Solutions to Chapter 3 Exercises

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3.05 Let $\widehat{\mathbf{e}}$ be the OLS residual from a regression of $\mathbf{Y}$ on $\mathbf{X}$. Find the OLS coefficient from a regression of $\widehat{\mathbf{e}}$ on $\mathbf{X}$.

Answer:

Step 1: Here, $\widehat{e}$ takes the place of Y in the formula for $\widehat{\beta}$.

$$\begin{aligned}
\widetilde{\beta} &= (X'X)^{-1}X'\widehat{e} \\
&= (X'X)^{-1}\left[X'Y - X'X\widehat{\beta}\right] \\
&= (X'X)^{-1}\left[X'Y - X'X(X'X)^{-1}X'Y\right] \\
&= (X'X)^{-1}\left[X'Y - X'Y\right] \\
&= 0
\end{aligned}$$

3.06 Let $\widehat{\mathbf{Y}} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$. Find the OLS coefficient from a regression of $\widehat{\mathbf{Y}}$ on $\mathbf{X}$.

Answer:

Step 1: Plug into regression formula and expand.

$$\begin{aligned}
\widetilde{\beta} &= (X'X)^{-1}X'\widehat{Y} \\
&= \left[(X'X)^{-1}X'X\right](X'X)^{-1}X'Y \\
&= I(X'X)^{-1}X'Y \\
&= \widehat{\beta}
\end{aligned}$$

3.11 Show that when $\mathbf{X}$ contains a constant, $n^{-1}\sum_{i=1}^{n}\widehat{Y}_i = \overline{Y}$.

Answer:

Step 1: Note that $Y_i = \widehat{Y}_i + \widehat{e}_i$. Also, $\overline{Y} := n^{-1}\sum_{i=1}^{n} Y_i$.

$$\begin{aligned}
& \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n}\widehat{Y}_i + \sum_{i=1}^{n}\widehat{e}_i \\
\Leftrightarrow \quad & n^{-1}\sum_{i=1}^{n} Y_i = n^{-1}\sum_{i=1}^{n}\widehat{Y}_i + n^{-1}\sum_{i=1}^{n}\widehat{e}_i \\
\Leftrightarrow \quad & \overline{Y} = n^{-1}\sum_{i=1}^{n}\widehat{Y}_i + n^{-1}\sum_{i=1}^{n}\widehat{e}_i
\end{aligned}$$

Step 2: The equation specified in the problem is then only true if $\sum\limits_{i=1}^{n} \widehat{e}_i = 0$. X is a matrix of the form:

$$\begin{bmatrix} x_{11} & x_{12} & \cdots x_{1,k-1} & 1 \\ x_{21} & x_{22} & \cdots x_{2,k-1} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots x_{n,k-1} & 1 \end{bmatrix}$$

Partition X into $\begin{bmatrix} X^* & O \end{bmatrix}$ such that X^* is a $n \times k - 1$ matrix of the x observations and O is a $n \times 1$ vector of ones. We want the sum of the residuals to be zero, and recall that we showed (in matrix form) in Exercise 3.05 that $X'\widehat{e} = 0$. Also note that the transpose of a partitioned matrix is like the transpose of a regular matrix except the elements are also transposed. So,

$$\begin{aligned} X'\widehat{e} &= \begin{bmatrix} X^{*\prime} \\ O' \end{bmatrix} \widehat{e} \\ &= \begin{bmatrix} X^{*\prime}\widehat{e} \\ O'\widehat{e} \end{bmatrix} \\ &= 0 \end{aligned}$$

Then $O'\widehat{e} = \sum\limits_{i=1}^{n} \widehat{e}_i = 0$.

