International Journal of Mathematical Education in Science and Technology, Vol. 37, No. 6, 15 September 2006, 707–720

Taylor & Francis
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The forced soft spring equation

T. H. FAY*
Tswane University of Technology Pretoria, South Africa

(*Received 15 January 2003*)

Through numerical investigations, this paper studies examples of the forced Duffing type spring equation

$$\ddot{x} + k\dot{x} + x + \varepsilon x^3 = F\cos\omega t$$

with ε negative. By performing trial-and-error numerical experiments, the existence is demonstrated of stability boundaries in the phase plane indicating initial conditions yielding bounded solutions. Subharmonic boundaries are discovered consisting of points in the phase plane giving rise to subharmonic solutions of fixed order, the existence of which appears to be new. With very small damping, subharmonic solutions may exist and in all cases the damped equations exhibit steady state trajectories identical with the harmonic trajectory and this harmonic solution trajectory is a limit cycle. Student investigations and computer laboratory research problems arise naturally and several are suggested.

1. Introduction

When a differential equation has a bounded solution is an important question, and when the bounded solution is periodic has even more practical importance. Finding a periodic solution to a nonlinear ordinary differential equation is in general a difficult task and only in a very few cases can direct methods be applied to an equation to find initial values leading to a solution that is periodic. Recently a technique was presented in this journal [1] (see also [2]) for finding periodic solutions having the same period as the forcing part of the equation, the so called harmonic solution; and in [3] a similar harmonic balance approximation technique is developed for finding periodic solutions having fundamental period three times that of the forcing, subharmonic solutions of order 1/3. In this article, we continue these investigations.

Duffing's equation, introduced in 1918 [4],

$$\ddot{x} + k\dot{x} + x + \varepsilon x^3 = F\cos\omega t \tag{1.1}$$

has become one of the most studied nonlinear equations: it is only mildly nonlinear and embodies much of the bad behaviour one can expect from a nonlinear equation (sensitivity to initial conditions, unbounded solutions) as well as considerable pleasing behaviour as we will illustrate. Duffing's equation has many solutions

*E-mail: thfay@hotmail.com

International Journal of Mathematical Education in Science and Technology
ISSN 0020–739X print/ISSN 1464–5211 online
http://www.tandf.co.uk/journals
DOI: 10.1080/00207390600990994

which are unbounded and recently the notion of nonlinear resonance, as discussed in Davis [5], was explored in [6].

The model of mechanical or electronic oscillations, typically that of a suspended spring with weight attached, is commonly presented in beginning courses on differential equations and the linear case is completely understood. The nonlinear case is more interesting as the range of resultant motions is much wider. By thinking this way when considering equation (1.1), we may interpret x as displacement, $\dot{x}$ as velocity, $\ddot{x}$ as acceleration (as is customary the 'dots' represent differentiation with respect to t) and t is thought of as time. The term $k\dot{x}$ represents viscous damping and $x + \varepsilon x^3$ represents a nonlinear restoring force. It is merely a matter of re-scaling the time variable to assure the coefficient of x is 1. We call $F \cos \omega t$ the (external) forcing. This model is called 'soft' when ε is negative and the associated homogeneous equation has three real critical values – two saddle points and a centre. For more information on hard and soft spring equations see [7].

A solution to (1.1) is called *harmonic* provided it is periodic with fundamental period $P = 2\pi/\omega$; a solution is called *subharmonic* provided it is periodic with fundamental period an integral multiple of the forcing period P. If the fundamental period of the solution is nP where n is a positive integer greater than 1, then the solution is called a *subharmonic solution of order* $1/n$.

The second-order harmonic balance approach used in [3] and [1] is limited and so we rely upon numerical explorations to study subharmonic solutions. In doing so, we discover the existence of subharmonic boundaries in the phase plane consisting of initial conditions giving rise to subharmonic solutions of a fixed order.

The behaviour in the damped case with respect to the nature of the steady state solutions is interesting as the steady state trajectory equals the harmonic solution trajectory and thus there is a limit cycle; in the hard spring ($\varepsilon > 0$) the steady state has period P but its trajectory and amplitude depend upon the initial displacement $x(0)$.

All initial value problems in this article were solved using *Mathematica*'s **ND-Solve** routine (for details see [8]). Solving the second-order initial value problem

$$\begin{aligned} &\ddot{x} + k\dot{x} + x + \varepsilon x^3 = F \cos \omega t \\ &x(0) = a, \;\; \dot{x}(0) = b \end{aligned}$$

is usually accomplished by solving the first-order system

$$\begin{aligned} &\dot{x} = y \\ &\dot{y} = -kx - x - \varepsilon x^3 + F \cos \omega t \\ &x(0) = a, \;\; y(0) = b \end{aligned}$$

However, for all initial value problems in this article, we solved the equivalent system

$$\begin{gathered} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -kz - y - 3\varepsilon x^2 y - \omega F \sin \omega t \\ x(0) = a, \;\; y(0) = b, \; z(0) = -kb - a - \varepsilon a^3 + F \end{gathered}$$

which produces equivalent and often superior numerical results and permits one to test the accuracy of the solution by 'substitution' as $z = \ddot{x}$. For more details on this procedure see [9].

