Homework 3

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Question 7:

1. zyBooks Exercise 3.1.1; a-g

   - $A = \{x \in \mathbb{Z} : \text{x is an integer multiple of 3}\}$
   - $B = \{x \in \mathbb{Z} : \text{x is a perfect square}\}$
   - $C = \{4, 5, 9, 10\}$
   - $D = \{2, 4, 11, 14\}$
   - $E = \{3, 6, 9\}$
   - $F = \{4, 6, 16\}$

   (a) $27 \in A$

   Answer:

   True, since 27 is a multiple of 3

   (b) $27 \in B$

   Answer:

   False, since 27 is not a perfect square

   (c) $100 \in B$

   Answer:

   True, since 100 is a perfect square

   (d) $E \subseteq C \vee C \subseteq E$

   Answer:

   False, since the statement translates to "Every element of E is also an element of C or every element of C is an element of E". The first part is False because $3 \notin C$, and the second part is False because $4, 5, 10 \notin E$.

   (e) $E \subseteq A$

   Answer:

   True, since all of $3, 6, 9$ are integer multiples of 3.

   (f) $A \subset E$

   Answer:

   False. A proper subset translates to $(A \subseteq E) \wedge (A \neq B)$. Since the cardinality of A is greater than the cardinality of E, it is impossible for every element of A to be in E, so the first condition of being a proper subset $A \subseteq E$ is not satisfied.

(g) $E \in A$

Answer:

False. E is a subset of A, but A does not contain the set E.

2. zyBooks Exercise 3.1.2; a-e

- $A = \{x \in \mathbb{Z} :$ x is an integer multiple of $3\}$
- $B = \{x \in \mathbb{Z} :$ x is a perfect square$\}$
- $C = \{4, 5, 9, 10\}$
- $D = \{2, 4, 11, 14\}$
- $E = \{3, 6, 9\}$
- $F = \{4, 6, 16\}$

(a) $15 \subset A$

Answer:

False. $15 \in A$, but it is an element so it cannot be a subset of another set.

(b) $\{15\} \subset A$

Answer:

True, since every element of the set (15) is in A, and $15 \neq A$.

(c) $\emptyset \subset A$

Answer:

True, since the empty set is a subset of every set.

(d) $A \subseteq A$

Answer:

True, since every element of A is in A.

(e) $\emptyset \in B$

Answer:

False, no element of B is a set.

3. zyBooks Exercise 3.1.5; b, d

Express each set using set builder notation. Then if the set is finite, give its cardinality. Otherwise, indicate that the set is infinite.

(b) $\{3, 6, 9, 12, \dots\}$

Answer:

This is the set of all positive integer multiples of 3.
So $A = \{x \in \mathbb{Z}^+ : x \mod 3 = 0\}$. This set is infinite because $\mathbb{Z}^+$ is infinite.

(d) $\{0, 10, 20, 30, \dots 1000\}$

Answer:

This is the set of non-negative integer multiples of 10, up to 1000.
So $A = \{x \in \mathbb{Z}, (0 \leq x \leq 1000) \wedge (x \mod 10 = 0)\}$, and $|A| = 101$.

4. zyBooks Exercise 3.2.1; a-k

Let $X = \{1, \{1\}, \{1, 2\}, 2, \{3\}, 4\}$.

(a) $2 \in X$

Answer:

True, since 2 is an element of X.

(b) $\{2\} \subseteq X$

Answer:

True, since every element of $\{2\}$ is in X.

(c) $\{2\} \in X$

Answer:

False, since X does not contain the set $\{2\}$.

(d) $3 \in X$

Answer:

False, since 3 is not an element of X.

(e) $\{1, 2\} \in X$

Answer:

True, since $\{1,\ 2\}$ is a set in X.

(f) $\{1, 2\} \subseteq X$

Answer:

True, since every element of $\{1,\ 2\}$ is in X.

(g) $\{2, 4\} \subseteq X$

Answer:

True, since every element of $\{2,\ 4\}$ is in X.

(h) $\{2, 4\} \in X$

Answer:

False, since X does not contain the set $\{2,\ 4\}$.

(i) $\{2, 3\} \subseteq X$

Answer:

False, since 3 is not an element of X.

(j) $\{2, 3\} \in X$

Answer:

False, since X does not contain the set $\{2,\ 3\}$

(k) $|X| = 7$

Answer:

False, X has 6 elements.

Question 8:

zyBooks Exercise 3.2.4; b

(b) Let $A = \{1, 2, 3\}$. What is $\{X \in \mathcal{P}(A) : 2 \in X\}$?

