PROBLEMS OF QUANTUM FIELD THEORIES IN CURVED SPACETIMES

A MASTER THESIS

Submitted in partial fulfillment of the requirements for the award of

Master's Degree in Physics of the Universe: Cosmology, Astrophysics, Particles and Astroparticles

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October 15, 2023

Abrstract

Quantum Field Theory is the fundamental theoretical structure of the Standard Model of elementary particles. This theory is formulated in a Minkowski space-time. However, the actual space-time metric is never of that type. On Earth, even at short distances, the metric is affected by both the force of the Earth's gravity and the solar force, but especially by the acceleration of the Earth's motion. At large distances the cosmological data lead one to think that, on average, the metric is of the Friedmann-Lemaitre-Roberson-Walker type.

The analysis of the quantization of fields in the presence of gravitational fields involves a number of theoretical connotations that we intend to explore in this paper. How Quantum Field Theory can be adapted when considering this gravitational background is the object of the proposal. In particular, how the structure of the quantum vacuum is affected when space-time is neither asymptotically Minkowskian, as is the case in the current Cosmological Model LCDM. There are a number of attempts to address the problem but to date none provides a satisfactory solution; the aim of the present work is to find the one that best fits the experimental observations.

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Conventions

The chosen convention for the metric signature will be $(+, -, -, -)$ as in [2] and most literature on particle physics. Common conventions and nomenclatures in mathematics and physics are used throughout the text, some of which are considered to be relevant and listed bellow alongside the first page of appearance:

x^μ, x	four-vector	??
x^i, $\mathbf{x}$	spacial vector	??
$g_{\mu\nu}$	general spacetime metric	??
$\eta_{\mu\nu}$	Minkowski spacetime metric	??
g	determinant of $g_{\mu\nu}$	??
∇_μ	covariant derivative	??
$S[\phi]$	action functional of a field ϕ and it's derivatives	??
$\bar{z}$	complex conjugate of z	??
$A^\dagger$	hermitian conjugate of A	??
$R^\alpha_{\ \beta\gamma\delta}$	Riemann tensor $\quad \equiv \nabla_\delta \Gamma^\alpha_{\ \beta\gamma} - \nabla_\delta \Gamma^\gamma_{\ \beta\alpha} + \ldots$	??
$f \overleftrightarrow{\nabla}_\mu g$	$\equiv f \nabla_\nu g - (\nabla_\mu f)\, g$	??
γ^μ	covariant Gamma matrices $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$	??
G, c, $\hbar$	standard universal constants, not necessarily in natural units	??

Other notation will be introduced as needed.

Preface

Preface

1 Introduction to QFT in curved spacetimes

$$\mathrm{d}l^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{1.1}$$

$$S = \int \left[\frac{1}{2\kappa}\left(R - 2\Lambda\right) + \mathcal{L}_\mathrm{M}\right]\sqrt{-g}\,\mathrm{d}^4x \tag{1.2}$$

$\kappa \equiv \frac{8\pi G}{c^4}$

Variation of S with respect to the inverse metric ($g^{\mu\nu}$) gives

$$\delta S = \int \left[\frac{\sqrt{-g}}{2\kappa}\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{2\kappa}\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} - \frac{\Lambda}{\kappa}\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta\mathcal{L}_\mathrm{M}}{\delta g^{\mu\nu}} + \frac{\mathcal{L}_\mathrm{M}}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x \tag{1.3}$$

$\delta S = 0$ and

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu} \qquad \frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}g_{\mu\nu} \tag{1.4}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -2\frac{8\pi G}{c^4}\left(\frac{\delta\mathcal{L}_\mathrm{M}}{\delta g^{\mu\nu}} - \frac{1}{2}\mathcal{L}_\mathrm{M}g_{\mu\nu}\right) \tag{1.5}$$

$$T_{\mu\nu} \equiv \mathcal{L}_\mathrm{M}g_{\mu\nu} - \frac{\delta\mathcal{L}_\mathrm{M}}{\delta g^{\mu\nu}} = \frac{-2}{\sqrt{-g}}\frac{\delta\left(\mathcal{L}_\mathrm{M}\sqrt{-g}\right)}{\delta g^{\mu\nu}} \tag{1.6}$$