We shall exhibit many interesting phenomena with our examples, many of which give rise to research problems suitable for students and computer laboratory research investigations.

2. The undamped case

In this section we investigate examples of

$$\ddot{x} + x + \varepsilon x^3 = F\cos\omega t \tag{2.1}$$

where ε is negative and small. We first limit our search for initial conditions leading to bounded solutions by determining (by trial-and-error) a stability boundary in the $x\dot{x}$-phase plane. Points within the boundary are initial conditions which lead to bounded solutions.

2.1. *The stability boundary*

The first example is one used in [1] and [6].

Example 2.1: Consider the equation

$$\ddot{x} + x - \frac{1}{6}x^3 = \frac{1}{3}\cos\frac{3}{5}t \tag{2.2}$$

The stability boundary shown in figure 1 was determined by trial-and-error to determine when a solution was bounded and when it was unbounded. A point (a, b) within this boundary gives rise to a bounded solution when solving equation (2.2) with initial conditions $x(0) = a, \dot{x}(0) = b$; points on or outside this boundary give rise to unbounded solutions. Note that the boundary is symmetric with respect to the x-axis.

The next example is used in [3] to study subharmonic solutions of order 1/3. It has a nice stability boundary.

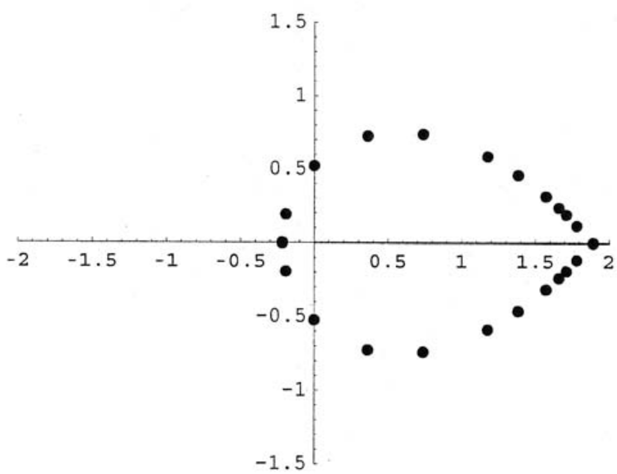

Figure 1. The stable/unstable boundary for $\ddot{x} + x - \dfrac{x^3}{6} = \dfrac{1}{3}\cos\dfrac{3}{5}$t.

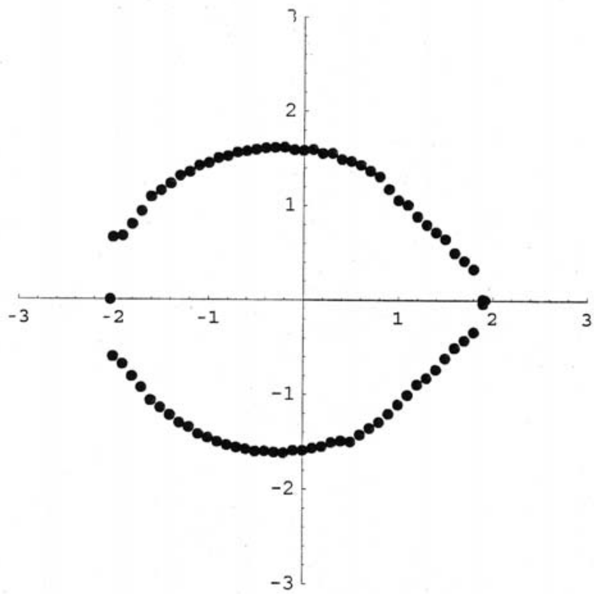

Figure 2. Stability boundary for example 2.2.

Example 2.2: Consider the equation

$$\ddot{x} + x - \frac{1}{6}x^3 = \frac{3}{2}\cos 2.85t \tag{2.3}$$

The stability boundary is shown in figure 2.

Stability boundaries for other examples are shown in figure 3. Not all the stability boundaries have the same appearance.

Example 2.3: Consider the equation

$$\ddot{x} + x - \frac{1}{6}x^3 = \frac{1}{3}\cos 0.76t \tag{2.4}$$

This example was used in [1]; the stability boundary is shown in figure 4.

2.2. *Subharmonic boundaries*

Example 2.1 was also used in [1] when investigating subharmonics of order 1/3. This example also produces many examples of subharmonic solutions of order 1/2 which are not supposed to exist [11, p. 258]. In figure 5 we plot the trajectories of two subharmonics for example 2.1. The first is a subharmonic of order 1/2 ($x(0) = 1.259$, $\dot{x}(0) = -0.311$) and the second is a subharmonic of order 1/3 ($x(0) = -0.2135$, $\dot{x}(0) = 0$). Both solutions are very stable and both plots are over the time interval [0, 100] representing nearly 5 periods and 3 periods respectively.

