Solutions to Chapter 2 Exercises

Cangyuan Li

June 13, 2021

2.01 Find $\mathbb{E}[\mathbb{E}[\mathbb{E}[Y|X_1, X_2, X_3]|X_1, X_2]|X_1]$

Answer:

Step 1: Apply the Law of Iterated Expectations (the smaller conditioning set wins), starting from the innermost expression.

$$\begin{aligned}\mathbb{E}[\mathbb{E}[\mathbb{E}[Y|X_1, X_2, X_3]|X_1, X_2]|X_1] &= \mathbb{E}[\mathbb{E}[Y|X_1, X_2]|X_1] \\ &= \mathbb{E}[Y|X_1]\end{aligned}$$

2.02 If $\mathbb{E}[Y|X] = a + bX$, find $\mathbb{E}[YX]$ as a function of the moments of X.

Answer:

Step 1: Write $\mathbb{E}[YX]$ in terms of conditional expecations and apply Conditioning Theorem, where $g(X) = X$.

$$\begin{aligned}\mathbb{E}[YX] &= \mathbb{E}[\mathbb{E}[YX|X]] \\ &= \mathbb{E}[X\mathbb{E}[Y|X]] \\ &= \mathbb{E}[X(a + bX)] \\ &= a\mathbb{E}[X] + b\mathbb{E}[X^2]\end{aligned}$$

2.15 Consider the intercept-only model $Y = \alpha + e$ with α the best linear predictor. Show that $\alpha = E[Y]$.

Answer:

Step 1: Use the formula for the best linear predictor. Note that in this case α takes the place of β and 1 takes the place of X.

$$\begin{aligned}\alpha &= X' \left(\mathbb{E}[XX']\right)^{-1} \mathbb{E}[XY] \\ &= 1 \cdot \frac{1}{1} \cdot \mathbb{E}[Y] \\ &= \mathbb{E}[Y]\end{aligned}$$

2.17 Let X be a random variable with $\mu = \mathbb{E}[X]$ and $\sigma^2 = var(X)$. Define

$$g(x, \mu, \sigma^2) = \begin{pmatrix} x - \mu \\ (x - \mu^2) - \sigma^2 \end{pmatrix}$$

Show that $\mathbb{E}[g(X, \mu, \sigma)] = 0$ if and only if $m = \mu$ and $s = \sigma^2$.

Answer:

Step 1: Show both set of conditions. Note that $\mu = \mathbb{E}[X]$ and $\sigma^2 = var(X)$. It is helpful to write in these terms in order to apply law of iterated expectations.

$$\begin{aligned}
\mathbb{E}[g(X, m, s)] &= \mathbb{E}\begin{pmatrix} X - \mathbb{E}[X] \\ (X - \mathbb{E}[X])^2 - var(X) \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{E}[X] - \mathbb{E}[\mathbb{E}[X]] \\ (\mathbb{E}[X - \mathbb{E}[\mathbb{X}]])^2 - \mathbb{E}[var(X)] \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{E}[X] - \mathbb{E}[X] \\ \mathbb{E}[var(X)] - \mathbb{E}[var(X)] \end{pmatrix} \\
&= 0
\end{aligned}$$

Step 2: Prove the other way around. For example, if $m \neq \mu$, then the law of iterated expectations would be violated.

2.18 Suppose that $X = (1, X_2, X_3)$ where $X_3 = \alpha_1 + \alpha_2 X_2$ is a linear function of X_2.

(a) Show that $\mathbf{Q}_{XX} = \mathbb{E}[XX']$ is not invertible.

Answer:

Step 1: The condition for invertibility is that there is no non-zero vector a such that $a'X = 0$. Writing things out with $a = a_1, a_2, a_3$,

$$\begin{aligned}
a'X &= \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} 1 \\ X_2 \\ \alpha_1 + \alpha 2 X_2 \end{bmatrix} \\
&= a_1 + a_2 X_2 + a_3 \alpha_1 + a_3 \alpha_2 X_2
\end{aligned}$$

Now, one can set $a_3 = -1$, $a_2 = \alpha_2$, and $a_1 = \alpha_1$, such that

$$\begin{aligned}
a'X &= \alpha_1 + \alpha_2 X_2 - \alpha_1 - \alpha_2 X_2 \\
&= 0
\end{aligned}$$

(b) Use a linear transformation of X to find an expression for the best linear predictor of Y given X.

Answer:

TODO