# Metamorphoses of Basin Boundaries in Nonlinear Dynamical Systems

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A basin boundary can undergo sudden changes in its character as a system parameter passes through certain critical values. In particular, basin boundaries can suddenly jump in position and can change from being smooth to being fractal. We describe these changes ("metamorphoses") and find that they involve certain special unstable orbits on the basin boundary which are *accessible* from inside one of the basins. The forced damped pendulum (Josephson junction) is used to illustrate these phenomena.

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Much work has been done on the mechanisms by which strange attractors arise and change as a parameter of the system is varied. These mechanisms include period-doubling cascades, intermittency, crises, etc. Recently it has been shown that basin boundaries[1] for typical nonlinear dynamical systems can be strange[2-6] (in the sense of having fractal dimension) and they occur in the simplest nonlinear systems (e.g., Jospehson junctions, clamped beams, electrical circuits, etc.). It is the purpose of this Letter to investigate how fractal basin boundaries arise and change as a parameter of the system is varied. For two-dimensional maps (e.g., those derived from Poincaré sections of three-dimensional continuous-time systems), basin boundaries change through sudden discrete events which we call basin-boundary metamorphoses, and we find that these metamorphoses are mediated by certain special unstable orbits on the basin boundary. These special orbits are characterized by being *accessible* from one side or the other of the boundary, and, in a sense, they completely determine the structure of the boundary. (A definition of "accessible" is given later.) As a parameter is varied, the metamorphosis occurs when, suddenly, an accessible unstable boundary orbit becomes inaccessible and, simultaneously, a different unstable orbit assumes the role of accessible orbit. As a result of a metamorphosis, a basin boundary can suddenly jump in size and can sometimes change from being smooth to being fractal. As an example illustrating this, Fig. 1 shows two computer-generated pictures of basins of attractions for the Josephson junction model equation, $\ddot{x} + \nu\dot{x} + \omega^2\sin x = f\cos t$, $-\infty < x < +\infty$. As the strength of the forcing $f$ is increased continuously, the basin boundary changes suddenly from being smooth, as in Fig. 1(a), to being fractal, as in Fig. 1(b).

As a specific illustrative numerical example, we shall use the Hénon map. We emphasize, however, that our conclusions apply to a broad class of nonlinear systems (e.g., see Ref. 6). The Hénon map is given by $x_{n+1} = A - x_n^2 - Jy_n$ and $y_{n+1} = x_n$, where $J$ is the determinant of the Jacobian matrix of the map. As the

parameter $A$ increases past the value $A_1 = -(J+1)^2/4$, a saddle-node bifurcation occurs in which one attracting fixed point and one saddle fixed point are created. For $A$ just slightly larger than $A_1$ there are two attractors, one at $(|x|, |y|) = \infty$, and the other the attracting fixed point. (For $A < A_1$, infinity is the only attractor). As $A$ increases past $A_1$, the period-one
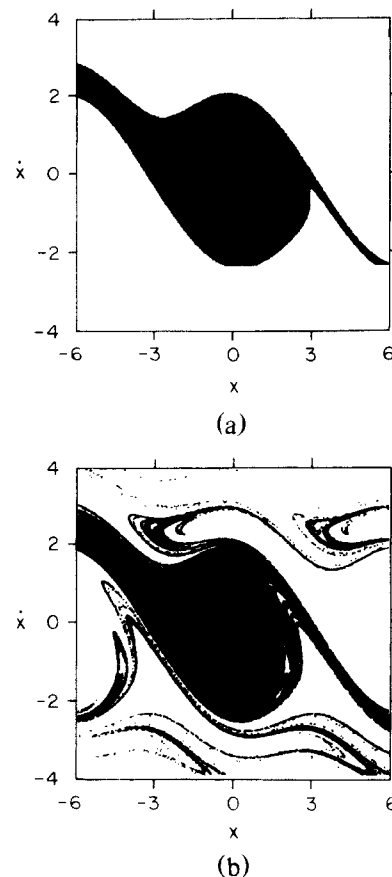


(a)



(b)

FIG. 1. The black region is the basin of attraction for a periodic attractor of the Josephson junction model (also forced damped pendulum), $\ddot{x} + 0.1\dot{x} + \sin x = f\cos t$, $-\infty < x < +\infty$. (a) Smooth basin boundary for $f = 0.3$. (b) Fractal basin boundary for $f = 0.5$.