A numerical search, again by trial-and-error, produced a boundary shown in figure 6; points on this boundary give rise to subharmonics of order 1/2 and thus it appears that there are infinitely many distinct subharmonic solutions of order 1/2.

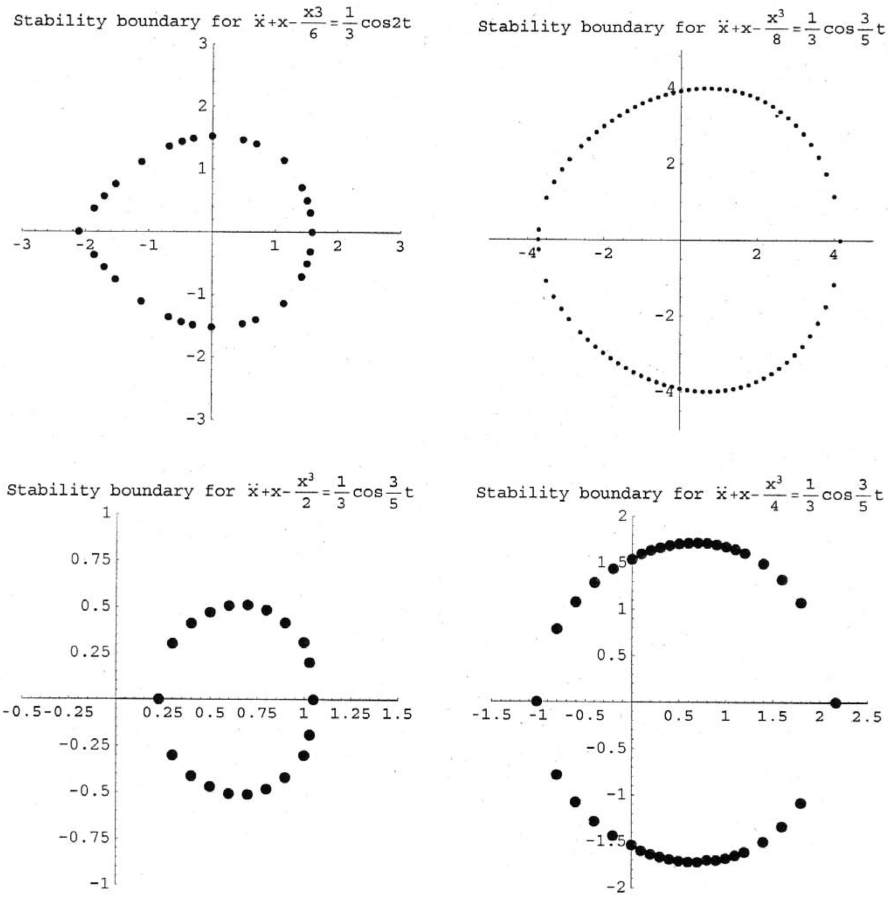

Figure 3. Stability boundaries.

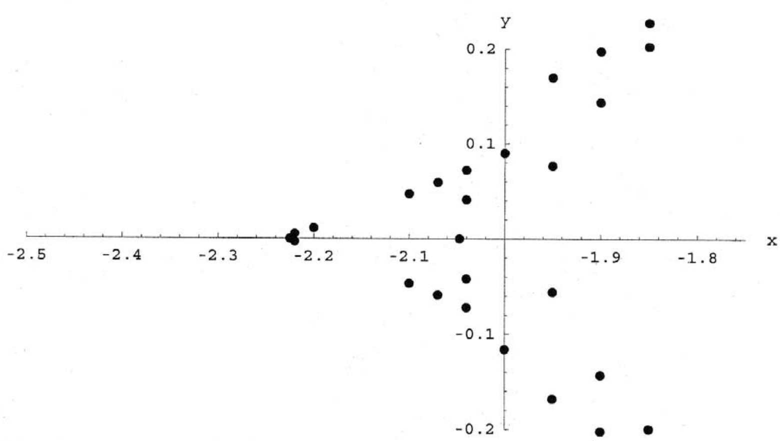

Figure 4. The stability boundary for example 2.3.

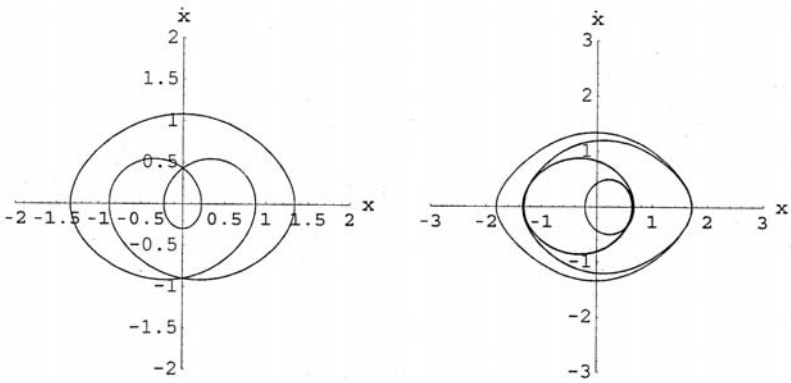

Figure 5. Subharmonic trajectories of orders 1/2 and 1/3 for example 2.1.

