Math Notes

Canine-Table

Github

POSIX Nexus serves as a comprehensive cross-language reference hub that explores the implementation and behavior of POSIX-compliant functionality across a diverse set of programming environments. Built atop the foundational IEEE Portable Operating System Interface (POSIX) standards, this project emphasizes compatibility, portability, and interoperability between operating systems.

Abstract

Contents

I Algebra

I Algebraic Identities

Difference of Squares

- **Definition** ⇒ Product of sum and difference equals difference of squares
- **General Formula** ⇒ $a^2 - b^2 = (a - b)(a + b)$
- **Pattern** ⇒ Always collapses into two linear factors
- **Example** ⇒ $x^2 - 9 = (x - 3)(x + 3)$
- **Extension** ⇒ Used in rationalizing denominators and factoring quadratics

Sum of Cubes

- **Definition** ⇒ Sum of cubes factors into binomial and trinomial
- **General Formula** ⇒ $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
- **Pattern** ⇒ Binomial carries the sum, trinomial balances with alternating signs
- **Example** ⇒ $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$
- **Extension** ⇒ Pairs with difference of cubes for full cubic factorization

Difference of Cubes

- **Definition** ⇒ Difference of cubes factors into binomial and trinomial
- **General Formula** ⇒ $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
- **Pattern** ⇒ Binomial carries the difference, trinomial balances with all positives
- **Example** ⇒ $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$
- **Extension** ⇒ Complements sum of cubes in cubic identities

Perfect Square Trinomial

- **Definition** $\Rightarrow$ Squaring a binomial produces a trinomial pattern
- **General Formula** $\Rightarrow$ $(a+b)^2 = a^2 + 2ab + b^2$
- **Negative Case** $\Rightarrow$ $(a-b)^2 = a^2 - 2ab + b^2$
- **Pattern** $\Rightarrow$ Always: square + double product + square
- **Example** $\Rightarrow$ $(x+3)^2 = x^2 + 6x + 9$
- **Extension** $\Rightarrow$ Recognizing this pattern speeds up factoring and simplification

Exponent and Radical Identities

- **Power of a Power** $\Rightarrow$ $(a^m)^n = a^{mn}$
- **Negative Exponent** $\Rightarrow$ $a^{-n} = \frac{1}{a^n}$
- **Fractional Exponent** $\Rightarrow$ $\sqrt[n]{a^m} = a^{m/n}$
- **Product Rule** $\Rightarrow$ $a^m \cdot a^n = a^{m+n}$
- **Quotient Rule** $\Rightarrow$ $\frac{a^m}{a^n} = a^{m-n}$
- **Extension** $\Rightarrow$ Links radicals, reciprocals, and powers into one unified system

Binomial Expansion

- **Definition** $\Rightarrow$ Expansion of $(a+b)^n$ into a sum of terms
- **General Formula** $\Rightarrow$ $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$
- **Binomial Coefficient** $\Rightarrow$ $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- **Pattern** $\Rightarrow$ Exponents of a decrease, exponents of b increase
- **Symmetry** $\Rightarrow$ Coefficients are symmetric: $\binom{n}{k} = \binom{n}{n-k}$
- **Connection** $\Rightarrow$ Coefficients form Pascal's Triangle
- **Example** $\Rightarrow$ $(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$

I Nomials

Nomial Lineage

- **Monomial** ⇒ One term only, e.g. $7x^2$
- **Binomial** ⇒ Two terms, e.g. $x + 2$
- **Trinomial** ⇒ Three terms, e.g. $x^2 + 3x + 2$
- **Polynomial** ⇒ Many terms, general family
- **Technique** ⇒ Prefix indicates number of terms
- **Outcome** ⇒ Classification helps organize algebraic expressions

$$\text{Monomial: } ax^n \tag{1}$$
$$\text{Binomial: } ax^n + bx^m \tag{2}$$
$$\text{Trinomial: } ax^n + bx^m + cx^k \tag{3}$$

Polynomial Forms

- **Monomial** ⇒ One term only, e.g. $7x^2$
- **Binomial** ⇒ Two unlike terms, e.g. $x + 2$
- **Trinomial** ⇒ Three terms, e.g. $x^2 + 3x + 2$
- **Polynomial** ⇒ General family with many terms
- **Technique** ⇒ Prefix indicates number of terms
- **Outcome** ⇒ Classification helps organize algebraic expressions

Like Terms

- **Definition** ⇒ Expressions with identical variable parts, e.g. $3x^2$ and $-5x^2$
- **Variable Match** ⇒ Same variables with same exponents
- **Coefficient** ⇒ Numbers in front may differ
- **Technique** ⇒ Combine by adding or subtracting coefficients
- **Outcome** ⇒ Simplifies polynomials by reducing to fewer terms

Coefficient

- **Definition** $\Rightarrow$ The number in front of a variable, e.g. in $7x$ the coefficient is 7
- **Variable Match** $\Rightarrow$ It scales the variable part without changing its type
- **Examples** $\Rightarrow$ $3x^2$ has coefficient 3, $-5y$ has coefficient -5
- **Constants** $\Rightarrow$ A constant term like 4 can be seen as coefficient 4 of x^0
- **Outcome** $\Rightarrow$ Coefficients tell how strongly each variable contributes to the polynomial

I Polynomial Exponents

Definition of Negative Exponent

- **Positive Exponent** $\Rightarrow$ x^n means multiply x by itself n times
- **Negative Exponent** $\Rightarrow$ x^{-n} means reciprocal of x^n
- **Rule** $\Rightarrow$ $x^{-n} = \frac{1}{x^n}$
- **Example** $\Rightarrow$ $x^{-3} = \frac{1}{x^3}$
- **Use** $\Rightarrow$ Negative exponents express division or reciprocals in algebra and calculus

$$x^3 = x \cdot x \cdot x \;\Rightarrow\; x^{-3} = \frac{1}{x^3}$$

Using Negative Exponents

- **Step 1** $\Rightarrow$ Recall exponent rules: $x^a \cdot x^b = x^{a+b}$
- **Step 2** $\Rightarrow$ Set $a = 3, b = -3$: $x^3 \cdot x^{-3} = x^0$
- **Step 3** $\Rightarrow$ But $x^0 = 1$
- **Step 4** $\Rightarrow$ So x^{-3} must equal $\frac{1}{x^3}$
- **Outcome** $\Rightarrow$ Negative exponent means reciprocal of positive power

Exponent Flip Examples

$$
\begin{aligned}
x^5 &= \frac{1}{x^{-5}} & x^{12} &= \frac{1}{x^{-12}} \quad (4)\\
x^{17} &= \frac{1}{x^{-17}} & 15x^9 &= \frac{15}{x^{-9}} \quad (5)\\
28x^5 &= \frac{28}{x^{-5}} & x^5 &= \frac{1}{x^{-5}} \quad (6)\\
x^{-3} &= \frac{1}{x^3} & x^3 &= \frac{1}{x^{-3}} \quad (7)\\
x^{-3} &= \frac{1}{x^3} & & \quad (8)
\end{aligned}
$$

Polynomial Exponent Rules Applied

- **Power of a Power** $\Rightarrow$ $(a^m)^n = a^{mn}$
- **Nested Powers** $\Rightarrow$ Combine: $8 \cdot 3 = 24$
- **Outer Flip** $\Rightarrow$ Apply (-9): $x^{768y^{5z^3}} \to x^{-6912y^{5z^3}}$
- **Technique** $\Rightarrow$ Multiply all exponents carefully, preserve inner structure
- **Outcome** $\Rightarrow$ Final simplified form: $x^{-6912y^{5z^3}}$

$$
\begin{aligned}
(((x^{32y^{5z^3}})^8)^3)^{-9} &= (x^{32y^{5z^3}})^{8\cdot 3} && \Rightarrow x^{768y^{5z^3}}\\
&= \left(x^{768y^{5z^3}}\right)^{-9} && \Rightarrow x^{-9\cdot 768y^{5z^3}}\\
&= x^{-6912y^{5z^3}}
\end{aligned}
$$

Exponent Rules Applied

- **Power of a Power** $\Rightarrow$ $(a^m)^n = a^{mn}$
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$$
\begin{aligned}
x^2 \cdot x^4 &= x^{2+4} && \Rightarrow\ x^6 \\
x^7 \cdot x^5 &= x^{5+7} && \Rightarrow\ x^{12} \\
x^8 \cdot x^9 &= x^{8+9} && \Rightarrow\ x^{17} \\
(3x^3)(5x^6) &= (3 \cdot 5)x^{3+6} && \Rightarrow 15x^9 \\
(4x^2)(7x^3) &= (4 \cdot 7)x^{2+3} && \Rightarrow 28x^5 \\
(4xy^2)(8x^2y^3) &= (4 \cdot 8)x^{1+2}(1 \cdot 1)y^{2+3} && \Rightarrow\ 32x^3y^5 \\
(5x^2y^3)(6x^3y^4) &= (5 \cdot 6)x^{2+3}(1 \cdot 1)y^{3+4} && \Rightarrow\ 30x^5y^7 \\
(7x^3y^4)(8x^5y^7) &= (7 \cdot 8)x^{3+5}(1 \cdot 1)y^{4+7} && \Rightarrow\ 56x^8y^{11}
\end{aligned}
$$

$$
\begin{aligned}
(x^3)^4 &= x^{3\cdot 4} = x^{12} \\
(x^4)^6 &= x^{4\cdot 6} = x^{24} \\
(x^3)^5 &= x^{3\cdot 5} = x^{15} \\
(3x^2)^4 &= 3^{1\cdot 4}x^{2\cdot 4} = 3^4x^8 = 81x^8 \\
(2x^3)^3 &= 2^{1\cdot 3}x^{3\cdot 3} = 2^3x^9 = 8x^9
\end{aligned}
$$

$$
\begin{aligned}
(3x^2)^2(2x^3)^3 &= 3^{1\cdot 2}x^{2\cdot 2}2^{1\cdot 3}x3 \cdot 3 \\
&= 3^2x^42^3x^9 \\
&= 9 \cdot 8x^{4+9} \\
&= 72x^{13}
\end{aligned}
$$

$$
\begin{aligned}
(3^2x^3y^4)^2(2^3x^2y^5)^3 &= (3^{2\cdot 2}x^{3\cdot 2}y^{4\cdot 2})(2^{3\cdot 3}x^{2\cdot 3}y^{5\cdot 3}) \\
&= (3^4x^6y^8)(2^9x^6y^{15}) \\
&= (81x^6y^8)(512x^6y^{15}) \\
&= 81 \cdot 512x6 + 6y^{8+15} \\
&= 41472x^{12}y^{23}
\end{aligned}
$$

$$
\begin{aligned}
-2^3 &= -2 \cdot -2 \cdot -2 = -8 \\
(-2)^3 &= -2 \cdot -2 \cdot -2 = -8 \\
-(-2)^3 &= -2 \cdot -2 \cdot -2 = 8
\end{aligned}
$$

$$
\begin{aligned}
(-7x^2y^3)^0 &= -7^{2\cdot 0}x^{2\cdot 0}y^{3\cdot 0} \\
&= -7^0x^0y^0 \\
&= 1 \cdot 1 \cdot 1 \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
3x(5x+8) &= 15x^2 + 24x \\
4x(x^2 - 2x + 3) &= 4x^3 - 8x^2 + 12x
\end{aligned}
$$

