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SOME PROBLEMS IN THE THEORY OF NONLINEAR OSCILLATIONS

(Nekotorye Zadachi Teorii Nelineinykh Kolebanii)

By
I. G. Malkin

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BOOK 1

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PREFACE

At the present time the so called "method of small parameter" is widely applied in the theory of nonlinear oscillations. In the present book this method is systematically applied to the solution of a number of problems in the theory of nonlinear oscillations. Herein is set forth the theory of periodic and almost periodic oscillations of quasilinear systems with one and several degrees of freedom, the theory of periodic and almost periodic oscillations of systems approximating arbitrary nonlinear system, and the theory of free and forced oscillations (periodic and almost periodic) of quasiharmonic systems, i.e. systems described by linear differential equations with periodic coefficients. A good deal of attention is at the same time devoted to the question of the stability of the oscillations.

Notwithstanding the rather large size of the book and the wide range of problems which it treats, the book in no wise pretends to an exhaustive presentation of all the problems to which the method of small parameter may successfully be applied (this is reflected in the title of the book). In particular, no consideration has been given to the establishment processes, the slowly varying processes in nonlinear systems and other problems that reduce to finding the solutions of the equations of motion in a finite interval of time.

All the problems considered in this book are set forth with sufficient mathematical rigor. In particular, much attention is given to the proofs of the existence of periodic or almost periodic solutions of the equations of the oscillations. The fundamental goal however which the author set himself is the presentation of practical procedures for computing the oscillations. To this question therefore the book devotes its primary attention. All the methods discussed are accompanied by examples with detailed computations. In the book is also given the solution of a number of concrete physical and technical problems. These too however serve principally for illustrating the computation methods presented.

The author has striven to make the book accessible to as wide a circle of readers as possible, which accounts for the somewhat concentrated method of presentation that is used. In particular, the reading of the first chapter requires a very limited amount of mathematical preparation. In order to give this chapter a completed form there is also presented in it an elementary discussion of the stability problem, the more thorough and detailed presentation of which is found in the third chapter. Similar repetitions are occasionally encountered also in other parts of the book. The author assumes that this shortcoming is compensated by the above indicated considerations of the book's methodical character.

In the writing of the book frequent use was made of the author's earlier published book, "The Methods of Lyapunov and Poincaré in the Theory of Nonlinear Oscillations".

The author desires to express his thanks to S.N. Shimanov and N.G. Bulgakov for the help he has received in reading the proofs.

I. Malkin

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CHAPTER I

QUASILINEAR OSCILLATIONS WITH ONE DEGREE OF FREEDOM

1. Idea of the Method of Poincaré. Small Parameter

We shall consider physical systems described by differential equations of the form

$$\frac{dx_s}{dt} = X_s(t, x_1, \ldots, x_r) \qquad (s = 1, 2, \ldots, r). \tag{1.1}$$

In this chapter we shall assume that if the right hand sides of these equations contain the time t explicitly they are continuous periodic functions of the latter. We shall assume the period of these functions to be equal to 2π, an assumption which can always be satisfied by a suitable choice of the unit of time. We shall not exclude from consideration those cases for which the right hand sides of (1.1) do not contain the time explicitly. In these cases we shall say that the system under consideration is AUTONOMOUS. Systems described by equations containing t explicitly we shall call NONAUTONOMOUS

The problem consists of obtaining the periodic oscillations of the system. To these oscillations correspond the periodic particular solutions of equations (1.1), i.e. such solutions

$$x_s = f_s(t)$$

of these equations for which $f_s(t)$ are periodic functions of t. It is assumed here that the general solution of equations (1.1) is not known.

In order to solve the problem posed it is natural to make use of a device, widely applied in practice, which consists of the following. In the majority of practical problems it is possible to separate out from the right hand sides of the investigated differential equations a certain group of terms which can be considered as small in comparison with the remaining leading terms. Assuming that these terms do not have essential significance for the problem considered (whether it be the problem of periodic solutions or some other problem) they are rejected and in this way a simpler system of equations is obtained. Having solved the problem for the simplified system we either restrict ourselves to the obtained solution or, taking it as the first approximation, we return to the initial equations and apply to them some special method of successive approximations that is specially worked out for the given problem.

This is the device that underlies the method of Poincaré for obtaining periodic solutions. In order to give the problem an exact mathematical formulation Poincaré introduced into consideration the so-called "small parameter". He assumed that the right hand sides of (1.1) depend not only on the variables $t, x_1, \ldots, x_r$, but also on a certain parameter μ, and are analytical functions of this parameter for its sufficiently small values. With this assumption, taking μ to be sufficiently small, equations (1.1) can be represented in the form

$$\frac{dx_s}{dt} = X_s^{(0)}(t, x_1, \ldots, x_r) + \mu X_s^{(1)}(t, x_1, \ldots, x_r) + \mu^2 X_s^{(2)}(t, x_1, \ldots, x_r) + \ldots \quad (s = 1, 2, \ldots, r), \tag{1.2}$$

where the magnitudes $X_s^{(o)}$, $X_s^{(1)}$, ... do not depend on μ. The expressions

$$\mu X_s^{(1)} + \mu^2 X_s^{(2)} + \ldots$$

represent the set of small terms in question. Rejecting these small terms in equations (1.2) we obtain the simplified system of equations

$$\frac{dx_s^0}{dt} = X_s^{(0)}(t, x_1^0, \ldots, x_r^0) \qquad (s = 1, 2, \ldots, r), \tag{1.3}$$

which, in what follows, we shall denote as the GENERATING system. Let

$$x_s^0 = \varphi_s(t) \qquad (s = 1, 2, \ldots, r) \tag{1.4}$$

be some periodic solution of the generating system. If we follow the above-mentioned procedure we shall have to take this solution as the first approximation of the periodic solution of the initial system. It has here been tacitly assumed that to the solution (1.4) there actually corresponds a periodic solution of the complete system (1.2) which differs little from the solution (1.4) for sufficiently small μ. Such assertion will, however, in general be incorrect. Poincaré was evidently the first to point out with sufficient clearness that a change of the right hand sides of the differential equations (in any problem), however small, may lead to sharp qualitative changes in the character of the solutions of these equations. In particular, Poincare showed that a periodic solution of the initial system does not always correspond to the periodic solution of the generating system. It may happen that for the periodic solution of the generating system no periodic solution of the initial system exists, but it may also happen that there may be several such solutions and even an infinite number of them. In other words, it may be the case that the problem of finding periodic solutions for the system (1.2) and the same problem for the system (1.3) do not have anything in common between them. Whence arises the following fundamental problem: to clarify the conditions under which to a given periodic solution of system (1.3) there corresponds one and only one periodic solution of system (1.2). It is evidently of capital interest to investigate also the cases where such unique correspondence between the periodic solutions of systems (1.2) and (1.3) do not exist. It may also be remarked that the question is here about those periodic solutions of system (1.2) which for sufficiently small μ differ little from the corresponding periodic solutions of system (1.3) or, more accurately, about those periodic solutions of system (1.2) which for $\mu = 0$ reduce to the corresponding solution of system (1.3).

It is to the study of the problems posed above to which the theory of Poincaré is devoted. We shall consider first the theory of Poincaré for that particular case for which the generating system (1.3) represents a system of linear equations with constant coefficients. In this particular case the initial system (1.2) is termed QUASILINEAR. The theory of this type of system has at the present time been worked out in detail. Notwithstanding the very special character of these systems they have been obtaining very wide application. With the aid of the theory of these systems it was succeeded in solving a whole series of very important problems.

In this chapter we shall present the theory of quasilinear systems with one degree of freedom. The fundamentals of this theory have been worked out by A.A. Andronov and A.A.Vitt[1]. Quasilinear systems with many degrees of freedom are considered in the following chapter.

2. Oscillations of a Nonautonomous System Not At Resonance

Let us consider the nonlinear system with one degree of freedom which for $\mu = 0$ becomes the usual linear oscillator.

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[1]Andronov A.A and Vitt A.A. K matematicheskoi teorii zakhvatyvaniya (On the Mathematical Theory of Capture), Zhurn. prikl. fiziki, vol. VII, no.4, 1930.

We have in view a system described by an equation of the form

$$\frac{d^2x}{dt^2}+k^2x+f(t)=\mu F(t, x, \dot{x}, \mu). \tag{2.1}$$

where k is a constant magnitude, which we shall assume different from an integer, f(t) is a continuous periodic function of the time of period 2π, that can be expanded into a Fourier series, $F(t, \dot{x}, \mu)$ is an analytic function of the variables x and $dx/dt = \dot{x}$ in a certain region and of the variable μ for its sufficiently small values. With respect to t the function F is continuous and periodic of period 2π, and capable of being expanded into a Fourier series. The parameter μ we shall here and everywhere in what follows assume to be positive, an assumption which evidently does not restrict the generality of the considerations.

The problem consists of obtaining the periodic solution of the equation (2.1). Since the period of the functions f and F with respect to t is equal to 2π, the period of the required periodic solution will either be equal to 2π or will be in a rational ratio with this number. [1] In the present section we shall consider only periodic solutions of period 2π; the periodic solutions of a period commensurable with 2π will be considered below in section 7.

According to the general idea of the method of Poincaré we must first of all find the periodic solution of the GENERATING equation

$$\frac{d^2x_0}{dt^2}+k^2x_0+f(t)=0, \tag{2.2}$$

which is obtained from equation (2.1) if we reject in it the terms containing μ.

Let

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt) \tag{2.3}$$

be the Fourier development of the function f(t). Since by assumption k differs from an integer, the generating equation (2.2) has a periodic solution of period 2π. This solution corresponds to the forced oscillations of the linear system determined by equation (2.2) and is computed by the well known formula

$$x_0=-\frac{a_0}{2k^2}-\sum_{n=1}^{\infty}\frac{a_n\cos nt+b_n\sin nt}{k^2-n^2}=\varphi(t). \tag{2.4}$$

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[1] In exceptional cases equation (2.1) can have periodic solutions with a period which is not commensurable with 2π. These solutions are not of interest to us and we shall not concern ourselves with them.

The solution (2.4) is the only periodic solution of equation (2.2) of period 2π. In fact, the general solution

$$x_0 = \varphi(t) + A\sin(kt + \alpha)$$

of this equation is composed of two purely periodic components of which one has the period 2π and the other the period $2\pi/k$. This solution will evidently be periodic of period 2π only in the case where k is an integer, which contradicts the condition.

The periodic solution (2.4) of the generating equation (2.2) we shall in what follows denote as the GENERATING solution. We must now establish under what conditions periodic solutions of the initial equation (2.1) exist for $\mu = 0$ and indicate methods for obtaining these solutions

For this purpose we denote by β_1 and β_2 deviations of the initial values of the magnitudes x and $\dot{x}$ in the required periodic solution of equation (2.1) from the initial values of these same magnitudes in the generating solution. This solution itself we shall denote by $x(t, \beta_1, \beta_2, \mu)$. We thus have by assumption

$$\left.\begin{aligned} x(0, \beta_1, \beta_2, \mu) &= \varphi(0) + \beta_1, \\ \dot{x}(0, \beta_1, \beta_2, \mu) &= \dot{\varphi}(0) + \beta_2. \end{aligned}\right\} \tag{2.5}$$

In order that the solution $x(t, \beta_1, \beta_2, \mu)$ be periodic of period 2π it is necessary, evidently, that the conditions be satisfied

$$\left.\begin{aligned} \psi_1(\beta_1, \beta_2, \mu) &= x(2\pi, \beta_1, \beta_2, \mu) - x(0, \beta_1, \beta_2, \mu) = 0, \\ \psi_2(\beta_1, \beta_2, \mu) &= \dot{x}(2\pi, \beta_1, \beta_2, \mu) - \dot{x}(0, \beta_1, \beta_2, \mu) = 0. \end{aligned}\right\} \tag{2.6}$$

Conversely, if conditions (2.6) are satisfied the solution $x(t, \beta_1, \beta_2, \mu)$ will be periodic. In fact, on satisfying these conditions the values of the magnitudes x and $\dot{x}$ at the instant $t = 2\pi$ will be the same at the initial instant of time. On the other hand, equation (2.1) by assumption does not change with change of t by $t + 2\pi$, and since the function x is completely determined by the initial values of x and $\dot{x}$, if we take for the initial instant of time $t = 0$ the value 2π we satisfy ourselves immediately that on the segment $[2\pi, 4\pi]$ the function x assumes the same values as on the segment $[0, 2\pi]$. Hence x will in fact be a periodic function of period 2π.

Equations (2.6) thus express the necessary and sufficient conditions for the periodicity of the solution $x(t, \beta_1, \beta_2, \mu)$. These equations serve to determine the unkown initial values of this solution and the question as to the existence of a periodic solution of equation (2.1) reduces to the question of the solvability of equations (2.6) with respect to β_1 and β_2. It is necessary here that β_1 and β_2, become zero for $\mu = 0$ since the required periodic solution must become the generating solution for $\mu = 0$.

Let us consider equation (2.6) in more detail. We shall assume that the region in which by assumption the function F is analytical with respect to x and $\dot{x}$ contains the generating solution (2.4). With this condition, as is shown in the theory of differential equations,[1] the function $x(t, \beta_1, \beta_2, \mu)$ will be analytic with respect to β_1, β_2 and μ. Since for $\beta_1 = \beta_2 = \mu = 0$ this function is to become the generating solution $\varphi(t)$, we can write

$$x(t, \beta_1, \beta_2, \mu) = \varphi(t) + A\beta_1 + B\beta_2 + C\mu + \ldots, \tag{2.7}$$

where A, B, C are unkown functions of time. To find these functions we substitute the series (2.7) in equation (2.1) and equate the coefficients of equal powers of the magnitudes β_1, β_2, μ. Equating the coefficients of the first powers of β_1 and β_2 we obtain

$$\frac{d^2A}{dt^2} + k^2A = 0, \qquad \frac{d^2B}{dt^2} + k^2B = 0. \tag{2.8}$$

Moreover, the initial conditions (2.5) give:

$$A(0) = 1, \quad \dot{A}(0) = 0, \quad B(0) = 0, \quad \dot{B}(0) = 1. \tag{2.9}$$

From (2.8) and (2.9) we obtain

$$A = \cos kt, \quad B = \frac{1}{k}\sin kt.$$

Hence equations (2.6) have the form

$$\left.\begin{aligned} &\psi_1(\beta_1, \beta_2, \mu) = [x] = \\ &\quad = (\cos 2k\pi - 1)\beta_1 + \frac{1}{k}\sin 2k\pi \cdot \beta_2 + [C]\mu + \ldots = 0, \\ &\psi_2(\beta_1, \beta_2, \mu) = [\dot{x}] = \\ &\quad = -k\sin 2k\pi \cdot \beta_1 + (\cos 2k\pi - 1)\beta_2 + [\dot{C}]\mu + \ldots = 0. \end{aligned}\right\} \tag{2.10}$$

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[1] Goursat, E. A Course in Mathematical Analysis, vol. III, Part I, Chap. XXIII.

Here, as in what follows, we make use of the notation

$$[F(t)] = F(2\pi) - F(0).$$

Equations (2.10) are identically satisfied for $\beta_1 = \beta_2 = \mu = 0$. Moreover, the functional determinant (or Jacobian) of the left sides of these equations with respect to β_1 and β_2 for $\beta_1 = \beta_2 = \mu = 0$ is different from zero.

In fact, for this determinant we have:

$$\left\{\frac{\partial(\psi_1, \psi_2)}{\partial(\beta_1, \beta_2)}\right\}_{\beta_1=\beta_2=\mu=0} = (\cos 2k\pi - 1)^2 + \sin^2 2k\pi \neq 0.$$

Hence, on the basis of the well known theorem on the existence of implicit functions, equations (2.10) have, for sufficiently small μ, one and only one solution $\beta_i = \beta_i(\mu)$ $(i=1,2)$, for which $\beta_i(0) = 0$, and this solution will be analytic with respect to μ. Substituting this solution in the function $x(t, \beta_1, \beta_2, \mu)$ we obtain one and only one periodic solution of the equation (2.1) which reduces to the generating solution for $\mu=0$, and this solution is evidently analytic with respect to μ.

Thus, FOR THE NONRESONANCE CASE CONSIDERED THERE EXISTS ONE AND ONLY ONE PERIODIC SOLUTION OF THE EQUATION (2.1) WHICH REDUCES TO THE GENERATING SOLUTION FOR $\mu=0$, AND THIS SOLUTION IS ANALYTIC WITH RESPECT TO μ.

We now proceed to the question of computing the required periodic solution. Since, by what has been proved, it is analytic with respect to μ, it has the form

$$x = \varphi(t) + x_1(t)\mu + x_2(t)\mu^2 + \ldots, \tag{2.11}$$

where $x_i(t)$ are certain unkown periodic functions of period 2π. To find these functions we substitute the series (2.11) in equation (2.1) and equate the coefficients of the same powers of μ. Then, as can easily be seen, there is obtained for x_1 the equation

$$\frac{d^2x_1}{dt^2} + k^2x_1 = F_1 = F(t, \varphi, \dot{\varphi}, 0), \tag{2.12}$$

and for $x_i(t)$ ($i > 1$) an equation of the form

$$\frac{d^2x_i}{dt^2}+k^2x_i=F_i, \tag{2.13}$$

where F_i are integral rational functions of the magnitudes φ, $x_1(t),\ldots,x_{i-1}(t)$ with periodic coefficients. The computation of the functions x_1, $x_2,\ldots$ thus reduces to the computation of the particular periodic solutions of equations (2.12) and(2.13).

Let us assume that all x_1, $x_2,\ldots,x_{i-1}$ have already been computed and are found to be periodic. Then F_i will be known as a periodic function of time, and since k is different from an integer, there exists one and only one periodic solution

$$x_i=\frac{a_{0i}}{2k^2}+\sum_{n=1}^{\infty}\frac{a_{ni}\cos nt+b_{ni}\sin nt}{k^2-n^2} \tag{2.14}$$

of the equation for x_i. Here a_{ni} and b_{ni} are the Fourier coefficients of the functions F_i. Since the function F_1 is known, we arrive at the conclusion that there exists one and only one series (2.11) with periodic coefficients that formally satisfies the equation (2.1). Consequently, this series represents the required periodic solution and it will therefore converge for sufficiently small μ.

We have assumed that k differs from an integer. If k were equal to an integer then both the generation solution (2.4) and the solutions (2.14) for x_i would lose all meaning due to the appearance of zeros in the denominators. An exception will be only that special case where not only the function f(t) but also all functions F_i do not contain the k-th harmonic in their expansions. Leaving this special case aside, we see that for k equal to an integer the solution (2.11) constructed by us ceases to exist.

We shall now assume that k is not equal to an integer but differs little from it. In this case in the generating solution and in the functions x_i terms appear with numerically very large coefficients. Since F_i are integral functions of the already earlier computed magnitudes x_j, these terms will appear in the expansion (2.11) in all higher powers. Nonetheless, the series (2.11) will converge for sufficiently small values of μ, that is, however little k differs from an integer there will always be found such small value of μ for which this series converges. However, in each physical problem the magnitude μ has an entirely definite, although possibly small value, and however small this value may be there will always be found for k such a sufficiently small neighborhood of an integer that the series (2.11) will be found to diverge.

Thus, the case where k differs little from an integer, like the case where k is an integer, requires special investigation. We shall denote such cases as RESONANCE cases. The solution (2.11) is valid only for cases far removed from resonance.

3. Oscillations of a Nonautonomous System at Resonance. Conditions for the Existence of a Periodic Solution.

Let us consider the system described by equation (2.1) for the existence of resonance, i.e. we shall assume that k is either an integer n or differs little from it. We shall assume that the 'mistuning' $n^2 - k^2$ is of the order of smallness of μ and set:

$$n^2 - k^2 = \mu a,$$

where a is a finite magnitude. Moreover, we shall assume that the coefficients of the n-th harmonic in the expansion of the function f(t) likewise is of the order of smallness of μ and set

$$a_n = \mu a'_n, \quad b_n = \mu b'_n.$$

Then, adding the terms

$$\mu a x, \; \mu (a'_n \cos nt + b'_n \sin nt)$$

to the function $\mu F(t, x, \dot{x}, \mu)$ and retaining for the new function thus obtained the initial designation, we can write equation (2.1) in the following form

$$\frac{d^2x}{dt^2} + n^2 x + f'(t) = \mu F(t, x, \dot{x}, \mu), \tag{3.1}$$

where

$$f'(t) = f(t) - a_n \cos nt - b_n \sin nt = \frac{a_0}{2} + \sum_{j \neq n} (a_j \cos jt + b_j \sin jt).$$

Let us consider the generating equation

$$\frac{d^2x_0}{dt^2} + n^2x_0 + f'(t) = 0. \tag{3.2}$$

The general solution of this equation has the form

$$x_0 = -\frac{a_0}{2n^2} - \sum_{j \mp n} \frac{a_j \cos jt + b_j \sin jt}{n^2 - j^2} + M_0 \cos nt + N_0 \sin nt =$$
$$= \varphi(t) + M_0 \cos nt + N_0 \sin nt, \tag{3.3}$$

where M_0 and N_0 are arbitrary constants. This solution will be periodic of the period 2π for any values of M_0 and N_0. Hence, in the resonance case considered the generating equation admits a family of periodic solutions depending on two arbitrary constants. In this respect the resonance solution differs sharply from the nonresonance solution, for which the generating equation admitted only one isolated periodic solution.

As the generating solution we shall take one of the periodic solutions of the family (3.3) corresponding to some fixed values of the constants M_0 and N_0 and investigate under what conditions the initial equation (3.1) admits a periodic solution that reduces to the generating solution for $\mu = 0$. Proceeding in the same manner as in the resonance case we denote the required periodic solution by $x(t, \beta_1, \beta_2, \mu)$, assuming the following initial conditions:

$$\left.\begin{aligned} x(0, \beta_1, \beta_2, \mu) &= x_0(0) + \beta_1, \\ \dot{x}(0, \beta_1, \beta_2, \mu) &= \dot{x}_0(0) + \beta_2. \end{aligned}\right\} \tag{3.4}$$

We shall here have:

$$x(t, \beta_1, \beta_2, \mu) =$$
$$= x_0(t) + A\beta_1 + B\beta_2 + C\mu + \mu(D\beta_1 + E\beta_2 + F\mu) + \dots \tag{3.5}$$

We write out in the expression $x(t, \beta_1, \beta_2, \mu)$ also the terms of the second order since we require them for what follows. The set of these terms, like the terms of higher orders, must reduce to zero for $\mu = 0$, since the expression (3.5) then reduces to the solution of the generating equation which, being linear, must contain the initial values only linearly.

In order that the solution (3.5) be periodic it is necessary and sufficient that the magnitudes β_1 and β_2 satisfy the following equations:

$$\left.\begin{aligned}\psi_1(\beta_1, \beta_2, \mu) = x(2\pi, \beta_1, \beta_2, \mu) - x(0, \beta_1, \beta_2, \mu) = 0,\\ \psi_2(\beta_1, \beta_2, \mu) = \dot{x}(2\pi, \beta_1, \beta_2, \mu) - \dot{x}(0, \beta_1, \beta_2, \mu) = 0.\end{aligned}\right\} \qquad (3.6)$$

As in the preceding section, for the coefficients A and B we obtain the differential equations

$$\frac{d^2A}{dt^2} + n^2A = 0, \qquad \frac{d^2B}{dt^2} + n^2B = 0$$

and the initial conditions

$$A(0) = 1, \qquad \dot{A}(0) = 0, \qquad B(0) = 0, \qquad \dot{B}(0) = 1,$$

whence we find

$$A = \cos nt, \quad B = \frac{1}{n}\sin nt.$$

Hence

$$[A] = [\dot{A}] = [B] = [\dot{B}] \equiv 0,$$

and equations (3.6) on the basis of (3.5) assume the form

$$\left.\begin{aligned}\psi_1(\beta_1, \beta_2, \mu) = \mu\{[C] + [D]\beta_1 + [E]\beta_2 + [F]\mu + \dots\} = 0,\\ \psi_2(\beta_1, \beta_2, \mu) = \mu\{[\dot{C}] + [\dot{D}]\beta_1 + [\dot{E}]\beta_2 + [\dot{F}]\mu + \dots\} = 0.\end{aligned}\right\} \quad (3.7)$$

The problem thus reduces to the determination of the conditions under which equations (3.7) can be solved for β_1 and β_2. It is here a question of a solution of these equations for which the magnitudes β_1 and β_2 become zero for $\mu = 0$, since for $\mu = 0$ the function (3.5) must reduce to the generating solution $x_0(t)$.

Equations (3.7), in contrast to the corresponding equations in the nonresonance case, do not have linear terms with β_1 and β_2. As a result, the functional determinant of these equations with respect to β_1 and β_2 becomes zero for $\beta_1 = \beta_2 = \mu = 0$, and the question of the solubility of these equations requires special investigation.

We can, first of all, reduce these equations by μ. But then the new equations obtained in this way will not be satisfied for $\beta_1 = \beta_2 = \mu = 0$ on account of the presence of the free terms $[C]$ and $[\dot{C}]$ in them. Hence, in order that for the solutions $\beta_i(\mu)$ of these equations our required conditions $\beta_i(0) = 0$ be satisfied, it is necessary first of all to require that the above-mentioned free terms reduce to zero. We thus obtain the two following conditions:

$$[C] = 0, \quad [\dot{C}] = 0. \tag{3.8}$$

Let us examine these conditions more closely. For this purpose let us find the coefficient C. To do this we substitute the solution (3.5) in equation (3.1) and equate the coefficients of μ to the first power. We thus obtain:

$$\frac{d^2C}{dt^2} + n^2C = F(t, x_0, \dot{x}_0, 0). \tag{3.9}$$

Moreover, the initial conditions (3.4) give

$$C(0) = \dot{C}(0) = 0. \tag{3.10}$$

The solution of equation (3.9) for the initial conditions (3.10) is, as is known, of the form

$$C = \frac{1}{n} \int_0^t F(\tau, x_0(\tau), \dot{x}_0(\tau), 0) \sin n(t - \tau)\, d\tau,$$

whence

$$\dot{C} = \int_0^t F(\tau, x_0(\tau), \dot{x}_0(\tau), 0) \cos n(t - \tau)\, d\tau.$$

Substituting in (3.10) and taking (3.3) into account we find that for the equations (3.7) to have the form of solutions we require it is necessary that the two following conditions be satisfied:

$$\left.\begin{aligned} P(M_0, N_0) = \int_0^{2\pi} F(\tau, M_0 \cos n\tau + N_0 \sin n\tau + \varphi(\tau),& \\ -M_0 n \sin n\tau + N_0 n \cos n\tau + \dot{\varphi}(\tau), 0) \sin n\tau\, d\tau = 0,& \\ Q(M_0, N_0) = \int_0^{2\pi} F(\tau, M_0 \cos n\tau + N_0 \sin n\tau + \varphi(\tau).& \\ -M_0 n \sin n\tau + N_0 n \cos n\tau + \dot{\varphi}(\tau), 0) \cos n\tau\, d\tau = 0.& \end{aligned}\right\} \tag{3.11}$$

These conditions are not satisfied, generally speaking, for any values of M_o and N_o, but are equations serving to determine these magnitudes. Consequently there will not, to any periodic solution of the generating equation (3.2), correspond a periodic solution of the initial equation (3.1). Such solutions can correspond only to those generating solutions for which the constants M_o and N_o satisfy equations (3.11).

Let us assume however that the constants M_o and N_o in the generating solution are actually chosen according to conditions (3.11). Then equations (3.7) assume the form

$$\left.\begin{aligned} \psi_1' &= [D]\beta_1 + [E]\beta_2 + [F]\mu + \cdots = 0, \\ \psi_2' &= [\dot{D}]\beta_1 + [\dot{E}]\beta_2 + [\dot{F}]\mu + \cdots = 0. \end{aligned}\right\} \qquad (3.12)$$

For the functional determinant of these equations with respect to β_1 and β_2 for $\beta_1 = \beta_2 = \mu = 0$ we obtain the expression

$$= \left\{ \frac{\partial(\psi_1', \psi_2')}{\partial(\beta_1, \beta_2)} \right\}_{\beta_1=\beta_2=\mu=0} = \begin{vmatrix} [D] & [E] \\ [\dot{D}] & [\dot{E}] \end{vmatrix}. \qquad (3.13)$$

If this expression is different from zero, then on the basis of the theorem on implicit functions equations (3.12), and therefore also equations (3.7), have one and only one solution $\beta_i = \beta_i(\mu)$ for which $\beta_i(0) = 0$, and this solution will be analytic with respect to μ. Substituting this solution in the function $x(t, \beta_1, \beta_2, \mu)$ we obtain a periodic solution of equation (3.1), and this solution is analytic with respect to μ. Thus, for each generating solution for which M_o and N_o satisfy equations (3.11) and for which the magnitude (3.13) is different from zero there exists one and only one periodic solution of the initial equation (3.1) and this solution will be analytic with respect to μ.

Let us now consider the computation of the magnitude (3.13). For this we must determine the coefficients D and E in the expansion (3.5). To determine these coefficients we substitute the series (3.5) in equation (3.1) and equate in both sides of this equation the coefficients $\beta_1\mu$ and $\beta_2\mu$.

Let $F^*(t, \beta_1, \beta_2, \mu)$ denote the expression to which the function $F(t, x, \dot{x}, \mu)$ reduces after replacing the magnitude x by the series (3.5). Expanding this expression into a Maclaurin series we can write:

$$F^*(t, \beta_1, \beta_2, \mu) =$$
$$= F^*(t, 0, 0, 0) + \left(\frac{\partial F^*}{\partial \beta_1}\right)\beta_1 + \left(\frac{\partial F^*}{\partial \beta_2}\right)\beta_2 + \left(\frac{\partial F^*}{\partial \mu}\right)\mu + \ldots =$$
$$= F(t, x_0, \dot{x}_0, 0) + \left\{\left(\frac{\partial F}{\partial x}\right)A + \left(\frac{\partial F}{\partial \dot{x}}\right)\dot{A}\right\}\beta_1 +$$
$$+ \left\{\left(\frac{\partial F}{\partial x}\right)B + \left(\frac{\partial F}{\partial \dot{x}}\right)\dot{B}\right\}\beta_2 + \left\{\left(\frac{\partial F}{\partial x}\right)C + \left(\frac{\partial F}{\partial \dot{x}}\right)\dot{C} + \left(\frac{\partial F}{\partial \mu}\right)\right\}\mu + \ldots, \tag{3.14}$$

where the parentheses denote that the derivatives are computed for the generating solution, i.e. that after differentiation it is necessary in the derivatives of the function F^* to put $\beta_1 = \beta_2 = \mu = 0$, and in the derivatives of the function F to put $x = x_0$, $\mu = 0$.

Using (3.14) we readily find that the coefficients D and E satisfy the differential equations

$$\frac{d^2D}{dt^2} + n^2D = \left(\frac{\partial F}{\partial x}\right)A + \left(\frac{\partial F}{\partial \dot{x}}\right)\dot{A} = \left(\frac{\partial F}{\partial x}\right)\cos nt - n\left(\frac{\partial F}{\partial \dot{x}}\right)\sin nt,$$
$$\frac{d^2E}{dt^2} + n^2E = \left(\frac{\partial F}{\partial x}\right)B + \left(\frac{\partial F}{\partial \dot{x}}\right)\dot{B} = \left(\frac{\partial F}{\partial x}\right)\frac{1}{n}\sin nt + \left(\frac{\partial F}{\partial \dot{x}}\right)\cos nt.$$

For the initial values of these coefficients we obtain from (3.4)

$$D(0) = \dot{D}(0) = E(0) = \dot{E}(0) = 0.$$

Whence we find

$$D = \frac{1}{n}\int_0^t \left\{\left(\frac{\partial F}{\partial x}\right)_{t=\tau}\cos n\tau - n\left(\frac{\partial F}{\partial \dot{x}}\right)_{t=\tau}\sin n\tau\right\}\sin n(t-\tau)\,d\tau,$$
$$\dot{D} = \int_0^t \left\{\left(\frac{\partial F}{\partial x}\right)_{t=\tau}\cos n\tau - n\left(\frac{\partial F}{\partial \dot{x}}\right)_{t=\tau}\sin n\tau\right\}\cos n(t-\tau)\,d\tau$$

and therefore

$$[D] = -\frac{1}{n}\int_0^{2\pi} \left\{\left(\frac{\partial F}{\partial x}\right)_{t=\tau}\cos n\tau - n\left(\frac{\partial F}{\partial \dot{x}}\right)_{t=\tau}\sin n\tau\right\}\sin n\tau\,d\tau,$$
$$[\dot{D}] = \int_0^{2\pi} \left\{\left(\frac{\partial F}{\partial x}\right)_{t=\tau}\cos n\tau - n\left(\frac{\partial F}{\partial \dot{x}}\right)_{t=\tau}\sin n\tau\right\}\cos n\tau\,d\tau.$$

Comparing with (3.11) we readily satisfy ourselves that

$$[D] = -\frac{1}{n}\frac{\partial P}{\partial M_0}, \quad [\dot{D}] = \frac{\partial Q}{\partial M_0}. \tag{3.15}$$

Similarly we obtain

$$[E] = -\frac{1}{n^2}\frac{\partial P}{\partial N_0}, \quad [\dot{E}] = \frac{1}{n}\frac{\partial Q}{\partial N_0} \tag{3.16}$$

and therefore

$$\begin{vmatrix} [D] & [\dot{D}] \\ [E] & [\dot{E}] \end{vmatrix} = -\frac{1}{n^2}\frac{\partial(P, Q)}{\partial(M_0, N_0)}.$$

Thus, the condition that the magnitude (3.13) does not reduce to zero is equivalent to the condition that the functional determinant of equations (3.11) with respect to the unkowns M_o and N_o does not reduce to zero.

Let the m magnitudes $x_1, \ldots, x_m$ satisfy the m equations

$$f_i(x_1, \ldots, x_m) = 0 \qquad (i = 1, \ldots, m). \tag{3.17}$$

By analogy with the case of a single equation with a single unkown we shall say that the magnitudes $m_1, \ldots x_m$ form a simple solution of equations (3.17) if they satisfy these equations and if at the same time the conditions are satisfied

$$\frac{\partial(f_1, \ldots, f_m)}{\partial(x_1, \ldots, x_m)} \neq 0.$$

The results we have obtained can be formulated in the form of the following fundamental proposition:

IN ORDER THAT THERE CORRESPOND TO THE GENERATING SOLUTION (3.3) A SOLUTION OF THE INITIAL EQUATION (3.1) IT IS NECESSARY THAT THE CONSTANTS M_o AND N_o SATISFY EQUATIONS (3.11). TO EACH SIMPLE SOLUTION OF THESE EQUATIONS THERE ACTUALLY CORRESPONDS FOR μ SUFFICIENTLY SMALL A PERIODIC SOLUTION OF EQUATION (3.1), AND THIS SOLUTION WILL BE ANALYTIC WITH RESPECT TO μ.

From this it is seen that the resonance case differs sharply from the case of nonresonance. In the nonresonance case both the generating equation and the initial equation have one and only one periodic solution. Hence, in the nonresonance case there exists a complete correspondence between the initial and simplified equation. In the resonance case , however, no matter how small the parameter μ may be, that is, however little the initial and simplified equations differ from each other, there exists a sharp qualitative difference between them. Whereas the generating equation possesses an infinite number of periodic solutions forming a continuous family, the initial equation possesses in general only a finite number of periodic solutions. As we shall see below, we shall always encounter this circumstance each time that the generating function admits a family of periodic solutions that depend on one or several parameters.

4. Oscillations of a Nonautonomous System at Resonance. Computation of Periodic Solution

We shall now show how practically to find the periodic solution of equation (3.1) the existence of which was established in the preceding section.

Thus, let us assume that in the generating solution (3.3) the constants M_0 and N_0 satisfy the equations (3.11) and are simple solutions of these equations. Then, as was shown, the required periodic solution of the initial equation (3.1) will exist and will be analytic with respect to μ. We can therefore seek to obtain this solution in the form of the formal series

$$x = x_0 + \mu x_1 + \mu^2 x_2 + \ldots, \tag{4.1}$$

where x_0 is the generating solution and x_1, x_2,... are certain unkown periodic functions of t of period 2π. To determine these functions we substitute the series (4.1) in equation (3.1) and equate the coefficients of like powers of μ. We first of all obtain

$$\frac{d^2 x_1}{dt^2} + n^2 x_1 = F(t, x_0, \dot{x}_0, 0). \tag{4.2}$$

Since n is an integer, equation (4.2), in contrast to the corresponding equation in the nonresonance case, either has no periodic solutions at all or, on the contrary, all its solutions are periodic of the period 2π. In fact, let

$$F(t, x_0, \dot{x}_0, 0) = \frac{a_{01}}{2} + \sum_{m=1}^{\infty} (a_{m1} \cos mt + b_{m1} \sin mt) \qquad (4.3)$$

be the Fourier expansion of the right hand side of equation (4.2). To each component of this expansion there corresponds a periodic component in the solution of equation (4.2) of the form

$$\frac{a_{m1} \cos mt + b_{m1} \sin mt}{n^2 - m^2}.$$

An exception is formed only by the component containing the n-th harmonic. To this component in the solution of equation (4.2) there corresponds the periodic term

$$\frac{t}{2n} (a_{n1} \sin nt - b_{n1} \cos nt).$$

Hence, in order that equation (4.2) admit a periodic solution it is necessary and sufficient that the equations be satisfied

$$a_{n1} = b_{n1} = 0,$$

or, in developed form,

$$\int_0^{2\pi} F(t, x_0, \dot{x}_0, 0) \cos nt\, dt = \int_0^{2\pi} F(t, x_0, \dot{x}_0, 0) \sin nt\, dt = 0. \quad (4.4)$$

Equations (4.4) agree with equations (3.11) which according to the choice of M_0 and N_0 are satisfied. Hence equation (4.2) for x_1 admits a particular periodic solution. But then also the general solution of this equation, determined by the formula

$$x_1 = \frac{a_{01}}{2n^2} + \sum_{m \neq n} \frac{a_{m1} \cos mt + b_{m1} \sin mt}{n^2 - m^2} + M_1 \cos nt + N_1 \sin nt,$$

where M_1 and N_1 are arbitrary constants, will likewise be periodic of period 2π.

We now proceed to the determination of x_2. For this magnitude there is obtained an equation of the type (4.2) but with a different expression on the right hand side. This expression will depend on x_1 and therefore will contain the arbitrary constants M_1 and N_1. Like equation (4.2), the equation for x_2 will admit a periodic solution only in the case that the coefficients of cos nt and sin nt in the Fourier expansion of its right hand side are equal to zero. This may be obtained by a suitable choice of the constants M_1 and N_1. We shall show how this can be done.

The equation for x_i ($i = 2,3,\ldots$) is of the form

$$\frac{d^2x_i}{dt^2} + n^2x_i = \left(\frac{\partial F}{\partial x}\right)x_{i-1} + \left(\frac{\partial F}{\partial \dot{x}}\right)\dot{x}_{i-1} + F_i, \qquad (4.5)$$

where the parentheses have the same meaning as in (3.14), that is, they indicate that the derivatives are computed for the generating solution, and F_i is an integral rational function of $x_o, x_1, \ldots, x_{i-2}$. We explicitly wrote out the terms containing x_{i-1} since they are required for what follows. Let us assume that all the functions $x_1, \ldots, x_{i-1}$ have already been computed and came out periodic. These functions are of the form

$$x_j = \varphi_j(t) + M_j \cos nt + N_j \sin nt \qquad (j = 1, 2, \ldots, i-1),$$

where φ_j are certain periodic functions of the time and M_j and N_j are constants. We shall assume that the constants $M_1, \ldots, M_{i-2}, N_1, \ldots, N_{i-2}$ have already been determined while the constants M_{i-1} and N_{i-1} still remain undetermined. Then, equating to zero the coefficients of cos nt and sin nt in the Fourier expansion of the right hand side of equation (4.5), we obtain the two equations:

$$\left.\begin{aligned}
&M_{i-1}\int_0^{2\pi}\left\{\left(\frac{\partial F}{\partial x}\right)\cos nt-n\left(\frac{\partial F}{\partial \dot{x}}\right)\sin nt\right\}\cos nt\,dt+\\
&+N_{i-1}\int_0^{2\pi}\left\{\left(\frac{\partial F}{\partial x}\right)\sin nt+n\left(\frac{\partial F}{\partial \dot{x}}\right)\cos nt\right\}\cos nt\,dt+\\
&\qquad\qquad+\int_0^{2\pi}F_i^*\cos nt\,dt=0,\\
&M_{i1}\int_0^{2\pi}\left\{\left(\frac{\partial F}{\partial x}\right)\cos nt-n\left(\frac{\partial F}{\partial \dot{x}}\right)\sin nt\right\}\sin nt\,dt+\\
&+N_{i-1}\int_0^{2\pi}\left\{\left(\frac{\partial F}{\partial x}\right)\sin nt+n\left(\frac{\partial F}{\partial \dot{x}}\right)\cos nt\right\}\sin nt\,dt+\\
&\qquad\qquad+\int_0^{2\pi}F_i^*\sin nt\,dt=0,
\end{aligned}\right\}\quad(4.6)$$

where

$$F_i^*=F_i+\left(\frac{\partial F}{\partial x}\right)\varphi_{i-1}+\left(\frac{\partial F}{\partial \dot{x}}\right)\dot{\varphi}_{i-1}$$

is a known periodic function of the time. Equations (4.6) expressing the necessary and sufficient conditions for the existence of periodic solutions for x_i determine the constants M_{i-1} and N_{i-1}. These equations are linear and their determinant agrees with the magnitude

$$\frac{\partial(P,Q)}{\partial(M_0,N_0)}.$$

Since the latter is by assumption different from zero, equations (4.6) are always solvable and have a single solution. Thus, for each set of values of the magnitudes M_0 and N_0 forming a simple solution of equations (3.11) there exists only one series (4.1) with periodic coefficients that formally satisfies equation (3.1). This series therefore represents the required periodic solution and converges for sufficiently small μ.

5. Application to the Theory of a Regenerative Receiver[1]

As a first example let us consider the oscillations of a regenerative receiver the circuit diagram of which is shown in fig. 1. Let an external electromotive force $E = P\sin\omega_1 t$ act on its grid circuit. The equation of the oscillations of such receiver, if the grid current is neglected, is of the form

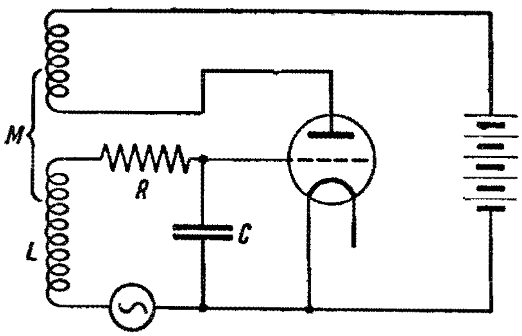

Fig. 1

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt - M\frac{di_a}{dt} = P\sin\omega_1 t, \quad (5.1)$$

where i is the current strength in the grid circuit, i_a the current strength in the anode circuit.

Assuming that the anode current depends only on the grid voltage v, where

$$v = \frac{1}{C}\int_0^t i\,dt, \quad (5.2)$$

we consider two cases of the characteristic of the tube: the characteristic of the "soft operation"

$$i_a(v) = S_0 v - \frac{1}{3} S_2 v^3 \quad (5.3)$$

and the characteristic of the "hard operation"

$$i_a(v) = S_0 v + \frac{1}{3} S_2 v^3 - \frac{1}{5} S_4 v^5. \quad (5.4)$$

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[1] See the work of A.A. Andronov and A.A.Vitt cited on p.3

Substituting (5.4), (5.3) and (5.2) in (5.1) and denoting the frequency of the circuit by ω_0 ($\omega_0^2 = 1/LC$), we obtain for the case of soft operation the equation

$$\frac{d^2v}{dt^2}+\left(\frac{R}{L}-\frac{MS_0}{LC}+\frac{MS_2}{LC}v^2\right)\frac{dv}{dt}+\omega_0^2 v=\frac{P}{LC}\sin\omega_1 t, \qquad (5.5)$$

and for the case of hard operation the equation

$$\frac{d^2v}{dt^2}+\left(\frac{R}{L}-\frac{MS_0}{LC}-\frac{MS_2}{LC}v^2+\frac{MS_4}{LC}v^4\right)\frac{dv}{dt}+\omega_0^2 v=\frac{P}{LC}\sin\omega_1 t.$$

If in these equations we introduce in place of the variable t the variable τ with the aid of the substitution $\tau = \omega_1 t$ they assume the following form:

$$\frac{d^2v}{d\tau^2}+\omega^2 v=(\alpha'-\gamma'^2v^2)\frac{dv}{d\tau}+\lambda\sin\tau,$$

$$\frac{d^2v}{d\tau^2}+\omega^2 v=(\alpha'+\gamma'^2v^2-\delta'^2v^4)\frac{dv}{d\tau}+\lambda\sin\tau,$$

where $\omega, \alpha', \gamma', \delta', \lambda$, are certain constants.

Setting, finally $\alpha'=\alpha\mu,\ \gamma'^2=\gamma^2\mu,\ \delta'^2=\delta^2\mu$, we obtain

$$\frac{d^2v}{d\tau^2}+\omega^2 v=\mu(\alpha-\gamma^2v^2)\frac{dv}{d\tau}+\lambda\sin\tau, \qquad (5.6)$$

$$\frac{d^2v}{d\tau^2}+\omega^2 v=\mu(\alpha+\gamma^2v^2-\delta^2v^4)\frac{dv}{d\tau}+\lambda\sin\tau. \qquad (5.7)$$

If the magnitude μ is sufficiently small, as we shall assume to be the case, equations (5.6) and (5.7) belong precisely to the type we have considered in detail in the preceding sections.

The magnitude α entering equations (5.6) and (5.7) can be both positive or negative[1] (we assume $\mu > 0$).

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[1] The pysical significance of the sign of the magnitude α will be explained below in sec. 12.

In the present section we shall consider in detail equation (5.6) on the assumption that $\alpha > 0$. In this case, by a suitable choice of the variable x, equation (5.6) may be reduced to the form

$$\frac{d^2x}{d\tau^2} + \omega^2 x = \mu(1 - x^2)\frac{dx}{d\tau} + \lambda \sin\tau, \tag{5.8}$$

which is the equation we shall consider. In order to obtain equation (5.8) from equation (5.5) we must set:

$$\left.\begin{aligned} x &= v\sqrt{\frac{MS_2}{MS_0 - RL}}, \qquad \mu = \frac{\omega_0^2}{\omega_1^2}(MS_0 - RC), \\ \lambda &= \frac{P\omega_0^2}{\omega_1^2}\sqrt{\frac{MS_2}{MS_0 - RL}}, \quad \omega^2 = \frac{\omega_0^2}{\omega_1^2}. \end{aligned}\right\} \tag{5.9}$$

OSCILLATIONS FAR FROM RESONANCE. Let us first assume that the magnitude ω^2 in equation (5.8) differs sufficiently from an integer. In this case the generating equation

$$\frac{d^2x_0}{d\tau^2} + \omega^2 x_0 = \lambda \sin\tau$$

has the following periodic solution

$$x_0 = \frac{\lambda}{\omega^2 - 1}\sin\tau.$$

To find the corresponding periodic solution of the complete equation (5.8) we substitute in it

$$x = x_0 + \mu x_1 + \dots$$

For x_1 we then obtain the equation

$$\frac{d^2x_1}{d\tau^2} + \omega^2 x_1 = (1 - x_0^2)\dot{x}_0 = \left(\frac{\lambda}{\omega^2 - 1} - \frac{\lambda^3}{4(\omega^2-1)^3}\right)\cos\tau - \frac{\lambda^3}{4(\omega^2-1)^3}\cos 3\tau.$$

This equation has the only periodic solution

$$x_1 = \left(\frac{\lambda}{(\omega^2-1)^2} - \frac{\lambda^3}{4(\omega^2-1)^4}\right)\cos\tau - \frac{\lambda^3}{4(\omega^2-1)^3(\omega^2-9)}\cos 3\tau.$$

In a similar manner we can compute also the further approximations, on which however we shall not dwell.

OSCILLATIONS NEAR THE PRINCIPAL RESONANCE. Let us now assume that ω^2 differs by a magnitude of the order of smallness of μ from an integer, namely unity. We shall say that in this case we are dealing with the principal resonance due to the fact that the disturbing force contains only the first harmonic.

Following the rule of the preceding section, we set

$$\omega^2 = 1 + \mu a, \qquad \lambda = \mu\lambda_0, \tag{5.10}$$

after which equation (5.8) assumes the form

$$\frac{d^2x}{d\tau} + x = \mu\left[\lambda_0 \sin\tau - ax + (1 - x^2)\dot{x}\right]. \tag{5.11}$$

The generating solution in the case considered contains two arbitrary constants and is of the form

$$x_0 = M_0 \cos\tau + N_0 \sin\tau_0, \tag{5.12}$$

where M_0 and N_0 are the arbitrary constants. Putting

$$x = x_0 + \mu x_1 + \mu^2 x_2 + \dots,$$

we obtain for x_1 and x_2 the following equations:

$$\frac{d^2x_1}{d\tau^2} + x_1 = \lambda_0 \sin\tau - a x_0 + (1 - x_0^2)\dot{x}_0, \tag{5.13}$$

$$\frac{d^2x_2}{d\tau^2} + x_2 = -a x_1 + (1 - x_0^2)\dot{x}_1 - 2x_0 x_1 \dot{x}_0. \tag{5.14}$$

Substituting in (5.13) for x_0 its value (5.12), after simple transformation we obtain

$$\begin{aligned}\frac{d^2x_1}{d\tau^2} + x_1 = &\left(-aM_0 + N_0 - \frac{1}{4}N_0M_0^2 - \frac{1}{4}N_0^3\right)\cos\tau + \\ &+ \left(\lambda_0 - M_0 - aN_0 + \frac{1}{4}M_0^3 + \frac{1}{4}M_0N_0^2\right)\sin\tau + \\ &+ \frac{1}{4}N_0(N_0^2 - 3M_0^2)\cos 3\tau + \frac{1}{4}M_0(M_0^2 - 3N_0^2)\sin 3\tau.\end{aligned} \tag{5.15}$$

For this equation to admit a periodic solution it is necessary and sufficient that the equations be satisfied

$$\left.\begin{aligned} P(M_0, N_0) &= -aN_0 - M_0\left[1-\tfrac{1}{4}(M_0^2+N_0^2)\right]+\lambda_0 = 0, \\ Q(M_0, N_0) &= -aM_0 + N_0\left[1-\tfrac{1}{4}(M_0^2+N_0^2)\right] = 0. \end{aligned}\right\} \quad (5.16)$$

These equations determine the constants M_o and N_o in the generating solution. We shall assume that M_o and N_o actually satisfy equations (5.16). The general solution of equation (5.15) will then be periodic. This solution will be:

$$\begin{aligned} x_1 = {} & M_1\cos\tau + N_1\sin\tau + \\ & + \tfrac{1}{32}N_0(3M_0^2-N_0^2)\cos 3\tau + \tfrac{1}{32}M_0(3N_0^2-M_0^2)\sin 3\tau, \end{aligned} \quad (5.17)$$

where M_1 and N_1 are arbitrary constants. These constants are determined from the condition of periodicity of the solution for x_2, namely, substituting in (5.14) for the magnitudes x_o and x_1 their expressions (5.12) and (5.17) we obtain:

$$\begin{aligned} \frac{d^2x_2}{d\tau^2} + x_2 = {} & \Big[\Big(-\tfrac{1}{2}M_0N_0 - a\Big)M_1 + \Big(1-\tfrac{1}{4}M_0^2-\tfrac{3}{4}N_0^2\Big)N_1 + \\ & + \tfrac{1}{128}M_0(M_0^2+N_0^2)^2\Big]\cos\tau + \Big[\Big(-1+\tfrac{3}{4}M_0^2+\tfrac{1}{4}N_0^2\Big)M_1 + \\ & + \Big(\tfrac{1}{2}M_0N_0 - a\Big)N_1 + \tfrac{1}{128}N_0(M_0^2+N_0^2)^2\Big]\sin\tau + \ldots, \end{aligned}$$

where the nonwritten down terms do not contain $\cos\tau$ and $\sin\tau$. Whence for M_1 and N_1 we obtain the following two equations:

$$\left.\begin{aligned} & \Big(-\tfrac{1}{2}M_0N_0 - a\Big)M_1 + \Big(1-\tfrac{1}{4}M_0^2-\tfrac{3}{4}N_0^2\Big)N_1 + \\ & \qquad + \tfrac{1}{128}M_0(M_0^2+N_0^2)^2 = 0, \\ & \Big(-1+\tfrac{3}{4}M_0^2+\tfrac{1}{4}N_0^2\Big)M_1 + \Big(\tfrac{1}{2}M_0N_0 - a\Big)N_1 + \\ & \qquad + \tfrac{1}{128}N_0(M_0^2+N_0^2)^2 = 0, \end{aligned}\right\} \quad (5.18)$$

expressing the conditions for the existence of periodic solutions for x_2. The obtained equations for M_1 and N_1, as follows from the general theory, turned out to be linear. The determinant of these equations identically agrees with the functional determinant

$$\frac{\partial (P, Q)}{\partial (M_0, N_0)} = -1 - a^2 + (M_0^2 + N_0^2) - \frac{3}{16}(M_0^2 + N_0^2)^2, \quad (5.19)$$

as is likewise in agreement with the general theory. Hence, if M_0 and N_0 are a simple solution for equations (5.16), we obtain a fully determined solution for M_1 and N_1 and therefore also for x_1.

Restricting ourselves to the obtained approximations let us examine more closely the equations (5.16) for M_0 and N_0. Passing to the amplitude and phase in the generating solution, which are determined by the relations

$$M_0 = A\sin\varphi, \quad N_0 = \cos\varphi,$$

we obtain from (5.16) the following equations for these magnitudes:

$$\operatorname{tg}\varphi = \frac{1 - \frac{1}{4}A^2}{a}, \quad (5.20)$$

$$A^2\left[a^2 + \left(1 - \frac{A^2}{4}\right)^2\right] = \lambda_0^2. \quad (5.21)$$

Thus, for determining the magnitude A^2 we shall have obtained a cubic equation. Hence, depending on the value of the mistuning a and the amplitude of the external action λ_0, we have obtained one or three real solutions. We are interested of course only in the positive solutions. In order to explain better the different cases that may here arise we shall construct the resonance curve, i.e. the curve of A^2 as a function of a for fixed value of λ_0. Formula (5.21) precisely represents the equations of this curve. In fig. 2 are shown a number of curves (5.21) corresponding to different values of λ_0^2. For $\lambda_0^2 = 0$ the curve (5.21) consists of the axis $A^2 = 0$ and the isolated (double) point (0,4). For λ_0^2 different from zero but sufficiently small the resonance curve consists of two branches: in place of the axis $A^2 = 0$ there

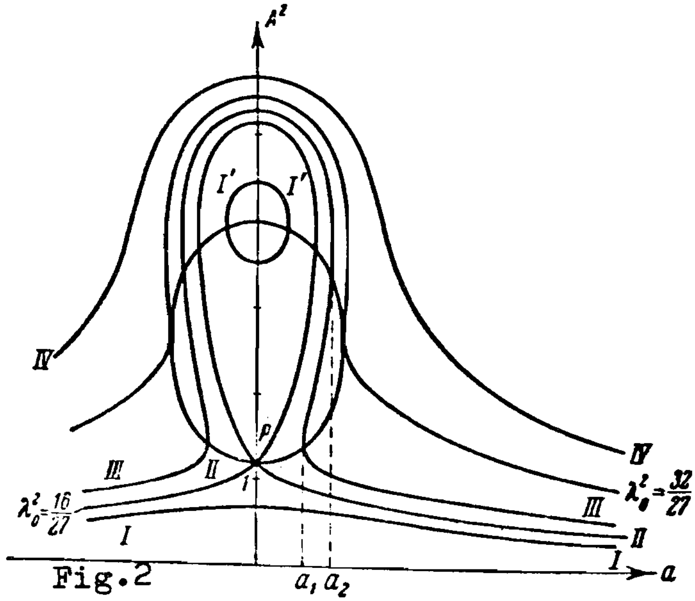

Fig.2

is obtained a curve of the form I - I, and in place of the double point (0,4) an oval I' - I' surrounding this point. With gradual increase of λ_0^2 the oval I' - I' and the branch I - I approach each other and for a certain value of λ_0^2 there is obtained only one branch of the form II - II with the double point P. This value of λ_0^2 is easily found from the condition that equation (5.21) for a = 0 must have a double root and therefore the expression (5.19) must also reduce to zero. Solving (5.19) and (5.21) simultaneously with a = 0 we obtain, in addition to the curve λ_0^2 with double point (0,4), already mentioned, the curve $\lambda_0^2 = 16/27$ with double point P(0,4/3). For $\lambda_0^2 > 16/27$ there will at first be obtained the curves of the form III - III and then, for sufficiently large λ_0^2 ($\lambda_0^2 > 32/27$), the curves of the type IV - IV intersected by each vertical in only one point. From this it is immediately evident that for $\lambda_0^2 < 16/27$ equation (5.21) will have, for a sufficiently small, three real solutions, while for a exceeding a certain limit depending on λ_0^2 it will have only one solution. In the case $16/27 < \lambda_0^2 < 32/27$ we shall obtain for $a < a_1$ one solution, for $a_1 < a < a_2$ three solutions, and for $a > a_2$ again one solution. For the amplitudes for the curves of the type IV - IV we shall obtain only one solution for any a . The values of a separating the regions for which equation (5.21) has three solutions from the regions for which this equation has one solution are determined from the condition that for these values the equation has a double root and therefore expression (5.19) reduces to zero. Thus, these values of a are found as abscissas of the points of intersection of the resonance curve with the ellipse

$$1+a^2-A^2+\frac{3}{16}A^4\equiv\frac{3}{16}\left(A^2-\frac{8}{3}\right)^2+a^2-\frac{1}{3}=0. \qquad (5.22)$$

Equation (5.22) can also be obtained as the equation of the geometic locus of the points at which the tangents to the resonance curves are vertical, i.e. from the condition $da/dA^2 = 0$. If a resonance curve with given value of λ_0 intersects the ellipse (5.22) equation (5.21) will have three real roots for the values of a included between the abscissas of the points of intersection, while for the remaining values of a it will have one solution. If the resonance curve does not intersect the ellipse (5.22) equation (5.21) will have only one solution for all values of a. Both these types of resonance curves are separated by the resonance curve passing through a vertex of the ellipse. From this the result is readily obtained that for this boundary curve $\lambda_0^2 = 32/27$.

Thus, equation (5.21) always has one or three real roots to which correspond one or three periodic solutions of equation (5.11). Whether, for the physical system under consideration, periodic oscillations, corresponding to these periodic solutions, will arise depends on whether these oscillations are stable or not. The question of the stability of these oscillations will be considered below in sec. 15.

6. Application to the Problem of Duffing

Let us now consider the equation

$$\frac{d^2x}{dt^2}+k^2x-\gamma x^3=\lambda\sin t, \qquad (6.1)$$

which was first studied by nonrigorous methods by Duffing. It may be considered as the equation of the forced vibrations of a conservative system with one degree of freedom with a nonlinear elastic restoring force. We shall assume that the magnitude γ is small and shall treat equation (6.1) as quasi-linear. For this purpose we substitute in it $\gamma = \mu\gamma'$, after which it assumes the form

$$\frac{d^2x}{dt^2}+k^2x=\lambda\sin t+\mu\gamma' x^3. \qquad (6.2)$$

We shall assume at first that we are dealing with the nonresonance case. The generating equation will then have a single periodic solution

$$x_0 = \frac{\lambda}{k^2 - 1} \sin t.$$

To this corresponds a single periodic solution of the equation (6.2) which we shall seek in the form

$$x = x_0 + \mu x_1 + \ldots \tag{6.3}$$

For x_1 we obtain the equation

$$\frac{d^2 x_1}{dt^2} + k^2 x_1 = \frac{\gamma' \lambda^3}{(k^2 - 1)^3} \sin^3 t,$$

which has the single periodic solution

$$x_1 = \frac{3\gamma' \lambda^3}{4\,(k^2 - 1)^4} \sin t - \frac{\gamma' \lambda^3}{4\,(k^2 - 1)^3\,(k^2 - 9)} \sin 3t.$$

The computation of the further approximations presents no difficulties. Let us now assume that we are dealing with the resonance case when k^2 is equal to unity or differs little from it. Setting in this case

$$1 - k^2 = \mu a, \qquad \lambda = \mu \lambda',$$

we reduce equation (6.1) to the form

$$\frac{d^2 x}{dt^2} + x = \mu\,(\lambda' \sin t + a x + \gamma' x^3). \tag{6.4}$$

The generating equation now has the periodic solution

$$x_0 = M_0 \cos t + N_0 \sin t,$$

that depends on the two arbitrary constants M_0 and N_0.

We shall seek the periodic solution of equation (6.4) in the form of the series

$$x = x_0 + \mu x_1 + \ldots \tag{6.5}$$

with periodic coefficients. We have:

$$\frac{d^2x_1}{dt^2} + x_1 = \lambda' \sin t + a(M_0 \cos t + N_0 \sin t) + \gamma'(M_0 \cos t + N_0 \sin t)^3.$$

For this equation to have a periodic solution it is necessary and sufficient that the equations be satisfied

$$aM_0 + \frac{3}{4}\gamma' M_0 (M_0^2 + N_0^2) = 0,$$

$$\lambda' + aN_0 + \frac{3}{4}\gamma' N_0 (M_0^2 + N_0^2) = 0.$$

These equations can be satisfied only on the assumption

$$M_0 = 0, \quad N_0 = A, \tag{6.6}$$

where A is a root of the cubic equation

$$\lambda' + aA + \frac{3}{4}\gamma' A^3 = 0. \tag{6.7}$$

The magnitude x_0 thus contains only the sine. This is due to the fact that equation (6.4) does not change when t is replaced by -t and x by -x. As a result, the further approximations will likewise contain only sines. We shall make use of this circumstance for the further computations where it will introduce certain simplifications.

With the arbitrary constants determined according to (6.6), we shall now have

$$x_1 = \frac{\gamma' A^3}{32} \sin 3t + N_1 \sin t,$$

where N_1 is a new arbitrary constant. This constant is determined from the condition of periodicity of the function x_2. For this function we have

$$\frac{d^2x_2}{dt^2} + x_2 = a\left(\frac{\gamma' A^3}{32} \sin 3t + N_1 \sin t\right) + \\ + 3\gamma' A^2 \sin^2 t \left(\frac{\gamma' A^3}{32} \sin 3t + N_1 \sin t\right),$$

and the condition of periodicity gives:

$$\left(a+\frac{9}{4}\gamma' A^2\right) N_1-\frac{3}{128}\gamma'^2 A^5=0. \tag{6.8}$$

As was to be expected, according to the general theory the equation for N_1 was found to be linear. With N_1 determined we now have:

$$x_2=\frac{1}{512}\gamma'^2 A^5 \sin 5t+$$
$$+\left(-\frac{1}{256}a\gamma' A^3+\frac{3}{32}\gamma'^2 A^2 N_1-\frac{3}{512}\gamma'^2 A^5\right)\sin 3t+N_2 \sin t,$$

where N_2 is a new arbitrary constant which is determined from the following approximation. We shall not however concern ourselves with its determination and restrict ourselves to the obtained approximations.

Let us consider more closely equation (6.7) which determines A. Returning to the initial notations we can rewrite it in the form:

$$\lambda+(1-k^2) A+\frac{3}{4}\gamma A^3=0. \tag{6.9}$$

The roots of this equation can be obtained graphically as the abscissas of the points of intersection of the straight line

$$y=(k^2-1) A-\lambda \tag{6.10}$$

and the cubical parabola

$$y=\frac{3}{4}\gamma A^3. \tag{6.11}$$

Let us assume for definiteness that $\gamma > 0$. Then, from fig.3 it is seen that when the magnitude $k^2 - 1$ is negative or when it is positive but does not exceed a certain magnitude α, for which the straight line (6.10) is tangent to the cubical parabola (6.9), equation (6.9) has only one real negative root. But if $k^2 - 1 > \alpha$ equation (6.9) will have three real roots of which one is negative and two positive.

We arrive at the same results if we construct the resonance curve $|A| = |A(k^2)|$. For this purpose we construct separately the parabola

$$k^2 = 1 + \frac{3}{4}\gamma A^2 \tag{6.12}$$

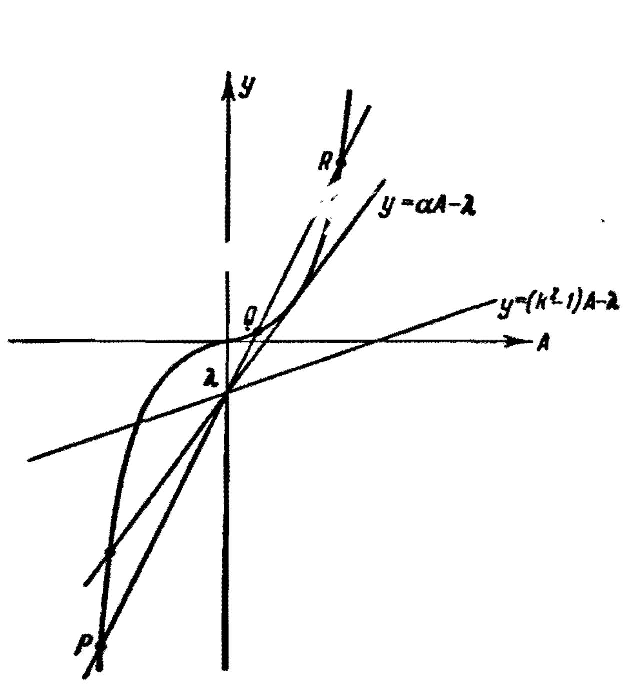

Fig.3

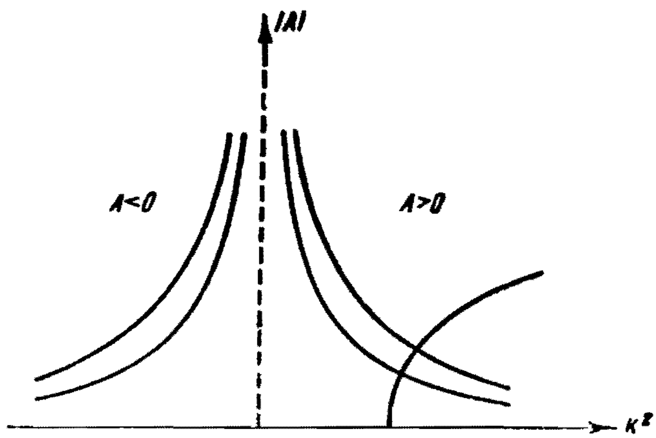

Fig. 4

and the family of curves (fig. 4)

$$k^2 = \frac{\lambda}{|A|} \, . \tag{6.13}$$

Adding the abscissas of both curves we obtain the resonance curves determined by equation (6.9) for different values of λ. The form of these curves is shown in fig. 5. From a consideration of this figure we at once arrive at the same conclusion with regard to the roots of equation (6.9) to which we arrived on the basis of fig. 3.

It is of interest to note that the parabola (6.12) represents the curve of frequency as a function of the amplitude of the free oscillations corresponding to the equation (6.1) for k = 1, i.e. the oscillations determined by the equation

$$\frac{d^2x}{dt^2} + x - \gamma x^3 = 0. \tag{6.14}$$

In fact, as will be shown in section 3 of chapter VII, the period of the solution of equation (6.14) with the initial conditions $x(0) = A$, $x(0) = 0$ is determined by the series

$$T = 2\pi\left(1 + \frac{3}{8}\gamma A^2 + \frac{57}{256}\gamma^2 A^4 + \dots\right).$$

Whence we easily obtain the result that the square of the frequency of these oscillations, with an accuracy up to the second order of magnitude relative to A, is determined by the right hand side of equation (6.12).

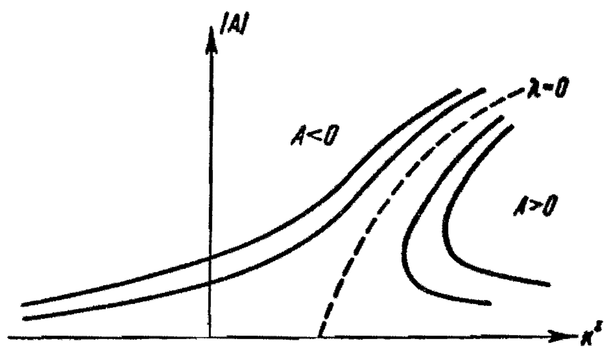

Fig. 5

We shall now compare the form of the resonance curves for the nonlinear problem under consideration with the same curves for the case of the corresponding linear problem, i.e. for the case $\gamma = 0$.

For the linear problem we have:

$$A = \frac{\lambda}{k^2 - 1}.$$

Hence the resonance curves can be obtained by adding the abscissas of the curves (6.13) (the same as for the nonlinear problem) to the abscissas of the straight line $k^2 = 1$ and are of the form shown in fig.6. These curves differ from the resonance curves in the nonlinear problem only in the circumstance that instead of the parabolic asymptote (6.12) a rectilinear asymptote is obtained.

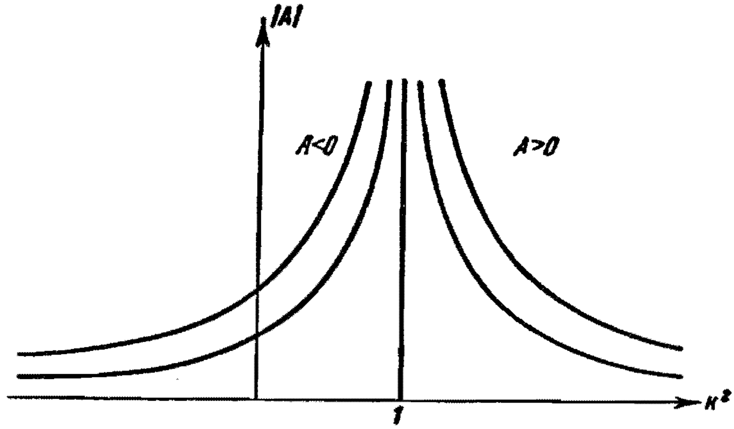

Fig. 6

From this however a very sharp qualitative difference results between the two systems. First of all, in the nonlinear system for accurate resonance, i.e. for $k^2 = 1$, in spite of the absence of damping, there is an entirely definite periodic solution with finite amplitude[1], determined on the basis of (6.9) by the formula

$$|A| = \sqrt[3]{\frac{4\lambda}{3\gamma}}.$$

Moreover, as was already noted above, beyond resonance, when the mistuning $k^2 = 1$ exceeds the magnitude of α, we obtain three different real values for the amplitude A, to which correspond three periodic solutions of equation (6.1).

The question of the stability of the obtained periodic solutions and the associated question of the character of the development of the oscillations will be considered below in section 16.

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[1] The amplitude under consideration is of course that of the principal term.

7. Resonance of the n-th Kind

In nonlinear systems under the action of a disturbing force intense forced oscillations may arise not only when the period of the natural oscillations is near the period T of the disturbing force but also when it is near the magnitude nT, where n is an integer. The period of the forced oscillations will then be equal to nT. For comparison we shall first consider the linear system described by the equation

$$\frac{d^2x}{dt^2} + k^2x + 2\delta\frac{dx}{dt} = \lambda \sin \omega t. \tag{7.1}$$

This equation has one and only one periodic solution determined by the formula

$$x = \frac{\lambda}{\sqrt{(k^2-\omega^2)^2 + 4\delta^2\omega^2}} \sin(\omega t + \varepsilon), \quad \operatorname{tg} \varepsilon = -\frac{2\delta\omega}{k^2-\omega^2}. \tag{7.2}$$

To this solution correspond the forced oscillations of the system. Every other solution of the system (7.1) for $t \to \infty$ approaches without limit the solution (7.2). But the solution (7.2) has the same period as the disturbing force. Oscillations with a period which is a multiple of the period of the disturbing force cannot arise in the system (7.1). However if the damping δ is equal to zero then for k exactly equal to ω/n the general solution of equation (7.1)

$$x = A \sin\left(\frac{\omega}{n} t + \alpha\right) + \frac{n^2\lambda}{(1-n^2)\,\omega^2} \sin \omega t$$

will be periodic with a period equal to n times the period of the disturbing force. Actually however, on account of the unavoidable damping in every real system of free oscillations, the oscillations that arise in it will have the period of the disturbing force.

The case would be otherwise if equation (7.1) contained nonlinear terms. In this case, both for the presence as well as the absence of damping in the system, oscillations may arise with a period $2\pi n/\omega$ not only when k is equal to ω/n but also when k differs little from this magnitude. This phenomenon, following L.I. Mandelshtam and N.D. Papaleksi, who first worked out its exact theory for a quasilinear system with one degree of freedom, is termed RESONANCE OF THE N-TH KIND. The oscillations are sometimes referred to as SUBHARMONIC of the $1/n$-th order.

Numerous studies have been devoted to the phenomenon of resonance of the n-th kind. In this section we shall present several fundamental results obtained by Mandelshtam and Papaleksi[1], where we follow however a somewhat different presentation.

Thus, let us consider the equation

$$\frac{d^2x}{dt^2}+k^2x=\mu F(t,\ x,\ \dot{x},\ \mu)-f(t), \tag{7.3}$$

where, as usual, the period of the functions f and F are assumed equal to 2π and the function F is analytic with respect to x and $\dot{x}$.

We shall assume that k differs little from $1/n$. Referring the term $(k^2 - 1/n^2)x$ to the function μF we can write equation (7.3) in the form

$$\frac{d^2x}{dt^2}+\frac{1}{n^2}x+f(t)=\mu F(t,\ x,\ \dot{x},\ \mu). \tag{7.4}$$

The problem consists in determining the conditions under which periodic solutions of period $2\pi n$ arise in the system.

We note first of all that since $1/n^2$ differs from an integer, equation (7.4), on the basis of the results of sec. 2, has a periodic solution of period 2π which for $\mu = 0$ reduces to the generating solution $x_o = \varphi(t)$, where function φ is defined by equation (2.4). We are interested however in those solutions of period $2\pi n$ which do not reduce to the solutions of period 2π. To obtain these solutions we note that the generating equation has a family of periodic solutions

$$x_0=\varphi(t)+M_0\cos\frac{t}{n}+N_0\sin\frac{t}{n} \tag{7.5}$$

of period $2\pi n$ that depend on the two arbitrary constants M_o and N_o. Since the right hand side of equation (7.4)

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[1] Mandelshtam L.I. and Papaleksi N.D., O yavleniakh rezonansa n-go roda. (On Resonance Phenomena of the n-th Kind), Zhurn. tekhn. Fiziki, vol. II, no 7-8, 1932.

is a periodic function of 2π relative to t, hence also of period $2\pi n$, the problem in no way differs from that studied in sec. 3 and 4, in which it is merely necessary to replace the period 2π by the period $2\pi n$. Hence, following the method of sec. 4, we shall seek the solution of interest in the form of the series

$$x = x_0 + \mu x_1 + \dots$$

For x_1 we have the equation

$$\frac{d^2 x_1}{dt^2} + \frac{1}{n^2} x_1 = F(t, x_0, \dot{x}_0, 0),$$

in which the right hand side is a known periodic function of period $2\pi n$. For this equation to have a periodic solution of period $2\pi n$ it is necessary and sufficient that in the expansion of the right hand side in terms of sines and cosines of the angles that are multiples of t/n, the terms with $\cos t/n$ and $\sin t/n$ should be absent. Consequently, the conditions must be satisfied

$$\left.\begin{aligned} P(M_0, N_0) &= \int_0^{2\pi n} F(t, x_0, \dot{x}_0, 0) \sin\frac{t}{n}\, dt = 0, \\ Q(M_0, N_0) &= \int_0^{2\pi n} F(t, x_0, \dot{x}_0, 0) \cos\frac{t}{n}\, dt = 0. \end{aligned}\right\} \quad (7.6)$$

We have thus obtained two equations for determining the arbitrary constants M_o and N_o. These equations are satisfied for $M_o = N_o = 0$. In fact, as was shown above, equation (7.4) has a periodic solution of period 2π corresponding to the generating solution $x_o = \varphi(t)$, i.e. the solution (7.5) with zero values of the magnitudes M_o and N_o. But this solution may evidently be considered as having a period $2\pi n$ and must therefore be included among the required solutions. It is also easily possible to satisfy oneself that equations (7.6) are satisfied for $M_o = N_o = 0$. In fact, for $M_o = N_o = 0$ the function $F(t, x_o, \dot{x}_o, 0)$ becomes periodic of period 2π and therefore its expansion contains sines and cosines only of whole multiples of t. Hence the Fourier expansions under the integral of the expressions in (7.6) will contain only trigonometric terms and these equations will be identically satisfied.

But equations (7.6) may have solutions for which at least one of the magnitudes M_0 and N_0 is different from zero. If for such solution the functional determinant

$$\frac{\partial (P, Q)}{\partial (M_0, N_0)}$$

is different from zero, then on the basis of the results of sec. 3, there actually exists a periodic solution of equation (7.4) and this solution will evidently have a period $2\pi n$ which does not reduce to the period 2π.

The further computation of the solution is conducted in the same way as for the usual resonance. It may be remarked that in practice it is more convenient to choose the unit of time in such manner that the right hand side of the equation has the period $2\pi/n$ and not 2π. The problem then reduces to finding a periodic solution of period 2π.

We may note, finally, that in the preceding considerations it was not required that the magnitude 2π be the LEAST PERIOD of the right hand side of equation (7.3). In particular, this period can be equal to $2\pi/m$, where m is an integer. The period of the obtained solution will then be n/m times as large as the period that figures in equation (7.3). In other words, in speaking of resonance of th n-th kind, we can assume that n is an arbitrary rational number and not necessarily an integer.

8. Examples of Resonance of the n-th Kind

EXAMPLE 1. RESONANCE OF THE SECOND KIND IN A REGENERATIVE RECEIVER. As a first example let us consider resonance of the second kind for a regenerative receiver. This problem was studied by L.I. Mandelshtam and N.D. Papaleksi who assumed a form of the tube characteristic for which the equation of the oscillations could be written as

$$\frac{d^2x}{dt^2} + x = \lambda \sin 2t + \mu \{ax + (\alpha + \beta x + \gamma x^2)\dot{x}\}. \tag{8.1}$$

The period of the external action is in the case under consideration equal to π. We shall therefore seek the periodic solution of period 2π. We shall here restrict ourselves to computing the zeroth approximation, i .e. to the generating solution. We have:

$$x_0 = M_0 \cos t + N_0 \sin t - \frac{\lambda}{3} \sin 2t,$$

where M_0 and N_0 are constants to be determined. They are found from the condition that the equation for x_1 admits a periodic solution.

This equation has the form

$$\frac{d^2x_1}{dt^2} + x_1 = ax_0 + (\alpha + \beta x_0 + \gamma x_0^2)\dot{x}_0 =$$
$$= \left\{M_0\left(a - \frac{1}{6}\lambda\beta\right) + N_0\left[\alpha + \frac{\gamma}{4}\left(M_0^2 + N_0^2 + \frac{2}{9}\lambda^2\right)\right]\right\}\cos t +$$
$$+ \left\{-M_0\left[\alpha + \frac{\gamma}{4}\left(M_0^2 + N_0^2 + \frac{2}{9}\lambda^2\right)\right] + N_0\left(a + \frac{1}{6}\lambda\beta\right)\right\}\sin t + \ldots,$$

where the terms not written down do not contain cos t and sin t. The condition of periodicity gives:

$$\left.\begin{aligned} M_0\left(a - \frac{1}{6}\lambda\beta\right) + N_0\left[\alpha + \frac{\gamma}{4}\left(M_0^2 + N_0^2 + \frac{2}{9}\lambda^2\right)\right] = 0, \\ -M_0\left[\alpha + \frac{\gamma}{4}\left(M_0^2 + N_0^2 + \frac{2}{9}\lambda^2\right)\right] + N_0\left(a + \frac{1}{6}\lambda\beta\right) = 0. \end{aligned}\right\} \quad (8.2)$$

These two equations, aside from the trivial solution $M_0 = N_0 = 0$, have additional solutions determined by the equations

$$\left.\begin{aligned} M_0^2 + N_0^2 = -\frac{4\alpha}{\gamma} - \frac{2}{9}\lambda^2 \pm \frac{2}{3\gamma}\sqrt{\lambda^2\beta^2 - 36a^2}, \\ \frac{M_0}{N_0} = \pm\sqrt{\frac{\lambda\beta + 6a}{\lambda\beta - 6a}}. \end{aligned}\right\} \quad (8.3)$$

The functional determinant of equations (8.2) for the solutions (8.3) is different from zero.

For sufficiently small a , these solutions are real and to them actually correspond periodic solutions of equation (8.1) of period 2π.

EXAMPLE 2. RESONANCE OF THE N-TH KIND FOR THE PROBLEM OF DUFFING.[1] We shall investigate resonance of the n-th

[1] All computations in this example were carried out by the student V.M. Mendelson.

kind for the equation of Duffing

$$\frac{d^2x}{dt^2}+k^2x-\gamma x^3=\lambda\sin t. \tag{8.4}$$

We shall assume therefore that k differs little from $1/n$ and set

$$k^2=\frac{1}{n^2}-\mu\frac{a}{n^2},\quad n^2\gamma=\mu\gamma_1,\quad t=n\tau,\quad n^2\lambda=\lambda_1. \tag{8.5}$$

Equation (8.4) then assumes the form

$$\frac{d^2x}{d\tau^2}+x=\lambda_1\sin n\tau+\mu(ax+\gamma_1x^3), \tag{8.6}$$

and the problem reduces to finding the periodic solutions of period 2π of this equation. We shall seek these solutions in the form of the series

$$x=x_0+\mu x_1+\dots, \tag{8.7}$$

where

$$x_0=\frac{\lambda_1}{1-n^2}\sin n\tau+M_0\cos\tau+N_0\sin\tau \tag{8.8}$$

and M_0 and N_0 are arbitrary constants. These constants are found from the conditions of the periodicity of the function x_1. For this function we have

$$\begin{aligned}\frac{d^2x_1}{d\tau^2}+x_1=ax_0+\gamma_1x_0^3=M_0&\left[a+\frac{3}{2}\frac{\gamma_1\lambda_1^2}{(1-n^2)^2}+\frac{3}{4}\gamma_1(M_0^2+N_0^2)\right]\cos\tau+\\&+N_0\left[a+\frac{3}{2}\frac{\gamma_1\lambda_1^2}{(1-n^2)^2}+\frac{3}{4}\gamma_1(M_0^2+N_0^2)\right]\sin\tau+\\&+\frac{1}{4}\gamma_1M_0(M_0^2-3N_0^2)\cos3\tau+\frac{1}{4}\gamma_1N_0(3M_0^2-N_0^2)\sin3\tau+\\&+\frac{3}{2}\frac{\gamma_1\lambda_1}{1-n^2}M_0N_0\cos(n-2)\tau+\frac{3}{4}\frac{\lambda_1\gamma_1}{1-n^2}(M_0^2-N_0^2)\sin(n-2)\tau+\\&+\frac{3}{2}\frac{\gamma_1\lambda_1^2}{(1-n^2)^2}\left(\frac{1}{2}\frac{\lambda_1}{1-n^2}+M_0+N_0\right)\sin n\tau-\\&-\frac{3}{2}\frac{\lambda_1\gamma_1}{1-n^2}M_0N_0\cos(n+2)\tau+\\&+\frac{3}{4}\frac{\lambda_1\gamma_1}{1-n^2}(M_0^2-N_0^2)\sin(n+2)\tau+\\&+\frac{3}{4}\frac{\gamma_1\lambda_1^2}{(1-n^2)^2}M_0\cos(2n-1)\tau+\frac{3}{4}\frac{\gamma_1\lambda_1^2}{(1-n^2)^2}N_0\sin(2n-1)\tau+\\&+\frac{3}{4}\frac{\gamma_1\lambda_1^2}{(1-n^2)^2}M_0\cos(2n+1)\tau-\frac{3}{4}\frac{\gamma_1\lambda_1^2}{(1-n^2)^2}N_0\sin(2n+1)\tau.\end{aligned} \tag{8.9}$$

The conditions of periodicity of the function x_1 are found different depending on whether $n \neq 3$ or $n = 3$. We shall first assume that $n \neq 3$. In this case the conditions of periodicity of x_1 assume the form

$$P(M_0, N_0) = M_0 \left[a + \frac{3}{2} \frac{\gamma_1 \lambda_1^2}{(1-n^2)^2} + \frac{3}{4} \gamma_1 (M_0^2 + N_0^2) \right] = 0,$$
$$Q(M_0, N_0) = N_0 \left[a + \frac{3}{2} \frac{\gamma_1 \lambda_1^2}{(1-n^2)^2} + \frac{3}{4} \gamma_1 (M_0^2 + N_0^2) \right] = 0.$$

The obtained equations, aside from the solution $M_0 = N_0 = 0$, have the further solution

$$M_0^2 + N_0^2 = -\frac{4}{3\gamma_1} \left[a + \frac{3}{2} \frac{\gamma_1 \lambda_1^2}{(1-n^2)^2} \right],$$

where one of the constants, M_0 or N_0, may be chosen arbitrarily. As a result, the functional determinant

$$\frac{\partial (P, Q)}{\partial (M_0, N_0)}$$

for this solution becomes zero and the method of the preceding section cannot be applied.

We shall now assume that $n = 3$. In this case the additional terms with $\cos \tau$ and $\sin \tau$ appear in equation (8.9) and the conditions of periodicity of the function x_1 assume the form

$$\left.\begin{aligned} P(M_0, N_0) &= \\ &= M_0 \left[a + \frac{3}{128} \gamma_1 \lambda_1^2 + \frac{3}{4} \gamma_1 (M_0^2 + N_0^2) \right] - \frac{3}{16} \gamma_1 \lambda_1 M_0 N_0 = 0, \\ Q(M_0, N_0) &= \\ &= N_0 \left[a + \frac{3}{128} \gamma_1 \lambda_1^2 + \frac{3}{4} \gamma_1 (M_0^2 + N_0^2) \right] - \frac{3}{32} \gamma_1 \lambda_1 (M_0^2 - N_0^2) = 0. \end{aligned}\right\} \tag{8.10}$$

These equations, aside from the solution $M_0 = N_0 = 0$, have other solutions. In particular, there is the solution

$$M_0 = 0, \; N_0 = \frac{-3\lambda_1 \gamma_1 \pm \sqrt{-3072 a \gamma_1 - 63 \gamma_1^2 \lambda_1^2}}{48 \gamma_1}. \tag{8.11}$$

In order that this solution be real it is necessary that the inequality be satisfied

$$-3072\,a\gamma_1-63\gamma_1^2\lambda_1^2>0.$$

Hence it is necessary that a and γ_1 be of different signs and that the condition be satisfied

$$|a|\geqslant\frac{21}{1024}\lambda_1^2|\gamma_1|$$

or, if with the aid of (8.5) we go over to the initial notations,

$$\left|k^2-\frac{1}{9}\right|\geqslant\frac{21\cdot81}{1024}\lambda^2|\gamma|,\quad \left(k^2-\frac{1}{9}\right)\gamma>0. \tag{8.12}$$

J. Stoker[1], in solving this problem by a less rigorous method, arrived at the inequality

$$\left|k^2-\frac{1}{9}\right|\geqslant\frac{21}{1024}\frac{1}{k^4}\lambda^2|\gamma|.$$

Besides the solution (8.11), equations (8.10) have additional solutions in which both magnitudes, M_0 and N_0, are different from zero. Thus, dividing equations (8.10) one by the other, we have:

$$\frac{M_0}{N_0}=\frac{2M_0N_0}{M_0^2-N_0^2},$$

after which from these equations we obtain:

$$\left.\begin{aligned} M_0&=\pm\sqrt{3}N_0=\pm\operatorname{tg}\frac{\pi}{3}N_0,\\ N_0&=\frac{3\lambda_1\gamma_1\pm\sqrt{-3072a\gamma_1-63\gamma_1^2\lambda_1^2}}{96\gamma_1}\end{aligned}\right\}. \tag{8.13}$$

If however account is taken of the fact that equation (8.6) does not change (for $n = 3$) on replacing x by $-x$ and τ by $\tau + \pi/3$, it is easily concluded that the solutions (8.13) in no way differ from the solutions (8.11).

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[1]Stoker, J., Nonlinear Vibrations in Mechanical and Electrical Systems, Interscience Publishers, 1950.

For the solutions(8.11) the functional determinant $\frac{\partial(P, Q)}{\partial(M_0, N_0)}$ is different from zero, since these solutions are simple. It is also easy to satisfy oneself of this by direct verification. Hence to the obtained solution M_0 and N_0 there actually correspond, for sufficiently small μ, periodic solutions of equation (8.6).

9. Autonomous Systems.

Conditions for the Existence of Periodic Solutions

We now pass on to the consideration of oscillations of autonomous systems. In this chapter we shall restrict ourselves to the consideration of the simple system with one degree of freedom described by the equation [1]

$$\frac{d^2x}{dt^2} + k^2 x = \mu f(x, \dot{x}, \mu), \tag{9.1}$$

where f is an analytic function of the variables x and $\dot{x} = dx/dt$ in a certain region G in which the generating solution under consideration lies, and of the parameter μ for its sufficiently small values.

Equation (9.1) is essentially a special case of the equations considered in the preceding sections. Autonomous systems however have their own characteristics which sharply differentiate them from nonautonomous systems, as a consequence of which equation (9.1) requires special investigation.

First of all, in the case of nonautonomous systems the period of the required periodic solution was always entirely definite, equal to or a multiple of the period of the equations themselves. In the case of equation (9.1) however that is now considered, since it does not contain the time explicitly, periodic solutions of arbitrary period are possible which, generally speaking, depend on μ.

Moreover, equation (9.1) does not change if t is replaced by $t + h$. Hence any solution of this equation, including a periodic solution, remains the same if in it t is replaced by $t + h$. This makes it possible, without disturbing the generality of the considerations, to assume that at the initial instant of time the magnitude dx/dt

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[1] Cf. Andronov A.A. and Khaikin S.E., Teoriya kolebanii (Theory of Oscillations), ONTI, 1937.

for the required periodic solution reduces to zero. In fact, let ω be the period of the required periodic solution. Hence $x(0) = x(\omega)$ and in the interval $(0, \omega)$ there must exist an instant of time t_1 for which the magnitude dx/dt becomes zero. We can make this instant of time to be the initial instant which, evidently, will only mean that the magnitude t in equation (9.1) is replaced by the magnitude $t + t_1$. Thus, in seeking the periodic solution of equation (9.1)[1] we can always assume that the magnitude dx/dt reduces to zero at $t = 0$.[1]

We now proceed to find the periodic solutions of equation (9.1) following the idea of Poincaré. For this purpose we consider first the generating equation

$$\frac{d^2x_0}{dt^2} + k^2x_0 = 0. \tag{9.2}$$

The general solution of this equation, for which $\dot{x}_0(0)=0$, is of the form

$$x_0 = M_0 \cos kt, \tag{9.3}$$

where M_0 is an arbitrary constant. This solution will be periodic of period $T = 2\pi/k$. Taking the solution corresponding to some fixed value of M_0 as the generating solution, we shall seek to obtain the periodic solution of equation (9.1) which for $\mu = 0$ reduces to the generating solution. Let this solution, which we shall denote by $x(t,\beta,\mu)$, be determined by the initial conditions

$$x(0, \beta, \mu) = x_0(0) + \beta = M_0 + \beta, \quad x_0(0, \beta, \mu) = 0, \tag{9.4}$$

where $\beta = \beta(\mu)$ is an unkown function of μ which becomes zero for $\mu = 0$. Thus, in contrast to autonomous systems, in the case considered the initial conditions of the required periodic solution contain only one unkown function. But on the other hand, as was pointed out above, the period of the required periodic solution for the system under consideration is not known beforehand. This period is equal to

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[1]Certain authors, among them A.A. Andronov and S.E. Khaikin (see also Minorsky N., Introduction to Non-Linear Mechanics, Ann Arbor, Mich., 1947) incorrectly recommend, in case it is not succeeded in finding a solution on the assumption $\dot{x}(0)=0$, trying to find it by assuming $x(0)=0$.

$$T+\alpha=\frac{2\pi}{k}+\alpha,$$

where $\alpha = \alpha(\mu)$ is likewise an unkown function of μ which, evidently, must reduce to zero for $\mu = 0$. Both unkown functions β and α are found from the conditions of periodicity of the solution $x(t, \beta, \mu)$.
These conditions are of the form

$$\left.\begin{aligned} x(T+\alpha, \beta, \mu)-x(0, \beta, \mu)\equiv x(T+\alpha, \beta, \mu)-M_0-\beta=0,\\ \dot{x}(T+\alpha, \beta, \mu)=0. \end{aligned}\right\} \tag{9.5}$$

Let us expand the left hand sides of these equations into a series in α and write out in the first equation the terms not higher than the second order and in the second equation the terms not higher than the first order. Then, taking (9.1) into account, we obtain

$$\left.\begin{aligned} x(T, \beta, \mu)+\dot{x}(T, \beta, \mu)\alpha+\\ +\frac{1}{2}[-k^2x(T, \beta, \mu)+\ldots]\alpha^2+\ldots-M_0-\beta=0,\\ \dot{x}(T, \beta, \mu)+[-k^2x(T, \beta, \mu)+\ldots]\alpha=0. \end{aligned}\right\} \tag{9.6}$$

Further, we have:

$$x(t, \beta, \mu)=x_0(t)+A\beta+\mu\{C+D\beta+E\mu+\ldots\}. \tag{9.7}$$

and also evidently,

$$\frac{d^2A}{dt^2}+k^2A=0, \quad A(0)=1, \quad \dot{A}(0)=0$$

so that

$$A=\cos kt.$$

Substituting in (9.6) and retaining in the first equation terms up to the second order and in the second equation terms that are linear with respect to α, β, μ we obtain:

$$\left.\begin{aligned} \mu\left\{C\left(\frac{2\pi}{k}\right)+D\left(\frac{2\pi}{k}\right)\beta+E\left(\frac{2\pi}{k}\right)\mu+\dot{C}\left(\frac{2\pi}{k}\right)\alpha+\ldots\right\}+\\ +\frac{1}{2}\alpha^2(-M_0k^2+\ldots)=0,\\ \mu\left\{\dot{C}\left(\frac{2\pi}{k}\right)+\ldots\right\}+\alpha(-M_0k^2+\ldots)=0. \end{aligned}\right\} \tag{9.8}$$

Assuming M_0 different from zero we can solve the second equation for α and this solution will be of the form

$$\alpha=\mu\left\{\frac{1}{M_0k^2}\dot{C}\left(\frac{2\pi}{k}\right)+\dots\right\},$$

where the terms not written down reduce to zero together with β and μ. Substituting α in the first of equations (9.8) we obtain for the determination of β the following equation:

$$\mu\left\{C\left(\frac{2\pi}{k}\right)+D\left(\frac{2\pi}{k}\right)\beta+Q\mu+\dots\right\}=0,$$

where Q is a certain coefficient which we do not need to write out.

For this equation to admit a solution for β that reduces to zero $\mu = 0$ it is necessary that the condition be satisfied

$$C\left(\frac{2\pi}{k}\right)=0. \tag{9.9}$$

If at the same time there is likewise satisfied the condition

$$D\left(\frac{2\pi}{k}\right)\neq 0, \tag{9.10}$$

then the solution for β will be the only one and analytic.

Let us compute the magnitudes $C(2\pi/k)$ and $D(2\pi/k)$. We have:

$$\frac{d^2C}{dt^2}+k^2C=f(x_0,\dot{x}_0,0),$$

$$\frac{d^2D}{dt^2}+k^2D=\frac{\partial f(x_0,\dot{x}_0,0)}{\partial x_0}\cos kt-k\frac{\partial f(x_0,\dot{x}_0,0)}{\partial\dot{x}_0}\sin kt,$$

$$C(0)=\dot{C}(0)=D(0)=\dot{D}(0)=0,$$

whence

$$C=\frac{1}{k}\int_0^t f(x_0(\tau),\dot{x}_0(\tau),0)\sin k(t-\tau)\,d\tau,$$

$$D=\frac{1}{k}\int_0^t\left\{\frac{\partial f(x_0,\dot{x}_0,0)}{\partial x_0}\cos kt-\right.$$

$$\left.-k\frac{\partial f(x_0,\dot{x}_0,0)}{\partial\dot{x}_0}\sin kt\right\}_{t=\tau}\sin k(t-\tau)\,d\tau\equiv\frac{\partial C}{\partial M_0}. \tag{9.11}$$

Hence, condition (9.9) has the form

$$\int_0^{2\pi/k} f(M_0 \cos k\tau, -kM_0 \sin k\tau, 0) \sin k\tau \, d\tau = 0$$

or, after change of variable under the integral sign,

$$P(M_0) = \int_0^{2\pi} f(M_0 \cos u, -kM_0 \sin u, 0) \sin u \, du = 0. \qquad (9.12)$$

As regards condition (9.10) it can, on the basis of (9.11), be rewritten in the following form:

$$\frac{dP(M_0)}{dM_0} \neq 0. \qquad (9.13)$$

We therefore arrive at the conclusion that to each nonmultiple root different from zero of equation (9.12) there corresponds only a single, analytic, solution of equations (9.5), and therefore a single periodic solution of equation (9.1), and this solution will be analytic with respect to μ. The period $2\pi/k + \alpha$ will likewise be analytic with respect to μ.

REMARK. We assume that M_0 is a simple root of equation (9.12). If M_0 makes not only the function $P(M_0)$ but also its derivative equal to zero, the question as to the existence of a periodic solution requires special investigation. Of particular interest is the case where the function $P(M_0)$ reduces to zero identically. Such case, as is easily seen, presents itself each time that the function $f(x, \dot{x}, \mu)$ does not contain $\dot{x}$ explicitly. The system considered will here be conservative and, as we shall see below, any solution of equation (9.1) will be periodic.

10. Computation of Periodic Solutions for an Autonomous System

We now proceed to the practical computation of the periodic solutions the existence of which was established in the preceding section.

Let M_0 be some nonmultiple root, different from zero, of equation (9.12). The periodic solution of equation (9.1) that corresponds to it, as was shown in the preceding section, will be analytic with respect to μ, and we could, as in analogous cases previously considered, seek this solution in the form of the series

$$x_0(t)+\mu x_1(t)+\mu^2x_2(t)+\ldots, \tag{10.1}$$

developed in terms of powers of μ. However, in the case under consideration, the representation of the solution in the form of the series (10.1) is found to be very unsuitable. The reason is that, since the period of the solution depends on μ, the coefficients of the expansion of this solution in powers of μ will not be periodic functions. In fact, since α depends on μ, therefore from the equation

$$x_0\left(t+\frac{2\pi}{k}+\alpha\right)+\mu x_1\left(t+\frac{2\pi}{k}+\alpha\right)+\ldots=x_0(t)+\mu x_1(t)+\ldots$$

the equations

$$x_s\left(t+\frac{2\pi}{k}+\alpha\right)=x_s(t).$$

by no means follow. On the contrary, the latter equations are evidently impossible, since their right hand sides do not depend on μ while the left hand sides do. Thus, for example, the expansion in powers of μ

$$\sin t+\mu t\cos t-\frac{\mu^2}{2}t^2\sin t+\ldots$$

of the function $\sin(1+\mu)t$, notwithstanding its peridicity, has nonperiodic coefficients. This arises from the circumstance that the period of this function , equal to $2\pi/(1+\mu)$, depends on μ.

Thus the form (10.1) of the required solution does not give any idea as regards the fundamental characteristic of this solution, namely its periodicity. We shall therefore seek to obtain the solution of interest in another form.[1] Let

$$\omega=\frac{2\pi}{k}+\alpha=\frac{2\pi}{k}(1+h_1\mu+h_2\mu^2+\ldots) \tag{10.2}$$

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[1]See Krylov N. and Bogolyubov N., Vvedenie v nelineinuyu mekhaniku. (Introduction to Nonlinear Mechanics), Kiev,1937,p.106. See also on the method of Liapounoff presented below in sec.3, ch. VII.

be the period of the required solution, where h_j are certain unkown but entirely definite constants. In equation (9.1) replace the variable t by the variable τ with the aid of the substitution

$$t = \frac{\tau}{k}(1 + h_1\mu + h_2\mu^2 + \ldots). \tag{10.3}$$

To the required periodic solution of equation (9.1) with period (10.2) there will then correspond the periodic solution of the new equation obtained in this manner

$$\frac{d^2x}{d\tau^2} + x(1 + h_1\mu + h_2\mu^2 + \ldots)^2 =$$
$$= \frac{\mu}{k^2} f\left[x, k(1 + h_1\mu + \ldots)^{-1}\frac{dx}{d\tau}, \mu\right](1 + h_1\mu + \ldots)^2 \tag{10.4}$$

with a period not depending on μ and equal to 2π. This new periodic solution will likewise be analytic with respect to μ. In fact, the solution considered, like every other solution of equation (10.4), will be an analytic function of the parameter μ, directly entering the equation, and of its initial values. But these initial values, coinciding evidently with (9.4), are in their turn certain entirely definite analytic functions of μ. From this the validity of our assertion immediately follows.

We shall therefore seek the periodic solution of equation (10.4) in the form of the series

$$x = x^{(0)}(\tau) + \mu x_1(\tau) + \mu^2 x_2(\tau) + \ldots \tag{10.5}$$

Since the period of the solution now does not depend on μ and is equal to 2π, the coefficients $x_s(\tau)$ will be periodic functions of τ of period 2π. Moreover, we must have:

$$\left(\frac{dx^{(0)}}{d\tau}\right)_{\tau=0} = \left(\frac{dx_j}{d\tau}\right)_{\tau=0} = 0 \qquad (j = 1, 2, \ldots), \tag{10.6}$$

since for the required periodic solution the derivative dx/dt reduces to zero for $t = 0$.

Since the required periodic solution undoubtedly exists, there must exist at least one series of the form (10.5) formally satisfying equation (10.4). If it turns out however that only one such series exists, then it will evidently represent the required solution and will, therefore, converge.

Substituting series (10.5) in equation (10.4) and equating coefficients with like powers of μ, we obtain an equation for finding the functions $x_j(t)$. First of all we have:

$$\frac{d^2x^{(0)}}{d\tau^2}+x^{(0)}=0,$$

whence, taking (10.6) into account, we obtain

$$x^{(0)}=M_0\cos\tau,$$

where M_o is an arbitrary constant. Further , we find

$$\frac{d^2x_1}{d\tau^2}+x_1=-2h_1M_0\cos\tau+\frac{1}{k^2}f(M_0\cos\tau,\ -kM_0\sin\tau,\ 0).$$

For this equation to have a periodic solution it is necessary and sufficient that the conditions be satisfied

$$\left.\begin{aligned} P(M_0)=\int_0^{2\pi} f(M_0\cos\tau,\ -kM_0\sin\tau,\ 0)\sin\tau\,d\tau=0,\\ -2h_1M_0+\frac{1}{\pi k^2}\int_0^{2\pi} f(M_0\cos\tau,\ -kM_0\sin\tau,\ 0)\cos\tau\,d\tau=0.\end{aligned}\right\}\quad(10.7)$$

The first of these conditions is already known to us as the equation determining the amplitude M_o of the generating solution. Having chosen the magnitude M_o and assuming, as we already have done above, that it is a nonmultiple root, different from zero, of the first of equations (10.7), we obtain from the second condition of (10.7) an entirely definite value for h_1. For x_1 we shall then have:

$$x_1=\varphi_1(\tau)+M_1\cos\tau+N_1\sin\tau,$$

where $\varphi_1(\tau)$ is a certain periodic function of τ of period 2π and M_1 and N_1 are arbitrary constants. These constants are determined from the initial conditions (10.6) and from the condition of periodicity of the function x_2. Together with these there is also determined the constant h_2 . We shall show how this is done.

For this purpose let us consider the equation determining the m-th approximation. As is easily seen, it is of the form

$$\frac{d^2x_m}{d\tau^2}+x_m=-2h_mM_0\cos\tau-2h_1x_{m-1}+$$
$$+\frac{1}{k^2}\left(\frac{\partial f(x,\dot{x},0)}{\partial x}\right)x_{m-1}+\frac{1}{k}\left(\frac{\partial f(x,\dot{x},0)}{\partial \dot{x}}\right)\frac{dx_{m-1}(\tau)}{d\tau}+F_m. \quad (10.8)$$

where F_m is an integral rational function of $x^{(0)}, x_1, \ldots, x_{m-2}$ with constant coefficients. This function depends also on $h_1, \ldots, h_{m-1}$, but does not depend on h. The parentheses around the derivatives indicate that after differentiation the magnitudes x and $\dot{x}$ should be replaced by the magnitudes $M_0\cos\tau$ and $-kM_0\sin\tau$.

If equation (10.8) has a periodic solution, it will be of the form

$$x_m=\varphi_m(\tau)+M_m\cos\tau+N_m\sin\tau,$$

where M_m and N_m are arbitrary constants and φ_m is a certain periodic function. For definiteness let us assume that all constants $h_1, \ldots, h_{m-1}$ and all functions $x_1, \ldots, x_{m-1}$ are already computed but that the constants M_{m-1} and N_{m-1} entering the expression x_{m-1} as yet remained undetermined. Equation (10.8) can then be represented in the form

$$\frac{d^2x_m}{d\tau^2}+x_m=-2h_mM_0\cos\tau-2h_1(M_{m-1}\cos\tau+N_{m-1}\sin\tau)+$$
$$+\frac{1}{k^2}\left(\frac{\partial f(x,\dot{x},0)}{\partial x}\right)(M_{m-1}\cos\tau+N_{m-1}\sin\tau)+$$
$$+\frac{1}{k}\left(\frac{\partial f(x,\dot{x},0)}{\partial \dot{x}}\right)(-M_{m-1}\sin\tau+N_{m-1}\cos\tau)+F_m^*(\tau),$$

where F_m^* is a certain known periodic function of τ of period 2π. The conditions of periodicity of the function

x_m give

$$-2h_m M_0 - 2h_1 M_{m-1} +$$
$$+\frac{M_{m-1}}{\pi k^2}\int_0^{2\pi}\left\{\left(\frac{\partial f(x,\dot{x},0)}{\partial x}\right)\cos\tau - k\left(\frac{\partial f(x,\dot{x},0)}{\partial \dot{x}}\right)\sin\tau\right\}\cos\tau\,d\tau +$$
$$+\frac{N_{m-1}}{\pi k^2}\int_0^{2\pi}\left\{\left(\frac{\partial f(x,\dot{x},0)}{\partial x}\right)\sin\tau + k\left(\frac{\partial f(x,\dot{x},0)}{\partial \dot{x}}\right)\cos\tau\right\}\cos\tau\,d\tau +$$
$$+\frac{1}{\pi}\int_0^{2\pi} F_m^*(\tau)\,d\tau = 0,$$

$$-2h_1 N_{m-1} +$$
$$+\frac{M_{m-1}}{\pi k^2}\int_0^{2\pi}\left\{\left(\frac{\partial f(x,\dot{x},0)}{\partial x}\right)\cos\tau - k\left(\frac{\partial f(x,\dot{x},0)}{\partial \dot{x}}\right)\sin\tau\right\}\sin\tau\,d\tau +$$
$$+\frac{N_{m-1}}{\pi k^2}\int_0^{2\pi}\left\{\left(\frac{\partial f(x,\dot{x},0)}{\partial x}\right)\sin\tau + k\left(\frac{\partial f(x,\dot{x},0)}{\partial \dot{x}}\right)\cos\tau\right\}\sin\tau\,d\tau +$$
$$+\frac{1}{\pi}\int_0^{2\pi} F_m^*\sin\tau\,d\tau = 0. \quad (10.9)$$

Moreover, from the initial conditions (10.6) we obtain:

$$\dot{\varphi}_{m-1}(0) + N_{m-1} = 0. \quad (10.10)$$

Equations (10.9) and (10.10) serve to determine M_{m-1}, N_{m-1} and h_m. Let us consider these equations in more detail. We have identically:

$$\int_0^{2\pi}\left\{\left(\frac{\partial f(x,\dot{x},0)}{\partial x}\right)\sin\tau + k\left(\frac{\partial f(x,\dot{x},0)}{\partial \dot{x}}\right)\cos\tau\right\}\sin\tau\,d\tau =$$
$$= -\frac{1}{M_0}\int_0^{2\pi}\frac{df(M_0\cos\tau, -kM_0\sin\tau, 0)}{d\tau}\sin\tau\,d\tau =$$
$$= \frac{1}{M_0}\int_0^{2\pi} f(M_0\cos\tau, \ -kM_0\sin\tau, 0)\cos\tau\,d\tau,$$

$$\int_0^{2\pi}\left\{\left(\frac{\partial f(x,\dot{x},0)}{\partial x}\right)\sin\tau + k\left(\frac{\partial f(x,\dot{x},0)}{\partial \dot{x}}\right)\cos\tau\right\}\cos\tau\,d\tau =$$
$$= -\frac{1}{M_0}\int_0^{2\pi} f(M_0\cos\tau, \ -kM_0\sin\tau, 0)\sin\tau\,d\tau,$$

$$\left(\frac{\partial f(x,\dot{x},0)}{\partial x}\right)\cos\tau - k\left(\frac{\partial f(x,\dot{x},0)}{\partial \dot{x}}\right)\sin\tau = \frac{\partial f(M_0\cos\tau, -kM_0\sin\tau, 0)}{\partial M_0}.$$

Hence, taking (10.7) into account, we can reduce equations (10.9) to the form

$$\left.\begin{aligned} -2h_m M_0 + 2M_0 M_{m-1}\frac{dh_1}{dM_0} + \frac{1}{\pi}\int_0^{2\pi} F_m^*(\tau)\cos\tau\, d\tau = 0,\\ \frac{1}{k^2}\frac{dP(M_0)}{dM_0}M_{m-1} + \int_0^{2\pi} F_m^*(\tau)\sin\tau\, d\tau = 0. \end{aligned}\right\} \quad (10.11)$$

Since the root M_0 is simple and different from zero, the second of equations (10.11) uniquely determines the magnitude M_{m-1} while the first of them uniquely determines the magnitude h_m. As regards equation (10.10), it uniquely determines the magnitude N_{m-1}.

Thus we have shown that there exists only one series of the form (10.5) formally satisfying equation (10.4). This series represents the required periodic solution in a practically suitable form. Furthermore, we have obtained also a suitable method of computing the coefficients of this series with the simultaneous computation of the coefficients h_j in the expansion for the period. In sec.12 we shall illustrate the entire course of the computation on a concrete example, but first of all we shall go on to the geometric interpretation of the obtained results.

11. Phase Plane for the System Considered in the Preceding Section. Limit Cycles . Self-Oscillations

Let us first consider the system described by equations (9.1). We shall put in this equation $\dot{x} = y$ and write it in the form of a system of two equations:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -k^2 x + \mu f(x, y, \mu). \qquad (11.1)$$

By considering x and y as the rectilinear coordinates of a certain point in the plane xy, we are able to represent all possible states of our system. The plane xy is called the plane of the states or the PHASE PLANE. To each motion of our system corresponds a certain curve in the phase plane, this curve is called the PHASE TRAJECTORY. Naturally, it has nothing in common with the trajectory of the motion. The phase trajectories are evidently the integral curves of the equation

$$\frac{dy}{dx} = \frac{-k^2x + \mu f(x, y, \mu)}{y}. \tag{11.2}$$

We shall discuss all possible forms of the phase trajectories.

We note first of all that for sufficiently small μ there exists one and only one point for which the right hand sides of equations (11.1) simultaneously reduce to zero. This point is situated on the x-axis at a distance of the order of smallness of μ from the origin. To it corresponds the only state of equilibrium of our system. It is evident that the state of equilibrium corresponds to the phase trajectory which degenerates into a point.

This point is a singular point of differential equation (11.2) . Through it may pass one integral curve, or there may pass several or even an infinite number of such curves, but it may also happen that not even one integral curve passes through this point. In fact, the theorem of Cauchy on the existence of solutions of differential equations is not applicable for the point considered because for this point the numerator and denominator in equation (11.2) simultaneously reduce to zero. For all other points in the phase plane that lie in the region G either dy/dx or dx/dy are analytic functions of x and y, hence all these points are ordinary points. Through each of them, according to the theorem of Cauchy, there passes one and only one integral curve.

Let us consider some periodic motion of our system. To it there evidently corresponds a closed phase trajectory. Conversely, to each closed phase trajectory there corresponds a periodic solution of equations (11.1). In fact, let T be the interval of time during which the point (x,y) describes completely a phase trajectory, starting from the initial instant of time t_o from any position (x_o, y_o) of this trajectory. We shall take $t_o + T$ as the initial instant. The values of x and y will as before be equal to x_o and y_o. But since equations (11.1) do not contain t explicitly, the initial instant of time does not play any part and the functions x and y will in the interval $(t_o + T, t_o + 2T)$ be the same as in the interval $(t_o, t_o + T)$, whence the periodicity of these functions immediately follows.

Having established this, let us assume at first that

$\mu = 0$, i.e. we consider the generating system

$$\frac{dx_0}{dt} = y_0, \quad \frac{dy_0}{dt} = -k^2 x_0.$$

The general solution

$$x_0 = M_0 \cos(kt + \alpha), \quad y_0 = -M_0 k \sin(kt + \alpha) \tag{11.3}$$

of this system is periodic. Hence all phase trajectories are closed curves. One of these curves degenerates into a point, the origin of coordinates, to which corresponds the state of equilibrium.

In the case considered it is easy of course to write down the equation of the phase trajectories. For this it is sufficient to eliminate t from the equations (11.3). Eliminating it we obtain

$$x_0^2 + \frac{y_0^2}{k^2} = M_0^2, \tag{11.4}$$

and therefore the phase trajectories form a family of similar ellipses which contract into the origin of coordinates.

The singular point of a differential equation of the first order possessing the property that all integral curves situated at a sufficiently small neighborhood of this point are closed is called a "CENTER". Thus, for $\mu = 0$ the singular point of equation (11.2), corresponding to the state of equilibrium, is a center.

Let us assume now that μ is different from zero but sufficiently small. In this case, as we have seen, the system will as before possess one position of equilibrium corresponding to the singular point of equation (11.2), which now, generally speaking, will not coincide with the origin but will be somewhat displaced from it on the x-axis. As regards the periodic solutions, equations (11.1), generally speaking, as was shown in sec. 9, have only a finite number of this type of solutions. Such for example will always be the case when the equation (9.12) has only simple roots for the amplitude M_0 of the generating solution.

Hence, excluding from consideration the special cases on which we spoke in the remarks in sec. 9, we arrive at the conclusion that in the phase plane in the region G there are only a finite number of closed trajectories. These closed trajectories will differ little from ellipses (11.4) corresponding to them and will therefore surround the origin and the singular point A corresponding to the point of equilibrium (fig.7)

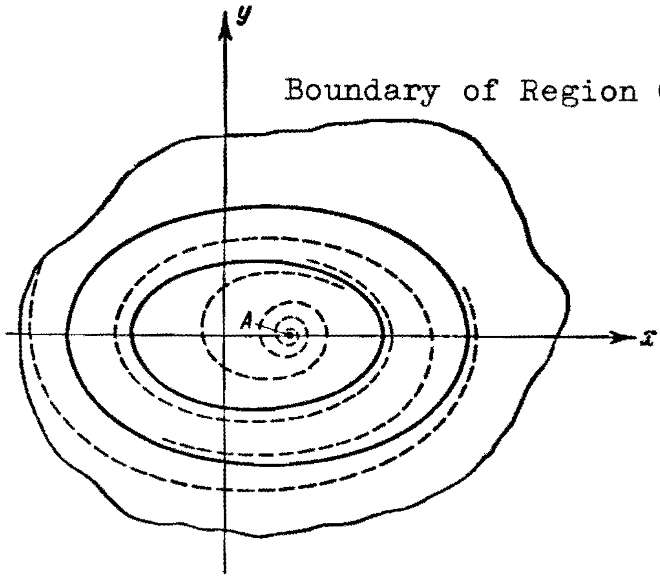

Fig. 7

Thus, in passing from the case $\mu = 0$ to the case $\mu \neq 0$ the singular point goes over into a singular point, while a finite number of ellipses (11.4) go over into closed trajectories differing little from them. But into what do the remaining ellipses (11.4) go over? In other words, what form will the remaining trajectories have? We have here in view only those phase trajectories which lie in the region G. We shall now show that the following important propositions hold true.

EACH PHASE TRAJECTORY LYING BETWEEN TWO CLOSED PHASE TRAJECTORIES IS A SPIRAL. THIS SPIRAL ASYMPTOTICALLY APPROACHES, AS IT MAKES AN INFINITE NUMBER OF TURNS, BOTH CLOSED TRAJECTORIES (ONE AT $t = +\infty$, THE OTHER AT $t = -\infty$) BETWEEN WHICH IT LIES.

EACH PHASE TRAJECTORY LYING WITHIN THE FIRST CLOSED TRAJECTORY ASYMPTOTICALLY APPROACHES THIS TRAJECTORY FROM ONE SIDE AND THE SINGULAR POINT FROM THE OTHER SIDE.

EACH PHASE TRAJECTORY LYING OUTSIDE THE LAST PHASE

TRAJECTORY AND SUFFICIENTLY CLOSE TO IT APPROACHES THIS TRAJECTORY FROM ONE SIDE AND LEAVES THE REGION G FROM THE OTHER SIDE.

The form of the phase trajectories in all three cases is show in fig. 7.

To prove these propositions let us consider some phase trajectory and follow it along some definite direction corresponding either to increasing or decreasing values of t . We shall show that for no finite value of t does the phase trajectory pass through the singular point. In fact, let

$$y=0, \quad x=a \tag{11.5}$$

be the coordinates of the singular point. If at any instant of time t the phase trajectory intersected the singular point, then, taking this instant of time as the initial one, we would have for the system (11.1) two different solutions with the same initial conditions: the solution corresponding to the trajectory considered and the solution (11.5) corresponding to the state of equilibrium. But the point (11.5) is the singular point for the equation (11.2). The system of equations (11.1) does not have singular points, since the right hand sides of these equations are regular in the entire region G. Hence for any initial values the system (11.1) has an entirely definite solution. In this way we arrived at a contradiction, showing that the trajectory considered does not pass through the singular point for any values of t. This trajectory furthermore does not intersect itself for any values of t nor any other phase trajectory, since the point of intersection would have to be a singular point of the system (11.1).

Having established this, let us in equation (11.1) go over to polar coordinates r and ϑ with the pole at the point (11.5) and with the polar axis coinciding with the x- axis. Taking positive angles in the clockwise direction, we shall have

$$x=a+r\cos\vartheta, \quad y=-r\sin\vartheta,$$

whence

$$\frac{d\vartheta}{dt}=\frac{1}{r^2}\left\{(a-x)\frac{dy}{dt}+y\frac{dx}{dt}\right\}.$$

Replacing dx/dt and dy/dt by the values from equations (11.1) and taking into account that

$$-k^2a+\mu f(a, 0, \mu)=0,$$

we obtain, after several transformations,

$$\frac{d\vartheta}{dt}=k^2\cos^2\vartheta+\sin^2\vartheta+\mu\theta(r, \vartheta, \mu), \tag{11.6}$$

where the function θ is regular for r = 0. Since

$$k^2\cos^2\vartheta+\sin^2\vartheta\begin{cases} >1 \text{ (for } k>1), \\ >k^2 \text{(for } k<1), \end{cases}$$

it follows from (11.6) that for sufficiently small μ a positive number α^2 exists such that for the entire region G the inequality holds

$$\frac{d\vartheta}{dt}>\alpha^2. \tag{11.7}$$

From this the result immediately follows that when $t\rightarrow+\infty$ then also $\vartheta\rightarrow+\infty$, and that when $t\rightarrow-\infty$, then also $\vartheta\rightarrow-\infty$. This of course will be true as long as the integral curve remains within the region G. Hence each integral curve lying in the region G, since by what was proven it does not intersect itself, must necessarily be either a spiral turning about the singular point or a closed curve.

Moreover, it follows from (11.7) that each such spiral (as well as closed trajectory) makes a complete turn during an interval of time not exceeding the magnitude $2\pi/\alpha^2$, independently of the initial conditions.

Let us now consider the phase trajectory L (fig.8), starting from sone point P of the x- axis.

We shall assume at first that the point P lies between two closed phase trajectories M_1 and M_2 between which there are no other closed phase trajectories. The integral curve considered will, by what has been proven above, be a spiral. We shall show that this spiral making an infinite number of turns asymptotically approaches both closed trajectories: one of them at $t\rightarrow+\infty$, and the other at $t\rightarrow-\infty$.

For this purpose let us follow along this spiral in a direction corresponding to increasing values of t. The spiral under consideration will not leave the region included between the closed trajectories since it cannot intersect these trajectories. For definiteness let us assume that the following point of intersection P_1 of the spiral with x- axis lies to the right of the point P. Then, if we denote by P_2, P_3,... the successive points of intersection of our spiral with the x-axis, each point P_n will lie to the right of point P_{n-1}. The sequence of points P_j will be infinite and therefore for $t \to \infty$ the spiral makes an infinite number of turns. This follows immediately from the proposition proved above that each turn is made in an interval of time not exceeding a certain fixed number.

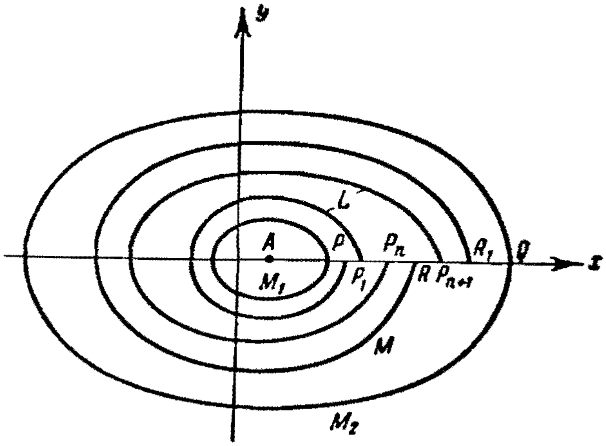

Fig. 8

Since all points P_j are situated to the left of the point of intersection Q of the outer closed trajectory with the x- axis, the sequence P_j will have a limiting point. Evidently there will be only one such limiting point and it either coincides with Q or is to the left of it.

We shall prove that the limiting point coincides with Q. Assume that the contrary is true, namely, that this point, which we shall denote by R, lies to the left of point Q. Consider the trajectory M passing through R. Since between M_1 and M_2 there are no closed trajectories, M is a spiral. Let R_1 be the first point of intersection after R of the trajectory M (if we follow along it in the direction of increasing t) with the x-axis. It can easily be proven that the point R_1 lies to the right of the point R, but we shall not give the proof since this has no significance for us. Simultaneously with M we shall consider the coil of the trajectory L connecting the points P_n and P_{n+1}. We shall show that if n is sufficiently large this coil will border arbitrarily closely upon a segment of the trajectory M. In fact, any solution of equations (11.1) is an analytic function of its initial conditions, and therefore, also of a continuous function of these magnitudes. This means that, whatever the finite magnitude T may be, two phase trajectories will remain arbitrarily close to each other during the entire time interval (t,t+T), provided at the instant of time t they issued from sufficiently close points of the phase plane. But since R is a limiting point of the sequence P_j, the point P_n will be as close as we please to the point R. Hence, as we asserted, the coil of the trajectory L issuing from the point P_n will lie arbitrarily close to the coil of the trajectory M issuing from the point R. In particular, the point P_{n+1} will be as close as we please to the point R_1.

In other words, when n increases without limit the point P_{n+1} approaches infinitely near the point R_1 situated at a definite distance from R_1 independently of the value of n. Thus we have arrived at a contradiction, which shows that the point R must coincide with the point Q.

Hence, as n increases without limit the point P_n approaches the point Q. But then, as we have already seen, the n-th coil of the spiral will likewise approach infinitely close to the trajectory issuing from Q, i.e. the trajectory M_2. Thus, for $t \to +\infty$ the trajectory issuing from P is a spiral asymptotically approaching the closed trajectory M_2. In exactly the same way it can be shown that for $t \to -\infty$ this spiral will wind on to the closed trajectory M_1.

We have assumed that the point P lies between two closed trajectories. Let us now assume that the point P lies within the first closed trajectory. Since, by what has been proved, any nonclosed trajectory lying in the region G is a spiral winding around a singular point , all our preceding considerations, as is not difficult to see, maintain their validity if the singular point is regarded as a degenerate closed trajectory. In fact, in order that in the preceding considerations it be possible to replace the inside closed trajectory by the singular point it is sufficient that the spiral wind around this point, as is the case, and that this spiral do not pass through this point for any value of t, as is likewise the case by our proof above. Thus, the phase trajectory under consideration is a spiral which on the one side asymptotically approaches the first closed trajectory and on the other asymptotically approaches the singular point.

In exactly the same way it can be shown that if the point P lies outside the last closed trajectory and is sufficiently near it, the corresponding trajectory will on the one side wind on to the closed trajectory and on the other side will leave the region G. Thus, our assertions in regard to the form of all phase trajectories may be considered as proved.

We saw that for $\mu = 0$ there corresponds to the state of equilibrium in the phase plane a singular point of the type of a center, i.e. a singular point such that all trajectories lying in a sufficiently small neighborhood of it are closed. In the case we shall now consider, on the contrary, there corresponds to the state of equilibrium a singular point of an entirely different type : all the trajectories lying in the neighborhood of this point form a family of spirals asymptotically approaching this point. A singular point of this type is called a "focus".

In exactly the same way the closed phase trajectories, corresponding to periodic motions, will now be of another type. Thus, for $\mu = 0$ all trajectories adjoining any closed trajectory will likewise be closed. In the case now considered however all trajectories adjoining a closed trajectory are spirals asymptotically approaching this closed trajectory. Closed trajectories of this type are called LIMIT CYCLES.

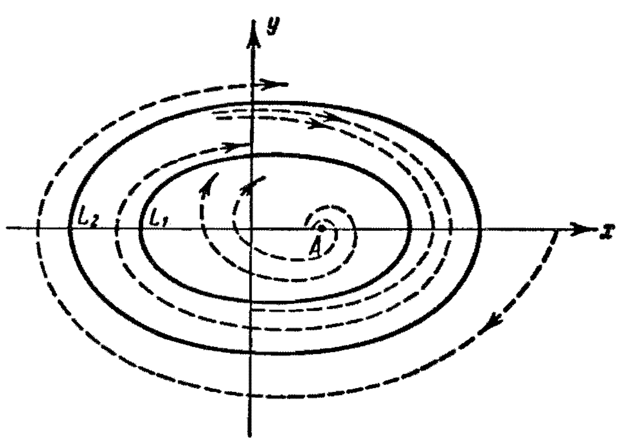

Fig. 9

Two kinds of limiting cycles are distinguished: STABLE and UNSTABLE. A limiting cycle is called stable if with increasing t all neighboring trajectories approach it. If however all trajectories adjoining a limiting cycle, or only those which lie on one side of it, move away from this cycle with increasing t such limiting cycle is called unstable. Thus, of the five limiting cycles shown in fig. 9 and 10 the cycles L_1 and M_2 will be stable while the others will be unstable.

TO STABLE LIMIT CYCLES CORRESPOND SELF-OSCILLATORY MOTIONS. In fact, let us assume for definiteness that we are dealing with the case represented in fig. 9. The state of equilibrium will in this case be unstable and, whatever the initial values of x and y, provided only that they are included in the region bounded by the second limiting cycle L_2, the motion of the system will asymptotically approach periodic oscillation corresponding to the limiting cycle L_1. Theoretically this oscillation is established at $t = \infty$. However, after the lapse of a definite interval of time the spirals approach so close to the limit cycle that it may practically be assumed that the motion takes place according to this cycle. Thus, for any initial values lying in a certain region, after the lapse of a certain interval of time a definite stable regime of oscillation is established which is entirely independent of these initial conditions, i.e. the regime of self-oscillations.

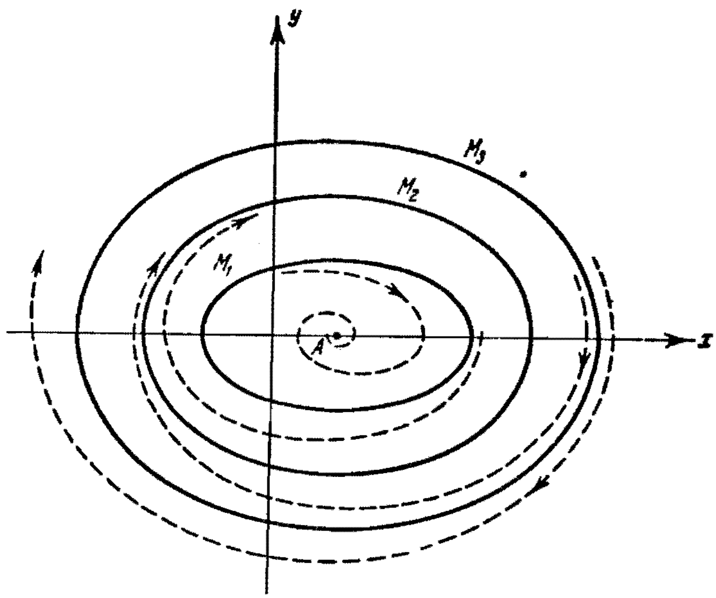

Fig. 10

Self-oscillations correspond to any stable limit cycle. Thus in the case represented in figure 10, to the stable limit cycle M_2 there likewise correspond self-oscillations, the region of initial values being bounded by the limit cycles M_1 and M_3. But in contrast to the case represented in figure 9, the state of equilibrium in the case under consideration is stable and in order that the system start to carry our self-oscillations it must be imparted a sufficiently large deviation from the position of equilibrium.

It follows from what has been said that the obtaining of stable limit cycles is one of the most important problems of the theory of nonlinear oscillations of autonomous systems. In this section a general procedure was indicated for obtaining the limit cycles of autonomous quasilinear systems. In section 16 a simple method will be given for the solution of the problem of the stability of these cycles.

To the nonstable limit cycles, as closed phase trajectories, there will likewise correspond periodic oscillations. These oscillations however do not have any physical significance, since on account of their instability, they will practically never be realized.

The theory of limit cycles, a particular case of which we have here considered, was first worked out by A. Poincaré [1] and then served as a subject for numerous further investigations. But the connection of this theory with the problem of self-oscillations was first considered by A.A. Andronov[2].

12. Self-Oscillations of a Tube Generator

As an example let us consider a tube generator working according to the scheme represented in fig. 11. It differs from the scheme considered in sec. 5 in the absence

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[1] Poincaré A., On Curves Determined by Differential Equations.

[2] Andronov A.A., Predelnye tsikly Puankare i teoriya kolebanii (Limit Cycles of Poincaré and the Theory of Oscillations), Sixth Congress of Russian Physicists , Gosisdat 1928. See also Andronov A.A., Les cycles limites de Poincare et la théorie des oscillations autoentretenues, Comptes Rendus, vol. 189, p.559, 1929.

of an externally applied action. Hence for the assumptions made in sec. 5, we can write down the following differential equations for the grid voltage v:

$$\frac{d^2v}{dt^2} + \omega^2 v = \mu(\alpha - \gamma^2 v^2)\frac{dv}{dt}. \qquad (12.1)$$

$$\frac{d^2v}{dt^2} + \omega^2 v = \mu(\alpha + \gamma^2 v^2 - \delta^2 v^4)\frac{dv}{dt}. \qquad (12.2)$$

The first of these equations holds for the case of a tube with soft operating characteristic, the second for a tube with hard operating characteristic.

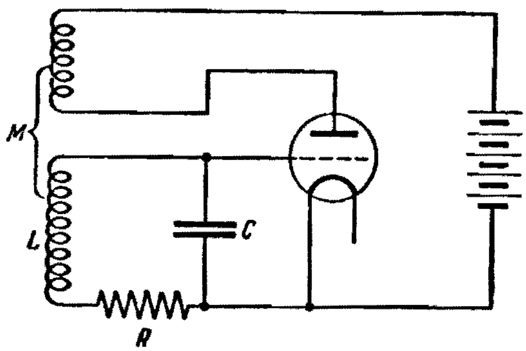

Fig. 11

To find the periodic solutions of equations (12.1) and (12.2) we proceed according to the rule of sec. 10. For this purpose we transform the equation with the aid of the substitution

$$t = \frac{\tau}{\omega}(1 + h_1\mu + h_2\mu^2 + \dots),$$

where h_1, h_2,... are certain unkown constants. The transformed equations assume the form

$$\frac{d^2v}{d\tau^2} + v(1 + h_1\mu + \dots)^2 = \frac{\mu}{\omega}(\alpha - \gamma^2 v^2)(1 + h_1\mu + \dots)\frac{dv}{d\tau}, \qquad (12.3)$$

$$\frac{d^2v}{d\tau^2} + v(1 + h_1\mu + \dots)^2 = \frac{\mu}{\omega}(\alpha + \gamma^2 v^2 - \delta^2 v^4)(1 + h_1\mu + \dots)\frac{dv}{d\tau}. \qquad (12.4)$$

We shall try to satisfy these equations formally by the series

$$v = M_0 \cos\tau + \mu v_1(\tau) + \mu^2 v_2(\tau) + \ldots, \tag{12.5}$$

where $v_i(\tau)$ is a periodic function of τ of period 2π for which the initial conditions are satisfied

$$\dot{v}_1(0) = \dot{v}_2(0) = \ldots = 0. \tag{12.6}$$

Let us consider first equation (12.3). For this equation we have

$$\frac{d^2 v_1}{d\tau^2} + v_1 + 2h_1 M_0 \cos\tau = -\frac{1}{\omega}(\alpha - \gamma^2 M_0^2 \cos^2\tau) M_0 \sin\tau =$$
$$= \left(-\frac{\alpha M_0}{\omega} + \frac{\gamma^2 M_0^3}{4\omega} \right) \sin\tau + \frac{\gamma^2 M_0^3}{4\omega} \sin 3\tau.$$

Equating to zero the coefficients of τ and $\sin\tau$, we obtain:

$$h_1 = 0, \; P(M_0) = -\frac{\alpha M_0}{\omega} + \frac{\gamma^2 M_0^3}{4\omega} = 0, \; M_0^2 = \frac{4\alpha}{\gamma^2}, \tag{12.7}$$

after which for v_1 we obtain the solution

$$v_1 = -\frac{\gamma^2 M_0^3}{32\omega} \sin 3\tau + M_1 \cos\tau + N_1 \sin\tau,$$

where M_1 and N_1 are arbitrary constants. The equation for v_2 is of the form

$$\frac{d^2 v_2}{d\tau^2} + v_2 = -2h_2 M_0 \cos\tau + \frac{\alpha}{\omega}\frac{dv_1}{d\tau} - \frac{\gamma^2 M_0^2}{\omega}\cos^2\tau \frac{dv_1}{d\tau} + \frac{\gamma^2 M_0^2}{\omega} v_1 \sin 2\tau =$$
$$= \left[M_0 \left(-2h_2 + \frac{\gamma^4 M_0^4}{128\omega^2} \right) + N_1 \left(\frac{\alpha}{\omega} - \frac{\gamma^2 M_0^2}{4\omega} \right) \right] \cos\tau +$$
$$+ M_1 \left(-\frac{\alpha}{\omega} + \frac{3\gamma^2 M_0^2}{4\omega} \right) \sin\tau + \ldots =$$
$$= M_0 \left(-2h_2 + \frac{\alpha^2}{8\omega^2} \right) \cos\tau + \frac{\gamma^2 M_0^2 M_1}{2\omega} \sin\tau + \ldots,$$

where the terms not written down do not contain $\cos\tau$ and

$\sin \tau$. Equating to zero the coefficients of $\cos \tau$ and $\sin \tau$ we obtain

$$h_2 = \frac{\alpha^2}{16\omega^2}, \qquad M_1 = 0$$

and from (12.6)

$$N_1 = \frac{3\gamma^2}{32\omega} M_0^3.$$

Thus, the required periodic solution of equation (12.1) has the form

$$\left.\begin{aligned} v &= M_0 \cos \tau + \left\{ \frac{3\gamma^2}{32\omega} M_0^3 \sin \tau - \frac{\gamma^2}{32\omega} M_0^3 \sin 3\tau \right\} \mu + \ldots, \\ \tau &= \omega t \left(1 - \frac{1}{16} \frac{\alpha^2}{\omega^2} \mu^2 + \ldots \right), \\ M_0 &= \frac{\sqrt{4\alpha}}{\gamma}. \end{aligned}\right\} \tag{12.8}$$

In order for this solution to be real it is necessary that α be positive. Thus, for $\alpha < 0$ there are no limit cycles in the system (12.1). The state of equilibrium (12.1) $v = 0$ will be here stable, as is immediately seen from equation (12.1) where the magnitude $-\alpha$ is the coefficient of linear damping. The form of the phase plane for α 0 is shown in fig. 12. No oscillations can arise in the system, all motions dampen out.

For $\alpha > 0$ the state of equilibrium will be unstable. As regards the periodic solution (12.8) , it will be stable, as will be shown in sec. 16, and to it will therefore correspond a stable limit cycle.

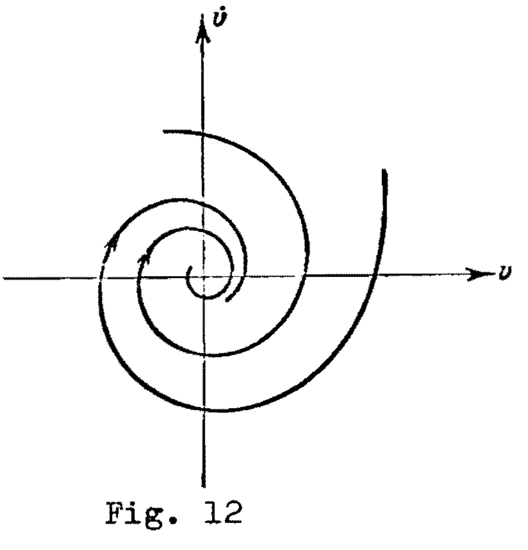

Fig. 12

The form of the phase plane for this case is shown in fig.13. The system will be a self-excited one. For any initial conditions a stable self-oscillation regime is established in it. The period of the oscillations, as follows from (12.8), is determined by the formula

$$T=\frac{2\pi}{\omega}\left(1+\frac{\alpha^2}{16\omega^2}\mu^2+\ldots\right).$$

We next consider the case of the tube with hard operating characteristic, restricting ourselves to the computation of the zeroth approximation. Substituting (12.5) in (12.4) we obtain:

$$\begin{aligned}\frac{d^2v_1}{d\tau^2}+v_1+2h_1M_0\cos\tau &= \\ &=\frac{1}{\omega}(\alpha+\gamma^2M_0^2\cos^2\tau-\delta^2M_0^4\cos^4\tau)(-M_0\sin\tau)= \\ &=\left(-\frac{\alpha}{\omega}M_0-\frac{\gamma^2}{4\omega}M_0^3+\frac{\delta^2}{8\omega}M_0^5\right)\sin\tau+\ldots,\end{aligned}$$

where the terms not written down do not contain cos τ and sin τ. Equating to zero the coefficients of cos τ and sin τ we obtain

$$h_1=0,\ P(M_0)=-\frac{\alpha}{\omega}M_0-\frac{\gamma^2}{4\omega}M_0^3+\frac{\delta^2}{8\omega}M_0^5=0, \qquad (12.9)$$

whence

$$M_0^2=\frac{\gamma^2\pm\sqrt{\gamma^4+32\alpha\delta^2}}{\delta^2}. \qquad (12.10)$$

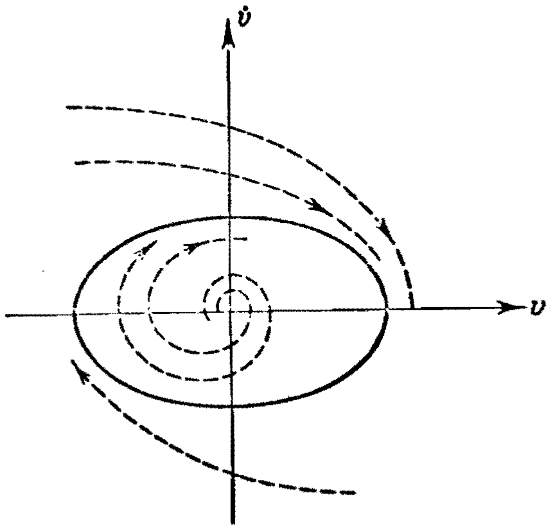

Fig. 13

In contrast to the preceding case, for the equation (12.2) periodic solutions may exist both for $\alpha > 0$ as well as for $\alpha < 0$. If $\alpha > 0$ we obtain (12.10) only one real value for M_o. The periodic solution corresponding to this value, as we shall show in sec. 16, will be stable. Since the state of equilibrium is at the same time unstable, the form of the phase plane will be that shown in fig. 13. Hence, as for the case of the soft characteristic tube, the system will be self-excited.

If $\alpha < 0$, then for $\gamma^4 + 32\alpha\delta^2 < 0$, the system will not have periodic solutions and the phase plane will be the same as in fig. 12. If $\alpha < 0$ and $\gamma^4 + 32\alpha\delta^2 > 0$ there will exist two periodic solutions. As we shall show in sec.16, the solutions corresponding to the smaller value of M_o will be unstable while that corresponding to the larger value of M_o will be stable. The phase plane will have the form shown in fig.14. The system will not be self-excited but a stable self-oscillation regime will be set up in it if the initial deviations from the state of equilibrium are sufficiently large.

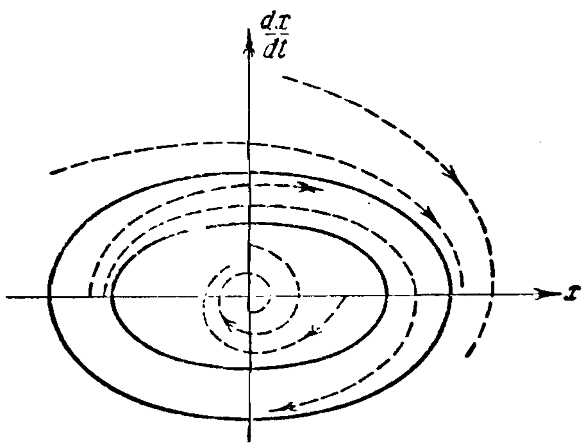

Fig. 14

13. Problem of the Stability of Periodic Motions

We have already several times remarked that for the theory of nonlinear oscillations the problem of fundamental significance is not only the determination of the periodic solutions but also the question of the stability of these solutions. Only those solutions can have physical significance which possess stability of the well known type. We shall now proceed to the investigation of the stability of periodic motions. In this chapter however we shall restrict ourselves to the presentation of only the elements of the theory of stability of periodic motions as applied to those simple cases of oscillations which were considered above. The general theory of the stability of periodic motions will be presented in chapter III.

All the systems considered in this chapter are described by an equation of the form[1]

$$\frac{d^2x}{dt^2} = X(t, x, \dot{x}), \tag{13.1}$$

where X either does not at all depend on the time (in the case of an autonomous system) or is a continuous periodic function of the time of period 2π. With respect to x and $\dot{x}$ the function X is analytic in a certain region.

Assume that in some way we have succeeded in finding a particular periodic solution of equation (13.1)

$$x = \varphi(t), \tag{13.2}$$

lying in the region of analyticity of the function X and let ω be the period of the function $\varphi(t)$. In the case of nonautonomous systems either $\omega = 2\pi$ or $\omega = 2p\pi$, where p is an integer. For determining whether the periodic solution (13.2) is stable let us consider some other entirely arbitrary solution of equation (13.1), which we shall write in the following form:

$$x = \varphi(t) + y(t). \tag{13.3}$$

Then, following A.M. Lyapunov , we shall say that the investigated periodic solution (13.2) is stable when the functions y and $\dot{y}$ remain small for all $t > 0$, provided they were sufficiently small at the initial instant of time. More accurately, THE PERIODIC SOLUTION

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[1] We do not here explicitly indicate that the equation of the oscillations contains the small parameter μ since for the question considered in the present section this circumstance is of no significance.

(13.2) OF EQUATION (13.1) IS TERMED STABLE IF FOR ANY ARBITRARILY SMALL POSITIVE NUMBER ε IT IS POSSIBLE TO FIND A POSITIVE NUMBER η SUCH THAT FOR ANY SOLUTION (13.3) OF THE ABOVE EQUATION THE INEQUALITIES $|y|<\varepsilon$, $|\dot{y}|<\varepsilon$ WILL BE SATISFIED FOR ALL $t>0$ AS SOON AS THE INEQUALITIES $|y(0)|\leqslant\eta, |\dot{y}(0)|\leqslant\eta$. ARE SATISFIED. It may happen that the investigated periodic solution will not only be stable but there will also be satisfied for it the conditions $\lim y = \lim \dot{y} = 0$ for $t\to\infty$. In this case the above periodic solution is termed ASYMPTOTICALLY STABLE.

Substituting solution (13.3) in equation (13.1) and expanding in a power series in y and $\dot{y}$ we obtain the following equation which the function y must satisfy:

$$\frac{d^2\varphi}{dt^2}+\frac{d^2y}{dt^2}=X(t,\varphi,\dot{\varphi})+\frac{\partial X(t,\varphi,\dot{\varphi})}{\partial\varphi}y+\frac{\partial X(t,\varphi,\dot{\varphi})}{\partial\dot{\varphi}}\dot{y}+\ldots, \quad (13.4)$$

where the terms not written down are of an order higher than the first with respect to y and $\dot{y}$. But since the function φ is a particular solution of equation (13.1) and therefore

$$\frac{d^2\varphi}{dt^2}=X(t,\varphi,\dot{\varphi}),$$

equation (13.4) assumes the form

$$\frac{d^2y}{dt^2}=\frac{\partial X(t,\varphi,\dot{\varphi})}{\partial\varphi}y+\frac{\partial X(t,\varphi,\dot{\varphi})}{\partial\dot{\varphi}}\dot{y}+\ldots \quad (13.5)$$

The obtained equation is termed the EQUATION OF THE DISTURBED MOTION and its first approximation, i.e. equation

$$\left.\begin{aligned}&\frac{d^2y}{dt^2}+q\frac{dy}{dt}+py=0,\\ &p=-\frac{\partial X(t,\varphi,\dot{\varphi})}{\partial\varphi},\quad q=-\frac{\partial X(t,\varphi,\dot{\varphi})}{\partial\dot{\varphi}},\end{aligned}\right\} \quad (13.6)$$

is termed the EQUATION IN VARIATIONS corresponding to the periodic solution under consideration. As has been shown by Lyapunov, in all cases, except special so-called "critical" cases, for judging the behavior of the function y from the point of view of the stability or instability it is sufficient to consider, instead of the complete equation (13.5), its first approximation, i.e. the equation in variations (13.6). We shall examine this equation more closely.

Equation (13.6) is a linear equation whose coefficients p and q both for nonautonomous as for autonomous systems are continuous periodic functions of t of period ω. An exception is constituted only by the case where the system under consideration is autonomous and the function φ degenerates into a constant, i.e. when the investigated periodic motion is a state of equilibrium. In this case the coefficients p and q are constants. We shall consider however the general case.

Let $y_1(t)$ and $y_2(t)$ be two particular solutions of equation (13.6) forming a fundamental system and determined by the initial conditions

$$y_1(0)=1, \quad \dot{y}_1(0)=0, \quad y_2(0)=0, \quad \dot{y}_2(0)=1. \tag{13.7}$$

If in these solutions $t+\omega$ is substituted for t then, since equation (13.6) does not change for such substitution, we again obtain solutions of this equation. These solutions will in general differ from the initial solutions but since for a linear equation every solution must be a linear combination of the fundamental system of solutions, we can write

$$\left.\begin{aligned} y_1(t+\omega)&=a_1y_1(t)+a_2y_2(t),\\ y_2(t+\omega)&=b_1y_1(t)+b_2y_2(t), \end{aligned}\right\} \tag{13.8}$$

where a_i, and b_1 are certain constants. Substituting in these relations and their derivatives t = 0 and taking (13.7) into account we obtain

$$\left.\begin{aligned} a_1&=y_1(\omega), \quad & a_2&=\dot{y}_1(\omega),\\ b_1&=y_2(\omega), \quad & b_2&=\dot{y}_2(\omega). \end{aligned}\right\} \tag{13.9}$$

Let us now consider the particular solution

$$y^*(t)=My_1(t)+Ny_2(t), \tag{13.10}$$

where M and N are certain constants which we try to choose in such manner that for the solution considered the relation is satisfied

$$y^*(t+\omega) = \rho y^*(t), \tag{13.11}$$

where ρ is likewise a certain constant. On the basis of (13.8) we have

$$y^*(t+\omega) = M[a_1 y_1(t) + a_2 y_2(t)] + N[b_1 y_1(t) + b_2 y_2(t)],$$

and we must therefore have

$$y_1(t)[(a_1-\rho)M + b_1 N] + y_2(t)[a_2 M + (b_2-\rho)N] = 0,$$

which on account of the independence of the solutions y_1 and y_2 gives:

$$\left.\begin{aligned} (a_1-\rho)M + b_1 N = 0, \\ a_2 M + (b_2-\rho)N = 0. \end{aligned}\right\} \tag{13.12}$$

We have thus obtained, for determining the constants M and N, a system of linear homogeneous equations. In order that this system have a nontrivial solution it is necessary and sufficient that the magnitude ρ satisfy the quadratic equation

$$\begin{vmatrix} a_1-\rho & b_1 \\ a_2 & b_2-\rho \end{vmatrix} = 0,$$

i.e. the equation

$$\rho^2 - 2A\rho + B = 0, \tag{13.13}$$

where on the basis of (13.9)

$$A = \frac{1}{2}[y_1(\omega) + \dot{y}_2(\omega)], \quad B = y_1(\omega)\dot{y}_2(\omega) - y_2(\omega)\dot{y}_1(\omega). \tag{13.14}$$

Equation (13.13) is termed the CHARACTERISTIC

EQUATION corresponding to the period ω . Let ρ_1 and ρ_2 be the roots of this equation. For simplicity we shall assume that these roots are different. Then for each of them we shall find from (13.12) the coefficients M and N on the substituting of which in (13.10) we shall obtain two different solutions y^* . Denote these solutions by y_1^* and y_2^*. We can then write :

$$y_1^*(t+\omega)=\rho_1 y_1^*(t),$$
$$y_2^*(t+\omega)=\rho_2 y_2^*(t).$$

Replacing t by $t+\omega$ m times we shall evidently have:

$$\left.\begin{aligned} y_1^*(t+m\omega)&=\rho_1^m y_1^*(t), \\ y_2^*(t+m\omega)&=\rho_2^m y_2^*(t). \end{aligned}\right\} \tag{13.15}$$

The solutions we have found y_1^* and y_2^* can be taken as a fundamental system so that any other solution of equation (13.6) will be a linear combination of these solutions.