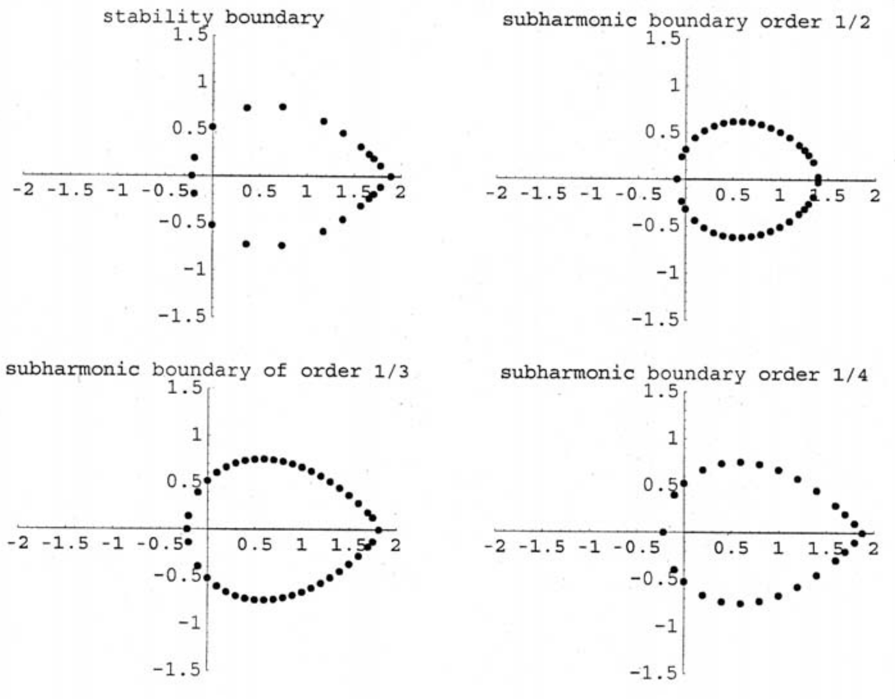

Figure 6. Stability boundary and subharmonic boundaries for example 2.1.

Based upon this success, we sought subharmonic solutions of order 1/3 and 1/4 and found the subharmonic boundaries shown in figure 6. These subharmonic boundaries turn out to be symmetric with respect to the x-axis and are nested within the stability boundary. The smaller the order of the subharmonic the closer the subharmonic boundary is to the stability boundary. Subharmonic solutions of order 1/5 exist for this example and presumably a subharmonic boundary exists for these solutions as well but this boundary would be very close to the stability boundary with the points differing in the fourth or possibly higher decimal place.

Student problem 1: Determine the subharmonic boundary for subharmonic solutions of order 1/5 for example 2.1.

Student problem 2: Determine the subharmonic boundaries of order 1/2, 1/3, and 1/4, if they exist for examples 2.2 and 2.3 and the examples shown in figure 3.

3. The damped case

In this section we investigate the damped Duffing equation

$$\ddot{x} + k\dot{x} + x + \varepsilon x^3 = F\cos\omega t \tag{3.1}$$

The positive damping parameter k plays an important role and we will see that the behaviour of solutions to some extent depends upon the size of k.

Example 3.1: Consider the equation

$$\ddot{x} + k\dot{x} + x - \frac{1}{6}x^3 = \frac{1}{3}\cos\frac{3}{5}t \tag{3.2}$$

where $k = 0.24$.

Using the harmonic balance technique in [1], initial conditions for the harmonic solution are

$$x(0) = 0.519674 \qquad \dot{x}(0) = 0.072267$$

Its trajectory is shown in figure 7.

With damping we expect to have transient terms affecting the trajectory for small values of t and for the trajectory to eventually stabilize in a 'steady state'. A typical trajectory is shown in figure 7 for initial conditions $x(0) = 1.0$ and $\dot{x}(0) = 0$. However, not all solutions are bounded and consequently we need to determine the stability boundary.

