

Introduction of Nonlinear Dynamics into a Undergraduate Intermediate Dynamics Course

Bongsu Kang

**Department of Mechanical Engineering
Indiana University – Purdue University Fort Wayne**

Abstract

This paper presents a way to introduce nonlinear dynamics and numerical analysis tools to junior and senior mechanical engineering students through an intermediate dynamics course. The main purpose of introducing nonlinear dynamics into a undergraduate dynamics course is to increase the student awareness of the rich dynamic behavior of physical systems that mechanical engineers often encounter in real world applications, while a more rigorous and quantitative treatment of this subject is left to graduate-level dynamics and vibration courses. Employing relatively simple mechanical systems, well known nonlinear dynamics phenomena such as the jump phenomenon due to stiffness hardening/softening, self excitation/limit cycle, parametric resonance, irregular motion, and the basic concepts of stability are introduced. In addition, the students are introduced to essential analysis techniques such as phase diagrams, Poincaré maps, and nondimensionalization. Equations of motion governing such nonlinear systems are numerically integrated with the Matlab[®] ODE solvers rather than analytically to ensure that the student's comprehension of the subject is not hindered by mathematics which are beyond the typical undergraduate engineering curricula. Discussions of the subject in the course underline the qualitative behavior of nonlinear dynamic systems rather than quantitative analysis.

Introduction

Typical undergraduate mechanical engineering curricula layout a sequence of dynamics and vibration courses, e.g., dynamics (first dynamics course), kinematics and kinetics of machinery or similar courses, system dynamics, and relevant technical electives. The Bachelor of Science in Mechanical Engineering program at Indiana University – Purdue University Fort Wayne (IPFW) requires three core courses and offers two technical elective courses in the areas of dynamics and vibration. A technical elective course titled “Intermediate Dynamics with Computer Applications” has been recently developed and taught by the author to meet a strong demand from mechanical engineering students for a dynamics course that is oriented to real world engineering applications. This course covers three major topics: three dimensional rigid body dynamics, time-response analysis of dynamic systems, and dynamics of one dimensional distributed parameter systems. Introduction to nonlinear dynamics is treated under the scope of the time-response analysis of dynamic systems.

In the first dynamics course, students learn how to apply the Newtonian mechanics to solve dynamics problems of particles and rigid bodies. However, more specifically, what they learn is a static version of dynamics. That is they solve a particular state, “a snap shot of the motion”, of a dynamic system for certain unknown kinematic quantities with the application of relatively simple algebra and trigonometry. From the author's perspective it is interesting to observe that

the students in their first course of dynamics do not recognize that what they deal with are mostly nonlinear dynamic systems. For example, most of the students can determine the acceleration and velocity of a simple pendulum when the position of the suspended mass and the tension of the cable are prescribed, or vice versa. However, they do not recognize that the pendulum is governed by a strong nonlinear differential equation and that there exists rich dynamic behavior and stability issues associated with such a simple dynamic system.

In general, students are exposed to a more mathematical approach to solving various dynamic systems governed by ordinary differential equations at the junior level in mechanical engineering curricula through a course in system dynamics. Mathematical modeling and understanding the dynamic nature of such systems through the transient and frequency response analyses are the main subject of this course. Although this course discusses the dynamic nature of physical systems, the dynamic systems treated in this course are limited to linear or linearized (usually about a stable equilibrium), time-invariant systems. The students in this course are generally *told* in a highly abstract manner by the textbook or instructor that the principle of superposition is not applicable to nonlinear systems, that procedures of finding the solutions of nonlinear dynamics problems are extremely complex, or even that nonlinear dynamics is a subject beyond undergraduate engineering curricula.

The importance of the student understanding of fundamental theories and principles in engineering education cannot be over stressed. Without eroding engineering fundamentals and the associated mathematical skills of students, computers and numerical analysis software (NAS) and modern computation tools can be used effectively to enhance the student learning of engineering subjects, particularly when tedious algebraic work exhausts the student confidence and patience in learning. In particular, the use of NAS can be justified when problems do not allow analytical solutions. A recent survey study¹ indicates that a large number of undergraduate mechanical engineering programs in the United States extensively use arithmetic computation software packages; e.g., Matlab[®] or Mathematica[®] in required (core) junior and senior mechanical engineering courses.

Problems seem to arise when students try to use NAS to solve a dynamics problem whose behavior is governed by a nonlinear differential equation(s). This situation often occurs when students are assigned design projects, or more importantly when students practice engineering in industry after graduation. Students are taught, in a series of dynamics courses, how to derive the equation of motion of a dynamic system whose resulting differential equation can be linear or nonlinear. At the same time, students these days are taught such that they are capable of using the above mentioned NAS to solve differential equations, even though their usage is mostly focused on solving ordinary linear differential systems. For most undergraduate mechanical engineering students, who have no experience with the complex nature of nonlinear dynamic systems, numerical solutions from NAS may cause confusion when interpreted from the linear dynamics point of view, or even lead the students to the wrong interpretation of the results. Therefore it is important for undergraduate mechanical engineering students to have a certain level of awareness of the typical behavior of nonlinear dynamic systems. The major objectives of introducing nonlinear dynamics into a undergraduate intermediate dynamics course are:

- to increase the student awareness of nonlinear dynamic systems and their behavior,
- to prepare students with basic analysis techniques for the use of NAS to identify typical nonlinear dynamic behavior,
- to motivate the students to pursue a more advanced study in the areas of dynamics and vibrations.

The topical prerequisites for this subject are fundamental dynamics, ordinary differential equations covered in typical undergraduate calculus courses, linear algebra, and the state space analysis. In addition, basic programming in Matlab[®] is required.

Discussion

Nondimensionalization

Consider a simple one degree of freedom pendulum shown in Fig. 1. The equation of motion governing the angular displacement θ is

$$\ddot{\theta} + \omega_n^2 \sin \theta = 0 \quad \omega_n = \sqrt{g/l} \quad (1)$$

where the dot (\bullet) denotes the time-derivative d/dt , ω_n the natural frequency of the associate linear system for a *small* value of θ , g the gravitational acceleration, and l the length of the pendulum. When the students are asked to find the time-response of the pendulum to initial conditions, the first question from the students is *What is the value of l ?* From the analysis point of view, this question may be acceptable. However, from the design point of view, the same question becomes meaningless since l is a system parameter that usually needs to be determined based on the given design criteria and kinematic outputs of the pendulum. This is probably because the student use of NAS has been oriented to *case-by-case* analyses of numerically well-defined problems. In other words, the students are not familiar with the process that brings out the important dimensionless parameters that dictates the behavior of the system; i.e., *nondimensionalization*. Upon introducing a nondimensional time variable defined as $\tau = t\omega_n$, Eq. (1) can be expressed in the following nondimensional form in which the pendulum length l is embedded

$$\ddot{\theta} + \sin \theta = 0 \quad (2)$$

where, note that the dot (\bullet) denotes the derivative with respect to the nondimensional time τ . Equation (2) can now be directly integrated with using one of the Matlab[®] ODE solvers^{*} to obtain the time-response curves for initial conditions. A brief lecture on the Matlab[®] ODE solvers² should be given if the students have no previous experience. Now, let $z_1 = \theta$ and $z_2 = \dot{\theta}$. Then, the state space representation of Eq. (2) is:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -\sin z_1 \end{aligned} \quad (3)$$

Nonlinear versus Linear Model

Probably one of the most effective ways to increase the student awareness of the nature of nonlinear dynamics is to let the students experience a case when a linear or linearized model of a physical system breaks down. The students are familiar with the linearized version of Eq. (2) for $|\theta| \ll 1$, $\ddot{\theta} + \theta = 0$, and its analytical solution

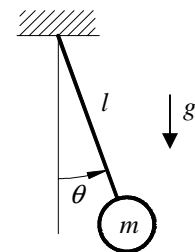


Figure 1. One DOF pendulum in planar motion.

* The state space model is the most natural model in Matlab[®]'s matrix environment, and also the Matlab[®] ODE solvers accept only first-order differential equations.