Having established this, let us first assume that both roots of the characteristic equation (these roots will in general be complex) have moduli less than unity. Then, as follows from (13.15), the two solutions y_1^* and y_2^* will approach zero as $t \to \infty$. But then the same property will be possessed by any other solution of equation (13.6) since it is a linear combination of the solutions y_1^* and y_2^*. It has been shown by Lyapunov that the same will hold true also for any solution of the complete equation (13.5) provided the initial values of the magnitudes y and y are sufficiently small. Hence if both roots of the characteristic equation have moduli less than unity the investigated periodic solution $\varphi(t)$ of equation (13.1) is asymptotically stable.

We shall now assume that the characteristic equation has at least one root - let this be the root ρ_1 the modulus of which is greater than one. In this case at least the solution y_1^* will be unrestricted. Hence the periodic solution considered will in the first approximation be unstable. It has been shown by Lyapunov that the same will be the case if instead of the equation of the first approximation (13.6) the complete equation (13.5) is considered.

Let us assume, finally, that the characteristic equation has no roots with moduli greater than one but has roots with moduli equal to one. In this case, as was shown by A.M. Lyapunov, the equation of the first approximation does not solve the problem of the stability and it is necessary in equation (13.5) to consider also the terms of higher orders.

Thus, for the investigated periodic solution to be stable it is necessary that the characteristic equation of the equation in variations do not have roots with moduli greater than one and it is sufficient that these roots have moduli less than one, the periodic solution in the latter case being asymptotically stable.

We shall now express these conditions through the coefficients A and B of the characteristic equation. For this purpose it is simplest to make in the characteristic equation the substitution

$$\rho=\frac{1+\lambda}{1-\lambda}, \tag{13.16}$$

after which it assumes the form

$$(2A+B+1)\lambda^2+2(-B+1)\lambda+(-2A+B+1)=0. \tag{13.17}$$

As is immediately seen from the substitution (13.16), $|\rho| \leqslant 1$ if $\mathrm{Re}(\lambda) \leqslant 0$, and $|\rho| > 1$ if $\mathrm{Re}(\lambda) > 0$. Hence for stability it is necessary that the real part of the roots of equation (13.17) be not positive, i.e. that the conditions be satisfied

$$2A+B+1\geqslant 0, \quad -2A+B+1\geqslant 0, \quad -B+1\geqslant 0. \tag{13.18}$$

These conditions will likewise be sufficient if they are satisfied with the inequality sign, and the stability will then be asymptotic. Combining the first two inequalities of (13.18) we obtain $B + 1 \geqslant 0$, which together with the third inequality of (13.18) gives:

$$|B|\leqslant 1. \tag{13.19}$$

In what follows we shall make use of this condition of stability instead of the third condition (13.18). It can of course at once be obtained from a direct examination of the equation (13.13).

Thus, for judging the stability of the periodic motion it is necessary to determine the coefficients A and B of the characteristic equation. According to (13.14) for this it is necessary to know the particular solutions of the equation in variations determined by the initial conditions (13.17). Since the equation in variations can not be integrated in closed form, these solutions must be determined by some approximate method. The problem is simplified considerably by the consideration that the above solutions must be known only for one value of the argument t, namely for $t = \omega$. In the following sections we shall show how the coefficient A is computed for all cases of oscillations considered in this chapter. As regards the coefficient B it can at once be expressed through the coefficients of the equation in the variations. In fact, according to the well known property of the Wronskian we can write for any linear equation

$$y_1(t)\dot{y}_2(t) - \dot{y}_1(t)y_2(t) = [y_1(0)\dot{y}_2(0) - \dot{y}_1(0)y_2(0)]\, e^{-\int_0^t q\,dt},$$

and therefore, on the basis of (13.14) and (13.7)

$$B = e^{-\int_0^\omega q\,dt}. \tag{13.20}$$

14. Stability of the Periodic Solutions of Autonomous Systems. Application to the Theory of the Self-Oscillations of a Tube Generator

Let us first consider the stability of the periodic solutions of quasilinear autonomous systems with one degree of freedom. Equation (13.1) in the case considered is of the form

$$\frac{d^2x}{dt^2} + k^2x = \mu f(x, \dot{x}, \mu) \tag{14.1}$$

with the generating solution

$$x_0 = M_0 \cos kt,$$

where M_0 is determined from the equation (9.12):

$$P(M_0)=\int_0^{2\pi} f(M_0\cos u,\ -kM_0\sin u,\ 0)\sin u\,du=0. \quad (14.2)$$

For the equation in variations (13.6) we find:

$$\frac{d^2y}{dt^2}+\left[k^2-\mu\frac{\partial f(\varphi,\dot\varphi,\mu)}{\partial\varphi}\right]y-\mu\frac{\partial f(\varphi,\dot\varphi,\mu)}{\partial\dot\varphi}\frac{dy}{dt}=0, \quad (14.3)$$

where

$$\varphi=M_0\cos kt+\mu(\ldots)+\ldots \quad (14.4)$$

is the investigated periodic solution of equation (14.1). The period of this solution, as was shown in sec.9 and 10, has the form

$$\omega=\frac{2\pi}{k}(1+h_1\mu+h_2\mu^2+\ldots).$$

where h_1, and h_2,... are certain constants. We shall show first of all that the characteristic equation of the equation in variations (14.3) has a root equal to unity.In fact, since φ is a solution of equation (14.1), we have identically

$$\frac{d^2\varphi}{dt^2}+k^2\varphi=\mu f(\varphi,\dot\varphi,\mu).$$

Differentiating this identity with respect to t we obtain

$$\frac{d^2}{dt^2}\left(\frac{d\varphi}{dt}\right)+\left[k^2-\mu\frac{\partial f(\varphi,\dot\varphi,\mu)}{\partial\varphi}\right]\frac{d\varphi}{dt}-\mu\frac{\partial f(\varphi,\dot\varphi,\mu)}{\partial\dot\varphi}\frac{d}{dt}\left(\frac{d\varphi}{dt}\right)=0.$$

Comparing with equation (14.3) we find that this equation has the particular solution

$$y^*=\frac{d\varphi}{dt}.$$

For this solution, in virtue of the periodicity of φ, the relation is satisfied $y^*(t+\omega)=y^*(t)$, i.e. the relation (13.11) with the value of ρ equal to unity. This proves that the characteristic equation for equation (14.3) has one root equal to unity. But then the second root of this equation is equal to the coefficient B, which is immediately computed by the formula (13.20). For the case under consideration this formula gives

$$B = e^{\int_0^{\omega} \mu \frac{\partial f(\varphi, \dot{\varphi}, \mu)}{\partial \dot{\varphi}} dt} \tag{14.5}$$

For stability it is necessary that both roots of the characteristic equation have moduli not greater than one. For the first root this condition is automatically satisfied and we find on the basis of (14.5) that for stability it is necessary that only the inequality

$$\int_0^{\omega} \frac{\partial f(\varphi, \dot{\varphi}, \mu)}{\partial \dot{\varphi}} dt \leqslant 0. \tag{14.6}$$

must be satisfied.

As was shown by A.A.Andronov and A.A. Vitt[1], condition (14.6) will in the case considered also be sufficient for stability if it is satisfied with the inequality sign. This will be the case notwithstanding the fact that one of the roots of the characteristic equation is equal to unity, which in the general case, as was shown in sec. 13, makes it necessary to consider terms of higher order than the first in the differential equation of the disturbed motion.

Rejecting in (14.6) the terms depending on μ we obtain in this manner the following sufficient condition of stability of the periodic solution (14.4) of equation (14.1):

$$\int_0^{2\pi/k} \frac{\partial f(M_0 \cos kt, -kM_0 \sin kt, 0)}{\partial \dot{x}} dt \equiv$$

$$\equiv \frac{1}{k} \int_0^{2\pi} \frac{\partial f(M_0 \cos u, -kM_0 \sin u, 0)}{\partial \dot{x}} du < 0. \tag{14.7}$$

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[1] See chapter III, sec. 9

The obtained condition may be given another form. For this purpose we transform the left hand side of equation (14.2) that determines M_0. After integrating by parts and elementary transformations we shall have:

$$P(M_0) \equiv -M_0 \int_0^{2\pi} \left\{ \frac{\partial f(M_0 \cos u, -kM_0 \sin u, 0)}{\partial x} \sin u + \right.$$

$$\left. + k \frac{\partial f(M_0 \cos u, -kM_0 \sin u, 0)}{\partial \dot{x}} \cos u \right\} \cos u \, du \equiv$$

$$\equiv -kM_0 \int_0^{2\pi} \frac{\partial f(M_0 \cos u, -kM_0 \sin u, 0)}{\partial \dot{x}} du -$$

$$- M_0 \int_0^{2\pi} \left\{ \frac{\partial f(M_0 \cos u, -kM_0 \sin u, 0)}{\partial x} \cos u - \right.$$

$$\left. - k \frac{\partial f(M_0 \cos u, -kM_0 \sin u, 0)}{\partial \dot{x}} \sin u \right\} \sin u \, du \equiv$$

$$\equiv -kM_0 \int_0^{2\pi} \frac{\partial f(M_0 \cos u, -kM_0 \sin u, 0)}{\partial \dot{x}} du - M_0 \frac{dP(M_0)}{dM_0} = 0.$$

Hence the condition of stability (14.7) can be represented in the following form:

$$\frac{dP(M_0)}{dM_0} > 0. \tag{14.8}$$

As an example let us investigate the stability of the self-oscillations of the tube generator studied in sec.12. Let us consider first the case of a tube with soft operating characteristic. The oscillations in this case are described by the equation (12.1) while the function $P(M_0)$ and the magnitude M_0 are determined by formulas (12.7):

$$P(M_0) = -\frac{\alpha M_0}{\omega} + \frac{\gamma^2 M_0^3}{4\omega}, \quad M_0^2 = \frac{4\alpha}{\gamma^2},$$

whence

$$\frac{dP(M_0)}{dM_0} = \frac{2\alpha}{\omega}.$$

The periodic solution will be real only for $\alpha > 0$ and for it condition (14.8) will be satisfied. Thus, the investigated periodic solution, as was indicated in sec.12, will be stable.

Let us now assume that the tube has a hard operating characteristic. The oscillations in this case are described by the equation (12.2) and the function $P(M_o)$ and the magnitude M_o are determined by the formulas (12.9) and (12.10):

$$P(M_0) = -\frac{\alpha}{\omega} M_0 - \frac{\gamma^2}{4\omega} M_0^3 + \frac{\delta^2}{8\omega} M_0^5,$$
$$M_0^2 = \frac{\gamma^2 \pm \sqrt{\gamma^4 + 32\alpha\delta^2}}{\delta^2}.$$

If $\alpha > 0$ there is obtained for M_o only one real value for which, as is easily seen, condition (14.8) is satisfied. If $\alpha < 0$ it is necessary, for the existence of periodic solutions, that $\gamma^4 + 32\alpha\delta^2 < 0$. There will here be ontained two positive values for M_o^2. It is easy to see that for the larger value of M_o^2 condition (14.8) is satisfied and therefore the corresponding periodic solution will be stable. On the contrary, for the smaller value of M_o^2 the magnitude dP/dM_o will be negative and therefore the corresponding periodic solution will be unstable.

15. Stability of the Periodic Motions of Nonautonomous Systems

For all periodic oscillations of the nonautonomous systems considered in this chapter equation (13.1) has the form

$$\frac{d^2x}{dt^2} + k^2x + f(t) = \mu F(t, x, \dot{x}, \mu), \qquad (15.1)$$

where the functions f and F have the period 2π with respect to t. The parameter k is here different from an integer if we are dealing with the nonresonance case; $k=n$, where n is an integer in the case of a simple resonance and $k=1/n$ for the resonance of the n-th kind. The investigated periodic motion has the form

$$x = x_0 + \mu x_1 + \mu^2 x_2 + \dots = \varphi(t), \tag{15.2}$$

where x_0 is the generating solution. The period ω of the solution (15.2) is equal to 2π for the nonresonance case and for simple resonance, and $\omega = 2n\pi$ in the case of resonance of the n-th kind.

The equation in variations (13.6) in the case considered is of the form

$$\frac{d^2y}{dt^2} + k^2 y - \mu \frac{\partial F(t, \varphi, \dot\varphi, \mu)}{\partial \varphi} y - \mu \frac{\partial F(t, \varphi, \dot\varphi, \mu)}{\partial \dot\varphi} \frac{dy}{dt} = 0. \tag{15.3}$$

We shall seek to obtain the particular solutions $y_1(t)$ and $y_2(t)$ of this equation, satisfying the initial conditions (13.7), in the form of series in powers of μ. We can write:

$$\left.\begin{aligned} y_1 &= y_1^{(0)} + y_1^{(1)}\mu + y_1^{(2)}\mu^2 + \dots, \\ y_2 &= y_2^{(0)} + y_2^{(1)}\mu + y_2^{(2)}\mu^2 + \dots \end{aligned}\right\} \tag{15.4}$$

For the functions $y_1^{(0)}, y_1^{(1)}, y_2^{(0)}, y_2^{(1)}$ we obtain the differential equations

$$\left.\begin{aligned} &\frac{d^2y_1^{(0)}}{dt^2} + k^2 y_1^{(0)} = 0, \qquad \frac{d^2y_2^{(0)}}{dt^2} + k^2 y_2^{(0)} = 0, \\ &\frac{d^2y_1^{(1)}}{dt^2} + k^2 y_1^{(1)} = \frac{\partial F(t, x_0, \dot x_0, 0)}{\partial x_0} y_1^{(0)} + \frac{\partial F(t, x_0, \dot x_0, 0)}{\partial \dot x_0} \dot y_1^{(0)}, \\ &\frac{d^2y_2^{(1)}}{dt^2} + k^2 y_2^{(1)} = \frac{\partial F(t, x_0, \dot x_0, 0)}{\partial x_0} y_2^{(0)} + \frac{\partial F(t, x_0, \dot x_0, 0)}{\partial \dot x_0} \dot y_2^{(0)} \end{aligned}\right\} \tag{15.5}$$

and the initial conditions

$$\left.\begin{aligned} &y_1^{(0)}(0) = 1, \quad \dot y_1^{(0)}(0) = 0, \quad y_1^{(1)}(0) = \dot y_1^{(1)}(0) = 0, \\ &y_2^{(0)}(0) = 0, \quad \dot y_2^{(0)}(0) = 1, \quad y_2^{(1)}(0) = \dot y_2^{(1)}(0) = 0. \end{aligned}\right\} \tag{15.6}$$

For the coefficients A and B of the characteristic equation we have, on the basis of (13.14), (13.20):

$$\left.\begin{aligned} 2A &= [y_1^{(0)}(\omega)+\dot{y}_2^{(0)}(\omega)]+[y_1^{(1)}(\omega)+\dot{y}_2^{(1)}(\omega)]\mu+\dots, \\ B &= e^{\mu\int\limits_0^{\omega}\frac{\partial F(t,\varphi,\dot{\varphi},\mu)}{\partial\dot{\varphi}}dt} = 1+\mu\int\limits_0^{2\pi}\frac{\partial F(t,x_0,\dot{x}_0,0)}{\partial\dot{x}_0}dt+\dots \end{aligned}\right\} \tag{15.7}$$

Having established this, we proceed to the consideration of the stability criteria. From (15.5) and (15.6) we find:

$$y_1^{(0)}=\cos kt,\quad y_2^{(0)}=\frac{1}{k}\sin kt. \tag{15.8}$$

Hence on the basis of (15.7) the first two conditions of stability (13.18) assume the form

$$\left.\begin{aligned} 2\cos k\omega+2+\mu(\dots)+\dots &\geqslant 0, \\ -2\cos k\omega+2+\mu(\dots)+\dots &\geqslant 0. \end{aligned}\right\} \tag{15.9}$$

We shall consider the following three cases:

1. NONRESONANCE CASE. In this case $\omega = 2\pi$ and k differs from an integer. For sufficiently small μ conditions (15.9) are therefore always satisfied with the inequality signs. Consequently, for stability it is necessary that only one condition of (13.18) be satisfied, and if it is satisfied with the sign of inequality the motion will actually be stable and moreover asymptotically stable. Taking (15.7) into account we arrive at the conclusion that in the nonresonance case it is sufficient , for the asymptotic stability of the periodic solution, that the inequality be satisfied

$$\int\limits_0^{2\pi}\frac{\partial F(t,x_0,\dot{x}_0,0)}{\partial\dot{x}_0}dt<0. \tag{15.10}$$

We may remark that if the expression in condition (15.10) were positive the periodic solution for sufficiently small μ would be unstable, while if the expression reduced to zero then it would be necessary, in judging the stability, to take into account higher order terms in the expansion of the magnitude B, and possibly also in the equation of the disturbed motion.

2. CASE OF SIMPLE RESONANCE. In this case $\omega = 2\pi$ and $k = n$, an integer. Hence on the basis of (15.8)

$$y_1^{(0)}(\omega) = 1, \quad \dot{y}_1^{(0)}(\omega) = 0, \quad y_2^{(0)}(\omega) = 0, \quad \dot{y}_2^{(0)}(\omega) = 1. \quad (15.11)$$

Substituting these values in conditions (15.9) we see that for sufficiently small μ the first of these conditions as before is satisfied with the inequality sign. As regards the second condition of (15.9), which corresponds to the second condition of (13.20), the term free from μ contained in it reduces to zero and therefore a more accurate computation of the coefficients A and B is necessary. We shall compute these coefficients with an accuracy up to second order magnitudes with respect to μ. For the coefficient A we have, on the basis of (15.7) and (15.11):

$$2A = 2 + [y_1^{(1)}(2\pi) + \dot{y}_2^{(1)}(2\pi)]\mu + [y_1^{(2)}(2\pi) + \dot{y}_2^{(2)}(2\pi)]\mu^2 + \dots \quad (15.12)$$

For computing the coefficient B it is now more convenient to use not expression (15.7) but the second equation of (13.14). Substituting in the latter for $y_1(\omega)$, $\dot{y}_1(\omega)$ and $y_2(\tau)$, $\dot{y}_2(\omega)$ their values obtained from (15.4) and taking into account (15.11) we obtain:

$$B = 1 + [y_1^{(1)}(\omega) + \dot{y}_2^{(1)}(\omega)]\mu + [y_1^{(2)}(\omega) + \dot{y}_2^{(2)}(\omega) +$$
$$+ y_1^{(1)}(\omega)\dot{y}_2^{(1)}(\omega) - \dot{y}_1^{(1)}(\omega)y_2^{(1)}(\omega)]\mu^2 + \dots,$$

and therefore, on the basis of (15.12), the second condition of (13.18) assumes the form

$$\mu^2[y_1^{(1)}(\omega)\dot{y}_2^{(1)}(\omega) - \dot{y}_1^{(1)}(\omega)y_2^{(1)}(\omega)] + \mu^3(\dots) + \dots \geqslant 0.$$

This condition will be satisfied for μ sufficiently small with the inequality sign if

$$y_1^{(1)}(\omega)\dot{y}_2^{(1)}(\omega) - \dot{y}_1^{(1)}(\omega)y_2^{(1)}(\omega) > 0. \quad (15.13)$$

We shall compute the magnitudes entering here. From the differential equations (15.5) and the initial conditions (15.6) we find:

$$y_1^{(1)} = \frac{1}{n}\int_0^t \left[\frac{\partial F(\tau, x_0, \dot{x}_0, 0)}{\partial x_0}\cos n\tau - n\frac{\partial F(\tau, x_0, \dot{x}_0, 0)}{\partial \dot{x}_0}\sin n\tau\right]\times$$
$$\times \sin n(t-\tau)\,d\tau,$$
$$y_2^{(1)} = \frac{1}{n^2}\int_0^t \left[\frac{\partial F(\tau, x_0, \dot{x}_0, 0)}{\partial x_0}\sin n\tau + n\frac{\partial F(\tau, x_0, \dot{x}_0, 0)}{\partial \dot{x}_0}\cos n\tau\right]\times$$
$$\times \sin n(t-\tau)\,d\tau$$

and therefore

$$\left.\begin{aligned}
y_1^{(1)}(\omega) &= -\frac{1}{n}\int_0^{2\pi}\left[\frac{\partial F(\tau, x_0, \dot{x}_0, 0)}{\partial x_0}\cos n\tau - \right.\\
&\qquad \left. - n\frac{\partial F(\tau, x_0, \dot{x}_0, 0)}{\partial \dot{x}_0}\sin n\tau\right]\sin n\tau\,d\tau,\\
\dot{y}_1^{(1)}(\omega) &= \int_0^{2\pi}\left[\frac{\partial F(\tau, x_0, \dot{x}_0, 0)}{\partial x_0}\cos n\tau - \right.\\
&\qquad \left. - n\frac{\partial F(\tau, x_0, \dot{x}_0, 0)}{\partial \dot{x}_0}\sin n\tau\right]\cos n\tau\,d\tau,\\
y_2^{(1)}(\omega) &= -\frac{1}{n^2}\int_0^{2\pi}\left[\frac{\partial F(\tau, x_0, \dot{x}_0, 0)}{\partial x_0}\sin n\tau + \right.\\
&\qquad \left. + n\frac{\partial F(\tau, x_0, \dot{x}_0, 0)}{\partial \dot{x}_0}\cos n\tau\right]\sin n\tau\,d\tau,\\
\dot{y}_2^{(1)}(\omega) &= \frac{1}{n}\int_0^{2\pi}\left[\frac{\partial F(\tau, x_0, \dot{x}_0, 0)}{\partial x_0}\sin n\tau + \right.\\
&\qquad \left. + n\frac{\partial F(\tau, x_0, \dot{x}_0, 0)}{\partial \dot{x}_0}\cos n\tau\right]\cos n\tau\,d\tau.
\end{aligned}\right\} \quad (15.14)$$

The generating solution x_0 entering these expressions, on the basis of the results of sec. 3, has the form

$$x_0(t) = M_0\cos nt + N_0\sin nt + \varphi^*(t),$$

where φ is a particular solution of the generating equation

$$\frac{d^2x_0}{dt^2}+n^2x_0+f(t)=0,$$

and M_0 and N_0 are determined by equations (3.11):

$$\left.\begin{aligned}P(M_0, N_0)=\int_0^{2\pi} F(\tau, x_0(\tau), \dot{x}_0(\tau), 0)\sin n\tau\, d\tau=0,\\ Q(M_0, N_0)=\int_0^{2\pi} F(\tau, x_0(\tau), \dot{x}_0(\tau), 0)\cos n\tau\, d\tau=0.\end{aligned}\right\} \quad (15.15)$$

whence we easily obtain

$$\left.\begin{aligned}y_1^{(1)}(\omega)=-\frac{1}{n}\frac{\partial P}{\partial M_0}, \qquad \dot{y}_1^{(1)}(\omega)=\frac{\partial Q}{\partial M_0},\\ y_2^{(1)}(\omega)=-\frac{1}{n^2}\frac{\partial P}{\partial N_0}, \qquad \dot{y}_2^{(1)}(\omega)=\frac{1}{n}\frac{\partial Q}{\partial N_0},\end{aligned}\right\} \quad (15.16)$$

and consequently condition (15.13) can be rewritten as

$$\frac{\partial(P, Q)}{\partial(M_0, N_0)}<0. \qquad (15.17)$$

To this condition, as in the resonance case, it is necessary to add condition (15.10) corresponding to condition (13.19).

Conditions (15.17) and (15.10) assure the asymptotic stability of the periodic solution in the resonance case. If at least in one of them the sign of inequality is changed into its opposite the condition of instability is obtained.

The obtained conditions can be given another form. In fact, from (15.14) we readily obtain:

$$\int_0^{2\pi}\frac{\partial F(\tau, x_0, \dot{x}_0, 0)}{\partial \dot{x}_0}\, d\tau=y_1^{(1)}(\omega)+\dot{y}_2^{(1)}(\omega),$$

and therefore on the basis of (15.16) condition (15.10) assumes the form

$$\left(-\frac{\partial P}{\partial M_0}+\frac{\partial Q}{\partial N_0}\right)<0. \qquad (15.18)$$

This condition together with condition (15.17) shows that for the periodic solution to be asymptotically stable it is sufficient that the roots of the equation

$$\begin{vmatrix} -\frac{\partial P}{\partial M_0}-\lambda & \frac{\partial Q}{\partial M_0} \\ -\frac{\partial P}{\partial N_0} & \frac{\partial Q}{\partial N_0}-\lambda \end{vmatrix}=0 \qquad (15.19)$$

have negative real parts. We may note that the magnitudes P and Q differ only by a multiple of π from the coefficients of sin nt and cos nt in the expansion of the expression

$$F(t,\ x_0(t),\ \dot{x}_0(t),\ 0), \qquad (15.20)$$

i.e. of the right hand side of equation (15.1) after substitution in it for the magnitude x the generating solution and for the magnitude μ zero. Let us denote these coefficients respectively by P^* and Q^*. In place of equation (15.19) we may then consider the equation

$$\begin{vmatrix} -\frac{\partial P^*}{\partial M_0}-\lambda & -\frac{\partial Q^*}{\partial M_0} \\ \frac{\partial P^*}{\partial N_0} & \frac{\partial Q^*}{\partial N_0}-\lambda \end{vmatrix}=0. \qquad (15.21)$$

To avoid possible confusion in the practical set-up of this equation it is useful to note that the magnitudes N_0 and $-M_0$ are the coefficients of cos nt and sin nt in the derivative dx_0/dt of the generating solution. Thus, the elements of the determinant (15.21) are the partial derivatives of the coefficients of the n-th harmonic of expression (15.20) with respect to the same harmonic in the expression dx_0/dt, the derivatives along the principal diagonal being with respect to similar coefficients (i.e. the coefficient of cos nt is differentiated with respect to the coefficient of cos nt and the coefficient of sin nt with respect to the coefficient of sin nt).

3. CASE OF RESONANCE OF THE N-TH KIND. In this case $k = 1/n$ and $\omega = 2n\pi$. Hence, on the basis of (15.8), as

in the preceding case, formulas (15.11) hold true. All the preceding calculations are therefore retained without any changes and we arrive at the conclusion that the investigated periodic solution will be asymptotically stable if both roots of the equation (15.19) have negative real parts and unstable if at least one root of this equation has a positive real part. The functions P and Q in equation (15.19) are now determined by the formulas (7.6):

$$P(M_0, N_0) = \int_0^{2n\pi} F(t, x_0, \dot{x}_0, 0) \sin \frac{t}{n} \, dt,$$

$$Q(M_0, N_0) = \int_0^{2n\pi} F(t, x_0, \dot{x}_0, 0) \cos \frac{t}{n} \, dt.$$

16. Stability of the Oscillations Considered in Sections 5 and 6

We shall apply the obtained results to investigate the stability of the oscillations of a regenerative receiver and of the oscillations in the problem of Duffing, that were considered in detail in sec. 5 and 6.

Regenerative receiver. The oscillations are described by the equation

$$\frac{d^2x}{dt^2} + \omega^2 x = \mu (1 - x^2) \frac{dx}{dt} + \lambda \sin t. \tag{16.1}$$

Let us consider first the nonresonance case, i.e. we assume that ω^2 is not near unity. In this case the generating solution is of the form

$$x_0 = \frac{\lambda}{\omega^2 - 1} \sin t \tag{16.2}$$

and for stability it is sufficient that there be satisfied the condition (15.10)

$$\int_0^{2\pi} \left[1 - \frac{\lambda^2}{(\omega^2 - 1)^2} \sin^2 t \right] dt < 0,$$

i.e.

$$2 - \frac{\lambda^2}{(\omega^2 - 1)^2} < 0. \tag{16.3}$$

If this condition is not satisfied, as will be the case when the system is sufficiently far from resonance, a periodic solution will not be physically realized, although equation (16.1) will admit such a solution. As will be shown below in sec. 18 of chapter IV, if condition (16.3) is not satisfied equation (16.1) will admit a particular solution to which corresponds a form of oscillation more complicated than periodic, the so-called "combinational" or "quasi-periodic" oscillation in which, together with the frequency of the external excitation, there will figure also a frequency close to the self-oscillation frequency of the system, and this oscillation will be stable.

Let us now consider the case of principal resonance, i.e. assume that the magnitude $\omega^2 - 1$ is of the order of smallness of μ. The oscillations will now be described by the equation (5.11):

$$\frac{d^2x}{dt^2} + x = \mu\left[\lambda_0 \sin t - ax + (1 - x^2)\frac{dx}{dt}\right], \tag{16.4}$$

where $\omega^2 - 1 = a\mu$ is the mistuning. The generating solution will be

$$x_0 = M_0 \cos t + N_0 \sin t, \tag{16.5}$$

and the functions $P(M_0, N_0)$ and $Q(M_0, N_0)$ are determined by equations (5.16). For the functional determinant $\frac{\partial(P, Q)}{\partial(M_0, N_0)}$ we obtain formula (5.19) and therefore the stability condition (15.17) assumes the form

$$1 + a^2 - (M_0^2 + N_0^2) + \frac{3}{16}(M_0^2 + N_0^2)^2 > 0. \tag{16.6}$$

As regards condition (15.18) it may , on the basis of (5.16), be written as follows:

$$2 - (M_0^2 + N_0^2) < 0. \tag{16.7}$$

This condition does not evidently differ from (16.3) since the magnitude $\lambda/(\omega^2-1)$ is the amplitude of the solution (16.2) and corresponds to the magnitude $M_0^2 + N_0^2$ in the solution (16.5).

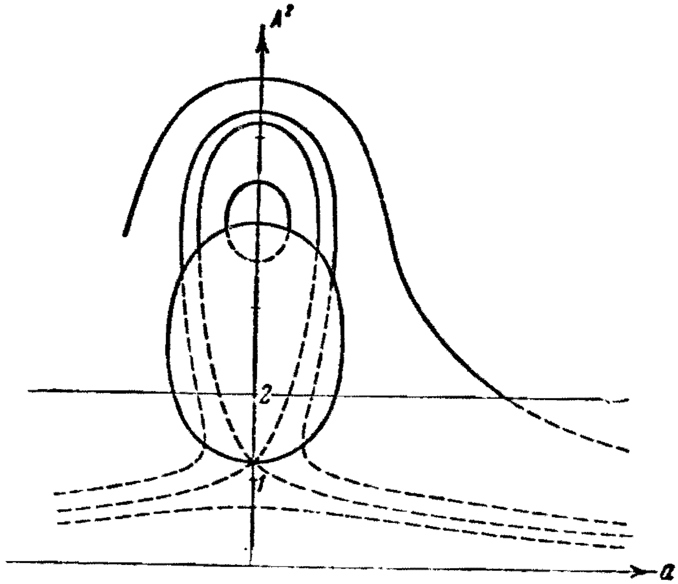

Fig.15

Let us consider the resonance curves giving the amplitude $A^2 = M_o^2 + N_o^2$ as a function of the mistuning a and which are represented in fig. 2. The conditions (16.6) and (16.7) show that stable oscillations correspond to those parts of these curves which lie outside the ellipse (5.22) and on the straight line $A^2 = 2$. In fig. 15 these parts of the resonance curves are represented by the continuous lines as distinquished from the dotted lines, to which correspond the unstable oscillations. From this figure it is seen that the stable periodic solutions are possible only for sufficiently small mistuning a. If the equation (16.4) admits three periodic solutions, which is possible only for $\lambda_o^2 < 32/27$, one of them will always be unstable and at least one stable. But in this case there are also possible two stable periodic solutions, as is seen from fig. 16, in which part of fig. 15 is shown to an enlarged scale. For $\lambda_o^2 > 32/27$, when equation (16.4) has only one periodic solution, the latter will always be stable if the mistuning a is sufficiently small. Thus, at SUFFICIENT NEARNESS TO RESONANCE EQUATION (16.4) WILL ALWAYS HAVE AT LEAST ONE STABLE PERIODIC SOLUTION.

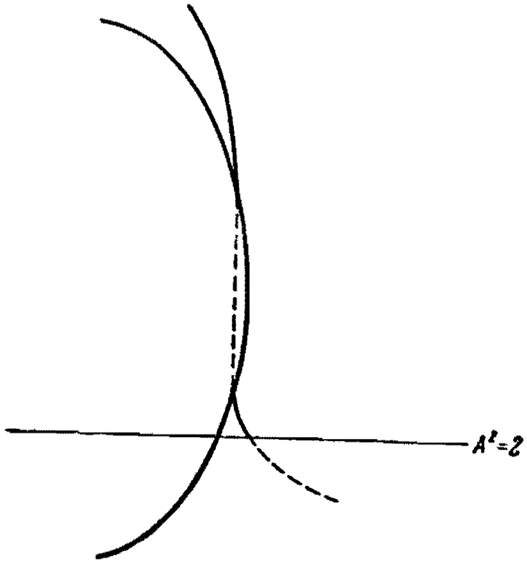

Fig. 16

The obtained results make it possible to compare the oscillations of the system (16.1) with the oscillations of the corresponding linear system. In the linear system, as a result of the unavoidable presence of resistances, the free oscillations rapidly die down and purely periodic oscillations with the frequency of the external excitations are established. The system (16.1) however is, as we have seen, a self-oscillatory one. In the absence of an external disturbance stable nondamping oscillations are established in it. In the presence of a disturbance these free oscillations will not, in general, dampen and oscillations with two frequencies will be established in the system. These oscillations however, as we shall see in sec. 18, chapter IV, will be stable only for a sufficiently far removal from resonance. Near resonance, as we have just shown, equation (16.1) admits a stable periodic solution. Consequently purely periodic oscillations with the frequency of the external excitation are established in the system. The self-oscillations will be suppressed by the external disturbance or, as we say, the frequency of the auto-oscillations will be "trapped" by the frequency of the external disturbance. This essentially is the phenomenon of trapping of the frequency that is observed in the self-oscillating systems.

Problem of Duffing. The oscillations are described by equation (6.2)

$$\frac{d^2x}{dt^2}+k^2x=\mu\gamma' x^3+\lambda\sin t \qquad (\gamma>0). \tag{16.8}$$

In the nonresonance case the generating solution will be

$$x_0 = \frac{\lambda}{k^2 - 1} \sin t$$

and for it the condition of stability (15.10) is satisfied with the equality sign. If we took into account resistance, i.e. considered the system

$$\frac{d^2x}{dt^2} + k^2 x = \mu \left(-2n \frac{dx}{dt} + \gamma' x^3 \right) + \lambda \sin t \quad (n > 0),$$

we should have for the corresponding generating solution in the nonresonance case

$$\frac{\partial F(t, x_0, \dot{x}_0, 0)}{\partial \dot{x}_0} = -2n$$

and therefore equation (15.10) would be satisfied with the inequality sign. As a result we must consider that the nonresonance solution is always stable.

For resonance, when $1-k^2=\mu a$, the equation of the oscillations will be of the form

$$\frac{d^2x}{dt^2} + x = \mu (\lambda' \sin t + ax + \gamma' x^3) \quad (\lambda = \mu\lambda'). \tag{16.9}$$

The generating solution will be

$$x_0 = M_0 \cos t + N_0 \sin t,$$

for which

$$P(M_0, N_0) = \lambda' + aN_0 + \frac{3}{4}\gamma' N_0 (M_0^2 + N_0^2),$$

$$Q(M_0, N_0) = aM_0 + \frac{3}{4}\gamma' M_0 (M_0^2 + N_0^2),$$

as a result of which we have $M_0 = 0$, $N_0 = A$, where A is a root of the cubic equation (6.7):

$$S(A) = \lambda' + aA + \frac{3}{4}\gamma' A^3 = 0. \tag{16.10}$$

Condition (15.10) as before is satisfied with the equality sign. As regards condition (15.17), it gives:

$$-\left(a+\frac{9}{4}\gamma' A^2\right)\left(a+\frac{3}{4}\gamma' A^2\right)<0,$$

or, remembering that A is a root of equation (16.10),

$$\frac{\lambda'}{A}\frac{dS(A)}{dA}<0. \tag{16.11}$$

The roots of equation (16.10) are graphically represented by the points P,Q,R in fig. 3 (p.31). Since $S(-\infty)<0$, it is easily seen that for the points P and R the magnitude dS/dA is positive while for the point Q it is negative. Hence, taking into account the signs of the magnitude A for these points, we find from (16.11) that the periodic solutions of equation (16.9) corresponding to the points P and Q are stable while the solution corresponding to the point R is unstable. In fig. 17, where the resonance curve constructed in fig. 6 has been reproduced, the solid lines correspond to the stable oscillations and the dotted line to the unstable oscillations.

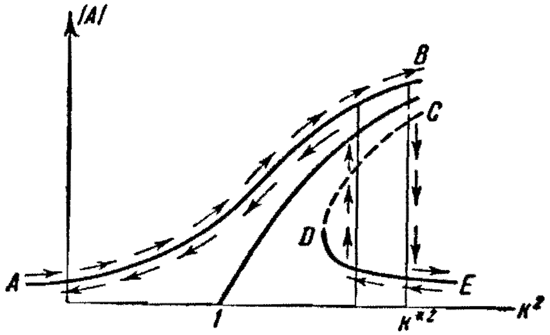

Fig. 17

From the obtained results the following conclusions in regard to the character of the development of the oscillations can be drawn.

Let $k^2<1$ and the difference k^2-1 be sufficiently large. The amplitude of the oscillations will then be represented by the corresponding point of the curve AB (fig. 17). If we now increase the frequency k the amplitude of the oscillations will likewise increase. When this amplitude becomes sufficiently large, the oscillations, as is clear from physical considerations, lose their stability. Let us assume that this occurs for $k = k^*$. Then for $k = k^*$ the system discontinuously goes over into another stable oscillation regime corresponding to the stable part DE of the curve CDE. With reverse change of k the oscillations discontinuously go over from the curve DE to the curve AB. This however occurs, in general, not for $k = k^*$, but for $k < k^*$. The amplitude of the oscillations at the same time increases and then, with further decrease of k, it will decrease. In fig. 18 the character of the development of the oscillations with change of k is indicated by the arrows.

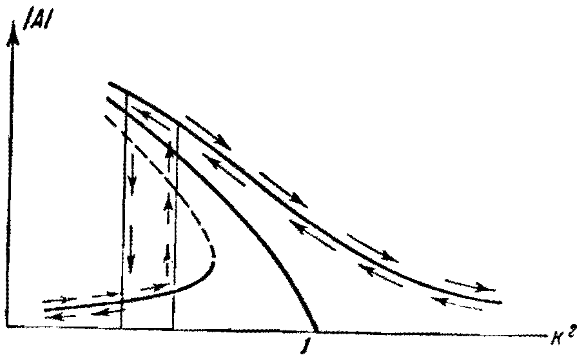

Fig. 18

We note that we assumed $\gamma > 0$. For $\gamma < 0$ the resonance curve will have the form shown in fig. 18, in which there is likewise indicated the character of the development of the oscillations with change of k.

The problem of Duffing for considerably more general assumptions will be considered in detail in sec. 7 of chapter VIII.

17. Systems with Nonanalytic Characteristic of the Nonlinearity

Up to now we have considered oscillations with only such quasilinear systems with one degree of freedom for which the right hand sides of the equations of motion were analytic with respect to the unkown function, its derivative and the parameter μ. In practice however it is often necessary to deal with cases for which these conditions are not satisfied. For cases of this kind the methods of constructing periodic solutions will not be analytic with respect to μ. It is possible however in the cases now to be considered to give another method of computing the periodic solutions with the aid of a simple process of successive approximations. In this section we shall restrict ourselves to the presentation of the method itself of computation. The proof of the convergence of the process of the successive approximations will be given in the following chapter together with the treatment of systems with an arbitrary number of degrees of freedom.

We shall consider separately nonautonomous and autonomous systems:

NONAUTONOMOUS SYSTEMS. Assume that the oscillations of the system are described by an equation of the form

$$\frac{d^2x}{dt^2}+k^2x+f(t)=\mu F(t,x,\dot{x},\mu), \tag{17.1}$$

where, as in sec. 2 and 4, the function f(t) is continuous, periodic with period 2π and can be expanded into a convergent Fourier series. The same properties with respect to t for all values of x and $\dot{x}$ lying in a certain region G and for sufficiently small μ are possessed by the function $F(t,x,\dot{x},\mu)$. However, in contrast to the cases considered in sec. 2 and 4, we shall assume that with respect to the variables x, $\dot{x}$ and μ the function F is not analytic but possesses only continuous partial derivatives of the first order.

Let us consider first the nonresonance case. In this case the generating equation

$$\frac{d^2x_0}{dt^2}+k^2x_0+f(t)=0 \tag{17.2}$$

has a single periodic solution $x_0 = \varphi(t)$. As will be shown in sec. 5 of chapter II, equation (17.1), as in the case of analytic F, admits for sufficiently small μ one and only one periodic solution which reduces to the generating solution for $\mu = 0$. To compute this solution we make use of the method of successive approximations. As the zeroth approximation we take the generating solution x_0, as the first approximation the periodic solution of the equation

$$\frac{d^2x_1}{dt^2}+k^2x_1+f(t)=\mu F(t,x_0,\dot{x}_0,\mu) \tag{17.3}$$

and in general as the l-th approximation the periodic solution of the equation

$$\frac{d^2x_l}{dt^2} + k^2x_l + f(t) = \mu F(t, x_{l-1}, \dot{x}_{l-1}, \mu) \quad (l = 2, 3, \ldots). \quad (17.4)$$

Since we are dealing with the nonresonance case and k is therefore different from an integer, equation (17.3), as well as equation (17.4) has only one periodic solution computed by the usual elementary methods, since the right hand sides of these equations will be known periodic functions of the time. Thus we are able to compute any number of approximations. The convergence of these approximations to the required periodic solution will be proven in sec. 9 of chapter II.

Let us now consider the resonance case, i.e. assume that k differs from a certain integer n by a magnitude of the order of smallness of μ. As in sec. 4, we reduce the equation of the oscillations to the form

$$\frac{d^2x}{dt^2} + n^2x + f(t) = \mu F(t, x, \dot{x}, \mu), \quad (17.5)$$

where the Fourier expansion of the function f(t) does not contain the n-th harmonic, as a result of which the generating equation as before admits a periodic solution, but one depending on two arbitary constants M_o and N_o and having the form

$$x_0 = \varphi(t) + M_0 \cos nt + N_0 \sin nt, \quad (17.6)$$

where φ is some particular solution of the generating equation.

It can be shown (see sec.6 of chapter II) that for a periodic solution of equation (17.5), which for $\mu = 0$ reduces to the generating solution, to exist it is necessary that the constants M_o and N_o satisfy equations (4.4)

$$\left.\begin{aligned} P(M_0, N_0) &= \int_0^{2\pi} F(t, x_0, \dot{x}_0, 0) \sin nt\, dt = 0, \\ Q(M_0, N_0) &= \int_0^{2\pi} F(t, x_0, \dot{x}_0, 0) \cos nt\, dt = 0. \end{aligned}\right\} \quad (17.7)$$

As in the case of analytic F, to each simple solution of these equations, i.e. to a solution for which

$$\frac{\partial(P, Q)}{\partial(M_0, N_0)} \neq 0, \tag{17.8}$$

there actually corresponds one and only one periodic solution of equation (17.5). Assuming that the constants M_o and N_o satisfy the above conditions we shall seek to obtain the periodic solution of equation (17.5) by successive approximations. For this purpose we take as the first approximation the function

$$x_1 = \varphi(t) + M_1 \cos nt + N_1 \sin nt,$$

where M_1 and N_1 are as yet undetermined constants. For the further approximations we take the periodic solution of the equations

$$\frac{d^2x_l}{dt^2} + n^2x_l + f(t) = \mu F(t, x_{l-1}, \dot{x}_{l-1}, \mu) \quad (l = 2, 3, \ldots). \tag{17.9}$$

Let us consider the equation determining x_2. Since n is an integer, in order that the equation for x_2 have a periodic solution it is necessary and sufficient that the equations be satisfied

$$P_1(M_1, N_1, \mu) = \int_0^{2\pi} F(t, x_1, \dot{x}_1, \mu) \sin nt\, dt = 0,$$

$$Q_1(M_1, N_1, \mu) = \int_0^{2\pi} F(t, x_1, \dot{x}_1, \mu) \cos nt\, dt = 0.$$

These equations determine the constants M_1 and N_1 of the first approximation. On the basis of (17.7) they are satisfied for $\mu = 0$, $M_1 = M_o$, $N_1 = N_o$. Since condition (17.8) is then satisfied, there exists for sufficiently small μ one and only one solution $M_1(\mu)$, $N_1(\mu)$ of these equations for which $M_1(0) = M_o$, $N_1(0) = N_o$. Having determined M_1 and N_1 in this manner, we obtain from (17.9)

$$x_2 = x_2^*(t) + M_2 \cos nt + N_2 \sin nt,$$

where M_2 and N_2 are as yet undetermined constants. These constants are determined from the condition of the existence of a periodic solution of equation (17.9) for $l = 3$. Proceeding further in analogous fashion we can compute any number of approximations. The convergence of the process will be proven in sec. 9 of chapter II.

AUTONOMOUS SYSTEMS. Let us consider the autonomous systems described by the equation

$$\frac{d^2x}{dt^2} + k^2x = \mu F(x, \dot{x}, \mu), \tag{17.10}$$

where in regard to the function F it is assumed only that it admits continuous partial derivations of the first order with respect to x and $\dot{x}$. In sec. 11 of chapter II it will be shown that, just as in the analytical case, for the equation (17.10) to admit a periodic solution, reducing for $\mu = 0$ to the generating solution

$$x_0 = M_0 \cos kt,$$

it is necessary that the constant M_0 satisfy the equation

$$P(M_0) = \int_0^{2\pi} F(M_0 \cos u, -kM_0 \sin u, 0) \sin u \, du = 0, \tag{17.11}$$

and that to each simple root of this equation there actually corresponds a periodic solution of equation (17.10). In virtue of the autonomy of the system it is here assumed that both in the required periodic solution as well as in the generating solution the magnitude dx/dt reduces to zero for $t = 0$. The period of the required periodic solution will be of the form

$$T = \frac{2\pi}{k}[1 + \mu\alpha(\mu)], \tag{17.12}$$

where $\alpha(\mu)$ is a certain function of μ, which now will not be analytic.

To compute this periodic solution it is convenient to replace equation (17.10) by a system of two equations of the first order, taking as the second unkown function the magnitude $y = -\frac{1}{k}\frac{dx}{dt}$. This system has the form

$$\frac{dx}{dt} = -ky, \quad \frac{dy}{dt} = kx - \frac{\mu}{k} F(x, -ky, \mu).$$

Let us substitute in it for the variable t the variable τ with the aid of the substitution

$$t=\frac{\tau}{k}[1+\mu\alpha(\mu)]$$

and seek to obtain the periodic solution with the given period 2π for the new system thus obtained

$$\left.\begin{aligned} \frac{dx}{d\tau} &= -y-\mu\alpha y, \\ \frac{dy}{d\tau} &= x+\mu\alpha x-\frac{\mu(1+\mu\alpha)}{k^2}F(x,-ky,\mu), \end{aligned}\right\} \qquad (17.13)$$

assuming that $y(0) = 0$. We take as the first approximation

$$x_1=M_1\cos\tau, \quad y_1=M_1\sin\tau, \quad \alpha_1=0.$$

Further, as the m-th approximation of the magnitudes x and y we take the periodic solution of the equations

$$\left.\begin{aligned} \frac{dx_m}{d\tau} &= -y_m-\mu\alpha_m y_{m-1}, \\ \frac{dy_m}{d\tau} &= x_m+\mu\alpha_m x_{m-1}- \\ &\quad -\frac{1}{k^2}\mu(1+\alpha_{m-1}\mu)F(x_{m-1},-ky_{m-1},\mu), \end{aligned}\right\} \qquad (17.14)$$

in which α_m denotes the m-th approximation of the magnitude α.

In order that the solutions of equations (17.14) be periodic it is necessary that the right hand sides satisfy certain conditions. We shall establish these conditions. Assume we have the system of equations

$$\left.\begin{aligned} \frac{du}{dt} &= -nv+A\cos nt+B\sin nt+\varphi(t), \\ \frac{dv}{dt} &= nu+C\cos nt+D\sin nt+\psi(t), \end{aligned}\right\} \qquad (17.15)$$

where n is an integer and φ and ψ are periodic functions of period 2π such that their Fourier expansions do not contain terms with $\cos nt$ and $\sin nt$. Eliminating v from these equations we obtain:

$$\frac{d^2u}{dt^2} + n^2u = n(B-C)\cos nt - n(A+D)\sin nt + \frac{d\varphi}{dt} - n\psi.$$

This equation will have a periodic solution if and only if the relations are satisfied

$$B - C = 0, \quad A + D = 0. \tag{17.16}$$

This will be the necessary and sufficient conditions for the existence of periodic solutions of equations (17.15).

Having established this, let us consider the equations determining x_2 and y_2, i.e. equations (17.14) for $m = 2$. These equations will be

$$\left.\begin{aligned} \frac{dx_2}{d\tau} &= -y_2 - \mu\alpha_2 M_1 \sin\tau, \\ \frac{dy_2}{d\tau} &= x_2 + \mu\alpha_2 M_1 \cos\tau - \frac{\mu}{k^2} F(M_1\cos\tau, -kM_1\sin\tau, \mu). \end{aligned}\right\} \tag{17.17}$$

Here $n = 1$ and the conditions of periodicity (17.16) give

$$\left.\begin{aligned} \int_0^{2\pi} F(M_1\cos\tau, -kM_1\sin\tau, \mu)\sin\tau\, d\tau = 0, \\ -2\alpha_2 M_1 + \frac{1}{\pi k^2}\int_0^{2\pi} F(M_1\cos\tau, -kM_1\sin\tau, \mu)\cos\tau\, d\tau = 0. \end{aligned}\right| \tag{17.18}$$

The first of these equations determines the arbitrary constant M_1 of the first approximation. This equation on the basis of (17.11) is satisfied for $\mu = 0$ and $M_1 = M_0$. Since by assumption $dP/dM_0 \neq 0$, this equation has one and only one solution $M_1 = M_1(\mu)$, which reduces to M_0 for $\mu = 0$. The second equation of (17.18) uniquely determines the magnitude α_2 (as in sec. 10, it is assumed that

$M_0 \neq 0$. Having in this way determined M_1 and α_2 from equations (17.17) we obtain

$$\left.\begin{aligned} x_2 &= \mu x_2^*(\tau,\ \mu) + M_2\cos\tau + N_2\sin\tau, \\ y_2 &= \mu y_2^*(\tau,\ \mu) + M_2\sin\tau - N_2\cos\tau, \end{aligned}\right\} \tag{17.19}$$

where μx_2^*, μy_2^* is some particular (periodic) solution of these equations while the terms containing the arbitrary constants M_2 and N_2 are the general solution of their homogeneous part. The constant N_2 is immediately determined from the initial condition $y_2(0) = 0$, which gives

$$N_2 = \mu y_2^*(0,\ \mu). \tag{17.20}$$

The constant M_2 on the other hand, together with the magnitude α_3 is determined from the conditions of periodicity of x_3 and y_3. In fact, from (17.14) on the basis of (17.19) and (17.20) we have

$$\begin{aligned} \frac{dx_3}{d\tau} &= -y_3 - \mu\alpha_3 M_2\sin\tau + \mu^2\alpha_3\varphi_3(\tau,\ \mu), \\ \frac{dy_3}{d\tau} &= x_3 + \mu\alpha_3 M_2\cos\tau + \mu^2\alpha_3\psi_3(\tau,\ \mu) - \frac{\mu(1+\mu\alpha_2)}{k^2}F(x_2,\ -ky_2,\ \mu), \end{aligned}$$

where φ_3 and ψ_3 are known periodic functions of τ, and from the conditions of periodicity (17.16) we obtain:

$$\left.\begin{aligned} &\frac{\alpha_3\mu}{\pi}\int_0^{2\pi}(\varphi_3\cos\tau + \psi_3\sin\tau)\,d\tau - \\ &\qquad - \frac{1+\alpha_2\mu}{\pi k^2}\int_0^{2\pi}F(x_2,\ -ky_2,\ \mu)\sin\tau\,d\tau = 0, \\ &-2\alpha_3 M_2 + \frac{\alpha_3\mu}{\pi}\int_0^{2\pi}(\varphi_3\sin\tau - \psi_3\cos\tau)\,d\tau + \\ &\qquad + \frac{1+\alpha_2\mu}{\pi k^2}\int_0^{2\pi}F(x_2,\ -ky_2,\ \mu)\cos\tau\,d\tau = 0. \end{aligned}\right\} \tag{17.21}$$

These equations serve to determine the magnitudes M_2 and α_3. For $\mu = 0$ they reduce to the equations

$$\left.\begin{aligned} P(M_2) = \int_0^{2\pi} F(M_2 \cos\tau, \ -kM_2 \sin\tau, 0) \sin\tau \, d\tau = 0, \\ 2\alpha_3 M_2 + \frac{1}{\pi k^2} \int_0^{2\pi} F(M_2 \cos\tau, \ -kM_2 \sin\tau, 0) \cos\tau \, d\tau = 0, \end{aligned}\right\} \tag{17.22}$$

which on the basis of (17.11) and (17.18) are satisfied for $\alpha_3 = \alpha_2(0)$ and $M_2 = M_1(0) \equiv M_0$. Since the functional determinant of the equations (17.22) with respect to α_3 and M_2 for $\alpha_3 = \alpha_2(0)$, $M_2 = M_0$ reduces to the magnitude $-2M_0 \frac{dP}{dM_0}$, different from zero, it follows that equations (17.21) for sufficiently small μ have one and only one solution $\alpha_3(\mu)$, $M_2(\mu)$, for which $M_2(0) = M_0$.

Continuing in a similar manner we can compute x_m, y_m and α_m for any m. As will be shown in chapter II, the above process of successive approximations converges and determines both the required periodic solution and its period (17.12).

In conclusion we may remark that all computations connected with the above described methods of successive approximations can be successfully applied also to the case of analytic equations.

CHAPTER II

PERIODIC OSCILLATIONS OF QUASILINEAR SYSTEMS WITH MANY DEGREES OF FREEDOM

1. Linear Equations with Constant Coefficients

Let us consider the system of linear equations

$$\frac{dx_s}{dt} = a_{s1}x_1 + \dots + a_{sn}x_n \qquad (s = 1, \dots, n), \tag{1.1}$$

where the coefficients a_{sj} are constants. The properties of such systems have been extensively studied.[1] We shall here dwell in detail however on certain properties which will play a very important part in all, that follows.

Let

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0 \tag{1.2}$$

be the FUNDAMENTAL equation[2] of the system (1.1). We shall show, first of all that this equation is invariant with

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[1] See, for example, Stepanov V.V., Kurs differentsial′nykh uravnenii,(Course in Differential Equations),6th ed., Gostekhizdat, 1953. Also Petrovskii I.G., Lektsii po teorii obyknovennykh differentsial′nykh uravnenii,(Lectures on the theory of Ordinary Differential Equations),Gostekhizdat, 4th ed.,1952.

[2] Equation (1.2) is most frequently called the characteristic equation. We shall however, to avoid confusion,call it the fundamental equation because in what follows we shall have occasion to consider linear equations with periodic coefficients for which there also exist a characteristic equation not agreeing with (1.2) for the special case when these coefficients are constant.

respect to any linear nonsingular transformation of the variables x_s. In other words, if in equations (1.1) we go over to variables y_s determined by the relations

$$y_s = \sum_{\alpha=1}^{n} b_{s\alpha} x_\alpha,$$

where $b_{s\alpha}$ are constants for which the determinant $|b_{s\alpha}|$ is different from zero, then for the new equations obtained in this manner

$$\frac{dy_s}{dt} = c_{s1}y_1 + \dots + c_{sn}y_n \tag{1.3}$$

the fundamental equation will be the same as equation (1.2). In fact, let us denote by a,b and c respectively the matrices $\|a_{sj}\|$, $\|b_{sj}\|$ and $\|c_{sj}\|$, and by E the unit matrix. Then, as is easy to see, we shall have

$$c = bab^{-1}$$

and therefore

$$|c - \rho E| \equiv |b^{-1}ab - \rho E| \equiv |b^{-1}(a - \rho E)b| \equiv$$
$$\equiv |b^{-1}|\cdot|a - \rho E|\cdot|b| \equiv |a - \rho E|,$$

which shows , that the fundamental equation of the system (1.3) agrees with the fundamental equation (1.2) of the system (1.1).

Now let λ be a root of equation (1.2). Then, as is known, the functions

$$x_s = A_s e^{\lambda t}, \tag{1.4}$$

where A_s are constants determined by the system of homogenous equations

$$a_{s1}A_1 + \dots + (a_{ss} - \lambda)A_s + \dots + a_{sn}A_n = 0, \tag{1.5}$$

are a solution of the system (1.1). If all the roots of equation (1.2) are simple, it is possible in this way to obtain n particular solutions of equations (1.1). These solutions will be linearly independent and will therefore form a fundamental system. Let us assume however that the root λ is multiple. In this case the system (1.1), in addition to the solution (1.4), will possess an additional solution of the form

$$x_s = f_s(t) e^{\lambda t}, \tag{1.6}$$

where f_s are polynomials the order of which does not exceed $p - 1$, if p is the multiplicity of the root. We shall here dwell in more detail on this problem, which plays a very important part in what follows.

We note first of all that if the system (1.1) has a solution of the form (1.6) it also has the solution

$$x_s = e^{\lambda t} \frac{df_s(t)}{dt}. \tag{1.7}$$

In fact, differentiating (1.1) we obtain

$$\frac{d}{dt}\left(\frac{dx_s}{dt}\right) = a_{s1}\left(\frac{dx_1}{dt}\right) + \ldots + a_{sn}\left(\frac{dx_n}{dt}\right),$$

whence it follows that the derivatives of any solution of the system (1.1) will likewise be solutions. Consequently, if the system (1.1) has a solution (1.6) it also has the solution

$$x_s = \lambda e^{\lambda t} f_s(t) + e^{\lambda t} \frac{df_s}{dt}.$$

Subtracting from it solution (1.6) multiplied by λ we obtain a new solution which is of the form (1.7).

With this established we proceed to the proof of a basic theorem with regard to solutions corresponding to a multiple root of the fundamental equation. This theorem consists in the following.

Let us assume first that the root λ of the fundamental equation having a multiplicity p does not reduce to zero at least one of the minors of the $(n-1)$th order of the determinant (1.2), so that the rank of this determinant is equal to $n-1$. In this case the system (1.1) admits a solution of the form (1.6) in which the degree of at least one of the polynomials f_s attains $p - 1$. Substituting for these polynomials their successive derivatives we obtain p particular solutions

$$x_{si} = e^{\lambda t} \frac{d^{i-1}}{dt^{i-1}} f_s(t) \qquad (i = 1, \ldots, p;\ s = 1, \ldots n), \tag{1.8}$$

which evidently will be independent. We shall say that in the case under consideration one set of solutions corresponds to the multiple root. The last solution of this group is evidently of the form (1.4).

Let us assume now that the rank of the fundamental determinant is equal to n - k[1]. In this case it is possible as before to construct p independent solutions that divide into k sets having the structure (1.8), namely, it is possible to find k systems of polynomials $f_s^{(1)}, \ldots, f_s^{(k)}$ the degrees of which do not exceed $p_1 - 1, \ldots p_k - 1$ respectively, where $p_1 + \ldots + p_k = p$, the degree of at least one of the polynomials $f_s^{(m)}$ being equal to $p_m - 1$. To each system $f_s^{(m)}$ there corresponds a set of p_m solutions of the form

$$x_{si}^{(m)} = e^{\lambda t} \frac{d^{i-1}}{dt^{i-1}} f_s^{(m)}(t) \tag{1.9}$$
$$(i = 1, \ldots, p_m;\quad m = 1, \ldots, k),$$

so that altogether there will be $p_1 + \ldots + p_m = p$ such solutions, all of them being independent.

To prove the above assertions we shall make use of the method of induction, since they are evidently true for a simple root. We shall assume therefore that the proposition has been proved for a root of multiplicity p - 1 and show that it is also true for a root of multiplicity p.

For the root under consideration λ it is possible in every case to construct a solution (1.4) where at least one of the constants A_s is different from zero. Let us assume that $A_1 \neq 0$ and in equations (1.1) make the substitution

$$x_1 = A_1 y_1,\ x_2 = A_2 y_1 + y_2, \ldots, x_n = A_n y_1 + y_n,$$

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[1] It is easy to show that the magnitude k cannot exceed p. In fact, if the rank of the determinant (1.2) is equal to n - k we can see by simple differentiation that $D'(\lambda) = D''(\lambda) = \ldots = D^{(k)}(\lambda) = 0$, and therefore the multiplicity of the root cannot be less than k.

after which they assume the form

$$\frac{dy_s}{dt} = b_{s1}y_1 + \ldots + b_{sn}y_n. \tag{1.10}$$

The solution (1.4) in the variables y_s will evidently be

$$y_1 = e^{\lambda t}, \quad y_2 = \ldots = y_n = 0, \tag{1.11}$$

and in order that equations (1.10) have such solution it is necessary that

$$b_{11} = \lambda, \quad b_{21} = \ldots = b_{n1} = 0.$$

Thus the system (1.10) breaks down into a system of n-1 equations with n-1 unknowns

$$\frac{dy_r}{dt} = b_{r2}y_2 + \ldots + b_{rn}y_n \qquad (r = 2, \ldots, n) \tag{1.12}$$

and the separate equation

$$\frac{dy_1}{dt} = \lambda y_1 + b_{12}y_2 + \ldots + b_{1n}y_n. \tag{1.12'}$$

Let $y_2(t), \ldots, y_n(t)$ be some solution of the system (1.12). Substituting it in (1.12') we find that the system of the functions $y_1(t)$, $y_2(t), \ldots, y_n(t)$, where

$$y_1(t) = e^{\lambda t}\int_0^t e^{-\lambda t}\left[b_{12}y_2(t) + \ldots + b_{1n}y_n(t)\right] dt \tag{1.13}$$

determines the solution of the complete system (1.12) and (1.12'). In particular, to the trivial solution $y_2 = \ldots = y_n = 0$ there corresponds in the complete system the solution (1.11) Hence, to any particular solution of the system (1.12), different from the trivial solution, there corresponds in the complete system a solution different from (1.11).

The fundamental determinant $\Delta(\rho)$ of the complete system (1.12) and (1.12′), coinciding by what has been proved with the fundamental determinant $D(\rho)$ of the system (1.1), is equal to $(\lambda - \rho)D'(\rho)$, where $D'(\rho)$ is the fundamental determinant of the system (1.12). From this it immediately follows that the fundamental equation of the system (1.12) has a root λ of multiplicity $p - 1$. To this root correspond by assumption $p - 1$ independent solutions of (1.12) that break down into a certain number of sets of the form (1.9). For simplicity let us assume that there will be two such sets. All statements remain valid for any number of sets. Let the first set consist of l solutions and the second set of $m = p - 1 - l$ solutions so that the system (1.12) admits $p - 1$ solutions of the form

$$y_{ri} = e^{\lambda t}\frac{d^{i-1}}{dt^{i-1}} f_r(t) = e^{\lambda t}\frac{d^{i-1}}{dt^{i-1}}(A_{r,l-1}t^{l-1} + \ldots + A_{r1}t + A_{r0}), \quad (1.14)$$

$$y^*_{rj} = e^{\lambda t}\frac{d^{j-1}}{dt^{j-1}}\varphi_r(t) = e^{\lambda t}\frac{d^{j-1}}{dt^{j-1}}(B_{r,m-1}t^{m-1} + \ldots + B_{r1}t + B_{r0}) \quad (1.15)$$

$$(i = 1, \ldots, l;\ j = 1, \ldots, m;\ r = 2, \ldots, n),$$

where at least one of the magnitudes $A_{r,\,l-1}$ and at least one of the magnitudes $B_{r,m-1}$ are different from zero. Substituting the functions (1.14) and (1.15) in (1.13) we obtain correspondingly

$$y_{1i} = e^{\lambda t}\frac{d^{i-1} f_1(t)}{dt^{i-1}}, \quad (1.14')$$

$$y^*_{1j} = e^{\lambda t}\frac{d^{j-1}\varphi_1(t)}{dt^{j-1}}, \quad (1.15')$$

where

$$f_1 = \int_0^t (b_{12}f_2 + \ldots + b_{1n}f_n)\,dt = At^l + \ldots + A_0, \quad (1.14'')$$

$$\varphi_1 = \int_0^t (b_{12}\varphi_2 + \ldots + b_{1n}\varphi_n)\,dt = A^*t^m + \ldots + A^*_0. \quad (1.15'')$$

Formulas (1.14) and (1.14′), (1.15) and (1.15′) determine $p - 1$ independent particular solutions of the complete system of equations (1.13) and (1.12′), different from the solutions (1.11). Together with the latter we obtain p independent particular solutions of the complete system corresponding to the root λ.

With this established, we now assume first that the two constants A and A^* are different from zero and for definiteness take $l \geqslant m$. Then if to the solutions y_{si} we add solutions (1.11) first multiplying it by the multiplier $Al\,!$, we obtain $l+1$ particular solutions

$$y_{s\alpha} = e^{\lambda t} \frac{d^{\alpha-1} f_s(t)}{dt^{\alpha-1}} \qquad (s=1, \ldots, n;\ \alpha = 1, \ldots, l+1),$$

forming a set of the required form. As for the solutions y^*_{sj}, they do not form a set in the above sense since in the function y^*_{1j} in the last of these solutions corresponding to $j=m$ the coefficient of $e^{\lambda t}$ is a linear function of t and not a constant. We can of course add to the solutions y^*_{sj} the further solution

$$y^*_{s,m+1} = e^{\lambda t} \frac{d^m \varphi_s(t)}{dt^m}$$

and obtain a set, but this solution on the basis of (1.15) and (1.15″) has the form $y^*_{1,m+1} = A^* m!\, e^{\lambda t}$, $y^*_{2,m+1} = \ldots = y^*_{n,m+1} = 0$ and agrees therefore with the solution (1.11), which has already been used. We therefore proceed otherwise. We replace the solutions y^*_{sj} of their linear combination by the last m solutions (1.14) and (1.14′) and form the m new solutions

$$\bar{y}_{sj}(t) = e^{\lambda t} \frac{d^{j-1} F_s(t)}{dt^{j-1}}, \qquad F_s = A^* \frac{d^{l-m} f_s}{dt^{l-m}} - A\varphi_s$$

$$(s=1, \ldots, n;\ j=1, \ldots, m).$$

The degrees of the polynomials F_s do not evidently exceed m - 1. It is easy however to see that the degree of at least one of these polynomials attains the value m - 1. In fact, if the degrees of all polynomials F_s were less than m-1, at least all functions $\bar{y}_{sm}$ would identically be equal to zero and consequently the solutions y_{sl} and y^*_{sm} would not be independent, which contradicts the assumption.

Thus, in the case where both constants A and A^* are different from zero we obtained for the complete system

(1.12) and (1.12$'$) p independent solutions corresponding to the root λ, which break down into two sets of the desired form. The same will be true if one of these constants, for example A^*, is equal to zero. In this case the solutions y^*_{sj} form a set. Another set is formed by the solutions y_{si} together with solution (1.11). If, finally, both constants A and A^* are equal to zero, both solutions Y_{si} and solutions y^*_{sj} form sets. Solution (1.11) can then be regarded as an independent third set consisting of a single solution. Thus, in all cases we obtain p independent solutions of the required form. Going over now to the variables x_s we obtain p independent solutions of the system (1.1), corresponding to the root λ that break down into sets of the form (1.9). It remains to be shown that there will be exactly k such sets if the rank of the fundamental determinant for the root λ is equal to k. This assertion is readily proved in the following manner.

The number of sets of solutions corresponding to the root under consideration is equal, evidently, to the number of independent solutions of the form (1.4) which this root possesses, since one such type of solution enters into the composition of each set. But this number is equal to the number of linearly independent solutions of the homogeneous system of algebraic equations (1.5), i.e. equal to k. Thus, the assertion in regard to the form of the solutions of the system (1.1) has been completely proven.

The actual computation of the above solution can be carried out most simply by the method of undetermined coefficients . The problem will here reduce to the solution of a system of linear algebraic equations. The simplest method of setting up these equations has been given by N.G. Chetaev.[1]

If root λ is complex the solutions of the form (1.4) and (1.6) will likewise be complex. It is easy however to represent them in a real form. For this purpose we note that since the coefficients a_{sj} are real, if equations (1.1) are satisfied by some complex solution they are also satisfied by the real and imaginary parts of this solution. Let $\lambda = \mu + i\nu$. Then in solution (1.4) or (1.6) the magnitudes A_s and f_s will likewise be complex. Let us put $A_s = B_s + iC_s$, $f_s = P_s + iQ_s$, where the constants A_s, B_s and the functions P_s, Q_s will be real.

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[1] Chetaev N.G. Ustoichivost dvizheniya, (The Stability of Motion), 2nd ed., Gostekhizdat, 1955.

Separating in (1.4) and (1.6) the real and imaginary parts we obtain the result that to the root $\lambda = \mu + i\nu$ correspond two solutions: either of the form

$$x_s = (B_s \cos \nu t - C_s \sin \nu t)\, e^{\mu t}, \quad x_s = (B_s \sin \nu t + C_s \cos \nu t)\, e^{\mu t},$$

or of the form

$$x_s = (P_s \cos \nu t - Q_s \sin \nu t)\, e^{\mu t}, \quad x_s = (P_s \sin \nu t + Q_s \cos \nu t)\, e^{\mu t}.$$

The same solutions correspond to the root $\lambda = \mu - i\nu$.

2. Periodic Solutions of Homogeneous Linear Systems with Constant Coefficients

From the above it follows that the system (1.1) will have periodic solutions if and only if the fundamental equation possesses either a zero root or purely imaginary roots.

Assume that the fundamental equation has a zero root of multiplicity p to which corresponds k sets of solutions. Then, by what has been said above, the system (1.17) will have k particular solutions of the form

$$x_{si} = A_{si} \qquad (s = 1, \ldots, n;\ i = 1, \ldots, k), \tag{2.1}$$

where A_{si} are constants. The remaining solutions corresponding to the zero root will be polynomials of t. The solutions (2.1) may evidently be regarded as periodic of arbitrary period.

Let us assume now that the fundamental equation has a pair of purely imaginary roots $\pm \nu i$. If these roots are multiple and there correspond to them k sets of solutions, system (1.1) has $2k$ solutions of the form

$$\left.\begin{aligned} x_{si} &= B_{si} \cos \nu t - C_{si} \sin \nu t, \\ x^*_{si} &= B_{si} \sin \nu t + C_{si} \cos \nu t \\ (s &= 1, \ldots, n;\ i = 1, \ldots, k), \end{aligned}\right\} \tag{2.2}$$

where B_{si} and C_{si} are constants. These solutions will be periodic of period $\omega = 2\pi/\nu$. The other solutions corresponding to the roots $\pm \nu i$, if they exist, i.e. if the multiplicity of these roots exceeds k, will not be periodic since they will contain terms of the form $t^m \sin \nu t$ and $t^m \cos \nu t$. However the system (1.1) may have periodic solutions of the same period ω that are different from (2.2). This will occur in the case where the fundamental equation has roots of the form $\pm p \nu i$, where p is an integer. The system (1.1) will then have solutions of period ω/p which may be considered as periodic of the period ω. Moreover, if the fundamental equation has a zero root, the solutions (2.1) may likewise be referred to the periodic solutions of period ω. For definiteness let us assume that the fundamental equation has a zero root to which corresponds k sets of solutions and r pairs of purely imaginary roots of the form $\pm p_j \nu i$, where p_j are integers, and to these roots correspond respectively k_j sets of solutions. The system (1.1) will then have $m = k + 2k_1 + \ldots + 2k_r$ periodic solutions of period $\omega = 2\pi/\nu$. Let us enumerate these solutions in some order and denote them by $\varphi_{s\alpha}$. The system (1.1) will then have the periodic solution of period ω:

$$x_s = C_1\varphi_{s1} + C_2\varphi_{s2} + \ldots + C_m\varphi_{sm},$$

containing m arbitrary constants $C_1, \ldots, C_m$.

3. Conjugate Systems. Reduction of Linear Equations with Constant Coefficients to Canonical Form

Let us consider the system of linear equations

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n \qquad (s = 1, \ldots, n). \tag{3.1}$$

where the coefficients p_{sj} are arbitrary continuous functions of t. The system of linear equations

$$\frac{dy_s}{dt} + p_{1s}y_1 + p_{2s}y_2 + \ldots + p_{ns}y_n = 0 \quad (s = 1, \ldots, n) \tag{3.2}$$

is called the conjugate of (3.1). Let $x_s(t)$ be an arbitrary solution of the system (3.1) and $y_s(t)$ an arbitrary solution of the system (3.2). We have

$$\frac{d}{dt}\sum_{\alpha=1}^{n} x_\alpha(t)\,y_\alpha(t) = \sum_{\alpha,\,\beta=1}^{n} y_\alpha p_{\alpha\beta} x_\beta - \sum_{\alpha,\,\beta=1}^{n} x_\alpha p_{\beta\alpha} y_\beta \equiv 0,$$

which leads us to the fundamental relation between arbitrary solutions of conjugate systems

$$\sum_{\alpha=1}^{n} y_\alpha(t)\,x_\alpha(t) = \text{const.} \tag{3.3}$$

From the obtained relations it follows in particular that the linear form of the variables $x_1,\ldots,x_n$

$$\sum_{\alpha=1}^{n} y_\alpha(t)\,x_\alpha \tag{3.4}$$

represents a first integral of the system (3.1).

Relation (3.3) is a special form of a more general relation. Let $u_s(t)$ be some particular solution of the nonhomogeneous system

$$\frac{du_s}{dt} = p_{s1}u_1 + \ldots + p_{sn}u_n + f_s(t).$$

Then, as is easy to see, we shall have

$$\frac{d}{dt}\sum_{\alpha=1}^{n} y_\alpha(t)\,u_\alpha(t) = \sum_{\alpha=1}^{n} y_\alpha(t)\,f_\alpha(t), \tag{3.5}$$

whence

$$\sum_{\alpha=1}^{n} y_\alpha(t)\,u_\alpha(t) = \int \sum_{\alpha=1}^{n} y_\alpha(t)\,f_\alpha(t)\,dt. \tag{3.6}$$

We shall frequently make use of relations (3.3) (3.6) in what follows.

Let us now turn to the system (1.1) with constant coefficients. The fundamental equation of the system conjugate to it

$$\frac{dy_s}{dt}+a_{1s}y_1+\ldots+a_{ns}y_n=0, \tag{3.7}$$

has roots differing only in sign from the roots of the fundamental equation (1.2). Let $-\lambda_1, -\lambda_2, \ldots, -\lambda_k$ be the roots of the fundamental equation of the system (3.7) so that the magnitudes $\lambda_1, \lambda_2, \ldots, \lambda_k$ are roots of the fundamental equation of system (1.1). We shall here write out each multiple root as many times as the sets of solutions which correspond to it in the system (3.7). Thus, among the magnitudes λ_i there may be identical ones but to each of them corresponds only one set of solutions of equations (3.7). We shall denote by n_p the number of solutions of the system (3.7) found in the set corresponding to the root $-\lambda_p$, so that $n_1+n_2+\ldots+n_k = n$. These solutions can be represented in the form

$$\left.\begin{aligned}
&y_{s1}^{(p)}=A_{s1}^{(p)}e^{-\lambda_p t},\\
&y_{s2}^{(p)}=(tA_{s1}^{(p)}+A_{s2}^{(p)})\,e^{-\lambda_p t},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&y_{sn_p}^{(p)}=\left(\frac{t^{n_p-1}}{(n_p-1)!}A_{s1}^{(p)}+\frac{t^{n_p-2}}{(n_p-2)!}A_{s2}^{(p)}+\ldots+A_{sn_p}^{(p)}\right)e^{-\lambda_p t}\\
&\qquad\qquad (s=1,\ \ldots,\ n;\ p=1,\ \ldots,\ k),
\end{aligned}\right\} \tag{3.8}$$

where at least one of the magnitudes $A_{si}^{(p)}$ $(s = 1,\ldots,n)$ is different from zero. We here choose the following system of denoting the solutions: the superscript in $y_{si}^{(p)}$ denotes the number of the set to which the solution (number of the root) belongs, while the subscript denotes the number of the solution in the set.

Substituting the obtained $n_1 + \ldots n_k = n$ independent particular solutions of system (3.7) in (3.4) we n independent first integrals of the system (1.1).

These integrals have the form

$$\left.\begin{array}{c} e^{-\lambda_p t} z_1^{(p)} = \text{const}, \\ e^{-\lambda_p t}\left(t z_1^{(p)} + z_2^{(p)}\right) = \text{const}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ e^{-\lambda_p t}\left(\frac{t^{n_p-1}}{(n_p-1)!} z_1^{(p)} + \frac{t^{n_p-2}}{(n_p-2)!} z_2^{(p)} + \ldots + z_{n_p}^{(p)}\right) = \text{const} \\ (p = 1, \ldots, k), \end{array}\right\} \tag{3.9}$$

where

$$z_i^{(p)} = \sum_{\alpha=1}^{n} A_{\alpha i}^{(p)} x_\alpha \tag{3.10}$$

$$(i = 1, \ldots, n_p;\ p = 1, \ldots, k)$$

are linear forms of the variables x_s with constant coefficients.

Relations (3.10) may be considered as the linear transformation of the variables x_s into the variables $z_i^{(p)}$. The determinant of this transformation

$$\begin{vmatrix} A_{11}^{(1)} & A_{21}^{(1)} & \ldots & A_{n1}^{(1)} \\ A_{12}^{(1)} & A_{22}^{(1)} & \ldots & A_{n2}^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ A_{1n_1}^{(1)} & A_{2n_1}^{(1)} & \ldots & A_{nn_1}^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ A_{11}^{(k)} & A_{21}^{(k)} & \ldots & A_{n1}^{(k)} \\ A_{12}^{(k)} & A_{22}^{(k)} & \ldots & A_{n2}^{(k)} \\ \cdot & \cdot & \cdot & \cdot \\ A_{1n_k}^{(k)} & A_{2n_k}^{(k)} & \ldots & A_{nn_k}^{(k)} \end{vmatrix}$$

differs, as is easily seen, only by the multiplier

$$e^{-(n_1\lambda_1 + \ldots + n_k\lambda_k)t}$$

which never becomes zero, from the determinant consisting of the n^2 functions $y_{si}^{(p)}$. The latter determinant however, in virtue of the independence of the solutions (3.8) is different from zero. From this it follows that the transformation (3.10) is not singular. Let us with the aid of it transform equation (1.1). For this purpose we form the derivatives with respect to t of the expressions (3.9). Then, taking into account the fact that these expressions are the first integrals of the system (1.1), we easily obtain

$$\left.\begin{aligned} &\frac{dz_1^{(p)}}{dt} = \lambda_p z_1^{(p)}, \\ &\frac{dz_i^{(p)}}{dt} = \lambda_p z_i^{(p)} - z_{i-1}^{(p)} \\ &(i = 2, \ldots, n_p;\ p = 1, \ldots, k). \end{aligned}\right\} \tag{3.11}$$

These will be the transformed equations. The form (3.11) of equations (1.1) is termed the CANONICAL FORM, and the variables $z_i^{(p)}$ CANONICAL VARIABLES.

The roots λ_p may also be complex and the corresponding variables $z_i^{(p)}$ will then also be complex. It is sometimes desirable to give equations (3.11) a real form. This can easily be done in the following manner.

Let $\lambda_p = \mu_p + i\nu_p$. We set $z_j^{(p)} = u_j^{(p)} + iv_j^{(p)}$. Then separating in (3.11) the real and imaginary parts, we obtain in place of the n_p equations (3.11), the $2n_p$ real equations

$$\left.\begin{aligned} &\frac{du_1^{(p)}}{dt} = \mu_p u_1^{(p)} - \nu_p v_1^{(p)}, \qquad \frac{dv_1^{(p)}}{dt} = \mu_p v_1^{(p)} + \nu_p u_1^{(p)}, \\ &\frac{du_i^{(p)}}{dt} = \mu_p u_i^{(p)} - \nu_p v_i^{(p)} - u_{i-1}^{(p)}, \qquad \frac{dv_i^{(p)}}{dt} = \mu_p v_i^{(p)} + \nu_p u_i^{(p)} - v_{i-1}^{(p)} \\ &(i = 2, \ldots, n_p). \end{aligned}\right\} \tag{3.12}$$

The same $2n_p$ equations are obtained also for the root $\mu_p - i\nu_p$, conjugate to λ_p, so that the total number of equations in the real and complex form will be the same.

REMARK. The system (3.11) has evidently n independent solutions, that break down into k sets such that the first solution in some p-th set has the form

$$z_{11}^{(p)} = e^{\lambda} \quad ,$$
$$z_{21}^{(p)} = -te^{\lambda_p t},$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$z_{n_p 1}^{(p)} = \frac{(-t)^{n_p-1}}{(n_p-1)!} e^{\lambda_p t},$$
$$z_{11}^{(r)} = z_{21}^{(r)} = \ldots = z_{n_p-1}^{(r)} = 0$$
$$(r = 1, \ldots, p-1, p+1, \ldots, k),$$

while the remaining solutions of this set are obtained from the first by successive differemtiation of the coefficients of $e^{\lambda_p t}$. This p-th set consists of n_p solutions and has a standard form. From this it follows that the roots λ and $-\lambda$ of the fundamental equations of conjugate systems have not only the same multiplicity, as immediately follows from the very form of these equations, but in both systems there correspond to these roots the same number of sets of solutions with the same number of solutions in the corresponding sets.

4. Periodic Solutions of Nonhomogeneous Linear Systems with Constant Coefficients

We proceed to the very important question, for what follows, in regard to the conditions of existence of periodic solutions for a system of linear nonhomogeneous equations

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n + f_s(t) \qquad (4.1)$$
$$(s = 1, \ldots, n),$$

where a_{sj} are, as before, constants while $f_s(t)$ are continuous periodic functions of period ω .

Let us denote by $x_{sj}(t)$ the fundamental system of solutions of the homogeneous equations

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n, \qquad (4.2)$$

determined by the initial conditions

$$x_{sj}(0) = \delta_{sj} \qquad (s, j = 1, \ldots, n), \qquad (4.3)$$

where δ_{sj} is the Kronecker symbol, i.e. $\delta_{ss} = 1$ and (for $s \neq j$) $\delta_{sj} = 0$. It is easy to show that the functions

$$x_s^*(t) = \int_0^t \sum_{\alpha=1}^n x_{s\alpha}(t-\tau) f_\alpha(\tau) d\tau \tag{4.4}$$

are a particular solution of the system (4.1). In fact, we have

$$\begin{aligned}\frac{dx_s^*}{dt} = & \sum_{\alpha=1}^n x_{s\alpha}(0) f_\alpha(t) + \\ & + \int_0^t \sum_{\alpha=1}^n [a_{s1}x_{1\alpha}(t-\tau) + \ldots + a_{sn}x_{n\alpha}(t-\tau)] f_\alpha(\tau) d\tau = \\ = & f_\alpha(t) + a_{s1}x_1^* + \ldots + a_{sn}x_n^*.\end{aligned}$$

Adding to this solution the general solution of the homogeneous system (4.2) we obtain the general solution of system (4.1) in the Cauchy form

$$x_s = C_1 x_{s1}(t) + \ldots + C_n x_{sn}(t) + \int_0^t \sum_{\alpha=1}^n x_{s\alpha}(t-\tau) f_\alpha(\tau) d\tau, \tag{4.5}$$

where $C_1, \ldots, C_n$ are arbitrary constants, which evidently are the initial values (for $t=0$) of the magnitudes m_s. We shall now choose the constants C_s in such manner that the solution (4.5) is periodic of period ω. For this it is necessary and sufficient that the relations $x_s(\omega) - x_s(0) = 0$ be satisfied. This gives for the determination of C_1 the following system of linear nonhomogeneous equations:

$$C_1 x_{s1}(\omega) + \ldots + C_n x_{sn}(\omega) - C_s + \int_0^\omega \sum_{\alpha=1}^n x_{s\alpha}(\omega-\tau) f_\alpha(\tau) d\tau = 0 \tag{4.6}$$
$$(s = 1, \ldots, n).$$

Setting in these equations $f_\alpha = 0$ we obtain the equations

$$C_1^* x_{s1}(\omega) + \ldots + C_n^* x_{sn}(\omega) - C_s^* = 0 \; (s = 1, \ldots, n) \tag{4.7}$$

for determining those values of the magnitudes C_s^* for which the functions

$$C_1^* x_{s1} + \ldots + C_n^* x_{sn}$$

are a periodic solution of period ω of the system (4.2).

As we saw in sec.2, the system (4.2) can have a periodic solution of period ω if and only if its fundamental equation has either a zero root or roots of the form $\pm 2\pi pi/\omega$, where p is an integer.

Let us now assume that the fundamental equation has neither a zero root nor roots of the form $\pm 2\pi pi/\omega$. This case we shall call the NONRESONANCE CASE. In the nonresonance case the system (4.7) cannot have any solutions except the trivial one $C_1^* \ldots C_n^* = 0$, and therefore the determinant

$$\begin{vmatrix} x_{11}(\omega)-1 & \ldots & x_{1n}(\omega) \\ \cdot & \cdot & \cdot \\ x_{n1}(\omega) & \ldots & x_{nn}(\omega)-1 \end{vmatrix} \tag{4.8}$$

is different from zero. But then equations (4.6) will have one and only one solution for $C_1, \ldots, C_n$, substituting which in (4.5) we obtain one and only one periodic solution of the system (4.1). We thus arrive at the conclusion:

IN THE NONRESONANCE CASE THE SYSTEM (4.1) ADMITS ONE AND ONLY ONE PERIODIC SOLUTION FOR ANY CHOICE OF THE FUNCTIONS $f_s(t)$.

Let us assume that the fundamental equation has either a zero root or roots of the form $\pm 2\pi pi/\omega$. This case we shall call the RESONANCE CASE. It plays a special role in the theory both of nonlinear and linear oscillations. In the resonance case the system (4.1) may also admit periodic solutions but for this it is necessary that the functions $f_s(t)$ satisfy certain conditions to the derivation of which we shall now proceed[1].

Let us assume that the fundamental equation of system (4.2) has a zero root of arbitrary multiplicity and to this root let there correspond k sets of solutions of

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[1] Malkin I.G., K teorii kolebanii kvazilineinykh system so mnogimi stepenyami svovody. (On the Theory of the Oscillations of Quasilinear Systems with Many Degrees of Freedom). Prikl. matem. i mekh., vol.XIV, no.4, 1950.

system (4.2). We shall assume, moreover, that the fundamenta equation has r pairs of purely imaginary roots of the form $\pm 2\pi p_j i/\omega$ ($j=1,\ldots,r$), where p_j are integers, likewise of arbitrary multiplicity, and to each root in the pair $\pm 2\pi p_j i/\omega$ let there correspond k_j sets of solutions. Then, as was shown in sec. 2, the system (4.2) will have m and only m periodic solutions of period ω, where the number m is determined by the formula

$$m = k + 2k_1 + \ldots + 2k_r.$$

Let us denote these periodic solutions of the system (4.2) by $\varphi_{s1}(t),\ldots,\varphi_{sm}(t)$. They are determined by formulas (2.1) and (2.2).

The fundamental equation of the system

$$\frac{dy_s}{dt} + a_{1s}y_1 + \ldots + a_{ns}y_n = 0, \tag{4.9}$$

conjugate to (4.2), as we saw in sec. 3, has roots differing only in sign from the roots of the fundamental equation of the system (4.2). The corresponding roots have the same multiplicity and the same number of sets of solutions corresponds to both systems. From this it follows that the fundamental equation of the system (4.9) has a zero root to which correspond k sets of solutions and r pairs of purely imaginary roots $\pm 2\pi p_j i/\omega$, to each of which corresponds respectively k_j sets of solutions. Consequently, the system (4.9) likewise has m and only m independent periodic solutions. We shall denote these solutions respectively by $\psi_{s1}(t),\ldots,\psi_{sm}(t)$.

On the basis of formula (3.3), which establishes the relation between the solutions of conjugate systems, we can write down for any h

$$\sum_{\alpha=1}^{n} x_{\alpha j}(t-h)\,\psi_{\alpha i}(t) = A_{ji}, \tag{4.10}$$

where A_{ji} are constants. To determine these constants we put $t = h$ in (4.10). On the basis of (4.3) we then have:

$$\sum_{\alpha=1}^{n} x_{\alpha j}(t-h)\,\psi_{\alpha i}(t) = \psi_{ji}(h). \tag{4.11}$$

Let us turn now to equations (4.7). Since the system (4.2) by assumption admits m independent periodic solutions, equations (4.7) admit m linearly independent solutions for $C_1^*, \ldots, C_n^*$. Hence, not only the determinant (4.8), but also all minors up to the order $n - m + 1$ inclusive are equal to zero and at least one of the minors of the (n-m)th order is different from zero. This means also that the left hand sides of equations (4.7) are connected by m linearly independent relations and for the solvability of equations (4.6) it is necessary and sufficient that the free terms of these equations be connected by the same relations. These relations can readily be obtained in the following manner.

Multiply the s-th equation of (4.6) by the magnitude $\psi_{si}(\omega)$ and sum over the index s from 1 to n. We obtain:

$$C_1 \sum_{s=1}^{n} x_{s1}(\omega)\psi_{si}(\omega) + \ldots + C_n \sum_{s=1}^{n} x_{sn}(\omega)\psi_{si}(\omega) - \\ - \sum_{s=1}^{n} \psi_{si}(\omega) C_s + \int_0^{\omega} \sum_{s,\,\alpha=1}^{n} x_{s\alpha}(\omega-\tau)\psi_{si}(\omega) f_\alpha(\tau)\,d\tau = 0. \quad (4.12)$$

But from (4.11), putting $t = \omega$, $h = 0$, and then $t = \omega$, $h = \tau$, we shall have the identities

$$\sum_{s=1}^{n} x_{sj}(\omega)\psi_{si}(\omega) \equiv \psi_{ji}(0),$$

$$\sum_{s=1}^{n} x_{s\alpha}(\omega-\tau)\psi_{si}(\omega) \equiv \psi_{\alpha i}(\tau).$$

Hence equations (4.12) assume the form

$$C_1[\psi_{1i}(0) - \psi_{1i}(\omega)] + \ldots + C_n[\psi_{ni}(0) - \psi_{ni}(\omega)] + \\ + \int_0^{\omega} \sum_{\alpha=1}^{n} f_\alpha(\tau)\psi_{\alpha i}(\tau)\,d\tau = 0.$$

But since $\psi_{si}(\omega) = \psi_{si}(0)$ on account of the periodicity of the function ψ_{si}, we obtain finally:

$$\int_0^{\omega} \sum_{\alpha=1}^{n} f_\alpha(\tau)\psi_{\alpha i}(\tau)\,d\tau = 0 \qquad (i = 1, \ldots, m). \qquad (4.13)$$

These evidently will be the required relations expressing the necessary and sufficient conditions for the solvability of (4.6) and, consequently, the necessary and sufficient conditions for the existence of real periodic solutions of equations (4.1). We thus arrive at the following conclusion:

IN ORDER THAT THE SYSTEM (4.1) ADMIT, FOR THE RESONANCE CASE, PERIODIC SOLUTIONS, IT IS NECESSARY AND SUFFICIENT THAT THE FUNCTIONS $f_s(t)$ SATISFY THE m CONDITIONS (4.13).

If conditions (4.13) are satisfied, the system (4.1) will admit, not one periodic solution, as was true in the resonance case, but an infinite number of such solutions. These solutions are determined by the formulas

$$x_s = M_1\varphi_{s1} + \ldots + M_m\varphi_{sm} + \omega_s(t),$$

where ω_s is some particular periodic solution of the equation (4.1) and $M_1, \ldots, M_m$ are arbitrary constants.

The functions ω_s determining the particular periodic solution of system (4.1) are evidently certain operators of $f_1, \ldots, f_n$. Since there is an infinite number of such particular solutions , there will likewise be an infinite number of such systems ofoperators. Of these we shall select one that possesses special properties.

For this purpose we again write down the general solution of equations (4.1) according to formulas (4.5), replacing however in the general integral of the homogeneous part the fundamental system x_{sj} by another system x^*_{sj}, namely, we shall assume that in the fundamental system x^*_{sj} are included the functions φ_{si} and form the first m solutions of this system. On account of the periodicity of the functions φ_{si} we then obtain for determining the constants C_s in the periodic solution , instead of equations (4.6), the equations

$$C_{m+1}[x^*_{s,m+1}(\omega) - x^*_{s,m+1}(0)] + \ldots + C_n[x^*_{sn}(\omega) - x^*_{sn}(0)] +$$

$$+ \int_0^\omega \sum_{\alpha=1}^n x_{s\alpha}(\omega - \tau) f_\alpha(\tau) d\tau = 0 \qquad (s = 1, \ldots, n). \quad (4.14)$$

Since the conditions (4.13) are assumed satisfied and therefore system (4.1) admits a periodic solution, these n equations with $n - m$ unknowns $C_{m+1},\dots,C_n$ have at least one solution . It is not difficult to see that there will be only one such solution.

In fact, if there are two systems of magnitudes $C^*_{m+1},\dots,C^*_n$ satisfying equations (4.14) then, substituting these magnitudes in the general integral, we obtain two periodic solutions of the system (4.1). Forming their difference we obtain the periodic functions

$$(C^*_{m+1} - C_{m+1})\,x^*_{s,\,m+1} + \dots + (C^*_n - C_n)\,x^*_{sn},$$

which evidently are a solution of the homogeneous system (4.2). But in this case these functions must necessarily be combinations of the functions φ_{si}, since the system (4.2) has no other periodic solutions. This however contradicts the condition of the independence of the chosen fundamental system of solutions.

Thus, assuming that $C_{m+1},\dots,C_n$ are chosen according to conditions (4.14) we obtain an entirely definite periodic solution of the system (4.1):

$$L_s(t,\ f_1,\ \dots,\ f_n) =$$
$$= C_{m+1}x^*_{s,\,m+1}(t) + \dots + C_n x^*_{sn}(t) + \int_0^t \sum_{\alpha=1}^{n} x_{s\alpha}(t-\tau)\,f_\alpha(\tau)\,d\tau \qquad (4.15)$$
$$(s = 1,\ \dots,\ n).$$

The operators L_s possess the following properties, of importance for what follows.

a) If f_s and F_s are two systems of periodic functions and C a constant, then

$$L_s(t,\ f_1+F_1,\ \dots,\ f_n+F_n) = L_s(t,\ f_1,\ \dots,\ f_n) + L_s(t,\ F_1,\ \dots,\ F_n), \qquad (4.16)$$

$$L_s(t,\ Cf_1,\ \dots,\ Cf_n) = CL_s(t,\ f_1,\ \dots,\ f_n). \qquad (4.17)$$

b) If the inequality $|f_s| < A$ is satisfied, where A is a certain constant, there will also be satisfied the inequality

$$|L_s(t,\ f_1,\ \dots,\ f_n)| < BA, \qquad (4.18)$$

where B is a constant that does not depend on the choice of the functions f_s.

c) If the functions f_s depend on certain parameters $\alpha_1,\ldots,\alpha_p$ and possesses with respect to these parameters a certain number of derivatives, the operators L will possess the same number of derivatives with respect to these parameters.

Properties a) and c) follow immediately from the form of L_s if account is taken of the fact that the coefficients $C_{m+1},\ldots,C_n$ on the basis of (4.14) are linear homogeneous functions of the magnitudes

$$\int_0^\omega x_{s\alpha}(\omega-\tau) f_\alpha(\tau)\,d\tau.$$

As regards property b) it is sufficient, in virtue of the periodicity of L_s, to prove its validity for $0 \leqslant t \leqslant \omega$. But in this interval of variation of t we evidently have:

$$|x_{s\alpha}(t)| < P,$$

$$\left|\int_0^\omega x_{s\alpha}(\omega-\tau) f_\alpha(\tau)\,d\tau\right| < \int_0^\omega |x_{s\alpha}(\omega-\tau)|\,|f_\alpha(\tau)|\,d\tau < \omega P A,$$

$$\left|\int_0^t x_{s\alpha}(t-\tau) f_\alpha(\tau)\,d\tau\right| \leqslant \int_0^\omega |x_{s\alpha}(t-\tau)|\,|f_\alpha(\tau)|\,d\tau < \omega P A,$$

where P is a constant. Whence taking into account the form of L_s and the properties just mentioned of the coefficients $C_{m+1},\ldots,C_n$, we immediately arrive at (4.18).

We shall now consider two examples illustrating conditions (4.13).

EXAMPLE 1. Let there be given the equation

$$\frac{d^2x}{dt^2} + k^2 x = f(t), \tag{4.19}$$

where $f(t)$ is an arbitrary continuous periodic function of t of period $\omega = 2\pi/k$. For this equation to admit periodic solutions it is necessary and sufficient, as we know, that the conditions be satisfied

$$\int_0^{2\pi/k} f(t)\cos kt\,dt = \int_0^{2\pi/k} f(t)\sin kt\,dt = 0. \tag{4.20}$$

We shall derive these conditions from the general conditions (4.13). For this purpose we write equation (4.19) in the form of the system

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -k^2 x_1 + f(t) \tag{4.21}$$

and form the system

$$\frac{dy_1}{dt} = k^2 y_2, \quad \frac{dy_2}{dt} = -y_1,$$

conjugate to the homogeneous part of the system (4.21). This system has the periodic solutions

$$\psi_{11} = \sin kt, \quad \psi_{21} = \frac{1}{k}\cos kt,$$
$$\psi_{12} = \cos kt, \quad \psi_{22} = -\frac{1}{k}\sin kt.$$

Substituting these solutions in (4.13) we obtain conditions (4.20).

EXAMPLE 2. Let us consider the system

$$\left.\begin{aligned} \frac{dx_1}{dt} &= -kx_2 + f(t), \\ \frac{dx_2}{dt} &= kx_1 + F(t), \end{aligned}\right\} \tag{4.22}$$

where f(t) and F(t) are continuous periodic functions of period $\omega = 2\pi/k$. Equations (4.9) for this system have the form

$$\frac{dy_1}{dt} = -ky_2, \quad \frac{dy_2}{dt} = ky_1,$$

and we can therefore put

$$\psi_{11} = \cos kt, \quad \psi_{21} = \sin kt,$$
$$\psi_{12} = \sin kt, \quad \psi_{22} = -\cos kt.$$

Substituting in (4.13) we obtain the following condition for the existence of periodic solutions of the system (4.22):

$$\left.\begin{aligned} \int_0^{2\pi/k} \{f(t)\cos kt + F(t)\sin kt\}\, dt = 0, \\ \int_0^{2\pi/k} \{f(t)\sin kt - F(t)\cos kt\}\, dt = 0. \end{aligned}\right\} \quad (4.2$$

These conditions for integral k were obtained by a different method in sec. 17 of the preceding chapter (conditions(17.16)).

5. Oscillations of Nonautonomous Systems Not at Resonance

We now pass on to the study of periodic oscillations of quasilinear nonautonomous systems with many degrees of freedom. [1] We shall assume that the oscillations of the system are described by equations of the form

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n + f_s(t) + \mu F_s(t, x_1, \ldots, x_n, \mu) \quad (5.1)$$
$$(s = 1, \ldots, n).$$

Here $f_s(t)$ are continuous periodic functions of t the period of which, as usual, we take equal to 2π. The functions F_s are defined for all values of t in the interval $[0, \infty]$, for $\mu < \mu_0$, where μ_0 is a positive constant, and for values of $x_1, \ldots, x_n$ lying in a certain region G of the space of these variables. In the indicated region of variation of the variables $t, x_1, \ldots, x_n, \mu$ the functions F_s are continuous and periodic with period 2π relative to t and admit with respect to the variables $x_1, \ldots x_n, \mu$ continuous derivatives up to an order k, which will be specified below.

Let us consider the generating system

$$\frac{dx_s^0}{dt} = a_{s1}x_1^0 + \ldots + a_{sn}x_n^0 + f_s(t) \quad (5.2)$$

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[1] See for example the texbooks on differential equation cited on p. 100.

and assume at first that we are dealing with the nonresonance case, i.e. that the fundamental question of the homogeneous part of this system has neither a zero root nor roots of the form $\pm pi$, where p is an integer. In this case, as was shown in the preceding section, the system (5.2) admits one and only one periodic solution

$$x_s^0 = \varphi_s(t),$$

which we shall take as the generating solution. We shall assume that this solution lies in the region G.

We shall seek to obtain the periodic solution of the complete system (5.1) that reduces to the generating solution for $\mu = 0$. For this purpose, following the method of Poincaré, we consider the solution of the system (5.1) determined by the initial conditions

$$x_s(0, \beta_1, \ldots, \beta_n, \mu) = \varphi_s(0) + \beta_s. \tag{5.3}$$

As is known from the theory of differential equations [1], the functions $x_s(t, \beta_1, \ldots, \beta_n, \mu)$ will admit derivatives with respect to the variables $x_1, \ldots, x_n, \mu$ up to the order k inclusive.

We shall try to find the magnitudes β_s as functions of the parameter μ in such manner that this solution is periodic and the magnitudes β_s reduce to zero for $\mu=0$. The problem reduces to finding the functions $\beta_s(\mu)$ reducing to zero for $\mu = 0$ and satisfying the equations

$$x_s(2\pi, \beta_1, \ldots, \beta_n, \mu) - x_s(0, \beta_1, \ldots, \beta_n, \mu) = 0 \tag{5.4}$$
$$(s = 1, \ldots, n).$$

We shall examine these equations more closely.

Substituting in the functions F_s in equations (5.1) the solution $x_s(t, \beta_1, \ldots, \beta_n, \mu)$ of these equations we obtain a system of linear equations with unknown right hand sides. Integrating these equations by the method of Cauchy we shall have

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[1] See for example the textbooks on differential equations cited on p. 100.

$$x_s = C_1 x_{s1}(t) + \dots + C_n x_{sn}(t) + \int_0^t \sum_{\alpha=1}^n x_{s\alpha}(t-\tau) f_\alpha(\tau)\, d\tau +$$
$$+ \mu \int_0^t \sum_{\alpha=1}^n x_{s\alpha}(t-\tau) F_\alpha(\tau, x(\tau, \beta, \mu), \mu)\, d\tau, \quad (5.5)$$

where C_s are arbitrary constants representing the initial values of the magnitudes x_s while x_{sj} are a fundamental system of solutions of the equations

$$\frac{dx_s}{dt} = a_{s1}x_1 + \dots + a_{sn}x_n, \quad (5.6)$$

determined by the initial conditions

$$x_{ss}(0) = 1, \quad x_{sj}(0) = 1 \quad (s \neq j).$$

the abbreviated notation being used

$$F_s(t, x, \mu) = F_s(t, x_1, \dots, x_n, \mu), \; x_s(t, \beta_1, \dots, \beta_n, \mu) = x_s(t, \beta, \mu).$$

Among the solutions (5.5) there is contained evidently also the solution $x_s(t, \beta_1, \dots, \beta_n, \mu)$ which corresponds to the values of the magnitudes C_s determined by the equations

$$C_s = \varphi_s(0) + \beta_s.$$

Substituting these values of C_s in (5.5) and taking into account that the expressions

$$\varphi_1(0) x_{s1}(t) + \dots + \varphi_n(0) x_{sn}(t) + \int_0^t \sum_{\alpha=1}^n x_{s\alpha}(t-\tau) f_\alpha(\tau)\, d\tau$$

are a solution of equations (5.2) with the initial conditions $\varphi_s(0)$, i.e. the generating solution $\varphi_s(t)$, we obtain

$$x_s(t, \beta_1, \dots, \beta_n, \mu) = \beta_1 x_{s1}(t) + \dots + \beta_n x_{sn}(t) + \varphi_s(t) +$$
$$+ \mu \int_0^t \sum_{\alpha=1}^n x_{s\alpha}(t-\tau) F_\alpha(\tau, x(\tau, \beta, \mu), \mu)\, d\tau. \quad (5.7)$$

Equations (5.4) can therefore be represented in the following form:

$$\beta_1 x_{s1}(2\pi) + \ldots + \beta_n x_{sn}(2\pi) - \beta_s + \\ + \mu \int_0^{2\pi} \sum_{\alpha=1}^{n} x_{s\alpha}(2\pi - \tau) F_\alpha(\tau, x(\tau, \beta, \mu), \mu)\, d\tau = 0 \quad (5.8)$$

$$(s = 1, \ldots, n).$$

These equations are satisfied for $\beta_1 = \ldots = \beta_n = \mu = 0$. Their functional determinant with respect to the magnitudes $\beta_1, \ldots, \beta_n$ reduces for $\beta_1 = \ldots = \beta_n = \mu = 0$ to the determinant (4.8) (for $\omega = 2\pi$) which, as was shown in the preceding section , is in the nonresonance case different from zero. Consequently, on the basis of the theorem on implicit functions, equations (5.8), for sufficiently small μ, have one and only one solution $\beta_s(\mu)$ for which $\beta_s(0) = 0$. Substituting the found values of $\beta_s(\mu)$ in $x_s(t, \beta_1, \ldots, \beta_n, \mu)$ we obtain one and only one periodic solution of the system (5.1) that reduces to the generating solution for $\mu = 0$. We thus arrive at the following theorem:

IN THE NONRESONANCE CASE THE SYSTEM (5.1) FOR SUFFICIENTLY SMALL VALUE OF μ HAS ONE AND ONLY ONE PERIODIC SOLUTION REDUCING TO THE GENERATING SOLUTION FOR $\mu = 0$.

We proceed to the question of the practical computation of the above indicated periodic solution. We shall here distinquish two cases depending on whether the functions F_s are analytic with respect to $x_1, \ldots, x_n$, μ or not.

Assume first that the functions F_s are analytic with respect to $x_1, \ldots, x_n$. The functions $x_s(t, \beta_1, \ldots, B_n, u)$ will then for sufficiently small values of $\beta_1, \ldots, \beta_n$, μ, be analytic with respect to these magnitudes. In precisely the same way the magnitudes $\beta_s(\mu)$, determined by equations (5.8), will be analytic functions of μ for its sufficiently small values. But then evidently the required periodic solutions will also be analytic functions of μ. We can therefore seek to obtain these functions in the form of the series

$$x_s = \varphi_s(t) + \mu x_s^{(1)}(t) + \ldots, \quad (5.9)$$

where $x_s^{(i)}$ are periodic functions of t of period 2π. Substituting these series in (5.1) and equating coefficients of like powers of μ we obtain for the determination of the coefficients $x_s^{(i)}$ equations of the form

$$\left.\begin{aligned} \frac{dx_s^{(1)}}{dt} &= a_{s1}x_1^{(1)} + \ldots + a_{sn}x_n^{(1)} + F_s(t, \varphi_1, \ldots, \varphi_n, 0), \\ \frac{dx_s^{(i)}}{dt} &= a_{s1}x_1^{(i)} + \ldots + a_{sn}x_n^{(i)} + F_s^{(i)} \quad (i = 2, 3, \ldots), \end{aligned}\right\} \quad (5.10)$$

where $F_s^{(i)}$ are integral rational functions with periodic coefficients of those $x_r^{(j)}$ for which $j < i$. If all these magnitudes already computed came out periodic, the equations for $x_s^{(i)}$ will admit one and only one periodic solution. From this it follows that equations (5.10) completely determine the coefficients $x_s^{(i)}$ and that, consequently, there exists one and only one system of series with periodic coefficients formally satisfying equations(5.1). These series will therefore represent the required periodic solution and will thus converge.

Let us assume now that the functions F_s are not analytic with respect to $x_1, \ldots, x_n, \mu$. Then for computing the required periodic solution the method of successive approximations may be applied, namely, taking the generating solution as the first approximation we determine the further approximations $x_s^{(i)}$ as periodic solutions of the equations

$$\frac{dx_s^{(i)}}{dt} = a_{s1}x_1^{(i)} + \ldots + a_{sn}x_n^{(i)} + f_s(t) + \mu F_s(t, x_1^{(i-1)}, \ldots, x_n^{(i-1)}, \mu).$$

The proof of the convergence of these approximations to the required periodic solution will be given below, in sec. 9. With regard to the functions F_s it will be sufficient to assume that they satisfy with respect to the variables $x_1, \ldots, x_n, \mu$ the Cauchy-Lipschitz conditions

$$|f_s(t, x_1', \ldots, x_n', \mu') - f_s(t, x_1'', \ldots, x_n'', \mu'')| < \\ < L\Big(\sum_{\alpha=1}^{n} |x_\alpha' - x_\alpha''| + |\mu' - \mu''|\Big), \quad (5.11)$$

where L is a certain constant.

Let us now consider the oscillations of a nonautonomous system described by the equation (5.1) in the case of resonance. We shall assume that the fundamental equation of the system (5.6) has either a zero root or roots of the form $\pm pi$, where p is an integer. We shall also consider it as a resonance case when the fundamental equation has roots different from the above mentioned critical values by a magnitude of the order of smallness of μ. This latter case can however be reduced to the preceding by referring the correction terms to the functions μF_s.

Let us assume for definiteness that the fundamental equation has a zero root of arbitrary multiplicity to which corresponds k sets of solutions and r pairs of purely imaginary roots of the form $\pm p_j i (j = 1,\ldots,r)$, where p_j are integers. These roots may likewise be of arbitrary multiplicity. We denote by k_j the number of sets of solutions corresponding to each of the roots in the pair $\pm p_j i$. Further, we denote by m the magnitude

$$m = k + 2k_1 + \ldots + 2k_r.$$

The system (5.6) will then admit m and only m periodic solutions which we shall denote, as in sec. 4, by $\varphi_{s1}(t),\ldots,\varphi_{sm}(t)$. Each of the functions φ_{sj} is either constant (for the solutions corresponding to the zero root), or has the form $A\cos pt + B\sin pt$. The system conjugate to (5.6) similarly has m independent periodic solutions which we denote by $\psi_{s1}(t),\ldots, \psi_{sm}(t)$. The functions ψ_{sj} have the same structure as the functions φ_{sj}.

Let us now consider the generating system (5.2). In order that this system in the case considered have a periodic solution it is necessary and sufficient that the functions $f_s(t)$ satisfy conditions (4.13). Taking the form of the functions ψ_{sj} into account it is easily seen that for this it is sufficient (but not necessary) that the Fourier expansions of the functions f_s do not contain the resonating harmonics $\cos p_j t$ and $\sin p_j t$ and also free terms if the fundamental equation has a zero root. We shall assume that the conditions (4.13) are in fact satisfied. The generating system will then have the family of periodic solutions

$$x_s^0 = x_s^{0*}(t) + M_1\varphi_{s1} + \ldots + M_m\varphi_{sm} \qquad (s = 1, \ldots, n), \qquad (6.1)$$

where $x_s^{0*}(t)$ is a particular periodic solution of this system while $M_1, \ldots, M_m^*$ are arbitrary constants. Taking some solution of this family corresponding to the values of the $M_i = M_i^*$ parameters as the generating solution we shall seek to obtain the periodic solution of the complete system (5.1) that reduces to this generating solution for $\mu = 0$.

For this purpose let us, as in the preceding section, denote by $x_s(t, \beta_1, \ldots, \beta_n, \mu)$ the solution of the complete system (5.1) determined by the initial conditions (5.3), where now however

$$\varphi_s(t) = x_s^{0*}(t) + M_1^*\varphi_{s1} + \ldots + M_m^*\varphi_{sm}. \qquad (6.2)$$

For this solution there is also valid formula (5.7) and the conditions of periodicity, determining the magnitudes β_s, have as before the form (5.8). But now however the value of the functional determinant of these equations with respect to $\beta_1, \ldots, \beta_n$ at the point $\beta_1 = \ldots = \beta_n = \mu = 0$, agreeing with (4.8) (for $\omega = 2\pi$), is equal to zero. All minors of this determinant up to the order $n - m + 1$ inclusive are also, as was shown in sec.4, equal to zero, but at least one minor of this determinant of $(n-m)$th order is different from zero. For definiteness we shall assume that

$$\begin{vmatrix} x_{11}(2\pi) - 1 & \ldots & x_{1,n-m}(2\pi) \\ \cdot & \cdot\,\cdot\,\cdot & \cdot \\ x_{n-m,1}(2\pi) & \ldots & x_{n-m,n-m}(2\pi) - 1 \end{vmatrix} \neq 0. \qquad (6.3)$$

It is easily seen that for this assumption there is likewise satisfied the condition

$$\begin{vmatrix} \varphi_{n-m+1,1}(0) & \ldots & \varphi_{n1}(0) \\ \cdot & \cdot\,\cdot\,\cdot & \cdot \\ \varphi_{n-m+1,m}(0) & \ldots & \varphi_{nm}(0) \end{vmatrix} \neq 0. \qquad (6.4)$$

In fact, any periodic solution of the homogeneous linear system (5.6), i.e. the expression

$$\bar{x}_1(t)=M_1\varphi_{s1}+\ldots+M_m\varphi_{sm} \qquad (s=1,\ldots,n), \qquad (6.5)$$

can be represented in the form

$$\bar{x}_s(t)=\bar{x}_1(0)x_{s1}(t)+\ldots+\bar{x}_n(0)x_{sn}(t),$$

where the magnitudes $\bar{x}_s(0)$ must satisfy the equations

$$x_1(0)x_{s1}(2\pi)+\ldots+\bar{x}_n(0)x_{sn}(2\pi)-\bar{x}_s(0)=0 \qquad (s=1,\ldots,n).$$

Since the rank of the determinant of this system is equal to $n-m$, it admits a solution for $\bar{x}_s(0)$ depending on m arbitrary constants for which, in virtue of (6.3), there may be taken the magnitudes $\bar{x}_{n-m+1}(0)\ldots x_n(0)$. But on the basis of (6.5)

$$\bar{x}_i(0)=M_1\varphi_{i1}(0)+\ldots+M_m\varphi_{im}(0) \qquad (i=n-m+1,\ldots,n),$$

and since these equations must be solvable for arbitrary $\bar{x}_i(0)$, condition (6.4) must be satisfied.

Having established this, we proceed to the consideration of the system (5.8). The problem is to determine the conditions under which this system admits a solution $\beta_s = \beta_s(\mu)$ for which $\beta_s(0) = 0$.

We multiply the s-th equation of (5.8) by $\psi_{si}(2\pi)$ and sum over the index s from 1 to n. Then, taking into account (4.11), we obtain, in exactly the same way as in sec. 4 (in deriving the equations (4.13)), the following m equations:

$$\mu\int_0^{2\pi}\sum_{\alpha=1}^{n}F_\alpha(\tau, x(\tau,\beta,\mu),\mu)\psi_{\alpha i}(\tau)\,d\tau=0 \qquad (i=1,\ldots,\omega). \quad (6.6)$$

We shall make use of these equations in place of the last m equations of (5.8). The remaining n - m equations of (5.8) , i.e. the equations

$$\beta_1 x_{j1}(2\pi)+\ldots+\beta_n x_{jn}(2\pi)-\beta_j+$$
$$+\mu\int_0^{2\pi}\sum_{\alpha=1}^{n} x_{j\alpha}(2\pi-\tau)F_\alpha(\tau, x(\tau,\beta,\mu),\mu)\,d\tau=0\ (j=1,\ldots,n-m), \quad (6.7)$$

we shall leave without change.

Equations (6.7) in virtue of (6.3) can be solved for the magnitudes $\beta_1,\ldots,\beta_{n-m}$ and will give the functions

$$\beta_j=\beta_j(\mu,\beta_{n-m+1},\ldots,\beta_n) \qquad (j=1,\ldots,n-m), \qquad (6.8)$$

which reduce to zero for $\beta_{n-m+1}=\beta_n=0$. Moreover, since the left hand sides of equations (6.7) have derivatives with respect to $\beta_1,\ldots,\beta_n,\mu$ up to the order k inclusive, the functions (6.8) will likewise have derivatives of the k-th order with respect to the variables $\beta_{n-m+1},\ldots,\beta_n,\mu$. Substituting the functions (6.8) in (6.6) and dividing by μ we obtain for the determination of $\beta_{n-m+1},\ldots,\beta_n$ the equations

$$Q_i(\beta_{n-m+1},\ldots,\beta_n,\mu)=\int_0^{2\pi}\sum_{\alpha=1}^{n}F_\alpha(\tau,\tilde{x}(\tau,\beta,\mu),\mu)\,\psi_{\alpha i}\,d\tau=0 \quad (6.9)$$
$$(i=1,\ldots,m),$$

where $\tilde{x}_s(\tau,\beta,\mu)=\tilde{x}_s(\tau,\beta_{n-m+1},\ldots,\beta_n,\mu)$ denotes the function $x_s(\tau,\beta_1,\ldots,\beta_n,\mu)$ after substituting in it for the magnitudes $\beta_1,\ldots,\beta_{n-m}$ their expressions (6.8). In this way the problem reduces to the determination of the functions $\beta_k(\mu)(k=n-m+1,\ldots,n)$ satisfying the equations (6.9) and conditions $\beta_k(0)=0$.

In order that equations (6.9) possess a solution of the required form it is evidently necessary that the equation be satisfied

$$P_i(M_1^*,\ldots,M_m^*)=Q_i(0,\ldots,0,0)=0, \qquad (6.10)$$

in which the magnitudes M_1^* enter on the basis of (5.7) and (6.2). If in satifying equations (6.10) there is satisfied the further condition

$$\left\{\frac{\partial(Q_1,\ldots,Q_m)}{\partial(\beta_{n-m+1},\ldots,\beta_n)}\right\}_{\beta=\mu=0}\neq 0, \tag{6.11}$$

equations (6.9), for μ sufficiently small, will in fact have a solution for $\beta_{n-m+1},\ldots,\beta_n$ of the required form and only one at that. After this relations (6.8) will give entirely definite values for $\beta_1,\ldots,\beta_{n-m}$ and substituting them in $x_s(t,\beta_1,\ldots,\beta_n,\mu)$ we obtain the required periodic solution of the system (5.1), which will be the only solution.

Let us consider more in detail equations (6.10) and condition (6.11). Setting in $x_s(t,\beta_1,\ldots,\beta_n,\mu)$ the magnitude μ equal to zero we obtain the solution of the generating system with the initial conditions (5.3). Then, substituting for the magnitudes $\beta_1,\ldots,\beta_{n-m}$ their expressions (6.8) for $\mu = 0$ we obtain a periodic solution of the generating system whatever the values of the magnitudes $\beta_{n-m+1},\ldots,\beta_n$ since for these substitutions the conditions of periodicity (6.6) and (6.7) are identically satisfied. We can therefore write:

$$\begin{aligned}\tilde{x}_s(t,\beta_{n-m+1},\ldots,\beta_n,0) &= x_s^{0*}(t)+M_1\varphi_{s1}+\ldots+M_m\varphi_{sm}=\\ &= x_s^0(t,M_1,\ldots,M_m),\end{aligned} \tag{6.12}$$

where x_s^{o*} have the same values as in (6.1) and $M_1\ldots,M_n$ are certain constants, functions of $\beta_{n-m+1},\ldots,\beta_n$. To obtain the explicit expressions of these functions we set $t = 0$ in (6.12). Then, taking (5.3) and (6.2) into account we obtain:

$$\beta_k=(M_1-M_1^*)\varphi_{k1}+\ldots+(M_m-M_m^*)\varphi_{km} \qquad (k=n-m+1,\ldots,n), \tag{6.13}$$

whence, in particular, it follows that to the zero values of the magnitudes $\beta_{n-m+1},\ldots,\beta_n$ there correspond the values M_1^* of the magnitudes M_1, since the determinant of the system (6.13) on the basis of (6.4) is different from zero.

From (6.9) and (6.12) we find:

$$Q_i(\beta_{n-m+1}, \ldots, \beta_n, 0) = \int_0^{2\pi} \sum_{\alpha=1}^{n} F_\alpha(\tau, x^0(\tau, M), 0)\,\psi_{\alpha i}(\tau)\,d\tau, \tag{6.14}$$

$$\left\{\frac{\partial(Q_1, \ldots, Q_m)}{\partial(\beta_{n-m+1}, \ldots, \beta_n)}\right\}_{\mu=0} = \left\{\frac{\partial(Q_1, \ldots, Q_m)}{\partial(M_1, \ldots, M_n)}\right\}_{\mu=0} : \frac{\partial(\beta_{n-m+1}, \ldots, \beta_n)}{\partial(M_1, \ldots, M_m)}.$$

Equations (6.10) therefore assume the form

$$P_i(M_1^*, \ldots, M_k^*) \equiv \int_0^{2\pi} \sum_{\alpha=1}^{n} F_\alpha(\tau, \varphi, 0)\,\psi_{\alpha i}(\tau)\,d\tau = 0 \tag{6.15}$$
$$(i = 1, \ldots, m),$$

since for $\beta_{n-m+1} = \ldots = \beta_n = 0$ the magnitudes M_1 become M_1^* and the functions $x_s^o(t, M_1, \ldots, M_m)$ become the generating solution $\varphi_s(t)$. By the same considerations, taking into account the fact that the determinant

$$\frac{\partial(\beta_{n-m+1}, \ldots, \beta_n)}{\partial(M_1, \ldots, M_m)},$$

agreeing, according to (6.13) , with (6.4), is different from zero, we can satisfy ourselves that condition (6.11) is equivalent to the condition

$$\frac{\partial(P_1, \ldots, P_m)}{\partial(M_1^*, \ldots, M_m^*)} \neq 0. \tag{6.16}$$

From all that was said above we obtain the following fundamental proposition:

IN ORDER THAT THE SYSTEM (5.1) FOR SUFFICIENTLY SMALL VALUE OF μ HAVE IN THE RESONANCE CASE A PERIODIC SOLUTION REDUCING TO THE GENERATING SOLUTION (6.2) FOR $\mu = 0$, IT IS NECESSARY THAT THE PARAMETERS M_i^* IN THIS GENERATING SOLUTION SATISFY EQUATION (6.15). FOR EACH SIMPLE SOLUTION OF THESE EQUATIONS, I.E. A SOLUTION FOR WHICH CONDITION (6.16) IS SATISFIED, THE INDICATED PERIODIC SOLUTION ACTUALLY EXISTS AND IS THE ONLY SOLUTION.

For the above proposition to hold true it is sufficient that the functions $P_i(M_1, \ldots, M_m)$ admit derivatives of the first order with respect to the variables $M_1, \ldots, M_m$, for which it is sufficient that the functions F_s admit derivatives of the first order with respect to the variables $x_1, \ldots, x_n, \mu$. Thus, in the case under consideration the number k, figuring in the conditions relative to the functions F_s and characterizing their degree of smoothness, can be taken equal to unity.

If for the solution M_1^* of equations (6.15) condition (6.16) is not satisfied the question as to the existence of a solution of equations (6.9), and therefore also of a periodic solution of the system (5.1), remains open. In practice the case may be of interest where equations (6.15) are satisfied identically. Assuming that the functions F_s have with respect to $x_1,\ldots,x_n,\mu$ derivatives at least of the first order, which condition the existence of derivatives of the first order of the function Q_i with respect to μ, we can, for the condition $Q_i(\beta_{n-m+1},\ldots\beta_n 0) \equiv 0$, write $Q_i = \mu Q_i'(\beta_{n-m+1},\ldots,\beta_n,\mu)$ and equations (6.9) assume the form

$$Q_i'(\beta_{n-m+1},\ldots,\beta_n,\mu)=0.$$

In order that these equations admit solutions of the required form it is necessary that the parameters M_i^* of the generating solution satisfy the equations

$$P_i'(M_1^*,\ldots,M_m^*)=Q_i'(0,\ldots,0)=0. \tag{6.17}$$

If at the same time the condition is satisfied

$$\frac{\partial(P_1',\ldots,P_m')}{\partial(M_1^*,\ldots,M_m^*)}\neq 0, \tag{6.18}$$

such solutions will actually exist and will be unique. In this case there will as before exist a single solution of the system (5.1) reducing for $\mu = 0$ to the generating solution. It is here necessary to assume that the functions F_s admit derivatives of the second order with respect to the variables $x_1,\ldots,x_n$, i.e. that k = 2, in order to assure the existence of the derivatives of the first order of the functions P_i' that figure in (6.18).

If equations (6.17) are likewise satisfied identically we can divide them by μ and proceed with the obtained new equations as with the preceding ones. If, continuing in this manner, we arrive at equations with respect to M_i^* not identically satisfied and having a simple solution, system (5.1) will admit a unique periodic solution reducing for $\mu=0$ to the generating solution corresponding to the found values of the parameters M_i^*.

We shall denote the case where equations (6.15) are satisfied identically as the SINGULAR case.

7. Practical Method of Computing the Periodic Solutions of Nonautonomous Systems at Resonance in the Case of Analytic Equations

Let us consider the system (5.1)

$$\frac{dx_s}{dt}=a_{s1}x_1+\dots+a_{sn}x_n+f_s(t)+\mu F_s(t, x_1, \dots, x_n, \mu) \qquad (7.1)$$
$$(s=1, \dots, n)$$

for the assumptions of the preceding section but for the particular case where the functions F_s are analytic with respect to $x_1,\dots,x_n,\mu$. We shall show how in this case to find practically for this system the periodic solution which for $\mu = 0$ reduces to the generating solution

$$x_s^0=x_s^0(t)+M_1^*\varphi_{s1}(t)+\dots+M_m^*\varphi_{sm}(t)=\varphi_s, \qquad (7.2)$$

in which the parameters M_i^* satisfy equations (6.15). In this section we shall restrict ourselves to the consideration of the nonsingular case, i.e. we shall assume that equations (6.15) are not identically satisfied. We shall, moreover, assume that condition (6.16) is satisfied. Then, as was shown in the preceding section, system (7.1) admits for sufficiently small μ a unique periodic solution reducing to (7.2) for $\mu = 0$. It is easily seen that in the case under consideration of analytic equations this solution will be analytic with respect to μ. In fact, the functions $x_s(t,\beta_1,\dots,\beta_n,\mu)$ will be analytic with respect to $\beta_1,\dots,\beta_n,\mu$. The magnitudes $\beta_1,\dots,\beta_{n-m}$ determined by formulas (6.8) will also be analytic with respect to $\beta_{n-m+1},\dots,\beta_n,\mu$ and the functions $\beta_{n-m+1}(\mu),\dots,\beta_n(\mu)$, determined by equations (6.9), will be analytic with respect to μ. But then, evidently, the required periodic function will also be analytic with respect to μ.

We shall therefore seek to obtain this solution in the form of the formal series

$$x_s = \varphi_s + \mu x_s^{(1)}(t) + \mu^2 x_s^{(2)}(t) + \dots \qquad (7.3)$$

with periodic coefficients. Such series necessarily exist and if they are unique they will represent the required periodic solution and consequently will converge for sufficiently small μ.

Substituting the series (7.3) in (7.1) and equating coefficients of like powers of μ we obtain the following equations for determining the functions $x_s^{(j)}$:

$$\frac{dx_s^{(1)}}{dt} = a_{s1}x_1^{(1)} + \dots + a_{sn}x_n^{(1)} + F_s(t, \varphi_1, \dots, \varphi_n, 0), \qquad (7.4)$$

$$\frac{dx_s^{(l)}}{dt} = a_{s1}x_1^{(l)} + \dots + a_{sn}x_n^{(l)} + \sum_{\alpha=1}^{n} \frac{\partial F_s(t, \varphi_1, \dots, \varphi_n, 0)}{\partial \varphi_\alpha} x_\alpha^{(l-1)} + F_s^{(l-2)}$$
$$(l = 2, 3, \dots). \qquad (7.5)$$

Here $F_s^{(l-2)}$ are integral rational functions with periodic coefficients of those $x_s^{(j)}$ for which $j \leq l - 2$.

In order that equations (7.4) admit a periodic solution it is necessary and sufficient that the relations be satisfied

$$\int_0^{2\pi} \sum_{\alpha=1}^{n} F_\alpha(t, \varphi_1, \dots, \varphi_n, 0)\psi_{\alpha i}\, dt = 0 \qquad (i = 1, \dots, m).$$

Again we have thus arrived at the equations (6.15) determining the parameters M_i^* in the generating solution. Since these equations are by assumption satisfied, there exists in fact a periodic solution for $x_s^{(1)}$. This solution has the form

$$x_s^{(1)} = x_s^{(1)*} + M_1^{(1)}\varphi_{s1} + \dots + M_m^{(1)}\varphi_{sm},$$

where $x_s^{(1)*}$ is some particular periodic solution of equations (7.4) and $M_i^{(1)}$ are arbitrary constants. These constants can be disposed of in such manner that the equations for $x_s^{(2)}$ admit a periodic solution. Having computed $x_s^{(2)}$ we can dispose of the constants entering these arbitrary functions for satisfying the conditions of existence of periodic solutions of equations (7.5) for $l = 3$. In a similar manner we proceed further. We shall show that for determining the constants that enter in each approximation linear algebraic equations will here be obtained which, if conditions (6.16) are satisfied, will possess a determinant different from zero.

In fact, let us assume that all functions $x_s^{(1)}, \ldots, x_s^{(l-2)}$ together with the constants that enter them have already been computed and came out periodic. We shall assume also that there have also been determined the functions $x_s^{(l-1)}$, having the form

$$x_s^{(l-1)} = x_s^{(l-1)*} + M_1^{(l-1)} \varphi_{s1} + \ldots + M_m^{(l-1)} \varphi_{sm}, \tag{7.6}$$

where $x_s^{(l-1)*}$ is some particular periodic solution for these functions but that the constants $M_i^{(l-1)}$ are still to be determined from the conditions of periodicity of the functions $x_s^{(l)}$. Substituting (7.6) in the equations for $x_s^{(l)}$ we obtain

$$\frac{dx_s^{(l)}}{dt} = a_{s1} x_1^{(l)} + \ldots + a_{sn} x_n^{(l)} + \\ + \sum_{i=1}^{m} M_i^{(l-1)} \sum_{\alpha=1}^{n} \frac{\partial F_s(t, \varphi_1, \ldots, \varphi_n, 0)}{\partial \varphi_\alpha} \varphi_{\alpha i} + f_s^{(l)}(t), \tag{7.7}$$

where $f_s^{(l)}$ is an entirely definite periodic function of time.

In order that equation (7.7) admit a periodic solution it is necessary and sufficient that the relations be satisfied

$$\sum_{i=1}^{m} M_i^{(l-1)} \int_0^{2\pi} \sum_{s, \alpha=1}^{n} \frac{\partial F_s(t, \varphi_1, \ldots, \varphi_n, 0)}{\partial \varphi_\alpha} \varphi_{\alpha i} \psi_{sj}\, dt + \int_0^{2\pi} \sum_{s=1}^{n} f_s^{(l)} \psi_{sj}\, dt = 0$$
$$(j = 1, \ldots, m)$$

or, on the basis of (6.15) and (7.2),

$$\sum_{i=1}^{m} \frac{\partial P_j(M_1^*, \ldots, M_m^*)}{\partial M_i^*} M_i^{(l-1)} + \int_0^{2\pi} \sum_{s=1}^{n} f_s^{(l)} \psi_{sj}\, dt = 0.$$

We have thus obtained for determining the constants a system of linear equations with a determinant agreeing with (6.16). Since this determinant is by assumption different from zero, the obtained equation will always be solvable. Thus, for the assumptions made, there exists one and only one system of series (7.3) formally satisfying equations (7.1). It is these series that represent the required periodic solution.

In practice the equations of motion are generally not obtained directly in the normal form (7.1). For the applicability of the method it is not necessary however to bring this system to the normal form. It is frequently more convenient to start immediately from the given equations and seek their periodic solution in the form of series developed in powers of μ.

EXAMPLE. Let us consider a system of the form

$$\frac{d^2 x_s}{dt^2} = \mu F_s(t, x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, \mu) \equiv \mu F_s(t, x, \dot{x}, \mu) \quad (7.8)$$
$$(s = 1, \ldots, n),$$

where the functions F_s are periodic with respect to t with period 2π and analytic with respect to $x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, \mu$. To a special case of this type of system is reduced the problem of the forced oscillations of conservative and in many cases also nonconservative systems with n degree of freedom, when the frequency of the disturbing forces is large as compared with the natural frequencies of the system. For the particular case of a single equation

$$\frac{d^2 x}{dt^2} = \mu F(t, x, \mu),$$

in which the function F does not contain the derivative dx/dt and is subject to certain additional restrictions the problem has been considered by K. Stellmacher [1], who

[1]Stellmacher K.L., Ueber erzwungene nichtlineare Schwingungen hoher Erregerfrequenz und ihre Stabilität, Zeitschr. f. angew. Math. und Mech., vol.34, no. 3, 1954.

proposed a rather cumbersome method of determining the periodic solution. The problem can be solved however very simply for a general system of the form (7.8).

In fact, the fundamental equation of the generating system

$$\frac{d^2x_s^0}{dt^2}=0$$

has in the case under consideration a zero root of multiplicity 2n, to which corresponds n sets of solutions. This system has a family of periodic solutions containing n arbitrary constants and having evidently the form $x_s^0 = M_s$. Hence, the periodic solution of interest to us must be sought in the form

$$x_s = M_s^* + \mu x_s^{(1)}(t) + \mu^2 x_s^{(2)}(t) + \dots$$

For the functions $x_s^{(1)}$ and $x_s^{(2)}$ there are here obtained the following equations

$$\frac{d^2x_s^{(1)}}{dt^2} = F_s(t, M^*, 0, 0),$$

$$\frac{d^2x_s^{(2)}}{dt^2} = \sum_{\alpha=1}^{n}\left[\frac{\partial F_s(t, M^*, 0, 0)}{\partial M_\alpha^*}x_\alpha^{(1)} + \left(\frac{\partial F_s(t, M^*, \dot{x}, 0)}{\partial \dot{x}_\alpha}\right)_{\dot{x}=0}\dot{x}_\alpha^{(1)}\right] + \\ + \left(\frac{\partial F_s(t, M^*, 0, \mu)}{\partial \mu}\right)_{\mu=0}. \quad (7.9)$$

In order that the equations for $x_s^{(1)}$ admit a periodic solution it is necessary and sufficient that the conditions be satisfied

$$\int_0^{2\pi} F_s(t, M^*, 0, 0)\,dt = 0. \quad (7.10)$$

In fact, if conditions (7.10) are satisfied the functions $dx_s^{(1)}/dt$ will be periodic and have the form

$$\frac{dx_s^{(1)}}{dt} = \int_0^t F_s(t, M^*, 0, 0)\,dt + A_s^{(1)},$$

where $A_s^{(1)}$ are constants. If these constants are determined by the formulas

$$A_s^{(1)} = -\frac{1}{2\pi}\int_0^{2\pi} dt \int_0^t F_s(t, M^*, 0, 0)\, dt,$$

then also the functions $x_s^{(1)}$ are obtained periodic and we shall have

$$x_s^{(1)} = \int_0^t dt \left[\int_0^t F_s(t, M^*, 0, 0)\, dt + A_s^{(1)} \right] + M_s^{(1)} \equiv x_s^{(1)*} + M_s^{(1)}, \quad (7.11)$$

where $M_s^{(1)}$ are arbitrary constants. Conditions (7.10) may of course be obtained also from the general formulas (4.13).

Equations (7.10) determine the values M_s^* in the generating solution. Let us assume that M_s^* are a simple solution of these equations. The magnitudes $x_s^{(1)}$ are then determined by formulas (7.11) and to find the constants $M_s^{(1)}$ we set up the conditions of periodicity of the functions $x_s^{(2)}$. These conditions on the basis of (7.9) and (7.11) give:

$$\sum_{\alpha=1}^{n} M_\alpha^{(1)} \int_0^{2\pi} \frac{\partial F_s(t, M^*, 0, 0)}{\partial M_\alpha^*}\, dt + \int_0^{2\pi} \left\{ \left(\frac{\partial F_s(t, M^*, 0, \mu)}{\partial \mu} \right)_{\mu=0} + \right.$$
$$\left. + \sum_{\alpha=1}^{n} \left[\frac{\partial F_s(t, M^*, 0, 0)}{\partial M_\alpha^*} x_\alpha^{(1)*} + \left(\frac{\partial F_s(t, M^*, \dot{x}, 0)}{\partial \dot{x}_\alpha} \right)_{\dot{x}=0} \dot{x}_\alpha^{(1)*} \right] \right\} dt = 0.$$

We have thus obtained, for determining the constants $M_s^{(1)}$, linear equations the determinant of which by stipulation is different from zero.

In a similar manner the further approximations are computed.

8. Practical Method of Computing the Periodic Solutions of Nonautonomous Systems at Resonance in the Case of Nonanalytic Equations

Assume that for the equations

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n + f_s(t) + \mu F_s(t, x_1 \ldots, x_n, \mu) \quad (8.1)$$
$$(s = 1, \ldots, n)$$

all conditions of the preceding section are satisfied with the exception of the condition of the analyticity of the functions F_s. Instead of this condition we shall assume that the functions F_s admit within the region of their definition continuous partial derivatives of the first order with respect to the variables $x_1, \ldots, x_n, \mu$. We shall show how in this case to find a periodic solution of the system (8.1) that reduces for $\mu = 0$ to the generating solution

$$x_s^0 = \varphi_s(t) = x_s^{(0)*} + M_1^* \varphi_{s1} + \ldots + M_m^* \varphi_{sm}. \quad (8.2)$$

As in the preceding section we here assume that the constants M_1^* are a solution of the equations

$$P_i(M_1^*, \ldots, M_m^*) = \int_0^{2\pi} \sum_{\alpha=1}^{n} F_\alpha(t, \varphi_1, \ldots, \varphi_n, 0) \psi_{\alpha i}(t)\, dt = 0 \quad (8.3)$$
$$(i = 1, \ldots, m),$$

for which the condition is satisfied

$$\frac{\partial(P_1, \ldots, P_m)}{\partial(M_1^*, \ldots, M_m^*)} \neq 0. \quad (8.4)$$

For these conditions , as was shown in sec. 6, the required periodic function actually exists and is moreover the only one.

To compute this periodic solution we make use of the method of successive approximations. For this purpose we take as the first approximation the functions

$$x_s^{(1)} = x_s^{(0)*} + M_1^{(1)}\varphi_{s1} + \ldots + M_m^{(1)}\varphi_{sm},$$

where $M_i^{(1)} = M_i^{(1)}(\mu)$ are certain as yet unkown functions of μ for which $M_i^{(1)}(0) = M_i^*$. As the further approximations $x_s^{(k)}$ we shall take the periodic solutions of the equations

$$\frac{dx_s^{(k)}}{dt} = a_{s1}x_1^{(k)} + \ldots + a_{sn}x_n^{(k)} + f_s(t) + \mu F_s(t, x_1^{(k-1)}, \ldots, x_n^{(k-1)}, \mu) \quad (8.5)$$

$$(k = 2, 3, \ldots).$$

Assume that the functions $x_s^{(k-1)}$ came out periodic. We can then write

$$x_s^{(k-1)} = M_1^{(k-1)}\varphi_{s1} + \ldots + M_m^{(k-1)}\varphi_{sm} + x_s^{(0)*} + \mu x_s^{(k-1)*}, \quad (8.6)$$

where $M_i^{(k-1)*}$ are arbitrary constants and $x_s^{(k-1)}$ is some particular periodic solution of the equations

$$\frac{dx_s^{(k-1)*}}{dt} = a_{s1}x_1^{(k-1)*} + \ldots + a_{sn}x_n^{(k-1)*} + F_s(t, x_1^{(k-2)}, \ldots, x_n^{(k-2)}, \mu),$$

which likewise must admit a periodic solution if such solution is admitted by the equations for $x_s^{(k-1)}$.

The constants $M_i^{(k-1)}$ are uniquely determined from the conditions of existence of periodic solutions for $x_s^{(k)}$. In fact, these conditions on account of (8.5) have the form

$$P_i^{(k)}(M_1^{(k-1)}, \ldots, M_m^{(k-1)}\mu) = \int_0^{2\pi} \sum_{\alpha=1}^{n} F_\alpha(t, x_1^{(k-1)}, \ldots, x_n^{(k-1)}, \mu)\,\psi_{\alpha i}\,dt = 0 \quad (8.7)$$

$$(i = 1, \ldots, m).$$

Equations (8.7) serve to define $M_i^{(k-1)}$. As is seen from (8.6), (8.2) and (8.3), these equations are identically satisfied for $M_i^{(k-1)} = M_i^*$, $\mu = 0$, and since by assumption the inequality (8.4) is satisfied, these equations admit for μ sufficiently small one and only one solution $M_i^{(k-1)}(\mu)$ for which $M_i^{(k-1)}(0) = M_i^*$.

Thus, we have obtained a completely determined expression for the successive approximations, and the functions $x_s^{(k)}$ for $\mu = 0$ reduce to the generating solution $\varphi_s(t)$.

The proof of the convergence of these approximations to the required periodic solution will be given in the following section.

We may remark that if the functions F_s do not contain μ explicitly the first approximation agrees with the generating solution. In fact, in this case equations (8.7) for $k = 2$ agree with equations (8.3).

EXAMPLE. As an example let us consider the forced oscillations of a follower system, the circuit diagram of which is shown in fig. 19.[1]

The contact plates connected with the follower shaft and turning with it are connected respectively with the armature of an electric motor M and with a d.c. generator G. The electric motor field is independently excited.

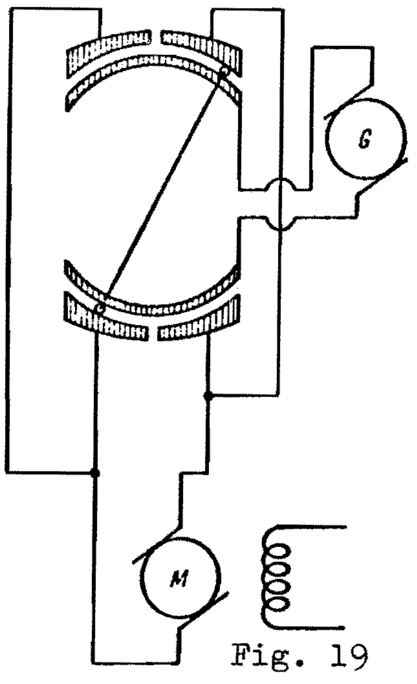

Fig. 19

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[1] This diagram and the equations of motion have been taken from the work papers : Bulgakov B.V., O primenenii metoda Puankare k svobodnym psevdolineinym kolebatel'nym sistemam,(On the Application of the Method of Poincaré to Free Pseudolinear Oscillating Systems), Prikl. Mat. i Mekh., vol. VI, no.4, 1942; Butenin N.V., K teorii prinuditel noi sinkhronizatsii v nelineinykh sledyashchnikh sistemakh. (On the Theory of Forced Synchronization in Nonlinear follower Systems), Trudy LKVVIA, no.XXXIX,1951.

The rotating arm closing the contacts is connected with the transmitter shaft. Let y be the angle of rotation of the transmitter shaft, x the angle of rotation of the follower shaft and

$$\psi = y - x \tag{8.8}$$

the error. From the diagram it is seen that when the error exceeds in magnitude a certain upper limit, the contacts are closed in such manner that the electric motor turns in the right or reverse direction depending on the sign of the magnitude ψ. This electric motor is connected with the follower shaft and turns it always in the direction of decreasing error. The equations of motion of the system consist of the equation of rotation of the armature

$$\theta \frac{d^2x}{dt^2} + n\frac{dx}{dt} = aI \tag{8.9}$$

and the equation of equilibrium of the electromotive forces

$$RI = E(\psi) - c\frac{dx}{dt} - L\frac{dI}{dt}. \tag{8.10}$$

Here θ is the reduced moment of inertia of the follower shaft, n the coefficient of viscous friction, a a constant coefficient, R, L, I the resistance, coefficient of self-induction and current respectively in the armature of the motor, c coefficient of counter-electromotive force and $E(\psi)$ the electromotive force externally applied to the armature of the electric motor. The magnitude $E(\psi)$ is a nonlinear, odd function of its argument, the graph of which has approximately the appearance shown in fig. 20.

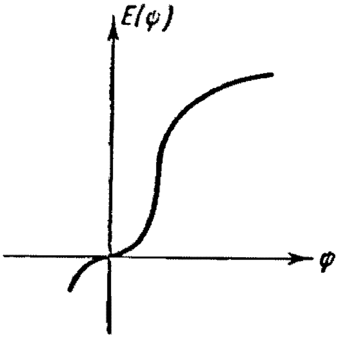

Fig. 20

We shall assume that the transmitter shaft performs the harmonic oscillations

$$y = A\sin\omega t, \tag{8.11}$$

and seek to the corresponding forced oscillations of the follower shaft. This problem has been solved by another method by N.V. Butenin for the case where the characteristic of the electric motor has the form shown in fig.21 and 22.

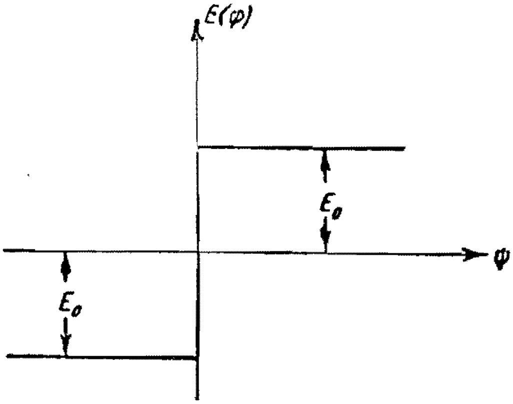

Fig. 21

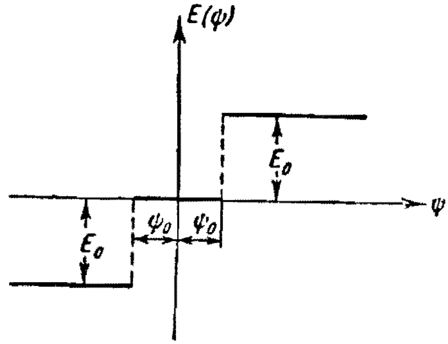

Fig. 22

Eliminating the magnitude I from (8.10) and (8.9) and then passing to the magnitude ψ with the aid of (8.8) and (8.11) we obtain the following equation for the oscillations

$$\frac{d^3\psi}{dt^3}+\frac{R\theta+Ln}{L\theta}\frac{d^2\psi}{dt^2}+\frac{Rn+ca}{L\theta}\frac{d\psi}{dt}=$$
$$=-\frac{a}{L\theta}E(\psi)-A\left(\omega^3-\frac{Rn+ca}{L\theta}\omega\right)\cos\omega t-\frac{R\theta+Ln}{L\theta}A\omega^2\sin\omega t.$$

Changing the time scale we rewrite this equation in the form

$$\frac{d^3\psi}{d\tau^3}+\frac{k^2}{\omega^2}\frac{d\psi}{d\tau}=-E^*(\psi)-\frac{R\theta+Ln}{\omega L\theta}\frac{d^2\psi}{d\tau^2}-$$
$$-A\left(1-\frac{k^2}{\omega^2}\right)\cos\tau-\frac{R\theta+Ln}{\omega L\theta}A\sin\tau, \quad (8.12)$$

where

$$\tau=\omega t, \quad k^2=\frac{Rn+ca}{L\theta}, \quad E^*(\psi)=\frac{a}{\omega^3L\theta}E(\psi).$$

We shall make the assumption that the system considered approximates a linear conservative system, for which, setting

$$\mu = \frac{R\theta + Ln}{\omega L\theta}, \qquad E^*(\psi) = \mu F(\psi), \tag{8.13}$$

we shall assume the magnitude μ small and take it as the fundamental parameter . We shall here be interested in the oscillations near resonance, i.e. we shall assume that the magnitude k differs from ω by a magnitude of the order of μ. Then, introducing the mistuning μb with the aid of the relation

$$\frac{k^2}{\omega^2} = 1 + \mu b,$$

we reduce the equation of oscillations (8.12), taking (8.13) into account, to the final form

$$\frac{d^3\psi}{d\tau^3} + \frac{d\psi}{d\tau} = \mu \left\{ -F(\psi) - b\frac{d\psi}{d\tau} - \frac{d^2\psi}{d\tau^2} + bA\cos\tau - A\sin\tau \right\}. \tag{8.14}$$

The fundamental equation of the generating system has one zero and a pair of purely imaginary roots. The general solution of this system

$$\psi_0 = L + M\cos\tau + N\sin\tau,$$

where L, M, N are arbitrary constants, will be periodic.