(i.e., fixed point) attractor moves away from the period-one saddle. The period-one saddle lies on the boundary of the basin of attraction of the period-one attractor. In fact, the stable manifold of the period-one saddle is the basin boundary [cf. Fig. 2(a)]. As $A$ increases further, the period-one attractor undergoes various bifurcations, most prominently a period-doubling cascade. We shall be interested in following the evolution of the basin boundary that is created at $A + A_1$ as $A$ is increased.

Figure 2 shows the basin of attraction of infinity in black. The three panels in this figure correspond to $J = 0.3$ and different values of $A$: $A = 1.150$, 1.395, and 1.405. Starting with a grid of initial conditions, those in the basin of attraction of infinity were determined by seeing which points on the grid yield orbits with large $x$ and $y$ values after a large number of iterations. Those which did are plotted as dots. These dots are so dense that they fill up the black region in the figure. For the case of Fig. 2(a) the boundary of the basin of infinity is apparently a smooth curve (the stable manifold of the period-one saddle). Initial conditions in the white region of Fig. 2(a) generate orbits which generally asymptotically approach the fixed-point attractor labeled in the figure. For the case of Fig. 2(b) the basin boundary shows considerably more structure, and, in fact, magnifications of it reveal that it is fractal (cf. Refs. 2–6 for discussion of fractal basin boundaries). Initial conditions in the white region generate orbits which generally asymptotically approach a period-two attractor (labeled by two dots in the figure) which results from a period-doubling bifurcation of the Fig. 2(a) fixed-point attractor.[7] One of our purposes in this paper will be to describe how a smooth basin boundary, as in Fig. 2(a), can become a fractal basin boundary, as in Fig. 2(b), as a system parameter is varied continuously. Henceforth, we shall denote by $A_{sf}$ the value of $A$ for which the boundary becomes fractal as $A$ increases through $A_{sf}$, and we shall call the accompanying conversion a *smooth-fractal basin-boundary metamorphosis*. (In addition, the basin boundary also suffers a discontinuous jump, in the sense discussed below, as $A$ passes through $A_{sf}$.) For our Hénon map example with $J = 0.3$ (Fig. 2) we find that $A_{sf} \cong 1.315$.

We also find another type of basin-boundary metamorphosis, in which the boundary is fractal both before and after the metamorphosis, but the extent of the basin jumps discontinuously. This is illustrated by a comparison of Figs. 2(b) and 2(c) which have slightly different values of the parameter $A$, and the same $J$ value; $A = 1.395$ in Fig. 2(b) and $A = 1.405$ in Fig. 2(c). Comparing these two figures we see that the basin of attraction of infinity (the black region) has enlarged by the addition of a set of thin filaments, some of which appear well within the interior of the

white region of Fig. 2(b) [in particular, note the region $-1.0 \leq x \leq -0.3$, $2.0 \leq y \leq 5.0$ in Fig. 2(c)]. We call the change exemplified by the transition from Fig.
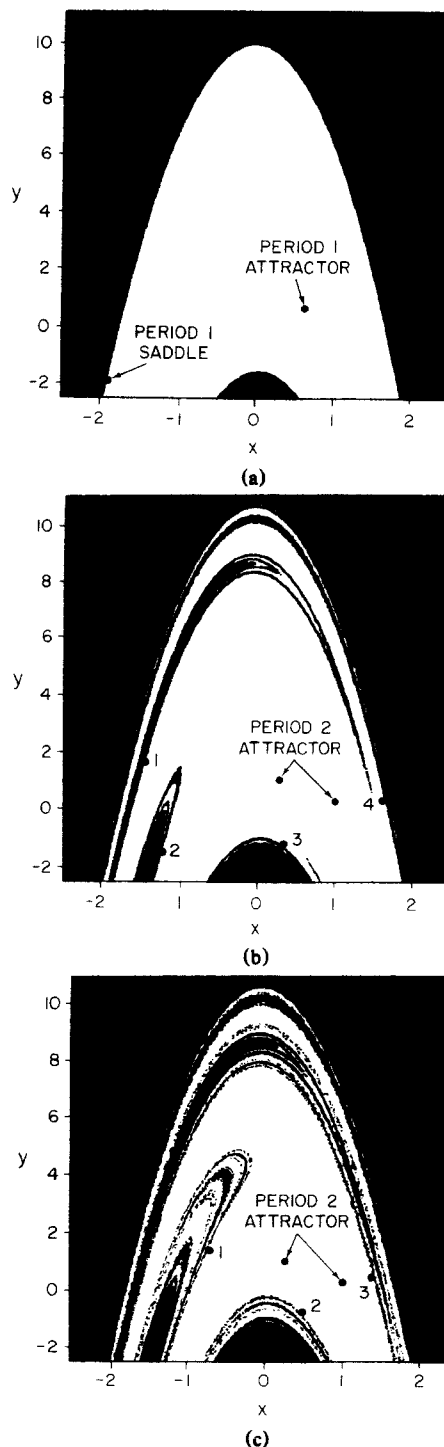