Answer:

Step 1: $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Step 2: We want the sets in the powerset of A that contain the element 2. So $\{X \in \mathcal{P}(A) : 2 \in X\} = \{\{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$

Question 9:

1. zyBooks Exercise 3.3.1; c-e

   Define the sets A, B, C, D as follows:

   - $A = \{-3, 0, 1, 4, 17\}$
   - $B = \{-12, -5, 1, 4, 6\}$
   - $C = \{x \in Z : x \text{ is odd}\}$
   - $D = \{x \in Z : x \text{ is positive}\}$

   (c) $A \cap C$

   Answer:

   All elements of A that are odd integers. So $\{-3, 1, 17\}$

   (d) $A \cup (B \cap C)$

   Answer:

   Step 1: $B \cap C$ is all elements of B that are odd integers. So $\{-5, 1\}$

   Step 2: $A \cup (B \cap C)$ is then $\{-3, 0, 1, 4, 17, -5\}$.

   (e) $A \cap B \cap C$

   Answer:

   From the previous question $B \cap C$ is $\{-5, 1\}$. By the associative property we can do this first. Then $A \cap B \cap C = \{1\}$.

2. zyBooks Exercise 3.3.3; a, b, e, f

   Use the following definitions to express each union or intersection given. You can use roster or set builder notation in your responses, but no set operations. For each definition, $i \in \mathbb{Z}^+$

   - $A_i = \left\{i^0, i^1, i^2\right\}$
   - $B_i = \left\{x \in \mathbb{R} : -i \leq x \leq \frac{1}{i}\right\}$
   - $C_i = \left\{x \in \mathbb{R} : \frac{-1}{i} \leq x \leq \frac{1}{i}\right\}$

   (a) $\bigcap_{i=2}^{5} A_i$

   Answer:

   - $A_2 = \{1, 4, 8\}$
   - $A_3 = \{1, 9, 27\}$

- $A_4 = \{1, 16, 64\}$
- $A_5 = \{1, 25, 125\}$

So $\bigcap_{i=2}^{5} A_i = \{1\}$

(b) $\bigcup_{i=2}^{5} A_i$

Answer:

This is the union of A_2 to A_5. So $\bigcup_{i=2}^{5} A_i = \{1, 4, 8, 9, 27, 16, 64, 25, 125\}$

(e) $\bigcap_{i=1}^{100} C_i$

Answer:

For for the first three choices if i, the set is all real numbers from -1 to 1, from -1/2 to 1/2, and from -1/3 to 1/3 respectively. The domain shrinks as i increases. Since this is the intersection of all the sets, we only look at the smallest possible domain. This is from -1 to $1/100$. So $\bigcap_{i=1}^{100} C_i = \left\{x \in \mathbb{R} : \frac{-1}{100} \leq x \leq \frac{1}{100}\right\}$

(f) $\bigcup_{i=1}^{100} C_i$

Answer:

Use the same logic as in (e), except we look at the largest possible domain. So $\bigcup_{i=1}^{100} C_i = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$.

3. zyBooks Exercise 3.3.4; b, d

Use the set definitions $A = \{a, b\}$ and $B = \{b, c\}$ to express each set below. Use roster notation in your solutions.

(b) $\mathcal{P}(A \cup B)$

Answer:

Step 1: $A \cup B = \{a, b, c\}$

Step 2: The powerset is then
$\mathcal{P}(A \cup B) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

(d) $\mathcal{P}(A) \cup \mathcal{P}(B)$

Answer:

Step 1:
$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
$\mathcal{P}(B) = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$

Step 2:
$\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$

Question 10:

1. zyBooks Exercise 3.5.1; b-c

   Use the definitions for A, B, and C to answer the questions. Express the elements using n-tuple notation, not string notation. The sets A, B, and C are defined as follows:

   - $A = \{\text{tall, grande, venti}\}$
   - $B = \{\text{foam, no-foam}\}$
   - $C = \{\text{non-fat, whole}\}$

   (b) Write an element from the set $B \times A \times C$.

   Answer:

   (foam, tall, non-fat)

   (c) Write the set $B \times C$ using roster notation.

   Answer:

   $\{(b, c) : b \in B \wedge c \in C\}$
   $\{$(foam, non-fat), (foam, whole), (no-foam, non-fat), (no-foam, whole)$\}$

2. zyBooks Exercise 3.5.3; b, c, e

   (b) $\mathbb{Z}^2 \subseteq \mathbb{R}^2$

   Answer:

   True, since $Z \subseteq R$.

   (c) $\mathbb{Z}^2 \cap \mathbb{Z}^3 = \emptyset$

   Answer:

   True, since an n-tuple of size 2 does not equal an n-tuple of size 3.

   (e) For any three sets A, B, C, if $A \subseteq B$, then $A \times C \subseteq B \times C$

   Answer:

   True, since if $A \subseteq B$ then every element of A is in B.

3. zyBooks Exercise 3.5.6; d-e

   Express the following sets using the roster method. Express the elements as strings, not n-tuples.

