PROBLEMS OF QUANTUM FIELD THEORIES IN CURVED SPACETIMES

A MASTER THESIS

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1 Introduction to QFT in curved spacetimes

$$\mathrm{d}l^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{1.1}$$

$$S = \int \left[\frac{1}{2\kappa}\left(R - 2\Lambda\right) + \mathcal{L}_{\mathrm{M}}\right]\sqrt{-g}\,\mathrm{d}^4x \tag{1.2}$$

$\kappa \equiv \frac{8\pi G}{c^4}$

Variation of S with respect to the inverse metric $(g^{\mu\nu})$ gives

$$\delta S = \int \left[\frac{\sqrt{-g}}{2\kappa}\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{2\kappa}\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} - \frac{\Lambda}{\kappa}\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta\mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}} + \frac{\mathcal{L}_{\mathrm{M}}}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\,\sqrt{-g}\,\mathrm{d}^4x \tag{1.3}$$

$\delta S = 0$ and

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu} \qquad \frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}g_{\mu\nu} \tag{1.4}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -2\frac{8\pi G}{c^4}\left(\frac{\delta\mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}} - \frac{1}{2}\mathcal{L}_{\mathrm{M}}g_{\mu\nu}\right) \tag{1.5}$$

$$T_{\mu\nu} \equiv \mathcal{L}_{\mathrm{M}}g_{\mu\nu} - \frac{\delta\mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}} = \frac{-2}{\sqrt{-g}}\frac{\delta\left(\mathcal{L}_{\mathrm{M}}\sqrt{-g}\right)}{\delta g^{\mu\nu}} \tag{1.6}$$

[1]

$$\nabla_\mu T^{\mu\nu} = 0 \tag{1.7}$$

(and its symmetric)

Now, in order to obtain the equations of motion for the matter fields, consider the lagrangian

$$\mathcal{L}_{\mathrm{M}} = \mathcal{L}_{\mathrm{M}}\big[\phi^\alpha(x),\,\nabla_\mu\phi^\alpha(x)\big] \tag{1.8}$$

variations of S with respect of ϕ^α result in

$$\delta S = \int \left[\frac{\partial\mathcal{L}_{\mathrm{M}}}{\partial\phi^\alpha}\delta\phi^\alpha + \frac{\partial\mathcal{L}_{\mathrm{M}}}{\partial\left(\nabla_\mu\phi^\alpha\right)}\nabla_\mu\left(\delta\phi^\alpha\right)\right]\sqrt{-g}\,\mathrm{d}^4x \tag{1.9}$$

and thus, after applying the generalized Gauss Theorem on a curved background, and considering that field variations vanish at the boundaries, one obtains

$$\frac{\partial\mathcal{L}_{\mathrm{M}}}{\partial\phi^\alpha} - \nabla_\mu\left[\frac{\partial\mathcal{L}_{\mathrm{M}}}{\partial\left(\nabla_\mu\phi^\alpha\right)}\right] = 0 \tag{1.10}$$

$$\Pi_\alpha \equiv \frac{\partial\mathcal{L}_{\mathrm{M}}}{\partial\left(\nabla_0\phi^\alpha\right)} \tag{1.11}$$

1.1 Scalar field

$$S[\phi] = \int \frac{1}{2}\Big[\nabla_\nu\phi\,\nabla^\nu\phi - \mu^2\phi^2 - \xi R\phi^2\Big]\sqrt{-g}\,\mathrm{d}^4x \tag{1.12}$$

equations of motion (Klein-Gordon)

$$\big[\nabla_\nu\nabla^\nu - \mu^2 - \xi R\big]\phi = 0 \tag{1.13}$$

$$T_{\mu\nu} = \nabla_\mu \phi \, \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \left[\nabla^\sigma \phi \nabla_\sigma \phi - \mu^2 \phi^2 \right] + \xi \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} + g_{\mu\nu} \nabla^\sigma \nabla_\sigma - \nabla_\mu \nabla_\nu \right] \phi^2 \quad (1.14)$$

note that for minimally coupled field ($\xi = 0$) the energy-momentum tensor is equivalent to the Noether energy-momentum tensor.