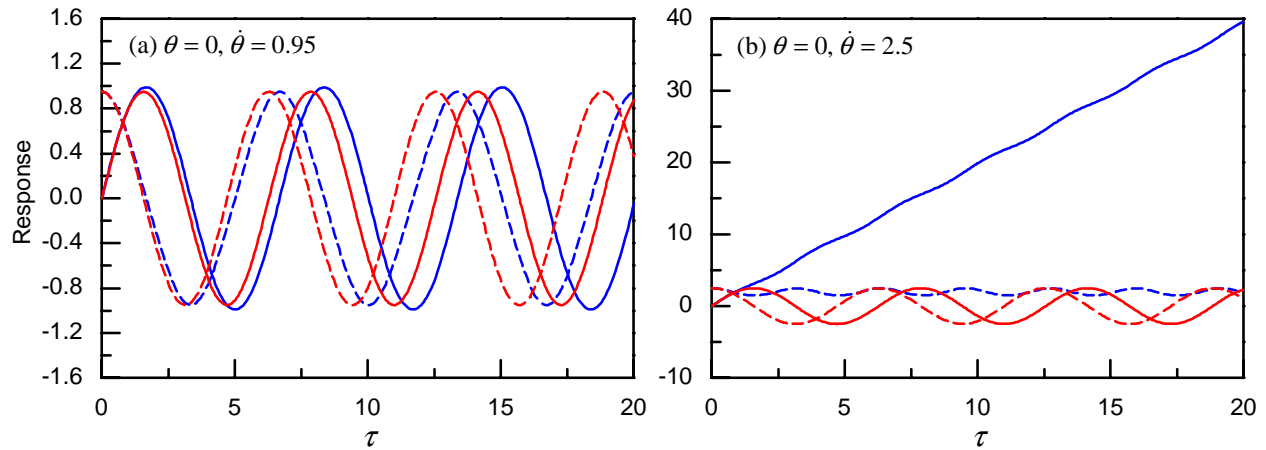


Figure 2. Comparison of the linear and nonlinear responses of the pendulum for different initial conditions. The red and blue curves denote the linear and nonlinear responses, respectively. The solid and dashed curves denote θ and $\dot{\theta}$, respectively.

$$\theta(\tau) = \theta_0 \cos \tau + \dot{\theta}_0 \sin \tau \quad (4)$$

where, θ_0 and $\dot{\theta}_0$ denote the initial displacement and velocity, respectively. As shown in Fig. 2, by comparing the linear and nonlinear solutions of Eq. (2) for two different cases of initial conditions, the students can better appreciate the limitation of the linear analysis. Figure 2 clearly demonstrates that the linear solution fails to predict the true motion of the pendulum when θ is not small due to a large initial velocity. In addition, it needs to be pointed out to the students that the period (or frequency) of the nonlinear motion is not constant and it depends on the amplitude as evidenced in Fig. 2(a). The students should be assigned to plot both linear and nonlinear response curves with different sets of initial conditions to compare. Sample Matlab[®] codes to generate the data for Fig 2 is presented in the Appendix.

Phase Diagram and Stability

The students should be now questioned about the global behavior of the pendulum and also the validity of the linear approximation. This can be best explained by introducing the phase diagram of the pendulum system, which is a common tool used for the qualitative analysis of a nonlinear dynamic system. To this end, apply the chain rule to rewrite Eq. (2) as

$$\dot{\theta} \frac{d\dot{\theta}}{d\theta} + \sin \theta = 0 \quad (5)$$

Integrating Eq. (5) yields

$$0.5\dot{\theta}^2 - \cos \theta = E \quad (6)$$

where E is a constant proportional to the system's total energy whose value depends on the initial conditions; i.e.,

$$E = 0.5\dot{\theta}_0^2 - \cos \theta_0 \quad (7)$$

Rewrite Eq. (6) to obtain the equation of the phase trajectories as

$$\dot{\theta} = \pm \sqrt{2(\cos \theta + E)} \quad (8)$$

In the same manner, the linearized system around $\theta = 0$ takes

$$\dot{\theta}^2 + \theta^2 = (\sqrt{2E_l})^2 \quad 2E_l = \dot{\theta}_0^2 + \theta_0^2 \quad (9)$$

which is the equation of a circle with the radius $\sqrt{2E_l}$. The phase diagram with varying values of E is shown in Fig. 3. There are some issues that need to be understood by the students.

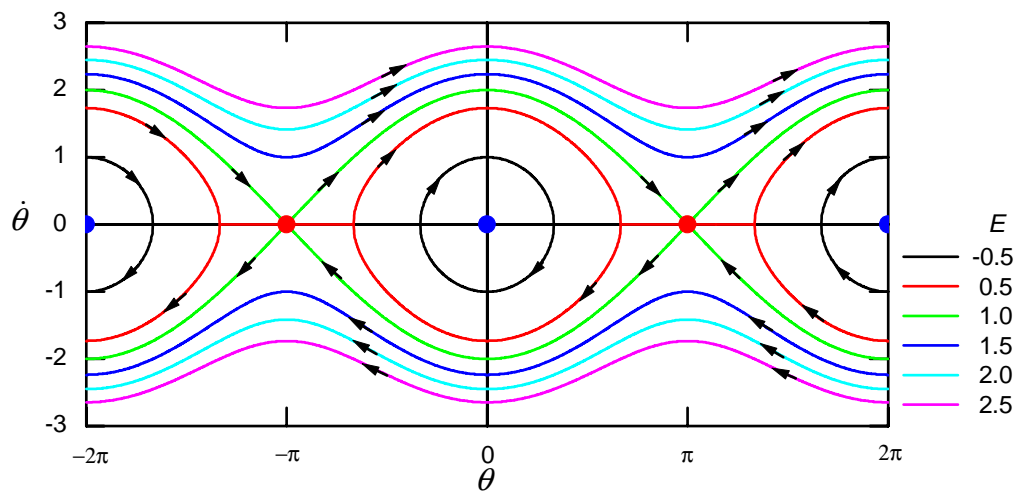


Figure 3. Phase diagram of the pendulum shown in Fig. 1.

- 1) For $E < 1$, the pendulum motion is characterized by closed trajectories, so that the motion repeats itself. Although the pendulum exhibits periodic motion, however, as already shown in Fig. 2(a), it is not necessarily harmonic as the linear model predicts.
- 2) For $E > 1$, the pendulum motion is characterized by open trajectories. The motion is rotary with continually increasing amplitude as shown in Fig. 2(b).
- 3) As E approaches to 0, the pendulum motion becomes harmonic, increasing the validity of the linear analysis. For $E = 0$, there is no motion.
- 4) $E = 1$ separates the two type of motions: oscillatory and rotary.
- 5) Associate with each trajectory is a direction, indicates by an arrow, showing how the state of the system changes as time increases. The direction of the arrows is determined by observing that when $\dot{\theta}$ is positive (negative), θ must be increasing (decreasing) with time.
- 6) The blue dots at $\theta = \pm 2n\pi$ ($n = 0, 1, 2, \dots$) and $\dot{\theta} = 0$ indicate stable equilibrium points with the pendulum pointing downward. When an equilibrium point is surrounded in its immediate neighborhood by closed trajectories, the equilibrium point is called *center*. A center is a stable equilibrium point.
- 7) The red dots at $\theta = \pm(2n+1)\pi$ ($n = 0, 1, 2, \dots$) and $\dot{\theta} = 0$ indicate unstable equilibrium points with the pendulum pointing upward. This equilibrium is unstable since a small displacement from the equilibrium state will generally involve a solution that progresses far from this initial state. An equilibrium point with trajectories of this type in its neighborhood is called a *saddle point*.
- 8) The above-mentioned equilibrium points can be obtained by solving Eq. (5) with $\dot{\theta} = 0$; i.e., $\sin \theta = 0$, which yields two physically distinct values of θ , 0 and π . The stability of each equilibrium point can be determined by examining the behavior of the linearized equation near the equilibrium point, called a *perturbed equation*. Let $\theta \equiv \theta^* + \bar{\theta}$, where θ^* denote the equilibrium point and $\bar{\theta}$ denote a small perturbation. Substituting this into the original nonlinear equation of motion, Eq. (1), yields

$$\ddot{\bar{\theta}} + \sin(\theta^* + \bar{\theta}) = 0 \quad (10)$$

Replacing $\sin(\theta^* + \bar{\theta})$ with its Taylor series and neglecting the nonlinear terms leads to the perturbed equation about θ^*

$$\ddot{\bar{\theta}} + \bar{\theta} \cos \theta^* = 0 \quad (11)$$

For $\theta^* = 0$, Eq. (11) becomes

$$\ddot{\bar{\theta}} + \bar{\theta} = 0 \quad (12)$$

which dictates a bounded motion. Thus the equilibrium point of $\theta^* = 0$ is (marginally) stable. However, for $\theta^* = \pi$, Eq. (11) becomes

$$\ddot{\bar{\theta}} - \bar{\theta} = 0 \quad (13)$$

which dictates an unbounded motion, and thus the equilibrium point of $\theta^* = \pi$ is unstable.

