at a frequency of resonance, each circuit presents an impedance at its terminals which is purely resistive. The negative-resistance element must likewise present a resistive impedance. It can then be argued that a possible solution must have the form $e = E_1 \cos \omega t + E_2 \cos 3\omega t$, where no additional phase angles are required.

Find possible equilibrium values for amplitudes E_1 and E_2, and determine which of these are orbitally stable.

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The literature on nonlinear differential equations and their applications to physical systems is both extensive and repetitious. In summarizing progress achieved in nonlinear circuitry during 1951, one anonymous author saw fit to comment[1] on the "usual nibbling at the standard problems in this field" and "the old chestnuts perennially resistant to all but brute-force procedures." Comments of this kind arise because mathematicians have been prone to discuss in print nonlinear problems carefully chosen to be at least partially solvable. Engineers have had to deal with problems for which solutions may not be forthcoming. Consequently, papers from engineers often include many experimental data, qualitative observations, and sometimes-sketchy mathematics. Nevertheless, the literature on nonlinear analysis does contain a number of references in which basic knowledge is nicely brought together. This bibliography contains certain of these references.

The list of books and papers given here is by no means intended to be exhaustive, nor is it intended to assign historical precedence to the origination of techniques of analysis. Rather, it is a collection of material which has been found useful in preparing the manuscript for the present book and which may well be helpful to the inquiring reader. All the material is in English. Additional references are given to certain topics, mentioned only briefly in the text, but which are of general interest. Certain of the references themselves include extensive bibliographies, and these are so indicated. Apologies are due those authors whose work is not specifically mentioned, even though it may be of high caliber.

A cause of considerable confusion is the fact that various authors, writing about essentially the same mathematical process, choose to use different terminology. Thus, what is discussed here as the perturbation method is described elsewhere as the method of Poincaré or the method of small parameters. Similarly, the method of harmonic balance is sometimes called the method of Kryloff and Bogoliuboff or equivalent linearization and is essentially the same as the use of equivalent impedances and describing functions. Many examples of this kind exist. The reader must be on constant guard for such variation in nomenclature.

Numbers in parentheses correspond to those in the List of References at the end of this bibliography.

General References

A good summary of the technical literature on the analysis of nonlinear control systems, existing in 1956, is given by Higgins (33), together with a brief historical background. Among the now classic, even though recent, books on the subject are those of Andronow and Chaikin (1), Hayashi (28), Kryloff and Bogoliuboff

[1] *Proc. IRE*, **40**:401 (1952).

(47), McLachlan (59), Minorsky (61), and Stoker (75). In addition, there is a shorter summary of a general nature by Minorsky (62). A general survey of certain nonlinear problems appears in papers by Bellin (3), Bennett (6), and Kármán (41). A number of nonlinear vibration problems are considered by Timoshenko and Young (78).

Chapter 2

There are many books which deal with the numerical solution of differential equations, a standard work being by Scarborough (73). The present rapid development of digital computing machinery has brought a flood of works on various aspects of numerical analysis, two of which are by Hildebrand (34) and Kunz (50). A book concerned wholly with differential equations is by Milne (60).

Chapter 3

The isocline method of graphical construction is basic to many other graphical methods. It is generally described in discussions of differential equations, as, for example, Kármán and Biot (42). It also appears in Andronow and Chaikin (1) and McLachlan (59). The delta method for attacking a second-order equation is described by Jacobsen (35) and a similar process by Ku (48). Extensions of this method to more complicated systems are suggested by Ku (49) and Buland (11). Some of the simpler techniques of graphical integration are discussed in a chapter concerning nonlinear elements in the book on electric circuits by the MIT staff (56). The graphical method of building up a solution for a reactive circuit with a nonlinear element is described, with extensions, by Preisman (72). Graphical procedures for getting the time scale from the phase plane are given by Jacobsen (35) and Cunningham (17). The classic analysis for the van der Pol equation is by van der Pol (69, 71), with additional discussion by Le Corbeiller (51). This latter paper includes the Lienard method.

Chapter 4

A great many books exist in the field of classical differential equations, among those written primarily for engineers being the works by Bronwell (10), Kármán and Biot (42), and Wylie (81). Certain nonlinear equations with exact solutions are discussed by McLachlan (59). The use of variation of parameters for nonlinear equations and material on elliptic functions are given by Keller (43). The elliptic integral and simple piecewise linear problems are considered by Hansen and Chenea (27). Convenient tables of elliptic integrals are those of Jahnke and Emde (36) and Flügge (23). Certain piecewise linear problems are discussed in some detail by Flügge-Lotz (24).

Chapter 5

The geometry associated with the several types of singularities is discussed by Andronow and Chaikin (1), Minorsky (61), Stoker (75), and Truxal (80). These same authors also consider analysis of mechanical systems in terms of their potential energies. Application to specific control problems is described by Kalman (37, 38) and to certain electronic circuits by Farley (21). Equations of biological growth are discussed by Lotka (53).

Chapter 6

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Chapter 7

A mathematical summary of driven systems is given by Bellin (3). Solutions for Duffing's equation by several methods are considered by McLachlan (59) and Stoker (75). The analysis of subharmonic generation given here is similar to that carried out by Levenson (52). Experimental work with subharmonic generation is described by Ludeke (54, 55) and McCrumm (57). Both theoretical and experimental work with a system similar to that leading to Duffing's equation is described by Hayashi (28 to 32), who considers both the steady-state and transient conditions. The lamp-bulb example is from Cunningham (14). The use of equivalent impedances in nonlinear circuits is discussed by Thomsen (76) and Thomsen and Schlesinger (77). The concept of the describing function is often attributed to Kochenburger (46), and its implications are discussed in some detail by Truxal (80). Examples of Fourier expansions for finding describing functions of systems with friction are given by Tou and Schultheiss (79). Application of the Ritz method to describing functions is discussed by Klotter (45). Forced oscillations in a negative-resistance circuit have been considered by Cartwright (12), van der Pol (70, 71), and Stoker (75).

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Chapter 9

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Chapter 10

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