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AN ALTERNATIVE ANALYSIS OF DUFFING'S EQUATION*

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Abstract. An analysis of the harmonic solutions of the Duffing equation can be accomplished by means of either the perturbation method or the method of iteration. However, this paper purports to study Duffing's equation by using the method of van der Pol. In addition to providing an independent derivation of the response curves, this approach leads to a simplified discussion of the stability question. The variational equation takes the form of a Hill's equation (in fact, the Mathieu equation). Hence, in order to investigate stability by the variational method one must study the boundedness properties of solutions of Mathieu's equation. This need is obviated by using van der Pol's method, since, in this case, the stability issue can be decided by applying the method of Andronow and Witt. That is, harmonic solutions are identified with the singular points of a first-order system of ordinary differential equations. Stability is then determined by utilizing Poincaré's criteria for the classification of singularities of such systems.

Key words. Duffing's equation, van der Pol method, method of Andronow and Witt

AMS(MOS) subject classifications. 34C15, 34D10, 70K05, 70K20, 73D35

1. Introduction.

We seek to study qualitatively the harmonic solutions of Duffing's equation

$$\ddot{s} + c\dot{s} + (\omega_0^2 s + \beta s^3) = F\cos(\omega t), \qquad c \geq 0 \tag{1.1}$$

by using the method of van der Pol. The harmonic solutions are those periodic solutions with the frequency of the excitation, ω. This equation models the forced oscillation of a single mass subjected to an elastic restoring force and linear damping. Note the following special cases [1]:

- $F = 0$, free oscillation,
- $c = 0$, undamped oscillation,
- $\beta = 0$, linear oscillation,
- $\beta > 0$, hard spring,
- $\beta < 0$, soft spring.

The first step in van der Pol's method is to assume that c, β, F, and $\omega - \omega_0$ are all small and of the same order, say $O(\epsilon)$. That is, we are in the neighborhood of the linear, undamped, free vibration. In what follows, $\omega_0, \omega = O(1)$.

The second and crucial step is the assume that

$$s(t) = b_1(t)\sin(\omega t) + b_2(t)\cos(\omega t), \tag{1.2}$$

where b_1, b_2 are slowly varying functions. Periodicity is enforced by requiring b_1, b_2 to be constant. That is to say, $s(t)$ is essentially an oscillation of frequency ω with slowly varying phase and amplitude. We express this analytically by setting

$$b_i = O(1), \quad \dot{b}_i = O(\epsilon), \quad \ddot{b}_i = O(\epsilon^2), \quad (i = 1, 2). \tag{1.3}$$

We assume that possible higher harmonics are $O(\epsilon^2)$.

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Hence, we have

$$
\begin{aligned}
\dot{s}(t) &= \dot{b}_1 \sin(\omega t) + \dot{b}_2 \cos(\omega t) + \omega b_1 \cos(\omega t) - \omega b_2 \sin(\omega t), \\
\ddot{s}(t) &= 2\omega \dot{b}_1 \cos(\omega t) - 2\omega \dot{b}_2 \sin(\omega t) \\
&\quad -\omega^2 b_1 \sin(\omega t) - \omega^2 b_2 \cos(\omega t) + \ddot{b}_1 \sin(\omega t) + \ddot{b}_2 \cos(\omega t), \\
s^3(t) &= \tfrac{3}{4}(b_1^2 + b_2^2)(b_1 \sin(\omega t) + b_2 \cos(\omega t)) + \text{terms of frequency } 3\omega.
\end{aligned} \tag{1.4}
$$

Inserting these expressions into Duffing's equation, rejecting terms of order greater than one, and equating coefficients of $\sin(\omega t)$ and $\cos(\omega t)$ results in the first-order system

$$
\begin{aligned}
2\dot{b}_1 + \left[\frac{\omega_0^2 - \omega^2}{\omega} + \frac{3}{4}\frac{\beta b^2}{\omega}\right] b_2 + c b_1 &= \frac{F}{\omega}, \\
-2\dot{b}_2 + \left[\frac{\omega_0^2 - \omega^2}{\omega} + \frac{3}{4}\frac{\beta b^2}{\omega}\right] b_1 - c b_2 &= 0,
\end{aligned} \tag{1.5}
$$

where $b^2 = b_1^2 + b_2^2$.