I Dividing Polynomials

$$
\begin{aligned}
\frac{x^8}{x^3} &= x^{8-3} \Rightarrow x^5 && (9) \\
\frac{x^5}{x^2} &= x^{5-2} \Rightarrow x^3 && (10) \\
\frac{x^5}{x^8} &= x^{5-8} \Rightarrow x^{-3} && (11) \\
\frac{x^4}{x^7} &= x^{4-7} \Rightarrow x^{-3} && (12) \\
& && (13)
\end{aligned}
$$

$$
\begin{aligned}
\frac{24x^9y^5}{8x^3y^{12}} &= \frac{24}{8}x^{9-3}\frac{1}{1}y^{5-12} \\
&= 3x^6y^{-7} \\
&= \frac{3x^6}{y^7}
\end{aligned}
$$

$$\begin{aligned}\frac{12x^5y^{-3}z^4}{36x^8y^{-4}z^{-8}} &= \frac{\frac{12}{3}x^{5-8}}{\frac{36}{3}}\frac{1}{1}y^{-3--4}\frac{1}{1}z^{4--8}\\ &= \frac{x^{-3}y^1z^{12}}{3}\\ &= \frac{yz^{12}}{3x^3}\end{aligned}$$

I Multiplying Polynomials

Multiplication Symbols

- **Dot** $\Rightarrow$ $\cdot$ is clean, algebraic, avoids confusion with x
- **Times** $\Rightarrow$ $\times$ is bold, arithmetic, or cross product
- **Context** $\Rightarrow$ Use $\cdot$ in algebra, $\times$ in arithmetic or vectors
- **Technique** $\Rightarrow$ Choose based on clarity and audience
- **Outcome** $\Rightarrow$ Both mean multiplication, but notation signals intent

$$x^2 \cdot x^4 = x^{2+4} \quad \Rightarrow \quad x^6$$

$$x^7 \cdot x^5 = x^{5+7} \quad \Rightarrow \quad x^{12}$$

$$x^8 \cdot x^9 = x^{8+9} \quad \Rightarrow \quad x^{17}$$

$$(3x^3)(5x^6) = (3 \cdot 5)x^{3+6} \quad \Rightarrow 15x^9$$

$$(4x^2)(7x^3) = (4 \cdot 7)x^{2+3} \quad \Rightarrow 28x^5$$

$$(4xy^2)(8x^2y^3) = (4 \cdot 8)x^{1+2}(1 \cdot 1)y^{2+3} \quad \Rightarrow \; 32x^3y^5$$

$$(5x^2y^3)(6x^3y^4) = (5 \cdot 6)x^{2+3}(1 \cdot 1)y^{3+4} \quad \Rightarrow \; 30x^5y^7$$

$$(7x^3y^4)(8x^5y^7) = (7 \cdot 8)x^{3+5}(1 \cdot 1)y^{4+7} \quad \Rightarrow \; 56x^8y^{11}$$

I Combining Polynomials

$$x + 4 = 7 \;\Rightarrow\; x + 4 - 4 = 7 - 4 \;\Rightarrow\; x + 0 = 3$$

$$x + 9 = 15 \;\Rightarrow\; x + 9 - 9 = 15 - 9 \;\Rightarrow\; x + 0 = 6$$

$$6 + x = 13 \;\Rightarrow\; 6 + x - 6 = 15 - 9 \;\Rightarrow\; 0 + x - 0 = 7$$

$$x - 3 = 9 \;\Rightarrow\; x - 3 + 3 = 9 - 3 \;\Rightarrow\; x - 0 = 6 \;\Rightarrow\; x = 6$$

$$x - 8 = 7 \;\Rightarrow\; x + 8 - 8 = 7 + 8 \;\Rightarrow\; x - 0 = 15 \;\Rightarrow\; x = 15$$

$$3x+5=11 \Rightarrow 3x+5-5=11-5 \Rightarrow \frac{3x}{3}=\frac{6}{3} \Rightarrow x=2$$

$$6.3=-2+x \Rightarrow 6.3+2=-2+2+x \Rightarrow 6.3=0+x \Rightarrow x=6.3$$

$$5=x-8 \Rightarrow 5+8=x-8+8 \Rightarrow 13=x-0 \Rightarrow$$
$$\hookrightarrow x=13$$

$$5-x=12 \Rightarrow 5-12-x+x=12-12+x \Rightarrow$$
$$\hookrightarrow -7=0+x \Rightarrow x=-7$$

$$-8=5-x \Rightarrow -8+8=5+8-x \Rightarrow 0+x=13-x+x \Rightarrow$$
$$\hookrightarrow x=13$$

$$3x=12 \Rightarrow \frac{3x}{3}=\frac{12}{3} \Rightarrow \frac{x}{1}=4 \Rightarrow x=4$$

$$7x=14 \Rightarrow \frac{7x}{7}=\frac{12}{7} \Rightarrow \frac{x}{1}=2 \Rightarrow x=2$$

$$-6x=-30 \Rightarrow \frac{-6x}{-6}=\frac{-30}{-6} \Rightarrow \frac{-x}{-1}=5 \Rightarrow x=5$$

$$-8x = 48 \Rightarrow \frac{-8x}{-8} = \frac{48}{-8} \Rightarrow \frac{-x}{-1} = -6 \Rightarrow x = -6$$

$$7x = -56 \Rightarrow \frac{7x}{7} = \frac{-56}{7} \Rightarrow \frac{x}{1} = -8 \Rightarrow x = -8$$

$$-8x = -72 \Rightarrow \frac{-8x}{-8} = \frac{-72}{-8} \Rightarrow \frac{x}{1} = 9 \Rightarrow x = 9$$

$$\begin{aligned} 4x + 3 = 6x - 15 \Rightarrow 4x - 4x + 3 + 15 = 6x - 4x - 15 + 15 \Rightarrow \\ \hookrightarrow \frac{18}{2} = \frac{2x}{2}; \Rightarrow x = 9 \end{aligned}$$

$$\begin{aligned} 3(2x-4) = 5(3x+2) - 3 \Rightarrow 6x - 12 = 15x + 10 - 3 \Rightarrow \\ \hookrightarrow 6x - 6x - 12 - 7 = 15x - 6x + 7 - 7 \Rightarrow \\ \hookrightarrow \frac{-19}{9} = \frac{9x}{9} \Rightarrow x = \frac{-19}{9} \end{aligned}$$

$$\begin{aligned} \frac{3}{4}x - \frac{2}{3} = 12 \Rightarrow (\frac{3}{4}x \cdot 4)3 - (\frac{2}{3} \cdot 3)4 = 12 \cdot 12 \\ \hookrightarrow (3x)3 - (2)4 = 144 \Rightarrow 9x - 8 + 8 = 144 + 8 \Rightarrow \\ \hookrightarrow \frac{9x}{9} = \frac{152}{9} \Rightarrow x = \frac{152}{9} \end{aligned}$$

$$\begin{aligned} \frac{2}{3}x + 5 = 8 \Rightarrow (\frac{2}{3}x + 5 = 8)3 \Rightarrow 2x + 15 - 15 = 24 - 15 \Rightarrow \\ \hookrightarrow \frac{2x}{2} = \frac{9}{2} \Rightarrow x = \frac{9}{2} \end{aligned}$$

$$
\begin{aligned}
(3x+5)+(4x-2) &= 7x+3 \\
(4x^2+3x+9)+(5x^2+7x-4) &= 9x^2+10x+5 \\
(5x^2-6x-12)-(7x^2+4x-13) &= 5x^2-6x-12-7x^2-4x+13 \\
&= 5x^2-7x^2-12+13-6x-4x \\
&= -2x^2+1-10x
\end{aligned}
$$

I FOIL Method

$$(a+b)(c+d) = ac+ad+bc+bd \tag{14}$$

FOIL Method

- **First** ⇒ Multiply first terms: $a \cdot c$
- **Outer** ⇒ Multiply outer terms: $a \cdot d$
- **Inner** ⇒ Multiply inner terms: $b \cdot c$
- **Last** ⇒ Multiply last terms: $b \cdot d$
- **Outcome** ⇒ Sum all four products to get the expanded expression

$$
\begin{aligned}
(2x+5)(4x^2-3x+6) &= (2\cdot 4)x^{1+2}+(2\cdot -3)^{1+1}+(2\cdot 6)x \\
&\hookrightarrow +(5\cdot 4)x^2+(5\cdot -3)x+(5\cdot 6) \\
&= 8x^3+(-6+20)^2+(12+-15)x+30 \\
&= 8x^3+14x^2+-3x+30
\end{aligned}
$$

$$
\begin{aligned}
(3x^2-2x+4)(4x^2+5x+-6) &= \\
\hookrightarrow (3\cdot 4)x^{2+2}+(3\cdot 5)x^{2+1}+(3\cdot -6)x^{2+0} \\
\hookrightarrow +(-2\cdot 4)x^{1+2}+(-2\cdot 5)x^{1+1}+(-2\cdot -6)x^{1+0} \\
\hookrightarrow +(4\cdot 4)x^{0+2}+(4\cdot 5)x^{0+1}+(4\cdot -6)x^{0+0} \\
=12x^4+(15+-8)x^3+(-18+-10+16)x^2+(12+20)x+-24 \\
=12x^4+7x^3+-12x^2+32x+-24
\end{aligned}
$$

I Factoring Polynomials

$4a^2+2ab–3a^2b+5$

Terms	Factors	Prime Factors
$4a^2$	$4, a^2$	$2, 2, a, a$
$2ab$	$3, a, b$	$2, a, b$
$-3a^2b$	$-3, a^2, b$	$-3, a, a, b$
5	5	5

$xy^2-3x^2y^2-6y+z$

Terms	Factors	Prime Factors
xy^2	$4, a^2$	$2, 2, a, a$
$-3x^2y$	$3, a, b$	$2, a, b$
$-6y$	$-6, y$	$-2, 3, y$
z	z	z

$-5+2(3a^2-3t)$

Terms	Factors	Prime Factors
-5	-5	-5
$2(3a^2-3t)$	$2.3a^2-3t$	$2.3a^2-3t$
$6t$	$6, t$	$3, 3, t$

$3x^2 + 5x - 2$

Terms	Factors	Prime Factors
$3x^2$	$3, x^2$	$3.x.x$
$5x$	$5, x$	$5, x$
-2	-2	-2

Definition of Prime Factor

- **Prime** ⇒ A number greater than 1 divisible only by 1 and itself
- **Factor** ⇒ A number that divides another evenly
- **Prime Factor** ⇒ A prime number that divides another number exactly
- **Example** ⇒ 60 = $2^2 \cdot 3 \cdot 5$; prime factors are 2, 3, 5
- **Use** ⇒ Prime factors are the building blocks of integers, used in LCM, GCD, and simplification

Find LCM of 12 and 18

$$\begin{aligned} 12 &= 2^2 \cdot 3 && \Rightarrow \\ 18 &= 2 \cdot 3^2 && \Rightarrow \\ \text{LCM} &= 2^2 \cdot 3^2 && \hookrightarrow 36 \end{aligned}$$

Using Prime Factors for LCM

- **Step 1** ⇒ Prime factorize each number
- **Step 2** ⇒ Collect all distinct primes
- **Step 3** ⇒ Take the highest power of each prime
- **Step 4** ⇒ Multiply them together
- **Outcome** ⇒ LCM(12,18) = 36

Euclidean Modulus

```
define nx_pt_mod(x, y) {
                x = nx_abs(x)
                if (x == 0)
                                return 0
                y = nx_abs(y)
```

```
                if (y > 0)
                        return x - y * nx_pt_trunc(x / y)
                print "<nx:impurity/>"
                return -1
}
```

The Greatest Common Factor

```
define nx_euc(x, y) {
                auto n
                if (x == y)
                        return x
                while (x > 0 && y > 0) {
                        n = x
                        x = nx_pt_mod(y, x)
                        y = n
                }
                return n
}
```

$$(8,12) \mapsto \gcd(8,12) = 4$$
$$8x + 12 \;\Rightarrow\; 4(\frac{8x}{2} + \frac{12}{4}) \;\Rightarrow\; 4(2x+3)$$

$$(4,2) \mapsto \gcd(4,2) = 2$$
$$4x^2 + 2x \;\Rightarrow\; 2x(\frac{4x^2}{2x} + \frac{2x}{2x}) \;\Rightarrow\; 2x(2x+1)$$

$$(12,18) \mapsto \gcd(12,18) = 6$$
$$12ab^2 + 18a^2b^3 \;\Rightarrow\; 6ab^2(2+3ab)$$

Definition of Perfect Square

- **Perfect Square** ⇒ A number that can be expressed as n^2 for some integer n
- **Integer Squared** ⇒ Formed by multiplying an integer by itself
- **Examples** ⇒ $1, 4, 9, 16, 25, 36, \ldots$
- **Non-Examples** ⇒ $2, 3, 5, 6, 7, 10, \ldots$
- **Use** ⇒ Perfect squares appear in factoring, radicals, and Pythagorean identities

Check if 49 is a perfect square

$$\begin{aligned} 49 &= 7 \cdot 7 && \Rightarrow \\ &= 7^2 && \hookrightarrow \end{aligned}$$

Therefore, 49 is a perfect square.