3.1. *The stability boundary*

The stability boundary in the damped case is quite different from that of the undamped case. In figure 8 we show the stability boundary for example 3.1. Note the 'wings' extending upper left and lower right. Points within the boundary lead

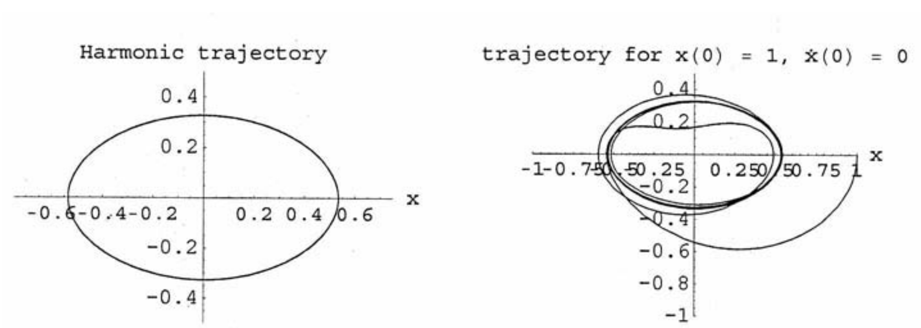

Figure 7. Harmonic trajectory and a typical tracjectory for example 3.1.

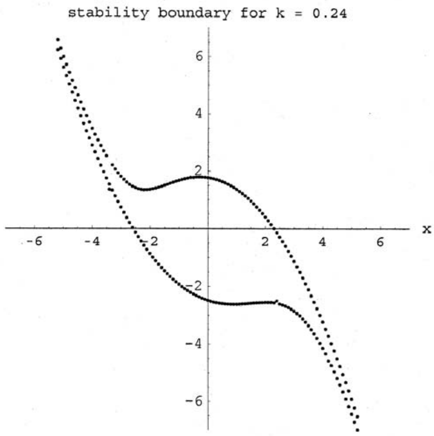

Figure 8. The stability boundary for example 3.1.

to bounded solutions and trajectories, while points on or outside the boundary lead to unbounded solutions and trajectories.

Example 3.2: Consider the equation

$$\ddot{x} + k\dot{x} + x - \frac{1}{6}x^3 = \frac{1}{3}\cos\frac{3}{5}t. \tag{3.3}$$

where $k = 0.002$, a very small parameter.

This example has a very stable harmonic solution for $x(0) = 0.55404958$ and $\dot{x}(0) = 0.0011051$; its trajectory is shown in the upper left of figure 9. There is a near subharmonic of order 1/2 for $x(0) = 1.0$ and $x(0) = -0.531$. The trajectory for the time interval $[0, 2P]$ is shown in the upper right in figure 9. The lower figures in figure 9 show the same trajectory only plotted over the intervals [0, 100] and [0, 300]. Since the damping coefficient is so small, the trajectory shown in the lower right has yet to stabilize at a steady state for t near 300.

With the damping coefficient so much reduced and very small, it should not be a surprise that the stability boundary has changed. The stability boundary for this example is shown in figure 10. Note that there is only one 'wing', which is relatively small. The slightly enlarged dot within the boundary is the initial condition for the harmonic solution.

Example 3.3: Consider the equation

$$\ddot{x} + k\dot{x} + x - \frac{1}{6}x^3 = \frac{3}{2}\cos 2.85t, \tag{3.4}$$

where $k = 0.002$.

This example was used in [11] (see also [3]) to demonstrate the existence of subharmonics of order 1/3 for a damped equation. In figure 11 we show the harmonic trajectory determined by $x(0) = -0.210436$ and $\dot{x}(0) = 0$. Also shown are subharmonics of order 1/3 determined by $(x(0), \dot{x}(0)) = (0.7344, 0)$ and $(x(0), \dot{x}(0)) = (-0.9422, 0)$. A typical trajectory for $(x(0), \dot{x}(0)) = (-1.5, 0.5)$ is shown as well.

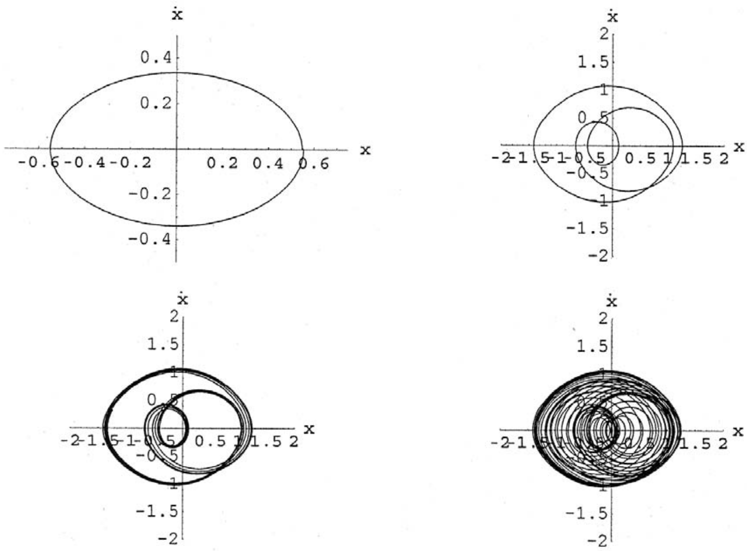

Figure 9. The harmonic trajectory and a near subharmonic of order 1/2.

