Solutions to Chapter 3 Exercises

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3.14 Let $\widehat{\beta}_n = (\mathbf{X_n'X_n})^{-1}\mathbf{X_n'Y_n}$ denote the OLS estimate where $\mathbf{Y_n}$ is $n \times 1$ and $\mathbf{X_n}$ is $n \times k$. Prove that the OLS estimate computed using an additional observation (Y_{n+1}, X_{n+1}) is

$$\widehat{\beta}_{n+1} = \widehat{\beta}_n + \frac{1}{1 + X'_{n+1}(X'_nX_n)^{-1}X_{n+1}}(X'_nX_n)^{-1}X_{n+1}(Y_{n+1} - X'_{n+1}\widehat{\beta}_n)$$

Answer:

Step 1: Apply the Woodbury Matrix Identity. Let

$$\begin{aligned} A &= X'_nX_n \\ B &= X_{n+1} \\ D &= X'_{n+1} \end{aligned}$$

The choice of D is a bit arbitrary. I think setting the last term equal to C would work, but this way the '1' is already present and the equation is a bit easier to manipulate. Then, since $C = I = 1$ because a single observation is 1 by 1.

$$\begin{aligned} (A + BCD)^{-1} &= A^{-1} - A^{-1}BC\left(C + CDA^{-1}BC\right)^{-1}CDA^{-1} \\ &= A^{-1} - A^{-1}B(1 + DA^{-1}B)DA^{-1} \end{aligned}$$

Step 2: Simplify. The terms of the expanded equation are:

$$\begin{aligned} Term1 &= A^{-1}X'_nY_n \\ &= (X'_nX_n)^{-1}X'_nY_n \\ &= \widehat{\beta}_n \end{aligned}$$

$$\begin{aligned} Term2 &= A^{-1}X_{n+1}Y_{n+1} \\ &= (X'_nX_n)^{-1}(X_{n+1}Y_{n+1}) \end{aligned}$$

$$\begin{aligned} Term3 &= (X'_nX_n)^{-1}X_{n+1}\left[1 + X'_{n+1}(X'_nX_n)^{-1}X_{n+1}\right]^{-1}X'_{n+1}(X'_nX_n)^{-1}X'_nY_n \\ &= (X'_nX_n)^{-1}X_{n+1}\left[1 + X'_{n+1}(X'_nX_n)^{-1}X_{n+1}\right]^{-1}X'_{n_1}\widehat{\beta}_n \end{aligned}$$

$$Term4 = (X_n'X_n)^{-1}X_{n+1}\left[1 + X_{n+1}'(X_n'X_n)^{-1}X_{n+1}\right]X_{n+1}'(X_n'X_n)^{-1}X_{n+1}Y_{n+1}$$

Step 3: Combine terms. Note that the term starting with '1 + ...' is a scalar, call it L. Call the term $Z := (X_n'X_n)^{-1}X_{n+1}$. Note also that $L = 1 + X_{n+1}'Z$. The observations are also scalars and may be rearranged. So:

$$\begin{aligned}
\widehat{\beta}_{n+1} &= Term1 + Term2 - Term3 - Term4 \\
&= \widehat{\beta}_n + ZY_{n+1} - \frac{1}{L}ZX_{n+1}'\widehat{\beta}_n - \frac{1}{L}ZX_{n+1}'ZY_{n+1} \\
&= \widehat{\beta}_n + ZY_{n+1}\frac{L}{L} - \frac{1}{L}ZZX_{n+1}'Y_{n+1} - -\frac{1}{L}ZX_{n+1}'\widehat{\beta}_n \\
&= \widehat{\beta}_n + \frac{1}{L}(ZY_{n+1} + ZZX_{n+1}'Y_{n+1} - ZZX_{n+1}'Y_{n+1} - ZX_{n+1}'\widehat{\beta}_n) \\
&= \widehat{\beta}_n + \frac{1}{L}Z(Y_{n+1-X_{n+1\widehat{\beta}_n}'})
\end{aligned}$$