Using Perfect Squares

- **Step 1** ⇒ Identify the number
- **Step 2** ⇒ Ask if it can be written as n^2
- **Step 3** ⇒ If yes, it is a perfect square
- **Step 4** ⇒ If no, it is not
- **Outcome** ⇒ 49 is a perfect square since $49 = 7^2$

Difference of Squares Applied

- **Identity** ⇒ $(a^2 - b^2) = (a-b)(a+b)$
- **Example** ⇒ $x^2 - 9$
- **Factorization** ⇒ Apply rule: $x^2 - 3^2 = (x-3)(x+3)$
- **Technique** ⇒ Recognize perfect squares and subtract
- **Outcome** ⇒ Final factored form: $(x-3)(x+3)$

$$x^2 - 9 \;\Rightarrow\; x^2 - 3^2 \;\Rightarrow\; (x-3)(x+3)$$

Perfect Squares Difference

- **Identity** $\Rightarrow$ $(a^2 - b^2) = (a-b)(a+b)$
- **Example** $\Rightarrow$ $x^2 - 9$
- **Factorization** $\Rightarrow$ $(x-3)(x+3)$
- **Technique** $\Rightarrow$ Spot the squares, apply the difference rule
- **Outcome** $\Rightarrow$ Factored polynomial form

$$(25) \mapsto \sqrt{(25)} = 5$$
$$x^2 - 25 \;\Rightarrow\; (x+5)(x-5)$$

$$(x-5)(x+5) \Rightarrow\; x^2 + 5x + -5x + -25 \Rightarrow$$
$$\hookrightarrow\; x^2 + (5x + -5x \Rightarrow\; 0) - 25 \Rightarrow\; x^2 - 25$$

$$(9) \mapsto \sqrt{(9)} = 3$$
$$x^2 - 9 \;\Rightarrow\; (x+3)(x-3)$$

$$(4) \mapsto \sqrt{(4)} = 2$$
$$x^2 - 4 \;\Rightarrow\; (x+2)(x-2)$$

$$4x^2 - 25 \;\Rightarrow\; (2x+5)(2x-5)$$

$$(81) \mapsto \sqrt{(81)} = 9$$
$$(16) \mapsto \sqrt{(16)} = 4$$
$$16x^2 - 25 \;\Rightarrow\; (4x+9)(4x-9)$$

$$25x^2 - 16y^2 \;\Rightarrow\; (5x+4y)(5x-4y)$$

$$81x^4 - 16y^8 \;\Rightarrow\; (9x^2+4y^4)(9x^2-4y^4) \;\Rightarrow\; (3x+2y^2)(3x-2y^2)$$

Factor by Grouping Applied

- **Setup** $\Rightarrow$ Polynomial with 4 terms
- **Grouping** $\Rightarrow$ Split into two pairs
- **Inner Factor** $\Rightarrow$ Factor each pair separately
- **Common Binomial** $\Rightarrow$ Extract the shared binomial
- **Outcome** $\Rightarrow$ Final factored form

$$\begin{aligned} &x^3+3x^2+2x+6 \;\Rightarrow\; (x^3+3x^2)+(2x+6) \;\Rightarrow \\ &\hookrightarrow x^2(x+3)+2(x+3) \;\Rightarrow\; (x^2+2)(x+3) \end{aligned}$$

$$\begin{aligned} &x^3-4x^2+3x-12 \;\Rightarrow\; x^2(x-4)+3(x-4) \;\Rightarrow \\ &\hookrightarrow \frac{x^2(x-4)}{x-4}+\frac{3(x-4)}{x-4} \;\Rightarrow \\ &\hookrightarrow (x-4)(x^2+3) \end{aligned}$$

$$
\begin{aligned}
2x^3 - 6x^2 + 4x - 12 \;\Rightarrow\; 2x^2(x-3) + 4(x-3) \;\Rightarrow& \\
\hookrightarrow \frac{2x^2(x-3)}{x-3} + \frac{4(x-3)}{x-3} \;\Rightarrow& \\
\hookrightarrow (x-3)(2x^2+4)&
\end{aligned}
$$

$$
\begin{aligned}
3x^3 + 8x^2 - 6x - 16 \;\Rightarrow\; x^2(3x+8) - 2(3x+8) \;\Rightarrow& \\
\hookrightarrow \frac{x^2(3x+8)}{3x+8} + \frac{-2(3x+8)}{3x+8} \;\Rightarrow& \\
\hookrightarrow (x^2-2)(3x+8) \;\Rightarrow& \\
\hookrightarrow 3x^3 - 6x + 8x^2 - 16 \;\Rightarrow\; 3x(x^2-2) + 8(x^2-2) \;\Rightarrow& \\
\hookrightarrow \frac{3x(x^2-2)}{x^2-2} + \frac{8(x^2-2)}{x^2-2} \;\Rightarrow& \\
\hookrightarrow (x^2-2)(3x+8)&
\end{aligned}
$$

Perfect Square Trinomial

- **Definition** $\Rightarrow$ A trinomial formed by squaring a binomial
- **General Form** $\Rightarrow$ $(a+b)^2 = a^2 + 2ab + b^2$
- **Expansion** $\Rightarrow$ First term squared, double product of terms, last term squared
- **Example** $\Rightarrow$ $(x+3)^2 = x^2 + 6x + 9$
- **Negative Case** $\Rightarrow$ $(a-b)^2 = a^2 - 2ab + b^2$
- **Pattern** $\Rightarrow$ Always: square + double product + square

I Expanding Polynomials

Binomial Expansion

- **Definition** ⇒ Expansion of $(a+b)^n$ into a sum of terms
- **General Formula** ⇒ $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$
- **Binomial Coefficient** ⇒ $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- **Pattern** ⇒ Exponents of a decrease, exponents of b increase
- **Symmetry** ⇒ Coefficients are symmetric: $\binom{n}{k} = \binom{n}{n-k}$
- **Connection** ⇒ Coefficients form Pascal's Triangle
- **Example** ⇒ $(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$

Pascal's Triangle Connection

- **Structure** ⇒ Pascal's Triangle is built by starting with 1 at the top, each entry below is the sum of the two above
- **Coefficients** ⇒ Row n of Pascal's Triangle gives the coefficients for $(a+b)^n$
- **Symmetry** ⇒ Coefficients are symmetric: $\binom{n}{k} = \binom{n}{n-k}$
- **Example Row 4** ⇒ Coefficients $1, 4, 6, 4, 1$ expand $(a+b)^4$
- **Combinatorics** ⇒ Each coefficient counts the number of ways to choose k items from n
- **Extension** ⇒ Pascal's Triangle also encodes identities: hockey-stick pattern, Fibonacci connections, binomial sums

Hockey-Stick Identity

- **Pattern** ⇒ Pick a diagonal in Pascal's Triangle, sum entries until a row
- **Result** ⇒ The sum equals the entry just below and to the right (like a hockey stick)
- **Formula** ⇒ $\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$
- **Example** ⇒ $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} = \binom{5}{3}$
- **Visual** ⇒ The diagonal entries form the "shaft," the final entry is the "blade"

Fibonacci Connection

- **Pattern** $\Rightarrow$ Sum shallow diagonals of Pascal's Triangle
- **Result** $\Rightarrow$ The sums produce Fibonacci numbers
- **Example** $\Rightarrow$ Row sums: $1, 1, 2, 3, 5, 8, 13, \ldots$
- **Formula** $\Rightarrow$ $\mathrm{Fib}(n) = \sum \binom{n-k}{k}$
- **Visual** $\Rightarrow$ Diagonal "paths" through Pascal's Triangle trace Fibonacci lineage

Binomial Sums

- **Row Sum** $\Rightarrow$ $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
- **Alternating Sum** $\Rightarrow$ $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$
- **Weighted Sum** $\Rightarrow$ $\sum_{k=0}^{n} k \binom{n}{k} = n \cdot 2^{n-1}$
- **Combinatorial Meaning** $\Rightarrow$ Row sum counts all subsets of an n-element set
- **Probability Link** $\Rightarrow$ Binomial distribution probabilities sum to 1 using these identities

I Lowest Common Denominator

refer to I to see the euc function.

The Lowest Common Denominator

```
define nx_lcd(x, y) {
            return x * y / nx_euc(x, y)
}
```

$$\frac{3}{4} + \frac{2}{5} \Rightarrow \frac{3 \cdot 5}{4 \cdot 5} + \frac{2 \cdot 4}{5 \cdot 4} \Rightarrow \frac{15}{20} + \frac{8}{20} \Rightarrow \frac{15 + 8}{20} \Rightarrow \frac{23}{20}$$

$$\frac{5}{6} + \frac{4}{7} \Rightarrow \frac{5 \cdot 7}{6 \cdot 7} + \frac{4 \cdot 6}{7 \cdot 6} \Rightarrow \frac{35}{42} + \frac{24}{42} \Rightarrow \frac{35 - 24}{42} \Rightarrow \frac{11}{42}$$

$$\frac{7}{5} \cdot \frac{4}{3} \Rightarrow \frac{7 \cdot 4}{5 \cdot 3} \Rightarrow \frac{28}{15} \Rightarrow 1\frac{13}{15}$$

$$(18, 20) \mapsto \gcd(18, 20) = 2$$
$$\frac{3}{5} \cdot \frac{6}{4} \Rightarrow \frac{3 \cdot 6}{5 \cdot 4} \Rightarrow \frac{18}{20} \Rightarrow \frac{\frac{18}{2}}{\frac{20}{2}} \Rightarrow \frac{9}{10}$$

$$(28, 63) \mapsto \gcd(28, 63) = 7$$
$$(56, 35) \mapsto \gcd(56, 35) = 7$$
$$\frac{28}{63} \cdot \frac{56}{35} \Rightarrow \frac{\frac{28}{7}}{\frac{63}{7}} \cdot \frac{\frac{56}{7}}{\frac{35}{7}} \Rightarrow \frac{4}{9} \cdot \frac{8}{5} \Rightarrow \frac{4 \cdot 8}{9 \cdot 5} \Rightarrow \frac{32}{45}$$