[1]

$$\nabla_\mu T^{\mu\nu} = 0 \tag{1.7}$$

(and its symmetric)

1.1 Construction of covariant actions

The equivalence principle says that field equations must be invariant with respect to local Lorentz transformations $\Lambda(x)$. To work on a local flat spacetime, we use the tetrad formalism

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \tag{1.8}$$

partial derivatives transform like

$$\partial_\mu \to \frac{\partial x^\nu}{\partial y^\mu}\partial_\nu \tag{1.9}$$

but, we see that $e^\mu_a\partial_\mu$ is not covariant, since

$$e^\mu_a\partial^a\phi(x) \to \Lambda^b_a e^\mu_b\partial_\mu\left[\rho\left(\Lambda\right)\phi(x)\right] = \Lambda^b_a e^\mu_b\left[\rho\left(\Lambda\right)\partial_\mu\phi + \partial_\mu\rho\left(\Lambda\right)\,\phi\right] \tag{1.10}$$

a covariant derivative D_μ should transform as

$$D_a\phi \to \Lambda^b_a\rho\left(\Lambda\right)D_b\phi \tag{1.11}$$

therefore we need to define a better derivative, a common option is

$$D_a \equiv e^\mu_a\left(\partial_\mu + \Gamma_\mu\right) \tag{1.12}$$

where, for the derivative to be covariant, the connection Γ_μ must transform as

$$\Gamma_\mu \to \rho\left(\Lambda\right)\Gamma_\mu\rho^{-1}\left(\Gamma\right) - \left[\partial_\mu\rho\left(\Lambda\right)\right]\rho^{-1}\left(\Lambda\right) \tag{1.13}$$

such connection can be written as

$$\Gamma_\mu = \frac{1}{2}\Sigma^{ab} e_a^\nu \nabla_\mu e_{b\nu} \tag{1.14}$$

where Σ^{ab} are the Lorentz generators and $e_a^\nu \nabla_\mu e_{b\nu} \equiv \omega_{ab\mu}$ is the torsion free spin connection.

Under the presence of a gauge field, we would also want the derivative to be gauge invariant, and thus, the gauge four-potential A must be considered in the definition of the derivative

$$D_a \equiv e_a^\mu \left(\partial_\mu + \Gamma_\mu - \frac{i}{\hbar} e A_\mu \right) \tag{1.15}$$

where e is the coupling constant.

This derivative of course will

Now, in order to obtain the equations of motion for the matter fields, consider the lagrangian

$$\mathcal{L}_\mathrm{M} = \mathcal{L}_\mathrm{M}\big[\phi^\alpha(x),\, D_\mu\phi^\alpha(x)\big] \tag{1.16}$$

variations of S with respect of ϕ^α result in

$$\delta S = \int \left[\frac{\partial \mathcal{L}_\mathrm{M}}{\partial \phi^\alpha} \delta\phi^\alpha + \frac{\partial \mathcal{L}_\mathrm{M}}{\partial \left(D_\mu \phi^\alpha\right)} D_a \left(\delta\phi^\alpha\right) \right] \sqrt{-g}\, \mathrm{d}^4x \tag{1.17}$$

and thus, after applying the generalized Gauss Theorem on a curved background, and considering that field variations vanish at the boundaries, one obtains

$$\frac{\partial \mathcal{L}_\mathrm{M}}{\partial \phi^\alpha} - D_\mu \left[\frac{\partial \mathcal{L}_\mathrm{M}}{\partial \left(D_\mu \phi^\alpha\right)} \right] = 0 \tag{1.18}$$

$$\Pi_\alpha \equiv \frac{\partial \mathcal{L}_\mathrm{M}}{\partial \left(D_0 \phi^\alpha\right)} \tag{1.19}$$