The qualitative nature of the solutions of this system varies greatly depending upon the values of the parameters c and β. Hence, this system is best treated by separate consideration of the following cases:

- Case I: $c = 0, \beta = 0$,
- Case II: $c = 0, \beta > 0$,
- Case III: $c = 0, \beta < 0$,
- Case IV: $c > 0, \beta = 0$,
- Case V: $c > 0, \beta > 0$,
- Case VI: $c > 0, \beta < 0$.

In what follows, we define $\Delta = 2(\omega_0 - \omega)$, i.e., Δ is proportional to the detuning $(\omega_0 - \omega)$. It then follows from $(\omega_0 - \omega) = O(\epsilon)$ that

$$
\omega_0^2 - \omega^2 = (\omega_0 - \omega)(\omega_0 + \omega) = 2\omega(\omega_0 - \omega) + O(\epsilon^2), \tag{1.6}
$$

and hence $(\omega_0^2 - \omega^2)/\omega = \Delta$ to within first-order terms. Consequently, we rewrite the above system as

$$
\begin{aligned}
2\dot{b}_1 + \left[\Delta + \frac{3}{4}\frac{\beta b^2}{\omega}\right] b_2 + c b_1 &= \frac{F}{\omega}, \\
-2\dot{b}_2 + \left[\Delta + \frac{3}{4}\frac{\beta b^2}{\omega}\right] b_1 - c b_2 &= 0.
\end{aligned} \tag{1.7}
$$

Since we are primarily interested in harmonic solutions, we require that b_1, b_2 be constant, i.e., $\dot{b}_1 = \dot{b}_2 = 0$. Equation (1.7) then reduces to a system of algebraic equations for the response curves. This restriction to constant b_1, b_2 also allows the application of the method of Andronow and Witt. That is, the harmonic solutions of Duffing's equation are identified with the singular points of the above first-order system of ordinary differential equations (ODEs). Stability is then determined by utilizing Poincaré's criteria for the classification of singularities of such systems [2].

Below, we adhere to the notation

$$
\frac{d\eta}{d\xi} = \frac{A\xi + B\eta}{C\xi + D\eta}. \tag{1.8}
$$

2. Case I: $c = 0$, $\beta = 0$. In this section, we analyze the case of an undamped, linear oscillator. Define

$$\sigma = \Delta, \quad x = b_1, \quad y = b_2, \quad r^2 = x^2 + y^2, \quad \tau = \frac{t}{2}, \quad F_1 = \frac{F}{\omega}. \tag{2.1}$$

Equation (1.7) then becomes

$$\begin{aligned} \frac{dx}{d\tau} + \sigma y &= F_1, \\ -\frac{dy}{d\tau} + \sigma x &= 0. \end{aligned} \tag{2.2}$$

The response curves for this case appear in Fig. 1.

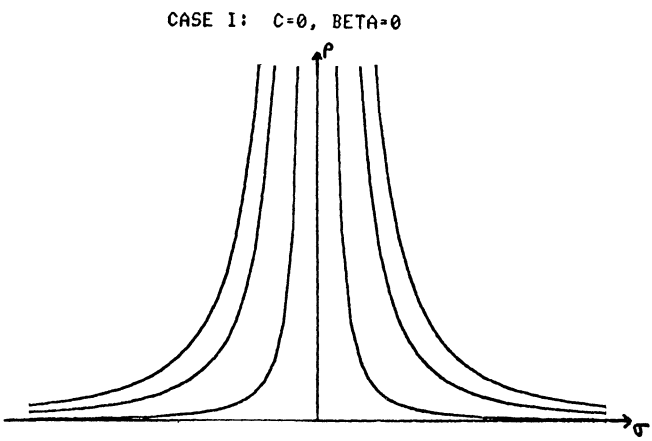

FIG. 1.

2.1. Response. At a singular point (x_0, y_0),

$$\begin{aligned} \sigma y_0 &= F_1, \\ \sigma x_0 &= 0. \end{aligned} \tag{2.3}$$

Letting $\rho = x_0^2 + y_0^2$, we deduce that

- $F_1 = 0$ (free vibration) $\Rightarrow$ either $\rho = 0$ or $\sigma = 0$,
- $F_1 \neq 0$ (forced vibration) $\Rightarrow$ $\rho = F_1^2/\sigma^2$.

