PROBLEMS OF QUANTUM FIELD THEORIES IN CURVED SPACETIMES

A MASTER THESIS

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by
Cano Jones, Alejandro

Supervised by
Asorey Carballeira, Manuel

Department of Theoretical Physics
University of Zaragoza
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Abrstract

Quantum Field Theory is the fundamental theoretical framework of the Standard Model of elementary particles. This theory is formulated in a Minkowski space-time. However, the actual space-time metric is never of that type. On Earth, even at short distances, the metric is affected by both the force of the Earth's gravity and the solar force, but especially by the acceleration of the Earth's motion. At large distances the cosmological data lead one to think that, on average, the metric is of the Friedmann-Lemaitre-Roberson-Walker type.

The analysis of the quantization of fields in the presence of gravitational fields involves a number of theoretical issues that we intend to explore in this paper. How Quantum Field Theory can be adapted to a gravitational background is the object of the proposal. In particular, how the structure of the quantum vacuum is affected when space-time is neither asymptotically Minkowskian, as is the case in the current Cosmological Model LCDM. There are a number of attempts to address the problem but to date none provides a satisfactory solution; the aim of the present work is to find the one that best fits the experimental observations.

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Conventions

The chosen convention for the metric signature will be $(+, -, -, -)$ as in [?] and most literature on particle physics. Common conventions and nomenclatures in mathematics and physics are used throughout the text, some of which are considered to be relevant:

x^μ, x	four-vector
x^i, $\mathbf{x}$	spacial vector
$g_{\mu\nu}$	general spacetime metric
$\eta_{\mu\nu}$	Minkowski spacetime metric
g	determinant of $g_{\mu\nu}$
∇_μ	covariant derivative
$S[\phi]$	action functional of a field ϕ and it's derivatives
$\bar{z}$	complex conjugate of z
$A^\dagger$	hermitian conjugate of A
$R^\alpha_{\ \beta\gamma\delta}$	Riemann tensor $\quad \equiv \nabla_\delta \Gamma^\alpha_{\ \beta\gamma} - \nabla_\delta \Gamma^\gamma_{\ \beta\alpha} + \ldots$
$f \overleftrightarrow{\nabla}_\mu g$	$\equiv f \nabla_\nu g - (\nabla_\mu f)\, g$
γ^μ	covariant Gamma matrices $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$
G, c, $\hbar$	standard universal constants, not necessarily in natural units

Other notation will be introduced as needed.

Preface

Preface

1 Introduction to QFT in Curved Spacetimes

1.1 Matter-Gravity Action

Consider a dynamic universe consisting of dark energy characterized by a cosmological constant Λ and some material content described by a Lagrangian density $\mathcal{L}_\mathrm{M}$. The action associated with such system would be

$$S = \int \left[\frac{1}{2\kappa} (R - 2\Lambda) + \mathcal{L}_\mathrm{M} \right] \sqrt{-g}\, \mathrm{d}^4 x, \tag{1.1}$$

where $\kappa \equiv \frac{8\pi G}{c^4}$ is known as the Einstein gravitational constant.

The equations that would describe the classical dynamics of the system can be obtained by variations of the action presented in eq. (1.1) and the stationary-action principle, which states that the path taken by the system will result in $\delta S = 0$. The equations of motion of the matter fields are the Euler-Lagrange equations, obtained from a Lagrangian density of matter be described by some set $\{\phi^\alpha(x)\}$ which depends on the fields and their covariant derivatives, i.e.

$$\mathcal{L}_\mathrm{M} = \mathcal{L}_\mathrm{M}\big[\phi^\alpha(x),\, \nabla_\mu \phi^\alpha(x)\big], \tag{1.2}$$

which can be derived from the variations of the action S with respect of ϕ^α

$$\delta S = \int \left[\frac{\partial \mathcal{L}_\mathrm{M}}{\partial \phi^\alpha} \delta\phi^\alpha + \frac{\partial \mathcal{L}_\mathrm{M}}{\partial\left(\nabla_\mu \phi^\alpha\right)} \nabla_\mu \left(\delta\phi^\alpha\right) \right] \sqrt{-g}\, \mathrm{d}^4 x. \tag{1.3}$$

After applying the generalized Gauss Theorem, the stationary-action principle leads to the aforementioned Euler-Lagrange equations

$$\frac{\partial \mathcal{L}_\mathrm{M}}{\partial \phi^\alpha} - \nabla_\mu \left[\frac{\partial \mathcal{L}_\mathrm{M}}{\partial\left(\nabla_\mu \phi^\alpha\right)} \right] = 0. \tag{1.4}$$

On the other hand, variations of S with respect to the inverse metric ($g^{\mu\nu}$) results in

$$\delta S = \int \left[\frac{1}{2\kappa} \frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{2\kappa} \frac{1}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} - \frac{\Lambda}{\kappa} \frac{1}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta \mathcal{L}_\mathrm{M}}{\delta g^{\mu\nu}} + \frac{\mathcal{L}_\mathrm{M}}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} \sqrt{-g}\, \mathrm{d}^4 x; \tag{1.5}$$

and again, by imposing $\delta S = 0$ and considering that (up to pure derivative terms)

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu}, \qquad \frac{1}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2} g_{\mu\nu}, \tag{1.6 a,b}$$

one obtains the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -2 \frac{8\pi G}{c^4} \left(\frac{\delta \mathcal{L}_\mathrm{M}}{\delta g^{\mu\nu}} - \frac{1}{2} \mathcal{L}_\mathrm{M} g_{\mu\nu} \right) \tag{1.7}$$

which are most commonly written in terms of the Hilbert energy-momentum tensor

$$T_{\mu\nu} \equiv \mathcal{L}_\mathrm{M} g_{\mu\nu} - 2 \frac{\delta \mathcal{L}_\mathrm{M}}{\delta g^{\mu\nu}} = \frac{-2}{\sqrt{-g}} \frac{\delta\left(\mathcal{L}_\mathrm{M}\sqrt{-g}\right)}{\delta g^{\mu\nu}}. \tag{1.8}$$

This tensor is the source of the spacetime curvature, and must not be confused with the Noether's energy-momentum tensor since the two are not, in general, equivalent [3], but upon integration of the corresponding conserved currents, results are the same [8]. In addition of being symmetric, the Hilbert energy-momentum tensor is covariantly conserved, i.e.

$$\nabla_\mu T^{\mu\nu} = 0; \tag{1.9}$$

this fact is of great use once the material Hamiltonian $\mathcal{H}_\mathrm{M}$ is defined:

$$\mathcal{H}_\mathrm{M} \equiv \int T^{00} c \sqrt{-g}\, \mathrm{d}^3\mathbf{x}, \tag{1.10}$$

since it will later be used to spawn the Fock space after the quantization procedure; and thus assure no energy loses will be present on the theory.

1.2 Construction of Covariant Actions

Standard quantum field theory alongside the Standard Model of particle physics is one of (if not the) best tested theories of physics, reason which, one doesn't need to reinvent the actions that are used on it, only a small tweak is needed to make the theory general covariant; it has been previously given for granted that the volume element will be $\sqrt{-g}\, \mathrm{d}^4x$; but there is another consideration, the derivatives cannot be simply ∂_μ since that is not (in general covariant). To correctly define a covariant derivative ∇_μ, one must introduce two elements: the first one is the known Christoffel symbols $\Gamma^\sigma_{\alpha\beta}$ which will contract the tensorial nature of the field; and the second one is the spin connection, given by

$$\Gamma_\nu \equiv \frac{1}{2}\Sigma^{AB}\omega_{AB\mu}; \tag{1.11}$$

where Σ^{AB} are to be understood as the Lorentz generators (the uppercase Latin indexes represent sums over a Minkowski background), and $\omega_{AB\mu}$ is the so called torsion free spin connection, defined as

$$\omega_{AB\mu} \equiv e^\nu_A \left(\partial_\mu e_{B\nu} - \Gamma^\sigma_{\nu\mu} e_{B\sigma}\right). \tag{1.12}$$

The new vector fields e^μ_A are known as the tetrad formalism coefficients, defined to transform general tensors to a local flat manifold, i.e.

$$g_{\mu\nu} = e^A_\mu e^B_\nu \eta_{ab}. \tag{1.13}$$

As a last note, if the field is coupled to a vector field A_μ the covariant derivative must be redefined as $\nabla'_\mu \equiv \nabla_\mu - \frac{i}{\hbar} e A_\mu$, where e would be the coupling constant.

1.2.1 Some Basic Examples

Scalar Field

The very first example given for a classical field is usually a (real) free scalar field $\phi(x)$ with some mass m; whose dynamics are given by the following action

$$S[\phi] = \int \frac{1}{2}\Big[\partial_\nu\phi\, \partial^\nu\phi - \mu^2\phi^2 - \xi R\phi^2\Big]\sqrt{-g}\, \mathrm{d}^4x. \tag{1.14}$$

The construction of such action arises from its Minkowskian counterpart (a primary study of the standard quantum field theory can be found in the appendix); since the field in question is scalar, the covariant derivative its simply ∂_ν, the massive term of the action is dependant on a parameter $\mu \equiv {}^{mc}/_{\hbar}$, and a term is added as a coupling to gravity (through the Ricci scalar R) with a coupling constant ξ [1].

The inclusion of such coupling is not a mere curiosity, since it's been proven [5] that a self interactive $\lambda\phi^4$ theory needs a term proportional to $R\phi^2$ to be renormalizable. Besides this, the

[1]The field is said to be minimally coupled to gravity if $\xi = 0$ and nonminimally coupled otherwise.

addition of a term proportional to $R\phi^2$ adds a new symmetry to de action, since for a massless field $\mu = 0$ with a coupling constant $\xi = 1/6$, the action is invariant under conformal transformations, i.e.

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} \equiv \Omega^2(x) g_{\mu\nu}, \tag{1.15}$$

to prove this, lets first obtain the equations of motion using the Euler-Lagrange equation 1.4, resulting in the generalized Klein-Gordon equation

$$\left[\partial_\nu \partial^\nu - \mu^2 - \xi R\right]\phi = 0, \tag{1.16}$$

and then, a conformal transformation can be made to then, considering that the field will transform as $\phi \to \tilde{\phi} = \Omega^\beta \phi$, resulting on the following expression:

$$\begin{aligned} 0 = \mu^2 \Omega^{\beta-2} \left(\Omega^2 - 1\right)\phi + 2\left(1+\beta\right)\Omega^{\beta-3}\partial^\nu \Omega \partial_\nu \phi + \\ + (6\xi + \beta)\Omega^{\beta-3}\left(\partial_\nu \partial^\nu \Omega\right)\phi + \beta(1+\beta)\Omega^{\beta-4}\partial_\nu \Omega \partial^\nu \Omega \, \phi. \end{aligned} \tag{1.17}$$

Considering a massless field, a solution of this equation corresponds to the following values:

$$\beta = -1, \qquad \xi = \frac{1}{6}, \tag{1.18 a,b}$$

proving the conformal invariance for such scenario.

From its definition in equation 1.8 and the action 1.14, one can obtain the expression for the energy momentum tensor[2]

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} g_{\mu\nu}\left[\partial^\sigma \phi \partial_\sigma \phi - \mu^2 \phi^2\right] + \xi\left[-R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu}\partial^\sigma \partial_\sigma + \partial_\mu \partial_\nu\right]\phi^2, \tag{1.19}$$

which has an interesting property of its trace,

$$T^\nu_\nu = \frac{1}{2}\left(6\xi - 1\right)\partial_\sigma \partial^\sigma \phi^2 + \mu^2 \phi^2, \tag{1.20}$$

as it is zero, for a conformal theory.