More rigorous treatments of phase diagrams and stability can be found in nonlinear dynamics textbooks, for example, Jordan and Smith³. Once the students have a clear understanding of stability, the students should be assigned with homework problems similar to the one presented below as an example.

As shown in Fig. E1, a bead of mass m slides on a smooth circular wire of radius R which rotates about the vertical axis with constant angular velocity Ω .

- using layman's terms, describe the motion of the bead,
- derive the equation governing the motion of the bead and nondimensionalize it,
- find all equilibrium points,
- construct the phase diagrams when $\Omega \leq \sqrt{g/R}$ and $\Omega > \sqrt{g/R}$,
- from the phase diagrams in (d), identify equilibrium points and examine the stability of each equilibrium point,
- analytically find all equilibrium points and examine the stability of each equilibrium point,
- what happens if Ω becomes very large?
- numerically integrate the equation of motion with different values of a and initial conditions and compare the results with the phase diagrams in (d).

Key

The kinetic energy T and the potential energy V are given by

$$T = \frac{1}{2} m R^2 (\dot{\theta}^2 + \Omega^2 \sin^2 \theta) \quad V = -mgR \cos \theta \quad (E1)$$

Lagrange's equation for the system is

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0 \quad (E2)$$

which leads to the equation of motion of the bead

$$\ddot{\theta} + \frac{g}{R} \sin \theta - \Omega^2 \sin \theta \cos \theta = 0 \quad (E3)$$

Let $\tau = t\omega_n$, where $\omega_n = \sqrt{g/R}$. Then, the nondimensional equation of motion is

$$\ddot{\theta} + (1 - a \cos \theta) \sin \theta = 0 \quad a = (\Omega/\omega_n)^2 \quad (E4)$$

It can be readily seen that if $\Omega = 0$, Eq. (E4) reduces to the same equation of motion for the pendulum in Fig. 1. The equation of trajectories of the bead can be obtained in the same manner as discussed before, which is

$$\dot{\theta}^2 + (a \cos \theta - 2) \cos \theta = 2E \quad \text{or} \quad \dot{\theta} = \pm \sqrt{2(\cos \theta + E) - a \cos^2 \theta} \quad (E5)$$

where E is a constant proportional to the system total energy. The (distinct) equilibrium points are given by,

$$(1 - a \cos \theta) \sin \theta = 0 \quad (E6)$$

Eq. (E6) yields two different sets of roots depending on the value of a ; i.e.,

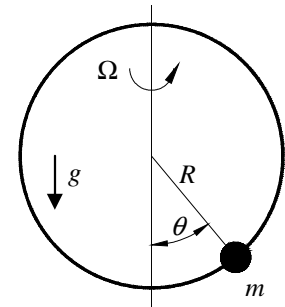


Figure E1. A bead sliding on a ring.

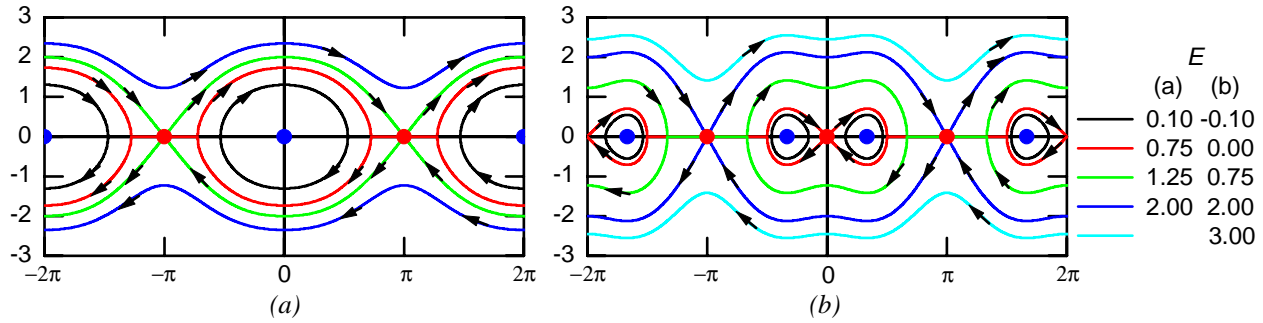


Figure E2. Phase diagrams for the system when (a) $a = 0.5$ and (b) $a = 2.0$.

$$\begin{cases} 0 \text{ and } \pi & \text{when } a \leq 1 \\ 0, \pi, \text{ and } \cos^{-1}(a^{-1}) & \text{when } a > 1 \end{cases} \quad (E7)$$

Figure E2 shows the phase diagrams indicating completely different behavior of the bead for different a . When $a \leq 1$, the bead behaves like a simple pendulum. In addition, the stable equilibrium point 0 when $a \leq 1$ is no longer stable if $a > 1$. Therefore the stability characteristic of the system completely changes as a increases beyond 1, and thus $a = 1$ is referred to as a bifurcation point. To determine analytically the stability of each equilibrium point, Eq. (E4) needs to be linearized about the equilibrium point as outlined in 8). The resulting perturbed equation is

$$\ddot{\bar{\theta}} + \{\cos \theta^* + a(1 - 2\cos^2 \theta^*)\}\bar{\theta} = 0 \quad (E8)$$

Substituting appropriate values for a and θ^* , the stability of the equilibrium points as shown in Fig. E2 can be confirmed. For example, when $a = 2$, Eq. (E8) reduced to

$$\ddot{\bar{\theta}} + 1.5\bar{\theta} = 0 \quad (E9)$$

which confirms a (marginally) stable motion as indicated with a blue dot at $\theta = \pm 1.047$ in Fig. E2(b). Numerical solutions should predict the same behavior of the system as the phase diagrams. For instance, Fig. E3 shows the responses of the bead to different initial conditions when $a = 0.5$ and $a = 2$. The green curve in Fig. E3(b) shows that when a becomes very large, the stable equilibrium points approaches $\pi/2$, which is expected by physic and also confirmed by $\lim_{a \rightarrow \infty} \cos^{-1}(a^{-1}) = \pi/2$.

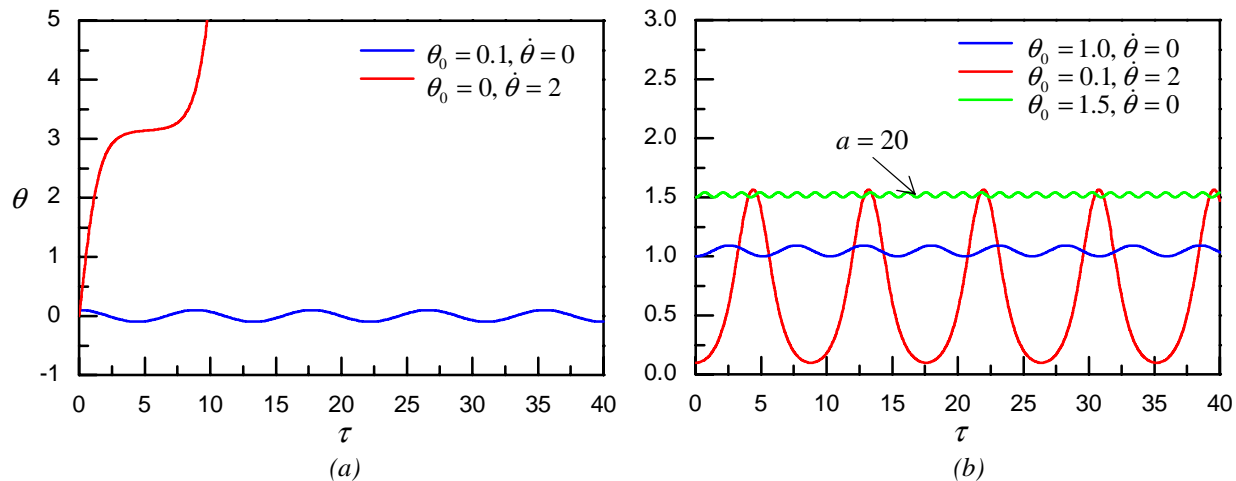


Figure E3. Responses of the bead to different initial conditions when (a) $a = 0.5$ and (b) $a = 2.0$.