2.2. Stability.

$$\frac{dy}{dx} = \frac{\sigma x}{F_1 - \sigma y}, \tag{2.4}$$

$$x = x_0 + \xi, \qquad y = y_0 + \eta \Rightarrow, \tag{2.5}$$

$$\frac{d\eta}{d\xi} = \frac{(\sigma)\xi + (0)\eta}{(0)\xi + (-\sigma)\eta} \Rightarrow, \tag{2.6}$$

$$\begin{aligned} (B - C)^2 + 4AD &= -4\sigma^2, \\ AD - BC &= -\sigma^2, \\ B + C &= 0. \end{aligned} \tag{2.7}$$

Hence, all singular points are centers since linearization was not necessary to obtain (2.6). All oscillations are stable in this case.

3. Case II: $c = 0$, $\beta > 0$. In this section, we analyze the case of an undamped, hard spring. Define $a_0^2 = 1/\left(\frac{3}{4}|\beta|\right)$. Then (1.7) can be rewritten as

$$
\begin{aligned}
2\dot{b}_1 + \left[\Delta \pm \frac{b^2}{a_0^2}\right] b_2 &= \frac{F}{\omega}, \\
-2\dot{b}_2 + \left[\Delta \pm \frac{b^2}{a_0^2}\right] b_1 &= 0,
\end{aligned} \tag{3.1}
$$

where the ambiguous sign is chosen to coincide with that of β. Letting

$$\sigma = \Delta, \quad x = \frac{b_1}{a_0}, \quad y = \frac{b_2}{a_0}, \quad r^2 = x^2 + y^2, \quad \tau = \frac{t}{2}, \quad F_1 = \frac{F}{(\omega a_0)}, \tag{3.2}$$

the above system becomes

$$
\begin{aligned}
\frac{dx}{d\tau} + (\sigma \pm r^2)y &= F_1, \\
-\frac{dy}{d\tau} + (\sigma \pm r^2)x &= 0.
\end{aligned} \tag{3.3}
$$

The response curves for this case appear in Fig. 2. The regions of stability and instability are therein depicted.

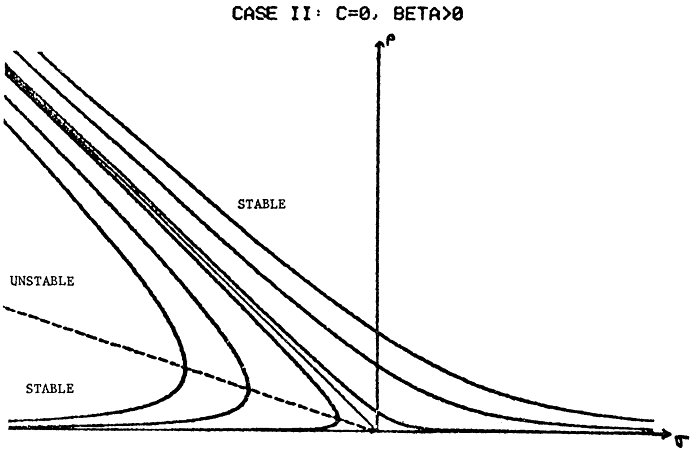

FIG. 2.

3.1. Response. At a singular point (x_0, y_0),

$$
\begin{aligned}
(\sigma \pm \rho)y_0 &= F_1, \\
(\sigma \pm \rho)x_0 &= 0.
\end{aligned} \tag{3.4}
$$

- $F_1 = 0$ (free vibration) $\Rightarrow$ either $\rho = 0$ or $\rho = \mp\sigma$,
- $F_1 \neq 0$ (forced vibration) $\Rightarrow$ $\rho = F_1^2/(\sigma \pm \rho)^2$.

Hence, for $\beta > 0$ we have

$$\rho(\sigma + \rho)^2 = F_1^2. \tag{3.5}$$

For F_1^2 small and $\rho \approx 0$ we have $\rho \approx F_1^2/\sigma^2$. For F_1^2 small and $\rho \approx 1$ we have $\rho \approx -\sigma \pm F_1$, depending on whether we are above or below the line $\rho = -\sigma$.