Dirac Field

For spin 1/2 particles, the Lorentz generators are

$$\Sigma^{AB} = -\frac{i}{2}\sigma^{AB} = \frac{1}{4}\left[\gamma^A, \gamma^B\right], \tag{1.21}$$

where Γ^A are the flat gamma matrices. Therefore the covariant derivative and the connection can be written as

$$\nabla_\mu \equiv \partial_\mu + \Gamma_\mu, \quad \Gamma_\mu = \frac{1}{8}\omega_{AB\mu}\left[\gamma^A, \gamma^B\right]. \tag{1.22a}$$

Taking the Dirac theory as inspiration, one could define de Dirac action in curved spacetimes as

$$S[\psi] = \int \bar{\psi}\left[i\gamma^\nu \nabla_\nu - \mu\right]\psi\sqrt{-g}\mathrm{d}^4x, \tag{1.23}$$

where $\Gamma^\nu \equiv \gamma^A e^\nu_A$ are the general gamma functions, which follow the next relation similar to their flat counterparts,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{1.24}$$

[2]Note that in a Minkowski background, the term $R\phi^2$ present in the action written in the equation 1.14, vanishes; but the energy momentum tensor $T_{\mu\nu}$ differs from the standard expression by a pure derivative term. The new tensor is known as the improved energy-momentum tensor.

From the Euler-Lagrange equation 1.4, its obtained the generalized Dirac equation

$$\left[i\gamma^{\mu}\left(\partial_{\mu}+\Gamma_{\mu}\right)-\mu\right]\psi=0. \tag{1.25}$$

And from its definition in equation 1.8, the energy momentum tensor will have [4, sec. 3.8] the following expression

$$T_{\mu\nu}=\frac{1}{4}i\left\{\bar{\psi}\left(\gamma_{\mu}\nabla_{\nu}-\gamma_{\nu}\nabla_{\mu}\right)-\left[\left(\nabla_{\mu}\bar{\psi}\right)\gamma_{\nu}-\left(\nabla_{\nu}\bar{\psi}\right)\gamma_{\mu}\right]\right\}\psi, \tag{1.26}$$

with a trace of the form $T^{\mu}_{\mu}=4m\bar{\psi}\psi$, which is traceless for a massless field.

A particularly interesting outcome of this field is the so called Schrödinger-Dirac equation, result of squaring the generalized Dirac operator $\left[i\gamma^{\mu}\left(\partial_{\mu}+\Gamma_{\mu}\right)-\mu\right]$ just as its done in a Minkowskian background to recover the Klein-Gordon equation,

$$\left[\nabla_{\nu}\nabla^{\nu}-\mu^{2}-\frac{1}{4}R\right]\psi=0, \tag{1.27}$$

this expression (known as the Weitzenböck formula) gives another "natural" choice for the scalar field coupling to gravity ξ; to obtain such value, one can compare it with the generalized Klein-Gordon equation 2.6 finding $\xi={}^{1}/_{4}$.

Electromagnetic Field

Having previously studied the Dirac field, it is expected to also include the electromagnetic field, which is described by the same action as in the Minkowskian background, that is,

$$S[A_{\mu}]=\int\left(-\frac{1}{4c}F_{\mu\nu}F^{\mu\nu}+\mathcal{L}_{\text{Gauge}}\right)\sqrt{-g}\,\mathrm{d}^{4}x, \tag{1.28}$$

where $\mathcal{L}_{\text{Gauge}}$ is a gauge fixing term of the form

$$\mathcal{L}_{\text{Gauge}}=-\frac{1}{2\alpha}\left(\nabla_{\nu}A^{\nu}\right)^{2}. \tag{1.29}$$

The Faraday tensor is defined as

$$F_{\mu\nu}=\nabla_{\mu}A_{\nu}-\nabla_{\nu}A_{\mu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}, \tag{1.30}$$

where the last equality is a result of the symmetry of the lower indices on the Christoffel symbols.

The equations of motion resulting from the action 1.28 and Euler-Lagrange 1.4 are

$$\nabla^{\nu}\nabla_{\nu}A_{\mu}+R^{\sigma}_{\mu}A_{\sigma}-(1-\alpha)\nabla_{\mu}\nabla^{\nu}A_{\nu}=0. \tag{1.31}$$

And finally, as for completeness, the energy-momentum tensor [4, sec. 3.8] given by 1.8 is

$$\begin{aligned}T_{\mu\nu}=&-\left(F_{\mu\alpha}F^{\alpha\nu}-\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)+\\&+\alpha\left\{A_{\mu}\left(\nabla_{\nu}\nabla_{\rho}A^{\rho}\right)+\left(\nabla_{\mu}\nabla_{\rho}A^{\rho}\right)A_{\nu}-g_{\mu\nu}\left[A^{\rho}\left(\nabla\rho\nabla_{\sigma}A^{\sigma}\right)+\frac{1}{2}\left(\nabla_{\rho}A^{\rho}\right)^{2}\right]\right\};\end{aligned} \tag{1.32}$$

with a trace $T^{\mu}_{\mu}=-2\alpha\nabla_{\nu}\left(A^{\nu}\nabla_{\rho}A^{\rho}\right)$, which is contributed by the gauge fixing term only.

1.3 Scalar field Quantization

Thanks to its simplicity, the scalar field is a great field to work with, with the intent of showing some properties of a theory. For that reason, for what follows, all work will be done considering a real scalar field described by the action 1.14.

Now, let $v(x)$ be a solution of the generalized Klein-Gordon equation 2.6, then its complex conjugated $\bar{v}$ will also be an independent solution. Now consider i to be a set of parameters that univocally describe a par of solutions v_i, $\bar{v}_i$ in such a way that the most general solution of 2.6 will be

$$\phi(x) = \sum_i \left[a_i v_i(x) + \bar{a}_i \bar{v}_i(x)\right], \tag{1.33}$$

where a_i and $\bar{a}_i$ are constant factors, determined by the following external binary operation

$$\langle \phi_1(x),\, \phi_2(x)\rangle \equiv \frac{i}{\hbar}\int g^{0\nu}\left(\phi_1 \overset{\leftrightarrow}{\partial}_\nu \bar{\phi}_2\right)\sqrt{-g}\,\mathrm{d}^3\mathbf{x}, \tag{1.34}$$

such that

$$a_i = \langle v_i(x),\, \phi(x)\rangle, \quad \bar{a}_i = \langle \bar{v}_i(x),\, \phi(x)\rangle. \tag{1.35 a,b}$$

The quantization procedure is done by promoting the field χ and its conjugate momentum $\Pi \equiv \partial_{ct}\chi$ to operators

$$\phi(x) \longrightarrow \hat{\phi}(x), \qquad \Pi(x) \longrightarrow \hat{\Pi}(x), \tag{1.36}$$

by promoting the constant factors to operators as well, that is

$$a_i \longrightarrow \hat{a}_i, \qquad \bar{a}_i \longrightarrow \hat{a}_i^\dagger, \tag{1.37}$$

and therefore

$$\hat{\phi}(x) = \sum_i \left[\hat{a}_i v_i(x) + \hat{a}_i^\dagger \bar{v}_i(x)\right]. \tag{1.38}$$

One the promotion of the field to operators have been done, commutation relations between those operators must be imposed; the easiest choice would be to assume canonical quantization relations, that is,

$$\left[\hat{\phi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = i\hbar\,\delta^3\left(\mathbf{x}-\mathbf{y}\right) \qquad \left[\hat{\phi}(\mathbf{x}),\, \hat{\phi}(\mathbf{y})\right] = \left[\hat{\Pi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = 0. \tag{1.39 a-c}$$

It would be desirable to obtain a formulation similar to the well known scalar field in a Minkoswskian background, where the Fock space is generated from a vacuum state and a set of creation and annihilation operators that follow some commutation rules. To do so, we will force the $\hat{a}_i$, $\hat{a}_i^\dagger$ operators to assume this roll, in such a way that

$$\left[\hat{a}_i,\, \hat{a}_j^\dagger\right] \propto \delta_{ij}, \qquad \left[\hat{a}_i,\, \hat{a}_j\right] = \left[\hat{a}_i^\dagger,\, \hat{a}_j^\dagger\right] = 0. \tag{1.40 a-c}$$

Thanks to the relation between the constant factors a_i and the operation $\langle v_i,\, \phi\rangle$, one can obtain the following relation

$$\begin{aligned}\left[\hat{a}_i,\, \hat{a}_j^\dagger\right] &= -\frac{1}{\hbar^2}\int\left[\left(v_i\hat{\Pi} - g^{0\nu}\left(\partial_\nu v_i\right)\hat{\phi}\sqrt{-g}\right)\Big|_{\mathbf{x}},\, \left(\bar{v}_j\hat{\Pi} - g^{0\nu}\left(\partial_\nu v_j\right)\hat{\phi}\sqrt{-g}\right)\Big|_{\mathbf{y}}\right]\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{y} = \\ &= \frac{i}{\hbar}\int g^{0\nu}\left(v_i \overset{\leftrightarrow}{\partial}_\nu \bar{v}_j\right)\sqrt{-g}\,\mathrm{d}^3\mathbf{x} = \langle v_i,\, v_j\rangle,\end{aligned} \tag{1.41}$$

where the field commutators where used. Equivalently

$$\left[\hat{a}_i,\, \hat{a}_j\right] = -\langle v_i,\, \bar{v}_j\rangle, \quad \left[\hat{a}_i^\dagger,\, \hat{a}_j^\dagger\right] = -\langle \bar{v}_i,\, v_j\rangle. \tag{1.42 a,b}$$

Therefore we must find a set of solutions $\{v_i(x),\, \bar{v}_i(x)\}$ such that

$$\langle v_i,\, v_j\rangle \propto \delta_{ij}, \quad \langle v_i,\, \bar{v}_j\rangle = \langle \bar{v}_i,\, v_j\rangle = 0. \tag{1.43 a-c}$$

With this, we can define the Fock space the usual way, starting with a vacuum state $|0\rangle$ such that the action of the annihilation operation fulfils

$$\hat{a}_i\, |0\rangle = 0 \qquad \forall i \tag{1.44}$$

where single particle states are formed from the creation operator

$$|i\rangle \equiv \hat{a}_i^\dagger\, |0\rangle \tag{1.45}$$

and multiparticle states like

$$|i,\, j,\, \ldots\rangle = \ldots \hat{a}_j^\dagger\, \hat{a}_i^\dagger\, |0\rangle \tag{1.46}$$

Since this is a scalar field, one might assume that the states are symmetric (describing boson particles), and this is easily confirmed, since

$$|i,\, j\rangle = \hat{a}_j^\dagger\, \hat{a}_i^\dagger\, |0\rangle = \left[\hat{a}_i^\dagger,\, \hat{a}_j^\dagger\right] |0\rangle + \hat{a}_i^\dagger\, \hat{a}_j^\dagger |0\rangle = |j,\, i\rangle \tag{1.47}$$

1.3.1 Bogoliubov Transformations

Consider now a second set $\{u_i(x),\, \bar{u}_i(x)\}$ of solutions to the Klein-Gordon equation 2.6 such that they meet the operational rules 1.43; the field would then be written as

$$\phi(x) = \sum_j \left[b_j u_j(x) + \bar{b}_j \bar{u}_j(x)\right]. \tag{1.48}$$

Quantization of the field is straightforward and equivalent to the method previously presented. The relation between the v and u solutions (mode functions) will be

$$v_i(x) \equiv \sum_j \left[\alpha_{ij} u_j(x) + \beta_{ij} \bar{u}_j(x)\right], \tag{1.49}$$

where α_{ij} and β_{ij} are known as Bogoliubov coefficients, that can be obtained as

$$\alpha_{ij} \propto \langle v_i,\, u_j\rangle \qquad \beta_{ij} \propto -\langle v_i,\, \bar{u}_j\rangle. \tag{1.50 a,b}$$