Definition of Keep-Change-Flip

- **Keep** $\Rightarrow$ Keep the first fraction exactly as it is
- **Change** $\Rightarrow$ Change the division sign to multiplication
- **Flip** $\Rightarrow$ Flip the second fraction (take its reciprocal)
- **Example** $\Rightarrow \frac{3}{4} \div \frac{2}{5}$
- **Outcome** $\Rightarrow \frac{3}{4} \times \frac{5}{2} = \frac{15}{8}$

$$\frac{3}{4} \div \frac{2}{5} \Rightarrow \frac{3}{4} \times \frac{5}{2}$$

Using Keep-Change-Flip

- **Step 1** $\Rightarrow$ Write the division problem
- **Step 2** $\Rightarrow$ Keep the first fraction
- **Step 3** $\Rightarrow$ Change division to multiplication
- **Step 4** $\Rightarrow$ Flip the second fraction
- **Step 5** $\Rightarrow$ Multiply across numerators and denominators
- **Outcome** $\Rightarrow$ Simplified fraction result

I Brackets

Curly Braces {}

- **Definition** $\Rightarrow$ Used to denote sets or grouping in mathematics, e.g. $\{1, 2, 3\}$
- **Programming** $\Rightarrow$ Common in code to enclose blocks of statements
- **LaTeX** $\Rightarrow$ Used to group arguments for commands
- **Example** $\Rightarrow$ $\{x \mid x > 0\}$ means the set of all positive x
- **Outcome** $\Rightarrow$ Curly braces signal structured grouping or set notation

Square Brackets []

- **Definition** $\Rightarrow$ Used for intervals, optional elements, or matrices
- **Interval** $\Rightarrow$ $[a, b]$ means all values between a and b, inclusive
- **Matrix** $\Rightarrow$ Brackets often enclose arrays of numbers
- **Example** $\Rightarrow$ $[2, 5]$ includes 2 and 5
- **Outcome** $\Rightarrow$ Square brackets emphasize inclusion or structured arrays

$$\begin{aligned} 8[-6+8(-2+4)] &= 8[-6+8\cdot 2] \\ &= 8[-6+16] \\ &= 8\cdot 10 \\ &= 80 \end{aligned}$$

Parentheses ()

- **Definition** ⇒ Used for grouping, order of operations, or function arguments
- **Math** ⇒ $(a+b)c$ ensures addition happens before multiplication
- **Functions** ⇒ $f(x)$ shows input to a function
- **Interval** ⇒ (a,b) means values strictly between a and b
- **Outcome** ⇒ Parentheses control grouping and precedence in math and logic

$$\begin{aligned}-(-(1-2)+(5)) &= -(-(-1)+5)\\ &= -(1+5)\\ &= -(6)\\ &= -6\end{aligned}$$

$$\begin{aligned}3\cdot(6-3+1)-4^2 &= 3\cdot 4-16\\ &= 12-16\\ &= -4\end{aligned}$$

$$(10,4)\mapsto \gcd(10,4)=2$$

$$\begin{aligned}\frac{2\cdot(5-1)+2}{4\cdot(2-1)} &\Leftrightarrow (2\cdot(5-1)+2)\div(4\cdot(2-1))\\ &= \frac{2\cdot 4+2}{4\cdot 1}\\ &= \frac{8+2}{4}\\ &= \frac{10}{4}\\ &= \frac{\frac{10}{2}}{\frac{4}{2}}\\ &= \frac{5}{2}\end{aligned}$$

$$
\begin{aligned}
5 + (1 \cdot 4 - (2 - 5) - 2) &= 4 - (-3) - 2 \\
&= 7 - 2 \\
&= 5
\end{aligned}
$$

Double Vertical Bars ||

- **Definition** ⇒ Used to denote absolute value or norm
- **Absolute Value** ⇒ $|x|$ is distance of x from 0
- **Norm** ⇒ $\|v\|$ is length of vector v
- **Example** ⇒ $|-5| = 5$, $\|(3,4)\| = 5$
- **Outcome** ⇒ Double bars measure magnitude or distance

$$
\begin{aligned}
4 + |3 - 9| &= 4 + |-6| && (15) \\
&= 4 + 6 && (16) \\
&= 10 && (17)
\end{aligned}
$$

Floor Brackets ⌊⌋

- **Definition** ⇒ Used for the floor function
- **Floor** ⇒ $\lfloor x \rfloor$ = greatest integer less than or equal to x
- **Example** ⇒ $\lfloor 3.7 \rfloor = 3$
- **Use** ⇒ Rounds down to nearest integer
- **Outcome** ⇒ Floor brackets capture downward rounding

Ceiling Brackets $\lceil\rceil$

- **Definition** ⇒ Used for the ceiling function
- **Ceiling** ⇒ $\lceil x \rceil$ = smallest integer greater than or equal to x
- **Example** ⇒ $\lceil 3.2 \rceil = 4$
- **Use** ⇒ Rounds up to nearest integer
- **Outcome** ⇒ Ceiling brackets capture upward rounding

I Dividing

Fraction Components

- **Numerator** ⇒ Top of the fraction, counts selected parts, e.g. in $\frac{3}{4}$ the numerator is 3
- **Denominator** ⇒ Bottom of the fraction, defines total equal parts, e.g. in $\frac{3}{4}$ the denominator is 4
- **Relationship** ⇒ Fraction = Numerator ÷ Denominator
- **Technique** ⇒ Numerator changes with quantity chosen, denominator fixes the partition size
- **Outcome** ⇒ Understanding both clarifies fraction meaning and operations

$$\frac{6-2^3}{2} = \frac{6-8}{2} \tag{18}$$
$$= \frac{-2}{2} \tag{19}$$
$$= -1 \tag{20}$$

Verbose Floating-Point Multiplication

- **Step 1** ⇒ Choose two floating-point numbers, e.g. 3.25 and 2.5
- **Step 2** ⇒ Express them as fractions: $3.25 = \frac{325}{100}$, $2.5 = \frac{25}{10}$
- **Step 3** ⇒ Multiply numerators: $325 \times 25 = 8125$
- **Step 4** ⇒ Multiply denominators: $100 \times 10 = 1000$
- **Step 5** ⇒ Form product fraction: $\frac{8125}{1000}$
- **Step 6** ⇒ Simplify fraction: $\frac{8125}{1000} = 8.125$
- **Step 7** ⇒ Restore decimal form: product is 8.125

$$
\begin{aligned}
10.4 &\Rightarrow \frac{10.4}{1}\frac{10.4}{1}\cdot 10 && \Rightarrow \Rightarrow \frac{104}{10}\\
1.3 &\Rightarrow \frac{1.3}{1} && \Rightarrow \frac{1.3}{1}\cdot 10 \Rightarrow \frac{13}{10}\\
1.5 &\Rightarrow \frac{1.5}{1} && \Rightarrow \frac{1.5}{1}\cdot 10 \Rightarrow \frac{15}{10}\\
-2.35 &\Rightarrow \frac{-2.35}{1} \Leftrightarrow \frac{2.35}{-1} && \Rightarrow \frac{-2.35}{1}\cdot 100 \Rightarrow \frac{-235}{100}
\end{aligned}
$$

$$(10056, 1950) \mapsto \gcd(10056, 1950) = 26$$

$$
\begin{aligned}
\frac{10.3^2 - (-2.35)^2}{1.3(1.5)} &= \frac{106.09 - 5.522}{1.95}\\
&= \frac{100.568}{1.95}\\
&= \frac{\frac{100568}{26}}{\frac{1950}{26}}\\
&= \frac{3868}{75}
\end{aligned}
$$

I Multiply

$$\left(\frac{103}{10}\right)^2 \Rightarrow \frac{103}{10}\cdot\frac{103}{10} \Rightarrow$$

			1	0	3
		×	1	0	3
			3	0	9
		0	0	0	0
+	1	0	3	0	0
	1	0	6	0	9

$$\frac{10609}{100} \Rightarrow 106.09$$

$$\left(\frac{-235}{100}\right)^2 \Rightarrow \frac{-235}{10} \cdot \frac{-235}{10} \Rightarrow$$

		−	2	3	5
	×	−	2	3	5
	1	0	5	1	5
		7	1	5	0
+	4	7	0	0	0
	5	7	6	6	5

$$\frac{57665}{100} \Rightarrow 576.65$$

I Radicals

Identity of $\sqrt[n]{a^n}$

- **Expression** $\Rightarrow$ $\sqrt[n]{a^n}$
- **Meaning** $\Rightarrow$ The n-th root of a^n
- **Simplification** $\Rightarrow$ Cancels the root and exponent, yielding a
- **Condition** $\Rightarrow$ Valid when $a \geq 0$ for real roots, or in complex domain otherwise
- **Example** $\Rightarrow$ $\sqrt[3]{2^3} = \sqrt[3]{8} = 2$
- **Extension** $\Rightarrow$ $\sqrt[n]{a^m} = a^{m/n}$

$$\sqrt[n]{a^n} = a$$

Definition of Radical

- **Radical** $\Rightarrow$ An expression that uses the root symbol $\sqrt{}$
- **Square Root** $\Rightarrow$ The most common radical, $\sqrt{x}$, meaning the number which squared gives x
- **Index** $\Rightarrow$ The small number above the radical, e.g. $\sqrt[3]{x}$ is the cube root
- **Radicand** $\Rightarrow$ The number or expression inside the radical sign
- **Example** $\Rightarrow$ $\sqrt{25} = 5$, $\sqrt[3]{8} = 2$
- **Use** $\Rightarrow$ Radicals are used to express roots, simplify algebraic expressions, and solve equations

$$\sqrt{50} \Rightarrow \sqrt{25 \cdot 2} \Rightarrow \sqrt{25} \cdot \sqrt{2} \Rightarrow 5\sqrt{2}$$

Using Radicals

- **Step 1** ⇒ Identify the radicand
- **Step 2** ⇒ Factor the radicand into perfect powers and leftovers
- **Step 3** ⇒ Simplify by taking the root of the perfect power
- **Step 4** ⇒ Leave the leftover inside the radical
- **Outcome** ⇒ $\sqrt{50} = 5\sqrt{2}$

Overview of nx_squares

- **Purpose** ⇒ Computes square roots using iterative scaling and subtraction
- **Input** ⇒ A single value x
- **Initialization** ⇒ Takes absolute value, sets scaling factor, prepares working variables
- **Scaling** ⇒ Doubles a seed until it exceeds the radicand
- **Iteration** ⇒ Subtracts scaled values, updates root approximation step by step
- **Condition** ⇒ Handles zero and invalid cases gracefully, returns early if needed
- **Output** ⇒ Final square root approximation, or truncated value if iteration fails

Square Roots

```
define nx_nr_sqrt(x) {
    auto y, p
    if (nx_xy_breach(x, 1) == -1)
        return -1
    y = x / 2
    p = 0
    while (y != p) {
        p = y
        y = (y + x / y) / 2
    }
    return y
}
```

Square Roots

```
define nx_squares(x) {
    auto a, s, b, y
    a = nx_abs(x)
    if (a == 0 || scale == 0)
        return x
    s = 1
    while (s < a)
        s = s * 2
    b = nx_scale(1)
    x = 0
    y = 0
    while (s > b) {
        if (s <= a) {
            a = a - s
            x = y
            y = nx_nr_sqrt(s)
            if (y == -1)
                return x
            if (x != 0)
                print x, ","
        }
        s = s / 3
    }
    return y
}
```

Definition of Vinculum

- **Vinculum** ⇒ The horizontal bar drawn over the radicand in a radical
- **Scope** ⇒ Shows exactly which terms are included under the radical
- **Fraction Use** ⇒ Also used as the bar separating numerator and denominator in fractions
- **Repeating Decimal** ⇒ Used to mark repeating digits, e.g. $0.\overline{3}$
- **Example** ⇒ In $\sqrt{a+b}$, the vinculum extends over $a+b$
- **Use** ⇒ Ensures clarity of grouping inside radicals, fractions, and repeating decimals