We obtain the locus of vertical tangencies as follows [3]:

$$\frac{d}{d\rho}[\rho(\sigma+\rho)^2]=0\Rightarrow, \tag{3.6}$$

$$(\sigma+\rho)^2+\rho\left[2(\sigma+\rho)\left(\frac{d\sigma}{d\rho}+1\right)\right]=0, \tag{3.7}$$

$$\frac{d\sigma}{d\rho}=0\Rightarrow\rho=-\frac{1}{3}\sigma. \tag{3.8}$$

3.2. Stability.

$$\frac{dy}{dx}=\frac{(\sigma\pm r^2)x}{F_1-(\sigma\pm r^2)y}, \tag{3.9}$$

$$x=x_0+\xi,\qquad y=y_0+\eta\Rightarrow r^2\approx\rho+2x_0\xi+2y_0\eta\Rightarrow, \tag{3.10}$$

$$\frac{d\eta}{d\xi}=\frac{(\sigma\pm(2x_0^2+\rho))\xi+(\pm 2x_0y_0)\eta}{(\mp 2x_0y_0)\xi+(-\sigma\mp(2y_0^2+\rho))\eta}\Rightarrow, \tag{3.11}$$

$$\begin{gathered}(B-C)^2+4AD=-4(3\rho\pm\sigma)(\rho\pm\sigma),\\ AD-BC=-(3\rho\pm\sigma)(\rho\pm\sigma),\\ B+C=0.\end{gathered} \tag{3.12}$$

The region above the line $\rho=-\sigma/3$, but below the line $\rho=-\sigma$ is seen to be composed of saddles and is hence a region of instability. However, the remaining area, where $(B-C)^2+4AD<0$, poses a bit of a problem. This is due to the fact that $B+C=0$ there, and hence Poincaré's theory breaks down. That is, we cannot conclude that a center of the linearized system corresponds to a center of the nonlinear system. We are, however, guaranteed that it corresponds to either a center or a spiral.

To settle this issue, we must once again study (3.9) in the neighborhood of the singular point but this time we must retain quadratic terms. This yields

$$\frac{d\eta}{d\xi}=\frac{\sigma\xi\pm(\rho\xi+2y_0\xi\eta+\xi^3+\xi\eta^2)}{-[\sigma\eta\pm(3\rho\eta+y_0\xi^2+3y_0\eta^2+\eta\xi^2+\eta^3)]}. \tag{3.13}$$

It is prudent to introduce polar coordinates at this point. Redefining $r^2=\xi^2+\eta^2$, we have

$$\begin{aligned}\frac{dr}{dt}&=\frac{1}{r}\left(\xi\frac{d\xi}{dt}+\eta\frac{d\eta}{dt}\right)\\ \frac{d\theta}{dt}&==\frac{1}{r^2}\left(\xi\frac{d\eta}{dt}-\eta\frac{d\xi}{dt}\right)\end{aligned}\quad\Rightarrow \tag{3.14}$$

(3.15)

$$\frac{dr}{dt}=-\frac{1}{r}(y_0\xi^3+\eta^2\xi y_0+2\xi\eta y_0^2)$$
$$\frac{d\theta}{dt}=\frac{1}{r^2}[\xi^2(\sigma\pm\rho)+\eta^2(\sigma\pm3\rho)\pm(\eta^4+\xi^4)+\xi^2\eta(\pm3y_0)+\xi^2\eta^2(\pm2)+\eta^3(\pm3y_0)].$$

The following observations should be made. The case that we are trying to study, i.e., $\sigma+\rho$ and $\sigma+3\rho$ of the same sign, is seen to be a center. This follows directly from the fact that (3.13) is odd in ξ and hence cannot have a spiral as an isolated singularity. Moreover, our previous conclusion that the region bounded by $\rho = -\sigma$ and $\rho = -\sigma/3$ was composed of saddles is again confirmed. In this case, as in general, dr/dt is odd in ξ, while for small r we have

$$\frac{d\theta}{dt} \approx \frac{\xi^2(\sigma \pm \rho) + \eta^2(\sigma \pm 3\rho)}{r^2}. \tag{3.15}$$

Hence, $d\theta/dt$ changes sign, suggestive of a saddle.