Since the field is the same independently of the mode set chosen:

$$\sum_i \left[\hat{a}_i v_i(x) + \hat{a}_i^\dagger v_i^*(x)\right] = \sum_j \left[\hat{b}_j u_j(x) + \hat{b}_j^\dagger u_j^*(x)\right] \tag{1.51}$$

and, as a result of the orthogonality of the mode functions, the relation between the creation and annihilation operators will be

$$\hat{a}_i = \sum_j \left(\bar{\alpha}_{ij}\hat{b}_j - \bar{\beta}_{ij}\hat{b}_j^\dagger\right), \quad \hat{a}_i^\dagger = \sum_j \left(-\beta_{ij}\hat{b}_j + \alpha_{ij}\hat{b}_j^\dagger\right). \tag{1.52 a,b}$$

Applying commutator rules present in equation 10, give new restrictions to the Bogoliubov coefficients

$$\left[\hat{a}_i,\, \hat{a}_j^\dagger\right] \propto \delta_{ij} \implies \sum_k \left(\bar{\alpha}_{ik}\alpha_{jk} - \bar{\beta}_{ik}\beta_{jk}\right) \propto \delta_{ij}, \tag{1.53}$$

$$\left[\hat{a}_i,\, \hat{a}_j\right] = 0 \implies \sum_k \left(\bar{\alpha}_{jk}\bar{\beta}_{ik} - \bar{\alpha}_{ik}\bar{\beta}_{jk}\right) = 0. \tag{1.54}$$

One might ask what the relevance of this transformation is, and it would be in its right to do so, since it is not a mere mathematical result. To see the reason of this transformation, one could compute the number of v particles that are present in the u vacuum; the computation is given by

$$\langle _u0|\hat{N}_v|_u0\rangle = \sum_i \langle _u0|\hat{a}_i^\dagger \hat{a}_i|_u0\rangle = \sum_i \left[\sum_{jk} \beta_{ij}\bar{\beta}_{ik} \langle _u0|\hat{b}_j \hat{b}_k^\dagger|_u0\rangle \right] \propto \sum_{ij} |\beta_{ij}|^2. \tag{1.55}$$

The usual expectation value of a term of the form $\langle 0|\hat{N}|0\rangle$ is to be zero, and yet, it has been proven that this is not the case (in general) for the current scenario. The interpretation of such result is that the notion of "particle" is dependent on the choice of solutions of the Klein-Gorton equation; and thus, one could define different vacuum states for different situations.

2 Scalar Fields in Expanding Universes

The methodology for the analysis of scalar fields in a general manifold was presented in the previous chapter as a preliminary for the rest of this work, and in particular of the present chapter. It is clear that the presence of symmetries of the theory will simplify computations, and thus, a great start might be an isotropic and homogeneous expanding universe, which is described by the so called Friedmann–Lemaître–Robertson–Walker metric. Using reduced-circumference polar coordinates, the line element associated with such metric is written as

$$\mathrm{d}l^2 = c^2\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\mathrm{d}\Omega^2\right], \quad \mathrm{d}\Omega \equiv \mathrm{d}\theta^2 + \sin^2\varphi\mathrm{d}\varphi^2, \tag{2.1 a,b}$$

where κ is the curvature of the space and $a(t)$ is the scale factor determining the expansion. The associated curvature scalar R is given by

$$R = \frac{6}{c^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right], \tag{2.2}$$

needed for the coupling of the field with gravity as previously stated.

2.1 Expanding scalar field action

The Weyl tensor associated to the metric presented in eq. (2.1) is identically zero, meaning that the metric is conformally flat, i.e. independently of the space curvature κ, and therefore there must exist a coordinate system where

$$\mathrm{d}l^2 = a^2(\eta)\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = a^2(\eta)\left[c^2\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2\right], \tag{2.3}$$

working in such coordinate system will give the opportunity to use some results of standard scalar field theory. To do so, the action presented in eq. (1.14) will be rewritten in terms of a new field $\chi(x) \equiv a(\eta)\,\phi(x)$ using the fact that $\sqrt{-g} = a^4$

$$S[\chi] = \int \frac{1}{2}\left[\partial_\nu\chi\,\partial^\nu\chi - \left(\mu^2a^2 + \xi Ra^2 - c^2\frac{a''}{a}\right)\chi^2 - \partial_\eta\left(c^2\chi^2\frac{a'}{a}\right)\right]\mathrm{d}^4x, \tag{2.4}$$

where $a' \equiv \partial_\eta a(\eta)$ and equivalently with a''.

Dropping the time derivative will result on the following action for the sacalar χ field

$$S[\chi] = \int \frac{1}{2}\Big[\partial_\nu\chi\,\partial^\nu\chi - \left(\mu^2a^2 + \xi Ra^2 - c^2\frac{a''}{a}\right)\chi^2\Big]\mathrm{d}^4x, \tag{2.5}$$

being the main source for the current study. In order to obtain the expressions describing the dynamics of this field, the Euler-Lagrange equations in eq. (1.4) will be used, resulting in the generalized Klein-Gordon equation

$$\left[\partial_\nu\partial^\nu + \mu^2_{\text{eff}}(t)\right]\chi = 0, \qquad \mu^2_{\text{eff}}(t) = \left(\mu^2 + \xi R\right)a^2 - \frac{a''}{ac^2}. \tag{2.6 a,b}$$

Solutions of the eq. (2.6) are dependent on an integration constant related to the momentum $\mathbf{k}$, and are of the form

$$\chi_{\mathbf{k}}(x) = \alpha_{\mathbf{k}}v_{\mathbf{k}}(\eta)e^{-i\mathbf{k}\mathbf{x}\hbar^{-1}} + \bar{\alpha}_{\mathbf{k}}\bar{v}_{\mathbf{k}}(\eta)e^{i\mathbf{k}\mathbf{x}\hbar^{-1}}, \tag{2.7}$$

and upon substitution in (2.6), one gets the following differential equation

$$v''_{\mathbf{k}}\,\hbar^2 + \omega^2_{\mathbf{k}}(\eta)\,v_{\mathbf{k}} = 0 \tag{2.8}$$

where the dispersion relation $\omega_{\mathbf{k}}(\eta)$ is defined as,

$$\omega_{\mathbf{k}}^2(\eta) = \mathbf{k}^2 + \hbar^2\mu_{\text{eff}}^2(\eta) = \mathbf{k}^2 + \left(m^2c^2 + \xi\hbar^2 R\right)a^2 - \hbar^2\frac{a''}{ac^2}. \tag{2.9}$$

Solving eq. (2.8) will in turn give the form of the set of solutions $\{\chi_{\mathbf{k}}\}$ needed to describe the general expression of $\chi(x)$; since we are currently considering general expansion parameters and curvature scalars, the following computations will be made using a general set of $v_{\mathbf{k}}$ functions. This functions nevertheless have some interesting properties, such as being capable of a choice of normalization, and a constant of motion: the imaginary part of $v_{\mathbf{k}}\bar{v}'_{\mathbf{k}}$. Lets check the last statement

$$\frac{\partial}{\partial\eta}\text{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}}) = \frac{\partial}{\partial\eta}\left(\frac{v_{\mathbf{k}}\bar{v}'_{\mathbf{k}} - \bar{v}_{\mathbf{k}}v'_{\mathbf{k}}}{2i}\right) = \frac{v_{\mathbf{k}}\bar{v}'_{\mathbf{k}} - \bar{v}_{\mathbf{k}}v''_{\mathbf{k}}}{2i} = 0 \tag{2.10}$$

last step is result from dispersion relation. Since the functions $v_{\mathbf{k}}$ are capable to a choice in normalization, we will choose a set of solutions of eq. (2.8) such that $\text{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}})$ its independent of the momentum $\mathbf{k}$, and equal for all modes, this constant of motion will simply be defined as

$$\text{Im}(v\bar{v}') \equiv \text{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}}), \quad \forall\,\mathbf{k}. \tag{2.11}$$

The most general solution $\chi(x)$ of equation eq. (2.6) can be written as a Fourier mode expansion

$$\chi(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\left[a_{\mathbf{k}}v_{\mathbf{k}}(\eta)e^{-i\mathbf{k}\mathbf{x}\hbar^{-1}} + \bar{a}_{\mathbf{k}}\bar{v}_{\mathbf{k}}(\eta)e^{i\mathbf{k}\mathbf{x}\hbar^{-1}}\right]. \tag{2.12}$$

2.2 Quantization

After promoting the fields to operators, and imposing commutation relations, one gets the following rules for the creation and annihilation operators from equations 10

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}^\dagger] = \frac{(2\pi\hbar)^3\hbar c}{2\text{Im}(v\bar{v}')}\delta^3(\mathbf{k}-\mathbf{q})\,, \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{q}}^\dagger] = 0, \tag{2.13 a-c}$$

to be able to verify the value of the proportional factor in 2.13.a, lets compute the commutator between $\hat{\chi}$ and $\hat{\Pi}$, i.e.

$$\begin{aligned}\left[\hat{\chi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = \frac{1}{c}\int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^6}\Big\{&[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}]\, v_{\mathbf{k}}v'_{\mathbf{q}}e^{-i(\mathbf{kx}+\mathbf{qy})\hbar^{-1}} + \left[\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{q}}^\dagger\right]\bar{v}_{\mathbf{k}}\bar{v}'_{\mathbf{q}}e^{-i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}} + \\ &+ \left[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}^\dagger\right]v_{\mathbf{k}}\bar{v}'_{\mathbf{q}}e^{-i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}} - \left[\hat{a}_{\mathbf{q}}, \hat{a}_{\mathbf{k}}^\dagger\right]\bar{v}_{\mathbf{k}}v'_{\mathbf{q}}e^{i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}}\Big\}\end{aligned} \tag{2.14}$$

using expresions 2.13 and considering that the proportional factor of 2.13.a to be α, previous expression simplify to the following one,

$$\left[\hat{\chi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = \frac{\alpha}{c}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^6}2i\text{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}})e^{-i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}}; \tag{2.15}$$

since $\text{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}})$ was considered to be momentum independent, this implies that

$$\left[\hat{\chi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = i\frac{2\alpha\text{Im}(v\bar{v}')}{c(2\pi\hbar)^3}\delta^3(\mathbf{x}-\mathbf{y}), \tag{2.16}$$

and, from equation 1.39.a one can solve for α, resulting in the value present in equation 2.13.

The next step in the quantization procedure is to obtain the Hamiltonian $\hat{\mathcal{H}}$ that spans the Fock space; to do so, we use the definition in equation 1.10 alongside the energy momentum tensor 1.19.