Definition of Radicand

- **Radicand** ⇒ The number or expression placed under the radical sign $\sqrt{}$
- **Role** ⇒ It is the quantity from which a root is extracted
- **Example** ⇒ In $\sqrt{25}$, the radicand is 25
- **Extended** ⇒ In $\sqrt[3]{8}$, the radicand is 8, and the index is 3
- **Scope** ⇒ The vinculum (bar) shows exactly which terms belong to the radicand
- **Outcome** ⇒ Identifying the radicand clarifies what is being rooted in the expression

Square Root Expansion

- **Radical** $\Rightarrow \sqrt{72}$
- **Factor** $\Rightarrow$ Break into perfect square and leftover: $72 = 36 \cdot 2$
- **Expand** $\Rightarrow \sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36} \cdot \sqrt{2}$
- **Simplify** $\Rightarrow \sqrt{36} = 6$
- **Outcome** $\Rightarrow \sqrt{72} = 6\sqrt{2}$

Square Root Truncation

- **Radical** $\Rightarrow \sqrt{72}$
- **Approximation** $\Rightarrow$ Leave as decimal: $\sqrt{72} \approx 8.485$
- **Truncation** $\Rightarrow$ Cut after two decimals: 8.48
- **Outcome** $\Rightarrow$ Truncated radical value ≈ 8.48

Types of Square Roots

- **Integer Square Root** $\Rightarrow$ The root of a whole number. It may simplify to a rational (e.g. $\sqrt{25} = 5$) or remain irrational (e.g. $\sqrt{2}$)
- **Fraction Square Root** $\Rightarrow$ The root of a ratio $\frac{p}{q}$, defined as $\sqrt{\frac{p}{q}} = \frac{\sqrt{p}}{\sqrt{q}}$
- **Rational Square Root** $\Rightarrow$ Occurs when both numerator and denominator are perfect squares, e.g. $\sqrt{\frac{9}{16}} = \frac{3}{4}$
- **Irrational Square Root** $\Rightarrow$ Occurs when either numerator or denominator is not a perfect square, e.g. $\sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3}$
- **Impurity Case** $\Rightarrow$ When the denominator equals 0, the expression is undefined, e.g. $\sqrt{\frac{p}{0}}$

I Linear Algebra

I Linear Equations

$$x + 4 = 7 \;\Rightarrow\; x + 4 - 4 = 7 - 4 \;\Rightarrow\; x + 0 = 3$$

$$x+9=15 \;\Rightarrow\; x+9-9=15-9 \;\Rightarrow\; x+0=6$$

$$6+x=13 \;\Rightarrow\; 6+x-6=15-9 \;\Rightarrow\; 0+x-0=7$$

$$x-3=9 \;\Rightarrow\; x-3+3=9-3 \;\Rightarrow\; x-0=6 \;\Rightarrow\; x=6$$

$$x-8=7 \;\Rightarrow\; x+8-8=7+8 \;\Rightarrow\; x-0=15 \;\Rightarrow\; x=15$$

$$6.3=-2+x \;\Rightarrow\; 6.3+2=-2+2+x \;\Rightarrow\; 6.3=0+x \;\Rightarrow\; x=6.3$$

$$5=x-8 \;\Rightarrow\; 5+8=x-8+8 \;\Rightarrow\; 13=x-0 \;\Rightarrow\; x=13$$

$$\begin{aligned} &5-x=12 \;\Rightarrow\; 5-12-x+x=12-12+x \;\Rightarrow \\ &\qquad\Leftrightarrow -7=0+x \;\Rightarrow\; x=-7 \end{aligned}$$

$$\begin{aligned} &-8=5-x \;\Rightarrow\; -8+8=5+8-x \;\Rightarrow \\ &\qquad\Leftrightarrow 0+x=13-x+x \;\Rightarrow\; x=13 \end{aligned}$$

$$3x=12 \;\Rightarrow\; \frac{3x}{3}=\frac{12}{3} \;\Rightarrow\; \frac{}{1}=4 \;\Rightarrow\; x=4$$

$$7x = 14 \Rightarrow \frac{7x}{7} = \frac{12}{7} \Rightarrow \frac{x}{1} = 2 \Rightarrow x = 2$$

$$-6x = -30 \Rightarrow \frac{-6x}{-6} = \frac{-30}{-6} \Rightarrow \frac{-x}{-1} = 5 \Rightarrow x = 5$$

$$-8x = 48 \Rightarrow \frac{-8x}{-8} = \frac{48}{-8} \Rightarrow \frac{-x}{-1} = -6 \Rightarrow x = -6$$

$$7x = -56 \Rightarrow \frac{7x}{7} = \frac{-56}{7} \Rightarrow \frac{x}{1} = -8 \Rightarrow x = -8$$

$$-8x = -72 \Rightarrow \frac{-8x}{-8} = \frac{-72}{-8} \Rightarrow \frac{x}{1} = 9 \Rightarrow x = 9$$

I Coordinate Plane

Domain

- **Definition** $\Rightarrow$ The set of all possible input values x for a function
- **Coordinate Plane** $\Rightarrow$ Represents the horizontal axis (x-axis)
- **Example** $\Rightarrow$ For $f(x) = \sqrt{x}$, the domain is $x \geq 0$
- **Relation** $\Rightarrow$ Domain specifies where the function is defined
- **Outcome** $\Rightarrow$ Determines the allowable values you can plug into the function

Range

- **Definition** ⇒ The set of all possible output values y from a function
- **Coordinate Plane** ⇒ Represents the vertical axis (y-axis)
- **Example** ⇒ For $f(x) = \sqrt{x}$, the range is $y \geq 0$
- **Relation** ⇒ Range shows the values the function can produce
- **Outcome** ⇒ Determines the spread of results plotted on the plane

Interval Notation

- **[a,b]** ⇒ Closed interval: includes both endpoints a and b
- **(a,b)** ⇒ Open interval: excludes both endpoints a and b
- **[a,b)** ⇒ Half-open interval: includes a but excludes b
- **(a,b]** ⇒ Half-open interval: excludes a but includes b
- **[0,x)** ⇒ All real numbers from 0 up to but not including x
- **(0,x)** ⇒ All real numbers greater than 0 and less than x

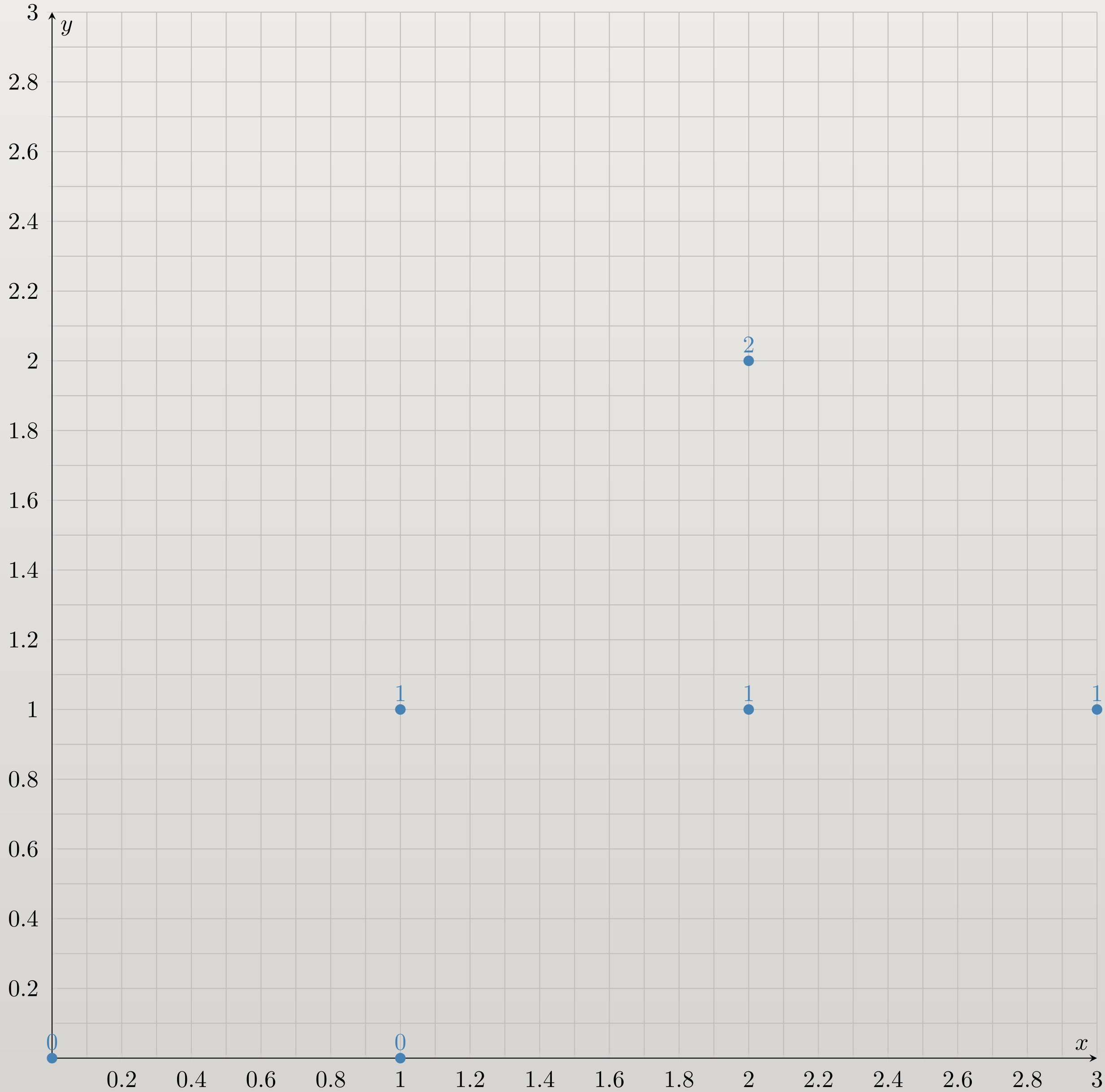
Relation of Ordered Pairs

- **Ordered Pair** ⇒ A pair written as (x, y), where x is the input and y is the output
- **Relation** ⇒ Any set of ordered pairs that connects elements from one set (domain) to another (range)
- **Domain** ⇒ The collection of all first elements x in the ordered pairs
- **Range** ⇒ The collection of all second elements y in the ordered pairs
- **Function** ⇒ A special type of relation where each x is paired with exactly one y
- **Example** ⇒ Relation: $\{(1, 2), (2, 3), (3, 4)\}$; Domain: $\{1, 2, 3\}$; Range: $\{2, 3, 4\}$

Ordered Pairs

Input (a, b)	**Output Value**
$(0, 0)$	5
$(1, 0)$	9
$(1, 1)$	8
$(2, 1)$	15
$(2, 2)$	17
$(3, 1)$	26

Ordered Pairs



I Distance Formula

$$\text{2D points} \qquad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{21}$$

$$\text{3D points} \qquad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{22}$$

Distance Formula

- **Definition** ⇒ Gives the length of the line segment between two points in the plane
- **Points** ⇒ Two points (x_1, y_1) and (x_2, y_2)
- **Formula** ⇒ $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- **Origin** ⇒ Derived from the Pythagorean Theorem
- **Domain** ⇒ Applies to all real coordinates in 2D space
- **Extension** ⇒ Generalizes to 3D: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Identity of $\sqrt[n]{a^n}$

- **Expression** ⇒ $\sqrt[n]{a^n}$
- **Meaning** ⇒ The n-th root of a^n
- **Simplification** ⇒ Cancels the root and exponent, yielding a
- **Condition** ⇒ Valid when $a \geq 0$ for real roots, or in complex domain otherwise
- **Example** ⇒ $\sqrt[3]{2^3} = \sqrt[3]{8} = 2$
- **Extension** ⇒ $\sqrt[n]{a^m} = a^{m/n}$

> refer to I to see the sqrt function.