We shall take as the generating solution

$$\varphi = L^* + M^*\cos\tau + N^*\sin\tau,$$

which we shall also consider as the zeroth approximation of the periodic solution of equation (8.14). Then, following the above mentioned method, we take as the first approximation $\psi_1(t)$ the periodic solution of the equation

$$\frac{d^3\psi_1}{d\tau^3} + \frac{d\psi_1}{d\tau} = \mu\{-F(\varphi) + (bM^* + N^* - A)\sin\tau + \\ + (-bN^* + M^* + bA)\cos\tau\}.$$

In order for this equation to admit a periodic solution it is necessary and sufficient that the Fourier expansion of its right hand side does not contain either a free term or terms with $\cos\tau$ and $\sin\tau$. This leads to the following equations:

$$\left.\begin{aligned} &\int_0^{2\pi} F(\varphi)\,d\tau = 0, \\ -\frac{1}{\pi}&\int_0^{2\pi} F(\varphi)\cos\tau\,d\tau - bN^* + M^* + bA = 0, \\ -\frac{1}{\pi}&\int_0^{2\pi} F(\varphi)\sin\tau\,d\tau + bM^* + N^* - A = 0, \end{aligned}\right\} \quad (8.15)$$

serving to determine the constants $\overset{*}{L}$, $\overset{*}{M}$, $\overset{*}{N}$ in the generating solution. Since the function $F(\psi)$ is odd, the first of these equations gives

$$\overset{*}{L} = 0$$

But then, denoting by $f(\psi)$ the original function of $F(\psi)$ we obtain

$$\int_0^{2\pi} F(\varphi)(N^*\cos\tau - M^*\sin\tau)\,d\tau = \Big|_0^{2\pi} f(\varphi) = 0.$$

From this we easily find instead of the second and third of equations (8.15) their two following combinations

$$\left.\begin{aligned} b(M^{*2} + N^{*2}) - AM^* - bAN^* = 0, \\ -\frac{1}{\pi}\int_0^{2\pi} \varphi F(\varphi)\,d\tau + M^{*2} + N^{*2} + bAM^* - AN^* = 0. \end{aligned}\right\} \quad (8.16)$$

These equations serve to determine the magnitudes $\overset{*}{M}$ and $\overset{*}{N}$. We shall not here concern ourselves with computing the further approximations.

The question of the stability of the periodic solution will be considered in sec. 12 of the following chapter.

9. Proof of the Convergence of the Successive Approximations

We proceed to prove the convergence of the process of successive approximations established in the preceding section. We recall that as the zeroth approximation there was taken the generating solution

$$x_s^{(0)} = \varphi_s(t) = x_s^{(0)*} + M_1^* \varphi_{s1} + \ldots + M_m^* \varphi_{sm}, \tag{9.1}$$

where M_i^* satisfy equations (8.3) and for the k-th approximation there is taken the periodic solution of the equations

$$\frac{dx_s^{(k)}}{dt} = a_{s1}x_1^{(k)} + \ldots + a_{sn}x_n^{(k)} + f_s(t) + \mu F_s(t, x_1^{(k-1)}, \ldots, x_n^{(k-1)}, \mu). \tag{9.2}$$

By assumption the functions F_s, for all values of t lying in the interval $0 \leqslant t < \infty$, for values of $x_1, \ldots, x_n$ lying in a certain region G of the space of these variables and for μ sufficiently small, possess continuous derivatives of the first order with respect to the variables $x_1, \ldots, x_n$. It is also assumed that the generating solution lies in the region G.

The periodic solution of equations (9.2) can be represented in the form

$$\left.\begin{aligned} x_s^{(k)} &= x_s^{(0)*} + M_1^{(k)}\varphi_{s1} + \ldots + M_m^{(k)}\varphi_{sm} + \mu L_s(t, F_1^{(k-1)}, \ldots, F_n^{(k-1)}), \\ F_s^{(i)} &= F_s(t, x_1^{(i)}, \ldots, x_n^{(i)}, \mu), \end{aligned}\right\} \tag{9.3}$$

where $M_i^{(k)}$ are constants satisfying equations (8.7) and L_s are the operators introduced in sec. 4. We introduce, for what follows, the notations

$$\left.\begin{aligned} \xi_s^{(k)}(t, M_1, \ldots, M_m, \mu) &= x_s^{(0)*} + M_1\varphi_{s1} + \ldots + M_m\varphi_{sm} + \\ &\quad + \mu L_s(t, F_1^{(k-1)}, \ldots, F_n^{(k-1)}), \\ P_i^{(k)}(M_1, \ldots, M_m, \mu) &= \sum_{\alpha=1}^{n} \int_0^{2\pi} F_\alpha(t, \xi_1^{(k)}, \ldots, \xi_n^{(k)}, \mu)\psi_{\alpha i}\, dt \end{aligned}\right\} \tag{9.4}$$

$$(s = 1, \ldots, n; \quad i = 1, \ldots, m; \quad k = 1, 2, \ldots).$$

Equations (8.7), determining the constants $M_i^{(k)}$, can then be represented in the form

$$P_i^{(k)}(M_1^{(k)}, \ldots, M_m^{(k)}, \mu) = 0 \quad (i = 1, \ldots, k), \tag{9.5}$$

since, evidently,

$$x_s^{(k)} = \xi_s^{(k)}(t, M_1^{(k)}, \ldots, M_m^{(k)}, \mu).$$

We shall show first of all that for sufficiently small μ all magnitudes $x_s^{(k)}$ lie in the region G. For this purpose we shall assume that this condition is satisfied for $x_s^{(1)}, \ldots, x_s^{(k-1)}$, and show that it will then be satisfied also for $x_s^{(k)}$.

In fact, on the basis of (4.18) we have first of all

$$|L_s(t, F_1^{(k-1)}, \ldots, F_n^{(k-1)})| < BA, \tag{9.6}$$

where A denotes the upper bound of the functions

$$F_s(t, x_1, \ldots, x_n, \mu)$$

in the region of their definition.

Let us consider, further, the functional determinant

$$\frac{\partial (P_1^{(k)}, \ldots, P_m^{(k)})}{\partial (M_1, \ldots, M_m)}.$$

This determinant, for $M_i = M_i^*$, $\mu = 0$, reduces to the magnitude (8.4) and therefore is different from zero. But then, as is easily seen from the expression of the partial derivatives

$$\frac{\partial P_i^{(k)}}{\partial M_j} = \sum_{\alpha, \beta=1}^{n} \int_0^{2\pi} \frac{\partial F_\alpha(t, \xi_1^{(k)}, \ldots, \xi_n^{(k)}, \mu)}{\partial \xi_\beta^{(k)}} \psi_{\alpha i} \varphi_{\beta j}\, dt,$$

there exist two positive numbers h and η, independent of the index k, such that for

$$|M_i - M_i^*| \leqslant h \tag{9.7}$$

and

$$\mu \leqslant \eta \tag{9.8}$$

the inequality holds

$$\left| \frac{\partial (P_1^{(k)}, \ldots, P_m^{(k)})}{\partial (M_1, \ldots, M_m)} \right| > \alpha, \tag{9.9}$$

where α is a certain positive number not dependent on k. We shall assume here that the constants h and η are so small that when conditions (9.7) and (9.8) are satisfied the magnitudes $\xi_s^{(k)}$ lie in the region G. This is possible, since the magnitudes (9.1) lie in the region G and the inequalities (9.6) are satisfied.

We require moreover that when (9.8) is satisfied the roots $M_i^{(k)}(\mu)$ of equations (9.5) lie in the region (9.7). The magnitudes $x_s^{(k)}$ will then lie in the region G. It therefore remains for us to prove that the magnitude η may actually be chosen so small that when condition (9.8) is satisfied there is satisfied also the condition

$$|M_i^{(k)}(\mu) - M_i^*| \leqslant h, \tag{9.10}$$

and at the same time η will not depend on k.

For this purpose let us denote by C the upper limit of the modulus of the functional determinant

$$\frac{\partial(P_1^{(k)}, \ldots, P_m^{(k)})}{\partial(M_1, \ldots, M_{i-1}, \mu, M_{i+1}, \ldots, M_m)}$$

in the entire region of its existence and assume that the number η in condition (9.8) satisfies the inequality

$$\eta < \frac{\alpha h}{C}.$$

We can evidently assume that C does not depend on k. We shall show that the inequalities (9.10) will then actually be satisfied.

In fact, since $M_i^{(k)}(0) = M_i^*$, the inequalities (9.10) will in any case be satisfied for sufficiently small value of μ. Let μ^* be the first value of μ for which at least one of inequalities (9.10) goes over into the equality. We must prove that $\mu^* > \eta$. Suppose we assume the contrary: $\mu^* \leqslant \eta$. We can write

$$M_i^{(k)}(\mu^*) - M_i^* = M_i^{(k)}(\mu^*) - M_i^{(k)}(0) = \mu^* \left(\frac{dM_i^{(k)}(\mu)}{d\mu}\right)_{\mu=\theta\mu^*} =$$

$$= \mu^* \left\{ \frac{\dfrac{\partial(P_1^{(k)}, \ldots, P_m^{(k)})}{\partial(M_1, \ldots, M_{i-1}, \mu, M_{i+1}, \ldots, M_m)}}{\dfrac{\partial(P_1^{(k)}, \ldots, P_m^{(k)})}{\partial(M_1, \ldots, M_m)}} \right\}_{M_j = M_j^{(k)}(\theta\mu^*),\ \mu=\theta\mu^*}$$

where θ is a proper positive function. Since $M_i^{(k)}(\theta\mu^*)$ still lies in the region (9.10), on the basis of (9.9) the estimates will hold

$$|M_i^{(k)}(\mu^*) - M_i^*| < \mu^* \frac{C}{\alpha} \leqslant \eta \frac{C}{\alpha} < h.$$

This however contradicts the assumption that for $\mu = \mu^*$ at least one of the inequalities (9.10) goes over into an equality.

Thus, we may regard it as proven that with condition (9.8) satisfied and with a suitable choice of the number η the inequalities (9.10) will be satisfied. Together with this it has been shown that all approximations $x_s^{(k)}$ lie in the region G.

We proceed now to the evaluation of the differences of the successive approximations. First of all we may write

$$|L_s(t, F_1^{(k-1)}, \ldots, F_n^{(k-1)}) - L_s(t, F_1^{(k-2)}, \ldots, F_n^{(k-2)})| < a_k,$$
$$|M_i^{(k)} - M_i^{(k-1)}| < b_k,$$

where a_k and b_k are constants for which, on the basis of the preceding, it is possible to fix certain upper limits not dependent on k.

Further, on the basis of (4.16) we have:

$$L_s(t, F_1^{(k)}, \ldots, F_n^{(k)}) - L_s(t, F_1^{(k-1)}, \ldots, F_n^{(k-1)}) = \\ = L_s(t, F_1^{(k)} - F_1^{(k-1)}, \ldots, F_n^{(k)} - F_n^{(k-1)}). \quad (9.11)$$

But the functions F_s evidently satisfy the Cauchy-Lipschitz conditions, and we can write:

$$|F_s^{(k)} - F_s^{(k-1)}| < P \sum_{\alpha=1}^{n} |x_\alpha^{(k)} - x_\alpha^{(k-1)}| \leqslant$$
$$\leqslant P \sum_{\alpha=1}^{n} \Big| \Big\{ \sum_{i=1}^{m} (M_i^{(k)} - M_i^{(k-1)}) \varphi_{\alpha i} + \mu [L_\alpha(t, F_1^{(k-1)}, \ldots, F_n^{(k-1)}) -$$
$$- L_\alpha(t, F_1^{(k-2)}, \ldots, F_n^{(k-2)})] \Big\} \Big| < nP(mMb_k + a_k\mu),$$

where P is a certain constant and M is the upper limit of the functions $|\varphi_{\alpha i}|$. Hence, from (9.11) on the basis of (4.18) we find:

$$|L_s(t, F_1^{(k)}, \dots, F_n^{(k)}) - L_s(t, F_1^{(k-1)}, \dots, F_n^{(k-1)})| < a_{k+1}, \quad (9.12)$$

where

$$a_{k+1} = nPB(mMb_k + \mu a_k). \quad (9.13)$$

We shall now estimate the difference $M_i^{(k+1)}(\mu) - M_i^{(k)}(\mu)$. For this purpose let us consider the auxiliary equations

$$R_i^{(k)}(N_1^{(k)}, \dots, N_m^{(k)}, \mu, \lambda) = 0 \quad (i = 1, \dots, m), \quad (9.14)$$

where there has been set

$$\left.\begin{aligned} &R_i^{(k)}(N_1, \dots, N_m, \mu, \lambda) = \\ &\qquad = \sum_{\alpha=1}^{n} \int_0^{2\pi} F_\alpha(t, \zeta_1^{(k)}, \dots, \zeta_n^{(k)}, \mu)\, \psi_{\alpha i}\, dt, \\ &\zeta_s^{(k)} = x_s^{0*} + N_1\varphi_{s1} + \dots + N_m\varphi_{sm} + \mu L_s(t, F_1^{(k-1)}, \dots, F_n^{(k-1)}) + \\ &\qquad + \lambda L_s(t, F_1^{(k)} - F_1^{(k-1)}, \dots, F_n^{(k)} - F_n^{(k-1)}) \end{aligned}\right\} \quad (9.15)$$

and the parameter λ varies on the segment $0 \leqslant \lambda \leqslant \mu$.

The magnitude η in the inequality (9.8) may evidently be chosen so small that the magnitudes $\zeta_s^{(k)}$ lie in the region G if N_i lie in the region

$$|N_i - M_i^*| \leqslant h, \quad (9.16)$$

and therefore the functions $R_i^{(k)}$ are completely determined. Evidently we have

$$M_i^{(k+1)}(\mu) = N_i^{(k)}(\mu, \mu), \qquad M_i^{(k)}(\mu) = N_i^{(k)}(\mu, 0),$$

where $N_i^{(k)}(\mu, \lambda)$ are the roots of the equations (9.14).

Hence

$$M_i^{(k+1)}(\mu)-M_i^{(k)}(\mu)=N_i^{(k)}(\mu,\mu)-N_i^{(k)}(\mu,0)=\mu\left(\frac{\partial N_i^{(k)}(\mu,\lambda)}{\partial\lambda}\right)_{\lambda=\theta\mu}=$$

$$=\mu\left\{\frac{\dfrac{\partial(R_1^{(k)},\ldots,R_m^{(k)})}{\partial(N_1,\ldots,N_{i-1},\lambda,N_{i+1},\ldots,N_m)}}{\dfrac{\partial(R_1^{(k)},\ldots,R_m^{(k)})}{\partial(N_1,\ldots,N_m)}}\right\}_{N_j=N_j^{(k)}(\mu,\theta\mu),\ \lambda=\theta\mu}, \quad (9.17)$$

where θ is a proper positive fraction.

In exactly the same way as was shown for the roots $M_i^{(k)}(\mu)$ of equations (9.5), it can be shown that the roots $N_i^{(k)}(\mu,\lambda)$ of equations (9.14) lie in the region (9.16), provided the magnitude η in the inequality (9.8) is sufficiently small. We shall assume that this condition is satisfied and that the magnitude η as before will not depend on k.

The functional determinant $\dfrac{\partial(R_1^{(k)},\ldots,R_m^{(k)})}{\partial(N_1,\ldots,N_m)}$ for $\lambda=\mu=0$, $N_i=M_i^*$ reduces to the magnitude (8.4) different from zero. Hence in the region determined by inequalities (9.8) and (9.16) the inequality holds

$$\left|\frac{\partial(R_1^{(k)},\ldots,R_m^{(k)})}{\partial(N_1,\ldots,N_m)}\right|>\beta, \quad (9.18)$$

where β is a positive constant independent of k provided the magnitudes h and η are sufficiently small. We shall assume that these conditions with respect to the magnitudes h and η are likewise satisfied, where , as before, these magnitudes will not depend on k.

We have further

$$\frac{\partial R_i^{(k)}}{\partial\lambda}=\sum_{\alpha,\beta=1}^{n}\int_0^{2\pi}\frac{\partial F_\alpha}{\partial x_\beta^{(k)}}L_\beta(t,F_1^{(k)}-F_1^{(k-1)},\ldots,F_n^{(k)}-F_n^{(k-1)})\psi_{\alpha i}\,dt.$$

Hence in the region of variation of the variables $N_1,\ldots,N_m$, λ, μ, determined by inequalities (9.16) and (9.8) , on the basis of (9.12) the estimate holds

$$\left|\frac{\partial H_i^{(k)}}{\partial\lambda}\right| < 2\pi n^2 Q N a_{k+1}. \qquad (9.19)$$

where Q is the upper limit of the partial derivatives $\partial F_s/\partial x_i$ in the region of their existence and N is the upper limit of the functions $|\psi_{sj}|$.

Since in the above indicated region of variations of the variables the partial derivatives $\partial R_i^{(k)}/\partial N_j$ evidently have a certain upper limit independent of k, there follows from (9.19):

$$\left|\left\{\frac{\partial\,(R_1^{(k)},\,\ldots,\,R_m^{(k)})}{\partial\,(N_1,\,\ldots,\,N_{i-1},\,\lambda,\,N_{i+1},\,\ldots,\,N_m)}\right\}_{N_j=N_j^{(k)}(\mu,\,\theta\mu),\;\lambda=\theta\mu}\right| < Ra_{k+1}, \quad (9.20)$$

where R is a certain positive number not depending on k.

Formulas (9.17), (9.18) and (9.20) give finally:

$$|M_i^{(k+1)}(\mu)-M_i^{(k)}(\mu)| < b_{k+1}, \qquad b_{k+1}=\frac{R}{\beta}a_{k+1}\mu. \qquad (9.21)$$

From (9.21) it follows that b_{k+1}/a_{k+1} does not depend on the index k. Hence it can be assumed that b_k/a_k likewise does not depend on k.

But then also the ratios b_{k+1}/b_k and a_{k+1}/a_k likewise do not depend on k, since on the basis of (9.13)

$$\left.\begin{aligned}\frac{a_{k+1}}{a_k}&=nPB\left(mM\frac{b_k}{a_k}+\mu\right),\\ \frac{b_{k+1}}{b_k}&=\frac{R}{\beta}\,\frac{a_{k+1}}{b_k}\,\mu=\frac{RnPB}{\beta}\left(mM+\frac{a_k}{b_k}\mu\right)\mu.\end{aligned}\right\} \qquad (9.22)$$

Since on the basis of (9.21) it may be assumed that b_k/a_k contains the multiplier μ, it follows from (9.22) that for sufficiently small μ the ratios a_{k+1}/a_k and b_{k+1}/b_k will be less than unity. From this it immediately follows that for sufficiently small μ the sequences $L_s(t,F_1^{(k)},\ldots,F_n^{(k)})$ and $M_i^{(k)}(\mu)$ absolutely and uniformly converge. Hence the sequences of the functions $x_s^{(k)}$

likewise uniformly converge to certain functions $x_s(t,\mu)$. Since all functions $x_s^{(k)}$ are periodic, the functions $x_s(t,\mu)$ will likewise be periodic. They will moreover satisfy the conditions $x_s(t,0) = \varphi_s(t)$. It remains to show that these functions will actually satisfy equations (8.1). This can easily be done the usual manner for the method of successive approximations.

We have

$$\lim_{k\to\infty} F_s(t, x_1^{(k)}, \ldots, x_n^{(k)}, \mu) = F_s(t, x_1(t,\mu), \ldots, x_n(t,\mu), \mu)$$

and therefore

$$\lim_{k\to\infty} L_s(t, F_1^{(k)}, \ldots, F_n^{(k)}) = \{L_s(t, F_1, \ldots, F_n)\}_{x_i = x_i(t,\mu)}.$$

Whence, taking (9.3) into account,

$$\begin{aligned}\mu\{L_s(t, F_1, \ldots, F_n)\}_{x_i=x_i(t,\mu)} &= \lim_{k\to\infty} \mu L_s(t, F_1^{(k)}, \ldots, F_n^{(k)}) = \\ &= \lim_{k\to\infty} (x_s^{(k)} - M_1^{(k)}\varphi_{s1} - \ldots - M_m^{(k)}\varphi_{sm}) - x_s^{(0)*} = \\ &= x_s(t,\mu) - M_1(\mu)\varphi_{s1} - \ldots - M_m(\mu)\varphi_{sm} - x_s^{(0)*},\end{aligned}$$

which shows that the functions $x_s(t,\mu)$ are a periodic solution of equations (8.1).

Thus, the convergence of the process of determining the periodic solutions of equations (8.1) has been completely proven. [1]

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[1] The proof given in the text is simpler than the proof initially given in the author's work cited on p.116. For the simplification use was made of the papers by Shimanov S.N., K teorii kolebanii kvazilineinykh sistem, (On the Theory of Oscillations of Quasilinear Systems), Prikl, mat. i mekh., vol XVIII, no. 2, 1954, and Bulgakov N.G., Kolebaniya kvazilineinykh avtonomnykh sistem so mnogimi stepenyami svobody, (Oscillations of Quasilinear Autonomous Systems with Many Degrees of Freedom), Dissertation, 1954.

10. Computation of the Periodic Solutions of Nonautonomous Systems at Resonance in the Singular Case

Let us again consider the quasilinear nonautonomous system

$$\frac{dx_s}{dt} = a_{s1}x_1 + \dots + a_{sn}x_n + f_s(t) + \mu F_s(t, x_1, \dots, x_n, \mu) \quad (10.1)$$

and assume that the generating system has the periodic solution

$$x_s^{(0)} = \varphi_s = x_s^{(0)*} + M_1^*\varphi_{s1} + \dots + M_m^*\varphi_{sm}, \quad (10.2)$$

depending on m arbitrary constants. We shall assume that we are dealing with the singular case, i.e. that the identity holds

$$P_i(M_1^*, \dots, M_m^*) = \int_0^{2\pi} \sum_{\alpha=1}^{n} F_\alpha(t, \varphi_1, \dots, \varphi_m, 0)\psi_{\alpha i}\,dt \equiv 0 \quad (10.3)$$
$$(i = 1, \dots, m).$$

This case may present itself in the solution of many practical problems. We shall not however here concern ourselves with this problem in its most general form but shall restrict ourselves to the consideration of one of the simplest cases. We shall here assume that the functions F_s are analytic with respect to $x_1, \dots, x_n, \mu$.

We shall seek to obtain the periodic solution of system (10.1) in the form of the series

$$x_s = \varphi_s(t) + \mu x_s^{(1)}(t) + \mu^2 x_s^{(2)} + \dots \quad (10.4)$$

with periodic coefficients. As in the nonsingular case we obtain for $x_s^{(1)}$ the equations

$$\frac{dx_s^{(1)}}{dt} = a_{s1}x_1^{(1)} + \dots + a_{sn}x_n^{(1)} + F_s(t, \varphi_1, \dots, \varphi_n, 0), \quad (10.5)$$

which however now have a periodic solution for any choice of the constants M_1^* in the generating solution, since equations (10.3) are now identically satisfied. This solution has the form

$$x_s^{(1)} = x_s^{(1)*}(t) + M_1^{(1)}\varphi_{s1} + \dots + M_m^{(1)}\varphi_{sm}, \qquad (10.6)$$

where $x_s^{(1)*}(t)$ is some particular periodic solution of equations (10.5) and $M_i^{(1)}$ are arbitrary constants. The functions $x_s^{(1)}$ thus contain 2m arbitrary constants: M_i^* and $M_i^{(1)}$. If we now set up the equations for $x_s^{(2)}$ and write the conditions for which they admit a periodic solution we obtain for the determination of the above constants m equations. It can easily be seen that these equations will not contain the constants $M_i^{(1)}$. In fact, as was shown in sec. 7, these equations will be linear with respect to $M_i^{(1)}$ and the coefficients of these magnitudes will be partial derivatives of the functions P_j with respect to M_i^* and therefore in the case considered will identically reduce to zero. Thus, the conditions of periodicity of the functions $x_s^{(2)}$ will contain only the constants M_i^* and will enable their determination. We shall consider these equations in more detail and shall show how the further computations are to be conducted.

The equations for $x_s^{(2)}$ are of the form

$$\frac{dx_s^{(2)}}{dt} = a_{s1}x_1^{(2)} + \dots + a_{sn}x_n^{(2)} + \sum_{\beta=1}^{n}\left(\frac{\partial F_s}{\partial x_\beta}\right)x_\beta^{(1)} + \left(\frac{\partial F_s}{\partial \mu}\right),$$

where parentheses indicate that the derivatives are computed at the point $x_1 = \varphi_1, \dots, x_n = \varphi_n, \mu = 0$. From this we obtain the following conditions of the periodicity of the functions $x_s^{(2)}$:

$$\int_0^{2\pi}\sum_{\alpha=1}^{n}\left\{\sum_{\beta=1}^{n}\left(\frac{\partial F_\alpha}{\partial x_\beta}\right)x_\beta^{(1)} + \left(\frac{\partial F_\alpha}{\partial \mu}\right)\right\}\psi_{\alpha i}\,dt = 0.$$

Substituting the expressions $x_\beta^{(1)}$ from (10.6) and taking into account the identities

$$\int_0^{2\pi} \sum_{\alpha,\beta=1}^{n} \left(\frac{\partial F_\alpha}{\partial x_\beta}\right) \varphi_{\beta j} \psi_{\alpha i}\, dt \equiv 0 \qquad (10.7)$$
$$(i,\ j = 1, \ldots, m),$$

obtained by differentiating identities (10.3) with respect to M_j^*, we obtain

$$Q_i(M_1^*, \ldots, M_m^*) = \\ = \int_0^{2\pi} \sum_{\alpha=1}^{n} \left\{ \sum_{\beta=1}^{n} \left(\frac{\partial F_\alpha}{\partial x_\beta}\right) x_\beta^{(1)*} + \left(\frac{\partial F_\alpha}{\partial \mu}\right) \right\} \psi_{\alpha i}\, dt = 0 \qquad (10.8)$$
$$(i = 1, \ldots, m).$$

These will be the equations for M_i^*. In contrast to the nonsingular case, they were obtained from the conditions of periodicity not of the first but of the second approximation.

Let us assume that equations (10.8) are not identically satisfied and let M_i^* be some simple solution of these equations, i.e. a solution for which the condition is satisfied

$$\frac{\partial(Q_1, \ldots, Q_m)}{\partial(M_1^*, \ldots, M_m^*)} \neq 0. \qquad (10.9)$$

We shall show that in this case there exists a unique system of series of the form (10.4) formally satisfying equations (10.1). For this purpose let us set up the equations for $x_s^{(k+2)}$ $(k=1,2,\ldots)$, separating out explicitly those which depend on $x_s^{(k)}$ and $x_s^{(k+1)}$. These equations, as is easy to see, for $k>1$, have the form

$$\frac{dx_s^{(k+2)}}{dt} = a_{s1} x_1^{(k+2)} + \ldots + a_{sn} x_n^{(k+2)} + \sum_{\beta=1}^{n} \left(\frac{\partial F_s}{\partial x_\beta}\right) x_\beta^{(k+1)} + \\ + \frac{1}{2} \sum_{\beta,\gamma=1}^{n} \left(\frac{\partial^2 F_s}{\partial x_\beta \partial x_\gamma}\right) (x_\gamma^{(1)} x_\beta^{(k)} + x_\beta^{(1)} x_\gamma^{(k)}) + \\ + \sum_{\beta=1}^{n} \left(\frac{\partial^2 F_s}{\partial x_\beta \partial \mu}\right) x_\beta^{(k)} + F_s^{(k+2)}, \qquad (10.10)$$

and for k = 1, the form

$$\frac{dx_s^{(3)}}{dt} = a_{s1}x_1^{(3)} + \ldots + a_{sn}x_n^{(3)} + \sum_{\beta=1}^{n}\left(\frac{\partial F_s}{\partial x_\beta}\right)x_\beta^{(2)} + $$
$$+ \frac{1}{2}\sum_{\beta,\gamma=1}^{n}\left(\frac{\partial^2 F_s}{\partial x_\beta\,\partial x_\gamma}\right)x_\beta^{(1)}x_\gamma^{(1)} + \sum_{\beta=1}^{n}\left(\frac{\partial^2 F_s}{\partial x_\beta\,\partial \mu}\right)x_\beta^{(1)} + F_s^{(3)}.$$

Here $F_s^{(k+2)}$ are integral rational functions with periodic coefficients of $s_j^{(0)},\ldots,x_j^{(k-1)}$.

Let us assume that all the functions $x_s^{(l)}$ up to the (k+1)th order have already been computed and came out periodic. These functions have the form

$$x_s^{(l)} = x_s^{(l)*} + M_1^{(l)}\varphi_{s1} + \ldots + M_m^{(l)}\varphi_{sm}\ (l = 1, 2, \ldots, k+1), \qquad (10.11)$$

where $x_s^{(l)*}$ is some particular periodic solution for $x_s^{(l)}$ and $M_i^{(l)}$ are constants. We assume that the functions $x_s^{(1)},\ldots,x_s^{(k-1)}$ have been determined completely together with the constants $M_i^{(1)},\ldots,M_i^{(k-1)}$ entering them but that in the functions $x_s^{(k)}$ and $x_s^{(k+1)}$ the constants $M_i^{(k)}$ and $M_i^{(k+1)}$ still remain to be determined. In order to obtain these constants we set up the conditions of the periodicity of the functions $x_s^{(k+2)}$. These conditions for $k > 1$ will be:

$$\int_0^{2\pi}\sum_{\alpha=1}^{n}\left\{\sum_{\beta=1}^{n}\left(\frac{\partial F_\alpha}{\partial x_\beta}\right)x_\beta^{(k+1)} + \sum_{\beta,\gamma=1}^{n}\left(\frac{\partial^2 F_\alpha}{\partial x_\beta\,\partial x_\gamma}\right)x_\beta^{(1)}x_\gamma^{(k)} + \right.$$
$$\left. + \sum_{\beta=1}^{n}\left(\frac{\partial^2 F_\alpha}{\partial x_\beta\,\partial \mu}\right)x_\beta^{(k)} + F_\alpha^{(k+2)}\right\}\psi_{\alpha i}\,dt = 0. \quad (10.12)$$

For k = 1 we shall have

$$\int_0^{2\pi}\sum_{\alpha=1}^{n}\left\{\sum_{\beta=1}^{n}\left(\frac{\partial F_\alpha}{\partial x_\beta}\right)x_\beta^{(2)} + \frac{1}{2}\sum_{\beta,\gamma=1}^{n}\left(\frac{\partial^2 F_\alpha}{\partial x_\beta\,\partial x_\gamma}\right)x_\beta^{(1)}x_\gamma^{(1)} + \right.$$
$$\left. + \sum_{\beta=1}^{n}\left(\frac{\partial^2 F_\alpha}{\partial x_\beta\,\partial \mu}\right)x_\beta^{(1)} + F_\alpha^{(3)}\right\}\psi_{\alpha i}\,dt = 0.$$

In equations (10.12) let us substitute for $x_s^{(k)}$ and

$x_s^{(k+1)}$ their expressions from (10.11). Then, taking into account identities (10.7) and the identities obtained from them by differentiation with respect to M_p^*:

$$\int_0^{2\pi} \sum_{\alpha,\beta,\gamma=1}^{n} \left(\frac{\partial^2 F_\alpha}{\partial x_\beta \partial x_\gamma}\right) \varphi_{\beta j} \varphi_{\gamma p} \psi_{\alpha i} \, dt \equiv 0$$
$$(i,\ j,\ p=1,\ \ldots,\ m).$$

we obtain

$$\sum_{p=1}^{m} \left\{ \int_0^{2\pi} \sum_{\alpha,\gamma=1}^{n} \left[\left(\frac{\partial^2 F_\alpha}{\partial x_\gamma \partial \mu}\right) + \sum_{\beta=1}^{n} \left(\frac{\partial^2 F_\alpha}{\partial x_\beta \partial x_\gamma}\right) x_\beta^{(1)*} \right] \varphi_{\gamma p} \psi_{\alpha i} \, dt \right\} M_p^{(k)} +$$
$$+ \int_0^{2\pi} \sum_{\alpha,\gamma=1}^{n} \left(\frac{\partial F_\alpha}{\partial x_\gamma}\right) x_\gamma^{(k+1)*} \psi_{\alpha i} \, dt + N_i'^{(k)} = 0 \quad (i=1,\ \ldots,\ m). \qquad (10.13)$$

where

$$N_i'^{(k)} = \int_0^{2\pi} \sum_{\alpha,\beta,\gamma=1}^{n} \left(\frac{\partial^2 F_\alpha}{\partial x_\beta \partial x_\gamma}\right) x_\gamma^{(k)*} \left(x_\beta^{(1)*} + \sum_{p=1}^{m} M_p^{(1)} \varphi_{\beta p} \right) \psi_{\alpha i} \, dt +$$
$$+ \int_0^{2\pi} \sum_{\alpha,\beta=1}^{n} \left(\frac{\partial^2 F_\alpha}{\partial x_\beta \partial \mu}\right) x_\beta^{(k)*} \psi_{\alpha i} \, dt + \int_0^{2\pi} \sum_{\alpha=1}^{n} F_\alpha^{(k+2)} \psi_{\alpha i} \, dt$$

are entirely definite constants not depending on $M_i^{(k)}$ and $M_i^{(k+1)}$.

Exactly the same equations are obtained also for k=1. The difference will be only in the fact that the magnitude N_i' will not contain $M_i^{(1)}$.

In equations (10.13) we explicitly separated out those terms which contain the constants $M_i^{(k)}$. The constants $M_i^{(k+1)}$, as was to be expected, are not at all contained in these equations. It is here necessary to bear in mind that the constants $M_i^{(k)}$ are contained also in the functions $x_s^{(k+1)*}$, since evidently any approximation depends on the preceding approximations and among them on the constants $M_i^{(l)}$ contained in these approximations. We shall establish the dependence of $x_s^{(k+1)*}$ on the constants $M_i^{(k)}$ in the explicit form.

From (10.10) and (10.11) it is seen that we can write

$$x_s^{(k+1)*} = \xi_s^{(k+1)} + \eta_s^{(k+1)}, \tag{10.14}$$

where $\xi_s^{(k+1)}$ is some particular periodic solution of the equations

$$\frac{d\xi_s^{(k+1)}}{dt} = a_{s1}\xi_1^{(k+1)} + \ldots + a_{sn}\xi_n^{(k+1)} + \sum_{p=1}^{m}\sum_{\beta=1}^{n}\left(\frac{\partial F_s}{\partial x_\beta}\right)\varphi_{\beta p} M_p^{(k)}, \tag{10.15}$$

if they permit such solution, while $\eta_s^{(k+1)}$ is some particular periodic solution of the equations [1]

$$\frac{d\eta_s^{(k+1)}}{dt} = a_{s1}\eta_1^{(k+1)} + \ldots + a_{sn}\eta_n^{(k+1)} + \sum_{\beta=1}^{n}\left(\frac{\partial F_s}{\partial x_\beta}\right)x_\beta^{(k)*} +$$
$$+ \frac{1}{2}\sum_{\beta,\gamma=1}^{n}\left(\frac{\partial^2 F_s}{\partial x_\beta \partial x_\gamma}\right)(x_\gamma^{(1)} x_\beta^{(k-1)} + x_\beta^{(1)} x_\gamma^{(k-1)}) +$$
$$+ \sum_{\beta=1}^{n}\left(\frac{\partial^2 F_s}{\partial x_\beta \partial \mu}\right)x_\beta^{(k-1)} + F_s^{(k+1)}.$$

The periodic solution for $\eta_s^{(k+1)}$ will here necessarily exist if the periodic solution for $\xi_s^{(k+1)}$ exists, since the functions $x_s^{(k+1)}$ are by assumption periodic. The functions $\eta_s^{(k+1)*}$ evidently do not depend on $M_i^{(k)}$. It is easy to see that

$$\xi_s^{(k+1)} = \sum_{p=1}^{m} M_p^{(k)} \frac{\partial x_s^{(1)*}}{\partial M_p^*} \tag{10.16}$$

will be a periodic solution of equations (10.15). In fact, the functions $x_s^{(1)*}$ will be periodic for any values of M_i^*. As a result, their derivatives with respect to M_i^* will likewise be periodic and together with these also the functions (10.16). Further, the functions $x_s^{(1)*}$ satisfy

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[1] For k = 1 and k = 2 these equations must be modified according to the equations for $x_s^{(2)}$ and $x_s^{(3)}$.

equations (10.5). Substituting these functions in these equations and differentiating the obtained identities with respect to M_p we shall on the basis of (10.4) have:

$$\frac{d}{dt}\left(\frac{\partial x_s^{(1)*}}{\partial M_p^*}\right)=a_{s1}\frac{\partial x_1^{(1)*}}{\partial M_p^*}+\ldots+a_{sn}\frac{\partial x_n^{(1)*}}{\partial M_p^*}+\sum_{\beta=1}^{n}\left(\frac{\partial F_s}{\partial x_\beta}\right)\varphi_{\beta p},$$

whence it immediately follows that the functions (10.16) satisfy equations (10.15).

From (10.14) and (10.16) it follows that equations (10.13) can be represented in the form

$$\sum_{p=1}^{m}\left\{\int_0^{2\pi}\sum_{\alpha,\gamma=1}^{n}\left[\left(\frac{\partial^2 F_\alpha}{\partial x_\gamma\partial\mu}\right)\varphi_{\gamma p}+\sum_{\beta=1}^{n}\left(\frac{\partial^2 F_\alpha}{\partial x_\beta\partial x_\gamma}\right)x_\beta^{(1)*}\varphi_{\gamma p}+\right.\right.$$
$$\left.\left.+\left(\frac{\partial F_\alpha}{\partial x_\gamma}\right)\frac{\partial x_\gamma^{(1)*}}{\partial M_p^*}\right]\varphi_{\alpha i}\,dt\right\}M_p^{(k)}+N_i^{(k)}=0\quad(i=1,\ldots,m),$$

where

$$N_i^{(k)}=N_i^{\prime(k)}+\int_0^{2\pi}\sum_{\alpha,\gamma=1}^{n}\left(\frac{\partial F_\alpha}{\partial x_\gamma}\right)\eta_\gamma^{(k+1)}\psi_{\alpha i}\,dt$$

are entirely definite constants not depending on $M_i^{(k)}$. On the basis of (10.8) the obtained equations may be represented in the following final form:

$$\sum_{j=1}^{m}\frac{\partial Q_i}{\partial M_j^*}M_j^{(k)}+N_i^{(k)}=0.$$

We thus obtained for the determination of the constants $M_i^{(k)}$ a system of linear algebraic equations the determinant of which on the basis of (10.9) is different from zero.

From all that has been said above it follows that for each generating solution for which the constants M_i are a simple solution of equations (10.8) there exists one and only one system of series of the form (10.4) formally satisfying equations (10.1). We shall not dwell here on the proof of convergence of these series, which is easily obtained from the general results of sec. 6.

All the computations in the singular case considered are obtained the same way as in the nonsingular case. The difference consists only in the fact that the arbitrary constants entering in each approximation are determined not from the succeeding approximation, as in the nonsingular case, but from an approximation higher by unity.[1]

EXAMPLE. As an example let us consider the oscillations of a pendulum the point of suspension of which performs rectilinear harmonic oscillations of large frequency under an angle δ with the vertical . Let A be the amplitude, ω the frequency and $A\cos\omega t$ the law of these oscillations. For the angle φ of deviation of the pendulum from the vertical we then obtain the differential equation

$$J\frac{d^2\varphi}{dt^2}+mgs\sin\varphi+ms\omega^2 A\cos\omega t\sin(\varphi-\delta),$$

where j is the moment of inertia of the pendulum with respect to the axis of suspension, s the distance from the center of gravity to this axis and m its mass.

We shall assume that the ratio A/l , in which $l = \sqrt{J/ms}$ is the reduced length of the pendulum, is a small magnitude, of the same order of k/ω , where $k = \sqrt{g/l}$, the frequency of small oscillations for motionless point of suspension. Then, setting

$$\mu=\frac{k}{\omega}=\frac{1}{\omega}\sqrt{\frac{g}{l}},\qquad \frac{A}{l}=\mu b,\ \ b=\frac{A\omega}{\sqrt{gl}},\qquad \tau=\omega t,$$

we obtain the following equation of motion:

$$\frac{d^2\varphi}{d\tau^2}=-\mu^2\sin\varphi-\mu b\sin(\varphi-\delta)\cos\tau. \tag{10.17}$$

The generating system $d^2\varphi_0/d\tau^2 = 0$ has a periodic solution φ_0 = const. depending on one arbitrary constant. We shall therefore seek the periodic solution of equation (10.17) in the form of the series

$$\varphi=\varphi_0+\mu\varphi_1+\mu^2\varphi_2+\dots \tag{10.18}$$

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[1] In regard to the singular case see also Merman G.A., Novyi klass periodicheskikh reshenii v ogranichennoi zadache trekh tel i v zadache Khilla, (New Class of Periodic Solutions in the Restricted Problem of Three Bodies and in the Problem of Hill). Tr. inst. teoret. astron., no.1, 1952.

For φ_1 we have:

$$\frac{d^2\varphi_1}{d\tau^2} = -b\sin(\varphi_0 - \delta)\cos\tau. \qquad (10.19)$$

This equation admits a periodic solution for any value of φ_0. Consequently we are dealing with the singular case. From (10.19) we find:

$$\varphi_1 = b\cos\tau\sin(\varphi_0 - \delta) + M_1, \qquad (10.20)$$

where M_1 is an arbitrary constant. Further, we can write

$$\frac{d^2\varphi_2}{d\tau^2} = -\sin\varphi_0 - b\cos(\varphi_0 - \delta)\cdot\varphi_1\cos\tau =$$
$$= -\sin\varphi_0 - b^2\sin(\varphi_0 - \delta)\cos(\varphi_0 - \delta)\cos^2\tau - M_1 b\cos(\varphi_0 - \delta)\cos\tau.$$

In order that this equation have a periodic solution it is necessary and sufficient that the Fourier expansion of its right hand side do not contain a free term. This gives:

$$Q(\varphi_0) = -\sin\varphi_0 - \frac{b^2}{4}\sin 2(\varphi_0 - \delta) = 0, \qquad (10.21)$$

after which we find

$$\varphi_2 = bM_1\cos(\varphi_0 - \delta)\cos\tau + \frac{b^2}{16}\sin 2(\varphi_0 - \delta)\cos 2\tau + M_2.$$

Equation (10.21) determines the magnitude φ_0. For b sufficiently small it has two solutions located near the lower and upper vertical positions. With increasing b two further solutions appear, and if b is sufficiently large these solutions differ little from the values $\delta, \delta+\pi, \delta+\pi/2, \delta-\pi/2$.

The magnitude M_1 which enters φ_1 is determined from the condition of periodicity of φ_3. The equation for φ_3 is of the form

$$\frac{d^2\varphi_3}{d\tau^2} = -\cos\varphi_0\cdot\varphi_1 - b\cos(\varphi_0 - \delta)\cdot\varphi_2\cos\tau + \frac{b}{2}\sin(\varphi_0 - \delta)\cdot\varphi_1^2\cos\tau$$

and the condition of periodicity gives:

$$\left[-\cos\varphi_0 - \frac{b^2}{2}\cos 2(\varphi_0 - \delta)\right]M_1 = Q'(\varphi_0)M_1 = 0.$$

Thus, $M_1 = 0$, after which from (10.18) and (10.20) we find:

$$\varphi = \varphi_0 + \mu b \cos\tau \sin(\varphi_0 - \delta) + \ldots = \varphi_0 + \frac{A}{l} \sin(\varphi_0 - \delta) \cos \omega t + \ldots$$

The further approximations we shall not here consider.

The investigation of the stability of the obtained periodic solutions will be given in sec. 15 of the following chapter.

11. Oscillations of Autonomous Systems [1]

Let us consider the autonomous system described by equations of the type

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n + \mu f_s(x_1, \ldots, x_n, \mu) \qquad (s = 1, \ldots, n), \tag{11.1}$$

where in regard to the functions f_s we shall assume that in a certain region G of the space of the variables $x_1, \ldots, x_n$ and for sufficiently small values of μ they admit continuous partial derivatives of the first order with respect to all their arguments.

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[1] Bulgakov N.G. Kolebaniya kvazilineinykh avtonomnykh sistem so mnogimimi stepenyami svobody i neanaliticheskoi kharakteristikoi nelineinosti, (Oscillations of Quasilinear Autonomous Systems with Many Degrees of Freedom and Non-analytic Characteristic of Nonlinearity), Prikl. matem. i mekh., vol XIX, no.3, 1955. See also Andronov A. and Vitt A., K matematicheskoi teorii avtokolebatel'nykh sistem s dvumya stepenyami svobody, (On the Mathematical Theory of Self-Oscillating Systems with Two Degrees of Freedom), Zhurn. Tekhn. Fiziki, vol. IV, no. 1, 1934 ; Coddington E.A. and Levinson N., Perturbations of Linear Systems with Constant Coefficients Possessing Periodic Solutions, Collected Papers : Contributions to the Theory of Nonlinear Oscillations, vol. II, Edited by S.Lefschetz, Princeton, 1952.

Let us assume that the generating solution

$$\frac{dx_s}{dt} = a_{s1}x_1 + \dots + a_{sn}x_n \tag{11.2}$$

admits m independent particular periodic solutions $\varphi_{s1}, \dots, \varphi_{sm}$ of the same period T, corresponding to the roots of the fundamental equation of the form $\pm p_j i \frac{2\pi}{T}$, where p_j are integers among which, in particular, may also be zero. We shall take the solution

$$x_s^{(0)} = \varphi_s(t) = M_1^* \varphi_{s1} + \dots + M_m^* \varphi_{sm},$$

where M_i^* are constants, as the generating solution [1] and assume that it lies in the region G. We shall explain the conditions under which the system (11.1) admits a periodic solution reducing for $\mu = 0$ to the generating solution.

We shall show first, that without disturbing the generality of the considerations, we can assume that $M_m^* = 0$.

In fact, assume that we have found the periodic solution of the complete system (11.1) which for $\mu = 0$ reduces to the generating solution with the value M_m^* different from zero. This solution has the form

$$x_s(t) = M_1^* \varphi_{s1} + \dots + M_m^* \varphi_{sm} + x_s^*(t, \mu), \tag{11.3}$$

where the functions x_s^* become zero for $\mu = 0$.

For definiteness we shall assume that the solutions $\varphi_{s,m-1}$ and φ_{sm} correspond to the pair of roots $\pm 2\pi pi/T \equiv \pm i\omega$. These solutions, as we saw in sec.2, have the form

$$\varphi_{s,m-1}(t) = A_s \sin \omega t + B_s \cos \omega t, \quad \varphi_{sm} = A_s \cos \omega t - B_s \sin \omega t.$$

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[1] The system (11.2) may have, besides the periodic solution considered, other periodic solutions with period different from T. These solutions can likewise be taken as the generating solutions.

Now let h be an arbitrary constant. We readily find:

$$M^*_{m-1}\varphi_{s,\,m-1}(t+h)+M^*_m\varphi_{sm}(t+h)=$$
$$=(M^*_{m-1}\cos\omega h-M^*_m\sin\omega h)\varphi_{s.\,m-1}(t)+$$
$$+(M^*_{m-1}\sin\omega h+M^*_m\cos\omega h)\varphi_{sm}(t).$$

Hence, if h is chosen according to the condition

$$\operatorname{tg}\omega h=-\frac{M^*_m}{M^*_{m-1}},$$

we obtain from (11.3)

$$x_s(t+h)=N_1\varphi_{s1}(t)+\ldots+N_{m-1}\varphi_{s,\,m-1}(t)+x^*_s(t+h,\mu),$$

where $N_1,\ldots,N_{m-1}$ are certain constants. In virtue of the autonomous character of the system the functions $x_s(t+h)$ are likewise a periodic solution of the system (11.1). Thus, if the equations (11.1) admit a periodic solution for which the corresponding generating solution has a parameter M^*_m different from zero this same system admits also a periodic solution for which the parameter M^*_m in the generating solution reduces to zero and this periodic solution can be obtained from the initial solution by simple displacement of the initial instant of time. From this it follows that by assuming $M_m = 0$ from the very beginning we do not risk losing the periodic solution of the system (11.1).

Our considerations lose their validity if all the magnitudes p_j are equal to zero, i.e. if all periodic solutions φ_{si} of equations (11.2) correspond to zero roots and reduce to constants. But in this case the generating solution corresponds to the positions of equilibrium. To the positions of equilibrium will also correspond the required periodic solutions of the system (11.1). The problem here reduces to the question of the solvability of the system of equations

$$a_{s1}x_1+\ldots+a_{sn}x_n+\mu f_s(x_1,\ldots,x_n)=0,$$

determining these positions of equilibrium. This problem differs from that which now concerns us.

Thus, we c an ssume that the generating solution has the form

$$x_s^{(0)} = \varphi_s(t) = M_1^* \varphi_{s1} + \ldots + M_{m-1}^* \varphi_{s,\, m-1}. \qquad (11.4)$$

For solving the proposed problem it would be possible, as in the case of nonautonomous systems, to make use of the general idea of the method of Poincaré. However, in the case considered this would lead to wearisome calculations so that it is more suitable to proceed by a different method.

We denote by $T(1+\mu\alpha)$ the period of the required periodic solution. This period, on account of the autonomous character of the system, will in general differ from the period of the generating solution and therefore the magnitude α will in general differ from zero and is one of the unknowns of the problem.

We introduce in place of the variable t the variable τ with the aid of the substitution

$$t = \tau(1 + \mu\alpha). \qquad (11.5)$$

The problem then reduces to seeking the periodic solution of period T of the new system of equations

$$\frac{dx_s}{d\tau} = a_{s1} x_1 + \ldots + a_{sn} x_n + \mu(1 + \mu\alpha) f_s(x_1, \ldots, x_n \mu) + \\ + \mu\alpha(a_{s1} x_1 + \ldots + a_{sn} x_n). \quad (11.6)$$

Let α^* be the value of the unknown magnitude α for $\mu = 0$. We shall show first of all that the magnitudes $M_1^*, \ldots, M_{m-1}^*$ and α^* must satisfy the system of equations

$$\int_0^T \sum_{\beta=1}^n f_\beta(\varphi_1, \ldots, \varphi_n, 0)\psi_{\beta i}\, dt + \alpha^*(A_{i1} M_1^* + \ldots + A_{im-1} M_{m-1}^*) \equiv \\ \equiv P_i(\alpha^*, M_1^*, \ldots, M_{m-1}^*) = 0 \quad (i = 1, \ldots, m), \quad (11.7)$$

where $\psi_{s1}(t), \ldots, \psi_{sm}(t)$, as in the preceding sections, denote the periodic solutions of the system conjugate to (11.2), and

$$A_{ij} = T \sum_{\beta=1}^n \frac{d\varphi_{\beta j}}{dt} \psi_{\beta i}. \qquad (11.8)$$

By the property of the solutions of conjugate systems the magnitudes A_{ij} are constants, since evidently the functions $d\varphi_{\beta j}/dt$ are solutions of the system (11.2).

For the proof we assume that equations (11.6) admit a periodic solution of period T and denote this solution by $x_s^*(\tau)$. We consider the system of linear nonhomogeneous equations

$$\frac{dx_s}{d\tau} = a_{s1}x_1 + \ldots + a_{sn}x_n + \mu(1+\mu\alpha)f_s(x_1^*, \ldots, x_n^*, \mu) + \\ + \mu\alpha(a_{s1}x_1^* + \ldots + a_{sn}x_n^*).$$

This system has evidently the solution x_s^* and therefore admits periodic solutions. But then the conditions must be satisfied

$$(1+\mu\alpha)\int_0^T \sum_{\beta=1}^n f_\beta(x_1^*, \ldots, x_n^*, \mu)\psi_{\beta i}\,dt + \\ + \alpha\int_0^T \sum_{\beta=1}^n (a_{\beta 1}x_1^* + \ldots + a_{\beta n}x_n^*)\psi_{\beta i}\,dt = 0.$$

Putting here $\mu = 0$ and taking into account that the functions $\varphi^*_{si}(t)$ satisfy equations (11.2) we obtain equations (11.7), since the functions x_s^* for $\mu = 0$ reduce to the generating solution φ_s.

We shall assume that the magnitudes $\alpha^*, M_1^*, \ldots, M_{m-1}^*$ are actually chosen according to equations (11.7) and let

$$\frac{\partial(P_1, \ldots, P_m)}{\partial(\alpha^*, M_1^*, \ldots, M_{m-1}^*)} \neq 0. \tag{11.9}$$

We shall show that in this case the system (11.6) for an entirely definite choice of magnitude $\alpha = \alpha(\mu)$ will actually admit a periodic solution of period T that reduces to the generating solution (11.4) for $\mu = 0$. For this purpose we shall seek to obtain this periodic solution by the method of successive approximations, taking as the zeroth approximation the generating solution (11.4) and as the k-th approximation the periodic solution of the equations

$$\frac{dx_s^{(k)}}{d\tau} = a_{s1} x_1^{(k)} + \ldots + a_{sn} x_n^{(k)} + \mu(1+\alpha^{(k)}\mu) f_s(x_1^{(k-1)}, \ldots, x_n^{(k-1)}, \mu) + \\ + \mu\alpha^{(k)}(a_{s1} x_1^{(k-1)} + \ldots + a_{sn} x_n^{(k-1)}). \quad (11.10)$$

where $\alpha^{(k)}$ denotes the k-th approximation of the magnitude α with $\alpha^{(k)}(0) = \alpha^*$.

Assume that all approximations up to the (k-1)th inclusive have already been computed and came out periodic. We can then assume that

$$x_s^{(k-1)} = M_1^{(k-1)} \varphi_{s1} + \ldots + M_{m-1}^{(k-1)} \varphi_{s, m-1} + \mu x_s^{(k-1)*}, \quad (11.11)$$

where $\mu x_s^{(k-1)*}$ is a particular periodic solution of the equations for $x_s^{(k-1)}$, while $M_1^{(k-1)}, \ldots, M_{m-1}^{(k-1)}$ are constants. These constants are uniquely determined from the conditions of the periodicity of the functions $x_s^{(k)}$, where we shall have $M_i^{(k-1)}(0) = M_i^*$, so that the functions $x_s^{(k-1)}$ for $\mu = 0$ reduce to the generating solution (11.4). Together with the constants $M_i^{(k-1)}$ there is also determined the magnitude $\alpha^{(k)}$.

In fact, the conditions of periodicity of the functions $x_s^{(k)}$ have the form

$$P_i^{(k)}(\alpha^{(k)}, M_1^{(k-1)}, \ldots, M_{m-1}^{(k-1)}, \mu) = \\ = (1+\mu\alpha^{(k)}) \int_0^T \sum_{\beta=1}^n f_\beta(x_1^{(k-1)}, \ldots, x_n^{(k-1)}, \mu) \psi_{\beta i}\, dt + \\ + \alpha^{(k)} \int_0^T \sum_{\beta=1}^n (a_{\beta 1} x_1^{(k-1)} + \ldots + a_{\beta n} x_n^{(k-1)}) \psi_{\beta i}\, dt = 0 \quad (11.12)$$

$$(i = 1, \ldots, m).$$

Since evidently

$$P_i^{(k)}(\alpha, M_1, \ldots, M_{m-1}, 0) \equiv P_i(\alpha, M_1, \ldots, M_{m-1}),$$

it follows from (11.7) and (11.9) that equations (11.12) have one and only one solution $M_i^{(k-1)}(\mu)$, $\alpha^{(k)}(\mu)$ for which $M_i^{(k-1)} = 0 = M_i^*$, $\alpha^{(k)}(0) = \alpha^*$.

Thus we have obtained an entirely definite process of successive approximations for computing both the periodic solution itself and the period $T(1 + \mu\alpha)$. The equations determining the functions $x_s^{(p)}$ and the constants $M_1^{(p)},\ldots,M_{m-1}^{(p)}$, $\alpha^{(p)}$ do not differ in their general structure from the equations determining the functions $x_s^{(p)}$ and the constants $M_1^{(p)},\ldots,M_m^{(p)}$ in the case of the nonautonomous systems which we considered in sec. 8, and therefore the proof of the convergence of the process, given in sec. 9, carries over also to the case now being considered.

Thus, to the generating solution (11.4) for which the parameters $M_1^*,\ldots,M_{m-1}^*$ satisfy relations (11.7) and conditions (11.9) there corresponds in fact for sufficiently small μ the periodic solution of the complete system (11.1). A more detailed analysis, which we shall not present here, shows that for the indicated assumptions the periodic solution we have found is the only one.

REMARK. Condition (11.9) will of course not be satisfied and the question of the existence of periodic solutions of interest to us will remain open if all the magnitudes A_{ij} determined by formulas (11.8) are equal to zero. This case will always occur if all roots of the fundamental equation of the system (11.2) to which correspond the solutions $\varphi_{s1},\ldots,\varphi_{sm}$ are multiple and the multiplicity of each of the roots exceeds the number of the sets of solutions corresponding to it.

In fact, for the assumptions made the system (11.2), besides the solutions φ_{si}, will admit also the solutions $t\varphi_{si} + \varphi'_{si}$, where φ'_{si} are periodic of functions of the time of period T. Substituting these solutions in (11.2) we easily find that the functions φ'_{si} satisfy the linear nonhomogeneous equations

$$\frac{d\varphi'_{si}}{dt} = a_{s1}\varphi'_{1i} + \ldots + a_{sn}\varphi'_{ni} - \varphi_{si},$$

which, therefore, have periodic solutions. But then the conditions must be satisfied

$$\int_0^T \sum_{\beta=1}^n \varphi_{\beta i}\psi_{\beta j}\,dt = T\sum_{\beta=1}^n \varphi_{\beta i}\psi_{\beta j} = 0$$

$$(i, j = 1, \ldots, m).$$

On the other hand, the functions $d\varphi_{si}/dt$ likewise satisfy equations (11.2) and since they are periodic they must be linear combinations of the functions φ_{si}. We thus have:

$$\frac{d\varphi_{si}}{dt} = \sum_{k=1}^{m} C_{ki}\,\varphi_{sk}$$

and therefore

$$\sum_{\beta=1}^{n} \frac{d\varphi_{\beta i}}{dt}\,\psi_{\beta j} = \sum_{k=1}^{n} C_{ki} \sum_{\beta=1}^{n} \varphi_{\beta k}\,\psi_{\beta j} = 0,$$

which shows that the magnitudes (11.8) reduce to zero.

12. Computation of the Periodic Solutions of Autonomous Systems in the Case of Analytic Equations

Assume that the functions f_s in equations (11.1) are analytic with respect to $x_1,\ldots,x_n,\mu$. The periodic solution of these equations and also its period $T(1 + \mu\alpha)$ will be analytic functions of the parameter μ.

In fact, let us assume that all the approximations $x_s^{(p)}$ of the magnitudes x_s and $\alpha^{(p)}$ of the magnitude α that were considered in the preceding section came out analytic with respect to μ up to the (k-1)th order inclusive. The equations determining the magnitudes $x_s^{(k)}$ and $\alpha^{(k)}$ will then be analytic with respect to μ as a consequence of which these magnitudes themselves will likewise be analytic with respect to μ. Hence, for any k all magnitudes $x_s^{(k)}$, $\alpha^{(k)}$ will be analytic with respect to μ, and since for sufficiently small μ the sequence of these magnitudes uniformly converge , the required periodic solution and its period will be analytic functions of μ for its sufficiently small values.

The above proven property of the required periodic solution and of its period makes it possible to compute them in the case under consideration not by the method of succesive approximations, established in the preceding section, but by the simpler device of expanding into a series in μ.

For this purpose we put

$$\alpha = \alpha^* + \mu\alpha_2 + \mu^2\alpha_3 + \dots,$$

where α^*, α_2, $\alpha_3, \dots$ are constants to be determined. The magnitudes x_s can likewise be represented in the form of series developed in powers of μ. Such representation however of the required periodic solution is practically very inconvenient. The reason is that the coefficients of the series for x_s, as was explained in detail in sec. 10 of chapter I, will not be obtained as periodic functions of μ, since the period of the required solution, determined by the formula

$$\omega = T(1 + \mu\alpha) = T(1 + \alpha^*\mu + \alpha_2\mu^2 + \dots), \tag{12.1}$$

depends on μ. We shall therefore proceed in the following manner.

We introduce into equations (11.1) in place of the variable t the variable τ with the aid of the substitution (11.5)

$$t = \tau(1 + \alpha^*\mu + \alpha_2\mu^2 + \dots). \tag{12.2}$$

The problem then reduces to seeking the periodic solutions of period T of the new system of equations

$$\frac{dx_s}{d\tau} = (a_{s1}x_1 + \dots + a_{sn}x_n)(1 + \alpha^*\mu + \alpha_2\mu^2 + \dots) + \\ + \mu(1 + \alpha^*\mu + \alpha_2\mu^2 + \dots) f_s(x_1, \dots, x_n, \mu). \tag{12.3}$$

This solution will likewise be analytic with respect to μ. In fact, the solution under consideration, like any other solution of equations (12.3), will be an analytic function of the parameter μ entering these equations directly and of their initial values. But these initial values, agreeing evidently with the initial values of the required periodic solution expressed through the variable t, will be analytic functions of μ. From this the validity of our assertion immediately follows.

We shall therefore seek also the periodic solution of equations (12.3) in the form of the series

$$x_s(\tau)=\varphi_s(\tau)+\mu x_s^{(1)}(\tau)+\mu^2 x_s^{(2)}(\tau)+\ldots=$$
$$=M_1^*\varphi_{s1}(\tau)+\ldots+M_{m-1}^*\varphi_{s,\,m-1}(\tau)+$$
$$+\mu x_s^{(1)}(\tau)+\mu^2 x_s^{(2)}(\tau)+\ldots, \quad (12.4)$$

where, as in the preceding section, the coefficient M_m^* in the generating solution is taken equal to zero, which as was shown does not affect the generalilty. Since the period the the functions $x_s(\tau)$ is equal to the magnitude T, not depending on μ, the functions $x_s^{(k)}$ will also be periodic of period T.

For the functions $x_s^{(1)}$ we have the equations

$$\frac{dx_s^{(1)}}{d\tau}=a_{s1}x_1^{(1)}+\ldots+a_{sn}x_n^{(1)}+f_s(\varphi_1,\ldots,\varphi_n,0)+$$
$$+\alpha^*(a_{s1}\varphi_1+\ldots+a_{sn}\varphi_n).$$

In order that these functions turn out periodic it is necessary and sufficient that the equations be satisfied

$$\int_0^T\sum_{\beta=1}^n f_\beta(\varphi_1,\ldots,\varphi_n,0)\psi_{\beta i}\,d\tau+\alpha^*(A_{i1}M_1^*+\ldots+A_{i,\,m-1}M_{m-1}^*)=$$
$$=P_i(\alpha^*,M_1^*,\ldots,M_{m-1}^*)=0 \qquad (i=1,\ldots,m), \quad (12.5)$$

where A_{ij} are determined by the formulas (11.8). We have thus arrived at equations (11.7). Let $M_1^*,\ldots,M_{m-1}^*$ and α^* actually be chosen according to these equations and let the conditions be simultaneously satisfied

$$\frac{\partial(P_1,\ldots,P_m)}{\partial(\alpha^*,M_1^*,\ldots,M_{m-1}^*)}\neq 0, \quad (12.6)$$

which, as was shown, assures the existence of the required periodic solution. We can here write :

$$x_s^{(1)}(\tau)=x_s^{(1)*}(\tau)+M_1^{(1)}\varphi_{s1}(\tau)+\ldots+M_{m-1}^{(1)}\varphi_{s,\,m-1}(\tau),$$

where $x_s^{(1)*}$ is some particular periodic solution of the equations for $x_s^{(1)}$ while $M_i^{(1)}$ are constants. As in the generating solution, we do not here introduce the functions φ_{sm} in $x_s^{(1)}$. The constants $M_i^{(m-1)}$ together with the constant α_2 are determined from the conditions of periodicity of the functions $x_s^{(2)}$, and as we shall presently show, the equations determining these magnitudes are found to be linear and their determinant will agree with (12.6).

For this purpose let us set up the equations determining the magnitudes $x_s^{(k)}$ for any k. For this it is necessary to substitute in equations (12.3) in place of the magnitudes x_s their expressions (12.4) and equate the coefficients of μ^k. In setting up these equations we write out in them explicitly the terms depending on $x_s^{(k-1)}$ and on α_k. We then obtain:

$$\frac{dx_s^{(k)}}{d\tau} = a_{s1}x_1^{(k)} + \ldots + a_{sn}x_n^{(k)} + \alpha_k(a_{s1}\varphi_1 + \ldots + a_{sn}\varphi_n) +$$
$$+ \alpha^*(a_{s1}x_1^{(k-1)} + \ldots + a_{sn}x_n^{(k-1)}) + \sum_{\gamma=1}^{n}\left(\frac{\partial f_s}{\partial x_\gamma}\right)x_\gamma^{(k-1)} + f_s^{(k)}, \quad (12.7)$$

where $f_s^{(k)}$ are integral rational functions with constant coefficients of $\varphi_j, x_j^{(1)}, \ldots, x_j^{(k-2)}$ ($j = 1, \ldots, n$), depending only on the constants α^*, $\alpha_1, \ldots, \alpha_{k-1}$ and the parentheses indicate that the derivatives are computed for the values $x_1 = \varphi_1, \ldots, x_n = \varphi_n$, $\mu = 0$.

From (12.7) it is seen that if the functions $x_s^{(p)}$ for any p are obtained as periodic they can be represented in the form

$$x_s^{(p)} = x_s^{(p)*} + M_1^{(p)}\varphi_{s1} + \ldots + M_{m-1}^{(p)}\varphi_{s,\,m-1} \quad (12.8)$$
$$(p = 1, 2, \ldots),$$

where $x_s^{(p)*}$ is a particular periodic solution of the equations for $x_s^{(p)}$ and $M_i^{(p)}$ are arbitrary constants. These constants are to be determined from the conditions of periodicity of the succeeding approximation.

Let us assume that all the constants $\alpha_1, \ldots, \alpha_{k-1}$, and also the functions $x_s^{(1)}, \ldots, x_s^{(k-2)}$ together with the constants $M_i^{(1)}, \ldots, M_i^{(k-2)}$ entering them have already been computed. We shall assume moreover that the functions $x_s^{(k-1)}$ have also been computed but that the constants $M_i^{(k-1)}$ entering them and the constants α_k are still to be determined from the conditions of the periodicity of the functions $x_s^{(k)}$. Setting up these conditions and substituting in them for the magnitudes $x_s^{(k-1)}$ their expressions from (12.8) we obtain

$$\alpha_k (A_{i1} M_1^* + \ldots + A_{i,\,m-1} M_{m-1}^*) + \\ + \alpha^* (A_{i1} M_1^{(k-1)} + \ldots + A_{i,\,m-1} M_{m-1}^{(k-1)}) + \\ + \sum_{j=1}^{m-1} M_j^{(k-1)} \int_0^T \sum_{\gamma,\beta=1}^{n} \left(\frac{\partial f_\beta}{\partial x_\gamma}\right) \varphi_{\gamma j} \psi_{\beta i}\, d\tau + N_i^{(k-1)} = 0,$$

where

$$N_i^{(k-1)} = \int_0^T \left[\sum_{\beta,\gamma=1}^{n} \left(\frac{\partial f_\beta}{\partial x_\gamma}\right) x_\gamma^{(k-1)*} \psi_{\beta i} + \sum_{\beta=1}^{n} f_\beta^{(k)} \psi_{\beta i} \right] d\tau$$

are known constants.

The obtained equations , as is easily seen, can be represented in the form

$$\frac{\partial P_i}{\partial \alpha^*} \alpha_k + \sum_{=1}^{m-1} \frac{\partial P_i}{\partial M_j^*} M_j^{(k-1)} + N_i^{(k-1)} = 0. \qquad (12.9)$$

We have thus obtained m linear equations determining the magnitudes α_k, $M_i^{(k-1)}$ the determinant of which on the basis of (12.6) is different from zero.

Thus, entirely definite series are obtained for $x_s(\tau)$ and α. These series therefore represent the required periodic solution and the magnitude α and consequently converge for sufficiently small μ. Going over to the variable t we obtain the required solution of equations (11.1) and its period (12.1).

13. Self-Oscillations in Two Coupled Circuits

We shall apply the method of the preceding section to computing the amplitude and frequency of the periodic self-oscillations which can arise in an electrical system with two degrees of freedom. As an example of such a system we shall consider two inductively coupled circuits one of which is excited by a cathode tube. The circuit diagram of this system is given in fig. 23. Systems of such type have been studied

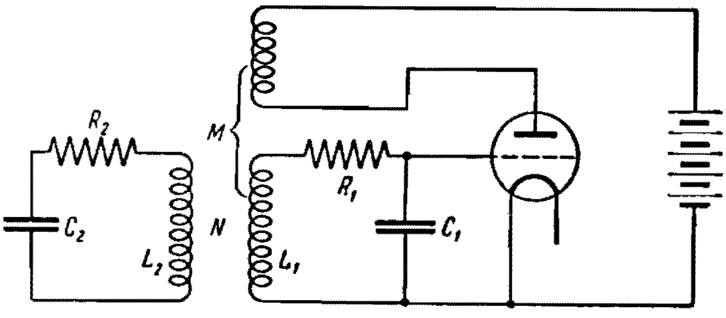

Fig. 23

by many authors using various methods.[1]

Neglecting the grid current we have the following equations connecting the currents i_1, i_2 and i_a:

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1} \int_0^t i_1 \, dt = M \frac{di_a}{dt} + N \frac{di_2}{dt},$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2} \int_0^t i_2 \, dt = N \frac{di_1}{dt}.$$

whence, going over to the voltages

$$v_1 = \frac{1}{C_1} \int_0^t i_1 \, dt, \quad v_2 = \frac{1}{C_2} \int_0^t i_2 \, dt$$

and setting

$$n_1^2 = \frac{1}{L_1 C_1}, \quad n_2^2 = \frac{1}{L_2 C_2}, \quad \varkappa_1 = N C_2 n_1^2, \quad \varkappa_2 = N C_1 n_2^2, \quad (13.1)$$

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[1] See for example Van der Pol, Phil. Mag., series 6, vol. 43, 1922 ; also the work of A.A. Andronov and A.A. Vitt cited on page 165.

we obtain

$$\frac{d^2v_1}{dt^2} - \varkappa_1 \frac{d^2v_2}{dt^2} + n_1^2 v_1 = n_1^2\left(-R_1C_1\frac{dv_1}{dt} + M\frac{di_a}{dt}\right),$$
$$\frac{d^2v_2}{dt^2} - \varkappa_2 \frac{d^2v_1}{dt^2} + n_2^2 v_2 = -n_2^2 R_2 C_2 \frac{dv_2}{dt}.$$

We shall neglect the reaction of the anode and assume that the tube has the soft operation characteristic so that

$$i_a = S_0 v_1 - \frac{1}{3} S_2 v_1^3.$$

Then introducing in place of v_1 and v_2 the nondimensional variables

$$x = v_1\sqrt{\frac{MS_2}{MS_0 - C_1R_1}},$$
$$y = v_2\sqrt{\frac{MS_2}{MS_0 - C_1R_1}}$$

and considering as the small parameter the nondimensional magnitude

$$\mu = n_1(MS_0 - C_1R_1), \tag{13.2}$$

we finally obtain the following equations of the oscillations

$$\left.\begin{aligned} &\frac{d^2x}{dt^2} - \varkappa_1\frac{d^2y}{dt^2} + n_1^2 x = \mu n_1(1-x^2)\frac{dx}{dt}, \\ &\frac{d^2y}{dt^2} - \varkappa_2\frac{d^2x}{dt^2} + n_2^2 y = -\mu\frac{n_2^2}{n_1}\delta\frac{dy}{dt}, \end{aligned}\right\} \tag{13.3}$$

where

$$\delta = \frac{C_2R_2}{MS_0 - C_1R_1}.$$

The general solution of the generating system

$$\left.\begin{aligned} &\frac{d^2x_0}{dt^2} - \varkappa_1\frac{d^2y_0}{dt^2} + n_1^2 x_0 = 0, \\ &\frac{d^2y_0}{dt^2} - \varkappa_2\frac{d^2x_0}{dt^2} + n_2^2 y_0 = 0 \end{aligned}\right\} \tag{13.4}$$

is of the form

$$\left.\begin{aligned} x_0 &= M_1 \cos \omega_1 t + M_2 \sin \omega_1 t + M_3 \cos \omega_2 t + M_4 \sin \omega_2 t, \\ y_0 &= M_1 k_1 \cos \omega_1 t + M_2 k_1 \sin \omega_1 t + M_3 k_2 \cos \omega_2 t + M_4 k_2 \sin \omega_2 t, \end{aligned}\right\} \quad (13.5)$$

where M_1, M_2, M_3, M_4 are arbitrary constants, k_1 and k_2 the values which are assumed by the magnitude

$$k = \frac{\omega^2 - n_1^2}{\varkappa_1 \omega^2} = \frac{\varkappa_2 \omega^2}{\omega^2 - n_2^2} \quad (13.6)$$

for $\omega = \omega_1$ and $\omega = \omega_2$, where ω_1 and ω_2 are roots of the fundamental equation

$$(1 - \varkappa_1 \varkappa_2)\, \omega^4 - (n_1^2 + n_2^2)\, \omega^2 + n_1^2 n_2^2 = 0. \quad (13.7)$$

It follows from (13.5) that the generating system has two families of periodic solutions depending on two arbitrary constants each and having respectively the periods $2\pi/\omega_0$ and $2\pi/\omega_2$. We shall seek the periodic solutions of the system (13.3) corresponding to each of these families. As we have seen in the preceding section, we can in the generating solution assume one of the arbitrary constants to be zero. We thus have the following generating solution:

$$x_0 = M^* \cos \omega t, \qquad y_0 = M^* k \cos \omega t, \quad (13.8)$$

where ω and k are respectively equal to ω_1 and k_1 or ω_2 and k_2.

Following the method of the preceding section we set in equations (13.3)

$$t = \tau (1 + \alpha^* \mu + \alpha_2 \mu^2 + \dots),$$

where α^*, $\alpha_2, \dots$ are certain entirely definite but as yet unknown constants. Equations (13.3) will assume the form

$$\left.\begin{aligned} \frac{d^2 x}{d\tau^2} - \varkappa_1 \frac{d^2 y}{d\tau^2} + n_1^2 x &= \mu \left\{ -2\alpha^* n_1^2 x + n_1 (1 - x^2) \frac{dx}{d\tau} \right\} + \dots, \\ \frac{d^2 y}{d\tau^2} - \varkappa_2 \frac{d^2 x}{d\tau^2} + n_2^2 y &= \mu \left\{ -2\alpha^* n_2^2 y - \frac{n_2^2}{n_1} \delta \frac{dy}{d\tau} \right\} + \dots, \end{aligned}\right\} \quad (13.9)$$

where the terms not written down are of an order higher than the first with respect to μ. We shall seek the periodic solution of period $2\pi/\omega$ of these equations in the form of the series

$$x = M^* \cos \omega\tau + \mu x_1(\tau) + \ldots,$$
$$y = kM^* \cos \omega\tau + \mu y_2(\tau) + \ldots$$

For x_1 and y_1 we then obtain the following equations

$$\left.\begin{aligned}
\frac{d^2 x_1}{d\tau^2} - \varkappa_1 \frac{d^2 y_1}{d\tau^2} + n_1^2 x_1 &= -2\alpha^* n_1^2 M^* \cos \omega\tau - \\
&\quad - n_1 (1 - M^{*2} \cos^2 \omega\tau) M^* \omega \sin \omega\tau, \\
\frac{d^2 y_1}{d\tau^2} - \varkappa_2 \frac{d^2 x_1}{d\tau^2} + n_2^2 y_1 &= -2\alpha^* n_2^2 M^* k \cos \omega\tau + \\
&\quad + \frac{n_2^2}{n_1} \delta k M^* \omega \sin \omega\tau.
\end{aligned}\right\} \quad (13.10)$$

We shall now write down the conditions of periodicity of the functions x_1 and y_1. For this, according to the general theory, it is necessary to know the periodic solution of the system conjugate to (13.4). In the case considered however it is simpler to proceed in the following manner.

The problem reduces to the determination of the conditions which must be satisfied by the coefficients P, Q, R and S in order that the system

$$\frac{d^2 x}{d\tau^2} - \varkappa_1 \frac{d^2 y}{d\tau^2} + n_1^2 x = P \cos \omega\tau + Q \sin \omega\tau,$$
$$\frac{d^2 y}{d\tau^2} - \varkappa_2 \frac{d^2 x}{d\tau^2} + n_2^2 y = R \cos \omega\tau + S \sin \omega\tau$$

admit a periodic solution. Setting here

$$x = A \cos \omega\tau + B \sin \omega\tau,$$
$$y = C \cos \omega\tau + D \sin \omega\tau,$$

we shall have

$$\left.\begin{aligned}
(n_1^2 - \omega^2) A + \varkappa_1 \omega^2 C = P, \qquad \varkappa_2 \omega^2 A + (n_2^2 - \omega^2) C = R, \\
(n_1^2 - \omega^2) B + \varkappa_1 \omega^2 D = Q, \qquad \varkappa_2 \omega^2 B + (n_2^2 - \omega^2) D = S.
\end{aligned}\right\} \quad (13.11)$$

Since ω^2 is a root of equation (13.7), equations (13.11) will have solutions only when the relations are satisfied

$$\varkappa_2\omega^2 P-(n_1^2-\omega^2)R=\varkappa_2\omega^2 Q-(n_1^2-\omega^2)S=0. \qquad (13.12)$$

These will be the required conditions of periodicity. Applying them to the equations (13.10) we obtain :

$$a^*M^*=0,$$
$$n_1\varkappa_2\omega^2\left(1-\frac{1}{4}M^{*2}\right)+\frac{n_2^2}{n_1}(n_1^2-\omega^2)\delta k=0,$$

whence, taking (13.6) into account, we obtain

$$a^*=0, \quad M^{*2}=4-\frac{4\delta n_2^2(n_1^2-\omega^2)}{n_1^2(n_2^2-\omega^2)}. \qquad (13.13)$$

We have thus found the amplitude of the generating solution and the first correction on the frequency. We shall limit ourselves to this approximation.

The investigation of the stability of the obtained periodic solution and the physical analysis will be given in sec. 14 of the following chapter.

CHAPTER III. STABILITY OF OSCILLATIONS

1. Statement of the Problem. Equations in Variations

The problem of the stability of motion in its most general form was first formulated by Lyapunov in his classical work "The General Problem of the Stability of Motion". In the same paper Lyapunov developed the fundamental methods of solving the stability problem. We shall here present several of the results obtained by Lyapunov.

Let us assume that the motion of the system under consideration is described by the differential equations

$$\frac{dx_s}{dt} = X_s(t, x_1, \ldots, x_n) \qquad (s = 1, \ldots, n), \tag{1.1}$$

where the right hand sides are in general not periodic with respect to t and are defined for all values $x_1, \ldots, x_n$ lying in a certain region G and for all values t lying either in the interval $[0, \infty]$ or in certain cases in the interval $[-\infty, +\infty]$. We shall here assume that the functions X_s in the indicated region of variation of the variables admit continuous and bounded derivatives of the first order with respect to the variables $x_1, \ldots, x_n$. We shall consider a certain motion of the system (1.1) corresponding to its particular solution

$$x_s = \varphi_s(t). \tag{1.2}$$

We shall judge the stability or instability of this motion in relation to the behavior of neighboring motions, i.e. motions for which the initial values (for $t = t_0$) of the magnitudes x_s differ little from $\varphi_s(t_0)$. All these motions we shall call DISTURBED motions while the motion (1.2), the stability of which is investigated, will be called UNDISTURBED. The differences $x_s - \varphi_s(t)$ we shall denote as DISTURBANCES. DEFINITION. AN UNDISTURBED MOTION IS CALLED STABLE IN THE SENSE OF LYAPUNOV IF FOR ANY POSITIVE NUMBER ε, NO MATTER HOW SMALL, ANOTHER POSITIVE NUMBER η CAN BE FOUND SUCH THAT FOR ALL DISTURBED MOTIONS FOR WHICH AT THE INITIAL INSTANT OF TIME t_0 THE INEQUALITIES ARE SATISFIED

$$|x_s(t_0) - \varphi_s(t_0)| \leqslant \eta,$$

THERE WILL, FOR ALL $t > t_0$, BE SATISFIED THE INEQUALITIES

$$|x_s(t) - \varphi_s(t)| < \varepsilon. \tag{1.3}$$

Motions which do not satisfy the conditions of stability are called UNSTABLE. Stable undisturbed motions for which for sufficiently small η not only conditions (1.3) but the stronger conditions

$$\lim_{t\to\infty} x_s(t) = \varphi_s(t),$$

are satisfied are called ASYMPTOTICALLY STABLE.

For investigating the stability of motion it is convenient to make use of the equations which are satisfied by the disturbances

$$y_s = x_s - \varphi_s(t). \tag{1.4}$$

These so-called EQUATIONS OF THE DISTURBED MOTION are obtained if the substitution (1.4) is made in equations (1.1). With this substitution equations (1.1) assume the form

$$\frac{dy_s}{dt} = X_s(t, \varphi_1 + y_1, \ldots, \varphi_n + y_n) - X_s(t, \varphi_1, \ldots, \varphi_n)$$

or

$$\frac{dy_s}{dt} = p_{s1}y_1 + \ldots + p_{sn}y_n + Y_s(t, y_1, \ldots, y_n). \tag{1.5}$$

where

$$p_{sj} = p_{sj}(t) = \frac{\partial X_s(t, \varphi_1, \ldots, \varphi_n)}{\partial \varphi_j}$$

and the functions $Y_s(t, y_1, \ldots, y_n)$ satisfy the conditions

$$\lim \frac{|Y_s(t, y_1, \ldots, y_n)|}{|y_1| + \cdots + |y_n|} = 0 \quad \text{for } |y_1| + \cdots + |y_n| \to 0.$$

Equations (1.5) have the trivial solution $y_1 = \ldots = y_n = 0$. To this solution corresponds evidently the undisturbed motion. In the variables $y_1, \ldots, y_n$ the conditions of stability is expressed as follows: The undisturbed motion is stable if for any positive number ε it is possible to choose a positive number η such that the inequality $|y_s(t)| < \varepsilon$ will be satisfied for all $t > t_0$ for any solution for which $|y_s(t_0)| \leq \eta$. If at the same time $\lim_{t \to \infty} y_s(t) = 0$ the undisturbed motion will be asymptotically stable.

If in equations (1.5) terms of higher orders are rejected the linear equations thus obtained

$$\frac{dy_s}{dt} = p_{s1} y_1 + \cdots + p_{sn} y_n \tag{1.6}$$

are called, following Poincare, the EQUATIONS IN VARIATIONS FOR THE SOLUTION (1.2). The equations in variations play a large part in the investigation of stability. As was shown by Lyapunov, in many cases the stability or instability of the undisturbed motion is entirely determined by the form of solutions of the equations in variations. These equations we shall frequently encounter in what follows also in the determination of oscillatory motions. We shall here derive two important properties of these equations, that have been established by Poincaré.

1. Let us assume that the initial system (1.1) admits the family of solutions

$$x_s = \varphi_s(t, h_1, \ldots, h_m). \tag{1.7}$$

depending on m arbitrary parameters h_i and that the undisturbed motion belongs to this family and corresponds to the values $h_i = h_i^*$ of the parameters. The equations in variations (1.6) then admit m solutions

$$y_{sj} = \frac{\partial \varphi_s}{\partial h_j} \qquad (j = 1, \ldots, m),$$

where the derivatives are computed for the undisturbed motion.

In fact, substituting (1.7) in (1.1), differentiating the obtained identities with respect to h_j and then passing to the undisturbed motion we shall have

$$\frac{d}{dt}\left(\frac{\partial \varphi_s}{\partial h_j}\right)_{h_i = h_i^*} = \left\{p_{s1}\frac{\partial \varphi_1}{\partial h_j} + \ldots + p_{sn}\frac{\partial \varphi_n}{\partial h_j}\right\}_{h_i = h_i^*},$$

which proves our assertion.

2. If the system (1.1) admits the first integral

$$F(t, x_1, \ldots, x_n) = \text{const},$$

the system in variations (1.6) admits the first integral

$$H = \left(\frac{\partial F}{\partial x_1}\right) y_1 + \ldots + \left(\frac{\partial F}{\partial x_n}\right) y_n = \text{const},$$

where the parentheses indicate that the derivatives are computed for the undisturbed motion.

In fact, we have identically

$$\frac{\partial F}{\partial t} + \sum_{\alpha=1}^{n} \frac{\partial F}{\partial x_\alpha} X_\alpha \equiv 0.$$

Differentiating this identity with respect to x_i we obtain

$$\frac{\partial^2 F}{\partial t\, \partial x_i} + \sum_{\alpha=1}^{n} \frac{\partial^2 F}{\partial x_\alpha \partial x_i} X_\alpha + \sum_{\alpha=1}^{n} \frac{\partial F}{\partial x_\alpha} \frac{\partial X_\alpha}{\partial x_i} \equiv 0. \tag{1.8}$$

On the other hand, taking the derivative of the function H with respect to t, assuming that the magnitudes y_s satisfy the equations in variations, we obtain

$$\frac{dH}{dt} = \sum_{\alpha=1}^{n} y_\alpha \frac{d}{dt}\left(\frac{\partial F}{\partial x_\alpha}\right) + \sum_{\alpha=1}^{n} \left(\frac{\partial F}{\partial x_\alpha}\right)(p_{\alpha 1} y_1 + \ldots + p_{\alpha n} y_n) =$$

$$= \sum_{\alpha=1}^{n} y_\alpha \left\{ \frac{d}{dt}\left(\frac{\partial F}{\partial x_\alpha}\right) + \sum_{\beta=1}^{n} \left(\frac{\partial F}{\partial x_\beta}\right)\left(\frac{\partial X_\beta}{\partial x_\alpha}\right) \right\}. \tag{1.9}$$

Since in the derivatives $(\partial F/\partial x_\alpha)$ the magnitudes $x_1, \ldots, x_n$ satisfy equations (1.1)

$$\frac{d}{dt}\left(\frac{\partial F}{\partial x_\alpha}\right) = \left(\frac{\partial^2 F}{\partial t\, \partial x_\alpha}\right) + \sum_{\beta=1}^{n} \left(\frac{\partial^2 F}{\partial x_\alpha \partial x_\beta}\right) X_\beta.$$

Substituting in (1.9) and taking (1.8) into account, we obtain

$$\frac{dH}{dt} = 0,$$

which proves our proposition.

2. Linear Equations with Periodic Coefficients. Characteristic Equation

If the right hand sides of the equations of motion (1.1) are periodic with respect to t and if there is investigated the stability of the periodic solution of these equations, i.e. the functions $\varphi_s(t)$ are likewise periodic, the right hand sides of the equations of the disturbed motion, and therefore of the equations in variations, will be periodic with respect to t.

Thus, the problem of the stability of the periodic motions reduces, first of all, to the investigation of a system of linear equations with periodic coefficients. Many other problems of oscillations are reduced to the investigation of this type of equations. These equations will play a very important part in all that follows. We shall therefore here give the fundamental aspects of the theory of linear equations with periodic coefficients.

Thus, let us consider the system of linear equations

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n \qquad (s = 1, \ldots, n), \tag{2.1}$$

where p_{sj} are continuous periodic functions of t with the same period ω. Let $x_{s1}, \ldots, x_{sm}$ be some fundamental system of solutions of these equations. As usual, the first index here denotes the number of the function in the solution and the second index the number of the solution.

If in all functions x_{sj} of some j-th solution we replace t by $t + \omega$, then in virtue of the periodicity of the coefficients p_{sj}, we again obtain a solution, since the functions $x_{sj}(t + \omega)$ will as before satisfy equations (2.1) if they are satisfied by the functions $x_{sj}(t)$. The obtained solution will not coincide with the initial solution x_{sj} but, like any solution of equations (2.1), it must necessarily be a linear combination of the fundamental system of solutions $x_{sj}(t)$. Hence we have:

$$x_{sj}(t+\omega) = a_{1j}x_{s1}(t) + a_{2j}x_{s2}(t) + \ldots + a_{nj}x_{sn}(t), \tag{2.2}$$

where a_{1j}, $a_{2j}, \ldots, a_{nj}$ are certain constants. By varying j from 1 to n we obtain the n^2 magnitudes a_{sj}

Let us form the equation

$$D(\rho) = \begin{vmatrix} a_{11}-\rho & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\rho & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\rho \end{vmatrix} = 0. \tag{2.3}$$

This equation, which plays a fundamental role in the theory of linear equations with periodic coefficients, is called the CHARACTERISTIC EQUATION CORRESPONDING TO THE PERIOD ω, or more briefly, the CHARACTERISTIC EQUATION.

We shall establish certain fundamental properties of the characteristic equation.

1. THE CHARACTERISTIC EQUATION DOES NOT DEPEND ON THE CHOICE OF THE FUNDAMENTAL SYSTEM.

In place of the fundamental system x_{sj} let us choose another fundamental system y_{sj}. For it we shall have:

$$y_{sj}(t+\omega)=c_{1j}y_{s1}(t)+c_{2j}y_{s2}(t)+\ldots+c_{nj}y_{sn}(t), \tag{2.4}$$

where c_{sj} are in general different from a_{sj}. We shall show however that the roots of the characteristic equation formed of the coefficients c_{sj} agree exactly with the roots of equation (2.3).

In fact, since the magnitudes y_{sj} constitute a fundamental system we must have

$$y_{sj}(t)=b_{1j}x_{s1}(t)+b_{2j}x_{s2}(t)+\ldots+b_{nj}x_{sn}(t), \tag{2.5}$$

where b_{sj} are certain constants. We shall denote by $x(t)$ the matrix of the functions $x_{sj}(t)$, by $y(t)$ the matrix of the functions $y_{sj}(t)$ and by a, b, c respectively the matrices of the coefficients a_{sj}, b_{sj}, c_{sj}. The relations (2.2), (2.4) and (2.5) can then be represented in the following manner:

$$x(t+\omega)=x(t)a, \quad y(t+\omega)=y(t)c, \quad y(t)=x(t)b.$$

Further, we have:

$$y(t+\omega)=x(t+\omega)b=x(t)ab=y(t)b^{-1}ab$$

and therefore

$$c\equiv b^{-1}ab.$$

Hence if E is the unit matrix the characteristic determinant of coefficients c_{sj} can be represented as follows:

$$|c-\rho E| \equiv |b^{-1}ab-\rho E| \equiv |b^{-1}(a-\rho E)b| \equiv$$
$$\equiv |b^{-1}|\cdot|a-\rho E|\cdot|b| \equiv |a-\rho E|,$$

which proves our proposition.

2. THE CHARACTERISTIC EQUATION DOES NOT CHANGE IF THE SYSTEM (2.1) IS SUBJECTED TO A NONSINGULAR LINEAR TRANSFORMATION WITH PERIODIC COEFFICIENTS OF PERIOD ω.

In fact, let us transform equations (2.1) with the aid of the substitution

$$y_s = q_{s1}(t)x_1 + \ldots + q_{sn}(t)x_n, \tag{2.6}$$

where q_{sj} are continuous periodic functions of t of period ω and such that the determinant $|q_{sj}|$ is different from zero for all values of t on the segment $[0, \omega]$.

Substituting the fundamental system $x_{sj}(t)$ in (2.6) we obtain the following fundamental system of solutions of the transformed equations:

$$y_{sj}(t) = q_{s1}(t)x_{1j}(t) + \ldots + q_{sn}(t)x_{nj}(t)$$

or in matrix notation

$$y(t) = q(t)x(t),$$

where q is the matrix of the coefficients q_{sj}. From this we find that

$$x(t) = q^{-1}(t)y(t),$$
$$y(t+\omega) = q(t+\omega)x(t+\omega) = q(t)x(t)a = q(t)q^{-1}(t)y(t)a = y(t)a$$

and therefore the characteristic equation of the transformed system agrees with the characteristic equation of the initial system

Let us assume that the fundamental system considered is determined by the initial conditions

$$x_{ss}(0)=1, \qquad x_{sj}(0)=0 \qquad (s\neq j).$$

Then, setting in (2.2) t = 0, we shall have:

$$x_{sj}(\omega)=a_{sj} \qquad (s,\ j=1,\ \ldots,\ n),$$

and therefore the characteristic equation can be represented in the following form

$$\begin{vmatrix} x_{11}(\omega)-\rho & x_{12}(\omega) & \ldots & x_{1n}(\omega) \\ x_{21}(\omega) & x_{22}(\omega)-\rho & \ldots & x_{2n}(\omega) \\ \cdot & \cdot & \cdot & \cdot \\ x_{n1}(\omega) & x_{n2}(\omega) & \ldots & x_{nn}(\omega)-\rho \end{vmatrix}=0. \tag{2.7}$$

We shall frequently make use of this form of the characteristic equation in what follows.

In particular, we shall make use of it for deriving an important formula giving the expression of the free term of the characteristic equation. For this purpose let us consider the Wronskian

$$\Delta(t)=\begin{vmatrix} x_{11}(t) & x_{12}(t) & \ldots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \ldots & x_{2n}(t) \\ \cdot & \cdot & \cdot & \cdot \\ x_{n1}(t) & x_{n2}(t) & \ldots & x_{nn}(t) \end{vmatrix}$$

As is known, for any t_o and t the relation holds

$$\Delta(t)=\Delta(t_0)\,e^{\int\limits_{t_0}^{t}\sum\limits_{s=1}^{n}p_{ss}dt}.$$

Setting $t_o = 0$, $t = \omega$, we obtain

$$\Delta(\omega)=e^{\int\limits_{0}^{\omega}\sum\limits_{s=1}^{n}p_{ss}dt}.$$

Comparing with (2.7) we find that if the characteristic equation is represented in the form

$$\rho^n + A_1\rho^{n-1} + \ldots + A_{n-1}\rho + A_n = 0,$$

the free term A_n is represented by the formula

$$(-1)^n A_n = e^{\int_0^\omega \sum_{s=1}^n p_{ss}dt}. \tag{2.8}$$

3. Analytic Form of the Solution of Linear Equations with Periodic Coefficients

The system (2.1) can not be integrated in closed form. It is possible however to indicate a general analytic form of its solution that is very similar to the analytic form of the solutions of linear equations with constant coefficients.

Making use of some definition of logarithms let us consider the magnitudes

$$\alpha_k = \frac{1}{\omega} \ln \rho_k, \tag{3.1}$$

where ρ_k are roots of the characteristic equation. These magnitudes are called the CHARACTERISTIC EXPONENTS of the system (2.1). The following fundamental propositions hold true.

For each root ρ_k of the characteristic equation a particular solution of equations (2.1) may be chosen of the form

$$x_s(t) = e^{\alpha_k t}\varphi_s(t) \qquad (s = 1, \ldots, n), \tag{3.2}$$

where φ_s are certain periodic functions of time of period ω. If the root ρ_k is simple there exists for it only one (with an accuracy up to a constant multiplier) solution of the form (3.2). If all the roots of the characteristic equation are simple there will exist for the system (2.1) n different particular solutions of the form (3.2). These solutions will evidently be independent and will therefore form a fundamental system.

Let us however assume that the root ρ_k is multiple and that its multiplicity is equal to μ. It is here necessary to distinguish two cases depending on the rank of the characteristic determinant for $\rho = \rho_k$. Let us assume first that this rank is equal to n - 1, so that the root ρ_k does not reduce to zero at least one of the minors of the (n-1)th order of the characteristic determinant. In this case there can be constructed for the root ρ_k a particular solution of the form

$$x_s = x_{s1}(t) = e^{\alpha_k t} P_s(t), \tag{3.3}$$

where P_s are polynomials in t with periodic coefficients of period ω. The degree of the polynomials P_s do not exceed $\mu - 1$ and the degree of at least one of them is equal to $\mu - 1$. Besides solution (3.3) there exist, for the root ρ_k, additional $\mu - 1$ independent particular solutions of equations (2.1) which can be obtained in the following manner.

Let

$$P(t) = t^m \varphi_1(t) + t^{m-1}\varphi_2(t) + \ldots + t\varphi_m(t) + \varphi_{m+1}(t)$$

be an arbitrary polynomial whose coefficients are periodic functions of t. We now differentiate this polynomial with respect to t, assuming that the coefficients φ_i are constant, and denote by D/Dt the operator of this kind of differentiation. Thus

$$\frac{DP}{Dt} = mt^{m-1}\varphi_1 + (m-1)t^{m-2}\varphi_2 + \ldots + 2t\varphi_{m-1} + \varphi_m.$$

It is found that for the root ρ_k in the case under consideration there exist additional $\mu - 1$ independent particular solutions obtained from (3.3) by the successive application of the operator D/Dt to the polynomials P_s. We can therefore write all solutions corresponding to the root ρ_k, including the solution (3.3), in the following manner

$$x_{si} = e^{\alpha_k t} \frac{D^{(i-1)}P_s}{Dt^{i-1}} \qquad (i = 1, \ldots, \mu). \tag{3.4}$$

We shall say that solutions (3.4) form one set and that in the case considered one set of solutions corresponds to the root ρ_k.

Assume now that the multiple root ρ_k reduces to zero all minors of the characteristic determinant up to the order n - p + 1 inclusive, not reducing to zero at least one of the minors of the (n-p)th order, so that the rank of the characteristic determinant for $\rho = \rho_k$ is equal to n - p. In this case there will correspond as before to the root under consideration μ solutions of equations (2.1), but these solutions break down into p independent sets. And if we denote by n_j the number of solutions in the j-th set $(n_1 + n_2 + \ldots + n_p = \mu)$ the solutions of this set are of the form

$$x_{si}^{(j)}(t) = e^{\alpha_k t} \frac{D^{(i-1)} P_s^{(j)}}{Dt^{i-1}} \qquad (i = 1, 2, \ldots, n_j). \tag{3.5}$$

Here $P_s^{(j)}$ are polynomials in t with periodic coefficients the degrees of which do not exceed $n_j - 1$, the degree of at least one of them being equal to $n_j - 1$. We can therefore write

$$P_s^{(j)} = \frac{t^{n_j-1}}{(n_j-1)!} \varphi_{s1}^{(j)} + \frac{t^{n_j-2}}{(n_j-2)!} \varphi_{s2}^{(j)} + \ldots + \varphi_{sn_j}^{(j)},$$

where $\varphi_{si}^{(j)}$ are periodic functions of t of period ω, at least one of the functions $\varphi_{s1}^{(j)}$, for any j, not reducing identically to zero.

The number p can not evidently exceed the multiplicity μ of the root considered but may attain this limit. In the latter case each set will consist of one solution. Each such solution will have the form (3.2).

The converse proposition is likewise true. If for the system (2.1) it is succeeded in finding μ particular solutions that divide into p sets of the form (3.5) the magnitude ρ_k is a root of the characteristic equation the multiplicity of which is not less than μ, this root reducing to zero all minors of the characteristic determinant up to an order of at least n - p + 1.

The proof of the above stated fundamental proposition on the analytic form of the solutions of equations (2.1) is given in the following section.

4. Proof of the Proposition of the Preceding Section

We proceed to the proof of the fundamental proposition with regard to the analytic form of the solutions of equations (2.1).

We shall show first of all that for any root ρ_k of the characteristic equation there exists a solution of equations (2.1) of the form (3.2) This solution possesses the property that for it the relations

$$x_s(t+\omega) = \rho_k x_s(t). \tag{4.1}$$

are satisfied. In fact, we have:

$$x_s(t+\omega) = e^{\alpha_k t} \cdot e^{\alpha_k \omega} \varphi_s(t+\omega) = \rho_k e^{\alpha_k t} \varphi_s(t) = \rho_k x_s(t).$$

Conversely, if for any solution $x_s(t)$ relations (4.1) are satisfied this solution must necessarily have the form (3.2). This follows immediately from the fact that if (4.1) are satisfied the functions $x_s e^{-\alpha_k t}$ will be periodic and therefore the functions x_s will have the form (3.2).

The problem thus reduces to determining the solution satisfying relations (4.1). This solution, if it exists, must be a linear combination of the fundamental system. We thus have:

$$x_s(t) = \beta_1 x_{s1}(t) + \ldots + \beta_n x_{sn}(t),$$

where β_s are certain constants. Substituting in (4.1) we obtain

$$\sum_{\alpha=1}^{n} \beta_\alpha x_{s\alpha}(t+\omega) = \rho_k \sum_{\alpha=1}^{n} \beta_\alpha x_{s\alpha}(t),$$

and therefore, on the basis of (2.2),

$$\sum_{\alpha,\gamma=1}^{n} \beta_\alpha a_{\gamma\alpha} x_{s\gamma}(t) = \rho_k \sum_{\gamma=1}^{n} \beta_\gamma x_{s\gamma}(t).$$

Equating the coefficients of $x_{s\gamma}(t)$ we obtain the result that the constants β_s satisfy the system of linear homogeneous equations.

$$a_{\gamma 1}\beta_1 + \ldots + (a_{\gamma\gamma} - \rho_k)\beta_\gamma + \ldots + a_{\gamma n}\beta_n = 0 \qquad (4.2)$$
$$(\gamma = 1, \ldots n).$$

Since ρ_k is a root of the characteristic equation the system (4.2) admits at least one solution different from the trivial one $\beta_1 = \ldots = \beta_n = 0$. To each root of the characteristic equation there thus corresponds at least one particular solution of the differential equations (2.1) that has the form (3.2). More than one solution of the form (3.2) may correspond to the root ρ_k. There will evidently be as many of these solutions as the number of independent solutions possessed by the linear system (4.2). There will consequently be p such solutions if the rank of the determinant $D(\rho_k)$ is equal to $n - p$. The rank of this determinant can be less than $n-1$ only in the case where the root ρ_k is multiple. Hence to each simple root of the characteristic equation there corresponds one and only one solution of the form (3.2).

We shall now assume that the root ρ_k is of multiplicity μ independent particular solutions of equations (2.1), breaking down into sets of the form (3.5). This assertion may be considered proved for $\mu = 1$. Hence in order to prove it in the general form we can apply the method of induction as follows: We assume that the assertion has been proved if the multiplicity of the root is equal to $\mu - 1$ and show that it remains valid if the multiplicity is equal to μ.

For this purpose we pass in equations (2.1) from the variables x_s to the variables y_s with the aid of the substitution

$$x_s = \varphi_s y_1 + b_{s2} y_2 + \ldots + b_{sn} y_n, \qquad (4.3)$$

where φ_s are the periodic functions of t that figure in the solution (3.2) and b_{sj} are arbitrary continuous i.e. that the determinant

$$\begin{vmatrix} \varphi_1 & \varphi_2 \cdots \varphi_n \\ b_{12} & b_{22} \ldots b_{2n} \\ \cdot & \cdot \cdot \cdot \cdot \cdot \\ b_{1n} & b_{2n} \ldots b_{nn} \end{vmatrix}$$

does not reduce to zero for any values of t. Such choice of the functions b_{sj} can be made in an infinite number of ways since the functions φ_s can not simultaneously reduce to zero for any values of t. In fact, if for some value $t = T$ all the functions φ_s became zero, then taking this value of t for the initial one we would have two different particular solutions of equations (2.1) with zero initial values: the solution (3.2) and the trivial solution $x_1 = \ldots = x_n = 0$, which is impossible.

The system (2.1) after the transformation (4.3) assumes the form

$$\frac{dy_s}{dt} = q_{s1}y_1 + q_{s2}y_2 + \ldots + q_{sn}y_n, \tag{4.4}$$

where q_{sj} are periodic functions of t of period ω. Since the system (2.1) admits the particular solution (3.2) the transformed system must admit the particular solution

$$y_1 = e^{\alpha_k t}, \quad y_2 = \ldots = y_n = 0. \tag{4.5}$$

Substituting this solution in (4.4) we find that all the coefficients q_{21}, $q_{31}, \ldots,$ q_{n1} are equal to zero while the coefficient q_{11} is equal to α_k. The system (4.4) consequently breaks down into a system

$$\frac{dy_s}{dt} = q_{s2}y_2 + q_{s3}y_3 + \ldots + q_{sn}y_n \quad (s = 2, 3, \ldots, n), \tag{4.6}$$

consisting of n-1 equations and the single equation

$$\frac{dy_1}{dt} = \alpha_k y_1 + q_{12}y_2 + \ldots + q_{1n}y_n. \tag{4.7}$$

Equations (4.6) form a self-contained system determining the n-1 functions $y_2, \ldots, y_n$. After these functions have been found we can find y_1 from equation (4.7) with the aid of a simple quadrature. In particular, if we find q ($q \leq n-1$) linearly independent solutions $y_{2i}(t), \ldots, y_{ni}(t)$ ($i = 1, 2, \ldots, q$) of equations (4.6) the functions $y_{1i}(t), \ldots, y_{ni}(t)$, where $y_{1i}(t)$ are determined by the formulas

$$y_{1i} = e^{\alpha_h t} \int_0^t e^{-\alpha_h t} (q_{12} y_{2i} + \ldots + q_{1n} y_{ni})\, dt, \qquad (4.8)$$

determine q independent solutions of the complete system (4.6) and (4.7). Adding to these solutions the already known solution (4.5) we obtain q + 1 solutions of this system which evidently will likewise be independent.

Let us form the characteristic equation of the complete system (4.6) and (4.7). For this purpose we consider the fundamental system of solutions

$$y_{21}, y_{31}, \ldots, y_{ni} \quad (i=1,2,\ldots,n-1)$$

of equations (4.6), determined by the initial conditions

$$y_{si}(0) = \begin{cases} 1 & (s = i+1), \\ 0 & (s \neq i+1). \end{cases}$$

The system of functions $y_{1i}, y_{2i}, \ldots, y_{ni}$, where y_{1i} are determined by formulas (4.8), together with solution (4.5) form a fundamental system of solutions of equations (4.6) and (4.7) of precisely the form which figures in equation (2.7) of the characteristic equation. Hence the characteristic equation of the system (4.6) and (4.7) can be represented in the form

$$D(\rho) = \begin{vmatrix} \rho_h - \rho & 0 & \ldots & 0 \\ y_{11}(\omega) & y_{21}(\omega) - \rho & \ldots & y_{n1}(\omega) \\ \ldots & \ldots & \ldots & \ldots \\ y_{1,n-1}(\omega) & y_{2,n-1}(\omega) & \ldots & y_{n,n-1}(\omega) - \rho \end{vmatrix} = (\rho_h - \rho) D'(\rho) = 0, \quad (4.9)$$

where $D'(\rho)$ is the characteristic determinant of the system (4.6).

As was shown in sec. 2, the characteristic equation remains invariant to a linear transformation of the variables. Equation (4.9) therefore agrees with the characteristic equation of the initial system (2.1). For the latter system ρ_k is a root of multiplicity μ. Hence it follows from (4.9) that ρ_k is a root of multiplicity $\mu - 1$ of the characteristic equation of system (4.6).

But then, by assumption, there correspond to this root $\mu - 1$ particular solutions of equations (4.6) that break down into sets of the above indicated type. Let us assume for definiteness that there are two sets of this kind. All our considerations however remain valid for any number of sets. Let the first set consist of the l solutions

$$y_{s\alpha} = e^{\alpha_k t} \frac{D^{(\alpha-1)}}{Dt^{\alpha-1}} \left(\frac{t^{l-1}}{(l-1)!} u_{s1} + \ldots + t u_{s,l-1} + u_{sl} \right) \qquad (4.10)$$
$$(s = 2, \ldots, n; \quad \alpha = 1, 2, \ldots, l),$$

and the second set of the m solutions

$$y^*_{s\beta} = e^{\alpha_k t} \frac{D^{(\beta-1)}}{Dt^{\beta-1}} \left(\frac{t^{m-1}}{(m-1)!} v_{s1} + \ldots + t v_{s,m-1} + v_{sm} \right) \qquad (4.11)$$
$$(s = 2, \ldots, n; \quad \beta = 1, 2, \ldots, m).$$

Here u_{sj}, v_{sj} are periodic functions of t and $l + m = \mu - 1$. As was shown above, the functions (4.10) and (4.11) together with the functions

$$\left.\begin{aligned} y_{1\alpha} &= e^{\alpha_k t} \int_0^t e^{-\alpha_k t} (q_{12} y_{2\alpha} + \ldots + q_{1n} y_{n\alpha})\, dt, \\ y^*_{1\beta} &= e^{\alpha_k t} \int_0^t e^{-\alpha_k t} (q_{12} y^*_{2\beta} + \ldots + q_{1n} y^*_{n\beta})\, dt \\ &(\alpha = 1, 2, \ldots, l; \quad \beta = 1, 2, \ldots, m) \end{aligned}\right\} \qquad (4.12)$$

form a system of $\mu - 1$ independent solutions of the equations (4.6) and (4.7).

On the basis of (4.10) and (4.11) the expressions under the integrals in the functions (4.12) do not contain exponential functions and we can write:

$$\left.\begin{aligned} y_{1\alpha} &= e^{\alpha_k t} \int_0^t \frac{D^{(\alpha-1)}}{Dt^{\alpha-1}} \left(\frac{t^{l-1}}{(l-1)!} u_1 + \ldots + t u_{l-1} + u_l \right) dt, \\ y^*_{1\beta} &= e^{\alpha_k t} \int_0^t \frac{D^{(\beta-1)}}{Dt^{\beta-1}} \left(\frac{t^{m-1}}{(m-1)!} v_1 + \ldots + t v_{m-1} + v_m \right) dt \\ &(\alpha = 1, 2, \ldots, l; \quad \beta = 1, 2, \ldots, m), \end{aligned}\right\} \tag{4.13}$$

where u_j , v_j are periodic functions of period ω .

Let $\varphi(t)$ be an arbitrary continuous periodic function of period ω . From the identity

$$\int_0^{t+\omega} \varphi(t)\,dt - \int_0^t \varphi(t)\,dt = \int_t^{t+\omega} \varphi(t)\,dt = \int_0^\omega \varphi(t)\,dt$$

it follows that for the integral of the functions φ to be periodic it is necessary and sufficient that its mean value be equal to zero. In the general case, setting

$$g = \frac{1}{\omega} \int_0^\omega \varphi(t)\,dt, \tag{4.14}$$

we shall have

$$\int_0^t \varphi\,dt = gt + \int_0^t (\varphi - g)\,dt = gt + \psi(t),$$

where $\psi(t)$ is a periodic function. Integrating by parts we readily obtain

$$\int_0^t \frac{t^p}{p!} \varphi(t)\,dt = \frac{g t^{p+1}}{(p+1)!} + \frac{t^p}{p!} \psi(t) - \int_0^t \frac{t^{p-1}}{(p-1)!} \psi(t)\,dt,$$

whence it follows that if P(t) is a polynomial of the p-th degree with periodic coefficients

$$P(t)=\frac{t^p}{p!}\varphi(t)+\frac{t^{p-1}}{(p-1)!}\varphi_2(t)+\ldots+\varphi_p(t),$$

its quadrature will be a polynomial of the (p + 1)-th degree of the form

$$Q(t)=\int_0^t P(t)\,dt=\frac{gt^{p+1}}{(p+1)!}+\frac{t^p}{p!}\psi_2+\ldots+\psi_{p+1} \tag{4.15}$$

where $\psi_2,\ldots,\psi_{p+1}$ are certain periodic functions and g is the constant determined by formula (4.14), i.e. the mean value of the coefficient of the greatest power of the polynomial $P(t)$.

We shall show that the identity holds

$$\frac{D^{(s)}}{Dt^s}\int_0^t P(t)\,dt-\int_0^t \frac{D^{(s)}P}{Dt^s}\,dt=A, \tag{4.16}$$

where A is a certain constant. In fact, differentiating the left hand side of (4.16) with respect to time and remembering that the operators $D^{(s)}/Dt^s$ and d/dt are evidently interchangeable we shall have

$$\frac{d}{dt}\left\{\frac{D^{(s)}}{Dt^s}\int_0^t P(t)\,dt-\int_0^t \frac{D^{(s)}P}{Dt^s}\,dt\right\}=\frac{D^{(s)}}{Dt^s}\left\{\frac{d}{dt}\int_0^t P(t)\,dt\right\}-\frac{D^{(s)}P}{Dt^s}=0,$$

whence the validity of (4.16) follows.

Taking (4.15) and (4.16) into account we find that the expressions (4.13) can be represented in the form

$$y_{1\alpha}=e^{\alpha_k t}\frac{D^{(\alpha-1)}}{Dt^{\alpha-1}}\left(G\frac{t^l}{l!}+U(t)\right)+Ae^{\alpha_k t},$$
$$y^*_{1\beta}=e^{\alpha_k t}\frac{D^{(\beta-1)}}{Dt^{\beta-1}}\left(G^*\frac{t^m}{m!}+U^*(t)\right)+A^*e^{\alpha_k t}$$
$$(\alpha=1,2,\ldots,l;\quad \beta=1,2,\ldots,m),$$

where A and A^* are constants, G and G^* are likewise constants determined by the formulas

$$G=\frac{1}{\omega}\int_0^\omega u_1\,dt,\quad G^*=\frac{1}{\omega}\int_0^\omega v_1\,dt,$$

while U and U^* are polynomials with periodic coefficients, the power of the first of them not exceeding $l-1$ and the power of the second not exceeding $m-1$.

If from the solutions thus obtained we subtract the solution (4.5) multiplied respectively by A and A^* the functions (4.10) and (4.11) do not change while in the functions $y_{1\alpha}$ and $y^*_{1\beta}$ the terms containing A and A^* drop out. Hence without affecting the generality we can put $A = A^* = 0$ and write:

$$\left.\begin{aligned} y_{1\alpha} &= e^{\alpha_k t}\frac{D^{(\alpha-1)}}{Dt^{\alpha-1}}\left(G\frac{t^l}{l!}+U(t)\right),\\ y^*_{1\beta} &= e^{\alpha_k t}\frac{D^{(\beta-1)}}{Dt^{\beta-1}}\left(G^*\frac{t^m}{m!}+U^*(t)\right)\\ &(\alpha=1,\ldots,l;\quad \beta=1,\ldots,m).\end{aligned}\right\}\tag{4.17}$$

Substituting (4.10), (4.11) and (4.17) in (4.3) we obtain for the system (2.1) l solutions of the form

$$x_{s\alpha}=e^{\alpha_k t}\frac{D^{(\alpha-1)}}{Dt^{\alpha-1}}\left(G\varphi_s\frac{t^l}{l!}+X_s(t)\right)\qquad(\alpha=1,\ldots,l)\tag{4.18}$$

and m solutions of the form

$$x^*_{s\beta}=e^{\alpha_k t}\frac{D^{(\beta-1)}}{Dt^{\beta-1}}\left(G^*\varphi_s\frac{t^m}{m!}+X^*_s(t)\right),\tag{4.19}$$

where $X_s(t)$ and $X^*_s(t)$ are certain polynomials with periodic coefficients the degrees of which do not exceed $l-1$ and $m-1$ respectively. Together with the solution (3.2) we thus obtain for the root under consideration $l+m+1=\mu$ independent solutions of equations (2.1).

Let us at first assume that both magnitudes G and G^* are different from zero. For definiteness we shall also assume that $l \geqslant m$. Then, if to solutions (4.18) we add the solution (3.2), first multiplying it by G, we obtain $l+1$ solutions constituting a set. In fact, we evidently have:

$$Ge^{\alpha_k t}\varphi_s(t)=e^{\alpha_k t}\frac{D^{(l)}}{Dt^l}\left(G\varphi_s\frac{t^l}{l!}+X_s(t)\right),$$

and therefore solution (3.2) multiplied by G belongs to the set (4.18) and corresponds to $\alpha = l \cdot 1$. As regards solutions (4.19), combining them with the m last solutions of (4.18) we obtain m new solutions

$$\bar{x}_{s\beta} = G^* x_{s,l-m+\beta} - G x^*_{s\beta} = e^{\alpha_k t} \frac{D^{(\beta-1)}}{Dt^{\beta-1}} \left(G^* \frac{D^{(l-m)}}{Dt^{l-m}} X_s - G X^*_s \right)$$
$$(\beta = 1, 2, \ldots, m),$$

similarly forming a set. In fact, the degree of at least one of the polynomials enclosed in the parentheses, in the expression for $\bar{x}_{s\beta}$, is equal to $m - 1$, for if all these polynomials were of smaller degree then in every case the identities would hold

$$\bar{x}_{sm} = G^* x_{sl} - G x^*_{sm} \equiv 0 \qquad (s = 1, \ldots, n)$$

and consequently not all solutions (4.18) and (4.19) would be independent, which contradicts the assumption.

Thus, in the case considered our assertion in regard to the form of the solutions corresponding to a multiple root is true.

We now assume that $G = 0$ but G^* is different from zero. In this case the degree of at least one of the polynomials $X_s(t)$ is equal to $l - 1$, since in the contrary case all functions x_{sl} would be equal to zero and therefore fewer than l solutions would be contained in (4.18). Hence equations (4.18) form a set of the required form. Adding the solution (3.2), multiplying it first by G^*, to the solutions (4.19) we obtain an additional set. Consequently, as in the preceding case, we shall have μ solutions that break down into two sets. If, finally, G^* is also equal to zero solutions (4.19) form a set and solution (3.2) must be regarded as a separate third set consisting of one solution.

Thus, in all cases our assertions as to the analytic form of the solutions of equations (2.1) can be considered as proved.

It still remains for us only to show that the number of sets of solutions corresponding to a multiple root is exactly equal to p, where $(n - p)$ is the rank of the characteristic determinant for the root under consideration. This assertion can easily be proved in the following manner.

The number of sets of solutions corresponding to the multiple root is equal, evidently, to the number of independent solutions of the form (3.2) (since in each set there is one such solution), which this solution possesses, and this number, as we have seen above, is exactly equal to p.

Thus, the fundamental assertion of the preceding section with regard to the analytic form of the solutions of equations (2.1) has been completely proved.

We shall not here give the proof of the converse proposition.[1]

5. Reductions of Linear Equations with Periodic Coefficients to Equations with Constant Coefficients

It was shown by A.M. Lyapunov that any system of linear equations with periodic coefficents can be transformed with the aid of a linear substitution with periodic coefficients to a system of equations with constant coefficients.

To carry out this transformation let us consider the system of linear equations

$$\frac{dy_s}{dt} + p_{1s}y_1 + \dots + p_{ns}y_n = 0, \tag{5.1}$$

conjugate to the system (2.1).

Let $\rho_1, \rho_2, \dots, \rho_m$ be the roots of the characteristic equation of the system (5.1) (and not of system (2.1) as in the preceding sections) and $\alpha_1, \alpha_2, \dots, \alpha_m$ the corresponding characteristic exponents. We shall here write out each multiple root as many times as the number of sets of solutions that correspond to it. Thus, among the numbers ρ_i there may be equal ones but to each

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[1] This proof can be found in our book, Teoria ustoichivosti dvizheniya (Theory of the Stability of Motion), Gostekhizdat, 1952.

of them corresponds only one set of solutions.

We denote by n_p the number of solutions in the set corresponding to the root ρ_p. Evidently

$$n_1 + n_2 + \ldots + n_m = n.$$

Under these conditions the system (5.1) has n independent solutions of the form

$$\left.\begin{aligned}
&y_{s1}^{(p)} = e^{\alpha_p t}\varphi_{s1}^{(p)}(t),\\
&y_{s2}^{(p)} = e^{\alpha_p t}(t\varphi_{s1}^{(p)}(t) + \varphi_{s2}^{(p)}(t)),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&y_{sn_p}^{(p)} = e^{\alpha_p t}\left(\frac{t^{n_p-1}}{(n_p-1)!}\varphi_{s1}^{(p)}(t) + \ldots + t\varphi_{s,\,n_p-1}^{(p)}(t) + \varphi_{sn_p}^{(p)}(t)\right)\\
&\qquad (s = 1, 2, \ldots, n; \quad p = 1, 2, \ldots, m),
\end{aligned}\right\} \tag{5.2}$$

where $\varphi_{sj}^{(p)}(t)$ are periodic functions of t of period ω. We here use the following system of notation of the solutions: the upper index in $y_{sj}^{(p)}$ denotes the number of the set (number of the root) to which the solution belongs, and the second lower index denotes the number of the solution in the set.

To each solution $y_s(t)$ of the system (5.1) corresponds, as was shown in sec. 3 of the preceding chapter, a first integral of the system (2.1) of the form

$$y_1(t)x_1 + y_2(t)x_2 + \ldots + y_n(t)x_n = \text{const}.$$

Substituting in the above solutions (5.2) we obtain n first integrals of the system (2.1). These integrals have the form

$$\left.\begin{aligned}
&e^{\alpha_p t}y_1^{(p)} = \text{const},\\
&e^{\alpha_p t}(ty_1^{(p)} + y_2^{(p)}) = \text{const},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&e^{\alpha_p t}\left(\frac{t^{n_p-1}}{(n_p-1)!}y_1^{(p)} + \frac{t^{n_p-2}}{(n_p-2)!}y_2^{(p)} + \ldots + y_{n_p}^{(p)}\right) = \text{const}\\
&\qquad (p = 1, 2, \ldots m),
\end{aligned}\right\} \tag{5.3}$$

where there has been put

$$y_j^{(p)} = \varphi_{1j}^{(p)}x_1 + \varphi_{2j}^{(p)}x_2 + \ldots + \varphi_{nj}^{(p)}x_n = \text{const} \tag{5.4}$$
$$(p = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n_p).$$

Relations (5.4) determine the linear substitution with periodic coefficients, the substitution not being singular for any values of t. In fact, the determinant made up of the n^2 functions (5.2) is different from zero since these functions constitute a fundamental system of solutions of linear equations. But this determinant, as can easily be seen, differs by a multiplier

$$e^{(n_1\alpha_1+n_2\alpha_2+\ldots+n_m\alpha_m)t}$$

which never reduces to zero, from the determinant of the substitution (5.4) and therefore the substitution (5.4) is not singular for any values of t.

Having established this, we transform the system (2.1) with the aid of the substitution (5.4). We take into account the fact that expressions (5.3) are first integrals of the system (2.1). Differentiating these integrals with respect to t and equating the derivatives to zero we readily obtain the following differential equations:

$$\left.\begin{array}{l} \dfrac{dy_1^{(p)}}{dt} = -\alpha_p y_1^{(p)}, \\ \dfrac{dy_2^{(p)}}{dt} = -\alpha_p y_2^{(p)} - y_1^{(p)}, \\ \cdots\cdots\cdots\cdots\cdots \\ \dfrac{dy_{n_p}^{(p)}}{dt} = -\alpha_p y_{n_p}^{(p)} - y_{n-1}^{(p)} \\ \quad (p=1, 2, \ldots, m). \end{array}\right\} \tag{5.5}$$

These will be the transformed equations possessing constant coefficients. Thus, with the aid of the nonsingula linear substitution with periodic coefficients (5.4) the system of equations (2.1) is transformed to the system of equations (5.5) with constant coefficients.

The obtained system (5.5) will have complex coefficients, since the magnitudes α_p will in general be complex. Hence, if we wish to deal only with real coefficients further transformations will be necessary. We shall show how this is done.

The magnitude α_p will be complex either when the corresponding root of the characteristic equation is complex or when this root is real but negative.

Let us consider first the former case. Assume that the root ρ_i is complex. Since the coefficients p_{sj} are real, all complex roots of the characteristic equation and all complex roots of the system (5.1) break down into pairs of conjugates. Let ρ_k be the complex conjugate of ρ_i. The solutions corresponding to the root ρ_k, i.e. the functions $y_{sj}^{(k)}$ will be the complex conjugates of the solutions $y_{sj}^{(i)}$, and therefore $n_k = n_i$ and the variables $y_j^{(k)}$ will be the complex conjugate of the variables $y_j^{(i)}$. Let

$$\alpha_i = \lambda_i + \sqrt{-1}\,\mu_i, \qquad \alpha_k = \lambda_k - \sqrt{-1}\,\mu_k,$$
$$y_j^{(i)} = u_j^{(i)} + \sqrt{-1}\,v_j^{(i)}, \qquad y_j^{(k)} = u_j^{(k)} - \sqrt{-1}\,v_j^{(k)}$$

and let us take $u_j^{(i)}$ and $v_j^{(i)}$ as new variables instead of $y_j^{(i)}$ and $y_j^{(k)}$. Then, separating out in the i-th and k-th sets of equations (5.5) the real and imaginary parts, we obtain, in place of the two indicated sets consisting of $n_i = n_k$ equations each possessing complex coefficients, one set consisting of $2n_i$ equations with real coefficients, of the following form

$$\left.\begin{aligned} \frac{du_1^{(i)}}{dt} &= \lambda_i u_1^{(i)} - \mu_i v_1^{(i)}, \\ \frac{dv_1^{(i)}}{dt} &= \lambda_i v_1^{(i)} + \mu_i u_1^{(i)}, \\ \frac{du_j^{(i)}}{dt} &= \lambda_i u_j^{(i)} - \mu_i v_j^{(i)} - u_{j-1}^{(i)}, \\ \frac{dv_j^{(i)}}{dt} &= \lambda_i v_j^{(i)} + \mu_i u_j^{(i)} - v_{j-1}^{(i)} \\ & (j = 2, \ldots, n_i). \end{aligned}\right\} \tag{5.6}$$

Assume now that ρ_i is a negative real number. In this case, taking the arithmetical value of the logarithm, we can write

$$\alpha_i = \frac{1}{\omega}\ln(-\rho_i) + \frac{(2q+1)\pi\sqrt{-1}}{\omega} = \lambda_i + \frac{(2q+1)\pi\sqrt{-1}}{\omega},$$

where q is an integer and the magnitude λ_i is real. The solutions corresponding to the root ρ_i will be obtained as complex. But since the coefficients of the equations (5.1) are real, the real parts of these solutions will likewise be solutions.

Hence to the root ρ_i correspond the solutions

$$\left.\begin{aligned}
&y_{s1}^{(i)} = e^{\lambda_i t}\psi_{s1}^{(i)}(t),\\
&y_{s2}^{(i)} = e^{\lambda_i t}\left(t\psi_{s1}^{(i)}(t) + \psi_{s2}^{(i)}(t)\right),\\
&\dots\dots\dots\dots\dots\\
&y_{sn_i}^{(i)} = e^{\lambda_i (t)}\left(\frac{t^{(n_i-1)}}{(n_i-1)!}\psi_{s1}^{(i)}(t) + \dots + t\psi_{s,n_i-1}^{(i)}(t) + \psi_{sn_i}^{(i)}(t)\right),
\end{aligned}\right\}$$

where

$$\psi_{sj}^{(i)} = \mathrm{Re}\ \left\{\left(\cos\frac{(2q+1)\pi t}{\omega} + \sqrt{-1}\sin\frac{(2q+1)\pi t}{\omega}\right)\varphi_{sj}^{(i)}\right\}, \quad (5.7)$$

and we obtain for this root the differential equations with real coefficients

$$\left.\begin{aligned}
&\frac{dz_1^{(i)}}{dt} = \lambda_i z_1^{(i)},\\
&\frac{dz_j^{(i)}}{dt} = \lambda_i z_j^{(i)} - z_j^{(i-1)}\\
&(j = 2, \dots, n_i).
\end{aligned}\right\} \quad (5.8)$$

Here the variables $z_j^{(i)}$ differ from the variables $y_j^{(i)}$, determined by formulas (5.4), only in that the functions $\varphi_{sj}^{(i)}$ have been replaced by the functions $\psi_{sj}^{(i)}$.

Thus, it can be considered as proven that the system of equations (2.1) can with the aid of a real nonsingular linear transformation be reduced to a system of equations with constant coefficients. If the characteristic equation of the system (5.1) has no real negative roots the coefficients of the substitution will be periodic functions of period ω. If however the characteristic equation has negative real roots the coefficients of the substitution will likewise be periodic functions but the period of these functions will in general be equal to 2ω. This follows directly from the circumstance that the period of the functions (5.7) will in general be 2ω since the multiplier

$$\cos\frac{(2q+1)\pi t}{\omega}+\sqrt{-1}\sin\frac{(2q+1)\pi t}{\omega}.$$

possesses this period.

6. Theorem of Lyapunov on the Roots of the Characteristic Equations of Conjugate Systems. Fundamental Equation of the Reduced System

A system of linear equations with constant coefficients can be considered as a special case of a system with periodic coefficients of arbitrary period ω. Hence for a system of this kind a characteristic equation in the sense of sec. 2 can be constructed. Let us consider what will be the characteristic equation of the system (5.5) of the preceding section, to which the system (2.1) with periodic coefficients was reduced.

For this purpose we note that equations (5.5) admit, evidently, a fundamental system of solutions that break down into m sets such that the first solution in some p-th set is of the form

$$\begin{gathered} y_{11}^{(p)}=e^{-\alpha_p t},\\ y_{21}^{(p)}=-te^{-\alpha_p t},\\ \cdots\cdots\\ y_{n_p 1}^{(p)}=\frac{(-t)^{n_p-1}}{(n_p-1)!}e^{-\alpha_p t},\\ y_{11}^{(k)}=y_{21}^{(k)}=\ldots=y_{n_k 1}^{(k)}=0\\ (k=1,2,\ldots,p-1,\ p+1,\ldots,m), \end{gathered}$$

while the remaining solutions of this set can be obtained from the first by successive differentiation of the coefficients of $e^{-\alpha_p t}$ or, what amounts to the same thing, by the successive application to these coefficients of the operator D/Dt. Thus, for example, the second solution of the above mentioned set is of the form

$$\begin{gathered} y_{12}^{(p)}=0,\\ y_{22}^{(p)}=-e^{-\alpha_p t},\\ \cdots\cdots\\ y_{n_p 2}^{(p)}=\frac{(-t)^{n_p-2}}{(n_p-2)!}e^{-\alpha_p t},\\ y_{12}^{(k)}=y_{22}^{(k)}=\ldots=y_{n_k 2}^{(k)}=0\\ (k=1,\ 2,\ldots,\ p-1,\ p+1,\ldots,m). \end{gathered}$$

Altogether there will be n_p solutions in the p-th set. Thus, for each magnitude $-\alpha_p$ there is obtained a set with n_p solutions. Among the magnitudes $-\alpha_1,\dots,-\alpha_m$ there can be some that are equal since by assumption each of the magnitudes α_p is written out as many times as the number of sets of solutions that correspond to the root ρ_k of the characteristic equation of system (5.1).

The obtained solutions of the system (5.5) will be precisely those that figure in the proposition established in sec. 4. Hence on the basis of the converse proposition we can assert that the magnitudes $-\alpha_p$ are characteristic exponents and therefore the magnitudes $1/\rho_k$ roots of the characteristic equation of the system (5.5). Moreover, from the same proposition it follows that the root $1/\rho_k$ of the characteristic equation of the system (5.5) is of the same multiplicity as the root of the characteristic equation of the system (5.1) and that to these roots in both systems corresponds the same number of sets with the same number of solutions in each set. But the system (5.5) was obtained from the system (2.1) by a nonsingular transformation with periodic coefficients, which, as was shown in sec. 5, does not change the roots of the characteristic equation. Consequently the roots of the characteristic equation of system (5.5) agree with the roots of the characteristic equation of system (2.1) and we arrive at the following theorem of Lyapunov:

THEOREM: IF ρ_k IS A ROOT OF THE CHARACTERISTIC EQUATION OF A SYSTEM OF LINEAR EQUATIONS WITH PERIODIC COEFFICIENTS THE MAGNITUDE $1/\rho_k$ IS A ROOT OF THE CHARACTERISTIC EQUATION OF THE CONJUGATE SYSTEM. THE MULTIPLICITY OF THE TWO ROOTS, THE NUMBER OF SETS OF SOLUTIONS CORRESPONDING TO THEM, AND THE NUMBER OF SOLUTIONS IN THE CORRESPONDING SETS ARE THE SAME.

Let us now consider the FUNDAMENTAL equation of system (5.5). It evidently has the form

$$D(\lambda)=\begin{vmatrix} \boxed{M_1} & & & \\ & \boxed{M_2} & & \\ & & \ddots & \\ & & & \boxed{M_m} \end{vmatrix}=0,$$

where

$$M_p = \begin{vmatrix} -a_p-\lambda & 0 & \dots & 0 & 0 \\ -1 & -a_p-\lambda & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & -1 & -a_p-\lambda \end{vmatrix}$$

From this the following theorem immediately follows:

THEOREM. IN TRANSFORMING A SYSTEM OF LINEAR EQUATIONS WITH PERIODIC COEFFICIENTS INTO A SYSTEM OF EQUATIONS WITH CONSTANT COEFFICIENTS THE ROOTS OF THE FUNDAMENTAL EQUATION WITH CONSTANT COEFFICIENTS ARE THE CHARACTERISTIC EXPONENTS OF THE INITIAL SYSTEM.

7. Some General Theorems on Stability of Motion

Let us assume that in investigating the stability of some motion we arrived at differential equations of the disturbed motion of the form

$$\frac{dx_s}{dt} = a_{s1}x_n + \dots + a_{sn}x_n + \varphi_s(t, x_1, \dots, x_n) \qquad (7.1)$$
$$(s = 1, \dots, n),$$

in which the coefficients a_{sj} of the equations of the first approximation

$$\frac{dx_s}{dt} = a_{s1}x_1 + \dots + a_{sn}x_n \qquad (7.2)$$

are constant. With respect to terms of higher order of smallness φ_s we shall only assume that these functions, which are defined in the region

$$t \geqslant 0, \quad |x_s| \leqslant H,$$

where H is a constant, are continuous with respect to t and satisfy with respect to the variables $x_1, \ldots, x_n$ the Cauchy-Lipschitz conditions

$$|\varphi_s(t, x_1', \ldots, x_n') - \varphi_s(t, x_1'', \ldots, x_n'')| < L \sum_{\alpha=1}^{n} |x_\alpha' - x_\alpha''|, \quad (7.3)$$

in which the coefficient L does not depend on t and is SUFFICIENTLY SMALL.

The following fundamental theorem of Lyapunov holds, which we here present without proof.[1]

THEOREM. IF THE DIFFERENTIAL EQUATIONS OF THE DISTURBED MOTION ARE OF THE FORM (7.1) AND IF ALL THE ROOTS OF THE FUNDAMENTAL EQUATION OF THE SYSTEM (7.2) HAVE NEGATIVE REAL PARTS, THE UNDISTURBED MOTION FOR SUFFICIENTLY SMALL L IS ASYMPTOTICALLY STABLE. IF THE INDICATED FUNDAMENTAL EQUATION HAS AT LEAST ONE ROOT WITH POSITIVE REAL PART THE UNDISTURBED MOTION FOR SUFFICIENTLY SMALL L IS UNSTABLE. IF THE FUNDAMENTAL EQUATION, NOT HAVING ROOTS WITH POSITIVE REAL PART, HAS ROOTS THE REAL PARTS OF WHICH ARE EQUAL TO ZERO, THE QUESTION OF STABILITY IS DECIDED BY THE TERMS φ_s, AS FOLLOWS: THE FUNCTIONS φ_s CAN BE CHOSEN IN SUCH MANNER THAT THE COEFFICIENT L IS ARBITRARILY SMALL AND THE UNDISTURBED MOTION IS STABLE OR UNSTABLE AT WILL.

We shall apply this theorem to the investigation of the stability of periodic motions. Let there be given a dynamic system described by the equations

$$\frac{dy_s}{dt} = Y_s(t, y_1, \ldots, y_n) \qquad (s = 1, \ldots, n), \quad (7.4)$$

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[1] The proof can be found in the books: Lyapunov A.M., Obshchaya zadacha ob ustoichivosti dvizheniya,(General Problem of the Stability of Motion), Gostekhizdat, 1950; Chetaev N.G., Ustoichivost dvizheniya, (Stability of Motion), 2nd ed., Gostekhizdat, 1955 ; Malkin ., Teoriya ustoichivosti dvizheniya, (Theory of the Stability of Motion), Gostekhizdat, 1952.

where the functions Y_s are periodic with respect to t with period ω, and let $y_s = f_s(t)$ be some periodic solution of this system, whose stability is investigated. We shall assume that the functions Y_s are continuous with respect to t and admit, in a certain region in which the solution f_s lies, continuous partial derivatives of the first order with respect to the variables $y_1, \ldots, y_n$. Forming the equations of the disturbed motion we obtain

$$\begin{aligned}\frac{dx_s}{dt} &= Y_s(t, f_1 + x_1, \ldots, f_n + x_n) - Y_s(t, f_1, \ldots, f_n) = \\ &= p_{s1}x_1 + \ldots + p_{sn}x_n + \varphi_s(t, x_1, \ldots, x_n). \qquad (7.5)\end{aligned}$$

Here x_s denote the disturbances $y_s - f_s(t)$, the functions p_{sj} are determined by the formulas

$$p_{sj} = \frac{\partial Y_s(t, f_1, \ldots, f_n)}{\partial f_j}$$

and the functions φ_s satisfy the relations

$$\lim \frac{\varphi_s(t, x_1, \ldots, x_n)}{|x_1| + \ldots + |x_n|} = 0 \quad \text{for} \quad (|x_1| + \ldots + |x_n|) \to 0. \qquad (7.6)$$

The linear equations

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n \qquad (7.7)$$

are the equations in variations of the disturbed motion investigated.

Since the functions Y_s in the neighborhood of the undisturbed motion have continuous derivatives with respect to $y_1, \ldots, y_n$, the functions φ_s satisfy, in the neighborhood of the origin, the Cauchy-Lipschitz conditions (7.3). In virtue of (7.6) the coefficient L for sufficiently small disturbances will here be as small as we please.

Thus the functions φ_s possess all the required properties for the applicability of the theorem of Lyapunov presented above. As regards the magnitudes p_{sj}, they will not be constants but periodic functions of t of ω. But, as was shown in sec. 5, the system (7.7) can, by a nonsingular linear substitution, be reduced to a system of linear equations with constant coefficients. If this substitution is applied to the nonlinear equations (7.5) they assume the form (7.1) and the new nonlinear terms will possess the same properties as the original. With such substitution the probelm of stability with respect to the new variables is equivalent to the same problem with respect to the old variables. Moreover, on the basis of the theorem proved in the preceding section, the fundamental equation of the system of the first approximation of the transformed equations will have as its roots the characteristic exponents of the system (7.7). We shall call the characteristic exponents of the system (7.7), i.e. the equations in variations for the periodic solution $f_s(t)$ under consideration, the characteristic exponents of this SOLUTION. From what was said above there is then obtained the following theorem, which similarly belongs to A.M. Lyapunov:

THEOREM. IF ALL CHARACTERISTIC EXPONENTS OF THE UNDISTURBED PERIODIC MOTION HAVE NEGATIVE REAL PARTS THIS MOTION IS ASYMPTOTICALLY STABLE. IF AT LEAST ONE OF THESE CHARACTERISTIC EXPONENTS HAS A POSITIVE REAL PART THE UNDISTURBED MOTION IS UNSTABLE. IF CERTAIN CHARACTERISTIC EXPONENTS HAVE REAL PARTS EQUAL TO ZERO, BUT NONE OF THEM HAS A POSITIVE REAL PART, THEN TO DECIDE THE PROBLEM OF STABILITY IT IS NECESSARY TO CONSIDER THE NONLINEAR TERMS IN THE EQUATIONS OF THE DISTURBED MOTION.

Let $\rho_1,\ldots,\rho_n$ be the roots of the characteristic equation of the system (7.7). Since the characteristic exponents $\alpha_1,\ldots,\alpha_n$ of this system are defined by the formulas

$$\alpha_s = \frac{1}{\omega}\ln\rho_s,$$

the preceding theorem may also be formulated as follows:

If all the roots of the characteristic equation of the system in variations of the periodic motion considered have moduli less than unity the investigated periodic motion is unstable. If some of the roots have moduli equal to unity and the other moduli less than unity the problem of the stability of the investigated periodic motion is decided by the nonlinear terms in the equations of the disturbed motion.

From the analytic form, established in sec. 3 and 4, of the solutions of the system (7.7) it follows that if all its characteristic exponents have negative real parts all its solutions approach zero, as the time increases without limit, like exponential functions and therefore the undisturbed motion is in the first approximation asymptotically stable. If at least one of the characteristic exponents of the system (7.7) has a positive real part the solutions of system (7.7) that correspond to this exponent are unbounded and the undisturbed motion is in the first approximation unstable. Hence, the theorems of Lyapunov show that in both these cases the answer to the problem of stability in considering the exact equations of the disturbed motion is obtained in the same way as in the first approximation.

The situation is otherwise in the case where the system (7.7) does not have characteristic exponents with positive real part but has characteristic exponents with real parts equal to zero. The solutions of the system (7.7) corresponding to these exponents can be bounded as well as unbounded and therefore in the first approximation the undisturbed motion can be stable as well as unstable. In fact, if the characteristic exponent with zero real part is simple or if it is multiple but the number of sets of solutions corresponding to it is equal to its multiplicity then all these solutions will be bounded. In the contrary case for the characteristic exponent under consideration αi there will exist a solution of the form

$$x_s = e^{i\alpha t}(t^p\varphi_{s1}(t) + \dots + t\varphi_{sp}(t) + \varphi_{s0}(t)),$$

where φ_{sj} are periodic functions, which evidently is not bounded. Thus, in the case under consideration the undisturbed motion in the first approximation can be stable as well as unstable. Moreover, the terms of higher order in the equations of the disturbed motion can be disposed of in such manner as to obtain stability or instability AT WILL. In other words, the answers to the problem of stability in considering the exact equations of the disturbed motion and in considering the equations of the first approximation may not agree.

The cases where the equations of the first approximation do not decide the question of stability we shall denote as CRITICAL.

8. Theorem of Lyapunov on the Characteristic Equation of Canonical Systems

According to the above considerations, for solving the question of stability of periodic motion it is necessary first of all to determine the roots of the characteristic equation of the corresponding system in variations. The coefficients of this equation can be found only approximately since according to sec. 2 they are expressed in terms of the fundamental system of solutions of the equations in variations. Below will be indicated several methods of approximately computing these coefficients, that enable a relatively simple determination of the stability conditions for the problems which we consider.

In certain cases, however, from the very form of the linear equations with periodic coefficients, it is possible to draw certain conclusions on the characteristic equation. One of the most important cases of this kind will be that in which the system of linear equations considered has the canonical form

$$\frac{dx_i}{dt}=\frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt}=-\frac{\partial H}{\partial x_i} \qquad (i=1, \ldots, n). \tag{8.1}$$

where $H(t, x_1,\ldots,x_n, y_1,\ldots,y_n)$ is a quadratic form of the variables $x_1,\ldots x_n$, $y_1,\ldots y_n$, whose coefficients are continuous periodic functions of t of period ω. In greater detail this system can be written in the following manner:

$$\left.\begin{aligned}\frac{dx_i}{dt}&=\sum_{\alpha=1}^{n}\frac{\partial^2 H}{\partial y_i\,\partial x_\alpha}x_\alpha+\sum_{\alpha=1}^{n}\frac{\partial^2 H}{\partial y_i\,\partial y_\alpha}y_\alpha,\\ \frac{dy_i}{dt}&=-\sum_{\alpha=1}^{n}\frac{\partial^2 H}{\partial x_i\,\partial x_\alpha}x_\alpha-\sum_{\alpha=1}^{n}\frac{\partial^2 H}{\partial x_i\,\partial y_\alpha}y_\alpha.\end{aligned}\right\} \tag{8.2}$$

It is easily seen that the equations in variations will have the canonical form if there is investigated the stability of an arbitrary motion likewise described by canonical equations of the form

$$\frac{du_i}{dt}=\frac{\partial K}{\partial v_i}, \qquad \frac{dv_i}{dt}=-\frac{\partial K}{\partial u_i} \tag{8.3}$$

(where K in general is not a quadratic form).

In fact, forming for equations (8.3) the equations in variations we obtain

$$\left.\begin{aligned} \frac{dx_i}{dt} &= \sum_{\alpha=1}^{n}\left(\frac{\partial^2 K}{\partial v_i\,\partial u_\alpha}\right)x_\alpha + \sum_{\alpha=1}^{n}\left(\frac{\partial^2 K}{\partial v_i\,\partial v_\alpha}\right)y_\alpha, \\ \frac{dy_i}{dt} &= -\sum_{\alpha=1}^{n}\left(\frac{\partial^2 K}{\partial u_i\,\partial u_\alpha}\right)x_\alpha - \sum_{\alpha=1}^{n}\left(\frac{\partial^2 K}{\partial u_i\,\partial v_\alpha}\right)y_\alpha, \end{aligned}\right\} \tag{8.4}$$

where the derivatives in the parentheses are computed for the undisturbed motion. Equations (8.4) have, as is easily seen, the form (8.1) , the coefficients of the quadratic form H being the corresponding partial derivatives of the second order of the function K , computed for the undisturbed motion.

The following theorem of Lyapunov holds true. [1]

THEOREM. LET ρ BE A ROOT OF THE CHARACTERISTIC EQUATION OF THE SYSTEM (8.1). THEN, IF $\rho = \pm 1$ THE MULTIPLICITY OF THIS ROOT WILL NECESSARILY BE EVEN. IF $\rho \neq \pm 1$ AND THIS ROOT HAS A MULTIPLICITY m AND THERE CORRESPOND TO IT p SETS OF SOLUTIONS, THE MAGNITUDE $1/\rho$ WILL LIKEWISE BE A ROOT OF THE CHARACTERISTIC EQUATION AND THIS ROOT WILL HAVE THE SAME MULTIPLICITY m AND TO IT WILL CORRESPOND THE SAME NUMBER p OF SETS OF SOLUTIONS.

PROOF. Let us consider the linear system conjugate to (8.1). This system has the form

$$\left.\begin{aligned} \frac{d\xi_i}{dt} &= -\sum_{\alpha=1}^{n}\frac{\partial^2 H}{\partial y_\alpha\,\partial x_i}\xi_\alpha + \sum_{\alpha=1}^{n}\frac{\partial^2 H}{\partial x_\alpha\,\partial x_i}\eta_\alpha, \\ \frac{d\eta_i}{dt} &= -\sum_{\alpha=1}^{n}\frac{\partial^2 H}{\partial y_\alpha\,\partial y_i}\xi_\alpha + \sum_{\alpha=1}^{n}\frac{\partial^2 H}{\partial x_\alpha\,\partial y_i}\eta_\alpha. \end{aligned}\right\} \tag{8.5}$$

Let ρ be some root of multiplicity m of the characteristic equation of the system (8.1). We assume at first that $\rho \neq \pm 1$. On the basis of the theorem on the roots of the characteristic equations of conjugate systems (sec. 6) the magnitude $1/\rho$ will be a root of multiplicity m of the characteristic equation of the system (8.5). Hence this system has m independent solutions of the form

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[1] We here present the theorem of Lyapunov in a somewhat more extended formulation that that given by Lyapunov. The proof given in the text likewise differs from the proof of Lyapunov.

$$\xi_{ij}=e^{-\alpha t}U_{ij}(t), \quad \eta_{ij}=e^{-\alpha t}V_{ij}(t) \quad \left(\alpha=\frac{1}{\omega}\ln\rho\right)$$
$$(j=1, \ldots, m),$$

that break down into a certain number of sets of known form. Here U_{ij}, V_{ij} are certain polynomials in t with periodic coefficients.

But the system (8.5), as is immediately evident from its structure, goes over into the system (8.1) if the magnitudes ξ_i are replaced by the magnitudes y_i and the magnitudes η_i by the magnitudes $-x_i$. Hence, if $\xi_i(t)$ and $\eta_i(t)$ are a solution of the system (8.5) the functions $x_i = -\eta_i(t)$, $y_i = \xi_i(t)$ determine the solution of the system (8.1). From this it immediately follows that the system (8.1) has m independent particular solutions

$$x_{ij}=-e^{-\alpha t}V_{ij}(t), \quad y_{ij}=e^{-\alpha t}U_{ij}(t)$$

and therefore the magnitude $-\alpha$ is a characteristic exponent while $1/\rho$ is a root of the characteristic equation of this system with a multiplicity not less than m. This multiplicity cannot evidently be greater than m since otherwise, by applying the proposition just proven to the root $1/\rho$, we obtain the result that, contrary to the assumption, the multiplicity of the root ρ exceeds m.

From our considerations it follows at once that the same number of sets of solutions and the same number of solutions in each set correspond to the root $1/\rho$ as correspond to the root ρ.

Thus, the theorem has been proven for each root different from $+1$ or -1. In order to complete the proof of the theorem it is sufficient to establish that if the characteristic equation of the system (8.1) has a root equal to $+1$ the multiplicity of such root is necessarily even and that the same holds true for a root equal to -1. To show this we note first of all that the sum of the multiplicities of the roots equal to ± 1 will necessarily be an even number, since on the basis of what has been proven the sum of the multiplicities of all the roots different from ± 1 will be even and the order $2n$ of the characteristic equation is likewise even.

Further, the product of all the roots of the characteristic equation on the basis of (2.8) is equal to the magnitude

$$e^{\int_0^\omega \sum p_{ss}\,dt}$$

But in the case considered

$$\sum p_{ss} = \sum_{i=1}^{n} \left(\frac{\partial^2 H}{\partial y_i\,\partial x_i} - \frac{\partial^2 H}{\partial x_i\,\partial y_i} \right) = 0$$

and therefore the product of all the roots of the characteristic equation is equal to $+1$. But since the product of all the roots different from ± 1, by what has been proven, is equal to 1, the prodcut of the roots equal to ± 1 is likewise equal to 1. Consequently, if the characteristic equation has a root equal to -1 the multiplicity of this root will necessarily be even. The same will then be true also with respect to a root equal to $+1$, if such root exists.

Thus the theorem has been completely proved.

From the theorem proved it follows that for equations of the form (8.1) stability can take place only in the case where all the roots of the characteristic equation have moduli equal to unity. But this case belongs to the critical cases and for a final judgment of the stability it is necessary to consider terms of higher orders in the equations of the disturbed motion.

9. Theorem of Andronov and Vitt on the Stability of the Periodic Motions of Autonomous Systems

We shall note another important case where it is possible beforehand to draw certain conclusions in regard to the characteristic equation of the system in variations.

Let us assume that it is required to investigate the stability of the periodic solution

$$x_s = \varphi_s(t) \tag{9.1}$$

of period ω of a certain AUTONOMOUS system. We shall show that at least one characteristic exponent of this solution has a real part equal to zero.

In fact, since the system considered is autonomous, the solution (9.1) belongs to a family depending on a single arbitrary constant h, which can be added to t. But then, as was shown in sec. 1, the equations in variations for the solution (9.1) admit the particular solution

$$\left(\frac{d\varphi_s(t+h)}{dh}\right)_{h=0} = \frac{d\varphi_s(t)}{dt}.$$

This solution is evidently periodic, whence it follows that at least one root of the characteristic equation of the system in variations is equal to 1, and therefore the corresponding characteristic exponent is equal to zero (or to the magnitude $\pm 2\pi pi/\omega$, where p is an integer, on account of the multiple valued character of the logarithmic function).