1.2 Scalar field

$$S\big[\phi\big] = \int \frac{1}{2}\Big[\nabla_\nu\phi\, \nabla^\nu\phi - \mu^2\phi^2 - \xi R \phi^2 \Big] \sqrt{-g}\, \mathrm{d}^4x \tag{1.20}$$

where $\mu \equiv {}^{mc}/_{\hbar}$, R is the curvature scalar of the manifold. For $\xi = 0$ the scalar field is said to be minimally coupled to gravity (nonminimally coupled otherwise); this is the simplest interactive term that couples the scalar field with the manifold; in curved spacetimes [3] a self interactive $\lambda\phi^4$ theory needs a term proportional to $R\phi^2$ to be renormalizable. For $\xi = {}^1/_6$ its said to be conformally coupled, and, in the case of a massless field ($\mu = 0$) the action will be invariant under conformal transformations, i.e.

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} \equiv \Omega^2(x) g_{\mu\nu}, \tag{1.21}$$

This can be seen from the equations of motion (Klein-Gordon)

$$\big[\nabla_\nu \nabla^\nu - \mu^2 - \xi R\big]\phi = 0 \tag{1.22}$$

considering $\phi \to \tilde{\phi} = \Omega^\beta \phi$ one gets that

$$\begin{aligned} 0 = \mu^2 \Omega^{\beta-2}\left(\Omega^2 - 1\right)\phi + 2\left(1+\beta\right)\Omega^{\beta-3} \nabla^\nu\Omega\nabla_\nu\phi + \\ + (6\xi + \beta)\Omega^{\beta-3}\left(\nabla_\nu\nabla^\nu\Omega\right)\phi + \beta(1+\beta)\Omega^{\beta-4}\nabla_\nu\Omega\nabla^\nu\Omega\,\phi \end{aligned} \tag{1.23}$$

this equation has a solution for massless fields, corresponding to

$$\beta = -1, \qquad \xi = \frac{1}{6} \tag{1.24 a,b}$$

The energy momentum tensor has the following form

$$T_{\mu\nu} = \nabla_\mu\phi\,\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left[\nabla^\sigma\phi\nabla_\sigma\phi - \mu^2\phi^2\right] + \xi\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} + g_{\mu\nu}\nabla^\sigma\nabla_\sigma - \nabla_\mu\nabla_\nu\right]\phi^2 \quad (1.25)$$

note that for minimally coupled field ($\xi = 0$) the energy-momentum tensor is equivalent to the Noether energy-momentum tensor.

Scalar product

$$\langle\phi_1(x),\,\phi_2(x)\rangle \equiv i\int g^{0\nu}\left(\phi_1\overleftrightarrow{\nabla}_\nu\phi_2^*\right)\sqrt{-g}\,\mathrm{d}^3\mathbf{x} \quad (1.26)$$

Let $v(x)$ be a solution of the Klein-Gordon equation, then $v^*(x)$ will also be an (linearly independent) solution. Let i represent the set of parameters that univocally describe a par of solutions $v_i(x)$, $v^*(x)$, therefore, the general solution of the Klein-Gordon equation will be of the form

$$\phi(x) = \sum_i\left[a_i v_i(x) + a_i^* v_i^*(x)\right] \quad (1.27)$$

where a_i, a_i^* are constant factors that can be written as

$$a_i = \langle v_i(x),\,\phi(x)\rangle \qquad a_i^* = \langle v_i^*(x),\,\phi(x)\rangle \quad (1.28)$$

Quantization of the field is done by promoting the fields to operators

$$\phi(x) \longrightarrow \hat{\phi}(x) \qquad \Pi(x) \longrightarrow \hat{\Pi}(x) \quad (1.29)$$

this is done by promoting the constant factors to operators as well, that is

$$a_i \longrightarrow \hat{a}_i \qquad a_i^* \longrightarrow \hat{a}_i^\dagger \quad (1.30)$$

and therefore

$$\hat{\phi}(x) = \sum_i\left[\hat{a}_i v_i(x) + \hat{a}_i^\dagger v_i^*(x)\right] \quad (1.31)$$

after the promotion of the fields to operators, commutation relations are imposed; the easiest choice would be to assume canonical quantization relations,

$$\left[\hat{\phi}(\mathbf{x}),\,\hat{\Pi}(\mathbf{y})\right] = i\hbar\,\delta^3\left(\mathbf{x}-\mathbf{y}\right) \qquad \left[\hat{\phi}(\mathbf{x}),\,\hat{\phi}(\mathbf{y})\right] = \left[\hat{\Pi}(\mathbf{x}),\,\hat{\Pi}(\mathbf{y})\right] = 0 \quad (1.32)$$