Jump Phenomenon

As shown in Fig. 4, consider the motion of a mass m restrained by a damper and a spring whose stiffness is characterized by the curve shown in the figure. When the mass is subjected to a periodic excitation, the equation of motion of the mass m in a nondimensional form is:

$$\ddot{\bar{x}} + 2\zeta\dot{\bar{x}} + \bar{x} + \epsilon\bar{x}^3 = f_0 \sin \alpha\tau \quad (14)$$

where the following nondimensional variables and parameters have been used:

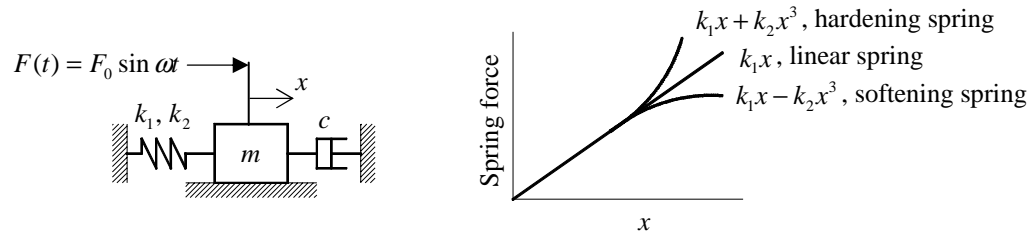


Figure 4. Mass restrained by a damper and a nonlinear spring.

$$\bar{x} = \frac{x}{l_0} \quad \tau = t\omega_n \quad \omega_n^2 = \frac{k_1}{m} \quad 2\zeta = \frac{c}{\sqrt{mk_1}} \quad \varepsilon = \frac{k_2 l_0^2}{k_1} \quad f_0 = \frac{F_0}{k_1 l_0} \quad \alpha = \frac{\omega}{\omega_n} \quad (15)$$

where l_0 denotes the unstretched length of the spring. Once the students are familiar with using Matlab[®] for solving differential equations, they are asked to find the maximum amplitudes of the frequency response of the mass for two different initial displacements which are very close to each other; i.e., $\bar{x}(0) = 3.419$ and $\bar{x}(0) = 3.420$ when $\alpha = 1.35$, $\zeta = 0.05$, $\varepsilon = 0.05$, and $\dot{\bar{x}}(0) = 0$. The students should find $\bar{x}_{\max} = 1.298$ for the first and $\bar{x}_{\max} = 5.140$ [†] for the latter initial displacement. When the students are curious or unsure about their findings due to the dramatically different results for such an insignificant difference in the initial conditions, it is a time to introduce a well-known nonlinear dynamic behavior called the *jump phenomenon*. Figure 5 shows the maximum amplitudes $|x|_{\max}$ versus excitation frequency α of the system for three different values of ε , where $\varepsilon = 0$ corresponds to the associated linear system. Sample Matlab[®] codes to generate the data for Fig. 5 is presented in the Appendix. The dashed curves are added to show the complete shapes of the response curves since it is extremely difficult or not possible for typical numerical analysis techniques to obtain the curves in these regions. Analytical solution techniques such as perturbation, averaging, or harmonic balance methods can be used to obtain a complete response curve, but these methods are beyond the scope of the current introductory discussions. Several important observations from Fig. 5 that need to be pointed out to the students are:

- 1) In contrast with linear systems, the mass-nonlinear spring system exhibits no resonance.
- 2) As α increases, the amplitude keeps increasing until it reaches to its maximum (point *B* for $\varepsilon = 0.025$), then it drops down to point *D*, not taking the path *BD* as indicated by the black arrows.
- 3) As α decreases, the amplitude keeps increasing until it reaches to point *D*, then it jumps up to point *A*, not taking the path *DB* as indicated by the pink arrows.
- 4) The behavior described in 2) and 3) is known as the jump phenomenon. In addition, two response amplitudes exist for a given α , for example when $\alpha = 1.35$,
- 5) Why are the amplitude curves slanted to the right? This is because as α approaches to 1, the amplitude will start to grow due to linear resonance. As the amplitude grows, the nonlinear effect starts becoming larger, resulting in increasing the stiffness of the system. With increased stiffness, the natural frequency also increases, and thus the linear resonance should

[†] Note that these values may be slightly different depending on the numerical computation environment.

occur at higher α . This behavior is reversed when the mass is restrained by a softening spring.

- 6) The portion of the curve that the system never traverses as α increases or decreases (e.g., curve between B and D for $\varepsilon = 0.025$) is unstable and practically unrealizable.
- 7) As in the case of linear systems, enough damping in the system can eliminate the jump phenomenon.
- 8) The governing equation of motion, Eq. (14), is called a damped *Duffing equation* with a forcing term.

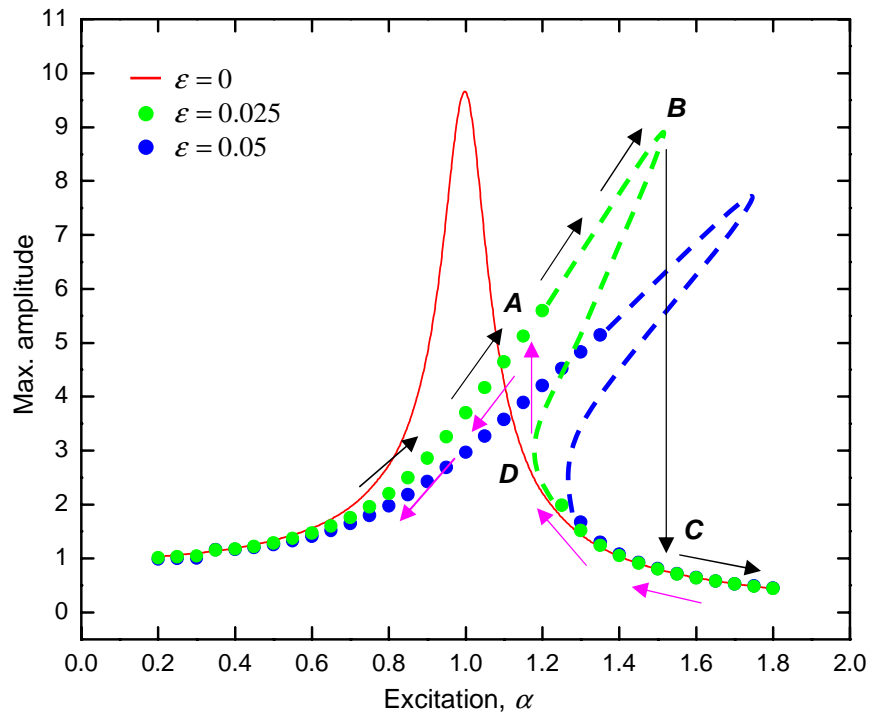


Figure 5. Frequency response curves of the mass-damper-hardening spring system when $\zeta = 0.05$ and $f_0 = 1$.

Once the students have a clear understanding of this example, they should be assigned a problem in which the nonlinear stiffness is due to geometry. The following example problems may be used for that purpose.

Show that the equation governing the motion of the simple pendulum in Fig 1 is also a Duffing equation and plot the frequency response curve. Assume that θ is measured from $\theta = 0$ is moderately large.

Key

$\sin \theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \dots$ near $\theta = 0$. Keeping the first two terms, the equation of motion becomes

$$\ddot{\theta} + \theta - \frac{1}{6}\theta^3 = 0 \quad (E10)$$

which is a form of Duffing equation with a softening stiffness term.

Drive the equation of motion in terms of x for the slider of mass m as shown in Fig. E4. The linear spring is initially under tension T_0 when $x = 0$ and the motion of the slider is guided by a smooth rail. Assuming $|x/L| \ll 1$, show that the equation of motion can be reduced to a Duffing equation.