Scalar product

$$\langle \phi_1(x), \, \phi_2(x) \rangle \equiv i \int g^{0\nu} \left(\phi_1 \overset{\leftrightarrow}{\nabla}_\nu \phi_2^* \right) \sqrt{-g} \, \mathrm{d}^3 \mathbf{x} \quad (1.15)$$

Let $v(x)$ be a solution of the Klein-Gordon equation, then $v^*(x)$ will also be an (linearly independent) solution. Let i represent the set of parameters that univocally describe a par of solutions $v_i(x)$, $v^*(x)$, therefore, the general solution of the Klein-Gordon equation will be of the form

$$\phi(x) = \sum_i \left[a_i v_i(x) + a_i^* v_i^*(x) \right] \quad (1.16)$$

where a_i, a_i^* are constant factors that can be written as

$$a_i = \langle v_i(x), \, \phi(x) \rangle \qquad a_i^* = \langle v_i^*(x), \, \phi(x) \rangle \quad (1.17)$$

Quantization of the field is done by promoting the fields to operators

$$\phi(x) \longrightarrow \hat{\phi}(x) \qquad \Pi(x) \longrightarrow \hat{\Pi}(x) \quad (1.18)$$

this is done by promoting the constant factors to operators as well, that is

$$a_i \longrightarrow \hat{a}_i \qquad a_i^* \longrightarrow \hat{a}_i^\dagger \quad (1.19)$$

and therefore

$$\hat{\phi}(x) = \sum \left[\hat{a}_i v_i(x) + \hat{a}_i^\dagger v_i^*(x) \right] \quad (1.20)$$

after the promotion of the fields to operators, commutation relations are imposed; the easiest choice would be to assume canonical quantization relations,

$$\left[\hat{\phi}(\mathbf{x}), \, \hat{\Pi}(\mathbf{y}) \right] = i\hbar \, \delta^3 (\mathbf{x} - \mathbf{y}) \qquad \left[\hat{\phi}(\mathbf{x}), \, \hat{\phi}(\mathbf{y}) \right] = \left[\hat{\Pi}(\mathbf{x}), \, \hat{\Pi}(\mathbf{y}) \right] = 0 \quad (1.21)$$

It would be desirable to obtain a formulation similar to the well known scalar field in a flat background, where the Fock space is generated from a vacuum state and a set of creation and annihilation operators that follow some commutation rules. To do so, we will force the $\hat{a}_i$, $\hat{a}_i^\dagger$ operators to assume this roll, in such a way that

$$\left[\hat{a}_i, \, \hat{a}_j^\dagger \right] \propto \delta_{ij} \qquad [\hat{a}_i, \, \hat{a}_j] = \left[\hat{a}_i^\dagger, \, \hat{a}_j^\dagger \right] = 0 \quad (1.22)$$

Thanks to the relation between the constant factors a_i and the scalar product $\langle v_i, \phi \rangle$, one can obtain

$$\left[\hat{a}_i, \, \hat{a}_j^\dagger \right] = - \int \left[\left(v_i \hat{\Pi} - g^{0\nu} (\nabla_\nu v_i) \hat{\phi} \sqrt{-g} \right) \Big|_{\mathbf{x}}, \, \left(v_j^* \hat{\Pi} - g^{0\nu} (\nabla_\nu v_j) \hat{\phi} \sqrt{-g} \right) \Big|_{\mathbf{y}} \right] \mathrm{d}^3\mathbf{x} \mathrm{d}^3\mathbf{y} =$$

$$= i\hbar \int g^{0\nu} \left(v_i \overset{\leftrightarrow}{\nabla}_\nu v_j^* \right) \sqrt{-g} \, \mathrm{d}^3\mathbf{x} = \hbar \langle v_i, \, v_j \rangle \quad (1.23)$$

where the field commutators where used. Equivalently

$$[\hat{a}_i, \, \hat{a}_j] = -\hbar \langle v_i, \, v_j^* \rangle \qquad \left[\hat{a}_i^\dagger, \, \hat{a}_j^\dagger \right] = -\hbar \langle v_i^*, \, v_j \rangle \quad (1.24)$$