It would be desirable to obtain a formulation similar to the well known scalar field in a flat background, where the Fock space is generated from a vacuum state and a set of creation and annihilation operators that follow some commutation rules. To do so, we will force the $\hat{a}_i$, $\hat{a}_i^\dagger$ operators to assume this roll, in such a way that

$$\left[\hat{a}_i,\,\hat{a}_j^\dagger\right] \propto \delta_{ij} \qquad \left[\hat{a}_i,\,\hat{a}_j\right] = \left[\hat{a}_i^\dagger,\,\hat{a}_j^\dagger\right] = 0 \quad (1.33)$$

Thanks to the relation between the constant factors a_i and the scalar product $\langle v_i,\,\phi\rangle$, one can obtain

$$\begin{aligned}\left[\hat{a}_i,\,\hat{a}_j^\dagger\right] &= -\int\left[\left(v_i\hat{\Pi} - g^{0\nu}\left(\nabla_\nu v_i\right)\hat{\phi}\sqrt{-g}\right)\Big|_{\mathbf{x}},\,\left(v_j^*\hat{\Pi} - g^{0\nu}\left(\nabla_\nu v_j\right)\hat{\phi}\sqrt{-g}\right)\Big|_{\mathbf{y}}\right]\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{y} = \\ &= i\hbar\int g^{0\nu}\left(v_i\overleftrightarrow{\nabla}_\nu v_j^*\right)\sqrt{-g}\,\mathrm{d}^3\mathbf{x} = \hbar\langle v_i,\,v_j\rangle \quad (1.34)\end{aligned}$$

where the field commutators where used. Equivalently

$$\left[\hat{a}_i,\,\hat{a}_j\right] = -\hbar\,\langle v_i,\,v_j^*\rangle \qquad \left[\hat{a}_i^\dagger,\,\hat{a}_j^\dagger\right] = -\hbar\,\langle v_i^*,\,v_j\rangle \quad (1.35)$$

Therefore we must find a set of solutions $\{v_i(x),\, v_i^*(x)\}$ such that

$$\langle v_i,\, v_j\rangle \propto \delta_{ij} \qquad \langle v_i,\, v_j^*\rangle = \langle v_i^*,\, v_j\rangle = 0 \tag{1.36}$$

With this, we can define the Fock space the usual way, starting with a vacuum state $|0\rangle$ such that the action of the annihilation operation fulfils

$$\hat{a}_i\,|0\rangle = 0 \qquad \forall i \tag{1.37}$$

where single particle states are formed from the creation operator

$$|i\rangle \equiv \hat{a}_i^\dagger\,|0\rangle \tag{1.38}$$

and multiparticle states like

$$|i,\, j,\, \ldots\rangle = \ldots\hat{a}_j^\dagger\,\hat{a}_i^\dagger\,|0\rangle \tag{1.39}$$

Since this is a scalar field, one might assume that the states are symmetric (describing boson particles), and this is easily confirmed, since

$$|i,\, j\rangle = \hat{a}_j^\dagger\,\hat{a}_i^\dagger\,|0\rangle = \left[\hat{a}_i^\dagger,\, \hat{a}_j^\dagger\right]|0\rangle + \hat{a}_i^\dagger\,\hat{a}_j^\dagger|0\rangle = |j,\, i\rangle \tag{1.40}$$