In summary,

- above $\rho = -\sigma \Rightarrow$ centers (stable),
- below $\rho = -\sigma/3 \Rightarrow$ centers (stable),
- between $\rho = -\sigma/3$ and $\rho = -\sigma \Rightarrow$ saddles (unstable).

Note that $\rho = -\sigma$, corresponding to the free vibration, is unstable being on the boundary of a stable and an unstable region. It should be noted that the locus of vertical tangencies, $\rho = -\sigma/3$, represents the transition from stability to instability as y_0 increases in magnitude through negative values.

4. Case III: $c = 0, \beta < 0$. In this section, we analyze the case of an undamped, soft spring. A quick perusal of Case II will convince us that Case III is merely a reflection of Case II about the ρ-axis. This situation is in stark contrast to the traditional treatment of the Duffing equation and represents a considerable advantage over such treatments. The response curves for this case appear in Fig. 3. The regions of stability and instability are therein depicted.

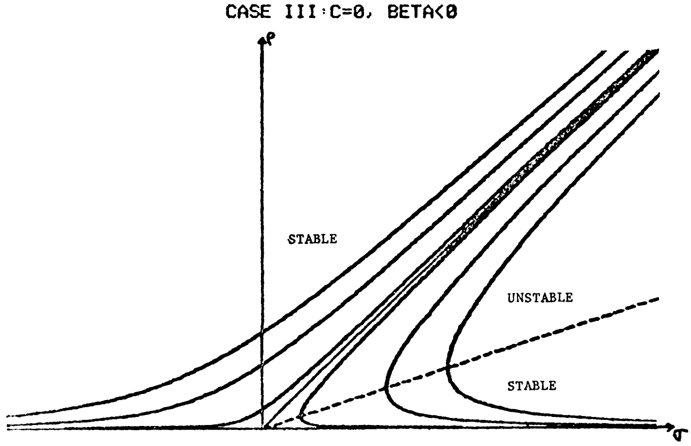

FIG. 3.

5. Case IV: $c > 0, \beta = 0$. In this section, we analyze the case of a damped, linear oscillator. Letting

$$\sigma = \frac{\Delta}{c}, \quad x = b_1, \quad y = b_2, \quad r^2 = x^2 + y^2, \quad \tau = \frac{ct}{2}, \quad F_1 = \frac{F}{\omega c}, \tag{5.1}$$

equation (1.7) becomes

$$\begin{aligned} \frac{dx}{d\tau} + \sigma y + x &= F_1, \\ -\frac{dy}{d\tau} + \sigma x - y &= 0. \end{aligned} \tag{5.2}$$

The response curves for this case appear in Fig. 4.

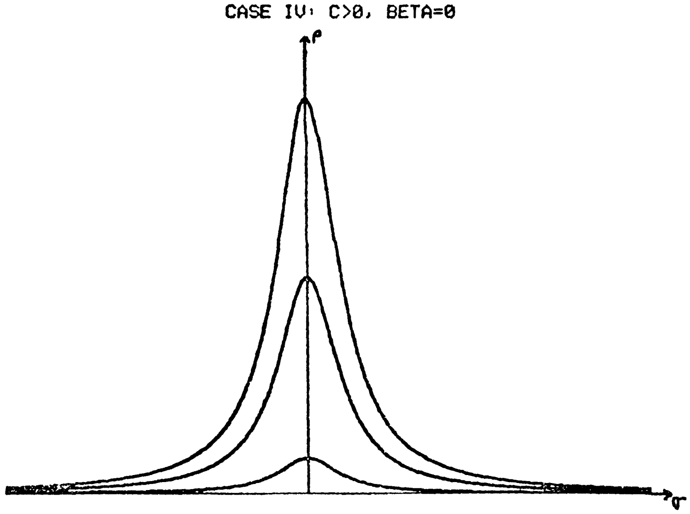

FIG. 4.

5.1. Response. At a singular point (x_0, y_0),

$$\begin{aligned} \sigma y_0 + x_0 &= F_1, \\ \sigma x_0 - y_0 &= 0, \end{aligned} \tag{5.3}$$

- $F_1 = 0$ (free vibration) $\Rightarrow \rho = 0$,
- $F_1 \neq 0$ (forced vibration) $\Rightarrow \rho = F_1^2/(\sigma^2 + 1)$.