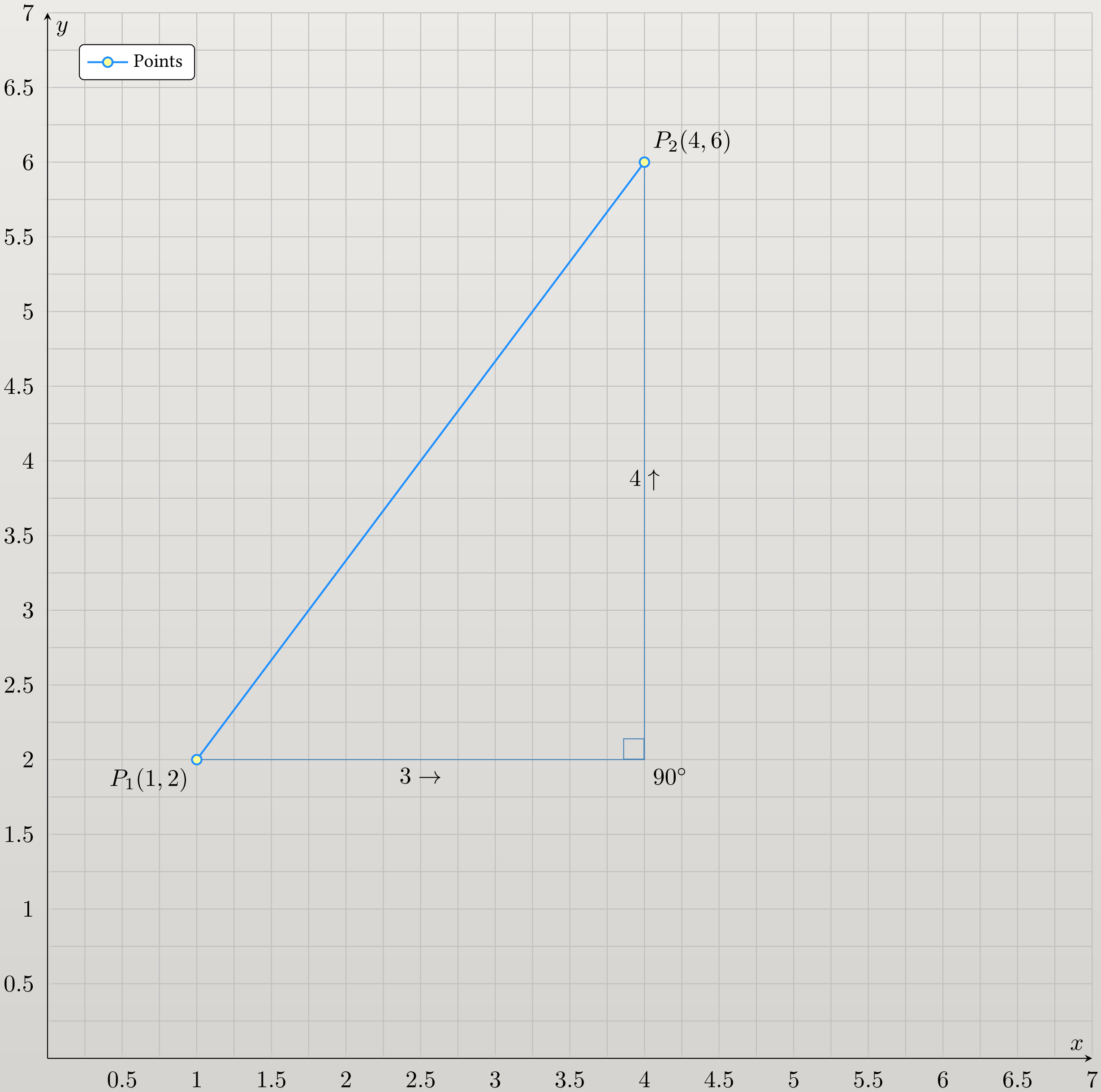
Distance Formula Functions

```
define nx_fma_dist1(x1, x2) {
        return nx_nr_sqrt((x2 - x1)^2)
}

define nx_fma_dist2(x1, x2, y1, y2) {
        return nx_nr_sqrt((x2 - x1)^2 + (y2 - y1)^2)
}

define nx_fma_dist3(x1, x2, y1, y2, z1, z2) {
        return nx_nr_sqrt((x2 - x1)^2 + (y2 - y1)^2  + (z2 - z1)^2)
```

```
}
```

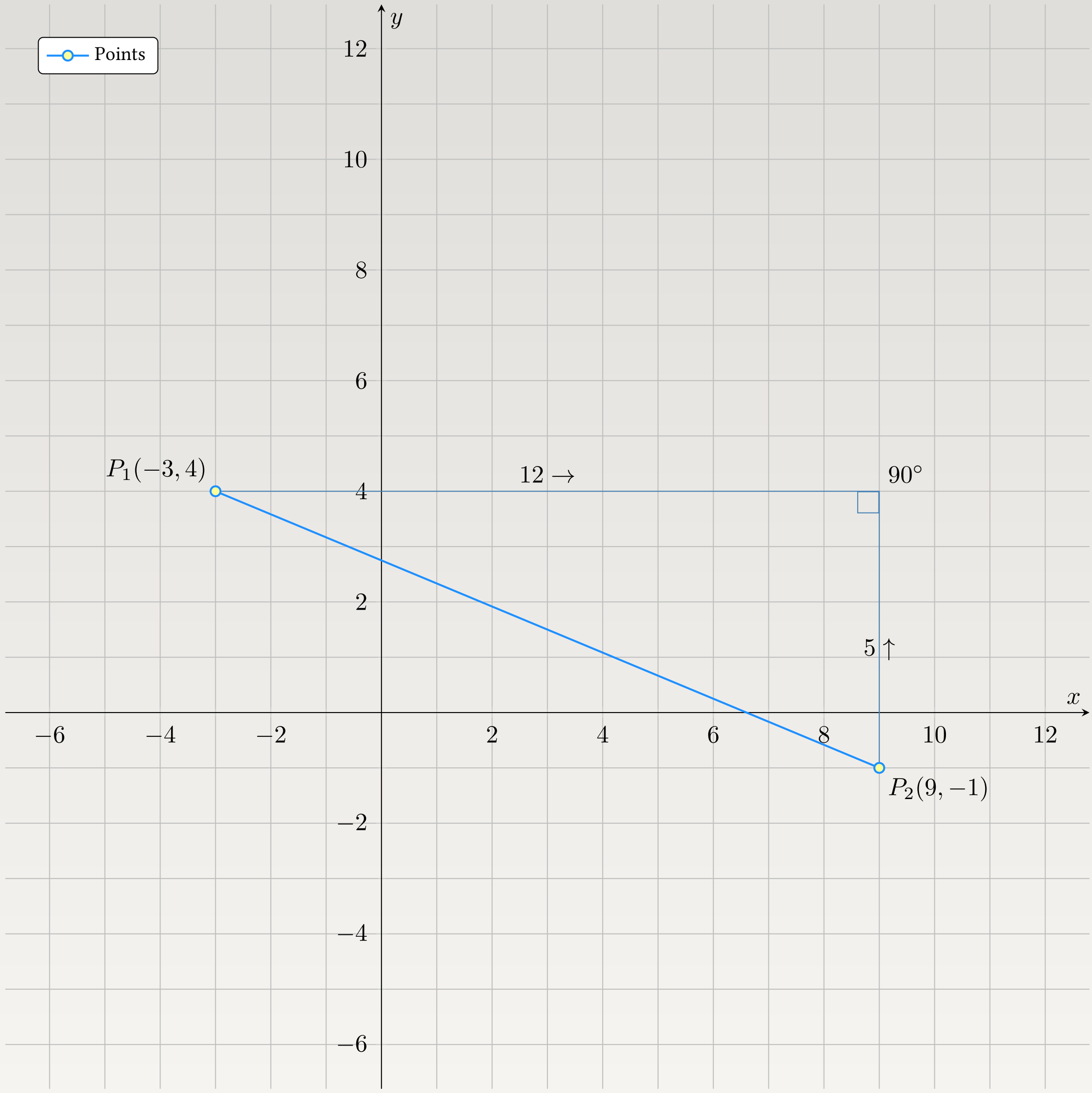
$$
\begin{aligned}
&& P_1(1,2) && P_2(4,6) \\
&& x_1 = 1 && x_2 = 4 \\
&& y_1 = 2 && y_2 = 6 \\
d = \sqrt{(4-1)^2 + (6-2)^2} \;\Rightarrow\; \sqrt{3^2+4^2} \;\Rightarrow\; \sqrt{9+16} \;\Rightarrow\; \sqrt{25} \;\Rightarrow\; 5
\end{aligned}
$$

$$d = \sqrt{(4-1)^2 + (6-2)^2}$$



$$P_1(-3,4) \qquad P_2(9,-1)$$
$$x_1 = -3 \qquad x_2 = 9$$
$$y_1 = 4 \qquad y_2 = -1$$
$$d = \sqrt{(9-(-3))^2 + ((-1)-4)^2} \Rightarrow \sqrt{12^2 + -5^2} \Rightarrow \sqrt{144+25} \Rightarrow \sqrt{169} \Rightarrow 13$$

$$d = \sqrt{(9-(-3))^2 + ((-1)-4)^2}$$



II Sets

$\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{R}^n$ and branches into $\mathbb{F}_p$

II Complex

Complex Numbers $\mathbb{C}$

- **Definition** $\Rightarrow$ All numbers of the form $a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$
- **Examples** $\Rightarrow$ $3 + 2i, -1 - i, 0 + 4i$
- **Relation** $\Rightarrow$ Contains all reals as the special case $b = 0$

II Finite Fields

Finite Fields $\mathbb{F}_p$

- **Definition** $\Rightarrow$ A set of integers modulo a prime p, with addition and multiplication defined mod p
- **Examples** $\Rightarrow$ $\mathbb{F}_2 = \{0, 1\}, \mathbb{F}_5 = \{0, 1, 2, 3, 4\}$
- **Relation** $\Rightarrow$ Finite fields are algebraic systems distinct from $\mathbb{R}$, but essential in number theory
- **Use** $\Rightarrow$ Cryptography, coding theory, error correction, and algebraic geometry

II Integers

Integers $\mathbb{Z}$

- **Definition** $\Rightarrow$ All whole numbers, both positive and negative, including zero
- **Examples** $\Rightarrow$ $\{-3, -2, -1, 0, 1, 2, 3, \dots\}$
- **Relation** $\Rightarrow$ Every integer can be expressed as a rational $\frac{n}{1}$, so $\mathbb{Z} \subset \mathbb{Q}$

II Natural Numbers

Natural Numbers $\mathbb{N}$

- **Definition** $\Rightarrow$ The counting numbers, starting from 1, sometimes including 0 depending on convention
- **Examples** $\Rightarrow$ $\{1, 2, 3, 4, \dots\}$ or $\{0, 1, 2, 3, \dots\}$
- **Relation** $\Rightarrow$ $\mathbb{N} \subset \mathbb{Z}$, since naturals are a subset of the integers
- **Use** $\Rightarrow$ Foundation for counting, arithmetic, and building larger number sets