1.3 Dirac Field

1.4 Electromagnetic field

1.5 Bogoliubov transformations

Consider now a second set $\{u_i(x),\, u_i^*(x)\}$ of solutions to the Klein-Gordon equation such that they meet the inner product rule; the field would then be written as

$$\phi(x) = \sum_j \left[b_j u_j(x) + b_j^* u_j^*(x)\right] \tag{1.41}$$

quantization of the field and creation and annihilation is straightforward. The relation between the v and u solutions would be

$$v_i(x) \equiv \sum_j \left[\alpha_{ij} u_j(x) + \beta_{ij} u_j^*(x)\right] \tag{1.42}$$

where α_{ij} and β_{ij} are known as Bogoliubov coefficients, that can be obtained as

$$\alpha_{ij} \propto \langle v_i,\, u_j\rangle \qquad \beta_{ij} \propto -\langle v_i,\, u_j^*\rangle \tag{1.43}$$

Since the field is the same independently of the mode set chosen

$$\sum_i \left[\hat{a}_i v_i(x) + \hat{a}_i^\dagger v_i^*(x)\right] = \sum_j \left[\hat{b}_j u_j(x) + \hat{b}_j^\dagger u_j^*(x)\right] \tag{1.44}$$

and, as a result of the orthogonality of the mode functions

$$\hat{a}_i = \sum_j \left(\alpha_{ij}^*\hat{b}_j - \beta_{ij}^*\hat{b}_j^\dagger\right) \qquad \hat{a}_i^\dagger = \sum_j \left(-\beta_{ij}\hat{b}_j + \alpha_{ij}\hat{b}_j^\dagger\right) \tag{1.45}$$

creation and annihilation commutation relations give new restrictions to the Bogoliubov coefficients

$$\left[\hat{a}_i,\, \hat{a}_j^\dagger\right] \propto \delta_{ij} \implies \sum_k \left(\alpha_{ik}^*\alpha_{jk} - \beta_{ik}^*\beta_{jk}\right) \propto \delta_{ij} \tag{1.46}$$

$$[\hat{a}_i,\, \hat{a}_j] = 0 \implies \sum_k \left(\alpha^*_{jk}\beta^*_{ik} - \alpha^*_{ik}\beta^*_{jk}\right) = 0 \tag{1.47}$$

Now, the relevance of the Bogoliubov transformations comes from the fact that the vacuum in the u solutions, have (in general) v particles,

$$\langle_u 0|\hat{N}_v|_u 0\rangle = \sum_i \langle_u 0|\hat{a}^\dagger_i \hat{a}_i|_u 0\rangle = \sum_i \left[\sum_{jk} \beta_{ij}\beta^*_{ik}\langle_u 0|\hat{b}_j\hat{b}^\dagger_k|_u 0\rangle\right] \propto \sum_{ij}|\beta_{ij}|^2 \tag{1.48}$$

therefore, there is not a unique vacuum.

1.6 A leap towards a continuum

Until now, it has been considered that the set of Klein-Gordon solutions could be categorised by a discrete set of parameters i, from a standard course in QFT, one of the main results is the fact that the solutions of the flat Klein-Gordon equations can be parametrised by a continuous 3-dimensional vector $\mathbf{k}$ (which is interpreted to be the momentum of the particle). Since all computations in this section where made by considering a discrete set of parameters, it is relevant to consider the continuum case.

A common computation in many fields of physics is the determination of the density of states $D(\mathbf{k})$ describing the number of modes with momentum between $\mathbf{k}$ and $\mathbf{k}+\mathrm{d}\mathbf{k}$. Consider a system with volume V, where the field goes to zero at its boundary; in this case, the permitted values of momenta must meet

$$k^i = n^i \frac{\pi\hbar}{V^{1/3}}\,, \qquad n^i \in \mathbb{Z} \tag{1.49}$$

Let $N(k)$ be the number of states with momentum modulus less than k, that is, the states such that

$$n = \sqrt{(n^1)^2 + (n^2)^2 + (n^3)^2} < k\frac{V^{1/3}}{\pi\hbar} \tag{1.50}$$

considering a flat momentum space[1] and a large enough volume, $N(k)$ will be essentially equal to an eight of the volume of a sphere with radius $kV^{1/3}/\pi\hbar$, that is

$$N(k) \approx \frac{1}{8}\frac{4}{3}\pi\left(k\frac{V^{1/3}}{\pi\hbar}\right)^3 = \frac{V}{6\pi^2\hbar^3}k^3 \tag{1.51}$$

meaning, that the density of states will be

$$D(\mathbf{k}) \equiv D(k) = \frac{\mathrm{d}N(k)}{\mathrm{d}k} \approx \frac{V}{2\pi^2\hbar^3}k^2 \tag{1.52}$$