Let us assume that this characteristic exponent is simple and that the real parts of the remaining characteristic exponents are negative. Then according to the theorem of Lyapunov the question of the stability of the motion must be decided by the terms of higher order in the equations of the disturbed motion. However, in the case considered these terms of higher order are not entirely arbitrary. The circumstance that the solution (9.1) belongs to a family depending on an arbitrary constant imposes certain relations not only on the equations in variations but also on the terms of higher order in the equations of the disturbed motion. These relations are precisely such that the following theorem, established by A. Andronov and A. Vitt [1], holds, and which we here present without proof.

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[1] Andronov A. and Vitt A., Ob ustoichivosti po Lyapunov (On Stability According to Lyapunov), Zhurn. eksp. i teor. fiziki, vol. 3, no. 5, 1933.

THEOREM. IF THE CHARACTERISTIC EXPONENT WITH REAL PART EQUAL TO ZERO OF SOME PERIODIC MOTION OF AN AUTONOMOUS SYSTEM IS SIMPLE AND IF THE REAL PARTS OF THE REMAINING CHARACTERISTIC EXPONENTS OF THIS MOTION ARE NEGATIVE THE MOTION IS STABLE. [2]

10. Approximate Computation of the Roots of the Characteristic Equation by the Method of Expanding in Powers of a Parameter

Let there be given a system of linear equations with periodic coefficients of period ω

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n \quad (s = 1, \ldots, n). \tag{10.1}$$

We denote by $x_{s1}(t),\ldots,x_{sn}(t)$ the fundamental system of solutions of these equations determined by the initial conditions

$$x_{sj}(0) = \delta_{sj}, \tag{10.2}$$

where δ_{sj} is the Kronecker symbol. Then, as was shown in sec. 2, the characteristic equation

$$\rho^n + A_1\rho^{n-1} + \ldots + A_{n-1}\rho + A_n = 0 \tag{10.3}$$

of this system can be represented in the form

$$D(\rho) = \begin{vmatrix} x_{11}(\omega) - \rho & x_{21}(\omega) & \ldots & x_{n1}(\omega) \\ x_{12}(\omega) & x_{22}(\omega) - \rho & \ldots & x_{n2}(\omega) \\ \ldots & \ldots & \ldots & \ldots \\ x_{1n}(\omega) & x_{2n}(\omega) & \ldots & x_{nn}(\omega) - \rho \end{vmatrix} = 0. \tag{10.4}$$

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[2] The proof of this theorem can be found in our book Teoriya ustoichivosti dvizheniya (Theory of the Stability of Motion), Gostekhizdat, 1952.

The coefficient A_n of the characteristic equation can at once be found by the formula (2.8):

$$(-1)^n A_n = e^{\int\limits_0^\omega \sum\limits_{s=1}^n p_{ss}\, dt} \qquad (10.5)$$

As regards the remaining coefficients A_i, it is necessary for their determination to know the fundamental system of solutions x_{sj}. But, as is seen from (10.4), it is not necessary to know this fundamental system for all values of t but only for one value $t = \omega$. Moreover, the conditions of stability are determined by inequalities and it is therefore sufficient to know only the approximate values of the coefficients of the characteristic equation. All this permits, for determining these coefficients, making use of the various approximate methods of integration. In the present section we shall consider one such method that is based on the expansion of the coefficients A_i in powers of a certain parameter.

In all the problems considered in this book the initial differential equations of motion and their investigated periodic solutions depend on a parameter μ. As a result, the coefficients p_{sj} of the equations (10.1) will likewise depend on the parameter μ. If the initial equations of motion and the periodic solution under consideration are analytic functions of μ the coefficients p_{sj} will be analytic functions of μ.

We shall make the more general assumption that the coefficients p_{sj} depend on p parameters $\mu_1, \ldots, \mu_p$ with respect to which they are analytic in the region defined by the inequalities

$$|\mu_i| \leqslant E_i \qquad (i = 1, \ldots, p), \qquad (10.6)$$

where E_i are certain positive numbers. We assume further that the period ω does not depend on these parameters.

Then, as is known, in any solution $x_s = x_s(t, \mu_1, \ldots \mu_p)$ of equations (10.1), the initial values of which do not depend on the parameters, the functions $x_s(t, \mu_1, \ldots \mu_p)$

will likewise by analytic with respect to μ_i in the same region (10.6) . In particular, the solutions x_{sj} that figure in the form (10.4) of the characteristic equation will also be analytic. This at once leads us to the following theorem of Lyapunov:

THEOREM. IF THE COEFFICIENTS p_{sj} OF THE SYSTEM (10.1) ARE ANALYTIC FUNCTIONS OF THE PARAMETERS $\mu_1,\ldots,\mu_p$ IN THE REGION (10.6) the coefficients A_i of the CHARACTERISTIC EQUATION (10.3) WILL BE ANALYTIC FUNCTIONS OF $\mu_1,\ldots,\mu_p$ IN THE REGION (10.6).

It is here necessary to note that the region of analyticity of the coefficients of the characteristic equation coincides with the region of analyticity of the coefficients of the investigated differential equations. In particular, if the coefficients of the investigated equations are integral functions of the parameters the coefficients of the characteristic equations will likewise be integral functions of the parameters.

The above proven theorem can be made use of for approximately computing the coefficients of the characteristic equation. We shall show how this is done.

For this purpose we assume that the coefficients of the system (10.1) depend only on one parameter μ, so that we can write

$$p_{si}=q_{si}(t)+\mu p_{si}^{(1)}(t)+\mu^2 p_{si}^{(2)}(t)+\ldots,$$

where $q_{si}(t)$, $p_{si}^{(1)}(t),\ldots$ are continuous periodic functions of period ω and the series converge for $\mu\leq E$.

Let us consider the fundamental system of solutions $x_{sj}(t,\mu)$ of equations (10.1) determined by the initial conditions.

$$x_{ss}(0,\mu)=1,\quad x_{sj}(0,\mu)=0\quad (s\neq j). \tag{10.7}$$

As has been shown above, we can write:

$$x_{sj} = x_{sj}^{(0)}(t) + \mu x_{sj}^{(1)}(t) + \dots, \tag{10.8}$$

where the series for all values of t converge in the region $\mu \leq E$. The initial conditions (10.7) give:

$$\left.\begin{array}{ll} x_{ss}^{(0)}(0) = 1, \quad x_{sj}^{(0)}(0) = 0 & (s \neq j), \\ x_{sj}^{(1)}(0) = x_{sj}^{(2)}(0) = \dots = 0 & (s \neq j,\ s = j). \end{array}\right\} \tag{10.9}$$

Substituting the series (10.8) in equations (10.1) and equating the coefficients of like powers of μ we obtain for determinging the unknown functions $x_{sj}^{(0)}$, $x_{sj}^{(1)}$, ... the following system of differential equations:

$$\left.\begin{array}{l} \dfrac{dx_{sj}^{(0)}}{dt} = q_{s1}x_{1j}^{(0)} + \dots + q_{sn}x_{nj}^{(0)}, \\ \dfrac{dx_{sj}^{(1)}}{dt} = q_{s1}x_{1j}^{(1)} + \dots + q_{sn}x_{nj}^{(1)} + \sum_{\alpha=1}^{n} p_{s\alpha}^{(1)}x_{\alpha j}^{(0)}, \\ \dfrac{dx_{sj}^{(k)}}{dt} = q_{s1}x_{1j}^{(k)} + \dots + q_{sn}x_{nj}^{(k)} + \sum_{\alpha=1}^{n}\sum_{\beta=0}^{k-1} p_{s\alpha}^{(k-\beta)}x_{\alpha j}^{(\beta)} \\ \qquad\qquad (k = 2, 3, \dots). \end{array}\right\} \tag{10.10}$$

All these systems of nonhomogeneous linear equations have the same homogeneous part. Assume that we can integrate in closed form the homogeneous system of equations

$$\frac{dy_s}{dt} = q_{s1}y_1 + \dots + q_{sn}y_n,$$

into which system (10.1) goes over for $\mu = 0$.

Equations (10.8) then make it possible to determine successively all the functions $x_{sj}^{(k)}$, starting with $k = 0$. The initial conditions (10.9) make them entirely definite.

Hence the solutions (10.8) can be computed with any desired degree of accuracy. Setting $t = \omega$ in these solutions and substituting in (10.4) we obtain approximate values of the coefficients of the characteristic equation.

We have thus obtained a method for the approximate computation of the coefficients of the characteristic equation for the case where the investigated equations contain analytically a certain parameter μ and for $\mu = 0$ are integrated in closed form. This procedure is particularly convenient when the magnitude μ is small, i.e. when the system under consideration differs little from the one to be accurately integrated. In this case to compute the coefficients of the characteristic equation it is possible to restrict oneself to a small number of approximations.

We meet with precisely a case of this kind when we investigate the stability of the periodic motions of quasilinear systems if the equations of the oscillations are analytic with respect to x_j and μ.

In fact, let the equations of the oscillations have the form

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n + F_s(t) + \mu f_s(t, x_1, \ldots, x_n, \mu),$$

where a_{sj} are constants and the functions f_s analytic with respect to μ and x_j. Forming the equations in variations for some periodic solution $x_s = \varphi_s(t,\mu)$ of these equations, analytic with respect to μ, we obtain:

$$\frac{dy_s}{dt} = a_{s1}y_1 + \ldots + a_{sn}y_n + \mu(f_{s1}y_1 + \ldots + f_{sn}y_n),$$

where

$$f_{sj} = \frac{\partial f_s(t, \varphi_1, \ldots, \varphi_n, \mu)}{\partial \varphi_j}$$

These equations are analytic with respect to μ and for $\mu = 0$ reduce to a system of linear equations with constant coefficients.

Thus, for the case of quasilinear systems with an analytic characteristic of the nonlinearity the method described of determining the coefficients of the characteristic equation leads to computations that can be practically carried out. We have already by this method established in sec. 14 of chapter I the criteria of stability of a quasilinear system with one degree of freedom. This method possesses considerable disadvantages.

It leads to cumbersome computations, especially in resonance cases. It is applicable only to analytic equations, and this makes it insufficient for the problems considered in this book. In the following section we present another method of solving the problem, that is applicable to analytic as well as to nonanalytic equations. Moreover, it immediately gives the characteristic exponent and not the coefficients of the characteristic equation and leads to more simple computations.

11. A Second Method of Approximate Computation of the Roots of the Characteristic Equation

Let us consider the system of linear equations with periodic coefficients

$$\frac{dx_s}{dt} = (a_{s1} + \mu f_{s1}(t, \mu))\, x_1 + \ldots + (a_{sn} + \mu f_{sn}(t, \mu))\, x_n \qquad (11.1)$$

$$(s = 1, \ldots, n),$$

where the functions $f_{sj}(t, \mu)$ for sufficiently small values of μ are continuous and periodic with respect to t with period ω and admit continuous derivatives of the first order with respect to μ.[1] The coefficients a_{sj} we shall assume as constant. We are thus considering systems that differ little from systems with constant coefficients. To systems of this kind, as we have seen in the preceding section, is reduced the investigation of the stability of the solutions of quasilinear systems. The more general systems where the coefficients a_{sj} are periodic functions of t similarly reduce to systems of this kind provided the system

$$\frac{dx_s^0}{dt} = a_{s1}x_1^0 + \ldots + a_{sn}x_n^0 \qquad (11.2)$$

can be integrated in closed form. In fact, in this case it is possible with the aid of a linear transformation that can be practically carried out to reduce the system (11.2) to one with constant coefficients. We shall thus assume that the coefficients a_{sj} in (11.1) and in (11.2) are constant.

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[1] This condition may be weakened

We saw in the preceding section that if the coefficients f_{sj} are analytic functions of μ the coefficients of the characteristic equation will be analytic functions of μ. The same considerations show that in the case now being considered the coefficients of the characteristic equations will be differentiable functions of μ and therefore the roots of this equation, and together with them the characteristic exponents, will in any case be continuous functions of μ. We can therefore assume that the characteristic exponents of the system (11.1) are of the form

$$\alpha_i = \lambda_i + \overline{a}_i(\mu). \tag{11.3}$$

where $\bar{a}_i$ are certain continuous functions of μ that reduce to zero for $\mu = 0$, and λ_i are the values of the characteristic exponents for $\mu=0$, i.e. the characteristic exponents of the system (11.2). But the characteristic exponents of a system of equations with constant coefficients are the roots of its fundamental equation.

Hence the magnitudes λ_i are roots of the equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0. \tag{11.4}$$

These roots can be taken as the zeroth approximation of the characteristic exponents. For the problem of stability we shall be interested only in the signs of the real parts of the characteristic exponents. These signs for sufficiently small value of μ will agree with the signs of the real parts of the magnitudes λ_i provided they are different from zero. Hence, if the real part of some magnitude λ_i is different from zero then for the problem of stability a more accurate value of the corresponding characteristic exponent need not be computed. But if among the magnitudes λ_i there are such whose real parts are equal to zero the corresponding characteristic exponents must be more accurately computed. We shall show how this can be done.

If the magnitude α_i, determined by formula (11.3), is a characteristic exponent of the system (11.1) this system must have at least one solution of the form

$$x_s = e^{(\lambda_i + \bar{a}_i)t}\,\varphi_s(t),$$

where φ_s is a periodic function of period ω. Hence, if in equations (11.1) the change of variables is made

$$x_s = e^{(\lambda_i+\bar{a}_i)t}\, y_s,$$

the transformed system

$$\frac{dy_s}{dt} = (a_{s1}+\mu f_{s1})\, y_1 + \ldots + (a_{sn}+\mu f_{sn})\, y_n - (\lambda_i+\bar{a}_i)\, y_s \quad (11.5)$$

must admit a periodic solution. It is this condition which serves to determine the magnitude $\bar{a}_i$.

We shall seek to obtain the periodic solution of the system (11.5) by the method of successive approximations. We shall take as the k-th approximation $y_s^{(k)}$ of the magnitudes y_s the periodic solution of the equations

$$\frac{dy_s^{(k)}}{dt} = a_{s1}y_1^{(k)} + \ldots + a_{sn}y_n^{(k)} - \lambda_i y_s^{(k)} + \\ + \mu\,(f_{s1}y_1^{(k-1)} + \ldots + f_{sn}y_n^{(k-1)}) - \bar{a}_i^{(k)}(\mu)\, y_s^{(k-1)}, \qquad (11.6)$$

where $\bar{a}_i^{(k)}$ denotes the k-th approximation of the magnitude $\bar{a}_i$. We here assume that $\bar{a}_i^{(o)} = 0$ and for the zeroth approximation of the magnitudes y_s we take the periodic solution of the equations

$$\frac{dy_s^0}{dt} = a_{s1}y_1^0 + \ldots + a_{sn}y_n^0 - \lambda_i y_s^0. \qquad (11.7)$$

If $\lambda_1, \ldots, \lambda_n$ are the roots of equation (11.4) the magnitudes $\lambda_1 - \lambda_i, \ldots, \lambda_n - \lambda_i$ determine the roots of the fundamental equation of the system (11.7). Hence this fundamental equation has at least one zero root and therefore the system (11.7) has at least one periodic solution. But the root λ_i of the equation (11.4) may be multiple and to it may correspond several periodic solutions of the system (11.7). Moreover, the equation (11.4) may have roots different from λ_i by a magnitude of the form $\pm\, 2p\pi\sqrt{-1}/\omega$, where p is an integer. The fundamental equation of system (11.7) will then have roots of the form $\pm\, 2p\pi\sqrt{-1}/\omega$ to which will likewise correspond a certain number of periodic solutions of this system.

We shall call all roots of the fundamental equation of the system (11.7) of the form $\pm 2p\pi\sqrt{-1}/\omega$, including zero roots, critical. We denote further by m the total number of sets of solutions of the system (11.7) that correspond to all the critical roots of its fundamental equation. The system (11.7) will then have m and only m periodic solutions, which we shall denote by $\varphi_{s1},\dots,\varphi_{sm}$. The periodic solutions of the system conjugate to (11.7) we shall respectively denote by $\psi_{sk},\dots, \psi_{sm}$.

We put

$$y_s^0 = M_1^* \varphi_{s1} + \dots + M_m^* \varphi_{sm}, \tag{11.8}$$

where M_j^* are arbitrary constants. We then obtain for $y_s^{(1)}$ the following equations:

$$\frac{dy_s^{(1)}}{dt} = a_{s1} y_1^{(1)} + \dots + a_{sn} y_n^{(1)} - \lambda_i y_s^{(1)} + \\ + \mu \sum_{j=1}^{m} M_j^* (f_{s1}\varphi_{1j} + \dots + f_{sn}\varphi_{nj}) - \bar{a}_i^{(1)} \sum_{j=1}^{m} M_j^* \varphi_{sj}. \tag{11.9}$$

For these equations to have a periodic solution it is necessary and sufficient that the relations be satisfied

$$(\mu B_{r1} - \bar{a}_i^{(1)} A_{r1}) M_1^* + \dots + (\mu B_{rm} - \bar{a}_i^{(1)} A_{rm}) M_m^* = 0 \quad (r = 1, \dots, m), \tag{11.10}$$

where there has been put

$$B_{rj} = \int_0^{\omega} \sum_{\alpha, \beta=1}^{n} f_{\alpha\beta} \varphi_{\beta j} \psi_{\alpha r}\, dt, \quad A_{rj} = \omega \sum_{\alpha=1}^{n} \varphi_{\alpha j} \psi_{\alpha r}. \tag{11.11}$$

From (11.10) it is seen that $\bar{a}_i^{(1)}$ must have an order of smallness μ provided not all magnitudes A_{rj} become identically zero, as we shall assume. In conformity with this we shall in what follows set:

$$\bar{a}_i(\mu) = \mu a_i(\mu), \quad \bar{a}_i^{(k)}(\mu) = \mu a_i^{(k)}(\mu). \tag{11.12}$$

From (11.10) we then obtain the following system of equations for determining the magnitudes M_j^* and $a_i^{(1)}$

$$(B_{r1}-a_i^{(1)}A_{r1})M_1^*+\ldots+(B_{rm}-a_i^{(1)}A_{rm})M_m^*=0 \qquad (11.13)$$
$$(r=1,\ \ldots,\ m).$$

Setting $\mu = 0$ we shall have

$$P_r=(B_{r1}^0-a_i^*A_{r1})M_1^*(0)+\ldots+(B_{rm}^0-a_i^*A_{rm})M_m^*(0)=0, \quad (11.14)$$

where

$$B_{rj}^0=B_{rj}(0), \quad a_i^*=a_i^{(1)}(0).$$

Hence the magnitude a_i^* must be a root of the algebraic equation

$$\begin{vmatrix} B_{11}^0-a_i^*A_{11} & B_{12}^0-a_i^*A_{12} & \ldots & B_{1m}^0-a_i^*A_{1m} \\ B_{21}^0-a_i^*A_{21} & B_{22}^0-a_i^*A_{22} & \ldots & B_{2m}^0-a_i^*A_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ B_{m1}^0-a_i^*A_{m1} & B_{m2}^0-a_i^*A_{m2} & \ldots & B_{mm}^0-a_i^*A_{mm} \end{vmatrix}=0, \quad (11.15)$$

the degree of which does not exceed m.

We shall assume that a_i^* is a simple root of this equation. From (11.14) there can then be found the magnitudes $M_j^*(0)$, one of which can be chosen entirely arbitrarily. For definiteness we shall assume that $M_m(0)$ may be chosen as this arbitrary magnitude. Having fixed the magnitude $M_m^*(0)$ in some way we obtain for $M_1^*(0),\ldots,M_{m-1}^*(0)$ entirely definite values $M_1^{*0},\ldots,M_{m-1}^{*0}$. The magnitudes $M_1^*(0),\ldots,M_{m-1}^*,a_i^*$ will then be simple solutions of the system (11.14) and for this solution the condition will be satisfied

$$\frac{\partial(P_1,\ \ldots,\ P_m)}{\partial(M_1^*(0),\ \ldots,\ M_{m-1}^*(0),\ a_i^*)}\neq 0. \qquad (11.16)$$

But then equations (11.13), if we set in them

$M_m^*(\mu) = M_m^*(0)$, will for sufficiently small value of μ admit one and only one solution $M_1(\mu), \ldots, M_{m-1}(\mu)$, $a_i^{(1)}(\mu)$ for which the conditions $M_j^*(0) = M_j^{*0}$, $a_i^{(1)}(0) = a_i^*$ are satisfied.

Having chosen the magnitudes M_j^* and $a_i^{(1)}$ in this manner we can find from (11.9) the periodic functions $y_s^{(1)}$. If $\mu y_s^{(1)*}$ is some particular periodic solution of equations (11.9) we can put:

$$y_s^{(1)} = M_1^{(1)}\varphi_{s1} + \ldots + M_{m-1}^{(1)}\varphi_{s,m-1} + M_m^*(0)\varphi_{sm} + \mu y_s^{(1)*},$$

where $M_1^{(1)}, \ldots, M_{m-1}^{(1)}$ are arbitrary constants. These constants are uniquely determined from the conditions of periodicity of the functions $y_s^{(2)}$. Together with these constants there is also determined the magnitude $a_i^{(2)}$. We shall show how this is done.

For this purpose we assume that for some value of k the equations for $y_s^{(k-1)}$ admit a periodic solution. We can then set

$$y_s^{(k-1)} = M_1^{(k-1)}\varphi_{s1} + \ldots + M_{m-1}^{(k-1)}\varphi_{s,m-1} + M_m^*(0)\varphi_{sm} + \mu y_s^{(k-1)*},$$

where $M_1^{(k-1)}, \ldots, M_{m-1}^{(k-1)}$ are arbitrary constants and $\mu y_s^{(k-1)*}$ is some particular solution for $y_s^{(k-1)}$. The conditions of periodicity of the functions $y_s^{(k)}$ have the form

$$(B_{r1} - a_i^{(k)}A_{r1})M_1^{(k-1)} + \ldots + (B_{r,m-1} - a_i^{(k)}A_{r,m-1})M_{m-1}^{(k-1)} + \\ + (B_{rm} - a_i^{(k)}A_{rm})M_m^*(0) + \mu B_r - \mu A_r a_i^{(k)} = 0 \quad (11.17)$$
$$(r = 1, \ldots, m),$$

where

$$B_r = \int_0^\omega \sum_{\alpha,\beta=1}^n f_{\alpha\beta}\, y_\beta^{(k-1)*}\psi_{\alpha r}\, dt, \qquad A_r = \int_0^\omega \sum_{\alpha=1}^n y_\alpha^{(k-1)*}\psi_{\alpha r}\, dt.$$

In virtue of (11.14) and (11.6) equations (11.7) admit one and only one solution $M_1^{(k-1)}(\mu), \ldots, M_{m-1}^{(k-1)}(\mu), a_i^{(k)}(\mu)$, satisfying the conditions $M_j^{(k-1)}(0) = M_j^*$, $a_i^{(k)}(0) = a_i^*$. We have thus obtained an entirely definite process of successive approximations for computing both the magnitudes a_i of interest to us and the functions y_s. The convergence of the process is proven in the same way as in sec. 9 of chapter II. From this convergence it follows that the magnitude $\lambda_i + \mu a_i$ that we have found is actually a characteristic exponent of the system (11.1) since the functions

$$e^{(\lambda_i + \mu a_i)t} \lim_{k\to\infty} y_s^{(k)}$$

form its particular solution.

In this way we have found as many different characteristic exponents corresponding to the root λ_i of equation (11.14) as the number of simple roots possessed by equation (11.15). It is easy however to see that the same values of a_i are obtained also for those roots of equation (11.4) which differ from λ_i by magnitudes of the form $\pm 2\pi p\sqrt{-1}/\omega$, where p is an integer.

In fact, let $\lambda_k = \lambda_i \pm 2\pi p\sqrt{-1}/\omega$. Then for the root λ_k we obtain in place of equations (11.5) the equations

$$\frac{dy_s}{dt} = (a_{s1} + \mu f_{s1}) y_1 + \ldots + (a_{sn} + \mu f_{sn}) y_n - \\ - \left(\lambda_i \pm \frac{2\pi p\sqrt{-1}}{\omega} + \bar{a}_i\right) y_s. \quad (11.18)$$

But if the equations (11.5) for some value of $\bar{a}_i$ admit a periodic solution $y_s = f_s(t)$ equations (11.18) FOR THE SAME VALUE OF a_i will admit the solution $e^{\pm 2\pi p t\sqrt{-1}/\omega} f_s(t)$, which is also periodic. Hence to the root λ_k there will correspond the characteristic exponent $\lambda_i \pm 2\pi p\sqrt{-1}/\omega + \bar{a}_i$. But according to the very definition of characteristic exponents by formula (3.1) any number of the form $\pm 2\pi p\sqrt{-1}/\omega$ can be discarded from them in consequence of the multiple valued character of the logarithmic function entering in (3.1). We can therefore assume that to the root λ_k there also corresponds the characteristic exponent $\lambda_i + \breve{a}_i$. In other words, for all sets of the critical roots of the fundamental equation of the system (11.7) we obtain with the aid of the above described method as many characteristic exponents as the number of simple roots possessed by the equation (11.15). If the total multiplicity of all the critical roots is equal to the total number of sets of solutions of equations (11.7) that correspond to them, i.e. to the number m, and if all the roots of equation (11.15) are simple we obtain all the characteristic exponents corresponding to the sets of critical roots. In the contrary case we obtain only part of the characteristic exponents corresponding to the critical roots and to determine the remaining exponents it is necessary to apply some other method. For the case of analytic equations it will be possible in certain cases to make use of the method described below in sec. 15.

We remark in conclusion that for each simple root of equation (11.4) the described method will give a unique characteristic exponent corresponding to the root. We may also remark that the roots of equation (11.15) give a first approximation of the required characteristic exponents which is usually entirely sufficient for practical purposes.

EXAMPLE. STABILITY OF THE FORCED OSCILLATIONS OF A FOLLOWER SYSTEM. We shall apply the method described above to the investigation of the stability of the forced oscillations of the follower system considered in sec.8 of the preceding chapter. The system is described by the equation

$$\frac{d^3\psi}{d\tau^3} + \frac{d\psi}{d\tau} = \mu\left\{ -F(\psi) - b\frac{d\psi}{d\tau} - \frac{d^2\psi}{d\tau^2} + bA\cos\tau - A\sin\tau \right\}, \quad (11.19)$$

and the forced oscillations of interest to us are described, in the zeroth approximation, by the equation

$$\psi^0 = \varphi(\tau) = M^*\cos\tau + N^*\sin\tau, \qquad (11.20)$$

where M^* and N^* are roots of equations (8.16) of chapter II.

Setting in (11.19) $x = \psi - \varphi(\tau)$ and rejecting terms of higher order than the first with respect to μ we obtain the equation in variations

$$\frac{d^3x}{d\tau^3}+\frac{dx}{d\tau}=\mu\left\{-F'(\varphi)x-b\frac{dx}{d\tau}-\frac{d^2x}{d\tau^2}\right\}, \qquad (11.21)$$

which for $\mu = 0$ reduces to the equation with constant coefficients

$$\frac{d^3x}{d\tau^3}+\frac{dx}{d\tau}=0. \qquad (11.22)$$

The fundamental equation for (11.22) has one zero root and two purely imaginary roots $\pm i$ differing from zero by a magnitude of the form $\pm 2\pi pi/\omega$. Hence, if we seek the characteristic exponent corresponding to zero root then simultaneously with it, according to what was said above, there will be also determined the characteristic exponents corresponding to the roots $\pm i$.

Following the established rule we set in (11.21)

$$x = e^{\mu\alpha(\mu)\tau}y.$$

We shall have

$$\frac{d^3y}{d\tau^3}+\frac{dy}{d\tau}=\mu\left\{-F'(\varphi)y-b\frac{dy}{d\tau}-\frac{d^2y}{d\tau^2}-3\alpha\frac{d^2y}{d\tau^2}-\alpha y\right\}. \qquad (11.23)$$

Here all the terms of order higher than the first with respect to μ have been rejected since we intend to restrict ourselves to the computation of only the first approximation of the characteristic exponents.

Let us now seek to obtain the periodic solution of equation (11.23) by the method of successive approximations. As the zeroth approximation we shall take the periodic solution

$$y^0 = M\cos\tau + N\sin\tau + P$$

of equation (11.22). Here M,N,P are arbitrary constants. The equation determining the first approximation y_1 is of the form

$$\begin{aligned}\frac{d^3y_1}{d\tau^3}+\frac{dy_1}{d\tau}=\mu\{&-F'(\varphi)(M\cos\tau+N\sin\tau+P)-\\&-b(-M\sin\tau+N\cos\tau)+(M\cos\tau+N\sin\tau)+\\&+2\alpha(M\cos\tau+N\sin\tau)-\alpha P\}.\end{aligned}$$

In order that the equation admit a periodic solution it is necessary and sufficient that in the Fourier expansion of its right hand side there be no terms with cos τ , sin τ and the free term. In this way, taking into account the fact that because of the oddness of the functions $F(\psi)$ the identities are satisfied

$$\int_0^{2\pi} F'(\varphi)\sin\tau\,d\tau = \int_0^{2\pi} F'(\varphi)\cos\tau\,d\tau = 0,$$

we obtain

$$P\left(2\pi\alpha + \int_0^{2\pi} F'(\varphi)\,d\tau\right) = 0,$$

$$M\left(\int_0^{2\pi} F'(\varphi)\cos^2\tau\,d\tau - \pi - 2\pi\alpha\right) + N\left(\int_0^{2\pi} F'(\varphi)\sin\tau\cos\tau d\tau + \pi b\right) = 0,$$

$$M\left(\int_0^{2\pi} F'(\varphi)\sin\tau\cos\tau\,d\tau - \pi b\right) + N\left(\int_0^{2\pi} F'(\varphi)\sin^2\tau\,d\tau - \pi - 2\pi\alpha\right) = 0.$$

Hence, for one characteristic exponent we obtain the value

$$\alpha = -\frac{1}{2\pi}\int_0^{2\pi} F'(\varphi)\,d\tau,$$

and the other two are roots of the quadratic equation

$$\begin{vmatrix} \int_0^{2\pi} F'(\varphi)\cos^2\tau\,d\tau - \pi - 2\pi\alpha & \int_0^{2\pi} F'(\varphi)\sin\tau\cos\tau\,d\tau + \pi b \\ \int_0^{2\pi} F'(\varphi)\sin\tau\cos\tau\,d\tau - \pi b & \int_0^{2\pi} F'(\varphi)\sin^2\tau\,d\tau - \pi - 2\pi\alpha \end{vmatrix} = 0.$$

The necessary conditions of stability, which state that the real parts of these characteristic exponents are not positive, are of the form

$$\int_0^{2\pi} F'(\varphi)\,d\tau \geqslant 0,$$

$$\int_0^{2\pi} F'(\varphi)\,d\tau - 2\pi \leqslant 0,$$

$$\left(\int_0^{2\pi} F'(\varphi)\cos^2\tau\,d\tau - \pi\right)\left(\int_0^{2\pi} F'(\varphi)\sin^2\tau\,d\tau - \pi\right) - \left(\int_0^{2\pi} F'(\varphi)\sin\tau\cos\tau\,d\tau\right)^2 + \pi^2 b^2 \geqslant 0.$$

If these conditions are satisfied with the inequality signs the motion considered, will for sufficiently small μ, actually be stable, and furthermore asymptotically stable. We may note that the first of these conditions is always satisfied with the inequality sign since the function $F(\psi)$ is from its physical meaning an always increasing function.

12. Application to the Problem of the Stability of Oscillations of Quasilinear Systems

In the preceding section we have shown on a particular example the application of the above described method of computing the characteristic exponents to the investigation of the stability of the periodic oscillations of quasi-linear systems. We shall now consider this problem for an arbitrary quasilinear nonautonomous system.

Let there be given the arbitrary nonautonomous quasi-linear system

$$\frac{dy_s}{dt} = a_{s1}y_1 + \dots + a_{sn}y_n + f_s(t) + \mu F_s(t, y_1, \dots, y_n, \mu) \quad (s = 1, \dots, n) \tag{12.1}$$

and let $y_s(t,\mu)$ be a periodic solution of this system for which it is necessary to find the characteristic exponents. We here assume that the functions f_s and F_s are continuous and periodic with respect to t with period 2π and that F_s for sufficiently small μ admit continuous partial derivatives with respect to y_i and μ in the neighborhood of the periodic solution under consideration.

The equations in variations for the solution considered are of the form

$$\frac{dx_s}{dt} = (a_{s1} + \mu f_{s1})x_1 + \dots + (a_{sn} + \mu f_{sn})x_n, \tag{12.2}$$

where

$$f_{si} = \frac{\partial F_s(t, y_1(t, \mu), \dots, y_n(t, \mu), \mu)}{\partial y_i}. \tag{12.3}$$

Let us consider the system

$$\frac{dx_s^0}{dt} = a_{s1}x_1^0 + \dots + a_{sn}x_n^0 \tag{12.4}$$

and assume at first that we are dealing with the nonresonance case. The fundamental equation of the system (12.4) will then have neither a zero root nor roots of the form $\pm pi$, where p is an integer. We shall assume that this fundamental equation does not have either multiple roots or roots differing from one another by a magnitude of the form $\pm pi$. Such restriction for the nonresonance case is not, evidently, essential.

Let λ_i be some root of the fundamental equation of system (12.4). The corresponding characteristic exponent α_i of the system (12.2) will then be of the form

$$\alpha_i = \lambda_i + \mu a_i(\mu). \tag{12.5}$$

We restrict ourselves here to the computation of this characteristic exponent with an accuracy up to magnitudes of the first order with respect to μ, for which it is sufficient to find the magnitude $a_i = a_i^* (0)$.

In the case considered the system

$$\frac{du_s}{dt} = a_{s1}u_1 + \dots + a_{sn}u_n - \lambda_i u_s$$

has one single periodic solution corresponding to the zero root of its fundamental equation. Hence the number m in the case considered will be equal to 1. The indicated periodic solution, which in correspondence with the notations of the preceding section must be denoted by φ_{s1}, has the form $\varphi_{s1} = Q_s$, where the constant magnitudes Q_s are the coefficients of the particular solution $x_s^0 = Q_s e^{\lambda_i t}$ of the system (12.4), corresponding to the root λ_i. For the functions ψ_{s1} we have $\psi_{s1} = R_s$, if $R_s e^{-\lambda_i t}$ is a particular solution of the system conjugate to (12.4), corresponding to the root $-\lambda_i$. Further, according to (11.11) and (12.3)

$$A_{11} = 2\pi \sum_{\alpha=1}^{n} Q_\alpha R_\alpha,$$

$$B_{11}^0 = B_{11}(0) = \sum_{\alpha,\beta=1}^{n} Q_\beta R_\alpha \int_0^{2\pi} \frac{\partial F_\alpha(\tau, \varphi_1, \dots, \varphi_n, 0)}{\partial \varphi_\beta}\, d\tau,$$

where $\varphi_s(t) = y_s(t,0)$ is the generating solution, and equation (11.15) now assumes the form

$$A_{11}a_i^* = B_{11}^0. \tag{12.6}$$

As we shall presently see, the coefficient of a_i^* in equation (12.6) for the assumptions made is always different from zero and this equation uniquely determines the magnitude a_i^*.

Let us now assume that we are dealing with the resonance case. We assume consequently that the fundamental equation of the system (12.4) has roots of the form $\pm$ pi among which in particular there may be zero roots. We here restrict ourselves to the consideration of the case where the number of sets of solutions of (12.4) corresponding to each root of this kind is equal to its multiplicity. Let the total multiplicity of all roots of the form $\pm$ pi be equal to m. The system (12.4) will then have m periodic solutions $\varphi_{s1}, \ldots, \varphi_{sm}$ and its conjugate system will have the m periodic solutions $\psi_{s1}, \ldots, \psi_{sm}$. We shall find the characteristic exponents of the system (12.2) that correspond to the roots of the form $\pm$ pi. These exponents we shall now seek to obtain in the form

$$\alpha_j = \mu a_j(\mu) \qquad (j = 1, \ldots, m), \tag{12.7}$$

since the magnitudes $\pm$ pi can be discarded from the characteristic exponent. As in the resonance case, we restrict ourselves to computing only the magnitudes $a_j^* = a_j(0)$.

For the magnitudes B_{rj}^0 we shall now have

$$B_{rj}^0 = \int_0^{2\pi} \sum_{\alpha, \beta = 1}^{n} \frac{\partial F_\alpha(t, \varphi_1, \ldots, \varphi_n, 0)}{\partial \varphi_\beta} \varphi_{\beta j} \psi_{\alpha r} \, dt,$$

where $\varphi_s(t)$ is the generating solution for the system (12.1). This generating solution is of the form

$$\varphi_s(t) = M_1^* \varphi_{s1} + \ldots + M_m^* \varphi_{sm} + y_s^{(0)*}(t),$$

where the magnitudes M_j^* satisfy the equations

$$P_i = \int_0^{2\pi} \sum_{\alpha=1}^{n} F_\alpha(t, \varphi_1, \ldots, \varphi_n, 0)\psi_{\alpha i}\, dt = 0 \qquad (i = 1, \ldots, m).$$

whence we readily find

$$B_{rj}^0 = \frac{\partial P_r}{\partial M_j^*},$$

Hence the equation (11.15) determining the magnitudes a_j^*, will be:

$$\begin{vmatrix} \frac{\partial P_1}{\partial M_1^*} - a_j^* A_{11} & \frac{\partial P_1}{\partial M_2^*} - a_j^* A_{12} & \ldots & \frac{\partial P_1}{\partial M_m^*} - a_j^* A_{1m} \\ \frac{\partial P_2}{\partial M_1^*} - a_j^* A_{21} & \frac{\partial P_2}{\partial M_2^*} - a_j^* A_{22} & \ldots & \frac{\partial P_2}{\partial M_m^*} - a_j^* A_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial P_m}{\partial M_1^*} - a_j^* A_{m1} & \frac{\partial P_m}{\partial M_2^*} - a_j^* A_{m2} & \ldots & \frac{\partial P_m}{\partial M_m^*} - a_j^* A_{mm} \end{vmatrix} = 0, \qquad (12.8)$$

where

$$A_{rj} = 2\pi \sum_{\alpha=1}^{n} \varphi_{\alpha j}\psi_{\alpha r}.$$

A more simple form can be given to the obtained equation if we choose the system of functions ψ_{sj} in a suitable manner. For this purpose we note first of all that for any fixed j at least one among the m magnitudes $A_{1j},\ldots,A_{mj}$ is different from zero. In fact in the contrary case the system of nonhomogeneous equations

$$\frac{du_s}{dt} = a_{s1}u_1 + \ldots + a_{sn}u_n + \varphi_{sj}$$

would admit a periodic solution $u_s = u_{sj}(t)$ and the system (12.4) would then admit the solution $x_s^0 = t\varphi_{sj} + u_{sj}(t)$ This contradicts however the condition that the number of sets of solutions of the system (12.4) corresponding to the roots of the form $\pm\, pi$ is equal to their multiplicity. In exactly the same way it can be asserted that among the magnitudes A_{rj} there is at least one different from zero for any fixed value of the index r. From the proven property of the coefficients A_{rj} it follows, in particular, that the coefficient a_i^* in equation (12.6) is different from zero.

We shall now show that the periodic solutions $\psi_{s1},\ldots,\psi_{sm}$ of the system conjugate (12.4) can be chosen in such manner that the magnitudes A_{rj} assume any initially given values. For this purpose we replace the solutions ψ_{sj} by another system of solutions ψ'_{sj}, defined by the relations

$$\psi'_{sj}=\sum_{p=1}^{m} a_{jp}\psi_{sp},$$

where a_{jp} are certain constants. For this system we shall have:

$$A'_{rj}=\sum_{\alpha=1}^{n}\varphi_{\alpha j}\psi'_{\alpha r}=\sum_{\alpha=1}^{n}\sum_{p=1}^{m}a_{rp}\varphi_{\alpha j}\psi_{\alpha p}=\sum_{p=1}^{m}a_{rp}A_{jp}.$$

Assuming here that A'_{rj} are initially given magnitudes we obtain m systems of linear equations for determining the coefficients a_{rp}. All these m systems have a common homogeneous part and, further, their determinant is different from zero. In fact, if this determinant $|A_{jp}|$ were equal to zero the equations

$$\sum_{p=1}^{m}a_{rp}A_{jp}=0$$

would admit at least one solution $a_{r1}, a_{r2},\ldots,a_{rm}$ different from the trivial solution $a_{r1}= a_{r2}=\ldots a_{rm}=0$ and therefore, at least for one value of the index r, we would have

$$A'_{r1}=A'_{r2}=\ \ldots\ =A'_{rm}=0;$$

which, by what has been proved, is impossible. Thus the coefficients a_{rp} may always be chosen such that the magnitudes A'_{rj} take initially given values. From this it follows that, without restricting generality, we can consider that the periodic solutions ψ_{sj} of the system conjugate to (12.4) have been chosen in such manner that the relations are satisfied:

$$2\pi\sum_{\alpha=1}^{n}\varphi_{\alpha r}\psi_{\alpha r}=1,\quad \sum_{\alpha=1}^{n}\varphi_{\alpha r}\psi_{\alpha j}=0 \qquad (r\neq j). \tag{12.9}$$

Equation (12.8) will then have the form

$$\begin{vmatrix} \frac{\partial P_1}{\partial M_1^*}-a_j^* & \frac{\partial P_1}{\partial M_2^*} & \cdots & \frac{\partial P_1}{\partial M_m^*} \\ \frac{\partial P_2}{\partial M_1^*} & \frac{\partial P_2}{\partial M_2^*}-a_j^* & \cdots & \frac{\partial P_3}{\partial M_m^*} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial P_m}{\partial M_1^*} & \frac{\partial P_m}{\partial M_2^*} & \cdots & \frac{\partial P_m}{\partial M_m^*}-a_j^* \end{vmatrix}=0. \tag{12.10}$$

If this equation has only simple solutions we obtain m different values for a_j^*, i.e. the approximate values of all the characteristic exponents corresponding to the critical roots of the fundamental equation of system (12.4). As regards the characteristic exponents corresponding to the noncritical roots of this equation they are computed in exactly the same way as in the nonresonance case.

From all that was said above it follows that if μ is sufficiently small and if the equation

$$\begin{vmatrix} \frac{\partial P_1}{\partial M_1^*}-\varkappa & \frac{\partial P_1}{\partial M_2^*} & \cdots & \frac{\partial P_1}{\partial M_m^*} \\ \frac{\partial P_2}{\partial M_1^*} & \frac{\partial P_2}{\partial M_2^*}-\varkappa & \cdots & \frac{\partial P_2}{\partial M_m^*} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial P_m}{\partial M_1^*} & \frac{\partial P_m}{\partial M_2^*} & \cdots & \frac{\partial P_m}{\partial M_m^*}-\varkappa \end{vmatrix}=0. \tag{12.11}$$

has only simple roots it is necessary, for the stability of the periodic solution considered of the system (12.1) in the resonance case, that the real parts of all the roots of equation (12.11) have signs opposite to the sign of μ. Moreover, it is necessary that the real parts of the characteristic exponents corresponding to the noncritical roots of the fundamental equation of the system (12.4) be not positive.

REMARK. We assumed that in the nonresonance case the root λ_i of the fundamental equation of system (12.4) is simple and that this equation has no roots differing from λ_i by magnitudes of the form $\pm p\sqrt{-1}$. The method of the preceding section permits however finding the characteristic exponents also in the case where these conditions are not satisfied provided that to the root λ_i

and to those roots which differ from it by magnitudes of the form $\pm\, p\sqrt{-1}$ there correspond as many sets of solutions of the system (12.4) as the multiplicity of these roots.

13. Application to the Case of Equations Analytic with Respect to the Parameter

The determination of the characteristic exponents by the method described in sec. 11 is simplified if the equations (11.1) are analytic with respect to the parameter.

Thus let us assume that in the equations (11.1)

$$\frac{dx_s}{dt} = [a_{s1} + \mu f_{s1}(t, \mu)]\, x_1 + \ \dots \ + [a_{sn} + \mu f_{sn}(t, \mu)]\, x_n \quad (13.1)$$
$$(s = 1, \ \dots, \ n)$$

the functions f_{sj} are analytic with respect to μ for its sufficiently small values. In this case on the basis of the theorem of Lyapunov established in sec. 10 the coefficients of the characteristic equation of system (13.1) will be analytic with respect to μ. From this however it does not follow that the roots of the characteristic equation will likewise be analytic with respect to μ.

In fact, let

$$D(\rho, \mu) = \rho^n + A_1(\mu)\rho^{n-1} + \ \dots \ + A_{n-1}(\mu)\rho + A_n(\mu) = 0, \quad (13.2)$$

where $A_j(\mu)$ are certain analytic functions of μ, be the characteristic equation of the system (13.1). Further, let ρ_0 be some root of this equation for $\mu = 0$. Then if this root is simple and consequently $dD(\rho_0, 0)/d\rho_0 \neq 0$, equation (13.2) for sufficiently small μ will admit one and only one solution $\rho(\mu)$ for which $\rho(0) = \rho_0$, and this solution will be analytic with respect to μ. The corresponding characteristic exponent $\alpha = \frac{1}{\varpi} \ln \rho(\mu)$ will be in this case likewise analytic with respect to μ. But if the root ρ_0 is multiple, and consequently $dD(\rho_0, 0/d\rho_0 = 0$, the question as to the roots of equation (13.2) for $\mu \neq 0$ becomes more complicated. Of course, this equation always has m roots but these roots will not, in general, be analytic with respect to μ. It can be shown that if equation (13.2) for $\mu = 0$ has a root ρ_0 whose multiplicity is equal to m this equation,[1] for μ sufficiently small, will have m roots reducing for $\mu = 0$ to ρ_0 and these roots can be expanded in series of integral powers of the magnitude $\mu^{1/k}$, where $1 \leqslant k \leqslant m$. The same will then hold true also for the corresponding characteristic exponents. The number k may in particular be equal to unity and then the indicated roots and the characteristic exponents corresponding to them will be analytic with respect to μ.

Having established this, let us consider the fundamental equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0 \qquad (13.3)$$

of the system (13.1) for $\mu = 0$. Let λ be some root of this equation. The magnitude $e^{\lambda\omega} = \rho_0$ will then be a root of the characteristic equation (13.2) for $\mu = 0$. This root will be multiple if λ is a multiple root of equation (13.3) or if this equation has roots differing from λ by magnitudes of the form $\pm 2p\pi i/\omega$. Hence to the critical roots of equation (13.3) will correspond the characteristic exponents of the system (13.1), which in general can be expanded in fractional powers of the magnitude μ.

Let us assume however that the number of sets of solutions of the system

$$\frac{dx_s^0}{dt} = a_{s1}x_1^0 + \dots + a_{sn}x_n^0, \qquad (13.4)$$

corresponding to the root λ of equation (13.3) and to each of the roots of this equation that differ from λ by magnitudes of the form $\pm 2p\pi i/\omega$ is equal to the multiplicity of these roots. We denote the total multiplicity of all these roots by m. The system

$$\frac{dy_s^0}{dt} = a_{s1}y_1^0 + \dots + a_{sn}y_n^0 - \lambda y_s^0 \qquad (13.5)$$

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[1] See for example Goursat E., Course in Mathematical Analysis, vol II, sec. 356.

will then have m periodic solutions $\varphi_{s1},\dots,\varphi_{sm}$ while the system conjugate to (13.5) will have m periodic solutions $\psi_{s1},\dots,\psi_{sm}$.

Let us assume further that the equation (11.15)

$$\begin{vmatrix} B^0_{11}-a^*A_{11} & B^0_{12}-a^*A_{12} & \dots & B^0_{1m}-a^*A_{1m} \\ B^0_{21}-a^*A_{21} & B^0_{22}-a^*A_{22} & \dots & B^0_{2m}-a^*A_{2m} \\ \dots & \dots & \dots & \dots \\ B^0_{m1}-a^*A_{m1} & B^0_{m2}-a^*A_{m2} & \dots & B^0_{mm}-a^*A_{mm} \end{vmatrix} = 0 \qquad (13.6)$$

has only simple roots. Then, applying the method of sec. 11, we obtain all m characteristic exponents corresponding to the root λ and to the entire aggregate of critical roots connected with it. It is easily seen that all these characteristic exponents will be capable of expansion in integral powers of the parameter μ.

In fact, if f_{sj} are analytic functions of μ then for any k equations (11.17), determining $M_j^{(k-1)}$, $a_i^{(k)}$, and equations (11.6), determining $y_s^{(k)}$, will be analytic with respect to μ if all preceding approximations came out analytic with respect to this magnitude. From this it follows immediately that all magnitudes $a_i^{(k)}, y_s^{(k)}$ will come out analytic with respect to μ and therefore the same will be true also for the limits a_i, y_s of the sequence of these magnitudes. In this way we arrive at the following proposition first established by a different method by S.N. Shimanov:

IF THE NUMBER OF SETS OF SOLUTIONS OF THE SYSTEM (13.4) CORRESPONDING TO THE ROOT λ OF EQUATION (13.3), AND ALSO TO ALL THE ROOTS OF THIS EQUATION DIFFERING FROM λ BY MAGNITUDES OF THE FORM $\pm\, 2p\pi i/\omega$, WHERE p IS AN INTEGER, IS EQUAL TO THE MULTIPLICITY OF THESE ROOTS, THE CHARACTERISTIC EXPONENTS OF THE SYSTEM (13.1) THAT CORRESPOND TO ALL THESE ROOTS CAN BE EXPANDED IN INTEGRAL POWERS OF THE MAGNITUDE μ UNDER CONDITION THAT ALL THE ROOTS OF EQUATION (13.6) ARE SIMPLE.

For the actual computation of the characteristic exponents when these conditions are satisfied the following procedure can be used by which the computations will be somewhat simpler as compared with the method of sec. 11.[1]

Let us in equations (13.1) make a change of variables $x_s = e^{\alpha t} y_s$, where

$$a = \lambda + \mu a^* + \mu^2 a_2 + \dots$$

is the required characteristic exponent, and try to choose the unknown constants a^*, $a_2, \dots$ in such manner that the transformed equations

$$\frac{dy_s}{dt} = a_{s1} y_1 + \dots + a_{sn} y_n - \lambda y_s + \\ + \mu (f_{s1} y_1 + \dots + f_{sn} y_n) - \mu (a^* + \mu a_2 + \dots) y_s$$

have a periodic solution of period ω. We shall seek to obtain this periodic solution in the form of a series

$$y_s = M_1^* \varphi_{s1} + \dots + M_m^* \varphi_{sm} + \mu y_s^{(1)} + \mu^2 y_s^{(2)} + \dots = \\ = y_s^0 + \mu y_s^{(1)} + \mu y_s^{(2)} + \dots,$$

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[1] The computation of the characteristic exponents for equations of the form (13.1) in the case where the functions are analytic with respect to μ with the aid of a procedure analogous to the one presented below was first given in the work of N.A. Artemyev, Metod opredeleniya khrakteristicheskikh pokazatelei i prilozhenie ego k dvum zadacham nebesnoi mekhaniki (Determining the Characteristic Exponents and its Application to Two Problems of Celestial Mechanics), Izv. AN SSSR, seriya mat., vol.8, No.2, 1944. However, Artemyev restricted himself to the consideration of only the case where equation (13.3) has neither multiple roots nor roots different from each other by magnitudes of the form $\pm 2p\pi i/\omega$. He thus excluded from consideration precisely the cases of most interest for the theory of nonlinear oscillations. Moreover, Artemyev failed to note that for the assumptions he made the characteristic exponents will always be capable of expansion in integral powers of μ by proposing in each individual case to investigate this question with the aid of some other method. The method here described for the general case here considered was pointed out by S.N. Shimanov.

where M_i^* are certain constants and $y_s^{(1)}$, $y_s^{(2)}, \ldots$ are periodic functions of t of period ω. For $y_s^{(1)}$ we have

$$\frac{dy_s^{(1)}}{dt} = a_{s1}y_1^{(1)} + \ldots + a_{sn}y_n^{(1)} - \lambda y_s^{(1)} + \\ + f_{s1}(t, 0)\, y_1^0 + \ldots + f_{sn}(t, 0)\, y_n^0 - a^* y_s^0,$$

and the conditions of periodicity of the functions $y_s^{(1)}$

$$\int_0^\omega \sum_{\alpha=1}^n [f_{\alpha 1}(t, 0)\, y_1^0 + \ldots + f_{\alpha n}(t, 0)\, y_n^0 - a^* y_\alpha^0]\, \psi_{\alpha r}\, dt = 0$$

$$(r = 1, \ldots, m)$$

give

$$P_r = (B_{r1}^0 - a^* A_{r1}) M_1^* + (B_{r2}^0 - a^* A_{r2}) M_2^* + \ldots + \\ + (B_{rm}^0 - a^* A_{rm}) M_m^* = 0 \quad (13.7)$$

$$(r = 1, \ldots, m),$$

where B_{rj}^0 have the same values as in sec. 11.

Consequently a^* must be a root of equation (13.6) which by assumption has only simple roots. Taking as a^* one of the roots of equation (13.6) we find from (13.7) the corresponding values of the magnitudes M_j^*, one of which, let it be M_m^*, can be chosen arbitrarily.

Having thus chosen the magnitudes M_j^*, a^* we obtain for $y_s^{(1)}$ a periodic solution. Let us assume that all magnitudes a^*, $a_2, \ldots, a_{k-1}$ and all functions $y_s^{(1)}, \ldots, y_s^{(k-1)}$ have already been computed and that the latter came out periodic. We can here set

$$y_s^{(k-1)} = M_1^{(k-1)}\varphi_{s1} + \ldots + M_{m-1}^{(k-1)}\varphi_{s,\, m-1} + M_m^* \varphi_{sm} + y_s^{(k-1)*},$$

where $M_1^{(k-1)}, \ldots, M_{m-1}^{(k-1)}$ are arbitrary constants and $y_s^{(k-1)*}$ certain periodic functions of t. We can then write for $y_s^{(k)}$ the following equations:

$$\frac{dy_s^{(k)}}{dt} = a_{s1}y_1^{(k)} + \ldots + a_{sn}y_n^{(k)} - \lambda y_s^{(k)} - a_k y_s^{(0)} + \\ + f_{s1}(t, 0)\, y_1^{(k-1)} + \ldots + f_{sn}(t, 0)\, y_n^{(k-1)} - a^* y_s^{(k-1)} + Y_s^{(k)}(t),$$

where $Y_s^{(k)}$ are known periodic functions of t of period ω.

The conditions of periodicity of the functions $y_s^{(k)}$ lead to the equations

$$(B^0_{r1} - a^* A_{r1}) M_1^{(k-1)} + \ldots + (B^0_{r,m-1} - a^* A_{r,m-1}) M_{m-1}^{(k-1)} - \\ - a_k \sum_{j=1}^{m} A_{rj} M_j^* + C_r = 0 \\ (r = 1, \ldots, m),$$

where

$$C_r = (B^0_{rm} - a^* A_{rm}) M_m^* + \\ + \int_0^\omega \sum_{\alpha=1}^{n} [f_{\alpha 1}(t, 0) y_1^{(k-1)*} + \ldots + f_{\alpha n}(t, 0) y_n^{(k-1)*} + Y_\alpha] \psi_{\alpha r} dt$$

are certain constants. We have obtained a system of linear equations determining the magnitudes

$$M_1^{(k-1)}, \ldots M_{m-1}^{(k-1)}, \ a_k$$

the determinant of which, agreeing with

$$\frac{\partial (P_1, \ldots, P_m)}{\partial (M_1^*, \ldots, M_{m-1}^*, a^*)},$$

according to the proof in sec. 11 is different from zero.

We can thus determine succesively all constants $a^*, a_2,$

Taking as a^* succesively all m roots of equation (13.6) we obtain all m characteristic exponents of system (13.1) that correspond to the root λ of equation (13.3) and all other roots of this equation differing from λ by magnitudes of the form $\pm 2p\pi/\omega$.

REMARK. Equation (13.3) actually has the degree m. In fact, the coefficient of a^{*m} in this equation is equal to the determinant $|A_{rj}|$ which, as was shown in the preceding section, for the conditions considered is different from zero.

14. Stability of the Self-Oscillations in Two Inductively Coupled Circuits

We shall apply the method of the preceding section to the investigation of the stability of self-oscillations in the two inductively coupled circuits that were considered in sec. 13 of chapter II.

The equations of motion after their reduction to the new independent variables τ, for which the period of the periodic solution under consideration does not depend on μ, are given by the formulas (13.9) of chapter II. But since according to formulas (13.13) of the same chapter the magnitude a^* is equal to zero we shall have

$$\left.\begin{aligned} &\frac{d^2x}{d\tau^2}-\varkappa_1\frac{d^2y}{d\tau^2}+n_1^2x=\mu n_1(1-x^2)\frac{dx}{d\tau}+\ldots,\\ &\frac{d^2y}{d\tau^2}-\varkappa_2\frac{d^2x}{d\tau^2}+n_2^2y=-\mu\frac{n_2^2}{n_1}\delta\frac{dy}{d\tau}+\ldots,\end{aligned}\right\}\tag{14.1}$$

where the terms not written down are of an order higher than the first with respect to μ. For definiteness we shall consider the self-oscillations corresponding to the frequency ω_1 of the linear system

$$\left.\begin{aligned} &\frac{d^2x_0}{d\tau^2}-\varkappa_1\frac{d^2y_0}{d\tau^2}+n_1^2x_0=0,\\ &\frac{d^2y_0}{d\tau^2}-\varkappa_2\frac{d^2x_0}{d\tau^2}+n_2^2y_0=0,\end{aligned}\right\}\tag{14.2}$$

for which the equation of the frequencies has the form

$$(1-\varkappa_1\varkappa_2)\,\omega^4-(n_1^2+n_2^2)\,\omega^2+n_1^2n_2^2=0.\tag{14.3}$$

These self-oscillations are, in the zeroth approximation, described by the equations

$$x_0=M_1^*\cos\omega_1\tau,\qquad y_0=M_1^*k_1\cos\omega_1\tau,\tag{14.4}$$

where according to formulas (13.6) and (13.13) of chapter II

$$k_1=\frac{\omega_1^2-n_1^2}{\varkappa_1\omega_1^2}=\frac{\varkappa_2\omega_1^2}{\omega_1^2-n_2^2},\qquad M_1^{*2}=4-\frac{4\delta n_2^2\,(n_1^2-\omega_1^2)}{n_1^2\,(n_2^2-\omega_1^2)}.\tag{14.5}$$

From (14.1) and (14.4) we obtain the following equations in variations:

$$\left.\begin{aligned} &\frac{d^2u}{d\tau^2}-\varkappa_1\frac{d^2v}{d\tau^2}+n_1^2u=\\ &\qquad=\mu n_1\left\{M_1^{*2}\omega_1\sin 2\omega_1\tau\cdot u+(1-M_1^{*2}\cos^2\omega_1\tau)\frac{du}{d\tau}\right\}+\ldots,\\ &\frac{d^2v}{d\tau^2}-\varkappa_2\frac{d^2u}{d\tau^2}+n_2^2v=-\mu\frac{n_2^2}{n_1}\delta\frac{dv}{d\tau}+\ldots,\end{aligned}\right\}\tag{14.6}$$

where the terms not written down are of an order higher than the first with respect to μ. For $\mu = 0$ these equations reduce to (14.2). The fundamental equation of the system (14.2) has the roots $\pm i\omega_1$ and $\pm i\omega_2$ of which the first two are critical. We shall first find the characteristic exponents corresponding to the critical roots.

For this purpose let us put in equations (14.6)

$$u = e^{\alpha\tau}\varphi, \quad v = e^{\alpha\tau}\psi, \tag{14.7}$$

where

$$\alpha = \mu\alpha^* + \mu^2\alpha_2 + \ldots, \tag{14.8}$$

and φ and ψ are periodic functions of τ of the period $2\pi/\omega_1$.

We obtain
$$\left.\begin{aligned} &\frac{d^2\varphi}{d\tau^2} - \varkappa_1 \frac{d^2\psi}{d\tau^2} + n_1^2 u = \mu \Big\{ M_1^{*2}\omega_1 n_1 \sin 2\omega_1\tau\cdot\varphi + \\ &\qquad + n_1(1 - M_1^{*2}\cos^2\omega_1\tau)\frac{d\varphi}{d\tau} - 2\alpha^*\frac{d\varphi}{d\tau} + 2\varkappa_1\alpha^*\frac{d\psi}{d\tau}\Big\} + \ldots, \\ &\frac{d^2\psi}{d\tau^2} - \varkappa_2\frac{d^2\varphi}{d\tau^2} + n_2^2\psi = \\ &\qquad = \mu\Big\{ -\frac{n_2^2}{n_1}\delta\frac{d\psi}{d\tau} - 2\alpha^*\frac{d\psi}{d\tau} + 2\alpha^*\varkappa_2\frac{d\varphi}{d\tau}\Big\} + \ldots \end{aligned}\right\} \tag{14.9}$$

We shall seek the periodic solutions of these equations in the form of the series

$$\left.\begin{aligned} \varphi &= \varphi_0 + \mu\varphi_1 + \ldots, \\ \psi &= \psi_0 + \mu\psi_1 + \ldots \end{aligned}\right\} \tag{14.10}$$

The functions φ_o and ψ_o must satisfy equations (14.2). These equations, as follows from their general solution (13.5) of chapter II, have the periodic solution

$$\varphi_0 = N_1\cos\omega_1\tau + N_2\sin\omega_1\tau, \quad \psi_0 = k_1(N_1\cos\omega_1\tau + N_2\sin\omega_1\tau),$$

containing the two arbitrary constants N_1 and N_2.

For φ_1 and ψ_1 we have

$$\begin{aligned} &\frac{d^2\varphi_1}{d\tau^2} - \varkappa_1\frac{d^2\psi_1}{d\tau^2} + n_1^2\varphi_1 = M_1^{*2}\omega_1 n_1\sin 2\omega_1\tau\cdot\varphi_0 + \\ &\qquad + n_1(1 - M_1^{*2}\cos^2\omega_1\tau)\frac{d\varphi_0}{d\tau} - 2\alpha^*\frac{d\varphi_0}{d\tau} + 2\varkappa_1\alpha^*\frac{d\psi_0}{d\tau}, \\ &\frac{d^2\psi_1}{d\tau^2} - \varkappa_2\frac{d^2\varphi_1}{d\tau^2} + n_2^2\psi_1 = -\frac{n_2^2}{n_1}\delta\frac{d\psi_0}{d\tau} - 2\alpha^*\frac{d\psi_0}{d\tau} + 2\alpha^*\varkappa_2\frac{d\varphi_0}{d\tau}, \end{aligned}$$

or

$$\left.\begin{aligned} \frac{d^2\varphi_1}{d\tau^2} - \varkappa_1 \frac{d^2\psi_1}{d\tau^2} + n_1^2\varphi_1 &= P\cos\omega_1\tau + Q\sin\omega_1\tau + \ldots, \\ \frac{d^2\psi_1}{d\tau^2} - \varkappa_2 \frac{d^2\varphi_1}{d\tau^2} + n_2^2\psi_1 &= Q\cos\omega_1\tau + R\sin\omega_1\tau + \ldots, \end{aligned}\right\} \qquad (14.11)$$

where the terms not written down do not contain cos $\omega_1\tau$ and sin $\omega_1\tau$ while the coefficients P,Q,R,S have the values

$$\begin{aligned} P &= \left\{ n_1\omega_1\left(1 - \frac{1}{4}M_1^{*2}\right) - 2a^*\omega_1(1 - k_1\varkappa_1)\right\} N_2, \\ Q &= \left\{ n_1\omega_1\left(\frac{3}{4}M_1^{*2} - 1\right) + 2a^*\omega_1(1 - k_1\varkappa_1)\right\} N_1, \\ R &= \left\{ -\frac{n_2^2}{n_1}\delta k_1\omega_1 - 2a^*\omega_1(k_1 - \varkappa_2)\right\} N_2, \\ S &= \left\{ \frac{n_2^2}{n_1}\delta k_1\omega_1 + 2a^*\omega_1(k_1 - \varkappa_2)\right\} N_1. \end{aligned}$$

In order that equations (14.11) admit periodic solutions it is necessary and sufficient, as was shown in sec.13 of chapter II, that the relations be satisfied (formulas(13.12))

$$\begin{aligned} (n_1^2 - \omega_1^2)R - \varkappa_2\omega_1^2 P &= 0, \\ (n_1^2 - \omega_1^2)S - \varkappa_2\omega_1^2 Q &= 0. \end{aligned}$$

These relations have the form

$$(Aa^* + B)M_2 = 0, \qquad (Ca^* + D)N_1 = 0,$$

whence we find the two following values for a^*:

$$a_1^* = -\frac{B}{A}, \qquad a_2^* = -\frac{D}{C}.$$

Computing the values of the coefficients A,B,C.D, after simple transformations we find:

$$a_1^* = 0, \qquad a_2^* = -\frac{n_1\varkappa_2\omega_1^2}{4(n_1^2\varkappa_2 + n_2^2k_1^2\varkappa_1)}. \qquad (14.12)$$

There were here used relations (14.5) and also relations

$$1-k_1\varkappa_1=\frac{n_1^2}{\omega_1^2}, \quad k_1-\varkappa_2=\frac{n_2^2 k_1}{\omega_1^2}. \tag{14.13}$$

Substituting (14.12) in (14.8) we obtain the first approximation of the characteristic exponents corresponding to the critical roots $\pm i\omega_1$ of fundamental equation (14.3). We shall restrict ourselves to this first approximation. One of the characteristic exponents was obtained equal to zero on account of the autonomous character of the initial system (14.1).

We shall now find the characteristic exponent corresponding to the root $i\omega_2$. For this purpose we transform, as before, equations (14.6) with the aid of the substitution (14.7) but where we now have

$$\alpha=i\omega_2+\mu a^*+\mu^2 a_2+\ldots \tag{14.14}$$

The transformed equations assume the form

$$\frac{d^2\varphi}{d\tau^2}-\varkappa_1\frac{d^2\psi}{d\tau^2}+2i\omega_2\frac{d\varphi}{d\tau}-2i\varkappa_1\omega_2\frac{d\psi}{d\tau}+(n_1^2-\omega_2^2)\varphi+\varkappa_1\omega_2^2\psi=$$
$$=\mu\left\{n_1M_1^{*2}\omega_1\sin 2\omega_1\tau\cdot\varphi+n_1(1-M_1^{*2}\cos^2\omega_1\tau)\left(\frac{d\varphi}{d\tau}+i\omega_2\varphi\right)-\right.$$
$$\left.-2a^*\frac{d\varphi}{d\tau}-2i\omega_2a^*\varphi+2a^*\varkappa_1\frac{d\psi}{d\tau}+2i\omega_2\varkappa_1a^*\psi\right\}+\ldots,$$

$$\frac{d^2\psi}{d\tau^2}-\varkappa_2\frac{d^2\varphi}{d\tau^2}+2i\omega_2\frac{d\psi}{d\tau}-2i\varkappa_2\omega_2\frac{d\varphi}{d\tau}+(n_2^2-\omega_2^2)\psi+\varkappa_2\omega_2^2\varphi=$$
$$=\mu\left\{-\frac{n_2^2}{n_1}\delta\left(\frac{d\psi}{d\tau}+i\omega_2\psi\right)-2a^*\frac{d\psi}{d\tau}-2i\omega_2a^*\psi+\right.$$
$$\left.+2a^*\varkappa_2\frac{d\varphi}{d\tau}+2i\omega_2a^*\varkappa_2\varphi\right\}+\ldots,$$

where the terms not written down have an order higher than the first with respect to μ. These equations we shall also try to satisfy by a periodic solution of the form (14.10). Now however there are obtained for φ_0 and ψ_0 the equations

$$\frac{d^2\varphi_0}{d\tau^2}-\varkappa_1\frac{d^2\psi_0}{d\tau^2}+2i\omega_2\frac{d\varphi_0}{d\tau}-2i\varkappa_1\omega_2\frac{d\psi_0}{d\tau}+(n_1^2-\omega_2^2)\varphi_0+\varkappa_1\omega_2^2\psi_0=0,$$

$$\frac{d^2\psi_0}{d\tau^2}-\varkappa_2\frac{d^2\varphi_0}{d\tau^2}+2i\omega_2\frac{d\psi_0}{d\tau}-2i\varkappa_2\omega_2\frac{d\varphi_0}{d\tau}+(n_2^2-\omega_2^2)\psi_0+\varkappa_2\omega_2^2\varphi_0=0,$$

which admit a single periodic solution and in this solution the magnitudes φ_0 and ψ_0 are constant. This solution will be

$$\varphi_0=-N\varkappa_1\omega_2^2, \quad \psi_0=N(n_1^2-\omega_2^2),$$

where N is an arbitrary constant. For φ_1 and ψ_1 we have the equations

$$\left.\begin{aligned}
&\frac{d^2\varphi_1}{d\tau^2}-\varkappa_1\frac{d^2\psi_1}{d\tau^2}+2i\omega_2\frac{d\varphi_1}{d\tau}-2i\varkappa_1\omega_2\frac{d\psi_1}{d\tau}+\\
&\qquad\qquad+(n_1^2-\omega_2^2)\varphi_1+\varkappa_1\omega_2^2\psi_1=\\
&=(n_1M_1^{*2}\omega_1\sin 2\omega_1\tau+in_1\omega_2(1-M_1^{*2}\cos^2\omega_1\tau)-2i\omega_2a^*)\varphi_0+\\
&\qquad\qquad+2i\omega_2\varkappa_1a^*\psi_0,\\
&\frac{d^2\psi_1}{d\tau^2}-\varkappa_2\frac{d^2\varphi_1}{d\tau^2}+2i\omega_2\frac{d\psi_1}{d\tau}-2i\varkappa_2\omega_2\frac{d\varphi_1}{d\tau}+\\
&\qquad\qquad+(n_2^2-\omega_2^2)\psi_1+\varkappa_2\omega_2^2\varphi_1=\\
&\qquad=2i\omega_2\varkappa_2a^*\varphi_0+\left(-i\frac{n_2^2}{n_1}\delta\omega_2-2i\omega_2a^*\right)\psi_0.
\end{aligned}\right\}\quad(14.15)$$

To set up the conditions of periodicity of the functions φ_1 and ψ_1 we may make use of the general formulas, which requires reduction of equations (14.15) to the normal form. It is possible however to proceed in a more simple fashion.

The fundamental equation of the homogeneous part of the system (14.15) has the roots $0,\ -2i\omega_2,\ \pm i\omega_1-i\omega_2$, and therefore the resonating terms in these equations will be only the free terms to which correspond the periodic solution with constant values for φ_1 and ψ_1. Hence, retaining in the right hand sides of equations (14.15) only the free terms we shall try to satisfy them by the solution $\varphi_1 = A$, $\psi_1 = B$, where A and B are constants. For these constants we then obtain the equations

$$\begin{aligned}
&\frac{1}{N}[(n_1^2-\omega_2^2)A+\varkappa_1\omega_2^2B]=-in_1\omega_2^3\varkappa_1\left(1-\frac{1}{2}M_1^{*2}\right)+\\
&\qquad\qquad+2i\omega_2^3\varkappa_1a^*+2i\omega_2\varkappa_1(n_1^2-\omega_2^2)a^*,\\
&\frac{1}{N}[\varkappa_2\omega_2^2A+(n_2^2-\omega_2^2)B]=-i\omega_2\frac{n_2^2\delta(n_1^2-\omega_2^2)}{n_1}-\\
&\qquad\qquad-2i\omega_2(n_1^2-\omega_2^2)a^*-2i\omega_2^3\varkappa_1\varkappa_2a^*.
\end{aligned}$$

The determinant of this system is equal to zero since ω_2 is a root of equation (14.3). Hence, for the written down equations to be solvable for A and B it is necessary that the condition of their compatibility be satisfied, which is the required condition of periodicity. To set up this condition we multiply by $n_2^2-\omega_1^2$ and $-\varkappa_1\omega_2^2$ respectively and combine. In this

$$a^* = a_3^* = \frac{n_1(n_2^2-\omega_2^2)(2M_1^{*2}-M_2^{*2})}{8(\omega_2^2-\omega_1^2)(1-\varkappa_1\varkappa_2)}, \qquad (14.16)$$

where M_2^* is the amplitude of the self-oscillations corresponding to the frequency ω_2.

This will be the required value of a_3^*. The amplitude a_4^* for the fourth characteristic exponent of the system (14.6), corresponding to the root $-i\omega_2$ of the fundamental equation of the system (14.2), is obtained by the simple substitution of $-\omega_2$ for ω_2.

We thus have

$$a_4^* = a_3^*$$

For sufficiently small u the required conditions of stability have the form

$$a_2^* < 0, \quad a_3^* < 0. \qquad (14.17)$$

On the basis of (14.12) the first of these conditions is always satisfied. As regards the second condition, it may, as can easily be seen, be represented in the following form:

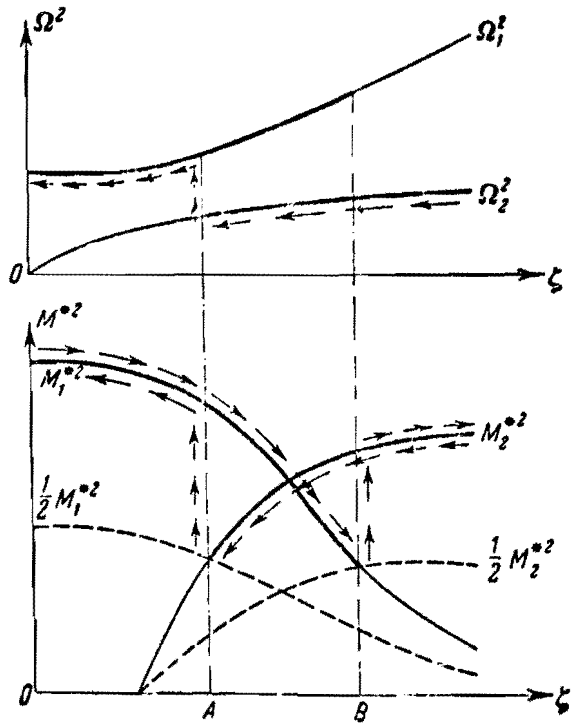
$$M_1^{*2} > \frac{1}{2} M_2^{*2}. \qquad (14.18)$$

In fact, as is seen from equation (14.3), one of its roots ω^2, is less than the two magnitudes n_1^2 and n_2^2 while the second root is greater than each of these magnitudes. As a result, the magnitudes $n_2^2 - \omega_2^2$ and $n_2^2 - \omega_1^2$ always have opposite signs. As regards the magnitude $1-\varkappa_1\varkappa_2$, it is assumed positive, since otherwise equation (14.3) would give imaginary values for one of the magnitudes ω_i .

The condition of stability of the self-oscillations corresponding to the frequency $_2$ will evidently have the form

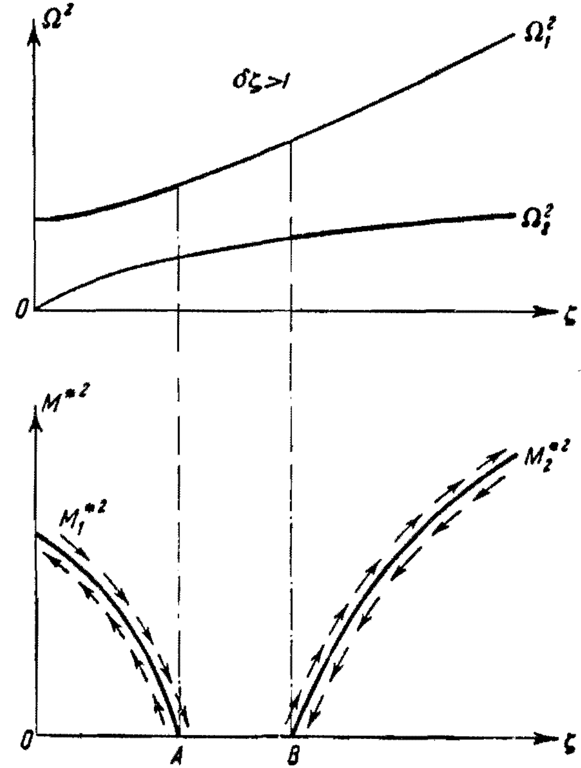
$$M_2^{*2} > \frac{1}{2} M_1^{*2}. \qquad (14.19)$$

In fig. 24 and 25 are given the graphs of the magnitude M_1^{*2}, M_2^{*2} and $\Omega^2 = \omega^2/n_1^2$ as functions of the 'mistuning'



Ω²
Ω₁²
Ω₂²
0
ζ
M*²
M₁*²
M₂*²
½ M₁*²
½ M₂*²
A
B

Fig. 24



Ω²
Ω₁²
Ω₂²
δζ>1
0
ζ
M*²
M₁*²
M₂*²
A
B

Fig. 25

$\zeta = n_2^2/n_1^2$ for fixed values of the magnitudes $\varkappa_1$ $\varkappa_2$ and $\delta\zeta$. Fig. 24 corresponds to the case $\delta\zeta < 1$, and fig. 25 to the case $\delta\zeta > 1$. The solid lines correspond to the stable oscillations. From these graphs it is seen that in the case $\delta\zeta < 1$, if the tuning is carried out from the right, i.e. from large values of ζ, self-oscillations with frequency ω_2 are first established in the system and then after a value of ζ corresponding to the point A the oscillations are disrupted at the frequency ω_1. In tuning from the reverse side the jump to the frequency ω_2 will be effected in another place, namely for ζ corresponding to the point B. This is what constitutes the phenonmenon of the so-called "pulling" effect.

In the case $\delta\zeta > 1$ for values of ζ situated to the left of point A self-oscillations with frequency ω_1 are set up in the system, while for values of ζ situated to the right of point B there are set up oscillations with frequency ω_2. For values of ζ situated on the segment AB there are no periodic oscillations of the system. It is easy to show that for these values of ζ the state of equilibrium of the system is stable, i.e. there are no self-oscillations.

15. Some Special Cases

Let us consider the equation of the second order

$$\frac{d^2x}{dt^2} = (\mu p_1 + \mu^2 p_2 + \ldots)x, \tag{15.1}$$

where p_1, $p_2, \ldots$ are continuous periodic functions of t of period ω and the series converges for sufficiently small values of μ.

For $\mu = 0$ the corresponding fundamental equation has a double zero root to which corresponds one set of solutions. Hence we shall not now be justified in asserting that the characteristic exponents of (15.1) are developable in integral powers of μ. It is easy to see that in the case considered these characteristic exponents can be expanded either in integral powers of μ or in integral powers of the magnitude $\sqrt{\mu}$. In fact, let

$$\rho^2 - 2A\rho + B = 0 \tag{15.2}$$

be the characteristic equation of some system of the second order with periodic coefficients analytic functions of the parameter μ so that A and B are likewise analytic functions of μ. We shall assume the magnitude $A^2 - B$ becomes zero for $\mu = 0$ so that equation (15.2) has a double root for $\mu = 0$. Let

$$A^2 - B = a_k\mu^k + a_{k+1}\mu^{k+1} + \ldots,$$

where $a_k \neq 0$. We can then write

$$\rho = A \pm (a_k\mu^k + a_{k+1}\mu^{k+1} + \ldots)^{1/2} =$$
$$= A \pm \sqrt{a_k\mu^k}\left(1 + \frac{a_{k+1}}{2a_k}\mu + \ldots\right),$$

and therefore the roots of equation (15.2) will expand in integral powers of the magnitude μ or of the magnitude $\sqrt{\mu}$ depending on whether the number k is even or odd.

Let us assume that for equation (15.1) the condition is satisfied

$$\int_0^\omega p_1\, dt = 0. \tag{15.3}$$

In this case if one more condition, indicated below, is satisfied the characteristic exponents will be expansible in integral powers of μ and these expansions can be found in the following manner:

In equation (15.1) we set $x = e^{\alpha t}y$, where

$$\alpha = \alpha_1\mu + \alpha_2\mu^2 + \ldots$$

is the required characteristic exponent. The equation thus obtained

$$\frac{d^2y}{dt^2} = (\mu p_1 + \mu^2 p_2 + \ldots)y - (\alpha_1\mu + \alpha_2\mu^2 + \ldots)^2 y - 2(\alpha_1\mu + \alpha_2\mu^2 + \ldots)\frac{dy}{dt}$$

must admit a periodic solution. We shall seek this periodic solution in the form of the series

$$y = y_0 + \mu y_1 + \mu^2 y_2 + \ldots, \tag{15.5}$$

where y_j is a periodic function of t of period ω. For y_0 we have $d^2y_0/dt^2 = 0$ and therefore this magnitude must be constant. Without affecting generality we can set $y_0 = 1$ since the solution (15.1) can be multiplied by an arbitrary constant multiplier. For y_1 we shall then have:

$$\frac{d^2 y_1}{dt^2} = p_1.$$

In virtue of condition (15.3) this equation admits a periodic solution.

Further we have:

$$\frac{d^2 y_2}{dt^2} = p_1 y_1 + p_2 - \alpha_1^2 - 2\alpha_1 \frac{dy_1}{dt}.$$

For this equation to admit a periodic solution it is necessary and sufficient that the condition be satisfied

$$\int_0^{\omega} \left(p_1 y_1 + p_2 - \alpha_1^2 - 2\alpha_1 \frac{dy_1}{dt} \right) dt = 0,$$

which gives

$$\alpha_1 = \pm \frac{1}{\sqrt{\omega}} \sqrt{\int_0^{\omega} (p_1 y_1 + p_2)\, dt}. \tag{15.6}$$

We shall assume that $\alpha_1 \neq 0$.

In a similar manner are computed all the remaining coefficients α_i and the equations for them will be linear so that having chosen as α_1 one of the magnitudes (15.6) we obtain an entirely definite characteristic exponent.

In fact, let us assume that all the constants $\alpha_2, \ldots, \alpha_{k-1}$ have been determined and all the functions $y_2, \ldots, y_k$ and that the latter are found periodic. We can then write:

$$\frac{d^2 y_{k+1}}{dt^2} = -2\alpha_1 \alpha_k - 2\alpha_k \frac{dy_1}{dt} + Y_{k+1}(t),$$

where Y_{k+1} is an entirely definite periodic function. The condition of periodicity of the function y_{k+1} gives

$$-2\alpha_1 \omega \alpha_k + \int_0^{\omega} Y_{k+1}(t)\, dt = 0,$$

which shows the correctness of our assertion, since $\alpha_1 \neq 0$.

As an example we shall investigate the stability of the oscillations of a pendulum the point of suspension of which performs harmonic oscillations of large frequency at an angle δ to the vertical (see sec. 10 of chapter II). The differential equation of motion has the form

$$\frac{d^2\varphi}{d\tau^2} = -\mu b \sin(\varphi - \delta)\cos\tau - \mu^2 \sin\varphi,$$

$$\mu b = \frac{A}{l}, \quad b = \frac{A\omega}{\sqrt{gl}}, \quad \tau = \omega t,$$

where A is the amplitude and ω the frequency of the oscillations of the point of suspension of the pendulum, while l is its reduced length. The periodic solution of this equation, the stability of which we must investigate, with an accuracy up to magnitudes of the first order with respect to μ is determined by the equation

$$\varphi = \varphi_0 + \mu b \sin(\varphi_0 - \delta)\cos\tau,$$

where φ_0 is a root of the equation

$$\sin\varphi_0 + \frac{b^2}{4}\sin 2(\varphi_0 - \delta) = 0. \qquad (15.7)$$

Forming the equation in variations we obtain:

$$\frac{d^2x}{d\tau^2} = \{-\mu b\cos(\varphi_0 - \delta)\cos\tau + \mu^2[b^2\sin^2(\varphi_0 - \delta)\cos^2\tau - \cos\varphi_0] + \dots\}\, x.$$

where the terms not written down are of an order higher than the second with respect to μ. This equation has the form (15.1) with

$$p_1 = -b\cos(\varphi_0 - \delta)\cos\tau, \quad y_1 = \int d\tau \int p_1\, d\tau = b\cos(\varphi_0 - \delta)\cos\tau,$$

$$p_2 = b^2\sin^2(\varphi_0 - \delta)\cos^2\tau - \cos\varphi_0.$$

In order that the motion be stable it is necessary that the real part of each of the magnitudes α_1 determined by formula (15.6) be equal to zero, for otherwise one of these magnitudes will have a positive real part and the motion will be unstable. Thus, we have the following condition of stability

$$-\int_0^{2\pi}(p_1y_1 + p_2)\, d\tau = 2\pi\left[\cos\varphi_0 + \frac{1}{2}b^2\cos 2(\varphi_0 - \delta)\right] > 0. \qquad (15.8)$$

Let us consider the particular case when $\delta = 0$, i.e. when the point of suspension performs oscillations along the vertical. In this case equation (15.7) has two solutions, $\varphi_0 = 0$ and $\varphi_0 = \pi$, to which correspond oscillations about thelower and upper positions of equilibrium of the pendulum. For the first of these solutions the condition of stability (15.8), which in the particular case considered assumes the form

$$\cos\varphi_0 + \frac{b^2}{2}\cos 2\varphi_0 > 0, \qquad (15.9)$$

is always satisfied. But it is satisfied also for the second of these solutions provided

$$b^2 = \frac{A^2\omega^2}{gl} > 2, \qquad (15.10)$$

i.e. if the frequency of the oscillations of the point of suspension is sufficiently large. We note that for the condition (15.10) there are obtained for φ_0 two further solutions determined by the formula

$$\cos\varphi_0 = -\frac{2}{b^2}.$$

For the oscillations corresponding to these two solutions condition (15.9) is not satisfied and these solutions will be unstable.

If the condition (15.3) for equation (15.1) is not satisfied the characteristic exponents will be developable in powers of the magnitude $\sqrt{\mu}$. They can be computed by a procedure similar to the one just considered setting in (15.1) $x = e^{\alpha t}y$, where

$$\alpha = \alpha_1\mu^{1/2} + \alpha_2\mu + \alpha_3\mu^{3/2} + \ldots, \qquad (15.11)$$

and seeking the periodic solution of the obtained equation in the form of the series

$$y = y_0 + y_1\mu^{1/2} + y_2\mu + \ldots \qquad (15.12)$$

We shall not here dwell on the question of the convergence of the series (15.4) and (15.11). We may note, finally, that in determining the magnitudes y_1, y_2,... arbitrary additive constants enter both in series (15.5) and in series (15.12). This corresponds to the fact that the solutions (15.5) and (15.12) can be multiplied by an arbitrary constant, which, in particular, can be chosen in the form of a series in μ or in $\sqrt{\mu}$ with constant coefficients.

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(Nekotorye Zadachi Teorii Nelineinykh Kolebanii)

By
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CHAPTER IV

ALMOST PERIODIC OSCILLATIONS OF QUASILINEAR SYSTEMS

1. Statement of the Problem. Basic Concepts

Let us consider the quasilinear system

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n + f_s(t) + \mu F_s(t, x_1, \ldots, x_n, \mu) \quad (1.1)$$
$$(s = 1, \ldots, n).$$

In the preceding chapters we considered the oscillations of quasilinear systems on the assumption that equations (1.1) either entirely do not depend on t or are periodic with respect to this variable. Moreover, we considered only periodic oscillations of these systems. In practice however it is necessary to deal with oscillations of a more general kind. Thus, the oscillations described by the function

$$x = A_1 \sin \omega_1 t + A_2 \sin \omega_2 t,$$

i.e. the superposition of two simple harmonic oscillations, will not be periodic if the frequencies ω_1 and ω_2 are irrational. This oscillation is a very particular case of special, so-called ALMOST PERIODIC OSCILLATIONS.

Let $f(t)$ be a continuous function defined for all values of t in the interval $-\infty < t < +\infty$. Then according to H. Bohr THE FUNCTION $f(t)$ IS TERMED ALMOST-PERIODIC IF FOR ANY ARBITRARILY SMALL POSITIVE NUMBER ε A POSITIVE NUMBER $l(\varepsilon)$ CAN BE FOUND SUCH THAT WITHIN EACH INTERVAL OF LENGTH l AT LEAST ONE NUMBER $\tau(\varepsilon)$ CAN BE FOUND FOR WHICH FOR ALL t THE INEQUALITY IS SATISFIED.

$$|f(t+\tau) - f(t)| < \varepsilon.$$

We shall here present without proof several basic properties of almost periodic functions.[1]

1) Every almost periodic function f(t) is bounded over the entire interval $(-\infty, +\infty)$.

2) Every almost periodic function f(t) is uniformly continuous over the entire interval $(-\infty, +\infty)$.

3) The sum of almost periodic functions is an almost periodic function.

Since periodic functions are a special case of almost periodic functions it follows that finite sums with constant coefficients of the form

$$f(t) = A_0 + \sum (A_p \cos \nu_p t + B_p \sin \nu_p t), \tag{1.2}$$

where $\nu_1, \nu_2, \ldots$ are entirely arbitrary numbers, in general incommensurable with each other, are almost periodic functions.

Finite or infinite sums of the form (1.2) we shall in what follows denote as TRIGONOMETRIC SUMS OR GENERALIZED FOURIER SERIES.

4) The product of almost periodic functions is an almost periodic function.

5) If an almost periodic function f(t) satisfies, over the entire interval $-\infty < t < +\infty$ the condition $|f(t)| > \alpha > 0$, where α is a constant, the function $1/f(t)$ will likewise be almost periodic.

6) The limit f(t) of a uniformly converging sequence of almost periodic functions $f_1(t), f_2(t), \ldots$ is an almost periodic function. From this follows, in particular, that not only finite trigonometric sums but also infinite sums of this kind, if they converge uniformly, are almost periodic functions.

7) For every almost periodic function f(t) there exists a limit

$$g = \lim_{t\to\infty} \frac{1}{t} \int_0^t f(t)\,dt. \tag{1.3}$$

----

[1] A detailed presentation of the theory of almost periodic functions can be found in the following books: Bohr, H., Almost Periodic Functions, Levitan B.M. Pochti- periodicheskie funktsii, (Almost Periodic Functions), Gostekhizdat, 1953.

The magnitude g is called the MEAN VALUE of the almost periodic function $f(t)$. If, in particular, the function $f(t)$ is periodic of period ω, it is possible, as is easily seen, to write:

$$g = \frac{1}{\omega} \int_0^{\omega} f(t)\, dt.$$

It is known that for any periodic function $f(t)$ the relation holds (see sec. 4 of chapter III)

$$f(t) = gt + \varphi(t), \tag{1.4}$$

where g is the mean value of this function and $\varphi(t)$ is a certain periodic function of the same period as $f(t)$. From this it follows that if $f(t)$ is an almost periodic function but is a sum of periodic functions of different periods (as, for example, finite Fourier series), then for it formula (1.4) will likewise hold, in which $\varphi(t)$ will now be an almost periodic function (the sum of periodic functions). The constant g will here be defined by formula (1.3). It is possible however to show that in the general case when $f(t)$ is an arbitrary almost periodic function formula (1.4) will not be true for it. In other words, in the general case there does not exist a constant g and an almost periodic function $\varphi(t)$ connected with the almost periodic function $f(t)$ by relation (1.4).

In this chapter we shall deal principally with those almost periodic functions that have the form (1.2), i.e. are finite (generalized) Fourier series. We shall be interested in almost periodic oscillations of systems of the form (1.1), which we shall assume, in case they are not autonomous, to be almost periodic with respect to t. For the purpose of simplifying the computations we shall most frequently assume that for the equations (1.1) the following conditions are satisfied:

1) The functions $f_s(t)$ are finite trigonometric sums.

2) The functions F_s can be expanded in the series

$$F_s(t, x_1, \ldots, x_n, \mu) = \sum_{i=0}^{\infty} F_s^{(i)}(t, x_1, \ldots, x_n)\mu^i, \tag{1.5}$$

converging for $\mu < \mu_0$, where μ_0 is a positive number not depending on $t, x_1, \ldots, x_n$.

3) The functions $F_s^{(i)}(t, x_1, \ldots, x_n)$ are polynomials in $x_1, \ldots, x_n$, almost periodic with respect to t, developable with respect to this variable in finite Fourier series. We can thus write:

$$F_s^{(i)} = A_{s0}^{(i)} + \sum_p (A_{sp}^{(i)} \cos \nu_p t + B_{sp}^{(i)} \sin \nu_p t).$$

where the sums contain only a finite number of terms and the coefficients are polynomials in $x_1, \ldots, x_n$.

2. Almost Periodic Solutions of Nonhomogeneous Linear Equations

Let us consider the system of linear nonhomogeneous equations

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n + f_s(t) \qquad (s = 1, \ldots, n), \tag{2.1}$$

where the coefficients a_{sj} are constant and f_s are almost periodic functions of t, with respect to which we do not for the present make any restrictions. We shall explain the conditions under which equations (2.1) admit almost periodic solutions and establish several properties of these solutions.

We shall assume first that the fundamental equation of the system (2.1)

$$\begin{vmatrix} a_{11} - \rho & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \rho & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \rho \end{vmatrix} = 0 \tag{2.2}$$

has no roots with real parts equal to zero. The following theorem, established by Neugebauer and Bohr, then holds:

THEOREM 1. IF EQUATION (2.2) HAS NO ROOTS WITH NULL REAL PARTS THE SYSTEM (2.1) ADMITS ONE AND ONLY ONE ALMOST PERIODIC SOLUTION $x_s(t)$ AND THIS SOLUTION SATISFIES THE INEQUALITIES

$$|x_s(t)| < AM, \tag{2.3}$$

WHERE M IS THE UPPER LIMIT OF THE FUNCTIONS $|f_s(t)|$ IN THE INTERVAL $(-\infty, +\infty)$ AND A IS A CERTAIN CONSTANT DEPENDING ONLY ON THE COEFFICIENTS a_{sj} AND NOT DEPENDING ON THE CHOICE OF THE FUNCTIONS f_s.

PROOF. Let us first assume that we have only one equation

$$\frac{dx}{dt} = \alpha x + f(t), \tag{2.4}$$

and let $\alpha = \lambda + i\nu$, with $\lambda \neq 0$. Then, putting $a = +\infty$ for $\lambda > 0$ and $a = -\infty$ for $\lambda < 0$, we can write down the following particular solution of equation (2.4):

$$x(t) = e^{\alpha t} \int_a^t e^{-\alpha t} f(t)\, dt = \int_a^t e^{\alpha(t-u)} f(u)\, du. \tag{2.5}$$

This solution is almost periodic. In fact, since the function $f(t)$ is almost periodic, given an arbitrary positive number ε we find a positive number $l(\varepsilon)$ such that within each interval of length l there exists an almost periodic τ for which over the entire real axis of t the inequality holds

$$|f(t+\tau) - f(t)| < \varepsilon |\alpha|.$$

But then we can write

$$|x(t+\tau) - x(t)| = \left| \int_a^{t+\tau} e^{\alpha(t+\tau-u)} f(u)\, du - \int_a^t e^{\alpha(t-u)} f(u)\, du \right| =$$

$$= \left| \int_a^t [f(u+\tau) - f(u)]\, e^{\alpha(t-u)}\, du \right| < \varepsilon |\alpha| \left| \int_a^t e^{\alpha(t-u)}\, du \right| = \varepsilon,$$

which shows that the solution $x(t)$ is almost periodic. Further, we have:

$$|x_s(t)| < M \left| \int_a^\infty e^{\alpha(t-u)}\, du \right| = \frac{M}{|\alpha|}.$$

Thus the theorem is true for n = 1. It remains therefore to show that it will be true for arbitrary n if it is true for a system of the (n-1)-th order.

For this purpose let us consider some root $\alpha = \lambda + i\nu$ of equation (2.2) and let

$$x_s = A_s e^{\alpha t}$$

be the solution, corresponding to this root, of the homogeneous part of the system (2.1). Here A_s are constants of which at least one is different from zero. For definiteness we shall assume that $A_1 \neq 0$ and transform system (2.1) with the aid of the substitution

$$x_1 = A_1 y_1, \quad x_2 = A_2 y_1 + y_2, \ldots, x_n = A_n y_1 + y_n.$$

The transformed equations have the form

$$\left.\begin{aligned} \frac{dy_1}{dt} &= \alpha y_1 + b_{12} y_2 + \ldots + b_{1n} y_n + F_1(t), \\ \frac{dy_k}{dt} &= b_{k2} y_2 + \ldots + b_{kn} y_n + F_k(t) \\ &\qquad (k = 2, \ldots, n), \end{aligned}\right\} \tag{2.6}$$

where b_{kj} are constants and

$$F_1 = \frac{1}{A_1} f_1, \quad F_k = f_k - \frac{A_k}{A_1} f_1 \qquad (k = 2, \ldots, n). \tag{2.7}$$

The roots of the equation

$$\begin{vmatrix} b_{22} - \rho & \ldots & b_{2n} \\ \ldots & \ldots & \ldots \\ b_{n2} & \ldots & b_{nn} - \rho \end{vmatrix} = 0 \tag{2.8}$$

are the n-1 roots of equation (2.1) and therefore equation (2.8) has no roots with null real parts. Hence the last n-1 equations of the system (2.6) admit one and only one almost periodic solution $y_k(t)$ for which by assumption the inequalities are satisfied

$$|y_k(t)| < A^* |F_k(t)|_{\max} \qquad (k = 2, \ldots, n), \tag{2.9}$$

where A^* is a certain constant depending only on b_{kj} and therefore, ultimately, only on a_{ij}. But then, by what has been proved, the first equation of system (2.6) admits an almost periodic solution $y_1(t)$ for which the inequality is satisfied

$$|y_1(t)| < \frac{1}{|\alpha|}\left\{A^* \sum_{k=2}^{n} |b_{1k}|\,|F_k(t)|_{\max} + |F_1(t)|_{\max}\right\}. \qquad (2.10)$$

From (2.9) and (2.10) on the basis of (2.7) we have:

$$|y_k| < A^*\left(1 + \left|\frac{A_k}{A_1}\right|\right) M \qquad (k = 2, \ldots, n),$$

$$|y_1| < \frac{1}{|\alpha|}\left\{A^* \sum_{k=2}^{n} |b_{1k}|\left(1 + \left|\frac{A_k}{A_1}\right|\right) + \frac{1}{|A_1|}\right\} M,$$

which proves the theorem.

Let us assume now that equation (2.2)has roots with nul real parts. In this case the system (2.1) will admit almost periodic solutions only under certain conditions with respect to the functions f_s. We shall establish these conditions.

Let us assume that equation (2.2) has the purely imaginary roots $\pm ik_1, \pm ik_2, \ldots, \pm ik_j$, where we write out each of these roots as many times as the number of corresponding sets of solutions of the homogeneous part of the system (2.1). We shall assume, moreover, that equation (2.2) has a zero root to which correspond l sets of solutions. Then the system

$$\frac{dy_s}{dt} + a_{1s}y_1 + \ldots + a_{ns}\,y_n = 0, \qquad (2.11)$$

conjugate to the homogeneous part of the system (2.1), will have $m = 2j + l$ periodic (of different periods) solutions of the form

$$\left.\begin{aligned} \psi^*_{si} &= P_{si}\cos k_i t - Q_{si}\sin k_i t, \\ \psi^*_{s,\,j+i} &= P_{si}\sin k_i t + Q_{si}\cos k_i t \\ & (i = 1, \ldots, j), \\ \psi^*_{s,\,2j+1} &= R_{s1}, \ldots, \psi^*_{sm} = R_{sl}, \end{aligned}\right\} \qquad (2.12)$$

where P_{si}, Q_{si}, $R_{s1}, \ldots, R_{sl}$ are constants. These solutions can be considered as almost periodic. Every other almost periodic solution of system (2.11) will be a linear combination of solutions (2.12). We shall consider, for generality, some other system $\psi_{s1}, \ldots, \psi_{sm}$ of almost periodic solutions of equations (2.11) determined by the relations

$$\psi_{sr} = a_{r1}\psi^*_{s1} + a_{r2}\psi^*_{s2} + \ldots + a_{rm}\psi^*_{sm} \quad (r = 1, \ldots, m), \qquad (2.13)$$

where a_{rp} are some constants for which the determinant $|a_{rp}|$ is different from zero.

We shall now assume that the functions $f_s(t)$ are finite trigonometric sums of the form

$$f_s(t) = A_{s0} + \sum (A_{sp} \cos \nu_p t + B_{sp} \sin \nu_p t),$$

and shall prove the following theorem:

THEOREM 2. IF IN THE SYSTEM (2.1) THE FUNCTIONS $f_s(t)$ ARE FINITE TRIGONOMETRIC SUMS THEN IN ORDER FOR THIS SYSTEM TO ADMIT AN ALMOST PERIODIC SOLUTION, WHEN THE EQUATION (2.2) HAS ROOTS WITH ZERO REAL PARTS, IT IS NECESSARY AND SUFFICIENT THAT THE RELATIONS BE SATISFIED

$$\lim_{t\to\infty} \int_0^t \sum_{\alpha=1}^n f_\alpha(t)\psi_{\alpha r}\, dt = 0 \qquad (r = 1, \ldots, m). \qquad (2.14)$$

PROOF. We shall first prove the necessity of conditions (2.14). For this purpose let us assume that the system (2.1) admits the almost periodic solution $x_s(t)$. We shall then have (formula 3.5) of Chapter II):

$$\frac{d}{dt} \sum_{\alpha=1}^n \psi_{\alpha r} x_\alpha(t) = \sum_{\alpha=1}^n f_\alpha(t)\psi_{\alpha r}.$$

whence, taking into account that the functions $x_\alpha(t)$ and $\psi_{\alpha r}$ as almost periodic are necessarily bounded, we find:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\alpha=1}^n f_\alpha(t)\psi_{\alpha r}\, dt = \lim_{t\to\infty} \frac{1}{t} \Big|_0^t \sum_{\alpha=1}^n x_\alpha(t)\psi_{\alpha r} = 0,$$

which proves the necessity of conditions (2.14).

In order to establish the sufficiency of conditions (2.14) we shall assume that they are satisfied and show that then system (2.1) admits an almost periodic solution.

From (2.13) it follows that if conditions (2.14) are satisfied there are satisfied also the conditions

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\sum_{\alpha=1}^n f_\alpha \psi^*_{\alpha r}\,dt=0 \qquad (r=1, \ldots, m), \tag{2.15}$$

since the determinant $|a_{rp}|$ is different from zero. We shall consider the system of equations

$$\frac{dx_s}{dt}=a_{s1}x_1+\ldots+a_{sn}x_n+A_{sp}\cos\nu_p t+B_{sp}\sin\nu_p t \tag{2.16}$$

and show that if (2.15) is satisfied the system admits a periodic solution of period $2\pi/\nu_p$. In fact, this assertion is evident if the magnitude ν_p is not equal to any one of the magnitudes $k_1, \ldots, k_j$. We shall assume however that ν_p is equal to certain of the magnitudes $k_1, \ldots, k_j$ (among the latter some may be equal). For definiteness let us assume that $\nu_p = k_1 = k_2$. Then as is easily seen, we shall have:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\sum_{\alpha=1}^n [f_\alpha - A_{\alpha p}\cos\nu_p t - B_{\alpha p}\sin\nu_p t]\,\psi^*_{\alpha l}\,dt=0$$
$$(l=1,\ 2,\ j+1,\ j+2),$$

since the expressions under the integral sign on the basis of (2.12) contain only trigonometric terms. Consequently equations (2.15) for $r = 1, 2, j + 1, j + 2$ assume the form

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\sum_{\alpha=1}^n (A_{\alpha p}\cos\nu_p t + B_{\alpha p}\sin\nu_p t)\,\psi^*_{\alpha l}\,dt=0$$
$$(l=1,\ 2,\ j+1,\ j+2).$$

But these conditions for the assumptions made are necessary and sufficient for the existence of a periodic solution (of period $2\pi/\nu_p$) for the system (2.16).

Similarly from the last l of equations (2.15) it is easily found that the system

$$\frac{dx_s}{dt} = a_{s1}x_1 + \dots a_{sn}x_n + A_{s0}$$

likewise admits a periodic solution (in which all the functions x_s are constants). Adding this solution to the periodic solutions of all the systems (2.16) corresponding to the different values which the index p may assume we obtain an almost periodic solution of system (2.1). This proves the sufficiency of conditions (2.14) for the existence of an almost periodic solution of the system (2.1).

Thus the theorem has been completely proven.[1]

3. Certain Properties of the Problem of Almost Periodic Oscillations. Small Divisors

We now proceed to the problem of almost periodic oscillations of quasilinear systems. The problem reduces to the question of the existence and computation of the almost periodic solutions of a system of differential equations of the form (1.1). We here encounter some very essential properties which we must first of all stop to consider.

As in the problem of periodic solutions we denote by $x_s(t,\beta_1,\dots\beta_n,\mu)$ the solution of system (1.1) with the initial conditions $x_s(0,\beta_1,\dots\beta_n,\mu) = \varphi_s(0) + \beta_s$, where

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[1] This theorem is a special case of a more general theorem established in the paper: Malkin I.G., O rezonanse v kvazigarmonicheskikh sistemakh, (On Resonance in Quasi-harmonic Systems), Prikl. mat. i mekh., vol. XVIII, no.4, 1954. See also sec. 9 of chapter V of this book.

$\varphi_s(t)$ is the generating solution. As before, the magnitudes x_s can be represented in the form of series developed in powers of the magnitudes $\beta_1, \ldots \beta_n, \mu$ whose coefficients can actually be computed. We shall now attempt to determine the magnitudes $\beta_1, \ldots, \beta_n$ in such manner that the solution under consideration is almost periodic. We then at once encounter a new difficulty thanks to which the problem under consideration differs sharply from the problem of periodic solutions. In fact, if it were a question of the periodic solutions of some period ω the magnitudes $\beta_1, \ldots, \beta_n$ would have to be determined from the equations

$$x_s(\omega, \beta_1, \ldots, \beta_n, \mu) - x_s(0, \beta_1, \ldots, \beta_n, \mu) = 0.$$

Thus, the problem of the periodic solutions reduces to that of the solution of a system of ordinary equations. It is in this manner that we have solved the problem in the preceding chapters. The problem of almost periodic solutions cannot however be considered by this method since the conditions of almost periodicity of the functions $x_s(t, \beta_1, \ldots, \beta_n, \mu)$ cannot be written in the form of a system of ordinary equations. Thus, the fundamental procedure which we made use of in the solution of the problem of periodic solutions is entirely inapplicable in the case of the problem of ALMOST periodic solutions.

It is possible however to proceed by a different method. We can try finding immediately an almost periodic solution of the system (1.1) in the form of formal series developed in powers of μ. By this method we have in the preceding chapters actually found periodic solutions and obtained convergent series. In the problem of almost periodic solutions however we here encounter another fundamental difficulty, which we can best explain with the aid of a concrete example.

Let us consider the oscillations of a system described by the equation

$$\frac{d^2x}{dt^2} + x = \mu(\gamma x^3 + A \sin t + B \sin \omega t). \tag{3.1}$$

If ω were equal to an integer we would be dealing with the simplest case of periodic oscillations of a system with one degree of freedom. We shall however assume that the number ω is irrational, so that the external force is not periodic and the problem can only involve almost periodic oscillations.

We shall seek to obtain an almost periodic solution of system (3.1) in the form of a series

$$x = x_0 + \mu x_1 + \mu^2 x_2 + \dots \tag{3.2}$$

with almost periodic coefficients. We have

$$\frac{d^2 x_0}{dt^2} + x_0 = 0,$$

$$\frac{d^2 x_1}{dt^2} + x_1 = \gamma x_0^3 + A \sin t + B \sin \omega t,$$

$$\frac{d^2 x_2}{dt^2} + x_2 = 3\gamma x_0^2 x_1.$$

Since equation (3.1) does not contain dx/dt, the required solution will contain only sines. Taking this into account we obtain for x_0:

$$x_0 = M_0 \sin t,$$

where M_0 is an arbitrary constant. This constant is determined from the condition of almost periodicity of the function x_1, for which reason the equation determining this function must not contain a term with $\sin t$. This condition gives:

$$M_0^3 = -\frac{4A}{3\gamma},$$

after which for x_1 we obtain:

$$x_1 = \frac{1}{32}\gamma M_0^3 \sin 3t + \frac{B}{1-\omega^2}\sin \omega t + M_1 \sin t,$$

where M_1 is an arbitrary constant. Equating to zero the coefficient of $\sin t$ in the equation for x_2 we obtain:

$$M_1 = \frac{1}{48}\gamma M_0^2,$$

after which from this equation we find:

$$x_2 = \frac{3}{1024}\gamma M_0^2 A \sin 3t - \frac{3}{2048}\gamma M_0^2 A \sin 5t +$$
$$+ \frac{3}{2(1-\omega^2)^2}\gamma M_0^2 B \sin \omega t - \frac{3\gamma M_0^2 B}{2(1-\omega^2)[1-(\omega-2)^2]}\sin(\omega-2)t -$$
$$- \frac{3\gamma M_0^2 B}{2(1-\omega^2)[1-(\omega+2)^2]}\sin(\omega+2)t + M_2 \sin t,$$

where M_2 is a new arbitrary constant. Continuing in this manner we obtain an entirely definite series (3.2) with almost periodic coefficients that formally satisfies equation (3.1).

Thus, the circumstance that the magnitude ω is irrational does not introduce any difficulties in the construction of a formal almost periodic solution and all computations are obtained in exactly the same way as in the determination of periodic solutions. Unfortunately however, the series so obtained will be divergent. This divergence is brought about by the following circumstances.

On the right hand side of the equation for x_k will be contained terms of the form

$$A_{mn}\sin(m\omega+n)t, \tag{3.3}$$

where m and n are integers the second of which may be positive or negative. For k sufficiently large the equations for x_k will contain terms of the form (3.3) with arbitrarily large values of m and $|n|$. To the terms (3.3) there will correspond in the function x_k the terms

$$\frac{A_{mn}}{1-(m\omega+n)^2}\sin(m\omega+n)t.$$

But whatever the irrational number ω, it is always possible to find integral values of m and n such that the number $m\omega + n$, without being equal to unity will differ from it by as little as we please. Consequently the function x_k will contain terms with arbitrarily small denominators. The presence of small divisors of this kind in the series (3.2) is what produces its divergence.

Similar circumstances are encountered also in the general case of equations (1.1). The construction of formal almost periodic solutions of these equations does not occasion greater difficulties than the construction of periodic solutions in the case of periodic equations. These series however are in general found to be divergent because their terms contain small divisors.

Because of the circumstances noted above the problem of almost periodic solutions is one of the most difficult in the theory of nonlinear oscillations. Nevertheless at the present time it was here succeeded in obtaining a number of fundamental results. They have been established principally in the numerous investigations of N.N. Krylov and N.N. Bogolyubov and, particularly, in the work of N.N. Bogolyubov: "On Certain Statistical Methods in Mathematical Physics".[1]

The methods presented in this chapter of finding almost periodic solutions rest largely on the investigations of N.M. Krylov and N.N. Bogolyubov.

4. Almost Periodic Oscillations of Nonautonomous Systems in the Absence of Critical Roots of the Fundamental Equation

Let us consider the nonautonomous quasilinear system

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n + f_s(t) + \mu F_s(t, x_1, \ldots, x_n, \mu) \quad (4.1)$$
$$(s = 1, \ldots, n)$$

for the general assumptions given in sec. 1 . We shall assume that the fundamental equation

$$\begin{vmatrix} a_{11}-\rho & a_{12} & \ldots & a_{1n} \\ a_{2n} & a_{22}-\rho & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\rho \end{vmatrix} = 0 \quad (4.2)$$

of the generating system

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n + f_s(t) \quad (4.3)$$

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[1] Bogolyubov N.N., O nekotorykh statisticheskikh metodakh v matematicheskoi fizike, (On Certain Statistical Methods in Mathematical Physics), Izd-vo Akad. nauk USSR, 1945

has no roots with zero real parts. Roots of this kind we shall denote as CRITICAL. In this case the generating system (4.3), as was shown in sec. 2, admits one and only one almost periodic solution $x_s = \varphi_s(t)$ which we take as the generating solution. We shall seek an almost periodic solution of the system (4.1) which for $\mu = 0$ reduces to the generating solution.

For this purpose we make use of the method of successive approximations. As the first approximation we shall take the generating solution, and as the k-th approximation $x_s^{(k)}$ of the functions x_s the almost periodic solution of the system

$$\frac{dx_s^{(k)}}{dt} = a_{s1}x_1^{(k)} + \dots + a_{sn}x_n^{(k)} + f_s(t) + \mu F_s(t,\ x_1^{(k-1)},\ \dots,\ x_n^{(k-1)},\ \mu) \quad (k = 2,\ 3,\ \dots). \tag{4.4}$$

Equations (4.4) uniquely determine the sequence of the almost periodic functions $x_s^{(k)}$. The inequalities (2.3) which the almost periodic solution of system (4.4) satisfies permits showing easily that for sufficiently small μ the sequence $x_s^{(k)}$ converges to certain almost periodic functions $x_s(t)$ satisfying equations (4.1). We shall not however give this proof here since a more general proposition will be proved below in sec. 9. It will then also be shown that for sufficiently small μ the solution thus obtained will be the only almost periodic solution of the system (4.1) that reduces for $\mu = 0$ to the generating solution. We thus arrive at the following theorem:

THEOREM. IF EQUATION (4.2) HAS NO ROOTS WITH ZERO REAL PARTS SYSTEM (4.1) FOR SUFFICIENTLY SMALL μ ADMITS ONE AND ONLY ONE ALMOST PERIODIC SOLUTION REDUCING FOR $\mu = 0$ TO THE GENERATING SOLUTION. [1]

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[1] This theorem with the aid of the simple transformation $x_s = \varphi_s(t) + y_s$ is obtained as a consequence of a considerably more general theorem established by N.N. Bogolyubov (see sec. 8 of the monograph cited on p. 273).

Thus, in the absence of critical roots of the fundamental equation an almost periodic solution of the system (4.1) exists and can be computed by quite the same procedure by which we have in the preceding chapters computed the periodic solutions in nonresonance cases for nonanalytic equations. The case where the fundamental equation has critical roots is considerably more complicated. It can however be reduced to the preceding with the aid of a special transformation introduced into the theory of nonlinear oscillations by N.M. Krylov and N.N. Bogolyubov.

5. Transformation of Krylov and Bogolyubov

Let us assume that in the system (4.1) satisfying the conditions of sec. 1 the constants a_{sj} and the functions $f_s(t)$ are equal to zero so that this system has the form

$$\frac{dx_s}{dt} = \mu F_s(t, x_1, \ldots, x_n, \mu) \qquad (s = 1, \ldots, n). \tag{5.1}$$

We shall denote systems of the form (5.1) as STANDARD. Many other systems are reduced to systems of this type with the aid of a transformation of the variables.

We shall present the system (5.1) in the form

$$\frac{dx_s}{dt} = \mu F_s^{(1)}(t, x_1, \ldots, x_n) + \mu^2 F_s^{(2)}(t, x_1, \ldots, x_n) + \ldots \tag{5.2}$$

Here, according to the general assumptions of sec. 1, the functions $F_s^{(j)}$ are finite sums of the form

$$F_s^{(j)} = A_{s0}^{(j)}(x_1, \ldots, x_n) + \\ + \sum_p [A_{sp}^{(j)}(x_1, \ldots, x_n) \cos \nu_p t + B_{sp}^{(j)}(x_1, \ldots, x_n) \sin \nu_p t], \tag{5.3}$$

where $A_{s0}^{(j)}$, $A_{sp}^{(j)}$, $B_{sp}^{(j)}$ are polynomials in $x_1, \ldots, x_n$.

It was shown by Krylov and Bogolyubov that, for any integer k, functions $u_s^{(1)}(t, y_1, \ldots, y_n)$, $u_s^{(2)}(t, y_1, \ldots, y_n), \ldots, u_s^{(k)}(t, y_1, \ldots, y_n)$, can be found such that the system (5.2) with the aid of the substitution

$$x_s = y_s + \mu u_s^{(1)}(t, y_1, \ldots, y_n) + \ldots + \mu^k u_s^{(k)}(t, y_1, \ldots, y_n) \tag{5.4}$$

transforms into

$$\frac{dy_s}{dt} = \mu Y_s^{(1)}(y_1, \ldots, y_n) + \ldots + \mu^k Y_s^{(k)}(y_1, \ldots, y_n) + \\ + \mu^{k+1} Y_s^*(t, y_1, \ldots, y_n, \mu), \quad (5.5)$$

where the functions $Y_s^{(1)}, \ldots Y_s^{(k)}$ do not contain t explicitly. The functions $u_s^{(j)}$ and Y_s^* have the same structure as the functions $F_s^i(t, x_1, \ldots, x_n)$, i.e. they can be represented in the form of finite trigonometric sums in the variable t with coefficients which are polynomials in $y_1, \ldots, y_n$. The functions Y_s^* depend, moreover, on the parameter μ with respect to which they are analytic. In fact, substituting (5.4) in (5.2) and taking (5.5) into account we shall have:

$$\mu \frac{\partial u_s^{(1)}}{\partial t} + \ldots + \mu^k \frac{\partial u_s^{(k)}}{\partial t} + \mu Y_s^{(1)} + \ldots + \mu^k Y_s^{(k)} + \mu^{k+1} Y_s^* = \\ = -\sum_{\alpha=1}^{n} \left\{ \left(\mu \frac{\partial u_s^{(1)}}{\partial y_\alpha} + \ldots + \mu^k \frac{\partial u_s^{(k)}}{\partial y_\alpha} \right) (\mu Y_\alpha^{(1)} + \ldots + \mu^k Y_\alpha^{(k)} + \mu^{k+1} Y_\alpha^*) \right\} + \\ + \mu F_s^{(1)}(t, y_1 + \mu u_1^{(1)} + \ldots, y_n + \mu u_n^{(1)} + \ldots) + \\ + \mu^2 F_s^{(2)}(t, y_1 + \mu u_1^{(1)} + \ldots, y_n + \mu u_n^{(1)} + \ldots) + \ldots \quad (5.6)$$

Equating coefficients of like powers of μ we obtain:

$$\left.\begin{aligned} &\frac{\partial u_s^{(1)}}{\partial t} + Y_s^{(1)} = F_s^{(1)}(t, y_1, \ldots, y_n), \\ &\frac{\partial u_s^{(2)}}{\partial t} + Y_s^{(2)} = -\sum_{\alpha=1}^{n} \frac{\partial u_s^{(1)}}{\partial y_\alpha} Y_\alpha^{(1)} + F_s^{(2)}(t, y_1, \ldots, y_n) + \\ &\quad + \sum_{\alpha=1}^{n} \frac{\partial F_s^{(1)}(t, y_1, \ldots, y_n)}{\partial y_\alpha} u_\alpha^{(1)} = R_s^{(2)}(t, y_1, \ldots, y_n) \end{aligned}\right\} \quad (5.7)$$

and in general

$$\frac{\partial u_s^{(j)}}{\partial t} + Y_s^{(j)} = R_s^{(j)}(t, y_1, \ldots, y_n) \quad (5.8)$$
$$(j = 3, \ldots, k),$$

where R_s^j are integral rational functions of those

$$u_r^{(i)}, \ \frac{\partial u_r^{(i)}}{\partial y_\alpha}, \ Y_r^{(i)}, \ F_r^{(i)}(t, y_1, \ldots, y_n), \ \frac{\partial F_r^{(i)}(t, y_1, \ldots, y_n)}{\partial y_\alpha}.$$

for which $i < j$.

Let $f(t)$ be some almost periodic function of the form

$$f(t) = A_0 + \sum (A_p \cos \nu_p t + B_p \sin \nu_p t).$$

We introduce the notations

$$\left. \begin{aligned} M\{f(t)\} &= A_0 = \lim_{t\to\infty} \frac{1}{t} \int_0^t f(t)\,dt, \\ J\{f(t)\} &= \sum \frac{A_p \sin \nu_p t - B_p \cos \nu_p t}{\nu_p}, \end{aligned} \right\} \tag{5.9}$$

so that

$$\int f(t)\,dt = M\{f(t)\}\cdot t + J\{f(t)\} + \alpha,$$

where α is an arbitrary constant. Equations (5.7) and (5.8) will then be satisfied if we put:

$$\left. \begin{aligned} Y_s^{(1)}(y_1, \ldots, y_n) &= \lim_{t\to\infty} \frac{1}{t} \int_0^t F_s^{(1)}(t, y_1, \ldots, y_n)\,dt = \\ &= M\{F_s^{(1)}(t, y_1, \ldots, y_n)\}, \\ u_s^{(1)}(t, y_1, \ldots, y_n) &= J\{F_s^{(1)}(t, y_1, \ldots, y_n)\} \end{aligned} \right\} \tag{5.10}$$

and

$$Y_s^{(j)} = M\{R_s^{(j)}(t, y_1, \ldots, y_n)\},\ u_s^{(j)} = J\{R_s^{(j)}(t, y_1, \ldots, y_n)\} \quad (j = 2, \ldots, k). \tag{5.11}$$

The functions $u_s^{(j)}(t, y_1, \ldots, y_n)$ will here evidently have the required structure

$$u_s^{(j)}(t, y_1, \ldots, y_n) = \sum [U_{sp}^{(j)}(y_1, \ldots, y_n) \cos \nu_p t + V_{sp}^{(j)}(y_1, \ldots, y_n) \sin \nu_p t],$$

where $U_{sp}^{(j)}$, $V_{sp}^{(j)}$ are polynomials in $y_1, \ldots, y_n$.

Assuming that in the substitution (5.4) the functions $u_s^{(j)}$ have been chosen by the method just indicated we reduce equations (5.1) to the form (5.5) , where the functions $\mu^{k+1} Y_s^*(t, y_1, \ldots, y_n)$ will represent the aggregate of terms of higher than the k-th order with respect to μ in the transformed equations. These functions will therefore be of the same form as the functions F_s in equations (5.1).

REMARK. In the preceding considerations great significance was possessed by the circumstance that the functions $F_s^{(j)}$ in equations (5.2) are finite trigonometric sums of the form (5.3). The requirement that the coefficients $A_{s0}^{(j)}$, $A_{sp}^{(j)}$, and $B_{sp}^{(j)}$, and therefore also the functions F_s, be polynomials in $x_1,\dots,x_n$ has no significance for the carrying out of the transformation itself and can be discarded.

6. Almost Periodic Solutions of Standard Systems

Let us consider the standard system

$$\frac{dx_s}{dt}=\mu F_s(t, x_1, \dots, x_n, \mu)=$$
$$=\mu F_s^{(1)}(t, x_1, \dots, x_n)+\mu^2 F_s^{(2)}(t, x_1, \dots, x_n)+\dots \quad (s=1, \dots, n), \tag{6.1}$$

in which the functions $F_s^{(j)}$ have the form (5.3). We shall make a change of the variables

$$x_s=y_s+\mu u_s(t, y_1, \dots, y_n), \tag{6.2}$$

where

$$u_s(t, y_1, \dots, y_n)=J\{F_s^{(1)}(t, y_1, \dots, y_n)\}.$$

Then, as was shown in the preceding section, equations (6.1) assume the form

$$\frac{dy_s}{dt}=\mu Y_s(y_1, \dots, y_n)+\mu^2 Y_s^*(t, y_1, \dots, y_n, \mu), \tag{6.3}$$

where

$$Y_s(y_1, \dots, y_n)=\lim_{t\to\infty}\frac{1}{t}\int_0^t F_s^{(1)}(t, y_1, \dots, y_n)\,dt\equiv$$
$$\equiv\lim_{t\to\infty}\frac{1}{t}\int_0^t F_s(t, y_1, \dots, y_n, 0)\,dt, \tag{6.4}$$

and Y_s^* are almost periodic functions of t of the same structure as the functions F_s.

Assume that the system of the first approximation

$$\frac{dy_s}{dt} = \mu Y_s(y_1, \ldots, y_n) \tag{6.5}$$

has the particular solution

$$y_s = y_s^0, \tag{6.6}$$

corresponding to the state of equilibrium of this system, i.e. that

$$Y_s(y_1^0, \ldots, y_n^0) = 0. \tag{6.7}$$

The equations in variation for this solution are of the form

$$\frac{d\xi_s}{dt} = p_{s1}\xi_1 + \ldots + p_{sn}\xi_n, \tag{6.8}$$

where the magnitudes p_{sj} are determined by the equations

$$p_{sj} = \frac{\partial Y_s(y_1^0, \ldots, y_n^0)}{\partial y_j^0} \tag{6.9}$$

and are constants. Let us assume that the fundamental equation of this system has no roots with real parts equal to zero. We shall show that in this case the system (6.3) has for sufficiently small μ an almost periodic solution which reduces to the solution (6.6) for $\mu = 0$.[1]

For this purpose let us set in equations (6.3)

$$y_s = y_s^0 + \mu z_s. \tag{6.10}$$

We then obtain:

$$\frac{dz_s}{dt} = Y_s(y_1^0, \ldots, y_n^0) + \mu(p_{s1}z_1 + \ldots + p_{sn}z_n) + \\ + \mu Y_s^*(t, y_1^0, \ldots, y_n^0, 0) + \mu^2 Z_s(t, z_1, \ldots, z_n, \mu),$$

----

[1] This proposition is a particular case of a more general one of N.N. Bogolyubov (see monograph cited on p. 273).

where the functions Z_s are analytic with respect to μ and have the same structure as the functions Y_s^*. Putting

$$\tau = \mu t, \quad Y_s^*(t, y_1^0, \ldots, y_n^0, 0) = f_s(t)$$

and taking (6.7) into account we shall finally have:

$$\frac{dz_s}{d\tau} = p_{s1}z_1 + \ldots + p_{sn}z_n + f_s\left(\frac{\tau}{\mu}\right) + \mu Z_s\left(\frac{\tau}{\mu}, z_1, \ldots, z_n, \mu\right). \tag{6.11}$$

Since by assumption the fundamental equation of system (6.8) does not have roots with zero real parts,[1] equations (6.1) satisfy all conditions of sec. 4.

These equations therefore possess the almost periodic solution $z_s = z_s^*(\tau)$, which is the limit of the sequence of almost periodic solutions $z_s^{(k)}(\tau)$ of the systems

$$\frac{dz_s^{(k)}}{d\tau} = p_{s1}z_1^{(k)} + \ldots + p_{sn}z_n^{(k)} + f_s\left(\frac{\tau}{\mu}\right) + \mu Z_s\left(\frac{\tau}{\mu}, z_1^{(k-1)}, \ldots, z_n^{(k-1)}, \mu\right)$$

$$(k = 2, 3, \ldots).$$

As the first approximation there is taken the almost periodic solution of the system

$$\frac{dz_s^{(1)}}{d\tau} = p_{s1}z_1^{(1)} + \ldots + p_{sn}z_n^{(1)} + f_s\left(\frac{\tau}{\mu}\right).$$

---

[1] As we shall see in sec. 9, where a proof will be given of the convergence of the successive approximations considered in sec. 4, the upper limit of the values of $|\mu|$ for which these approximations converge does not depend on the particular choice of the functions f_s and F_s that figure in equations (4.1) but depends on the upper limits of the functions $|f_s|$, $|F_s|$, $|\partial F_s/\partial x_j|$ in a certain neighborhood of the generating solution. Hence the circumstance that the variable τ in equation (6.11) enters in the combination τ/μ has no significance.

Substituting $z_s^*(\tau)$ in (6.10) we obtain the almost periodic solution of system (6.3) reducing for $\mu = 0$ to (6.6). Since the functions u_s in the substitution (6.2) are almost periodic with respect to t, there will correspond to the obtained solution the almost periodic oscillation of the system (6.1) that reduces for $\mu = 0$ to the state of equilibrium of the system (6.5).

We shall now investigate the question of the stability of the obtained almost periodic solution of system (6.1). This evidently is equivalent to the problem of the stability of the almost periodic solution $z_s^*(\tau)$ of system (6.11). To solve this problem we form the equations of the disturbed motion, setting in (6.11)

$$z_s = z_s^* + v_s.$$

We shall have:

$$\frac{dv_s}{d\tau} = p_{s1}v_1 + \ldots + p_{sn}v_n + \mu V_s(\tau, v_1, \ldots, v_n, \mu), \tag{6.12}$$

where

$$V_s = Z_s\left(\frac{\tau}{\mu}, z_1^* + v_1, \ldots, z_n^* + v_n, \mu\right) - Z_s\left(\frac{\tau}{\mu}, z_1^*, \ldots, z_n^*, \mu\right). \tag{6.13}$$

Let M be the upper limit of the magnitude $|\partial Z_s / \partial v_j|$ in the region

$$-\infty < \tau < +\infty, \quad |v_s| \leqslant H,$$

where H is a positive constant. Then, on the basis of (6.13), in this region will be satisfied the Cauchy-Lipschitz conditions

$$\mu |V_s(\tau, v_1', \ldots, v_n', \mu) - V_s(\tau, v_1'', \ldots, v_n'', \mu)| < L \sum_{\alpha=1}^{n} |v_\alpha' - v_\alpha''|,$$

in which the constant $L = \mu M$ for sufficiently small μ will be arbitrarily small. Equations (6.12) therefore possess all the properties of the equations of the disturbed motion there were considered in sec. 7 of chapter III, and on the basis of the theorem of Lyapunov there given we arrive at the following conclusion:

IF ALL THE ROOTS OF THE FUNDAMENTAL EQUATION OF SYSTEM (6.8) HAVE NEGATIVE REAL PARTS THE ALMOST PERIODIC SOLUTION UNDER CONSIDERATION OF SYSTEM (6.1) IS ASYMPTOTICALLY STABLE. IF THIS FUNDAMENTAL EQUATION HAS AT LEAST ONE ROOT WITH POSITIVE REAL PART THE ALMOST PERIODIC SOLUTION IN QUESTION IS UNSTABLE.

7. Almost Periodic Oscillations of Nonautonomous Systems for the Case of the Presence of Critical Roots of the Fundamental Equation. Special Case

Let us now consider the quasilinear nonautonomous system of the general form

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n + f_s(t) + \mu F_s(t, x_1, \ldots, x_n, \mu) \quad (7.1)$$
$$(s = 1, \ldots, n),$$

where the functions f_s and F_s satisfy the general conditions indicated in sec. 1. In sec. 4 we considered the question of the almost periodic solution of the system (7.1) on the assumption that the fundamental equation of the system

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n \quad (7.2)$$

does not have critical roots, i.e. roots with zero real parts. We now pass on to the investigation of the problem on the assumption that this equation has critical roots. In this section we shall consider the special case where all the roots of the indicated equation are critical. We shall here assume that to each of these roots correspond as many sets of solutions of equations (7.2) as its multiplicity. All the solutions of system (7.2) will then be almost periodic.

Let $\varphi_{s1}, \ldots, \varphi_{sn}$ be some fundamental system of solutions of equations (7.2). All the functions φ_{sj} will be almost periodic. We denote further by $\psi_{si}, \ldots, \psi_{sn}$ the fundamental system of solutions of the equations conjugate to (7.2). The functions ψ_{sj} will likewise be almost periodic. Both φ_{sj} and ψ_{sj} are finite trigonometric sums.

In order that the system (7.1) for $\mu = 0$ admit an almost periodic solution it is necessary and sufficient, as was shown in sec. 2, that the conditions be satisfied

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\alpha=1}^n f_\alpha(t)\psi_{\alpha i}\,dt = 0 \qquad (i = 1, \ldots, n). \tag{7.3}$$

We shall assume that conditions (7.3) are in fact satisfied. The general solution of the generating system will then also be almost periodic and it will have the form

$$x_s = M_1\varphi_{s1} + \ldots + M_n\varphi_{sn} + x_s^{0*}(t), \tag{7.4}$$

where $M_1, \ldots, M_n$ are arbitrary constants and x_s^{0*} is some particular almost periodic solution of this system. By the property of the functions $f_s(t)$ the functions x_s^{0*} will be finite trigonometric sums.

For the assumptions made the system (7.1) can at once be reduced to the standard form. For this purpose let us transform equations (7.1) to the variables $M_1, \ldots, M_n$ with the aid of the substitution (7.4). Then, taking into account that (7.4) is the general solution of the generating system, we shall have

$$\frac{dM_1}{dt}\varphi_{s1} + \ldots + \frac{dM_n}{dt}\varphi_{sn} = \mu F_s(t, x_1, \ldots, x_n, \mu), \tag{7.5}$$

where on the right hand sides the magnitudes x_s are to be replaced by their expressions (7.4).

For the determinant $|\varphi_{si}(t)|$ on the basis of the theorem of Lyapunov we can write:

$$|\varphi_{si}(t)| = |\varphi_{si}(0)|\, e^{(a_{11}+\ldots+a_{nn})t} = |\varphi_{si}(0)|\, e^{i\alpha t},$$

where according to the condition on the roots of the fundamental equation of the system (7.2) the magnitude α is real. It follows that if equations (7.5) are solved for the derivatives then in the obtained "standard" equations

$$\frac{dM_s}{dt} = \mu\Phi_s(t, M_1, \ldots, M_n, \mu) \tag{7.6}$$

the functions Φ_s will have the same structure with respect to the variables $t, M_1, \ldots, M_n, \mu$ as the functions F_s with respect to variables $t, x_1, \ldots, x_n, \mu$.

Equations (7.6) satisfy all conditions of the preceding section. Hence on the basis of the results there obtained we can state that these equations have an almost periodic solution reducing for $\mu = 0$ to certain constants M_s^*. These constants must satisfy the equations corresponding to equations (6.7) of the preceding section. On the basis of (6.4) these equations have the form

$$P_i(M_1, \ldots, M_n) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \Phi_i(t, M_1, \ldots, M_n, 0)\, dt = 0 \quad (7.7)$$

It is necessary here to assume that the fundamental equation of the system corresponding to (6.8) does not have roots with zero real parts. This fundamental equation on the basis of (6.9) has the form

$$\begin{vmatrix} \frac{\partial P_1}{\partial M_1} - \varkappa & \frac{\partial P_1}{\partial M_2} & \ldots & \frac{\partial P_1}{\partial M_n} \\ \frac{\partial P_2}{\partial M_1} & \frac{\partial P_2}{\partial M_2} - \varkappa & \ldots & \frac{\partial P_2}{\partial M_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial P_n}{\partial M_1} & \frac{\partial P_n}{\partial M_2} & \ldots & \frac{\partial P_n}{\partial M_n} - \varkappa \end{vmatrix} = 0. \quad (7.8)$$

Substituting the values of M_i in (7.4) we obtain an almost periodic solution of system (7.1). This solution for $\mu = 0$ reduces to the generating solution

$$x_s = M_1^* \varphi_{s1} + \ldots + M_n^* \varphi_{sn} + x_s^{0*}(t).$$

The parameters M_i^* of the generating almost periodic solution, to which corresponds the almost periodic solution of the complete system (7.1), satisfies equations (7.7), the analogous corresponding equations in the problem of periodic solutions. It is however necessary to bear in mind that in contrast to the problem of periodic solutions conditions (7.7) are not necessary in order that to the generating solution there correspond an almost periodic solution of the complete system.

From the results of the preceding section it follows further that the obtained almost periodic solution of system (7.1) will be asymptotically stable if all the roots of equation (7.8) have negative real parts, and unstable if at least one of the indicated roots has a positive real part.

8. Almost Periodic Oscillations of Nonautonomous Systems for the Case of the Presence of Critical Roots of the Fundamental Equation. General Case

Let us now consider the system (7.1) on the assumption that the characteristic equation of the system (7.2) has $m < n$ critical roots. We shall here as before assume that the number of sets of solutions of the system (7.2) corresponding to each critical root is equal to its multiplicity. The system (7.2) will then have m almost periodic solutions which we shall denote by $\varphi_{s1}, \ldots, \varphi_{sm}$ and the system conjugate to (7.2) will likewise have m almost periodic solutions which we denote by $\psi_{s1}, \ldots \psi_{sm}$. The system (7.2) will not have solutions of the form $t\varphi_{si} + f_{si}$, where f_{si} are almost periodic functions. As a consequence, exactly as in sec. 12 of Chapter III, it can be shown that the almost periodic solutions ψ_{si} of the system conjugate to (7.2) can be chosen in such manner that the relations are satisfied

$$\sum_{\alpha=1}^{n} \varphi_{\alpha i} \psi_{\alpha j} = \delta_{ij}, \; \delta_{ii} = 1, \; \delta_{ik} = 0 \quad (i \neq k) \qquad (8.1)$$
$$(i, j = 1, \ldots, m).$$

We shall assume that the functions ψ_{si} are actually chosen according to these conditions. We shall assume further that the functions $f_s(t)$ satisfy the conditions

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\alpha=1}^{n} f_\alpha(t) \psi_{\alpha i}(t)\, dt = 0 \qquad (i = 1, \ldots, m). \qquad (8.2)$$

The generating system

$$\frac{dx_s^0}{dt} = a_{s1} x_1^0 + \ldots + a_{sn} x_n^0 + f_s(t) \qquad (s = 1, \ldots, n) \qquad (8.3)$$

will then admit the almost periodic solution

$$x_s^0 = M_1 \varphi_{s1} + \dots + M_m \varphi_{sm} + x_s^{0*}(t), \tag{8.4}$$

depending on m arbitrary constants $M_1, \dots, M_m$. Here x_s^{0*} is some particular almost periodic solution of the system (8.3). For this, as for every other particular solution of the system (8.3), the relations hold

$$\frac{d}{dt} \sum_{\alpha=1}^{n} x_\alpha^{0*} \psi_{\alpha i} = \sum_{\alpha=1}^{n} f_\alpha \psi_{\alpha i}. \tag{8.5}$$

We shall now seek to obtain the almost periodic solution of the system (7.1) that reduces for $\mu = 0$ to the generating solution. For this it is necessary to transform equation (7.1) to certain new variables.

We introduce first of all the m variables $\xi_1, \dots, \xi_m$ with the aid of the substitution

$$\xi_i = \sum_{\alpha=1}^{n} x_\alpha \psi_{\alpha i} \qquad (i = 1, \dots, m). \tag{8.6}$$

The remaining $k = n - m$ new variables we introduce in the following manner.

The equation

$$\begin{vmatrix} a_{11} - \rho & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \rho & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \rho \end{vmatrix} = 0 \tag{8.7}$$

has n - m roots with real parts different from zero. W note these roots by $\rho_1, \dots, \rho_l$ respectively. Each roo is here written out as many times as the number of its corresponding sets of solutions of equations (7.2). Thus among the numbers ρ_j there may be some that are equal but to each of them corresponds one set of solutions.

The fundamental equation of the system

$$\frac{dx_s}{dt} + a_{1s} x_1 + \dots + a_{ns} x_n = 0, \tag{8.8}$$

conjugate to (7.2) has the roots $-\rho_1, \ldots, -\rho_l$, to each of which likewise corresponds one set of solutions. We shall denote by n_p the number of solutions of system (8.8) corresponding to the root $-\rho_p$. Evidently we have

$$n_1 + n_2 + \ldots + n_l = n - m = k.$$

Then, as was shown in sec. 1 of chapter II, the system (8.8) will have k particular solutions of the form

$$\left.\begin{aligned} x_{s1}^{(p)} &= C_{s1}^{(p)} e^{-\lambda_p t}, \\ x_{s2}^{(p)} &= (tC_{s1}^{(p)} + C_{s2}^{(p)}) e^{-\lambda_p t}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{sn_p}^{(p)} &= \left(\frac{t^{n_p-1}}{(n_p-1)!} C_{s1}^{(p)} + \frac{t^{n_p-2}}{(n_p-2)!} C_{s2}^{(p)} + \ldots + C_{sn_p}^{(p)}\right) e^{-\lambda_p t} \\ &\qquad (p = 1, \ldots, l), \end{aligned}\right\} \quad (8.9)$$

where $C_{sj}^{(p)}$ are constants.

With this established we now introduce into consideration k variables $\eta_j^{(p)}$ with the aid of a linear substitution with constant coefficients

$$\eta_j^{(p)} = \sum_{\alpha=1}^{n} C_{\alpha j}^{(p)} x_\alpha \qquad (p = 1, \ldots, l;\ j = 1, \ldots, n_p). \quad (8.10)$$

Together with (8.6) we obtain n new variables. We shall find the determinant of the complete substitution (8.6) and (8.10), which we shall denote by D(t). This determinant has the form

$$D(t) = \begin{vmatrix} \psi_{11} & \psi_{21} & \ldots & \psi_{n1} \\ \cdots & \cdots & \cdots & \cdots \\ \psi_{1m} & \psi_{2m} & \ldots & \psi_{nm} \\ C_{11}^{(1)} & C_{21}^{(1)} & \ldots & C_{n1}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ C_{1n_1}^{(1)} & C_{2n_1}^{(1)} & \ldots & C_{nn_1}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ C_{11}^{(l)} & C_{21}^{(l)} & \ldots & C_{n1}^{(l)} \\ \cdots & \cdots & \cdots & \cdots \\ C_{1n_l}^{(l)} & C_{2n_l}^{(l)} & \ldots & C_{nn_l}^{(l)} \end{vmatrix}.$$

On the other hand, the functions (8.9) together with the functions $\psi_{s1}, \ldots, \psi_{sm}$ form a system of n independent particular solutions of equations (8.8). The Wronskian $\Delta(t)$ of this fundamental system, after elementary transformations on it, can be represented in the form

$$\Delta(t) = D(t)\, e^{-(n_1\lambda_1+\ldots+n_l\lambda_l)\,t}. \tag{8.11}$$

But by the formula of Liouville for the Wronskian we can write:

$$\Delta(t) = \Delta(0)\, e^{-(a_{11}+\ldots+a_{nn})\,t} = \Delta(0)\, e^{-(n_1\lambda_1+\ldots+n_l\lambda_l+i\nu_1+\ldots+i\nu_m)\,t},$$

where $i\nu_1, i\nu_2, \ldots, i\nu_m$ is the aggregate of all critical roots of equation (8.7) (among the magnitudes ν_j there may be some equal to each other or equal to zero). Comparing with (8.11) we find:

$$D(t) = \Delta(0)\, e^{i(\nu_1+\ldots+\nu_m)\,t}.$$

From this it follows that the substitution (8.6), (8.10) is not singular and that the coefficients of the inverse transformation are almost periodic functions of t, being finite trigonometric sums.

We shall now transform equations (7.2) with the aid of the substitution thus introduced. Taking into account that the right hand sides of (8.6) and also the expressions

$$\eta_1^{(p)} e^{-\lambda_p t}, \quad (t\eta_1^{(p)} + \eta_2^{(p)})\, e^{-\lambda_p t},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\left(\frac{t^{n_p-1}}{(p-1)!}\eta_1^{(p)} + \frac{t^{n_p-2}}{(n_p-2)!}\eta_2^{(p)} + \ldots + \eta_{n_p}^{(p)}\right) e^{-\lambda_p t}$$

are the first integrals of equations (7.2) we easily find that these equations after transformation assume the form

$$\left.\begin{gathered}
\frac{d\xi_1}{dt} = \frac{d\xi_2}{dt} = \ldots = \frac{d\xi_m}{dt} = 0, \\
\frac{d\eta_1^{(p)}}{dt} = \lambda_p \eta_1^{(p)}, \qquad \frac{d\eta_j^{(p)}}{dt} = \lambda_p \eta_j^{(p)} - \eta_{j-1}^{(p)} \\
(p = 1, \ldots, l;\ j = 1, \ldots, n).
\end{gathered}\right\} \tag{8.12}$$

Among the variables $\eta_j^{(p)}$ there may be some that are complex. The latter break down into pairs of complex conjugates. If we wish to retain the equations in the real form we take in place of each pair of conjugate complex variables $\eta_j^{(p)}$ their real and imaginary parts as the new variables. In this way we obtain $k = n - m$ real variables. With the object of simplifying the writing we shall renumber these variables in some order and denote them by $\eta_1, \ldots, \eta_k$. The last k equations of (8.12) can then be written in the form

$$\frac{d\eta_r}{dt} = b_{r1}\eta_1 + \ldots + b_{rk}\eta_k \qquad (r = 1, \ldots, k). \tag{8.13}$$

The roots of the fundamental equation

$$\begin{vmatrix} b_{11} - \lambda & b_{12} & \ldots & b_{1k} \\ b_{21} & b_{22} - \lambda & \ldots & b_{2k} \\ \ldots & \ldots & \ldots & \ldots \\ b_{k1} & b_{h2} & \ldots & b_{kk} - \lambda \end{vmatrix} = 0 \tag{8.14}$$

of the system (8.13) will evidently be the magnitudes $\rho_1, \ldots, \rho_l$ and consequently this equation does not have roots with zero real parts.

We shall express the initial variables x_s in terms of the new variables ξ_i, η_i. We can write:

$$x_s = F_{s1}\xi_1 + \ldots + F_{sm}\xi_m + f_{s1}\eta_1 + \ldots + f_{sk}\eta_k, \tag{8.15}$$

where F_{si}, f_{sr} are certain almost periodic functions of t, developable in finite Fourier series. We shall show that $F_{si} \equiv \varphi_{si}$. In fact, substituting (8.15) in (8.6) and equating the coefficients of ξ_j we obtain

$$\sum_{\alpha=1}^{n} F_{\alpha j}\psi_{\alpha i} = \delta_{ij} \qquad (i, j = 1, \ldots m), \tag{8.16}$$

where δ_{ij} is the Kronecker symbol. From this, on the basis of the relations between the solutions of conjugate systems of linear differential equations, we conclude that the functions F_{si} determine the particular solutions of the system (7.2). But since these solutions are almost periodic we must necessarily have:

$$F_{\alpha j} = \sum_{i=1}^{m} A_{ji}\varphi_{\alpha i}. \tag{8.17}$$

where A_{ji} are certain constants. Substituting these relations in (8.16) and taking into account (8.1) we obtain:

$$\delta_{ij} = A_{j1}\delta_{1i} + \ldots + A_{jm}\delta_{mi} = A_{ji},$$

and therefore relations (8.17) give $F_{\alpha j} = A_{\alpha j}$. Thus we can write:

$$x_s = \varphi_{s1}\xi_1 + \ldots + \varphi_{sm}\xi_m + f_{s1}\eta_1 + \ldots + f_{sk}\eta_k. \tag{8.18}$$

We now proceed to the investigation of equations (7.1). In place of the variables x_s we introduce the variables $\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_k$. We then evidently obtain:

$$\left.\begin{aligned} \frac{d\xi_i}{dt} &= \varphi_i(t) + \mu\Phi_i(t, \xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_k, \mu), \\ \frac{d\eta_r}{dt} &= b_{r1}\eta_1 + \ldots + b_{rk}\eta_k + \psi_r(t) + \\ &\qquad + \mu\Psi_r(t, \xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_k, \mu) \\ &(i = 1, \ldots, m;\ r = 1, \ldots, k). \end{aligned}\right\} \tag{8.19}$$

Here $\varphi_i(t)$ and $\psi_r(t)$ are finite trigonometric sums. The functions Φ_i and Ψ_i have the same structure as the functions F_s. These functions can therefore be expanded in series of integral powers of μ, the terms of these expansions constituting finite trigonometric sums with respect to t the coefficients of which are polynomials in ξ_i, η_r. Moreover, on the basis of (8.6) we have:

$$\varphi_i(t) = \sum_{\alpha=1}^{n} f_\alpha(t)\,\psi_{\alpha i}, \tag{8.20}$$

$$\Phi_i(t, \xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_k, \mu) = \sum_{\alpha=1}^{n} F_\alpha(t, x_1, \ldots, x_n, \mu)\,\psi_{\alpha i}. \tag{8.21}$$

Setting in (8.19) $\mu = 0$ we obtain the equations of the first approximation

$$\left.\begin{aligned} \frac{d\xi_i}{dt} &= \varphi_i(t) = \sum_{\alpha=1}^{n} f_\alpha\psi_{\alpha i}, \\ \frac{d\eta_r}{dt} &= b_{r1}\eta_1 + \ldots + b_{rk}\eta_k + \psi_r(t) \\ &(i = 1, \ldots, m;\quad r = 1, \ldots, k), \end{aligned}\right\} \tag{8.22}$$

which are no other than the generating system expressed in the new variables.

The general solution of the first group of these equations on the basis of (8.5) can be represented in the form

$$\xi_i = M_i + \sum_{\alpha=1}^{n} x_\alpha^{0*}(t)\,\psi_{\alpha i} = \xi_i^0(t), \tag{8.23}$$

where M_i are arbitrary constants. This solution is almost periodic. The second group of equations (8.22) has one and only one almost periodic solution which we denote by $\eta_r^0(t)$. This is a consequence of the fact that equation (8.14) does not have roots with zero real parts.

Thus, the system (8.22) has an almost periodic solution depending on m arbitrary constants. It corresponds to the almost periodic solution of (8.4) of the generating system. As is easily seen, we are justified in assuming that the same arbitrary constants enter in (8.4) and (8.23). In fact, substituting (8.4) in (8.6) and taking (8.1) into account we obtain exactly (8.23), which proves our statement.

Substituting ξ_i^0 and η_i^0 in (8.18) and comparing with (8.4) we obtain the following identities:

$$x_s^{0*} = \sum_{i=1}^{m} \sum_{\alpha=1}^{n} x_\alpha^{0*}\psi_{\alpha i}\varphi_{si} + \sum_{r=1}^{k} f_{sr}\eta_r^0, \tag{8.24}$$

which we shall make use of in what follows.

Let us now subject equations (8.19) to a transformation analogous to the transformation of Krylov and Bogolyubov for the "standard" equations. For this purpose we introduce the functions

$$P_i(M_1, \ldots, M_m) \text{ и } u_i(t, M_1, \ldots, M_m),$$

determined by the relations

$$\int_0^t \Phi_i(t, \xi_1^0, \ldots, \xi_m^0, \eta_1^0, \ldots, \eta_k^0, 0)\,dt = P_i t + u_i, \tag{8.25}$$

so that

$$u_i(t, M_1, \ldots, M_m) = J\{\Phi_i(t, \xi_1^0, \ldots, \eta_k^0, 0)\} \tag{8.26}$$

and

$$P_i(M_1, \ldots, M_m) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \Phi_i(t, \xi_1^0, \ldots, \eta_k^0, 0)\, dt. \tag{8.27}$$

Taking (8.21) into account the functions P_i can also be represented in the form

$$P_i(M_1, \ldots, M_m) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\alpha=1}^{n} F_\alpha(t, x_1^0, \ldots, x_n^0, 0)\, \psi_{\alpha i}\, dt, \tag{8.28}$$

since to the functions ξ_i^0 and η_r^0 in the variables $x_1, \ldots, x_n$ correspond the functions (8.4). The magnitudes u_i will evidently be almost periodic functions of t, being finite trigonometric sums. Both the functions u_i and the functions P_i are polynomials with respect to $M_1, \ldots, M_m$.