Therefore we must find a set of solutions $\{v_i(x), v_i^*(x)\}$ such that

$$\langle v_i^*, \, v_j \rangle \propto \delta_{ij} \qquad \langle v_i, \, v_j^* \rangle = \langle v_i^*, \, v_j \rangle = 0 \quad (1.25)$$

With this, we can define the Fock space the usual way, starting with a vacuum state $|0\rangle$ such that the action of the annihilation operation fulfils

$$\hat{a}_i \, |0\rangle = 0 \qquad \forall i \tag{1.26}$$

where single particle states are formed from the creation operator

$$|i\rangle \equiv \hat{a}_i^\dagger \, |0\rangle \tag{1.27}$$

and multiparticle states like

$$|i,\, j,\, \ldots\rangle = \ldots \hat{a}_j^\dagger \, \hat{a}_i^\dagger \, |0\rangle \tag{1.28}$$

Since this is a scalar field, one might assume that the states are symmetric (describing boson particles), and this is easily confirmed, since

$$|i,\, j\rangle = \hat{a}_j^\dagger \, \hat{a}_i^\dagger \, |0\rangle = \left[\hat{a}_i^\dagger,\, \hat{a}_j^\dagger \right] |0\rangle + \hat{a}_i^\dagger \, \hat{a}_j^\dagger |0\rangle = |j,\, i\rangle \tag{1.29}$$

2 Scalar field in an expanding universe

FLRW metric

$$\mathrm{d}l^2 = c^2 \mathrm{d}t^2 - a^2(t) \left[\frac{\mathrm{d}r^2}{1 - \kappa r^2} + r^2 \mathrm{d}\Omega^2 \right] \tag{2.1}$$

Weyl tensor $=0$ therefore the metric is conformally flat, i.e. independently of the curvature κ there must exist a coordinate system where

$$\mathrm{d}l^2 = a(t)\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a(t) \left[c^2 \mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2 \right] \tag{2.2}$$

the standard action describing the dynamics of a (non-minimally coupled to gravity) real scalar field is

$$s = \int \frac{1}{2} \left[\nabla_\nu \phi \, \nabla^\nu \phi - \mu^2 \phi^2 - \xi R \phi^2 \right] \sqrt{-g} \; \mathrm{d}^4 x \tag{2.3}$$

$\sqrt{-g} = a^4 \; \chi = a\phi$

$$s = \int \frac{1}{2} \left[\partial_\nu \chi \, \partial^\nu \chi - \left(\mu^2 a^2 + \xi R a^2 - c^2 \frac{a''}{a} \right) \chi^2 - \partial_t \left(c^2 \chi^2 \frac{a'}{a} \right) \right] \mathrm{d}^4 x \tag{2.4}$$

dropping the time derivative

$$s = \int \frac{1}{2} \left[\partial_\nu \chi \, \partial^\nu \chi - \left(\mu^2 a^2 + \xi R a^2 - c^2 \frac{a''}{a} \right) \chi^2 \right] \mathrm{d}^4 x \tag{2.5}$$

by Euler-Lagrange

$$\left[\partial_\nu \partial^\nu + \mu_{\mathrm{eff}}^2(t) \right] \chi = 0 \tag{2.6}$$

where

$$\mu_{\mathrm{eff}}^2(t) = \left(\mu^2 + \xi R \right) a^2 - c^2 \frac{a''}{a} \tag{2.7}$$

solutions of previous equation have the form

$$\chi = a \, v(t) e^{\pm i \mathbf{k} \mathbf{x} \hbar^{-1}} \tag{2.8}$$

meaning that, the dispersion relation is

$$v'' \hbar^2 + \omega^2(t) \, v = 0 \tag{2.9}$$

where $\omega(t)$ is defined as

$$\omega^2(t) = \mathbf{k}^2 + \hbar^2 \mu_{\text{eff}}^2(t) = \mathbf{k}^2 + \left(m^2 c^2 + \xi \hbar^2 R\right) a(t) - \hbar^2 c^2 \frac{a''}{a} \tag{2.10}$$