With this, one could approximate a discrete sum over a parameter i to an integral over a continuum $\mathbf{k}$

$$\sum_i f_i = \int_0^\infty D(k) f_k \mathrm{d}k \approx \int_0^\infty \frac{V}{2\pi^2\hbar^3} f_k k^2 \mathrm{d}k \equiv \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3} f_{\mathbf{k}} \tag{1.53}$$

where it has been defined.

$$4\pi V f_k k^2 \equiv \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} f_{\mathbf{k}} \sin\varphi \mathrm{d}\theta \mathrm{d}\varphi \tag{1.54}$$

therefore $\mathrm{d}^3\mathbf{k}/(2\pi\hbar)^3$ is to be understood as the volume element of the momentum space.

[1]In contrast to modified theories of relativity in which this is not the case, like the κ-Poincaré relativity.

2 Scalar field in an expanding universe

FLRW metric

$$\mathrm{d}l^2 = c^2\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\mathrm{d}\Omega^2\right] \tag{2.1}$$

Weyl tensor =0 therefore the metric is conformally flat, i.e. independently of the curvature κ there must exist a coordinate system where

$$\mathrm{d}l^2 = a(t)\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = a(t)\left[c^2\mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2\right] \tag{2.2}$$

the standard action describing the dynamics of a (non-minimally coupled to gravity) real scalar field is

$$s = \int \frac{1}{2}\left[\nabla_\nu\phi\,\nabla^\nu\phi - \mu^2\phi^2 - \xi R\phi^2\right]\sqrt{-g}\,\mathrm{d}^4x \tag{2.3}$$

$\sqrt{-g} = a^4$ $\chi = a\phi$

$$s = \int \frac{1}{2}\left[\partial_\nu\chi\,\partial^\nu\chi - \left(\mu^2a^2 + \xi Ra^2 - c^2\frac{a''}{a}\right)\chi^2 - \partial_t\left(c^2\chi^2\frac{a'}{a}\right)\right]\mathrm{d}^4x \tag{2.4}$$

dropping the time derivative

$$s = \int \frac{1}{2}\left[\partial_\nu\chi\,\partial^\nu\chi - \left(\mu^2a^2 + \xi Ra^2 - c^2\frac{a''}{a}\right)\chi^2\right]\mathrm{d}^4x \tag{2.5}$$

by Euler-Lagrange

$$\left[\partial_\nu\partial^\nu + \mu^2_{\text{eff}}(t)\right]\chi = 0 \tag{2.6}$$

where

$$\mu^2_{\text{eff}}(t) = \left(\mu^2 + \xi R\right)a^2 - \frac{a''}{ac^2} \tag{2.7}$$

solutions of previous equation have the form

$$\chi = a\,v(t)e^{\pm i\mathbf{kx}\hbar^{-1}} \tag{2.8}$$

meaning that, the dispersion relation is

$$v''\hbar^2 + \omega^2(t)\,v = 0 \tag{2.9}$$

where $\omega(t)$ is defined as

$$\omega^2_{\mathbf{k}}(t) = \mathbf{k}^2 + \hbar^2\mu^2_{\text{eff}}(t) = \mathbf{k}^2 + \left(m^2c^2 + \xi\hbar^2R\right)a^2(t) - \hbar^2\frac{a''}{ac^2} \tag{2.10}$$

now, proof that $\mathrm{Im}(vv'^*)$ is constant through time

$$\frac{\partial}{\partial t}\mathrm{Im}(vv'^*) = \frac{\partial}{\partial t}\left(\frac{vv'^* - v^*v'}{2i}\right) = \frac{vv''^* - v^*v''}{2i} = 0 \tag{2.11}$$

last step is result from dispersion relation. Since dispersion relation is scalable by a time independent function, $\mathrm{Im}(v'v^*)$ can be determined to be a chosen value, a particular useful choice is to consider it momentum independent. $\mathrm{Im}(v'v^*) = W[v, v^*]$ therefore, if its not equal to 0, they are linearly independent solutions to dispersion relation.