5.2. Stability.

$$\frac{dy}{dx} = \frac{\sigma x - y}{F_1 - \sigma y - x}, \tag{5.4}$$

$$x = x_0 + \xi,\ y = y_0 + \eta \Rightarrow, \tag{5.5}$$

$$\frac{d\eta}{d\xi} = \frac{(\sigma)\xi + (-1)\eta}{(-1)\xi + (-\sigma)\eta} \Rightarrow, \tag{5.6}$$

$$\begin{aligned} (B - C)^2 + 4AD &= -4\sigma^2, \\ AD - BC &= -1 - \sigma^2, \\ B + C &= -2. \end{aligned} \tag{5.7}$$

Hence, all the oscillations correspond to stable spirals.

6. Case V: $c > 0$, $\beta > 0$. In this section, we analyze the case of a damped, hard spring. Letting

$$\sigma = \frac{\Delta}{c}, \quad x = \frac{b_1}{a_0}, \quad y = \frac{b_2}{a_0}, \quad r^2 = x^2 + y^2, \quad \tau = \frac{ct}{2}, \quad F_1 = \frac{F}{\omega c}, \tag{6.1}$$

equation (1.7) becomes

$$\begin{aligned} \frac{dx}{d\tau} + (\sigma \pm r^2)y + x &= F_1, \\ -\frac{dy}{d\tau} + (\sigma \pm r^2)x - y &= 0. \end{aligned} \tag{6.2}$$

The response curves for this case appear in Fig. 5. The regions of stability and instability are therein depicted.

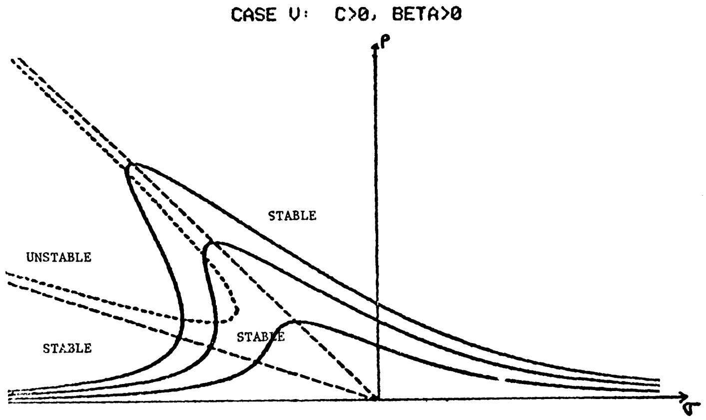

FIG. 5.

6.1. Response. At a singular point (x_0, y_0),

$$(\sigma \pm \rho)y_0 + x_0 = F_1, \quad (\sigma \pm \rho)x_0 - y_0 = 0, \tag{6.3}$$

- $F_1 = 0$ (free vibration) $\Rightarrow \rho = 0$,
- $F_1 \neq 0$ (forced vibration) $\Rightarrow \rho = F_1^2/[(\sigma \pm \rho)^2 + 1]$.

Hence, we have

$$\rho[(\sigma \pm \rho)^2 + 1] = F_1^2. \tag{6.4}$$

We obtain the locus of vertical tangencies as follows [3]:

$$\frac{d}{d\rho}\left\{\rho[(\sigma \pm \rho)^2 + 1]\right\} = 0 \Rightarrow, \tag{6.5}$$

$$[(\sigma \pm \rho)^2 + 1] + \rho\left[2(\sigma \pm \rho)\left(\frac{d\sigma}{d\rho} \pm 1\right)\right] = 0, \tag{6.6}$$

$$\frac{d\sigma}{d\rho} = 0 \Rightarrow (\sigma \pm 3\rho)(\sigma \pm \rho) + 1 = 0. \tag{6.7}$$

This is the equation of a hyperbola with asymptotes of

$$\rho = \mp\sigma, \qquad \rho = \mp\sigma/3, \tag{6.8}$$

the choice of sign agreeing with that of β.

