Solutions to Chapter 13 Exercises

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13.21 Take the linear equation $Y = X'\beta + e$ and consider the following estimators of β.

(a) $\widehat{\beta}$: 2SLS using the instruments Z_1.

(b) $\widetilde{\beta}$: 2SLS using the instruments Z_2.

(c) $\overline{\beta}$: GMM using the instruments $Z = (Z_1, Z_2)$ and the weight matrix

$$W = \begin{pmatrix} \left(\mathbf{Z_1'}\ \mathbf{Z_2'}^{-1}\right)\lambda & 0 \\ 0 & \left(\mathbf{Z_1'}\ \mathbf{Z_2'}^{-1}\right)(1-\lambda) \end{pmatrix}$$

for $\lambda \in (0,1)$.

Find an expression for $\overline{\beta}$ which shows that it is a specific weighted average of $\widehat{\beta}$ and $\widetilde{\beta}$.

Answer:

Step 1: $\overline{\beta} = (\mathbf{X'ZWZ'X})^{-1}(\mathbf{X'ZWZ'Y})$

$$\begin{aligned}\mathbf{ZWZ'} &= \begin{bmatrix}\mathbf{Z_1} & \mathbf{Z_2}\end{bmatrix}\begin{bmatrix}(\mathbf{Z_1}'\mathbf{Z_1})^{-1}\lambda & \mathbf{O} \\ \mathbf{O} & (\mathbf{Z_2}'\mathbf{Z_2})^{-1}(1-\lambda)\end{bmatrix}\begin{bmatrix}\mathbf{Z_1}' \\ \mathbf{Z_2}'\end{bmatrix} \\ &= \begin{bmatrix}\mathbf{Z_1}(\mathbf{Z_1}'\mathbf{Z_1})^{-1}\lambda & \mathbf{Z_2}(\mathbf{Z_2}'\mathbf{Z_2})^{-1}(1-\lambda)\end{bmatrix}\begin{bmatrix}\mathbf{Z_1}' \\ \mathbf{Z_2}'\end{bmatrix} \\ &= \mathbf{Z_1}(\mathbf{Z_1}'\mathbf{Z_1})^{-1}\mathbf{Z_1}'\lambda + \mathbf{Z_2}(\mathbf{Z_2}'\mathbf{Z_2})^{-1}\mathbf{Z_2}'(1-\lambda)\end{aligned}$$

$$\begin{aligned}\overline{\beta} = &\left(\mathbf{X'Z_1}(\mathbf{Z_1}'\mathbf{Z_1})^{-1}\mathbf{Z_1}'\mathbf{X}\lambda + \mathbf{X'Z_2}(\mathbf{Z_2}'\mathbf{Z_2})^{-1}\mathbf{Z_2}'\mathbf{X}(1-\lambda)\right)^{-1} \\ &\qquad \left(\mathbf{X'Z_1}(\mathbf{Z_1}'\mathbf{Z_1})^{-1}\mathbf{Z_1}'\mathbf{Y}\lambda + \mathbf{X'Z_2}(\mathbf{Z_2}'\mathbf{Z_2})^{-1}\mathbf{Z_2}'\mathbf{Y}(1-\lambda)\right)\end{aligned}$$

Step 2:

$$\widehat{\beta} = (\mathbf{X'Z_1}(\mathbf{Z_1'Z_1})^{-\mathbf{1}}\mathbf{Z_1'X})^{-\mathbf{1}}\mathbf{X'Z_1}(\mathbf{Z_1'Z_1})^{-\mathbf{1}}\mathbf{Z_1'Y}$$

$$\widetilde{\beta} = (\mathbf{X'Z_2}(\mathbf{Z_2'Z_2})^{-\mathbf{1}}\mathbf{Z_2'X})^{-\mathbf{1}}\mathbf{X'Z_2}(\mathbf{Z_2'Z_2})^{-\mathbf{1}}\mathbf{Z_2'Y}$$

Step 3: Let $A_j := \mathbf{X'Z_j(Z_j'Z_j)^{-1}Z_j'X}$ and $B_j := \mathbf{X'Z_j(Z_j'Z_j)^{-1}Z_j'Y}$

$$\begin{aligned}
\overline{\beta} &= \Big(\mathbf{A_1}\lambda + \mathbf{A_2}(1-\lambda)\Big)^{-1}\Big(\mathbf{A_1A_1^{-1}B_1}\lambda + \mathbf{A_2A_2^{-1}B_2}(1-\lambda)\Big) \\
&= \Big(\mathbf{A_1}\lambda + \mathbf{A_2}(1-\lambda)\Big)^{-1}\Big(\mathbf{A_1}\widehat{\beta}\lambda + \mathbf{A_2}\widetilde{\beta}(1-\lambda)\Big)
\end{aligned}$$