As a simplification, lets consider a minimally coupled theory, i.e. $\xi = 0$; then the Hamiltonian will be

$$\hat{\mathcal{H}}(t) = \int \frac{c}{2}\left[\hat{\Pi}^2 + \left(\nabla\hat{\chi}\right)^2 + \mu_{\text{eff}}^2(t)\hat{\chi}^2\right]\mathrm{d}^3\mathbf{x}. \tag{2.17}$$

Substitution of the general expression of $\hat{\chi}$ (from equation 2.12) and $\hat{\Pi}$ (remembering that $\Pi \equiv \partial_0\chi$) one will get the following expansion,

$$\hat{\mathcal{H}}(\eta) = \frac{c}{2}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\left[\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}F_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\bar{F}_{\mathbf{k}} + \left(2\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\delta^3(\mathbf{0})\right)E_{\mathbf{k}}\right], \tag{2.18}$$

where the functions $F_{\mathbf{k}}(t)$ and $E_{\mathbf{k}}(t)$ are defined as

$$F_{\mathbf{k}}(\eta) = \left(\frac{1}{\hbar c}\right)^2\left[\hbar^2 v_{\mathbf{k}}'^2 + \omega_{\mathbf{k}}^2(t)\,c^2 v_{\mathbf{k}}^2\right],\quad E_{\mathbf{k}}(\eta) = \left(\frac{1}{\hbar c}\right)^2\left[\hbar^2\left|v_{\mathbf{k}}'\right|^2 + \omega_{\mathbf{k}}^2(t)\,c^2\left|v_{\mathbf{k}}\right|^2\right]. \tag{2.19 a,b}$$

2.3 Instantaneous Vacuum State

Note that the only way a vacuum state $|0\rangle$ could remain an eigenstate of the Hamiltonian 2.18 at all times, would be if $F_{\mathbf{k}}(\eta) = 0$, at all times, i.e.

$$F_{\mathbf{k}}(\eta) = \left(\frac{1}{\hbar c}\right)^2\left[\hbar^2 v_{\mathbf{k}}'^2 + \omega_{\mathbf{k}}^2(\eta)\,c^2 v_{\mathbf{k}}^2\right] = 0, \tag{2.20}$$

solving for $v_{\mathbf{k}}$ gives the following expression

$$v_{\mathbf{k}}(\eta) = \mathrm{C}\exp\left[\pm\frac{c}{i\hbar}\int\omega_{\mathbf{k}}(\eta)\,\mathrm{d}\eta\right], \tag{2.21}$$

which is not compatible with 2.8 except for a time independent dispersion relation $\omega_{\mathbf{k}}$.

The last result implies that, at different times, one can (and should) define different vacuum states; and thus, we will define the *instantaneous vacuum state* $|_{(\eta_0)}0\rangle$ as the one that at some time t_0 will minimize the energy density. Since all possible states are related by Bogolyubov transformations, finding the instantaneous vacumm state is the same as finding the set of functions $v_{\mathbf{f}}$ that are simultaneously solution of 2.8 and minimize

$$\langle_{(\eta_0)}0|\hat{\mathcal{H}}(\eta_0)|_{(\eta_0)}0\rangle = \rho(\eta_0)\delta^3(\mathbf{0}) = \frac{\hbar c^2\,\delta^3(\mathbf{0})}{4\mathrm{Im}(v\bar{v}')}\int\mathrm{d}^3\mathbf{k}\,E_{\mathbf{k}} \tag{2.22}$$

To minimise the energy density of the vacuum state is to find the set of functions $v_{\mathbf{k}}$ that minimise $E_{\mathbf{k}}$. Suppose that $v_{\mathbf{k}}$ can be written as

$$v_{\mathbf{k}} = r_{\mathbf{k}}e^{i\alpha_{\mathbf{k}}} \tag{2.23}$$

since $\mathrm{Im}(v\bar{v}')$ was constant through time

$$r_{\mathbf{k}}^2\alpha_{\mathbf{k}}' = -\mathrm{Im}(v\bar{v}') \tag{2.24}$$

this means

$$E_{\mathbf{k}} = \left(\frac{1}{\hbar c}\right)^2\left\{\hbar^2\left[r_{\mathbf{k}}'^2 + \mathrm{Im}^2\left(v\bar{v}'\right)\frac{1}{r_{\mathbf{k}}^2}\right] + \omega_{\mathbf{k}}^2\,c^2 r_{\mathbf{k}}^2\right\} \tag{2.25}$$

the minimum of this function must fulfil $r_{\mathbf{k}}'(\eta_0) = 0$. Now, if $\omega_{\mathbf{k}}^2(\eta_0)$ and $\mathrm{Im}(v\bar{v}')$ have the same sign, the minimum of $E_{\mathbf{k}}$ happens when $r_{\mathbf{k}}(\eta_0) = \left[\frac{\hbar\mathrm{Im}(v\bar{v}')}{\omega_{\mathbf{k}}(\eta_0)\,c}\right]^{1/2}$.

If there is a minimum, then

$$v_{\mathbf{k}}(\eta_0) = \left[\frac{\hbar\mathrm{Im}(v\bar{v}')}{\omega_{\mathbf{k}}(\eta_0)\,c}\right]^{1/2}e^{i\alpha_{\mathbf{k}}(\eta_0)}\qquad v_{\mathbf{k}}'(\eta_0) = -c\frac{\omega_{\mathbf{k}}(\eta_0)}{ih}\,v_{\mathbf{k}}(\eta_0) \tag{2.26}$$

under these functions,

$$E_{\mathbf{k}}(\eta_0) = 2\frac{\mathrm{Im}(v\bar{v}')}{\hbar c}\omega_{\mathbf{k}}(\eta_0) \qquad F_{\mathbf{k}}(\eta_0) = 0 \tag{2.27}$$

meaning

$$\hat{\mathcal{H}}(\eta_0) = \mathrm{Im}(v\bar{v}')\frac{1}{\hbar}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\left(2\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\delta^3(\mathbf{0})\right)\omega_{\mathbf{k}}(t_0) \tag{2.28}$$

which is equivalent to the standard Hamiltonian for a scalar field without the presence of gravity.

But what is the energy of the instantaneous vacuum at a different time? The relation of such energies can be computed considering that the field at some time η can be described as some Bogoliubov transformation of the same field at a time η_0.

From equations 1.50 one gets that the Bogoliubov coefficients will be given by

$$\alpha_{\mathbf{kp}} = \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\langle\chi_{\mathbf{k}}(\eta_0),\, \chi_{\mathbf{p}}(\eta)\rangle, \qquad \beta_{\mathbf{kp}} = -\frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\langle\chi_{\mathbf{k}}(\eta_0),\, \bar{\chi}_{\mathbf{p}}(\eta)\rangle, \tag{2.29 a,b}$$

and, since the field can be written as $\chi_{\mathbf{k}} = v_{\mathbf{k}}e^{i\mathbf{kx}/\hbar}$; from the definition of the binary operation $\langle\cdot,\cdot\rangle$ in expression 1.34, one can see that

$$\alpha_{\mathbf{kp}} \propto \delta^3(\mathbf{k}-\mathbf{p}), \qquad \beta_{\mathbf{kp}} \propto \delta^3(\mathbf{k}+\mathbf{p}), \tag{2.30 a,b}$$

and thus, it is possible to write $v_{\mathbf{k}}$ at an arbitrary time t as

$$v_{\mathbf{k}}(\eta) = \alpha_{\mathbf{k}}v_{\mathbf{k}}(\eta_0) + \beta_{\mathbf{k}}\bar{v}_{\mathbf{k}}(\eta_0); \tag{2.31}$$

where, recalling that $\mathrm{Im}(v\bar{v}')$ is constant through time, the relation between $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ must be

$$|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1. \tag{2.32}$$

The energy of the instantaneous vacuum state $|_{(\eta_0)}0\rangle$ at a time η is given by $\langle_{(\eta_0)}0|\hat{\mathcal{H}}(\eta)|_{(\eta_0)}0\rangle$, were the hamiltonian $\hat{\mathcal{H}}(\eta)$ is given by expression 2.18; to compute that, lets first obtain

$$\langle_{(\eta_0)}0|\hat{a}^\dagger_{\mathbf{k}}(\eta)\hat{a}_{\mathbf{k}}(\eta)|_{(\eta_0)}0\rangle = \left|\beta_{\mathbf{k}}\right|^2\frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\delta^3(\mathbf{0}) \tag{2.33}$$

therefore the energy of the instantaneous vacuum state $|_{(\eta_0)}0\rangle$ at a time η is given by

$$\langle_{(\eta_0)}0|\hat{\mathcal{H}}(\eta)|_{(\eta_0)}0\rangle = \delta^3(\mathbf{0})\int \mathrm{d}^3\mathbf{k}\left(\frac{1}{2}+\left|\beta_{\mathbf{k}}\right|^2\right)c\,\omega_{\mathbf{k}}(\eta) \geq \langle_{(\eta_0)}0|\hat{\mathcal{H}}(\eta_0)|_{(\eta_0)}0\rangle. \tag{2.34}$$

As expected, the energy density will be bigger at a different time η if $\beta_{\mathbf{k}} \neq 0$ for all $\mathbf{k}$, since the definition of $|_{(\eta_0)}0\rangle$ is such state that minimizes the energy density at some particular time η_0; meaning that at a different time, other state must have a lower energy density.

2.4 Case of Study: de Sitter Universe

The de Sitter Universe is a flat FLRW metric with no matter or radiation, but it does have a a positive cosmological constant Λ. Per the Friedmann equations,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho + \Lambda c^2}{3} - \frac{\kappa c^2}{a^2}, \quad (\rho = \kappa = 0) \tag{2.35}$$

the expansion parameter $a(t)$ will be equal to

$$a(t) = a_1 e^{H_\Lambda t} + a_2 e^{-H_\Lambda t}\,, \tag{2.36}$$

where $H_\Lambda = \sqrt{\Lambda c^2/3}$ it's the Hubble-Lemaître constant. The most common choice is to set $a_2 = 0$, and thus, consider an always expanding universe.

The line element describing the motion of particles through this universe is given by

$$\mathrm{d}l^2 = c^2\mathrm{d}t^2 - a^2(t)\mathrm{d}\mathbf{x}^2, \tag{2.37}$$

more commonly expressed as a function of the conformal time η defined as

$$\eta \equiv -\int_t^\infty \frac{\mathrm{d}t'}{a(t')} = -\frac{1}{a_1 H_\Lambda} e^{-H_\Lambda t} = -\frac{1}{a(t)H_\Lambda}; \tag{2.38}$$

and thus, the line element will be given by

$$\mathrm{d}l^2 = \frac{1}{H_\Lambda \eta^2}\left[c^2\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2\right], \tag{2.39}$$

which has the same form as 2.3.