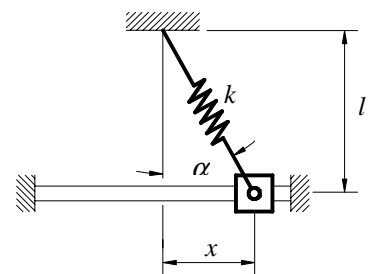


Figure E4. A slider and spring system.

Key

When the slider moves an amount x , the spring force $F = T_0 + k(\sqrt{x^2 + l^2} - l)$ and its component in the x -direction is

$$F_x = F \sin \alpha = \frac{T_0 x}{\sqrt{x^2 + l^2}} + kx(1 - \frac{l}{\sqrt{x^2 + l^2}}) \quad (E11)$$

$(1 + (x/l)^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}\left(\frac{x}{l}\right)^2 + \frac{3}{8}\left(\frac{x}{l}\right)^4 - \dots$ for $x/l \ll 1$. Keeping the first two terms, $F_x = \frac{T_0 x}{l} + \frac{1}{2l^3}(kl - T_0)x^3$.

Then, the equation of motion of the slider is

$$m\ddot{x} + \frac{T_0 x}{l} + \frac{1}{2l^3}(kl - T_0)x^3 = 0 \quad (E12)$$

Self Excited Motion and Limit Cycle

Consider the system shown in Fig. 6. A continuous belt is being driven by rollers at a constant speed v_0 . A block of mass m restrained a spring and a dashpot rests on the belt.

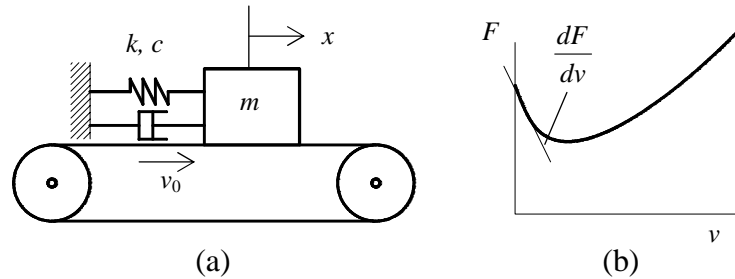


Figure 6. A mass block resting on a continuously moving belt.

Denoting F as the friction force between the block and belt, the equation of motion is

$$m\ddot{x} + c\dot{x} = F(v_0 - \dot{x}) \quad (16)$$

where, the friction force F is assumed to be a function of the slip velocity between the two bodies, $v_0 - \dot{x}$. The friction force is usually obtained experimentally and a typical curve of the friction force versus the slip velocity v is illustrated in Fig. 6(b). Note that the slope of the curve has negative slope over small values of v . If F is expanded into a Taylor series; i.e.,

$$F(v_0 - x) = F(v_0) - \frac{dF}{dv}\dot{x} + \frac{1}{2}\frac{d^2F}{dv^2}\dot{x}^2 - \dots \quad (17)$$

and let

$$y = x - \frac{F(v_0)}{k} \quad \tau = \omega_n t \quad \omega_n^2 = \frac{k}{m} \quad 2\zeta = \frac{c}{\sqrt{mk}} \quad (18)$$

then, Eq. (16) takes the following nondimensional form

$$\ddot{y} + (2\zeta + \frac{1}{\omega_n} \frac{dF}{dv})\dot{y} + y = \frac{1}{2m} \frac{d^2F}{dv^2} \dot{y}^2 - \dots \quad (19)$$

Equation (19) indicates that the stability of the equilibrium position $y=0$ of the linearized system depends on the sign of the damping coefficient $(2\zeta + \omega_n^{-1} dF/dv)$. The sign of this term depends mainly on the sign of the friction force slope dF/dv . If $(2\zeta + \omega_n^{-1} dF/dv) < 0$, dynamic instability occurs. In other words, the amplitude of the block keeps increasing. This is another

important class of nonlinear dynamic phenomenon known as *self-excited oscillations* since the motion itself produces the exciting force. A classical example of a system known to have self-excited oscillations is van der Pol's oscillator governed by the following equation

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0 \quad \alpha > 0 \quad (20)$$

At this point, the students are asked to construct the phase diagrams of Eq. (20) numerically (since no analytical approach is available) for different values of α and initial conditions.

Figure 7 shows typical phase diagrams of the system in Eq. (20). Sample Matlab[®] codes can be found in the Appendix. Important observations that need to be made by the students are:

- 1) Regardless of the initial conditions, all phase trajectories drift toward a single closed curve, called a *limit cycle*. Since a closed phase trajectory corresponds to a periodic solution, this implies that every solution of Eq. (20) tends to be a periodic solution as $\tau \rightarrow \infty$ regardless of the initial conditions.
- 2) If the initial conditions are within the limit cycle, the solution curve spirals outward. If the initial conditions reside outside the limit cycle, the solution curve spirals inward.
- 3) The shape of the limit cycle depends on the value of α .
- 4) A linearized analysis would have predicted instability of the origin, which is not correct. For systems of this type linearization needs to be done about the limit cycle.
- 5) Systems that have self-excited motion and limit cycles typically possess variable damping. For example, the second term in Eq. (20) can be regarded as an amplitude-dependent damping. Note that self-excited motion can also occur in a linear system, while a limit cycle occurs only in nonlinear systems.

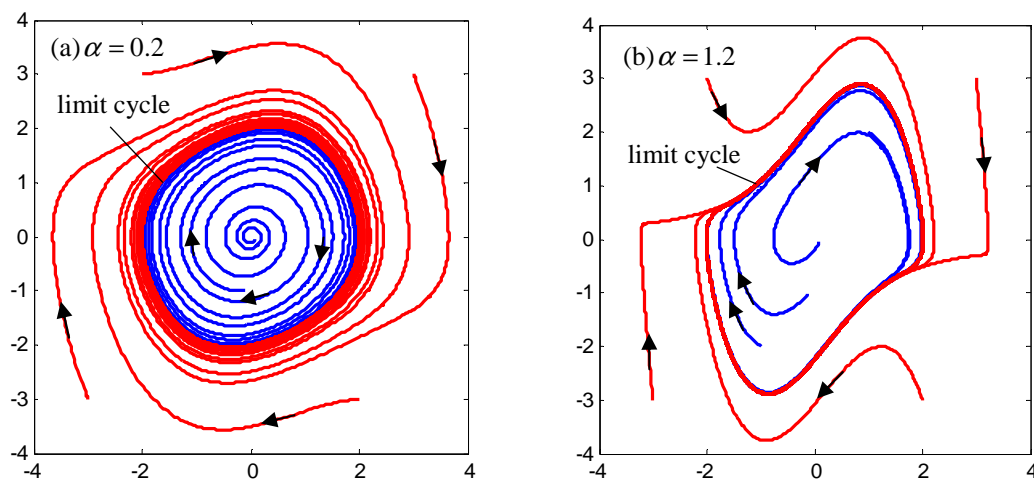


Figure 7. Phase diagram of van der Pol's equation.

Parametric Resonance: Mathieu Equation

Consider the pendulum subjected to an external excitation as shown in Fig. 8. The equation of motion is

$$\ddot{\theta} + \left(\frac{g}{l} + \frac{F_0}{ml} \cos \omega t\right) \sin \theta = 0 \quad (21)$$

This pendulum system can be viewed as a swing pushed by the rider by changing his/her center of gravity during the motion. Let

$$\omega_n = \sqrt{g/l} \quad \tau = \omega_n t \quad \varepsilon = F_0/(mg) \quad \alpha = \omega/\omega_n \quad (22)$$

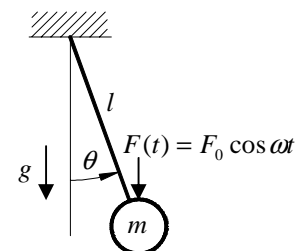


Figure 8. Simple pendulum under vertical excitation.

Then, the nondimensionalized equation of motion is

$$\ddot{\theta} + (1 + \varepsilon \cos \alpha \tau) \sin \theta = 0 \quad (23)$$

For small θ in the neighborhood of $\theta = 0$,

$$\ddot{\theta} + (1 + \varepsilon \cos \alpha \tau) \theta = 0 \quad (24)$$

It can be seen from Eq. (24) the coefficient of the stiffness term is not constant, but time-dependent. Such an equation is known as the *Mathieu equation*. Although it is a linear ODE, it does not render an analytical solution because of the time-dependent coefficient.