FIG. 2. Basin of attraction of infinity in black for the Hénon map with $J = 0.3$ and (a) $A = 1.150$, (b) $A = 1.395$, and (c) $A = 1.405$.

2(b) to Fig. 2(c) a *fractal-fractal basin-boundary metamorphosis*, and we denote the value of $A$ at which it occurs by $A_{ff}$; we find $A_{ff} \cong 1.396$. As $A$ is decreased toward $A_{ff}$, the Cantor set of thin filaments sent into the old basin become ever thinner, their area in the depicted region going to zero, but they remain essentially fixed in the indicated position, *not* contracting to the position of the basin boundary shown in Fig. 2(b). Hence, the position of the boundary is changing discontinuously at $A_{ff}$, although the *area* of the white and black regions apparently changes continuously.

Both types of metamorphoses are accompanied by a simultaneous change in which saddle periodic orbit on the boundary is accessible from the white basin.

Definition: A boundary point $p$ is accessible from a region if one can construct a finite-length curve connecting $p$ to a point in the interior of the region such that no point of the curve lies in the boundary of the region except for $p$.

As an illustration consider the stable and unstable manifolds of the period-one saddle after they have crossed, $A > A_1^*$, where $A_1^*$ denotes the value of $A$ at which these manifolds become tangent. A schematic illustration is given in Fig. 3. This figure shows that, as a result of the homoclinic intersection, a series of progressively longer and thinner tongues of the basin of the attractor at infinity accumulate on the right-hand side of the stable manifold through the period-one saddle. (Each tongue is the preimage of the preceeding tongue.) A finite-length curve connecting a point in the interior of the white region and the
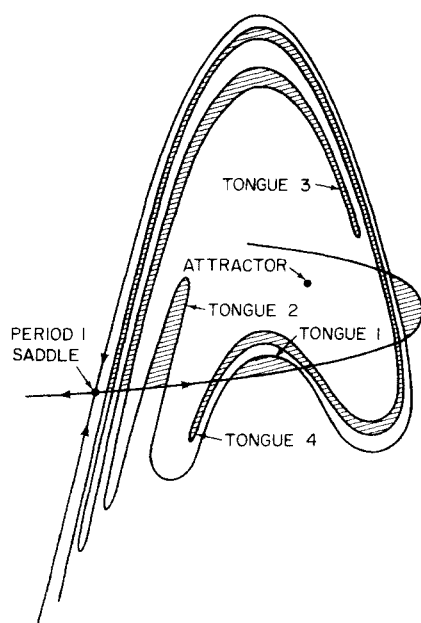
period-one saddle cannot be constructed, since to do this one would have to circumvent all the tongues accumulating on the period-one stable manifold and the length of the $n$th tongue approaches infinity as $n \to \infty$. Thus the period-one saddle is not accessible from the white region. Note, however, that it is accessible from the other side; that is, it is accessible from the black region [see also Fig. 2(b)].

In Fig. 2(b), we have also labeled the elements of a period-four saddle orbit with the numerals 1–4 corresponding to the order in which they cycle $(1 \to 2 \to 3 \to 4 \to 1 \to \ldots)$. Evidently this saddle lies on the basin boundary and is accessible from the white region.[8] We also note that there are tongues accumulating on the stable manifold of the period-four saddle, but they do so on the side away from the finite attractor. Hence, a white region is left open for a curve to join the period-four saddle (and points on its stable manifold) to the interior of the white region. There are also infinitely many other saddle periodic orbits lying in the basin boundary.[6] These, however, are not accessible either from the infinity basin or from the finite attractor because they have tongues of the two basins accumulating on them from *both* sides.[9] (Such saddles might be said to be "buried" in the boundary.)