We obtain the locus of horizontal tangencies as follows [3]:

$$\frac{d}{d\sigma}\left\{\rho[(\sigma \pm \rho)^2 + 1]\right\} = 0 \Rightarrow, \tag{6.9}$$

$$\frac{d\rho}{d\sigma}[(\sigma \pm \rho)^2 + 1] + \rho\left[2(\sigma \pm \rho)\left(1 \pm \frac{d\rho}{d\sigma}\right)\right] = 0, \tag{6.10}$$

$$\frac{d\rho}{d\sigma} = 0 \Rightarrow \quad \text{either } \rho = 0, \quad \text{or} \quad \rho = \mp\sigma, \tag{6.11}$$

the choice of sign agreeing with that of β.

6.2. Stability.

$$\frac{dy}{dx} = \frac{(\sigma \pm r^2)x - y}{F_1 - (\sigma \pm r^2)y - x}, \tag{6.12}$$

$$x = x_0 + \xi,\ y = y_0 + \eta \Rightarrow r^2 \approx \rho + 2x_0\xi + 2y_0\eta \Rightarrow, \tag{6.13}$$

$$\frac{d\eta}{d\xi} = \frac{(\sigma \pm (2x_0^2 + \rho))\xi + (-1 \pm 2x_0y_0)\eta}{(-1 \mp 2x_0y_0)\xi + (-\sigma \mp (2y_0^2 + \rho))\eta} \Rightarrow, \tag{6.14}$$

$$\begin{aligned} (B - C)^2 + 4AD &= -4(\sigma \pm 3\rho)(\sigma \pm \rho), \\ AD - BC &= -(\sigma^2 \pm 4\sigma\rho + 3\rho^2 + 1), \\ B + C &= -2. \end{aligned} \tag{6.15}$$

Hence, the following situation prevails:
- above $\rho = -\sigma \Rightarrow$ stable spirals,
- below $\rho = -\sigma/3 \Rightarrow$ stable spirals,
- between $\rho = -\sigma/3$ and $\rho = -\sigma$ but "outside" hyperbola $\Rightarrow$ stable nodes,
- between $\rho = -\sigma/3$ and $\rho = -\sigma$ but "inside" hyperbola $\Rightarrow$ saddles (unstable).

Note that, once again, the locus of vertical tangencies corresponds to transition from stability to instability.

7. Case VI: $c > 0$, $\beta < 0$. In this section, we analyze the case of a damped, soft spring. As before, this case is seen upon inspection to be the reflection of Case V about the ρ-axis. The response curves for this case appear in Fig. 6. The regions of stability and instability are therein depicted.

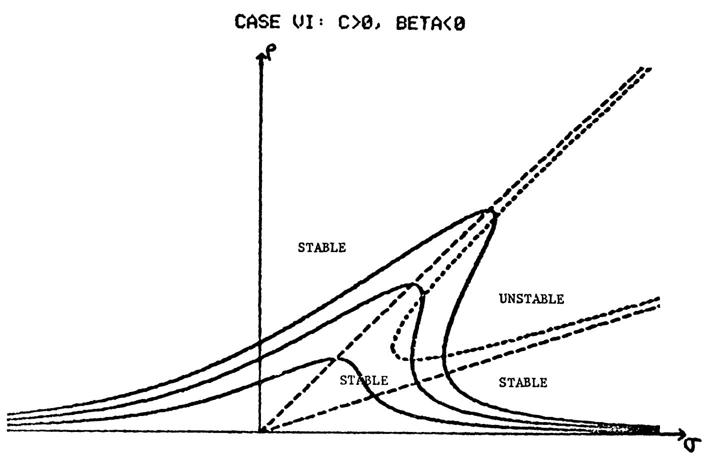

FIG. 6.

8. Conclusion. It has been shown that Duffing's equation can be successfully analyzed in great detail by the method of van der Pol and the method of Andronow and Witt. We were required to assume that certain parameters in the differential equation were small and that the detuning was also small. These requirements are in fact more restrictive than those for alternative methods of analysis. However, within the stated assumptions, this method has two very attractive features. The first is simplicity, this of course being largely a matter of personal taste. The second is the symmetric roles played by $\beta > 0$ and $\beta < 0$, which significantly reduces the overall analytical burden.

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