NYU Computer Science Bridge to Tandon Course Spring 2022

Homework 8

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Question 5:

Section A:

Use mathematical induction to prove that for any positive integer n, 3 divides $n^3 + 2n$. We use the formula $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Answer:

Step 1: The base case is $n = 1$. $1^3 + 2(1) = 3$, which is divisible by 3. We can find an integer m such that $3m = 3$ ($m = 1$).

Step 2: For all $k \geq 1$, we want to show that 3 divides $(k+1)^3 + 2(k+1)$.

$$\begin{aligned}
(k+1)^3 + 2(k+1) &= k^3 + 3k^2(1) + 3k(1^2) + 1^3 + 2k + 2 \\
&= k^3 + 3k^2 + 3k + 2k + 1 + 2 \\
&= k^3 + 2k + 3k^2 + 3k + 3 \\
&= 3z + 3k^2 + 3k + 3, \text{ by the inductive hypothesis} \\
&= 3(z + k^2 + k + 1)
\end{aligned}$$

Since k, z are integers, the expression $z + k^2 + k + 1$ is an integer. So we can find an integer $m = z + k^2 + k + 1$ such that $(k+1)^3 + 2(k+1) = 3m$.

Section B:

Use strong induction to prove that any positive integer $n \geq 2$ can be written as a product of primes.

Answer:

Step 1: The base case is $k = 2$ and $k = 3$. For $k = 2, 2 \times 1 = 2$. For $k = 3, 3 \times 1 = 3$. So the base case is satisfied.

Step 2: For all $k \geq 2$, we want to show that $k + 1$ can be written as a product of two primes. If the number is prime, then it can be expressed as that number times 1. If that number is not prime, then

$$k + 1 = ab$$

where a, b are integers. By the inductive hypothesis a, b can be expressed as a product of primes or are primes themselves. So $k+1$ can be expressed as a product of primes.

Question 6:

Section A: zyBooks Exercise 7.4.1; a-g

Define $P(n)$ to be the assertion that

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) Verify that $P(3)$ is true.

Answer:

$$\begin{aligned}\sum_{j=1}^{3} j^2 &= 1^2 + 2^2 + 3^3 \\ &= 1 + 4 + 9 \\ &= 14\end{aligned}$$

$$\begin{aligned}\frac{n(n+1)(2n+1)}{6} &= \frac{3(3+1)(2 \cdot 3 + 1)}{6} \\ &= \frac{3(4)(7)}{6} \\ &= 14\end{aligned}$$

(b) Express $P(k)$.

Answer:

$$\begin{aligned}P(k) &= \sum_{j=1}^{k} j^2 \\ &= \frac{k(k+1)(2k+1)}{6} \\ &= \frac{(k^2+k)(2k+1)}{6} \\ &= \frac{2k^3 + k^2 + 2k^2 + k}{6} \\ &= \frac{2k^3 + 3k^2 + k}{6}\end{aligned}$$

(c) Express $P(k+1)$

Answer:

$$\begin{aligned}
P(k+1) &= \sum_{j=1}^{k+1} j^2 \\
&= \frac{(k+1)(k+1+1)[2(k+1)+1]}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{2k^3+9k^2+13k+6}{6}
\end{aligned}$$

(d) In an inductive proof that for every positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

what must be proven in the base case?

Answer:

We must prove $n = 1$, so

$$\begin{aligned}
\sum_{j=1}^{1} j^2 &= 1^2 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\frac{1(1+1)(2\cdot 1+1)}{6} &= \frac{1(2)(3)}{6} \\
&= 1
\end{aligned}$$

(e) In an inductive proof that for every positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

what must be proven in the inductive step?

Answer:

For any positive integer k, $P(k) \implies P(k+1)$. So for every $k \geq 1$, if

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

, then

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+1+1)[2(k+1)+1]}{6}$$

(f) What would be the inductive hypothesis in the inductive step from your previous answer?

Answer:

The inductive hypothesis would be

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

(g) Prove by induction that for any positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

Answer:

In part (d) we proved the base case. In part (e) we set up the inductive step. So,

$$\begin{aligned}
\sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^{k} +(k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2, \text{ by the induction hypothesis} \\
&= \frac{2k^3+3k^2+k}{6} + (k+1)^2, \text{ by part (b)} \\
&= \frac{2k^3+3k^2+k}{6} + \frac{6}{6}(k^2+2k+1) \\
&= \frac{2k^3+3k^2+k+6k^2+12k+6}{6} \\
&= \frac{2k^3+9k^2+13k+6}{6}
\end{aligned}$$

And we know that $P(k+1) = \frac{2k^3+9k^2+13k+6}{6}$ from part (c).

Section B: zyBooks Exercise 7.4.3; c

(c) Prove that for n geq 1,

$$\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$$

Answer:

Step 1: The base case is $n = 1$.

$$\begin{aligned}\sum_{j=1}^{1} \frac{1}{1^2} &= \frac{1}{1} \\ &= 1\end{aligned}$$

$$2 - \frac{1}{1} = 1$$

$1 \leq 1$, so the base case is satisfied.

Step 2: For $k \geq 1$, we want to show that

$$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$$

$$\begin{aligned}\sum_{j=1}^{k+1} \frac{1}{j^2} &= \frac{1}{(k+1)^2} + \sum_{j=1}^{k} \frac{1}{j^2} \\ &= \frac{1}{(k+1)^2} + \left(2 - \frac{1}{k}\right)\end{aligned}$$

So

$$
\begin{aligned}
\sum_{j=1}^{k+1} \frac{1}{j^2} &\leq \frac{1}{(k+1)^2} + \left(2 - \frac{1}{k}\right) \\
&\leq \frac{1}{k(k+1)} + \left(2 - \frac{1}{k}\right) \\
&\leq \frac{1}{k}\frac{1}{k+1} - \frac{1}{k} + 2 \\
&\leq \frac{1}{k}\left(\frac{1}{k+1} - 1\right) + 2 \\
&\leq 2 - \frac{1}{k}\left(1 - \frac{1}{k+1}\right) \\
&\leq 2 - \frac{1}{k}\left(\frac{k+1}{k+1} - \frac{1}{k+1}\right) \\
&\leq 2 - \frac{1}{k}\left(\frac{k+1-1}{k+1}\right) \\
&\leq 2 - \frac{1}{k}\left(\frac{k}{k+1}\right) \\
&\leq 2 - \left(\frac{1}{k+1}\right)
\end{aligned}
$$

We know that $\frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)}$ since $k \geq 1$. So we can make that replacement in line 2.

Section C: zyBooks Exercise 7.5.1; a

(a) Prove that for any positive integer n, 4 evenly divides $3^2n - 1$.

Answer:

Step 1: The base case is $n = 1$. $3^{2(1)} - 1 = 8$, which is divisible by four.

Step 2: In the inductive step, we want to prove that for $k \geq 1$, 4 evenly divides $3^{2(k+1)} - 1$. The induction hypothesis $3^2k - 1 = 4m$ is equivalent to $(3^2)^k - 1 = 9^k - 1 = 4m$.

$$\begin{aligned}
3^{2(k+1)} - 1 &= 9^{k+1} - 1 \\
&= 9 \cdot 9^k - 1 \\
&= 8 \cdot 9^k + 9^k - 1 \\
&= 8 \cdot 9^k + 4m, \text{ by the induction hypothesis} \\
&= 4(2 \cdot 9^k + m)
\end{aligned}$$

Since $k \geq 1$ and m are integers, the inner expression is an integer as well.