Thus as $A$ increases past $A_1^*$, the boundary saddle which is accessible from the finite attractor suddenly changes from being the period-one saddle to being the period-four saddle. Both orbits, the period one and the period four, exist before and after this metamorphosis. However, the period-four saddle is in the interior of the white region before $A$ reaches $A_1^*$, and, as $A$ increases past $A_1^*$, *the basin boundary suddenly jumps inward* to the period-four saddle.

The homoclinic crossing of Fig. 3 makes the basin boundary fractal; i.e., $A_1^* = A_{sf}$. This follows from the facts that after such a homoclinic crossing, the intersection of a typical line with the closure of the stable manifold of the period-one saddle is a Cantor set (and hence "fractal"), and that the closure of the period-one stable manifold is identical to the basin boundary.[6] Indeed, the fractal dimension of the boundary for the case shown in Fig. 2(b) has been numerically determined to be $d = 1.530 \pm 0.006$. (We have done this by using the numerical technique of "uncertainty exponent measurement" introduced in Ref. 4.) In the case of the dynamical system corresponding to Fig. 1, the accessible orbit from the white regions in Fig. 1(a) (smooth basin boundary) is a period-one orbit, while for Fig. 1(b) (fractal basin boundary) it is a period-two orbit.

Next consider the transition from Fig. 2(b) to Fig. 2(c). We find that the essential difference between these two figures is again due to a change in accessible orbits. In particular, for the parameters of Fig. 2(b), the accessible saddle is the period four, while for Fig.



FIG. 3. Schematic illustration of tongues accumulating on the period-one-saddle stable manifold.

2(c) the period-four saddle is no longer accessible from the white region (it becomes buried in the boundary), but a period-three saddle is accessible, as indicated in Fig. 2(c) $(1 \to 2 \to 3 \to 1 \to \ldots)$. Furthermore, the transition between the two cases occurs at $A = A_4^*$ when the stable and unstable manifolds of the period-four saddle cross; i.e., $A_4^* = A_{ff}$. Numerically we find $A_{ff} \cong 1.396$.

A very striking consequence of basin-boundary metamorphoses is that they lead to changes in the character of the final crisis which destroys the attractor. In particular, such a crisis is due to the collision of the attractor with an unstable orbit on the basin boundary.[2] The collision can occur only with a boundary orbit which is accessible. Since basin-boundary metamorphoses can change the accessible boundary orbits, they also change the type of crisis.[6]

In this paper we have investigated how basin boundaries in two-dimensional maps change with variation of a system parameter. Our conclusions are as follows: (i) A basin boundary can suddenly jump in size and change its character as a system parameter passes through certain critical values. (We call such changes *basin-boundary metamorphoses*.) (ii) Basin-boundary metamorphoses can occur as a result of homoclinic intersections of the stable and unstable manifolds of a saddle periodic orbit on the basin boundary.[10] (iii) The structure of the basin boundaries which we investigate is, to a large extent, determined by the accessible saddles which lie on the basin boundary. (iv) The character changes, referred to in (i), are changes in the accessible saddle orbits on the basin boundary and sometimes a change of the boundary from being smooth to being fractal [as in the transition from Fig. 1(a) to Fig. 1(b) and from Fig. 2(a) to Fig. 2(b)]. (v) The saddle which a chaotic attractor collides with at its final crisis is a boundary saddle accessible to that attractor.

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[1]In general, the character of the motion of a dynamical system, as determined by the set in phase space to which it tends for long time (the attractor), depends on the initial conditions. The set of initial conditions for which trajectories asymptotically approach a given attractor is its *basin of attraction*. The boundary of such a set is here called the *basin boundary*.

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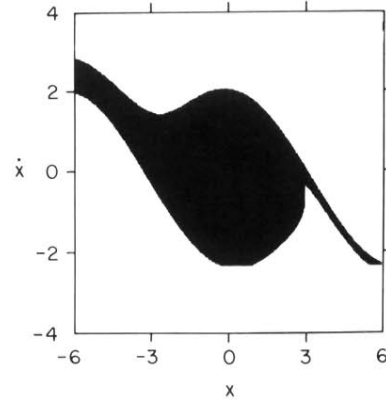
[6]C. Grebogi, E. Ott, and J. A. Yorke, "Basin Boundary Metamorphoses: Changes in Accessible Boundary Orbits" (to be published).