now, proof that $\text{Im}(vv'^*)$ is constant through time

$$\frac{\partial}{\partial t} \text{Im}(vv'^*) = \frac{\partial}{\partial t} \left(\frac{vv'^* - v^* v'}{2i} \right) = \frac{vv''^* - v^* v''}{2i} = 0 \tag{2.11}$$

last step is result from dispersion relation. Since dispersion relation is scalable by a time independent function, $\text{Im}(v'v^*)$ can be determined to be a chosen value, a particular useful choice is to consider it momentum independent. $\text{Im}(v'v^*) = W[v, v^*]$ therefore, if its not equal to 0, they are linearly independent solutions to dispersion relation.

The most general solution to the main equation is

$$\chi = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3} \left[a_{\mathbf{k}} v_{\mathbf{k}}(t) e^{i\mathbf{k}\mathbf{x}\hbar^{-1}} + a_{\mathbf{k}}^* v_{\mathbf{k}}^*(t) e^{-i\mathbf{k}\mathbf{x}\hbar^{-1}} \right] \tag{2.12}$$

The field χ and its conjugate momentum $\Pi = \partial_{ct}\chi$ are promoted to operators on the quantum Hilbert space, with the standar canonical conmutation relations

$$\left[\hat{\chi}(t, \mathbf{x}), \hat{\Pi}(t, \mathbf{y}) \right] = i\hbar \, \delta^3(\mathbf{x} - \mathbf{y}) \tag{2.13}$$

$$\left[\hat{\chi}(t, \mathbf{x}), \hat{\chi}(t, \mathbf{y}) \right] = \left[\hat{\Pi}(t, \mathbf{x}), \hat{\Pi}(t, \mathbf{y}) \right] = 0 \tag{2.14}$$

where the operational nature of the fields arrise from the promotion of the mode amplitudes, i.e.

$$a_{\mathbf{k}} \longrightarrow \hat{a}_{\mathbf{k}} \qquad a_{\mathbf{k}}^* \longrightarrow \hat{a}_{\mathbf{k}}^\dagger \tag{2.15}$$

this operators fulfill the following conmutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}^\dagger] = \frac{(2\pi\hbar)^3 \hbar c}{2\text{Im}(v'v^*)} \delta^3(\mathbf{k} - \mathbf{q}), \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{q}}^\dagger] = 0 \tag{2.16}$$

(note that $\hat{a}_{\mathbf{k}} \neq \hat{a}_{-\mathbf{k}}$)

To prove this, consider that

$$\left[\hat{\chi}(\mathbf{x}), \hat{\Pi}(\mathbf{y}) \right] = \frac{1}{c} \int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^6} \left\{ [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] v_{\mathbf{k}} v_{\mathbf{q}}' e^{i(\mathbf{k}\mathbf{x}+\mathbf{q}\mathbf{y})\hbar^{-1}} + \left[\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{q}}^\dagger \right] v_{\mathbf{k}}^* v_{\mathbf{q}}^{*\prime} e^{i(\mathbf{k}\mathbf{x}-\mathbf{q}\mathbf{y})\hbar^{-1}} + \right.$$
$$\left. + \left[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}^\dagger \right] v_{\mathbf{k}} v_{\mathbf{q}}^{*\prime} e^{i(\mathbf{k}\mathbf{x}-\mathbf{q}\mathbf{y})\hbar^{-1}} - \left[\hat{a}_{\mathbf{q}}, \hat{a}_{\mathbf{k}}^\dagger \right] v_{\mathbf{k}}^* v_{\mathbf{q}}' e^{-i(\mathbf{k}\mathbf{x}-\mathbf{q}\mathbf{y})\hbar^{-1}} \right\} \tag{2.17}$$

if the operators $\hat{a}$ and $\hat{a}^\dagger$ are to be understood as creation and annihilation operators, they must fulfill