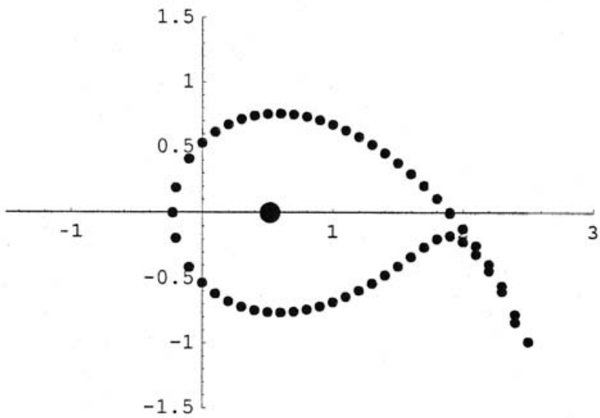

Figure 10. The stability boundary for example 3.2.

With such a small damping parameter we might expect that matters haven't changed much from the situation in example 2.2. However, the stability boundary is quite different, as we see from figure 12. The slightly enlarged dot within the boundary is the initial condition for the harmonic solution.

Things become quite different when we increase the damping coefficient.

Example 3.4: Consider the equation

$$\ddot{x} + k\dot{x} + x - \frac{1}{6}x^3 = \frac{3}{2}\cos 2.85t \tag{3.5}$$

where $k = 0.24$.

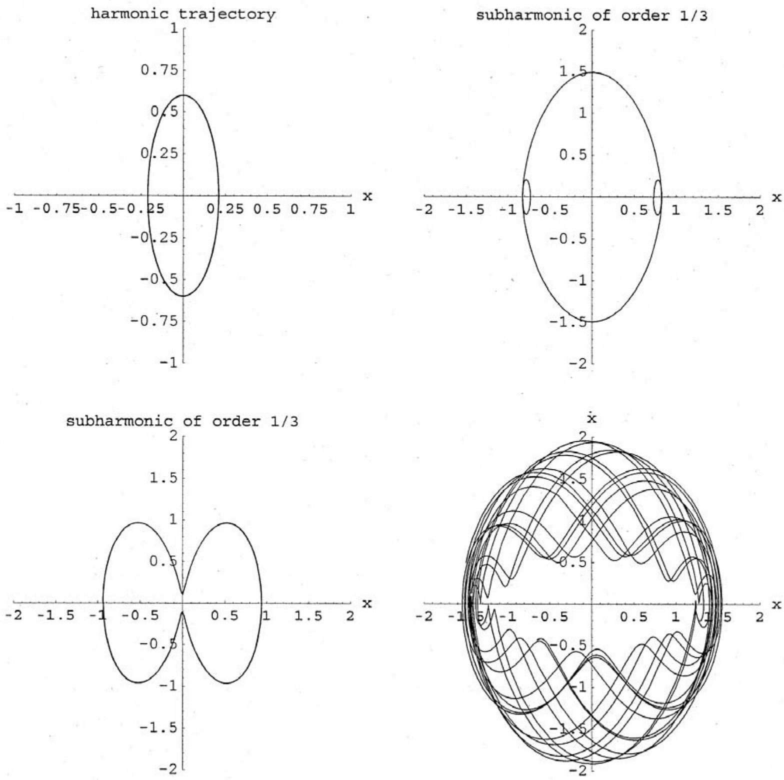

Figure 11. Trajectories for example 3.3.

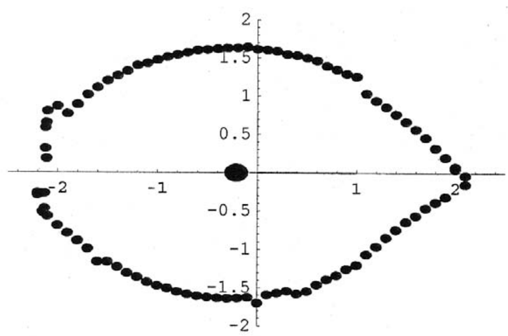

Figure 12. The stability boundary for example 3.3.

With the damping coefficient much larger, the stability boundary changed dramatically and resembles that for example 3.1 (figure 13).

Student problem 3: Determine the stability boundary for other values of k.

Student problem 4: Determine an analytic equation for the stability of equation (3.1).

Student problem 5: Determine the subharmonic boundary of order 1/3 for example 3.3.

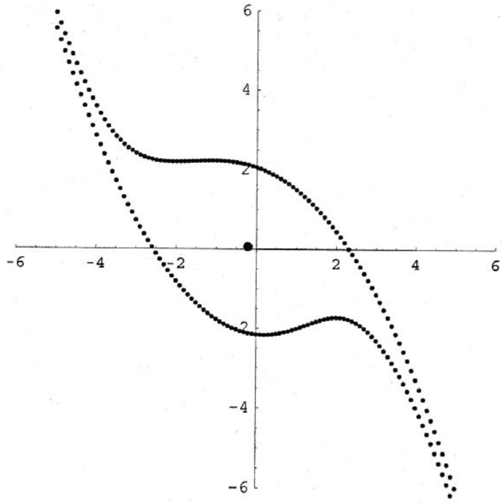

Figure 13. The stability boundary for example 3.4.