Now, from equation 2.2 one can compute the curvature scalar $R = {}^{12}/_{c^2} H_\Lambda^2$, and thus, the dispersion relation 2.9 will be given by the following expression

$$\omega_{\mathbf{k}}^2(\eta) = \mathbf{k}^2 + \left[\left(\frac{mc^2}{H_\Lambda}\right)^2 + 2\left(6\xi - 1\right)\hbar^2\right]\frac{1}{c^2\eta^2}, \tag{2.40}$$

from which one might obtain the solutions of the differential equation 2.8; to do so it is best to use the following change of variables,

$$s \equiv -k\eta\frac{c}{\hbar}, \qquad v_{\mathbf{k}} \equiv \sqrt{s}f(s), \tag{2.41 a,b}$$

obtaining the Bessel's differential equation

$$s^2\frac{\mathrm{d}^2 f}{\mathrm{d}s^2} + s\frac{\mathrm{d}f}{\mathrm{d}s} + \left(s^2 - \nu^2\right)f(s) = 0, \tag{2.42}$$

with a parameter

$$\nu^2 \equiv (3 - 16\xi)\frac{3}{4} - \left(\frac{mc^2}{H_\Lambda \hbar}\right)^2. \tag{2.43}$$

The solutions of such differential equation are given by the so called Bessel functions of the first kind $J_\nu(s)$ and $Y_\nu(s)$; therefore the $v_{\mathbf{k}}$ functions can be deduced as

$$f(s) = AJ_\nu(s) + BY_\nu(s) \implies v_{\mathbf{k}}(\eta) = \sqrt{k|\eta|\frac{c}{\hbar}}\left[A_{\mathbf{k}}J_\nu(k|\eta|\frac{c}{\hbar}) + B_{\mathbf{k}}Y_\nu(k|\eta|\frac{c}{\hbar})\right] \tag{2.44}$$

For $\nu^2 \geq 0$, both J_ν and Y_ν will be real functions, but for $\nu < 0$ they will be complex functions [6]; for simplicity, we will focus on the $\nu \geq 0$ case. In addition, consider that the choice of $\mathrm{Im}(v\bar{v}')$ will translate in a restriction on the relation between the $A_{\mathbf{k}}$, $B_{\mathbf{k}}$ parameters;

$$\mathrm{Im}\left(v\bar{v}'\right) = ik^2|\eta|\frac{c}{\hbar}\left(A_{\mathbf{k}}\bar{B}_{\mathbf{k}} - \bar{A}_{\mathbf{k}}B_{\mathbf{k}}\right)W\left[J_\nu\left(k|\eta|\right), Y_\nu\left(k|\eta|\right)\right], \tag{2.45}$$

where $W[f,g] \equiv fg' - f'g$ its the Wronskian, particularly [1, 10.5.2] for the Bessel functions, one gets $W\left[J_\nu\left(s\right), Y_\nu\left(s\right)\right] = \frac{2}{\pi s}$ and thus, the relation between the $A_{\mathbf{k}}$, $B_{\mathbf{k}}$ parameters will be

$$\left(A_{\mathbf{k}}\bar{B}_{\mathbf{k}} - \bar{A}_{\mathbf{k}}B_{\mathbf{k}}\right) = -i\frac{\pi}{2k}\mathrm{Im}\left(v\bar{v}'\right). \tag{2.46}$$

As previously seen, the choice of a set of solutions of the Klein-Gordon equation, defines the choice of a vacuum; an example is the instantaneous vacuum, but for a de Sitter background there is a special choice, the Bunch-Davies vacuum[1].

[1] Also known as euclidean vacuum.

2.4.1 Bunch-Davies Vacuum

Consider the behaviour of the field on the distant past, that is, $k|\eta| \to \infty$; at this moment, the dispersion relation 2.40 is given by $\omega_{\mathbf{k}} \approx k$, therefore, the solutions of 2.8 are given by

$$v_{\mathbf{k}}(\eta) \approx \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} e^{i\omega_{\mathbf{k}}\eta}, \tag{2.47}$$

which is the usual flat temporal mode function. The same limit can be taken over the solution 2.44, considering the asymptotic expressions for the Bessel functions [1, 10.7.8], one obtains

$$v_{\mathbf{k}}(\eta) \approx \sqrt{\frac{2}{\pi}} \left[A_{\mathbf{k}} \cos\lambda_\nu + B_{\mathbf{k}} \sin\lambda_\nu\right], \quad \lambda_\nu \equiv k|\eta|\frac{c}{\hbar} - \frac{\pi}{2}\nu - \frac{\pi}{4}. \tag{2.48 a,b}$$

One could easily recover the flat temporal mode function by imposing the relation $B_{\mathbf{k}} = iA_{\mathbf{k}}$; this choice, alongside the relation 2.46, meaning that $|A_{\mathbf{k}}| = \sqrt{\pi/4k \mathrm{Im}\,(v\bar{v}')}$; and therefore, the Bunch-Davies temporal mode functions are given (except for an irrelevant phase) by

$$v_{\mathbf{k}}(\eta) = \sqrt{\frac{\pi}{4}|\eta| \mathrm{Im}\,(v\bar{v}')} H_\nu^{(1)}\left(k|\eta|\right), \tag{2.49}$$

where $H_\nu^{(1)}(s) \equiv J_\nu(s) + iY_\nu(s)$ is known as the Hankel function of the first kind.

This vacuum state is of particular interest [2] since it is not time dependent, is invariant under the de Sitter symmetry group and, for a conformal theory (i.e. $m = 0$ and $\xi = 1/6$) the temporal mode function is given by[2]

$$v_{\mathbf{k}}(\eta) = -i\sqrt{\frac{\hbar \mathrm{Im}\,(v\bar{v}')}{2\omega_{\mathbf{k}} c}} e^{i\omega_{\mathbf{k}}|\eta|c/\hbar} \tag{2.50}$$

which is (up to an irrelevant phase) the standard plane wave temporal mode function.

Any other vacuum state can be obtained by a Bogoliubov transformation of the Bunch-Davies vacuum; from the relation of the Bogoliubov parameters in expression 2.32, one could define a new set of temporal mode functions $\{u_{\mathbf{k}}\}$ as

$$u_{\mathbf{k}}(\eta) \equiv \cosh\alpha\, v_{\mathbf{k}}(\eta) + e^{i\beta} \sinh\alpha\, \bar{v}_{\mathbf{k}}(\eta), \tag{2.51}$$

where $0 \leq \alpha < \infty$ and $-\pi \leq \beta < \pi$ are constant parameters defining the transformation. This new vacuum state denoted by $|_{(\alpha,\beta)}0\rangle$ is known as Mottola-Allen vacuum.

[2]For such theory, $\omega_{\mathbf{k}} = k$ and $\nu = 1/2$; and for such value, the Hankel function can be written as

$$H_{1/2}^{(1)}(s) = -i\sqrt{\frac{2}{\pi s}} e^{is}.$$

3 The Unruh Effect

3.1 Accelerated Observers and Unruh Temperature

In contrast to previous chapters, in this section there will be no effects of gravitation, and a flat $1+1$ spacetime (for simplicity) will be considered; the difference from the standard flat QFT, here a non inertial (accelerated) observed will be considered. Let an observer measure a self constant two-acceleration $\alpha^\mu \equiv {}^{\mathrm{d}^2 x^\mu}/{}_{\mathrm{d}\tau^2}$, and thus

$$\alpha^2 = \left(c\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2}\right)^2 - \left(\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2}\right)^2. \tag{3.1}$$

Solutions of such differential equation can be written as

$$t(\tau) = t_0 + t_1\tau \pm \frac{c}{\alpha}\sinh\left(\frac{\alpha\tau}{c}\right), \quad x(\tau) = x_0 + x_1\tau \pm \frac{c^2}{\alpha}\cosh\left(\frac{\alpha\tau}{c}\right), \tag{3.2 a,b}$$

where t_0, t_1, x_0, x_1 are constant real parameters. Since the trajectory must ensure that $c^2\mathrm{d}\tau^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2$, then the "velocities" must be such that $ct_1 = x_1 = 0$; furthermore, for simplicity we consider some coordinates in which $ct_0 = x_0 = 0$. Such trajectory can be described as the hyperbola

$$x^2 - c^2t^2 = \frac{c^2}{\alpha^2}. \tag{3.3}$$

There exists a coordinate system known as Rindler coordinates, that is specially useful in the description of accelerated observers, such coordinates do not map the whole spacetime, and must be divided into two maps: whereas $x > c|t|$ or otherwise. The coordinates used will be named as (η, ξ) $((\tilde{\eta}, \tilde{\xi})$ if working in the second chart), the first could be understood as some sort of temporal coordinate, while the second as a parameter determining the acceleration of the observer. Such coordinate system can be written[1] as:

- Chart $x > c|t|$ (3.1 blue section),

$$t(\eta,\, \xi) \equiv \frac{c}{\alpha}\sinh\left(\frac{\alpha\eta}{c}\right)e^{\alpha\xi/c^2}, \tag{3.4 a}$$

$$x(\eta,\, \xi) \equiv \frac{c^2}{\alpha}\cosh\left(\frac{\alpha\eta}{c}\right)e^{\alpha\xi/c^2}, \tag{3.4 b}$$

- Chart $x < c|t|$ (3.1 purple section),

$$t(\tilde{\eta},\, \tilde{\xi}) \equiv -\frac{c}{\alpha}\sinh\left(\frac{\alpha\tilde{\eta}}{c}\right)e^{\alpha\tilde{\xi}/c^2}, \tag{3.5 a}$$

$$x(\tilde{\eta},\, \tilde{\xi}) \equiv -\frac{c^2}{\alpha}\cosh\left(\frac{\alpha\tilde{\eta}}{c}\right)e^{\alpha\tilde{\xi}/c^2}, \tag{3.5 b}$$

One can check that, these coordinates meet the following hyperbolic relation,

$$x^2 - c^2t^2 = \frac{c^2}{\alpha^2}e^{2\alpha\xi/c^2}, \tag{3.6}$$

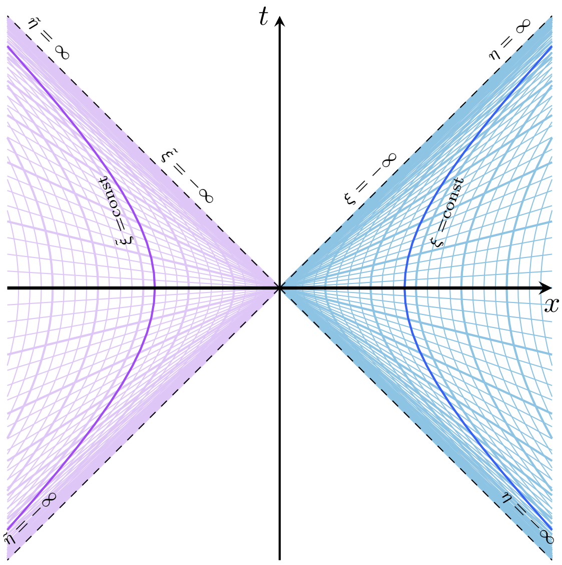

Figure 3.1: Rindler Coordinates (Left and Right charts).

[1] The choice is not unique, depending on the definition of ξ the exponential factor can be exchanged by a different form, such as $(1+\xi)$ or simply by ξ.

and thus, an observer with some coordinates (η, ξ) can be said to experience an acceleration $\alpha e^{-\xi/c^2}$. Using such coordinate system, the line element will be[2],

$$c^2 \mathrm{d}\tau^2 = e^{\alpha\xi/c^2} \left[c^2 \mathrm{d}\eta^2 - \mathrm{d}\xi^2\right]. \tag{3.7}$$

The fact that a non-inertial reference system will experience some deviations of a theory in relation to an inertial one should not surprise any reader, since its a basic result of elementary physics, and thus, it is also expected in a relativistic quantum theory to be true. To be able to demonstrate this, we will consider the simplest possible case, a massless and minimally coupled scalar field ϕ; whose equation of motion given by the Klein-Gordon equation 1.16 will be

$$e^{-2\alpha\xi/c^2} \left[\partial^2_{c\eta} - \partial^2_{\xi}\right] \phi = 0. \tag{3.8}$$

Solutions of such equation are the usual plane waves[3], just as it would for an inertial observer. Since the solutions of an accelerated reference frame and an inertial observer are the same, it would seem that all phenomena will be described equally by both, but this is not the same, since their metric description will be different, and thus, the field will be related by a Bogoliubov transformation; that is the two observers might disagree on their definition of the vacuum state.

