

PROBLEMS OF QUANTUM FIELD THEORIES IN CURVED SPACETIMES

A MASTER THESIS

Submitted in partial fulfillment of the requirements for the award of

Master's Degree in Physics of the Universe: Cosmology, Astrophysics, Particles
and Astroparticles

by

Cano Jones, Alejandro

Supervised by

Asorey Carballeira, Manuel



Department of Theoretical Physics
UNIVERSITY OF ZARAGOZA
August 21, 2023

Contents

1	First Chapter	1
	Notas sobre unidades	I
	Bibliography	II

1 First Chapter

FLRW metric

$$dl^2 = c^2 dt^2 - a^2(t) \left[\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right] \quad (1.1)$$

Weyl tensor =0 therefore the metric is conformally flat, i.e. independently of the curvature κ there must exist a coordinate system where

$$dl^2 = a(t) \eta_{\mu\nu} dx^\mu dx^\nu = a(t) [c^2 dt^2 - d\mathbf{x}^2] \quad (1.2)$$

the standard action describing the dynamics of a (non-minimally coupled to gravity) real scalar field is

$$s = \int \frac{1}{2} \left[\nabla_\nu \phi \nabla^\nu \phi - \mu^2 \phi^2 - \xi R \phi^2 \right] \sqrt{-g} d^4x \quad (1.3)$$

$$\sqrt{-g} = a^4 \quad \chi = a\phi$$

$$s = \int \frac{1}{2} \left[\partial_\nu \chi \partial^\nu \chi - \left(\mu^2 a^2 + \xi R a^2 - c^2 \frac{a''}{a} \right) \chi^2 - \partial_t \left(c^2 \chi^2 \frac{a'}{a} \right) \right] d^4x \quad (1.4)$$

dropping the time drivative

$$s = \int \frac{1}{2} \left[\partial_\nu \chi \partial^\nu \chi - \left(\mu^2 a^2 + \xi R a^2 - c^2 \frac{a''}{a} \right) \chi^2 \right] d^4x \quad (1.5)$$

by Euler-Lagrange

$$[\partial_\nu \partial^\nu + \mu_{\text{eff}}^2(t)] \chi = 0 \quad (1.6)$$

where

$$\mu_{\text{eff}}^2(t) = (\mu^2 + \xi R) a^2 - c^2 \frac{a''}{a} \quad (1.7)$$

solutions of previous equation have the form

$$\chi = a v(t) e^{\pm i \mathbf{k} \mathbf{x} \hbar^{-1}} \quad (1.8)$$

meaning that, the dispersion relation is

$$v'' \hbar^2 + \omega^2(t) v = 0 \quad (1.9)$$

where $\omega(t)$ is defined as

$$\omega^2(t) = \mathbf{k}^2 + \hbar^2 \mu_{\text{eff}}^2(t) = \mathbf{k}^2 + (m^2 c^2 + \xi \hbar^2 R) a(t) - \hbar^2 c^2 \frac{a''}{a} \quad (1.10)$$

now, proof that $\text{Im}(vv'^*)$ is constant through time

$$\frac{\partial}{\partial t} \text{Im}(vv'^*) = \frac{\partial}{\partial t} \left(\frac{vv'^* - v^*v'}{2i} \right) = \frac{vv''^* - v^*v''}{2i} = 0 \quad (1.11)$$

last step is result from dispersion relation. Since dispersion relation is scalable by a time independent function, $\text{Im}(v'v^*)$ can be determined to be a chosen value, a particular useful choice is to consider it momentum independent.

The most general solution to the main equation is

$$\chi = \int \frac{d^3 \mathbf{k}}{(2\pi \hbar)^3} \left[a_{\mathbf{k}} v_{\mathbf{k}}(t) e^{i \mathbf{k} \mathbf{x} \hbar^{-1}} + a_{\mathbf{k}}^* v_{\mathbf{k}}^*(t) e^{-i \mathbf{k} \mathbf{x} \hbar^{-1}} \right] \quad (1.12)$$

1 First Chapter

The field χ and its conjugate momentum $\Pi = \partial_{ct}\chi$ are promoted to operators on the quantum Hilbert space, with the standar canonical conmutation relations

$$[\hat{\chi}(t, \mathbf{x}), \hat{\Pi}(t, \mathbf{y})] = i\hbar \delta^3(\mathbf{x} - \mathbf{y}) \quad (1.13)$$

$$[\hat{\chi}(t, \mathbf{x}), \hat{\chi}(t, \mathbf{y})] = [\hat{\Pi}(t, \mathbf{x}), \hat{\Pi}(t, \mathbf{y})] = 0 \quad (1.14)$$

where the operational nature of the fields arrise from the promotion of the mode amplitudes, i.e.