Again, the students should be asked to integrate both Eqs. (23) and (24) numerically using Matlab[®] for various values of ε (< 1) and α including $\alpha = 1$ and $\alpha = 2$, and to compare the results. Figure 9 shows the responses of the pendulum when $\varepsilon = 0.3$, where the results from the nonlinear and linearized models are shown in blue and red, respectively. For $\alpha = 1$, it can be seen from Fig. 9(a) that the amplitude of the pendulum gradually increases with increasing τ , indicating a resonance for the linearized model. When $\alpha = 2$, in Fig. 9(b), the amplitude of the pendulum drastically increases; much faster than when $\alpha = 1$. This unstable motion is due to what is known as a *principal parametric resonance* that occurs when the frequency of the parametric excitation is close to twice of the natural frequency of the system. At this point, the students should be warned that the linearized model, Eq. (24), cannot be used for a quantitative assessment of the pendulum behavior when a resonance occurs, as shown by the comparison with the solution of the original nonlinear equation, Eq. (23). In other words, the linear model can be only used to predict the onset of instability, which is still very important though, and thus the post-resonance behavior of the system requires the nonlinear analysis.

The stability characteristics of the Mathieu equation can be examined conveniently by means of *transition curves*³ that demarcate stable and unstable regions on the parameter plane (α , ε), also referred to as a *Strutt diagram*. Construction of this diagram requires more advanced analytic solution techniques that are beyond the scope of the course objectives.

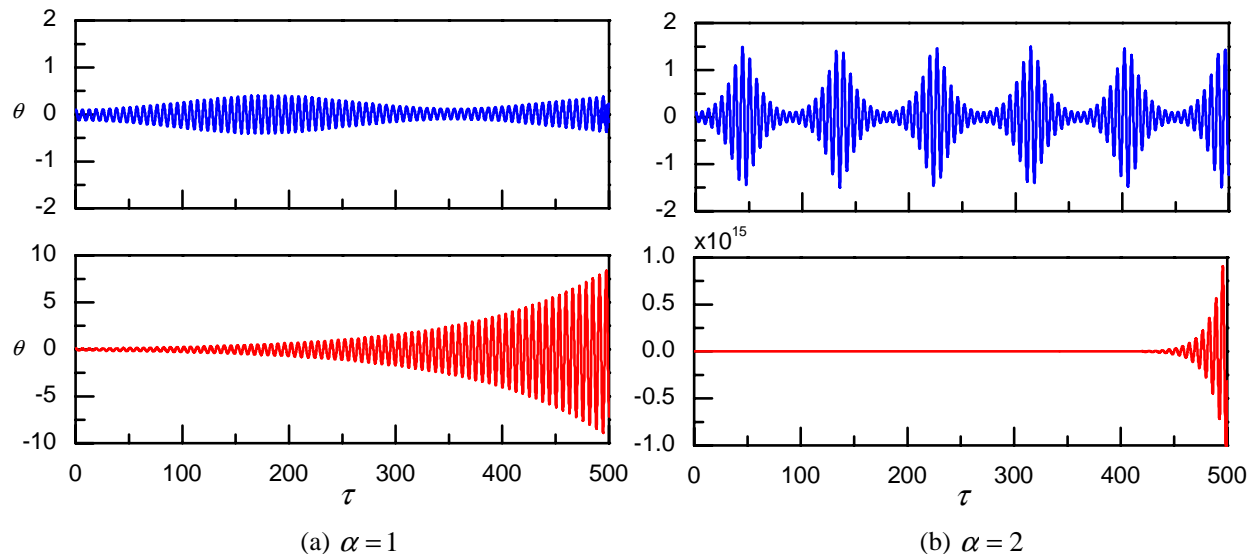


Figure 9. Response of the pendulum due to a parametric excitation.

Irregular Motion: Chaos

There is a class of nonlinear dynamic behavior that is referred to as *chaos* or *chaotic motion*

in mechanical dynamic systems⁴. Linear and nonlinear oscillatory motions are in general characterized by periodicity. Periodicity reflects a high degree of regularity and order. Dynamic behavior that is not stationary, periodic, or quasi-periodic is called *aperiodic* or *irregular*. A chaotic motion differs from a random motion by the fact it can take place even in simple deterministic dynamic systems with periodic inputs. There is no precise definition for a chaotic motion because it cannot be represented through standard mathematical functions. However, from a practical point of view, chaos can be defined as a bounded steady state behavior that is not an equilibrium, a periodic, or a quasi-periodic solution. Chaos is nonlinear behavior; hence a chaotic system must have nonlinear properties.

In order to give a brief introduction of chaotic motions, the students are assigned the homework problem described below before any discussion in class. In this way, the students experience by themselves a rather peculiar, but interesting, nonlinear dynamic behavior, and also they become more curious about the topic to be discussed in class.

Consider the mass-damper-nonlinear spring system in Fig. 4. If the mass block is excited by a function $F(t) = F_0 \sin \omega_1 t \cos \omega_2 t$, the equation of motion simply becomes

$$\ddot{x} + 2\zeta\dot{x} + \bar{x} + \varepsilon x^3 = f_0 \sin \alpha_1 \tau \cos \alpha_2 \tau \quad (E13)$$

- Numerically solve Eq. (E13) and plot the time history of x and the corresponding phase diagram up to $\tau = 60,000$ when $\zeta = 0.01$, $\varepsilon = 0.2$, $f_0 = 0.1$, $\alpha_1 = 0.8$, and $\alpha_2 = 0$. Discard the first 20% of the original time history data (to eliminate the transient response), and then sample x and \dot{x} with the excitation frequency (i.e., at every $\tau = 2\pi/\alpha_1$). Using the sampled data, construct a phase diagram (called a Poincaré map). What do you see on the phase diagram and also on the Poincaré map?
- Do the same as in (a), but with $\alpha_1 = 0.8$ and $\alpha_2 = 0.04$. Sample x and \dot{x} with each of the excitation frequencies (i.e., one set at every $\tau = 2\pi/(\alpha_1 - \alpha_2)$ and another set at every $\tau = 2\pi/(\alpha_1 + \alpha_2)$), and then using the sampled data, construct a phase diagram for each data set. What do you see on the phase diagram? What do you see on each Poincaré map?
- Do the same as in (b), but with $\alpha_1 = 0.94$ and $\alpha_2 = 0.021$. What do you see on each Poincaré map?
- Do the same as in (b), but with $\alpha_1 = 0.88$ and $\alpha_2 = 0.02$. What do you see on each Poincaré map?
- Do FFT analysis of your time series and present the Fourier spectrum for each of the above results.

Key

- The students should find that the response is periodic with a single frequency as evidenced by the Poincaré map and the frequency spectrum shown in Fig. E4. The Poincaré map can be considered as a stroboscopic version of the phase diagram. If one observes or records a motion at discrete times, instead of observing it continuously in time, then the motion appears as a sequence of dots. In particular, as in this case, if the motion is periodic with a single frequency, it will be represented by a point in the Poincaré map. Sample Matlab[®] codes to construct the Poincaré map in Fig. E4 are presented in the Appendix.

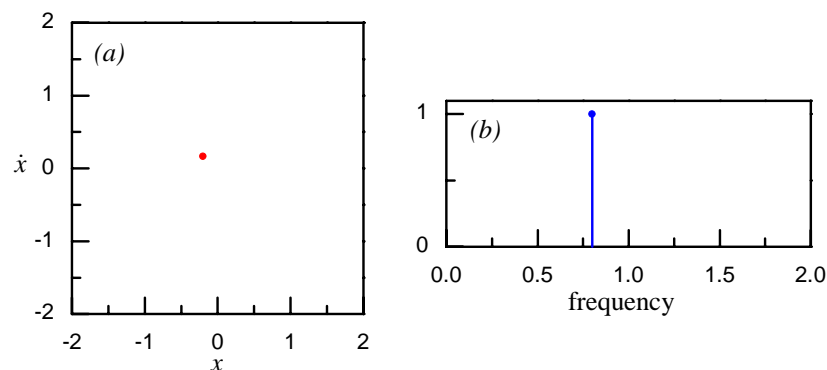


Figure E4. (a) Poincaré map (b) frequency spectrum indicating that the motion is periodic.