In order to explicitly compute this, it will be useful to use the so called null coordinates $u \equiv c\eta - \xi$ and $v \equiv c\eta + \xi$ for the accelerated observer, and $U \equiv ct - x$ and $V \equiv ct + x$; through which the solution of the equation of motion will be (depending on the needed chart)

$$\phi^u_\omega \equiv e^{i\omega u\, \hbar^{-1}}, \quad \phi^v_\omega \equiv e^{i\omega v\, \hbar^{-1}}, \quad \phi^{\tilde{u}}_\omega \equiv e^{i\omega \tilde{u}\, \hbar^{-1}}, \quad \phi^{\tilde{v}}_\omega \equiv e^{i\omega \tilde{v}\, \hbar^{-1}}. \tag{3.9 a-d}$$

In addition, null coordinates will also be used for the inertial observer those being $U \equiv ct - x$ and $V \equiv ct + x$; and the relation to the accelerated null coordinates will be

$$U(u,\, v) = -\frac{c^2}{\alpha} e^{-\alpha u/c^2}, \qquad V(u,\, v) = \frac{c^2}{\alpha} e^{\alpha v/c^2}. \tag{3.10 a,b}$$

Using null coordinates the field can be described as two independent non interactive fields (since there is no autointeration term), that is, $\phi \equiv \phi_u + \phi_v$; in what follows, we will only consider the ϕ_u field (defined bellow), but the same can be done for the ϕ_v field. Not, lets expand the ϕ_u field as a set of Rindler modes 3.9.a,c meaning

$$\phi_u \equiv \int_0^\infty \frac{\mathrm{d}\omega}{(2\pi\hbar)\sqrt{2\omega}} \left\{\Theta(-U)\left[a^u_\omega \phi^u_\omega + \bar{a}^u_\omega \bar{\phi}^u_\omega\right] + \Theta(U)\left[a^{\tilde{u}}_\omega \phi^{\tilde{u}}_\omega + \bar{a}^{\tilde{u}}_\omega \bar{\phi}^{\tilde{u}}_\omega\right]\right\}; \tag{3.11}$$

clearly, it can also be written in terms of planar waves in U coordinates for the inertial observer.

Since we were interested in the difference in vacuum states, we would need to obtain the Bogoliubov coefficients connecting the Rindler modes, with the inertial ones; to do so, we compute the coefficient $\beta_{\Omega,\omega}$ using 1.50.b, obtaining

$$\beta^u_{\Omega\omega} \propto \langle \phi^u_\Omega, \bar{\phi}^U_\omega \rangle = \frac{1}{2\hbar^2}\sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{\infty} \exp\left\{i\left[\omega U(u) + \Omega u\right]\hbar^{-1}\right\}\mathrm{d}u, \tag{3.12}$$

where the expression for $U(u)$ is given by 3.10.a. This integral is solvable using the change of variables $z \equiv i\,({}^{\omega\hbar c^2}/_{\alpha})\exp(-{}^{au}/_{c^2})$ which will result on

$$\int_{-\infty}^{\infty} \exp\left\{i\left[\omega U(u) + \Omega u\right]\hbar^{-1}\right\}\mathrm{d}u = \frac{1}{\alpha}\left(-i\frac{\alpha\hbar}{\omega c^2}\right)^{ic^2\Omega/\alpha\hbar} \int_0^\infty z^{ic^2\Omega/\alpha\hbar - 1} e^{-z}\mathrm{d}z; \tag{3.13}$$

[2] Note that as it would be expected, the metric is conformally flat.

[3] This is a direct result of the fact that, for a $1+1$ scalar theory, conformal symmetry is obtained for $m = \xi = 0$.

where the integral is the Gamma function $\Gamma\left(ic^2\Omega/\alpha\hbar\right)$. This means that the Bogoliubov coefficient can be written as

$$\beta^u_{\Omega\omega} \propto \Gamma\left(i\frac{\Omega c^2}{\alpha\hbar}\right)\sqrt{\frac{\Omega}{\omega}}\left(\frac{c^2\omega}{\alpha\hbar}\right)^{-i\frac{\omega c^2}{\alpha\hbar}} e^{-\frac{\pi\Omega c^2}{2\alpha\hbar}}. \tag{3.14}$$

Now, from 1.55 we know how to compute the number of particles of some momentum Ω that an accelerated observer would measure in the inertial vacuum; that is

$$N_\Omega \equiv \langle_U 0|\left(\hat{a}^u_\Omega\right)^\dagger \hat{a}^u_\Omega|_U 0\rangle \propto \int_0^\infty \frac{\mathrm{d}\Omega}{(2\pi\hbar)\sqrt{2\Omega}}|\beta^u_{\Omega\Omega}| \propto \frac{1}{e^{\frac{2\pi\Omega c}{\alpha\hbar}}-1}\delta(0). \tag{3.15}$$

This expression has some resemblance with the expected value of particles with energy Ω for Bose-Einstein statistics; if we were to interpret it as such (which is not an overreach, considering the bosonic nature of the scalar field particles), a temperature might be defined as

$$T_0 \equiv \frac{\alpha\hbar}{2\pi c k_B}. \tag{3.16}$$

This so called temperature though, cannot be interpreted as the temperature that an accelerated Rindler observer would measure (since it is dependent on the coordinate variable α), to obtain such value of the temperature, one should consider the Tolman's law[4], stating that the proper temperature T_{Unruh} is given by

$$T_{\text{Unruh}} = \sqrt{g_{00}}\,T_0 = \left(\alpha e^{-\alpha/c^2\xi}\right)\frac{\hbar}{2\pi c k_B} \equiv \frac{a\hbar}{2\pi c k_B}; \tag{3.17}$$

where a is the acceleration measured by the non-inertial observer, as stated on equation 3.6.

3.2 Application to Black Holes: Hawking Radiation

According with the no hair conjecture, black holes can be univocally describe only by three parameters: its mass M, its angular momentum J and its electric charge Q. For simplicity, lets consider a Schwarzschild black hole[5], which considers $J = Q = 0$; the line element describing the spacetime of such black hole [9] can be written in spherical coordinates as

$$c^2\mathrm{d}\tau^2 = \left(1-\frac{2GM}{c^2 r}\right)c^2\mathrm{d}t^2 - \left(1-\frac{2GM}{c^2 r}\right)^{-1}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2\right), \tag{3.18}$$

where $R_S \equiv 2GM/c^2$ is the so called *Schwarzschild radius*; the apparent singularity at $r = R_S$ is (as we will show shortly) not a physical one, but a by-product of the coordinate system.

In order to simplify even more the problem (and to be able to create bridge to the previous section), lets consider a $1+1$ Schwarzschild black hole; to obtain such solution, simply take the limit $\mathrm{d}\theta = \mathrm{d}\varphi = 0$ at the solution 3.18. There are two interesting coordinate systems to describe this spacetime; the "tortoise" coordinates and the Kruskal–Szekeres coordinates; each with some precise interest. The first of the two comes from the use of the so called tortoise coordinate, given by the expression

$$\mathrm{d}r^* \equiv \left(1-\frac{R_S}{r}\right)^{-1}\mathrm{d}r, \tag{3.19}$$

[4] An sketch of a proof goes as follows: Consider some conserved energy E_0 measured by an observer in an stationary gravitational field, such energy relates to the energy measured by another observer by $E_0 = \sqrt{g_{00}}E$; since the thermodynamic relation between energy and temperature comes from the entropy S as $T_0 = \partial S/\partial E_0$, then the proper temperature must follow the relation given by the Tolman's law.

[5] This is the first found solution of the Einstein field equations 1.7 by Karl Schwarzschild in 1916; Einstein himself was quoted to be amazed by the simplicity and the speed of Schwarzschild's derivation.

meaning that the line element will be

$$c^2 \mathrm{d}\tau^2 = \left[1 - \frac{R_S}{r\,(r^*)}\right]\left[c^2 \mathrm{d}t^2 - \mathrm{d}r^*\right]; \tag{3.20}$$

similarly to the previous section, it is useful to write it using null coordinates $u \equiv ct - r^*$ and $v \equiv ct + r^*$,

$$c^2 \mathrm{d}\tau^2 = \left[1 - \frac{R_S}{r\,(u, v)}\right]\left[\mathrm{d}^2 u - \mathrm{d}^2 v\right]. \tag{3.21}$$

Last expression can be rewritten considering that[6]

$$1 - \frac{R_S}{r\,(u, v)} = \frac{R_S}{r} \exp\left(1 - \frac{r}{R_S}\right) \exp\left(\frac{v - u}{2R_S}\right) \tag{3.22}$$

and, defining the Kruskal–Szekeres null coordinates $(U,\, V)$ as

$$U \equiv -2R_S e^{-u/2R_S}, \qquad V \equiv 2R_S e^{v/2R_S}, \tag{3.23 a,b}$$

the line element can be written as

$$c^2 \tau^2 = \frac{R_S}{r(U, V)} e^{1 - r(U,\,V)/R_S} \mathrm{d}U \mathrm{d}V. \tag{3.24}$$

Note that both coordinate systems make the metric conformally flat, and the relation between the null tortoise and null Kruskal–Szekeres coordinates given by 3.23 is the same as in 3.10 considering $\alpha = {c^2}/{2R_S}$; therefore the procedure of the previous section can be used here, and thus, according to 3.16, a tortoise observer located at $r \to \infty$ will measure that the black hole emits a thermal bath of massless boson particles at a temperature

$$T_{\text{Hawking}} = \frac{\hbar c^3}{8\pi G M k_B}. \tag{3.25}$$

If one where to consider a $3+1$ black hole [11, sec. 9.1.4] described by the line element given by 3.18, the Klein-Gordon equation would not be $\partial^\mu \partial_\nu \phi = 0$, but would have an effective potential

$$\mathcal{V}_{\text{eff}} \equiv \left(1 - \frac{R_S}{r}\right)\left[\frac{R_S}{r^3} + \frac{l(l+1)}{r^2}\right], \tag{3.26}$$

l being the quantum orbital angular momentum of the state. The effect of such potential is such that an escaping wave, upon reaching $r \to \infty$ would have changed its frequency Ω, reason for which 3.15 would be affected by some greybody factor $\Gamma(\Omega)$

$$N_\Omega \propto \frac{\Gamma(\Omega)}{e^{\Omega/k_B T_{\text{Hawking}}} - 1} \delta(\mathbf{0}). \tag{3.27}$$

3.2.1 Black Hole Thermodynamics

Once it has been proven that an observer can detect an emission of particles from a black hole, the question of the origin of the required energy for the creation of such particles arises. One can consider that the energy required to create Unruh particles can come from the energy used for the continuous acceleration, but this cannot be the answer for the Hawking radiation, since both observers are free falling. Therefore the only possible answer must be that the particles extract

[6] To check this, one must use the integrated form of the tortoise coordinate 3.19 given by

$$r^*(r) = r - R_S + R_S \ln\left(\frac{r}{R_S} - 1\right).$$

their energy directly from the black hole; meaning that a black hole must be in thermodynamic equilibrium with the field. This reasoning can be used to deduce the evolution of black holes; as stated before, black holes can be described univocally by three parameters, and thus, its entropy fundamental relation can be written as

$$\mathrm{d}S = \left(\frac{\partial S}{\partial M}\right)\mathrm{d}M + \left(\frac{\partial S}{\partial J}\right)\mathrm{d}J + \left(\frac{\partial S}{\partial Q}\right)\mathrm{d}Q. \tag{3.28}$$

For a Schwarzschild black hole, this relation simplifies (since $Q = J = 0$); and considering that the energy must equal Mc^2, then one can deduce the so called Bekenstein-Hawking [7] entropy

$$\mathrm{d}S = \frac{c^2}{T_{\text{Hawking}}}\mathrm{d}M \implies S_{\text{BH}} = \frac{4\pi G k_B}{\hbar c}M^2. \tag{3.29}$$

Since we are considering that the field extracts energy from the black hole, and the energy of the black hole is proportional to its mass; it is then clear that the black hole must be loosing mass. One can easily deduce the temporal mass expression considering the black hole as a black body, and thus following the Stefan-Boltzmann law for the luminosity L, i.e.

$$L \equiv -c^2\frac{\mathrm{d}M}{\mathrm{d}t} = \epsilon A\sigma T^4_{\text{Hawking}} \tag{3.30}$$

where σ its the Stefan-Boltzmann constant and ϵ is a factor of correction for possible greybody effects and deviations from the use of the electromagnetic field (mostly lose of degrees of freedom). Solutions of such differential equation are

$$M(t) = M_0\left(1 - \frac{t}{t_{BH}}\right)^{1/3}, \quad t_{BH} \equiv 5120\frac{\pi G^2}{\epsilon\hbar c^4}M_0^3. \tag{3.31 a,b}$$

To get an idea of the strength of such radiation, lets consider an average stellar black hole (the most numerous type), which masses around $100M_{\odot}$; considering that at $t = 0$, a black hole has that mass, it would have lost all of it (it is said to have "evaporated") in about $\sim 2.1 \cdot 10^{73}$ years, an unfathomable magnitude comparable to the age of the universe ($13.7 \cdot 10^9$ years).