$$a_{\mathbf{k}} \longrightarrow \hat{a}_{\mathbf{k}} \quad a_{\mathbf{k}}^* \longrightarrow \hat{a}_{\mathbf{k}}^\dagger \quad (1.15)$$

this operators fulfill the following conmutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}^\dagger] = \frac{(2\pi\hbar)^3 \hbar c}{2\text{Im}(v'v^*)} \delta^3(\mathbf{k} - \mathbf{q}), \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{q}}^\dagger] = 0 \quad (1.16)$$

(note that $\hat{a}_{\mathbf{k}} \neq \hat{a}_{-\mathbf{k}}$)

To prove this, consider that

$$\begin{aligned} [\hat{\chi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})] &= \frac{1}{c} \int \frac{d^3\mathbf{k} d^3\mathbf{q}}{(2\pi\hbar)^6} \left\{ [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] v_{\mathbf{k}} v'_{\mathbf{q}} e^{i(\mathbf{k}\mathbf{x} + \mathbf{q}\mathbf{y})\hbar^{-1}} + [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{q}}^\dagger] v_{\mathbf{k}}^* v'_{\mathbf{q}}^* e^{i(\mathbf{k}\mathbf{x} - \mathbf{q}\mathbf{y})\hbar^{-1}} + \right. \\ &\quad \left. + [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}^\dagger] v_{\mathbf{k}} v_{\mathbf{q}}^* e^{i(\mathbf{k}\mathbf{x} - \mathbf{q}\mathbf{y})\hbar^{-1}} - [\hat{a}_{\mathbf{q}}, \hat{a}_{\mathbf{k}}^\dagger] v_{\mathbf{k}}^* v'_{\mathbf{q}} e^{-i(\mathbf{k}\mathbf{x} - \mathbf{q}\mathbf{y})\hbar^{-1}} \right\} \end{aligned} \quad (1.17)$$

if the operators \hat{a} and \hat{a}^\dagger are to be understood as creation and annihilation operators, they must fulfill

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}^\dagger] = \alpha \delta^3(\mathbf{k} - \mathbf{q}), \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{q}}^\dagger] = 0 \quad (1.18)$$

where $\alpha \in \mathbb{C}$, and thus

$$[\hat{\chi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})] = \frac{\alpha}{c} \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^6} 2i\text{Im}(v_{\mathbf{k}} v_{\mathbf{k}}^*) e^{i(\mathbf{k}\mathbf{x} - \mathbf{q}\mathbf{y})\hbar^{-1}} \quad (1.19)$$

considering $\text{Im}(v'v^*)$ momentum independent, and remembering the canonical conmutation relations, one finds that

$$\alpha \text{Im}(vv^*) = \frac{1}{2} \hbar c (2\pi\hbar)^3 \quad (1.20)$$

The hamiltonian

$$\hat{\mathcal{H}}(t) = \int \frac{c}{2} \left[\hat{\Pi}^2 + (\nabla \hat{\chi})^2 + \mu_{\text{eff}}^2(t) \hat{\chi}^2 \right] d^3\mathbf{x} \quad (1.21)$$

$$\begin{aligned} \hat{\Pi}^2 &= \frac{1}{c^2} \int \frac{d^3\mathbf{k} d^3\mathbf{q}}{(2\pi\hbar)^6} \left[\hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}} v'_{\mathbf{k}} v'_{\mathbf{q}} e^{i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}}^\dagger v'_{\mathbf{k}} v'_{\mathbf{q}}^* e^{i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} + \right. \\ &\quad \left. + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}} v_{\mathbf{k}}^* v'_{\mathbf{q}} e^{-i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}}^\dagger v_{\mathbf{k}}^* v_{\mathbf{q}}^* e^{-i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} \right] \end{aligned} \quad (1.22)$$

$$\begin{aligned} (\nabla \hat{\chi})^2 &= -\frac{1}{\hbar^2} \int \frac{d^3\mathbf{k} d^3\mathbf{q}}{(2\pi\hbar)^6} \mathbf{k}\mathbf{q} \left[\hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}} v_{\mathbf{k}} v_{\mathbf{q}} e^{i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} - \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}}^\dagger v_{\mathbf{k}} v_{\mathbf{q}}^* e^{i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} - \right. \\ &\quad \left. - \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}} v_{\mathbf{k}}^* v_{\mathbf{q}} e^{-i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}}^\dagger v_{\mathbf{k}}^* v_{\mathbf{q}}^* e^{-i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} \right] \end{aligned} \quad (1.23)$$