The most general solution to the main equation is

$$\chi = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\left[a_{\mathbf{k}}v_{\mathbf{k}}(t)e^{i\mathbf{kx}\hbar^{-1}} + a^*_{\mathbf{k}}v^*_{\mathbf{k}}(t)e^{-i\mathbf{kx}\hbar^{-1}}\right] \tag{2.12}$$

The field χ and its conjugate momentum $\Pi = \partial_{ct}\chi$ are promoted to operators on the quantum Hilbert space, with the standar canonical conmutation relations

$$\left[\hat{\chi}(t,\mathbf{x}), \hat{\Pi}(t,\mathbf{y})\right] = i\hbar\,\delta^3(\mathbf{x}-\mathbf{y}) \tag{2.13}$$

$$\left[\hat{\chi}(t,\mathbf{x}), \hat{\chi}(t,\mathbf{y})\right] = \left[\hat{\Pi}(t,\mathbf{x}), \hat{\Pi}(t,\mathbf{y})\right] = 0 \tag{2.14}$$

where the operational nature of the fields arrise from the promotion of the mode amplitudes, i.e.

$$a_{\mathbf{k}} \longrightarrow \hat{a}_{\mathbf{k}} \qquad a^*_{\mathbf{k}} \longrightarrow \hat{a}^\dagger_{\mathbf{k}} \tag{2.15}$$

this operators fulfill the following conmutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v'v^*)}\delta^3(\mathbf{k}-\mathbf{q})\,, \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = 0 \tag{2.16}$$

(note that $\hat{a}_{\mathbf{k}} \neq \hat{a}_{-\mathbf{k}}$)

To prove this, consider that

$$\begin{aligned}\left[\hat{\chi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = \frac{1}{c}\int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^6}\Big\{&[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}]\, v_{\mathbf{k}}v'_{\mathbf{q}}e^{i(\mathbf{kx}+\mathbf{qy})\hbar^{-1}} + \left[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}\right] v^*_{\mathbf{k}}v^{*\prime}_{\mathbf{q}}e^{i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}} + \\ &+ \left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}\right] v_{\mathbf{k}}v^{*\prime}_{\mathbf{q}}e^{i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}} - \left[\hat{a}_{\mathbf{q}}, \hat{a}^\dagger_{\mathbf{k}}\right] v^*_{\mathbf{k}}v'_{\mathbf{q}}e^{-i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}}\Big\}\end{aligned} \tag{2.17}$$

if the operators $\hat{a}$ and $\hat{a}^\dagger$ are to be understood as creation and annihilation operators, they must fulfill

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = \alpha\delta^3(\mathbf{k}-\mathbf{q})\,, \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = 0 \tag{2.18}$$

where $\alpha \in \mathbb{C}$, and thus

$$\left[\hat{\chi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = \frac{\alpha}{c}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^6}2i\mathrm{Im}(v_{\mathbf{k}}v^{*\prime}_{\mathbf{k}})e^{i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}} \tag{2.19}$$

considering $\mathrm{Im}(v'v^*)$ momentum independent, and remembering the canonical conmutation relations, one finds that

$$\alpha\mathrm{Im}(vv^{*\prime}) = \frac{1}{2}\hbar c(2\pi\hbar)^3 \tag{2.20}$$

Let $\xi = 0$, i.e. work in the nonminimally coupled scalar field; then the Hamiltonian will be

$$\hat{\mathcal{H}}(t) = \int \frac{c}{2}\left[\hat{\Pi}^2 + \left(\nabla\hat{\chi}\right)^2 + \mu^2_{\text{eff}}(t)\hat{\chi}^2\right]\mathrm{d}^3\mathbf{x} \tag{2.21}$$