[7] In reality, they presented their result not as a function of the mass M, but as a function of the surface area A, and thus, the proper Bekenstein-Hawking entropy would be $S_{BH} = {}^{c^3 k_B}/_{\hbar G}A$.

4 Problems Relating to $T_{\mu\nu}$

4.1 The Effective Action W

For simplicity we consider a scalar field described by the action presented at equation 1.14 (...)

According to the path integral formulation of quantum field theory, the time-ordered vacuum expectation value of $T_{\mu\nu}$ is given by

$$\langle T_{\mu\nu}\rangle = \frac{\int \mathcal{D}[\phi]\, T_{\mu\nu}\, e^{iS_{\mathrm{M}}[\phi]\hbar^{-1}}}{\int \mathcal{D}[\phi]\, e^{iS_{\mathrm{M}}[\phi]\hbar^{-1}}} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta W}{\delta g^{\mu\nu}}, \tag{4.1}$$

where the last definition mimics the energy momentum tensor describe in equation 1.8, where the quantity W is said to be the effective action of the field for a semi-classic treatment.

Now, would be useful to have a closed expression for W, this will be done using the generating functional $Z[J] \equiv \int \mathcal{D}[\phi]\, \exp\{iS_{\mathrm{M}}[\phi]\hbar^{-1} + i\int \mathrm{d}^4 J(x)\phi(x)\}$, where $J(x)$ is some external current, that will be consider zero for the following treatment. Substituting the definition of $T_{\mu\nu}$ (again, expression 1.8) on the previous equation, one finds the following relation

$$\langle T_{\mu\nu}\rangle = \frac{-2}{Z[0]\sqrt{-g}}\int \mathcal{D}[\phi]\,\frac{\delta S_{\mathrm{M}}[\phi]}{\delta g^{\mu\nu}}\, e^{iS_{\mathrm{M}}[\phi]\hbar^{-1}} = \frac{2i\hbar}{Z[0]\sqrt{-g}}\frac{\delta Z[0]}{\delta g^{\mu\nu}}, \tag{4.2}$$

meaning that the effective action can be written as

$$W = -i\hbar \ln\left(\frac{Z[0]}{\mu_0 \hbar^{1/2}}\right) + \mathrm{C}; \tag{4.3}$$

where we have introduced an integral constant C, and a constant μ_0 with the same units as μ (that is, inverse of length), in order to maintain the argument of the logarithm dimensionless.

The expression for W can be simplified even further in terms of (supposedly) known parameters; to obtain such representation, consider that the action $S_{\mathrm{M}}[\phi]$ given by equation 1.14 can be written as

$$S_{\mathrm{M}}[\phi] = \frac{1}{2}\int \partial_\nu\left[\sqrt{-g}\,\phi\,\partial^\nu\phi\right]\mathrm{d}^4x - \int \frac{1}{2}\phi\left[\partial_\nu\partial^\nu + \mu^2 + \xi R\right]\phi\,\sqrt{-g}\,\mathrm{d}^4x, \tag{4.4}$$

where the first term is a total derivative and thus can be dropped. Next we could use the Dirac delta function to write

$$\phi(x) = \int \phi(y)\frac{\delta^4(x-y)}{\sqrt{-g(x)}}\sqrt{-g(y)}\,\mathrm{d}^4y; \tag{4.5}$$

meaning that the action might be expressed as

$$S_{\mathrm{M}}[\phi] = -\frac{1}{2}\int \sqrt{-g(x)}\,\mathrm{d}^4x \int \phi(x)K(x,y)\phi(y)\sqrt{-g(y)}\,\mathrm{d}^4y; \tag{4.6}$$

where we defined the function $K(x,y) \equiv \left[\partial_\nu\partial^\nu + \mu^2 + \xi R(x)\right]\delta^4(x-y)/\sqrt{-g(x)}$. But what exactly represents $K(x,y)$? Well, considering the definition of an inverse function

$$\int K(x,y)K^{-1}(y,z)\,\sqrt{-g(y)}\,\mathrm{d}^4y = \frac{\delta^4(x-z)}{\sqrt{-g(z)}}, \tag{4.7}$$

one ends up with the following relation

$$\left[\partial_\nu\partial^\nu + \mu^2 + \xi R(x)\right]K^{-1}(x,z) = \frac{\delta^4(x-z)}{\sqrt{-g(z)}}; \tag{4.8}$$

which is the definition of the Feynman propagator $G_F(x,z)$, meaning that

$$K(x,y) = -G_F^{-1}(x,y). \tag{4.9}$$

Using this relation, we can write the action as

$$S_{\mathrm{M}}[\phi] = \frac{1}{2}\int \sqrt{-g(x)}\,\mathrm{d}^4x \int \phi(x) G_F^{-1}(x,y)\phi(y)\sqrt{-g(y)}\,\mathrm{d}^4y, \tag{4.10}$$

which can be interpreted as the product of matrices of continuous index $\phi^\dagger G_F\,\phi$ (where G_F is the operator related to the propagator $G_F(x,y)$ through $G_F(x,y) = \langle x|G_F|y\rangle$), and thus, one can compute the value of $Z[0]$ as

$$Z[0] = \int \mathcal{D}[\phi]\,\exp\left\{ i\frac{\phi^\dagger G_F^{-1}\phi}{2\hbar}\right\} \propto \left[\hbar^{-1}\det\left(-G_F\right)\right]^{-1/2}. \tag{4.11}$$

Substituting this value in equation 4.3, and by appropriately choosing the value of the constant C to compensate for the proportionality factor [1], one can deduce the following expression for the effective action

$$W = -\frac{i\hbar}{2}\ln\left[\mu_0^{-2}\det\left(-G_F\right)\right] = -\frac{i\hbar}{2}\mathrm{Tr}\left[\ln\left(-\mu_0^{-2}G_F\right)\right]. \tag{4.12}$$

Here we introduced the trace of the operator $-\mu_0^{-2}G_F$, this can be computed using the definition of the trace,

$$\mathrm{Tr}[A] \equiv \int A(x,x)\,\sqrt{-g(x)}\,\mathrm{d}^4x = \int \langle x|A|x\rangle\,\sqrt{-g}\,\mathrm{d}^4x; \tag{4.13}$$

which will be relevant in what follows.

4.1.1 Renormalization

$$G_F^{\mathrm{DS}}(x,y) \equiv -i\frac{\Delta^{1/2}(x,y)}{(4\pi)^{(d+1)/2}}\int_0^\infty F(x,y;is)\exp\left[-is\mu^2 + \frac{\sigma(x,y)}{2is}\right](is)^{-(d+1)/2}\,\mathrm{d}(is) \tag{4.14}$$

$$G_F = -K^{-1} = -\int_0^\infty e^{-isK}\mathrm{d}(is) \tag{4.15}$$

$$\left\langle x\left|e^{-isK}\right|y\right\rangle = i\frac{\Delta^{1/2}(x,y)}{(4\pi)^{(d+1)/2}}F(x,y;is)\exp\left[-is\mu^2+\frac{\sigma(x,y)}{2is}\right](is)^{-(d+1)/2} \tag{4.16}$$

$$\left\langle x\left|(is)^{-1}e^{-isK}\right|y\right\rangle = i\int_{\mu^2}^\infty \frac{\Delta^{1/2}(x,y)}{(4\pi)^{(d+1)/2}}F(x,y;is)\exp\left[-is\mu^2+\frac{\sigma(x,y)}{2is}\right](is)^{-(d+1)/2}\,\mathrm{d}\left(\mu^2\right) \tag{4.17}$$

$$\int_0^\infty (is)^{-1}e^{-isK}\mathrm{d}(is) = -\ln\left(\mu_0^2\,K\right) + \mathrm{C}' = \ln\left(-\mu_0^{-2}\,G_F\right) + \mathrm{C}' \tag{4.18}$$

$$W = \frac{i\hbar}{2}\int_{\mu^2}^\infty \mathrm{d}\left(\mu^2\right)\int G_F(x,x)\,\sqrt{-g(x)}\,\mathrm{d}^4x \tag{4.19}$$

$$\mathcal{L}_{\mathrm{eff}} \equiv \lim_{y\to x}\frac{i\hbar c}{2}\int_{\mu^2}^\infty G_F^{\mathrm{DS}}(x,y)\,\mathrm{d}\left(\mu^2\right) \tag{4.20}$$

$$\mathcal{L}_{\mathrm{eff}} = \lim_{y\to x}\frac{\hbar c}{2\mu_0^{d-3}}\frac{\Delta^{1/2}(x,y)}{(4\pi)^{(d+1)/2}}\int_0^\infty F(x,y;is)\exp\left[-is\mu^2+\frac{\sigma(x,y)}{2is}\right](is)^{-(d+3)/2}\,\mathrm{d}(is) \tag{4.21}$$

μ_0^{d-3} was introduced so that the units of $\mathcal{L}_{\mathrm{eff}}$ remain to be the same as for $d = 3$. Blablabla convergence for any d and thus we take the limit

[1] Nevertheless, if another choice were to be made, the sum of both the constant C and the logarithm of the proportionality factor, is not a function of the metric, and thus, it is irrelevant under variations of $g_{\mu\nu}$.