4. Steady states

One might think that these nonlinear equations with damping should behave in a similar manner as a linear equation (forced with damping) in that there will be a combination of components in the solution: a transient component and a steady state component. This is true but the mixing of these components in the nonlinear case is much more complex. For the soft spring, we will show that the steady state trajectory in the phase plane coincides with the trajectory for the harmonic solution and thus the harmonic trajectory is a limit cycle. This is different behaviour from that for a hard spring ($\varepsilon > 0$) where the steady states are more complicated (see [10]).

Example 4.1: Consider the equation

$$\ddot{x} + k\dot{x} + x - \frac{1}{6}x^3 = \frac{3}{2}\cos 2.85t \qquad (4.1)$$

Typical trajectories are shown in figure 14; the top row figures are for $k = 0.002$ and the bottom row for $k = 0.24$. Clearly, limit cycles are evident. Because $k = 0.002$ is so small, sometimes a trajectory has to be plotted over a long time interval such as $[0, 400]$ before the steady state becomes evident; with $k = 0.24$ there is no such problem and often an interval of $[0, 100]$ suffices. The plots shown in figure 14 were done for a time interval of $[0, 300]$. The harmonic solutions for these cases are determined by the initial conditions given above. Plots of just the steady state are obtained by plotting again, restricting t to be between $300 - P$ and 300. By plotting separately the harmonic solutions and the steady states and overlaying the plots, one discovers that they coincide. Perhaps the surprising thing is that these steady state trajectories and harmonic trajectories coincide with the harmonic trajectory for the undamped case $k = 0$. The solutions are not the same, but are the $k = 0$ harmonic solution with a phase shift.