The transformation now under consideration is of the form

$$\left.\begin{aligned} \xi_i &= \xi_i^0 + \mu u_i = M_i + \sum_{\alpha=1}^{n} x_\alpha^{0*} \psi_{\alpha i} + \mu u_i, \\ \eta_r &= \eta_r^0 + \mu z_r, \end{aligned}\right\} \tag{8.29}$$

the magnitudes M_i and z_r being taken as the new variables For the first group of equations (8.19) we find:

$$\frac{dM_i}{dt} + \mu \sum_{j=1}^{m} \frac{\partial u_i}{\partial M_j} \frac{dM_j}{dt} + \mu \frac{\partial u_i}{\partial t} = \mu \Phi_i(t, \xi_1^0, \ldots, \eta_k^0, 0) + \ldots,$$

where the terms not written down are of an order higher than the first with respect to μ. Whence, taking (8.25) into account , we obtain:

$$\frac{dM_i}{dt} + \mu \sum_{j=1}^{m} \frac{\partial u_i}{\partial M_j} \frac{dM_j}{dt} = \mu P_i(M_1, \ldots, M_m) + \ldots \tag{8.30}$$

For the second group of equations (8.19), taking into account that the functions η_r^o are a particular solution of the system (8.22) (these functions do not depend on M_i), we find:

$$\mu \frac{dz_r}{dt} = \mu (b_{r1} z_1 + \ldots + b_{rk} z_k) + \mu \Psi_r (t, \xi_1^0, \ldots, \eta_k^0, 0) + \ldots \quad (8.31)$$

Solving (8.30) for dM_i/dt (which for sufficiently small μ is evidently possible) and setting in (8.31)

$$\Psi_r (t, \xi_1^0, \ldots, \eta_k^0, 0) = Z_r^0 (t, M_1, \ldots, M_m),$$

we shall have

$$\left.\begin{aligned} \frac{dM_i}{dt} &= \mu P_i (M_1, \ldots, M_m) + \mu^2 Q_i (t, M_1, \ldots, M_m, z_1, \ldots, z_k, \mu), \\ \frac{dz_r}{dt} &= b_{r1} z_1 + \ldots + b_{rk} z_k + Z_r^0 (t, M_1, \ldots, M_m) + \\ &\qquad + \mu Z_r (t, M_1, \ldots, M_m, z_1, \ldots, z_k, \mu) \\ &(i = 1, \ldots, m;\ r = 1, \ldots, k = n - m). \end{aligned}\right\} \quad (8.32)$$

Hence the functions Q_i and Z_r have the same structure as the functions F_s in equations (7.1), namely, they are in the form of series which converge for sufficiently small μ

$$\left.\begin{aligned} \mu^2 Q_i &= \sum_{p=2}^{\infty} Q_i^{(p)} (t, M_1, \ldots, z_k) \mu^p, \\ \mu Z_r &= \sum_{q=1}^{\infty} Z_r^{(q)} (t, M_1, \ldots, z_k) \mu^q, \end{aligned}\right\} \quad (8.33)$$

in which $Q_i^{(p)}$ and $Z_r^{(q)}$ are almost periodic functions of t (finite trigonometric sums) and polynomials with respect to the remaining arguments. The same is true for the functions Z_r^o.

For equations (8.32) an almost periodic solution can be constructed in the following manner:

Assume that the equations

$$P_i (M_1, \ldots, M_m) = 0 \qquad (i = 1, \ldots, m) \quad (8.34)$$

admit a solution $M_i = M_i^*$ for which equation

$$\begin{vmatrix} \frac{\partial P_1}{\partial M_1} - \varkappa & \frac{\partial P_1}{\partial M_2} & \dots & \frac{\partial P_1}{\partial M_m} \\ \frac{\partial P_2}{\partial M_1} & \frac{\partial P_2}{\partial M_2} - \varkappa & \dots & \frac{\partial P_2}{\partial M_m} \\ \dots & \dots & \dots & \dots \\ \frac{\partial P_m}{\partial M_1} & \frac{\partial P_m}{\partial M_2} & \dots & \frac{\partial P_m}{\partial M_m} - \varkappa \end{vmatrix} = 0 \qquad (8.35)$$

has no roots with zero real parts. Let us set

$$M_i = M_i^* + \mu N_i. \qquad (8.36)$$

Equations (8.32) then assume the form

$$\left.\begin{aligned} \frac{dN_i}{dt} &= \mu(c_{i1}N_1 + \dots + c_{im}N_m) + \\ &\quad + \mu Q_i(t, M_1^*, \dots, M_m^*, z_1, \dots, z_k, 0) + \\ &\quad + \mu^2 Q_i^*(t, N_1, \dots, N_m, z_1, \dots, z_k, \mu), \\ \frac{dz_r}{dt} &= b_{r1}z_1 + \dots + b_{rk}z_k + Z_r^0(t, M_1^*, \dots, M_m^*) + \\ &\quad + \mu Z_r^*(t, N_1, \dots, N_m, z_1, \dots, z_r), \end{aligned}\right\} \qquad (8.37)$$

where

$$c_{ij} = \frac{\partial P_i(M_1^*, \dots, M_m^*)}{\partial M_j^*}$$

and the functions Q_i^* and Z_r^* have the same structure as the functions Q_i and Z_r.

We shall now seek an almost periodic solution of the system (8.37) by the method of successive approximations. As the first approximation we shall take an almost periodic solution of the system

$$\left.\begin{aligned} \frac{dz_r^{(1)}}{dt} &= b_{r1}z_1^{(1)} + \dots + b_{rk}z_k^{(1)} + Z_r^{(0)}(t, M_1^*, \dots, M_m^*), \\ \frac{dN_i^{(1)}}{dt} &= \mu(c_{i1}N_1^{(1)} + \dots + c_{im}N_m^{(1)}) + \\ &\quad + \mu Q_i(t, M_1^*, \dots, z_k^{(1)}, 0) \end{aligned}\right\} \qquad (8.38)$$

and as the p-th approximation an almost periodic solution of the system

$$
\left.\begin{aligned}
\frac{dz_r^{(p)}}{dt} &= b_{r1}z_1^{(p)} + \ldots + b_{rk}z_k^{(p)} + Z_r^0(t, M_1^*, \ldots, M_m^*) + \\
&\qquad + \mu Z_r^*(t, N_1^{(p-1)}, \ldots, z_k^{(p-1)}, \mu), \\
\frac{dN_i^{(p)}}{dt} &= \mu(c_{i1}N_1^{(p)} + \ldots + c_{im}N_m^{(p)}) + \mu Q_i(t, M_1^*, \ldots, z_k^{(p)}, 0) + \\
&\qquad + \mu^2 Q_i^*(t, N_1^{(p-1)}, \ldots, z_k^{(p-1)}, \mu).
\end{aligned}\right\} \tag{8.39}
$$

Since equation (8.14) and the equation

$$
\begin{vmatrix}
c_{11} - \varkappa & c_{12} & \ldots & c_{1m} \\
c_{21} & c_{22} - \varkappa & \ldots & c_{2m} \\
\cdot & \cdot & \cdot & \cdot \\
c_{m1} & c_{m2} & \ldots & c_{mm} - \varkappa
\end{vmatrix} = 0
$$

do not have roots with real parts equal to zero the system (8.38) admits one and only one almost periodic solution. In exactly the same way equations (8.39) also uniquely determine sequences of almost periodic functions $N_i^{(p)}$, $z_r^{(p)}$ that determine a particular almost periodic solution of system (8.37). Returning to the initial variables x_s we obtain the almost periodic solution of the system (7.1). As is seen from (8.36) and (8.29), this solution for $\mu = 0$ reduces to the generating solution (8.4) for which the parameters M_i have the values M_i^* i.e. are roots of equations (8.34). This leads to the following theorem: [1]

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[1] This theorem was established in the paper: Malkin I.G., O pochti-periodicheskikh kolebaniyakh nelineinykh neavtonomnykh sistem, (On Almost Periodic Oscillations of Nonlinear Nonautonomous Systems), Prikl. matem. i mekh., vol. XVIII, no. 6, 1954. The problem considered in this section was investigated also from another point of view in the paper: Biryuk G.I., K voprosu o sushchestvovanii pochti-periodicheskikh reshenii nelineinykh sistem s malym parametrom v sluchae vyrozhdeniya, (On the Question of the Existence of Almost Periodic Solutions of Nonlinear Systems with Small Parameter in the Case of Degeneration), DAN SSSR, vol. XCVII, no. 4, 1954.

THEOREM. ASSUME THAT FOR THE EQUATIONS

$$\frac{dx_s}{dt} = a_{s1}x_1 + \dots + a_{sn}x_n + f_s(t) + \mu F_s(t, x_1, \dots, x_n, \mu) \quad (7.1)$$

THE GENERATING SYSTEM

$$\frac{dx_s^0}{dt} = a_{s1}x_1^0 + \dots + a_{sn}x_n^0 + f_s(t)$$

ADMITS AN ALMOST PERIODIC SOLUTION

$$x_s^0 = M_1\varphi_{s1} + \dots + M_m\varphi_{sm} + x_s^{0*}(t)$$

AND THAT THE FUNDAMENTAL EQUATION OF THIS GENERATING SYSTEM HAS $n - m$ ROOTS WITH REAL PARTS DIFFERENT FROM ZERO. THEN, IF THE PARAMETERS M_i SATISFY THE EQUATIONS

$$P_i(M_1, \dots, M_m) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\alpha=1}^{n} F_\alpha(t, x_1^0, \dots, x_n^0, 0)\psi_{\alpha i}\, dt = 0 \quad (8.40)$$

AN EQUATION (8.35) HAS NO ROOTS WITH ZERO REAL PARTS THE SYSTEM (7.1) FOR SUFFICIENTLY SMALL μ ADMITS AN ALMOST PERIODIC SOLUTION REDUCING TO THE GENERATING SOLUTION FOR $\mu = 0$.

The theorem established is an immediate generalization of the corresponding theorem in the problem of periodic solutions. In particular, equations (8.40), which by assumption are satisfied by the magnitudes M_i, are an immediate generalization of the corresponding equations in the problem of periodic solutions. However we here have also some essential differences. First of all, in the theorem just proven an essential assumption is that the fundamental equation of the generating system has $n - m$ roots with real parts different from zero, a condition not required in the problem of periodic solutions. Further, in the problem of periodic solutions it was sufficient to assume that the magnitudes M_i are a simple solution of equations (8.40) or, what amounts to the same thing, that the equation (8.35) has no zero root. In the problem of almost periodic solutions however the stronger assumption is made that equation (8.35) has neither a zero root nor purely imaginary roots. Finally, in the problem of almost periodic solutions conditions (8.40) are not necessary.

As a result of the special features noted above the fundamental results on periodic solutions that were established in chapter II cannot be considered as a simple consequence of the results of the present section. The problem of periodic solutions for those general assumptions for which the problem was treated in chapter II, if it is considered as a special case of the problem of almost periodic solutions, belongs to those specific cases that require special investigation.

9. Proof of the Convergence of the Successive Approximations

We now go on to the proof that the sequences of almost periodic solutions $z_r^{(p)}$ and $N_i^{(p)}$ considered in the preceding section converge and actually determine an almost periodic solution of the system (8.37).

These sequences are determined, as we have seen, as the almost periodic solutions of the system of equations

$$\left.\begin{aligned}
&\frac{dz_r^{(p)}}{dt} = b_{r1}z_1^{(p)} + \ldots + b_{rk}z_k^{(p)} + u_r(t) + \\
&\qquad + \mu Z_r^{(*)}(t, N_1^{(p-1)}, \ldots, N_m^{(p-1)}, z_1^{(p-1)}, \ldots, z_k^{(p-1)}, \mu), \\
&\frac{dN_i^{(p)}}{dt} = \mu(c_{i1}N_1^{(p)} + \ldots + c_{im}N_m^{(p)}) + \mu\overline{Q}_i(t, z_1^{(p)}, \ldots, z_k^{(p)}) + \\
&\qquad + \mu^2 Q_i^*(t, N_1^{(p-1)}, \ldots, N_m^{(p-1)}, z_1^{(p-1)}, \ldots, z_k^{(p-1)}, \mu), \\
&u_r(t) = Z_r^0(t, M_1^*, \ldots, M_m^*), \\
&\overline{Q}_i = Q_i(t, M_1^*, \ldots, M_m^*, z_1^{(p)}, \ldots, z_k^{(p)}, 0),
\end{aligned}\right\} \quad (9.1)$$

where

$$\left.\begin{aligned}
&\frac{dz_r^{(1)}}{dt} = b_{r1}z_1^{(1)} + \ldots + b_{rk}z_k^{(1)} + u_r(t), \\
&\frac{dN_i^{(1)}}{dt} = \mu(c_{i1}N_1^{(1)} + \ldots + c_{im}N_m^{(1)}) + \\
&\qquad + \mu\overline{Q}_i(t, z_1^{(1)}, \ldots, z_k^{(1)}).
\end{aligned}\right\} \quad (9.2)$$

Let $f_1(t), \ldots, f_m(t)$ be arbitrary almost periodic functions and f the upper limit of their moduli for all real values of t. Since equation (8.35) has no roots with zero real parts, the system of equations

$$\frac{dv_i}{d\tau} = c_{i1}v_1 + \ldots + c_{im}v_m + f_i(\tau)$$

as shown in sec. 2, admits a single almost periodic solution $v_i = v_i(\tau)$ and this solution satisfies the inequalities

$$|v_i| < Pf,$$

where P is a certain constant not depending on the choice of the functions f_i .

Let us now consider the system

$$\frac{dv_i}{dt} = \mu(c_{i1}v_1 + \ldots + c_{im}v_m) + f_i(t). \tag{9.3}$$

Setting in it $\tau = \mu t$ we obtain:

$$\frac{dv_i}{d\tau} = c_{i1}v_1 + \ldots + c_{im}v_m + \frac{1}{\mu} f_i\left(\frac{\tau}{\mu}\right),$$

whence it follows immediately that the almost periodic solution $v_i(t)$ of system (9.3) satisfies the inequalities

$$|v_i(t)| < \frac{1}{\mu} Pf. \tag{9.4}$$

Since equation (8.14) likewise has no roots with zero real parts we can say that, whatever the system of almost periodic functions $F_1(t), \ldots, F_k(k)$ for which $|F_r(t)| < F$, for the almost periodic solution of the system

$$\frac{dw_r}{dt} = b_{r1}w_1 + \ldots + b_{rk}w_h + F_r(t) \tag{9.5}$$

the estimates hold

$$|w_r(t)| < QF, \tag{9.6}$$

where Q does not depend on the functions $F_r(t)$.

Let us now choose a positive constant H and consider the region determined by the inequalities

$$|z_r - z_r^{(1)}(t)| \leqslant H, \quad |N_i - N_i^{(1)}(t)| \leqslant H. \tag{9.7}$$

It is evident that in this region the inequalities hold

$$|Z_r^*| < A, \quad |\overline{Q}_i| < B, \quad |Q_i^*| < C \tag{9.8}$$

and the Cauchy-Lipschitz conditions are satisfied

$$\left.\begin{aligned}
&|Z_r^*(t, N_1', \ldots, N_m', z_1', \ldots, z_k', \mu) - \\
&\qquad - Z_r^*(t, N_1'', \ldots, N_m'', z_1'', \ldots, z_m'', \mu)| \leqslant \\
&\qquad\qquad \leqslant L\Big(\sum_{\alpha=1}^{m} |N_\alpha' - N_\alpha''| + \sum_{\beta=1}^{k} |z_\beta' - z_\beta''|\Big), \\
&|\overline{Q}_i(t, z_1', \ldots, z_k') - \overline{Q}_i(t, z_1'', \ldots, z_k'')| \leqslant M \sum_{\beta=1}^{k} |z_\beta' - z_\beta''|, \\
&|Q_i^*(t, N_1', \ldots, N_m', z_1', \ldots, z_k', \mu) - \\
&\qquad - Q_i^*(t, N_1'', \ldots, N_m'', z_1'', \ldots, z_k'', \mu)| \leqslant \\
&\qquad\qquad \leqslant N\Big(\sum_{\alpha=1}^{m} |N_\alpha' - N_\alpha''| + \sum_{\beta=1}^{k} |z_\beta' - z_\beta''|\Big),
\end{aligned}\right\} \quad (9.9)$$

where A,B,C,L,M, N are certain constants. It is here, as in all that follows, assumed that μ is sufficiently small ($\mu < h$) and t varies in the range $-\infty < t < +\infty$.

We shall show that for sufficiently small μ all approximations lie in the region (9.7). For this purpose we assume that the functions $z_r^{(p-1)}$, $N_i^{(p-1)}$ lie in the region (9.7) and show that the same will be true also for the functions $z_r^{(p)}$, $N_i^{(p)}$. From the equations

$$\frac{d\,(z_r^{(p)} - z_r^{(1)})}{dt} = b_{r1}(z_1^{(p)} - z_1^{(1)}) + \ldots + b_{rk}(z_k^{(p)} - z_k^{(1)}) + $$
$$+ \mu Z_r^*(t, N_1^{(p-1)}, \ldots, z_k^{(p-1)}, \mu),$$

which evidently are satisfied by the functions $z_r^{(p)} - z_r^{(1)}$, on the basis of (9.8) and (9.6) we find:

$$|z_r^{(p)} - z_r^{(1)}| < \mu AQ, \quad (9.10)$$

whence follows that for sufficiently small μ the functions $z_r^{(p)} - z_r^{(1)}$ actually lie in region (9.7). But then on the basis of (9.8) and (9.9) the estimates also hold

$$|\overline{Q}_i(t, z_1^{(p)}, \ldots, z_k^{(p)}) - \overline{Q}_i(t, z_1^{(1)}, \ldots, z_k^{(1)})| + $$
$$+ |\mu Q_i^*(t, N_1^{(p-1)}, \ldots, z_k^{(p-1)}, \mu)| \leqslant \mu kAQM + \mu C,$$

and therefore the equations

$$\frac{d(N_i^{(p)}-N_i^{(1)})}{dt}=\mu\sum_{\alpha=1}^{m}c_{i\alpha}(N_\alpha^{(p)}-N_\alpha^{(1)})+\mu\overline{Q}_i(t, z_1^{(p)}, \ldots, z_k^{(p)})-$$
$$-\mu\overline{Q}_i(t, z_1^{(1)}, \ldots, z_k^{(1)})+\mu^2 Q_i^*(t, N_1^{(p-1)}, \ldots, z_k^{(p-1)}, \mu)$$

on the basis of (9.4) give:

$$|N_i^{(p)}-N_i^{(1)}|\leqslant\mu(kAQM+C)P. \tag{9.11}$$

From this it follows that for sufficiently small μ the functions $N_i^{(p)} - N_i^{(1)}$ likewise lie in region (9.7).

We shall now estimate the differences of the successive approximations. From what has been proven it follows that for all p we can write:

$$|z_r^{(p)}-z_r^{(p-1)}|\leqslant a_p, \quad |N_i^{(p)}-N_i^{(p-1)}|\leqslant a_p, \tag{9.12}$$

where a_p are arbitrary constants. From this on the basis of (9.9) we find:

$$Z_r^*(t, N_1^{(p)}, \ldots, z_k^{(p)}, \mu)-Z_r^*(t, N_1^{(p-1)}, \ldots, z_k^{(p-1)}, \mu)|\leqslant$$
$$\leqslant L(k+m)a_p=nLa_p.$$

Hence the equations

$$\frac{d(z_r^{(p+1)}-z_r^{(p)})}{dt}=\sum_{\beta=1}^{k}b_{r\beta}(z_r^{(p+1)}-z_r^{(p)})+$$
$$+\mu Z_r^*(t, N_1^{(p)}, \ldots, z_k^{(p)}, \mu)-\mu Z_r^*(t, N_1^{(p-1)}, \ldots, z_k^{(p-1)}, \mu)$$

on the basis of (9.6) give:

$$|z_r^{(p+1)}-z_r^{(p)}|<\mu nLQa_p. \tag{9.13}$$

Taking this into account we find from (9.9):

$$|\overline{Q}_i(t, z_1^{(p+1)}, \ldots, z_k^{(p+1)})-\overline{Q}_i(t, z_1^{(p)}, \ldots, z_k^{(p)})|+$$
$$+\mu|Q_i^*(t, N_1^{(p)}, \ldots, z_k^{(p)}, \mu)-Q_i^*(t, N_1^{(p-1)}, \ldots, z_k^{(p-1)}, \mu)|<$$
$$<\mu knMLQa_p+\mu Nna_p,$$

and therefore the equations determining $N_i^{(p+1)} - N_i^{(p)}$ on the basis of (9.4) give:

$$|N_i^{(p+1)} - N_i^{(p)}| < \mu n P(kMLQ + N) a_p. \tag{9.14}$$

Inequalities (9.13) and (9.14) show that for sufficiently small μ we can put:

$$a_{p+1} = \theta a_p,$$

where θ is a certain positive proper function that does not depend on p. From this it follows immediately that the sequences $z_r^{(p)}$ and $N_i^{(p)}$ uniformly converge to certain almost periodic functions $z_r^*(t)$ and $N_i^*(t)$.

We shall show that the functions z_r^* and N_i^* actually satisfy equations (8.37). For this purpose let us consider the functions z_r' and N_i'. which are almost periodic solutions of the linear equations

$$\frac{dz_r'}{dt} = b_{r1} z_1' + \ldots + b_{rk} z_k' + u_r(t) + \mu Z_r^*(t, N_1^*, \ldots, z_k^*, \mu),$$

$$\frac{dN_i'}{dt} = \mu (c_{i1} N_1' + \ldots + c_{im} N_m') + \mu \overline{Q}_i (t, z_1^*, \ldots, z_k^*) + \\ + \mu^2 Q_i^* (t, N_1^*, \ldots, z_k^*, \mu)$$

with known right hand sides. Evidently it is sufficient for us to show that $z_r' = z_r^*$, $N_i' = N_i^*$.

In fact, we have:

$$\frac{d(z_r' - z_r^{(p)})}{dt} = \sum_{\beta=1}^{k} b_{r\beta} (z_r' - z_r^{(p)}) + \\ + \mu Z_r^* (t, N_1^*, \ldots, z_k^*, \mu) - \mu Z_r^* (t, N_1^{(p-1)}, \ldots, z_k^{(p-1)}, \mu).$$

But

$$|Z_r^*(t, N_1^*, \ldots, z_k^*, \mu) - Z_r^*(t, N_1^{(p-1)}, \ldots, z_k^{(p-1)}, \mu)| < nLA_{p-1},$$

where A_p are the upper limits of the functions
$|z_r^{(p)} - z_r^*|$, $|N_i^{(p)} - N_i^*|$.
Whence on the basis of (9.6) we obtain:

$$|z_r' - z_r^{(p)}| < \mu n L Q A_{p-1}. \tag{9.15}$$

Similarly we obtain the estimates

$$|\overline{Q}_i(t, z_1^*, \ldots, z_k^*) - \overline{Q}_i(t, z_1^{(p)}, \ldots, z_k^{(p)})| + \\ + \mu |Q_i^*(t, N_1^*, \ldots, z_k^*, \mu) - Q_i^*(t, N_1^{(p-1)}, \ldots, z_k^{(p-1)}, \mu)| < \\ < kMA_p + \mu nNA_{p-1},$$

and therefore the equations determining $N_i' - N_i^{(p)}$, on the basis of (9.4) give:

$$|N_i' - N_i^{(p)}| < kMPA_p + \mu nNPA_{p-1}. \tag{9.16}$$

But since we can consider that

$$\lim_{p\to\infty} A_p = 0,$$

we obtain from (9.15) and (9.16):

$$z_r' = \lim_{p\to\infty} z_r^{(p)} = z_r^*, \qquad N_i' = \lim_{p\to\infty} N_i^{(p)} = N_i^*,$$

which proves our proposition.

As is shown by inequalities (9.10) and (9.11), the almost periodic solution we have found, for sufficiently small μ differs as little as we please from the almost periodic solution $z_r^{(1)}(t)$, $N_i^{(1)}(t)$ of the system (9.2). We shall show that the system (8.37) has a unique almost periodic solution possessing this property. In fact, let us assume that the system (8.37), besides the solution z_r^*, N_r^*, has a further almost periodic solution $z_r^{*\prime}$, $N_i^{*\prime}$ possessing the indicated property. Let ε be some positive number less than unity. By assumption we can assert that a positive number μ_0 exists such that for $\mu < \mu_0$ we shall have:

$$|z_r^* - z_r^{*\prime}| < \varepsilon, \qquad |N_i^* - N_i^{*\prime}| < \varepsilon.$$

But the functions $z_r^* - z_r^{*\prime}$ satisfy equations

$$\frac{d(z_r^* - z_r^{*\prime})}{dt} = \sum_{\rho=1}^{k} b_{r\rho}(z_\rho^* - z_\rho^{*\prime}) + \mu Z_r^*(t, N_1^*, \ldots, z_k^*, \mu) - \\ - \mu Z_r^*(t, N_1^{*\prime}, \ldots, z_k^{*\prime}, \mu).$$

From these equations and the corresponding equations for $N_i^* - N_i^{*\prime}$, by the computations which led us to the estimates (9.13) and (9.14) for the functions $|z_r^{(p+1)} - z_r^{(p)}|$ and $|N_i^{(p+1)} - N_i^{(p)}|$, we obtain:

$$|z_r^* - z_r^{*\prime}| < \mu n L Q \varepsilon,$$
$$|N_i^* - N_i^{*\prime}| < \mu n P (kMLQ + N)\varepsilon.$$

Hence, if μ_o is so small that for $\mu < \mu_o$ the inequalities are satisfied

$$\mu n L Q < \varepsilon, \quad \mu n P (kMLQ + N) < \varepsilon,$$

we shall have:

$$|z_r^* - z_r^{*\prime}| < \varepsilon^2, \quad |N_i^* - N_i^{*\prime}| < \varepsilon^2.$$

Whence by repeating the estimates we arrive at the inequalities

$$|z_r^* - z_r^{*\prime}| < \varepsilon^l, \quad |N_i^* - N_i^{*\prime}| < \varepsilon^l,$$

where l is an arbitrary integer. But since $\varepsilon < 1$ it follows that $z_r^{*\prime} = z_r^*$, $N_i^{*\prime} = N_i^*$, as was required to be proved.

REMARK. Equations (4.1), for the assumptions made in sec. 4, represent a particular case of equations (8.37) corresponding to $m = 0$. Hence from the results just obtained follows the convergence of the process of successive approximations considered in sec. 4 and the validity of the theorem there established.

10. Practical Methods of Computing the Almost Periodic Solutions Considered in Sec. 8

We now proceed to the question of the practical computation of the almost periodic solutions considered in sec. 8. Notwithstanding the fact that the equations considered are analytic with respect to μ the obtained almost periodic solutions will not be analytic with respect to μ. However there is no need in practical computations to repeat all the cumbersome computations which we conducted in sec. 8 and which we required for obtaining certain general formulas and results. For practical purposes all these computations can be considerably simplified. We shall here present two methods for the practical computation of an almost periodic solution.

1. The most cumbersome part of the computations in determining almost periodic solutions is associated with the reduction of the equations of the oscillations to the form (8.32). This transformation can be carried out simply in the following manner.

With the aid of a linear substitution with constant coefficients we separate out in the equations of the oscillations

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n + f_s(t) = \mu F_s(t, x_1, \ldots, x_n, \mu) \quad (10.1)$$

the critical variables. In other words, we transform them to the form

$$\left.\begin{aligned} \frac{dy_i}{dt} &= p_{i1}y_1 + \ldots + p_{im}y_m + p_i(t) + \\ &\quad + \mu Y_i(t, y_1, \ldots, y_m, \eta_1, \ldots, \eta_k, \mu) \\ \frac{d\eta_r}{dt} &= b_{r1}\eta_1 + \ldots + b_{rk}\eta_k + \psi_r(t) + \\ &\quad + \mu \Psi_r(t, y_1, \ldots, y_m, \eta_1, \ldots, \eta_k, \mu) \\ &(i = 1, \ldots, m;\ r = 1, \ldots, k = n - m), \end{aligned}\right\} \quad (10.2)$$

where the constants p_{ij} are such that the equation

$$\begin{vmatrix} p_{11} - \lambda & p_{12} & \ldots & p_{1m} \\ p_{21} & p_{22} - \lambda & \ldots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \ldots & p_{mm} - \lambda \end{vmatrix} = 0$$

has roots with only zero parts, while the equation

$$\begin{vmatrix} b_{11} - \rho & b_{12} & \ldots & b_{1k} \\ b_{21} & b_{22} - \rho & \ldots & b_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ b_{k1} & b_{k2} & \ldots & b_{kk} - \rho \end{vmatrix} = 0$$

on the contrary has roots with real parts different from zero. We assume at the same time that equations (10.1) satisfy the conditions indicated in sec. 8.

In particular, the generating system has an almost periodic solution depending on m arbitrary constants. For these conditions the general solution of the system

$$\frac{dy_i}{dt} = p_{i1}y_1 + \dots + p_{im}y_m + p_i(t)$$

has the form

$$y_i = \overline{M}_1\varphi^*_{i1} + \dots + \overline{M}_m\varphi^*_{im} + y^*_i(t),$$

where φ^*_{ij} and y^*_i are almost periodic functions and $\overline{M}_i$ are constants.

We now take as the new variables in place of $y_1,\dots,y_m$ the variables $\overline{M}_1,\dots,\overline{M}_m$ (not changing the variables $\eta_1,\dots,\eta_k$). Equations (10.2) then assume the form

$$\left.\begin{aligned} &\frac{d\overline{M}_i}{dt} = \mu\Phi_i(t, \overline{M}_1, \dots, \overline{M}_m, \eta_1, \dots, \eta_k, \mu),\\ &\frac{d\eta_r}{dt} = b_{r1}\eta_1 + \dots + b_{rk}\eta_k + \psi_r(t) + \mu\Psi^*_r(t, \overline{M}_1, \dots, \eta_k, \mu). \end{aligned}\right\} \quad (10.3$$

We subject the obtained equations to a transformation analogous to the transformation of Krylov and Bogolyubov. For this purpose we introduce into consideration the functions $P_i(M_1,\dots,M_m)$ and $u_i(t,M_1,\dots,M_m)$ determined by the equations

$$\left.\begin{aligned} &P_i(M_1, \dots, M_m) =\\ &\qquad = \lim_{t\to\infty}\frac{1}{t}\int_0^t \Phi_i(t, M_1, \dots, M_m, \eta^0_1, \dots, \eta^0_k, 0)\,dt,\\ &u_i(t, M_1, \dots, M_m) =\\ &\qquad = \int \Phi_i(t, M_1, \dots, M_m, \eta^0_1, \dots, \eta^0_k, 0)\,dt - P_i t, \end{aligned}\right\} \quad (10.$$

where $\eta^0_r(t)$ is an almost periodic solution of the system

$$\frac{d\eta_r}{dt} = b_{r1}\eta_1 + \dots + b_{rk}\eta_k + \psi_r(t). \quad (10.5$$

Since the functions $\Phi_i(t, M_1, \ldots, M_m, \eta_1^o, \ldots, \eta_k^o, 0)$ will be obtained as almost periodic with respect to t, developable in finite Fourier series, the functions u_i will likewise be almost periodic with respect to t. We set

$$\overline{M}_i = M_i + \mu u_i(t, M_1, \ldots, M_m), \quad \eta_r = \eta_r^0 + \mu z_r. \tag{10.6}$$

Then we finally obtain the equations

$$\frac{dM_i}{dt} = \mu P_i(M_1, \ldots, M_m) + \mu^2 Q_i(t, M_1, \ldots, z_k, \mu),$$
$$\frac{dz_r}{dt} = b_{r1}z_1 + \ldots + b_{rk}z_k + Z_r^0(t, M_1, \ldots, M_m) + \mu Z_r(t, M_1, \ldots, z_k, \mu),$$

i.e. equations of the form (8.32).

For practical purposes it is generally possible to restrict oneself to the setting up of only the equations of the first approximation, i.e. equations (10.5) and the equations

$$\frac{dM_i}{dt} = \mu P_i(M_1, \ldots, M_m). \tag{10.7}$$

For setting up the latter there is no need of the transformation (10.6). In fact, as is seen from (10.4), equations (10.7) are obtained at once from the first group of equations (10.3) by averaging with respect to t after which the magnitudes η_r will be replaced by the functions $\eta_r^o(t)$. For improving the approximation it is possible, in place of equations (10.5), to take the equations

$$\frac{dz_r}{dt} = b_{r1}z_1 + \ldots + b_{rk}z_k + Z_r^0(t, M_1, \ldots, M_m),$$

which in the variables η_r corresponds to the equations

$$\frac{d\eta_r}{dt} = b_{r1}\eta_1 + \ldots + b_{rk}\eta_k + \psi_r(t) +$$
$$+ \mu\Psi_r(t, M_1, \ldots, M_m, \eta_1^0, \ldots, \eta_k^0, 0). \tag{10.8}$$

The required almost periodic solutions in the improved first approximation are described by the equations

$$M_i = M_i^*, \quad \eta_r = \eta_r^*(t),$$

where $M_1^*,\ldots,M_m^*$ are the roots of the equations $P_i(M_1^*,\ldots,M_m^*) = 0$, while $\eta_r^*(t)$ is an almost periodic solution of equations (10.8) in which the magnitudes M_i are replaced by the values M_i^*.

The almost periodic solutions thus obtained will satisfy differential equations differing from the exact equations of the oscillations by magnitudes of the second order with respect to μ.

2. To compute the first approximation of the almost periodic solution it is possible to start out directly from equations (10.1). If we restrict ourselves to computing the generating solution it can at once be determined by the formulas

$$x_s^0 = M^*\varphi_{s1} + \ldots + M_m^*\varphi_{sm} + x_s^{0*}(t),$$

where x_s^{0*} is a particular almost periodic solution of the generating system and the magnitudes M_i^* are roots of the equations

$$P_i(M_1,\ldots,M_m) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \sum_{\alpha=1}^n F_\alpha(t, x_1^0, \ldots, x_n^0, 0)\psi_{\alpha i}\,dt = 0. \quad (10.9)$$

To set up equations (10.9) it is necessary to compute first the functions $\psi_{\alpha i}$, i.e. the almost periodic solutions of the system

$$\frac{dx_s}{dt} + a_{1s}x_1 + \ldots + a_{ns}x_n = 0. \quad (10.10)$$

The functions $\psi_{\alpha i}$ must be chosen such that the normalizing conditions (8.1) are satisfied. Let us, however, consider some system of m other independent almost periodic solution of the system (10.10) not connected by relations (8.1). Let us denote these solutions by $\bar\psi_{s1},\ldots\bar\psi_{sm}$ and the functions $\bar P_i$ corresponding to them by $\bar P_i$. Then, evidently, we shall have:

$$\bar\psi_{si} = A_{1i}\psi_{s1} + \ldots + A_{mi}\psi_{sm},$$

where A_{ji} are certain constants for which the determinant $|A_{ji}|$ is different from zero. For the functions $\overline{P}_i$ we shall therefore obtain:

$$\overline{P}_i = A_{1i}P_1 + \ldots + A_{mi}P_m,$$

and since the determinant $|A_{ji}|$ is different from zero the system of equations (10.9) is equivalent to the system of equations

$$\overline{P}_i(M_1, \ldots, M_m) = 0.$$

Thus, to find the magnitudes M_i^* there is no need to assume that the functions ψ_{si} are connected by relations (8.1). It is possible to take as these functions any system of almost periodic solutions of equations (10.10).

It is however necessary to bear in mind that, according to the results of sec. 8, we can be assured that an almost periodic solution of the system (10.1) will actually correspond to the obtained values of the magnitudes M_i^*, if all the roots of equation (8.35) have real parts different from zero. Hence, besides equations (10.9), it is necessary also to set up equation (8.35). This equation however, as we shall see in the next section, will necessarily be required also in the investigation of the stability of the found almost periodic solution. But equation (8.35) will have another form if the functions ψ_{si} are not connected by relations (8.1). We shall now show how we can find both equation (8.35) and equations (10.9) and the first approximation of the required almost periodic function without in general resorting to the setting up of the functions ψ_{si}.

The required (improved) first approximation must for $\mu = 0$ reduce to the generating solution. Hence this first approximation is of the form

$$x_s = M_1\varphi_{s1} + \ldots + M_m\varphi_{sm} + x_s^{0*} + \mu x_s^{(1)}(t), \tag{10.11}$$

where $x_s^{(1)}$ are certain unkown almost periodic functions. This solution satisfies equations (10.1) with an accuracy up to magnitudes of the first order with respect to μ. Hence if we substitute in both sides of system (10.1) for the magnitudes x_s the expressions (10.11) there should be obtained in both sides both the same free terms (which is identically satisfied) and the same coefficients of the first power of μ. In this way there is obtained the following system of equations for the determination of the functions $x_s^{(1)}$:

$$\frac{dx_s^{(1)}}{dt} = a_{s1}x_1^{(1)} + \ldots + a_{sn}x_n^{(1)} + F_s(t, x_1^0, \ldots, x_n^0, 0). \quad (10.12)$$

In order that these equations admit an almost periodic solution it is necessary and sufficient that equations (10.9) be satisfied (where the functions φ_{si} need not necessarily satisfy relations (8.1). We have thus again arrived at equations (10.9). However, in each individual problem the conditions for the existence of almost periodic solutions for the system (10.12) can easily be set up directly without turning to the general formulas (10.9). Up to now we have proceeded in this way in all problems when it was necessary for us to set up the conditions for the existence of periodic solutions of equations of the type (10.1). We can now proceed in an analogous manner also in setting up the conditions for the existence of almost periodic solutions and this problem will not in the least be any more complicated since the right hand sides of equations (10.12) represent finite trigonometric sums.

Having determined the constants M_i from the conditions of almost periodicity of the functions $x_s^{(1)}$ we can then determine the functions $x_s^{(1)}$ themselves, which will give us an improved first approximation of the required almost periodic solution. We may note that the functions $x_s^{(1)}$ will contain m arbitrary constants since we can write

$$x_s^{(1)} = M_1^{(1)}\varphi_{s1} + \ldots + M_m^{(1)}\varphi_{sm} + x_s^{(1)*}(t),$$

where $x_s^{(1)*}$ is some particular almost periodic solution of the system (10.12).

The constants $M_s^{(1)}$ can not be determined if we restrict ourselves to the computation of only the first approximation.

We now proceed to the setting up of equation (8.35) which we require not only for clarification of the question of the actual existence of the required almost periodic solution but also for investigating its stability. For this purpose let us, for the obtained almost periodic solution, set up the equations in variations, restricting ourselves to the terms of the first order with respect to μ. Setting in equations (10.1)

$$x_s = x_s^0 + \mu x_s^{(1)} + \xi_s$$

and rejecting magnitudes of order higher than the first we obtain the required equations in variations in the form

$$\frac{d\xi_s}{dt}=\left(a_{s1}+\mu\frac{\partial F_s^0}{\partial x_1^0}\right)\xi_1+\ldots+\left(a_{sn}+\mu\frac{\partial F_s^0}{\partial x_n^0}\right)\xi_n,$$
$$F_s^0=F_s(t, x_1^0, \ldots, x_n^0, 0).$$

Let us now replace the obtained equations by the equation

$$\frac{d\zeta_s}{dt}=\left(a_{s1}+\mu\frac{\partial F_s^0}{\partial x_1^0}\right)\zeta_1+\ldots+\left(a_{sn}+\mu\frac{\partial F_s^0}{\partial x_n^0}\right)\zeta_n-\mu\varkappa\zeta_s, \quad (10.1$$

which corresponds to the substituion $\xi_s=e^{\mu\varkappa t}\zeta_s$, and try to choose the magnitude $\varkappa$ in such manner that equations (10.13) can be satisfied with an accuracy up to magnitudes of the first order by an almost periodic solution of the form

$$\zeta_s=\zeta_s^0+\mu\zeta_s^{(1)}.$$

We shall have

$$\zeta_s^0=N_1\varphi_{s1}+\ldots+N_m\varphi_{sm},$$

where $N_n,\ldots,N_m$ are arbitrary constants and

$$\frac{d\zeta_s^{(1)}}{dt}=a_{s1}\zeta_1^{(1)}+\ldots+a_{sn}\zeta_n^{(1)}+\sum_{\alpha=1}^{n}\sum_{i=1}^{m}\frac{\partial F_s^0}{\partial x_\alpha^0}\varphi_{\alpha i}N_i-\varkappa\sum_{i=1}^{m}\varphi_{si}N_i. \quad (10.14)$$

For this system to admit an almost periodic solution it is necessary and sufficient that the conditions be satisfied

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\sum_{s=1}^{n}\left\{\sum_{\alpha=1}^{n}\sum_{i=1}^{m}\frac{\partial F_s^0}{\partial x_\alpha^0}\varphi_{\alpha i}N_i-\varkappa\sum_{i=1}^{m}\varphi_{si}N_i\right\}\psi_{sj}\,dt=0$$
$$(j=1, \ldots, m)$$

or, comparing with (10.9),

$$\left(\frac{\partial P_j}{\partial M_1}-\varkappa A_{1j}\right)N_1+\ldots+\left(\frac{\partial P_j}{\partial M_m}-\varkappa A_{mj}\right)N_m=0, \quad (10.15)$$

where A_{ij} are constants determined by the equations

$$A_{ij}=\sum_{s=1}^{n}\varphi_{si}\psi_{sj}.$$

For equations (10.15) to admit a nontrivial solution it is necessary and sufficient that the magnitude $\varkappa$ be a root of the equation

$$\begin{vmatrix} \frac{\partial P_1}{\partial M_1} - \varkappa A_{11} & \frac{\partial P_1}{\partial M_2} - \varkappa A_{21} & \cdots & \frac{\partial P_1}{\partial M_m} - \varkappa A_{m1} \\ \frac{\partial P_2}{\partial M_1} - \varkappa A_{12} & \frac{\partial P_2}{\partial M_2} - \varkappa A_{22} & \cdots & \frac{\partial P_2}{\partial M_m} - \varkappa A_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial P_m}{\partial M_1} - \varkappa A_{1m} & \frac{\partial P_m}{\partial M_2} - \varkappa A_{2m} & \cdots & \frac{\partial P_m}{\partial M_m} - \varkappa A_{mm} \end{vmatrix} = 0. \quad (10.16)$$

If the functions ψ_{si} are chosen such that relations (8.1) are satisfied equation (10.16) will agree with equation (8.35). Thus equation (8.35) determines those values of $\varkappa$ for which the system (10.14) admits an almost periodic solution. But the conditions of existence of almost periodic solutions of the system (10.14) can be set up directly, which permits setting up equation (8.35) without first computing the functions ψ_{si}. From what has been said it also follows that if in equations (10.9) it is not assumed that relations (8.1) are satisfied it will be necessary to take equation (10.16) in place of equation (8.35).

We may remark in conclusion that the method just presented of computing the first approximation for the almost periodic solution essentially in no way differs from the method of computing periodic solutions. As to the method described of setting up equation (8.35), it is entirely analogous to the method of setting up the equations for the characteristic exponents, established in sec. 12 of chapter III.

Examples illustrating the methods described will be given below. But first we shall establish the criteria of stability for the obtained almost periodic solutions.

11. Stability Criteria

For establishing the stability criteria of the almost periodic motions considered in the preceding sections we shall make use of certain known results of A. M. Lyapunov.

Let there be given a certain system of linear equations with constant coefficients

$$\frac{dx_s}{dt} = p_{s1}x_1 + \dots + p_{sn}x_n. \tag{11.1}$$

If the fundamental equation of this system has no pair of roots the sum of which is equal to zero there always exists a quadratic form $V(x_1,\dots,x_n)$ of the variables $x_1,\dots,x_n$ that satisfies the equation

$$\sum_{\alpha=1}^{n} \frac{\partial V}{\partial x_\alpha}(p_{\alpha 1}x_1 + \dots + p_{\alpha n}x_n) = -\sum_{\alpha=1}^{n} x_\alpha^2.$$

If at the same time all the roots of the fundamental equation of system (11.1) have negative real parts the form V will necessarily be positive definite. If however among the roots there is at least one with positive real part the form V may take negative values.

We shall now prove the following theorem:

THEOREM. IF, WHEN THE CONDITIONS OF THE THEOREM OF SEC. 8 ARE SATISFIED, ALL THE ROOTS OF EQUATION (8.35) AND THE n - m ROOTS OF THE FUNDAMENTAL EQUATION OF THE GENERATING SYSTEM HAVE NEGATIVE REAL PARTS THE ALMOST PERIODIC SOLUTION CONSIDERED IN THIS THEOREM, FOR SUFFICIENTLY SMALL μ, IS ASYMPTOTICALLY STABLE. IF, ON THE CONTRARY, THE REAL PART OF AT LEAST ONE OF THESE ROOTS IS POSITIVE THE ALMOST PERIODIC SOLUTION UNDER CONSIDERATION IS UNSTABLE.

PROOF. The problem reduces to the investigation of the stability of the almost periodic solution of equations (8.37). Let $N_i = \bar{N}_i(t)$, $z_r = \bar{z}_r(t)$ be this solution. Taking it as the undistrubed motion we set up the equations of the disturbed motion, setting

$$N_i = \bar{N}_i(t) + \varphi_i, \quad z_r = \bar{z}_r(t) + \psi_r$$
$$(i = 1, \dots, m;\ r = 1, \dots, k).$$

We shall have:

$$\left.\begin{aligned}
\frac{d\varphi_i}{dt} &= \mu(c_{i1}\varphi_1 + \ldots + c_{im}\varphi_m) + \\
&\quad + \mu[Q_i(t, M_1^*, \ldots, M_m^*, \bar{z}_1 + \psi_1, \ldots, \bar{z}_k + \psi_k, 0) - \\
&\qquad - Q_i(t, M_1^*, \ldots, M_m^*, \bar{z}_1, \ldots, \bar{z}_k, 0)] + \\
&\quad + \mu^2[Q_i^*(t, \bar{N}_1 + \varphi_1, \ldots, \bar{N}_m + \varphi_m, \bar{z}_1 + \psi_1, \ldots, \bar{z}_k + \psi_k, \mu) - \\
&\qquad - Q_i^*(t, \bar{N}_1, \ldots, \bar{N}_m, \bar{z}_1, \ldots, \bar{z}_k, \mu)], \\
\frac{d\psi_r}{dt} &= b_{r1}\psi_1 + \ldots + b_{rk}\psi_k + \\
&\quad + \mu[Z_r^*(t, \bar{N}_1 + \varphi_1, \ldots, \bar{N}_m + \varphi_m, \bar{z}_1 + \psi_1, \ldots, \bar{z}_k + \psi_k, \mu) - \\
&\qquad - Z_r^*(t, \bar{N}_1, \ldots, \bar{N}_m, \bar{z}_1, \ldots, \bar{z}_k, \mu)].
\end{aligned}\right\} \quad (11.2)$$

We now form the two quadratic forms $V_1(\varphi_1, \ldots, \varphi_m)$ and $V_2(\psi_1, \ldots, \psi_k)$ with the aid of the equations

$$\sum_{i=1}^{m} \frac{\partial V_1}{\partial \varphi_i}(c_{i1}\varphi_1 + \ldots + c_{im}\varphi_m) = -\sum_{i=1}^{m} \varphi_i^2,$$

$$\sum_{r=1}^{k} \frac{\partial V_2}{\partial \psi_r}(b_{r1}\psi_1 + \ldots + b_{rk}\psi_k) = -\sum_{r=1}^{k} \psi^2.$$

We assume here, for simplifying the proof, that neither equation (8.35) nor equation (8.14) have pairs of roots giving the sum zero. We set

$$V = \frac{1}{\mu} V_1(\varphi_1, \ldots, \varphi_m) + a V_2(\psi_1, \ldots, \psi_k),$$

where α is a positive constant and compute the total derivative of V with respect to t. In virtue of equations (11.2) we shall have:

$$\begin{aligned}
\frac{dV}{dt} &= -\sum_{i=1}^{m} \varphi_i^2 - a\sum_{r=1}^{k} \psi_r^2 + \\
&\quad + \sum_{i=1}^{m} \frac{\partial V_1}{\partial \varphi_i}[Q_i(t, M_1^*, \ldots, \bar{z}_k + \psi_k, 0) - Q_i(t, M_1^*, \ldots, \bar{z}_k, 0)] + \\
&\quad + \mu \sum_{i=1}^{m} \frac{\partial V_1}{\partial \varphi_i}[Q_i^*(t, \bar{N}_1 + \varphi_1, \ldots, \bar{z}_k + \psi_k, \mu) - Q_i^*(t, \bar{N}_1, \ldots, \bar{z}_k, \mu)] + \\
&\quad + \mu a \sum_{r=1}^{k} \frac{\partial V_2}{\partial \psi_r}[Z_r^*(t, \bar{N}_1 + \varphi_1, \ldots, \bar{z}_k + \psi_k, \mu) - Z_r^*(t, \bar{N}_1, \ldots, \bar{z}_k, \mu)].
\end{aligned}$$

Since the inequalities are satisfied (in the neighborhood of the undisturbed motion)

$$|Q_i(t, \overline{M}_1^*, \ldots, \bar{z}_k + \psi_k, 0) - Q_i(t, M_1^*, \ldots, z_k, 0)| < A \sum_{s=1}^{k} |\psi_s|,$$

$$|Q_i^*(t, \overline{N}_1 + \varphi_1, \ldots, \bar{z}_k + \psi_k, \mu) - Q_i^*(t, \overline{N}_1, \ldots, \bar{z}_k, \mu)| < $$
$$< B\left(\sum_{j=1}^{m} |\varphi_j| + \sum_{s=1}^{k} |\psi_s|\right),$$

$$|Z_r^*(t, \overline{N}_1 + \varphi_1, \ldots, \bar{z}_k + \psi_k, \mu) - Z_r^*(t, \overline{N}_1, \ldots, \bar{z}_k, \mu)| <$$
$$< C\left(\sum_{j=1}^{m} |\varphi_j| + \sum_{s=1}^{k} |\psi_s|\right),$$

where A, B, C are certain constants, it is easily seen that for μ sufficiently small and α sufficiently large, dV/dt will be a definite negative function of the variables $\varphi_1, \ldots, \varphi_m$, $\psi_1, \ldots, \psi_k$.

With this established, let us assume first that the real parts of all roots of equation (8.35) and the n- m roots of the fundamental equation of the generating system have negative real parts. This means, in particular, that all roots of equation (8.14) have negative real parts. In this case both quadratic forms V_1 and V_2 will be positive definite with respect to the variables that enter them and therefore the function V will be positive definite with respect to all the n variables $\varphi_1, \ldots, \varphi_m$, $\psi_1, \ldots, \psi_k$. But in this case the function V satisfies all the conditions of the well known theorem of Lyapunov on asymptotic stability and consequently the undisturbed motion is asymptotically stable.

If at least one of the roots figuring in the conditions of the theorem has a positive real part at least one of the forms V_1 and V_2, and therefore also the form V, can assume negative values. In this case the function V satisfies all the conditions of the theorem of Lyapunov on instability and the undisturbed motion will be unstable. Thus, the theorem has been completely proven.

12. Application to the Problem of the Forced Oscillations of a Regenerative Receiver

We shall consider the system described by the equation

$$\frac{d^2x}{dt^2} + k^2 x = \mu (1 - x^2) \frac{dx}{dt} + A \sin \omega_1 t + B \sin \omega_2 t, \qquad (12.1)$$

and assume that the ratio ω_1/ω_2 is irrational. To an equation of this kind is reduced, under certain simplifying assumptions, the problem of the forced oscillations of a regenerative receiver when the external excitation consists of two harmonics with incommensurable frequencies. This problem has been considered by us in detail in sec. 5 of chapter I for the case of periodic excitation, i.e. on the assumption that one of the constants, A or B, is equal to zero. With this assumption it was shown that the system possesses a stable periodically oscillating regime when the frequency k is near the frequency of the external excitation. This occurs notwithstanding the fact that the system considered is self-oscillating and in the absence of external excitation there arise in it stable self-oscillations with a frequency differing little from k. The frequency of the self-oscillations is "trapped" by the frequency of the external excitation. The situation is otherwise if k differs considerably from the frequency of the external excitation. In this case the system (12.1) as before admits a periodic solution (for periodic excitation) but the motion corresponding to it is unstable and therefore does not physically arise. It is found that in this case there arise in the system stable, almost periodic oscillations in which, together with the frequency of the external excitation, there is present also the frequency of the self-oscillations. In the first chapter, where the methods of determining only the periodic solutions were set forth, we naturally did not have the possibility of finding these combined oscillations. We shall see whether we are able to solve this problem with the aid of the methods described in the preceding sections. We shall assume, for the sake of generality, that the external excitation is likewise almost periodic, i.e. that neither of the constants A and B is equal to zero.

As in the problem of periodic solutions, two cases must here be distinguished: The nonresonance and the resonance. We shall say that we are dealing with a nonresonance case when none of the magnitudes $mk + m_1\omega_1 + m_2\omega_2$ is of the order of smallness of μ. Here m, m_1, m_2 are some integral positive or negative numbers for which $|m| + |m_1| + |m_2| \leq 4$ and $m \neq 0$. We shall first investigate the nonresonance case.

We shall use the first method of sec. 9 and restrict ourselves to the first approximation. The general solution of equation (12.1) (which we consider as a system of two equations with respect to x and dx/dt) for $\mu = 0$ has the form

$$\left.\begin{aligned} x &= M_1 \cos kt + M_2 \sin kt + \frac{A}{k^2-\omega_1^2}\sin\omega_1 t + \frac{B}{k^2-\omega_2^2}\sin\omega_2 t,\\ \frac{dx}{dt} &= -kM_1\sin kt + kM_2\cos kt + \frac{A\omega_1}{k^2-\omega_1^2}\cos\omega_1 t + \frac{B\omega_2}{k^2-\omega_2^2}\cos\omega_2 t, \end{aligned}\right\} \tag{12.2}$$

where M_1 and M_2 are arbitrary constants. We shall take these magnitudes as the new variables in place of x and dx/dt. We shall then have:

$$\cos kt\,\frac{dM_1}{dt} + \sin kt\,\frac{dM_2}{dt} = 0,$$

$$-k\sin kt\,\frac{dM_1}{dt} + k\cos kt\,\frac{dM_2}{dt} = \mu(1-x^2)\frac{dx}{dt}.$$

whence

$$\left.\begin{aligned} \frac{dM_1}{dt} &= -\frac{\mu}{k}(1-x^2)\frac{dx}{dt}\sin kt,\\ \frac{dM_2}{dt} &= \frac{\mu}{k}(1-x^2)\frac{dx}{dt}\cos kt. \end{aligned}\right\} \tag{12.3}$$

Averaging the right hand sides with respect to t we obtain the following equations of the first approximation:

$$\frac{dM_1}{dt} = \mu P_1(M_1, M_2), \quad \frac{dM_2}{dt} = \mu P_2(M_1, M_2), \tag{12.4}$$

where

$$P_1 = -\frac{1}{k}\lim_{t\to\infty}\frac{1}{t}\int_0^t (1-x^2)\frac{dx}{dt}\sin kt\,dt,$$

$$P_2 = \frac{1}{k}\lim_{t\to\infty}\frac{1}{t}\int_0^t (1-x^2)\frac{dx}{dt}\cos kt\,dt.$$

The functions P_1 and P_2 are the free terms of the expansions of the right hand sides of equations (12.3) in trigonometric sums. Taking (12.2) into account we readily obtain

$$\left.\begin{aligned} 8P_1 &= M_1\left[4-(M_1^2+M_2^2)-\frac{2A^2}{(k^2-\omega_1^2)^2}-\frac{2B^2}{(k^2-\omega_2^2)^2}\right], \\ 8P_2 &= M_2\left[4-(M_1^2+M_2^2)-\frac{2A^2}{(k^2-\omega_1^2)^2}-\frac{2B^2}{(k^2-\omega_2^2)^2}\right]. \end{aligned}\right\} \qquad (12.5)$$

The required almost periodic solution of equation (12.1) is determined to the first approximation by formulas (12.2) in which M_1 and M_2 are constant magnitudes satisfying the equations

$$P_1(M_1, M_2) = P_2(M_1, M_2) = 0. \qquad (12.6)$$

It is here assumed that the equation

$$\begin{vmatrix} \frac{\partial P_1}{\partial M_1} - \varkappa & \frac{\partial P_1}{\partial M_2} \\ \frac{\partial P_2}{\partial M_1} & \frac{\partial P_2}{\partial M_2} - \varkappa \end{vmatrix} = 0 \qquad (12.7)$$

has no roots with zero real parts.

Equations (12.6) have the solution $M_1 = M_2 = 0$ and an infinite manifold of solutions satisfying the equation

$$4-(M_1^2+M_2^2)-\frac{2A^2}{(k^2-\omega_1^2)^2}-\frac{2B^2}{(k^2-\omega_2^2)^2}=0. \qquad (12.8)$$

But for these latter solutions the determinant $\frac{\partial(P_1, P_2)}{\partial(M_1, M_2)}$ evidently reduces to zero and therefore equation (12.7) has a zero root. Equations (12.8) are therefore not suitable for our purpose and must be discarded. As regards the solution $M_1 = M_2 = 0$, there actually corresponds to it an almost periodic solution of the system (12.1) since for it the roots of equation (12.7) will be the magnitudes

$$\varkappa_1 = \varkappa_2 = \frac{1}{4}\left(2-\frac{A^2}{(k^2-\omega_1^2)^2}-\frac{B^2}{(k^2-\omega_2^2)^2}\right). \qquad (12.9)$$

The obtained almost periodic solution for $\mu = 0$ reduces to the generating solution

$$x^0 = \frac{A}{k^2 - \omega_1^2} \sin \omega_1 t + \frac{B^2}{k^2 - \omega_2^2} \sin \omega_2 t,$$

in which as before the frequency k is absent. In this way the theory developed in the preceding sections permits finding only those almost periodic solutions of equation (12.1) in which the frequency of the self-oscillations is "trapped" by the frequencies of the external excitation. As we shall see in the next section, this circumstance holds true not only for the special problem under consideration but also in the general case. The methods of the preceding sections enable finding only those almost periodic solutions in which only the frequencies contained in the functions f_s and F_s prevail. For finding almost periodic solutions in which, together with the indicated frequencies, there prevail also the frequencies of the free oscillations of the linearized system (more accurately, near the system) other methods will be indicated below.

Let us continue however with the consideration of equation (12.1). On the basis of the results of sec. 7 and 11 the almost periodic solution that we have found will be stable if the roots (12.9) of equation (12.7) are negative. We thus obtain the following condition of stability:

$$\frac{A^2}{(k^2 - \omega_1^2)^2} + \frac{B^2}{(k^2 - \omega_2^2)^2} > 2. \tag{12.10}$$

This condition will not of course be satisfied if the magnitude k differs by a sufficiently large value from the frequencies ω_1 and ω_2.

Let us now consider the resonance case in which the magnitude k differs from ω_1 by a magnitude of the order of smallness of μ. Setting in this case, as in the problem of periodic solutions,

$$k^2 = \omega_1^2 - \mu a, \qquad A = \mu\lambda, \tag{12.11}$$

we obtain in place of equation (12.1) the equation

$$\frac{d^2x}{dt^2} + \omega_1^2 x = \mu(1 - x^2)\frac{dx}{dt} + \mu a x + \mu\lambda \sin \omega_1 t + B \sin \omega_2 t, \tag{12.12}$$

which in the variables M_1 and M_2 is equivalent to the system

$$\left.\begin{aligned} \frac{dM_1}{dt} &= -\frac{\mu}{\omega_1}\left\{(1-x^2)\frac{dx}{dt}+ax+\lambda\sin\omega_1 t\right\}\sin\omega_1 t,\\ \frac{dM_2}{dt} &= \frac{\mu}{\omega_1}\left\{(1-x^2)\frac{dx}{dt}+ax+\lambda\sin\omega_1 t\right\}\cos\omega_1 t. \end{aligned}\right\} \tag{12.13}$$

with

$$x = M_1\cos\omega_1 t + M_2\sin\omega_1 t + \frac{B}{\omega_1^2-\omega_2^2}\sin\omega_2 t,$$
$$\frac{dx}{dt} = -\omega_1 M_1\sin\omega_1 t + \omega_1 M_2\cos\omega_1 t + \frac{B\omega_2}{\omega_1^2-\omega_2^2}\cos\omega_2 t.$$

For the functions $P_1(M_1,M_2)$ and $P_2(M_1,M_2)$ we have

$$2\omega_1 P_1(M_1,M_2) = -\lambda + aM_2 + \omega_1 M_1\left(1-\frac{B^2}{2(\omega_1^2-\omega_2^2)^2}-\frac{M_1^2+M_2^2}{4}\right),$$
$$2\omega_1 P_2(M_1,M_2) = -aM_1 + \omega_1 M_2\left(1-\frac{B^2}{2(\omega_1^2-\omega_2^2)^2}-\frac{M_1^2+M_2^2}{4}\right),$$

and equations (12.6) determining the parameters M_1 and M_2 differ little from the corresponding equations (5.16) of chapter I in the problem of periodic solutions. These equations can be investigated in exactly the same way as in the problem of periodic solutions. We shall restrict ourselves however to the consideration of the case of exact resonance, i.e. we shall assume that $\alpha = 0$. In this case the equations have the solution $M_2 = 0$, $M_1 = M$, where M is a root of the cubic equation

$$P(M) = -\frac{1}{8}M^3 + \left(\frac{1}{2}-\frac{B^2}{4(\omega_1^2-\omega_2^2)^2}\right)M - \frac{\lambda}{2\omega_1} = 0. \tag{12.14}$$

The roots of equation (12.7) will now be the magnitudes

$$\varkappa_1 = \frac{dP(M)}{dM}, \quad \varkappa_2 = \frac{1}{2}-\frac{B^2}{4(\omega_1^2-\omega_2^2)^2}-\frac{M^2}{8} = \frac{\lambda}{2\omega_1 M}.$$

Hence, if equation (12.14) has one or three different real solutions equation (12.7) will have real roots different from zero. In this case equation (12.12) will have one or three almost periodic solutions, reducing for $\mu = 0$ to the solutions

$$x^0 = M\cos\omega_1 t + \frac{B}{\omega_1^2-\omega_2^2}\sin\omega_2 t$$

of the generating system. Since equation (12.14) cannot have a triple root at least one almost periodic solution always exists.

The conditions of stability of the obtained almost periodic solutions have the form

$$M < 0, \quad \frac{dP(M)}{dM} < 0.$$

Hence only those almost periodic solutions can be stable which correspond to negative roots of equation (12.14). But this equation, as is seen from the expression of its free term, always has at least one negative root. If there is only one such root the almost periodic solution corresponding to it will actually be stable since $P(-\infty) > 0$ and therefore for this root $P'(M) < 0$. If however equation (12.14) has three negative roots the almost periodic solutions corresponding to the least and greatest of them will be stable while the solution corresponding to the middle root will be unstable.

13. Analysis of the Equations Determining the Parameters of the Generating System. Resonance and Nonresonance Frequencies

Let us consider again the quasilinear system

$$\frac{dx_s}{dt} = a_{s1}x_1 + \dots + a_{sn}x_n + f_s(t) + \mu F_s(t, x_1, \dots, x_n, \mu) \quad (13.1)$$
$$(s = 1, \dots, n)$$

for the assumptions of sec. 8. The generating system will then have the almost periodic solution

$$x_s^0 = M_1\varphi_{s1} + \dots + M_m\varphi_{sm} + x_s^{0*}(t), \quad (13.2)$$

depending on the arbitrary sonstants $M_1, \dots, M_M$.

The fundamental equation

$$\begin{vmatrix} a_{11}-\rho & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\rho & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\rho \end{vmatrix} = 0 \quad (13.3)$$

of the generating system has m roots with real parts equal to zero. The remaining $n-m$ roots of this equation have real parts different from zero. Let us assume that among the m roots of equation (13.3) with zero real parts there are $2q$ purely imaginary roots $\pm \lambda_1 i, \ldots, \pm \lambda_q i$, so that the magnitudes λ_i are the frequencies of the free oscillations of the generating system.

Let $\nu_1, \ldots, \nu_N$ be the aggregate of all frequencies figuring in the expansions in trigonometric sums with respect to t of the functions $F_s(t, x_1, \ldots, x_n, 0)$ and also of the functions $f_s(t)$. We shall call λ_p a RESONANCE frequency if at least one of the magnitudes

$$m_1\lambda_1 + \ldots + m_p\lambda_p + \ldots + m_q\lambda_q + n_1\nu_1 + \ldots + n_N\nu_N \qquad (13.4)$$

is of the order of smallnessof μ. Here $m_1, \ldots, m_q, n_1, \ldots, n_N$ are positive or negative integers for which

$$|m_1| + \ldots + |m_q| + |n_1| + \ldots + |n_N| \leqslant r + 2,$$

with $m_p \neq 0$, and r is the greatest order with respect to $x_1, \ldots, x_n$ of the functions $F_s(t, x_1, \ldots, x_n, 0)$. From this definition it is seen that if the frequency λ_p is not a resonance frequency the roots $\pm \lambda_p i$ of equation (13.3) will necessarily be simple. In the preceding section we have seen on a particular example that with the methods given above it is not possible to construct an almost periodic solution in which among the principal terms there was a harmonic with nonresonance frequency. We shall show that this holds true in the general case.

For this purpose let us assume that λ_1 is one of the resonance frequencies. We can then assume that

$$\varphi_{s1} = A_s \cos \lambda_1 t - B_s \sin \lambda_1 t,$$
$$\varphi_{s2} = A_s \sin \lambda_1 t + B_s \cos \lambda_1 t$$

and that the remaining functions φ_{sj} do not contain terms with $\cos \lambda_1 t$ and $\sin \lambda_1 t$.

The principal terms of the almost periodic solutions of the system (13.1) are the functions (13.2) in which the constants M_i satisfy the equations

$$P_i(M_1, \ldots, M_m) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \sum_{\alpha=1}^{n} F_\alpha(t, x_1^0, \ldots, x_n^0, 0) \psi_{\alpha i}\, dt = 0 \qquad (13.5)$$

$$(i = 1, \ldots, m)$$

on the assumption that equation (8.35) has no roots with zero real parts. We shall show that these conditions can not be satisfied otherwise than on the assumption $M_1 = M_2 = 0$.

In fact, the functions ψ_{sj} figuring in (13.5) are some linear combinations of the solutions of the system

$$\frac{dy_s}{dt} + a_{1s}y_1 + \ldots + a_{ns}y_n = 0, \tag{13.6}$$

corresponding to the roots with zero real parts of equation (13.3). As was remarked in sec. 10, it is not necessary here for computing the roots of equations (13.5) to assume that the functions ψ_{sj} satisfy conditions (8.1). We can, in particular, set

$$\left.\begin{aligned}\psi_{s1} &= C_s \cos\lambda_1 t - D_s \sin\lambda_1 t,\\ \psi_{s2} &= C_s \sin\lambda_1 t + D_s \cos\lambda_1 t\end{aligned}\right\} \tag{13.7}$$

and consider that the remaining functions ψ_{sj} do not contain terms with $\cos\lambda_1 t$ and $\sin\lambda_1 t$. But then, as is easily seen, the first two equations (13.5) will be satisfied if we set $M_1 = M_2 = 0$.

In fact, for the assumption stated we can write [1]:

$$\sum_{\alpha=1}^{n} F_\alpha(t, x_1^0, \ldots, x_n^0, 0)\,\psi_{\alpha j} = \sum_p (M_{jp}\cos u_p + N_{jp}\sin u_p) \tag{13.8}$$
$$(j = 1, 2),$$

where the sum consists of a finite number of terms. The magnitudes u_p here have the form

$$u_p = (\lambda_1 + m_2^{(p)}\lambda_2 + \ldots + m_q^{(p)}\lambda_q + n_1^{(p)}\nu_1 + \ldots + n_N^{(p)}\nu_N)\,t,$$

where $m_2^{(p)}, \ldots, m_q^{(p)}, n_1^{(p)}, \ldots, n_N^{(p)}$ - are integers for which $|m_2^{(p)}| + \ldots + |m_q^{(p)}| + |n_1^{(p)}| + \ldots + |n_N^{(p)}| \leqslant r+1$. But since λ_1 is not a resonance frequency not one of the magnitudes u_p is equal to zero. From this it follows that the free terms

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[1] We can assume that the functions x_s^0 contain only those frequencies which enter in f_s, i.e. $\nu_1, \ldots, \nu_N$.

in the expansions in trigonometric sums of the functions (13.8), i.e. the left hand sides of the first two equations (13.5), reduce to zero. Putting then in the remaining equations (13.5), $M_1 = M_2 = 0$ we obtain $m - 2$ equations for determining the magnitudes $M_3, \ldots M_m$. Let us assume that these equations have the solution $M_3 = M_3^*, \ldots, M_m = M_m^*$ and that equation (8.35) for $M_1 = M_2 = 0,\ M_3 = M_3^*, \ldots, M_m = M_m^*$ has no roots with zero real parts. The obtained quantities $M_1, \ldots, M_m$ then satisfy all the required conditions.

We shall now show, conversely, that if equations (13.5) have a solution $M_i = M_i^*$ for which equation (8.35) has no roots with zero real parts we must necessarily have $M_1^* = M_2^* = 0$. For this purpose let us put

$$M_1 = M\cos\alpha, \qquad M_2 = M\sin\alpha,$$

so that

$$x_s^0 = M\left[A_s\cos(\lambda_1 t - \alpha) - B_s\sin(\lambda_1 t - \alpha)\right] + \\ + M_3\varphi_{s3} + \ldots + M_m\varphi_{sm} + x_s^{0*}(t).$$

The functions $\psi_{s3}, \ldots, \psi_{sm}$ do not contain terms with $\cos\lambda_1 t$ and $\sin\lambda_1 t$. We can therefore write for $i > 2$:

$$\sum_{\alpha=1}^{n} F_\alpha(t, x_1^0, \ldots, x_n^0, 0)\psi_{\alpha i} = \sum_p (M_{ip}\cos v_p + N_{ip}\sin v_p) \quad (13.9)$$
$$(i = 3, \ldots, m),$$

where

$$v_p = m^{(p)}(\lambda_1 t - \alpha) + (m_2^{(p)}\lambda_2 + \ldots + m_q^{(p)}\lambda_q + n_1^{(p)}\nu_1 + \ldots + n_N^{(p)}\nu_N)t$$

and $m^{(p)}, m_2^{(p)}, \ldots, n_N^{(p)}$ are integers for which $|m^{(p)}| + |m_2^{(p)}| + \ldots + |n_N^{(p)}| \leqslant r + 2$. Since the frequency λ_1 is nonresonance the coefficient of t in v_p can become zero only for the condition that $m^{(p)} = 0$. But then v_p will also not contain α. From this it follows that the free terms in (13.9), i.e. the left hand sides of the last $m - 2$ equations of (13.5), will not depend on α and we can write them in the following form:

$$P_i(M_1, \ldots, M_m) \equiv Q_i(M_3, \ldots, M_m, M) = 0 \qquad (13.10)$$
$$(i = 3, \ldots, m).$$

Further we have :

$$C_s \cos\lambda_1 t - D_s \sin\lambda_1 t =$$
$$= [C_s \cos(\lambda_1 t - \alpha) - D_s \sin(\lambda_1 t - \alpha)]\cos\alpha -$$
$$- [C_s \sin(\lambda_1 t - \alpha) + D_s \cos(\lambda_1 t - \alpha)]\sin\alpha,$$
$$C_s \sin\lambda_1 t + D_s \cos\lambda_1 t =$$
$$= [C_s \cos(\lambda_1 t - \alpha) - D_s \sin(\lambda_1 t - \alpha)]\sin\alpha +$$
$$+ [C_s \sin(\lambda_1 t - \alpha) + D_s \cos(\lambda_1 t - \alpha)]\cos\alpha.$$

The first two of equations (13.5) therefore have the form

$$\left.\begin{aligned} P(M_3, \ldots, M_m, M)\cos\alpha - Q(M_3, \ldots, M_m, M)\sin\alpha = 0, \\ P(M_3, \ldots, M_m, M)\sin\alpha + Q(M_3, \ldots, M_m, M)\cos\alpha = 0, \end{aligned}\right\} \quad (13.11)$$

where the functions

$$P = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\beta=1}^n F_\beta(t, x_1^0, \ldots, x_n, 0) \times$$
$$\times [C_\beta \cos(\lambda_1 t - \alpha) - D_\beta \sin(\lambda_1 t - \alpha)]\, dt,$$
$$Q = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\beta=1}^n F_\beta(t, x_1^0, \ldots, x_n^0, 0) \times$$
$$\times [C_\beta \sin(\lambda_1 t - \alpha) + D_\beta \cos(\lambda_1 t - \alpha)]\, dt$$

likewise do not depend on α. From (13.11) we obtain:

$$P(M_3, \ldots, M_m, M) = 0, \quad Q(M_3, \ldots, M_m, M) = 0. \quad (13.12)$$

The system (13.5) is thus equivalent to the system of m equations (13.10) and (13.12) with $m - 1$ unknowns $M_3, \ldots, M_m, M$. Let us assume that these equations are simultaneous and have the solution $M = M^*, M_3 = M_3^*, \ldots, M_m = M_m^*$. Then, if $M \neq 0$, we obtain for M_1^* and M_2^* an infinite manifold of values connected by the relation $M_1^{*2} + M_2^{*2} = M^{*2}$, and for the solution thus found of equations (13.5) the functional determinant

$$\frac{\partial(P_1, \ldots, P_m)}{\partial(M_1, \ldots, M_m)}$$

will reduce to zero. Hence, for the obtained solution the condition of the roots of equation (8.35) will not be satisfied. In order that this condition be satisfied it is necessary that M^* be equal to zero, which will give for M_1^* and M_2^* the entirely definite values $M_1^* = M_2^* = 0$. At the same time, as was shown, equations (13.12) are identically satisfied and the number of equations will be equal to the number of unknowns.

Thus, the proposition on the nonresonance frequencies has been completely proven.

Let us now make a somewhat more general assumption, namely that among the frequencies $\lambda_1, \ldots, \lambda_q$ there exist $p \leq q$ frequencies, let them be $\lambda_1, \ldots, \lambda_p$, such that for no integral $m_{p+1}, \ldots, m_q, n_1, \ldots, n_N$ for which $|m_{p+1}| + \ldots + |m_q| + |n_1| + \ldots + |n_N| \leqslant r+1$ are the relations satisfied

$$\lambda_j = m_{p+1}\lambda_{p+1} + \ldots + m_q\lambda_q + n_1\nu_1 + \ldots + n_N\nu_N$$
$$(j = 1, \ldots, p).$$

Let us set

$$\begin{aligned} \varphi_{sj} &= A_{sj}\cos\lambda_j t - B_{sj}\sin\lambda_j t, & \varphi_{s,p+j} &= A_{sj}\sin\lambda_j t + B_{sj}\cos\lambda_j t, \\ \psi_{sj} &= C_{sj}\cos\lambda_j t - D_{sj}\sin\lambda_j t, & \psi_{s,p+j} &= C_{sj}\sin\lambda_j t + D_{sj}\cos\lambda_j t. \end{aligned}$$

Then it follows from the preceding considerations that the first 2p equations of (13.5) will be identically satisfied for $M_1 = \ldots = M_{2p} = 0$. It is necessary however to note that, in contrast to the case where the frequencies $\lambda_1, \ldots, \lambda_p$ are resonance frequencies, it follows from the preceding considerations that the system (13.5) has no solution for which the conditions with respect to the roots of equation (8.35) are satisfied and for which $M_1^2 + \ldots + M_{2p}^2 \neq 0$. Thus for the case under consideration the possibility is not excluded of obtaining by the methods described an almost periodic solution for which the generating solution contains the frequencies $\lambda_1, \ldots, \lambda_p$.

We shall note an important special case. Let us assume that the system considered is an autonomous one. In this case all magnitudes $\nu_1, \ldots, \nu_N$ are equal to zero. Then, if between the frequencies of the oscillations of the generating linear system no linear relations with integral coefficients exist all these frequencies will be nonresonance frequencies. Hence we shall have $M_1 = M_2 = \ldots = M_m = 0$ and therefore the methods of the preceding sections do not enable the determination of any almost periodic oscillations whatever, including also periodic. This is in agreement with the results of the preceding chapters according to which the periodic oscillations of autonomous systems have frequencies different from the frequencies of a linearized system.

14. Certain Simplifications of the Computation of Almost Periodic Solutions in the Presence of Nonresonance Frequencies

Let us assume that the system under consideration possesses nonresonance frequencies. In this case, as was shown in the preceding section, the parameters M_i corresponding to the nonresonance frequencies are found to be equal to zero. This introduces considerable simplifications in computing almost periodic solutions if we restrict ourselves to the first approximation and make use of the second method of sec. 10. In fact, in this case it is at once possible to include in the generating solution terms corresponding to the nonresonance frequencies and this solution will contain a smaller number of unknowns. To determine these unknowns it is necessary to set up only the part of equations (13.5) corresponding to the resonance frequencies.

EXAMPLE. Let us consider again equation (12.1), studied in detail in sec. 12, and assume that we are dealing with the nonresonance case. In this case the frequency k will be a nonresonance one and for the generating solution we can at once write

$$x^0 = \frac{A}{k^2-\omega_1^2}\sin\omega_1 t + \frac{B}{k^2-\omega_2^2}\sin\omega_2 t \tag{14.1}$$

without having first to compute the functions (12.5). To set up equations (12.7) we proceed according to the second method of sec. 10, in the following manner.

We set up for the obtained almost periodic solution the equation in variations restricting ourselves to magnitudes of the first order with respect to μ.

This gives:

$$\frac{d^2\xi}{dt^2}+k^2\xi=-2\mu x_0\frac{dx_0}{dt}\xi+\mu(1-x_0^2)\frac{d\xi}{dt}.$$

We then make the substitution $\xi = e^{\mu \varkappa t}\zeta$, similarly restricting ourselves to magnitudes of the first order. We obtain:

$$\frac{d^2\zeta}{dt^2}+k^2\zeta=-2\mu x_0\frac{dx_0}{dt}\zeta+\mu(1-x_0^2)\frac{d\zeta}{dt}-2\mu\varkappa\frac{d\zeta}{dt}$$

We now set up the existence conditions of an almost periodic solution of the equation

$$\frac{d^2\zeta}{dt^2}+k^2\zeta=-2x_0\frac{dx_0}{dt}\zeta_0+(1-x_0^2)\frac{d\zeta_0}{dt}-2\varkappa\frac{d\zeta_0}{dt},$$

where

$$\zeta_0=M\cos kt+N\sin kt.$$

For this we equate to zero the coefficients of cos kt and sin kt on the right hand side of the equation. Taking (14.1) into account we obtain

$$\left(1-\frac{A^2}{2(k^2-\omega_1^2)^2}-\frac{B^2}{2(k^2-\omega_2^2)^2}-2\varkappa\right)M=0,$$
$$\left(1-\frac{A^2}{2(k^2-\omega_1^2)^2}-\frac{B^2}{2(k^2-\omega_2^2)^2}-2\varkappa\right)N=0,$$

which gives for the roots of equation (12.7) the values (12.9).

15. Oscillations with Nonresonance Frequencies. Properties of the First Approximation

We proceed now to the methods of determining those oscillations of system (13.1) in which, together with the resonance frequencies, there are also present nonresonance frequencies.

Let us assume that the generating system has $q \leqslant m/2$

natural frequencies $\lambda_1, \ldots, \lambda_q$ of which $p \leq q$ are nonresonance. Let these be the frequencies $\lambda_1, \ldots, \lambda_p$. We can then put

$$\left.\begin{aligned} \varphi_{sj}(t) &= A_{sj}\cos\lambda_j t - B_{sj}\sin\lambda_j t, \\ \varphi_{s,j+p}(t) &= A_{sj}\sin\lambda_j t + B_{sj}\cos\lambda_j t, \\ \psi_{sj}(t) &= C_{sj}\cos\lambda_j t - D_{sj}\sin\lambda_j t, \\ \psi_{s,j+p}(t) &= C_{sj}\sin\lambda_j t + D_{sj}\cos\lambda_j t \\ &(j = 1, \ldots, p) \end{aligned}\right\} \tag{15.1}$$

and consider that the remaining functions φ_{sj} and ψ_{si} do not contain the frequencies $\lambda_1, \ldots, \lambda_p$. We shall assume, moreover, that for all functions $\psi_{s1}, \ldots, \psi_{sm}$ relations (8.1) are satisfied. This requirement is compatible with (15.1).

Let us now write equations (13.1) in the form (8.32):

$$\begin{aligned} \frac{dM_i}{dt} &= \mu P_i(M_1, \ldots, M_m) + \mu^2 Q_i(t, M_1, \ldots, M_m, z_1, \ldots, z_k, \mu), \\ \frac{dz_r}{dt} &= b_{r1}z_1 + \ldots + b_{rk}z_k + Z_r^0(t, M_1, \ldots, M_m) + \\ &\quad + \mu Z_r(t, M_1, \ldots, M_m, z_1, \ldots, z_k, \mu) \qquad (15.2) \\ &(i = 1, \ldots, m;\ r = 1, \ldots, k = n - m), \end{aligned}$$

where

$$P_i = \lim_{t\to\infty}\frac{1}{t}\int_0^t \sum_{\alpha=1}^{n} F_\alpha(t, x_1^0, \ldots, x_n^0, 0)\,\psi_{\alpha i}\, d', \tag{15.3}$$

$$x_s^0 = M_1\varphi_{s1} + \ldots + M_m\varphi_{sm} + x_s^{0*}(t). \tag{15.4}$$

The variables $M_1, \ldots, M_m, z_1, \ldots, z_k$ are connected with the original variables $x_1, \ldots, x_n$ by the relations (8.18) and (8.29):

$$\left.\begin{aligned} x_s &= \varphi_{s1}\xi_1 + \ldots + \varphi_{sm}\xi_m + f_{s1}\eta_1 + \ldots + f_{sk}\eta_k, \\ \xi_i &= M_i + \sum_{\alpha=1}^{n}\psi_{\alpha i}x_\alpha^* + \mu u_i(t, M_1, \ldots, M_m), \\ \eta_r &= \eta_r^0 + \mu z_r, \end{aligned}\right\} \tag{15.5}$$

where

$$u_i = \int_0^t \sum_{\alpha=1}^n F_\alpha(t, x_1^0, \ldots, x_n^0, 0)\psi_{\alpha i}\,dt - P_i t \tag{15.6}$$

and η_r^o are certain almost periodic functions. Moreover the identities (8.24) are satisfied:

$$x_s^{0*} = \sum_{i=1}^m \sum_{\alpha=1}^n \varphi_{si}\psi_{\alpha i} x_\alpha^{0*} + \sum_{r=1}^k f_{sr}\eta_r^0. \tag{15.7}$$

We set for $j = 1,\ldots,p$

$$M_j = A_j \cos\alpha_j, \quad M_{p+j} = A_j \sin\alpha_j \quad (j=1, \ldots, p) \tag{15.8}$$

and introduce in equations (15.2) in place of the 2p variables $M_1,\ldots,M_{2p}$ the variables $A_1,\ldots,A_p$ and the variables

$$\theta_j = \lambda_j t - \alpha_j. \tag{15.9}$$

Moreover, for convenience of writing we introduce the notations $M_{2p+1} = A_{p+1},\ldots,M_m = A_{m-p}$. For the expression x_s^o, on the basis of (15.1) and (15.4), we then obtain:

$$x_s^0 = A_1\varphi_{s1}\left(\frac{\theta_1}{\lambda_1}\right) + \ldots + A_p\varphi_{sp}\left(\frac{\theta_p}{\lambda_p}\right) + A_{p+1}\varphi_{s,p+1}(t) + \ldots + A_{m-p}\varphi_{sm}(t) + x_s^{0*}(t). \tag{15.10}$$

From this, on the basis of (15.3) we find that the functions $P_{2p+1}(M_1,\ldots,M_m),\ldots,P_m(M_1,\ldots,M_m)$ can be represented in the form

$$P_l(M_1, \ldots, M_m) = R_{l-p}(A_1, \ldots, A_{m-p}) \quad (l = 2p+1, \ldots, m), \tag{15.11}$$

where the magnitudes R_{l-p} do not depend on $\theta_1,\ldots,\theta_p$. This is a consequence of the fact that the functions $\psi_{s,2p+1},\ldots, \psi_{sm}$ do not contain the frequencies $\lambda_1,\ldots, \lambda_p$ and that these are nonresonance frequencies.

As regards the functions $P_1(M_1,\ldots,M_m),\ldots,P_{2p}(M_1,\ldots M$ they have, on account of (15.1), the form

$$\left.\begin{aligned} &P_j(M_1,\ldots,M_m)=\\ &\qquad=R_j(A_1,\ldots,A_{m-p})\cos\alpha_j-S_j(A_1,\ldots,A_{m-p})\sin\alpha_j,\\ &P_{j+p}(M_1,\ldots,M_m)=\\ &\qquad=R_j(A_1,\ldots,A_{m-p})\sin\alpha_j+S_j(A_1,\ldots,A_{m-p})\cos\alpha_j\\ &\qquad\qquad(j=1,\ldots,p), \end{aligned}\right\}\tag{15.12}$$

where the functions

$$\left.\begin{aligned} &R_j(A_1,\ldots,A_{m-p})=\\ &\qquad=\lim_{t\to\infty}\frac{1}{t}\int_0^t\sum_{\alpha=1}^n F_\alpha(t,x_1^0,\ldots,x_n^0,0)\,\psi_{\alpha j}\left(\frac{\theta_j}{\lambda_j}\right)dt,\\ &S_j(A_1,\ldots,A_{m-p})=\\ &\qquad=\lim_{t\to\infty}\frac{1}{t}\int_0^t\sum_{\alpha=1}^n F_\alpha(t,x_1^0,\ldots,x_n^0,0)\,\psi_{\alpha,p+j}\left(\frac{\theta_j}{\lambda_j}\right)dt\\ &\qquad\qquad(j=1,\ldots,p) \end{aligned}\right\}\tag{15.13}$$

likewise do not depend on $\Theta_1,\ldots,\Theta_p$.

From (15.8) and (15.9) there follow:

$$\left.\begin{aligned} &\frac{dA_j}{dt}=\cos\alpha_j\frac{dM_j}{dt}+\sin\alpha_j\frac{dM_{p+j}}{dt},\\ &\frac{d\theta_j}{dt}=\lambda_j+\frac{1}{A_j}\left(\sin\alpha_j\frac{dM_j}{dt}-\cos\alpha_j\frac{dM_{p+j}}{dt}\right)\\ &\qquad\qquad(j=1,\ldots,p). \end{aligned}\right\}\tag{15.14}$$

Whence, taking (15.11) and (15.12) into account and writing for briefness

$$F(t,A_1,\ldots,A_{m-p},\theta_1,\ldots,\theta_p,z_1,\ldots,z_k,\mu)=F(t,A,\theta,z,\mu),$$

we easily find that after transformation equations (15.2) assume the form [1]

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[1] These equations for k = 0 reduce to the system studied by N.N. Bogolyubov (see monograph cited on p. 273) for more general assumptions with regard to the right hand sides.

$$\left.\begin{aligned}&\frac{dA_g}{dt}=\mu R_g(A_1,\ldots,A_{m-p})+\mu^2R_g^*(t,\theta,A,z,\mu),\\&\frac{d\theta_j}{dt}=\lambda_j-\frac{1}{A_j}[\mu S_j(A_1,\ldots,A_{m-p})+\mu^2S_j^*(t,\theta,A,z,\mu)],\\&\frac{dz_r}{dt}=b_{r1}z_1+\ldots+b_{rk}z_k+Z_r^{0*}(t,A,\theta)+\mu Z_r^*(t,\theta,A,z,\mu)\\&\qquad(g=1,\ldots,m-p;\ j=1,\ldots,p;\ r=1,\ldots,k).\end{aligned}\right\}\quad(15.15)$$

Here the functions R_g^*, S_j^*, Z_r^{o*}, Z_r^* are periodic with respect to each of the variables $\theta_1,\ldots,\theta_p$ with period 2π. With respect to the remaining variables these functions have the same structure as the functions Q_i, Z_r with respect to the corresponding arguments. In particular, they are almost periodic with respect to t.

We shall reject in equations (15.15) terms of higher order and consider the equations of the first approximation thus obtained

$$\left.\begin{aligned}&\frac{dA_g}{dt}=\mu R_g(A_1,\ldots,A_{m-p}),\\&\frac{d\theta_j}{dt}=\lambda_j-\frac{\mu}{A_j}S_j(A_1,\ldots,A_{m-p}),\\&\frac{dz_r}{dt}=b_{r1}z_1+\ldots+b_{rk}z_k+Z_r^{0*}(t,A,\theta).\end{aligned}\right\}\quad(15.16)$$

Assume that the system of m - p equations

$$R_g(A_1,\ldots,A_{m-p})=0\qquad(g=1,\ldots,m-p)\qquad(15.17)$$

with m - p unknowns $A_1,\ldots,A_{m-p}$ has the solution $A_g = A_g^o$. The system (15.16) then has the stationary solution

$$\left.\begin{aligned}&A_g=A_g^0,\\&\theta_j=\theta_j^0=\left[\lambda_j-\frac{\mu}{A_j^0}S_j(A_1^0,\ldots,A_{m-p}^0)\right]t+\varepsilon_j,\\&z_r=z_r^0(t),\\&(g=1,\ldots,m-p;\ j=1,\ldots,p;\ r=1,\ldots,k),\end{aligned}\right\}\quad(15.18)$$

depending on p arbitrary constants $\varepsilon_1,\ldots,\varepsilon_p$. Here $z_r^o(t)$ is an almost periodic solution of the system

$$\frac{dz_r}{dt} = b_{r1}z_1 + \ldots + b_{rk}z_k + Z_r^{0*}(t, A^0, \theta^0). \qquad (15.19)$$

This system actually admits an almost periodic solution, which is furthermore unique, since the equation

$$\begin{vmatrix} b_{11}-\lambda & b_{12} & \ldots & b_{1k} \\ b_{21} & b_{22}-\lambda & \ldots & b_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ b_{k1} & b_{k2} & \ldots & b_{kk}-\lambda \end{vmatrix} = 0 \qquad (15.20)$$

does not by assumption have roots with zero real parts.

The stationary solution (15.18) in the initial variables $x_1, \ldots, x_n$, as follows from (15.5) and (15.10), has the form

$$x_s = A_1^0\varphi_{s1}\left(\frac{\theta_1^0}{\lambda_1}\right) + \ldots + A_p^0\varphi_{sp}\left(\frac{\theta_p^0}{\lambda_p}\right) + \\ + A_{p+1}^{(0)}(t)\varphi_{s,\,2p+1}(t) + \ldots + A_{m-p}^0\varphi_{sm}(t) + x_s^{0*}(t) + \mu x_s^{(1)}(t), \qquad (15.21)$$

where

$$x_s^{(1)}(t) = \sum_{i=1}^{m} \varphi_{si}(t)\, u_i(t, M_1, \ldots, M_m) + \sum_{r=1}^{k} f_{sr} z_r^0,$$

and the magnitudes M_i in the functions u_i must be replaced by the values

$$M_j = A_j^0 \cos(\lambda_j t - \theta_j^0),$$
$$M_{p+j} = A_j^0 \sin(\lambda_j t - \theta_j^0) \qquad (j = 1, \ldots, p),$$
$$M_{2p+1} = A_{p+1}^0, \ \ldots, \ M_m = A_{m-p}^0.$$

This solution is almost periodic. It can be replaced, with sufficient accuracy for practical purposes, by the more simple solution:

$$x_s = A_1^0\varphi_{s1}\left(\frac{\theta_1^0}{\lambda_1}\right) + \ldots + A_p^0\varphi_{sp}\left(\frac{\theta_p^0}{\lambda_p}\right) + \\ + A_{p+1}^0\varphi_{s,\,2p+1}(t) + \ldots + A_{m-p}^0\varphi_{sm}(t) + x_s^{0*}(t). \qquad (15.22)$$

The system (13.1) thus admits in the first approximation a family of almost periodic solutions for which the parameters A_g in the generating solution satisfy the equations (15.17). This family contains p arbitrary constants $\varepsilon_1, \ldots, \varepsilon_p$, i.e. as many constants as the number of nonresonance solutions possessed by the generating system. In the obtained almost periodic solution of system (13.1) there are present not only the resonance frequencies but also the frequencies

$$\lambda_j' = \lambda_j - \frac{\mu}{A_j^0} S_j(A_1^0, \ldots, A_{m-p}^0) \qquad (j = 1, \ldots, p), \qquad (15.23)$$

which differ by small magnitudes of the order of μ from the nonresonance frequencies.

Let us now consider the question of the stability of the family of almost periodic motions thus obtained. For this purpose we set up the equations of the disturbed motion for the magnitudes A_g and Θ_g. Putting in (15.16)

$$A_g = A_g^0 + \alpha_g, \qquad \theta_j = \theta_j^0 + \vartheta_j,$$

we obtain:

$$\left.\begin{aligned} \frac{d\alpha_g}{dt} &= \mu(d_{g1}\alpha_1 + \ldots + d_{g,\,m-p}\alpha_{m-p}) + \mu\rho_g(\alpha_1, \ldots, \alpha_{m-p}), \\ \frac{d\vartheta_j}{dt} &= -\mu\left[\frac{1}{A_j^0 + \alpha_j} S_j(A_1^0 + \alpha_1, \ldots, A_{m-p}^0 + \alpha_p) - \right. \\ &\qquad\qquad \left. - \frac{1}{A_j^0} S_j(A_1^0, \ldots, A_{m-p}^0)\right], \end{aligned}\right\} \qquad (15.24)$$

where

$$d_{gq} = \frac{\partial R_g(A_1^0, \ldots, A_{m-p}^0)}{\partial A_q^0}, \qquad (15.25)$$

and $\rho_g(\alpha_1, \ldots, \alpha_{m-p})$ are analytic functions of the variables α_g (polynomials) the expansions of which in powers of these variables start with terms of not lower than the second order.

We shall assume first of all that all roots of the equation

$$\left|\begin{matrix} \frac{\partial R_1}{\partial A_1} - \omega & \frac{\partial R_1}{\partial A_2} & \cdots & \frac{\partial R_1}{\partial A_{m-p}} \\ \frac{\partial R_2}{\partial A_1} & \frac{\partial R_2}{\partial A_2} - \omega & \cdots & \frac{\partial R_2}{\partial A_{m-p}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial R_{m-p}}{\partial A_1} & \frac{\partial R_{m-p}}{\partial A_2} & \cdots & \frac{\partial R_{m-p}}{\partial A_{m-p}} - \omega \end{matrix}\right|_{A_g = A_g^0} = 0, \qquad (15.26)$$

and also all roots of equation (15.20) have negative real parts. Then from the first group of equations (15.24) it follows immediately that all the solutions $\alpha_g(t)$ of these equations exponentially approach zero as t increases without limit, provided the initial values of the functions $\alpha_g(t)$ are sufficiently small. But then from the second group of equations (15.24) it follows that

$$\lim_{t\to\infty} \vartheta_j(t) = \delta_j, \tag{15.27}$$

where

$$\delta_j = \vartheta_j(0) - \mu \int_0^\infty \left[\frac{1}{A_j^0 + \alpha_j(t)} S_j(A_1^0 + \alpha_1, \ \ldots, \ A_{m-p}^0 + \alpha_{m-p}) - \right.$$
$$\left. - \frac{1}{A_j^0} S_j(A_1^0, \ \ldots, \ A_{m-1}^0) \right] dt = \text{const},$$

since the integrals figuring in δ_j evidently converge.

Turning now to the functions $z_r(t)$ of equations (15.16) we readily find that for arbitrary $z_r(0)$ the relations are satisfied

$$\lim_{t\to\infty} z_r(t) = \bar{z}_r(t),$$

where $\bar{z}_r(t)$ is an almost periodic solution of the equations

$$\frac{dz_r}{dt} = b_{r1}z_1 + \ldots + b_{rk}z_k + Z_r^{0*}(t, A^0, \theta^0 + \delta).$$

From what has been said it follows that if all the roots of equation (15.26) and equation (15.20) have negative real parts each almost periodic solution of equations (15.16) belonging to the family (15.18) is stable. If the disturbances are sufficiently small any disturbed motion for $t \to \infty$ asymptotically approaches one of the almost periodic motions of the above mentioned family (not coinciding in virtue of (15.27) with the undisturbed motion.)

Let us now assume that either equation (15.26) or equation (15.20) has roots with positive real parts. Then either the trivial solution $\alpha_1 = \ldots = \alpha_{m-p} = 0$ of equations (15.24) will be unstable or there exist solutions of the equations for z_r with arbitrarily small values $z_r(0)$ for which

$$\lim_{t\to\infty} (z_1^2 + \ldots + z_k^2) = \infty.$$

Hence, in the case considered each of the almost periodic solutions of the family (15.18) is unstable.

From the results of sec. 13 it follows that, for the assumption made in the present section that the frequencies $\lambda_1,\ldots,\lambda_p$ are nonresonance frequencies, the first 2p functions of (15.3) necessarily reduce to zero for $M_1=\ldots=M_{2p}=0$. From this it follows that the first p equations of (15.17) are necessarily satisfied for $A_1=\ldots=A_p=0$. It is easy however to satisfy oneself directly that for $j = 1,\ldots,p$ it is possible to write:

$$\left.\begin{aligned} R_j(A_1, \ldots, A_{m-p}) &= A_j R'_j(A_1, \ldots, A_{m-p}), \\ S_j(A_1, \ldots, A_{m-p}) &= A_j S'_j(A_1, \ldots, A_{m-p}) \end{aligned}\right\} \qquad (15.28)$$
$$(j=1, \ldots, p),$$

where R'_j, S'_j are polynomials in $A_1,\ldots,A_{m-p}$. In fact, the functions $P_i(M_1,\ldots,M_m)$ are polynomials and consequently rational, which, on the basis of (15.8) and (15.12) is possible only when relations (15.28) are satisfied.

Thus, the system of equations (15.17) can have a solution in which $A_1=\ldots=A_p=0$. This evidently is that solution to which correspond the almost periodic oscillations of the system (13.1) in which there are no nonresonance frequencies and which have been investigated in detail in the preceding sections. If we therefore restrict ourselves to the first approximation the method of the present section makes it possible to determine not only the oscillations with nonresonance frequencies but also the oscillations investigated in the preceding sections.

For investigating the stability of the above oscillations and also for clarifying the question of their actual existence it is necessary to know the signs of the real parts of the roots of equation (8.35). We shall show that this can be done with the aid of equation (15.26) (the degree of which is less by p units than the degree of equation (8.35)) so that there is no need of setting up equation (8.35).

In fact, on the basis of (15.12) and (15.28) we have:

$$\left.\begin{aligned}&\left(\frac{\partial P_j}{\partial M_j}\right)=R'_i\equiv\left(\frac{\partial R_j}{\partial A_j}\right),\ \left(\frac{\partial P_j}{\partial M_{p+j}}\right)=-S'_j\equiv-\left(\frac{\partial_j}{\partial A_j}\right),\\&\left(\frac{\partial P_{p+j}}{\partial M_j}\right)=S'_j\equiv\left(\frac{\partial S_j}{\partial A_j}\right),\ \left(\frac{\partial P_{p+j}}{\partial M_{p+j}}\right)=R'_j\equiv\left(\frac{\partial R_j}{\partial A_j}\right),\\&\qquad\left(\frac{\partial P_j}{\partial M_i}\right)=\left(\frac{\partial P_{p+j}}{\partial M_i}\right)=0\\&(i\neq j,\ i\neq p+j;\ j=1,\ \ldots,\ p;\ i=1,\ \ldots,\ m),\end{aligned}\right\}\quad(15.29)$$

where the parentheses denote that the derivatives have been computed for $M_1 = \ldots = M_{2p} = 0$ and therefore for $A_1 = \ldots = A_p = 0$. From this it follows that equation (8.35) for $M_1 = \ldots = M_{2p} = 0$ assumes the form

$$D(x)=\prod_{j=1}^{p}\begin{vmatrix}\dfrac{\partial R_j}{\partial A_j}-x & -\left(\dfrac{\partial S_j}{\partial A_j}\right)\\ \left(\dfrac{\partial S_j}{\partial A_j}\right) & \left(\dfrac{\partial R_j}{\partial A_j}\right)-x\end{vmatrix}\times$$

$$\times\begin{vmatrix}\left(\dfrac{\partial P_{2p+1}}{\partial M_{2p+1}}\right)-x & \ldots & \left(\dfrac{\partial P_{2p+1}}{\partial M_m}\right)\\ \ldots & \ldots & \ldots\\ \left(\dfrac{\partial P_m}{\partial M_{2p+1}}\right) & \ldots & \left(\dfrac{\partial P_m}{\partial M_m}\right)-x\end{vmatrix}=0$$

and has therefore the 2p roots

$$\left(\frac{\partial R_j}{\partial A_j}\right)\pm\sqrt{-1}\left(\frac{\partial S_j}{\partial A_j}\right)$$

and m - 2p roots determined by the equation

$$\begin{vmatrix}\left(\dfrac{\partial P_{2p+1}}{\partial M_{2p+1}}\right)-x & \ldots & \left(\dfrac{\partial P_{2p+1}}{\partial M_m}\right)\\ \ldots & \ldots & \ldots\\ \dfrac{\partial P_m}{\partial M_{2p+1}} & \ldots & \left(\dfrac{\partial P_m}{\partial M_m}\right)-x\end{vmatrix}=0. \qquad (15.30)$$

On the other hand, from (15.28) we have:

$$\left(\frac{\partial R_j}{\partial A_k}\right)=0\qquad(j,\ k=1,\ \ldots,\ p;\ j\neq k),$$

whence we easily find that equation (15.26) for $A_1 = \ldots = A_p = 0$ has the p roots

$$\omega_j = \left(\frac{\partial R_j}{\partial A_j}\right) \qquad (j = 1, \ldots, p)$$

and m - 2p roots determined by the equation

$$\begin{vmatrix} \left(\frac{\partial R_{p+1}}{\partial A_{p+1}}\right) - \omega & \ldots & \left(\frac{\partial R_{p+1}}{\partial A_{m-p}}\right) \\ \ldots & \ldots & \ldots \\ \left(\frac{\partial R_{m-p}}{\partial A_{p+1}}\right) & \ldots & \left(\frac{\partial R_{m-p}}{\partial A_{m-p}}\right) - \omega \end{vmatrix} = 0,$$

which agrees with equation (15.30) (since

$$R_{p+1} = P_{2p+1}, \; \ldots, \; A_{p+1} = M_{2p \cdot 1}, \; \ldots).$$

Hence the real parts of the roots of equation (8.35) for $M_1 = \ldots = M_{2p} = 0$ agree with the real parts of the roots of equation (15.26). The problem of stability therefore for the oscillations corresponding to the solution of equations (15.17), for which $A_1 = \ldots = A_p = 0$, is solved by the same equation (15.26) by which is solved the problem of stability for the oscillations corresponding to the other solutions of equations (15.17).

We may note in conclusion that the results of the present section are equally applicable to nonautonomous as well as to autonomous systems.

16. Oscillations with Nonresonance Frequencies. Properties of Exact Solutions

In the preceding section we considered in detail the oscillations with nonresonance frequencies determined by the equations of the first approximation. We showed that these oscillations are found to be almost periodic and can be described by equations (15.22). Let us now consider what character will be possessed by the oscillations described by the exact equations of the motion. We saw earlier that oscillations in which nonresonance frequencies are absent will be found almost periodic also in the case where the exact equations of the motion are considered and we established a converging process of successive approximations for determining these oscillations.

The situation is otherwise in the case of oscillations with nonresonance frequencies. In this case the exact equations of the oscillations, in contrast to the equations of the first approximation, do not have a family of almost periodic solutions. Nevertheless, for sufficiently small μ, we can, with an accuracy sufficient for practical purposes, consider that the equations (15.22) describe actual oscillations of the system and therefore these oscillations can be considered almost periodic.

Thus, let us assume that equations (15.17) have a solution $A_g = A_g^o$ for which all roots of equation (15.26) have negative real parts. For the assumptions made, the system (13.1) admits in the first approximation a family of stable almost periodic solutions described by the equations (15.21), or by the more simple but for practical purposes sufficiently accurate equations (15.22). Let $x_s(t)$ be the exact solutions of equations (13.1). We shall show that the following theorem holds true:

THERE EXISTS A NUMBER μ^* AND A CERTAIN REGION G OF THE VARIABLES $x_1, \ldots, x_n$ SUCH THAT FOR $t \geq 0$, $\mu \leq \mu^*$, $x_s(0) \in$ (THE FUNCTIONS $x_s(t)$ CAN BE REPRESENTED IN THE FORM

$$\left.\begin{aligned} x_s(t) = A_1\varphi_{s1}\left(\frac{\theta_1}{\lambda_1}\right) + \ldots + A_p\varphi_{sp}\left(\frac{\theta_p}{\lambda_p}\right) + \\ + A_{p+1}\varphi_{s,2p+1}(t) + \ldots + A_{m-p}\varphi_{sm}(t) + \mu\bar{x}_s(t), \\ \theta_j(t) = \theta_j(0) + \left[\lambda_j - \frac{\mu}{A_j^0} S_j(A_1^0, \ldots, A_{m-p}^0) + \mu\varphi_j(t)\right] t \\ (s = 1, \ldots, n;\ j = 1, \ldots, p), \end{aligned}\right\} \quad (16.$$

WHERE THE FUNCTIONS $A_g(t)$, $\varphi_j(t)$, $x_s(t)$ POSSESS THE FOLLOWING PROPERTIES:

1) HOWEVER SMALL THE POSITIVE NUMBERS ε_1 AND ε_2 TWO OTHER POSITIVE NUMBERS η AND μ_o CAN BE FOUND SUCH THAT THE FUNCTIONS A_g AND φ_j WILL FOR ALL $t > 0$ SATISFY THE INEQUALITIES

$$|\alpha_g| = |A_g(t) - A_g^0| < \varepsilon_1, \quad (16.2$$

$$|\varphi_j(t)| < \varepsilon_2, \quad (16.3$$

PROVIDED THE INEQUALITIES ARE SATISFIED

$$\mu < \mu_0 < \mu^*, \tag{16.4}$$

$$|\alpha_g(0)| = |A_g(0) - A_g^0| < \eta; \tag{16.5}$$

2) FOR ALL $t \geq 0$ THE INEQUALITIES ARE SATISFIED

$$|\bar{x}_s(t)| < B, \tag{16.6}$$

WHERE THE CONSTANT B IS DETERMINED BY THE EQUATIONS OF MOTION.

PROOF. We shall start from the equations of motion transformed to the form (15.15). In these equations we put

$$A_g = A_g^0 + \alpha_g.$$

On the basis of (15.24) we then obtain:

$$\frac{d\alpha_g}{dt} = \mu(d_{g1}\alpha_1 + \ldots + d_{g,m-p}\alpha_{m-p}) + \mu\rho_g(\alpha_1, \ldots, \alpha_{m-p}) + \\ + \mu^2 R_g^*(t, \theta, A^0 + \alpha, z, \mu), \tag{16.7}$$

$$\frac{dz_r}{dt} = b_{r1}z_1 + \ldots + b_{rk}z_k + Z_r^{(*)}(t, A^0 + \alpha, \theta) + \\ + \mu Z_r^*(t, \theta, A^0 + \alpha, z, \mu), \tag{16.8}$$

$$\frac{d\theta_j}{dt} = \lambda_j - \frac{\mu}{A_j}[S_j(A_1^0 + \alpha_1, \ldots, A_{m-p}^0 + \alpha_p) + \\ + \mu S_j^*(t, \theta, A^0 + \alpha, z, \mu)]. \tag{16.9}$$

We now form the two quadratic forms $V(a_1, \ldots, a_{m-p})$ and $W(z_1, \ldots, z_k)$ with the aid of the equations

$$\sum_{g=1}^{m-p} \frac{\partial V}{\partial \alpha_g}(d_{g1}\alpha_1 + \ldots + d_{g,m-p}\alpha_{m-p}) = -\sum_{g=1}^{m-p} \alpha_g^2,$$

$$\sum_{r=1}^{k} \frac{\partial W}{\partial z_r}(b_{r1}z_1 + \ldots + b_{rk}z_k) = -\sum_{r=1}^{k} z_r^2.$$

Since by assumption the roots of equations (15.26) and (15.20) have negative real parts, both forms V and W will, on the basis of the theorem of Lyapunov mentioned in sec.11, be positive definite. For the total derivatives of these forms with respect to time, formed in virtue of equations (16.7) and (16.8), we shall have:

$$\frac{dV}{dt} = \mu\left(-\sum_{g=1}^{m-p}\alpha_g^2 + \sum_{g=1}^{m-p}\frac{\partial V}{\partial \alpha_g}\rho_g + \mu\sum_{g=1}^{m-p}\frac{\partial V}{\partial \alpha_g}R_g^*\right). \quad (16.10)$$

$$\frac{dW}{dt} = -\sum_{r=1}^{k} z_r^2 + \sum_{r=1}^{k}\frac{\partial W}{\partial z_r}Z_r^{0*}(t,\ A^0+\alpha,\ 0) + \mu\sum_{r=1}^{k}\frac{\partial W}{\partial z_r}Z_r^*. \quad (16.11)$$

Let there be assigned a positive number h and let us consider the region of variation of the variables $\alpha_1,\dots,\alpha_{m-p}$ bounded by the ellipsoid $V(\alpha_1,\dots,\alpha_{m-p}) = h$. In this region the functions $|Z_r^{0*}(t, A^0 + \alpha, \theta)|$ have constant upper bounds. Taking this into account let us consider the expression dW/dt at the points $(z_1,\dots,z_k)$ lying on the ellipsoid $W(z_1,\dots,z_k) = C$, where C is likewise a positive number. It can easily be shown that if $\mu < \mu^*$, where μ^* is sufficiently small, and the number C is sufficiently large, the inequality will hold

$$\left(\frac{dW}{dt}\right)_{W=C} < 0 \quad \text{for} \quad V(\alpha_1,\dots,\alpha_{m-p}) \leqslant h. \quad (16.12)$$

In fact, the second component in (16.11) increases, with increasing C, more slowly than the first component since it is linear with respect to $z_1,\dots,z_k$. Hence for sufficiently large C the second component in (16.11) will be small in comparison with the first. The smallness however of the third component is assured by the choice of the number μ^*.

Let us now consider the derivative dV/dt. Since the expansion of the functions $\rho_g(\alpha_1,\dots,\alpha_{m-p})$ starts with terms of not lower than the second order, the expression

$$-\sum_{g=1}^{m-p}\alpha_g^2 + \sum_{g=1}^{m-p}\frac{\partial V}{\partial \alpha_g}\rho_g$$

will be a negative definite function. Hence if $c > 0$ is sufficiently small, $\mu_0 < \mu^*$ can be chosen so small that for (16.4) there is satisfied on the ellipsoid $V(\alpha_1, \ldots, \alpha_{m-p})$ the condition

$$\left(\frac{dV}{dt}\right)_{V=c} < 0 \quad \text{for} \quad W(z_1, \ldots, z_k) \leqslant C. \tag{16.13}$$

The number c can here be taken arbitrarily small. We shall assume that $c < h$ and that the ellipsoid $V=c$ lies entirely in the region (16.2).

Having established this, let us consider some solution of the equations (16.7)-(16.9) the initial values of which satify the inequalities (16.5). We shall assume that the point $(z_1(0), \ldots, z_k(0))$ lies within the ellipsoid $W=C$ and η is such that the point $(\alpha_1(0), \ldots, \alpha_{m-p}(0))$ lies within the ellipsoid $V = c$. We shall show that then the points $(z_1, \ldots, z_k)$ and $(\alpha_1, \ldots, \alpha_{m-p})$ will for all $t > 0$ remain within these ellipsoids and, therefore, conditions (16.2) will be satisfied.

In fact, the point $(z_1, \ldots, z_k)$ will remain within the ellipsoid $W = C$ at least during a certain finite interval of time. Assume that it leaves this ellipsoid and that this occurs for the first time at the instant of time $t = T$. Then for this instant of time the conditions will be satisfied

$$W(z_1(T), \ldots, z_k(T)) = C, \quad \left(\frac{dW}{dt}\right)_{t=T} > 0. \tag{16.14}$$

But during the entire time interval $(0,T)$ the derivative dV/dt on the ellipsoid $V = c$ will, on the basis of (16.13), remain negative. Hence the point $(\alpha_1, \ldots, \alpha_{m-p})$ will during this interval of time remain within this ellipsoid and we shall have:

$$V(\alpha_1(T), \ldots, \alpha_{m-p}(T)) < c < h.$$

From this it follows that for $t = T$ the inequality (16.12) holds, which contradicts (16.14). Thus, the point $(z_1, \ldots, z_k)$ will all the time remain within the ellipsoid $W = C$. But then there will all the time be satisfied the condition (16.13) and the point $(\alpha_1, \ldots, \alpha_{m-p})$ will remain within the ellipsoid $V < c$. Consequently, for all $t > 0$

inequalities (16.2) will be satisfied.

We shall now prove the validity of the assertions in regard to the structure of the functions $\Theta_j(t)$. From equations (16.9) we actually find:

$$\theta_j(t)=\theta_j(0)+\left[\lambda_j-\frac{\mu}{A_j^0}S_j(A_1^0,\ \ldots,\ A_{m-p}^0)+\mu\varphi_j(t)\right]t,$$

where

$$t\varphi_j(t)=\int_0^t\left\{\left[-\frac{1}{A_j^0+\alpha_j}S_j(A_1^0+\alpha_1,\ \ldots,\ A_{m-p}^0+\alpha_p)+\right.\right.$$
$$\left.\left.+\frac{1}{A_j^0}S_j(A_1^0,\ \ldots,\ A_{m-p}^0)\right]-\frac{\mu}{A_j^0+\alpha_j}S_j^*\right\}dt,$$

whence it is at once seen that if in (16.2) and (16.4) the numbers ε_1 and μ_0 are sufficiently small inequalities (16.3) will be satisfied.

We now go over to the initial variables x_s with the aid of the formulas of sec. 15. The solution $A_g(t), z_r(t), \Theta_j($ then assumes the form (16.1) with the functions $\bar{x}_s(t)$ satisfying the inequalities (16.6) in which B is determined by the constants μ^*, h and C and depends therefore only on the structure of the equations of motion. The region G of the allowed values of the magnitudes $x_s(0)$ is likewise determined by the constants μ^*, h and C.

In this way our theorem has been completely proven.

It follows from the proven theorem that for small vallues of μ the equations of the first approximation sufficiently accurately characterize the oscillations of the system both from the quantitative and qualitative points of view.

We have assumed that the oscillations under consideratio were stable since only oscillations of this kind can be of practical interest. But it could have been shown that even in the case where the almost periodic solutions of the first approximation are unstable this first approximation gives a correct picture of the actual motion of the system in the neighborhood of these almost periodic solutions.

17. Practical Procedures for Obtaining Oscillations with Nonresonance Frequencies

For the practical computation of the stationary solutions of system (13.1) in the presence of nonresonance frequencies the first approximation is generally sufficient. It can be at once found from the general formulas

$$\left.\begin{aligned} x_s &= A_1^0\varphi_{s1}\left(\frac{\theta_1^0}{\lambda_1}\right)+\ldots+A_p^0\varphi_{sp}\left(\frac{\theta_p^0}{\lambda_p}\right)+ \\ &\quad + A_{p+1}^0\varphi_{s,\,2p+1}(t)+\ldots+A_{m-p}^0\varphi_{sm}(t)+x_s^{0*}(t), \\ \theta_j^0 &= \left[\lambda_j - \frac{\mu}{A_j^0}S_j(A_1^0,\,\ldots,\,A_{m-p}^0)\right]t+\varepsilon_j, \end{aligned}\right\} \quad (17.1)$$

where the magnitudes A_g^o are roots of equations (15.17) while the functions R_g and S_j are determined by formulas (15.13). We here take into account also the first approximations of the corrections on the nonresonance frequencies, which may be of practical interest.

Occasionally however in order to avoid computing the functions ψ_{si}, and also where there is need of a more accurate determination of the oscillations, it is necessary to set up equations (15.15) or at least their first approximation (15.16). For this purpose the procedures of sec. 10 may be used and the equations first brought to the form (15.2) after which, making the transformation (15.8), we obtain equations (15.15). It is convenient however to reduce the equations to the form (15.15) at once, which can be done, in particular, in the following manner.

In equations (13.1) we first of all separate out the critical variables, i.e. we reduce them with the aid of a linear substitution with constant coefficients to the form

$$\left.\begin{aligned} \frac{d\xi_i}{dt} &= p_{i1}\xi_1+\ldots+p_{im}\xi_m+p_i(t)+ \\ &\quad +\mu X_i(t,\,\xi_1,\,\ldots,\,\xi_m,\,\eta_1,\,\ldots,\,\eta_k,\,\mu), \\ \frac{d\eta_r}{dt} &= b_{r1}\eta_1+\ldots+b_{rk}\eta_k+q_r(t)+ \\ &\quad +\mu Y_r(t,\,\xi_1,\,\ldots,\,\xi_m,\,\eta_1,\,\ldots,\,\eta_k,\,\mu) \\ &(i=1,\,\ldots,\,m;\ r=1,\,\ldots,\,k=n-m), \end{aligned}\right\} \quad (17.2)$$

where the constants p_{ij} and b_{rj} are such that the equation

$$\begin{vmatrix} b_{11}-\lambda & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22}-\lambda & \dots & b_{2k} \\ \dots & \dots & \dots & \dots \\ b_{k1} & b_{k2} & \dots & b_{kk}-\lambda \end{vmatrix} = 0$$

has no roots with zero real parts while the real parts of all roots of the equation

$$\begin{vmatrix} p_{11}-\rho & p_{12} & \dots & p_{1m} \\ p_{21} & p_{22}-\rho & \dots & p_{2m} \\ \dots & \dots & \dots & \dots \\ p_{m1} & p_{m2} & \dots & p_{mm}-\rho \end{vmatrix} = 0$$

are on the contrary equal to zero.

For the assumptions made with respect to equations (13.1) the general solution of the linear part of the first group of equations (17.2) can be represented in the form

$$\begin{gathered} \xi_i = \bar{A}_1 \varphi_{i1}^* \left(\frac{\bar{\theta}_1}{\lambda_1}\right) + \dots + \bar{A}_p \varphi_{ip}^* \left(\frac{\bar{\theta}_p}{\lambda_p}\right) + \bar{A}_{p+1} \varphi_{i,\,2p+1}^*(t) + \dots + \\ + \bar{A}_{m-p} \varphi_{im}^*(t) + \xi_i^{0*}(t), \quad (17.3) \\ \bar{\theta}_j = \lambda_j t + \varepsilon_j \qquad (j = 1, \dots, p), \end{gathered}$$

where $\bar{A}_g$ and ε_j are arbitrary constants, while the functions $\varphi^*_{i1}, \dots, \varphi^*_{ip}$ have the form

$$\varphi_{ij}^*(t) = A_i^{*(j)} \cos \lambda_j t - B_i^{*(j)} \sin \lambda_j t \qquad (j = 1, \dots, p),$$

and the functions $\varphi^*_{i,2p+1}, \dots, \varphi^*_{im}, \xi_i^{0*}$ are almost periodic and do not contain the nonresonance frequencies $\lambda_1, \dots, \lambda_p$. If we now in equations (17.2) make the substitution (17.3), taking the magnitudes $\bar{A}_g$, $\bar{\theta}_j$ as new variables in place of ξ_i, the transformed equations will have the form

$$\left.\begin{aligned}&\frac{d\bar{A}_g}{dt}=\mu\bar{R}_g(t,\bar{\theta},\bar{A},\eta,\mu),\\&\frac{d\bar{\theta}_j}{dt}=\lambda_j+\mu\bar{S}_j(t,\bar{\theta},\bar{A},\eta,\mu),\\&\frac{d\eta_r}{dt}=b_{r1}\eta_1+\ldots+b_{rk}\eta_k+q_r(t)+\mu\bar{Y}_r(t,\bar{\theta},\bar{A},\eta,\mu)\\&\quad(g=1,\ldots,m-p;\ j=1,\ldots,p;\ r=1,\ldots,k),\end{aligned}\right\}\tag{17.4}$$

where the functions $\bar{R}_g$, $\bar{S}_j$, $\bar{Y}_r$ are periodic with respect to each of the variables $\bar{\theta}_1,\ldots,\bar{\theta}_p$ with periodi 2π.

Let us consider the functions $\bar{R}_g(t,\bar{\theta},\bar{A},\eta^o,0)$, $\bar{S}_j(t,\bar{\theta},\bar{A},\eta^o,0)$, where $\eta_r^o(t)$ is an almost periodic solution of the system

$$\frac{d\eta_r^o}{dt}=b_{r1}\eta_1^o+\ldots+b_{rk}\eta_k^o+q_r(t).$$

By the property of the right hand sides of equations (13.1) we can write:

$$\left.\begin{aligned}&\bar{R}_g(t,\bar{\theta},\bar{A},\eta^o,0)=\\&\qquad=R_g(\bar{A}_1,\ldots,\bar{A}_{m-p})+\sum_\sigma(A_{g\sigma}\cos u_\sigma+B_{g\sigma}\sin u_\sigma),\\&\bar{S}_j(t,\bar{\theta},\bar{A},\eta^o,0)=\\&\qquad=S_j(\bar{A}_1,\ldots,\bar{A}_{m-p})+\sum_\sigma(C_{j\sigma}\cos u_\sigma+D_{j\sigma}\sin u_\sigma),\\&u_\sigma=m_1\bar{\theta}_1+\ldots+m_p\bar{\theta}_p+\\&\qquad+(m_{p+1}\lambda_{p+1}+\ldots+m_q\lambda_q+n_1\nu_1+\ldots+n_N\nu_N)\,t,\end{aligned}\right\}\tag{17.5}$$

where the coefficients $A_{g\sigma}$, $B_{g\sigma}$, $C_{j\sigma}$, $D_{j\sigma}$ depend only on $\bar{A}_1,\ldots,\bar{A}_{m-p}$, the sums consist of a finite number of terms and $m_1,\ldots,m_q,n_1,\ldots,n_N$ are (positive or negative) integers. We now replace in equations (17.4) the variables $\bar{A}_g$, θ_j and η_r by the variables A_g, θ_r and z_r with the aid of the substitution

$$\left.\begin{aligned}&\bar{A}_g=A_g+\mu U_g(t,\theta,A),\\&\bar{\theta}_j=\theta_j+\mu V_j(t,\theta,A),\\&\eta_r=\eta_r^o+\mu z_r,\end{aligned}\right\}\tag{17.6}$$

where

$$U_g(t, \bar{\theta}, \bar{A}) = \sum_s \frac{A_{gs} \sin u_s - A_{gs} \cos u_s}{m_1\lambda_1 + \ldots + m_q\lambda_q + n_1\nu_1 + \ldots + n_N\nu_N},$$

$$V_j(t, \bar{\theta}, \bar{A}) = \sum_s \frac{C_{js} \sin u_s - D_{js} \cos u_s}{m_1\lambda_1 + \ldots + m_q\lambda_q + n_1\nu_1 + \ldots + n_N\nu_n}.$$

Then, taking into account that for the functions U_g and V_j the identities hold

$$\frac{\partial U_g(t, \theta, A)}{\partial t} + \sum_{j=1}^{p} \frac{\partial U_g(t, \theta, A)}{\partial \theta_j} \lambda_j = \\ = \bar{R}_g(t, \theta, A, \eta^0, 0) - R_g(A_1, \ldots, A_{m-p}),$$

$$\frac{\partial V_j(t, \theta, A)}{\partial t} + \sum_{j=1}^{p} \frac{\partial V_j(t, \theta, A)}{\partial \theta_j} \lambda_j = \\ = \bar{S}_j(t, \theta, A, \eta^0, 0) - S_j(A_1, \ldots, A_{m-p}),$$

we easily find that after the transformation equations (17.4) assume the form

$$\left.\begin{aligned} &\frac{dA_g}{dt} = \mu R_g(A_1, \ldots, A_{m-p}) + \mu^2 R_g^*(t, \theta, A, z, \mu), \\ &\frac{d\theta_j}{dt} = \lambda_j + \mu S_j(A_1, \ldots, A_{m-p}) + \mu^2 S_j^*(t, \theta, A, z, \mu), \\ &\frac{dz_r}{dt} = b_{r1}z_1 + \ldots + b_{rk}z_k + \bar{Y}_r(t, \theta, A, \eta^0, 0) + \\ &\qquad\qquad + \mu Z_r(t, \theta, A, z, \mu) \\ &(g = 1, \ldots, m-p;\ j = 1, \ldots, p;\ r = 1, \ldots, k), \end{aligned}\right\} \quad (17.7)$$

differing only by a small change of notation from (15.15). As is seen from (17.5), the functions $R_g(A_1, \ldots, A_{m-p})$ and $S_j(A_1, \ldots, A_{m-p})$ are obtained from $\bar{R}_g(t, \Theta, A\eta^0, 0)$ and $\bar{S}_j(t, \Theta, A, \eta^0, 0)$ by averaging with respect to the variable T, $\Theta_1, \ldots, \Theta_p$. Hence if it is necessary to set up only the equations of the first approximation

$$\left.\begin{aligned} &\frac{dA_g}{dt} = \mu R_g(A_1, \ldots, A_{m-p}), \\ &\frac{d\theta_j}{dt} = \lambda_j + \mu S_j(A_1, \ldots, A_{m-p}), \\ &\frac{d\eta_r}{dt} = b_{r1}\eta_1 + \ldots + b_{r2}\eta_k + q_r(t), \end{aligned}\right\} \quad (17.8)$$

the transformation (17.6) need not be carried out. Instead, it is sufficient to discard in equations (17.4) all terms of higher order relative to μ (in the equations for η_r the terms of the first order) and average the right hand sides of the equations for $\bar{A}_g$ and $\bar{\theta}_j$ with respect to the variables $t, \theta_1, \ldots, \theta_p$, first replacing in them the magnitudes η_r by the functions $\eta_r^o(t)$. Instead of the last group of equations (17.8) the more accurate equations can be taken

$$\frac{d\eta_r}{dt} = b_{r1}\eta_1 + \ldots + b_{rk}\eta_k + q_r(t) + \mu \bar{Y}_r(t, \theta, A, \eta^0, 0). \quad (17.9)$$

Since the frequencies $\lambda_1, \ldots, \lambda_p$ are nonresonance frequencies, it is also, as can easily be seen, possible to write:

$$\left.\begin{aligned} R_g(A_1, \ldots, A_{m-p}) &= \lim_{t\to\infty} \frac{1}{t}\int_0^t \bar{R}_g(t, \theta^0, A, \eta^0, 0)\,dt, \\ S_j(A_1, \ldots, A_{m-p}) &= \lim_{t\to\infty} \frac{1}{t}\int_0^t \bar{S}_j(t, \theta^0, A, \eta^0, 0)\,dt, \end{aligned}\right\} \quad (17.10)$$

where

$$\theta_j^0 = \lambda_j t + \varepsilon_j.$$

18. Examples

We shall consider examples of oscillation with nonresonance frequencies.

1. FORCED VIBRATIONS OF A REGENERATIVE RECEIVER. We shall take the equations of the oscillations in the form

$$\frac{d^2x}{dt^2} + k^2x = A\sin\omega_1 t + B\sin\omega_2 t + \mu(1 - x^2)\frac{dx}{dt}, \quad (18.1)$$

where the ratio ω_1/ω_2 is irrational and the frequency k is assumed a nonresonance frequency. The oscillations with frequencies ω_1 and ω_2 were studied in detail in sec. 12.

It was there shown that such oscillations always exist and will be stable if the condition is satisfied

$$\frac{A^2}{(k^2-\omega_1^2)^2}+\frac{B^2}{(k^2-\omega_2)^2}-2>0. \tag{18.2}$$

This condition will of course not be satisfied if k differs by a considerable amount from ω_1 and ω_2. We shall show that when the condition (18.2) is not satisfied there will always exist oscillations with frequencies ω_1, ω_2 and l and these oscillations will be stable.

The general solution of the linear part of equation (18.1) (which we consider as a system of two equations with unknowns x and dx/dt) has the form

$$\left.\begin{aligned} x &= M\sin\theta+\frac{A^2}{k^2-\omega_1^2}\sin\omega_1 t+\frac{B^2}{k^2-\omega_2^2}\sin\omega_2 t,\\ \frac{dx}{dt} &= kM\cos\theta+\frac{A^2\omega_1}{k^2-\omega_1^2}\cos\omega_1 t+\frac{B^2\omega_2}{k^2-\omega_2^2}\cos\omega_2 t,\\ \theta &= kt+\varepsilon, \end{aligned}\right\} \tag{18.3}$$

where M and ε are arbitrary constants. These relations we regard as the transformation of the variables x and dx/dt into the variables M and Θ. Differentiating the first of these relations and equating to the second we first of all obtain:

$$\frac{dM}{dt}\sin\theta+M\cos\theta\left(\frac{d\theta}{dt}-k\right)=0.$$

The substitution in equation (18.1) gives:

$$k\frac{dM}{dt}\cos\theta-kM\sin\theta\left(\frac{d\theta}{dt}-k\right)=\mu(1-x^2)\frac{dx}{dt}.$$

whence we obtain

$$\begin{aligned} \frac{dM}{dt} &= \frac{\mu}{k}(1-x^2)\frac{dx}{dt}\cos\theta,\\ \frac{d\theta}{dt} &= k-\frac{\mu}{kM}(1-x^2)\frac{dx}{dt}\sin\theta, \end{aligned}$$

where the magnitudes x and dx/dt must be replaced by their values (18.3). The equations of the first approximation have the form

$$\frac{dM}{dt} = \mu R(M), \qquad \frac{d\theta}{dt} = k - \frac{\mu}{M} S(M), \tag{18.4}$$

where R(M) and S(M) are the mean values of the functions

$$\frac{1}{k}(1 - x^2)\frac{dx}{dt}\cos\theta, \qquad \frac{1}{k}(1 - x^2)\frac{dx}{dt}\sin\theta,$$

i.e. the free terms of their expansions in trigonometric sums. Simple computations give:

$$2R(M) = M\left[1 - \frac{M^2}{4} - \frac{A^2}{2(k^2-\omega_1^2)^2} - \frac{B^2}{2(k^2-\omega_2^2)^2}\right], \qquad S(M) = 0.$$

Equations (18.4) have the stationary solution

$$M^2 = M^{*2} = 2\left[2 - \frac{A^2}{(k^2-\omega_1^2)^2} - \frac{B^2}{(k^2-\omega_2^2)^2}\right],$$
$$\theta = kt + \varepsilon,$$

where M^* is a root of the equation R(M) = 0 and ε is an arbitrary constant.

Thus, equation (18.1) has a family of stationary solutions which in the first approximation is determined by the equation

$$x = M^*\sin(kt + \varepsilon) + \frac{A}{k^2-\omega_1^2}\sin\omega_1 t + \frac{B}{k^2-\omega_2^2}\sin\omega_2 t \tag{18.5}$$

and is almost periodic with the three frequencies ω_1, ω_2, k.

The obtained solutions, as is seen from the expression for M^*, will exist precisely when

$$2 - \frac{A^2}{(k^2-\omega_1^2)^2} + \frac{B^2}{(k^2-\omega_2^2)^2} > 0, \tag{18.6}$$

i.e. when the oscillation with frequencies ω_1 and ω_2 will be unstable. We shall show that the oscillations (18.5) will then be stable.

In fact, equation (15.26) now takes the form

$$\frac{dR(M^*)}{dM^*} - \varkappa = 0$$

and the condition of stability, consisting in the fact that the root $\varkappa = R'(M^*)$ of this equation must have a negative real part, is expressed by inequality (18.6).

2. FORCED OSCILLATIONS OF A GYROSCOPIC CAR.[1] Let us consider the oscillations which may arise in a mechanical system with two degrees of freedom possessing a gyroscopic stabilizer. As such a system we shall consider a monorail gyroscopic car the scheme of which is shown in fig. 26. In this scheme the gyroscope is mounted in a frame which rotates around the horizontal axis AB of the car. To the top of the frame is attached an additional load E. We shall consider the problem taking the forces of the viscous friction into account. Then, as is known,[2] for stabilizing the car it is necessary to apply to the frame a certain moment M which accelerates the precession of the frame.

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1 See Butenin N.V., K teorii vynuzhdennykh kolebanii v nelineinoi mekhanicheskoi sisteme s dvumya stepenyami svobody (On the Theory of Forced Vibrations in a Nonlinear Mechanical System with Two Degrees of Freedom), Prikl. matem. i mekh., vol XIII, no. 4, 1949; Butenin N.V., K teorii rezonansa v mekhanicheskoi avtokolebatel'noi sisteme s giroskopicheskimi chlenami (On the Theory of Resonance in a Mechanical Self-Oscillating System with Gyroscopic Terms), Prikl. matem. i mekh. vol.XIV, no.1, 1950.

2 See Loitsianskii L.G. and Lurye A.I., Kurs teoreticheskoi mekhaniki (Course in Theoretical Mechanics), section 174, vol. II, 1948.

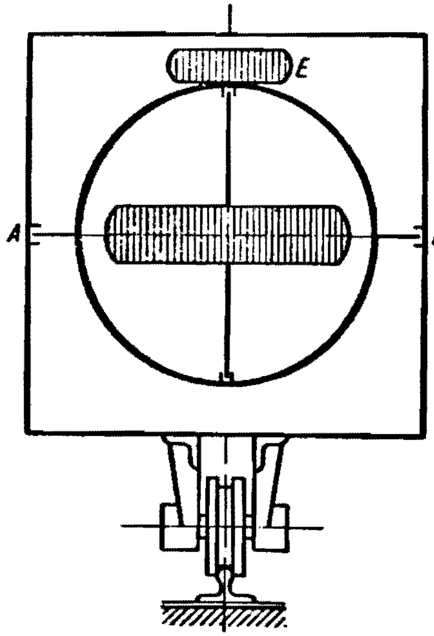

Fig. 26

We set

$$J_0 = J_b + \frac{Q_2 + Q_1}{g} a^2 + A + C_1 + \frac{p}{g}(a+b)^2,$$

$$A_0 = A + B_1 + \frac{p}{g} b^2,$$

where J_b is the moment of inertia of the car with respect to the rail, A the equatorial moment of inertia of the gyroscope, B_1 the equatorial moment of inertia of the frame, C_1 its polar moment of inertia, Q_2 the weight of the gyroscope, Q_1 the weight of the frame, p the weight of the load E, a the distance from the axis AB to the rail, b the distance from the axis AB to the center of gravity of the load E. We denote further by h the distance from the center of gravity of the entire system without the load to the rail, C the polar moment of inertia of the gyroscope, Ω its angular velocity, γ' and γ'' the coefficients of viscous friction of the car and frame respectively and P the weight of the entire system without the load. We assume, moreover, that the car is acted upon by an external disturbing moment M_1 sin t. For the angle of rotation y of the car and the angle of rotation x of the frame there are then obtained the following differential equations.

$$A_0 \frac{d^2x}{dt^2} - C\Omega \frac{dy}{dt} - pbx = -\gamma'' \frac{dx}{dt} + M,$$

$$J_0 \frac{d^2y}{dt^2} + C\Omega \frac{dx}{dt} - Phy = -\gamma' \frac{dy}{dt} + M_1 \sin \nu t.$$

Let us assume that the servo-moment M is determined by the formula

$$M = \left[\alpha' - \beta' \left(\frac{dx}{dt}\right)^2\right] \frac{dx}{dt} \qquad (\alpha' > 0,\ \beta' > 0).$$

Then, setting

$$\varkappa_1 = \frac{C\Omega}{A_0}, \quad \varkappa_2 = \frac{C\Omega}{J_0}, \quad n_1^2 = \frac{pb}{A_0}, \quad n_2^2 = \frac{Ph}{J_0},$$

We shall assume that the nondimensional magnitudes

$$\mu = \frac{\alpha' - \gamma''}{A_0 n_1}, \quad \frac{\gamma'}{J_0 n_1}, \quad \frac{n_1 \beta'}{A_0}$$

are small in comparison with unity. In this case the differential equations of the oscillations can be represented in the form

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - \varkappa_1 \frac{dy}{dt} - n_1^2 x &= \mu n_1 \left[\left(\frac{dx}{dt}\right) - \beta \left(\frac{dx}{dt}\right)^3\right] = \mu f, \\ \frac{d^2y}{dt^2} + \varkappa_2 \frac{dx}{dt} - n_2^2 y &= -\mu n_1 \lambda \frac{dy}{dt} + Q \sin \nu t = \mu F + Q \sin \nu t, \end{aligned}\right\} \quad (18.7)$$

where

$$\beta = \frac{\beta'}{\alpha' - \gamma''}, \quad \lambda = \frac{A_0 \gamma'}{J_0 (\alpha' - \gamma'')}, \quad Q = \frac{M_1}{J_0}.$$

We shall take these as the initial equations.

Let us first consider the nonresonance case where between the frequencies ω_1 and ω_2 of the free oscillations of the linearized system and the frequency ν no relation exists of the form

$$m_1\omega_1 + m_2\omega_2 + n\nu = 0$$

with integral m_1, m_2, n. We here do not exclude from consideration also the case where $Q = 0$, so that the analysis given below is valid also for free oscillations.

The general solution of the linear part of equations (18.7) can be represented in the form

$$\left.\begin{aligned}
x &= A_1 \sin\theta_1 + A_2 \sin\theta_2 + d\cos\nu t,\\
\frac{dx}{dt} &= A_1\omega_1\cos\theta_1 + A_2\omega_2\cos\theta_2 - d\nu\sin\nu t,\\
y &= A_1 k_1\cos\theta_1 + A_2 k_2\cos\theta_2 + c\sin\nu t,\\
\frac{dy}{dt} &= -A_1\omega_1 k_1\sin\theta_1 - A_2\omega_2 k_2\sin\theta_2 + c\nu\cos\nu t,\\
\theta_1 &= \omega_1 t + \alpha_1,\quad \theta_2 = \omega_2 t + \alpha_2.
\end{aligned}\right\} \tag{18.8}$$

where A_1, A_2, α_1, α_2 are arbitrary constants and the frequencies ω_1 and ω_2 satisfy the equation

$$\omega^4 + (n_1^2 + n_2^2 - \varkappa_1\varkappa_2)\,\omega^2 + n_1^2 n_2^2 = 0.$$

It is here assumed that the system is stabilized so that the equation of the frequencies has real roots. Further, in the solution (18.8) the magnitudes d and c have the values

$$d = \frac{\varkappa_1\nu Q}{\nu^4 + (n_1^2 + n_2^2 - \varkappa_1\varkappa_2)\,\nu^2 + n_1^2 n_2^2)},$$

$$c = \frac{-(\nu^2 + n_1^2)\,Q}{\nu^4 + (n_1^2 + n_2^2 - \varkappa_1\varkappa_2)\,\nu^2 + n_1^2 n_2^2}$$

and for the coefficients k_1 and k_2 the relations hold

$$k_1 = \frac{\omega_1^2 + n_1^2}{\varkappa_1\omega_1} = \frac{\varkappa_2\omega_1}{\omega_1^2 + n_2^2}, \qquad k_2 = \frac{\omega_2^2 + n_1^2}{\varkappa_1\omega_2} = \frac{\varkappa_2\omega_2}{\omega_2^2 + n_2^2}, \tag{18.9}$$

We transform equations (18.7) to the new variables A_1, A_2, Θ_1, Θ_2 with the aid of the substitution (18.8). We shall have:

$$\sin\theta_1\frac{dA_1}{dt}+\sin\theta_2\frac{dA_2}{dt}+A_1\cos\theta_1\left(\frac{d\theta_1}{dt}-\omega_1\right)+$$
$$+A_2\cos\theta_2\left(\frac{d\theta_2}{dt}-\omega_2\right)=0,$$
$$k_1\cos\theta_1\frac{dA_1}{dt}+k_2\cos\theta_2\frac{dA_2}{dt}-A_1k_1\sin\theta_1\left(\frac{d\theta_1}{dt}-\omega_1\right)-$$
$$-A_2k_2\sin\theta_2\left(\frac{d\theta_2}{dt}-\omega_2\right)=0,$$
$$\omega_1\cos\theta_1\frac{dA_1}{dt}+\omega_2\cos\theta_2\frac{dA_2}{dt}-A_1\omega_1\sin\theta_1\left(\frac{d\theta_1}{dt}-\omega_1\right)-$$
$$-A_2\omega_2\sin\theta_2\left(\frac{d\theta_2}{dt}-\omega_2\right)=\mu f,$$
$$-\omega_1k_1\sin\theta_1\frac{dA_1}{dt}-\omega_2k_2\sin\theta_2\frac{dA_2}{dt}-A_1k_1\omega_1\cos\theta_1\left(\frac{d\theta_1}{dt}-\omega_1\right)-$$
$$-A_2k_2\omega_2\cos\theta_2\left(\frac{d\theta_2}{dt}-\omega_2\right)=\mu F.$$

whence, taking into account that from (18.9) the relations follow

$$k_1\omega_2-k_2\omega_1=\frac{n_1(\omega_2^2-\omega_1^2)}{\varkappa_1n_2},\qquad k_1\omega_1-k_2\omega_2=\frac{\omega_1^2-\omega_2^2}{\varkappa_1},$$

we readily find:

$$\frac{dA_1}{dt}=-\mu(F^*\sin\theta_1+k_2f^*\cos\theta_1),$$
$$\frac{dA_2}{dt}=\mu(F^*\sin\theta_2+k_1f^*\cos\theta_2),$$
$$\frac{d\theta_1}{dt}=\omega_1+\frac{\mu}{A_1}(-F^*\cos\theta_1+k_2f^*\sin\theta_1)$$
$$\frac{d\theta_2}{dt}=\omega_2+\frac{\mu}{A_2}(F^*\cos\theta_2-k_1f^*\sin\theta_2)$$

where

$$F^*=-\frac{\varkappa_1F}{\omega_2^2-\omega_1^2},\qquad f^*=\frac{\varkappa_1n_2f}{n_1(\omega_2^2-\omega_1^2)}.\tag{18.10}$$

Expanding the right hand sides of the obtained equations in trigonometric sums and discarding all except the free terms we obtain the following equations of the first approximation:

$$\left.\begin{aligned} \frac{dA_1}{dt} &= \mu b_1 (c_1 - A_1^2\omega_1^2 - 2A_2^2\omega_2^2 - 2d^2\nu^2) A_1 = \mu R_1 (A_1, A_2), \\ \frac{dA_2}{dt} &= \mu b_2 (-c_2 + 2A_1^2\omega_1^2 + A_2^2\omega_2^2 + 2d^2\nu^2) A_2 = \mu R_2 (A_1, A_2), \\ \frac{d\theta_1}{dt} &= \omega_1, \quad \frac{d\theta_2}{dt} = \omega_2 \end{aligned}\right\} \tag{18.11}$$

where

$$\left.\begin{aligned} b_1 &= \frac{3\beta k_2 n_2 \varkappa_1 \omega_1}{8(\omega_1^2 - \omega_2^2)}, & b_2 &= \frac{3\beta k_1 n_2 \varkappa_1 \omega_2}{8(\omega_1^2 - \omega_2^2)}, \\ c_1 &= \frac{4}{3\beta}\left(1 - \frac{k_1 n_1 \lambda}{k_2 n_2}\right), & c_2 &= \frac{4}{3\beta}\left(1 - \frac{k_2 n_1 \lambda}{k_1 n_2}\right). \end{aligned}\right\} \tag{18.12}$$

Thus, the oscillations of the system are described in the first approximation by the equations

$$\left.\begin{aligned} x &= A_1^* \sin(\omega_1 t + \varepsilon_1) + A_2^* \sin(\omega_2 t + \varepsilon_2) + d\cos\nu t, \\ y &= A_1^* k_1 \cos(\omega_1 t + \varepsilon_1) + A_2^* k_2 \cos(\omega_2 t + \varepsilon_2) + c\sin\nu t, \end{aligned}\right\} \tag{18.13}$$

where ε_1 and ε_2 are arbitrary constants and A_1^* and A_2^* are the roots of the equations

$$\left.\begin{aligned} (c_1 - A_1^2\omega_1^2 - 2A_2^2\omega_2^2 - 2d^2\nu^2) A_1 &= 0, \\ (-c_2 + 2A_1^2\omega_1^2 + A^2\omega_2^2 + 2d^2\nu^2) A_2 &= 0. \end{aligned}\right\} \tag{18.14}$$

Let us first consider the free oscillations. i.e. we assume that $Q = 0$, which gives $d = 0$. Equations (18.14) will then have the solution

$$A_2^* = 0, \qquad A_1^{*2} = \frac{c_1}{\omega_1^2}. \tag{18.15}$$

To this solution correspond periodic oscillations with frequency ω_1. The roots of the equation

$$\begin{vmatrix} \frac{\partial R_1}{\partial A_1} - \varkappa & \frac{\partial R_1}{\partial A_2} \\ \frac{\partial R_2}{\partial A_1} & \frac{\partial R_2}{\partial A_2} - \varkappa \end{vmatrix} = 0 \tag{18.16}$$

for the solution (18.15) are respectively

$$\varkappa_1 = -2b_1 A_1^* \omega_1^2, \quad \varkappa_2 = (2c_1 - c_2) b_2,$$

and therefore on the basis of (18.12) the conditions of stability of this solution have the form

$$\omega_1^2 > \omega_2^2, \quad 2c_1 - c_2 < 0.$$

Equations (18.14) have also the solution (for d = 0)

$$A_1^* = 0, \quad A_2^{*2} = \frac{c_2}{\omega_2^2},$$

to which correspond periodic oscillations with the frequency ω_2. Moreover, equations (18.14) for d = 0 have the further solution

$$\left.\begin{aligned} A_1^{*2} &= \frac{2c_2 - c_1}{3\omega_1^2}, \\ A_2^{*2} &= \frac{2c_1 - c_2}{3\omega_2^2}. \end{aligned}\right\} \tag{18.17}$$

To this solution correspond almost periodic oscillations with the frequencies ω_1 and ω_2. Equation (18.16) for this solution has the form

$$\varkappa^2 + 2\varkappa (b_1 A_1^{*2}\omega_1^2 - b_2 A_2^{*2}\omega_2^2) + 12 b_1 b_2 A_1^{*2} A_2^{*2} \omega_1^2 \omega_2^2 = 0.$$

Hence if this solution is real, i.e. if the right hand sides of (18.17) are positive, the oscillations corresponding to it will be stable if the inequality is satisfied

$$3(b_1 A_1^{*2}\omega_1^2 - b_2 A_2^{*2}\omega_2^2) = b_1(2c_2 - c_1) - b_2(2c_1 - c_2) > 0.$$

We shall now assume that $Q \neq 0$. In this case equation (18.14) have a solution

$$\left.\begin{aligned} A_1^* &= 0, \\ A_2^{*2}\omega_2^2 &= c_2 - 2d^2\nu^2, \end{aligned}\right\} \tag{18.18}$$

to which correspond almost periodic oscillations with frequencies ω_2 and ν, a solution

$$\left.\begin{aligned} A_2^* &= 0, \\ A_1^{*2}\omega_1^2 &= c_1 - 2d^2\nu^2, \end{aligned}\right\} \tag{18.19}$$

to which correspond almost periodic oscillations with frequencies ω_1 and ν, and a solution

$$\left.\begin{aligned} A_1^{*2}\omega_1^2 &= \frac{1}{3}(2c_2 - c_1 - 2d^2\nu^2), \\ A_2^{*2}\omega_2^2 &= \frac{1}{3}(2c_1 - c_2 - 2d^2\nu^2), \end{aligned}\right\} \tag{18.20}$$

to which correspond almost periodic oscillations with the three frequencies ω_1, ω_2 and ν. Computing the roots of equation (18.16) for these solutions we obtain the following condition of stability: for the solution (18.18)

$$\omega_1^2 - \omega_2^2 < 0, \quad c_1 - 2c_2 + 2d^2\nu^2 > 0, \tag{18.21}$$

for the solution (18.19)

$$\omega_1^2 - \omega_2^2 > 0, \quad 2c_1 - c_2 - 2d^2\nu^2 < 0 \tag{18.22}$$

and for the solution (18.20)

$$b_1(2c_2 - c_1 - 2d^2\nu^2) - b_2(2c_1 - c_2 - 2d^2\nu^2) > 0. \tag{18.23}$$

Let us now consider one of the possible resonance cases. We assume that $\nu = \omega_2$. Moreover, we assume, as we usually do in resonance cases, that $Q = \mu Q^*$, since otherwise the generating system will not have almost periodic solutions.