3.14 Let $\widehat{\beta}_n = (\mathbf{X_n'X_n})^{-\mathbf{1}}\mathbf{X_n'Y_n}$ denote the OLS estimate where $\mathbf{Y_n}$ is $n \times 1$ and $\mathbf{X_n}$ is $n \times k$. Prove that the OLS estimate computed using an additional observation (Y_{n+1}, X_{n+1}) is

$$\widehat{\beta}_{n+1} = \widehat{\beta}_n + \frac{1}{1 + X_{n+1}'(X_n'X_n)^{-1}X_{n+1}}(X_n'X_n)^{-1}X_{n+1}(Y_{n+1} - X_{n+1}'\widehat{\beta}_n)$$

Answer:

Step 1: Apply the Woodbury Matrix Identity. Let

$$\begin{aligned} A &= X_n'X_n \\ B &= X_{n+1} \\ D &= X_{n+1}' \end{aligned}$$

The choice of D is a bit arbitrary. I think setting the last term equal to C would work, but this way the '1' is already present and the equation is a bit easier to manipulate. Then, since $C = I = 1$ because a single observation is 1 by 1.

$$\begin{aligned}(A + BCD)^{-1} &= A^{-1} - A^{-1}BC\left(C + CDA^{-1}BC\right)^{-1}CDA^{-1} \\ &= A^{-1} - A^{-1}B(1 + DA^{-1}B)DA^{-1}\end{aligned}$$

Step 2: Simplify. The terms of the expanded equation are:

$$\begin{aligned}Term1 &= A^{-1}X_n'Y_n \\ &= (X_n'X_n)^{-1}X_n'Y_n \\ &= \widehat{\beta}_n\end{aligned}$$

$$\begin{aligned}Term2 &= A^{-1}X_{n+1}Y_{n+1} \\ &= (X_n'X_n)^{-1}(X_{n+1}Y_{n+1})\end{aligned}$$

$$\begin{aligned}Term3 &= (X_n'X_n)^{-1}X_{n+1}\left[1 + X_{n+1}'(X_n'X_n)^{-1}X_{n+1}\right]^{-1}X_{n+1}'(X_n'X_n)^{-1}X_n'Y_n \\ &= (X_n'X_n)^{-1}X_{n+1}\left[1 + X_{n+1}'(X_n'X_n)^{-1}X_{n+1}\right]^{-1}X_{n_1}'\widehat{\beta}_n\end{aligned}$$

$$Term4 = (X_n'X_n)^{-1}X_{n+1}\left[1 + X_{n+1}'(X_n'X_n)^{-1}X_{n+1}\right]X_{n+1}'(X_n'X_n)^{-1}X_{n+1}Y_{n+1}$$

Step 3: Combine terms. Note that the term starting with '1 + ...' is a scalar, call it L. Call the term $Z := (X_n'X_n)^{-1}X_{n+1}$. Note also that $L = 1 + X_{n+1}'Z$. The observations are also scalars and may be rearranged. So:

$$
\begin{aligned}
\widehat{\beta}_{n+1} &= Term1 + Term2 - Term3 - Term4 \\
&= \widehat{\beta}_n + ZY_{n+1} - \frac{1}{L} ZX'_{n+1}\widehat{\beta}_n - \frac{1}{L} ZX'_{n+1}ZY_{n+1} \\
&= \widehat{\beta}_n + ZY_{n+1}\frac{L}{L} - \frac{1}{L} ZZX'_{n+1}Y_{n+1} - -\frac{1}{L} ZX'_{n+1}\widehat{\beta}_n \\
&= \widehat{\beta}_n + \frac{1}{L}(ZY_{n+1} + ZZX'_{n+1}Y_{n+1} - ZZX'_{n+1}Y_{n+1} - ZX'_{n+1}\widehat{\beta}_n) \\
&= \widehat{\beta}_n + \frac{1}{L} Z(Y_{n+1} - X'_{n+1}\widehat{\beta}_n)
\end{aligned}
$$

3.19 For the intercept-only model $Y_i = \beta + e_i$, show that the leave-one-out prediction error is

$$
\widetilde{e} = \left(\frac{n}{n-1}\right)\left(Y_i - \overline{Y}\right)
$$