(b) For this case, the students should find that the response is quasi-periodic as evidenced by the Poincaré map and the frequency spectrum with couple dominant frequencies which are $\alpha_1 - \alpha_2$ and $\alpha_1 + \alpha_2$ shown in Fig. E5. A quasi-period motion is represented by a finite set of points or a closed curve depending on the number of frequency components in the motion and algebraic relations between the frequencies. However, for a complex quasi-periodic motion, the Poincaré map is not enough to determine the nature of the motion, thus additional analysis tools such as Fourier spectra or Lyapunov exponents are typically used. In this discussion, Fourier spectra are used as an additional tool since they are relatively simple to understand for the students. See (d) for more discussion on a complex quasi-periodic motion.

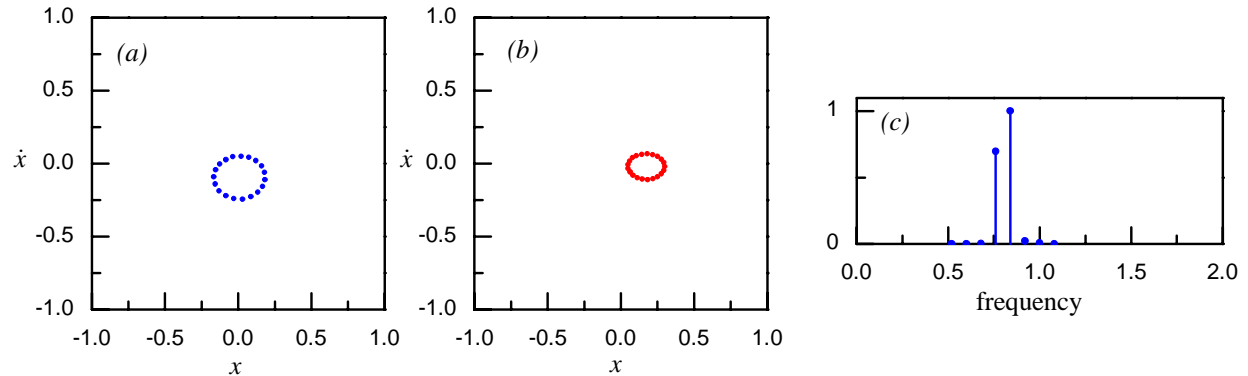


Figure E5. (a) Poincaré map sampled at $\tau = 2\pi/(\alpha_1 - \alpha_2)$, (b) Poincaré map sampled at $\tau = 2\pi/(\alpha_1 + \alpha_2)$, and (c) frequency spectrum indicating that the motion is quasi-periodic.

(c) For this case, the students should find that the response is chaotic. This is evidenced by the Poincaré map and the frequency spectrum shown in Fig. E6. When a Poincaré map does not show either a finite set of points or a closed curve as in this case, the motion is likely to be chaotic. One of the important properties of chaotic motions is the appearance of a continuous broadband spectrum of frequencies in the Fourier spectrum. The frequency spectra of random signals also have a continuous broadband character, but in most cases, chaotic signals can be distinguished from random signals by the presence of predominant spikes and the broadness of the distributed frequency spectrum.

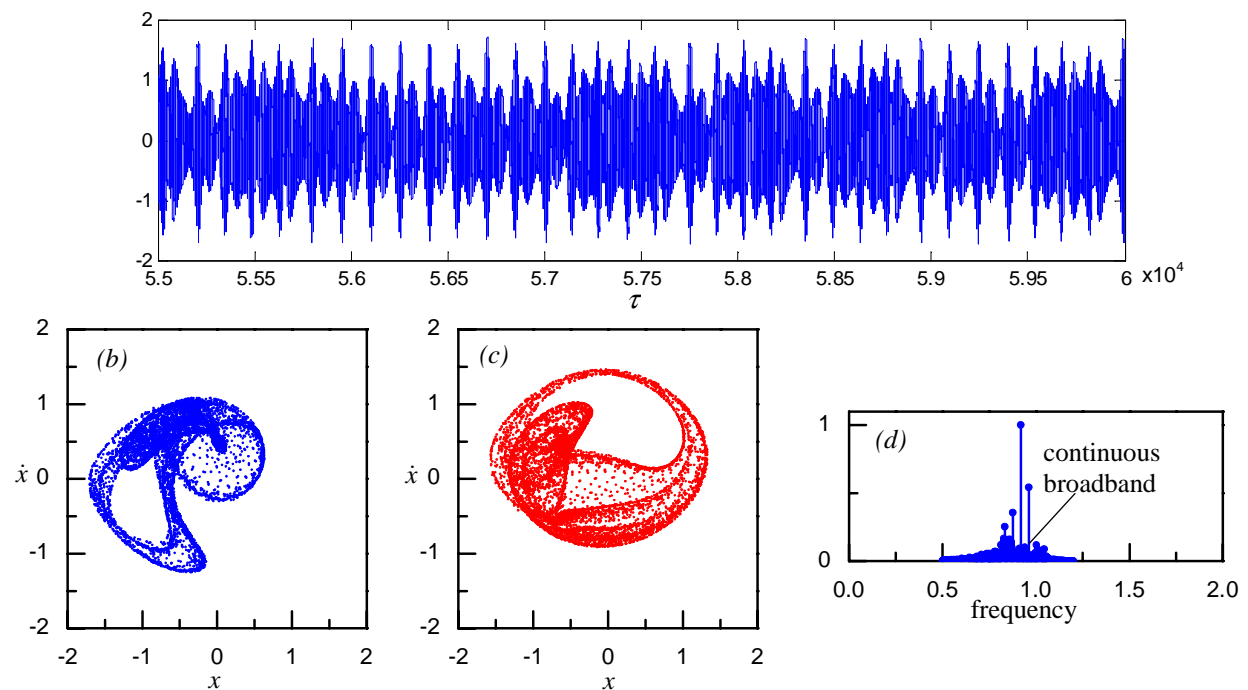


Figure E6. Chaotic response indicated by its (a) time history, (b) Poincaré map sampled at $\tau = 2\pi/(\alpha_1 - \alpha_2)$, (c) Poincaré map sampled at $\tau = 2\pi/(\alpha_1 + \alpha_2)$, and (d) frequency spectrum.

(d) For this case, the students should be cautioned. As shown in Fig. E7, although the Poincaré maps do not seem to show either a finite set of points or a closed curve, the motion is not chaotic as evidenced by the frequency spectrum which does not show a continuous broadband spread over the dominant frequencies as in the previous case. Hence, the motion is complex quasi-periodic, not chaotic.

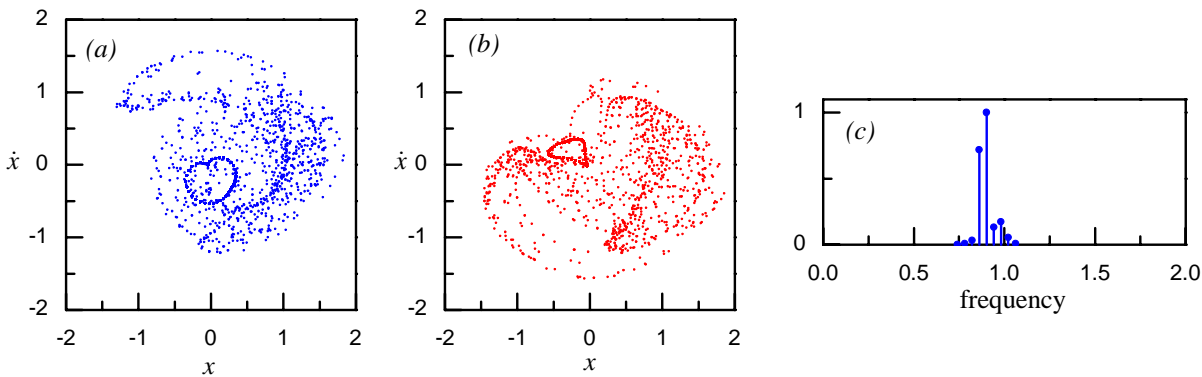


Figure E7. Quasi-periodic response that could be mistakenly considered as a chaotic motion. Shown are: (a) Poincaré map sampled at $\tau = 2\pi/(\alpha_1 - \alpha_2)$, (c) Poincaré map sampled at $\tau = 2\pi/(\alpha_1 + \alpha_2)$, and (d) frequency spectrum.