According to the general theory we now represent the solution of system (18.7) for $\mu = 0$ in the form

$$\left.\begin{aligned} x &= A\sin\theta + M_1\cos\omega_2 t + M_2\sin\omega_2 t, \\ \frac{dx}{dt} &= A\omega_1\cos\theta - M_1\omega_2\sin\omega_2 t + M_2\omega_2\cos\omega_2 t, \\ y &= Ak_1\cos\theta - M_1k_2\sin\omega_2 t + M_2k_2\cos\omega_2 t, \\ \frac{dy}{dt} &= -A\omega_1k_1\sin\theta - M_1k_2\omega_2\cos\omega_2 t - M_2\omega_2k_2\sin\omega_2 t, \\ \theta &= \omega_1 t + \alpha, \end{aligned}\right\} \tag{18.24}$$

where A, M_1, M_2, α are arbitrary constants. Taking the magnitudes A, M_1, M_2, Θ as new variables we shall have

$$\left.\begin{aligned}
&\frac{dA}{dt}\sin\theta+\frac{dM_1}{dt}\cos\omega_2 t+\frac{dM_2}{dt}\sin\omega_2 t+A\cos\theta\left(\frac{d\theta}{dt}-\omega_1\right)=0,\\
&\frac{dA}{dt}k_1\cos\theta-\frac{dM_1}{dt}k_2\sin\omega_2 t+\frac{dM_2}{dt}k_2\cos\omega_2 t-\\
&\qquad\qquad -Ak_1\sin\theta\left(\frac{d\theta}{dt}-\omega_1\right)=0,\\
&\frac{dA}{dt}\omega_1\cos\theta-\frac{dM_1}{dt}\omega_2\sin\omega_2 t+\frac{dM_2}{dt}\omega_2\cos\omega_2 t-\\
&\qquad\qquad -A\omega_1\sin\theta\left(\frac{d\theta}{dt}-\omega_1\right)=\mu f,\\
&-\frac{dA}{dt}\omega_1 k_1\sin\theta-\frac{dM_1}{dt}\omega_2 k_2\cos\omega_2 t-\frac{dM_2}{dt}\omega_2 k_2\sin\omega_2 t-\\
&\qquad\qquad -A\omega_1 k_1\cos\theta\left(\frac{d\theta}{dt}-\omega_1\right)=\mu F+\mu Q^*,
\end{aligned}\right\} \quad (18.2$$

whence

$$\begin{aligned}
\frac{dM_1}{dt}&=\mu\left(F_*\cos\omega_2 t-k_1 f^*\sin\omega_2 t\right),\\
\frac{dM_2}{dt}&=\mu\left(F_*\sin\omega_2 t+k_1 f^*\cos\omega_2 t\right),\\
\frac{dA}{dt}&=\mu\left(-F_*\sin\theta-k_2 f^*\cos\theta\right),\\
\frac{d\theta}{dt}&=\omega_1+\mu\left(-F_*\cos\theta+k_2 f^*\sin\theta\right),
\end{aligned}$$

where

$$F_*=\frac{\varkappa_1 Q^*}{\omega_1^2-\omega_2^2}+F^*,$$

and F^* and f^* as before are determined by formulas (18.10)

Substituting in the right hand sides of equations (18.25) for x, dx/dt, y, dy/dt their values from (18.24) and averaging over the variables we obtain the following equations of the first approximation:

$$\begin{aligned}
\frac{dM_1}{dt}&=\mu b_2\left(-c_2+M_1^2\omega_2^2+M_2^2\omega_2^2+2A^2\omega_1^2\right)M_1=\mu R_1(M_1, M_2, A),\\
\frac{dM_2}{dt}&=\mu b_2\left(-c_2+M_1^2\omega_2^2+M_2^2\omega_2^2+2A^2\omega_1^2\right)M_2+\frac{\varkappa_1 Q^*}{\omega_1^2-\omega_2^2}=\mu R_2(M_1, M_2, A),\\
\frac{dA}{dt}&=\mu b_1\left(c_1-2M_1^2\omega_2^2-2M_2^2\omega_2^2-A^2\omega_1^2\right)A=\mu R(M_1, M_2, A),\\
\frac{d\theta}{dt}&=\omega_1,
\end{aligned}$$

where b_1, b_2, c_1, c_2 have the same values as in (18.12).

Equations (18.7) have stationary solutions determined in the first approximation by the equations

$$x = A^* \sin(\omega_1 t + \varepsilon) + M_2^* \sin \omega_2 t,$$
$$y = A^* k_1 \cos(\omega_1 t + \varepsilon) + M_2^* k_2 \cos \omega_2 t,$$

where ε is an arbitrary constant and A^*, M_2^* the roots of the equations

$$b_2(-c_2 + M_2^2\omega_2^2 + 2A^2\omega_1^2) M_2 + \frac{\varkappa_1 Q^*}{\omega_1^2 - \omega_2^2} = 0,$$
$$(c_1 - 2M_2^2\omega_2^2 - A^2\omega_1^2) A = 0,$$

since from the equations $R_1 = R_2 = 0$ it follows that $M_1 = 0$.

These stationary solutions will be stable if for $M_1 = 0$, $M_2 = M_2^*$, $A = A^*$ the roots of the equation

$$\begin{vmatrix} \frac{\partial R_1}{\partial M_1} - \varkappa & \frac{\partial R_1}{\partial M_2} & \frac{\partial R_1}{\partial A} \\ \frac{\partial R_2}{\partial M_1} & \frac{\partial R_2}{\partial M_2} - \varkappa & \frac{\partial R_2}{\partial A} \\ \frac{\partial R}{\partial M_1} & \frac{\partial R}{\partial M_2} & \frac{\partial R}{\partial A} - \varkappa \end{vmatrix} = 0$$

have negative real parts. We shall not here dwell on the analysis of the stationary solutions thus obtained and their stability, referring to the reader the works of N.V. Butenin cited on p. 350.

19. Averaging Principle

Let us consider the system of the standard form

$$\frac{dx_s}{dt} = \mu X_s(t, x_1, \ldots, x_n, \mu) \qquad (s = 1, \ldots, n), \tag{19.1}$$

where the right hand sides are almost periodic with respect to t and satisfy the general conditions indicated in sec. 6. Let us average the right hand sides of the system with respect to t and replace it by the more simple system

$$\frac{dy_s}{dt} = \mu \lim_{t\to\infty} \frac{1}{t} \int_0^t X_s(t, y_1, \ldots, y_n, 0)\,dt. \qquad (19.2)$$

In sec. 6 we saw that between the systems (19.1) and (19.2) a close relation exists which permits in certain oscillation problems replacing the system (19.1) by the system (19.2). We showed, namely, that if certain conditio are satisfied there correspond to the stationary states of system (19.2), which are states of equilibrium, stationary states of the system (19.1), which are almost periodic oscillations. To the stationary positions of equilibrium of system (19.2) correspond stable almost periodic oscillations of system (19.1) while to the unstable positions of equilibrium there correspond unstable almost periodic oscillations. Moreover, for sufficiently small μ the system (19.1), for almost periodic oscillations, will remain arbitrarily near the corresponding position of equilibrium of the system (19.2).

Let us now consider a more general system of the form

$$\left.\begin{aligned} \frac{dx_s}{dt} &= \mu X_s(t, \theta_1, \ldots, \theta_p, x_1, \ldots, x_n, \mu), \\ \frac{d\theta_j}{dt} &= \lambda_j + S_j^*(t, \theta_1, \ldots, \theta_p, x_1, \ldots, x_n, \mu) \\ & (j = 1, \ldots, p), \end{aligned}\right\} \qquad (19.3)$$

where the functions X_s, S_j^* are almost periodic with respect to t and periodic with period 2π with respect to each of the angular variables $\Theta_1,\ldots,\Theta_p$. The system (19.3) we can consider as a particular case of the system (17.4). We shall assume that for equations (19.3) the same general conditions are satisfied as for equations (17.4) and, in particular, that the frequencies $\lambda_1,\ldots,\lambda_p$ are nonresonance frequencies.

We shall replace the system (19.3) by a simpler one by averaging the right hand sides with respect to t and the variables $\Theta_1,\ldots,\Theta_p$, or, what amounts to the same thing with respect to the variable t after substituting for the variables Θ_j the functions $\lambda_j t$. In this way the system is obtained

$$\left.\begin{aligned} \frac{dy_s}{dt} &= \mu Y_s(y_1, \ldots, y_n), \\ \frac{d\vartheta_j}{dt} &= \lambda_j + \mu S_j(y_1, \ldots, y_n), \end{aligned}\right\} \qquad (19.4)$$

where

$$Y_s = \lim_{t\to\infty} \frac{1}{t} \int_0^t X_s(t, \lambda t, y, 0)\, dt,$$

$$S_j = \lim_{t\to\infty} \frac{1}{t} \int_0^t S_j(t, \lambda t, y, 0)\, dt$$

(in the same way as for the system (19.1), the terms of higher order are discarded). This simplified system, as we saw, is similarly closely connected with the initial system (19.3). Thus, let

$$y_s = y_s^0, \quad \vartheta_j(t) = \vartheta_j(0) + [\lambda_j + \mu S_j(y_1^0, \ldots, y_n^0)]\, t,$$

where y_s^o is a solution of the equations $Y_s(y_1, \ldots, y_n) = 0$, be the stationary solution of system (19.4) and assume that it is stable. Then, as follows from the results of sec. 16, all solutions of equations (19.3) for which $|x_s(0) - y_s^o|$ are sufficiently small can be represented in the form

$$x_s(t) = y_s^0 + \alpha_s(t), \quad \vartheta_j(t) = \vartheta_j(0) + [\lambda_j + \mu S_j(y_1^0, \ldots, y_n^0)]\, t + \mu t \varphi_j(t),$$

where $|\alpha_s|$ and $|\varphi_j|$ for all $t > 0$ remain arbitrarily small provided μ is sufficiently small.

The replacing of equations of the form (19.1) or (19.3) by the more simple equations (19.2) and (19.4) is called the "averaging principle". This principle is very widely applied in the theory of nonlinear oscillations. It is employed for the study of not only steady motions but also the processes of establishment. In other words, the exact equations are replaced by the simplified not only in considering steady state motions of the system but also other motions (for a finite interval of time).

For the simplest systems with one degree of freedom the "averaging principle" was first applied for the study of nonlinear oscillations by van der Pol. The principle received further development in the works of L. I. Mandelshtam, N.D. Papaleksi, A. A. Andronov, B. V. Bulgakov and many others. The most important results however in this direction were obtained by N. M. Krylov and N. N. Bogolyubov. They were namely the first to explain the nature of the simplification itself, by showing, with the aid of a transformation specially introduced by them in the theory of nonlinear oscillations, that the question reduces to the rejection of terms of higher than the first order with respect to μ in the initial equations. Krylov and Bogolyubov also gave a method of constructing simplified equations differing from the initial equations by magnitudes of arbitrarily high order.[1] Of particular importance are their results on the establishing of a relation between the stationary solutions of the complete and simplified systems. They proved,[2] namely, the existence of almost periodic solutions for the equation

$$\frac{d^2x}{dt^2}+k^2x=\mu f\left(t,\ x,\ \frac{dx}{dt},\ \mu\right),$$

that correspond to the stationary solutions of the simplified system. Finally, in the monograph "On Certain Statistical Methods in Mathematical Physics", frequently cited by us, Bogolyubov established a correspondence between the stationary solutions of systems of the form (19.1) and (19.3) and the simplified systems for very general assumptions made with regard to the right hand sides

Many cases of quasilinear systems are with the aid of transformations reduced to systems of the form (19.1) and (19.3). In particular, all the examples considered by us in the preceding section belong to this class. However, as we have seen, in the case where the fundamental equation of the generating system has noncritical roots, the equations of the oscillations reduce to the form (17.4)

$$\left.\begin{aligned}
&\frac{dx_s}{dt}=\mu X_s(t,\ \theta_1,\ \ldots,\ \theta_p,\ x_1,\ \ldots,\ x_n,\ z_1,\ \ldots,\ z_k,\ \mu),\\
&\frac{d\theta_j}{dt}=\lambda_j+\mu S_j^*(t,\ \theta_1,\ \ldots,\ \theta_p,\ x_1,\ \ldots,\ x_n,\ z_1,\ \ldots,\ z_k,\ \mu),\\
&\frac{dz_r}{dt}=b_{r1}z_1+\ldots+b_{rk}z_k+f_r(t+\mu Z_r(t,\ \theta,\ x,\ z,\ \mu)\\
&\qquad (s=1,\ \ldots,\ n;\ j=1,\ \ldots,\ p;\ r=1,\ \ldots,\ k)
\end{aligned}\right\}\quad (19.5$$

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1 This method for equations of the form (19.1) was set forth in section 5.

2 Krylov N.M. and Bogolyubov N.N., Prilozhenie metodov nelineinoi mekhaniki k teorii statsionarnykh kolebanii (Application of the Methods of Nonlinear Mechanics to the Theory of Stationary Oscillations), Izd.-vo Vseukrainskoi Akademii Nauk, Kiev, 1934.

or, in the case of the absence of nonresonance frequencies, to their particular case

$$\left.\begin{aligned}\frac{dx_s}{dt}&=\mu X_s(t, x_1, \ldots, x_n, z_1, \ldots, z_k, \mu),\\ \frac{dz_r}{dt}&=b_{r1}z_1+\ldots+b_{rk}z_k+f_r(t)+\mu Z_r(t, x, z, \mu).\end{aligned}\right\} \quad (19.6)$$

For systems (19.5) and (19.6) the generalized averaging principle, as we have seen, is valid; according to the results, namely, of sec. 15 and 16 the system (19.5) can be replaced by the simplified system

$$\left.\begin{aligned}\frac{dy_s}{dt}&=\mu \lim_{t\to\infty}\frac{1}{t}\int_0^t X_s(t, \lambda_1 t, \ldots, \lambda_p t, y_1, \ldots, y_n, \zeta_1^0, \ldots, \zeta_k^0, 0)\,dt,\\ \frac{d\vartheta_j}{dt}&=\lambda_j+\mu \lim_{t\to\infty}\frac{1}{t}\int_0^t S_j^*(t, \lambda t, y, \zeta^0, 0)\,dt,\\ \frac{d\zeta_r}{dt}&=b_{r1}\zeta_1+\ldots+b_{rk}\zeta_k+f_r(t)+\mu Z_r(t, \vartheta, y, \zeta^0, 0),\end{aligned}\right\} \quad (19.7)$$

where $\zeta_r^0(t)$ is an almost periodic solution of the system

$$\frac{d\zeta_r^0}{dt}=b_{r1}\zeta_1^0+\ldots+b_{rk}\zeta_k^0+f_r(t).$$

The relation between the stationary solutions of system (19.5) and the simplified system (19.7) will be of the same type as between the systems (19.3) and (19.4). In the particular case where the complete system has the form (19.6) its stationary solutions, as has been shown will be almost periodic.

CHAPTER V

QUASIHARMONIC SYSTEMS

A. FREE OSCILLATIONS OF QUASIHARMONIC SYSTEMS

1. Parametric Resonance. Statement of the Problem

QUASIHARMONIC SYSTEMS are systems the oscillations of which are described by linear equations with periodic coefficients. Systems of this kind play a large part in various problems of physics and engineering. We shall consider first the free oscillations of quasiharmonic systems, i.e. oscillations described by homogeneous equations. The phenomenon of most interest here from the physical point of view is that of so-called PARAMETRIC RESONANCE. To explain the essential nature of this phenomenon and the mathematical problems associated with it we shall first consider several examples.

1. PENDULUM WITH OSCILLATING POINT OF SUSPENSION. Let us consider a pendulum of length l the axis of suspension of which performs vertical harmonic oscillations with small amplitude a and frequency ω. The differential equation of motion of the pendulum has the form

$$\frac{d^2\varphi}{dt^2} = -\frac{g}{l}\left(1+\frac{a\omega^2}{g}\sin\omega t\right)\sin\varphi,$$

where φ is the angle of displacement from the lower vertical position. Setting $\omega t = \tau$ and discarding the nonlinear terms we obtain

$$\frac{d^2\varphi}{dt^2} = -\frac{g}{l\omega^2}\left(1+\frac{a\omega^2}{g}\sin\tau\right)\varphi. \tag{1.1}$$

We have already in sec. 10 of chapter II considered the problem of the motion of a pendulum with oscillating point of suspension, in particular the case where this point oscillates along a vertical. We there assumed that the frequency ω was very large and were interested in the possible periodic motions of the pendulum and their stability. Here we shall not assume that the frequency ω is large and the problem that interests us will be somewhat different.

In the same way as for the pendulum with fixed point of suspension, the vertical is the position of equilibrium. But in contrast to the case of the pendulum with fixed point of suspension this position of equilibrium can be both stable or unstable. As we shall namely see below, for certain values of the frequency ω the characteristic equation for the system (1.1) may have a root with a modulus greater than one. In this case transverse oscillations of the pendulum with infinitely increasing amplitude will occur.[1] These oscillations, as we shall see, are excited not by the disturbing force but by the periodic change of a parameter of the system (the reduced length of the pendulum). In this case there is said to be parametric resonance. The problem that interests us consists precisely in separating out those values of the frequency ω for which parametric resonance occurs.

2. PARAMETRIC EXCITATION OF OSCILLATIONS IN AN ELECTRICAL CIRCUIT. Let us consider an electrical oscillating circuit consisting of a capacitance C, a self-inductance L and a resistance R. If we denote the charge by q the equation of the oscillations will be

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=0. \tag{1.2}$$

Let us now assume that one of the parameters of the system, for example the capacitance C, varies periodically. For example, let

$$\frac{1}{C}=\frac{1}{C_0}(1+m\cos\omega t).$$

Equation (1.2) then assumes the form

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C_0}(1+m\cos\omega t)q=0.$$

----

[1] Actually the amplitude of the oscillations will remain finite since the exact equation of the oscillations is nonlinear.

Setting in this equation

$$q = xe^{-\frac{R}{2L}t}, \qquad \tau = \omega t, \qquad \mu = \frac{4Lm}{4L - R^2C_0},$$

we obtain:

$$\frac{d^2x}{d\tau^2} + \frac{4L - R^2C_0}{4L^2C_0\omega^2}(1 + \mu\cos\tau)x = 0. \tag{1.3}$$

We have again arrived at an equation of the second order with periodic coefficients. Its characteristic equation for a suitable choice of the frequency ω may have a root with a modulus greater than one. In this case there will be parametric resonance: intense electrical oscillations will arise in the system notwithstanding the absence in it of current sources. This phenomenon may be utilized for the construction of an electrical current generator differing entirely from the usual kind and based on the mechanical variation of the capacitance (or self-inductance). A generator of such type was first realized by L. I. Mandelshtam and N. D. Papaleksi.

3. LONGITUDINAL BENDING OF PRISMATIC RODS UNDER THE ACTION OF LONGITUDINAL PERIODIC FORCES.[1] Let us investigate the problem of the transverse oscillations of a straight prismatic rod (fig. 27) hinge supported at the ends and compressed by a longitudinal force $P = P_0 \sin t$ sinusoidally varying with the

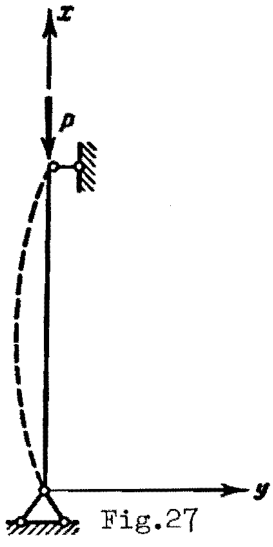

Fig.27

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[1] Belyaev N.M., Ustoichivost' prizmaticheskikh sterzhnei pod deistviem peremenykh prodol'nykh sil (Stability of Prismatic Rods under the Action of Variable Longitudinal Forces), Collection of articles "Inzhenernye sooruzheniya i stroitel'naya mekhanika" (Engineering Structures and Structural Mechanics), Izd-vo Put, 1924.

time.

The equation of the bent axis of the rod, in the coordinates indicated in the sketch, has the form

$$EJ\frac{\partial^2 y}{\partial x^2} = -M_1 - M_2,$$

where EJ is the rigidity in bending, $M_1 = yP_0 \sin \omega t$ the bending moment in the cross-section x of the force P and M_2 the bending moment in the same cross-section of the inertia forces. Differentiating twice with respect to x and taking into account that the magnitude $\partial^2 M_2 / \partial x^2$ is the intensity of the forces of inertia we obtain the following equation of the transverse oscillations of the rod:

$$EJ\frac{\partial^4 y}{\partial x^4} + \rho\frac{\partial^2 y}{\partial t^2} + P_0 \sin \omega t \frac{\partial^2 y}{\partial x^2} = 0.$$

where ρ is the density.

Since the rod is hinge-supported the boundary conditions will be

$$y(0,\ t) = y(l,\ t) = \left(\frac{\partial^2 y}{\partial x^2}\right)_{x=0} = \left(\frac{\partial^2 y}{\partial x^2}\right)_{x=l} = 0,$$

where l is the length of the rod. Taking these boundary conditions into account, we set:

$$y = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi}{l} x.$$

For the functions T_n we then obtain the equations

$$\frac{d^2 T_n}{dt^2} + \frac{n^4 \pi^4 EJ}{\rho l^4}\left(1 - \frac{P_0 l^2}{n^2 \pi^2 EJ} \sin \omega t\right) T_n = 0 \quad .$$

or

$$\frac{d^2 T_n}{d\tau^2} + \frac{\nu_n^2}{\omega^2}\left(1 - \frac{P_0}{Q_n} \sin \tau\right) T_n = 0, \qquad (1.4)$$

where

$$\tau = \mu t, \quad \nu_n^2 = \frac{n^4\pi^4 EJ}{\rho l^4}, \quad Q_n = \frac{n^2\pi^2 EJ}{l^2},$$

the magnitudes ν_n and Q_n denoting the n-th natural frequency of the transverse vibrations and the n-th critical force in the problem of static stability.

We have again arrived to an equation with periodic coefficients, which for a suitable choice of the frequency ω of the force P may have unstable solutions. Thus, in this problem too parametric resonance may occur.

In all examples considered the problem reduced to the investigation of an equation of the second order of the form

$$\frac{d^2x}{dt^2} + \lambda p(t) x = 0, \tag{1.5}$$

where p(t) is a periodic function of the time and λ a certain parameter. It is necessary to determine those values of the parameter λ for which stability or instability will occur for this equation. Since the conditions of stability and instability are determined by inequalities, the values of λ for which stability or instability will occur will in general lie in certain intervals. Those intervals of the values of λ for which stability takes place we shall call REGIONS OF STABILITY of equation (1.5), REGIONS OF INSTABILITY being analogously defined.

Thus, in all preceding examples the problem reduced to the determination of the regions of stability and instability for an equation of the form (1.5). To the same problem are reduced many other important problems of engineering, physics and astronomy. This problem was first considered by A. M. Lyapunov,[1] who obtained in connection with it a number of fundamental results.

We shall consider the above mentioned problem in detail for certain special assumptions, namely, we shall assume that the function p(t) in equation (1.5) differs little from its mean value so that we can write

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[1] Lyapunov A., 1) Sur une équation différentielle linéaire du second ordre, Comptes Rendus de l'Acad. des sciences de Paris, vol. 128, 1899, p. 910-913, 2) Sur une équation transcendente et les équations différentielles linéaires du second ordre a coefficientes périodiques, ibid., vol. 128, 1899, p. 1085-1088.

$$p(t) = a(1 + \mu f(t)),$$

where f(t) is a periodic function of t of period ω for which

$$\int_0^\omega f(t)\,dt = 0,$$

a is a constant and μ also a constant which is small in comparison with unity.

We shall show first of all that if $\lambda a < 0$, then for μ sufficiently small instability always takes place for equation (1.5). In fact, let us construct the derivative with respect to t of the function $V = x\,dx/dt$. In virtue of (1.5) we shall have

$$\frac{dV}{dt} = \left(\frac{dx}{dt}\right)^2 - \lambda a(1 + \mu f(t))\,x^2,$$

whence it follows that if $\lambda a < 0$ and μ sufficiently small this derivative will be a positive definite function of the variables x and dx/dt. But since the function V changes sign with respect to these variables, it satisfies all the conditions of the theorem of Lyapunov on instability, whence it follows that the solution of equation (1.5) is unstable.

Thus, we need investigate only that case where $\lambda a > 0$. We can therefore write equation (1.5) in the following form:

$$\frac{d^2x}{dt^2} + \lambda^2(1 + \mu f(t))\,x = 0. \tag{1.6}$$

It may be assumed with a sufficient degree of accuracy that all examples of parametric resonance considered above are described by equations of the type (1.6). For this equation, as we shall see, instability may occur for arbitrarily small value of the magnitude μ. This circumstance has a particularly important significance from the practical point of view. In fact, in practice, if μ is sufficiently small, the term $\mu f(t)$ is often neglected in the equation of oscillations and this may lead to obviously erroneous results, since for $\mu = 0$ the equation (1.6) always has only stable solutions.

In the following sections below we shall first take up the detailed investigation of equation (1.6). After this we shall pass on to consider the question of the regions of stability and instability for systems with many degrees of freedom.

2. Regions of Stability and Instability for Equations of the Second Order

Let us consider the equation of the second order

$$\frac{d^2x}{dt^2}+\lambda^2(1+\mu f)x=0, \tag{2.1}$$

where f is a periodic function of the time. For convenience of the further computations we shall assume that the period of this function is equal to π, which evidently can always be attained by a suitable choice of the unit of time. We shall assume for the sake of generality that the parameter μ enters not only as a multiplier of f but that the function f itself depends on it. This dependence we shall assume as analytic so that

$$f=f_1(t)+\mu f_2(t)+\mu^2 f_3(t)+\dots, \tag{2.2}$$

where the functions $f_i(t)$ do not depend on μ and are periodic of period π and the series converges for $\mu \leqslant a$, where a is a certain constant magnitude.

We shall set up the characteristic equation for equation (2.1). If the equation (2.1) is written in the form of the system

$$\frac{dx}{dt}=y, \quad \frac{dy}{dt}=-\lambda^2(1+\mu f)x,$$

formula (2.8) of chapter III shows at once that the free term of the required characteristic equation is equal to 1. Hence this equation has the form

$$\rho^2-2A(\lambda,\mu)\rho+1=0. \tag{2.3}$$

Here the coefficient A, according to the theorem of Lyapunov proved in sec. 10 of chapter III, is an analytic function of the parameters λ and μ for all values of λ and for all $\mu \leqslant a$.

We proceed to the determination of the regions of stability and instability for equation (2.1) as a function of the values of λ. Let us vary this parameter by giving it all possible real values. It is here sufficient to consider only positive values since equation (2.1) does not change on substituting $-\lambda$ for λ and consequently the distribution of the regions of interest to us will for $\lambda < 0$ be the same as for $\lambda > 0$.

From (2.3) we obtain:

$$\rho(\lambda, \mu) = A \pm \sqrt{A^2 - 1}.$$

Hence, if $A^2 > 1$ equation (2.3) will have two real roots of which one is numerically greater than 1 and the other less. If however $A^2 < 1$ both roots of equation (2.3) will be complex with moduli equal to one. Thus, to the regions of stability of equation (2.1) those values of λ correspond for which $A^2 < 1$ and to the regions of instability those values for which $A^2 > 1$. From this it follows immediately that the regions of stability and instability are separated by those values of λ for which there are satisfied either the equation

$$A(\lambda, \mu) = +1, \tag{2.4}$$

or the equation

$$A(\lambda, \mu) = -1. \tag{2.5}$$

Let us examine these equations more closely. The characteristic exponents of equation (2.1) for $\mu = 0$ will evidently be the magnitudes $\pm\lambda i$. Passing to the roots of the characteristic equation we find that for $\mu = 0$ equation (2.3) has the following roots:

$$\rho_1(\lambda, 0) = e^{\pi\lambda i}, \quad \rho_2(\lambda, 0) = e^{-\pi\lambda i}$$

and therefore

$$A(\lambda, 0) = \frac{1}{2}(\rho_1 + \rho_2) = \cos \pi\lambda,$$

so that

$$A(\lambda, \mu) = \cos \pi\lambda + \mu F(\lambda, \mu), \tag{2.6}$$

where $F(\lambda, \mu)$ is an analytic function of the parameters λ and μ. Having established this, let us consider equations

(2.4) and (2.5). From (2.6) it is at once seen that these equations are satisfied for $\mu = 0$ and $\lambda = n$, where n is an integer. For n odd equation (2.5) is satisfied and for n even, equation (2.4). It may thus be expected that for $\mu \neq 0$ but sufficiently small equation (2.5) has a λ solution for λ in the neighborhood of any odd integer while equation (2.4) has a solution in the neighborhood of any even integer, the solutions reducing to these integers for $\mu = 0$. For clarifying the question of the existence of these solutions let us set in the expression for A

$$\lambda = n + \alpha \tag{2.7}$$

and equate the obtained expression for n odd to -1 and for n even to $+1$. We then obtain the following equation for the magnitude α:

$$(-1)^{n+1}\left\{\frac{\alpha^2\pi^2}{2!} - \frac{\alpha^4\pi^4}{4!} + \ldots\right\} + \mu F(n+\alpha, \mu) = 0. \tag{2.8}$$

The left hand side of this equation is an analytic function of the magnitudes α and μ, reducing to zero for $\alpha = \mu = 0$. If the derivative of this function with respect to α did not become zero for $\alpha = \mu = 0$ then, on the basis of the theorem of the existence of implicit functions, equation (2.8), for μ different from zero but sufficiently small, would admit one and only one solution $\alpha(\mu)$ that reduces to zero for $\mu = 0$. However the derivative of the left hand side of (2.8) with respect to α reduces to zero for $\alpha = \mu = 0$ and therefore the above mentioned existence theorem is not applicable to the case under consideration.

But the general theory of implicit functions determined by analytic equations[1] shows that in the case considered equation (2.8) admits two and two solutions reducing to zero for $\mu = 0$ and these solutions for μ sufficiently small are analytic functions of either the magnitude μ or of the magnitude $\sqrt{\mu}$.

Substituting the solutions for α in (2.7) we obtain solutions of equations (2.4) and (2.5). By giving n all possible integral values we obtain all solutions of equations (2.4) and (2.5). For n odd we obtain the solutions of equation (2.5) and for n even the solutions of equation(2.4

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[1] See Goursat E., Course in Mathematical Analysis, vol. II, chapter XVII, sec. 355, 356.

We shall now show that all the solutions thus obtained of equations (2.4) and (2.5) are real. For this purpose we note first of all that if λ satisfies equation (2.4), equation (2.1) has a periodic solution with a period equal to the period of the coefficient, i.e. π. In fact, in this case the roots of the characteristic equation are equal to unity, whence follows immediately the existence of the indicated solution. If λ satisfies equation (2.5) the roots of the characteristic equation will be equal to - 1 and therefore there will exist a solution $x = \varphi(t)$ satisfying the condition

$$\varphi(t+\pi) = -\varphi(t).$$

The function $\varphi(t)$ will likewise be periodic but with period 2π. Having established this, let us assume that $\lambda = \lambda^*$ is a root of equation (2.4) and let $\varphi(t)$ be the corresponding periodic solution of equation (2.1). We have:

$$\frac{d^2\varphi}{dt^2} + \lambda^{*2}(1+\mu f)\varphi \equiv 0.$$

Let us multiply this identity by $\bar{\varphi}dt$, where $\bar{\varphi}$ is the complex conjugate to φ, and integerate between the limits 0 and π. Then, integrating by parts and taking into account that in virtue of the periodicity

$$\Big|_0^\pi \bar{\varphi}\frac{d\varphi}{dt} = 0,$$

we obtain:

$$-\int_0^\pi \frac{d\bar{\varphi}}{dt}\frac{d\varphi}{dt}\,dt + \lambda^{*2}\int_0^\pi (1+\mu f)\varphi\bar{\varphi}\,dt \equiv 0. \tag{2.9}$$

The expressions under the integral signs are real and positive and therefore the integrals themselves will be the same. But then identity (2.9) shows that the magnitude λ^{*2} will also be real and positive and therefore the magnitude λ^* will be real.

In exactly the same way is proven the reality of the roots of equation (2.5). The difference consists only in the circumstance that in the identity (2.9) the integrals will now be taken between the limits 0 and 2π.

We shall now show that all the roots of equations

(2.4) and (2.5) for μ sufficiently small will be analytic functions of this magnitude. In fact, from the preceding it follows that all the roots of equations (2.4) and (2.5) are either analytic functions of the magnitude μ or analytic functions of the magnitude $\sqrt{\mu}$. But if at least one of these roots is an analytic function of $\sqrt{\mu}$ it will necessary be complex either for $\mu > 0$ or for $\mu < 0$. But we have just shown that all roots of equations (2.4) and (2.5) are real and this will be true independently of whether the parameter μ is positive or negative. Thus, all roots of equations (2.4) and (2.5) will be analytic functions of the magnitude μ in a certain neighborhood of the point $\mu = 0$, i.e. they will be developable in series of integral positive powers of μ converging for μ sufficiently small.

We shall now consider all possible values of λ and construct the graph of the curve $A(\lambda)$ (regarding μ as fixed). For this purpose we lay off on the λ-axis all roots of equations (2.4) and (2.5). As we have seen, all these roots break down into pairs lying near integral values of λ. Let us denote by $\bar{\lambda}'_n, \bar{\lambda}''_n$ the roots lying near an odd integer n and by λ'_n, λ''_n the roots lying near an even integer n. All these roots divide up the λ-axis into intervals of two types (fig. 28). The intervals of the first type are bounded on both sides by roots of one form, i.e. which either both satisfy equation (2.4) or both satisfy equation (2.5). To intervals of this type belong for example the intervals $(\bar{\lambda}'_1, \bar{\lambda}''_1)$ and $(\lambda'_2, \lambda''_2)$. We shall call intervals of the first type HOMOGENEOUS. Intervals of the second type are bounded on one side by a root of equation (2.4) and on the other side by a root of equation (2.5). Intervals of such type for example as the interval $(\bar{\lambda}''_1, \lambda'_2)$ we shall call NONHOMOGENEOUS. As is seen from the sketch, homogeneous alternate with nonhomogeneous intervals. Certain homogeneous intervals can degenerate into a point. This will be the case if equation (2.4) or (2.5) has multiple roots.

It is now easy to see that in each homogeneous (non-degenerate) interval the inequality holds $A^2 > 1$, and in each nonhomogeneous interval the inequality $A^2 < 1$. In other words, the regions of instability coincide with the homogeneous intervals and the regions of stability with the nonhomogeneous intervals. In fact, let us consider some nonhomegenous interval, for example $(\bar{\lambda}''_1, \lambda'_2)$.

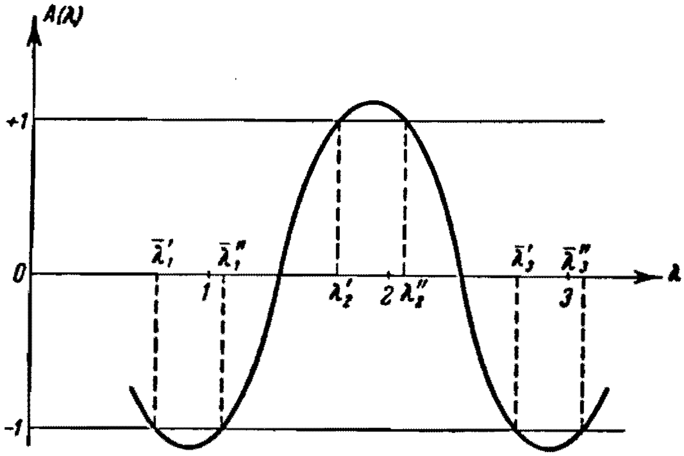

Fig. 28

In this interval the magnitude A changes from -1 at the start to $+1$ at the end of the interval. And since the magnitude A can nowhere become ± 1 within the interval, because all roots of equations (2.4) and (2.5) are situated at the ends of the intervals, the inequality $A^2 < 1$ must necessarily be satisfied at all points within the interval considered. Let us now consider some homogeneous nondegenerate interval at the ends of which A = +1. Let this be for example the interval (λ'_2 , λ''_2). Since in this interval A varies from + 1 to +1 there must at all points within the interval be satisfied either the inequality $A^2 > 1$ or the inequality $A^2 < 1$. In order to explain which of these cases actually holds let us consider the points lying near one of the ends of the interval. Let this end be λ'_2 . Since the interval is by assumption not degenerate, the root λ'_2 is simple and therefore $dA/d\lambda$ does not become zero for $\lambda = \lambda'_2$. And since to the left of the point λ'_2 the magnitude A is less than one the magnitude A will be greater than one to the right of this point. The proposition has thus be proved. In the same way this proposition is proved also for the homogeneous intervals at the ends of which A = -1.

Thus by giving the parameter λ in equation (2.1) all possible values we obtain an infinite sequence of alternating regions of stability and instability. The boundaries of these regions are constituted by the roots of equations (2.4) and (2.5), which divide the λ-axis into intervals of two kinds: homogeneous, which are bounded at both ends either by the roots of equation (2.4) or by the roots of equation (2.5), and nonhomogeneous, which are bounded at one end by a root of equation (2.4) and at the other end by a root of equation (2.5). The regions of stability coincide with the nonhomogeneous intervals and the regions of instability with the homogeneous interval. The roots of equation (2.5) are located near odd integers[1], with two roots near each such integer, these roots running together for $\mu = 0$ and becoming equal to the integer.[2] The roots of equation (2.4) are in a similar manner located near even integers. All these roots for μ sufficiently small are analytic functions of this magnitude.

3. Practical Method of Determining the Regions of Stability and Instability for Equations of the Second Order

From the results of the preceding section it follows that all regions of instability of equation (2.1) are situated in the neighborhood of integers, one region of instability being situated in the neighborhood of each integer n. This region for even n is bounded by the roots of equation (2.4) and for odd n by the roots of equation (2.5). For $\mu = 0$ each region of instability contracts to a point.

Let us assume that in equation (2.1)

$$\frac{d^2x}{dt^2} + \lambda^2(1+\mu f)\,x = 0, \qquad (3.1)$$

----

[1] It is necessary to bear in mind that the period of equation (2.1) was taken equal to π. If this period were equal to some other number ω the roots of equations (2.4) and (2.5) would be situated near numbers of the form $n\pi/\omega$, where n is an integer.

[2] It should not be supposed that each such pair of roots is invariably separated by its corresponding integer. It may happen that both roots lie on one side of the integer in question.

where

$$f = f_1(t) + \mu f_2(t) + \dots,$$

and λ is a root of equation (2.4) or (2.5). For definiteness let it be a root which for $\mu = 0$ reduces to a given integer n. As we have seen in the preceding section, for μ sufficiently small this root is an analytic function of μ and we can write

$$\lambda^2 = n^2 + \alpha_1\mu + \alpha_2\mu^2 + \dots \tag{3.2}$$

For the assumption made, equation (3.1), as already pointed out in the preceding section, has a periodic solution with a period equal to 2π, if n is odd, or π if n is even. Will this solution likewise be an analytic function of μ?.

Since the coefficients of equation (3.1) are analytic functions of μ, every solution of this equation the initial values of which do not depend on μ will be analytic with respect to μ. The same will be true also with respect to any solution whose initial values depend on μ but are analytic functions of this magnitude. Let $x = x(t)$ be a solution of equation (3.1) determined by the initial conditions

$$x(0) = 1, \qquad \dot{x}(0) = 0. \tag{3.3}$$

The characteristic equation for (3.1), for the assumption made with regard to λ, has a double root equal to 1 (for n even) or -1 (for n odd). Hence any solution of equation (3.1), and in particular the solution $x(t)$ under consideration, will either be periodic, or of the form

$$x(t) = t\varphi(t) + \psi(t), \tag{3.4}$$

where φ and ψ are periodic functions of the time.[1] In

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[1] The period of these functions is equal to π if n is an even number and 2π if n is odd. In general, in this section, when speaking of periodic solutions of equation (3.1) we shall have in mind, without specifying so especially, that we are considering solutions with period π or 2π depending on whether n is even or odd.

the latter case the function $\varphi(t)$ also determines a solution of equation (3.1), which is the required periodic solution. In virtue of the fact that the initial conditions (3.3) of solution (3.4) do not depend on μ this solution is analytic with respect to μ. But then, as can easily be seen, the function $\varphi(t)$ determining the required periodic solution is analytic with respect to μ.

Thus, we have shown that in the case considered the equation (3.1) admits a periodic solution analytic with respect to μ. Let this solution have the form

$$x = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \cdots, \qquad (3.5)$$

where $x_i(t)$ is a periodic function of time and the series converges for sufficiently small μ. In order to render the solution definite it will be necessary to give certain additional conditions determining the arbritrary constant multiplier which is contained in this solution. This can be done in the following manner.

If the solution (3.5) does not reduce to zero for $t = 0$ then by multiplying it by a suitably chosen multiplier we can obtain a solution reducing for $t = 0$ to an initially given magnitude

$$M = M_0 + \mu M_1 + \mu^2 M_2 + \ldots \qquad (3.6)$$

In other words, in the case considered equation (3.1) admits a periodic solution of the form (3.5) in which the functions $x_i(t)$ satisfy the initial conditions

$$x_i(0) = M_i, \qquad (3.7)$$

where M_i are constants satisfying only the single condition that the series (3.6) converge. We can therefore choose the first N constants M_i, where N is an arbitrarily large number, in an entirely arbitrary manner.

If the solution (3.5) becomes zero for $t = 0$ its derivative for $t = 0$ will be different from zero and we can require that the initial conditions be satisfied

$$\dot{x}_i(0) = N_i, \qquad (3.8)$$

where N_i are arbitrary constants for which the series

$$N_0 + \mu N_1 + \mu^2 N_2 + \ldots \qquad (3.9)$$

converges.

In order to clarify in each concrete case which of the initial conditions (3.7) or (3.8) can be satisfied it is sufficient to determine the function $x_o(t)$. If $x_o(0) \neq 0$ the solution (3.5) for μ sufficiently small will not become zero for t= 0 and we shall have initial conditions (3.7). If however $x_o(0) = 0$ but $\dot{x}_o(0) \neq 0$ conditions (3.8) can be satisfied. If $x_o(0) = \dot{x}_o(0) = 0$ then, as will be seen below, from the form of $x_o(t)$ we shall have identically $x_o(t) = 0$. This case may be excluded since, by dividing, in case it is necessary, the solution (3.5) by a suitable power of μ we can always bring about that it has a free term.

Conditions (3.7) or (3.8) uniquely determine the solution (3.5) provided we are not dealing with the exceptional case where all solutions of equation (3.1) are periodic. This case is possible since the characteristic equation has double roots to which can correspond two sets of solutions. In this latter case in order that the solution of (3.1) be completely defined it is necessary to give the initial values of the solution itself as well as of its derivative.

With this established, we substitute in equation (3.1) for λ^2 the series (3.2) and try to satisfy it by a formal series of the form(3.5) with periodic coefficients. From what was said above it follows that such series can always be found provided the coefficients α_i in (3.2) are chosen in a definite manner, namely, so that the series (3.2) satisfies either equation (2.4) or equation (2.5). It will then be possible to satisfy either the initial conditions (3.7) or the initial conditions (3.8), or both simultaneously. Let us consider the equations which must be satisfied by the functions $x_i(t)$.

Substituting the series (3.2) and (3.5) in 3.1) we shall have:

$$\sum_{i=0}^{\infty} \frac{d^2x_i}{dt^2}\mu^i + \left(n^2 + \sum_{j=1}^{\infty} \alpha_j \mu^j\right)\left(1 + \sum_{j=1}^{\infty} f_j \mu^j\right)\sum_{i=0}^{\infty} x_i \mu^i = 0.$$

Equating the coefficients of like powers of μ we obtain the following differential equations:

$$\left.\begin{aligned} &\frac{d^2x_0^2}{dt^2}+n^2x_0=0, \\ &\frac{d^2x_1}{dt^2}+n^2x_1=-n^2f_1x_0-\alpha_1x_0, \\ &\frac{d^2x_2}{dt^2}+n^2x_2=-(n^2f_1+\alpha_1)x_1-(n^2f_2+\alpha_1f_1)x_0-\alpha_2x_0 \end{aligned}\right\} \tag{3.10}$$

and in general

$$\frac{d^2x_k}{dt^2}+n^2x_k=-(n^2f_1+\alpha_1)x_{k-1}-\alpha_kx_0+F_k(t, x_0, \ldots, x_{k-2}), \tag{3.11}$$

where F_k are linear functions of $x_o,x_1,\ldots,x_{k-2}$ with periodic coefficients. These functions depend on the constants $\alpha_1,\ldots,\alpha_{k-1}$ and do not contain the constant α_k.

Equations (3.10) and (3.11) make it possible to determine successively the unknown functions $x_k(t)$. At the same time from the conditions of periodicity of these functions the unknown coefficients α_i in the expression for λ^2 are successively determined.

The general solution for x_o has the form

$$x_0=A_0\cos nt+B_0\sin nt,$$

Where A_o and B_o are arbitrary constants. Further, we can write:

$$\left.\begin{aligned} n^2f_1\cos nt&=\sum_{m=0}^{\infty}(a_m\cos mt+b_m\sin mt), \\ n^2f_1\sin nt&=\sum_{m=0}^{\infty}(c_m\cos mt+d_m\sin mt), \end{aligned}\right\} \tag{3.12}$$

since the function f_1 is periodic of period π; also

$$\left.\begin{aligned} a_n=\frac{\beta}{2},\quad b_n=\frac{\gamma}{2}, \\ c_n=\frac{\gamma}{2},\quad d_n=-\frac{\beta}{2}; \end{aligned}\right\} \tag{3.13}$$

where β and γ are the coefficients of cos2nt and sin2nt in theexpansion of the function f_1. In fact, the Fourier expansion of the function f_1 does not contain the free term since by assumption the mean value of the function f is equal to zero. As a result, the terms containing cos nt and sin nt in expressions (3.12) can appear only because of the terms with cos2nt and sin2nt in the expansion of the function f_1.

From formulas (3.12) and (3.13) we at once find the coefficients of cos nt and sin nt in the right hand side of the equation for x_1. Equating these coefficients to zero we obtain the following conditions of the periodicity of the function x_1:

$$\left.\begin{aligned}(\beta+2\alpha_1)A_0+\gamma B_0=0,\\ \gamma A_0+(2\alpha_1-\beta)B_0=0.\end{aligned}\right\} \tag{3.14}$$

In order that these equations have a solution for A_0 and B_0 other than trivial it is necessary and sufficient that the magnitude α_1 satisfy the quadratic equation

$$(2\alpha_1+\beta)(2\alpha_1-\beta)-\gamma^2=0.$$

whence we obtain:

$$\alpha_1=\pm\frac{1}{2}\sqrt{\beta^2+\gamma^2}.$$

It is here necessary to consider two cases depending on whether $\beta^2+\gamma^2\neq 0$ or $\beta^2+\gamma^2=0$. We shall first consider the former of these cases. In this case the equation for α_1 has two simple real roots. We equate α_1 to one of these roots. Since it is simple, it does not reduce to zero at least one of the minors of the determinant of the system (3.14), as a result of which only one of the magnitudes A_0 and B_0 can be chosen arbitrarily. Let us assume that this magnitude is A_0, for which it is necessary that it be obtained different from zero. This condition will for example be satisfied if $\gamma\neq 0$. Forthis assumption the magnitude $x_0(0)$ will be different from zero. Hence on the basis of what was said above we can, in computing the functions $x_i(t)$, satisfy the initial condtions (3.7). In particular, we can set $A_0 = 1$.

Equations (3.14) will then give a definite value for B_o.

Having thus chosen α_1, A_o, and B_o we shall have:

$$x_1 = x_1^*(t) + A_1 \cos nt + B_1 \sin nt,$$

where x_1^* is a particular periodic solution and A_1 and B_1 are arbitrary constants. The magnitude A_1 we can set equal to zero. This will mean that in the conditions (3.7) the magnitude M_1 has been taken equal to $x_1^*(0)$.

We now proceed to the computation of the further approximations. Equating to zero the coefficients of cos nt and sin nt in the equation for x_2 we obtain a system of linear nonhomogeneous equations

$$2A_0\alpha_2 + \gamma B_1 + 2p_2 = 0,$$
$$2B_0\alpha_2 + (2\alpha_1 - \beta) B_1 + 2q_2 = 0,$$

where p_2 and q_2 are entirely definite constants representing the coefficients of cos nt and sin nt in the expression $n^2 f_1 x_1^* + (\alpha_1 f_1 + n^2 f_2) x_o$. The determinant of this system , equal to

$$x_2 = x_2^*(t) + A_2 \cos nt + B_2 \sin nt,$$

is different from zero and therefore this system uniquely determines the magnitudes α_2 and B_1.

After the magnitudes α_2 and B_1 have been computed by the above method we obtain from the equation for x_2 the solution

$$\Delta = 2(2\alpha_1 - \beta) A_0 - 2\gamma B_0 \equiv 8\alpha_1 A_0 = 8\alpha_1,$$

where $x_2^*(t)$ is a certain periodic function and A_2 and B_2 are arbitrary constants. The constant A_2, like A_1, can be set equal to zero. As to the constant B_2, it is determined together with α_3 from the condition of periodicity of the function x_3.

Continuing further in an analogous fashion we assume that the constants $\alpha_1, \ldots, \alpha_{k-1}$ and the functions $x_o, x_1, \ldots, x_{k-1}$ have already been determined and that these functions came out periodic. We can here write:

$$x_{k-1} = x_{k-1}^*(t) + B_{k-1} \sin nt,$$

where x^*_{k-1} is a certain periodic function and B_{k-1} is a still remaining undetermined constant which must be computed together with the constant α_k from the condition of periodicity of the function x_k. These conditions we obtain by equating to zero the coefficients of cos nt and sin nt in the equation for x_k. In this way, as is easily seen, we obtain the following equations:

$$\left.\begin{aligned} 2A_0\alpha_k + \gamma B_{k-1} + 2p_k = 0, \\ 2B_0\alpha_k + (2\alpha_1 - \beta) B_{k-1} + 2q_k = 0, \end{aligned}\right\} \tag{3.15}$$

where p_k and q_k are entirely definite constants. Equations (3.15) uniquely determine the magnitudes α_k and B_{k-1}.

We have assumed that equations (3.14) give for A_o a magnitude different from zero. Let us assume that $A_o = 0$ and consequently $B_o \neq 0$. In this case $x_o(0) = 0$ but $\dot{x}_o(0) \neq 0$, and therefore instead of the initial conditions (3.7) there will figure the initial conditions (3.8). As a result it will be possible in the expressions for x_k to reject the terms with sin nt and these expressions will have the form

$$x_k = x_k^* + A_k \cos nt,$$

where A_k are arbitrary constants. The rejection of the terms $B_k$sin nt is equivalent to the assumptions that in the initial conditions (3.8) the magnitudes N_k are taken equal to $\dot{x}^*_k(0)$. The constant A_k, as in the preceding case, is determined together with the constant α_{k+1} from the condition of periodicity of x_{k+1}.

Thus, in the case considered, starting from some root of the quadratic equation for α_1, we obtain one and only one formal expansion of the form (3.2) for λ^2 for which equation (3.1) admits a formal periodic solution of the form (3.5) satisfying the initial conditions (3.7) or (3.8). By considering the two values for α_1 we obtain two different formal expansions for λ^2 and the periodic solution. But on the other hand, by what has been proven, there exist for each integer n two and only two values for λ^2, bounding a corresponding region of instability and representing CONVERGENT series of the form (3.2) for which the equation (3.1) admits a periodic solution of the form(3.5) satisfying initial conditions (3.7) or (3.8). From this it follows immediately that the expansions obtained above for λ^2 precisely represent the required boundaries of the regions of instability and therefore converge.

Let us assume now that $\beta^2 + \gamma^2 = 0$, which will hold true in the case where the expansion of the function $f_1(t)$ does not contain terms with cos2nt and sin2nt. In this case the quadratic equation for α_1 will have a double root (equal to zero) which reduces to zero all the minors of the determinant of the system of equations (3.14). These equations do not therefore determine any relation between A_o and B_o. It must not however be supposed that A_o and B_o can be taken entirely arbitrarily. These magnitudes, as in the case $\beta^2 + \gamma^2 \neq 0$, are in general connected with each other, but this relation is established in considering the succeeding approximations.

In fact, let us consider the equation for x_1. The right hand side of this equation contains the multiplier x_o. Hence the general solution of this equation, which in accordance with the choice of α_1 will be periodic, has the form

$$x_1 = A_1 \cos nt + B_1 \sin nt + A_0 \varphi_1(t) + B_0 \psi_1(t),$$

where φ_1 and ψ_1 are periodic functions and A_1 and B_1 arbitrary constants.

Let us now consider the equation for x_2 and equate to zero the coefficients of cos nt and sin nt in the right hand side of this equation. The equations thus obtained will necessarily be homogeneous with respect to A_o and B_o and will not contain A_1 and B_1. These equations will have the form

$$\left.\begin{aligned}(P-\alpha_2) A_0+QB_0=0,\\ RA_0+(S-\alpha_2) B_0=0,\end{aligned}\right\} \tag{3.16}$$

where P, Q, R, S are certain arbitrary constants. In fact, since $\alpha_1 = 0$ the only term in the right hand side of the equation for x_2 not containing A_o and B_o will be

$$-n^2 f_1 (A_1 \cos nt + B_1 \sin nt),$$

and this term by assumption will not contain either cos nt or sin nt.

Equating the determinant of equations (3.16) to zero we obtain a quadratic equation for α_2. If this equation has simple roots there is obtained for each of them an entirely definite series for λ^2, exactly as in the case $\beta^2 + \gamma^2 \neq 0$. We can here put either $A_1 = 0$ or $B_1 = 0$ depending on which of the magnitudes A_o and B_o determined by equations (3.16) is known to be different from zero. The second of these constants together with α_3 is determined from the condition of periodicity of x_3. The equations for these constants will be found linear and will give entirely definite values for them. The further approximations are computed in similar fashion. The computations will here be quite the same as in the case $\beta^2 + \gamma^2 \neq 0$. The difference will lie only in that the constants A_k or B_k entering the k-th approximation will be determined from the condition of the periodicity, not of the (k+1)-th but of the (k+2)-th approximation.

If it turns out that the equation for α_2 also has multiple roots the investigation becomes more complicated. We shall not here consider these more complicated cases in the general form since in each particular individual case their investigation presents no difficulties. However diverse these special cases may be we can on the basis of the above considerations be sure that there will always be obtained at least two formal expansions for the magnitude λ^2 of interest to us. It can be shown (we shall not here dwell on the proof) that no more than two such expansions are ever obtained and that consequently they will be the required expansions and will converge. It may of course happen, as follows from the general theory, that the two expansions coincide. The character of the computations in all cases differs little from the more simple cases considered above and is sufficiently well clarified with the aid of the examples given in the following section.

4. Examples of the Application of the Method of the Preceding Section

We shall consider several examples that illustrate the method of the preceeding section.

EXAMPLE 1. Let us determine the first region of instability (i.e. corresponding to n = 1) for the equation

$$\frac{d^2x}{dt^2}+\lambda^2[1+(a_1\cos 2t+a_2\cos 4t)\mu+$$
$$+(a_3\cos 2t+a_4\cos 4t+a_5\cos 6t+a_6\cos 8t)\mu^2+\dots]x=0, \quad (4.1)$$

where a_i are constants. For this purpose we put in equation (4.1)

$$\lambda^2=1+\alpha_1\mu+\alpha_2\mu^2+\dots \quad (4.2)$$

and try to satisfy it formally by a series of the form

$$x=A_0\cos t+B_0\sin t+\mu x_1(t)+\mu^2 x_2(t)+\dots=$$
$$=x_0+\mu x_1(t)+\mu^2 x_2(t)+\dots,$$

where A_0 and B_0 are constants and x_i are periodic functions of t (of period 2π). For x_1 and x_2 we obtain the following equations:

$$\frac{d^2x_1}{dt^2}+x_1=-(a_1\cos 2t+a_2\cos 4t)x_0-\alpha_1 x_0=$$
$$=-\left(\frac{a_1}{2}+\alpha_1\right)A_0\cos t+\left(\frac{a_1}{2}-\alpha_1\right)B_0\sin t-$$
$$-\frac{1}{2}(a_1+a_2)A_0\cos 3t-\frac{1}{2}(a_1-a_2)B_0\sin 3t-$$
$$-\frac{A_0a_2}{2}\cos 5t-\frac{B_0a_2}{2}\sin 5t,$$

$$\frac{d^2x_2}{dt^2}+x_2=-(a_1\cos 2t+a_2\cos 4t)x_1-$$
$$-(a_3\cos 2t+a_4\cos 4t+a_5\cos 6t+a_6\cos 8t)x_1-$$
$$-\alpha_1 x_0-\alpha_1(a_1\cos 2t+a_2\cos 4t)x_0-\alpha_2 x_0.$$

From the condition of periodicity of the function x_1 we find:

$$\left(\frac{a_1}{2}+\alpha_1\right)A_0=0, \quad \left(\frac{a_1}{2}-\alpha_1\right)B_0=0. \tag{4.3}$$

Equating the determinant of this system to zero we obtain for α_1 the two different solutions

$$\alpha_1=-\frac{a_1}{2}, \quad \alpha_1=\frac{a_1}{2}.$$

Each of these solutions gives the start of the series (4.2). By considering both of these solutions we obtain two series (4.2) which, by what was proven, converge and determine the boundaries of the region of instability.

Let us first take $\alpha_1 = - a_1/2$. In this case the equations (4.3) give for B_o a value equal to zero. The magnitude A_o, remaining arbitrary, we can according to the general theory take equal to unity. For these assumptions the equation for x_1 assumes the form

$$\frac{d^2x_1}{dt^2}+x_1=-\frac{a_1+a_2}{2}\cos 3t-\frac{a_2}{2}\cos 5t,$$

whence we obtain

$$x_1=A_1\cos t+B_1\sin t+\frac{a_1+a_2}{16}\cos 3t+\frac{a_2}{48}\cos 5t. \tag{4.4}$$

Since $x_o(0) = A_o \neq 0$, we can, according to the general theory, put $A_1 = 0$. As to the constant B_1, it is determined together with α_2 from the condition of periodicity of x_2. This condition gives:

$$B_1=0, \quad \alpha_2=-\frac{(a_1+a_2)^2}{32}-\frac{a_2^2}{96}+\frac{a_1^2}{4}-\frac{a_3}{2}.$$

The further approximations can be obtained in this way but we restrict ourselves to the first two.

Let us now put $\alpha_2 = a_1/2$. In this case from equations (4.3) we find $A_o = 0$, $B_o = 1$. Since now $A_o = 0$ but $B_o \neq 0$, we can not put $A_1 = 0$ in the expression (4.4) for x_1 but can put $B_1 = 0$. If the found values of x_o, x_1 and α_1 are now substituted in the equation for x_2 then equating the coefficients of cos t and sin t to zero we shall have

$$A_1 = 0,$$
$$\alpha_2 = -\frac{(a_2 - a_1)^2}{32} + \frac{a_1^2}{4} - \frac{a_2^2}{96} + \frac{a_3}{2}.$$

Thus, the first region of instability is determined by the inequalities

$$1 - \frac{a_1}{2}\mu + \frac{1}{96}(21a_1^2 - 4a_2^2 - 6a_1a_2 - 48a_3)\mu^2 + \ldots \leqslant$$
$$\leqslant \lambda^2 \leqslant 1 + \frac{a_1}{2}\mu + \frac{1}{96}(21a_1^2 - 4a_2^2 + 6a_1a_2 + 48a_3)\mu^2 + \ldots$$

EXAMPLE 2. Let us determine the second region of instability for the so-called Mathieu equation

$$\frac{d^2x}{dt^2} + \lambda^2(1 + \mu\cos 2t)\,x = 0. \tag{4.5}$$

To determine the region of instability near $= 2$ we set

$$\lambda^2 = 4 + \alpha_1\mu + \alpha_2\mu^2 + \ldots$$

and try to satisfy equation (4.5) by the series

$$x = x_0 + \mu x_1 + \mu^2 x_2 + \ldots = A_0\cos 2t + B_0\sin 2t + \mu x_1 + \mu^2 x_2 + \ldots$$

with periodic coefficients (of period π). We have

$$\frac{d^2x_1}{dt^2} + 4x_1 = -4x_0\cos 2t - \alpha_1 x_0 =$$
$$= -2A_0 - 2A_0\cos 4t - 2B_0\sin 4t - \alpha_1(A_0\cos 2t + B_0\sin 2t),$$
$$\frac{d^2x_2}{dt^2} + 4x_2 = -4x_1\cos 2t - \alpha_2 x_0,$$
$$\frac{d^2x_3}{dt^2} + 4x_3 = -4x_2\cos 2t - \alpha_2 x_1 - \alpha_2 x_0\cos 2t - \alpha_3 x_0 - \alpha_1 x_2 - \alpha_1 x_1\cos 2t.$$

Equating the coefficients of cos2t and sin2t in the equation for x_1 to zero we find:

$$\alpha_1 = 0, \tag{4.6}$$

the magnitudes A_0 and B_0 remaining arbitrary. We here encounter precisely the case remarked on in the preceding section where the magnitude $\beta^2 + \gamma^2$ there introduced becomes zero. As we there pointed out, the magnitudes A_0 and B_0 will, in general, be connected with each other but this connection is established in considering the succeeding approximations. We shall compute these approximations. On the basis of (4.6) we have:

$$x_1 = -\frac{A_0}{2} + \frac{A_0}{6}\cos 4t + \frac{B_0}{6}\sin 4t + A_1\cos 2t + B_1\sin 2t, \tag{4.7}$$

where A_1 and B_1 are arbitrary constants. Since we do not know yet which of the magnitudes A_0 and B_0 is known to be different from zero we cannot as yet take one of the constants A_1 and B_1 equal to zero. Substituting x_0 and x_1 in the equation for x_2 we obtain:

$$\frac{d^2x_2}{dt^2} + 4x_2 = \left(\frac{5}{3} - \alpha_2\right)A_0\cos 2t - \left(\frac{1}{3} + \alpha_2\right)B_0\sin 2t -$$
$$- 2A_1\cos 4t - 2B_1\sin 4t - \frac{A_0}{3}\cos 6t - \frac{B_0}{3}\sin 6t - 2A_1.$$

The conditions of periodicity give:

$$\left(\frac{5}{3} - \alpha_2\right)A_0 = 0, \quad \left(\frac{1}{3} + \alpha_2\right)B_0 = 0,$$

and therefore two different values are obtained for α_2. We thus have two variants. For the first variant

$$\alpha_2 = \frac{5}{3}, \quad A_0 = 1, \quad B_0 = 0,$$

and for the second variant

$$\alpha_2 = -\frac{1}{3}, \quad A_0 = 0, \quad B_0 = 1.$$

Let us first consider the first variant. Since in this variant $A_0 \neq 0$, we can in (4.7) set A_1 equal to zero. After this we obtain:

$$x_2 = \frac{1}{6} B_1 \sin 4t + \frac{1}{96} \cos 6t + B_2 \sin 2t,$$

where B_2 is an arbitrary constant. Substituting the obtained approximations in the equation for x_3 and writing out the conditions of periodicity of this function we obtain two linear nonhomogeneous equations for α_3 and B_1. These equations have the form

$$B_1 = 0, \quad \alpha_3 = 0.$$

In analogous manner the further approximations are computed. In contrast to the preceding example, corresponding to the case $\beta^2 + \gamma^2 \neq 0$, the constants B_k entering x_k are determined not from the condition of periodicity of the functions x_{k+1} but from the condition of the periodicity of the functions x_{k+2}.

Let us now consider the second variant. In this case

$$x_1 = \frac{1}{6} \sin 4t + A_1 \cos 2t,$$

$$x_2 = -\frac{1}{2} A_1 + \frac{1}{6} A_1 \cos 4t + \frac{1}{96} \sin 6t + A_2 \cos 2t$$

and the conditions of periodicity for x_3 give:

$$\alpha_3 = 0, \quad A_1 = 0.$$

Thus, with an accuracy up to magnitudes of the third order with respect to μ, the second region of instability for equation (4.5) is determined by the inequalities

$$4 - \frac{1}{3}\mu^2 + \ldots \leqslant \lambda^2 \leqslant 4 + \frac{5}{3}\mu^2 + \ldots$$

EXAMPLE 3. Let us determine the third region of instability for the Mathieu equation (4.5). Setting

$$\lambda^2 = 9 + \alpha_1\mu + \alpha_2\mu^2 + \ldots,$$

$$x = x_0 + \mu x_1 + \mu^2 x_2 + \ldots =$$

$$= A_0 \cos 3t + B_0 \sin 3t + \mu x_1 + \mu^2 x_2 + \ldots,$$

we shall now have:

$$\frac{d^2x_1}{dt^2}+9x_1=-9x_0\cos 2t-\alpha_1x_0,$$

$$\frac{d^2x_2}{dt^2}+9x_2=-9x_1\cos 2t-\alpha_1(x_1+x_0\cos 2t)-\alpha_2x_0,$$

$$\frac{d^2x_3}{dt^2}+9x_3=-9x_2\cos 2t-\alpha_1(x_2+x_1\cos 2t)-\alpha_2(x_1+x_0\cos 2t)-\alpha_3x_0,$$

$$\frac{d^2x_4}{dt^2}+9x_4=-9x_3\cos 2t-\alpha_1(x_3+x_2\cos 2t)-$$
$$-\alpha_2(x_2+x_1\cos 2t)-\alpha_3(x_1+x_0\cos 2t)-\alpha_4x_0.$$

The condition of periodicity of x_1 gives:

$$\alpha_1=0,$$

A_0 and B_0 remaining undetermined. For x_1 we obtain the expression

$$x_1=-\frac{9}{16}A_0\cos t-\frac{9}{16}B_0\sin t+\frac{9}{32}A_0\cos 5t+$$
$$+\frac{9}{32}B_0\sin 5t+A_1\cos 3t+B_1\sin 3t,$$

where A_1 and B_1 are arbitrary constants. As in the preceding example we cannot as yet equate any one of these constants equal to zero. Substituting the obtained values of α_1 and x_1 in the equation for x_2 we obtain:

$$\frac{d^2x_2}{dt^2}+9x_2=\frac{9}{2}\left(\frac{9}{16}A_0-A_1\right)\cos t-\frac{9}{2}\left(\frac{9}{16}B_0+B_1\right)\sin t-$$
$$-\frac{9}{2}A_1\cos 5t-\frac{9}{2}B_1\sin 5t-\frac{81}{64}A_0\cos 7t-\frac{81}{64}B_0\sin 7t+$$
$$+\left(\frac{81}{64}-\alpha_2\right)A_0\cos 3t+\left(\frac{81}{64}-\alpha_2\right)B_0\sin 3t, \tag{4.8}$$

and therefore the conditions of periodicity of x_2 have the form

$$\left(\frac{81}{64}-\alpha_2\right)A_0=0, \quad \left(\frac{81}{64}-\alpha_2\right)B_0=0.$$

The solution obtained from this for α_2

$$\alpha_2=\frac{81}{64}$$

will again be double, with A_0 and B_0 as before remaining undetermined. From (4.8) we find:

$$x_2 = \frac{9}{16}\left(\frac{9}{16}A_0 - A_1\right)\cos t - \frac{9}{16}\left(\frac{9}{16}B_0 + B_1\right)\sin t +$$
$$+\frac{9}{32}A_1\cos 5t + \frac{9}{32}B_1\sin 5t + \frac{81}{2560}A_0\cos 7t + \frac{81}{2560}B_0\sin 7t +$$
$$+A_2\cos 3t + B_2\sin 3t,$$

where A_2 and B_2 are constants remaining as yet undetermined.

The equation for x_3 after substituting the found values for x_1, x_2, α_1 and α_2 assume the form

$$\frac{d^2x_3}{dt^2} + 9x_3 = -\frac{9}{2}\left(\frac{153}{512}A_0 - \frac{9}{16}A_1 + A_2\right)\cos t -$$
$$-\frac{9}{2}\left(\frac{153}{512}B_0 + \frac{9}{16}B_1 + B_2\right)\sin t - \left(\alpha_3 + \frac{729}{512}\right)A_0\cos 3t +$$
$$+\left(\frac{729}{512} - \alpha_3\right)B_0\sin 3t - \frac{5103}{10\,240}A_0\cos 5t - \frac{5103}{10\,240}\sin 5t -$$
$$-\frac{9}{2}A_2\cos 5t - \frac{9}{2}B_2\sin 5t + a_7\cos 7t + b_7\sin 7t +$$
$$+a_9\cos 9t + b_9\sin 9t,$$

where a_7, b_7, a_9, b_9 are certain entirely definite coefficients which we need not write out. The conditions of periodicity of x_3 therefore have the form

$$\left(\alpha_3 + \frac{729}{512}\right)A_0 = 0, \qquad \left(\alpha_3 - \frac{729}{512}\right)B_0 = 0.$$

These conditions give two different values for α_3 and the further computations must be conducted in two variants. For the first variant we have:

$$\alpha_3 = -\frac{729}{512}, \qquad B_0 = 0, \qquad A_0 = 1.$$

Now we can put $A_1 = A_2 = 0$ and in the further approximations not introduce terms with $\cos 3t$ in the expressions for x_k. For these assumptions we shall have

$$x_3 = -\frac{1377}{8192}\cos t - \frac{9}{16}\left(\frac{9}{16}B_1 + B_2\right)\sin t + \\ + \frac{5103}{163\,840}\cos 5t + \frac{9}{32}B_2 \sin 5t - \frac{a_7}{40}\cos 7t - \frac{b_7}{40}\sin 7t - \\ - \frac{a_9}{72}\cos 9t - \frac{b_9}{72}\sin 9t + B_3 \sin 3t$$

where B_3 is an undetermined constant. Substituting the found approximations in the equation for x_4 and setting up the conditions of periodicity of this function we obtain two linear nonhomogeneous equations for the determination of B_1 and α_4. We obtain:

$$B_1 = 0, \quad \alpha_4 = -\frac{235\,467}{327\,680}.$$

If the computation of the approximations is continued there are successively determined the constants α_5, α_6,... Each constant α_k is determined simultaneously with the constant B_{k-2} from the conditions of the periodicity of the function x_k, which will give for these two magnitudes a solvable system of linear nonhomogeneous equations.

For the second variant we shall have

$$\alpha_3 = \frac{729}{512}, \quad B_0 = 1, \quad A_0 = 0$$

and all further computations will be the same as for the first variant with the difference only that it is now necessary to set all magnitudes B_k equal to zero and to determine the magnitudes A_k from the conditions of periodicity of the functions x_{k+2}. For the magnitudes A_1 and α_4 we obtain the following values:

$$A_1 = 0, \quad \alpha_4 = \frac{260\,953}{327\,680}.$$

Thus, the required region of instability is determined by the following inqualities:

$$9+\frac{81}{64}\mu^2-\frac{729}{512}\mu^3-\frac{235\,467}{327\,680}\mu^4+\ldots\leqslant\lambda^2\leqslant$$
$$\leqslant 9+\frac{81}{64}\mu^2+\frac{729}{512}\mu^3+\frac{260\,953}{327\,680}\mu^4+\ldots \quad (4.9)$$

EXAMPLE 4. The computations presented above make it possible to determine the region of instability for all the examples of parametric resonance given in sec. 1. Let us consider for example the problem of the parametric excitation of oscillations in an electrical circuit. The problem reduces to the determination of the regions of instability for the equation (1.3) which, by setting $\tau = 2\tau_1$, we write in the following form:

$$\left.\begin{aligned} &\frac{d^2x}{d\tau_1^2}+\frac{4L-R^2C_0}{L^2C_0\omega^2}(1+\mu\cos 2\tau_1)\,x=0,\\ &\mu=\frac{4Lm}{4L-R^2C_0}. \end{aligned}\right\} \quad (4.10)$$

The region of values of the frequency ω of the change of capacitance, corresponding to the first region of instability, can be indicated at once on the basis of the computations for the equation (4.1). Setting in them $a_1 = 1$, $a_2 = a_3 = a_4 = a_5 = a_6 = 0$ we can write

$$1-\frac{1}{2}\mu+\frac{7}{32}\mu^2+\ldots\leqslant\frac{4L-R^2C_0}{L^2C_0\omega^2}\leqslant 1+\frac{1}{2}\mu+\frac{7}{32}\mu^2+\ldots \quad (4.11)$$

For the second region of instability, on the basis of the results obtained for example 2, we shall have:

$$4-\frac{1}{3}\mu^2+\ldots\leqslant\frac{4L-R^2C_0}{L^2C_0\omega^2}\leqslant 4+\frac{5}{3}\mu^2+\ldots \quad (4.12)$$

From inequalities (4.11) and(4.12) it is seen that in order that the above mentioned regions of instability exist it is necessary first of all that the condition be satisfied

$$4L-R^2C_0>0.$$

We shall assume that this condition is satisfied. but in satisfying this condition it cannot yet be assured that in the above mentioned regions unstable oscillations actually arise in the electrical circuit. The reason is that the variable x figuring in equation (1.3) is an auxiliary one. For the current q we have:

$$q = e^{-\frac{R}{2L}t} x.$$

Consequently, on the boundaries of the regions (4.11) and (4.12), where the real parts of the characteristic exponents of equation (4.10) are equal to zero, the oscillations of the current will be damped. If the magnitude $R/2L$ is sufficiently small, namely, less than the greatest value of the real parts of the characteristic exponents of equation (4.10) in the regions (4.11) and (4.12), regions of instability will exist for the current q, situated within the regions (4.11) and (4.12). If this condition with regard to $R/2L$ is not satisfied the current oscillations in regions (4.11) and (4.12) will be damped.

We may remark that the width of the second region of instability (4.12) is of the order of smallness of μ^2. As a result, unstable current oscillations can arise in it only for very small values of $R/2L$. The values of this magnitude must be still smaller for the possibility of unstable oscillations within the third region of instability since the width of the latter, on the basis of (4.9) is of the order of smallness of μ^3. Hence for practical purposes the first region of instability is of the most significance. Similar circumstances apply also in other cases of parametric resonance. Since in practice resistances are unavoidable, the regions of instability corresponding to $n > 1$ generally have no practical significance.

5. Problem of Parametric Resonance for Canonical Systems with Many Degrees of Freedom

Let us now consider the problem of parametric resonance for mechanical systems with many degrees of freedom. We shall assume that these systems are described by linear canonical equations of the form

$$\frac{dy_k}{d\tau} = \frac{\partial \overline{H}}{\partial z_k}, \quad \frac{dz_k}{d\tau} = -\frac{\partial \overline{H}}{\partial y_k} \qquad (k = 1, \ldots, m), \tag{5.1}$$

where $H = H(\tau, y_1, \ldots, y_m, z_1, \ldots, z_m, \mu)$ is a quadratic form of the variables $y_1, \ldots, y_m, z_1, \ldots, z_m$ whose coefficients are continuous periodic functions of τ of period

$T = 2\pi/\omega$ and analytic functions of the parameter μ for its sufficiently small values.

Particular cases of the systems (5.1) are the systems

$$\frac{d^2 z_k}{d\tau^2} = q_{k1} z_1 + \ldots + q_{km} z_m, \tag{5.2}$$

provided the conditions $q_{ki} = q_{ik}$ are satisfied. In fact, the system (5.1) for

$$2\overline{H} = -\sum_{\alpha=1}^{m} y_\alpha^2 + \sum_{\alpha,\beta=1}^{m} q_{\alpha\beta} z_\alpha z_\beta$$

has the form

$$\frac{dy_k}{d\tau} = q_{k1} z_1 + \ldots + q_{km} z_m, \qquad \frac{dz_k}{d\tau} = y_k$$

and therefore coincides with the system (5.2). For $m = 1$ the system (5.2) reduces to one equation of the second order the particular case of which was considered in detail in the preceding sections.

Let us put in equations (5.1)

$$\left.\begin{array}{c} t = \frac{1}{2}\omega\tau = \frac{\pi}{T}\tau; \qquad \lambda = \frac{2}{\omega}; \\ H(t, y_1, \ldots, z_m, \mu) = \overline{H}\left(\frac{2t}{\omega}, y_1, \ldots, z_m, \mu\right), \end{array}\right\} \tag{5.3}$$

after which they will assume the form

$$\frac{dy_k}{dt} = \lambda \frac{\partial H}{\partial z_k}, \qquad \frac{dz_k}{dt} = -\lambda \frac{\partial H}{\partial y_k} \qquad (k = 1, \ldots, m), \tag{5.4}$$

where $H(t, y_1, \ldots, z_m, \mu)$ will have the period π with respect to t.

The problem of parametric resonance for the system (5.4) consists in determining those values of the frequency ω or of the values of the parameter λ for which the characteristic equation of this system has roots with moduli greater than unity. This problem for certain restrictions on the function H was considered by M. G. Krein [1], who established several important general theorems. In particular, Krein succeeded in extending to systems of the form (5.4) certain fundamental theorems of Lyapunov on a second order equation. We shall here consider the problem for more particular assumptions, which will permit us to obtain certain more concrete results and also procedures for practically determining the regions of stability and instability. We shall assume, as in the case of one equation of the second order, that the system of equations (5.4) differs little from a system of equations with constant coefficients. In other words, we shall assume that H is of the form

$$H = H_0 + \mu H_1 + \mu^2 H_2 + \ldots \doteq H_0 + \mu H^*, \tag{5.5}$$

where H_0, H_1,...are quadratic forms of the variables y_i, z_i, the coefficients of the form H_0 being constants, while the coefficients of the forms H_1, H_2,... are periodic functions of t of period π.

In what follows a fundamental part will be played by the fact that for the system (5.4) the theorem of Lyapunov, proven in sec. 8 of chapter II, holds according to which for each root ρ of the characteristic equation of this system there is a root $1/\rho$ of the same multiplicity.

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[1] Krein M. G., Osnovnye polozheniya λ-zon ustoichivosti kanonicheskoi sistemy lineinykh differentsial'nykh uravnenii s periodicheskimi koeffitsientami (Fundamental Aspects of the λ Zones of Stability of a System of Linear Differential Equations with Periodic Coefficients). Collection of Articles, "Pamayati Aleksandra Aleksandrovicha Andronova" (In Memory of Alexander Alexandrovich Andronov), Izd-vo AN SSSR, 1955.

We can therefore assume that the characteristic exponents of systems of the form (5.4) are pairwise equal and opposite in sign. From this, in particular, it follows that all the roots of the fundamental equation of the system of linear equations with constant coefficients

$$\frac{dy_k}{dt} = \frac{\partial H_0}{\partial z_k}, \qquad \frac{dz_k}{dt} = -\frac{\partial H_0}{\partial y_k} \tag{5.6}$$

will be pairwise equal and opposite in sign. But then, evidently, the same will be true for any values of λ for the fundamental equation of the system

$$\frac{dy_k}{dt} = \lambda \frac{\partial H_0}{\partial z_k}, \qquad \frac{dz_k}{dt} = -\lambda \frac{\partial H_0}{\partial y_k}. \tag{5.7}$$

Having established this, let us assume that the fundamental equation of the system (5.6) has roots with real parts different from zero. Then this equation and together with it the fundamental equation of the system (5.7) will have roots with positive real parts. As a result, for sufficiently small μ, the system (5.4) will for any values of λ have characteristic exponents with positive real parts and for it instability will take place. This case is of no interest to us.

We shall therefore assume that the fundamental equation of the system (5.6) has only purely imaginary roots. For this, as is known, it is sufficient[1] that the form H_o be of determinate sign. We shall assume that this condition is actually satisfied and for definiteness take $H_o > 0$. For sufficiently small μ the form H will then likewise be positive definite, as we shall also assume.

Let $\pm \omega_1 i, \ldots, \pm \omega_m i$ be the roots of the fundamental equation of the system (5.6), so that the magnitudes ω_j are its proper frequencies. We shall assume that all the magnitudes $m_1 \omega_1 + \ldots + m_m \omega_m$ for any integral (positive and negative) $m_1, \ldots, m_m$ are different from zero. From this it follows, in particular, that all the magnitudes ω_j are different.

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[1] This condition is not however necessary.

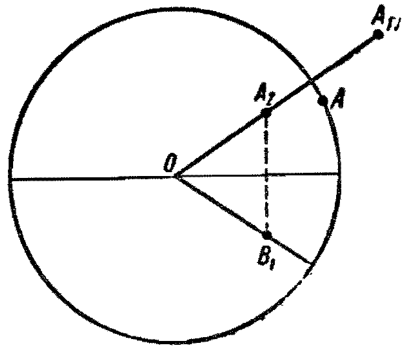

Fig. 29

Let us now pass on to the determination of those values of λ near which can be found regions of instability. For this purpose let us assume that there exists a number $\lambda^*(\mu)$ such that for $\lambda = \lambda^*$ all roots of the characteristic equation of the system (5.4) have moduli equal to unity while for $\lambda^* < \lambda < \lambda^* + \alpha(\mu)$ certain of these roots have moduli different from unity. Here $\alpha(\mu)$ is a positive number which evidently approaches zero as $\mu \to 0$. Let $\rho_1, \ldots, \rho_{2m}$ be the roots of the characteristic equation of system (5.4). We shall construct these roots in the plane of the complex variable ρ. Then for $\lambda = \lambda^*$ all these roots will be situated on a circle of radius 1 while for $\lambda^* < \lambda < \lambda^* + \alpha$ certain of the roots will not lie on this circle. The number of the latter will by the theorem of Lyapunov necessarily be even and half of them will lie within the circle $|\rho| = 1$ and half outside this circle. Let $\lambda^* < \lambda < \lambda^* + \alpha$ and the point A_1 (fig 29) represent one of the roots outside the circle; the point B_1 that corresponds to it by Lyapunov's theorem on the roots lies within the circle. We here have $OA_1 \cdot OB_1 = 1$ and the rays OA_1 and OB_1 are symmetrical with respect to the real axis. But since the coefficients of the characteristic equation of system (5.4) are real, all its complex roots will be pairwise conjugate and there exists another root, represented by the point A_2, symmetrical to B_1 with respect to the real axis.

Now let $\lambda \to \lambda^*$. The points A_1 and A_2 will then approach the circle and coalesce in a certain point A.

Hence, for $\lambda = \lambda^*$ the characteristic equation of the system (5.4) will have at least one multiple root with modulus equal to one.

We obtain the same result if for $\lambda^* - \alpha < \lambda < \lambda^*$ certain of the roots have a modulus greater than one. Thus, if for the system (5.4) regions of instability exist, they are necessarily bounded by the same values $\lambda = \lambda^*(\mu)$ for which the characteristic equation of this system has multiple roots with moduli equal to one.

Now let $a_1, \ldots, a_{2m}$ be the characteristic exponents of the system (5.4) so that

$$\rho_j = e^{\pi a_j} \qquad (j = 1, \ldots, 2m). \tag{5.8}$$

Let us assume that for $\lambda = \lambda^*(\mu)$ the equality $\rho_j = \rho_i$ holds. It then follows from (5.8) that for $\lambda = \lambda^*(\mu)$ the relation must be satisfied

$$a_j = a_i \pm 2N\sqrt{-1},$$

where N is an integer. Putting here $\mu = 0$ we obtain

$$\begin{gathered} \lambda_0(\omega_j \pm \omega_i) = \pm 2N, \qquad \lambda_0 = \lambda^*(0) \\ (i, j = 1, \ldots, m;\ N = 1, 2, \ldots), \end{gathered} \tag{5.9}$$

since the characteristic exponents of the system (5.7) are the roots $\pm\lambda\omega_1 i, \ldots, \pm\lambda\omega_m i$ of its fundamental equation. From this we arrive at the following result:

The regions of instability of the system (5.4) can be situated only near those values of $\lambda = \lambda_o$ for which relations (5.9) are satisfied.

Replacing the magnitude λ by its value from (5.3) we can write this condition also in the form:

$$\frac{\omega_i \pm \omega_j}{N} \approx \pm\omega \qquad (i, j = 1, \ldots, m;\ N = 1, 2, \ldots), \tag{5.10}$$

where ω is the frequency of the excitation. Thus, parametric resonance can arise only for that frequency of excitation for which relations (5.10) are satisfied.

In particular, parametric resonance is possible for

$$\omega \approx \frac{2\omega_i}{N} \qquad (i=1, \ldots, m;\ N=1, 2, \ldots). \tag{5.11}$$

In this case we shall speak of SIMPLE parametric resonance. Parametric resonance for which relations (5.9) or (5.10) are satisfied with $i \neq j$ we shall call COMBINATION resonance.

We may remark that for our conditions the characteristic equation of the system (5.4) cannot for $\mu = 0$ have roots the multiplicity of which exceeds 2.

In fact, if for the above mentioned equation for $\mu = 0$ there existed a triple root then for certain values of i, j, r relations would be satisfied of the form

$$\lambda\omega_i = \pm\lambda\omega_j \pm 2N = \pm\lambda\omega_r + 2N',$$

where N, N' are integers and therefore the nagnitudes $\omega_i, \omega_j, \omega_r$ would be connected by the relation

$$N(\omega_j \pm \omega_r) = \pm N'(\omega_i \pm \omega_j).$$

which contradicts the assumption.

6. Regions of Simple Parametric Resonance for Canonical Systems with Many Degrees of Freedom

Let us consider first simple parametric resonance, i.e. that for which (5.11) is satisfied. It is possible only near values of $\lambda = \lambda_o$ satisfying relations of the form

$$\lambda_0\omega_i = N \qquad (i=1, \ldots, m;\ N=\pm 1, \pm 2, \ldots). \tag{6.1}$$

We shall clarify whether there actually exists a region of instability near $\lambda = \lambda_o$ (i.e. contracting for $\mu = 0$ to the point $\lambda = \lambda_o$) and shall try to determine its boundaries. If such region exists it should, by what has been proven in the preceding section, be bounded by those values of $\lambda = \lambda^*(\mu)$ for which the characteristic equation of the system (5.4) has a double root $\rho = \rho^*(\mu)$

with modulus equal to 1. We must here have $\lambda^*(0) = \lambda_0$ so that we can put:

$$\lambda^* = \lambda_0 + \alpha(\mu) = \frac{N}{\omega_i} + \alpha(\mu), \tag{6.2}$$

where $\alpha(0) = 0$.

We shall show first of all that $\rho = \pm 1$, for which it is sufficient to show that ρ^* is a real magnitude. In fact, if the magnitude ρ^* were complex, the magnitude $\bar{\rho}^*$, its conjugate, would likewise be a root of the characteristic equation. But

$$\left.\begin{aligned} \rho^*(0) &= e^{\pi\lambda_0\omega_i\sqrt{-1}} = e^{\pi N\sqrt{-1}} = (-1)^N, \\ \bar{\rho}^*(0) &= e^{-\pi N\sqrt{-1}} = (-1)^N \end{aligned}\right\} \tag{6.3}$$

and therefore $\rho^*(0) = \bar{\rho}^*(0)$. Since each of the roots $\rho^*(\mu)$ and $\bar{\rho}^*(\mu)$ is double it follows that the characteristic equation of system (5.4) has for $\mu = 0$ roots the multiplicity of which is not less than four, which, as was shown at the end of the preceding section, is impossible.

Thus $\rho^* = \pm 1$. As follows from (6.3), we can here write more definitely

$$\rho^* = (-1)^N. \tag{6.4}$$

From this it follows that the required region of instability is bounded by those values of $\lambda = \lambda^*$ for which the characteristic equation of the system (5.4) has double roots of the form (6.4). But if this equation has a root of the form (6.4) then according to the theorem of Lyapunov this root must necessarily be double. Hence, if we denote by $D(\rho, \lambda, \mu)$ the characteristic determinant of the system (5.4), the equation determining the magnitude λ^* will have the form

$$D((-1)^N, \lambda^*, \mu) = 0. \tag{6.5}$$

Let us examine this equation more closely. Its left hand side is an analytic function of λ^* and μ. Moreover we have:

$$D(\rho, \lambda, \mu) = \prod_{j=1}^{m} (\rho - e^{\lambda\omega_j\pi i})(\rho - e^{-\lambda\omega_j\pi i}) + \mu F^*(\rho, \lambda, \mu),$$

where $F^*(\rho, \lambda, \mu)$ is an analytic function of λ and μ. This is a consequence of the fact that for $\mu = 0$ the roots of the characteristic equation of the system (5.4) are the magnitudes $e^{\pm\lambda\omega_j\pi i}$ Whence on the basis of (6.2) and (6.5) we find the following equation for $\alpha(\mu)$:

$$2^m(1-\cos\alpha\pi\omega_i)\prod_{j\neq i}^{1,m}[1-(-1)^N\cos(\lambda_0+\alpha)\pi\omega_j]+\mu F(\alpha, \mu)=0. \quad (6.6)$$

where F is an analytic function of α and μ. Equation (6.6) for $\mu = 0$ has the double root $\alpha = 0$. Hence on the basis of the implicit function theorem, which we already made use of in sec. 2, equation (6.6), for $\mu \neq 0$ but sufficiently small, admits two and only two solutions which are analytic functions either of the magnitude μ or of the magnitude $\sqrt{\mu}$. We shall show that the first case holds true, i.e. that both roots $\alpha(\mu)$ are analytic functions of the magnitude μ. For this we note that if these roots are analytic functions of $\sqrt{\mu}$ they will be complex either for $\mu > 0$ or for $\mu < 0$. Hence it is sufficient for us to show that the two roots $\alpha(\mu)$ are real both for $\mu > 0$ and for $\mu < 0$. This can be shown in the following manner.[1]

Let us assume that $\alpha(\mu)$, and therefore also λ^*, are complex. Since for $\lambda = \lambda^*$ the characteristic equation of the system (5.4) has roots of the form (6.4) this system has for $\lambda = \lambda^*$ a periodic solution of period 2π or π. Let the functions

$$y_k(t)=u_k^{(1)}(t)+iu_k^{(2)}(t), \qquad z_k(t)=v_k^{(1)}(t)+iv_k^{(2)}(t) \qquad (6.7)$$

determine this solution.

Denote by $\bar{\lambda}^*$, $\bar{x}_k$ and $\bar{y}_k$ the magnitudes complex conjugate to λ^*, x_k and y_k and let $\bar{H}=H(t, \bar{y}_1, \ldots, \bar{y}_m, \bar{z}_1, \ldots, \bar{z}_m)$. Then from the identities

$$\frac{dy_k(t)}{dt}\equiv\lambda^*\frac{\partial H}{\partial z_k}, \qquad \frac{dz_k(t)}{dt}\equiv-\lambda^*\frac{\partial H}{\partial y_k},$$
$$\frac{d\bar{y}_k}{dt}\equiv\bar{\lambda}^*\frac{\partial\bar{H}}{\partial z_k}, \qquad \frac{d\bar{z}_k}{dt}\equiv-\lambda^*\frac{\partial\bar{H}}{\partial y_k}$$

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[1] In the same way an analogous theorem is proven by M. G. Krein (see sec. 3 of the work cited on p. 397).

we find

$$\frac{d}{dt}\sum_{\alpha=1}^{m}(y_\alpha \bar{z}_\alpha - z_\alpha \bar{y}_\alpha) \equiv$$

$$\equiv \lambda^* \sum_{\alpha=1}^{m}\left(\frac{\partial H}{\partial z_\alpha}\bar{z}_\alpha + \frac{\partial H}{\partial y_\alpha}\bar{y}_\alpha\right) - \bar{\lambda}^* \sum_{\alpha=1}^{m}\left(\frac{\partial \bar{H}}{\partial \bar{y}_\alpha} y_\alpha + \frac{\partial \bar{H}}{\partial \bar{z}_\alpha} z_\alpha\right). \quad (6.8)$$

But for any quadratic form

$$W = \sum_{\alpha, \beta=1}^{n} a_{\alpha\beta}\, x_\alpha x_\beta$$

of the magnitudes $x_s = \xi_s + i\eta_s$ the identity holds

$$\sum_{\alpha=1}^{n} \frac{\partial W}{\partial x_\alpha}\bar{x}_\alpha \equiv 2 \sum_{\alpha, \beta=1}^{n} a_{\alpha\beta}(\xi_\alpha \xi_\beta + \eta_\alpha \eta_\beta) + 2i \sum_{\alpha, \beta=1}^{n} a_{\alpha\beta}(\eta_\alpha \xi_\beta - \eta_\beta \xi_\alpha) \equiv$$
$$\equiv 2W(\xi_1, \dots, \xi_n) + 2W(\eta_1, \dots, \eta_n).$$

Hence from (6.8) we obtain:

$$\frac{d}{dt}\sum_{\alpha=1}^{m}(y_\alpha \bar{z}_\alpha - z_\alpha \bar{y}_\alpha) =$$
$$= 2(\lambda^* - \bar{\lambda}^*)[H(t, u_1^{(1)}, \dots, v_m^{(1)}) + H(t, u_1^{(2)}, \dots, v_m^{(2)})],$$

whence

$$\Big|_0^{2\pi} \sum_{\alpha=1}^{m}(y_\alpha \bar{z}_\alpha - z_\alpha \bar{y}_\alpha) \equiv$$

$$\equiv 2(\lambda^* - \bar{\lambda}^*)\int_0^{2\pi}[H(t, u_1^{(1)}, \dots, v_m^{(1)}) + H(t, u_1^{(2)}, \dots, v_m^{(2)})]\,dt.$$

But the left hand side of this identity reduces to zero on account of the periodicity of the solution (6.7). Since moreover the integral on the right hand side is different from zero because of the definite sign of H we obtain $\lambda^* - \bar{\lambda}^* = 0$, which proves our proposition.

Thus, equation (6.6) always has two real roots and these roots are analytic functions of the parameter μ. The question now arises as to the practical computation of these roots and also whether they actually bound the region of instability. These problems can be solved in the following manner.

We note first of all that if near $\lambda = \lambda_0$ there actually is a region of instability it coincides with the interval $(\lambda_0 + \alpha^{(1)}, \lambda_0 + \alpha^{(2)})$, where $\alpha^{(1)}$ and $\alpha^{(2)}$ are the two roots of equation (6.6). The two regions outside this interval and extending to the nearest critical values of λ (which, as was shown, are near the values satisfying conditions (5.9)), will necessarily be regions of instability. This follows from the fact that the regions of instability contract to a point for $\mu = 0$.

We shall now seek to obtain the magnitude of interest to us λ^* in the form of the formal series

$$\lambda^* = \frac{N}{\omega_i} + \alpha_1\mu + \alpha_2\mu^2 + \dots \qquad (6.9)$$

We shall seek the coefficients of this series from the condition that the system (5.4) for $\lambda = \lambda^*$ has a periodic solution of period 2π (or even π, if N is an odd number). For convenience of writing we shall not in what follows explicitly indicate the canonical character of the system and denote the variables y_k, z_k by $x_1, \dots, x_n$, where $n = 2m$. The system (5.4) for $\lambda = \lambda^*$ can then be written in the following general form:

$$\frac{dx_s}{dt} = (\lambda_0 + \mu\alpha_1 + \dots) \sum_{\beta=1}^{n} (a_{s\beta} + \mu f_{s\beta}^{(1)}(t) + \dots) x_\beta \qquad (6.10)$$

$$(s = 1, \dots, n = 2m).$$

where $f_{s\beta}^{(p)}$ are certain periodic functions of time of period π. The fundamental equation of the system

$$\frac{dx_s^0}{dt} = \lambda_0 (a_{s1} x_1^0 + \dots + a_{sn} x_n^0) \qquad (6.11)$$

will have the roots $\pm\lambda_0\omega_1 i, \dots, \pm\lambda_0\omega_m i$, of which two and only two will, on the basis of the choice of λ_0, have the form $\pm$ Ni. Consequently, the system (6.11) admits two and only two periodic solutions of period 2π, which we denote by $\varphi_{s1}(t)$ and $\varphi_{s2}(t)$. We denote by $\psi_{s1}(t)$ and $\psi_{s2}(t)$ the periodic solutions of period 2π of the system conjugate to (6.11).

We shall show that the required periodic solution of system (6.10) can be assumed analytic with respect to μ. For this we consider the fundamental system of solutions $x_{sj}(t)$ of equations (6.10) satisfying the initial conditions.

$$x_{ss}(0)=1, \quad x_{sj}(0)=0 \qquad (s \neq j).$$

These solutions will be analytic functions of μ. Let, further, ρ be some root of the characteristic equation of the system (6.10) and $x_s(t)$ the solution corresponding to it of this system, i.e. the solution for which

$$x_s(t+\pi)=\rho x_s(t).$$

This solution is determined by the equations

$$x_s(t)=\beta_1 x_{s1}+\ldots+\beta_n x_{sn},$$

in which the magnitudes β_s, as was shown in sec. 4 of chapter III, are a solution of the homogeneous linear system

$$x_{s1}(\pi)\beta_1+\ldots+(x_{ss}(\pi)-\rho)\beta_s+\ldots+x_{sn}(\pi)\beta_n=0,$$

i.e. the corresponding minors of the characteristic determinant. Hence the magnitudes β_s, and together with them the magnitudes x_s, will be analytic functions of μ provided the magnitude ρ possesses this property. For the periodic solution of interest to us $\rho = \pm 1$, which shows that this solution will be analytic with respect to μ.

With this established, we shall seek the periodic solution of the system (6.10) in the form of the series

$$x_s = x_s^0 + \mu x_s^{(1)}(t) + \ldots = M_0\varphi_{s1} + N_0\varphi_{s2} + \mu x_s^{(1)}(t) + \ldots, \qquad (6.12)$$

where M_o and N_o are arbitrary constants and $x_s^{(p)}$ are periodic functions of t of period π. For the functions $x_s^{(1)}$ we have the equations

$$\frac{dx_s^{(1)}}{dt} = \lambda_0 \sum_{\beta=1}^{n} a_{s\beta} x_\beta^{(1)} + \sum_{\beta=1}^{n} (\alpha_1 a_{s\beta} + \lambda_0 f_{s\beta}^{(1)}(t)) (M_0 \varphi_{\beta 1} + N_0 \varphi_{\beta 2}), \quad (6.13)$$

while the conditions of periodicity of these functions give

$$(\alpha_1 A_{1i} + B_{1i}) M_0 + (\alpha_1 A_{2i} + B_{2i}) N_0 = 0 \qquad (i = 1,\ 2), \qquad (6.14)$$

where

$$A_{ji} = \int_0^{2\pi} \sum_{\beta,\,\gamma=1}^{n} a_{\gamma\beta} \varphi_{\beta j} \psi_{\gamma i}\, dt, \quad B_{ji} = \lambda_0 \int_0^{2\pi} \sum_{\beta,\,\gamma=1}^{n} f_{\gamma\beta}^{(1)} \varphi_{\beta j} \psi_{\gamma i}\, dt.$$

In order that this system admit a solution for M_o and N_o other than a trivial solution it is necessary and sufficient that α_1 be a root of the quadratic equation

$$\begin{aligned} D(\alpha_1) &= (\alpha_1 A_{11} + B_{11})(\alpha_1 A_{22} + B_{22}) - (\alpha_1 A_{21} + B_{21})(\alpha_1 A_{12} + B_{12}) = \\ &= A\alpha_1^2 + B\alpha_1 + C = 0. \end{aligned} \qquad (6.15)$$

We shall assume that this equation has simple roots. These roots will necessarily be real since, by what has been proven, the magnitude $\alpha(\mu)$ must be real. We shall equate α_1 to one of these roots. Since it is simple it does not reduce to zero at least one of the minors of the determinant of the system (6.14). Hence the system (6.14) gives a solution for M_c and N_o in which only one of these magnitudes is arbitrary. Let this magnitude be M_o, for which it is necessary that it result different from zero.

Having chosen the magnitudes α_1, M_o and N_o in this manner we obtain for $x_s^{(1)}$ a periodic solution which has the form

$$x_s^{(1)} = M_1 \varphi_{s1} + N_1 \varphi_{s2} + x_s^{(1)*}(t),$$

where $x_s^{(1)*}$ is some particular periodic solution ofthe system (6.13) and M_1 and N_1 are arbitrary constants. The constant M_1, as we shall presently show, can be chosen quite arbitrarily and, in particular, can be set equal to zero. As regards the constant N_1, it is, together with the constant α_2, uniquely determined from the condition of periodicity of the functions $x_s^{(2)}$.

In fact, the equation determining an arbitrary approximation $x_s^{(p)}$ is of the form

$$\frac{dx_s^{(p)}}{dt} = \lambda_0 \sum_{\beta=1}^{n} a_{s\beta} x_\beta^{(p)} + \sum_{\beta=1}^{n} (\alpha_1 a_{s\beta} + \lambda_0 f_{s\beta}^{(1)}) x_\beta^{(p-1)} + \\ + \alpha_p \sum_{\beta=1}^{n} a_{s\beta} (M_0 \varphi_{\beta 1} + N_0 \varphi_{\beta 2}) + X_s^{(p)}, \quad (6.16)$$

where $X_s^{(p)}$ is a linear function with known periodic coefficients of x_s^0, $x_s^{(1)}$, ..., $x_s^{(p-2)}$, $\alpha_1, \ldots, \alpha_{p-1}$. Hence if all functions $x_s^{(1)}, \ldots, x_s^{(p-1)}$ have already been computed and turned out periodic, there are obtained for $x_s^{(p)}$ equations with periodic nonhomogeneous terms. The periodic solutions of these equations, if they exist, are of the form

$$x_s^{(p)} = M_p \varphi_{s1} + N_p \varphi_{s2} + x_s^{(p)*}(t), \quad (6.17)$$

where $x_s^{(p)*}$ are periodic functions and M_p and N_p arbitrary constants. We shall show first of all that the constant M_p can be chosen quite arbitrarily. For this purpose let us assume that the periodic solution (6.12) has already been computed. Multiplying this solution by the arbitrary convergent series

$$1 + A_1\mu + A_2\mu^2 + \ldots,$$

we obtain a new periodic solution

$$\bar{x}_s = x_s^0 + \mu \bar{x}_s^{(1)} + \ldots,$$

in which the functions $\bar{x}_s^{(p)}$ will evidently have the form

$$\bar{x}_s^{(p)} = \overline{M}_p \varphi_{s1} + \overline{N}_p \varphi_{s2} + \bar{x}_s^{(p)*}(t),$$

where $\bar{x}_s^{(p)*}$ are certain periodic functions of t, while

$$\overline{M}_p = M_p + M_{p-1} A_1 + \ldots + M_0 A_p,$$
$$\overline{N}_p = N_p + N_{p-1} A_1 + \ldots + N_0 A_p.$$

Since by assumption $M_0 \neq 0$ the constants A_p can be chosen in such manner that the constants $\bar{M}_p$ have previously assigned values.

We shall therefore assume that in equations (6.17) $M_p = 0$ so that the periodic solutions of equations (6.16) have the form

$$x_s^{(p)} = N_p \varphi_{s2} + x_s^{(p)*}(t) \qquad (p = 1, 2, \ldots). \qquad (6.18)$$

With this established, let us assume that all the constants $\alpha_1, \ldots, \alpha_{p-1}$ and $N_1, \ldots, N_{p-2}$, and also all the functions $x_s^{(1)}, \ldots, x_s^{(p-1)}$ have already been computed but that the constant N_{p-1} entering the expression $x_s^{(p-1)}$ is not yet determined. The conditions of periodicity of the functions $x_s^{(p)}$ will then give

$$\alpha_p (M_0 A_{1i} + N_0 A_{2i}) + N_{k-1} (\alpha_1 A_{2i} + B_{2i}) + C_i = 0, \qquad (6.19)$$

where

$$C_i = \int_0^{2\pi} \sum_{\beta=1}^{n} X_\beta^{(p)} \psi_{\beta i}\, dt + \int_0^{2\pi} \sum_{\beta,\gamma=1}^{n} (\alpha_1 a_{\gamma\beta} + \lambda_0 f_{\gamma\beta}^{(1)}) x_\beta^{(p-1)*} \psi_{\gamma i}\, dt$$

are known constants. As is easily seen, the determinant of this linear system of equations for α_p and N_{p-1} is equal to the magnitude $M_o \, dD(\alpha_1)/d\alpha_1$ and therefore different from zero, since by assumption the root α_1 of equation (6.15) is simple.

Thus, for each of the roots $\alpha_1^{(1)}$ and $\alpha_2^{(2)} > \alpha_1^{(1)}$ of the equation (6.15) we obtain an entirely definite series (6.9) which on account of this will converge. We shall denote the magnitudes λ^* obtained in this manner by λ^*_1 and λ^*_2 and shall show that the interval $(\lambda^*_1, \lambda^*_2)$ is actually a region of instability.

For this purpose we put in the equations investigated $\lambda = \lambda_o + \mu\varkappa$, where $\alpha_1^{(1)} < \varkappa < \alpha_1^{(2)}$ and show that for sufficiently small μ one of the characteristic exponents of the system has a positive real part. For this we compute the characteristic exponents of the system that reduce to $\pm\lambda_o \omega_i \sqrt{-1}$ for $\mu = 0$. These exponents have the form

$$a = \lambda_0 \omega_i \sqrt{-1} + \mu a_1 + \mu^2 a_2 + \ldots =$$
$$= N \sqrt{-1} + \mu a_1 + \mu^2 a_2 + \ldots$$

To determine them, following the method of sec. 13 of chapter III, we put in the equations (6.10) (for $\lambda = \lambda_o + \mu\varkappa$)

$$x_s = e^{(\mu a_1 + \mu^2 a_2 + \ldots) t} y_s$$

and determine the constants $a_1, a_2, \ldots$ from the condition of the existence of periodic solutions for the system thus obtained

$$\frac{dy_s}{dt} = (\lambda_0 + \mu\varkappa) \sum_{\beta=1}^{n} (a_{s\beta} + \mu f_{s\beta}^{(1)} + \ldots) y_\beta - (\mu a_1 + \ldots) y_s. \quad (6.20)$$

For our purpose it is sufficient to determine only the magnitude a_1.

If the periodic solution of system (6.20) is sought in the form of the series

$$y_s = P_0 \varphi_{s1} + Q_0 \varphi_{s2} + \mu y_s^{(1)} + \ldots,$$

where P_o and Q_o are arbitrary constants, we obtain for $y_s^{(1)}$ the equations

$$\frac{dy_s^{(1)}}{dt}=\lambda_0\sum_{\beta=1}^{n} a_{s\beta}\, y_\beta^{(1)}+\sum_{\beta=1}^{n}\left(\varkappa a_{s\beta}+\lambda_0 f_{s\beta}^{(1)}\right)\left(P_0\varphi_{\beta 1}+Q_0\varphi_{\beta 2}\right)-$$
$$-a_1\left(P_0\varphi_{s1}+Q_0\varphi_{s2}\right),$$

and the conditions of periodicity will give:

$$\left.\begin{aligned}(\varkappa A_{11}+B_{11}-a_1)P_0+(\varkappa A_{21}+B_{21})Q_0=0,\\ (\varkappa A_{12}+B_{12})P_0+(\varkappa A_{22}+B_{22}-a_1)Q_0=0.\end{aligned}\right\}\qquad(6.21)$$

To simplify the computations we here assume that the functions ψ_{s1} and ψ_{s2} are chosen such that the conditions are satisfied

$$\sum_{\alpha=1}^{n}\psi_{\alpha i}\varphi_{\alpha i}=1,\qquad \sum_{\alpha=1}^{n}\psi_{\alpha i}\varphi_{\alpha k}=0\qquad(i\neq k).\qquad(6.22)$$

Equating to zero the determinant of system (6.21) we obtain a quadratic equation for a_1. One of the roots of this equation corresponds to the characteristic exponent reducing for $\mu = 0$ to $+Ni$ and the other to the exponent reducing for $\mu = 0$ to $-Ni$. Since the system (6.10) is canonical the coefficient of the first degree of α_1 in the quadratic equation, determining this magnitude, should be equal to zero. Taking this circumstance into account we easily find from (6.21):

$$a_1=\pm\sqrt{-(A\varkappa^2+B\varkappa+C)},\qquad(6.23)$$

where A, B, C have the same values as in (6.15).

As was pointed out above, the regions of instability are situated on either side of the interval $(\lambda_1^*, \lambda_2^*)$. As a result, if the magnitude $\varkappa$ lies outside the interval $(\alpha_1^{(1)}, \alpha_1^{(2)})$ both roots (6.23) must be purely imaginary and therefore the quadratic function $A\varkappa^2 + B\varkappa + C$ is positive. But then for $\alpha_1^{(1)} < \varkappa < \alpha_1^{(2)}$ this function must be negative since the magnitudes $\alpha_1^{(1)}$ and $\alpha_1^{(2)}$ are its roots. From this it immediately follows that for $\alpha_1^{(1)} < \varkappa < \alpha_1^{(2)}$ and μ sufficiently small one of the characteristic exponents of the system under consideration will have a positive real part. This shows that the interval $(\lambda^*_1, \lambda^*_2)$ is a region of instability.

We had assumed that equation (6.15) has simple roots. When this condition is not satisfied the computations become somewhat complicated. We shall not concern ourselves here with the detailed investigation of this case. It is practically of less interest since the width of the region of instability will have with respect to μ an order of smallness higher than the first. However, if the need arises of investigating this case for some concrete example this does not present any difficulties if one is guided by the general considerations here presented.

7. A Second Method of Determining the Regions of Instability of Parametric Resonance for Canonical Systems. Regions of Combined Resonance

Another method may be given for determining the regions of instability that is equally suitable both for simple and for combination resonance. This method is based on the direct computation of the characteristic exponents near the critical values of the parameter .

Let us consider the system (5.4) which, as in the preceding section, we write in the general form

$$\frac{dx_s}{dt} = \lambda \sum_{\alpha=1}^{n} (a_{s\alpha} + \mu f_{s\alpha}) x_\alpha \tag{7.1}$$

$$(s = 1, \ldots, n = 2m).$$

Here $f_{s\alpha}$ are periodic functions of t of period π, depending also on the parameter μ. In contrast to the preceding sections we shall not assume that these functions are analytic with respect to μ but we shall restrict ourselves by the weaker assumption that they admit a continuous partial derivative with respect to μ. In everything else we shall adhere to the assumptions of sec. 5.

We shall choose λ_0 in such manner that there is satisfied one of the conditions of the form

$$\lambda_0\omega_j = \pm\lambda_0\omega_k + 2N \qquad (j, k = 1, \ldots, m;\ N = \pm 1, \pm 2, \ldots) \quad (7.2)$$

and shall seek the region of instability which for $\mu = 0$ contracts to the point $\lambda = \lambda_0$. The magnitude k in relation (7.2) may here be different from j so that the required region of instability can be a region of simple as well as of combination parametric resonance. For the solution of the problem we shall set in equations (7.1) $\lambda = \lambda_0 + \mu\varkappa$ and seek the interval of variation of the magnitude $\varkappa$ within which these equations have a characteristic exponent with positive real part. For this purpose we shall determine, by the method of sec. 11 of chapter III, the characteristic exponent of the system (7.1) for $\lambda = \lambda_0 + \mu\varkappa$, reducing for $\mu = 0$ to the magnitude $\lambda_0\omega_j i$. This exponent has the form[1]

$$\alpha = \lambda_0\omega_j i + \mu a(\mu), \qquad (7.3)$$

where the magnitude $a(\mu)$ is determined from the condition of the existence of a periodic solution of period 2π for the system of equations

$$\frac{dy_s}{dt} = (\lambda_0 + \mu\varkappa)\sum_{\alpha=1}^{n}(a_{s\alpha} + \mu f_{s\alpha})\,y_\alpha - (\lambda_0\omega_j i + \mu a)\,y_s. \qquad (7.4)$$

Let us first consider the system

$$\frac{dy_s^0}{dt} = \lambda_0\sum_{\alpha=1}^{n} a_{s\alpha}y_\alpha^0 - \lambda_0\omega_j i y_s^0. \qquad (7.5)$$

The fundamental equation of this system has the roots $(\pm\lambda_1 - \lambda_j)\omega_1 i, \ldots, (\pm\lambda_m - \lambda_j)\omega_m i$. One of these roots is equal to zero and one and only one has, on the basis of (7.2), the form $2Ni$. Hence the system (7.5) has two and

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[1] In the case of simple resonance the component $\lambda_0\omega_j i$ may be discarded since it will have the form Ni.

and only two periodic solutions of period 2π: a solution φ_{s1} in which all magnitudes φ_{s1} are constants, and a solution φ_{s2} in which the magnitudes φ_{s2} are periodic functions of t of period 2π.

Following the method of sec. 11 of chapter III we shall seek the periodic solution of the system (7.4) by the method of successive approximations. For this purpose we set

$$y_s^0 = M_0\varphi_{s1} + N_0\varphi_{s2},$$

where M_o and N_o are arbitrary constants and determine the p-th approximation $y_s^{(k)}$ as the periodic solution of the system

$$\frac{dy_s^{(p)}}{dt} = \lambda_0 \sum_{\alpha=1}^{n} a_{s\alpha} y_\alpha^{(p)} - \lambda_0 \omega_j i y_s^{(p)} + \\ + \mu \sum_{\alpha=1}^{n} (\varkappa a_{s\alpha} + \lambda_0 f_{s\alpha} + \mu\varkappa f_{s\alpha})\, y_\alpha^{(p-1)} - \mu a^{(p)} y_s^{(p-1)},$$

where $a^{(p)}$ denotes the p-th approximation ofthe magnitude a.

Further, let ψ_{s1} and ψ_{12} be the periodic solutions of period 2π of the system conjugate to (7.5). We shall assume for simplicity of computation that relations (6.22) are satisfied. The conditions of periodicity of the functions $y_s^{(1)}$ then have the form

$$\left.\begin{aligned} &\left(\varkappa A_{11} + B_{11} + \frac{\mu\varkappa}{\lambda_0} B_{11} - a^{(1)}\right) M_0 + \\ &\qquad + \left(\varkappa A_{21} + B_{21} + \frac{\mu\varkappa}{\lambda_0} B_{21}\right) N_0 = 0, \\ &\left(\varkappa A_{12} + B_{12} + \frac{\mu\varkappa}{\lambda_0} B_{12}\right) M_0 + \\ &\qquad + \left(\varkappa A_{22} + B_{22} + \frac{\mu\varkappa}{\lambda_0} B_{22} - a^{(1)}\right) N_0 = 0, \end{aligned}\right\} \tag{7.6}$$

where

$$A_{ij} = A_{ij}(\mu) = \sum_{\alpha, \beta=1}^{n} \int_0^{2\pi} a_{\beta\alpha}\varphi_{\alpha i}\psi_{\beta j}\,dt,$$

$$B_{ij} = B_{ij}(\mu) = \sum_{\alpha, \beta=1}^{n} \int_0^{2\pi} \lambda_0 f_{\beta\alpha}\varphi_{\alpha i}\psi_{\beta j}\,dt.$$

Putting here $\mu = 0$ and using the notations

$$A_{ij}^0 = A_{ij}(0), \quad B_{ij}^0 = B_{ij}(0),$$
$$M^* = M_0(0), \quad N^* = N_0(0), \quad a^* = a^{(1)}(0),$$

we obtain:

$$\left.\begin{aligned} (\varkappa A_{11}^0 + B_{11}^0 - a^*) M^* + (\varkappa A_{21}^0 + B_{21}^0) N^* = 0, \\ (\varkappa A_{12}^0 + B_{12}^0) M^* + (\varkappa A_{22}^0 + B_{22}^0 - a^*) N^* = 0. \end{aligned}\right\} \tag{7.7}$$

whence we obtain for a^* the quadratic equation

$$\begin{aligned} a^{*2} - [\varkappa(A_{11}^0 + A_{22}^0) + B_{11}^0 + B_{22}^0]\, a^* + \\ + (\varkappa A_{11}^0 + B_{11}^0)(\varkappa A_{22}^0 + B_{22}^0) - (\varkappa A_{21}^0 + B_{21}^0)(\varkappa A_{12}^0 + B_{12}^0) = 0. \end{aligned} \tag{7.8}$$

Let us assume that this equation has the simple roots a_1^* and a_2^*. For each of these roots we obtain a nontrivial solution for M^* and N^*, where one and only one of these magnitudes can be chosen quite arbitrarily. We shall assume that for the root a_1^* this magnitude is M^*. Then, as was shown in sec. 11 of chapter III, equations (7.6), for $\mu \neq 0$ but sufficiently small, if $M_0 = M^*$ is substituted in them admit one and only one solution $a_1^{(1)}(\mu)$, $N_0(\mu)$ for which $a_1^{(1)}(0) = a_1^*(0)$, $N_0(0) = N^*$. Having in this way determined the magnitude $a_1^{(1)}$ the further approximations are found as indicated in sec. 11 of chapter III. In a similar way we determine the characteristic exponent corresponding to the root a_2^*.

We thus obtain the two characteristic exponents $\alpha_1(\mu)$ and $\alpha_2(\mu)$ of the system (7.1). One of these exponents corresponds to the root $\lambda_0\omega_j i$ of the fundamental equation of the constant part of this system and the other, not to the root $-\lambda_0\omega_j i$, but to that of the two roots $\pm\lambda_0\omega_k i$ of this equation for which relation (7.2) is satisfied. Hence, notwithstanding the canonicity of the system considered, the magnitude $\alpha_1 + \alpha_2$ is different from zero, in consequence of which also the coefficient of a^* in equation (7.8) is in general likewise different from zero.

The obtained characteristic exponents depend on the magnitude $\varkappa$. The required region of instability corresponds to that interval $(\varkappa_1, \varkappa_2)$ of variation of the magnitude $\varkappa$ within which the real part of one of the magnitudes α_1, α_2 is positive. The equation determining the boundaries of this interval are obtained by setting this real part equal to zero.

For practical purposes it is usually possible to restrict oneself to the computation of only the first approximation of the magnitudes α_1 and α_2, which are obtained if the magnitudes a_1 and a_2 are replaced by the values a^*_1 and a^*_2. We then arrive at the quite elementary problem of determining a region in which at least one of the roots of the quadratic equation (7.8) has a positive real part.

We had assumed that equation (7.8) has simple roots. The case where this equation, for any value of $\varkappa$, has multiple roots is more complicated and is not here considered.

If the equations are analytic with respect to μ the characteristic exponents are more conveniently determined by the method of sec. 13 of chapter III.

8. Example. Theorem of M. G. Krein

Let us consider as an example a system of equations of the form

$$\frac{d^2x_s}{dt^2}+\lambda^2\left(\omega_s^2x_s+\mu\sum_{\beta=1}^{m}f_{s\beta}x_\beta\right)=0 \tag{8.1}$$
$$(s=1,\ \ldots,\ m),$$

where the frequencies ω_s satisfy the general conditions indicated in sec. 5 and $f_{s\beta}=f_{\beta s}$ are arbitrary periodic functions of t of period π, developable in Fourier series

$$f_{s\beta}=\sum_{p=1}^{\infty}(P_{s\beta}^{(p)}\cos 2pt+Q_{s\beta}^{(p)}\sin 2pt). \tag{8.2}$$

To equations of the form (8.1) are reduced many practical problems on parametric resonance. We shall determine for these equations the first regions of instability for simple and combination parametric resonance. We shall therefore seek the regions of instability situated near the values $\lambda=\lambda_0$ satisfying the equations

$$\lambda_0\omega_j=1, \tag{8.3}$$
$$\lambda_0\omega_j=-\lambda_0\omega_k+2, \tag{8.4}$$
$$\lambda_0\omega_j=+\lambda_0\omega_k+2. \tag{8.5}$$

For the solution of the problem in the case (8.3) we shall apply the method of sec. 6 and in the cases (8.4) and (8.5) the method of the preceding section. In all cases we shall restrict ourselves to the first approximation.

Let us first consider the case (8.3). Following the method of sec. 6 we seek the value $\lambda=\lambda^*$, corresponding to the boundary of the region of instability, in the form of the series

$$\lambda^*=\lambda_0+\mu\alpha_1+\ldots=\frac{1}{\omega_j}+\mu\alpha_1+\ldots \tag{8.6}$$

the coefficients of which we determine from the condition that for $\lambda=\lambda^*$ the system (8.1) admits a periodic solution of period 2π, developable in the series

$$x_s=x_s^0+\mu x_s^{(1)}+\ldots$$

For the functions x_s^o we have the equations

$$\frac{d^2x_j^0}{dt^2}+x_j^0=0, \qquad \frac{d^2x_r^0}{dt^2}+\lambda_0^2\omega_r^2x_r^0=0 \qquad (r\neq j).$$

The periodic solution (of period 2π) of these equations has the form

$$x_j^0=M_0\cos t+N_0\sin t, \qquad x_r^0=0 \qquad (r\neq j),$$

where M_o and N_o are arbitrary constants. From this we obtain for $x_s^{(1)}$ the equations

$$\frac{d^2x_j^{(1)}}{dt}+x_j^{(1)}+(\omega_j^2\alpha_1+\lambda_0^2f_{jj})(M_0\cos t+N_0\sin t)=0,$$
$$\frac{d^2x_r^{(1)}}{dt^2}+\lambda_0^2\omega_r^2x_r^{(1)}+\lambda_0^2f_{ii}(M_0\cos t+N_0\sin t)=0$$
$$(r\neq j).$$

The second group of these equations admits a periodic solution without any supplementary conditions. As to the magnitude $x_j^{(1)}$, in order that it come out periodic it is necessary that in the equation which determines it the coefficients of cos t and sin t reduce to zero. In this way, taking (8.2) into account, we obtain the following equations:

$$\left(\omega_j^2\alpha_1+\frac{1}{2}\lambda_0^2P_{jj}^{(1)}\right)M_0+\frac{1}{2}\lambda_0^2Q_{jj}^{(1)}N_0=0,$$
$$\frac{1}{2}\lambda_0^2Q_{jj}^{(1)}M_0+\left(\omega_j^2\alpha_1-\frac{1}{2}\lambda_0^2P_{jj}^{(1)}\right)N_0=0,$$

whence we find:

$$\alpha_1=\pm\frac{\lambda_0^2\sqrt{P_{jj}^{(1)2}+Q_{jj}^{(1)2}}}{2\omega_j^2}=\frac{\pm\sqrt{P_{jj}^{(1)2}+Q_{jj}^{(1)2}}}{2\omega_j^4}.$$

Hence the required boundaries of the region of instability are determined in the first approximation by the following formulas:

$$\lambda_{1,2}^{*2} = \frac{1}{\omega_j^2} \pm \mu \frac{\sqrt{P_{jj}^{(1)2} + Q_{jj}^{(1)2}}}{2\omega_j^4}. \tag{8.7}$$

Thus in the first approximation the functions $f_{s\beta}$ do not show either the mutual effect of the coordinates or the higher harmonics.

Let us now consider the cases (8.4) and (8.5). Following the method of the preceding section we set in equations (8.1)

$$\lambda^2 = \lambda_0^2 + \mu\varkappa$$

and seek the characteristic exponent of this system corresponding to the root $\lambda_0 \omega_j i$. Since in the case considered the equations are analytic with respect to μ this characteristic exponent may be sought by the method of sec. 13 of chapter III. For this purpose we make the substitution

$$x_s = e^{(\lambda_0\omega_j i + \mu a_1 + \dots)t} y_s$$

and determine the coefficients $a_1, \dots$ from the condition that the obtained equations have a periodic solution of period π. These equations, if we discard in them all terms of order of smallness higher than the first, have the form

$$\frac{d^2 y_s}{dt^2} + \lambda_0^2 (\omega_s^2 - \omega_j^2) y_s + 2\lambda_0 \omega_j i \frac{dy_s}{dt} +$$

$$+ \left\{ (\varkappa\omega_s^2 + 2a_1\lambda_0\omega_j i) y_s + \lambda_0^2 \sum_{\alpha=1}^{n} f_{s\alpha} y_\alpha + 2a_1 \frac{dy_s}{dt} \right\} \mu = 0.$$

We shall seek to obtain the periodic solution of these equations in the form of the series

$$y_s = y_s^0 + \mu y_s^{(1)} + \dots$$

For y_s^0 we obtain the system

$$\frac{d^2 y_s^0}{dt^2} + \lambda_0^2 (\omega_s^2 - \omega_j^2) y_s^0 + 2\lambda_0 \omega_j i \frac{dy_s^0}{dt} = 0, \tag{8.8}$$

which breaks down into n separate equations of the second order. The roots of the fundamental equation for the s-th equation will be the magnitudes $\pm\lambda_0\omega_s i - \lambda_0\omega_j i$.

Hence only the j-th and k-th equations have critical roots 0 and -2i. The periodic solution of period π of the system (8.8) will therefore have the form

$$y_j^0 = M_0, \quad y_k^0 = N_0 e^{-2it}, \quad y_r^0 = 0 \quad (r \neq j,\ r \neq k),$$

where M_0 and N_0 are arbitrary constants. Having in this manner determined the functions y_s^0 we obtain for $y_s^{(1)}$ the following equations:

$$\frac{d^2 y_s^{(1)}}{dt^2} + \lambda_0^2(\omega_s^2 - \omega_j^2)\, y_s^{(1)} + 2\lambda_0\omega_j i \frac{dy_s^{(1)}}{dt} +$$
$$+ (\varkappa\omega_s^2 + 2a_1\lambda_0\omega_j i)\, y_s^0 + 2a_1 \frac{dy_s^0}{dt} + \lambda_0^2 (f_{sj} M_0 + f_{sk} N_0 e^{-2it}) = 0.$$

All equations of this system, with the exception of the j-th and k-th, admit periodic solutions without any supplementary conditions on their right hand sides. In order that the j-th and k-th equations admit periodic solutions it is necessary and sufficient that there be no free term in the right hand side of the j-th equation and no term with e^{-2it} in the right hand side of the k-th equation. In this way, taking (8.2) into account, we obtain the following conditions of periodicity of the functions $y_s^{(1)}$:

$$(\varkappa\omega_j^2 + 2a_1\lambda_0\omega_j i)\, M_0 + \frac{1}{2}\lambda_0^2 (P_{jk}^{(1)} - iQ_{jk}^{(1)})\, N_0 = 0,$$
$$\frac{1}{2}\lambda_0^2 (P_{jk}^{(1)} + iQ_{jk}^{(1)})\, M_0 + (\varkappa\omega_k^2 + 2a_1\lambda_0\omega_j i - 4a_1 i)\, N_0 = 0.$$

whence we find:

$$16\lambda_0\omega_j(\lambda_0\omega_j - 2)\, a_1^2 - 8i\varkappa\lambda_0\omega_j [\omega_k^2 + \omega_j(\lambda_0\omega_j - 2)]\, a_1 - 4\varkappa^2\omega_j\omega_k^2 +$$
$$+ \lambda_0^4 (P_{jk}^{(1)2} + Q_{jk}^{(1)2}) = 0. \quad (8.9)$$

Having established this, let us assume first that we are dealing with the case (8.4). In this case equation (8.9) assumes the form

$$16\lambda_0^2\omega_j\omega_k a_1^2 - 8i\varkappa\omega_j\omega_k(\omega_j - \omega_k)\lambda_0 a_1 + 4\varkappa^2\omega_j^2\omega_k^2 - \lambda_0^4(P_{jk}^{(1)\,2} + Q_{jk}^{(1)\,2}) = 0.$$

One of its roots will have a positive real part when and only when the inequality is satisfied

$$\varkappa^2 < \frac{\lambda_0^4(P_{jk}^{(1)\,2} + Q_{jk}^{(1)\,2})}{\omega_j\omega_k(\omega_j + \omega_k)^2}.$$

From this we arrive at the following approximate values of $\lambda = \lambda^*$ on the boundaries of the region of instability

$$\lambda_{1,2}^* = \lambda_0 \pm \frac{\lambda_0^2\sqrt{P_{jk}^{(1)\,2} + Q_{jk}^{(1)\,2}}}{(\omega_j + \omega_k)\sqrt{\omega_j\omega_k}}\mu = \frac{2}{\omega_j + \omega_k} \pm \frac{4\sqrt{P_{jk}^{(1)\,2} + Q_{jk}^{(1)\,2}}}{(\omega_j + \omega_k)^3\sqrt{\omega_j\omega_k}}\mu.$$

These formulas for j = k go over into (8.7).

Let us now assume that condition (8.5) is satisfied. In this case equation (8.9) assumes the form

$$16\lambda_0^2\omega_j\omega_k a_1^2 - 8i\varkappa\omega_j\omega_k(\omega_j + \omega_k)\lambda_0 a - 4\varkappa^2\omega_j^2\omega_k^2 + \lambda_0^4(P_{jk}^{(1)\,2} + Q_{jk}^{(1)\,2}) = 0.$$

This equation has purely imaginary roots for any values of $\varkappa$. The obtained result is not accidental. It is in agreement with an important theorem established by M. G. Krein [1], which we here present without proof:

IN ORDER THAT FOR THE SYSTEM (8.1) THERE EXIST A REGION OF INSTABILITY CONTRACTING FOR $\mu = 0$ TO THE POINT $\lambda = \lambda_0$ IT IS NECESSARY THAT THE MAGNITUDE λ_0 SATISFY A RELATION OF THE FORM

$$\lambda_0\omega_j = -\lambda_0\omega_k + 2N \qquad (j,\,k = 1,\,\ldots,\,m;\; N = 1,\,2,\,\ldots).$$

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[1] See the work cited on p.397 . This theorem was formulated by Krein in a more general form.

In other words, for systems of the form (8.1) the plus sign before the term $\lambda_o \omega_k$ in relations (8.5) must be discarded.

B. FORCED OSCILLATIONS OF QUASIHARMONIC SYSTEMS

9. Conditions for the Existence of Almost Periodic Solutions of Systems of Linear Equations with Periodic Coefficients

We now go on to consider systems described by linear nonhomogeneous equations of the form

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n + f_s(t) \qquad (s = 1, \ldots, n), \qquad (9.1)$$

where P_{sj} are certain continuous periodic functions of t of period ω, and $f_s(t)$ are almost periodic functions of t.

The general solution of system (9.1) is made up of the general solution of the homogeneous system

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n, \qquad (9.2)$$

describing the free oscillations and investigated by us in detail, and some particular solution characterizing the forced oscillations. In this part we shall take up the computation ofthese forced oscillations, the explanation of their properties, the investigation of the possibilities of resonance, etc. In all these cases a question of fundamental importance is that of the existence for the system (9.1) of almost periodic solutions. To this question we shall now turn our attention.

In what follows we shall assume that the functions $f_s(t)$ in equations (9.1) are of the form

$$f_s(t) = \sum_{j=1}^{N} e^{i\nu_j t} f_{sj}(t), \qquad (9.3)$$

where $\nu_1, \ldots, \nu_N$ are arbitrary real numbers and f_{sj} continuous periodic functions of t of period ω developable into absolutely and uniformly converging Fourier series. Let us consider the characteristic equation of system (9.2).

We shall assume that this equation can have both simple and multiple roots, that to each multiple root can correspond one or several sets of solutions. We denote by $\lambda_1,\ldots,\lambda_k$ all roots of the characteristic equation having a modulus equal to one and by $\mu_1,\ldots,\mu_m$ all roots having a modulus different from one. We shall here write out each multiple root as many times as the sets of solutions that correspond to it. Therefore $k + m \leq n$ and among the magnitudes λ_i and μ_α some may be equal but to each of them corresponds only one set of solutions. Let, further,

$$i\alpha_j = \frac{1}{\omega}\ln\lambda_j, \quad \omega_\alpha = \frac{1}{\omega}\ln\mu_\alpha = \beta_\alpha + i\gamma_\alpha \tag{9.4}$$

$$(j = 1, \ldots, k;\ \alpha = 1, \ldots, m)$$

be the characteristic exponents of the system (9.2). Here α_j, β_α and γ_α are real numbers, all β_α being different from zero.

For the assumptions made, the system (9.2) has k and only k almost periodic solutions which we shall denote by $\varphi_{s1},\ldots,\varphi_{sk}$. Each of the functions φ_{sj} is here equal to the product of a periodic function of period ω, developable into a uniformly and absolutely converging Fourier series, and of a periodic function $e^{i\alpha_j t}$

Together with the system (9.2) we consider the system

$$\frac{dy_s}{dt} + p_{1s}y_1 + \ldots + p_{1s}y_n = 0, \tag{9.5}$$

conjugate to it. According to the theorem of Lyapunov the characteristic equation of this system has the roots $1/\lambda_1,\ldots,1/\lambda_k$, $1/\mu_1,\ldots,1/\mu_m$, the same number of sets of solutions corresponding to corresponding roots in the systems (9.2) and (9.5). System (9.5) will therefore have k and only k almost periodic solutions, which we shall denote by $\psi_{s1},\ldots,\psi_{sk}$. The functions ψ_{sj} have the same structure as the functions φ_{sj}.

We shall now prove the following theorem: [1]

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[1] See the work cited on p.295.

THEOREM. IN ORDER THAT SYSTEM (9.1) ADMIT ALMOST PERIODIC SOLUTIONS IT IS NECESSARY AND SUFFICIENT THAT THE FOLLOWING CONDITIONS BE SATISFIED:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\sum_{\alpha=1}^n f_\alpha\psi_{\alpha j}\,dt=0\qquad (j=1,\ldots,k). \tag{9.6}$$

PROOF. Let us denote by p_j and q_α the number of solutions corresponding in (9.2) and (9.5) to the roots λ_j and μ_α so that $p_1+\ldots+p_k+q_1+\ldots+q_m=n$. The system (9.5) will then have n particular solutions of the form

$$\left.\begin{aligned}
&y_{s1}^{(j)}=\psi_{sj},\\
&y_{s2}^{(j)}=t\psi_{sj}+\psi_{sj}^{(2)},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&y_{sp_j}^{(j)}=\frac{t^{p_j-1}}{(p_j-1)!}\psi_{sj}+\frac{t^{p_j-2}}{(p_j-2)!}\psi_{sj}^{(2)}+\ldots+\psi_{sj}^{(p_j)}\\
&\qquad (j=1,\ldots,k;\quad s=1,\ldots n),\\
&y_{s1}^{(k+\alpha)}=e^{-\omega_\alpha t}F_{s\alpha},\\
&y_{s2}^{(k+\alpha)}=e^{-\omega_\alpha t}(tF_{s\alpha}+F_{s\alpha}^{(2)}),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&y_{sq_\alpha}^{(k+\alpha)}=e^{-\omega_\alpha t}\left(\frac{t^{q_\alpha-1}}{(q_\alpha-1)!}F_{s\alpha}+\frac{t^{q_\alpha-2}}{(q_\alpha-2)!}F_{s\alpha}^{(2)}+\ldots+F_{s\alpha}^{(q_\alpha)}\right)\\
&\qquad (\alpha=1,\ldots,m;\quad s=1,\ldots,n),
\end{aligned}\right\} \tag{9.7}$$

where $\psi_{sj}^{(p)}$ are almost periodic functions having the same structure as the functions ψ_{sj} while $F_{s\alpha}^{(p)}$ are simple periodic functions of t of period ω. Making use of these solutions, in the same way as in sec. 5 of chapter III, we transform the system (9.1) to equations with constant coefficients. For this purpose we introduce in the system (9.1) in place of the variables $x_1,\ldots,x_n$ the variables $u_1^{(j)},\ldots,u_{pj}^{(j)}$, $v_1^{(\alpha)},\ldots,v_{q\alpha}^{(\alpha)}$ with the aid of the substitution

$$\left.\begin{aligned}
&u_1^{(j)}=\sum_{s=1}^n\psi_{sj}x_s,\qquad u_\beta^{(j)}=\sum_{s=1}^n\psi_{sj}^{(\beta)}x_s,\\
&v_1^{(\alpha)}=\sum_{s=1}^n F_{s\alpha}x_s,\qquad v_\gamma^{(\alpha)}=\sum_{s=1}^n F_{s\alpha}^{(\gamma)}x_s\\
&(j=1,\ldots,k;\quad \alpha=1,\ldots,m;\\
&\beta=2,\ldots,p_j;\quad \gamma=2,\ldots,q_\alpha).
\end{aligned}\right\} \tag{9.8}$$

Then, taking into account that the expressions

$$u_1^{(j)},\ tu_1^{(j)}+u_2^{(j)},\ \ldots,\ \frac{t^{p_j-1}}{(p_j-1)!}u_1^{(j)}+\ldots+u_{p_j}^{(j)}$$

and

$$e^{-\omega_a t}v_1^{(a)},\ e^{-\omega_a t}(tv_1^{(a)}+v_2^{(a)}),\ \ldots,\ e^{-\omega_a t}\left(\frac{t^{q_a-1}}{(q_a-1)!}v_1^{(a)}+\ldots+v_{q_a}^{(a)}\right)$$

are the first integrals of equations (9.2) we obtain:

$$\frac{du_1^{(j)}}{dt}=\sum_{s=1}^{n}\psi_{sj}f_s,\qquad \frac{du_\beta^{(j)}}{dt}=-u_{\beta-1}^{(j)}+\sum_{s=1}^{n}\psi_{sj}^{(\beta)}f_s$$
$$(j=1,\ \ldots,\ k;\qquad \beta=2,\ \ldots,\ p_i); \tag{9.9}$$

$$\frac{dv_1^{(a)}}{dt}=\omega_a v_1^{(a)}+\sum_{s=1}^{n}F_{sa}f_s,\qquad \frac{dv_\gamma^{(a)}}{dt}=\omega_a v_\gamma^{(a)}-v_{\gamma-1}^{(a)}+\sum_{s=1}^{n}F_{sa}^{(\gamma)}f_s$$
$$(a=1,\ \ldots,\ m;\qquad \gamma=2,\ \ldots,\ q_a). \tag{9.10}$$

We shall denote by D(t) the determinant of the substitution (9.8) and by $\Delta(t)$ the Wronskian determinant of the fundamental system of solutions (9.7) of equations (9.5). By elementary transformations of the determinant $\Delta(t)$ we easily arrive at the relation

$$\Delta(t)=e^{-(q_1\omega_1+\ldots+q_m\omega_m)t}D(t),$$

from which it follows that $|D(t)|>0$, since the solutions (9.7) are independent and therefore the magnitude $\Delta(t)$ does not reduce to zero for any values of t. But, as follows directly from the form of the substitution (9.8), we can write

$$D(t)=e^{i(p_1a_1+\ldots+p_ka_k)t}\varphi(t),$$

where $\varphi(t)$ is a periodic function of period ω. Hence $|D(t)| = |\varphi(t)|$ and from the inequality $|D(t)|>0$ it follows that $|D(t)|>l$, where l is the exact lower limit of the function $\varphi(t)$ on the segment $[0,\omega]$. From this it follows that the transformation inverse to (9.8) will have almost periodic coefficients and therefore the problem of the almost periodic solutions for system (9.1) is equivalent to the same problem for systems (9.9) and (9.10). We shall turn to this last problem.

Equations (9.10) form a self-contained system of linear equations with constant coefficients and almost periodic nonhomogeneous parts. The fundamental equation of this system has roots ω_α with real parts different from zero. Equations (9.10) therefore admit one and only one almost periodic solution for any choice of the functions f_s.

We now go on to equations (9.9). By the assumption with regard to f_s the right hand side of the equation for $u_1^{(j)}$ represents a finite sum of terms of the form $e^{i\alpha t}F(t)$, where α is real and $F(t)$ is a periodic function of t of period ω, developable in an absolutely and uniformly converging Fourier series. Hence the indefinite integral of this right hand side and therefore the functions $u_1^{(j)}$ will have the form $u_1^{(j)} = g(t) + \psi(t)$, where g is the average value of the right hand side and $\psi(t)$ is an almost periodic function. Hence, in order that the function $u_1^{(j)}$ come out almost periodic it is necessary and sufficient that the equation be satisfied $g = 0$, which gives the j-th condition (9.6).

Let us assume that this condition is actually satisfied. We shall then have

$$u_1^{(j)} = \int_0^t \sum_{s=1}^n \psi_{sj} f_s \, dt + c_1^{(j)},$$

where $c_1^{(j)}$ is an arbitrary constant. This constant may be disposed of in such manner that the solution for $u_2^{(j)}$ also comes out almost periodic. For this it is sufficient to set:

$$c_1^{(j)} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{s=1}^n \left\{ \psi_{sj}^{(2)} f_s - \int_0^t \psi_{sj} f_s \, dt \right\} dt.$$

Continuing in this manner we arrive at the conclusion that if the j-th condition (9.6) is satisfied the j-th set of equations (9.9) admits an almost periodic solution, which evidently proves the theorem.

10. Conditions of Resonance. Form of the Forced Oscillations

The theorem proved in the preceding section permits solving the question of the presence or absence of resonance for quasiharmonic systems. For the case of a single equation of the second order the theory of resonance has been investigated in detail in the studies of G. S. Gorelik.[1] We here consider this problem for the general case of systems of the form (9.1).

We remark first of all that if the characteristic equation of the system (9.2) has no roots with moduli equal to unity the system (9.1) admits one and only one almost periodic solution for any choice of the functions f_s. This follows immediately from the proven theorem. Let us assume however that the characteristic equation of the system (9.2) has roots with moduli equal to unity. In this case the system (9.1) will as before admit almost periodic solutions for any choice of the functions f_s with a given spectrum of frequencies if no relation is satisfied of the form

$$\nu_q = \alpha_j \pm \frac{2\pi r}{\omega} \tag{10.1}$$

$$(q = 1, \ldots, N; \quad j = 1, \ldots, k; \quad r = 0, 1, 2, \ldots),$$

where ν_q are the frequencies figuring in the expressions (9.3) for f_s. In fact, if the above conditions are satisfied the Fourier expansion of the expressions under the integral sign in conditions (9.6) will not contain free terms and therefore these conditions will be identically satisfied.

Let us assume that relations of the form (10.1) exist. In this case resonance takes place. Almost periodic solutions of system (9.1) will exist only for such choice of the functions f_s for which conditions (9.6) are satisfied.

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[1] Gorelik G.S., Rezonansnye yavleniya v lineinykh sistemakh s peridodicheski menyayushchimisya parametrami (Resonance Phenomona in Linear Systems with Periodically Varying Parameters), Zhurn. tekhn. fiziki, vol. IV, no. 10, 1934, vol. V. no. 2, 1935 and vol. V, no. 3, 1935.

In the resonance case, with conditons (9.6) satisfied, the system (9.1) will admit particular solutions in the form of polynomials in t with almost periodic coefficients. The degree of these polynomials is equal to $p_i - 1$, where i is the value of the index j for which conditions (9.6) are not satisfied and for which the magnitude p_i is the maximum. We easily arrive at this conclusion from a consideration of equations (9.9) for the nonsatisfying of conditions (9.6).

We shall now show that if the system (9.1) admits almost periodic solutions it may be assumed that these solutions have the form

$$x_s(t) = \sum_{j=1}^{N} e^{i\nu_j t}\Phi_{sj}(t), \tag{10.2}$$

where Φ_{sj} are periodic functions of t of period ω. We here have in view all cases of existence of almost periodic solutions, i.e. both the resonance cases and nonresonance cases with condition (9.6) satisfied.

For the proof it is evidently sufficient to consider the case where in (9.3) $N = 1$, i.e. where the functions $f_s(t)$ have the form

$$f_s(t) = e^{i\nu t}F_s(t).$$

In this case the following uniformly convergent expansions will be valid:

$$\sum_{s=1}^{n} f_s F_{s\alpha} = \sum_{r=-\infty}^{+\infty} A_{r\alpha}\, e^{i\left(\nu+\frac{2\pi r}{\omega}\right)t},$$

and equations (9.10) give

$$v_1^{(\alpha)} = \sum_{r=-\infty}^{+\infty} \frac{A_{r\alpha}\, e^{i\left(\nu+\frac{2\pi r}{\omega}\right)t}}{-\omega_\alpha + i\nu + \frac{2\pi r i}{\omega}} = e^{i\nu t}\, V_1^{(\alpha)}(t),$$

where $V_1^{(\alpha)}$ are periodic functions of period ω. After this we find successively

$$v_\gamma^{(\alpha)} = e^{i\nu t}\, V_\gamma^{(\alpha)}(t) \qquad (\gamma = 2, \ldots, q_\alpha),$$

where $V_\gamma^{(\alpha)}$ are likewise periodic functions of period ω.

Further we have:

$$\psi_{sj} = e^{-i\alpha_j t} \psi^*_{sj}, \tag{10.3}$$

and from equations (9.9), with conditions (9.6) satisfied, we easily obtain:

$$u_r^{(j)} = e^{i(-\alpha_j + \nu)t} U_r^{(j)} \qquad (r = 1, \ldots, p_j). \tag{10.4}$$

Substituting the obtained expressions for $u_r^{(j)}$, $v_\gamma^{(\alpha)}$ in (9.8) and taking (10.3) into account we obtain finally:

$$x_s = e^{i\nu t} X_s(t). \tag{10.5}$$

In equations (10.3), (10.4) and (10.5) the functions ψ^*_{sj}, $U_r^{(j)}$, X_s are periodic with period ω.

From the proved assertion it follows, in particular, that if the functions f_s are purely periodic of the same period ω as the coefficients p_{sj}, the system (9.1), if in general it admits almost periodic solutions, admits a purely periodic solution. Let us consider the case where the functio f_s are periodic with period ω in more detail. In this case in the representation (9.3) $\nu_q = 0$ and conditions (10.1) assume the form

$$\alpha_j = \pm \frac{2\pi r}{\omega} \qquad (r = 0, 1, 2, \ldots),$$

i.e. $\lambda_j = 1$. Consequently, if the characteristic equation of the system (9.2) does not have a root equal to unity, then whatever the periodic functions f_s of period ω may be, the system (9.1) will admit a periodic solution. This solution will be the only one. In fact, if there existed two such solutions their difference would be a periodic solution of system (9.2), which is impossible in the case of absence of unit roots of its characteristic equation. Let us assume however that the characteristic equation has a root equal to unity and that to this root correspond $p \leq k$ sets of solutions. The system (9.5) will then admit p and only p periodic solutions of period ω. Let these solutions be $\psi_{s1}, \ldots, \psi_{sp}$. The first p of conditions (9.6) can then be rewritten in the following form:

$$\int_0^{\omega} \sum_{\alpha=1}^{n} f_\alpha \psi_{\alpha i} \, dt = 0 \qquad (i = 1, \ldots, p). \tag{10.6}$$

As to the remaining k- p of conditions (9.6) they will be identically satisfied. We thus arrive at the following theorem:

IN ORDER FOR SYSTEM (9.1), FOR PERIODIC FUNCTIONS f_s OF PERIOD ω, ADMIT A PERIODIC SOLUTION IT IS NECESSARY AND SUFFICIENT THAT CONDITIONS (10.6) BE SATISFIED, WHERE $\psi_{s1}, \ldots, \psi_{sp}$ ARE PERIODIC SOLUTIONS OF THE SYSTEM (9.5).

We have assumed that the functions f_s can be represented in the form of the finite sums (9.3). Such restriction is essential for the validity of the theorem of the preceding section and the conclusions obtained from it on the resonance for the systems (9.1). If the sums entering in (9.3) do not consist of a finite number of terms but represent absolutely and uniformly converging series it may turn out that system (9.1) will not admit bounded solutions, i.e. resonance will take place even if relations of the form (10.1) do not exist and therefore conditions (9.6) are satisfied. For simplicity let us consider the case where all the functions f_s are periodic with the same period ω^*, different from ω. We shall assume further that the Fourier series of the functions f_s converge absolutely and uniformly but are not finite sums. In this case we have $\nu_q = \pm 2\pi qi/\omega^*$ $(q = 1,2,\ldots)$. We shall assume that no relation of the form (10.1) is satisfied and therefore conditions (9.6) are identically satisfied. For these conditions the mean values of the functions

$$\sum_{s=1}^{n} \psi_{sj} f_s$$

are equal to zero. Nevertheless the indefinite integrals of these functions, i.e. the functions $u_1^{(j)}$, as is known from the general theory of almost periodic functions, will not in general be bounded.

The possibility of the existence of resonance even in the case where no relation of the form (10.1) is satisfied was first pointed out by V. V. Stepanov [1], in investigating the problem of resonance for the case of a single equation of the second order.

We may remark in conclusion that if the characteristic equation of the system (9.2) has roots with moduli equal to unity and this system has an almost periodic solution there will be an infinite number of such solutions and they will all be determined by the formula

$$x_s = A_1\varphi_{s1} + \ldots + A_k\varphi_{sk} + x_s^*(t),$$

where x_s^* is some particular almost periodic solution and $A_1, \ldots, A_k$ are arbitrary sonstants.

11. Dependence of the Forced Oscillations on a Parameter

Let us assume that equations (9.1) can be represented in the form

$$\frac{dx_s}{dt} = (p_{s1} + \mu q_{s1})\,x_1 + \ldots + (p_{sn} + \mu q_{sn})\,x_n + f_s(t, \mu) \qquad (s = 1, \ldots, n), \tag{11.1}$$

where p_{sj} and q_{sj} are continuous periodic functions of t of period ω, the functions q_{sj} depending also on the parameter μ with respect to which they are analytic for $\mu < \mu_0$. As regards the functions f_s, they have, as before, the form (9.3) but the functions f_{sj} can also contain the parameter μ with respect to which they are analytic for $\mu < \mu_0$.

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[1] Stepanov, V.V., O resheniyakh lineinogo uravneniya s periodicheskimi koeffitsientami pri nalichii periodicheskoi vozmushchayushchei sily (On the Solutions of a Linear Equation with Periodic Coefficients in the Presence of a Periodic Disturbing Force), Prikl. matem. i mekh., vol.XIV, no. 3, 1950.

We shall consider first the particular case where all the functions f_s are purely periodic of period ω. In this case, as was shown in the preceding section, we can assume that the almost periodic solution of equations (11.1), if it exists, is periodic. We shall denote this solution by $x_s(t,\mu)$. Further, we shall denote by $\bar{x}_s(t,\mu)$ the particular solution of the system (11.1) with the initial conditions $\bar{x}_s(0,\mu) = 0$ and by $x_{sj}(t,\mu)$ a fundamental system of solutions of the homogeneous part of these equations with initial conditions $x_{ss}(0,\mu)=1, x_{sj}(0,\mu)=0 (s\neq j)$. All functions $\bar{x}_s$ and x_{sj} are analytic with respect to μ for $\mu < \mu_o$. Evidently we have:

$$x_s(t,\mu)=\beta_1 x_{s1}(t,\mu)+\ldots+\beta_n x_{sn}(t,\mu)+\bar{x}_s(t,\mu), \qquad (11.2)$$

Where the constants β_s satisfy the equations

$$x_{s1}(\omega,\mu)\beta_1+\ldots+[x_{ss}(\omega,\mu)-1]\beta_s+\ldots+x_{sn}(\omega,\mu)\beta_n+$$
$$+\bar{x}_s(\omega,\mu)-\bar{x}_s(0,\mu)=0. \quad (11.3)$$

The determinant $\Delta(\mu)$ of this linear system agrees with the value of the characteristic determinant $D(\rho,\mu)$ of the system (11.1) for $\rho = 1$ (see formula (2.7) of chapter III).

With this established, let us at first assume that $D(1,0) \neq 0$, i.e. that the characteristic equation of the homogeneous part of the system

$$\frac{dx_s^0}{dt}=p_{s1}x_1^0+\ldots+p_{sn}x_n^0+f_s(t,\mu) \qquad (11.4)$$

has no roots equal to unity, or what amounts to the same thing that the system (11.4) is not at resonance. For sufficiently small μ we shall then have $\Delta \neq 0$ and therefore equations (11.3) will admit a unique solution for $\beta_s(\mu)$. The magnitudes β_s, and therefore also the functions $x_s(t,\mu)$, will be analytic with respect to μ for $\mu < |\mu^*|$, where μ^* is the smallest root of the equation $\Delta(\mu) = 0$.