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}^\dagger] = \alpha \delta^3(\mathbf{k} - \mathbf{q}), \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{q}}^\dagger] = 0 \tag{2.18}$$

where $\alpha \in \mathbb{C}$, and thus

$$\left[\hat{\chi}(\mathbf{x}), \hat{\Pi}(\mathbf{y}) \right] = \frac{\alpha}{c} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^6} 2i\text{Im}(v_{\mathbf{k}} v_{\mathbf{k}}^{*\prime}) e^{i(\mathbf{k}\mathbf{x}-\mathbf{q}\mathbf{y})\hbar^{-1}} \tag{2.19}$$

considering $\text{Im}(v'v^*)$ momentum independent, and remembering the canonical conmutation relations, one finds that

$$\alpha \text{Im}(vv^{*\prime}) = \frac{1}{2}\hbar c (2\pi\hbar)^3 \tag{2.20}$$

The hamiltonian

$$\hat{\mathcal{H}}(t) = \int \frac{c}{2} \left[\hat{\Pi}^2 + \left(\nabla \hat{\chi} \right)^2 + \mu_{\text{eff}}^2(t)\hat{\chi}^2 \right] \mathrm{d}^3\mathbf{x} \tag{2.21}$$

$$\hat{\Pi}^2 = \frac{1}{c^2} \int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^6} \left[\hat{a}_\mathbf{k}\hat{a}_\mathbf{q} v'_\mathbf{k} v'_\mathbf{q} e^{i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_\mathbf{k}\hat{a}_\mathbf{q}^\dagger v'_\mathbf{k} v'^{*}_\mathbf{q} e^{i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} + \right.$$
$$\left. + \hat{a}_\mathbf{k}^\dagger \hat{a}_\mathbf{q} v'^{*}_\mathbf{k} v'_\mathbf{q} e^{-i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_\mathbf{k}^\dagger \hat{a}_\mathbf{q}^\dagger v'^{*}_\mathbf{k} v'^{*}_\mathbf{q} e^{-i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}} \right] \tag{2.22}$$

$$\left(\nabla \hat{\chi} \right)^2 = -\frac{1}{\hbar^2} \int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^6} \mathbf{k}\mathbf{q} \left[\hat{a}_\mathbf{k}\hat{a}_\mathbf{q} v_\mathbf{k} v_\mathbf{q} e^{i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}} - \hat{a}_\mathbf{k}\hat{a}_\mathbf{q}^\dagger v_\mathbf{k} v^{*}_\mathbf{q} e^{i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} - \right.$$
$$\left. - \hat{a}_\mathbf{k}^\dagger \hat{a}_\mathbf{q} v^{*}_\mathbf{k} v_\mathbf{q} e^{-i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_\mathbf{k}^\dagger \hat{a}_\mathbf{q}^\dagger v^{*}_\mathbf{k} v^{*}_\mathbf{q} e^{-i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}} \right] \tag{2.23}$$

$$\hat{\chi}^2 = \int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^6} \left[\hat{a}_\mathbf{k}\hat{a}_\mathbf{q} v_\mathbf{k} v_\mathbf{q} e^{i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_\mathbf{k}\hat{a}_\mathbf{q}^\dagger v_\mathbf{k} v^{*}_\mathbf{q} e^{i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} + \right.$$
$$\left. + \hat{a}_\mathbf{k}^\dagger \hat{a}_\mathbf{q} v^{*}_\mathbf{k} v_\mathbf{q} e^{-i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_\mathbf{k}^\dagger \hat{a}_\mathbf{q}^\dagger v^{*}_\mathbf{k} v^{*}_\mathbf{q} e^{-i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}} \right] \tag{2.24}$$