(d) $\{xy : x \in \{0\} \cup \{0\}^2 \wedge y \in \{1\} \cup \{1\}^2\}$

Answer:

Step 1: $\{0\}^2 = \{00\}$ and $1^2 = \{11\}$. So x is in $\{0, 00\}$ and y is in $\{1, 11\}$

Step 2: $\{01, 011, 001, 0011\}$

(e) $\{xy : x \in \{aa, ab\} \wedge y \in \{a\} \cup \{a\}^2\}$

Answer:

$\{aaa, aaaa, aba, abaa\}$

4. zyBooks Exercise 3.5.7; c, f, g

Use the following set definitions to specify each set in roster notation. Except where noted, express elements of Cartesian products as strings.

- $A = \{a\}$
- $B = \{b, c\}$
- $C = \{a, b, d\}$

(c) $(A \times B) \cup (A \times C)$

Answer:

$$\begin{aligned}(A \times B) \cup (A \times C) &= \{ab, ac\} \cup \{aa, ab, ad\} \\ &= \{ab, ac, aa, ab, ad\} \\ &= \{ab, ac, aa, ad\}\end{aligned}$$

(f) $\mathcal{P}(A \times B)$

Answer:

$$\begin{aligned}\mathcal{P}(A \times B) &= \mathcal{P}(\{ab, ac\}) \\ &= \{\emptyset, \{ab\}, \{ac\}, \{ab, ac\}\}\end{aligned}$$

(g) $\mathcal{P}(A) \times \mathcal{P}(B)$

Answer:

$$\begin{aligned}\mathcal{P}(A) \times \mathcal{P}(B) &= \{\emptyset, \{a\}\} \times \{\emptyset, \{b\}, \{c\}, \{b, c\}\} \\ &= \{(\emptyset, \emptyset), (\emptyset, \{b\}), (\emptyset, \{c\}), (\emptyset, \{b, c\}), \\ &\qquad (\{a\}, \emptyset), (\{a\}, \{b\}), (\{a\}, \{c\}), (\{a\}, \{b, c\})\}\end{aligned}$$

Question 11:

1. zyBooks Exercise 3.6.2; b-c

   Use the set identities given in the table to prove the following new identities. Label each step in your proof with the set identity used to establish that step.

   (b) $(B \cup A) \cap (\overline{B} \cup A)$

   Answer:

$$\begin{aligned}(B \cup A) \cap (\overline{B} \cup A) &= (B \cap \overline{B}) \cup A, \text{ by Distributive Law} \\ &= \emptyset \cup A, \text{ by Complement Law} \\ &= A, \text{ by Identity Law}\end{aligned}$$

   (c) $\overline{A \cap \overline{B}} = \overline{A} \cup B$

   Answer:

$$\begin{aligned}\overline{A \cap \overline{B}} &= \overline{A} \cup \overline{\overline{B}}, \text{ by De Morgan's Law} \\ &= \overline{A} \cup B, \text{ by Double Complement}\end{aligned}$$

2. zyBooks Exercise 3.6.3; b, d

   A set equation is not an identity if there are examples for the variables denoting the sets that cause the equation to be false. Show that each set equation given below is not a set identity.

   (b) $A - (B \cap A) = A$

   Answer:

   Let A be any set except the empty set, and let $B = A$. Then $B \cap A = A$ by idempotency. Then $A - A$ is the empty set. As a concrete example, $A = \{1\}, B = \{1\}$ is a counterexample.

   (d) $(B - A) \cup A = A$

   Answer:

   Let $B = \{1, 2\}$ and $A = \{1\}$. Then:

$$\begin{aligned}(B - A) \cup A &= (\{1, 2\} - \{1\}) \cup \{1\} \\ &= \{2\} \cup \{1\} \\ &= \{2, 1\}\end{aligned}$$

   Which does not equal A.

3. zyBooks Exercise 3.6.4; b-c

   The set subtraction law states that $A - B = A \cap \overline{B}$. Use the set subtraction law as well as the other set identities given in the table to prove each of the following new identities. Label each step in your proof with the set identity used to establish that step.

   (b) $A \cap (B - A) = \emptyset$

   Answer:

$$\begin{aligned} A \cap (B - A) &= A \cap (B \cap \overline{A}), \text{ by Set Subtraction Law} \\ &= (B \cap \overline{A}) \cap A, \text{ by Commutative Law} \\ &= B \cap (\overline{A} \cap A), \text{ by Associative Law} \\ &= B \cap \emptyset, \text{ by Complement Law} \\ &= \emptyset, \text{ by Domination Law} \end{aligned}$$

   (c) $A \cup (B - A) = A \cup B$

   Answer:

$$\begin{aligned} A \cup (B - A) &= A \cup (B \cap \overline{A}), \text{ by Set Subtraction Law} \\ &= (A \cup B) \cap (A \cup \overline{A}), \text{ by Distributive Law} \\ &= (A \cup B) \cap \mathcal{U}, \text{ by Complement Law} \\ &= A \cup B, \text{ by Identity Law} \end{aligned}$$