$$\mathcal{L}_{\text{eff}} = \frac{\hbar c}{2\mu_0^{d-3}} \frac{1}{(4\pi)^{(d+1)/2}} \sum_{n=0}^{\infty} a_n(x) \int_0^{\infty} \exp\left(-is\mu^2\right)(is)^{n-(d+3)/2}\, \mathrm{d}(is) =$$

$$= \frac{\hbar c}{2(4\pi)^{(d-1)/2}} \left(\frac{\mu}{\mu_0}\right)^{(d-3)} \sum_{n=0}^{\infty} a_n(x)\mu^{2(2-n)}\Gamma\left(n - \frac{d+1}{2}\right) \quad (4.22)$$

$$\Gamma\left(-\frac{d+1}{2}\right) = \frac{4}{d^2-1}\left(\frac{2}{3-d} - \gamma\right) + \mathcal{O}(d-3), \quad (4.23a)$$

$$\Gamma\left(1-\frac{d+1}{2}\right) = \frac{2}{1-d}\left(\frac{2}{3-d} - \gamma\right) + \mathcal{O}(d-3), \quad (4.23b)$$

$$\Gamma\left(2-\frac{d+1}{2}\right) = \frac{2}{3-d} - \gamma + \mathcal{O}(d-3). \quad (4.23c)$$

$$\left(\frac{\mu}{\mu_0}\right)^{d-3} = 1 + (d-3)\ln\left(\frac{\mu}{\mu_0}\right) + \mathcal{O}\left[(d-3)^2\right] \quad (4.24)$$

$$\mathcal{L}_{\text{eff}}^{\infty} = -\lim_{d\to 3} \frac{\hbar c}{2(4\pi)^2} \left\{\frac{1}{d-3} + \frac{1}{2}\left[\gamma + 2\ln\left(\frac{\mu}{\mu_0}\right)\right]\right\} \left[\mu^4 a_0(x) - \mu^2 a_1(x) + 2a_2(x)\right] \quad (4.25)$$

$$a_0(x) = 1, \quad a_1(x) = \left(\frac{1}{6} - \xi\right) R, \quad (4.26\text{ a,b})$$

$$a_2(x) = \frac{1}{180}\left(R_{\alpha\beta\gamma\sigma}R^{\alpha\beta\gamma\sigma} - R_{\alpha\beta}R^{\alpha\beta}\right) - \frac{1}{6}\left(\frac{1}{5} - \xi\right)\partial_\nu\partial^\nu R + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2. \quad (4.26\text{ c})$$

$$\mathcal{L}'_{\text{G}} \equiv \mathcal{L}_{\text{G}} - \mathcal{L}_{\text{eff}} = \left(A + \frac{1}{2\kappa}\right) R - \left(B + \frac{1}{\kappa}\Lambda\right) - \lim_{d\to 3} \frac{a_2(x)\hbar c}{(4\pi)^2} \left\{\frac{1}{d-3} + \frac{1}{2}\left[\gamma + 2\ln\left(\frac{\mu}{\mu_0}\right)\right]\right\} \quad (4.27)$$

$$A \equiv \lim_{d\to 3} \frac{\mu^2\,\hbar c}{2(4\pi)^2}\left(\frac{1}{6} - \xi\right)\left\{\frac{1}{d-3} + \frac{1}{2}\left[\gamma + 2\ln\left(\frac{\mu}{\mu_0}\right)\right]\right\}, \quad (4.28a)$$

$$B \equiv \lim_{d\to 3} \frac{\mu^4\,\hbar c}{2(4\pi)^2}\left\{\frac{1}{d-3} + \frac{1}{2}\left[\gamma + 2\ln\left(\frac{\mu}{\mu_0}\right)\right]\right\} \quad (4.28b)$$

$$G' \equiv G\,(1 + 2\kappa A)^{-1}, \quad \Lambda' \equiv (\Lambda + B\kappa)\,. \quad (4.29a)$$

$S[g] = \int \frac{1}{2\kappa} f(R)\sqrt{-g}\,\mathrm{d}^4x$

$$S[g,\phi] = \int \left[\frac{1}{2\kappa} f(R) - \mathcal{L}_{\text{eff}}\right]\sqrt{-g}\,\mathrm{d}^4x + W_{\text{ren}} \quad (4.30)$$

$$W_{\text{ren}} = \int \left[\mathcal{L}_{\text{eff}} - \mathcal{L}_{\text{eff}}^{\infty}\right]\sqrt{-g}\mathrm{d}^4x = -\frac{\hbar}{64\pi^2}\int\int_0^{\infty} \ln(is)\partial_{is}^3\left[F(x;is)e^{-is\mu^2}\right]\sqrt{-g}\,\mathrm{d}(is)\mathrm{d}^4x. \quad (4.31)$$

4.2 The Conformal Anomaly

TBD

5 Final Discussions

TBD

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Scalar field in Minkowski background

Throughout this thesis, the main field used to introduce the theory of quantum field in curved spacetimes, was the scalar field, reason for which it might be of good practice the introduction of the theory of scalar fields on a Minkowski background.

Introduction

The simplest action for a (real) scalar field without interactions, might be

$$S[\phi] = \int \frac{1}{2}\Big[\partial_\nu\phi\,\partial^\nu\phi - \mu^2\phi^2\Big]\,\mathrm{d}^4x. \tag{1}$$

From this action, one can obtain the equations of motion of the field from the Euler-Lagrange equations, which would yield as a result the so called Klein-Gordon equation,

$$\left(\partial_\nu\partial^\nu - \mu^2\right)\phi = 0, \tag{2}$$

with solutions of the form

$$\phi_{\mathbf{k}} = a_{\mathbf{k}}e^{ikx\,\hbar^{-1}} + \bar{a}_{\mathbf{k}}e^{-ikx\,\hbar^{-1}}. \tag{3}$$

Substitution of this solution on the Klein-Gordon equation, gives the following dispersion relation

$$k_\nu k^\nu = \hbar^2\mu^2, \tag{4}$$

which gives a relation between the parameter μ and the mass of the field m through $\mu = {}^{mc}/_{\hbar}$.

The most general solution of the Klein-Gordon equation can be written as a mode expansion of solutions of the form 3, this would be

$$\phi(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3 2k_0}\phi_{\mathbf{k}} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3 2k_0}\left(a_{\mathbf{k}}e^{ikx\,\hbar^{-1}} + \bar{a}_{\mathbf{k}}e^{-ikx\,\hbar^{-1}}\right); \tag{5}$$

where $\frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3 2k_0}$ is a Lorentz-invariant measure. It is convenient to redefine $a_{\mathbf{k}} \to a_{\mathbf{k}}(2k_0)^{-1/2}$, and thus, the field would be described as,

$$\phi(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3\sqrt{2k_0}}\left(a_{\mathbf{k}}e^{ikx\,\hbar^{-1}} + \bar{a}_{\mathbf{k}}e^{-ikx\,\hbar^{-1}}\right). \tag{6}$$

Quantization

The methodology to canonically quantize a field comes from the promotion of the field $\phi(x)$ and its conjugated momenta $\Pi(x) \equiv \partial_{ct}$ unto quantum operators,

$$\phi(x) \longrightarrow \hat{\phi}(x), \qquad \Pi(x) \longrightarrow \hat{\Pi}(x),$$

to do so, the most common procedure is to promote the mode constant factors to quantum operators, which will be known as annihilation and creator operators,

$$a_{\mathbf{k}} \longrightarrow \hat{a}_{\mathbf{k}}, \qquad \bar{a}_{\mathbf{k}} \longrightarrow \hat{a}^\dagger_{\mathbf{k}}.$$

This will imply that the quantum field operator $\hat{\phi}(x)$ will have the form

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3\sqrt{2k_0}}\left(\hat{a}_{\mathbf{k}}e^{ikx\,\hbar^{-1}} + \hat{a}^\dagger_{\mathbf{k}}e^{-ikx\,\hbar^{-1}}\right). \tag{7}$$

In addition of the promotion, some commutation rules must be imposed, to do so, Dirac proposes the following procedure: to replace the Poisson Brackets of the phase space with commutators, as

$$\{A, B\} \to \frac{1}{i\hbar}\left[\hat{A}, \hat{B}\right]; \tag{8}$$

the Poisson Bracket for a coordinate q_i and a conjugate momentum p_j is $\{q_i, p_j\} = \delta_{ij}$, therefore it is natural to consider as commutation rules the following,

$$\left[\hat{\phi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = i\hbar\, \delta^3\left(\mathbf{x} - \mathbf{y}\right) \qquad \left[\hat{\phi}(\mathbf{x}),\, \hat{\phi}(\mathbf{y})\right] = \left[\hat{\Pi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = 0.. \tag{9 a-c}$$

Substitution of the quantum field expression on these commutators, will give the needed commutation rules for the annihilation and creation operators:

$$\left[\hat{a}_{\mathbf{k}},\, \hat{a}^\dagger_{\mathbf{q}}\right] = \left(2\pi\hbar\right)^3 \hbar^2 \delta^3\left(\mathbf{k} - \mathbf{q}\right), \qquad \left[\hat{a}_{\mathbf{k}},\, \hat{a}_{\mathbf{q}}\right] = \left[\hat{a}^\dagger_{\mathbf{k}},\, \hat{a}^\dagger_{\mathbf{q}}\right] = 0. \tag{10 a-c}$$

Hamiltonian and Fock space

The most relevant quantum operator is without doubt, the Hamiltonian $\hat{\mathcal{H}}$, which is given by Noether's theorem as the conserved current

$$\hat{\mathcal{H}} = \int \left[\frac{\partial \hat{\mathcal{L}}}{\partial\left(\partial_0 \hat{\phi}\right)} - \hat{\mathcal{L}}\right] \mathrm{d}^3\mathbf{x}; \tag{11}$$

for the given field, one obtains the following expression for the Hamiltonian,

$$\hat{\mathcal{H}} = \int \left(\hat{\Pi}\partial_0\hat{\phi} - \hat{\mathcal{L}}\right) \mathrm{d}^3\mathbf{x} = \int \frac{c}{2}\left[\hat{\Pi}^2 + \left(\nabla\hat{\phi}\right)^2 + \mu^2\hat{\phi}^2\right] \mathrm{d}^3\mathbf{x}; \tag{12}$$

and once the quantum field expansion is substituted, it will be simplified as

$$\hat{\mathcal{H}} = \int E_{\mathbf{p}} \left[\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \frac{1}{2}(2\pi\hbar)^3\hbar^2\delta(\mathbf{0})\right] \frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3\hbar^2}. \tag{13}$$

Note that the constant term

$$\hat{\mathcal{H}}_{\mathrm{div}} = \frac{1}{2}(2\pi\hbar)^3\hbar^2\delta(\mathbf{0}) \int E_{\mathbf{p}} \frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3\hbar^2}, \tag{14}$$

is divergent, and known as vacuum energy; it is useful to define a "normal" ordered Hamiltonian without this term, which will be zero for a vacuum state

$$:\hat{\mathcal{H}}:= \int E_{\mathbf{p}}\ \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} \frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3\hbar^2}. \tag{15}$$

The Fock space $\{|\mathbf{p}\rangle\}$ generated by the Hamiltonian is formed from a vacuum (no particles) state $|0\rangle$ which is annihilated by $\hat{a}_{\mathbf{p}}$, i.e.

$$\hat{a}_{\mathbf{p}}|0\rangle = 0; \tag{16}$$

other one particle states, with some momentum $\mathbf{p}$, are formed from the vacuum state after applying the creator operator,

$$|\mathbf{p}\rangle \equiv \hat{a}^\dagger_{\mathbf{p}}|0\rangle; \tag{17}$$

similar to a quantum oscillator. Multiparticle states are formed after the chain use of the creator operators,

$$|\mathbf{p}_1, \mathbf{p}_2, \ldots\rangle \equiv \ldots \hat{a}^\dagger_{\mathbf{p}_2}\hat{a}^\dagger_{\mathbf{p}_1}|0\rangle; \tag{18}$$

note that this quantum states are bosonic, since

$$|\mathbf{p}_1, \mathbf{p}_2\rangle = \hat{a}^\dagger_{\mathbf{p}_2}\hat{a}^\dagger_{\mathbf{p}_1}|0\rangle = \left[\hat{a}^\dagger_{\mathbf{p}_2}, \hat{a}^\dagger_{\mathbf{p}_1}\right]|0\rangle + \hat{a}^\dagger_{\mathbf{p}_1}\hat{a}^\dagger_{\mathbf{p}_2}|0\rangle = |\mathbf{p}_2, \mathbf{p}_1\rangle. \tag{19}$$

Units

- $[S] = [\hbar]$
- $[a] = [\xi] = 1$
- $[\mu] = [L]^{-1}$
- $[R] = [L]^{-2}$
- $[\phi] = [\chi] = [\hbar]^{1/2}[L]^{-1}$
- $[\Pi] = [\hbar]^{1/2}[L]^{-2}$
- $[a_{\mathbf{k}}] = [\hbar]^{1/2}[L]^{2}$

Questions & To-Do

1 To-Do

- Minkowski scalar field.
- Move Units appendix to conventions.
- Preface: "Natura abhorret vacuum" (René Descartes), *Horror vacui*, "gravity of the situation of current research".