Let us assume now that $D(1,0) = 0$ but the magnitude $D(1,\mu)$ for sufficiently small $\mu \neq 0$ is different from zero, i.e. that the system (11.4) is at resonance while for system (11.4) there is no resonance. This case is very important in practice. To this case is reduced for example the problem of the effect of small resistances or other small changes of the parameters of the system on the resonance. In this case we shall have $\Delta(\mu) = \mu^p \Delta'(\mu)$, where p is some integer and $\Delta'(0) \neq 0$.

The functions $x_s(t,\mu)$ will now have the form

$$x_s(t, \mu) = \mu^{-p} x_s'(t, \mu), \tag{11.5}$$

where the functions $x_s'(t,\mu)$ are analytic with respect to μ for its sufficiently small values.

Let us assume, finally, that both $D(1,0) = 0$ and $D(1,\mu) = 0$, i.e. that both for system (11.1) and system (11.4) resonance occurs. This case may for example occur if the system (11.1) in the absence of disturbances is situated on the boundary of the region of stability. We shall here assume that system (11.1) still admits periodic solutions, i.e. that conditions (10.6) are satisfied. In this case the system of linear equations (11.3) will admit a solution for β_s notwithstanding the fact that its determinant is equal to zero. But in this case it is evidently possible, as before, to assume that the functions x_s have the form (11.5).

Thus in all cases where the system (11.1) for periodic f_s admits periodic solutions these can be represented in the form (11.5). We shall show that the same is true also for any functions f_s having the form (9.3). An almost periodic solution of equations (11.1), in all cases where it exists, can be represented in the form (11.5) where the functions x_s' are almost periodic with respect to t and analytic with respect to μ. Evidently, it is sufficient for us to prove our assertion for the case in which $f_s = e^{i\nu t}F_s$, where F_s are periodic functions of t of period ω. In this case, as was shown in the preceding section, we can assume that the required almost periodic solution, if it at all exists, has the form $x_s = e^{i\nu t}y_s$, where y_s are periodic functions of period ω.

But the functions y_s evidently satisfy the equations

$$\frac{dy_s}{dt} = (p_{s1} + \mu q_{s1}) y_1 + \dots + (p_{sn} + \mu q_{sn}) y_n - i\nu y_s + F_s,$$

the right hand sides of which are periodic with respect to t with period ω. Hence, by what has been proven, the functions y_s necessarily have the form $y_s = \mu^{-p} y_s$, where y_s are analytic with respect to μ and consequently the functions

x_s have the form (11.5). In this way our statement has been proven.

12. Practical Method of Computing the Forced Oscillations

The theorem proven in the preceding section permits a simple computation of the forced oscillation for systems, the coefficients of which differ little from constants.

Let us consider a system of equations of the form

$$\frac{dx_s}{dt} = \sum_{\alpha=1}^{n} (a_{s\alpha} + \mu q_{s\alpha}^{(1)} + \ldots) x_s + f_s^{(1)} + \mu f_s^{(2)} + \ldots \qquad (12.1)$$
$$(s = 1, \ldots, n),$$

where the series converge uniformly for $\mu < \mu_0$, $-\infty < t < +\infty$, a_{sj} are constants, $q_{sj}^{(r)}$ continuous periodic functions of t of period ω, and $f_s^{(r)}$ have the form

$$f_s^{(r)} = \sum_{j=1}^{N} e^{i\nu_j t} f_{sj}^{(r)} \qquad (r = 1, 2, \ldots), \qquad (12.2)$$

where $f_{sj}^{(r)}$ are also continuous periodic functions of t of period ω. We shall seek an almost periodic solution of system (12.1). For this purpose we replace system (12.1) by the system

$$\frac{dy_s}{dt} = \sum_{\alpha=1}^{n} (a_{s\alpha} + \mu q_{s\alpha}^{(1)} + \ldots) y + \mu^p (f_s^{(1)} + \mu f_s^{(2)} + \ldots), \quad (12.3)$$

where p is a certain integer. If we find an almost periodic solution of system (12.3) the almost periodic solution we are interested in of the system (12.1) will be obtained, evidently, by simply dividing the found solution by μ^p.

From the results of the preceding section it follows that if the number p is sufficiently large the almost periodic solution of system (12.3), if it at all exists, will be analytic with respect to μ. We shall therefore seek this solution in the form of the series

$$y_s = y_s^0 + \mu y_s^{(1)} + \dots \tag{12.4}$$

with almost periodic coefficients. As was shown, it is here possible to assume that the functions $y_s^{(r)}$ have the form

$$y_s^{(r)} = \sum_{j=1}^{N} e^{i\nu_j t} \Phi_{sj}^{(r)}(t), \tag{12.5}$$

where the functions $\Phi_{sj}^{(r)}$ are periodic with period ω.

We shall distinguish three cases:

1. We first consider the NONRESONANCE case. We shall assume that the fundamental equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0 \tag{12.6}$$

of the system

$$\frac{du_s}{dt} = a_{s1} u_1 + \dots + a_{sn} u_n \tag{12.7}$$

has no roots equal to $i\nu_j + 2\pi ki/\omega$ ($j = 1, \dots, N$, $k = 0, 1, \dots$). In this case we can set $p = 0$ and for the functions $y_s^{(r)}$, in place of which we can now write $x_s^{(r)}$, we shall obtain the equations

$$\frac{dx_s^0}{dt} = \sum_{\alpha=1}^{n} a_{s\alpha} x_\alpha^0 + f_s^{(1)},$$

$$\frac{dx_s^{(r)}}{dt} = \sum_{\alpha=1}^{n} a_{s\alpha} x_\alpha^{(r)} + \sum_{\alpha=1}^{n} q_{s\alpha}^{(1)} x_\alpha^{(r-1)} + \dots + \sum_{\alpha=1}^{n} q_{s\alpha}^{(r)} x_\alpha^0 + f_s^{(r)}$$

$$(r = 1, 2, \dots).$$

For the assumptions made on the roots of equation (12.6) both the equations for x_s^o and the equations for $x_s^{(r)}$ ($r = 1,2,\ldots$) admit a unique almost periodic solution of the form (12.5). As a result, the series (12.4) will converge and actually represent an almost periodic solution of system (12.1). This solution will be unique if equation (12.6) has no purely imaginary or zero roots.

2. Let us assume now that some of the magnitudes $i\nu_j + 2\pi ki/\omega$ ($j = 1,\ldots, N$, $k = 0,1,\ldots$) are roots of equation (12.6) so that the system (12.1) for $\mu = 0$ is at resonance. We shall assume that the total number of sets of solutions of system (12.7) that correspond to these roots and by $\psi_{s1},\ldots,\psi_{sm}$ the almost periodic solutions of the system conjugate to it. The system (12.7) may here have also other almost periodic solutions different form $\varphi_{s1},\ldots,\varphi_{sm}$, since equation (12.6) may have, besides the above mentioned roots, also other purely imaginary roots.

Putting

$$y_s^0 = M_1^0\varphi_{s1} + \ldots + M_m^0\varphi_{sm}, \tag{12.8}$$

where M_i are arbitrary constants, we shall immediately have:

$$\frac{dy_s^{(1)}}{dt} = \sum_{\alpha=1}^{n} a_{s\alpha}\, y_\alpha^{(1)} + M_1^0 \sum_{\alpha=1}^{n} q_{s\alpha}^{(1)}\varphi_{\alpha 1} + \ldots + M_m^0 \sum_{\alpha=1}^{n} q_{s\alpha}^{(1)}\varphi_{\alpha m} + \varkappa f_s^{(1)}, \tag{12.9}$$

where $\varkappa = 1$ if $p = 1$ and $\varkappa = 0$ if $p > 1$. Which of there two cases holds we shall presently establish from a consideration of the obtained equations.

The conditions of existence of almost periodic solutions of equations (12.9) are of the form[1]

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[1] The remaining conditions (9.6), if the system conjugate to (12.7) has almost periodic solutions different from $\psi_{s1},\ldots,\psi_{sm}$, are satisfied in themselves.

$$M_1^0 A_{1i} + \ldots + M_m^0 A_{mi} + \varkappa A_i = 0 \qquad (i = 1, \ldots m), \quad (12.10)$$

where

$$A_{ji} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\alpha,\beta=1}^n q_{\beta\alpha}^{(1)} \varphi_{\alpha j} \psi_{\beta i}\, dt, \quad A_i = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\alpha=1}^n f_\alpha^{(1)} \psi_{\alpha i}\, dt$$
$$(j, i = 1, \ldots, m).$$

Let us assume that the determinant $|A_{ji}|$ of the system (12.10) is different from zero. We put p = 1 and therefore $\varkappa = 1$. The system (12.10) then uniquely determines the constants M_i^0. At the same time the system (12.9) will admit almost periodic solutions of the form (12.5) for which we can write:

$$y_s^{(1)} = M_1^{(1)} \varphi_{s1} + \ldots + M_m^{(1)} \varphi_{sm} + y_s^{(1)*},$$

where $y_s^{(1)*}$ are almost periodic functions and $M_i^{(1)}$ are arbitrary constants. These constants, in exactly the same way as M_i^0, are uniquely determined from the conditions of periodicity of the functions $y_s^{(2)}$. Continuing in this manner we shall obtain an entirely definite system of series (12.4) with coefficients of the form (12.5). These series will converge and after dividing by μ will give the required almost periodic solution of the system (12.1). This solution will thus start with the terms of the (-1)-th order with respect to μ.

3. Let us assume that for the same conditions of resonance as in the preceding case the determinant $|A_{ji}|$ of system (12.10) is equal to zero. For definiteness we shall assume that the rank of this determinant is equal to $k < m-1$ and that the system of equations (12.10) is incompatible. In this case it will be necessary for us to assume $p \geqslant 2$ and therefore $\varkappa = 0$. For the determination of M_i^0 we then obtain a system of homogeneous equations from which it is possible to determine k constants M_i^0 in terms of the m - k remaining ones. For definiteness we shall assume that as the

independent constants may be taken $M^0_{k+1}, \ldots, M^0_m$ so that we can write:

$$M^0_q = \alpha_{q1} M^0_{k+1} + \ldots + \alpha_{q,\,m-k} M^0_m \quad (q = 1, \ldots, k), \qquad (12.11)$$

where α_{qj} are certain entirely definite constants.

Having in this manner chosen the constant M^0_i we obtain an almost periodic solution of equations (12.9) of the following form:

$$y_s^{(1)} = M_1^{(1)} \varphi_{s1} + \ldots + M_m^{(1)} \varphi_{sm} + M^0_{k+1} y^{(1)}_{s,k+1} + \ldots + M^0_m y^{(1)}_{sm}. \qquad (12.12)$$

Here $M_i^{(1)}$ are arbitrary constants and $y_{sj}^{(1)}$ are certain almost periodic functions. The functions $y_{sj}^{(1)}$ will here have the structure (12.5). For y^0_s on the basis of (12.11) we obtain:

$$y^0_s = M^0_{k+1} \varphi^*_{s,k+1} + \ldots + M^0_m \varphi^*_{sm}, \qquad (12.13)$$

where

$$\varphi^*_{sr} = \varphi_{sr} + \sum_{q=1}^{k} \alpha_{q,\,r-k} \varphi_{sq} \qquad (r = k+1, \ldots, m).$$

For $y_s^{(2)}$ we immediately obtain the following equations:

$$\frac{dy_s^{(2)}}{dt} = \sum_{\alpha=1}^{n} a_{s\alpha} y_\alpha^{(2)} + \sum_{\alpha=1}^{n} q_{s\alpha}^{(1)} y_\alpha^{(1)} + \sum_{\alpha=1}^{n} q_{s\alpha}^{(2)} y^0_\alpha + \varkappa f_s^{(1)}. \quad (12.14)$$

where $\varkappa = 1$, if $p = 2$ and $\varkappa = 0$ if $p > 2$.

The conditions of periodicity of the functions $y_s^{(2)}$ on the basis of (12.12) and (12.13) have the form

$$A_{1i} M_1^{(1)} + \ldots + A_{mi} M_m^{(1)} + B_{k+1,\,i} M^0_{k+1} + \ldots + B_{mi} M^0_m + \varkappa A_i = 0 \qquad (12.15)$$

$$(i = 1, \ldots, m),$$

where

$$B_{ri} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\beta,\alpha=1}^{n} (q_{\beta\alpha}^{(1)} y_{\alpha r}^{(1)} + q_{\beta\alpha}^{(2)} \varphi_{\alpha r}^{*}) \psi_{\beta i}\, dt$$

$$(r = k+1, \ldots, m).$$

We have obtained for the determination of $M_1^{(1)}, \ldots, M_m^{(1)}$ a system of linear algebraic equations. Since the determinant $|A_{ji}|$ of this system is equal to zero and has the rank k, in order that this system be compatible it is necessary that the known terms in these equations be connected by $m - k$ relations, which are obtained by the elimination of the magnitudes $M_1^{(1)}, \ldots, M_m^{(1)}$ from these equations.

We thus arrive at equations of the form

$$\delta_{k+1,j} M_{k+1}^0 + \ldots + \delta_{mj} M_m^0 + \delta_j \varkappa = 0 \qquad (12.16)$$

$$(j = 1, \ldots, m-k),$$

where δ_{rj}, δ_j are certain known constants. Let us assume that the determinant $|\delta_{rj}|$ of this system of equations is different from zero. Then from these equations the magnitudes $M_{k+1}^o, \ldots, M_m^{(1)}$ are uniquely determined. Substituting them in (12.15) we obtain a solvable system of equations for $M_i^{(1)}$ from which we find:

$$M_q^{(1)} = a_{q1} M_{k+1}^{(1)} + \ldots + a_{q,m-k} M_m^{(1)} + M_q^{(1)*} \qquad (q = 1, \ldots, k),$$

where $M_q^{(1)*}$ are known constants. The constants $M_{k+1}^{(1)}, \ldots, M_m^{(1)}$ can be determined from the conditions of almost periodicity of the functions $y_s^{(3)}$, in exactly the same way that the constants M_{k+1}^o, M_m^o were determined from the conditions of almost periodicity of the functions $y_s^{(2)}$.

By computing the further approximations in the same way as the preceding we arrive at a completely determined system of series (12.4) which will therefore converge and represent the required almost periodic solution of the system (12.3). The corresponding almost periodic solution of the system (12.1) will now start with the terms of the (-2)-th order with respect to μ.

We had assumed that the system of linear nonhomogeneous equations (12.10) was not compatible. However nothing essentially changes if this system is capable of solution. In this case it will be necessary to put $p = 1$ and $\varkappa = 1$. Instead of (12.11) we then obtain

$$M_q^0 = \alpha_{q1} M_{k+1}^0 + \ldots + \alpha_{q,\, m-k} M_m^0 + M_q^{0*} \qquad (q = 1, \ldots, k),$$

where M_q^0 are certain known constants. This entails only a change of the free terms in equations (12.15).

If the determinant $|\delta_{rj}|$ is also equal to zero it will be necessary to put $k \geqslant 3$ and proceed in the same manner as above. We shall not however dwell here on the detailed investigation of this more complicated case.

S. N. Shimanov [1] proposed another method of finding a particular solution of system (12.1) that is based on a special "method of auxiliary systems" worked out by him. Shimanov considered only the case where the functions $f_s^{(r)}$ are periodic of the same period as the coefficients of the equations. To this case however, as we saw in the preceding section, can be reduced also the general case where the functions $f_s^{(r)}$ have the form (12.2).

We may remark in conclusion that the system (12.1) is a special case of nonautonomous quasilinear systems. In particular, if the functions $f_s^{(r)}$ are purely periodic of period ω the general theory of the periodic solutions of

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[1] Shimanov S.N., K teorii kvazigarmonicheskikh kolebanii (On the Theory of Quasiharmonic Oscillations), Prikl. matem. i mekh., vol. XVI, no. 2, 1952.

quasilinear systems, worked out in detail in chapter II, can be directly applied. In cases 1 and 2 this leads us to the methods discussed in this section. As regards case 3, it belongs to those special cases that were excluded from consideration in Chapter II. If the functions $f_s^{(r)}$ are not periodic but have the general form (12.2) the almost periodic solutions could have been sought by the general methods of chapter IV. But this would have been unsuited and would have let to considerable restrictions. The problem just considered was considerably more simple since the linearity of the equations conditions the possibility of expansion of the required solutions in power series in μ, which in the general case is not possible.

13. Examples of Computation of Forced Oscillations

We shall consider two simple examples.

EXAMPLE 1. We shall compute the forced oscillations of a system described by the equation

$$\frac{d^2x}{dt^2} + (\omega_1^2 + \mu \cos 2\omega_1 t)\, x = A \cos \omega_1 t + B \cos \omega_2 t, \qquad (13.1)$$

where the ratio ω_1/ω_2 is irrational. Replacing equation (13.1) by the equation

$$\frac{d^2y}{dt^2} + (\omega_1^2 + \mu \cos 2\omega_1 t)\, y = \mu^p (A \cos \omega_1 t + B \cos \omega_2 t),$$

we shall try to satisfy it by the series

$$y = M_0 \cos \omega_1 t + N_0 \sin \omega_1 t + \mu y_1 + \dots$$

with almost periodic coefficients. For y_1 we have:

$$\frac{d^2y_1}{dt^2} + \omega_1^2 y_1 + (M_0 \cos \omega_1 t + N_0 \sin \omega_1 t) \cos 2\omega_1 t =$$
$$= \varkappa (A \cos \omega_1 t + B \cos \omega_2 t), \qquad (13.2)$$

where $\varkappa = 1$ if $p = 1$ and $\varkappa = 0$ if it is necessary to put $p \geq 2$. In the given case $\varkappa = p = 1$, since the conditions of almost periodicity of the function y_1, having evidently the form

$$M_0 = 2\varkappa A, \quad N_0 = 0,$$

are solvable. From (13.2) we find

$$y_1 = M_1 \cos \omega_1 t + N_1 \cos \omega_1 t + \frac{B}{\omega_1^2 - \omega_2^2} \cos \omega_2 t + \frac{A}{8\omega_1^2} \cos 3\omega_1 t,$$

where the arbitrary constants M_1 and N_1 are determined from the conditions of almost periodicity of the function y_2. The further computations are conducted in the usual manner and we shall not present them here. Thus, the almost periodic solution of equation (13.1) has the form

$$x = 2\mu^{-1} A \cos \omega_1 t +$$
$$+\left(M_1 \cos \omega_1 t + N_1 \sin \omega_1 t + \frac{B}{\omega_1^2 - \omega_2^2} \cos \omega_2 t + \frac{A}{8\omega_1^2} \cos 3\omega_1 t \right) + \mu(\ldots) + \ldots$$

EXAMPLE 2. Let us find the particular periodic solution of the equation

$$\frac{d^2x}{dt^2} + [1 + \mu(5 + 6\cos 2t + 8\sin 2t)]\, x = A\cos t + B\sin t. \quad (13.3)$$

For this purpose we replace this equation by the equation

$$\frac{d^2y}{dt^2} + [1 + \mu(5 + 6\cos 2t + 8\sin 2t)]\, y = \mu^p (A\cos t + B\sin t)$$

and set:

$$y = M_0 \cos t + N_0 \sin t + \mu y_1 + \mu^2 y_2 + \ldots$$

We shall have:

$$\frac{d^2y_1}{dt^2} + y_1 + (5 + 6\cos 2t + 8\sin 2t)(M_0 \cos t + N_0 \sin t) =$$
$$= \varkappa(A\cos t + B\sin t), \quad (13.4)$$

where $\varkappa = 1$ if it is possible to put $p = 1$, and $\varkappa = 0$ if it is necessary to assume that $p \geqslant 2$. The conditions of periodicity of the function y_1 have the form

$$8M_0 + 4N_0 = \varkappa A, \quad 4M_0 + 2N_0 = \varkappa B. \quad (13.5)$$

It is here necessary to consider two different cases depending on whether the magnitude A - 2B is different from or equal to zero.

Let us first assume that $A - 2B \neq 0$. In this case equations (13.5) will be incompatible and it is necessary to put $\varkappa = 0$, i.e. $p \geqslant 2$. On this assumptions conditions (13.5) will give only one equation

$$2M_0 + N_0 = 0. \tag{13.6}$$

The second equation connecting the magnitudes M_0 and N_0 is obtained from the conditions of compatibility of the equations expressing the conditions of periodicity of the function y_2.

Let us however continue our computations. From (13.4) we shall now have:

$$y_1 = M_1 \cos t + N_1 \sin t + \alpha_1 \cos 3t + \beta_1 \sin 3t,$$

where

$$\alpha_1 = \frac{1}{8}(3M_0 - 4N_0) \quad \beta_1 = \frac{1}{8}(4M_0 + 3N_0), \tag{13.7}$$

and for y_2 the equation is obtained

$$\begin{aligned}\frac{d^2y_2}{dt^2} + y_2 &+ \\ &+ (5 + 6\cos 2t + 8\sin 2t)(M_1 \cos t + N_1 \sin t + \alpha_1 \cos 3t + \beta_1 \sin 3t) = \\ &= \varkappa (A \cos t + B \sin t).\end{aligned} \tag{13.8}$$

where $\varkappa = 1$ if $p = 2$ and $\varkappa = 0$ if $p \geqslant 3$.

The conditions of periodicity of the function y_2 have the form

$$8M_1 + 4N_1 + 3\alpha_1 + 4\beta_1 = A\varkappa, \quad 4M_1 + 2N_1 - 4\alpha_1 + 3\beta_1 = B\varkappa, \tag{13.9}$$

and from the conditions of compatibility of these equations we obtain:

$$11\alpha_1 - 2\beta_1 = (A - 2B)\varkappa \tag{13.10}$$

or, taking (13.7) into account,

$$25M_0 - 50N_0 = 8(A-2B)\varkappa.$$

This will be the second equation of M_0 and N_0. It does not contradict condition (13.6) and we may therefore assume $\varkappa = 1$ and consequently $p = 2$. We thus obtain:

$$M_0 = \frac{8}{125}(A-2B), \quad N_0 = \frac{16}{25}(2B-A), \tag{13.11}$$

and for M_1 and N_1 there remains only one of equations (13.9), which after substituting the values α_1 and β_1 assumes the form

$$20M_1 + 10N_1 = 2A + B. \tag{13.12}$$

From (13.8) we now find:

$$y_2 = M_2 \cos t + N_2 \sin t + \alpha_2 \cos 3t + \beta_2 \sin 3t + a_2 \cos 5t + b_2 \sin 5t,$$

where

$$\left.\begin{aligned} \alpha_2 &= \frac{1}{8}(3M_1 - 4N_1 + 5\alpha_1), \\ \beta_2 &= \frac{1}{8}(4M_1 + 3N_1 + 5\beta_1), \end{aligned}\right\} \tag{13.13}$$

and a_2 and b_2 are certain constants which we need not write out.

The equation for y_3 has the form

$$\frac{d^2 y_3}{dt^2} + y_3 + (5 + 6\cos 2t + 8 \sin 2t)\, y_2 = 0,$$

and the conditions of periodicity give:

$$8M_2 + 4N_2 + 3\alpha_2 + 4\beta_2 = 0, \quad 4M_2 + 2N_2 - 4\alpha_2 + 3\beta_2 = 0,$$

whence we obtain

$$11\alpha_2 - 2\beta_2 = 0$$

or, on the basis of (13.13) and (13.10),

$$5M_1 - 10N_1 = -A + 2B. \tag{13.14}$$

From (13.14) and (13.12) we find:

$$M_1 = \frac{1}{25}(A + 3B), \qquad N_1 = \frac{1}{50}(6A - 7B).$$

Subsituting the found values of M_o, N_o, M_1, N_1, in y_o and y_1 we obtain the following particular periodic solution of equation (13.3):

$$x = \mu^{-2}\left\{\frac{8}{125}(A - 2B)\cos t + \frac{16}{125}(2B - A)\sin t + \right.$$
$$+ \mu\left[\frac{1}{25}(A + 3B)\cos t + \frac{1}{50}(6A - 7B)\sin t + \right.$$
$$\left.\left. + \frac{11}{125}(A - 2B)\cos 3t + \frac{2}{125}(2B - A)\sin 3t\right] + \ldots\right\}. \tag{13.15}$$

Let us now assume that A - 2B = 0. In this case equations (13.5) will be compatible and it is necessary to put $\varkappa = p = 1$. Instead of (13.6) we shall now have:

$$2M_0 + N_0 = B. \tag{13.16}$$

The expression for y_1 is found the same as before while for y_2 we obtain, instead of (13.8), the equation

$$\frac{d^2y_2}{dt^2} + y_2 + (5 + 6\cos 2t + 8\sin 2t)(M_1\cos t + N_1\sin t +$$
$$+ \alpha_1\cos 3t + \beta_1\sin 3t) = 0.$$

As a result, in place of (13.10) we shall now have:

$$11\alpha_1 - 2\beta_1 = 0$$

or, on the basis of (13.7),

$$M_0 - 2N_0 = 0. \tag{13.17}$$

From (13.16) and (13.17) we find:

$$M_0 = \frac{1}{5}B, \qquad N_0 = \frac{1}{10}B.$$

The further approximations are computed in the same way as before. Thus, for the periodic solution of equation (13.3) for A - 2B = 0 we have:

$$x = \mu^{-1}\left\{\frac{1}{5}B\cos t + \frac{1}{10}B\sin t + \mu(\ldots) + \ldots\right\}.$$

It agrees with expression (13.15) if we put in the latter A = 2B.

CHAPTER VI

APPROXIMATIONS TO ARBITRARY NONLINEAR SYSTEMS

1. Periodic Solutions of Nonautonomxous Systems in the Case of an Isolated Generating Solution

In the preceding chapters we considered systems approximating linear systems. Many general results there obtained may however be immediately extended also to systems of a more general form.

Let us assume that the oscillations of a system are described by equations of the form

$$\frac{dx_s}{dt} = X_s(t, x_1, \ldots, x_n) + \mu f_s(t, x_1, \ldots, x_n, \mu) \qquad (1.1)$$
$$(s = 1, \ldots, n),$$

where the right hand sides are defined for all real values of t, for values of μ lying on a certain segment $[0, \mu_0]$, and the values of $x_1, \ldots, x_n$ lying in a certain closed region G of the space of these variables. We shall assume that in this region of variation of the variables the right hand sides of equations (1.1) are continuous and periodic with respect to t with period ω, f_s admit continuous partial derivatives of the first order with respect to the variables $x_1, \ldots, x_n$, μ, while the functions X_s do not depend on μ and admit continuous derivatives of the second order with respect to $x_1, \ldots, x_n$.

Let us consider the generating system

$$\frac{dx_s^0}{dt} = X_s(t, x_1^0, \ldots, x_n^0) \qquad (1.2)$$

and assume that it admits a periodic solution

$$x_s^0 = \varphi_s(t) \tag{1.3}$$

of period ω, lying in the region G. Taking it as the generating solution we shall seek the conditions for which there corresponds to it a periodic solution of the complete system (1.1). In other words, we shall seek the conditions for which the system (1.1) admits a periodic solution reducing for $\mu = 0$ to (1.3).

For this purpose let us denote by

$$x_s(t, \gamma_1, \ldots, \gamma_n, \mu)$$

a solution of system (1.1) with the initial conditions

$$x_s(0, \gamma_1, \ldots, \gamma_n, \mu) = \gamma_s. \tag{1.4}$$

We here evidently have:

$$x_s(t, \varphi_1(0), \ldots, \varphi_n(0), 0) \equiv \varphi_s(t). \tag{1.5}$$

For this solution to be periodic it is necessary and sufficient that the equations be satisfied

$$\psi_s(\gamma_1, \ldots, \gamma_n, \mu) = x_s(\omega, \gamma_1, \ldots, \gamma_n, \mu) - \gamma_s = 0, \tag{1.6}$$

where the functions ψ_s in the neighborhood of the point $\mu = 0$, $\gamma_1, \ldots, \gamma_n$, μ, since the right hand sides of equations (1.1) admit continuous derivatives with respect to $x_1, \ldots, x_n$, μ.

In virtue of (1.5) conditions (1.6) are satisfied for $\mu = 0$, $\gamma_s = \varphi_s(0)$, since the generating solution is periodic. Hence if also the condition is satisfied

$$\left\{ \frac{\partial(\psi_1, \ldots, \psi_n)}{\partial(\gamma_1, \ldots, \gamma_n)} \right\}_{\gamma_s = \varphi_s(0),\ \mu = 0} \neq 0, \tag{1.7}$$

equations (1.6) will admit for sufficiently small μ one and only one solution $\gamma_s = \gamma_s(\mu)$, for which $\gamma_s(0) = \varphi_s(0)$. Substituting this solution in the functions $x_x(t, \gamma_1, \ldots, \gamma_n, \mu)$ we obtain the periodic solution of system (1.1) reducing for $\mu = 0$ to the generating solution. We thus arrive at the following theorem, by Poincaré:[1]

IF CONDITION (1.7) IS SATISFIED THE SYSTEM (1.1) FOR μ SUFFICIENTLY SMALL ADMITS ONE AND ONLY ONE PERIODIC SOLUTION REDUCING TO THE GENERATING SOLUTION (1.3) FOR $\mu = 0$.

The theorem of Poincare can be formulated in still another way. For this purpose let us consider the system of linear equations with periodic coefficients

$$\frac{dy_s}{dt} = p_{s1}y_1 + \ldots + p_{sn}y_n \qquad (s = 1, \ldots, n), \tag{1.8}$$

which are the equations in variations of system (1.2) for the solution (1.3). Here the coefficients p_{sj} are continuous periodic functions of t of period ω, that are determined by the formulas

$$p_{sj} = \left(\frac{\partial X_s}{\partial x_j}\right) = \frac{\partial X_s(t, \varphi_1, \ldots \varphi_n)}{\partial \varphi_j}, \tag{1.9}$$

where the parentheses denote that the derivatives have been computed for the generating solution (1.3).

Since the functions $x_s(t, \gamma_1, \ldots, \gamma_n, 0)$ are a solution of equations (1.2), depending on the n arbitrary constants $\gamma_1, \ldots, \gamma_n$ and reducing to the solution (1.3) for $\gamma_s = \varphi_s(0)$, then, as was shown in sec. 1 of chapter III, the derivatives of these functions with respect to the arbitrary constants, computed for $\gamma_s = \varphi_s(0)$, determine the particular solutions of the equations in variations (1.8). The system (1.8) thus has n particular solutions $y_{s1}, \ldots, y_{sn}$, determined by the formulas

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[1] Poincaré A., Les méthodes nouvelles de la méchanique céleste, vol. 1, ch. III, Paris, 1892. Poincaré assumed that the right hand sides of equations (1.1) are analytic with respect to $x_1, \ldots, x_n$, μ.

$$y_{si}(t) = \left\{ \frac{\partial x_s(t, \gamma_1, \ldots, \gamma_n, 0)}{\partial \gamma_i} \right\}_{\gamma_j = \varphi_j(0)} \qquad (s, i = 1, \ldots, n). \qquad (1.10)$$

Here, as usual, the first index denotes the number of the function in the solution and the second index the number of the solution. From (1.10) and (1.4) we find:

$$y_{sj}(0) = \delta_{sj}, \qquad (1.11)$$

where δ_{sj} is the Kronecker symbol. From this it follows that the determinant

$$D(\rho) = \begin{vmatrix} y_{11}(\omega) - \rho & y_{21}(\omega) & \ldots y_{n1}(\omega) \\ y_{12}(\omega) & y_{22}(\omega) - \rho & \ldots y_{n2}(\omega) \\ \ldots & \ldots & \ldots \\ y_{1n}(\omega) & y_{2n}(\omega) & \ldots y_{nn}(\omega) - \rho \end{vmatrix} \qquad (1.12)$$

is the characteristic determinant of the system (1.8).

Further, from (1.6) and (1.10) we obtain:

$$\left(\frac{\partial \psi_s}{\partial \gamma_j} \right)_{\gamma_i = \varphi_i(0),\, \mu = 0} = y_{sj}(\omega) - \delta_{sj}. \qquad (1.13)$$

Hence

$$\left\{ \frac{\partial(\psi_1, \ldots, \psi_n)}{\partial(\gamma_1, \ldots, \gamma_n)} \right\}_{\gamma_i = \varphi_i(0),\, \mu = 0} = \begin{vmatrix} y_{11}(\omega) - 1 & y_{21}(\omega) & \ldots y_{n1}(\omega) \\ y_{12}(\omega) & y_{22}(\omega) - 1 & \ldots y_{n2}(\omega) \\ \ldots & \ldots & \ldots \\ y_{1n}(\omega) & y_{2n}(\omega) & \ldots y_{nn}(\omega) - 1 \end{vmatrix} = D(1) \qquad (1.14)$$

and condition (1.7) is equivalent to the requirement that the characteristic equation of system (1.8) does not have a root equal to 1 or, what amounts to the same thing, that this system does not have periodic solutions.

We shall say that the periodic solution (1.3) of system (1.2) is an ISOLATED solution if the equations in variations corresponding to it do not have periodic solutions. We can therefore express the theorem of Poincaré also in the following form:

FOR ANY ISOLATED GENERATING PERIODIC SOLUTION THE SYSTEM (1.1) FOR SUFFICIENTLY SMALL μ ADMITS ONE AND ONLY ONE PERIODIC SOLUTION WHICH REDUCES TO THIS GENERATING SOLUTION FOR $\mu = 0$.

If equations (1.1) are analytic with respect to μ, $x_1, \ldots, x_n$ the above periodic solution, as is easily seen, will be analytic with respect to μ.

2. Periodic Solutions of Nonautonomous Systems in the Case of a Family of Generating Solutions

From the theorem of Poincaré established in the preceding section it follows that in the case of an isolated generating solution a complete correspondence exists between the complete system (1.1) and the simplified system (1.2). For the theory of nonlinear oscillations however precisely those cases are of greatest interest in which such correspondence between the complete and simplified systems does not exist, which is possible only when the generating solution is not an isolated one. The most important case of this kind, as we found in the theory of quasilinear systems, is that where the generating solution under consideration belongs to a family depending on a certain number of parameters. We shall now turn to consider this case.

Let us assume that the generating system (1.2) admits the family of periodic solutions of period ω

$$x_s^0 = \varphi_s(t, h_1, \ldots, h_k) \qquad (s = 1, \ldots, n), \tag{2.1}$$

depending on $k \leqslant n$ arbitrary parameters h_i and that the generating solution in question corresponds to the values $h_i = h_i^*$ of these parameters. We shall here assume that the generating solution lies in the region G. In correspondence with the assumed condition that the functions X_s in equations (1.2) admit continuous partial derivatives of the second order with respect to $x_1, \ldots, x_n$ we shall assume that in the neighborhood of the generating solution the functions φ_s are twice differentiable with respect to the parameters h_i. Moreover, we shall assume that the independence of these parameters is assured by the circumstance that at least one of the determinants of the k-th order contained in the matrix

$$\left\| \begin{matrix} \frac{\partial \varphi_1}{\partial h_1} \cdots \frac{\partial \varphi_1}{\partial h_k} \\ \cdots\cdots \\ \frac{\partial \varphi_n}{\partial h_1} \cdots \frac{\partial \varphi_n}{\partial h_k} \end{matrix} \right\| \tag{2.2}$$

does not reduce to zero for $h_i = h_i^*$, $t = 0$ and therefore these parameters can be expressed in terms of k initial values of the magnitudes x_s^0

We shall denote, as in the preceding section, by

$$x_s(t, \gamma_1, \ldots, \gamma_n, \mu)$$

the solution of equations (1.1) with the initial conditions (1.4). Then, as before, the conditions of periodicity of this solution will have the form (1.6). These conditions will be satisfied for the generating solution, i.e. for $\mu = 0$, $\gamma_s = \varphi_s(0, h_1^*, \ldots, h_k^*)$. But in contrast to the case of the isolated generating solution conditions (1.6) will also be satisfied for $\mu = 0$, $\gamma_s = \varphi_s(0, h_1, \ldots, h_k)$, since all solutions (2.1) of the generating system are periodic. In other words, the equations (1.6) have for $\mu = 0$ not one solution, as was the case before, but a family of solutions $\gamma_s = \varphi_s(0, h_1, \ldots, h_k)$ depending on k arbitrary constants. As a result the functional determinant

$$\left\{ \frac{\partial(\psi_1, \ldots, \psi_n)}{\partial(\gamma_1, \ldots, \gamma_n)} \right\}_{\mu=0,\quad \gamma_s=\varphi_s(0, h_1^*, \ldots, h_k^*)} \tag{2.3}$$

necessarily reduces to zero. Moreover, all the minors of this determinant up to the $(n - k + 1)$-th order inclusive must likewise necessarily reduce to zero. Of this we may convince ourselves also in the following manner.

Since the system (1.2) admits the solution $x_s^0 = \varphi_s(t, h_1, \ldots, h_k)$ depending on k arbitrary parameters, the functions

$$\varphi_{si}(t)=\left\{\frac{\partial\varphi_s(t, h_1, \ldots, h_k)}{\partial h_i}\right\}_{h_j=h_j^*} \tag{2.4}$$

$$(s=1, \ldots, n;\ i=1, \ldots, k)$$

determine k particular solutions of the equations in variations (1.8). But these functions are evidently periodic of period ω. Thus the system of equations in variations (1.8) has k particular periodic solutions. Consequently the characteristic equation of this system has a root equal to 1 which has a multiplicity not less than k and which reduces to zero all the minors of the characteristic determinant at least up to the $(n - k + 1)$-th order. From this on the basis of (1.14) we convince ourselves of the correctness of our assertion in regard to the functional determinant (2.3).

We shall assume that at least one of the minors of the $(n-k)$-th order of the determinant (2.3) is different from zero or, what amounts to the same thing, that the system in variations (1.8) has precisely k periodic solutions. Let us assume for definiteness that

$$\left\{\frac{\partial(\psi_1, \ldots, \psi_{n-k})}{\partial(\gamma_1, \ldots, \gamma_{n-k})}\right\}_{\mu=0,\quad \gamma_s=\varphi_s(0, h_1^*, \ldots, h_k^*)} \neq 0. \tag{2.5}$$

For this condition the first $n - k$ of equations (1.6) have a solution for $\gamma_1,\ldots,\gamma_{n-k}$, in which these magnitudes are functions of $\gamma_{n-k+1},\ldots,\gamma_n$, μ reducing to

$$\varphi_1(0, h_1^*, \ldots, h_k^*), \ldots, \varphi_{n-k}(0, h_1^*, \ldots, h_k^*)$$

for

$$\mu=0, \quad \gamma_j=\varphi_j(0, h_1^*, \ldots, h_k^*) \qquad (j=n-k+1, \ldots, n)$$

and possessing in the neighborhood of this point continuous partial derivatives of the first order. Substituting these magnitudes in the last k of equations (1.6) we obtain for the determination of $\gamma_{n-k+1},\ldots, \gamma_n$ a system of k equations. The left hand sides of the equations thus obtained will have in the neighborhood of the point $\mu = 0$, $\gamma_j = \varphi_j(0,h_1^*,\ldots,h_k^*)$

$(j = n-k+1,\dots,n)$ continuous partial derivatives of the first order.

The above equations can therefore be represented in the form

$$\Phi_i = F_i(\gamma_{n-k+1}, \dots, \gamma_n) + \mu\Psi_i(\gamma_{n-k+1}, \dots, \gamma_n, \mu) = 0 \qquad (2.6)$$
$$(i = 1, \dots, k),$$

where F_i and Ψ_i have near the values $\mu = 0$, $\gamma_j = \varphi_j(0, h_1^*, \dots, h_k^*)$ continuous partial derivatives of the first order with respect to the variables $\gamma_{n-k+1}, \dots, \gamma_n$.

Since the system (1.6) has for $\mu = 0$ the solution

$$\gamma_s = \varphi_s(0, h_1, \dots, h_k) \qquad (s = 1, \dots, n),$$

depending on k arbitrary parameters the system (2.6) should also for the same condition have a solution for $\gamma_{n-k+1}, \dots, \gamma_n$ depending on k arbitrary parameters, which is possible only in the case where the functions F_i reduce identically to zero. Thus, equations (2.6) after dividing by μ assume the form

$$\Psi_i(\gamma_{n-k+1}, \dots, \gamma_n, \mu) = 0 \quad (i = 1, \dots, k). \qquad (2.7)$$

For the existence of the required periodic solution it is necessary and sufficient that system (2.7) admit a solution with respect to $\gamma_j (j = n-k+1,\dots,n)$ in which these magnitudes reduce to $\varphi_j(0, h_1, \dots, h_k^*)$ for $\mu = 0$. And for this it is necessary first of all that the relations be satisfied

$$P_i(h_1^*, \dots, h_k^*) =$$
$$= \Psi_i(\varphi_{n-k+1}(0, h_1^*, \dots, h_k^*), \dots, \varphi_n(0, h_1^*, \dots, h_k^*), 0) = 0 \quad (2.8)$$
$$(i = 1, \dots, k).$$

Thus, the necessary conditions have been obtained which the values of the parameters $h_1^*, \dots, h_k^*$ of the generating solution must satisfy in order that there correspond to it a periodic solution of the complete system (1.1). If, with conditions (2.8) satisfied, there will also be satisfied the condition

$$\Delta = \left\{ \frac{\partial(\Psi_1, \ldots, \Psi_k)}{\partial(\gamma_{n-k+1}, \ldots, \gamma_n)} \right\}_{\mu=0, \quad \gamma_j=\varphi_j(0, h_1^*, \ldots, h_k^*)} \neq 0, \qquad (2.9)$$

the system (2.7) will actually admit a solution, the only one for $\gamma_j (j = n-k+1, \ldots, n)$, of the required form, and then, as was shown above, there will also exist the required periodic solution.

We shall show that condition (2.9) is equivalent to the condition

$$\frac{\partial(P_1, \ldots, P_k)}{\partial(h_1^*, \ldots, h_k^*)} \neq 0. \qquad (2.10)$$

We have, first of all:

$$\frac{\partial(P_1, \ldots, P_k)}{\partial(h_1^*, \ldots, h_k^*)} = \Delta \left\{ \frac{\partial(\varphi_{n-k+1}, \ldots, \varphi_n)}{\partial(h_1, \ldots, h_k)} \right\}_{t=0, h_j=h_j^*},$$

and therefore it is sufficient for us to show that the functional determinant

$$A = \left\{ \frac{\partial(\varphi_{n-k+1}, \ldots, \varphi_n)}{\partial(h_1, \ldots, h_k)} \right\}_{t=0, h_j=h_j^*}$$

is different from zero. Of the correctness of the latter assertion we can convince ourselves in the following way.

Let us consider the system of linear algebraic equations

$$\left(\frac{\partial\psi_s}{\partial\gamma_1}\right) A_1 + \ldots + \left(\frac{\partial\psi_s}{\partial\gamma_n}\right) A_n = 0 \qquad (s = 1, \ldots, n), \quad (2.11)$$

where the parentheses denote that the derivatives are computed at the point $\mu = 0$, $\gamma_s = \varphi_s(0, h_1^*, \ldots, h_k^*)$ ($s = 1, \ldots, n$). Since the determinant together with all its minors up to the $(n - k + 1)$-th order reduce to zero but at least one minor of the $(n-k)$-th order is different from zero, the system (2.11) admits k independent solutions $A_{s1}, \ldots, A_{sk}$. On the basis of (2.5) this system of equations can be chosen in the following way. We set

$$A_{n-k+i,j}=\begin{cases}1 & \text{for } i=j,\\ 0 & \text{for } i\neq j\end{cases} \qquad (i,\ j=1,\ 2,\ \ldots,\ k), \qquad (2.12)$$

and determine the magnitudes $A_{1j},\ldots,A_{n-k,j}$ from the first $n - k$ of equations (2.11). It is possible however to indicate another system of linearly independent solutions of equations (2.11).

For this we recall that equations (1.6) are identically satisfied for $\mu = 0$, $\gamma_s = \varphi_s(0,h_1,\ldots,h_k)$. Differentiating these identities with respect to h_j and then putting $h_i = h_i^*$ we shall have:

$$\left(\frac{\partial\psi_s}{\partial\gamma_1}\right)\frac{\partial\varphi_1(0,\ h_1^*,\ \ldots,\ h_k^*)}{\partial h_j^*}+\ldots+\left(\frac{\partial\psi_s}{\partial\gamma_n}\right)\frac{\partial\varphi_n(0,\ h_1^*,\ \ldots,\ h_k^*)}{\partial h_j^*}=0.$$

System (2.11) therefore has the solutions

$$B_{sj}=\frac{\partial\varphi_s(0,\ h_1^*,\ \ldots,\ h_k^*)}{\partial h_j^*} \qquad (s=1,\ \ldots,\ n; \quad j=1,\ \ldots,\ k).$$

But then we must have:

$$B_{sj}=\sum_{\alpha=1}^{k} a_{\alpha j}A_{s\alpha} \qquad (s=1,\ \ldots,\ n; \quad j=1,\ \ldots,\ k), \qquad (2.13)$$

where $a_{\alpha j}$ are certain constants. From these relations it follows that all possible determinants of the k-th order of the matrix (2.2) are equal for $t = 0$, $h_i = h_i^*$ to the product of the determinant $|a_{\alpha j}|$ by the corresponding determinant of the magnitudes $A_{s\alpha}$. But since by assumption at least one determinant of the k-th order of the matrix (2.2) for $t = 0$, $h_i = h_i^*$ is different from zero, the determinant $|a_{\alpha j}|$ is also not equal to zero. Putting now in (2.13)

$$s = n - k + 1,\ldots, n$$

and taking (2.12) into consideration we shall have

$$a_{\alpha j}=\frac{\partial\varphi_{n-k+\alpha}(0,\ h_1^*,\ \ldots,\ h_k^*)}{\partial h_j^*} \qquad (\alpha,\ j=1,\ \ldots,\ k),$$

and therefore the determinant A is equal to $|a_{\alpha j}|$, and, as was stated, is different from zero.

We shall now find the developed form of equations (2.8). As was already remarked, the functions ψ_r admit in the neighborhood of the point $\gamma_s = \varphi_s(0, h_1^*, \ldots, h_k^*)$, $\mu = 0$ continuous partial derivatives of the first order. Hence, setting $\beta_s = \gamma_s - \varphi_s(0, h_1^*, \ldots, h_k^*)$, we can represent equations (1.6) in the form

$$\psi_s = (u_{s1} + U_{s1})\beta_1 + \ldots + (u_{sn} + U_{sn})\beta_n + (v_s + V_s)\mu = 0, \quad (2.14)$$

where

$$u_{si} = \left(\frac{\partial \psi_s}{\partial \gamma_i}\right), \quad v_s = \left(\frac{\partial \psi_s}{\partial \mu}\right), \quad (2.15)$$

and U_{si} and V_s are continuous functions of $\beta_1, \ldots, \beta_n$, μ reducing to zero for $\beta_1 = \ldots = \beta_n = \mu = 0$. From (2.14) it follows immediately that equations (2.8) are obtained as a result of the elimination of the magnitudes $\beta_1, \ldots, \beta_n$ from the linear equations

$$u_{s1}\beta_1 + \ldots + u_{sn}\beta_n + v_s\mu = 0. \quad (2.16)$$

Let us consider these equations in more detail. On the basis of (1.13) we can write:

$$u_{si} = y_{si}^*(\omega, 0) - \delta_{si}, \quad (2.17)$$

where $y_{si}^*(t, \tau)$ are a fundamental system of solutions of equations (1.8) with the initial conditions (for $t = \tau$)

$$y_{si}^*(\tau, \tau) = \delta_{si}. \quad (2.18)$$

As to the magnitudes v_s, for them we find on the basis of (1.6)

$$v_s = \left(\frac{\partial x_s(\omega, \gamma_1, \ldots, \gamma_n, \mu)}{\partial \mu}\right). \quad (2.19)$$

But the functions $x_s(t, \gamma_1, \ldots, \gamma_n, \mu)$ satisfy equations (1.1). Substituting them in these equations and differentiating the obtained identities with respect to μ we shall have

$$\frac{d}{dt}\frac{\partial x_s}{\partial \mu} \equiv \sum_{\alpha=1}^{n} \frac{\partial X_s}{\partial x_\alpha}\frac{\partial x_\alpha}{\partial \mu} + \frac{\partial}{\partial \mu}\mu f_s(t, x_1, \ldots, x_n, \mu).$$

Putting here $\mu = 0$, $\gamma_s = \varphi_s(0, h_1^*, \ldots, h_k^*)$ and taking (1.9) into account we obtain:

$$\frac{d}{dt}\left(\frac{\partial x_s}{\partial \mu}\right) = \sum_{\alpha=1}^{n} p_{s\alpha}\left(\frac{\partial x_\alpha}{\partial \mu}\right) + f_s(t, \varphi_1, \ldots, \varphi_n, 0)_{h_i=h_i^*}. \quad (2.20)$$

Moreover, form (1.4) we find:

$$\frac{\partial x_s(0, \gamma_1, \ldots \gamma_n, \mu)}{\partial \mu} = 0.$$

The functions $\partial x_s(t, \gamma_1, \ldots, \gamma_n, \mu)/\partial \mu$ therefore form a particular solution of equations (2.20) with zero initial values. But then, by simple verification, we easily convince ourselves that

$$\left(\frac{\partial x_s(t, \gamma_1, \ldots, \gamma_n, \mu)}{\partial \mu}\right) =$$

$$= \int_0^t \sum_{\alpha=1}^{n} [f_\alpha(\tau, \varphi_1, \ldots, \varphi_n, 0)]_{t=\tau,\, h_i=h_i^*}\, y_{s\alpha}^*(t, \tau)\, d\tau.$$

Having established this we can, on the basis of (2.17) and (2.19), write equations (2.16) in the following manner:

$$\sum_{\alpha=1}^{n} y_{s\alpha}^*(\omega, 0)\beta_\alpha - \beta_s +$$

$$+ \mu \int_0^\omega \sum_{\alpha=1}^{n} [f_\alpha(\tau, \varphi_1, \ldots, \varphi_n, 0)]_{t=\tau,\, h_i=h_i^*}\, y_{s\alpha}^*(\omega, \tau)\, d\tau = 0, \quad (2.21)$$

and the problem reduces to the elimination of the magnitudes $\beta_1, \ldots, \beta_n$ from these equations. For this purpose let us consider the system of equations

$$\frac{dz_s}{dt} + p_{1s}z_1 + \ldots + p_{ns}z_n = 0, \quad (2.22)$$

conjugate to the system of variations (1.8). As we have seen, the system (1.8) admits the k periodic solutions (2.4). Hence the system (2.22) likewise admits k periodic solutions, which we shall denote by $\psi_{s1}(t),\ldots,\psi_{sk}(t)$. From the relation between the solutions of conjugate systems we easily find on the basis of (2.18) the identities

$$\sum_{\alpha=1}^{n} y^{*}_{s\alpha}(t,\tau)\,\psi_{\alpha i}(t)=\psi_{si}(\tau), \tag{2.23}$$

which are what we use for eliminating $\beta_1,\ldots,\beta_n$ from (2.21). Thus we obtain from (2.21):

$$\sum_{s,\alpha=1}^{n} y^{*}_{s\alpha}(\omega,0)\,\psi_{si}(\omega)\,\beta_\alpha-\sum_{s=1}^{n}\beta_s\psi_{si}(\omega)+$$
$$+\mu\int_0^{\omega}\sum_{s,\alpha=1}^{n}[f_\alpha(\tau,\ldots)]_{t=\tau,\,h_i=h_i^*}\,y^{*}_{s\alpha}(\omega,\tau)\,\psi_{si}(\omega)\,d\tau=0,$$

and identities (2.23) give:

$$\mu\int_0^{\omega}\sum_{\alpha=1}^{n}[f_\alpha(\tau,\ldots)]_{t=\tau,\,h_i=h_i^*}\,\psi_{\alpha i}(\tau)\,d\tau=0\qquad(i=1,\ldots,k).$$

This will be the required result of the elimination of $\beta_1,\ldots,\beta_n$ from equations (2.16). Equations (2.8) determining the parameters of the generating solution can thus be represented in the following form:

$$P_i(h_1^*,\ldots,h_k^*)=$$
$$=\int_0^{\omega}\sum_{\alpha=1}^{n} f_\alpha(t,\varphi_1(t,\,h_1^*,\ldots,h_k^*),\ldots,\varphi_n(t,\,h_1^*,\ldots,h_k^*),\,0)\,\psi_{\alpha i}(t)\,dt=0$$
$$(i=1,\ldots;k), \tag{2.24}$$

and we arrive finally at the following theorem: [1]

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[1] Malkin I.G., K teorii periodicheskikh reshenii Puankare (On Poincaré's Theory of Periodic Solutions), Prikl. matem. i mekh., vol. XIII, no. 6, 1949.

THEOREM. IN ORDER THAT THE SYSTEM (1.1) ADMIT A PERIODIC SOLUTION REDUCING FOR $\mu = 0$ TO A GENERATING SOLUTION BELONGING TO THE FAMILY (2.1) IT IS NECESSARY THAT THE PARAMETERS h_i OF THIS GENERATING SOLUTION SATISFY THE SYSTEM OF EQUATIONS (2.24). TO EACH SIMPLE SOLUTION OF THIS SYSTEM OF EQUATIONS, i.e. TO EACH SOLUTION FOR WHICH CONDITION (2.10) IS SATISFIED, THERE ACTUALLY CORRESPONDS ONE AND ONLY ONE PERIODIC SOLUTION OF SYSTEM (1.1).

From the preceding analysis it follows also that in the periodic solution thus obtained of system (1.1) the functions x_s will possess continuous derivatives of the first order with respect to μ. If the right hand sides of equations (1.1) admit with respect to the variables $x_1,\ldots,x_n,\mu$ partial derivatives up to the order p the functions x_s in the periodic solution will admit derivatives with respect to μ also up to order p. If the right hand sides of equations (1.1) are analytic with respect to $x_1,\ldots,x_n$, μ the obtained periodic solutions will also be analytic with respect to μ. In fact, in this case the functions $x_s(t,\gamma_1,\ldots,\gamma_n,\mu)$ will be analytic with respect to $\gamma_1\ldots,\gamma_n$, μ and the solutions $\gamma_s = \gamma_s(\mu)$ of equations (2.7) and (1.6) will be analytic with respect to μ.

In practice the case may also be of interest where equations (2.24) are satisfied identically. In this case the functions Ψ_i in equations (2.7) have the form

$$\Psi_i = \mu^p \Psi_i^*(\gamma_{n-k+1}, \ldots, \gamma_n, \mu),$$

where p is a certain integer. It is here assumed that the functions Ψ_i have derivatives with respect to μ of an order not less than p. For this it is sufficient that the right hand sides of equations (1.1) have partial derivatives with respect to μ, $x_1,\ldots,x_n$ up to order $p + 1$. Equations (2.8) are now replaced by the equations

$$P_i^*(h_1^*, \ldots, h_k^*) = \\ = \Psi_i^*(\varphi_{n-k+1}(0, h_1^*, \ldots, h_k^*), \ldots, \varphi_n(0, h_1^*, \ldots, h_k^*), 0) = 0 \quad (2.25)$$
$$(i = 1, \ldots, k)$$

and for each solution of these equations for which

$$\frac{\partial (P_1^*, \ldots, P_k^*)}{\partial (h_1^*, \ldots, h_k^*)} \neq 0, \tag{2.26}$$

there will in fact exist a periodic solution of the system (1.1), and which is moreover unique.

3. Case of Analytic Equations

Let us assume that the right hand sides of equations (1.1) are analytic with respect to $x_1, \ldots, x_n$, μ so that these equations can be represented in the form

$$\frac{dx_s}{dt} = X_s(t, x_1, \ldots, x_n) + \mu f_s^{(1)} + \mu^2 f_s^{(2)} + \ldots \quad (s = 1, \ldots, n), \tag{3.1}$$

where the functions $f_s^{(p)} = f_s^{(p)}(t, x_1, \ldots, x_n)$ are analytic with respect to $x_1, \ldots, x_n$ in the region G. We shall assume the generating system (1.2) has a family of periodic solutions (2.1) depending on k arbitrary constants $h_1, \ldots, h_k$. Further, let h_i^* be some solution of equations (2.24) for which condition (2.10) is satisfied. Then, as was shown in the preceding section, the system (3.1) admits only one periodic solution reducing for $\mu = 0$ to the solution

$$\varphi_s(t, h_1^*, \ldots, h_k^*)$$

of the family (2.1) and this periodic solution will be analytic with respect to μ. We can therefore write for this solution

$$x_s = \varphi_s(t, h_1^*, \ldots, h_k^*) + \mu x_s^{(1)} + \mu^2 x_s^{(2)} + \ldots, \tag{3.2}$$

where $x_s^{(p)}$ are certain periodic functions of t of period ω. We shall show that there exists only one system of series of the form (3.2) satisfying equations (3.1) and shall indicate the method of their computation.

Substituting the series (3.2) in equations (3.1) and equating coefficients of like powers of μ we obtain:

$$\frac{dx_s^{(p)}}{dt} = p_{s1}x_1^{(p)} + \ldots + p_{sn}x_n^{(p)} + F_s^{(p)}, \tag{3.3}$$

where p_{sj} are the coefficients of the equations in variations determined by the formulas

$$p_{sj} = \left(\frac{\partial X_s(t, \varphi_1, \ldots, \varphi_n)}{\partial \varphi_j}\right)_{h_i = h_i^*}, \tag{3.4}$$

and $F_s^{(p)}$ are polynomials with periodic coefficients of $x_s^{(1)}, \ldots, x_s^{(p-1)}$

In particular,

$$F_s^{(1)} = f_s^{(1)}\left(t, \varphi_1(h_1^*, \ldots, h_k^*), \ldots, \varphi_n(t, h_1^*, \ldots, h_k^*)\right). \tag{3.5}$$

In order that equations (3.3) admit a periodic solution for $x_s^{(p)}$ it is necessary and sufficient that the conditions be satisfied

$$\int_0^{\omega} \sum_{\alpha=1}^{n} F_\alpha^{(p)} \psi_{\alpha i}\, dt = 0 \qquad (i = 1, \ldots, k), \tag{3.6}$$

where $\psi_{s1}, \ldots, \psi_{sk}$ are periodic solutions of the system conjugate to the system of equations in variations.

For p = 1 equations (3.6) agree with equations (2.24) determining the parameters of the generating solution. Since by assumption the latter are satisfied, the equations for $x_s^{(1)}$ admit a periodic solution. This solution is of the form

$$x_s^{(1)} = M_1^{(1)}\varphi_{s1} + \ldots + M_k^{(1)}\varphi_{sk} + x_s^{(1)*},$$

where $x_s^{(1)*}$ is some particular solution of the equations for $x_s^{(1)}$, $\varphi_{s1}, \ldots, \varphi_{sk}$ are the periodic solutions of the equations in variations, determined by equations (2.4), and $M_i^{(1)}$ are arbitrary constants. These constants are determined from the conditions of periodicity of the functions $x_s^{(2)}$.

We shall assume in general that all the functions $x_s^{(1)}, \ldots, x_s^{(l)}$ have already been computed and turned out periodic. The functions $x_s^{(l)}$ will here have the form

$$x_s^{(l)} = M_1^{(l)} \varphi_{s1} + \ \ldots \ + M_k^{(l)} \varphi_{sk} + x_s^{(l)*},$$

where $M_i^{(l)}$ are constants and $x_s^{(l)*}$ is some particular periodic solution of the equations for $x_s^{(l)}$. The constants $M_i^{(l)}$ are determined from the conditions of periodicity of the functions $x_s^{(l+1)}$. Let us examine more closely the equations determining these constants. We shall show that they are linear and their determinant agrees with the functional determinant (2.10).

It is easy to see that the functions $F_s^{(l+1)}$ have the form

$$\begin{aligned}
F_s^{(l+1)} = \frac{1}{2} \sum_{j=1}^{k} \sum_{\alpha,\beta=1}^{n} \left(\frac{\partial^2 X_s}{\partial x_\alpha \partial x_\beta} \right) (x_\alpha^{(1)} \varphi_{\beta j} + x_\beta^{(1)} \varphi_{\alpha j}) M_j^{(l)} + \\
+ \sum_{j=1}^{k} \sum_{\alpha=1}^{n} \left(\frac{\partial f_s^{(1)}}{\partial x_\alpha} \right) \varphi_{\alpha j} M_j^{(l)} + R_s^{(l+1)} = \\
= \sum_{j=1}^{k} \sum_{\alpha,\beta=1}^{n} \left(\frac{\partial^2 X_s}{\partial x_\alpha \partial x_\beta} \right) x_\alpha^{(1)} \varphi_{\beta j} M_j^{(l)} + \\
+ \sum_{j=1}^{k} \sum_{\alpha=1}^{n} \left(\frac{\partial f_s^{(1)}}{\partial x_\alpha} \right) \varphi_{\alpha j} M_j^{(l)} + R_s^{(l+1)},
\end{aligned}$$

where $R_s^{(l+1)}$ are entirely definite periodic functions of time that do not depend on $M_i^{(1)}$ and the parentheses denote that the derivatives are taken for $x_s = \varphi_s(t, h_1^*, \ldots, h_k^*)$.

On the basis of (3.4), (3.5) and (2.4) we have:

$$F_s^{(l+1)} = \sum_{j=1}^{k} \left\{ \sum_{\alpha=1}^{n} \frac{\partial p_{s\alpha}}{\partial h_j^*} x_\alpha^{(1)} + \frac{\partial F_s^{(1)}}{\partial h_j^*} \right\} M_j^{(l)} + R_s^{(l+1)}. \tag{3.7}$$

For $l > 1$ the functions $F_s^{(l+1)}$ are linear with respect to M_i^l but the functions $F_s^{(2)}$ will be quadratic with respect to $M_i^{(1)}$ since the constants $M_i^{(1)}$ enter also in $x_s^{(1)}$. We can here write:

$$F_s^{(2)} = \sum_{\alpha=1}^{n} \sum_{i,\, j=1}^{k} \frac{\partial p_{s\alpha}}{\partial h_j^*} \varphi_{\alpha i} M_i^{(1)} M_j^{(1)} +$$

$$+ \sum_{j=1}^{n} \left\{ \sum_{\alpha=1}^{n} \frac{\partial p_{s\alpha}}{\partial h_j^*} x_\alpha^{(1)*} + \frac{\partial F_s^{(1)}}{\partial h_j^*} \right\} M_j^{(1)} + R_s^{(2)}.$$

From (3.7) we find that equations (3.6) for $p = l + 1$, determining the constants $M_i^{(l)}$, have the form

$$A_{1i} M_1^{(l)} + \ldots + A_{ki} M_k^{(l)} + B_i^{(l)} = 0 \quad (i = 1, \ldots, k), \tag{3.8}$$

where the magnitudes $B_i^{(l)}$ do not depend on $M_j^{(l)}$ and

$$A_{ji} = \int_0^\omega \sum_{\beta=1}^{n} \left(\sum_{\alpha=1}^{n} \frac{\partial p_{\beta\alpha}}{\partial h_j^*} x_\alpha^{(1)} + \frac{\partial F_\beta^{(1)}}{\partial h_j^*} \right) \psi_{\beta i} \, dt. \tag{3.9}$$

For $l = 1$ equations (3.8) will contain additional quadratic terms the set of which A_i is of the form

$$A_i = \sum_{j,\, m=1}^{k} A_{jmi} M_j^{(1)} M_m^{(1)},$$

$$A_{jmi} = \int_0^\omega \sum_{\alpha,\, \beta=1}^{n} \frac{\partial p_{\beta\alpha}}{\partial h_j^*} \varphi_{\alpha m} \psi_{\beta i} \, dt.$$

The functions $x_\beta^{(1)}$ (as also $x_\beta^{(1)*}$) satisfy the equations

$$\frac{dx_\beta^{(1)}}{dt} = \sum_{\alpha=1}^{n} p_{\beta\alpha} x_\alpha^{(1)} + F_\alpha^{(1)}, \tag{3.10}$$

whence by differentiation with respect to h_j^* we find:

$$\frac{d}{dt}\left(\frac{\partial x_\beta^{(1)}}{\partial h_j^*}\right) = \sum_{\alpha=1}^{n} p_{\beta\alpha} \frac{\partial x_\alpha^{(1)}}{\partial h_j^*} + \sum_{\alpha=1}^{n} \frac{\partial p_{\beta\alpha}}{\partial h_j^*} x_\alpha^{(1)} + \frac{\partial F_\alpha^{(1)}}{\partial h_j^*}.$$

As a result the magnitudes A_{ji} can be represented in the form

$$A_{ji} = \int_0^\omega \sum_{\beta=1}^{n} \left\{ \frac{d}{dt}\left(\frac{\partial x_\beta^{(1)}}{\partial h_j^*}\right) - \sum_{\alpha=1}^{n} p_{\beta\alpha} \frac{\partial x_\alpha^{(1)}}{\partial h_j^*} \right\} \psi_{\beta i}\, dt.$$

Integrating by parts and taking into account that the functions $\psi_{\beta i}$ satisfy equations (2.22) we find successively:

$$A_{ji} = \Big|_0^\omega \sum_{\beta=1}^{n} \frac{\partial x_\beta^{(1)}}{\partial h_j^*} \psi_{\beta i} - \int_0^\omega \sum_{\beta=1}^{n} \left\{ \frac{\partial x_\beta^{(1)}}{\partial h_j^*} \frac{d\psi_{\beta i}}{dt} + \sum_{\alpha=1}^{n} p_{\beta\alpha} \frac{\partial x_\alpha^{(1)}}{\partial h_j^*} \psi_{\beta i} \right\} dt =$$

$$= \Big|_0^\omega \sum_{\beta=1}^{n} \frac{\partial x_\beta^{(1)}}{\partial h_j^*} \psi_{\beta i} - \int_0^\omega \sum_{\beta=1}^{n} \frac{\partial x_\beta^{(1)}}{\partial h_j^*} \left\{ \frac{d\psi_{\beta i}}{dt} + \sum_{\gamma=1}^{n} p_{\gamma\beta} \psi_{\gamma i} \right\} dt =$$

$$= \Big|_0^\omega \sum_{\beta=1}^{n} \frac{\partial x_\beta^{(1)}}{\partial h_j^*} \psi_{\beta i}.$$

We may remark that the functions $x_\beta^{(1)}$ are periodic only for definite values of the parameters h_j^* and therefore the functions $\partial x_\beta^{(1)}/\partial h_j^*$ will not, in general, be periodic and the expressions A_{ji} will be different from zero. But we can write:

$$A_{ji} = \frac{\partial}{\partial h_j^*} \Big|_0^\omega \sum_{\beta=1}^{n} x_\beta^{(1)} \psi_{\beta i},$$

since the functions $\psi_{\beta i}$ will be periodic for any values of the parameters h_j^* and therefore the expressions $x_\beta^{(1)} \partial\psi_{\beta i}/\partial h_j^*$ will also be periodic. Further we have:

$$\frac{d}{dt} \sum_{\beta=1}^{n} x_\beta^{(1)} \psi_{\beta i} = \sum_{\beta=1}^{n} F_\beta^{(1)} \psi_{\beta i}.$$

This is an immediate consequence of the fact that the functions $\psi_{\beta i}$ satisfy the equations conjugate to the homogeneous part of equations (3.10). Hence we finally obtain:

$$A_{ji} = \frac{\partial}{\partial h_j^*} \int_0^\omega \sum_{\beta=1}^n F_\beta^{(1)} \psi_{\beta i}\, dt = \frac{\partial P_i}{\partial h_j^*}.$$

As to the coefficients A_{jmi}, they identically reduce to zero. We can convince ourselves of this by repeating for them the same computations as for the coefficients A_{ji} and taking into account the fact that the functions $\varphi_{\alpha m}$ satisfy the equations in variations which, in contrast to the equations for $x_\beta^{(1)}$, are homogeneous.

Thus, the equations for $M_i^{(l)}$ are always obtained linear. The homogeneous part of these equations does not depend on the index i and has a determinant agreeing with the functional determinant (2.10). Hence to any simple solution of equations (2.24) corresponds one and only one system of series (3.2) formally satisfying equations (3.1). These series will therefore converge and actually represent the required periodic solution.

4. Practical Computation of the Periodic Solutions

We now turn to the question of the practical computation of the obtained periodic solutions.

If we restrict ourselves to only the generating solution it is necessary for its computation to know the values of the h_j^* parameters on which this solution depends, i.e. the roots of equations (2.24). To set up these equations it is necessary to know the functions ψ_{si} forming k independent periodic solutions of the system (2.22). The solution of this last problem requires in general finding the general solution of the system of linear equations (1.8) with periodic coefficients, the equations in variations for the generating system. It is in this that the entire difficulty of the problem consists since general methods for the solution of linear systems with periodic coefficients do not, as we know, exist. This is the general defect of the method presented above, as a consequence of which the region of its practical applicability is considerably narrowed, since it is necessary to restrict the choice of the generating systems, namely, the generating system must be chosen in such manner that its general solution is known, and not only the periodic. In the latter case the general solution of the equations in variations can be determined. In fact, if

$$x_s^0 = x_s^0(t, \alpha_1, \ldots, \alpha_n),$$

where $\alpha_1, \ldots, \alpha_n$ are arbitrary constants, is the general solution of the generating system, the derivatives

$$\frac{\partial x_s^0}{\partial \alpha_i} \qquad (i = 1, \ldots, n),$$

computed for the values of the constants corresponding to the generating solution, determine n independent particular solutions of the equations in variations. We may remark that for the complete integration of the equations in variations it is sufficient to find $n - 1$ of their particular solutions, for which it is sufficient to know, not the general solution of the generating system, but a solution depending on $n - 1$ arbitrary constants. In particular, the equations in variations can be completely integrated in the case where the number k of parameters h_i in the generating solution is equal to $n - 1$.

In certain cases however, which may frequently be encountered in practice, it is not necessary for finding the functions ψ_{si}, and therefore for setting up equations (2.24), to seek the general solution of the equations in variations and these functions can be determined DIRECTLY FROM THE GENERATING SOLUTION. The simplest case of this kind will be that for which the system in variations is self-conjugate, i.e. when the relations are satisfied

$$p_{sj} = -p_{js}.$$

In this case we can put $\psi_{si} = \varphi_{si}$ and the functions φ_{si} are determined by equations (2.4) directly from the generating solution.

Let us consider another more general and in practice considerably more important case. We shall assume that the system (1.1) approximates a canonical, i.e. has the form

$$\frac{dx_s}{dt} = \frac{\partial H}{\partial x_{n+s}} + \mu f_s, \quad \frac{dx_{n+s}}{dt} = -\frac{\partial H}{\partial x_s} + \mu f_{n+s}$$
$$(s = 1, \ldots, n),$$

where the functions $H = H(t, x_1, \ldots, x_{2n})$, $f_r = f_r(t, x_1, \ldots, x_{2n}, \mu)$ are such that the general conditions are satisfied which are satisfied by equations (1.1). We shall assume that the generating system admits a periodic solution

$$x_r^0 = x_r^0(t, h_1, \ldots, h_k) \qquad (r = 1, \ldots, 2n),$$

depending on k arbitrary constants. The equations in variations (1.8) for some solution of this family now have the form

$$\frac{dy_s}{dt} = \sum_{\alpha=1}^{2n} \left(\frac{\partial^2 H}{\partial x_{n+s} \partial x_\alpha}\right) y_\alpha, \quad \frac{dy_{n+s}}{dt} = -\sum_{\alpha=1}^{2n} \left(\frac{\partial^2 H}{\partial x_s \, \partial x_\alpha}\right) y_\alpha \qquad (4.1)$$
$$(s = 1, \ldots, n),$$

where the derivatives are computed for $x_r = x_r^0(t, h_1, \ldots, h_k)$. These equations have k periodic solutions

$$\varphi_{ri} = \frac{\partial x_r^0(t, h_1, \ldots, h_k)}{\partial h_i} \qquad (i = 1, \ldots, k).$$

Let us now set up the system conjugate to (4.1). We shall have:

$$\frac{dz_r}{dt} = -\sum_{\beta=1}^{n} \left(\frac{\partial^2 H}{\partial x_r \, \partial x_{n+\beta}}\right) z_\beta + \sum_{\beta=1}^{n} \left(\frac{\partial^2 H}{\partial x_r \, \partial x_\beta}\right) z_{n+\beta}.$$

These equations go over into (4.1) on replacing z_s by y_{n+s} and z_{n+s} by $-y_s$ ($s = 1, \ldots, n$). From this it follows immediately that these equations have the k periodic solutions

$$\psi_{si} = \frac{\partial x_{s+n}^0(t, h_1, \ldots, h_k)}{\partial h_i}, \quad \psi_{n+s,i} = -\frac{\partial x_s^0(t, h_1, \ldots, h_k)}{\partial h_i}$$
$$(s = 1, \ldots, n; \quad i = 1, \ldots, k),$$

and therefore equations (2.24) assume the form

$$P_i(h_1, \ldots, h_k) = \int_0^\omega \sum_{\beta=1}^{n} \left\{ f_\beta(t, x_1^0, \ldots, x_{2n}^0, 0) \frac{\partial x_{n+\beta}^0}{\partial h_i} - \right.$$
$$\left. - f_{n+\beta}(t, x_1^0, \ldots, x_{2n}^0, 0) \frac{\partial x_\beta^0}{\partial h_i} \right\} dt = 0. \quad (4.2)$$

We shall note another important case. Let us assume that for the generating system the first integrals are known

$$F_i(t, x_1^0, \ldots, x_n^0) = \text{const} \qquad (i = 1, \ldots, k),$$

the number of which is equal to the number k of parameters entering in the generating solution. Here F_i are periodic functions with respect to t. In this case we may also at once write down equations (2.24); we have, namely:

$$P_i(h_1, \ldots, h_k) = \\ = \int_0^\omega \sum_{\alpha=1}^n f_\alpha(t, \varphi_1, \ldots, \varphi_n, 0) \frac{\partial F_i(t, \varphi_1, \ldots, \varphi_n)}{\partial \varphi_\alpha} dt = 0. \qquad (4.3)$$

In fact, according to the properties, proven in sec. 1 of chapter III, of the equations in variations the functions

$$\sum_{\alpha=1}^n \frac{\partial F_i(t, \varphi_1, \ldots, \varphi_n)}{\partial \varphi_\alpha} y_\alpha \qquad (i = 1, \ldots, k)$$

give the first integrals of the system (1.8) and therefore the functions $\partial F_i(t, \varphi_1, \ldots, \varphi_n) / \partial \varphi_s$ form k independent solutions of equations (2.22), and since these functions are periodic they can be taken as the functions ψ_{si}.

We go on to the question of computation of the further approximations. If the system (1.1) is analytic the required periodic solution, as we have seen, can be expanded in the series (3.2) and the determination of the coefficients $x_s^{(p)}$ of these series reduces to finding the periodic solution of a system of nonhomogeneous equations of the form

$$\frac{du_s}{dt} = p_{s1} u_1 + \ldots + p_{sn} u_n + f_s(t), \qquad (4.4)$$

where $f_s(t)$ are known periodic functions of t of period ω and the homogeneous part agrees with the equations in variations.

If the right hand sides of equations (1.1) are not analytic with respect to $x_1,\ldots,x_n$, μ, it is possible, for computing the periodic solution, to use the method of successive approximations. We shall not however here dwell on this question but only point out that in this case the problem reduces to the finding of periodic solutions of equations of the form (4.4).

The problem of finding periodic solutions of a system of linear nonhomogeneous equations of the form (4.4) requires, generally speaking, a knowledge of the general solution of the corresponding homogeneous part, i.e. of the equations in variations. However, in many cases it is possible to find approximate expressions for the required solutions of equations (4.4) without having recourse to the integration of the homogeneous part of these equations. Thus for example in many cases the coefficients p_{sj} differ little from constants and then for the solution of the problem the methods of sec. 12 of the preceding chapter may be used. In general, since it is a question of computing, not the principal terms of the required solution, which are determined by the generating solution, but of the correction terms, it is possible to restrict oneself to a rough approximation and apply some elementary procedures.

5. Stability Criteria of the Periodic Solutions Considered

Let us se up the necessary conditions of stability of the periodic solutions considered in the preceding sections, restricting ourselves to the case of analytic equations.

We shall start from equations of motion in the form (3.1).

$$\frac{dx_s}{dt} = X_s(t, x_1, \ldots, x_n) + \mu f_s^{(1)}(t, x_1, \ldots, x_n) + \ldots \qquad (5.1)$$
$$(s = 1, \ldots, n).$$

The investigated periodic motion has the form

$$x_s(t) = \varphi_s(t, h_1^*, \ldots, h_k^*) + \mu x_s^{(1)}(t) + \ldots, \qquad (5.2)$$

where $h_1^*,\ldots, h_k^*$ is a solution of equations (2.24). We shall take this solution as the undisturbed and set up for it the equations in variations of the system (5.1). For this purpose we set in equations (5.1) $x_s = x_s(t) + z_s$ and in the obtained equations discard terms of higher than the first order with respect to $z_1, \ldots, z_n$. Then, with an accuracy up to magnitudes of the first order with respect to μ, we shall have:

$$\frac{dz_s}{dt} = (p_{s1} + \mu q_{s1}) z_1 + \ldots + (p_{sn} + \mu q_{sn}) z_n, \tag{5.3}$$

where

$$p_{sj} = \left(\frac{\partial X_s}{\partial x_j}\right), \quad q_{sj} = \sum_{\alpha=1}^{n} \left(\frac{\partial^2 X_s}{\partial x_j \partial x_\alpha}\right) x_\alpha^{(1)}(t) + \left(\frac{\partial f_s^{(1)}}{\partial x_j}\right), \tag{5.4}$$

the derivatives being computed at the point $x_s = \varphi_s(t, h_1^*, \ldots, h_k^*)$.

The characteristic exponents of the system (5.3) by which are determined the stability or instability of solution (5.2) reduce for $\mu = 0$ to the characteristic exponents of the equations

$$\frac{dy_s}{dt} = p_{s1} y_1 + \ldots + p_{sn} y_n, \tag{5.5}$$

which are the equations in variations of the generating system for the generating solution. Hence, if μ is sufficiently small, there may be considered, for the problem of stability, instead of the characteristic exponents of the system (5.3) the characteristic exponents of system (5.5) with the exception of those exponents which have zero real parts. In the presence of the latter the corresponding characteristic exponents of system (5.3) must be more accurately computed. This can be done in exactly the same way as in the case of constant coefficients p_{sj} provided the general solution of equations (5.5) is known.

The system (5.5) has, as we know, k periodic solutions

$$\varphi_{si} = \left(\frac{\partial \varphi_s(t, h_1, \ldots, h_k)}{\partial h_i}\right)_{h_j = h_j^*} \tag{5.6}$$

and therefore at least k of its characteristic exponents are equal to zero. Let us assume that the number of zero characteristic exponents of the system (5.5) is exactly equal to k. Then their corresponding characteristic exponents of the system (5.3) can be computed by the method of sec. 11 of chapter III.

Following this method, let us in equations (5.3) put

$$y_s = e^{\mu a t}\eta_s,$$

where μa is the first term of the expansion of the required characteristic exponents. Equations (5.3) then assume the form

$$\frac{d\eta_s}{dt} = (p_{s1} + \mu q_{s1})\eta_1 + \ldots + (p_{sn} + \mu q_{sn})\eta_n - \mu a \eta_s. \tag{5.7}$$

The required magnitude a is determined from the conditions of existence of a periodic solution for the system (5.7). This periodic solution has the form

$$\eta_s = M_1\varphi_{s1} + \ldots + M_k\varphi_{sk} + \mu\eta_s^{(1)} + \ldots,$$

wher M_i are constants. For $\eta_s^{(1)}$ we obtain the equations

$$\frac{d\eta_s^{(1)}}{dt} = p_{s1}\eta_1^{(1)} + \ldots + p_{sn}\eta_n^{(1)} + \sum_{\alpha=1}^{n}\sum_{j=1}^{k} q_{s\alpha}\varphi_{\alpha j}M_j - a\sum_{j=1} \varphi_{sj}M_j$$

or, taking (5.4) and (5.6) into account,

$$\frac{d\eta_s^{(1)}}{dt} = p_{s1}\eta_1^{(1)} + \ldots + p_{sn}\eta_n^{(1)} + \\ + \sum_{j=1}^{k}\left(\sum_{\alpha=1}^{n}\frac{\partial p_{s\alpha}}{\partial h_j^*}x_\alpha^{(1)} + \frac{\partial F_s^{(1)}}{\partial h_j^*}\right)M_j - a\sum_{j=1}^{k}\varphi_{sj}M_j,$$

where

$$F_s^{(1)} = f_s^{(1)}\left(t, \varphi_1(t, h_1^*, \ldots, h_k^*), \ldots, \varphi_n(t, h_1^*, \ldots, h_k^*)\right).$$

The conditions of periodicity of the functions $\eta_s^{(1)}$ have the form

$$(A_{1i} - a\alpha_{1i})M_1 + \ldots + (A_{ki} - a\alpha_{ki})M_k = 0 \tag{5.8}$$
$$(i = 1, \ldots, k),$$

where

$$A_{ji} = \int_0^{\omega} \sum_{\beta=1}^{n} \left(\sum_{\alpha=1}^{n} \frac{\partial p_{\beta\alpha}}{\partial h_j^*} x_\alpha^{(1)} + \frac{hF_\beta^{(1)}}{\partial h_j^*} \right) \psi_{\beta i} dt,$$

$$a_{ji} = \omega \sum_{\alpha=1}^{n} \varphi_{\alpha j} \psi_{\alpha i}.$$

The coefficients A_{ji} agree accurately with the expressions (3.9) and therefore, as was shown in section 3, we can write:

$$A_{ji} = \frac{\partial P_i}{\partial h_j^*}.$$

Setting the determinant of system (5.8) equal to zero we obtain the following equation:

$$\begin{vmatrix} \frac{\partial P_1}{\partial h_1^*} - a\alpha_{11} & \frac{\partial P_1}{\partial h_2^*} - a\alpha_{21} & \cdots & \frac{\partial P_1}{\partial h_k^*} - a\alpha_{k1} \\ \frac{\partial P_2}{\partial h_1^*} - a\alpha_{12} & \frac{\partial P_2}{\partial h_2^*} - a\alpha_{22} & \cdots & \frac{\partial P_2}{\partial h_k^*} - a\alpha_{k2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial P_k}{\partial h_1^*} - a\alpha_{1k} & \frac{\partial P_k}{\partial h_2^*} - a\alpha_{2k} & \cdots & \frac{\partial P_k}{\partial h_k^*} - a\alpha_{kk} \end{vmatrix} = 0, \qquad (5.9)$$

which determines the first terms of the expansion of k characteristic exponents of the system (5.3), provided all the roots of this equation are different, as we shall assume.

Thus, for the stability of the periodic solution (5.2) it is necessary that all roots of equation (5.9) have non-positive real parts.

If all these real parts are negative and if the real parts of the remaining $n - k$ characteristic exponents of system (5.3) are also negative, then for sufficiently small μ the solution (5.2) will actually be stable, and furthermore asymptotically stable.

The equation (5.9) exactly agrees with equation (12.8) of chapter III, that was established for quasilinear systems. If the functions ψ_{si} are chosen such that the relations are satisfied

$$\sum_{\alpha=1}^{n} \psi_{\alpha i}\varphi_{\alpha i} = 1, \qquad \sum_{\alpha=1}^{n} \psi_{\alpha i}\varphi_{\alpha j} = 0 \qquad (i \neq j),$$

equation (5.9) can be replaced by the more simple equation

$$\begin{vmatrix} \frac{\partial P_1}{\partial h_1^*} - \lambda & \frac{\partial P_1}{\partial h_2^*} & \dots & \frac{\partial P_1}{\partial h_k^*} \\ \frac{\partial P_2}{\partial h_1^*} & \frac{\partial P_2}{\partial h_2^*} - \lambda & \dots & \frac{\partial P_2}{\partial h_k^*} \\ \dots & \dots & \dots & \dots \\ \frac{\partial P_k}{\partial h_1^*} & \frac{\partial P_k}{\partial h_2^*} & \dots & \frac{\partial P_k}{\partial h_k^*} - \lambda \end{vmatrix} = 0 \qquad (\lambda = a\omega). \qquad (5.10)$$

We may remark in conclusion that for setting up equation (5.9) it is necessary merely to know the functions P_i. Hence, as we saw in the preceding section, this equation can be written also when the general solution of equations (5.5) is not known.

6. Nonautonomous System with One Degree of Freedom Approximating a Conservative System

Let us consider the special case where the system (3.1) has the form

$$\frac{d^2x}{dt^2} + F(x) = \mu f_1(x, \dot{x}, t) + \mu^2 f_2(x, \dot{x}, t) + \dots, \qquad (6.1)$$

where the functions $F(x)$ and $f_i(x, \dot{x}, t)$ are analytic with respect to their arguments x and $\dot{x} = dx/dt$, the function $F(x)$ not containing $\dot{x}$.

The generating system

$$\frac{d^2x_0}{dt^2} + F(x_0) = 0 \qquad (6.2)$$

is conservative and may be written in the canonical form

$$\frac{dx_0}{dt} = \frac{\partial H}{\partial \dot{x}_0}, \qquad \frac{d\dot{x}_0}{dt} = -\frac{\partial H}{\partial x_0},$$

where

$$H = \frac{1}{2}\dot{x}_0^2 + V(x_0), \qquad V(x_0) = \int F(x_0)\,dx_0.$$

The energy integral

$$\frac{1}{2}\dot{x}_0^2+V(x_0)=E \tag{6.3}$$

determines the phase tranjectories of the system (6.2) and makes it possible to find its general solution with the aid of quadratures. Let us write this general solution of equation (6.2) in the form

$$x_0=x_0(t+h,\, c), \tag{6.4}$$

where the arbitrary constants h and c are so chosen that the conditions are satisfied

$$x_0(0,\, c)=c, \qquad \dot{x}_0(0,\, c)=0. \tag{6.5}$$

The function x_o is then determined by the equation

$$\sqrt{2}(t+h)=\int_c^{x_0}\frac{dx_0}{\pm\sqrt{V(c)-V(x_0)}}\,. \tag{6.6}$$

Let us investigate this function more closely. We shall assume that the point $x_s = c$ is not a state of equilibrium of the system (6.2) so that $F(c) \neq 0$. For definiteness we shall assume that $F(c) > 0$ and consider the point x_o moving along the x- axis according to the law (6.4). At the initial instant $t + h = 0$ the velocity of this point is equal to zero and its acceleration is negative. Hence the point x_o for $t + h > 0$ moves at first to the left and in formula (6.6) it will be necessary to take the minus sign before the radical while the quantity under the radical sign will be positive. If at the same time the function

$$f(x_0)=V(c)-V(x_0) \tag{6.7}$$

to the left of the point $x_o = c$ in the region G does not become zero the motion of the point x_o will continue to the left until it leaves the region G. Let us assume however that there is an interval of values of c for which the function (6.7) has in the region G a root less than c and that the value of c under consideration belongs to this interval. Let $c^* = c^*(c)$ be the root of the function (6.7) nearest to c and assume at first that this root is simple. The point x_o then attains the value c^* after a finite interval of time T/2, where

$$T = T(c) = \sqrt{2}\int_{c}^{c^*} \frac{dx_0}{\left|\sqrt{V(c) - V(x_0)}\right|}. \tag{6.8}$$

In fact, the expression under the integral in (6.8) is continuous over the entire segment $[c^*, c]$ with the exception of its ends where it becomes infinity of the order 1/2 (the root $x_o = c$ of the function (6.7) is likewise simple since $F(c) = V'(c) \neq 0$). The integral (6.8) is therefore finite.

After the point x_o atains the value c^*, where now evidently $F(c^*) < 0$, it turns to the right and will move thus until it reaches the value c after an interval of time likewise equal to T/2. For this stage of the motion it will be necessary in (6.6) to take the minus sign before the radical. After the point x_o reaches the value c its motion will repeat periodically.

Let us assume now that the root c^* of the function (6.7) will be multiple. Since we must then have $F(c^*) = V'(c^*) = 0$, there will correspond to the value $x_o = c^*$ the state of equilibrium of the system (6.2).

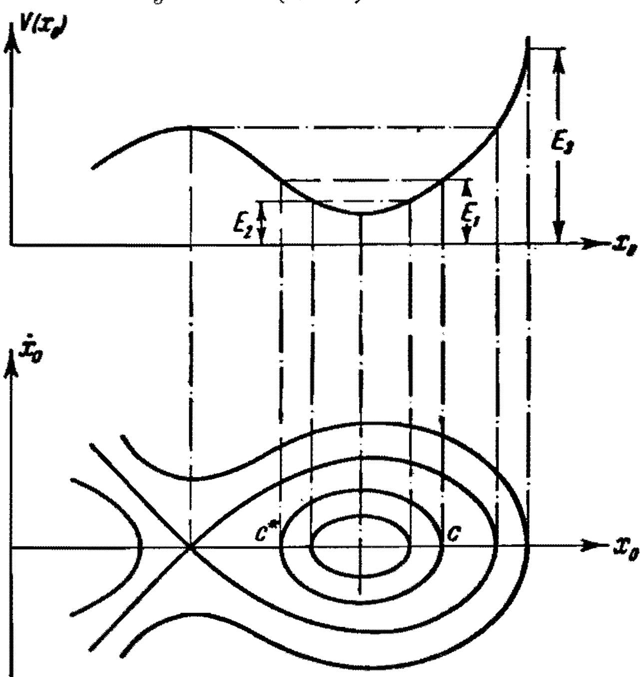

Fig. 30

For this assumption the integral (6.8) will diverge and the point x_0 will asymptotically approach the state of equilibrium as $t \to \infty$.

Fig. 30 shows the phase trajectories of the system (6.2) for the case of the existence of values of c for which the function (6.7) has roots less than c. These phase trajectories are determined by equation (6.3) and will be symmetrical with respect to the x_0- axis. On the sketch are indicated several trajectories corresponding to different values of E. A graph is also given of the function $V(x_0)$. To the points where the function $V(x_0)$ has a maximum or minimum correspond states of equilibrium of the system, the state of unstable equilibrium corresponding to the maximum and stable equilibrium to the minimum. To the stable states of equilibrium correspond the singular points of equation (6.2) of the type of a 'center' and to the unstable the singular points called "saddle" points.

Thus, if there exists a region of values of c for which the function (6.7) has a root less than c (for $F(c)>0$), we shall suppose that for these values of c the function (6.4) will be periodic. Hence, for the assumption made, equation (6.2) will admit a family of periodic solutions of equations (6.4) depending on two arbitrary constants. The period of these solutions, determined by formula (6.8), will depend on c.

Let us denote by c_m the value of c satisfying the equation

$$T(c_m) = \frac{\omega}{m}. \tag{6.9}$$

Here m is an arbitrary integer, ω a period of the right hand side of equation (6.1) (which is not necessarily the least period of this right hand side). For each value of c_m we obtain a family of periodic solutions

$$x_0 = x_0(t+h, c_m) \tag{6.10}$$

of equation (6.2), having a period ω depending on ONE arbitrary parameter h. These solutions can be taken as the generating solutions and for them can be found the periodic solutions of the complete equation (6.1). For this it is necessary however to consider first in more detail the corresponding equation in variations.

The equation in variations for any solution of the family (6.10) has the form

$$\frac{d^2y}{dt^2} + F'(x_0)\,y = 0 \qquad \left(F'(x) = \frac{dF(x)}{dx}\right). \tag{6.11}$$

This equation has the two particular solutions

$$y_1 = \dot{x}_0(t+h,\, c_m) = u(t+h), \tag{6.12}$$

$$y_2^* = \left(\frac{\partial x_0(t+h,\, c)}{\partial c}\right)_{c=c_m}, \tag{6.13}$$

since the function (6.4) is the generating solution of equation (6.2). The solution u is periodic of period ω. As to the solution y_2^*, it will not be periodic due to the fact that the period of the function (6.4) depends on c. We shall explain in more detail the form of y_2^*.

For this purpose let us replace in the function $x_o(t+h,\ c)$ the variable t by the variable τ with the aid of the relation

$$t + h = \frac{(\tau + h)\,T(c)}{\omega} \tag{6.14}$$

and denote the function thus obtained by $x_s^*(\tau,\ c)$. This function will be periodic with resptct to τ of period ω. Differentiating this function with respect to c we obtain:

$$\begin{aligned}\frac{\partial x_0^*}{\partial c} = \frac{\partial}{\partial c} x_0\left(\frac{(\tau+h)\,T}{\omega},\ c\right) &= \frac{\partial x_0(t+h,\, c)}{\partial c} + \frac{\tau+h}{\omega}\frac{dT}{dc}\dot{x}_0(t+h,\, c) = \\ &= \frac{\partial x_0(t+h,\, c)}{\partial c} + \frac{t+h}{T}\frac{dT}{dc}\dot{x}_0(t+h,\, c),\end{aligned}$$

where the left hand side is a periodic function of τ of period ω. Putting $c = c_m$, we find from (6.12),(6.13) and (6.9):

$$y_2^* = -\frac{m}{\omega}\frac{dT}{dc}\{(t+h)\,u(t+h) + v(t+h)\}, \tag{6.15}$$

where

$$v(t+h) = -\frac{\omega}{m}\left\{\frac{1}{\frac{dT}{dc}}\frac{\partial x_0^*}{\partial c}\right\}_{c=c_m} = -\frac{\omega}{m}\left\{\frac{1}{\frac{dT}{dc}}\frac{\partial}{\partial c}x_0\left(\frac{(\tau+h)T}{\omega}, c\right)\right\}_{c=c_m} \qquad (6.16)$$

is a periodic function of time of period ω. From (6.15) and (6.12) we find that equation (6.11) has also a particular solution

$$y_2 = tu(t+h) + v(t+h), \qquad (6.17)$$

which, together with (6.12), we shall take as the initial. We note down the following properites of the functions u and v.

1. The function u satisfies the equation in variations and the initial conditions (for t + h = 0)

$$u(0) = 0, \quad \dot{u}(0) = -F(c_m). \qquad (6.18)$$

These conditions are directly obtained from (6.12), (6.5) and (6.2).

2. The function v(t+h) satisfies the equation

$$\frac{d^2v}{dt^2} + F'(x_0)v = -2\dot{u}(t+h). \qquad (6.19)$$

This is directly obtained from the circumstnce that the function tu + v satisfies the equation in variations.

3. The equation holds

$$v(0) = -\frac{\omega}{m\left(\frac{dT}{dc}\right)_{c=c_m}}. \qquad (6.20)$$

In fact, from (6.16) and (6.5) we find:

$$v(0) = -\frac{\omega}{m}\left\{\frac{1}{\frac{dT}{dc}}\frac{\partial x_0(0, c)}{\partial c}\right\}_{c=c_m} = -\frac{\omega}{m\left(\frac{dT}{dc}\right)_{c=c_m}}.$$

4. The equation also holds

$$u^2 + u\dot{v} - \dot{u}v = K = \text{const.} \qquad (6.21)$$

In fact, the left hand side of (6.21) represents the Wronskian determinant

$$\Delta(t) = \begin{vmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{vmatrix}$$

of the solutions y_1 and y_2 of the equation in variations. Since the latter does not contain a term with first derivative, on the basis of the formula of Liouville, which for a linear equation of the second order has the form

$$\Delta(t) = \Delta(t_0)\, e^{-\int_{t_0}^{t} q\, dt},$$

where q is the coefficient of the first derivative, we conclude that (6.21) holds true.

From (6.18) and (6.20) we find directly that

$$K = -\frac{\omega F(c_m)}{\omega \left(\frac{dT}{dc}\right)_{c=c_m}}. \qquad (6.22)$$

Let us now consider the nonhomogeneous equation

$$\frac{d^2 z}{dt^2} + F'(x_0)\, z = f(t). \qquad (6.23)$$

As one can easily convince oneself by direct verification or by applying the method of variation of the arbitrary constants, the general solution of this equation can be represented in the form

$$Kz = u \int_0^t \left(\int_0^t uf\, dt - vf - \beta \right) dt + v \left(\int_0^t uf\, dt - \beta \right) + \alpha u, \qquad (6.24)$$

where β and α are arbitrary constants. If the function $f(t)$ is periodic of period ω, then in order for equation (6.23) to admit a periodic solution it is necessary and sufficient that the condition be satisfied

$$\int_0^\omega uf\,dt = 0. \tag{6.25}$$

In fact, when (6.25) is satisfied the coefficient of v in the expression (6.24) will be periodic. The expression under the outer integral in the first term will also be periodic. Hence if the constant β is so chosen that the mean value of this expression reduces to zero, i.e. if we assume that

$$\beta = -\frac{1}{\omega}\int_0^\omega vf\,dt + \frac{1}{\omega}\int_0^\omega dt\int_0^t uf\,dt, \tag{6.26}$$

the solution (6.24) will be periodic. The constant α here remains arbitrary. Condition (6.25) may of course be obtained at once also from the general conditions of periodicity if account is taken of the fact that equation (6.11) is self-conjugate and therefore the function u is also a periodic solution of the equation conjugate to (6.11).

With this established, we shall seek a periodic solution of equation (6.1) reducing for $\mu = 0$ to the generating solution (6.10). Following the method of sec. 3, we shall seek this solution in the form of the series

$$x = x_0(t+h, c_m) + \mu x_1(t) + \dots \tag{6.27}$$

For x_1 we have:

$$\frac{d^2x_1}{dt^2} + F'(x_0)\,x_1 = f_1(x_0, \dot{x}_0, t) = F_1(t), \tag{6.28}$$

and the condition of periodicity (6.25) on the basis of (6.12) gives:

$$P(h) = \int_0^\omega f_1[x_0(t+h, c_m), \; \dot{x}_0(t+h, c_m), t]\,\dot{x}_0(t+h, c_m)\,dt = 0. \tag{6.29}$$

This equation serves to determine the constant h. Let us assume that h has been chosen according to this equation, with $F'(h) \neq 0$. For x_1, according to (6.24) and (6.26), we shall then have:

$$x_1 = \frac{1}{K}\left[u\int_0^t\left(\int_0^t uF_1\,dt - vF_1 - \beta_1\right)dt + v\left(\int_0^t uF_1\,dt - \beta_1\right)\right] + \alpha_1 u, \tag{6.30}$$

where

$$\beta_1 = -\frac{1}{\omega}\int_0^\omega vF_1\,dt + \frac{1}{\omega}\int_0^\omega dt\int_0^t uF_1\,dt$$

and α_1 is an arbitrary constant. This constant is determined from the condition of periodicity of the function x_2. The equation for α_1, as was shown in sec. 3, will be linear. The further approximations are computed in an analogous manner.

We may remark that for the actual computation of the function $x_0(t + h, c)$ and its period $T(c)$ there is no need of invariably using formulas (6.6) and (6.8). In practice it is usually more convenient to use approximating procedures of some kind that give the function x_0 already developed in a Fourier series. One procedure of this kind for systems of special form will be discussed in the following chapter.

A detailed investigation of the problem of periodic solutions of systems of the form (6.1) has been given by A. M. Kats [1], who considered in detail also the singular case when the equation (6.28) determining h is identically satisfied. Kats also gave stability criteria both for the nonsingular and singular cases. We shall now turn to a discussion of these criteria for the nonsingular case which we here consider.

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[1] Kats A.M., Vynuzhdennye kolebaniya nelineinykh sistem s odnoi stepen'yu svobody (Forced Oscillations of Nonlinear Systems with One Degree of Freedom), Prikl. matem. i mekh., vol. XIX, no. 1, 1955.

7. Stability Criteria of the Periodic Solution Considered in the Preceding Section

To solve the problem of the stability of the periodic solution (6.27) we shall set up for it the corresponding equation in variations. For this purpose, putting in equations (6.1)

$$x = x_0(t+h, c_m) + \mu x_1(t) + \dots + y$$

and discarding magnitudes of higher than the first order with respect to y we obtain the following equation in variations

$$\frac{d^2y}{dt^2} + F'(x_0)\,y + \mu\left\{\left[F''(x_0)\,x_1 - \left(\frac{\partial f_1}{\partial x}\right)\right] y - \left(\frac{\partial f_1}{\partial \dot{x}}\right)\frac{dy}{dt}\right\} + \dots = 0, \tag{7.1}$$

where the terms not written down are of an order higher than the first with respect to μ and the derivatives of the function f_1 are computed for the generating solution $x = x_o(t + h, c_m)$.

Equation (7.1) for $\mu = 0$ reduces to equation (6.11) which, as we saw, has the two independent solutions (6.12) and (6.17). Hence the two characteristic exponents of equation (7.1) for $\mu = 0$ reduce to zero and we must therefore compute them more accurately. We can not here make use of the results of sec. 5 since the form of the solutions (6.12) and (6.17) shows that to a zero characteristic exponent of equation (6.11) corresponds, not two, but one set of solutions. As a result, the characteristic exponents of equation (7.1) will in general be developable in integral powers, not of μ, but of $\sqrt{\mu}$.

Let

$$\lambda_1 = \alpha_1\sqrt{\mu} + \alpha_2\mu + \alpha_3(\sqrt{\mu})^3 + \dots \tag{7.2}$$

be the expansion of one of the characteristic exponents of equation (7.1). It is easy to see that if this expansion actually contains fractional powers of μ the expansion of the second characteristic exponent λ_2 will have the form

$$\lambda_2 = -\alpha_1\sqrt{\mu} + \alpha_2\mu - \alpha_3(\sqrt{\mu})^3 + \dots \tag{7.3}$$

In fact, let

$$\rho^2 - 2A\rho + B = 0$$

be the characteristic equation for (7.1). Since for $\mu = 0$ this equation must have a double root equal to 1 the expansions must hold of the form

$$A = 1 + a_1\mu + \ldots, \quad B = 1 + b_1\mu + \ldots, \quad A^2 - B = c_k\mu^k + \ldots$$

and therefore

$$\rho_{1,2} = 1 + a_1\mu + \ldots \pm \sqrt{c_k}\,(\sqrt{\mu})^k (1 + \beta_1\mu + \ldots), \tag{7.4}$$

where the terms not written down contain only integral powers of μ. Hence, if these expansions contain fractional powers of μ, i.e. if the number k is odd, one expansion is obtained from the other by a simple substitution of $-\sqrt{\mu}$ for $\sqrt{\mu}$. But then the same will be true also for the expansions (7.2) and (7.3) of the characteristic exponents.

From (7.2) and (7.3) it is immediately seen that for stability it is necessary that the real part of the coefficient α_1 reduce to zero. If this condition is satisfied the sign of the real parts of the magnitudes λ_1 and λ_2 will be determined by the coefficient α_2. From (7.4) it is seen that the coefficient α_2 will always be real while the coefficient α_1 will be either real or purely imaginary. From this we arrive at the following necessary conditions of stability

$$\alpha_1^2 \leqslant 0, \quad \alpha_2 \leqslant 0. \tag{7.5}$$

If these conditions are satisfied with the signs of inequality then for sufficiently small μ stability will actually hold and will moreover be asymptotic.

Let us now proceed to determine the coefficients α_1

and α_2. For this purpose we transform equation (7.1) with the aid of the substitution

$$y = e^{\lambda_1 t} z$$

and seek to obtain the magnitudes α_1 from the condition of existence of a periodic solution for the equation thus obtained

$$\frac{d^2 z}{dt^2} + F'(x_0) z + 2\sqrt{\mu}\,\alpha_1 \frac{dz}{dt} + \\ + \mu \left\{ \left[F''(x_0) x_1 - \left(\frac{\partial f_1}{\partial x}\right) + \alpha_1^2 \right] z + \\ + \left[-\left(\frac{\partial f_1}{\partial \dot{x}}\right) + 2\alpha_2 \right] \frac{dz}{dt} \right\} + \ldots = 0.$$

We shall seek this periodic solution in the form of the series

$$z = z_0 + \sqrt{\mu}\, z_1 + \mu z_2 + \ldots$$

We shall then have:

$$\frac{d^2 z_0}{dt^2} + F'(x_0) z_0 = 0,$$

$$\frac{d^2 z_1}{dt^2} + F'(x_0) z_1 = -2\alpha_1 \frac{dz_0}{dt},$$

$$\frac{d^2 z_2}{dt^2} + F'(x_0) z_2 = \varphi(t) = \\ = -2\alpha_1 \frac{dz_1}{dt} - \left[F''(x_0) x_1 - \left(\frac{\partial f_1}{\partial x}\right) + \alpha_1^2 \right] z_0 - \\ - \left[-\left(\frac{\partial f_1}{\partial \dot{x}}\right) + 2\alpha_2 \right] \frac{dz_0}{dt}.$$

For z_0 and z_1 on the basis of (6.19) we find:

$$z_0 = M_0 u, \quad z_1 = \alpha_1 M_0 v + M_1 u,$$

where M_0 and M_1 are arbitrary constants. The condition of periodicity of the function z_2, which on the basis of (6.25) has the form

$$\int_0^{\omega} \varphi(t)\, u\, dt \equiv \int_0^{\omega} \varphi(t)\, \dot{x}_0\, dt = 0,$$

therefore gives

$$\alpha_1^2 \int_0^{\omega} (u^2 + 2\alpha_1 u\dot{v})\,dt +$$

$$+ \int_0^{\omega} \left[F''(x_0)\, x_1\, \dot{x}_0^2 - \left(\frac{\partial f_1}{\partial x}\right) \dot{x}_0^2 - \left(\frac{\partial f_1}{\partial \dot{x}}\right) \dot{x}_0 \ddot{x}_0 \right] dt = 0. \quad \textbf{(7.6)}$$

But on the basis of (6.21)

$$\int_0^{\omega} (u^2 + 2\dot{u}v)\,dt \equiv \int_0^{\omega} (u^2 + u\dot{v} - \dot{u}v)\,dt = \omega K. \qquad \textbf{(7.7)}$$

Further, taking (6.28) into account, we can write:

$$\int_0^{\omega} F''(x_0)\, x_1 \dot{x}_0^2\,dt = -\int_0^{\omega} F'(x_0)\,(\dot{x}_1 \dot{x}_0 + x_1 \ddot{x}_0)\,dt =$$

$$= -\int_0^{\omega} [F'(x_0)\, x_1 \ddot{x}_0 - F(x_0)\, \ddot{x}_1]\,dt =$$

$$-\int^{\omega} \ddot{x}_0)\, \ddot{x}_1 + F'(x_0)\, x_1)\,dt = -\int_0^{\omega} f_1(x_0,\ \dot{x}_0,\ t)\, \ddot{x}_0\,dt. \qquad \textbf{(7.8)}$$

Hence equation (7.6) assumes the form

$$\omega K \alpha_1^2 = \int_0^{\omega} \left[f_1(x_0,\ \dot{x}_0,\ t)\, \ddot{x}_0 + \left(\frac{\partial f_1}{\partial x}\right) \dot{x}_0^2 + \left(\frac{\partial f_1}{\partial \dot{x}}\right) \dot{x}_0 \ddot{x}_0 \right] dt,$$

whence, taking into consideration (6.29), we finally obtain

$$\alpha_1^2 = \frac{1}{\omega K} \frac{dP(h)}{dh}. \qquad \textbf{(7.9)}$$

The coefficient α_2 is determined from the condition of periodicity of the function z_3.

It is possible however to determine it more simply. In fact, the sum of the characteristic exponents of any system of linear equations with periodic coefficients

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n$$

on the basis of formula (2.8) of chapter III is determined by the equality

$$\omega \sum_{\alpha=1}^{n} \lambda_\alpha = \int_0^\omega \sum_{\alpha=1}^{n} p_{\alpha\alpha}\, dt.$$

Applying this equality to equation (7.1) we obtain:

$$\omega(\lambda_1 + \lambda_2) = \mu \int_0^\omega \left(\frac{\partial f_1}{\partial \dot{x}}\right) dt,$$

whence, comparing with (7.2) and (7.3), we find:

$$\alpha_2 = \frac{1}{2\omega} \int_0^\omega \left(\frac{\partial f_1}{\partial \dot{x}}\right) dt. \tag{7.10}$$

Substituting (7.9) and (7.10) in (7.5) we finally obtain the following conditions of stability of solution (6.27) of equation (6.1):

$$K \frac{dP(h)}{dh} \leqslant 0, \tag{7.11}$$

$$\int_0^\omega \left(\frac{\partial f_1}{\partial \dot{x}}\right) dt \leqslant 0. \tag{7.12}$$

With this we finish our discussion of the theory of periodic solutions of systems of the form (6.1). A concrete example will be considered below in sec. 7 of chapter VIII.

8. Periodic Solutions of Autonomous Systems

We shall now consider the problem of the periodic solutions of autonomous systems. Let there be given the system

$$\frac{dx_s}{dt} = X_s(x_1, \ldots, x_n) + \mu f_s(x_1, \ldots, x_n, \mu) \qquad (8.1)$$
$$(s = 1, \ldots, n),$$

where the functions X_s in a certain region G of the space of the variables $x_1, \ldots, x_n$ admit continuous partial derivatives of the second order with respect to $x_1, \ldots, x_n$ and the functions f_s in the same region G and for $0 \leq \mu \leq \mu_0$ admit continuous derivatives of the first order with respect to $x_1, \ldots, x_n$, μ.

Let us assume that the generating system

$$\frac{dx_s^0}{dt} = X_s^0(x_1, \ldots, x_n) \qquad (8.2)$$

admits a periodic solution

$$x_s^0 = \varphi_s(t) \qquad (8.3)$$

of a certain period T. We shall seek the conditions under which the system (8.1) for μ sufficiently small likewise admits a periodic solution reducing for $\mu = 0$ to the generating solution, and the period τ of this solution. Since the system (8.1) is autonomous, the period τ does not coincide in general with the period T of the generating solution and is one of the unknowns of the problem.

We shall denote by $\gamma_1, \ldots, \gamma_n$ the initial values of the required periodic solution. The magnitudes γ_s are to be determined together with τ. We may remark however that our problem contains a certain indefiniteness. In fact, if the system (8.1) admits some periodic solution then in virtue of its autonomous character it necessarily admits an entire family of such solutions which are obtained by replacing t by $t+h$, where h is an arbitrary constant. Evidently it is sufficient for us to find some one solution of this family. We can distinguish it by the fact that we assign one of the magnitudes γ_s. In fact, if this magnitude γ_s is correctly given, namely if the corresponding coordinate x_s actually assumes in some solution of the required family at some instant of time the value γ_s, we can take this instant of time as the initial one. Hence, if system (8.1) at all admits a periodic solution it also admits such solution for which one of the magnitudes x_s assumes for $t = 0$ the initially given value and therefore the additional restriction made does not affect the generality of the problem. In practice difficulties never arise in the question of the correct choice of one of the magnitudes γ_s since usually there are always known in advance certain values which at least one of the magnitudes x_s must assume in the process of oscillation.

We shall in what follows for definiteness assume that the magnitude γ_n is known. Moreover, we shall assume that the magnitude $\varphi_n(0)$ is also equal to γ_n. This fixes also the value of the constant h, which can be added to t also in the generating solution. The problem will thus contain n unknowns $\tau, \gamma_1, \ldots, \gamma_{n-1}$. All these unknowns will play the same part in the further considerations and therefore we shall for convenience denote this set of unkowns by $\alpha_1, \ldots, \alpha_n$.

We shall denote by $x_s(t, \gamma_1, \ldots, \gamma_n, \mu)$ the solution of system (8.1) determined by the initial conditions

$$x_s(0, \gamma_1, \ldots, \gamma_n, \mu) = \gamma_s. \tag{8.4}$$

The equations determining $\alpha_1, \ldots, \alpha_n$ will then have the following form:

$$\psi_s(\alpha_1, \ldots, \alpha_n, \mu) = x_s(\tau, \gamma_1, \ldots, \gamma_n, \mu) - \gamma_s = 0. \tag{8.5}$$

Let $\alpha_1^0, \ldots, \alpha_n^0$ be the values of $\alpha_1, \ldots, \alpha_n$ corresponding to the generating solution, i.e. the set of magnitudes $T, \varphi_1(0), \ldots, \varphi_{n-1}(0)$. Equations (8.5) will then satisfy for $\mu = 0$, $\alpha_s = \alpha_s^0$, since the generating solution is periodic of period T. Hence if the condition is satisfied

$$\left\{ \frac{\partial(\psi_1, \ldots, \psi_n)}{\partial(\alpha_1, \ldots, \alpha_n)} \right\}_{\alpha_s = \alpha_s^0}, \tag{8.6}$$

equations (8.5) will admit for sufficiently small μ one and only one solution $\alpha_s = \alpha_s(\mu)$ for which $\alpha_s(0) = \alpha_s^0$. Substituting this solution in $x_s(t, \gamma_1, \ldots, \gamma_n, \mu)$ we obtain the unique periodic solution of system (8.1) reducing for $\mu = 0$ to the generating solution, and its period τ.

We shall say that the generating solution (8.3) is isolated if condition (8.6) is satisfied for it. We thus arrive at the following theorem of A. Poincaré:

IF THE GENERATING SOLUTION IS ISOLATED THE SYSTEM (8.1) ADMITS FOR SUFFICIENTLY SMALL μ ONE AND ONLY ONE PERIODIC SOLUTION REDUCING FOR $\mu = 0$ TO THIS GENERATING SOLUTION.

For the theory of nonlinear oscillations the greatest interest however is presented by the case where such unique correspondence between the complete and generating system does not exist. The most important case of this kind will be that for which the generating solution belongs to a family depending on a certain number of parameters. This case we shall now consider.[1]

Assume that the generating system admits a family of periodic solutions

$$x_s^0 = \varphi_s(t, h_1, \ldots, h_k), \tag{8.7}$$

depending, besides on the constant h, which can be added to t, on k additional constants $h_1, \ldots, h_k$. The period $T = T(h_1, \ldots, h_k)$ will in the general case also depend on these constants. Let us assume that the generating solution under consideration corresponds to the values $h_i = h_i^*$ of the parameters.

Denoting, as before, by $\alpha_s^0 = \alpha_s^0(h_1^*, \ldots, h_k^*)$ the values of α_s in generating solution, i.e. the set of magnitudes $T^* = T^*(h_1^*, \ldots, h_k^*)$, $\varphi_1(0, h_1^*, \ldots, h_k^*), \ldots, \varphi_{n-1}(0, h_1^*, \ldots, h_k^*)$,

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[1] A very important special case of this problem was considered in the work: Zhevakin S.A., Ob otyskanii predel'nykh tsiklov v sistemakh blizkikh k nekotorym nelineinym (On the Obtaining of Limit Cycles in Systems Approximating Certain Nonlinear Systems), Prikl. matem.i mekh, vol. XV, no.2, 1951.

we shall, as before, have the result that the system of equations (8.5) is satisfied for $\mu = 0$, $\alpha_s = \alpha_s^o$. But in the case considered the system (8.5) has for $\mu = 0$ also the solution $\tau = T(h_1,\ldots,h_k)$, $\gamma_1 = \varphi_1(0,h_1,\ldots,h_k)$, $\gamma_{n-1} = \varphi_{n-1}(0,h_1,\ldots,h_k)$, depending on k arbitrary constants. As a result, both the determinant (8.6) and all of it minors up to the order $n - k + 1$ inclusive reduce to zero and the question as to the existence of solutions of the system (8.5) requires further investigation

Let us assume that at least one of the minors of the $(n - k)$-th order of the determinant (8.6) is different from zero. Let this be the minor

$$\left\{\frac{\partial(\psi_1, \ldots, \psi_{n-k})}{\partial(\alpha_1, \ldots, \alpha_{n-k})}\right\}_{\alpha_s=\alpha_s^0}. \tag{8.8}$$

Then the first $n - k$ of equations (8.5) can be solved for the magnitudes $\alpha_1,\ldots,\alpha_{n-k}$ so that we shall have $\alpha_j = \alpha_j(\alpha_{n-k+1},\ldots,\alpha_n,\mu)$ $(j=1,\ldots,n-k)$ with $\alpha_j(\alpha^o_{nik+1},\ldots,\alpha_n^o,0) = \alpha_j^o$. Substituting these magnitudes in the last k of equations (8.5) we obtain for the determination of $\alpha_{n-k+1},\ldots,\alpha_n$ equations of the form

$$F_i(\alpha_{n-k+1}, \ldots, \alpha_n, \mu) = 0 \qquad (i = 1, \ldots, k), \tag{8.9}$$

where the functions F_i reduce to zero for $\mu = 0$, $\alpha_r = \alpha_r^o$.

As we already said, the system (8.5) admits for $\mu = 0$ a solution depending on k arbitrary constants. Hence the system (8.9) must also admit for $\mu = 0$ a solution containing k arbitrary constants, which evidently is possible only in that case in which for $\mu = 0$ they are identically satisfied. Hence we must have $F_i = \mu\Psi_i$ so that equations (8.9) assume the form

$$\Psi_i(\alpha_{n-k+1}, \ldots, \alpha_n, \mu) = 0 \qquad (i = 1, \ldots, k), \tag{8.9a}$$

where now the functions Ψ_i in general do not reduce to zero for $\mu = 0$, $\alpha_{n-k+1} = \alpha^o_{n-k+1},\ldots,\alpha_n = \alpha_n^o$. Hence in order that equations (8.9a) admit a solution $\alpha_r = \alpha_r(\mu)$ $(r = n-k+1,\ldots,n)$ for which conditions $\alpha_r(0) = \alpha_r^o$

are satisfied it is necessary first of all that the parameters h_i of the generating solution satisfy the equations

$$P_i(h_1^*, \ldots, h_k^*) = \\ = \Psi_i\left(\alpha_{n-k+1}^0(h_1^*, \ldots, h_k^*), \ldots, \alpha_n^0(h_1^*, \ldots, h_k^*), 0\right) = 0 \quad (8.10) \\ (i = 1, \ldots, k).$$

If, moreover, the condition is satisfied

$$\Delta = \left\{ \frac{\partial(\Psi_1, \ldots, \Psi_k)}{\partial(\alpha_{n-k+1}, \ldots, \alpha_n)} \right\}_{\alpha_r = \alpha_r^0,\ \mu = 0} \neq 0, \qquad (8.11)$$

then for sufficiently small μ the system (8.9) will actually admit a solution for α_r of the required form. Substituting the found values of $\alpha_1, \ldots, \alpha_n$ in the function $x_s(t, \gamma_1, \ldots, \gamma_n, \mu)$, we obtain a periodic solution of system (8.1) reducing for $\mu = 0$ to the generating solution $\varphi_s(t, h_1^*, \ldots, h_k^*)$, and its period τ.

From (8.10) we find:

$$\frac{\partial(P_1, \ldots, P_k)}{\partial(h_1^*, \ldots, h_k^*)} = \Delta \frac{\partial(\alpha_{n-k+1}^0, \ldots, \alpha_n^0)}{\partial(h_1^*, \ldots, h_k^*)}.$$

Hence if the condition is satisfied

$$\frac{\partial(P_1, \ldots, P_k)}{\partial(h_1^*, \ldots, h_k^*)} \neq 0, \qquad (8.12)$$

condition (8.11) will also be satisfied. It can be shown that, as in the case of nonautonomous systems, from condition (8.11) follows, conversely, condition (8.12), the proof of which however we shall not dwell on here. We arrive at the conclusion that to each generating solution for which the parameters h_i are a simple solution of equations (8.10) there actually corresponds for small μ a periodic solution of the complete system (8.1).

In the preceding considerations we have frequently used the partial derivatives of the various functions with respect to the various arguments. It is easy to see that, as in the case of nonautonomous systems, for the assumptions made with regard to equations (8.1) these derivatives actually exist.

9. Periodic Solutions of Autonomous Systems. Equations for the Parameters of the Generating Solutions

Let us find the developed form of the equations (8.10). For this purpose we put

$$\tau = T^* + \alpha = T(h_1^*, \ldots, h_k^*) + \alpha, \quad \gamma_s = \varphi_s(0, h_1^*, \ldots, h_k^*) + \beta_s,$$

where $\beta_n = 0$ according to the condition on the choice of the magnitude γ_n. Since the functions Ψ_s are continuously differentiable and reduce to zero for $\mu = \alpha = \beta_1 = \ldots = \beta_{n-1} = 0$, we can write equations (8.5) in the following form:

$$\left[\left(\frac{\partial\psi_s}{\partial\gamma_1}\right) + U_{s1}\right]\beta_1 + \ldots + \left[\left(\frac{\partial\psi_s}{\partial\gamma_{n-1}}\right) + U_{s,\,n-1}\right]\beta_{n-1} + \\ + \left[\left(\frac{\partial\psi_s}{\partial\mu}\right) + V_s\right]\mu + \left[\left(\frac{\partial\psi_s}{\partial\tau}\right) + W_s\right]\alpha = 0. \quad (9.1)$$

Here U_{sj}, V_s, W_s are continuous functions of $\beta_1, \ldots, \beta_{n-1}, \alpha, \mu$, reducing to zero for $\beta_1 = \ldots = \beta_{n-1} = \alpha = \mu = 0$, and the parentheses denote that the derivatives are computed for the generating solution, i.e. for

$$\tau = T^*, \ \mu = 0, \ \gamma_s = \varphi_s(0, h_1^*, \ldots, h_k^*).$$

From (9.1) it follows that equations (8.10) are obtained as a result of the elimination of the magnitudes α, $\beta_1, \ldots, \beta_{n-1}$ from the linear equations

$$\left(\frac{\partial\psi_s}{\partial\gamma_1}\right)\beta_1 + \ldots + \left(\frac{\partial\psi_s}{\partial\gamma_{n-1}}\right)\beta_{n-1} + \left(\frac{\partial\psi_s}{\partial\mu}\right)\mu + \left(\frac{\partial\psi_s}{\partial\tau}\right)\alpha = 0. \quad (9.2)$$

As a result of the fact that the functions $\Psi_s(\tau, \gamma_1, \ldots, \gamma_{n-1}, \mu)$ for $\mu = 0, \gamma_s = \varphi_s(0, h_1^*, \ldots, h_k^*)$ reduce to $\varphi_s(\tau, h_1^*, \ldots, h_k^*) - \gamma_s$ we have:

$$\left(\frac{\partial\psi_s}{\partial\tau}\right) = \dot{\varphi}_s(T^*, h_1^*, \ldots, h_k^*). \quad (9.3)$$

Let us further consider the equations

$$\frac{dy_s}{dt} = p_{s1}y_1 + \ldots + p_{sn}y_n \quad (9.4)$$

with periodic coefficients of period $T^* = T^*(h_1^*,\dots,h_k^*)$

$$p_{sj} = \left(\frac{\partial X_s}{\partial x_j}\right)_{x_r=\varphi_r(t,\, h_1^*,\, \dots,\, h_k^*)},$$

which are the equations in variations for the generating solution. We denote by $y_{s1}^*(t, t_o),\dots,y_{sn}^*(t, t_o)$ a fundamental system of equations determined by the initial conditions

$$y_{sj}^*(t_0, t_0) = \delta_{sj}, \tag{9.5}$$

where δ_{sj} is the Kronecker symbol. Then, as was shown in sec. 2, we can write:

$$\left(\frac{\partial \psi_s}{\partial \gamma_j}\right) = y_{sj}^*(T^*, 0) - \delta_{sj},$$

$$\left(\frac{\partial \psi_s}{\partial \mu}\right) = \int_0^{T^*} \sum_{\beta=1}^{n} [f_\beta(\varphi_1, \dots, \varphi_n, 0)]_{t=t_0,\, h_i=h_i^*} y_{s\beta}^*(T^*, t_0)\, dt_0,$$

and therefore equations (9.2) can be written in the form

$$\sum_{j=1}^{n-1} y_{sj}^*(T^*, 0)\beta_j - \beta_s + \dot{\varphi}_s(T^*, h_1^*, \dots, h_k^*)\alpha +$$
$$+\mu \int_0^{T^*} \sum_{\beta=1}^{n} [f_\beta(\varphi_1, \dots, \varphi_n, 0)]_{t=t_0,\, h_i=h_i^*} y_{s\beta}^*(T^*, t_0)\, dt_0 = 0, \tag{9.6}$$

and the problem reduces to the elimination of the magnitudes $\beta_1,\dots,\beta_{n-1},\alpha$ from these equations. For the solution of this problem let consider more closely equation (9.4). These equations have k particular solutions

$$y_{si} = \left(\frac{\partial \varphi_s(t, h_1, \dots, h_k)}{\partial h_i}\right)_{h_j=h_j^*} \quad (i = 1, \dots, k). \tag{9.7}$$

Moreover, since in the solution (8.7) an arbitrary constant h may be added to t, equations (9.4) have also the solution

$$y_{s,\, k+1} = \dot{\varphi}_s(t, h_1^*, \dots, h_k^*). \tag{9.8}$$

This solution is periodic of period T^*. As to the solutions (9.7), they will not in general be periodic. It is here necessary to distinguish two cases: when the period T of the solutions (8.7) depends on the parameters h_i and when it does not depend on them.

Let us assume first that T does not depend on $h_1, \ldots, h_k$. All the solutions (9.7) will then be periodic of period T^*. Thus, in the case considered the system (9.4) will have $k + 1$ periodic solutions (9.7) and (9.8). We shall assume that the system (9.4) has no other periodic solutions.

Let us assume now that the period T depends at least on one of the magnitudes $h_1, \ldots, h_k$. Let us consider the functions

$$u_s(t, h_1, \ldots, h_k) = \varphi_s\left(\frac{T}{T^*}t, h_1, \ldots, h_k\right).$$

These functions will evidently be periodic with respect to t with a period T^* not depending on h_i. As a result the partial derivatives of these functions with respect to h_i will also be periodic of period T^*. Forming these derivatives and then setting $h_j = h_j^*$ we shall have:

$$\left(\frac{\partial u_s}{\partial h_i}\right)_{h_j=h^*} = \left(\frac{\partial \varphi_s(t, h_1, \ldots, h_k)}{\partial h_i}\right)_{h_j=h_j^*} + \\ + \frac{1}{T^*}\left(\frac{\partial T}{\partial h_i}\right)_{h_j=h_j^*} t\dot{\varphi}_s(t, h_1^*, \ldots, h_k^*).$$

Hence, the solutions (9.7) have the form

$$y_{sj} = -\frac{1}{T^*}\left(\frac{\partial T}{\partial h_i}\right)_{h_j=h_j^*} t\dot{\varphi}_s(t, h_1^*, \ldots, h_k^*) + v_{si}(t), \qquad (9.9)$$

where

$$v_{si} = \left[\frac{\partial}{\partial h_i}\varphi_s\left(\frac{T}{T^*}t, h_1, \ldots, h_k\right)\right]_{h_j=h_j^*}$$

are periodic functions of t of period T^*. From this it follows that the equations (9.4) have, besides (9.8), additional $k - 1$ periodic solutions

$$v_{sr}(t)\left(\frac{\partial T}{\partial h_1}\right)_{h_j=h_j^*} - v_{s1}(t)\left(\frac{\partial T}{\partial h_r}\right)_{h_j=h_j^*} \quad (r = 2, \ldots, k). \qquad (9.10)$$

Thus, in the case where the period T depends on h_i equations (9.4) have the k periodic solutions (9.8) and (9.10). We shall assume that the system (9.4) has no other periodic solutions.

With this established, let us proceed to the elimination of the magnitudes α, $\beta_1, \ldots, \beta_{n-1}$ from equations (9.6). We shall assume first that the period T depends on the parameters h_i. In this case the system conjugate to (9.4) will admit k periodic solutions, which we shall denote by $\psi_{s1}, \ldots, \psi_{sk}$. It is easy to see that

$$\sum_{\gamma=1}^{n} \dot{\varphi}_\gamma (T^*, h_1^*, \ldots, h_k^*) \psi_{\gamma i} (T^*) = 0 \qquad (9.11)$$
$$(i = 1, \ldots, k).$$

In fact, since y_{sj} and ψ_{si} are solutions of conjugate systems we must have:

$$\sum_{\gamma=1}^{n} y_{\gamma j}(t)\, \psi_{\gamma i}(t) = \text{const.}$$

Hence

$$\sum_{\gamma=1}^{n} y_{\gamma j} (T^*)\, \psi_{\gamma i} (T^*) = \sum_{\gamma=1}^{n} y_{\gamma i}(0)\, \psi_{\gamma i}(0),$$

whence on the basis of (9.9) equations (9.11) follow, since the functions v_{sj} are periodic.

Let us now multiply equations (9.6) by $\psi_{si}(T^*)$ and sum with respect to the index s. Then, taking into account (9.11) and (2.23), we obtain, exactly as in sec. 2, after dividing by μ. the equations

$$P_i (h_1^*, \ldots, h_k^*) =$$
$$= \int_0^{T^*} \sum_{\beta=1}^{n} f_\beta [\varphi_1 (t, h_1^*, \ldots, h_k^*), \ldots, \varphi_n(t, h_1^*, \ldots, h_k^*), 0]\, \psi_{\beta i}(t)\, dt = 0. \qquad (9.12)$$

This will be the result of eliminating the magnitudes $\alpha, \beta_1, \ldots, \beta_{n-1}$ from equations (9.6), i.e. the developed form of equations (8.10).

Let us now assume that the period T of the generating solution does not depend on the parameters h_i. In this case the system conjugate to(9.4) will have k + 1 periodic solutions, which we shall denote by $\psi_{s1}, \ldots, \psi_{s,k+1}$, The identities (9.11) will not in general be satisfied and, proceeding in the same manner as in the previous case, we obtain the k + 1 equations

$$Q_r(h_1^*, \ldots, h_k^*, \alpha) = \alpha \sum_{\beta=1}^{n} \dot{\varphi}_\beta (T^*, h_1^*, \ldots, h_k^*) \psi_{\beta r}(T^*) + \\ + \int_0^{T^*} f_\beta [\varphi_1(t, h_1^*, \ldots, h_k^*), \ldots, \varphi_n(t, h_1^*, \ldots, h_k^*), 0] \psi_{\beta r}(t)\, dt = 0 \tag{9.13}$$

$$(r = 1, \ldots, k+1).$$

In order to obtain the functions P_i it remains only to eliminate the magnitude α from the obtained equations. There is however no need of this. In our case it is more expedient to consider directly equations (9.13) that permit determining both the magnitudes $h_1^*, \ldots, h_k^*$ and the magnitude α, the first approximation of the correction of the period. Condition (8.12) is now replaced by the equivalent condition

$$\frac{\partial (Q_1, \ldots, Q_{k+1})}{\partial (h_1^*, \ldots, h_k^*, \alpha)} \neq 0. \tag{9.14}$$

Equations (9.13) agree with those which we obtained for quasilinear systems for which the case under consideration precisely occurs where the period of the generating solutions does not depend on the parameters.

We have thus obtained the equations determining the zeroth approximation of the required periodic solutions of the system (8.1). For computing the further approximations there may be used, as for quasilinear systems, either the method of successive approximations or the method of expansion into series in μ, if the equations under consideration are analytic. We shall not however here dwell on this question.

10. Almost Periodic Solutions of Nonautonomous Systems in the Case of an Isolated Generating Solution

The fundamental results of sec. 1 and 2 on the periodic solutions of nonautonomous system can be generalized to almost periodic solutions.[1]

Let there be given the nonautonomous system

$$\frac{dx_s}{dt} = X_s(t, x_1, \ldots, x_n) + \mu f_s(t, x_1, \ldots, x_n, \mu) \qquad (10.1)$$
$$(s = 1, \ldots, n),$$

where with regard to the functions X_s and f_s we make the following general assumtions. We shall assume that the variable t varies in the interval $(-\infty, +\infty)$, the variables $x_1, \ldots, x_n$ in a certain region G of the space of these variables and the parameter μ on the segment $[0, \mu_o]$. The functions X_s possess derivatives of the second order with respect to the variables $x_1, \ldots, x_n$, and the functions f_s derivatives of the first order with respect to the variables $x_1, \ldots, x_n$ and μ, and these derivatives satisfy with respect to these variables the Cauchy-Lipschitz conditions with coefficients not depending on t. With respect to t the functions X_s are continuous and periodic with given period ω, while the functions f_s are almost periodic and such that for any fixed μ the functions $f_s(t, x_1(t), \ldots, x_n(t), \mu)$, where $x_s(t)$ are arbitrary almost periodic functions of t lying in the region G, will also be almost periodic.

Let us consider the generating solution

$$\frac{dx_s^0}{dt} = X_s(t, x_1^0, \ldots, x_n^0) \qquad (10.2)$$

and assume that it admits in the region G the periodic solution

$$x_s^0 = \varphi_s(t). \qquad (10.3)$$

Taking this solution as the generating solution we

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[1] Malkin I.G., O pochti-periodicheskikh kolebaniyakh neavtonomnykh sistem (On Almost Periodic Oscillations of Nonautonomous Systems), Prikl. matem. i mekh., vol. XVIII, no. 6, 1954.

shall seek the conditions for which the complete system (10.1) admits an almost periodic solution which reduces for $\mu = 0$ to this generating solution.

Let

$$\frac{dy_s}{dt} = p_{s1}y_1 + \dots + p_{sn}y_n \qquad \left(p_{sj} = \left(\frac{\partial X_s}{\partial x_j}\right)_{x_i=\varphi_i(t)}\right) \quad (10.4)$$

be the equation in variations for the generating solution. We shall say that the solution (10.3) is an isolated one if the equations in variations (10.4) do not have almost periodic solutions, for which it is necessary that their characteristic equation have no roots equal to unity. This condition is more rigorous than in sec. 1, where we considered a solution as isolated if its equations in variations have no periodic solutions. This more rigorous requirement is connected with the fact that we are now considering periodic solutions as a particular case of the wider class of almost periodic solutions.

In the present section we consider the problem for the case of an isolated generating solution. The following theorem holds here:

THEOREM. IF THE GENERATING SOLUTION IS ISOLATED THE SYSTEM (10.1) FOR SUFFICIENTLY SMALL μ ADMITS ONE AND ONLY ONE ALMOST PERIODIC SOLUTION REDUCING TO THE GENERATING SOLUTION FOR $\mu = 0$.

PROOF. Let us put in equations (10.1)

$$x_s = \varphi_s(t) + y_s, \quad (10.5)$$

after which they will assume the form

$$\frac{dy_s}{dt} = \sum_{\alpha=1}^{n} p_{s\alpha}y_\alpha + Y_s(t, y_1, \dots, y_n) + \mu f_s(t, \varphi_1, \dots, \varphi_n, 0) + \\ + \mu F_s(t, y_1, \dots, y_n, \mu), \quad (10.6)$$

where

$$\left.\begin{aligned} Y_s &= X_s(t, \varphi_1 + y_1, \dots, \varphi_n + y_n) - \\ &\quad - X_s(t, \varphi_1, \dots, \varphi_n) - \sum_{\alpha=1}^{n} p_{s\alpha}y_\alpha, \\ F_s &= f_s(t, \varphi_1 + y_1, \dots, \varphi_n + y_n, \mu) - \\ &\quad - f_s(t, \varphi_1, \dots, \varphi_n, 0). \end{aligned}\right\} \quad (10.7)$$

According to the conditions imposed on the right hand sides of equations (10.1), for sufficiently small values of the magnitudes y_s, y'_s, y''_s, μ, μ', μ'' the functions Y_s satisfy the inequalities

$$|Y_s(t, y_1, \ldots, y_n)| < A \sum_{\alpha=1}^{n} |y_\alpha|^2, \tag{10.8}$$

$$|Y_s(t, y'_1, \ldots, y'_n) - Y_s(t, y''_1, \ldots, y''_n)| < \\ < B \sum_{\alpha=1}^{n} (|y'_\alpha| + |y''_\alpha|) \sum_{\alpha=1}^{n} |y'_\alpha - y''_\alpha|, \tag{10.9}$$

and the functions F_s the inequalities

$$|F_s(t, y'_1, \ldots, y'_n, \mu') - F_s(t, y''_1, \ldots, y''_n, \mu'')| < \\ < C \left(\sum_{\alpha=1}^{n} |y'_\alpha - y''_\alpha| + |\mu' - \mu''| \right), \tag{10.10}$$

where A, B, C are certain constants. From (10.10) it follows that the inequality also holds

$$|F_s(t, y_1, \ldots, y_n, \mu)| < C \left(\sum_{\alpha=1}^{n} |y_\alpha| + \mu \right), \tag{10.11}$$

since the functions F_s reduce to zero for $y_1 = \ldots = y_n = \mu = 0$.

We shall seek the almost periodic solution of system (10.6) by the method of successive approximations, taking as the first approximation the almost periodic solution of the system

$$\frac{dy_s^{(1)}}{dt} = \sum_{\alpha=1}^{n} p_{s\alpha} y_\alpha + \mu f_s(t, \varphi_1, \ldots, \varphi_n, 0) \tag{10.12}$$

and as the k-th approximation the almost periodic solution of the system

$$\frac{dy_s^{(k)}}{dt} = \sum_{\alpha=1}^{n} p_{s\alpha} y_\alpha^{(k)} + \mu f_s(t, \varphi_1, \ldots, \varphi_n, 0) + \\ + Y_s(t, y_1^{(k-1)}, \ldots, y_n^{(k-1)}) + \mu F_s(t, y_1^{(k-1)}, \ldots, y_n^{(k-1)}, \mu). \tag{10.13}$$

Let us consider the nonhomogeneous system

$$\frac{dy_s}{dt} = \sum_{\alpha=1}^{n} p_{s\alpha} y_\alpha + p_s(t), \tag{10.14}$$

where p_s are arbitrary almost periodic functions of t. Since by assumption the characteristic equation of the system (10.4) has no roots with moduli equal to one, system (10.14) admits one and only one almost periodic solution. This solution satisfies the inequalities

$$|y_s| < QL, \tag{10.15}$$

where Q is the upper limit of the functions $|p_s|$ and L is a certain constant not depending on p_s. These assertions were proved by us in sec. 2 of chapter IV for the special case when the coefficients p_{sj} are constant. The general case is however reduced to this special case since any system of linear equations with periodic coefficients can with the aid of a substitution also with periodic coefficients be transformed into a system of equations with constant coefficients.

Let M be the upper limit of the functions $f_s(t, \varphi_1, \ldots, \varphi_n, 0)$. Then it follows, from what has been said above, that the system (10.12) admits one and only one almost periodic solution $y_s^{(1)}$ and this solution satisfies the inequalities

$$|y_s^{(p)}| < \mu ML < \mu K \qquad (p = 1, 2, \ldots), \tag{10.16}$$

where K is a certain constant, the magnitude of which will be chosen below. From (10.16) it follows that if μ is sufficiently small the magnitudes $\varphi_s + y_s^{(1)}$ will lie in the region G and therefore the magnitudes $y_s^{(1)}$ in the region of definition of the functions Y_s and F_s.

Having in this manner determined the functions $y_s^{(1)}$ we can determine $y_s^{(2)}$ from (10.13). Let us assume for definiteness that all the functions $y_s^{(2)}, \ldots, y_s^{(k-1)}$ have already been computed, that they all turned out almost periodic and that they satisfy inequalities (10.16). Then by the property of the functions X_s and f_s the right hand sides of equations (10.13) will be known almost periodic functions of t and from these equations we find one and only one system of almost periodic functions $y_s^{(k)}$. For these functions on the basis of (10.8), (10.11), (10.15) and (10.16) the estimates are valid

$$|y_s^{(k)}| < \mu ML + nA\mu^2K^2L + \mu^2C(nK+1)L,$$

and therefore inequalities (10.16) are satisfied also for $p = k$ provided K is chosen in such manner that the inequalities are satisfied

$$ML + \mu nAK^2L + \mu C(nK+1)L < K,$$

which evidently for sufficiently small μ is always possible.

Let us consider the differences $y_s^{(k+1)} - y_s^{(k)}$. They satisfy the equations

$$\frac{d(y_s^{(k+1)} - y_s^{(k)})}{dt} = \sum_{a=1}^{n} p_{sa}(y_a^{(k+1)} - y_a^{(k)}) +$$
$$+ Y_s(t, y^{(k)}) - Y_s(t, y^{(k-1)}) + \mu\{F_s(t, y^{(k)}, \mu) - F_s(t, y^{(k-1)}, \mu)\}.$$

Hence, denoting by a_p the upper limit of the magnitudes $\left| y_s^{(p)} - y_s^{(p-1)} \right|$, we shall, on the basis of (10.9), (10.10) and (10.15) have:

$$|y_s^{(k+1)} - y_s^{(k)}| < 2\mu n^2 BKLa_k + \mu CnLa_k.$$

We can therefore put:

$$a_{k+1} = \mu(2n^2BK + Cn)La_k,$$

whence it follows immediately that for sufficiently small μ

the sequence $y_s^{(k)}$ uniformly converges to certain almost periodic functions $y_s^*(t)$.

We shall show that these functions satisfy equations (10.6). For this purpose we denote by $u_s(t)$ the unique almost periodic solution of the equations

$$\frac{du_s}{dt} = \sum_{\alpha=1}^{n} p_{s\alpha} u_\alpha + Y_s(t, y_1^*, \ldots, y_n^*) + \mu f_s(t, \varphi_1, \ldots, \varphi_n, 0) + \\ + \mu F_s(t, y_1^*, \ldots, y_n^*, \mu).$$

Then, precisely as for the magnitudes $y_s^{(k+1)} - y_s^{(k)}$, we can obtain the inequalities

$$|u_s - y_s^{(k)}| < \mu(2n^2BK + Cn)L|y_s^* - y_s^{(k-1)}|_{\max}.$$

But since the right hand sides of these inequalities for $k \to \infty$ tend to zero, we find:

$$u_s = \lim_{k\to\infty} y_s^{(k)} = y_s^*,$$

which evidently shows that the functions y_s^* satisfy equations (10.6).

It remains to show that for sufficiently small μ the functions y_s^* determine a unique almost periodic solution of equations (10.6). For this let us assume that the functions y_s' are also an almost periodic solution of equations (10.6). Then, on the basis of the preceding, the estimates will hold

$$|y_s' - y_s^*| < \mu(2n^2BK + Cn)L|y_s' - y_s^*|_{\max},$$

repeating which we find also the inequalities

$$|y_s' - y_s^*| < \{\mu(2n^2BK + Cn)L\}^m |y_s' - y_s^*|_{\max},$$

where m is an arbitrarily large integer. From this it at once follows that $y_s' = y_s^*$, which proves the uniqueness of the solutions y_s^*.

From (10.16) it follows that the functions y_s^* reduce to zero for $\mu = 0$. Substituting these functions in (10.5) we obtain one and only one almost periodic solution of the system (10.1) reducing for $\mu = 0$ to the generating solution. In this way the theorem has been completely proven.

REMARK. By slightly altering the preceding considerations it is easy to show that the theory remains valid for certain weaker restrictions on the functions X_s and f_s, namely, the conditions that the functions f_s admit continuous derivatives of the first order with respect to the variables $x_1, \ldots, x_n, \mu$ while the functions X_s admit continuous derivatives of the second order can be replaced by the condition that the functions f_s and the first derivatives of the functions X_s satisfy with respect to the variables the Cauchy-Lipschitz inequalities with coefficients not depending on t.

11. Almost Periodic Solutions of Nonautonomous Systems in the Case of a Family of Generating Solutions

Passing on to the case where the generating system admits a family of periodic solutions we shall impose certain additional restrictions on the right hand sides of equations (10.1), namely, we shall assume that the functions f_s can be represented in the form of the finite sums

$$f_s(t, x_1, \ldots, x_n, \mu) = \sum_p f_{sp}(t, x_1, \ldots, x_n, \mu)\, e^{i\nu_p t} \qquad (i = \sqrt{-1}),$$

where ν_p are real numbers and the functions f_{sp} are continuous with respect to t and periodic with period ω. Moreover, it is assumed that the functions f_s satisfy all conditions stated for them in the preceding section.

With this supposition let us assume that the system (10.2) admits a family of periodic solutions of period ω

$$x_s^0 = \varphi_s(t, h_1, \ldots, h_k) \qquad (s = 1, \ldots, n), \tag{11.1}$$

depending on $k \leqslant n$ arbitrary parameters h_i. The system in variations (10.4) has in this case k independent particular solutions

$$\varphi_{si} = \frac{\partial \varphi_s(t, h_1, \ldots, h_k)}{\partial h_i} \qquad (i = 1, \ldots, k), \tag{11.2}$$

which will evidently be periodic of period ω. In consequence of this the characteristic equation of this system has k roots equal to unity to which correspond k sets of solutions. We shall assume that the remaining $n - k$ roots of this characteristic equation have moduli different from unity.

For these conditions the system conjugate to (10.4) has k and only k periodic solutions, which we shall denote as usual by $\psi_{s1}, \ldots, \psi_{sk}$. We shall here assume that the functions ψ_{si} have been chosen in such manner that the conditions are satisfied

$$\sum_{\alpha=1}^{n} \varphi_{\alpha i} \psi_{\alpha j} = \delta_{ij}, \tag{11.3}$$

where δ_{ij} is the Kronecker symbol. For the assumptions made with respect to the characteristic equation of system (10.4) such choice of the functions ψ_{si} is always possible. Of this we can convince ourselves in precisely the same way as in sec. 12 of chapter III for the case of the constant coefficients p_{sj}.

We shall put

$$P_i(h_1, \ldots, h_k) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\alpha=1}^{n} f_\alpha(t, \varphi_1, \ldots, \varphi_n, 0)\, \psi_{\alpha i}\, dt \tag{11.4}$$
$$(i = 1, \ldots, k)$$

and prove the following theorem:

THEOREM. FOR EACH GENERATING SOLUTION OF THE FAMILY (11.1), LYING IN THE REGION G, FOR WHICH THE PARAMETERS h_j SATISFY THE EQUATIONS

$$P_i(h_1, \ldots, h_k) = 0 \qquad (i = 1, \ldots, k) \tag{11.5}$$

AND THE EQUATION WITH RESPECT TO ρ

$$\begin{vmatrix} \frac{\partial P_1}{\partial h_1} - \rho & \frac{\partial P_1}{\partial h_2} & \cdots & \frac{\partial P_1}{\partial h_k} \\ \frac{\partial P_2}{\partial h_1} & \frac{\partial P_2}{\partial h_2} - \rho & \cdots & \frac{\partial P_2}{\partial h_k} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial P_k}{\partial h_1} & \frac{\partial P_k}{\partial h_2} & \cdots & \frac{\partial P_k}{\partial h_k} - \rho \end{vmatrix} = 0 \tag{11.6}$$

HAS NO ROOTS WITH REAL PARTS EQUAL TO ZERO THERE EXISTS FOR SUFFICIENTLY SMALL μ AN ALMOST PERIODIC SOLUTION OF SYSTEM (10.1) REDUCING FOR $\mu = 0$ TO THE GENERATING SOLUTION.

THIS ALMOST PERIODIC SOLUTION OF SYSTEM (10.1) WILL BE ASYMPTOTICALLY STABLE IF ALL ROOTS OF EQUATION (11.6) AND $n - k$ CHARACTERISTIC EXPONENTS OF SYSTEM (10.4) HAVE NEGATIVE REAL PARTS, AND UNSTABLE IF AT LEAST ONE OF THESE MAGNITUDES HAS A POSITIVE REAL PART.

PROOF. 1. Let us put in equations (10.1)

$$x_s = \varphi_s(t, h_1, \ldots, h_k) + \mu y_s, \tag{11.7}$$

after which they assume the form

$$\frac{dy_s}{dt} = \sum_{\alpha=1}^{n} p_{s\alpha} y_\alpha + f_s(t, \varphi_1, \ldots, \varphi_n, 0) + \mu Y_s(t, y, \mu), \tag{11.8}$$

where

$$Y_s(t, y, \mu) = \frac{1}{2} \sum_{\alpha, \beta=1}^{n} \left(\frac{\partial^2 X_s}{\partial x_\alpha \partial x_\beta}\right) y_\alpha y_\beta + \\ + \sum_{\alpha=1}^{n} \left(\frac{\partial f_s}{\partial x_\alpha}\right) y_\alpha + \left(\frac{\partial f_s}{\partial \mu}\right) + Y_s^*(t, y, \mu) \tag{11.9}$$

and the functions Y_s^* reduce to zero for $\mu = 0$, so that

$$Y_s(t, y, 0) = \frac{1}{2} \sum_{\alpha, \beta=1}^{n} \left(\frac{\partial^2 X_s}{\partial x_\alpha \partial x_\beta}\right) y_\alpha y_\beta + \sum_{\alpha=1}^{n} \left(\frac{\partial f_s}{\partial x_\alpha}\right) y_\alpha + \left(\frac{\partial f_s}{\partial \mu}\right). \tag{11.10}$$

In (11.9) and (11.10) the derivatives are computed for $x_r = \varphi_r$, $\mu = 0$.

The question of the almost periodic solutions of a system of the form (11.8) has been considered in detail in sec. 8 of chapter IV on the assumption that the coefficients p_{sj} are constants, and for more severe restrictions on the structure of the functions f_s and Y_s. Nonetheless it is possible for equations (11.8) to repeat without essential changes the basic computations of the above mentioned section

We shall assume that for the generating solution under consideration conditions (11.5) are satisfied. The system of linear equations

$$\frac{dy_s^0}{dt} = \sum_{\alpha=1}^{n} p_{s\alpha} y_\alpha^0 + f_s(t, \varphi_1, \ldots, \varphi_n, 0), \tag{11.11}$$

then admits, as shown in sec. 9 and 10 of the preceding chapter, an almost periodic solution and this solution has the form

$$y_s^0 = M_1 \varphi_{s1} + \ldots + M_k \varphi_{sk} + y_s^{0*}(t), \tag{11.12}$$

where M_i are arbitrary constants and y_s^{0*} is some particular almost periodic solution of equations (11.11). The functions y_s^{0*} are finite sums of the products of purely periodic functions of period ω and functions of the form $e^{i\alpha t}$, where α is real.

