NYU Computer Science Bridge to Tandon Course Spring 2022

Homework 4

Name(s)	NetID(s)
Rishie Nandhan Babu	rb5291@nyu.edu
Amulya Thota	ast9920@nyu.edu
Cangyuan Li	cl4220@stern.nyu.edu
Daniel Lim	dl3009@nyu.edu

Question 9:

Section A: zyBooks Exercise 4.1.3; b-c

Which of the following functions are from $\mathbb{R}$ to $\mathbb{R}$? If f is a function, give its range.

(b) $f(x) = \dfrac{1}{x^2 - 4}$

Answer:

This is not a function from $\mathbb{R}$ to $\mathbb{R}$ because for $x = 2$ or $x = -2$, there is no corresponding y.

(c) $f(x) = \sqrt{x^2}$

Answer:

This is a function from $\mathbb{R}$ to $\mathbb{R}$ because the square root is undefined only for negative numbers, and $\forall x, x^2 >= 0$. f is a function because there is exactly one y that corresponds to an x. The range is $[0, \infty)$ since $\forall x, f(x) >= 0$.

Section B: zyBooks Exercise 4.1.5; b, d, h, i, l

(b) Let $A = \{2, 3, 4, 5\}$. $f : A \to \mathbb{Z}$ s.t. $f(x) = x^2$

Answer:

{ 4, 9, 16, 25 }

(d) Let $f : \{0, 1\}^5 \to \mathbb{Z}$. For $x \in \{0, 1\}^5$, $f(x)$ is the number of 1's that occur in x.

Answer:

The range is { 0, 1, 2, 3, 4, 5 }. There can be at most 5 1's in a string 11111, and the lowest possible is 0 1's in a string 00000.

(h) Let $A = \{1, 2, 3\}$. $f : A \times A \to \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (y, x)$

Answer:

Step 1: $A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

Step 2: Since $A = A$, $(x, y) = (y, x)$, so the range is the set in Step 1.

(i) Let $A = \{1, 2, 3\}$. $f : A \times A \to \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (x, y + 1)$

Answer:

From (h) we have $A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$. Then the range is:

$$\{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$$

(l) Let $A = \{1, 2, 3\}$. $f : \mathcal{P}(A) \to \mathcal{P}(A)$. For $X \subseteq A, f(X) = X - \{1\}$

Answer:

Step 1: $\mathcal{P}(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Step 2: All elements of the powerset are by definition subsets of A, so the range is:

$$\{\varnothing, \{2\}, \{3\}, \{2, 3\}\}$$

Question 10:

Part I.

Section A: zyBooks Exercise 4.2.2; c, g, k

Indicate if the function is one-to-one or not, and if it is onto or not. If not one-to-one or onto, give an example showing why.

(c) $f : \mathbb{Z} \to \mathbb{Z}$. $h(x) = x^3$

Answer:

One-to-one, but not Onto.

One-to-one: no two integers have the same result when the raised to the power 3.

Not onto: there are examples for y in target set for which there is no x in the domain such that $x^3 = y$. For example: 2, 3, 4 etc have no integer cube roots.

(g) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$. $f(x, y) = (x + 1, 2y)$

Answer: One-to-one, but not Onto.

One-to-one: no two elements of the domain can be mapped to the same element in the target.

Not onto: for example , for $(2, 3)$ in the Target, there is no corresponding (x, y) in the Domain as there is no y such that $2y = 3$.

(k) $f : \mathbb{Z}^+ \to \mathbb{Z}^+$. $f(x, y) = 2^x + y$

Answer: Neither one-to-one or onto.

Not one-to-one: Example: Let us consider the integer 7 in the target. For two elements in the Domain $(1, 5)$ and $(2, 3)$, the function maps to 7 in the Target.

Not onto: Example: For integer 1 in the target, there is no (x, y) in the Domain such that the function maps it to 1 in the Target.

Section B: zyBooks Exercise 4.2.4; b, c, d, g

Indicate if the function is one-to-one, onto, neither, or both. If not one-to-one or onto, give an example showing why.

(b) $f\{0,1\}^3 \rightarrow \{0,1\}^3$

Answer:

Neither one-to-one nor Onto

Not one-to-one: For example, two elements of the domain: 000 and 100 are mapped to the same element 100 in the target.
$f\,(000) = f\,(100) = 100$
Not onto: For example, the element (000) in the Target does not have a corresponding element in the domain.

(c) $f\{0,1\}^3 \rightarrow \{0,1\}^3$: The output of f is obtained by taking the input string and reversing the bits.

Answer:

Both one-to-one and onto.

One-to-one: Since each element in the domain is unique, its reverse, i.e, the corresponding element in the Target is also unique.