$$\hat{\mathcal{H}} = \frac{c}{2} \int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^3} \left\{ \hat{a}_\mathbf{k}\hat{a}_\mathbf{q} \left[\frac{1}{c^2} v'_\mathbf{k} v'_\mathbf{q} - \left(\frac{1}{\hbar^2} \mathbf{k}\mathbf{q} - \mu_{\text{eff}}^2 \right) v_\mathbf{k} v_\mathbf{q} \right] \delta^3(\mathbf{k}+\mathbf{q}) + \right.$$
$$+ \hat{a}_\mathbf{k}\hat{a}_\mathbf{q}^\dagger \left[\frac{1}{c^2} v'_\mathbf{k} v'^{*}_\mathbf{q} + \left(\frac{1}{\hbar^2} \mathbf{k}\mathbf{q} + \mu_{\text{eff}}^2 \right) v_\mathbf{k} v^{*}_\mathbf{q} \right] \delta^3(\mathbf{k}-\mathbf{q}) +$$
$$+ \hat{a}_\mathbf{k}^\dagger \hat{a}_\mathbf{q} \left[\frac{1}{c^2} v'^{*}_\mathbf{k} v'_\mathbf{q} + \left(\frac{1}{\hbar^2} \mathbf{k}\mathbf{q} + \mu_{\text{eff}}^2 \right) v^{*}_\mathbf{k} v_\mathbf{q} \right] \delta^3(\mathbf{k}-\mathbf{q}) +$$
$$\left. + \hat{a}_\mathbf{k}^\dagger \hat{a}_\mathbf{q}^\dagger \left[\frac{1}{c^2} v'^{*}_\mathbf{k} v'^{*}_\mathbf{q} - \left(\frac{1}{\hbar^2} \mathbf{k}\mathbf{q} - \mu_{\text{eff}}^2 \right) v^{*}_\mathbf{k} v^{*}_\mathbf{q} \right] \delta^3(\mathbf{k}+\mathbf{q}) \right\} \tag{2.25}$$

$$\hat{\mathcal{H}} = \frac{c}{2} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3} \left\{ \hat{a}_\mathbf{k}\hat{a}_{-\mathbf{k}} \left[\frac{1}{c^2} v'_\mathbf{k} v'_\mathbf{k} + \frac{1}{\hbar^2} \omega_\mathbf{k}^2(t) v_\mathbf{k} v_\mathbf{k} \right] + \right.$$
$$+ \hat{a}_\mathbf{k}\hat{a}_\mathbf{k}^\dagger \left[\frac{1}{c^2} v'_\mathbf{k} v'^{*}_\mathbf{k} + \frac{1}{\hbar^2} \omega_\mathbf{k}^2(t) v_\mathbf{k} v^{*}_\mathbf{k} \right] +$$
$$+ \hat{a}_\mathbf{k}^\dagger \hat{a}_\mathbf{k} \left[\frac{1}{c^2} v'^{*}_\mathbf{k} v'_\mathbf{k} + \frac{1}{\hbar^2} \omega_\mathbf{k}^2(t) v^{*}_\mathbf{k} v_\mathbf{k} \right] +$$
$$\left. + \hat{a}_\mathbf{k}^\dagger \hat{a}_{-\mathbf{k}}^\dagger \left[\frac{1}{c^2} v'^{*}_\mathbf{k} v'^{*}_\mathbf{k} + \frac{1}{\hbar^2} \omega_\mathbf{k}^2(t) v^{*}_\mathbf{k} v^{*}_\mathbf{k} \right] \right\} \tag{2.26}$$

$$\hat{\mathcal{H}} = \frac{c}{2} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3} \left[\hat{a}_\mathbf{k}\hat{a}_{-\mathbf{k}} F_\mathbf{k} + \hat{a}_\mathbf{k}^\dagger \hat{a}_{-\mathbf{k}}^\dagger F_\mathbf{k}^* + \left(2\hat{a}_\mathbf{k}^\dagger \hat{a}_\mathbf{k} + \frac{(2\pi\hbar)^3 \hbar c}{2\mathrm{Im}(v'v^*)} \delta^3(\mathbf{0}) \right) E_\mathbf{k} \right] \tag{2.27}$$

where

$$F_\mathbf{k}(t) = \left(\frac{1}{\hbar c} \right)^2 \left[\hbar^2 v'^2_\mathbf{k} + \omega_\mathbf{k}^2(t) c^2 v_\mathbf{k}^2 \right] \tag{2.28}$$

$$E_\mathbf{k}(t) = \left(\frac{1}{\hbar c} \right)^2 \left[\hbar^2 |v'_\mathbf{k}|^2 + \omega_\mathbf{k}^2(t) c^2 |v_\mathbf{k}|^2 \right] \tag{2.29}$$