[7]When we use the term "the basin boundary" it should usually be understood to mean the boundary of the basin of the attractor at infinity. Note that saddle-node bifurcations in the white region do occur and can lead to the presence of more than one attractor there. When this happens, then the white region is the union of the basins of the attractors not at infinity. Most commonly, however, additonal attractors created by saddle-node bifurcations only exist over a comparatively small range in $A$ before they are destroyed by a crisis. Thus it is common for the white region to be the basin of a single attractor. We have not observed any saddle-node bifurcations occurring in the black region.
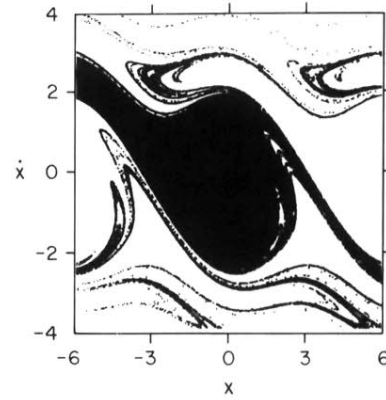
[8]A systematic numerical procedure to find accessible saddles will be given in Ref. 6. Extensive application of this technique yields only one boundary saddle orbit accessible from the white region and one from the black, and we believe that these are the only accessible boundary saddles.

[9]The mechanisms by which an infinity of unstable saddles become inaccessibly "buried" under an infinite number of layers of the boundary will be discussed in Ref. 6.

[10]Structural changes of basin boundaries due to homoclinic intersections have also been discussed by F. C. Moon and G.-X. Li [Phys. Rev. Lett. 55, 1439 (1985)] and by J. Guckenheimer and P. J. Holmes [*Nonlinear Oscillation, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, New York, 1983), p. 114]. Also S. M. Hammel and C. Jones (private communication) have given a mathematical proof of the occurrence of sudden jumps in the basin boundary.
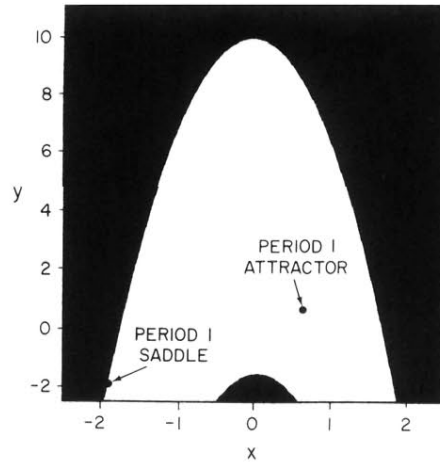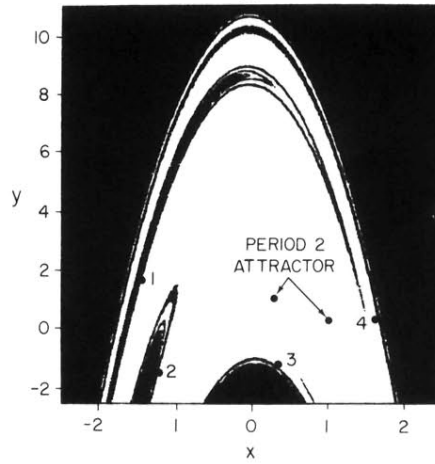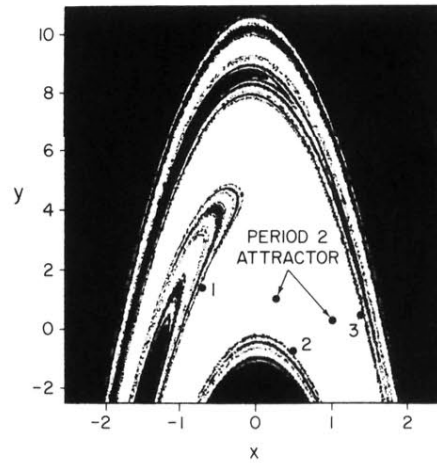
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