II Primes

Primes $\mathbb{P}$

- **Definition** $\Rightarrow$ Natural numbers greater than 1 that have no divisors other than 1 and themselves
- **Examples** $\Rightarrow$ $\{2, 3, 5, 7, 11, 13, \dots\}$
- **Relation** $\Rightarrow$ $\mathbb{P} \subset \mathbb{N}$, primes are a special subset of the naturals
- **Use** $\Rightarrow$ Foundation of number theory, factorization, and cryptography

II Quaternions

Quaternions $\mathbb{H}$

- **Definition** $\Rightarrow$ Numbers of the form $a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$ and $i^2 = j^2 = k^2 = ijk = -1$
- **Examples** $\Rightarrow$ $1 + 2i + 3j + 4k,\ -i + j$
- **Relation** $\Rightarrow$ $\mathbb{C} \subset \mathbb{H}$, since complex numbers are a special case with $c = d = 0$
- **Use** $\Rightarrow$ Applied in 3D rotations, computer graphics, and physics for representing orientation

II Rationals

Rationals $\mathbb{Q}$

- **Definition** $\Rightarrow$ Numbers expressible as a fraction $\frac{p}{q}$ with integers p, q and $q \neq 0$
- **Examples** $\Rightarrow$ $\frac{2}{3}, -\frac{5}{4}, 7 = \frac{7}{1}$
- **Relation** $\Rightarrow$ All rationals are contained in the reals, so $\mathbb{Q} \subset \mathbb{R}$

II Reals

Reals $\mathbb{R}$

- **Definition** $\Rightarrow$ All rationals plus irrationals (non-repeating, non-terminating decimals)
- **Examples** $\Rightarrow$ π, $\sqrt{2}$, $0.333\ldots$
- **Relation** $\Rightarrow$ Every real is a complex number with imaginary part 0, so $\mathbb{R} \subset \mathbb{C}$

II Vector Spaces

Vector Spaces $\mathbb{R}^n$

- **Definition** $\Rightarrow$ Ordered tuples of real numbers, representing points or vectors in n-dimensional space
- **Examples** $\Rightarrow$ $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$, $\mathbb{R}^3 = \{(x, y, z)\}$
- **Relation** $\Rightarrow$ Built on $\mathbb{R}$, extending reals into higher dimensions
- **Use** $\Rightarrow$ Geometry, physics, linear algebra, and computer graphics for modeling multidimensional systems

III Definitions

III Pairs

Factorial (n!)

Concept	Factorial (n!)
Definition	The product of all positive integers up to n. Defined as $n! = n \times (n-1) \times \ldots \times 1$.
Core Idea	Factorial counts permutations and combinations — it grows extremely fast.
Example	$5! = 120$.
Applications	Used in combinatorics, probability, and series expansions.
Pair	Inverse gamma function (not elementary).

Logarithm Base 2 ($\log_2$)

Concept	Logarithm Base 2 ($\log_2(x)$)
Definition	The inverse of the power of 2. Defined as the exponent y such that $2^y = x$.
Core Idea	$\log_2(x)$ measures how many times you multiply 2 to reach x.
Example	$\log_2(8) = 3$.
Applications	Widely used in computer science, information theory, and binary systems.
Pair	Power of 2 function (2^x).

Natural Logarithm ($\ln$)

Concept	Natural Logarithm ($\ln(x)$)
Definition	The inverse of the exponential function. Defined as the power to which e must be raised to equal x.
Core Idea	$\ln(x)$ undoes exponentiation with base e.
Example	$\ln(e^3) = 3$.
Applications	Used in calculus, growth/decay models, and solving exponential equations.
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Tangent and Cotangent

Concept	Tangent ($\tan(x)$) and Cotangent ($\cot(x)$)
Definition	$\tan(x) = \frac{\sin(x)}{\cos(x)}$, while $\cot(x) = \frac{\cos(x)}{\sin(x)}$. They are reciprocals: $\cot(x) = \frac{1}{\tan(x)}$.
Core Idea	Tangent measures slope (rise/run). Cotangent flips that slope (run/rise).
Example	At 45°, $\tan(45°) = 1$ and $\cot(45°) = 1$.
Applications	Used in trigonometry, calculus, and geometry — especially for slope and angle analysis.
Pair	Reciprocal functions: $\tan(x) \leftrightarrow \cot(x)$.

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III Notation

III Delta in Mathematics

Finite Difference

- **Definition** ⇒ Delta denotes change or difference between two values
- **Formula** ⇒ $\Delta x = x_{\text{final}} - x_{\text{initial}}$
- **Usage** ⇒ Discrete counterpart to derivative: $\Delta y / \Delta x \approx dy/dx$
- **Example** ⇒ $\Delta y = f(x_2) - f(x_1)$
- **Extension** ⇒ Forms the basis of finite difference methods in numerical analysis

Discriminant

- **Definition** ⇒ In quadratic equations, Δ denotes the discriminant
- **Formula** ⇒ $\Delta = b^2 - 4ac$
- **Usage** ⇒ Determines the nature of roots of $ax^2 + bx + c = 0$
- **Example** ⇒ $\Delta > 0$: two real roots; $\Delta = 0$: one real root; $\Delta < 0$: complex roots
- **Extension** ⇒ Generalized discriminants exist for higher-degree polynomials

Triangle Symbol

- **Definition** ⇒ Δ also denotes a triangle in geometry
- **Usage** ⇒ ΔABC means triangle with vertices A, B, C
- **Connection** ⇒ Links algebraic glyph with geometric figure lineage
- **Extension** ⇒ Area of a triangle often denoted by Δ

III Theta in Mathematics

Angle Representation

- **Definition** ⇒ Greek letter θ used to denote an angle
- **Trigonometry** ⇒ Appears in sine, cosine, tangent: e.g. $\sin(\theta), \cos(\theta), \tan(\theta)$
- **Geometry** ⇒ Used in right triangles to relate sides and angles
- **Polar Coordinates** ⇒ Point (r, θ) defined by radius and angle
- **Example** ⇒ In a right triangle, $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$

Other Mathematical Uses

- **Statistics** ⇒ θ often denotes parameters in probability distributions
- **Complex Numbers** ⇒ Angle of rotation in Euler's formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$
- **Calculus** ⇒ Variable of integration in polar coordinates
- **Numerical Value** ⇒ In Greek numerals, θ has value 9
- **Connection** ⇒ Symbol of rotation, periodicity, and parameterization across math

III Epsilon in Mathematics

Infinitesimal Bound

- **Definition** ⇒ Greek letter ε used to denote a very small positive quantity
- **Limit Definition** ⇒ Appears in ε-δ proofs: for every $\varepsilon > 0$, there exists $\delta > 0$
- **Usage** ⇒ Measures closeness of a function to a limit
- **Example** ⇒ $|f(x) - L| < \varepsilon$ whenever $|x - c| < \delta$
- **Connection** ⇒ Core of rigorous calculus and real analysis

Error and Approximation

- **Definition** ⇒ ε often denotes error tolerance or margin
- **Numerical Analysis** ⇒ Used to bound approximation error
- **Example** ⇒ If $|x - x_0| < \varepsilon$, then x is within tolerance
- **Connection** ⇒ Links exact mathematics with numerical computation

Other Uses

- **Set Theory** ⇒ ε sometimes used as membership symbol, though ∈ is standard
- **Complexity** ⇒ ε denotes arbitrarily small constants in algorithm analysis
- **Probability** ⇒ ε used in inequalities like Chebyshev's or ε-nets
- **Connection** ⇒ Universal glyph for "smallness" across math disciplines

III Delta in Mathematics

Infinitesimal Change

- **Definition** ⇒ Greek letter δ denotes a very small positive quantity
- **Limit Proofs** ⇒ Appears in ε-δ definitions of limits
- **Formula** ⇒ For every $\varepsilon > 0$, there exists $\delta > 0$
- **Usage** ⇒ Controls how close x must be to c for $f(x)$ to be within ε of L
- **Example** ⇒ If $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$
- **Connection** ⇒ δ measures input closeness, ε measures output closeness

Variation and Error

- **Definition** ⇒ δ often denotes small variation or tolerance
- **Numerical Analysis** ⇒ Used to bound input error
- **Example** ⇒ If $|x - x_0| < \delta$, then x is within tolerance of x_0
- **Connection** ⇒ Pairs with ε to formalize precision in analysis and computation

Other Mathematical Uses

- **Geometry** ⇒ δ sometimes used for small angles
- **Statistics** ⇒ δ may denote deviation or perturbation
- **Complexity** ⇒ δ used for small constants in algorithm analysis
- **Connection** ⇒ Universal glyph for “small input change” across math disciplines

III Proportional Symbol

Definition

- **Glyph** ⇒ $\propto$ resembles the left half of ∞
- **Meaning** ⇒ Denotes proportionality between two quantities
- **Formula** ⇒ $y \propto x$ means $y = kx$ for some constant k
- **Usage** ⇒ Used in algebra, physics, and statistics to show direct proportionality
- **Example** ⇒ Gravitational force: $F \propto \frac{1}{r^2}$

Key Properties

- **Constant of Proportionality** ⇒ Always exists: $y = kx$
- **Direct Proportionality** ⇒ If one doubles, the other doubles
- **Inverse Proportionality** ⇒ Written as $y \propto \frac{1}{x}$
- **Scaling** ⇒ Proportionality preserves ratios
- **Connection** ⇒ Symbol bridges ratios, scaling laws, and functional dependence

III Infinity in Mathematics

Concept of Infinity

- **Definition** ⇒ ∞ denotes an unbounded quantity, larger than any real number
- **Calculus** ⇒ Appears in limits: $\lim_{x\to\infty} f(x)$
- **Set Theory** ⇒ Represents cardinalities of infinite sets
- **Geometry** ⇒ Used to mark points at infinity in projective geometry
- **Connection** ⇒ Symbol of endlessness, beyond finite measurement

Infinity in Calculus

- **Improper Integrals** $\Rightarrow$ $\int_1^\infty \frac{1}{x^2}dx$
- **Limits** $\Rightarrow$ $\lim_{x\to\infty} \frac{1}{x} = 0$
- **Series** $\Rightarrow$ Infinite sums: $\sum_{n=1}^\infty \frac{1}{n^2}$
- **Connection** $\Rightarrow$ ∞ marks the boundary of convergence and divergence
- **Example** $\Rightarrow$ Harmonic series diverges: $\sum_{n=1}^\infty \frac{1}{n}$

Infinity in Set Theory

- **Countable Infinity** $\Rightarrow$ Size of natural numbers, denoted $\aleph_0$
- **Uncountable Infinity** $\Rightarrow$ Size of real numbers, larger than $\aleph_0$
- **Comparison** $\Rightarrow$ Not all infinities are equal
- **Connection** $\Rightarrow$ ∞ as a concept differs from cardinal numbers
- **Example** $\Rightarrow$ $|\mathbb{N}| = \aleph_0$, but $|\mathbb{R}| > \aleph_0$

III Perpendicular Symbol

Definition

- **Glyph** $\Rightarrow$ $\perp$ is the mathematical symbol for perpendicularity
- **Meaning** $\Rightarrow$ Two lines, segments, or planes meet at a right angle (90°)
- **Notation** $\Rightarrow$ $AB \perp CD$ means line AB is perpendicular to line CD
- **Geometry** $\Rightarrow$ Used to denote orthogonality in Euclidean space
- **Example** $\Rightarrow$ In a square, adjacent sides are $\perp$ to each other