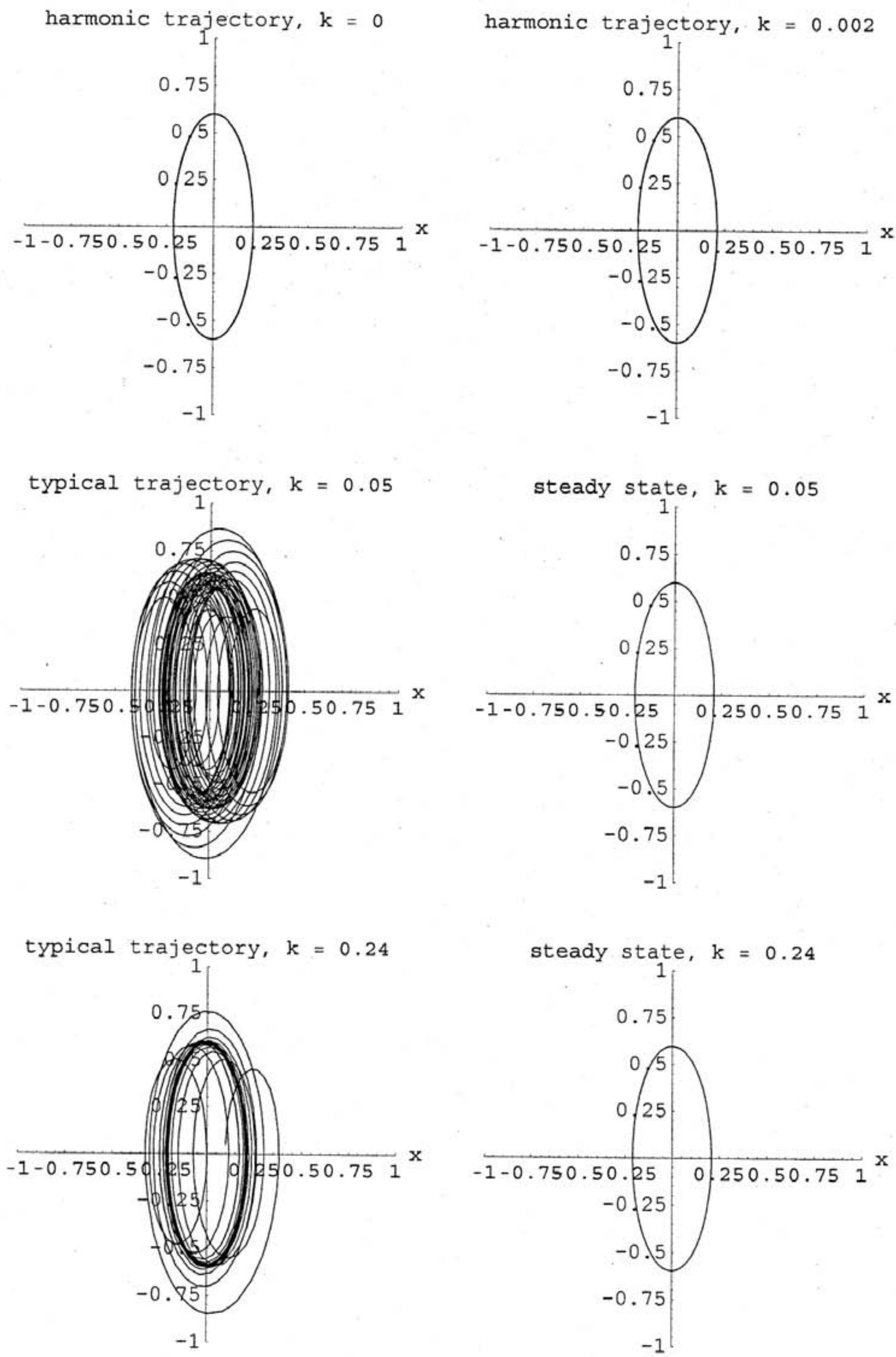

Figure 14. The steady states are the harmonic trajectory for any k.

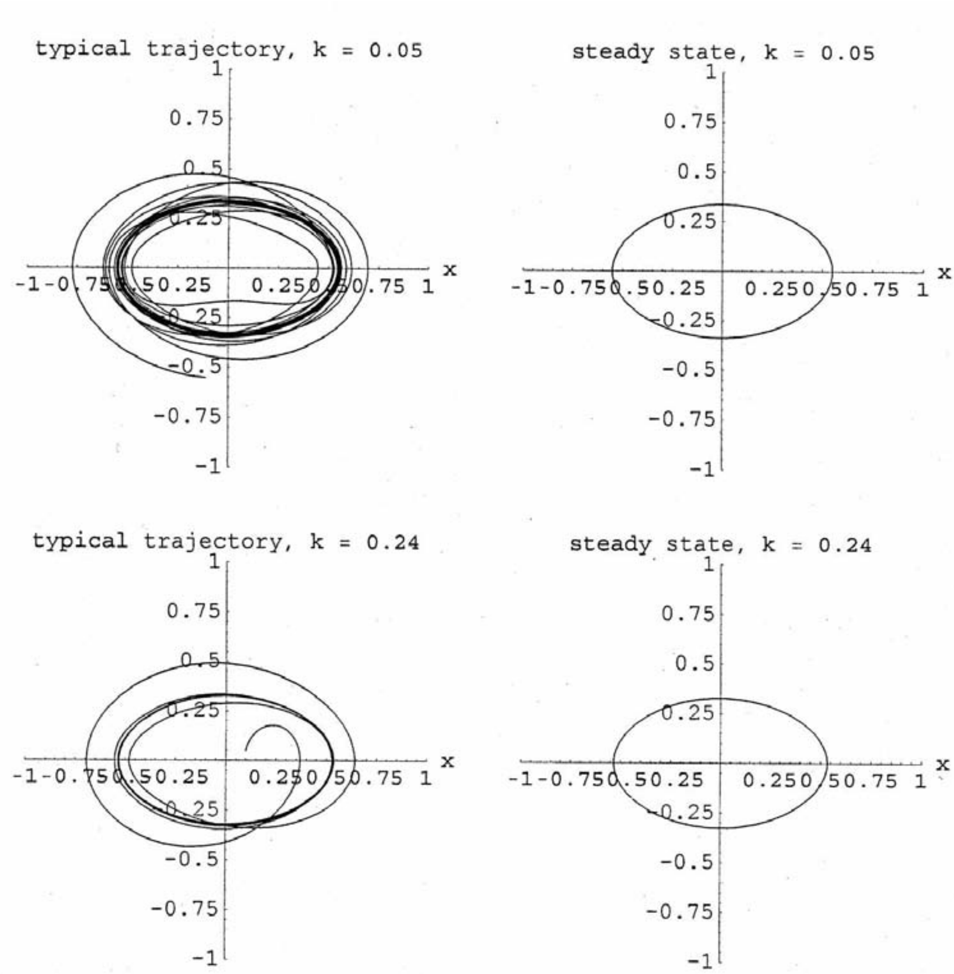

Figure 15. The steady state is a limit cycle for any k.

Example 4.2: Consider the equation

$$\ddot{x} + k\dot{x} + x - \frac{1}{6}x^3 = \frac{1}{3}\cos\frac{3}{5}t \tag{4.2}$$

Typical trajectories for this example are shown in figure 15; the top row figures are for $k = 0.05$ and the bottom row is for $k = 0.24$. Again the steady states are evident and again they coincide with the harmonic trajectory for $k = 0$.

Student problem 6: Verify analytically that equations (4.1) and (4.2) have limit cycles.

Student problem 7: Prove the limit cycle is the harmonic trajectory regardless of the value of k.

Student problem 8: Prove the limit cycle is the harmonic solution trajectory for $k = 0$ in equation (3.1).

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