$$\begin{aligned} \hat{\chi}^2 &= \int \frac{d^3\mathbf{k} d^3\mathbf{q}}{(2\pi\hbar)^6} \left[\hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}} v_{\mathbf{k}} v_{\mathbf{q}} e^{i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}}^\dagger v_{\mathbf{k}} v_{\mathbf{q}}^* e^{i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} + \right. \\ &\quad \left. + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}} v_{\mathbf{k}}^* v_{\mathbf{q}} e^{-i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}}^\dagger v_{\mathbf{k}}^* v_{\mathbf{q}}^* e^{-i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} \right] \end{aligned} \quad (1.24)$$

$$\begin{aligned}
\hat{\mathcal{H}} = \frac{c}{2} \int \frac{d^3\mathbf{k}d^3\mathbf{q}}{(2\pi\hbar)^3} \left\{ \hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{q}} \left[\frac{1}{c^2}v'_{\mathbf{k}}v'_{\mathbf{q}} - \left(\frac{1}{\hbar^2}\mathbf{kq} - \mu_{\text{eff}}^2 \right) v_{\mathbf{k}}v_{\mathbf{q}} \right] \delta^3(\mathbf{k} + \mathbf{q}) + \right. \\
+ \hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{q}}^\dagger \left[\frac{1}{c^2}v'_{\mathbf{k}}v'_{\mathbf{q}}^* + \left(\frac{1}{\hbar^2}\mathbf{kq} + \mu_{\text{eff}}^2 \right) v_{\mathbf{k}}v_{\mathbf{q}}^* \right] \delta^3(\mathbf{k} - \mathbf{q}) + \\
+ \hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{q}} \left[\frac{1}{c^2}v_{\mathbf{k}}^*v'_{\mathbf{q}} + \left(\frac{1}{\hbar^2}\mathbf{kq} + \mu_{\text{eff}}^2 \right) v_{\mathbf{k}}^*v_{\mathbf{q}} \right] \delta^3(\mathbf{k} - \mathbf{q}) + \\
\left. + \hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{q}}^\dagger \left[\frac{1}{c^2}v_{\mathbf{k}}^*v_{\mathbf{q}}^* - \left(\frac{1}{\hbar^2}\mathbf{kq} - \mu_{\text{eff}}^2 \right) v_{\mathbf{k}}^*v_{\mathbf{q}}^* \right] \delta^3(\mathbf{k} + \mathbf{q}) \right\} \quad (1.25)
\end{aligned}$$

$$\begin{aligned}
\hat{\mathcal{H}} = \frac{c}{2} \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3} \left\{ \hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} \left[\frac{1}{c^2}v'_{\mathbf{k}}v'_{\mathbf{k}} + \frac{1}{\hbar^2}\omega_{\mathbf{k}}^2(t)v_{\mathbf{k}}v_{\mathbf{k}} \right] + \right. \\
+ \hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k}}^\dagger \left[\frac{1}{c^2}v'_{\mathbf{k}}v'_{\mathbf{k}}^* + \frac{1}{\hbar^2}\omega_{\mathbf{k}}^2(t)v_{\mathbf{k}}v_{\mathbf{k}}^* \right] + \\
+ \hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}} \left[\frac{1}{c^2}v_{\mathbf{k}}^*v'_{\mathbf{k}} + \frac{1}{\hbar^2}\omega_{\mathbf{k}}^2(t)v_{\mathbf{k}}^*v_{\mathbf{k}} \right] + \\
\left. + \hat{a}_{\mathbf{k}}^\dagger\hat{a}_{-\mathbf{k}}^\dagger \left[\frac{1}{c^2}v_{\mathbf{k}}^*v_{-\mathbf{k}}^* + \frac{1}{\hbar^2}\omega_{\mathbf{k}}^2(t)v_{\mathbf{k}}^*v_{-\mathbf{k}}^* \right] \right\} \quad (1.26)
\end{aligned}$$

$$\hat{\mathcal{H}} = \frac{c}{2} \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3} \left[\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}F_{\mathbf{k}} + \hat{a}_{\mathbf{k}}^\dagger\hat{a}_{-\mathbf{k}}^\dagger F_{\mathbf{k}}^* + \left(2\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}} + \frac{(2\pi\hbar)^3\hbar c}{2\text{Im}(v'v^*)}\delta^3(\mathbf{0}) \right) E_{\mathbf{k}} \right] \quad (1.27)$$

where

$$F_{\mathbf{k}}(t) = \left(\frac{1}{\hbar c} \right)^2 \left[\hbar^2 v_{\mathbf{k}}'^2 + \omega_{\mathbf{k}}^2(t) c^2 v_{\mathbf{k}}^2 \right] \quad (1.28)$$

$$E_{\mathbf{k}}(t) = \left(\frac{1}{\hbar c} \right)^2 \left[\hbar^2 |v_{\mathbf{k}}'|^2 + \omega_{\mathbf{k}}^2(t) c^2 |v_{\mathbf{k}}|^2 \right] \quad (1.29)$$