Linear Algebra Connection

- **Orthogonality** $\Rightarrow$ $\perp$ denotes vectors with dot product zero
- **Formula** $\Rightarrow$ $\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0$
- **Usage** $\Rightarrow$ Defines orthogonal bases and projections
- **Example** $\Rightarrow$ $(1,0) \perp (0,1)$ in $\mathbb{R}^2$
- **Extension** $\Rightarrow$ Orthogonality generalizes perpendicularity to higher dimensions

Other Uses

- **Logic** $\Rightarrow$ $\perp$ sometimes denotes contradiction or falsity
- **Probability** $\Rightarrow$ $\perp$ used to denote independence in some texts
- **Connection** $\Rightarrow$ Symbol bridges geometry, algebra, and logic
- **Visual** $\Rightarrow$ Always evokes the right-angle lineage

III Plus-Minus and Minus-Plus

Plus-Minus (±)

- **Definition** $\Rightarrow$ Symbol ± means “plus or minus”
- **Usage** $\Rightarrow$ Represents two possible values: $a + b$ or $a - b$
- **Example** $\Rightarrow$ Quadratic formula: $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
- **Connection** $\Rightarrow$ Encodes duality in solutions, symmetry in expansions
- **Extension** $\Rightarrow$ Used in error bounds and approximations: $x \pm \varepsilon$

Minus-Plus (∓)

- **Definition** $\Rightarrow$ Symbol ∓ means “minus or plus,” paired with ±
- **Usage** $\Rightarrow$ Ensures opposite choice when ± is used earlier
- **Example** $\Rightarrow$ If first term is $+$, second term takes −; if first is −, second takes $+$
- **Connection** $\Rightarrow$ Keeps expressions consistent in paired signs
- **Extension** $\Rightarrow$ Common in trigonometric identities and vector formulas

Combined Expression

- **Notation** $\Rightarrow$ $a \pm b \mp c$
- **Meaning** $\Rightarrow$ Two cases: $a + b - c$ or $a - b + c$
- **Pattern** $\Rightarrow$ ± and ∓ always paired to flip signs consistently
- **Example** $\Rightarrow$ In trig: $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
- **Connection** $\Rightarrow$ Encodes dual solutions in compact symbolic form

III Parallel Symbol

Definition

- **Glyph** ⇒ ∥ is the mathematical symbol for parallelism
- **Meaning** ⇒ Two lines, segments, or planes never intersect and remain equidistant
- **Notation** ⇒ $AB \parallel CD$ means line AB is parallel to line CD
- **Geometry** ⇒ Used in Euclidean geometry to denote parallel lines and planes
- **Example** ⇒ In a rectangle, opposite sides are ∥ to each other

Linear Algebra Connection

- **Vectors** ⇒ ∥ denotes vectors that are scalar multiples of each other
- **Formula** ⇒ $\vec{u} \parallel \vec{v} \iff \vec{u} = k\vec{v}$
- **Usage** ⇒ Defines direction equivalence in vector spaces
- **Example** ⇒ $(2, 4)$ ∥ $(1, 2)$ since $(2, 4) = 2(1, 2)$
- **Extension** ⇒ Parallelism generalizes beyond geometry into linear algebra and physics

Other Mathematical Uses

- **Analysis** ⇒ ∥x∥ sometimes denotes norm of a vector
- **Logic** ⇒ ∥ used in some texts for "parallel execution" or independence
- **Connection** ⇒ Symbol bridges geometry, algebra, and analysis
- **Visual** ⇒ Always evokes equidistant, non-intersecting lineage

III Angle Symbol

Definition

- **Glyph** ⇒ ∠ is the mathematical symbol for an angle
- **Meaning** ⇒ Represents the measure of rotation between two intersecting lines or rays
- **Notation** ⇒ $\angle ABC$ means the angle formed at vertex B by rays BA and BC
- **Units** ⇒ Measured in degrees (°) or radians
- **Example** ⇒ $\angle ABC = 90^\circ$ denotes a right angle

Geometry Usage

- **Right Angle** $\Rightarrow \angle = 90^\circ$
- **Acute Angle** $\Rightarrow \angle < 90^\circ$
- **Obtuse Angle** $\Rightarrow 90^\circ < \angle < 180^\circ$
- **Straight Angle** $\Rightarrow \angle = 180^\circ$
- **Reflex Angle** $\Rightarrow 180^\circ < \angle < 360^\circ$

Other Mathematical Uses

- **Trigonometry** $\Rightarrow$ $\angle$ used inside sine, cosine, tangent functions
- **Polar Coordinates** $\Rightarrow$ Point (r, θ) defined by radius and angle
- **Complex Numbers** $\Rightarrow$ Angle defines argument of a complex number
- **Vector Analysis** $\Rightarrow$ Angle between vectors via dot product
- **Connection** $\Rightarrow$ Symbol bridges geometry, trigonometry, and analysis

III Similar vs Equivalent

Similar Symbol ($\sim$)

- **Glyph** $\Rightarrow$ $\sim$is the symbol for similarity
- **Geometry** $\Rightarrow$ $\triangle ABC \sim \triangle DEF$ means triangles have equal angles and proportional sides
- **Algebra** $\Rightarrow$ Sometimes used to denote asymptotic equivalence: $f(x) \sim g(x)$ as $x \to \infty$
- **Pattern** $\Rightarrow$ Similarity preserves shape but not necessarily size
- **Example** $\Rightarrow$ $\triangle ABC \sim \triangle DEF$ if $\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$
- **Connection** $\Rightarrow$ Symbol of proportionality in geometry and asymptotics in analysis

Equivalent Symbol (≡)

- **Glyph** ⇒ ≡ is the symbol for equivalence
- **Congruence** ⇒ $\triangle ABC \equiv \triangle DEF$ means triangles are identical in shape and size
- **Number Theory** ⇒ Used for modular congruence: $a \equiv b \pmod{n}$
- **Logic** ⇒ Denotes logical equivalence: $P \equiv Q$
- **Pattern** ⇒ Equivalence preserves both shape and size, or exact relation
- **Example** ⇒ $17 \equiv 5 \pmod{12}$
- **Connection** ⇒ Symbol of exact sameness across geometry, algebra, and logic

III Approximate Symbol

Definition

- **Glyph** ⇒ ≈ is the mathematical symbol for approximation
- **Meaning** ⇒ Indicates two values are close but not exactly equal
- **Notation** ⇒ $a \approx b$ means a is approximately equal to b
- **Usage** ⇒ Common in numerical analysis, applied math, and physics
- **Example** ⇒ $\pi \approx 3.1416$

Contexts of Use

- **Numerical** ⇒ Used when rounding or truncating decimals
- **Physics** ⇒ Marks measured values close to theoretical ones
- **Statistics** ⇒ Denotes approximate probabilities or estimates
- **Analysis** ⇒ Signals asymptotic closeness in limits
- **Connection** ⇒ Symbol bridges exact math with practical computation

Related Symbols

- **Equal Sign** $\Rightarrow$ $=$ denotes exact equality
- **Tilde** $\Rightarrow$ $\sim$ denotes similarity or asymptotic equivalence
- **Congruence** $\Rightarrow$ $\equiv$ denotes exact equivalence or modular congruence
- **Approx** $\Rightarrow$ $\approx$specifically signals numerical closeness
- **Extension** $\Rightarrow$ Each glyph stages a different level of sameness

III Similar or Equal Symbol

Definition

- **Glyph** $\Rightarrow$ $\simeq$is the symbol for "similar or equal"
- **Meaning** $\Rightarrow$ Indicates two quantities are nearly equal and share structural similarity
- **Usage** $\Rightarrow$ Common in analysis, approximation, and asymptotic notation
- **Example** $\Rightarrow$ $f(x) \simeq g(x)$ means functions are close in value and form
- **Connection** $\Rightarrow$ Bridges similarity ($\sim$) and equality (=)

Contexts of Use

- **Approximation** $\Rightarrow$ Used when values are not exactly equal but very close
- **Asymptotics** $\Rightarrow$ Signals functions behave similarly as $x \to \infty$
- **Geometry** $\Rightarrow$ Sometimes used to denote figures nearly congruent
- **Physics** $\Rightarrow$ Marks quantities equal within experimental tolerance
- **Extension** $\Rightarrow$ $\simeq$is less strict than $\equiv$(equivalent) but stronger than $\approx$(approximate)

III Congruent Symbol

Definition

- **Glyph** ⇒ ≅ is produced in LaTeX with `\cong`
- **Meaning** ⇒ Denotes congruence: "is congruent to"
- **Geometry** ⇒ $\triangle ABC \cong \triangle DEF$ means triangles are identical in shape and size
- **Pattern** ⇒ Congruence preserves both angles and side lengths
- **Example** ⇒ $\triangle$ with sides 3,4,5 is $\cong$ to another 3,4,5 triangle
- **Connection** ⇒ Stronger than similarity (∼), exact match without scaling

Other Mathematical Uses

- **Number Theory** ⇒ Sometimes used interchangeably with ≡ for modular congruence
- **Analysis** ⇒ Can denote "is approximately congruent" in some texts
- **Logic** ⇒ Rarely used for structural equivalence
- **Extension** ⇒ ≅ bridges geometry congruence with algebraic congruence
- **Visual** ⇒ Glyph resembles equality with a tilde, staging sameness plus shape relation

III Limit Symbol (lim)

Definition

- **Glyph** ⇒ `lim` denotes the limit of a function or sequence
- **Meaning** ⇒ Describes the value a function approaches as the input approaches some point
- **Notation** ⇒ $\lim_{x \to c} f(x)$
- **Example** ⇒ $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- **Connection** ⇒ Core concept in calculus, analysis, and continuity

Types of Limits

- **Finite Limit** ⇒ Function approaches a finite value as input approaches a point
- **Infinite Limit** ⇒ Function grows without bound: $\lim_{x\to 0^+} \frac{1}{x} = \infty$
- **One-Sided Limit** ⇒ $\lim_{x\to c^-} f(x)$ or $\lim_{x\to c^+} f(x)$
- **At Infinity** ⇒ $\lim_{x\to\infty} f(x)$
- **Sequence Limit** ⇒ $\lim_{n\to\infty} a_n$

Formal Definition

- **ε-δ Definition** ⇒ For every $\varepsilon > 0$, there exists $\delta > 0$
- **Condition** ⇒ If $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$
- **Meaning** ⇒ $f(x)$ gets arbitrarily close to L as x approaches c
- **Connection** ⇒ Defines continuity and rigor in calculus
- **Example** ⇒ $\lim_{x\to 2}(3x + 1) = 7$

Limits

- **Usage** ⇒ Appears in limit notation: $\lim_{x\to c} f(x)$
- **Meaning** ⇒ “to” means the variable approaches a value
- **Example** ⇒ $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- **Connection** ⇒ Glyph of approach, not exact arrival
- **Extension** ⇒ Also used in one-sided limits: $x \to c^+$, $x \to c^-$

Mappings

- **Usage** ⇒ Appears in function notation: $f : A \to B$
- **Meaning** ⇒ “to” denotes mapping from domain to codomain
- **Example** ⇒ $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$
- **Connection** ⇒ Glyph of transformation, linking sets
- **Extension** ⇒ Used in category theory: arrows $A \to B$

Ranges and Intervals

- **Usage** ⇒ Appears in describing ranges: “from ... to ...”
- **Meaning** ⇒ Marks boundaries of intervals or sums
- **Example** ⇒ $\sum_{i=1}^{n} a_i$ means i runs from 1 to n
- **Connection** ⇒ Glyph of span, marking start and end
- **Extension** ⇒ Used in integrals: $\int_a^b f(x)\,dx$