Now, the expectation value of the hamiltonian at time t_0 in the state $|_{(v)}0\rangle$

$$\langle_{(v)}0|\hat{\mathcal{H}}(t_0)|_{(v)}0\rangle = \rho(t_0)\delta^3(\mathbf{0}) = \frac{\hbar c^2 \, \delta^3(\mathbf{0})}{4\mathrm{Im}(v'v^*)} \int \mathrm{d}^3\mathbf{k} \, E_\mathbf{k} \tag{2.30}$$

To minimise the energy density of de vacuum state is to fin the set of functions $v_{\mathbf{k}}$ that minimise $E_{\mathbf{k}}$. Suppose that $v_{\mathbf{k}}$ can be written as

$$v_{\mathbf{k}} = r_{\mathbf{k}} e^{i\alpha_{\mathbf{k}}} \tag{2.31}$$

since $\text{Im}(vv^{'*})$ was constant through time

$$\text{Im}(v_{\mathbf{k}} v_{\mathbf{k}}^{'*}) = -r_{\mathbf{k}}^2 \alpha_{\mathbf{k}}' \tag{2.32}$$

this means

$$E_{\mathbf{k}} = \left(\frac{1}{\hbar c}\right)^2 \left\{ \hbar^2 \left[r_{\mathbf{k}}^{'2} + \text{Im}^2\left(v_{\mathbf{k}} v_{\mathbf{k}}^{'*}\right) \frac{1}{r_{\mathbf{k}}^2} \right] + \omega_{\mathbf{k}}^2 c^2 r_{\mathbf{k}}^2 \right\} \tag{2.33}$$

the minimum of this function must fulfil $r_{\mathbf{k}}'(t_0) = 0$. Now, if $\omega_{\mathbf{k}}^2(t_0)$ and $\text{Im}(v_{\mathbf{k}} v_{\mathbf{k}}^{'*})$ have the same sign, the minimum of $E_{\mathbf{k}}$ happens when $r_{\mathbf{k}}(t_0) = \left[\frac{\hbar \text{Im}(v_{\mathbf{k}} v_{\mathbf{k}}^{'*})}{\omega_{\mathbf{k}}(t_0) c} \right]^{1/2}$.

If there is a minimum, then

$$v_{\mathbf{k}}(t_0) = \left[\frac{\hbar \text{Im}(v_{\mathbf{k}} v_{\mathbf{k}}^{'*})}{\omega_{\mathbf{k}}(t_0) c} \right]^{1/2} e^{i\alpha_{\mathbf{k}}(t_0)} \qquad v_{\mathbf{k}}'(t_0) = -c \frac{\omega_{\mathbf{k}}(t_0)}{ih} v_{\mathbf{k}}(t_0) \tag{2.34}$$

under these functions,

$$E_{\mathbf{k}}(t_0) = 2 \frac{\text{Im}(v_{\mathbf{k}} v_{\mathbf{k}}^{'*})}{\hbar c} \omega_{\mathbf{k}}(t_0) \qquad F_{\mathbf{k}}(t_0) = 0 \tag{2.35}$$

meaning

$$\hat{\mathcal{H}}(t_0) = \text{Im}(vv^{'*}) \frac{1}{\hbar} \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3} \left(2\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \frac{(2\pi\hbar)^3 \hbar c}{2\text{Im}(v'v^*)} \delta^3(\mathbf{0}) \right) \omega_{\mathbf{k}}(t_0) \tag{2.36}$$

which is equivalent to the standard Hamiltonian for a scalar field without the presence of gravity.