Let us now transform equations (11.8) with the aid of the substitution

$$y_s = y_s^{0*} + v_s. \tag{11.13}$$

We shall have:

$$\frac{dv_s}{dt} = \sum_{\alpha=1}^{n} p_{s\alpha} v_\alpha + \mu Y_s(t, y, \mu). \tag{11.14}$$

The linear part of the system (11.14) can be transformed into a system of linear equations with constant coefficients. Since the system (10.4) has k characteristic exponents equal to zero and n - k characteristic exponents with real parts different from zero, the fundamental equation of the transformed system with constant coefficients will have k roots equal to zero and n - k roots with real parts different from zero. The transformation itself can be chosen such that the critical variables are separated out in the transformed equations. Thus, the transformed equations have the form

$$\frac{d\xi_i}{dt} = 0, \quad \frac{d\eta_j}{dt} = b_{j1}\eta_1 + \ldots + b_{jm}\eta_m \qquad (11.15)$$
$$(i = 1, \ldots, k; \; j = 1, \ldots, m),$$

where $m = n - k$, ξ_i and η_i are the new variables and b_{jr} are certain constants for which all the roots of the equation

$$\begin{vmatrix} b_{11} - \lambda & b_{12} & \ldots & b_{1m} \\ b_{21} & b_{22} - \lambda & \ldots & b_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ b_{m1} & b_{m2} & \ldots & b_{mm} - \lambda \end{vmatrix} = 0 \qquad (11.16)$$

have real parts different from zero. These roots determine n - k characteristic exponents of the system (10.4). The first group of equations (11.15) has such a simple structure because of the fact that to the zero characteristic exponent of the system (10.4) correspond k sets of solutions.

As follows from sec. 5 of chapter III, where a developed form of this transformation is given, the variables ξ_i and η_i are connected with the variables v_s by the relations

$$\xi_i = \sum_{\alpha=1}^{n} v_\alpha \psi_{\alpha i}, \quad \eta_j = \sum_{\alpha=1}^{n} v_\alpha \psi^*_{\alpha j} \qquad (11.17)$$
$$(i = 1, \ldots, k; \quad j = 1, \ldots, m),$$

where ψ_{si} are the periodic functions figuring in (11.3) and ψ^*_{sj} are also periodic functions of period ω which we need not give explicitly. Exactly in the same way as in sec. 8 of chapter IV we shall show that the inverse transformation, when (11.3) is satisfied, has the form

$$v_s = \varphi_{s1}\xi_1 + \ldots + \varphi_{sk}\xi_h + F_{s1}\eta_1 + \ldots + F_{sm}\eta_m, \qquad (11.18)$$

where F_{sj} are certain periodic functions of period ω.

Let us subject equations (11.14) to the transformation (11.17). We shall have:

$$\left.\begin{aligned} &\frac{d\xi_i}{dt} = \mu \sum_{\alpha=1}^{n} Y_\alpha(t, y, \mu)\,\psi_{\alpha i}, \\ &\frac{d\eta_j}{dt} = b_{j1}\eta_1 + \ldots + b_{jm}\eta_m + \mu \sum_{\alpha=1}^{n} Y_\alpha(t, y, \mu)\,\psi^*_{\alpha j} \\ &(i = 1, \ldots, k; \quad j = 1, \ldots, m). \end{aligned}\right\} \qquad (11.19)$$

To the obtained equations there can now be applied a transformation analogous to the transformation of Krylov and Bogolyubov. For this purpose we put

$$\xi_i = M_i + \mu u_i(t, M_1, \ldots, M_k), \ \eta_j = \mu z_j, \qquad (11.20)$$

where

$$\left.\begin{aligned} &u_i = \int_0^t \sum_{\alpha=1}^{n} Y_\alpha(t, y_1^0, \ldots, y_n^0, 0)\,\psi_{\alpha i}\,dt - A_i t, \\ &A_i = A_i(M_1, \ldots, M_k) = \\ &\quad = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\alpha=1}^{n} Y_\alpha(t, y_1^0, \ldots, y_n^0, 0)\,\psi_{\alpha i}\,dt \end{aligned}\right\} \qquad (11.21)$$

and y_s^o are the almost periodic solutions (11.12) of equations (11.11). The expressions under the integral in the functions u_i represent finite sums of terms of the form $\varphi(t)e^{i\alpha t}$, where α is real and $\varphi(t)$ is a periodic function of t of period ω. Consequently the functions u_i will be almost periodic.

Equations (11.19) after the transformation (11.20) will assume the form

$$\left.\begin{aligned} &\frac{dM_i}{dt} + \mu \sum_{r=1}^{k} \frac{\partial u_i}{\partial M_r}\frac{dM_r}{dt} = \mu A_i(M_1, \ldots, M_k) + \\ &\quad + \mu \sum_{\alpha=1}^{n} \{Y_\alpha(t, y_1, \ldots, y_n, \mu) - Y_\alpha(t, y_1^0, \ldots, y_n^0, 0)\}\,\psi_{\alpha i}, \\ &\frac{dz_j}{dt} = b_{j1}z_1 + \ldots + b_{jm}z_m + \sum_{\alpha=1}^{n} Y_\alpha(t, y, \mu)\,\psi^*_{\alpha i}. \end{aligned}\right\} \qquad (11.22)$$

But from (11.13), (11.18), (11.12) and (11.20) we have:

$$y_s = y_s^0 + \mu(\varphi_{s1}u_1 + \ldots + \varphi_{sk}u_k + F_{s1}z_1 + \ldots + F_{sm}z_m),$$

and since the functions Y_α satisfy with respect to the variables $y_1, \ldots, y_n$, μ the Cauchy-Lipschitz conditions, after solving equations (11.22) for dM_i/dt we finally obtain

$$\left.\begin{aligned} &\frac{dM_j}{dt} = \mu A_i(M_1, \ldots, M_k) + \mu^2 B_i(t, M, z, \mu), \\ &\frac{dz_j}{dt} = \sum_{r=1}^{m} b_{jr}z_r + F_j(t, M_1, \ldots, M_k) + \mu Z_j(t, M, z, \mu) \\ &\qquad\qquad (i = 1, \ldots, k;\ j = 1, \ldots, m), \end{aligned}\right\} \quad (11.23)$$

where

$$F_j = \sum_{\alpha=1}^{m} Y_\alpha(t, y_1^0, \ldots, y_n^0, 0)\,\psi_{\alpha j}^*.$$

The functions B_i, F_j and Z_j are almost periodic with respect to t while with respect to $M_1, \ldots, M_k$, $z_1, \ldots, z_m$ they satisfy the Cauchy-Lipschitz conditions.

All the transformations made were such that to any almost periodic solution of system (11.23) there corresponds an almost periodic solution of system (10.1) that reduces for $\mu = 0$ to the generating solution under consideration. Moreover, the problem of stability for the almost periodic solution of system (11.23) agrees with the same problem for the corresponding almost periodic solution of system (10.1).

2. Let us consider the functions $A_i(M_1, \ldots, M_k)$ in more detail. We shall show that they are linear with respect to $M_1, \ldots, M_k$:

$$A_i = A_{i1}M_1 + \ldots + A_{ik}M_k + A_{i0} \quad (11.24)$$

where

$$A_{ij} = \frac{\partial P_i(h_1, \ldots, h_k)}{\partial h_j}. \quad (11.25)$$

In fact, substituting (11.10) in (11.21) we obtain:

$$A_i = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\alpha=1}^{n} \left\{ \frac{1}{2} \sum_{\beta,\gamma=1}^{n} \left(\frac{\partial^2 X_\alpha}{\partial x_\beta \partial x_\gamma} \right) y_\beta^0 y_\gamma^0 + \right.$$
$$\left. + \sum_{\beta=1}^{n} \left(\frac{\partial f_\alpha}{\partial x_\beta} \right) y_\beta^0 + \left(\frac{\partial f_\alpha}{\partial \mu} \right) \right\} \psi_{\alpha i}\, dt$$

and therefore, on the basis of (11.12)

$$\frac{\partial A_i}{\partial M_j} = A_{ij} + \sum_{r=1}^{k} A_{ijr} M_r, \tag{11.26}$$

where

$$A_{ijr} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\alpha,\beta,\gamma=1}^{n} \left(\frac{\partial^2 X_\alpha}{\partial x_\beta \partial x_\gamma} \right) \varphi_{\beta j} \varphi_{\gamma r} \psi_{\alpha i}\, dt, \tag{11.27}$$

$$A_{ij} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\alpha=1}^{n} \left\{ \sum_{\beta,\gamma=1}^{n} \left(\frac{\partial^2 X_\alpha}{\partial x_\beta \partial x_\gamma} \right) \varphi_{\beta j} y_\gamma^{0*} + \sum_{\beta=1}^{n} \left(\frac{\partial f_\alpha}{\partial x_\beta} \right) \varphi_{\beta j} \right\} \psi_{\alpha i}\, dt, \tag{11.28}$$

On the basis of (11.2) and (10.4) we have:

$$A_{ij} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\alpha=1}^{n} \left\{ \sum_{\gamma=1}^{n} \frac{\partial p_{\alpha\gamma}}{\partial h_j} y_\gamma^{0*} + \frac{\partial f_\alpha(t, \varphi, 0)}{\partial h_j} \right\} \psi_{\alpha i}\, dt,$$

or, taking into account that y_s^{0*} is the particular solution of equations (11.11) we find:

$$A_{ij} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\alpha=1}^{n} \left\{ \frac{d}{dt} \frac{\partial y_\alpha^{0*}}{\partial h_j} - \sum_{\gamma=1}^{n} p_{\alpha\gamma} \frac{\partial y_\gamma^{0*}}{\partial h_j} \right\} \psi_{\alpha i}\, dt =$$
$$= \lim_{t\to\infty} \frac{1}{t} \left\{ \left| \sum_{\alpha=1}^{n} \frac{\partial y_\alpha^{0*}}{\partial h_j} \psi_{\alpha i} \right|_0^t - \int_0^t \sum_{\gamma=1}^{n} \left[\frac{\partial y_\gamma^{0*}}{\partial h_j} \left(\frac{d\psi_{\gamma i}}{dt} + \sum_{\alpha=1}^{n} p_{\alpha\gamma} \psi_{\alpha i} \right) \right] dt \right\} =$$
$$= \lim_{t\to\infty} \frac{1}{t} \left| \sum_{\alpha=1}^{n} \frac{\partial y_\alpha^{0*}}{\partial h_j} \cdot \psi_{\alpha i} \right|_0^t, \tag{11.29}$$

since the functions $\psi_{\gamma i}$ satisfy the equations conjugate to (10.4). Further, we have identically

$$\frac{d}{dt} \sum_{\alpha=1}^{n} y_\alpha^{0*} \psi_{\alpha i} = \sum_{\alpha=1}^{n} f_\alpha(t, \varphi, 0) \psi_{\alpha i},$$

whence follows

$$\frac{1}{t}\Big|_0^t\left(\sum_{\alpha=1}^{n}\frac{\partial y_\alpha^{0*}}{\partial h_j}\psi_{\alpha i}+\sum_{\alpha=1}^{n}y_\alpha^{0*}\frac{\partial\psi_{\alpha i}}{\partial h_j}\right)=\frac{\partial}{\partial h_j}\frac{1}{t}\int_0^t\sum_{\alpha=1}^{n}f_\alpha(t,\varphi,0)\,\psi_{\alpha i}\,dt,$$

and therefore, on the basis of (11.4)

$$\lim_{t\to\infty}\frac{1}{t}\Big|_0^t\sum_{\alpha=1}^{n}\frac{\partial y_\alpha^{0*}}{\partial h_j}\psi_{\alpha i}=\frac{\partial P_i(h_1,\ldots,h_k)}{\partial h_j},$$

since the functions $\psi_{\alpha i}$ are periodic for any values of h_i and therefore the functions $\partial\psi_{\alpha i}/\partial h_j$ will likewise be periodic and hence bounded. From this and from (11.29) we convince ourselves of the validity of (11.25).

Let us now proceed to the computation of A_{ijr}. For this we note that the magnitudes $\varphi_{\gamma r}$ satisfy the equations in variations, i.e. equations (11.11) for $f_s = 0$. Hence, comparing (11.27) with (11.28) we see that the magnitudes $A_{i\gamma r}$ are obtained from the magnitudes A_{ij} if in the latter there is put $f_s = 0$, which on the basis of (11.25) gives $A_{i\gamma r} = 0$. Thus, taking (11.26) into account, we can consider as proven that the functions A_i have the form (11.24), with relations (11.25) satisfied.

3. Since by the conditions of the theorem equation (11.6) has no zero root, the determinant of the magnitudes A_{ij} is different from zero. As a result we can, without loss of generality, assume that in (11.24) the magnitudes A_{io} are equal to zero. In fact, the general case can be reduced to this particular case by the substitution of variables $M_i = M_i^* + N_i$, where M_i^* is a solution of the equations $A_i = 0$. In this way we can consider that the equations (11.23) have the form

$$\left.\begin{aligned}\frac{dM_i}{dt}&=\mu\left(\frac{\partial P_i}{\partial h_1}M_1+\ldots+\frac{\partial P_i}{\partial h_k}M_k\right)+\mu^2B_i(t,M,z,\mu),\\ \frac{dz_j}{dt}&=\sum_{r=1}^{m}b_{jr}z_r+F_j(t,M_1,\ldots,M_k)+\mu Z_j(t,M,z,\mu).\end{aligned}\right\}\quad(11.30)$$

The almost periodic solution of these equations can be found, exactly the same way as the almost periodic solution of equations (8.37) of chapter IV, by the method of successive approximations. We shall take in the first approximation $M_i^{(1)} = 0$, and as the first approximation of the magnitudes z_j we shall take the almost periodic solution of the equations

$$\frac{dz_j^{(1)}}{dt} = \sum_{r=1}^{m} b_{jr} z_r^{(1)} + F_j\ (t,\ 0,\ \ldots,\ 0).$$

The further approximations $M_i^{(p)}$, $z_j^{(p)}$ $(p > 1)$ we determine with the aid of the equations

$$\left.\begin{aligned} \frac{dM_i^{(p)}}{dt} &= \mu\left(\frac{\partial P_i}{\partial h_1} M_1^{(p)} + \ldots + \frac{\partial P_i}{\partial h_k} M_k^{(p)}\right) + \\ &\qquad\qquad + \mu^2 B_i\ (t,\ M^{(p-1)},\ z^{(p-1)},\ \mu), \\ \frac{dz_j^{(p)}}{dt} &= \sum_{r=1}^{m} b_{jr} z_r^{(p)} + F_j\ (t,\ M_1^{(p)},\ \ldots,\ M_k^{(p)}) + \\ &\qquad\qquad + \mu Z_j\ (t,\ M^{(p-1)},\ z^{(p-1)},\ \mu). \end{aligned}\right\} \quad (11.31)$$

By the property of the roots of equations (11.6) and (11.16) each of the systems of equations

$$\begin{aligned} \frac{dM_i}{dt} &= \mu\left(\frac{\partial P_i}{\partial h_1} M_1 + \ldots + \frac{\partial P_i}{\partial h_k} M_k\right) + p_i(t), \\ \frac{dz_j}{dt} &= b_{j1} z_1 + \ldots + b_{jm} z_m + q_j(t) \end{aligned}$$

admits one and only one almost periodic solution for any choice of the almost periodic functions p_i and q_j. Hence the system (11.31) determines in a unique manner the sequence of almost periodic functions $M_i^{(p)}$, $z_j^{(p)}$. The proof of the convergence of these sequences to the required almost periodic solution of equations (11.30) was given in sec. 9 of chapter IV and differs little from the proof of convergence of the sequence $y_s^{(p)}$ for equations (10.6) of the present chapter.

To the obtained almost periodic solution of equations (11.30) corresponds, as was already remarked above, an almost periodic solution of the system (10.1) reducing for $\mu = 0$ to the generating solution investigated. The problem of stability for this almost periodic solution of system (10.1) agrees with the problem of stability for the found almost periodic solution of system (11.30). The latter problem was considered in sec. 11 of chapter IV and the stability criteria there obtained agree with those which are indicated in the formulation of the theorem. The theorem can therefore be considered as completely proven.

12. Case in which the Number of Parameters of the Generating Solution is Equal to the Order of the System

In the preceding section we assumed that both the generating solution and the functions X_s are periodic with respect to t. A consideration of the more general case where these magnitudes are almost periodic leads to great difficulties connected with the circumstance that the corresponding system in variations will have almost periodic coefficients, which greatly complicates the problem. These difficulties can however be easily overcome in the case where the number of parameters of the generating solution is equal to the order of the system. To this case we shall now turn our attention.

Let us assume that the equations of the oscillations have the form

$$\frac{dx_s}{dt} = X_s(t, x_1, \ldots, x_n) + \mu f_s(t, x_1, \ldots, x_n, \mu) \qquad (12.1)$$
$$(s = 1, \ldots, n),$$

where the functions X_s and f_s satisfy the general conditions indicated in sec. 10 for equations (10.1). We shall however assume that not only the functions f_s but also the functions X_s are almost periodic with respect to t and furthermore such that for any choice of the arbitrary functions $x_s(t)$ lying in the region G the functions $X_s(t, x_1(t), \ldots, x_n(t))$ are likewise almost periodic. We shall assume further that the generating system

$$\frac{dx_s}{dt} = X_s(t, x_1, \ldots, x_n) \qquad (12.2)$$

admits the family of solutions

$$x_s = \varphi_s\,(t, h_1, \ldots, h_n), \tag{12.3}$$

depending on n arbitrary constants h_s for which the magnitudes φ_s, for all values $h_1, \ldots, h_n$ lying in a certain region H, are almost periodic functions of t. We shall, moreover, assume that for these values of h_s the functions

$$\varphi_{si} = \frac{\partial \varphi_s}{\partial h_i} \qquad (s, i = 1, \ldots, n) \tag{12.4}$$

are likewise almost periodic with respect to t. For this, in particular, it is sufficient that the functions φ_s be almost periodic UNIFORMLY WITH RESPECT TO h_i, i.e. that for each positive ε a positive $l(\varepsilon)$ exist such that within each interval of length $l(\varepsilon)$ there will exist a number τ for which

$$|\varphi_s(t+\tau, h_1, \ldots, h_n) - \varphi_s(t, h_1, \ldots, h_n)| < \varepsilon$$

for all values of t and h_i considered.

The functions φ_{si} determine n independent almost periodic solutions of the equations in variations

$$\frac{dy_s}{dt} = p_{s1}y_1 + \cdots + p_{sn}y_n \qquad \left(p_{sj} = \left(\frac{\partial X_s}{\partial x_j}\right)_{x_i = \varphi_i}\right). \tag{12.5}$$

The Wronskian determinant

$$\frac{\partial(\varphi_1, \ldots, \varphi_n)}{\partial(h_1, \ldots, h_n)}$$

of these solutions is different from zero for all values of t and h_i. We shall assume that this determinant numerically exceeds a certain positive number for all values of t and h_i considered. In this case the system of equations

$$\sum_{\alpha=1}^{n} \varphi_{\alpha i}\psi_{\alpha j} = \delta_{ij} \qquad (i, j = 1, \ldots, n), \tag{12.6}$$

where δ_{ij} is the Kronecker symbol, determines n^2 almost periodic functions ψ_{sj}. These functions for each fixed j are evidently a particular solution of the system conjugate to (12.5). Thus, in the case considered the system conjugate to (12.5) likewise admits n almost periodic solutions $\psi_{s1},\ldots,\psi_{sn}$.

We shall now consider relations (12.3) as a transformation of the variables $x_1,\ldots,x_n$ into the variables $h_1,\ldots,h_n$. Then, taking into account that the functions φ_s are a particular solution of equations (12.2), we obtain:

$$\sum_{s=1}^{n} \frac{\partial \varphi_\alpha}{\partial h_s} \frac{dh_s}{dt} = \mu f_s(t, \varphi_1, \ldots, \varphi_n, \mu).$$

whence on the basis of (12.6) we find:

$$\frac{dh_s}{dt} = \mu \sum_{\alpha=1}^{n} f_\alpha(t, \varphi_1, \ldots, \varphi_n, \mu)\, \psi_{\alpha i}\, dt. \tag{12.7}$$

The obtained equations can be considered as a particular case of the equations studied in the preceding section and the theorem there proven can be applied to them. The generating solution now has the form h_s = const. and the equations determining the values of h_s will be:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\alpha=1}^{n} f_\alpha(t, \varphi_1, \ldots, \varphi_n, 0)\, \psi_{\alpha i}\, dt = 0. \tag{12.8}$$

These equations agree exactly with equations (11.5) for the system (12.1). Hence if the roots of equation (11.6) have real parts different from zero the system (12.7) will admit for sufficiently small μ an almost periodic solution $h_s = h_s(t, \mu)$ for which $h_s(t,0) = h_s^*$, where h_s^* is a solution of equations (12.8). To this almost periodic solution of equations (12.7) corresponds the almost periodic solution of equations (12.1) that reduces for $\mu = 0$ to the generating solution $\varphi_s(t, h_1^*,\ldots,h_n^*)$. From this the following important supplement to the theorem of the preceding section is obtained.

Let us assume that with the conditions of the above mentioned theorem satisfied the number k of parameters h_i

is equal to n. Then this theorem remains valid if not only the functions f_s but also the functions X_s and the generating solution φ_s are almost periodic with respect to t, provided the above indicated supplementary conditions are satisfied.

We may remark that the theorem of the preceding section was proved for the assumption that the functions f_s have the special structure that was indicated for them. The right hand sides of equations (12.7) do not in general have this structure. Nevertheless the supplementary theorem here formulated remains valid. The reason is that the structure in question was required only for the possibility of reducing the equations to the form (11.23). In the case now considered the equations (12.7) already have the form (11.19) and to reduce them to the form (11.23) it is sufficient to subject them to the Krylov and Bogolyubov transformation. For this transformation to be possible it is sufficient to assume in addition that the functions

$$u_i = \int_0^t \sum_{\alpha=1}^n f_\alpha (t, \varphi_1, \ldots, \varphi_n, 0)\, \psi_{\alpha i}\, dt - \\ - t \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{\alpha=1}^n f_\alpha (t, \varphi_1, \ldots, \varphi_n, 0)\, \psi_{\alpha i}\, dt$$

are almost periodic. However, as follows from the investigations of N.N. Bogolyubov [1], such additional requirement is superfluous.

Let us now assume that the generating system has the form

$$\frac{dx_s}{dt} = p_{s1}x_n + \ldots + p_{sn}x_n + F_s(t), \tag{12.9}$$

where p_{sj} are periodic functions of t of some period ω and F_s are almost periodic functions. In this case the theorem of the preceding section remains also valid if the generating solution is almost periodic. The case in which the

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1 See the monograph, already frequently referred to, "O nekotorykh statisticheskikh metodakh matematicheskoi fiziki" (On Certain Statistical Methods of Mathematical Physics).

generating system has the form (12.9) was investigated by us in detail in chapter IV on the assumption that the coefficients p_{sj} are constants. The fundamental results there obtained can easily be carried over to the general case in which the coefficients are not constants but periodic functions of t. We shall not however here dwell on this question, referring the reader to our work cited on p.498.

CHAPTER VII

LYAPUNOV SYSTEMS

1. Statement of the Problem

In connection with the problem of the stability of motion in one of the special cases when this problem is not solved by the linear terms in the equations of the disturbed motion A. M. Lyapunov proved the existence of a family of periodic solutions for one special system of nonlinear differential equations and indicated a method for their effective computation. These systems, which have received the name of LYAPUNOV SYSTEMS, have found wide application in practice, since to them are reduced many problems of mechanics and astronomy. In particular, to the computation of periodic solutions of Lyapunov systems is reduced the problem of the free oscillations of conservative systems about a position of equilibrium or about the state of steady motion, when account is taken also of the higher order terms in the expansions of the kinetic and potential energies.

We may note also a further important significance of Lyapunov systems. We saw in the preceding section that the effectiveness of the methods there presented of determining the oscillations of systems approximating those that are arbitrary nonlinear is based on the possibility of the actual computation, if not of the general, at least of the periodic solution of the generating system. This to a considerable extent restricts the choice of the generating system. As such generating system of sufficiently general form Lyapunov systems may be taken.

In this chapter we present the theory of Lyapunov systems. In the following chapter will be presented the theory of systems approximating Lyapunov systems.

Let us consider the system of differential equations of the n-th order of the form

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n + X_s^*(x_1, \ldots, x_n) \qquad (1.1)$$
$$(s = 1, \ldots, n),$$

where a_{sj} are constants and X_s^* are analytic functions of the variables $x_1,\ldots,x_n$, not dependent on t, in the neighborhood of the point $x_1 = \ldots = x_n = 0$, the expansions of which in powers of these variables start with terms of not lower than the second order. With respect to this system we shall make the following fundamental assumptions.

1. We shall assume that the fundamental equation

$$\begin{vmatrix} a_{11}-\rho & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\rho & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\rho \end{vmatrix} = 0 \qquad (1.2)$$

has at least one pair of purely imaginary roots $\pm \lambda i$.

With such assumption the system (1.1) can be transformed to the form

$$\left.\begin{aligned} &\frac{dx}{dt} = -\lambda y + X, \qquad \frac{dy}{dt} = \lambda x + Y, \\ &\frac{dx_s}{dt} = b_{s1}x_1 + \ldots + b_{sm}x_m + a_s x + b_s y + X_s \\ &\qquad (s = 1, \ldots, m). \end{aligned}\right\} \qquad (1.3)$$

Here $m = n - 2$, X, Y, X_s are analytic functions of the variables x, y, $x_1,\ldots,x_m$, the expansions of which in powers of these variables start with terms of not lower than the second order, and b_{sj}, a_s, b_s are certain real constants.

The actual transformation of equations (1.1) to the form (1.3) can be carried out in the following manner. The fundamental equation of the system conjugate to the linear part of system (1.1) also has a pair of purely imaginary roots $\pm\lambda i$, which are satisfied by two particular solutions of the form

$$(A_s + iB_s)\, e^{-i\lambda t}, \quad (A_s - iB_s)\, e^{i\lambda t},$$

where A_s and B_s are real constants. Hence, the linear part of the system (1.1) has the two first integrals

$$(x + iy)\, e^{-i\lambda t}, \quad (x - iy)\, e^{i\lambda t},$$

where

$$x = \sum_{\alpha=1}^{n} A_\alpha x_\alpha, \quad y = \sum_{\alpha=1}^{n} B_\alpha x_\alpha.$$

Taking the magnitudes x and y as new variables in place of x_{n-1} and x_n we arrive at a system of the form (1.3).

2. We shall assume that the roots $\pm\lambda i$ are simple, that their integral multiples are not roots of equation (1.2) and that this equation does not also have a zero root.

Since equation (1.2) must agree with the fundamental equation of the linear part of system (1.3), which evidently breaks down into an equation of the second order

$$\rho^2 + \lambda^2 = 0$$

and an equation of the m-th order

$$D(\rho) = \begin{vmatrix} b_{11} - \rho & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} - \rho & \dots & b_{2m} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mm} - \rho \end{vmatrix} = 0, \qquad (1.4)$$

it follows from the assumption made that equation (1.4) does not have either a zero root or a root of the form $\pm p\lambda i$, where p is an integer. (This does not exclude however the possibility of the presence of purely imaginary roots different from $\pm p\lambda i$).

Moreover, from the assumption made it follows that the magnitudes a_s and b_s in equations (1.3) can be taken equal to zero. In fact, making the additional transformation of variables

$$\bar{x}_s = x_s + \alpha_s x + \beta_s y \qquad (s = 1, \ldots, m),$$

where α_s and β_s are certain constants, we shall have:

$$\frac{d\bar{x}_s}{dt} = \sum_{\alpha=1}^{m} b_{s\alpha}\bar{x}_\alpha - $$
$$- \left(\sum_{\alpha=1}^{m} b_{s\alpha}\alpha_\alpha - \lambda\beta_s - a_s \right) x - \left(\sum_{\alpha=1}^{n} b_{s\alpha}\beta_\alpha + \lambda\alpha_s - b_s \right) y + \ldots$$

Hence, if we choose the constants α_s and β_s according to the equations

$$\left.\begin{aligned} b_{s1}\alpha_1 + \ldots + b_{sm}\alpha_m - \lambda\beta_s = a_s, \\ b_{s1}\beta_1 + \ldots + b_{sm}\beta_m + \lambda\alpha_s = b_s, \end{aligned}\right\} \qquad (1.5)$$

the magnitudes in the transformed equations playing the part of α_s and β_s will be equal to zero. From equations (1.5), setting $\alpha_s + i\beta_s = \gamma_s$, we find:

$$b_{s1}\gamma_1 + \ldots + b_{sm}\gamma_m + i\lambda\gamma_s = a_s + ib_s.$$

The determinant of this system, evidently equal to $D(-i\lambda)$, is different from zero since $-\lambda$ is not a root of equation (1.4). In this way we obtain entirely definite values for the magnitudes γ_s and hence also for the magnitudes α_s and β_s.

Thus, without loss of generality, we can assume that in equations (1.3) the magnitudes a_s and b_s are equal to zero and these equations have the form

$$\left.\begin{aligned} &\frac{dx}{dt} = -\lambda y + X(x, y, x_1, \ldots, x_m), \\ &\frac{dy}{dt} = \lambda x + Y(x, y, x_1, \ldots, x_m), \\ &\frac{dx_s}{dt} = b_{s1}x_1 + \ldots + b_{sm}x_m + X_s(x, y, x_1, \ldots, x_m) \\ &\qquad\qquad (s = 1, \ldots, m). \end{aligned}\right\} \qquad (1.6)$$

We shall use this form of the equations of the oscillations in what follows.

3. We shall assume, finally, that the system considered admits a first integral of the form

$$H(x, y, x_1, \ldots, x_m) = \text{const}, \tag{1.7}$$

where H is an analytic function of the variables x, y, $x_1, \ldots, x_m$, the expansion of which in powers of these variables contains terms of the second order, these not becoming zero for $x_1, = \ldots = x_m = 0$.

Let us determine the functions H more precisely. Let

$$H = Ax + By + \sum_{\alpha=1}^{m} A_\alpha x_\alpha + \\ + Cx^2 + Dxy + Ey^2 + \sum_{\alpha=1}^{m} C_\alpha x x_\alpha + \sum_{\alpha=1}^{m} D_\alpha y x_\alpha + \\ + W(x_1, \ldots, x_m) + S(x, y, x_1, \ldots, x_m),$$

where W is a quadratic form of the variables $x_1, \ldots, x_m$ and S is an analytic function of x, y, $x_1, \ldots, x_m$, the expansions of which start with terms of not lower than the third order. Since the function H determines the first integral of equations (1.6) the identity holds

$$\frac{\partial H}{\partial x}(-\lambda y + X) + \frac{\partial H}{\partial y}(\lambda x + Y) + \\ + \sum_{\alpha=1}^{m} \frac{\partial H}{\partial x_\alpha}(\beta_{\alpha 1} x_1 + \ldots + b_{\alpha m} x_m + X_\alpha) \equiv 0. \tag{1.8}$$

Equating in this identity the coefficients of the first degrees of the magnitudes x, y, x_s to zero we obtain:

$$A = 0, \quad B = 0, \quad \sum_{\alpha=1}^{m} b_{\alpha s} A_\alpha = 0 \quad (s = 1, \ldots, m).$$

whence it follows that

$$A_1 = \ldots = A_m = 0,$$

since the determinant of the coefficients $b_{\alpha s}$, in virtue of the condition on the roots of equation (1.4), is different from zero. Thus the function H does not contain terms of the first order.

Equating in identity (1.8) to zero the coefficients of xx_s and yx_s we obtain the result that the magnitudes C_s and D_s satisfy the linear homogeneous equations

$$b_{1s}C_1 + \dots + b_{ms}C_m + \lambda D_s = 0,$$
$$b_{1s}D_1 + \dots + b_{ms}D_m - \lambda C_s = 0$$

or setting $E_s = D_s + iC_s$,

$$b_{s1}E_1 + \dots + b_{sm}E_m + i\lambda E_s = 0.$$

The determinant of this system of equations, as we already indicated above, is different from zero. Hence all magnitudes E_s, and therefore also the magnitudes D_s and C_s, are equal to zero. As to the magnitudes C, D, E, according to the assumption, at least one of them is different from zero. On the other hand, equating to zero the coefficients of x^2, y^2 and xy in the identity (1.8) we find:

$$D = 0, \; C = E,$$

and therefore the magnitudes C and E, which are equal to each other, must be different from zero. Without loss of generality we can take these magnitudes equal to unity since the integral (1.7) can be divided by an arbitrary factor.

Thus, the integral (1.7) has the form

$$H = x^2 + y^2 + W(x_1, \dots, x_m) + S = \text{const.} \tag{1.9}$$

Systems of equations of the type (1.1) satisfying the above three conditions [1] we shall call Lyapunov systems.

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[1] Periodic solutions of systems of the form (1.1) for more general assumptions have been considered in the two papers: Sokolov V.M., O periodicheskikh kolebaniyakh sistem Lyapunova v odnom osobom sluchae (On the Periodic Oscillations of Lyapunov Systems in One Particular Case, Trudy Ural'skogo politekhn. in-ta, 1954, Ryabov Yu. A., Obobshchenie odnoi teoremy A.M. Lyapunova (Generalization of a Theorem of Lyapunov), Uch. zap. MGU, no. 165, Matematika, 1954.

We shall similarly denote mechanical systems described by these equations.

As an example let us consider the system of equations

$$\frac{dx_i}{dt} = -\lambda_i y_i + \frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt} = \lambda_i x_i - \frac{\partial F}{\partial x_i} \qquad (1.10)$$
$$(i = 1, \ldots, k),$$

Where F is an analytic function of the variables x_i, y_i the expansion of which starts with terms of now lower than the third order. By equations of this type are described the nonlinear oscillations of a conservative system with k degrees of freedom about the position of stable equilibrium or about the steady state of motion.

If not all magnitudes λ_i/λ_j are integers the sysetm (1.10) satisfies all the above stated conditions and is a particular case of Lyapunov systems.

2. Periodic Solutions of Lyapunov Systems

We pass on to the investigation of the question of the existence of periodic solutions of Lyapunov systems. For this purpose we transform equations (1.6) with the aid of the substitution

$$x = \rho\cos\vartheta, \quad y = \rho\sin\vartheta, \quad x_s = \rho z_s \quad (s = 1, \ldots, m). \qquad (2.1)$$

The transformed system has the form

$$\left.\begin{aligned} &\frac{d\rho}{dt} = \rho^2 R(\rho, z_j, \vartheta), \\ &\frac{d\vartheta}{dt} = \lambda + \rho\Theta(\rho, z_j, \vartheta), \\ &\frac{dz_s}{dt} = b_{s1}z_1 + \ldots + b_{sm}z_m + \rho Z_s(\rho, z_j, \vartheta) \\ &\qquad (s = 1, \ldots, m). \end{aligned}\right\} \qquad (2.2)$$

Here R, Θ and Z_s are analytic functions of the variables ρ, $z_1, \ldots z_m$, the expansions of which in powers of these variables converge when the moduli of these magnitudes do not exceed certain constant limits. The coefficients of these expansions are periodic functions of ϑ of period 2π, which are polynomials in cos ϑ and sin ϑ.

By eliminating t from the system (2.2) and using the integral (1.9) we can reduce this system of the (m+2)-th order to a system of the m-th order. In fact, we have, first of all:

$$\frac{dz_s}{d\vartheta} = c_{s1}z_1 + \dots + c_{sm}z_m + \rho Z_s^*(\rho, z_j, \vartheta), \tag{2.3}$$

where

$$c_{sj} = \frac{b_{sj}}{\lambda}, \tag{2.4}$$

and Z_s^* are functions of the same type as Z_s. We denote further by μ^2 the arbitrary constant in the integral (1.9) and going over to the new variables, we obtain:

$$\rho^2[1 + F(\rho, z_j, \vartheta)] = \mu^2, \tag{2.5}$$

where F is an analytic function of the variables ρ, $z_1, \dots, z_m$, reducing to zero for $\rho = z_1 = \dots = z_m = 0$, the coefficients of whose expansion are polynomials in cos ϑ and sin ϑ. Equation (2.5) can be solved for ρ. For this it is necessary first of all to take the square root, which gives

$$\rho[1 + F^*(\rho, z_j, \vartheta)] = \pm\mu, \tag{2.6}$$

where F^* is a function of the same type as F. Since the magnitude μ is an arbitrary constant, we can without loss of generality, take the plus sign in the obtained equation. Now solving (2.6) for ρ we obtain:

$$\rho = \mu[1 + G(\mu, z_j, \vartheta)],$$

where G is an analytic function of the variables $\mu, z_1, \dots, z_m$, the expansion of which in powers of these variables converges when the moduli of these magnitudes do not exceed certain sufficiently small constant limits. The coefficients of the expansion are polynomials in cos ϑ and sin ϑ.

Substituting ρ in equations (2.3) we obtain finally the following system of the m-th order:

$$\frac{dz_s}{d\vartheta} = c_{s1}z_1 + \ldots + c_{sm}z_m + \mu U_s(\mu, z_j, \vartheta) \tag{2.7}$$
$$(s = 1, \ldots, m),$$

determining the magnitudes z_s as functions of ϑ. Here U_s are functions of the same type as G, i.e. analytic with respect to μ, $z_1, \ldots, z_m$ and periodic with respect to ϑ.

Since equation (1.4) does not by assumption have roots of the form $\pm p\lambda i$, where p is any integer, including zero, the equation

$$\begin{vmatrix} c_{11} - \rho & c_{12} & \ldots & c_{1m} \\ c_{21} & c_{22} - \rho & \ldots & c_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ c_{m1} & c_{m2} & \ldots & c_{mm} - \rho \end{vmatrix} = 0 \tag{2.8}$$

on the basis of (2.4) does not have roots of the form $\pm pi$. From this it follows that the system (2.7) is a particular case of the systems considered in sec. 5 of chapter II. Hence it admits for sufficiently small values of μ a periodic solution reducing to the generating solution for $\mu = 0$. (In the particular case considered the generating solution is the trivial solution $z_1^o = \ldots = z_m^o = 0$ of the system

$$\frac{dz_s^0}{d\vartheta} = c_{s1}z_1^0 + \ldots + c_{sm}z_m^0,$$

to which (2.7) reduces for $\mu = 0$). This solution will be analytic with respect to μ and for it we can write:

$$z_s = \mu z_s^{(1)}(\vartheta) + \mu^2 z_s^{(2)}(\vartheta) + \ldots \qquad (s = 1, \ldots, m), \tag{2.9}$$

where $z_s^{(k)}$ are periodic functions of ϑ of period 2π.

Substituting expressions (2.9) for z_s in the expression for ρ we obtain the result that ρ is likewise a periodic function of ϑ of the form

$$\rho = \mu + \mu^2\rho^{(2)}(\vartheta) + \mu^3\rho^3(\vartheta) + \ldots, \tag{2.10}$$

where $\rho^{(k)}$ are periodic functions of ϑ of period 2π,

Thus, we have proved the existence of a periodic solution of the system (2.2) if we take the magnitude ϑ as the variable. This solution depends analytically on the arbitrary constant μ.

Substituting (2.9) and (2.10) in (2.1) we obtain a particular solution of equations (1.6) in which the magnitudes x, y, x_s are periodic functions of ϑ of period 2π. These magnitudes will likewise be analytic functions of the constant μ. We shall show that if we again go over to the independent variable t the obtained solution will as before be periodic and analytic with respect to μ and that the period of the solution will likewise be an analytic function of μ. For this purpose let us consider the second of equations (2.2) that determines ϑ. Substituting in this equation the values of ρ and z_s and separating the variables we obtain:

$$t=\int_0^{\vartheta}\frac{d\vartheta}{\lambda+\rho\vartheta}=\frac{1}{\lambda}\int_0^{\vartheta}[1+\theta_1(\vartheta)\mu+\theta_2(\vartheta)\mu^2+\dots]\,d\vartheta,$$

where Θ_k are periodic functions of ϑ of period 2π and the series under the integral sign converges for sufficiently small values of μ. The constant of integration is here chosen such that at the initial instant $t = 0$ the magnitude ϑ also reduces to zero.

Forming the difference

$$t(\vartheta+2\pi)-t(\vartheta)=\frac{1}{\lambda}\int_{\vartheta}^{\vartheta+2\pi}(1+\theta_1\mu+\theta_2\mu^2+\dots)\,d\vartheta,$$

we see that it is a constant magnitude, independent of ϑ, equal to

$$T=\frac{2\pi}{\lambda}(1+\alpha_1\mu+\alpha_2\mu^2+\dots), \tag{2.11}$$

where α_k are constants determined by the formulas

$$\alpha_k=\frac{1}{2\pi}\int_0^{2\pi}\theta_k(\vartheta)\,d\vartheta \qquad (k=1, 2, \dots).$$

From this it follows that every magnitude which is a periodic function of ϑ of period 2π is also periodic with respect to t with period equal to T. In particular, the magnitudes x, y, x_s in the above mentioned particular solution are also periodic functions of t of period T. At the same time we see from the expression for the period that it is an analytic function of μ reducing to $2\pi/\lambda$ for $\mu = 0$.

We shall show that the magnitudes x, y, x_s expressed in terms of t are likewise analytic functions of μ. In fact, the magnitudes x, y, x_s in the periodic solution under consideration, and in every other solution of equations (1.6), are analytic functions of their initial values. As to these initial values, by setting the magnitude ϑ equal to zero in formulas (2.1), (2.9) and (2.10) we see that they in their turn are analytic functions of the magnitude μ. From this the validity of our assertion immediately follows.

We shall denote by c the initial value of the magnitude ρ. From (2.10) we have:

$$c = \mu + \mu^2 \rho^{(2)}(0) + \dots, \tag{2.12}$$

i.e. c is an analytic function of the magnitude μ reducing to zero together with it. Solving (2.12) for μ we see that μ in turn is an analytic function of c reducing to zero for $c = 0$. In the obtained periodic solution we can replace the magnitude μ by its expression in terms of c and then this solution, like its period, will be analytic with respect to c.

We may remark that since for $t = 0$ the magnitude ϑ also reduces to zero the magnitude c agrees with the initial value of the magnitude x while the initial value of the magnitude y is equal to zero. Moreover, from (2.1) it follows that for $c = 0$ the magnitudes x, y, x_s likewise reduce to zero. As to the period T, as shown by (2.11), for $c = 0$ it reduces to $2\pi/\lambda$.

Summarizing, we can state the following fundamental theorem:

THEOREM. THE LYAPUNOV SYSTEM (1.6) ADMITS IN A SUFFICIENTLY SMALL NEIGHBORHOOD OF THE ORIGIN A PERIODIC SOLUTION DEPENDING ON AN ARBITRARY PARAMETER. THIS PARAMETER IS THE INITIAL VALUE OF THE MAGNITUDE x, WHICH IS ASSUMED SUFFICIENTLY SMALL. THE INITIAL VALUE OF y IS EQUAL TO ZERO.

THE INITIAL VALUES OF x_s ARE CERTAIN ANALYTIC FUNCTIONS OF THE MAGNITUDE c. THE PERIOD OF THE SOLUTION IS AN ANALYTIC FUNCTION OF c, REDUCING TO $2\pi/\lambda$ FOR $c = 0$. THE SOLUTION ITSELF IS LIKEWISE ANALYTIC WITH RESPECT TO c AND FOR $c = 0$ REDUCES TO THE TRIVIAL SOLUTION $x = y = x_1 = \ldots = x_m = 0$.

Since the system (1.6) is autonomous we may in the periodic solution considered replace t by t + h, where h is an arbitrary constant. In this way there is obtained a periodic solution depending on two arbitrary constants.

3. Practical Method of Computing the Periodic Solutions of Lyapunov Systems

For the actual computation of the periodic solutions, the existence of which we have established in the preceding section, Lyapunov proposed a very convenient method a certain modification of which we had already used in the theory of quasilinear autonomous systems.

Let

$$T = \frac{2\pi}{\lambda}(1 + h_1 c + h_2 c^2 + \ldots) \tag{3.1}$$

be the period of the required solution. Here h_j are certain unknown but entirely definite constants. We shall introduce in equations (1.6), in place of the independent variable t, the variable τ with the aid of the substitution

$$t = \frac{\tau}{\lambda}(1 + h_1 c + h_2 c^2 + \ldots).$$

Equations (1.6) assume the form

$$\left.\begin{aligned} \frac{dx}{d\tau} &= \left(-y + \frac{1}{\lambda}X\right)(1 + h_1 c + h_2 c^2 + \ldots), \\ \frac{dy}{d\tau} &= \left(x + \frac{1}{\lambda}Y\right)(1 + h_1 c + h_2 c^2 + \ldots), \\ \frac{dx_s}{d\tau} &= \left(c_{s1}x_1 + \ldots + c_{sm}x_m + \frac{1}{\lambda}X_s\right)(1 + h_1 c + h_2 c^2 + \ldots) \\ &\qquad (s = 1, \ldots, m), \end{aligned}\right\} \tag{3.2}$$

where c_{sj} have the same values as in formulas (2.4).

To the required periodic solution of equations (1.6) of period T corresponds a periodic solution of equations (3.2) of period 2π. This solution will likewise be analytic with respect to c. In fact, the periodic solution considered will be analytic with respect to the parameter c, entering directly into the equations, and of its initial values. The initial values of the periodic solution will be the same as before the transformation, since the magnitudes t and τ simultaneously reduce to zero. As we saw in the preceding section, these initial values will be : for x the magnitude c, for y zero, and for x_s certain unknown but entirely definite analytic functions. Thus, the periodic solution under consideration, being analytic with respect to c and its initial values, will also be analytic with respect to c. We can therefore write:

$$\left.\begin{aligned} x &= cx^{(1)}(\tau)+c^2x^{(2)}(\tau)+\ldots, \\ y &= cy^{(1)}(\tau)+c^2y^{(2)}(\tau)+\ldots, \\ x_s &= cx_s^{(1)}(\tau)+c^2x_s^{(2)}(\tau)+\ldots, \end{aligned}\right\} \tag{3.3}$$

where $x^{(k)}$, $y^{(k)}$, $x_s^{(k)}$ are certain periodic functions of τ of period 2π. The initial values give:

$$\left.\begin{gathered} x^{(1)}(0)=1, \quad x^{(2)}(0)=x^{(3)}(0)=\ldots=0, \\ y^{(1)}(0)=y^{(2)}(0)=\ldots=0. \end{gathered}\right\} \tag{3.4}$$

Substituting expansions (3.3) in (3.2) and equating coefficients of like powers of c we obtain the equations for finding these coefficients. In particular, we have:

$$\frac{dx_1}{d\tau}=-y^{(1)}, \quad \frac{dy^{(1)}}{d\tau}=x^{(1)},$$
$$\frac{dx_s^{(1)}}{d\tau}=c_{s1}x_1^{(1)}+\ldots+c_{sm}x_m^{(1)}.$$

Since the fundamental equation (2.8) has no roots of the form $\pm$ pi, where p is an integer or zero, the equations for $x_s^{(1)}$ have a unique periodic solution: $x_1^{(1)} = \ldots = x_m^{(1)} = 0$. For $x^{(1)}$ and $y^{(1)}$ on the basis of (3.4) we find:

$$x^{(1)} = \cos\tau, \qquad y^{(1)} = \sin\tau.$$

Further we have

$$\left.\begin{aligned}\frac{dx^{(2)}}{d\tau} &= -y^{(2)} - h_1 \sin\tau + X^{(2)}(\tau),\\ \frac{dy^{(2)}}{d\tau} &= x^{(2)} + h_1 \cos\tau + Y^{(2)}(\tau),\\ \frac{dx_s^{(2)}}{d\tau} &= c_{s1}x_1^{(2)} + \ldots + c_{sm}x_m^{(2)} + X_s^{(2)},\end{aligned}\right\} \tag{3.5}$$

where $X^{(2)}$, $Y^{(2)}$ and $X_s^{(2)}$ are certain known periodic functions of τ of period 2π. In virtue of the already noted property of the roots of the fundamental equation (2.8) the equations for $x_s^{(2)}$ admit for these magnitudes an entirely definite periodic solution. As to the equations for $x^{(2)}$ and $y^{(2)}$, in order that they admit a periodic solution it is necessary and sufficient that the conditions be satisfied

$$\left.\begin{aligned}2h_1 &= \frac{1}{\pi}\int_0^{2\pi} (X^{(2)}(\tau)\sin\tau - Y^{(2)}(\tau)\cos\tau)\, d\tau,\\ 0 &= \int_0^{2\pi} (X^{(2)}(\tau)\cos\tau + Y^{(2)}(\tau)\sin\tau)\, d\tau.\end{aligned}\right\} \tag{3.6}$$

The first of these conditions uniquely determines the constant h_1. As to the second condition, we do not as yet have one undetermined constant of which we can dispose in order to satisfy this condition. Hence if this condition is satisfied by itself equations (3.5) have a periodic solution for $x^{(1)}$ and $y^{(1)}$, otherwise there will be no solution. Since however by what has been proven there exists at least one system of series (3.3) formally satisfying equations (3.2) the condition (3.6) is necessarily satisfied. But in this case equations (3.5) admit for $x^{(2)}$ and $y^{(2)}$ a periodic solution of the form

$$x^{(2)} = A\cos\tau + B\sin\tau + \varphi_2(\tau),$$
$$y^{(2)} = A\sin\tau - B\cos\tau + \psi_2(\tau),$$

where φ_2 and ψ_2 are certain periodic functions of τ of period 2π while A and B are arbitrary constants. These constants are completely determined by the initial conditions (3.4).

We have thus obtained entirely definite functions for $x^{(2)}$ and $y^{(2)}$ and an entirely definite value for the constant h_1. We may remark that the constant h_1 is always found equal to zero, because the functions $X^{(2)}$ and $Y^{(2)}$ represent, as can be easily seen, quadratic forms of cos τ and sin τ. Below it will be shown that the first magnitude h_j not equal to zero always has an even index.

Let us now assume that all the functions $x^{(j)}, y^{(j)}, x_s^{(j)}$ for which $j < k$ and all the magnitudes h_j for which $j < k - 1$ have already been computed and entirely definite values obtained. Then, for finding $x^{(k)}$, $y^{(k)}$ and $x_s^{(k)}$ we obtain equations of the form

$$\frac{dx^{(k)}}{d\tau} = -y^{(k)} - h_{k-1}\sin\tau + X^{(k)}(\tau),$$

$$\frac{dy^{(k)}}{d\tau} = x^{(k)} + h_{k-1}\cos\tau + Y^{(k)}(\tau),$$

$$\frac{dx_s^{(k)}}{d\tau} = c_{s1}x^{(k)} + \ldots + c_{sm}x_m^{(k)} + X_s^{(k)}(\tau),$$

where $X^{(k)}$, $Y^{(k)}$, $X_s^{(k)}$ are certain known periodic functions of τ of period 2π.

The equations for $x_s^{(k)}$ have a unique solution in which these magnitudes are periodic functions of τ. As to the equations for $x^{(k)}$ and $y^{(k)}$, in order that they admit a periodic solution it is necessary and sufficient that the conditions be satisfied

$$2h_{k-1} = \frac{1}{\pi}\int_0^{2\pi}\left(X^{(k)}(\tau)\sin\tau - Y^{(k)}(\tau)\cos\tau\right)d\tau,$$

$$0 = \int_0^{2\pi}\left(X^{(k)}(\tau)\cos\tau + Y^{(k)}(\tau)\sin\tau\right)d\tau.$$

The first of these conditions uniquely determines the magnitude h_{k-1} and the second condition, in virtue of the fact that the series (3.3) necessarily exist, is satisfied by itself. As a result, the equations for $x^{(k)}$ and $y^{(k)}$ admit a periodic solution of the form

$$x^{(k)} = A_k \cos\tau + B_k \sin\tau + \varphi_k(\tau),$$
$$y^{(k)} = A_k \sin\tau - B_k \cos\tau + \psi_k(\tau),$$

where φ_k and ψ_k are periodic functions of τ of period 2π and A_k and B_k are arbitrary constants. For these constants we obtain from the initial conditions (3.4) the entirely definite values:

$$A_k = -\varphi_k(0), \quad B_k = \psi_k(0).$$

From what was said it follows that there exists one and only one system of series of the form (3.3) formally satisfying equations (3.2). These series will consequently converge and actually represent the required periodic solution. At the same time we have obtained a simple and practically convenient method of computing both the solution itself and its period. Moreover, the solution is obtained in a form very convenient for practical purposes.

EXAMPLE 1. Let us consider as an example the differential equation of the second order

$$\frac{d^2x}{dt^2} + k^2x - \gamma x^3 = 0, \tag{3.7}$$

describing many important cases of the free oscillations of conservative mechanical systems with one degree of freedom. Equation (3.7) is evidently a very special case of Lyapunov systems and therefore the preceding theory can be applied to it.

Setting

$$t = \frac{\tau}{k}(1 + h_2c^2 + h_3c^3 + \dots), \tag{3.8}$$

We obtain the equation

$$\frac{d^2x}{d\tau^2}+\left(x-\frac{\gamma}{k^2}x^3\right)(1+h_2c^2+\dots)^2.$$

We shall try to satisfy this equation by the series

$$x=c\cos\tau+c^3x_3(\tau)+\dots, \tag{3.9}$$

where $x_j(\tau)$ are periodic functions of τ of period 2π which satisfy the initial conditions

$$x_3(0)=x_5(0)=\dots=\dot{x}_3(0)=\dot{x}_5(0)=\dots=0. \tag{3.10}$$

The series (3.9) can contain only odd powers of c and the series (3.8) only even powers of c since equation (3.7) does not change on substituting -x for x. We have:

$$\frac{d^2x_3}{d\tau^2}+x_3=-2h_2\cos\tau+\frac{\gamma}{k^2}\cos^3\tau=$$
$$=\left(\frac{3}{4}\frac{\gamma}{k^2}-2h_2\right)\cos\tau+\frac{1}{4}\frac{\gamma}{k^2}\cos 3\tau. \tag{3.11}$$

The condition of periodicity gives:

$$h_2=\frac{3}{8}\frac{\gamma}{k^2},$$

after which from equation (3.11), taking the initial conditions (3.10) into account, we obtain:

$$x_3=-\frac{\gamma}{32k^2}\cos 3\tau+\frac{\gamma}{32k^2}\cos\tau.$$

For x_5 we have the equation

$$\frac{d^2x_5}{d\tau^2}+x_5=\left(-2h_4+\frac{57}{128}\frac{\gamma^2}{k^4}\right)\cos\tau+\frac{3}{16}\frac{\gamma^2}{k^4}\cos 3\tau-\frac{3}{128}\frac{\gamma^2}{k^4}\cos 5\tau,$$

from which we find:

$$h_4=\frac{57}{256}\frac{\gamma^2}{k^4},$$
$$x_5=\frac{23}{1024}\frac{\gamma^2}{k^4}\cos\tau-\frac{3}{128}\frac{\gamma^2}{k^4}\cos 3\tau+\frac{1}{1024}\frac{\gamma^2}{k^4}\cos 5\tau.$$

We restrict ourselves to this approximation. Thus, the periodic solution of equation (3.7) and its period are expressed approximately by the following formulas:

$$
\left.\begin{aligned}
t &= \frac{\tau}{k}(1 + h_2 c^2 + h_4 c^4),\\
x &= c\cos\tau + c^3 x_3(\tau) + c^5 x_5(\tau),\\
x_3(\tau) &= \frac{\gamma}{32k^2}\cos\tau - \frac{\gamma}{32k^2}\cos 3\tau,\\
x_5(\tau) &= \frac{23}{1024}\frac{\gamma^2}{k^4}\cos\tau - \frac{3}{128}\frac{\gamma^2}{k^4}\cos 3\tau + \frac{1}{1024}\frac{\gamma^2}{k^4}\cos 5\tau,\\
h_2 &= \frac{3}{8}\frac{\gamma}{k^2}, \qquad h_4 = \frac{57}{256}\frac{\gamma^2}{k^4},\\
T &= \frac{2\pi}{k}(1 + h_2 c^2 + h_4 c^4).
\end{aligned}\right\} \tag{3.12}
$$

Replacing t by $t + h$, where h is an arbitrary constant, we obtain a periodic solution of equation (3.7) containing two arbitrary constants, which will therefore be its general solution.

EXAMPLE 2. Let us also find the periodic solution of the equation

$$\frac{d^2x}{dt^2} + k^2 x - \beta x^2 = 0,$$

which we shall use in what follows. Making the substitution (3.8) we obtain the equation

$$\frac{d^2x}{d\tau^2} + \left(x - \frac{\beta}{k^2}x^2\right)(1 + h_2c^2 + h_3c^3 + \ldots)^2.$$

We shall try to satisfy it by the series

$$x = c\cos\tau + c^2 x_2(\tau) + c^3 x_3(\tau) + \ldots$$

with the initial conditions

$$x_2(0) = x_3(0) = \ldots = \dot{x}_2(0) = \dot{x}_3(0) = \ldots = 0.$$

$$\frac{d^2x_2}{d\tau^2}+x_2=\frac{\beta}{k^2}\cos^2\tau,$$

whence we find:

$$x_2=\frac{\beta}{2k^2}-\frac{\beta}{3k^2}\cos\tau-\frac{\beta}{6k^2}\cos 2\tau.$$

From the equation for x_3

$$\frac{d^2x_3}{d\tau^2}+x_3=-2h_2\cos\tau+\frac{2\beta}{k^2}x_2\cos\tau;$$

we further obtain:

$$h_2=\frac{5}{12}\frac{\beta^2}{k^4},$$
$$x_3=-\frac{\beta^2}{3k^4}+\frac{29}{144}\frac{\beta^2}{k^4}\cos\tau+\frac{\beta^2}{9k^4}\cos 2\tau+\frac{\beta^2}{48k^4}\cos 3\tau,$$

after which the condition of periodicity of the function x_4, determined by the equation

$$\frac{d^2x_4}{d\tau^2}+x_4=-2h_3\cos\tau-2h_2x_2+\frac{\beta}{k^2}x_2^2+\frac{2\beta}{k^2}x_3\cos\tau+\frac{2\beta}{k^2}h_2\cos^2\tau,$$

gives:

$$h_3=-\frac{5}{18}\frac{\beta^3}{k^6}.$$

The required periodic solution and its period T are thus determined by the formulas

$$\left.\begin{aligned}
\tau&=k\left(1+\frac{5}{12}\frac{\beta^2}{k^4}c^2-\frac{5}{18}\frac{\beta^3}{k^6}c^3+\dots\right)^{-1}t,\\
x&=c\cos\tau+\left(\frac{\beta}{2k^2}-\frac{\beta}{3k^2}\cos\tau-\frac{\beta}{6k^2}\cos 2\tau\right)c^2+\\
&\quad+\left(-\frac{\beta^2}{3k^4}+\frac{29}{144}\frac{\beta^2}{k^4}\cos\tau+\frac{\beta^2}{9k^4}\cos 2\tau+\frac{\beta^2}{48k^4}\cos 3\tau\right)c^3+\dots,\\
T&=\frac{2\pi}{k}\left(1+\frac{5}{12}\frac{\beta^2}{k^4}c^2-\frac{5}{18}\frac{\beta^3}{k^6}c^3+\dots\right).
\end{aligned}\right\}\tag{3.13}$$

4. Some Properties of the Periodic Solutions of Lyapunov Systems

We shall here note several properties of the periodic solutions of Lyapunov systems which we shall require in what follows. Let us first consider a system of the second order

$$\frac{dx}{dt} = -\lambda y + X(x, y),$$
$$\frac{dy}{dt} = \lambda x + Y(x, y), \tag{4.1}$$

and let

$$H = x^2 + y^2 + S(x, y) = \text{const} \tag{4.2}$$

be its first integral. Here S is an analytic function of x and y, the expansion of which starts with terms of not lower than the third order. The system (4.1) admits, by what has been proven, a periodic solution depending on two arbitrary constants of which one is the initial value of the magnitude x and the other the initial value of the time. Since in the case considered the differential equations are of the second order this periodic solution is the general solution of these equations. Hence, whatever the initial values a and b of the magnitudes x and y, the solution of the equations (4.1) with these initial values will be periodic. It is required only that the numerical values of the magnitudes a and b be sufficiently small since the periodic solutions of the Lyapunov systems that we are considering exist in a sufficiently small neighborhood of the origin.

Let us consider the phase plane xy for the equations (4.1). All the phase trajectories situated in a sufficiently small neighborhood of the origin will be closed since the corresponding solutions of equations (4.1) are periodic. Hence, as for the corresponding linear equations

$$\frac{dx}{dt} = -\lambda y, \quad \frac{dy}{dt} = \lambda x, \tag{4.3}$$

the origin is a singular point of the center type. The equation of the phase trajectories is determined by the integral (4.2) from which it follows that these phase trajectories in a sufficiently small neighborhood of the origin differ little from circles, which are the phase trajectories of the linear equations (4.3). Thus, in a sufficiently small neighborhood of the origin the phase planes of equations (4.1) and of the linear equations (4.3) differ little from each other. There is however here one principal difference: the period of a circuit for all phase trajectories of equations (4.3) is equal to $2\pi/\lambda$, whereas for equations (4.1) each phase trajectory has its own circuit period. This difference, as we shall see below, has considerable significance.

Let us now consider a Lyapunov system of arbitrary order. We shall show that this system, like the system of second order, has a periodic solution in which the initial values of x and y are arbitrary sufficiently small magnitudes a and b while the initial values of x_s are found to be entirely definite analytic functions of a and b. For this purpose let us seek a particular solution of equations (1.6) in which x_s are analytic functions of x and y. These functions, if they exist, must evidently satisfy the system of partial differential equations

$$\frac{\partial x_s}{\partial x}(-\lambda y+X)+\frac{\partial x_s}{\partial y}(\lambda x+Y)=b_{s1}x_1+\ldots+b_{sm}x_m+X_s \qquad (4.4)$$

$$(s=1,\ldots,m).$$

Let us try to satisfy these equations by formal series of the form

$$x_s(x,y)=A_sx+B_sy+C_sx^2+D_sxy+E_sy^2+\ldots \qquad (4.5)$$

with undetermined coefficients. Substituting these series in equations (4.4) and equating coefficients of the first degrees of x and y we obtain, for finding A_s and B_s, a system of linear homogeneous equations

$$\lambda B_s=b_{s1}A_1+\ldots+b_{sm}A_m,$$
$$-\lambda A_s=b_{s1}B_1+\ldots+b_{sm}B_m.$$

The determinant of this system, as we already know, is different from zero and therefore all the magnitudes A_s and B_s are equal to zero. Thus, the functions $x_s(x,y)$ cannot contain linear terms.

Equating, in equations (4.4), the coefficients of the terms of second order we similarly obtain a system of linear equations for the determination of the magnitudes C_s, D_s and E_s. However these equations will be nonhomogeneous and therefore, if their determinant is different from zero, we obtain entirely definite values for the required coefficients, at least one of them being different from zero. In general, if in series (4.5) the coefficients of all the terms of an order lower than k are already computed then for the coefficients of the terms of the k-th order a system of linear nonhomogeneous equations is obtained the number of which is equal to the number of these coefficients. It was shown by A. M. Lyapunov that the determinants of all these systems of equations, for the assumptions made with regard to equations (1.6), are different from zero and consequently there exists one and only one system of series (4.5) formally satisfying equations (4.4). Moreover, it was shown by Lyapunov that the series thus obtained converge for sufficiently small values of x and y and therefore actually represent a solution of the system (4.4). We shall not here present this proof.

Substituting the functions $x_s(x,y)$ thus obtained in the first two of equations (1.6) we obtain for determining x and y the equations

$$\left.\begin{aligned} \frac{dx}{dt} &= -\lambda y + X(x, y, x_1(x, y), \ldots, x_m(x, y)), \\ \frac{dy}{dt} &= \lambda x + Y(x, y, x_1(x, y), \ldots, x_m(x, y)). \end{aligned}\right\} \tag{4.6}$$

Let $x(t, a, b)$, $y(t, a,b)$ be the general solution of these equations, where a and b are the initial values of x and y, which we consider sufficiently small. Then the functions

$$x(t, a, b),\ y(t, a, b),\ x_s(x(t, a, b),\ y(t, a, b)) \tag{4.7}$$

determine a particular solution of equations (1.6) containing two arbitrary constants. This solution will be periodic. In fact, since the system (1.6) admits the first integral (1.9) the system (4.6) likewise admits the first integral

$$x^2 + y^2 + W(x_1(x, y), \ldots, x_m(x, y)) + \ldots = \text{const.} \qquad (4.8)$$

This integral will be of the form (4.2) since the expansions of the functions $x_s(x,y)$ start with terms of not lower than the second order. Hence the system (4.6) is a Lyapunov system and therefore its general solution for sufficiently small a and b will be periodic. But then the solution (4.7) of equations (1.6) will likewise be periodic, the initial values of the magnitudes x_s, equal evidently to $x_s(a, b)$, being analytic functions of a and b.

Thus, the Lyapunov system (1.6) admits a periodic solution depending on the two parameters a and b, which are the initial values of the magnitudes x and y. These parameters can be chosen entirely arbitrarily provided they are sufficiently small.

If in the solution (4.7) we put $a = c$ and $b = 0$ the periodic solution is obtained which we considered in the two preceding sections. In fact, for this substitution in solution (4.7) x at the initial instant becomes c, y becomes zero, and x_s certain analytic functions of c. But these conditions, as we know, completely determine the periodic solution considered in the preceding sections. Now replacing t by $t + h$, where h is an arbitrary constant, we obtain a periodic solution depending again on two arbitrary constants. This solution differs evidently from solution (4.7) only in the choice of parameters.

We shall now investigate what form the period of the solution will take if as parameters we take the magnitudes a and b. Let us first consider the system of the second order (4.1). Setting in it

$$x = \rho\cos\vartheta, \qquad y = \rho\sin\vartheta,$$

we obtain:

$$\left.\begin{aligned} \frac{d\rho}{dt} &= \cos\vartheta\, X(\rho\cos\vartheta, \rho\sin\vartheta) + \sin\vartheta\, Y(\rho\cos\vartheta, \rho\sin\vartheta), \\ \frac{d\vartheta}{dt} &= \lambda + \frac{1}{\rho}[\cos\vartheta\, Y(\rho\cos\vartheta, \rho\sin\vartheta) - \sin\vartheta\, X(\rho\cos\vartheta, \rho\sin\vartheta)]. \end{aligned}\right\} \qquad (4.9)$$

As we saw in sec. 2, the period of the solution is determined by the formula

$$T=\int_0^{2\pi} \frac{\rho d\vartheta}{\lambda\rho+\cos\vartheta Y(\rho\cos\vartheta, \rho\sin\vartheta)-\sin\vartheta X(\rho\cos\vartheta, \rho\sin\vartheta)}. \qquad (4.10)$$

Here ρ must be expressed as a function of ϑ. For this we can use either the first of equations (4.9) or, as we did in sec. 2, the integral (4.2), which in the variables ρ and ϑ has the form

$$\rho^2\left[1+\frac{1}{\rho^2} S(\rho\cos\vartheta, \rho\sin\vartheta)\right]=\mu^2, \qquad (4.11)$$

where μ is an arbitrary constnat. From equation (4.11) we obtain for ρ the two solutions:

$$\rho=\pm\mu+u^{(2)}\mu^2\pm u^{(3)}\mu^3+\ldots, \qquad (4.12)$$

where $u^{(2)}$, $u^{(3)}$,... are periodic functions of ϑ of period 2π. Substituting some one of these solutions in (4.10) we obtain an expression of the period in the form of an analytic function of μ. We shall show that this expression contains only even powers of μ. For this purpose we note that equation (4.11) does not change on replacing ρ by $-\rho$ and ϑ by $\vartheta+\pi$. Hence if in one of the solutions (4.12) ϑ is replaced by $\vartheta+\pi$ all the functions $u^{(k)}$ with odd k do not change while those with even k assume their original value with opposite sign. Hence each of the functions ρ with change of μ to $-\mu$ and ϑ to $\vartheta+\pi$ assumes its original value with opposite sign. But then the period T, as is seen from the structure of the expression under the integral sign in (4.10), for this substitution does not change at all. On the other hand, the substitution in the expression for T of $\vartheta+\pi$ for ϑ, evidently equivalent to the displacement of both limits of integration by π, does not change the magnitude of T since the function to be integrated is periodic and the integration extends over a complete period. Hence the period T does not change on replacing μ by $-\mu$ and therefore its expansion contains only even powers of μ.

In this way we have:

$$T=\frac{2\pi}{\lambda}(1+H_2\mu^2+H_4\mu^4+\ldots). \qquad (4.13)$$

The magnitude μ^2, expressed in terms of a and b, has evidently the form

$$\mu^2 = a^2 + b^2 + S(a, b). \tag{4.14}$$

Let us now consider the case of a system of any order. As was shown above, the finding of a periodic solution of this system reduces to finding the periodic solution of a system of the second order. From this it follows that the period T will as before be expressed by formula (4.13), but μ^2, on the basis of (4.8), will have the form

$$\begin{aligned}\mu^2 &= a^2 + b^2 + W(x_1(a, b), \ldots, x_m(a, b)) + \ldots = \\ &= a^2 + b^2 + S^*(a, b),\end{aligned} \tag{4.15}$$

where the expansion of S^* starts with terms of not lower than the third order.

If in (4.13) we set $a = c$ and $b = 0$ the obtained expression of the period must agree with (3.1). From this it follows that the smallest power of c in expression (3.1) for the period will necessarily be even.

5. Principal Oscillations of Conservative Systems

The theory of A. M. Lyapunov finds application in many problems of mechanics. Let us consider, in particular, equations (1.10) describing the oscillations of a conservative system about the position of stable equilibrium or steady state of motion. If all numbers λ_s are such that the ratio of any pair of them is not an integer each pair of variables x_s, y_s can play the part of x and y in equations (1.6). Hence equations (1.10) admit k periodic solutions containing each two arbitrary constants. These constants are the initial values a_s and b_s of the corresponding pair of variables x_s and y_s. The periods of these oscillations are magnitudes reducing to $2\pi/\lambda_s$ for $a_s = b_s = 0$.

If the function F is equal to zero, and consequently equations (1.10) reduce to linear, the periodic solutions under consideration assume the form

$$x_i = y_i = 0 \qquad (i \neq s),$$
$$x_s = a_s \cos \lambda_s t - b_s \sin \lambda_s t,$$
$$y_s = a_s \sin \lambda_s t + b_s \cos \lambda_s t$$
$$(s = 1, \ldots, k).$$

These solutions are the principal oscillations of the obtained linear system.

When the function F is not identically equal to zero we can also speak of principal oscillations of the system, thus calling the above mentioned k periodic solutions. Thus, principal oscillations occur also when the higher order terms are taken into account in equations of oscillations of the form (1.10). For nonlinear systems however the general solution of the equations of the oscillations is not a combination of the solutions representing the principal oscillations. Moreover, the period of the principal oscillations of nonlinear systems, in contrast to linear, depend on the initial conditions.

In the following chapter we shall concern ourselves with the investigation of approximations to Lyapunov systems. In other words, we shall take the Lyapunov systems as generating systems. The class of the systems thus obtained is considerably wider than the class of quasilinear systems. This generality is attained principally through the fact that the periods of the periodic solutions of Lyapunov systems depend on the initial conditions. It is physically evident that this property of Lyapunov systems is particularly brought out in the action of external periodic forces on them. Because of this we shall in the following chapter consider only nonautonomous systems.

CHAPTER VIII

APPROXIMATIONS TO LYAPUNOV SYSTEMS

1. Generating Solutions

In this chapter we shall consider periodic oscillations of systems described by the equations

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n + X_s^*(x_1, \ldots, x_n) + \\ + \mu f_s(t, x_1, \ldots, x_n, \mu) \quad (1.1)$$
$$(s = 1, \ldots, n),$$

where a_{sj} are constants, X_s^* are analytic functions of $x_1,\ldots,x_n$, the expansions of which start with terms of not lower than the second order, f_s are analytic functions of the variables $x_1,\ldots,x_n$ and of the small parameter μ, the expansions of which in powers of $x_1,\ldots,x_n$ can contain both linear and free terms. The functions f_s depend also on t with respect to which they are continuous, periodic and developable in Fourier series. The period of these functions we shall for convenience take to be equal to 2π. The region of analyticity of the functions X_s and f_s with respect to the variables $x_1,\ldots,x_n$ is a certain neighborhood of the origin.

We assume, moreover, that the generating system

$$\frac{dx_s^0}{dt} = a_{s1}x_1^0 + \ldots + a_{sn}x_n^0 + X_s^*(x_1^0, \ldots, x_n^0) \quad (1.2)$$

is a Lyapunov system.

Let $\pm\lambda i$ be a pair of purely imaginary roots of the fundamnetal equation

$$\begin{vmatrix} a_{11}-\rho & a_{12} \dots a_{1n} \\ a_{21} & a_{22}-\rho \dots a_{2n} \\ \dots & \dots \\ a_{n1} & a_{n2} \dots a_{nn}-\rho \end{vmatrix} = 0, \tag{1.3}$$

satisfying the conditions of sec. 1 of the preceding chapter, so that this pair of roots can be taken as the leading pair and equations (1.2) can be represented in the form

$$\left.\begin{aligned} &\frac{d\xi}{dt} = -\lambda\eta + X(\xi, \eta, \xi_j), \\ &\frac{d\eta}{dt} = \lambda\xi + Y(\xi, \eta, \xi_j), \\ &\frac{d\xi_s}{dt} = \sum_{\alpha=1}^{m} b_{s\alpha}\xi_\alpha + X_s(\xi, \eta, \xi_j) \\ &\qquad (s = 1, \dots, m), \end{aligned}\right\} \tag{1.4}$$

where m = n - 2. We recall that the fundamental equation

$$\begin{vmatrix} b_{11}-\rho & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22}-\rho & \dots & b_{2m} \\ \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn}-\rho \end{vmatrix} = 0 \tag{1.5}$$

here has neither a zero root nor roots of the form $\pm p\lambda i$, where p is an integer. Moreover, the system (1.4) admits a first integral of the form

$$H = \xi^2 + \eta^2 + W(\xi_1, \dots, \xi_m) + S(\xi, \eta, \xi_1, \dots, \xi_m) = \text{const}, \tag{1.6}$$

where W is a quadratic form of the variables $\xi_1, \dots, \xi_m$ while S is an analytic function of the variables $\xi, \eta, \xi_1, \dots, \xi_m$, the expansions of which start with terms of not lower than the third order.

Having represented the generating system in the form (1.4) and thereby separated out the leading variables we can write equations (1.1) in the form

$$
\left.\begin{aligned}
\frac{dx}{dt} &= -\lambda y + X(x, y, x_j) + \mu f(t, x, y, x_j, \mu), \\
\frac{dy}{dt} &= \lambda x + Y(x, y, x_j) + \mu F(t, x, y, x_j, \mu), \\
\frac{dx_s}{dt} &= \sum_{\alpha=1}^{m} b_{s\alpha} x_\alpha + X_s(x, y, x_j) + \mu F_s(t, x, y, x_j, \mu) \\
&\qquad (s = 1, \ldots, m),
\end{aligned}\right\} \tag{1.7}
$$

where the functions X,Y, and X_s have the same structure as the functions X_s^* and the functions f,F and F_s the same structure as the functions f_s.

We usually denote the variables in the generating system of equations by the same letters as in the complete system, providing them only with the index zero. In the investigation however of equations of oscillations in the form (1.7) this manner of writing is found to be too cumbersome and we shall therefore, for the generating system (1.4), use a different notation. In what follows we shall use the equations of the oscillations both in the form (1.1) and in the form (1.7).

As was shown in the preceding chapter, the system (1.4) admits a periodic solution depending on two arbitrary constants c, h and having the form

$$
\left.\begin{aligned}
\xi &= c\cos\tau + c^2\xi_2(\tau) + \ldots, \\
\eta &= c\sin\tau + c^2\eta_2(\tau) + \ldots, \\
\xi_s &= c\xi_{s1}(\tau) + c^2\xi_{s2}(\tau) + \ldots, \\
\tau &= \lambda(t + h)(1 + h_2c^2 + \ldots)^{-1}.
\end{aligned}\right\} \tag{1.8}
$$

where ξ_k, η_k, ξ_{sk} are periodic functions of τ of period 2π satisfying the conditions

$$
\xi_k(0) = \eta_k(0) = 0 \qquad (k = 2, 3, \ldots),
$$

and $h_2, h_3, \ldots$ are certain constants of which the first not equal to zero has an even index. The series (1.8) converge for all values of h and for sufficiently small values of c. The period of solution (1.8) is determined by the formula

$$
T = \frac{2\pi}{\lambda}(1 + h_2c^2 + \ldots). \tag{1.9}
$$

According to the general method for finding periodic solutions of the complete system of equations we must first of all find all solutions of equations of the generating system that have a period agreeing with the period of the right hand sides of the complete system, i.e. 2π. From the expression for T it follows that to find these solutions we must, while leaving the magnitude h arbitrary, determine the magnitude c from the relation

$$T = \frac{2\pi}{p}, \tag{1.10}$$

where p is an arbitrary integer. The number c thus obtained we shall denote by c_p and the corresponding generating solution by $\{\xi^{(p)}(t+h),\ \eta^{(p)}(t+h),\ \xi_s^{(p)}(t+h)\}$.
This solution on the basis of (1.8) is determined by the formulas

$$\left.\begin{aligned} \xi^{(p)}(t+h) &= c_p \cos\tau + c_p^2\, \xi_2(\tau) + \ldots, \\ \eta^{(p)}(t+h) &= c_p \sin\tau + c_p^2\, \eta_2(\tau) + \ldots, \\ \xi_s^{(p)}(t+h) &= c_p \xi_{s1}(\tau) + c_p^2\, \xi_{s2}(\tau) + \ldots, \\ \tau &= p(t+h). \end{aligned}\right\} \tag{1.11}$$

The equation determining c_p has the form

$$h_{2r}c^{2r} + h_{2r+1}c^{2r+1} + \ldots = \frac{\lambda - p}{p}; \tag{1.12}$$

where h_{2r} is the first of the constants $h_2, h_3, \ldots,$ not reducing to zero. Taking the 2r-th root we obtain:

$$c + \frac{h_{2r+1}}{2rh_{2r}}c^2 + \ldots = a, \tag{1.13}$$

where

$$a = \sqrt[2r]{\frac{\lambda - p}{ph_{2r}}}. \tag{1.14}$$

Equation (1.13) has one and only one solution for c reducing to zero for $a = 0$ and this solution will be analytic with respect to a. But since the magnitude a has $2r$ values we obtain $2r$ solutions of equation (1.12) reducing to zero for $a = 0$, i.e. for $\lambda = p$. If $h_{2r}(\lambda - p) < 0$, all these solutions will be complex. If $h_{2r}(\lambda - p) > 0$ two and only two solutions will be real, one of them being positive the other negative. Hence, for $h_{2r} > 0$ we can give the number p any integral values less than λ while for $h_{2r} < 0$ any integral values greater than λ. Choosing the number p in this manner we obtain for each p two generating solutions

$$\{\xi^{(p)}(t+h), \quad \eta^{(p)}(t+h), \quad \xi_s^{(p)}(t+h)\}.$$

Besides $\{\xi^{(p)}(t + h), \eta^{(p)}(t + h), \xi_s^{(p)}(t + h)\}$ a generating solution will also be the trivial solution $\xi = \eta = \xi_1 = \dots = \xi_m = 0$ of the system (1.4), which can be considered as a periodic solution of period 2π.

2. The Periodic Solution $\{x_s^{(o)}\}$

We proceed to the determination of the periodic solutions of system (1.1). We shall first find a periodic solution reducing for $\mu = 0$ to the trivial solution $x_1^o = \dots = x_n^o = 0$ of the generating system. This solution we shall denote be the symbol $\{x_s^o\}$. If we shall make use of the equations of oscillations in the form (1.7) we shall denote the same solution by $\{x^{(o)}, y^{(o)}, x_s^{(o)}\}$.

The equations in variations of the generating system (1.2) for the trivial solution $x_1^o = \dots = x_n^o = 0$ have evidently the form

$$\frac{du_s}{dt} = a_{s1}u_1 + \dots + a_{sn}u_n. \tag{2.1}$$

Let us assume first that the fundamental equation (1.3) of this system does not have roots of the form $\pm pi$, where p is an integer. In this case the system (2.1) will not have periodic solutions of period 2π and therefore the generating solution considered will be an isolated one. On the basis of the general Poincaré's theorem established in sec. 1 of chapter VI the system (1.1) will then have one and only one periodic solution reducing for $\mu = 0$ to the generating solution and this solution will be analytic with respect to μ. For the practical computation of this solution we shall seek it in the form of the series

$$x_s^{(0)}(t) = \mu x_{s1}(t) + \mu^2 x_{s2}(t) + \dots \tag{2.2}$$

with periodic coefficients, formally satisfying equations (1.1). For the coefficients of these series we obtain equations of the form

$$\frac{dx_{sk}}{dt} = a_{s1}x_{1k} + \dots + a_{sn}x_{nk} + f_{sk}(t), \tag{2.3}$$

wher f_{sk} are integral rational functions with periodic coefficients for all x_{sj} for which $j < k$. These equations make it possible to determine successively the coefficients x_{sk} in the form of periodic functions of time, entirely definite coefficients being obtained.

If the fundamental equation (1.3) has purely imaginary roots of the form $\pm$ pi, where p is an integer, resonance will take place. We must also consider as resonance cases those for which the fundamental equation has roots differing from $\pm$ pi by a magnitude of the order of smallness of μ. The theory of resonance we shall take up in the following section.

3. Periodic Solution for Resonance

We proceed with the question of the existence of a periodic solution of system (1.1) that reduces for $\mu = 0$ to the trivial solution $x_1^o = \dots = x_n^o = 0$ of the generating system in the case of resonance. Let us therefore assume that the fundamental equation (1.3) has a pair of roots differing from $\pm$ pi, where p is an integer, by a magnitude of the order of smallness of μ. We shall assume that equation (1.3) has no other roots of this type. In regard to the correction terms to the functions f_s we can assume, as in the earlier investigated cases of resonance, that the above mentioned critical roots are equal to ± pi. We shall assume that this pair of purely imaginary roots satisfies all conditions stated in sec. 1 of the preceding chapter so that these roots can be taken for the fundamental pair of purely imaginary roots which figures in the form (1.7) of the equations of the oscillatiions

The equations of the oscillations can thus be represented in the form

$$\left.\begin{aligned}\frac{dx}{dt} &= -py + X + \mu f(t, x, y, x_j, \mu),\\ \frac{dy}{dt} &= px + Y + \mu F(t, x, y, x_j, \mu),\\ \frac{dx_s}{dt} &= \sum_{\alpha=1}^{m} b_{s\alpha}x_\alpha + X_s + \mu F_s(t, x, y, x_j, \mu),\\ &\qquad (s = 1, \ldots, m).\end{aligned}\right\} \tag{3.1}$$

If we try to satisfy these equations by formal series of the form

$$\begin{aligned}x &= \mu x^{(1)}(t) + \mu^2 x^{(2)}(t) + \ldots,\\ y &= \mu y^{(1)}(t) + \mu^2 y^{(2)}(t) + \ldots,\\ x_s &= \mu x_s^{(1)}(t) + \mu^2 x_s^{(2)}(t) + \ldots\end{aligned}$$

with periodic coefficients, we obtain for the coefficients of the first approximation the equations

$$\left.\begin{aligned}\frac{dx^{(1)}}{dt} &= -py^{(1)} + f(t, 0, \ldots, 0),\\ \frac{dy^{(1)}}{dt} &= px^{(1)} + F(t, 0, \ldots, 0),\\ \frac{dx_s^{(1)}}{dt} &= \sum_{\alpha=1}^{m} b_{s\alpha}x_\alpha + F_s(t, 0, \ldots, 0).\end{aligned}\right\} \tag{3.2}$$

Since by assumption the fundamental equation (1.5) has no roots of the form ± ki, where k is an integer, this system admits a periodic solution for $x_s^{(1)}$. As to the magnitudes $x^{(1)}$ and $y^{(1)}$, in order that they be periodic it is necessary and sufficient that the two magnitudes

$$\delta_1 = B_{1p} - A_{2p}, \qquad \delta_2 = A_{1p} + B_{2p} \tag{3.3}$$

reduce to zero. There has here been put

$$\left.\begin{aligned} f(t, 0, \ldots, 0) &= A_{10} + \sum_{n=1}^{\infty} (A_{1n} \cos nt + B_{1n} \sin nt), \\ F(t, 0, \ldots, 0) &= B_{10} + \sum_{n=1}^{\infty} (A_{2n} \cos nt + B_{2n} \sin nt). \end{aligned}\right\} \tag{3.4}$$

We shall assume that at least one of the magnitudes (3.3) is different from zero. In this case we shall say that PRINCIPAL RESONANCE takes place.

For principal resonance the following theorem is true:

THEOREM. LET $2l$ BE THE SMALLEST DEGREE OF THE MAGNITUDE c IN THE EXPANSION OF THE PERIOD

$$T = \frac{2\pi}{p}(1 + h_{2l}c^{2l} + \ldots)$$

OF THE PERIODIC SOLUTION OF THE GENERATING SYSTEM

$$\left.\begin{aligned} \frac{d\xi}{dt} &= -p\eta + X(\xi, \eta, \xi_j), \\ \frac{d\eta}{dt} &= p\xi + Y(\xi, \eta, \xi_j), \\ \frac{d\xi_s}{dt} &= \sum_{\alpha=1}^{m} b_{s\alpha}\xi_\alpha + X_s(\xi, \eta, \xi_j) \end{aligned}\right\} \tag{3.5}$$

WITH THE INITIAL CONDITIONS

$$\xi(0) = c, \qquad \eta(0) = 0.$$

THEN, FOR PRINCIPAL RESONANCE, THERE EXISTS ONE AND ONLY ONE PERIODIC SOLUTION $\{x^{(res)}, y^{(res)}, x_s^{(res)}\}$ OF EQUATIONS (3.1) FOR WHICH THE FUNCTIONS $x^{(res)}, y^{(res)}, x_s^{(res)}$ REDUCE TO ZERO FOR $\mu = 0$ AND THESE FUNCTIONS CAN BE EXPANDED IN SERIES OF POSITIVE INTEGRAL POWERS OF THE MAGNITUDE $\nu = \mu^{\frac{1}{2l+1}}$, CONVERGING FOR SUFFICIENTLY SMALL VALUE OF μ.

PROOF. Let us consider the generating system (3.5). As was shown in sec. 4 of chapter VII, for this system there exists a periodic solution containing two arbitrary constants a and b, which are the initial values of the magnitudes ξ and η. The magnitudes ξ_s in this solution are functions of $w_s(\xi, \eta)$ of ξ and η satisfying the partial differential equations

$$\frac{\partial w_s}{\partial \xi}[-p\eta + X(\xi, \eta, w_j)] + \frac{\partial w_s}{\partial \eta}[p\eta + Y(\xi, \eta, w_j)] = \\ = \sum_{\alpha=1}^{m} b_{s\alpha} w_\alpha + X_s(\xi, \eta, w_j).$$

Let us in equations (3.1) replace the variables x_s by the variables u_s with the aid of the substitution

$$u_s = x_s - w_s(x, y) \quad (s = 1, \ldots, m).$$

As was shown in sec. 4 of chapter VII, the expansions of the functions w_s start with terms of not lower than the second order and therefore the linear terms in equations (3.1) do not change. The functions X, Y, X_s, f, F, F_s however are replaced by other functions of analogous form, which we shall denote respectively by X^*, Y^*, X_s^*, f^*, F^*, F_s^*. Equations (3.1) thus go over into the equations

$$\left.\begin{aligned} \frac{dx}{dt} &= -py + X^*(x, y, u_j) + \mu f^*(t, x, y, u_j, \mu), \\ \frac{dy}{dt} &= px + Y^*(x, y, u_j) + \mu F^*(t, x, y, u_j, \mu), \\ \frac{du_s}{dt} &= \sum_{\alpha=1}^{m} b_{s\alpha} x_\alpha + X_s^*(x, y, u_j) + \mu F_s^*(t, x, y, u_j, \mu), \end{aligned}\right\} \quad (3.6)$$

and equations (3.5) assume the form

$$\left.\begin{aligned} \frac{d\xi}{dt} &= -p\eta + X^*(\xi, \eta, \xi_j), \\ \frac{d\eta}{dt} &= p\xi + Y^*(\xi, \eta, \xi_j), \\ \frac{d\xi_s}{dt} &= \sum_{\alpha=1}^{m} b_{s\alpha}\xi_\alpha + X_s^*(\xi, \eta, \xi_j). \end{aligned}\right\} \quad (3.7)$$

The transformation made is such that the problem of seeking the periodic solution of system (3.1) is equivalent to the same problem for the system (3.6). We here evidently have:

$$f^*(t, 0, \ldots, 0) = f(t, 0, \ldots, 0),$$
$$F^*(t, 0, \ldots, 0) = F(t, 0, \ldots, 0),$$

so that the magnitudes (3.3) do not change and therefore principal resonance also takes place for the system (3.6).

The periodic solution of the generating system (3.5) goes over into the periodic solution of the generating system (3.7). The latter solution possesses the property that for it the magnitudes ξ_s are identically equal to zero. This follows immediately from the transformation formulas. Hence the solution of equations (3.7) with the initial conditions

$$\xi(0) = a, \quad \eta(0) = b, \quad \xi_s(0) = 0, \tag{3.8}$$

where a and b are arbitrary constants, will be periodic. The period of the solution on the basis of formulas (4.13) and (4.15) of chapter VII has the form

$$T = \frac{2\pi}{p}\{1 + H_{2l}(a^2 + b^2)^l + \ldots\}, \tag{3.9}$$

where the magnitude $H_{2l} = h_{2l}$ by the condition of the theorem is different form zero.

From the fact that equations (3.7) have a solution in which all ξ_s are equal to zero it follows also that

$$\{X_s^*(x, y, \xi_j)\}_{\xi_j=0} = 0. \tag{3.10}$$

Having established this, let us denote by

$$x(t, a, b, \beta_j, \mu), \quad y(t, a, b, \beta_j, \mu), \quad u_s(t, a, b, \beta_j, \mu) \tag{3.11}$$

the solution of equations (3.6) with the initial conditions

$$x(0, a, b, \beta_j, \mu) = a, \quad y(0, a, b, \beta_j, \mu) = b, \quad u_s(0, a, b, \beta_j, \mu) = \beta_s.$$

We have identically:

$$\left.\begin{array}{l}\{x(pT, a, b, \beta_j, 0)\}_{\beta_j=0} \equiv a, \\ \{y(pT, a, b, \beta_j, 0)\}_{\beta_j=0} \equiv b.\end{array}\right\} \tag{3.12}$$

In fact, the solution under consideration for $\mu = o$ reduces to the solution of system (3.7) which for the initial conditions (3.8) will be periodic with period T and therefore also with period pT. Further, on the basis of (3.10)

$$\{u_s(t, a, b, \beta_j, 0)\}_{\beta_j=0} \equiv 0. \tag{3.13}$$

In order that the solution in question of the complete system be periodic of period 2π it is necessary and sufficient that the equations be satisfied

$$\left.\begin{array}{c}[x] = x(2\pi, a, b, \beta_j, \mu) - a = 0, \\ [y] = y(2\pi, a, b, \beta_j, \mu) - b = 0, \\ [u_s] = u_s(2\pi, a, b, \beta_j, \mu) - \beta_s = 0, \\ (s = 1, \ldots, m),\end{array}\right\} \tag{3.14}$$

where the notation was introduced

$$[f_s(t)] \equiv f_s(2\pi) - f_s(0).$$

Let us consider equations (3.14) in more detail. Writing out only the linear terms we have:

$$\left.\begin{array}{l}x(t, a, b, \beta_j, \mu) = A_1 a + B_1 b + C_1 \mu + \ldots, \\ y(t, a, b, \beta_j, \mu) = A_2 a + B_2 b + C_2 \mu + \ldots, \\ u_s(t, a, b, \beta_j, \mu) = A_{s1}\beta_1 + \ldots + A_{sm}\beta_m + D_s\mu + \ldots\end{array}\right\} \tag{3.15}$$

(the remaining linear terms, as can easily be seen from equations (3.6), identically reduce to zero). Hence the functional determinant of the last m equations of (3.14) with respect to the magnitudes $\beta_1, \ldots, \beta_m$ for $a = b = \beta_1 = \ldots = \beta_m = \mu = 0$ has the value

$$\Delta = \begin{vmatrix} A_{11}(2\pi) - 1 & A_{12}(2\pi) & \ldots & A_{1m}(2\pi) \\ A_{21}(2\pi) & A_{22}(2\pi) - 1 & \ldots & A_{2m}(2\pi) \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1}(2\pi) & A_{m2}(2\pi) & \ldots & A_{mm}(2\pi) - 1 \end{vmatrix}. \tag{3.16}$$

But the magnitudes A_{sj} satisfy the equations

$$\frac{dA_{sj}}{dt} = b_{s1}A_{1j} + \ldots + b_{sm}A_{mj} \qquad (3.17)$$

and the initial conditions

$$A_{ss}(0) = 1, \quad A_{sj}(0) = 0 \ (s \neq j).$$

From this it follows that if the system (3.17) is considered as a special case of a system with periodic coefficients of period 2π and $D(\rho)$ denotes its characteristic determinant, we obtain from (3.16)

$$\Delta = D(1).$$

But the characteristic exponents of the system (3.17), considered as a system with periodic coefficients, are the roots of its fundamental equation (1.5). Among the latter by assumption there are no magnitudes of the form $\pm ki$, where k is an integer, and therefore the characteristic equation $D(\rho)$ has no roots equal to unity. Hence we have $\Delta \neq 0$. The last m equations of (3.14) are therefore solvable for the magnitudes β_s and give for them the analytic functions

$$\beta_s = \beta_s^*(a, b, \mu) \qquad (s = 1, \ldots, m), \qquad (3.18)$$

reducing to zero for $a = b = \mu = 0$. However the functions (3.18) will reduce to zero for $\mu = 0$ and a and b different from zero. This immediately follows from the fact that the functions $u_s(t, a, b, \beta_j, \mu)$ on the basis of (3.13) necessarily reduce to zero for $\beta_1 = \ldots = \beta_m = \mu = 0$ whatever the values of t, a, b.

Thus,

$$\beta_j^*(a, b, 0) = 0. \qquad (3.19)$$

Let us now substitute the obtained magnitudes β_j in the first two of equations (3.14). We shall have:

$$x(2\pi, a, b, \beta_j^*(a, b, \mu), \mu) - a = 0,$$
$$y(2\pi, a, b, \beta_j^*(a, b, \mu), \mu) - b = 0.$$

Separating out the terms not depending on μ and taking (3.15) into account we find:

$$x(2\pi, a, b, \beta_j^*(a, b, 0), 0) - a + \mu\{[C_1] + \Phi_1(a, b, \mu)\} = 0,$$
$$y(2\pi, a, b, \beta_j^*(a, b, 0), 0) - b + \mu\{[C_2] + \Phi_2(a, b, \mu)\} = 0,$$

where Φ_1 and Φ_2 are analytic functions reducing to zero for $a = b = \mu = 0$; whence, taking (3.19) into account, we obtain:

$$\{x(2\pi, a, b, \beta_j, 0)\}_{\beta_j=0} - a + \mu\{[C_1] + \Phi_1(a, b, \mu)\} = 0,$$
$$\{y(2\pi, a, b, \beta_j, 0)\}_{\beta_j=0} - b + \mu\{[C_2] + \Phi_2(a, b, \mu)\} = 0.$$

Replacing in these equations the magnitude 2π by its value from (3.9)

$$2\pi = pT + h, \quad h = -2\pi H_{2l}(a^2 + b^2)^l + \dots$$

and expanding in a series in h we shall have:

$$\{x(pT, a, b, \beta_j, 0)\}_{\beta_j=0} + h\left\{\frac{dx(t, a, b, \beta_j, 0)}{dt}\right\}_{\beta_j=0,\, t=pT} + \dots - \\ - a + \mu\{[C_1] + \Phi_1\} = 0,$$
$$\{y(pT, a, b, \beta_j, 0)\}_{\beta_j=0} + h\left\{\frac{dy(t, a, b, \beta_j, 0)}{dt}\right\}_{\beta_j=0,\, t=pT} + \dots - \\ - b + \mu\{[C_2] + \Phi_2\} = 0.$$

Taking into account (3.12) and the periodicity of the functions

$$\{x(t, a, b, \beta_j, 0)\}_{\beta_j=0}, \quad \{y(t, a, b, \beta_j, 0)\}_{\beta_j=0},$$

and also the fact that these functions satisfy equations (3.5) we find finally:

$$
\left.\begin{aligned}
2\pi p H_{2l}(a^2+b^2)^l b + \ldots + \mu\{[C_1]+\Phi_1(a, b, \mu)\} = 0,\\
2\pi p H_{2l}(a^2+b^2)^l a + \ldots + \mu\{[C_2]+\Phi_2(a, b, \mu)\} = 0,
\end{aligned}\right\} \quad (3.20)
$$

where the terms not written down depend only on a and b and with respect to these magnitudes are of an order higher than $2l + 1$.

The magnitudes C_1 and C_2 satisfy the equations

$$
\begin{aligned}
\frac{dC_1}{dt} &= -pC_2 + f(t, 0, \ldots, 0),\\
\frac{dC_2}{dt} &= pC_1 + F(t, 0, \ldots, 0)
\end{aligned}
$$

and the initial conditions

$$
C_1(0) = C_2(0) = 0,
$$

which on the basis of (3.4) gives:

$$
\begin{aligned}
C_1 &= \frac{1}{2}(B_{1p} - A_{2p})\, t \sin pt + \frac{1}{2}(A_{1p} + B_{2p})\, t \cos pt + C_1^*(t),\\
C_2 &= \frac{1}{2}(A_{1p} + B_{2p})\, t \sin pt - \frac{1}{2}(B_{1p} - A_{2p})\, t \cos pt + C_2^*(t),
\end{aligned}
$$

where C_1^* and C_2^* are certain periodic functions of t of period 2π. From this on the basis of (3.3) we obtain:

$$
[C_1] = \delta_2\pi, \quad [C_2] = -\delta_1\pi
$$

and therefore at least one of the magnitudes $[C_1]$ and $[C_2]$ is different from zero.

With this established, let us pass on to the solution of equations (3.20). We must find the roots a and b of these equations that reduce to zero for $\mu = 0$. For this purpose we set:

$$
\nu = \mu^{\frac{1}{2l+1}}, \quad a = \nu\alpha_1, \quad b = \nu\alpha_2.
$$

Then, dividing by ν^{2l+1}, we obtain:

$$\left.\begin{aligned} 2\pi p H_{2l}(\alpha_1^2+\alpha_2^2)^l\alpha_2+[C_1]+\nu\Phi_1^*(\alpha_1,\ \alpha_2,\ \nu)=0,\\ -2\pi p H_{2l}(\alpha_1^2+\alpha_2^2)^l\alpha_1+[C_2]+\nu\Phi_2^*(\alpha_1,\ \alpha_2,\ \nu)=0. \end{aligned}\right\} \quad (3.21)$$

These equations for $\nu = 0$ have the unique real solution

$$\left.\begin{aligned} \alpha_1^{(0)}&=\frac{[C_2]}{\sqrt[2l+1]{2p\pi H_{2l}\{[C_1]^2+[C_2]^2\}^l}}\,,\\ \alpha_2^{(0)}&=\frac{-[C_1]}{\sqrt[2l+1]{2p\pi H_{2l}\{[C_1]^2+[C_2]^2\}^l}}\,. \end{aligned}\right\} \quad (3.22)$$

Since the corresponding functional determinant, equal to

$$4(1+2l)\pi^2p^2H_{2l}^2(\alpha_1^{(0)2}+\alpha_2^{(0)2})^{2l},$$

is different from zero, equations (3.21) admit in the neighborhood of $\nu = 0$ only one real solution and this solution will have the form

$$\begin{aligned} \alpha_1&=\alpha_1^{(0)}+\nu\alpha_1^{(1)}+\ldots,\\ \alpha_2&=\alpha_2^{(0)}+\nu\alpha_2^{(1)}+\ldots, \end{aligned}$$

where the right hand sides are analytic functions of ν in the neighborhood of $\nu = 0$.

Equations (3.20) will therefore have the real solution

$$\left.\begin{aligned} a&=\alpha_1^{(0)}\nu+\alpha_1^{(1)}\nu^2+\ldots,\\ b&=\alpha_2^{(0)}\nu+\alpha_2^{(1)}\nu^2+\ldots \end{aligned}\right\} \quad (3.23)$$

It is not difficult to see that this is the only real solution of equations (3.20) for which a and b reduce to zero for $\mu = 0$. In fact, for each such solution, as is known from the theory of equations of the type of (3.20), the magnitudes a and b are analytic functions of $\mu^{1/m}$, where m is an integer. But, as is easy to verify, equations (3.20) can be satisfied by series developable in $\mu^{1/m}$ provided only that $m = 2l + 1$.

Substituting the found values of a and b in (3.18) we obtain β_s, which will likewise be analytic functions of the magnitude ν. Substituting a, b and β_s in (3.11) we obtain a periodic solution of equations (3.6) which satisfies all conditions of the theorem.

In this way the theorem has been proven.

4. Practical Method of Computing the Resonance Solution

For practically computing the resonance solution we proceed in the following manner.

We shall seek series with periodic coefficients

$$\left.\begin{aligned} x^{(res)} &= x^{(1)}\mu^{\frac{1}{2l+1}} + x^{(2)}\mu^{\frac{2}{2l+1}} + \dots, \\ y^{(res)} &= y^{(1)}\mu^{\frac{1}{2l+1}} + y^{(2)}\mu^{\frac{2}{2l+1}} + \dots, \\ x_s^{(res)} &= x_s^{(1)}\mu^{\frac{1}{2l+1}} + x_s^{(2)}\mu^{\frac{2}{2l+1}} + \dots, \end{aligned}\right\} \tag{4.1}$$

formally satisfying equations (3.1). On the basis of the proven theorem such series can always be found. For the coefficients of these series we shall obtain equations of the form

$$\left.\begin{aligned} &\frac{dx^{(k)}}{dt} = -py^{(k)} + f^{(k)}, \quad \frac{dy^{(k)}}{dt} = px^{(k)} + F^{(k)}, \\ &\frac{dx_s^{(k)}}{dt} = \sum_{\alpha=1}^{m} b_{s\alpha}x_\alpha^{(k)} + F_s^{(k)}, \end{aligned}\right\} \tag{4.2}$$

where $f^{(k)}$, $F^{(k)}$ and $F_s^{(k)}$ are integral rational functions with periodic coefficients for those $x^{(i)}$, $y^{(i)}$, $x_s^{(i)}$ for which $i < k$. Let us assume that all $x^{(i)}$, $y^{(i)}$, $x_s^{(i)}$ for which $i < k$ have already been computed and turned out periodic. Equations (4.2) will then give for $x_s^{(k)}$ entirely definite expressions. As to the magnitudes x^k

and y^k, for these we shall have expressions of the form

$$\left.\begin{aligned} x^{(k)} &= M_k \cos pt - N_k \sin pt + x_k^*(t), \\ y^{(k)} &= M_k \sin pt + N_k \cos pt + y_k^*(t), \end{aligned}\right\} \tag{4.3}$$

where x_k^* and y_k^* is a certain particular solution of the equations for $x^{(k)}$ and $y^{(k)}$ while M_k and N_k are arbitrary constants. In order that the equations for $x^{(k)}$ and $y^{(k)}$ actually admit a periodic solution it is necessary and sufficient that the conditions be satisfied

$$a_{1p}^{(k)} + b_{2p}^{(k)} = 0, \qquad b_{1p}^{(k)} - a_{2p}^{(k)} = 0, \tag{4.4}$$

where $a_{1p}^{(k)}$ and $b_{1p}^{(k)}$ denote the coefficients of cos pt and sin pt in the Fourier expansion of the function f_k and $a_{2p}^{(k)}$ and $b_{2p}^{(k)}$ denote the same coefficients for the function F_k. The equations (4.4) make it possible to determine the arbitrary constants M_j and N_j entering the successive approximations.

Without presenting here a detailed analysis we shall state the following properties of equations (4.4):

1. These equations are identically satisfied for $k < 2l + 1$.

2. For $k = 2l + 1$ these equations contain only the constants M_1 and N_1 and will be nonlinear but having a unique real solution which is easily obtained in closed form.

3. For $k = 2l + j$ $(j > 1)$ these equations contain only M_j and N_j (besides the magnitudes $M_1, \ldots, M_{j-1}$, $N_1, \ldots, N_{j-1}$, already computed from the preceding equations) and will be linear with respect to these magnitudes.

It is easy also to see that M_1 and N_1 are respectively equal to $\alpha_1^{(o)}$ and $\alpha_2^{(o)}$ and can therefore be computed by formulas (3.22).

EXAMPLE. As an example let us consider the equation

$$\frac{d^2x}{dt^2} + n^2x - \beta x^2 = \mu a \cos nt, \tag{4.5}$$

for which evidently there is principal resonance. From the expression (3.13) found in the preceding chapter for the period of the free oscillations determined by the equation

$$\frac{d^2x}{dt^2} + n^2x - \beta x^2 = 0,$$

It is seen that the magnitude h_2 already is different from zero. We therefore put

$$x = x_1\mu^{\frac{1}{3}} + x_2\mu^{\frac{2}{3}} + \dots$$

For the coefficients of the expansions we have the equations

$$\frac{d^2x_1}{dt^2} + n^2x_1 = 0,$$

$$\frac{d^2x_2}{dt^2} + n^2x_2 = \beta x_1^2,$$

$$\frac{d^2x_3}{dt^2} + n^2x_3 = 2\beta x_1x_2 + a\cos nt,$$

$$\frac{d^2x_4}{dt^2} + n^2x_4 = \beta x_2^2 + 2\beta x_1x_3.$$

Since the equation considered does not contain either sines or dx/dt the Fourier expansion of the function x will contain only cosines. Hence

$$x_1 = M_1 \cos nt,$$
$$x_2 = M_2 \cos nt + \frac{\beta M_1^2}{2n^2} - \frac{\beta M_1^2}{6n^2}\cos 2nt.$$

In order that the equation for x_3 admit a periodic solution it is necessary and sufficient that the coefficient of cos nt on the right hand side of this equation reduce to zero. This gives

$$M_1 = -\sqrt[3]{\frac{6an^2}{5\beta^2}}$$

and

$$x_3 = \frac{\beta M_1 M_2}{n^2} - \frac{\beta M_1 M_2}{3n^2}\cos 2nt + \frac{\beta^2 M_1^3}{48n^4}\cos 3nt + M_3 \cos nt.$$

From the condition of periodicity of the function x_4 we find $M_2 = 0$ and therefore

$$x = -\sqrt[3]{\frac{6an^2\mu}{5\beta^2}}\cos nt + \left(\sqrt[3]{\frac{6an^2\mu}{5\beta^2}}\right)^2\left(\frac{\beta}{2n^2} - \frac{\beta}{6n^2}\cos 2nt\right) + \ldots$$

We restrict ourselves to this approximation.

We shall now try to solve the same problem with the aid of the theory of quasilinear systems. For this purpose we set in equation (4.5) $\beta = \mu\beta'$ and reduce it to the quasilinear form

$$\frac{d^2x}{dt^2} + n^2x = \mu(\beta' x^2 + a\cos nt).$$

Following the general theory of quasilinear systems we set:

$$x = M_0 \cos nt + N_0 \sin nt + \mu x_1 + \ldots,$$

where M_0 and N_0 are arbitrary constants. We shall have:

$$\frac{d^2x_1}{dt^2} + n^2x_1 = \beta'(M_0 \cos nt + N_0 \sin nt)^2 + a\cos nt,$$

and the conditions of periodicity lead to the equations

$$P(M_0, N_0) \equiv a = 0, \quad Q(M_0, N_0) \equiv 0.$$

These equations have no solution. Thus, the periodic solution of system (4.5) cannot be found with the aid of the theory of quasilinear systems.

5. The Periodic Solution $\left\{x_s^{(p)}\right\}$

We now proceed to the investigation of the periodic solution of system (1.4) that reduces for $\mu = 0$ to the generating solution $\left\{\xi^{(p)}(t + h),\ \eta^{(p)}(t + h),\ \xi_s^{(p)}(t + h)\right\}$. This solution we shall denote by $\left\{x^{(p)},\ y^{(p)},\ x_s^{(p)}\right\}$ or simply by $\left\{x_s^{(p)}\right\}$, if we use the equations of the oscillations in the form (1.1). In the latter case we shall denote the generating solution in question by $\left\{\xi_s^{(p)}(t + h)\right\}$.

Since the generating solution depends on the arbitrary constant h the problem under consideration is a special case of the general problem investigated in detail in sec. 2- 4 of chapter VI. According to the results there obtained in order that a periodic solution of the system (1.1) correspond to the generating solution it is necessary that the constant h be a root of the equation

$$P(h) = \int_0^{2\pi} \sum_{\alpha=1}^{n} f_\alpha(t,\ \xi_1^{(p)}(t+h),\ \ldots,\ \xi_n^{(p)}(t+h),\ 0)\,\psi_\alpha\,dt = 0. \tag{5.1}$$

Here $\psi_s(t)$ is a periodic solution of the equations conjugate to the equations in variations of the generating system

$$\frac{d\xi_s}{dt} = a_{s1}\xi_1 + \ldots + a_{sn}\xi_n + X_s^*(\xi_1, \ldots, \xi_n) \tag{5.2}$$

for the solution $\left\{\xi_s^{(p)}(t + h)\right\}$. Since the system (5.2) admits the first integral

$$H(\xi_1, \ldots, \xi_n) = \text{const}, \tag{5.3}$$

it is possible to apply the results of sec. 6 of chapter VI and write equation (5.1) in the explicit form:

$$P(h) = \int_0^{2\pi} \sum_{\alpha=1}^{n} f_\alpha(t,\ \xi_1^{(p)}(t+h),\ \ldots,\ \xi_n^{(p)}(t+h),\ 0)\,\frac{\partial H}{\partial \xi_\alpha}\,dt = 0, \tag{5.4}$$

where the partial derivatives of the function H are computed for the generating solution.

It was shown also that to each nonmultiple root of equation (5.4) corresponds one and only one periodic solution of system (1.1) and this solution is analytic with respect to μ.

For the validity of the above assertion it is necessary however still to verify that the equations in variations of the generating system admit only one periodic solution. These equations have the form

$$\frac{du_s}{dt}=\sum_{\alpha=1}^{n}\left(a_{s\alpha}+\frac{\partial X_s^*}{\partial \xi_\alpha}\right)u_\alpha=\sum_{\alpha=1}^{n}p_{s\alpha}u_\alpha, \tag{5.5}$$

where in the derivatives of the functions X_s^* the magnitudes ξ_j are to be replaced by their values in the generating solution, i.e. by the functions $\xi_s^{(p)}(t + h)$. For the latter we can write:

$$\xi_s^{(p)}(t+h)=c_p\xi_{s1}(t+h)+c_p^2\xi_{s2}(t+h)+\dots, \tag{5.6}$$

where ξ_{sj} are periodic functions of t of period $2\pi/p$.

The generating system is autonomous and admits a family of periodic solutions depending, besides on the parameter h, on the essential parameter c, the generating solution in question belonging to this family. Hence on the basis of the general results of sec. 9 of chapter VI we can assert that the system (5.5) admits a particular periodic solution, which we shall denote by $\varphi_s(t)$, and another particular solution which, because of the fact that the period T of the generating family depends on c, will not be periodic but will have the form $t\varphi_s + \varphi_s^*$, where φ_s^* is also a periodic function. The characteristic equation of the system (5.5) consequently has two roots equal to unity. We shall show that the remaining roots of this equation are different from unity.

In fact, for $c_p = 0$, as is seen from (5.6), the system (5.5) goes over into the system with constant coefficients

$$\frac{du_s}{dt} = \sum_{\alpha=1}^{n} a_{s\alpha} u_\alpha. \tag{5.7}$$

The fundamental equation of this system according to the fundamental theorem has a pair of purely imaginary roots $\pm \lambda i$. But since we assumed that $c_p = 0$ therefore $\lambda = p$. Moreover, by assumption the fundamental equation of the system (5.7) does not have roots of the form $\pm k\lambda i = kpi$, where k is an integer. We shall make the somewhat stronger assumption that the above equation does not in general have roots of the form $\pm ki$, different from the roots $\pm pi$. The system (5.5) for $c_p = 0$ will then have only two characteristic exponents of the form $\pm pi$, i.e. the characteristic equation of this system will have only two roots equal to unity. The same will however be true also for c_p different from zero but sufficiently small, since the roots of the characteristic equation of system (5.5) will be continuous functions of the magnitude c_p. We may remark also that even if the system (5.5) had for $c_p = 0$ characteristic exponents of the form $\pm ki$, different from $\pm pi$, only in very exceptional cases would these exponents retain their form also for $c_p \neq 0$. We can thus assume that the characteristic equation of system (5.5) has only two roots equal to unity to which, according to what was said above, corresponds only one periodic solution $\varphi_s(t)$.

Let us assume that h is a simple root of equation (5.4). Then, as was shown, the periodic solution $\left\{x_s^{(p)}\right\}$ will actually exist and will be analytic with respect to μ. We can therefore write:

$$x_s^{(p)} = \xi_s^{(p)}(t+h) + \mu x_{s1}(t) + \ldots \tag{5.8}$$

For finding the functions x_{sj} we obtain equations of the form

$$\frac{dx_{sj}}{dt} = p_{s1}x_{1j} + \ldots + p_{sn}x_{nj} + F_{sj}, \tag{5.9}$$

where F_{sj} will be known periodic functions of t if all functions $x_{s1},\ldots,x_{s,j-1}$ have already been computed and turned out periodic. The periodic solution of equations (5.9), if it exists, will have the form

$$x_{sj} = M_j \varphi_s(t) + x^*_{sj}(t),$$

where x^*_{sj} is a particular periodic solution of system (5.9) and M_j is an arbitrary constant. This constant is determined from the condition of the existence of a periodic solution of the equations for $x_{s,j+1}$. As was shown in sec. 3 of chapter VI, the equation determining M_j has the form

$$\frac{dP(h)}{dh} M_j = L_j,$$

where L_j is a certain known constant.

Since the expansion of the function H starts with terms of the second order the function dP/dh, as is seen from (5.6) and (5.4), reduces to zero for $c_p = 0$. Because of this the magnitudes M_j, and together with them also the functions x_{sj} in the series (5.8), will contain the magnitude c_p in the denominators. Hence for fixed μ the solution $\left\{x_s^{(p)}\right\}$ will cease to exist if the magnitude c_p is very small, i.e. if λ differs little from an integer p. In this case resonance will take place and if the conditions of principal resonance are satisfied the periodic solution of system (1.1) can be found by the method of the preceding section.

For the actual computation of the coefficients x_{sj} it is necessary to know the general solution of the equations in variations (5.5). This will always be possible for systems of the second order since we know the two particular solutions φ_s and $t\varphi_s + \psi_s$. In particular, this will be possible for the system described by the equation

$$\frac{d^2x}{dt^2} + k^2x + F(x) = \mu f\left(t, x, \frac{dx}{dt}\right), \tag{5.10}$$

where F(x) is an analytic function the expansion of which starts with terms of not lower than the second order.

The system (5.10) is a special case of the systems considered in sec. 4 of chapter VI and for it the solution $\{x^{(p)}\}$ can be computed by the general formulas there established.

6. Stability Criteria

The stability of the obtained periodic solutions of the system (1.1) may be investigated by the general methods described in chapter III. Serious difficulties however arise here in investigating the stability of the resonance solution since for this it is necessary to consider $2l + 1$ approximations. We shall therefore now consider the establishment of the stability criteria of the resonance solution by a somewhat different method, restricting ourselves to systems of the second order.

Let us consider the system of the second order

$$\left.\begin{aligned} \frac{dx}{dt} &= -py + X(x, y) + \mu f(t, x, y, \mu), \\ \frac{dy}{dt} &= px + Y(x, y) + \mu F(t, x, y, \mu), \end{aligned}\right\} \tag{6.1}$$

reducing for $\mu = 0$ to a Lyapunov system on satisfying the resonance conditions. For it we set up the equations in variations

$$\left.\begin{aligned} \frac{du}{dt} &= \left(\frac{\partial X}{\partial x} + \mu \frac{\partial f}{\partial x}\right)u + \left(-p + \frac{\partial X}{\partial y} + \mu \frac{\partial f}{\partial y}\right)v, \\ \frac{dv}{dt} &= \left(p + \frac{\partial Y}{\partial x} + \mu \frac{\partial F}{\partial x}\right)u + \left(\frac{\partial Y}{\partial y} + \mu \frac{\partial F}{\partial y}\right)v, \end{aligned}\right\} \tag{6.2}$$

where the derivatives are computed for the investigated solution, i.e. for the resonance solution $\{x^{(res)}, y^{(res)}\}$. Let us form the characteristic equation of this system.

For this purpose we shall denote by $x(t, a, b, \mu)$, $y(t, a, b, \mu)$ the solution of system (6.1) with initial conditions

$$x(0, a, b, \mu) = a, \quad y(0, a, b, \mu) = b. \tag{6.3}$$

The system in variations (6.2) will then have the fundamental system of solutions

$$\left.\begin{aligned} u_1(t) &= \frac{\partial x(t, a, b, \mu)}{\partial a}, \quad & v_1(t) &= \frac{\partial y(t, a, b, \mu)}{\partial a}, \\ u_2(t) &= \frac{\partial x(t, a, b, \mu)}{\partial b}, \quad & v_2(t) &= \frac{\partial y(t, a, b, \mu)}{\partial b}, \end{aligned}\right\} \tag{6.4}$$

where after differentiation the magnitudes a and b must be replaced by their values $a(\mu)$ and $b(\mu)$ in the solution $\{ x^{(res)}, y^{(res)} \}$. On the basis of (6.3) the initial conditions are satisfied

$$\begin{aligned} u_1(0) &= 1, \quad & v_1(0) &= 0, \\ u_2(0) &= 0, \quad & v_2(0) &= 1, \end{aligned}$$

and therefore the characteristic equation of system (6.2) has the form

$$\begin{vmatrix} u_1(2\pi) - \rho & v_1(2\pi) \\ u_2(2\pi) & v_2(2\pi) - \rho \end{vmatrix} = 0. \tag{6.5}$$

But

$$\left.\begin{aligned} x(2\pi, a, b, \mu) &= \psi_1(a, b, \mu) + a, \\ y(2\pi, a, b, \mu) &= \psi_2(a, b, \mu) + b, \end{aligned}\right\} \tag{6.6}$$

where ψ_1 and ψ_2 are the left hand sides of the first two of equations (3.14) for the case now considered, which, as was shown, have the form (3.20), so that we can write:

$$\left.\begin{aligned} \psi_1 &= 2\pi p H_{2l} (a^2 + b^2)^l b + \ldots + \mu \{[c_1] + \Phi_1(a, b, \mu)\}, \\ \psi_2 &= -2\pi p H_{2l} (a^2 + b^2)^l a + \ldots + \mu \{[c_2] + \Phi_2(a, \beta, \mu)\}. \end{aligned}\right\} \tag{6.7}$$

From (6.4) and (6.7) for the characteristic equation (6.5) we finally have:

$$\begin{vmatrix} \dfrac{\partial \psi_1}{\partial a} + 1 - \rho & \dfrac{\partial \psi_1}{\partial b} \\ \dfrac{\partial \psi_2}{\partial a} & \dfrac{\partial \psi_2}{\partial b} + 1 - \rho \end{vmatrix} = \rho^2 + 2A\rho + B = 0, \tag{6.8}$$

where the magnitudes a and b must be replaced by their values for the solution $\left\{x^{(res)}, y^{(res)}\right\}$, i.e. by the series (3.23).

The coefficient B in the characteristic equation may be expressed explicitly in terms of the coefficients of the system (6.2) with the aid of formula (2.8) of chapter III, which gives:

$$B = \exp\left\{\int_0^{2\pi}\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \mu\left(\frac{\partial f}{\partial x} + \frac{\partial F}{\partial y}\right)\right] dt. \qquad (6.9)$$

For stability of the investigated solution it is necessary that the condition be satisfied

$$\int_0^{2\pi}\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \mu\left(\frac{\partial f}{\partial x} + \frac{\partial F}{\partial y}\right)\right] dt \leqslant 0. \qquad (6.10)$$

If the roots ρ_1 and ρ_2 of the characteristic equation are complex we already have, when (6.10) is satisfied, $|\rho_1| = |\rho_2| \leqslant 1$. If these roots are real then, as is easy to see, they will for sufficiently small μ be positive. This follows from the fact that the functions ψ_1 and ψ_2, as is seen from (6.7) and (3.23), reduce to zero for $\mu = 0$ and consequently the roots ρ_1 and ρ_2 for μ sufficiently small will differ little from unity. Hence if condition (6.10) is satisfied and ρ_1 and ρ_2 are real they will be less than unity if the further inequality is satisfied

$$(1-\rho_1)(1-\rho_2) > 0,$$

which on the basis of (6.8) has the form

$$\frac{\partial(\psi_1, \psi_2)}{\partial(a, b)} > 0. \qquad (6.11)$$

Substituting here for ψ_1 and ψ_2 their expressions (6.7) and taking (3.23) into account we obtain:

$$\frac{\partial(\psi_1, \psi_2)}{\partial(a, b)} = 4\pi^2 p^2 H_{2l}^2 (1+2l)(a^2+b^2)^{2l} + \ldots =$$
$$= 4\pi^2 p^2 H_{2l}^2 (1+2l)\left(\alpha_1^{(0)^2} + \alpha_2^{(0)^2}\right)^{2l} \mu^{\frac{4l}{1+2l}} + \ldots,$$

where the terms not written down have a higher order of smallness. From this it follows that for sufficiently small μ the inequality (6.11) is always satisfied. Hence there remains only the one condition of stability (6.10) and if it is satisfied with the inequality sign the solution $\{x^{(res)}, y^{(res)}\}$ will actually be stable and furthermore asymptotically so.

7. Application to Duffing's Problem

This section is devoted to a detailed investigation of the forced oscillations of a system described by the equation

$$\frac{d^2x}{dt^2} + k^2x - \gamma x^3 = \mu\left(a\cos mt + b\cos nt - 2H\frac{dx}{dt}\right) \qquad (7.1)$$
$$(a > 0,\ b > 0,\ H > 0,\ \gamma > 0),$$

where m and n are integers. We are thus considering the problem of Duffing in the case where there is resistance and the disturbing force contains two harmonics. The case where there is no resistance and the disturbing force contains only one harmonic was considered by us in chapter I with the aid of the quasilinear theory. We now consider the general case of equation (7.1) treating it as an approximation to a Lyapunov system. We shall investigate for equation (7.1) the solutions $\{x^{(m)}\}$ and $\{x^{(n)}\}$, the solution $\{x^{(o)}\}$, the resonance solution, the stability of all these solutions and the character of the development of the oscillations. We shall not here consider the solutions $\{x^{(p)}\}$ for p different from m and n, which evidently determine the subharmonic oscillations of various orders (integral and fractional). We shall consider subharmonic oscillations in the next section for another system.

We shall start with a consideration of the solution $\{x^{(m)}\}$. The solution $\{x^{(n)}\}$ is obtained by a simple interchange of the numbers m and n.

a) SOLUTION $\{x^{(m)}\}$. The generating equation

$$\frac{d^2\xi}{dt^2} + k^2\xi - \gamma\xi^3 = 0 \qquad (7.2)$$

was considered by us in the preceding chapter. The general solution of this equation is determined by the formulas (3.12) there obtained. Hence on the basis of (1.11) we have:

$$\left.\begin{aligned} \xi^{(m)}(t+h) &= c_m \cos\tau + c_m^3 \xi_3(\tau) + c_m^5 \xi_5(\tau) + \ldots, \\ \tau &= m(t+h), \end{aligned}\right\} \tag{7.3}$$

where

$$\left.\begin{aligned} \xi_3(\tau) &= \frac{1}{32}\frac{\gamma}{k^2}\cos\tau - \frac{1}{32}\frac{\gamma}{k^2}\cos 3\tau, \\ \xi_5(\tau) &= \frac{23}{1024}\frac{\gamma^2}{k^4}\cos\tau - \frac{3}{128}\frac{\gamma^2}{k^4}\cos 3\tau + \frac{1}{1024}\frac{\gamma^2}{k^4}\cos 5\tau. \end{aligned}\right\} \tag{7.4}$$

The period T is determined by the formula

$$T = \frac{2\pi}{k}(1 + h_2 c^2 + h_4 c^4 + \ldots),$$
$$h_2 = \frac{3}{8}\frac{\gamma}{k^2}, \qquad h_4 = \frac{57}{256}\frac{\gamma^2}{k^4},$$

and therefore the equation (1.12) for c_m assumes the form

$$\frac{3}{8}\frac{\gamma}{k^2}c_m^2 + \frac{57}{256}\frac{\gamma^2}{k^4}c_m^4 + \ldots = \frac{k-m}{m}. \tag{7.5}$$

This equation has real solutions only for $k > m$. There will be two such solutions and they will be numerically equal to each other and of opposite sign.

To find the solution $\{x^{(m)}\}$ we use the general formulas of sec. 6 of chapter VI. First, by the formulas (6.12) and (6.16) there established, we find the functions u and v, for which we obtain:

$$\left.\begin{aligned} u(t+h) &= -c_m \sin m(t+h) + c_m^3 \dot{\xi}_3[m(t+h)] + \\ &\qquad + c_m^5 \dot{\xi}_5[m(t+h)] + \ldots, \\ v &= \frac{1}{Ac_m}\{\cos m(t+h) + 3c_m^2 \xi_3[m(t+h)] + \\ &\qquad + 5c_m^4 \xi_5[m(t+h)]\} + \ldots, \end{aligned}\right\} \tag{7.6}$$

where

$$A = -\frac{m}{k}(2h_2 + 4h_1c_m^2 + \ldots), \tag{7.7}$$

and by formula (6.22) of chapter VI we find the constant K,

$$K = -\frac{k(k^2 - \gamma c_m^2)}{m(2h_2 + 4h_4c_m^2 + \ldots)}. \tag{7.8}$$

Then, setting

$$x^m = \xi^{(m)}(t+h) + \mu x_1(t) + \mu^2 x_2(t) + \ldots, \tag{7.9}$$

we determine the magnitude x_1 by the general formula (6.30) of chapter VI. Carrying out the computations with an accuracy up to magnitudes of the first order with respect to c_m we shall have:

$$x_1 = x_1^* + \alpha_1 u, \tag{7.10}$$

where

$$x_1^* = \frac{a\cos mh}{2KA^2c_m^2}\left(-1 + \frac{3}{32}\frac{\gamma^2}{k^2}c_m^2\right)\cos m(t+h) + \\ + \frac{mHc_m}{2KA}\sin m(t+h) + \frac{a}{4KA}\cos mt + \frac{bm^2}{(m^2-n^2)KA}\cos nt, \tag{7.11}$$

and the constant α_1 must be determined from the condition of periodicity of the function x_2.

For the constant h we have on the basis of formula (6.29) of chapter VI the following equation:

$$P(h) = \int_0^{2\pi}\left[a\cos mt + b\cos nt - 2H\frac{d\xi^{(m)}(t+h)}{dt}\right]\frac{d\xi^{(m)}(t+h)}{dt}\,dt = 0.$$

Let us consider this equation more closely. On the basis of (7.3) and (7.4) we can write:

$$\xi^{(m)}(t+h) = \sum_{j=0}^{\infty} A_{2j+1}\cos(2j+1)m(t+h),$$

where

$$A_1 = c_m + \frac{1}{32}\frac{\gamma}{k^2}c_m^3 + \frac{23}{1024}\frac{\gamma^2}{k^4}c_m^5 + \dots,$$

$$A_3 = -\frac{1}{32}\frac{\gamma}{k^2}c_m^3 - \frac{3}{128}\frac{\gamma^2}{k^4}c_m^5 + \dots$$

We shall assume that the ratio n/m is not an odd integer. With this assumption the equation for h assumes the form

$$P(h) = -a\pi m A_1 \sin mh - 2\pi H m^2 \sum_{j=0}^{\infty}(2j+1)^2 A_{2j+1}^2 = 0.$$

This equation determines two values for the angle mh, the sum of these angles being equal to π. The magnitude $P'(h)$ for these values of h is different from zero. If we now take into consideration that we have two different values for c_m of equal magnitude but opposite sign we must conclude that there exist four periodic solutions $\{x^{(m)}\}$. Actually however there are only two different solutions $\{x^{(m)}\}$. In fact, since the function $\xi^{(m)}(t+h)$ contains only odd powers of c_m and only odd multiples of the angle $m(t+h)$ the substitution of $\pi - mh$ for mh is equivalent, as can easily be noted from the form of $P(h)$, to passing from the generating solution $\{\xi^{(m)}(t+h)\}$ with the chosen value of c_m to the generating solution $\{\xi^{(m)}(t+h)\}$ with the value of c_m of opposite sign. We can therefore assume that the angle mh is acute.

If only the first powers of c_m are retained in the function $P(h)$ we arrive at the following approximate formula:

$$\sin mh = \frac{2mHc_m}{a}. \tag{7.12}$$

This approximate formula will evidently hold also for the case that n/m is an odd integer.

We shall proceed to the computation of the magnitude α_1. For x_2 we obtain the equation

$$\frac{d^2x_2}{dt^2} + (k^2 - 3\gamma\xi^{(m)2})\,x_2 = 3\gamma\xi^{(m)}x_1^2 - 2H\frac{dx_1}{dt},$$

and the condition of periodicity of the function x_2 has the form

$$\int_0^{2\pi}\left(3\gamma\xi^{(m)}x_1^2-2H\frac{dx_1}{dt}\right)u\,dt=0.$$

Substituting for x_1 its value (7.10) and noting that the coefficient of α_1^2, by what was earlier proven, necessarily reduces to zero and that

$$\int_0^{2\pi}u\frac{du}{dt}\,dt=0,$$

we obtain:

$$6\gamma\alpha_1\int_0^{2\pi}x_1^*\xi^{(m)}u^2dt=\int_0^{2\pi}\left(2H\frac{dx_1^*}{dt}-3\gamma\xi^{(m)}x_1^{*2}\right)u\,dt.$$

If we now restrict ourselves in the computation to only the smallest term (with respect to c_m) in the function x_1^*, i.e. the term

$$-\frac{a\cos mh}{2KA^2c_m^2}\cos m(t+h),$$

and the smallest terms in the expressions for $\xi^{(m)}$ and u we shall have:

$$\alpha_1=\frac{4H}{3\gamma c_m^2}. \tag{7.13}$$

Formulas (7.3) - (7.13) determine the solution $\{x^{(m)}\}$.

b) SOLUTION $\{x^{(o)}\}$. For this solution setting

$$x^{(0)}=\mu x_1+\mu^2x_2+\dots \tag{7.14}$$

and proceeding as indicated in sec. 2 we find:

$$\left.\begin{aligned}
x_1&=\frac{a}{k^2-m^2}\cos mt+\frac{b}{k^2-n^2}\cos nt,\\
x_2&=\frac{2Hma}{(k^2-m^2)^2}\sin mt+\frac{2Hnb}{(k^2-n^2)^2}\sin nt,\\
x_3&=A_m\cos mt+A_n\cos nt+A_{3m}\cos 3mt+\\
&\quad+A_{3n}\cos 3nt+A_{2m+n}\cos(2m+n)t+\\
&\quad+A_{2m-n}\cos(2m-n)t+A_{2n+m}\cos(2n+m)t+\\
&\quad+A_{2n-m}\cos(2n-m)t,
\end{aligned}\right\} \tag{7.15}$$

where

$$\left.\begin{aligned}
A_m &= \frac{3}{2}\gamma\left[\frac{a^3}{2(k^2-m^2)^4}+\frac{ab^2}{(k^2-m^2)^2(k^2-n^2)^2}\right]-\frac{4Hm^2a}{(k^2-m^2)^3},\\
A_n &= \frac{3}{2}\gamma\left[\frac{b^3}{2(k^2-n^2)^4}+\frac{a^2b}{(k^2-m^2)^2(k^2-n^2)^2}\right]-\frac{4Hn^2b}{(k^2-n^2)^3},\\
A_{3m} &= \frac{\gamma a^3}{4(k^2-m^2)^3(k^2-9m^2)},\\
A_{3n} &= \frac{\gamma b^3}{4(k^2-n^2)^3(k^2-9n^2)},\\
A_{2m\pm n} &= \frac{3\gamma a^2 b}{4(k^2-m^2)^2(k^2-n^2)[k^2-(2m\pm n)^2]},\\
A_{2n\pm m} &= \frac{3\gamma ab^2}{4(k^2-n^2)^2(k^2-m^2)[k^2-(2n\pm m)^2]}.
\end{aligned}\right\}\qquad (7.16)$$

Formulas (7.14) - (7.16) determine the solution $\{x^{(o)}\}$.

c) RESONANCE SOLUTION. We shall find the oscillations of the system in the neighborhood of resonance k = m. For this purpose we set:

$$m^2-k^2=\mu\lambda,$$

after which equation (7.1) assumes the form

$$\frac{d^2x}{dt^2}+m^2x-\gamma x^3=\mu\left(a\cos mt+b\cos nt-2H\frac{dx}{dt}+\lambda x\right). \qquad (7.17)$$

Following the method of sec. 4 we seek the solution of this equation in the form of the series

$$x^{(res)}=x_1\mu^{\frac{1}{3}}+x_2\mu^{\frac{2}{3}}+x_3\mu+\dots \qquad (7.18)$$

We have:

$$\frac{d^2x_1}{dt^2}+m^2x_1=0, \qquad \frac{d^2x_2}{dt^2}+m^2x_2=0,$$

$$\frac{d^2x_3}{dt^2}+m^2x_3=\gamma x_1^3+a\cos mt+b\cos nt,$$

$$\frac{d^2x_4}{dt^2}+m^2x_4=3\gamma x_1^2x_2-2H\frac{dx_1}{dt}+\lambda x_1,$$

$$\frac{d^2x_5}{dt^2}+m^2x_5=3\gamma x_1^2x_3+3\gamma x_1x_2^2-2H\frac{dx_2}{dt}+\lambda x_2.$$

whence we find:

$$\left.\begin{aligned}
x_1 &= A_1\cos mt+B_1\sin mt,\\
x_2 &= A_2\cos mt+B_2\sin mt,
\end{aligned}\right\}\qquad (7.19)$$

where A_1, A_2, B_1, B_2 are constants. Equating the coefficients of cos mt and sin mt to zero in the equation for x_3 we obtain:

$$A_1 = -\sqrt[3]{\frac{4a}{3\gamma}}, \quad B_1 = 0, \tag{7.20}$$

after which we shall have:

$$x_3 = \frac{a}{24m^2}\cos 3mt + \frac{b}{m^2 - n^2}\cos nt + A_3 \cos mt + B_3 \sin mt. \tag{7.21}$$

From the conditions of periodicity of the functions x_4 and x_5 we find the following values of the coefficients A_2, B_2, A_3 and B_3:

$$\left.\begin{aligned} A_2 &= \frac{4\lambda}{9\gamma}\sqrt[3]{\frac{3\gamma}{4a}}, & B_2 &= \frac{8Hm}{3\gamma}\sqrt[3]{\frac{3\gamma}{4a}}, \\ A_3 &= -\frac{a}{72m^2} + \frac{32H^2m^2}{a\gamma}, & B_3 &= -\frac{16Hm\lambda}{9a\gamma}. \end{aligned}\right\} \tag{7.22}$$

Formulas (7.18) - (7.22) determine the resonance solution.

d) STABILITY OF THE SOLUTIONS. We shall investigate the obtained periodic solutions of equations (7.1) for stability.

The equation in variations for the solution $\{x^{(o)}\}$ has the form

$$\frac{d^2u}{dt^2} + k^2u + 2H\mu\frac{dv}{dt} + \ldots = 0,$$

where the terms not written down are of an order higher than the first with respect to μ. Hence the characteristic exponents of the solution $\{x^{(0)}\}$ are determined by the formula

$$\alpha_{1,2} = \pm ik - H\mu + \ldots$$

From this it follows that for sufficiently small μ the solution $\{x^{(o)}\}$ is asymptotically stable.

For the solution $\{x^{(m)}\}$ the conditions of stability, as was shown in sec. 7 of chapter VI, are given by the inequalities

$$\int_0^{2\pi} \frac{\partial f}{\partial \dot{x}}\, dt \leqslant 0, \qquad K\frac{dP(h)}{dh} \leqslant 0, \tag{7.23}$$

where f is the right hand side of equation (7.1). The first of these conditions is evidently satisfied with the inequality sign. Further, on the basis of (7.8) and the expression for P(h) we have:

$$K\frac{dP(h)}{dh} = \frac{ak\pi \cos mh}{m\,(2h_2 + 4h_4 c_m^2 + \ldots)}\,(k^2 - \gamma c_m^2)\left(1 + \frac{1}{32}\frac{\gamma}{k^2}c_m^2 + \ldots\right)c_m,$$

whence it follows that for sufficiently small c_m the second of conditions (7.23) is satisfied with the inequality sign if $c_m < 0$, and is not satisfied if $c_m > 0$. Consequently, of the two solutions $\{x^{(m)}\}$ the solution corresponding to the negative value of c_m is asymptotically stable while the solution corresponding to the positive c_m is unstable .

Let us consider, finally, the resonance solution. The stability condition (6.10) established for it in the preceding section agrees in the case under consideration with the first of conditions (7.23) and is therefore satisfied with the inequality sign. The resonance solution is thus asymptotically stable.

e) CHARACTER OF THE DEVELOPMENT OF THE OSCILLATIONS. From all the foregoing it is now possible to draw definite conclusions in regard to the character of the development of the oscillations.

Let $n > m$ and assume at first that $k < m$. The oscillations will then develop according to the stable solution $\{x^{(o)}\}$. As k approaches m the series determining the solution $\{x^{(o)}\}$ begins to diverge. From this however it does not follow that the solution $\{x^{(o)}\}$ ceases to exist. It can be retained but has a different analytic expression. A more detailed analysis shows that as k approaches m the solution $\{x^{(o)}\}$ continuously goes over into the resonance solution (in the neighborhood of $k = m$) while the latter, with further increase in k, continuously goes over into the solution $\{x^{(m)}\}$, arising after the resonance, corresponding to the negative value of c_m. For such value of k (or somewhat greater) there will also exist the solution corresponding to the positive value of c_m, and again the solution $\{x^{(o)}\}$. The former of these solutions, as has already been shown, in unstable.

With further increase in k the magnitude c_m, as follows from its defining equation, will increase numerically, i.e. the amplitude of the oscillations will increase, as a result of which these oscillations will at a certain instant become unstable. Let us assume that this occurs for $k = k^*$. How will the oscillations develop for $k > k^*$? It is here necessary to consider two distinct cases depending on whether the magnitude $n - m$ is sufficiently large or not.

Let us assume first that the difference $n - m$ is sufficiently large, namely we assume that $n > k^*$. In this case for $k = k^*$ the only stable solution is $\{x^{(o)}\}$. As a result the oscillations go over discontinuously into this solution. With further increase of k the solution $\{x^{(o)}\}$ continuously goes over, through the resonance solution in the neighborhood of $k = n$, into the solution $\{x^{(m)}\}$ corresponding to the negative value of c_n, and then, after loss of stability, again into the solution $\{x^{(o)}\}$ (discontinuously).

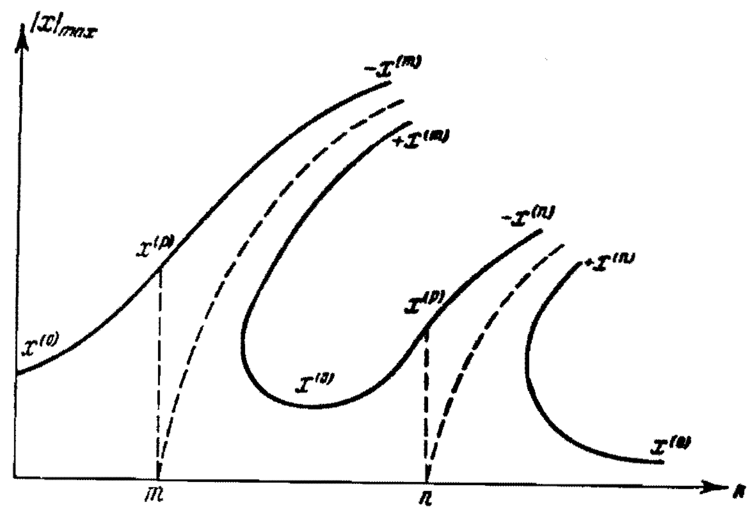

Fig. 31

Let us now assume that the difference n - m is small, namely we assume that $n < k^*$. In this case the oscillations will develop according to the solution $\{x^{(m)}\}$ with negative c_m for values of k not only

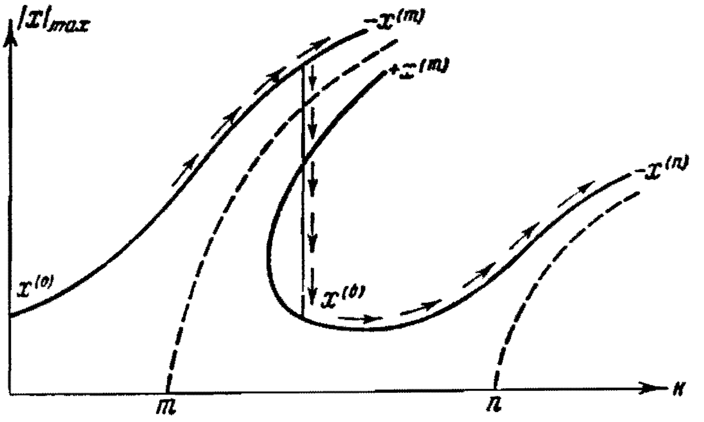

Fig. 32

less than but even equal to and exceeding n. Hence, notwithstanding the fact that the second harmonic will be in resonance the first harmonic will prevail in the forced oscillations. For $k > k^*$ the oscillations will break down either into the solution $\{x^{(o)}\}$ or the solution $\{x^{(n)}\}$ (with negative c_n), which can still be stable.

All that was said above becomes very clear if we construct the resonance curves. Fig. 31 shows the relation existing between the different periodic solutions. The dotted curves correspond to the generating solutions $\{\xi^{(m)}(t + h)\}$ and $\{\xi^{(n)}(t + h)\}$ ("the skeleton curves"). In fig. 32 the arrows indicate the

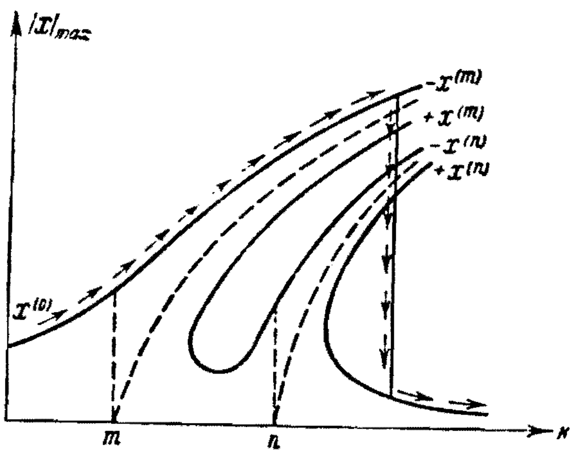

Fig. 33

development of the oscillations with gradual increase of k for the case where the difference n - m is sufficiently large. In fig. 33 is shown one of the possible variants of the development of the oscillations for the case in which the magnitude n - m is small.

8. Example of Determination of Subharmonic Oscillations

The problem which we have considered in the preceding section could have been solved also with the aid of the quasilinear theory. In determining however the oscillations near resonance it would have been necessary for us to solve a system of two cubic equations for determining the amplitudes of the generating solution and then to investigate the nine periodic solutions thus obtained. If the order of the characteristic of the nonlinearity in equation (7.1) were higher than three the degrees of the equations which determine the amplitudes in the generating solution, if this equation is regarded as quasilinear, would likewise have been higher than three. This would have resulted in the practical ineffectiveness of the method. On the other hand, an increase in the order of the characteristic of the nonlinearity in equation (7.1) in applying the method of the preceding section would in no way have complicated the computations.

Examples may however be given for which the quasilinear treatment, in general, leads to erroneous results. Thus, in particular, we saw that if the equation (4.5) considered in sec. 4 is treated as quasilinear we arrive at the erroneous result of the absence for this equation of a periodic solution, which however actually exists and can be computed with the aid of the procedure there indicated. We shall now, with the aid of the methods developed in this chapter, determine for a system of the type (4.5) the subharmonic oscillations and shall then show that these oscillations cannot be determined with the aid of the quasilinear theory.

We shall consider the equation

$$\frac{d^2x}{dt^2} + k^2x - \beta x^2 = \mu a \cos nt \tag{8.1}$$

and compute for it the solution $x^{(1)}$, which evidently determines the subharmonic oscillations of order 1/n.

The general solution of the generating equation

$$\frac{d^2\xi}{dt^2} + k^2\xi - \beta\xi^2 = 0$$

is given by the formulas (3.13) of the preceding chapter. We therefore have:

$$\xi^{(1)}(t+h) = \sum_{m=0}^{\infty} A_m \cos m(t+h),$$

where

$$A_0 = \frac{1}{2}\frac{\beta}{k^2}c_1^2 - \frac{1}{3}\frac{\beta^2}{k^4}c_1^3 + \ldots,$$

$$A_1 = c_1 - \frac{1}{3}\frac{\beta}{k^2}c_1^2 + \frac{29}{144}\frac{\beta^2}{k^4}c_1^3 + \ldots,$$

$$A_2 = -\frac{1}{6}\frac{\beta}{k^2}c_1^2 + \frac{1}{9}\frac{\beta^2}{k^4}c_1^3 + \ldots,$$

$$A_3 = \frac{1}{48}\frac{\beta^2}{k^4}c_1^3 + \ldots$$

For the period T of the free oscillations we have:

$$T = \frac{2\pi}{k}(1 + h_2 c^2 + h_3 c^3 + \ldots),$$

$$h_2 = \frac{5}{12}\frac{\beta^2}{k^4}, \quad h_3 = -\frac{5}{18}\frac{\beta^3}{k^6},$$

and therefore the equation determining c_1 has the form

$$\frac{5}{12}\frac{\beta^2}{k^4}c_1^2 - \frac{5}{18}\frac{\beta^3}{k^6}c_1^3 + \ldots = k - 1.$$

This equation determines two real values of c_1 if the condition is satisfied

$$k - 1 > 0, \tag{8.2}$$

these values being equal in magnitude and of opposite sign.

We shall restrict ourselves to determining only the generating solution, for which it is necessary for us to compute only the value of the constant h. For this magnitude we have the equation

$$P(h) = -a\int_0^{2\pi} \cos nt \sum_{m=0}^{\infty} A_m m \sin m(t+h)\,dt \equiv -an\pi A_n \sin nh = 0,$$

from which we find for it the two values

$$h_1 = 0, \; h_2 = \pi/n$$

To each of these values of h there actually corresponds a periodic solution $\{x^{(1)}\}$ since

$$\left(\frac{dP}{dh}\right)_{h=h_1} = -an^2\pi A_n \neq 0, \quad \left(\frac{dP}{dh}\right)_{h=h_2} = an^2\pi A_n \neq 0. \quad (8.3)$$

Since we have two different values for c_1, we thus obtain four different solutions $\{x^{(1)}\}$, i.e. for equation (8.1), with condition (8.2) satisfied, there exist four forms of subharmonic oscillations of order $1/n$ for any n.

For the magnitude K, determined by formula (6.22) of chapter VI, we now have:

$$K = -\frac{k(k^2-\beta c_1)}{2h_2+3h_3c_1+\ldots}.$$

From this, taking (8.3) into account, we see that the second of the conditions of stability (7.23) is satisfied (with the inequality sign) only for two of the solutions $\{x^{(1)}\}$. As to the first of conditions (7.23), it is always satisfied with the equality sign. Thus, of the four subharmonic oscillations found two are stable and two unstable.

Let us now try to solve the same problem with the aid of the quasilinear theory. For this it is necessary to put $\beta = \mu\beta'$ and consider the equation (8.1) for the values of k near unity, i.e. it is necessary to put also $k^2 = 1 - \mu b$. It is not necessary however to assume that the amplitude of the disturbing force has the order of smallness of μ. Thus, with the quasilinear treatment we must consider an equation of the form

$$\frac{d^2x}{dt^2} + x = a\cos nt + \mu(bx + \beta' x^2).$$

Setting

$$x = \frac{a}{1-n^2}\cos nt + M_0\cos t + N_0 \sin t + \mu x_1 + \mu^2 x_2 + \ldots,$$

we obtain for x_1 the following equation:

$$\frac{d^2x_1}{dt^2} + x_1 = \frac{1}{2}\beta'\left(M_0^2 + N_0^2 + \frac{a^2}{(1-n^2)^2}\right) + bM_0\cos t +$$
$$+ bN_0\sin t + \frac{\beta' a}{1-n^2}M_0\cos(n-1)t + \frac{\beta' a}{1-n^2}N_0\sin(n-1)t + \ldots,$$

where the terms not written down contain the cosines and sines of the arguments 2t, nt, (n + 1)t, 2nt. The equations determining M_o and N_o will therefore either have the form

$$P(M_0, N_0) = bM_0 = 0, \quad Q(M_0, N_0) = bN_0 = 0,$$

if n $\neq$ 0, or the form

$$P(M_0, N_0) = \left(b - \frac{\beta' a}{3}\right) M_0 = 0, \quad Q(M_0, N_0) = \left(b - \frac{\beta' a}{3}\right) N_0 = 0,$$

if n = 2. In either case these equations will have solutions different from the trivial solution $M_o = N_o = 0$ only in case the condition is satisfied

$$\frac{\partial(P, Q)}{\partial(M_0, N_0)} \equiv 0.$$

But with this condition satisfied the methods discussed in chapter I are inapplicable. Thus, at least with the aid of the general procedures of the quasilinear theory, we are not able to determine the subharmonic oscillations for the system considered. With the quasilinear treatment the problem requires special investigation since it belongs to the class of singular cases.[1]

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[1] All the computations in this section are taken from the graduate thesis of the student V. M. Mendelson.

AUTHOR INDEX [1]

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[1] The parentheses indicate alternative transcriptions of the names of the more familiar authors that appear in the literature -Translator

SUBJECT INDEX