Onto: Every element in the target can be identified as a reverse of an element in the domain.

(d) $f\{0,1\}^3 \rightarrow \{0,1\}^4$: The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string.

Answer:

One-to-one, but not Onto.

One-to-one: every element in the domain has a unique corresponding element in the target.

Not onto: The elements the target that have non-identical first and last bits: such as 0001

(g) Let A be defined as the set {1,2,3,4,5,6,7,8} and B be defined as {1}.
$f : P(A) \to P(A)$. For $X \subseteq A$, $f(X) = X - B$

Answer:

Neither One-to-one nor Onto.

Not one-to-one:
Example: Consider these two subsets of A $\{2\} and \{1,2\}$.
Since $f(X) = X - B$,
$f\{2\} = \{2\} - \{1\} = \{2\}$
$f\{1,2\} = \{1,2\} - \{1\} = \{2\}$
Both $\{2\} and \{1,2\}$ in the Domain set map to the same corresponding element $\{2\}$ in the Target set. Hence not one-to-one.

Not onto:
Example : Consider $\{1\}$ in the target set. There is no corresponding X in the Domain set such that $f(X) = \{1\}$. Hence, not onto.

Part II.

Give an example of a function from the set of integers to the set of positive integers ($\mathbb{Z} \to \mathbb{Z}^+$) that is:

(a) One-to-one, but not onto.

$$f(x) = \begin{cases} \text{if x is negative,} & |x| \cdot 2 \\ \text{if x is non-negative,} & 2x + 3 \end{cases}$$

If x is negative, $f(x)$ will always be positive and even.
If x is positive, the lowest possible $f(x)$ is 5.
If x is zero, $f(x)$ is 3.
All non-negative x outputs an odd/positive $f(x)$.
But there is no x such that $f(x) = 1$.
$\therefore$ One-to-one, but not onto.

(b) Onto, but not one-to-one.

$f(x) = |x| + 1$

Not one-to-one because $x = 2 and x = -2$ can produce the same $f(x) = 3$.
Onto, because for every $f(x)$ value, there exists some corresponding x that outputs it.
$\therefore$ Onto, but not one-to-one.

(c) One-to-one and onto.

$$f(x) = \begin{cases} \text{if x is negative,} & |x| \cdot 2 \\ \text{if x is non-negative,} & 2x + 1 \end{cases}$$

If x is 0, $f(x) = 1$
If x is positive, $f(x) = 3, 5, 7, ...$ (all positive odd integers)
If x is negative, $f(x) = 2, 4, 6, ...$ (all positive even integers)
$\therefore$ One-to-one and onto (bijection).

(d) Neither one-to-one nor onto.

$f(x) = x^2 + 1$ If $x = -1$ or 1, then $f(x) = 2$ for either. Hence, not one-to-one. Considering the case where $f(x) = 7$ or 11 or any other positive integer that is not a perfect square $+1$, there does not exist an integer solution of x.
$\therefore$ Neither one-to-one nor onto.

Question 11:

zyBooks Exercise 4.3.2; c, d, g, i.

For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}.

(c) $f : \mathbb{R} \to \mathbb{R}$. $f(x) = 2x + 3$

Answer:

$\Rightarrow$ Both one-to-one and onto, so the function has a well-defined inverse.
$\Rightarrow$ Let $f(x) = y$, so $y = 2x + 3$
$\Rightarrow 2x = y - 3$
$\Rightarrow x = \frac{y-3}{2}$
$\therefore f^{-1}(y) = \frac{y-3}{2}$.

(d) Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$
$f : P(A) \to \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$
For $X \subseteq A, f(x) = |X|$. Recall that for a finite set $A, P(A)$ denotes the power set of A which is the set of all subsets of A.

Answer:

$\Rightarrow$ Not one to one, as $f(\{1\})$ and $f(\{2\})$ have the same cardinality of $f(x) = |X| = 1$.
$\Rightarrow$ Not onto, since there is no $X \subseteq P(A)$ that has a cardinality of 0. The smallest $|X| = 1$, for $\{1\}$ or $\emptyset$
$\therefore$ Well-defined inverse does not exist.

(g) $f : \{0, 1\}^3 \to f : \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whetehr the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$.

Answer:

$\Rightarrow$ Function is both one-to-one and onto, and hence has a well-defined inverse.
$\Rightarrow f^{-1}(x) =$ The inverse output is obtained by simply taking the input string and reversing, which is the same as the function itself.
$\therefore f^{-1} = f$.

(i) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, f(x, y) = (x + 5, y - 2)$.