Bogolyubov Transformation

$$u_{\mathbf{k}}(t) = \alpha_{\mathbf{k}} v_{\mathbf{k}}(t) + \beta_{\mathbf{k}} v_{\mathbf{k}}^*(t) \tag{2.37}$$

$\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \mathbb{C}$ (time independent)

$$\text{Im}(u_{\mathbf{k}}' u_{\mathbf{k}}^*) = \text{Im}(v_{\mathbf{k}}' v_{\mathbf{k}}^*)\left(|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 \right) \tag{2.38}$$

Changing the v functions would entail a change in the creation and annihilation, therefore if we could write the field as

$$\hat{\chi} = \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3} \left[\hat{b}_{\mathbf{k}} u_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}\hbar^{-1}} + \hat{b}_{\mathbf{k}}^\dagger u_{\mathbf{k}}^* e^{-i\mathbf{k}\mathbf{x}\hbar^{-1}} \right] \tag{2.39}$$

the field must be tha same as if it was written with de v functions and $\hat{a}$ operators, that means that

$$\hat{b}_{\mathbf{k}} u_{\mathbf{k}} + \hat{b}_{-\mathbf{k}}^\dagger u_{\mathbf{k}}^* = \hat{a}_{\mathbf{k}} v_{\mathbf{k}} + \hat{a}_{-\mathbf{k}}^\dagger v_{\mathbf{k}}^* \tag{2.40}$$

and thus, the relation between the operators would be

$$\hat{a}_{\mathbf{k}} = \alpha_{\mathbf{k}} \hat{b}_{\mathbf{k}} + \beta_{\mathbf{k}}^* \hat{b}_{-\mathbf{k}}^\dagger \qquad \hat{a}_{\mathbf{k}}^\dagger = \beta_{\mathbf{k}} \hat{b}_{-\mathbf{k}} + \alpha_{\mathbf{k}}^* \hat{b}_{\mathbf{k}}^\dagger \tag{2.41}$$

now, there are 'a' particles in the 'b' vacuum

$$\langle_{(b)}0|\hat{\mathcal{N}}_{\mathbf{k}}^{(a)}|_{(b)}0\rangle = \langle_{(b)}0|\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}|_{(b)}0\rangle = \left|\beta_{\mathbf{k}}\right|^2 \frac{(2\pi\hbar)^3 \hbar c}{2\text{Im}(u'u^*)} \delta^3(\mathbf{0}) \tag{2.42}$$

therefore

$$\langle_{(t_0)}0|\hat{\mathcal{H}}(t)|_{(t_0)}0\rangle = \delta^3(\mathbf{0}) \int \mathrm{d}^3\mathbf{k} \left(\frac{|\beta_\mathbf{k}|^2}{|\alpha_\mathbf{k}|^2 - |\beta_\mathbf{k}|^2} + \frac{1}{2} \right) c\,\omega_\mathbf{k}(t) \tag{2.43}$$

meaning, if $\beta_\mathbf{k} \neq 0$ for all $\mathbf{k}$ then, at a time $t > t_0$ the energy density will be different in relation to the original vacuum.

3 de Sitter Universe

The de Sitter Universe is a flat FLRW metric with no matter or radiation, but it does have a a positive cosmological constant Λ. Per the Friedmann equations,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G + \Lambda c^2}{3} - \frac{\kappa c^2}{a^2} \tag{3.1}$$

the expansion parameter $a(t)$ will be equal to

$$a(t) = a_1 e^{H_\Lambda t} + a_2 e^{-H_\Lambda t} \,, \qquad H_\Lambda = \sqrt{\frac{\Lambda c^2}{3}} \tag{3.2}$$

Scalar field in Minkowski background

$$\eta_{\mu\nu} \tag{1}$$

Units

- $[s] = [\hbar]$
- $[a] = [\xi] = 1$
- $[\mu] = [L]^{-1}$
- $[R] = [L]^{-2}$
- $[\phi] = [\chi] = [\hbar]^{1/2}[L]^{-1}$
- $[\Pi] = [\hbar]^{1/2}[L]^{-2}$
- $[a_{\mathbf{k}}] = [\hbar]^{1/2}[L]^2$

Questions

- How do you know that there is a set of solutions of Klein Gordon such that the inner product fulfils the given results?

- It's the Hamiltonian well defined?

- Can you always write a FLRW metric as a flat one with a coordinate change?

Bibliography

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