Now, the expectation value of the hamiltonian at time t_0 in the state $|(v)0\rangle$

$$\langle (v)0 | \hat{\mathcal{H}}(t_0) | (v)0 \rangle = \rho(t_0)\delta^3(\mathbf{0}) = \frac{\hbar c^2 \delta^3(\mathbf{0})}{4\text{Im}(v'v^*)} \int d^3\mathbf{k} E_{\mathbf{k}} \quad (1.30)$$

To minimise the energy density of de vacuum state is to fin the set of functions $v_{\mathbf{k}}$ that minimise $E_{\mathbf{k}}$. Suppose that $v_{\mathbf{k}}$ can be written as

$$v_{\mathbf{k}} = r_{\mathbf{k}} e^{i\alpha_{\mathbf{k}}} \quad (1.31)$$

since $\text{Im}(vv'^*)$ was constant through time

$$\text{Im}(v_{\mathbf{k}}v_{\mathbf{k}}'^*) = -r_{\mathbf{k}}^2 \alpha_{\mathbf{k}}' \quad (1.32)$$

this means

$$E_{\mathbf{k}} = \left(\frac{1}{\hbar c} \right)^2 \left\{ \hbar^2 \left[r_{\mathbf{k}}'^2 + \text{Im}^2(v_{\mathbf{k}}v_{\mathbf{k}}'^*) \frac{1}{r_{\mathbf{k}}^2} \right] + \omega_{\mathbf{k}}^2 c^2 r_{\mathbf{k}}^2 \right\} \quad (1.33)$$

the minimum of this function must fulfil $r_{\mathbf{k}}'(t_0) = 0$. Now, if $\omega_{\mathbf{k}}^2(t_0)$ and $\text{Im}(v_{\mathbf{k}}v_{\mathbf{k}}'^*)$ have the same sign, the minimum of $E_{\mathbf{k}}$ happens when $r_{\mathbf{k}}(t_0) = \left[\frac{\hbar \text{Im}(v_{\mathbf{k}}v_{\mathbf{k}}'^*)}{\omega_{\mathbf{k}}(t_0) c} \right]^{1/2}$.

If there is a minimum, then

$$v_{\mathbf{k}}(t_0) = \left[\frac{\hbar \text{Im}(v_{\mathbf{k}}v_{\mathbf{k}}'^*)}{\omega_{\mathbf{k}}(t_0) c} \right]^{1/2} e^{i\alpha_{\mathbf{k}}(t_0)} \quad v_{\mathbf{k}}'(t_0) = -c \frac{\omega_{\mathbf{k}}(t_0)}{i\hbar} v_{\mathbf{k}}(t_0) \quad (1.34)$$

1 First Chapter

under these functions,

$$E_{\mathbf{k}}(t_0) = 2 \frac{\text{Im}(v_{\mathbf{k}} v_{\mathbf{k}}'^*)}{\hbar c} \omega_{\mathbf{k}}(t_0) \quad F_{\mathbf{k}}(t_0) = 0 \quad (1.35)$$

meaning

$$\hat{\mathcal{H}}(t_0) = \text{Im}(v v'^*) \frac{1}{\hbar} \int \frac{d^3 \mathbf{k}}{(2\pi \hbar)^3} \left(2 \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \frac{(2\pi \hbar)^3 \hbar c}{2 \text{Im}(v' v^*)} \delta^3(\mathbf{0}) \right) \omega_{\mathbf{k}}(t_0) \quad (1.36)$$

which is equivalent to the standard Hamiltonian for a scalar field without the presence of gravity.

Notas sobre unidades

- $[s] = [\hbar]$
- $[a] = [\xi] = 1$
- $[\mu] = [L]^{-1}$
- $[R] = [L]^{-2}$
- $[\phi] = [\chi] = [\hbar]^{1/2}[L]^{-1}$
- $[\Pi] = [\hbar]^{1/2}[L]^{-2}$
- $[a_{\mathbf{k}}] = [\hbar]^{1/2}[L]^2$

Bibliography

- [1] Viatcheslav Mukhanov and Sergei Winitzki. *Introduction to Quantum Effects in Gravity*. Cambridge University Press, 2007.