Conclusions

This paper has presented a way to extend students knowledge in nonlinear dynamics to a fundamental level, and also to increase the student confidence and ability of using numerical analysis software (NAS), specifically Matlab[®], to solve challenging dynamics problems. NAS has become an essential analysis and design tool for engineers, in both industry and academia, and accordingly the student's ability to use such software to solve engineering problems has become an important issue in undergraduate engineering education. As a result, various NAS tools have been incorporated (if not, at least incorporable) in undergraduate engineering courses with diverse learning objectives and approaches. However, in general, the introduction of such NAS tools appears to be in such a way that students do not discover the full potential of these powerful engineering tools. In particular, the application of such NAS tools is mostly limited to linear systems. This seems to somewhat defeat the main purpose of using the NAS tools since these tools are developed to solve problems whose analytical solutions are not feasible, such as nonlinear dynamics problems. Nonlinear dynamics is one of the engineering subjects that benefits from the advancement of modern digital computer hardware and software because of the mathematical complexity involved in finding analytical solutions. Nonlinear dynamic systems differ from linear systems in their behavior and the associated solution approaches as well. The student ability of using the NAS tools and interpreting the numerical results representing the behavior linear dynamic systems do not mean that the same ability applies to analyze a nonlinear dynamic system. Students should be familiarized with the behavior of nonlinear dynamic systems so that, when they encounter a nonlinear dynamic phenomenon of a physical system, they can correctly identify the problem and take a proper approach to solve the problem. Through this concise introductory coverage of nonlinear dynamics in an intermediate dynamics course, the students are:

- aware of the existence of *non*-linear behavior of physical dynamic systems,

- aware of the limitation of linear analyses of physical dynamic systems,
- able to identify typical nonlinear dynamic behavior such as the jump phenomenon, self-excited motion, limit cycle, parametric resonance, and irregular motions,
- able to apply the NAS tools to analyze relatively simple nonlinear dynamics problems.

The overall response of the student to the subject was found positive in spite of the concern that the subject might be a little difficult for undergraduate engineering students to digest. Some of the student answers to the question *What did you like most about this course?* from the course evaluation are:

- *Applied dynamics. The problems presented appear to reflect actual problems, not necessarily simplified examples,*
- *Interesting and practical information. I liked using Matlab® to solve complex problems,*
- *The material is challenging, but still practical,*
- *The material was new to me,*
- *I learned to look at systems in dynamic situations in a much more educated and realistic fashion.*

References

1. B.K. Hodge and W.G. Steels, 2001, "Computational Paradigms in Undergraduate Mechanical Engineering Education," *Proceedings of the 2001 ASEE Annual Conference and Exposition*, Session 2266, Albuquerque, New Mexico.
2. R. Pratap, 2002, *Getting Started with MATLAB A Quick Introduction for Scientists and Engineers*, Oxford University Press, Oxford.
3. D.W. Jordan and P. Smith, 1987, *Nonlinear Ordinary Differential Equations*, Oxford University Press, Oxford.
4. F.C. Moon, 1987, *Chaotic Vibrations An Introduction for Applied Scientist and Engineers*, John Wiley & Sons, New York.

BONGSU KANG

Bongsu Kang is currently an Assistant Professor of Mechanical Engineering at Indiana University-Purdue University Fort Wayne. Dr. Kang received a B.S. in Mechanical Engineering from Yonsei University, Seoul, in 1988, and a Ph.D. in Mechanical Engineering from Wayne State University in 2000. After receiving his B.S.M.E., he worked for five years in the area of automotive braking systems at the Hyundai Technical Research and Development Center, Seoul. He teaches Statics, Dynamics, Kinematics and Kinetics of Machinery, System Dynamics, Vibration, and C/C++ Programming, and advises the IPFW SAE student chapter. His research interests include applied mathematics, elastodynamics and vibrations, elastic wave motion analysis, and vehicle-ground interactions. Member of ASME, SAE, and ASEE.

Appendix

Sample Matlab® codes for Figure 2:

Script file

```
tspan=[0 20]; % time span
z0=[0;1.2]; % initial conditions
options=odeset('reltol',1e-6,'abstol',1e-8);

% solving eom21 using ode45 solver
[t1 sol1]=ode45('eom21',tspan,z0,options);
theta1=sol1(:,1); % extract displacement
theta_dot1=sol1(:,2); % Extract velocity
% solving eom22 using ode45 solver
[t2 sol2]=ode45('eom22',tspan,z0,options);
theta2=sol2(:,1); % extract displacement
theta_dot2=sol2(:,2); % extract velocity
```

Associated M-file eom21.m

```
function zdot=eom21(t,z);
% Call syntax: zdot=eom21(t,z);
% Inputs are: t=time;
% z=[z(1);z(2)]=[theta;theta_dot];
% Output is: zdot=[z(2);-sin(z(1))];
zdot=[z(2);-sin(z(1))];
```

Associated M-file eom22.m

```
function zdot=eom22(t,z);
% Call syntax: zdot=eom22(t,z);
% Inputs are: t=time;
% z=[z(1);z(2)]=[theta;theta_dot];
% Output is: zdot=[z(2);-z(1)];
zdot=[z(2);-z(1)];
```

Sample Matlab® codes for Figure 5:

Script file

```
global alp;
ftime=500;
tspan=[0 ftime]; % time span
z0=[0;0]; % initial conditions
options=odeset('reltol',1e-6,'abstol',1e-8);
amp=[];
alpha=[];
% downward marching
for i=32:-1:0
alp=0.2+i/20;
[t,sol]=ode45('eom5',tspan,z0,options);
x=sol(:,1); % extract displacement
x_dot=sol(:,2); % extract velocity

% discard transient response
st=floor(0.8*length(x));
sample=x(st:end);
% take max. amplitude
amp(i+1)=max(sample);
alpha(i+1)=alp;
% update initial conditions
z0=[amp(i+1);0];%
end;
result=[alpha' amp'];
```

Associated M-file eom5.m

```
function zdot=eom5(t,z);
eps=0.05;
zeta=0.05;
f0=1;
```

```
global alp;
```

```
zdot=[z(2);-2*zeta*z(2)-z(1)-...
eps*(z(1).^3)+f0*sin(alp*t)];
```

Sample Matlab® codes for Figure 7:

Script file

```
tspan=[0 100];
options=odeset('reltol',1e-6,'abstol',1e-8);
% initial displacement
id=[0.1 -0.1 1 -1 3 2 -2 -3];
% initial velocity
iv=[0 -1 2 -2 3 -3 3 -3];
for i=1:8
z0=[id(i);iv(i)];
[t sol]=ode45('eom7',tspan,z0,options);
x=sol(:,1);
x_dot=sol(:,2);
plot(x,x_dot);
hold on;
end
hold off
```

Associated M-file eom7.m

```
function zdot=eom7(t,z);
alp=1.2;
zdot=[z(2);-(alp*(z(1)^2-1)*z(2)+z(1))];
```

Sample Matlab® codes for Figure E4:

Script file

```
global alp1;
global alp2;
tf=60000;
tspan=[0 tf]; % time span
z0=[0.01;0]; % initial Conditions
options=odeset('reltol',1e-6,'abstol',1e-8);
sol=ode45(@eomE4,tspan,z0,options);
sf1=2*pi/(alp1-alp2); %sampling interval
sf2=2*pi/(alp1+alp2); %sampling interval
st1=0.2*tf:sf1:tf; %sampling time
st2=0.2*tf:sf2:tf; %sampling time
d1=deval(sol,st1,1); % displacement at sf1
v1=deval(sol,st1,2); % velocity at sf1
d2=deval(sol,st2,1); % displacement at sf2
v2=deval(sol,st2,2); % velocity at sf2
subplot(1,2,1);
plot(d1,v1,'.','MarkerSize',1);
xlabel('displacement');ylabel('velocity');
subplot(1,2,2);
plot(d2,v2,'.','MarkerSize',1);
xlabel('displacement');ylabel('velocity');
```

Associated M-file eomE4.m

```
function zdot=eom3(t,z);
zeta=0.01;
f0=0.1;
ep=0.2;
global alp1;
global alp2;
alp1=0.88;
alp2=0.02;
zdot=[z(2);-2*zeta*z(2)-...
z(1)+ep*z(1)^3+f0*sin(alp1*t)*cos(alp2*t)];
```