Answer:

$\Rightarrow$ Function is both one-to-one and onto, and hence has a well-defined inverse.
$\Rightarrow$ Let $f(x, y)$ be P, Q, so $P = x + 5$ and $Q = y - 2$
$\Rightarrow$ Solve for x, y with respect to P, Q
$\Rightarrow$ So $x = P - 5$, $y = Q + 2$
$\therefore f^{-1}(x, y) = (x - 5, y + 2)$.

Section B: zyBooks Exercise 4.4.8; c, d

The domain and target set of functions f, g, and h are $\mathbb{Z}$. The functions are defined as:

- $f(x) = 2x + 3$
- $g(x) = 5x + 7$
- $h(x) = x^2 + 1$

(c) $f \circ h$

Answer:

$$\begin{aligned} f(h(x)) &= 2(x^2 + 1) + 3 \\ &= 2x^2 + 2 + 3 \\ &= 2x^2 + 5 \end{aligned}$$

(d) $h \circ f$

Answer:

$$\begin{aligned} h(f(x)) &= (2x + 3)^2 + 1 \\ &= (2x)^2 + 2 \cdot 2x \cdot 3 + 3^2 + 1 \\ &= 4x^2 + 12x + 10 \end{aligned}$$

Section C: zyBooks Exercise 4.4.2; b-d

(b) **Answer:**

$$
\begin{aligned}
(f \circ h)(x) &= f(h(x)) \\
&= f\left(\left\lceil \frac{x}{5} \right\rceil\right) \\
&= \left(\left\lceil \frac{x}{5} \right\rceil\right)^2
\end{aligned}
$$

$$
\begin{aligned}
\therefore (f \circ h)(52) &= \left(\left\lceil \frac{52}{5} \right\rceil\right)^2 \\
&= (\lceil 10.4 \rceil)^2 \\
&= (11)^2 = 121
\end{aligned}
$$

(c) **Answer:**

$$
\begin{aligned}
(g \circ h \circ f)(x) &= g(h(f(x))) \\
&= g\left(h\left(x^2\right)\right) \\
&= g\left(\left\lceil \frac{x^2}{5} \right\rceil\right) \\
&= 2^{\left\lceil \frac{x^2}{5} \right\rceil}
\end{aligned}
$$

$$
\begin{aligned}
\therefore (g \circ h \circ f)(4) &= 2^{\left\lceil \frac{16}{5} \right\rceil} \\
&= 2^4 \\
&= 16
\end{aligned}
$$

(d) **Answer:**

$$
\begin{aligned}
(h \circ f)(x) &= h(f(x)) \\
&= h\left(x^2\right) \\
&= \left\lceil \frac{x^2}{5} \right\rceil
\end{aligned}
$$

Section D: zyBooks Exercise 4.4.6; c-e

Define the following functions f, g, and h:

- $f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1.
- $g : \{0,1\}^3 \to \{0,1\}^3$. The output of g is obtained by taking the input string and reversing the bits.
- $h : \{0,1\}^3 \to \{0,1\}^3$. The output of h is obtained by taking the input string x, and replacing the last bit with a copy of the first bit.

(c) What is $(h \circ f)(010)$?

Answer:

$$\begin{aligned}(h \circ f)(010) &= h(f(010)) \\ &= h(110) \\ &= 111\end{aligned}$$

(d) What is the range of $h \circ f$?

Answer:

Domain	f(x)	h(f(x))
000	100	101
001	101	101
010	110	111
011	111	111
100	100	101
101	101	101
110	110	111
111	111	111

$\therefore$ Range of $h \circ f = \{101, 111\}$

(e) What is the range of $g \circ f$?

Answer:

Domain	f(x)	g(f(x))
000	100	001
001	101	101
010	110	011
011	111	111
100	100	001
101	101	101
110	110	011
111	111	111

$\therefore$ Range of $g \circ f = \{001, 101, 011, 111\}$

Section E: Extra Credit, zyBooks Exercise 4.4.4; c, d

Let $f : X \to Y$ and $g : Y \to X$ be two functions.

(c) Is it possible that f is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

Answer:

Let us assume $g \circ f$ is one-to-one. We will show that f has to be one-to-one.

Assume $x_1 \in X, x_2 \in X$ such that $x_1 \neq x_2$.

Since $(g \circ f)(x)$ is one-to-one,

$$g(f(x_1)) \neq g(f(x_2))$$

Taking inverse of g on both sides, we get,

$$g^{-1}(g(f(x_1))) \neq g^{-1}(g(f(x_2)))$$
$$\Rightarrow f(x_1) \neq f(x_2) \quad [as\ h^{-1}(h(x)) = x]$$

Hence, f is one-to-one.

i.e. If $g \circ f$ is one-to-one, we must have f as one-to-one. ■

(d) Is it possible that g is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

Answer:

Following example depicts that g is not one-to-one and $g \circ f$ is one-to-one.