$$\begin{aligned}\hat{\Pi}^2 = \frac{1}{c^2}\int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^6}\Big[&\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{q}}v'_{\mathbf{k}}v'_{\mathbf{q}}e^{i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{q}}v'_{\mathbf{k}}v'^*_{\mathbf{q}}e^{i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} + \\ &+ \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{q}}v'^*_{\mathbf{k}}v'_{\mathbf{q}}e^{-i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{q}}v'^*_{\mathbf{k}}v'^*_{\mathbf{q}}e^{-i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}}\Big]\end{aligned} \tag{2.22}$$

$$\begin{aligned}\left(\nabla\hat{\chi}\right)^2 = -\frac{1}{\hbar^2}\int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^6}\mathbf{kq}\Big[&\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{q}}v_{\mathbf{k}}v_{\mathbf{q}}e^{i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}} - \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{q}}v_{\mathbf{k}}v^*_{\mathbf{q}}e^{i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} - \\ &- \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{q}}v^*_{\mathbf{k}}v_{\mathbf{q}}e^{-i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{q}}v^*_{\mathbf{k}}v^*_{\mathbf{q}}e^{-i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}}\Big]\end{aligned} \tag{2.23}$$

$$\hat{\chi}^2 = \int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^6} \Big[\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{q}} v_{\mathbf{k}} v_{\mathbf{q}} e^{i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{q}} v_{\mathbf{k}} v^*_{\mathbf{q}} e^{i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} + + \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{q}} v^*_{\mathbf{k}} v_{\mathbf{q}} e^{-i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{q}} v^*_{\mathbf{k}} v^*_{\mathbf{q}} e^{-i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}}\Big] \tag{2.24}$$

$$\begin{aligned}\hat{\mathcal{H}} = \frac{c}{2}\int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^3} \Big\{ &\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{q}} \left[\frac{1}{c^2} v'_{\mathbf{k}} v'_{\mathbf{q}} - \left(\frac{1}{\hbar^2}\mathbf{kq} - \mu^2_{\text{eff}}\right) v_{\mathbf{k}} v_{\mathbf{q}}\right] \delta^3(\mathbf{k}+\mathbf{q}) + \\ &+ \hat{a}_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{q}} \left[\frac{1}{c^2} v'_{\mathbf{k}} v'^*_{\mathbf{q}} + \left(\frac{1}{\hbar^2}\mathbf{kq} + \mu^2_{\text{eff}}\right) v_{\mathbf{k}} v^*_{\mathbf{q}}\right] \delta^3(\mathbf{k}-\mathbf{q}) + \\ &+ \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{q}} \left[\frac{1}{c^2} v'^*_{\mathbf{k}} v'_{\mathbf{q}} + \left(\frac{1}{\hbar^2}\mathbf{kq} + \mu^2_{\text{eff}}\right) v^*_{\mathbf{k}} v_{\mathbf{q}}\right] \delta^3(\mathbf{k}-\mathbf{q}) + \\ &+ \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{q}} \left[\frac{1}{c^2} v'^*_{\mathbf{k}} v'^*_{\mathbf{q}} - \left(\frac{1}{\hbar^2}\mathbf{kq} - \mu^2_{\text{eff}}\right) v^*_{\mathbf{k}} v^*_{\mathbf{q}}\right] \delta^3(\mathbf{k}+\mathbf{q}) \Big\}\end{aligned} \tag{2.25}$$

$$\begin{aligned}\hat{\mathcal{H}} = \frac{c}{2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3} \Big\{ &\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} \left[\frac{1}{c^2} v'_{\mathbf{k}} v'_{\mathbf{k}} + \frac{1}{\hbar^2}\omega^2_{\mathbf{k}}(t) v_{\mathbf{k}} v_{\mathbf{k}}\right] + \\ &+ \hat{a}_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{k}} \left[\frac{1}{c^2} v'_{\mathbf{k}} v'^*_{\mathbf{k}} + \frac{1}{\hbar^2}\omega^2_{\mathbf{k}}(t) v_{\mathbf{k}} v^*_{\mathbf{k}}\right] + \\ &+ \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} \left[\frac{1}{c^2} v'^*_{\mathbf{k}} v'_{\mathbf{k}} + \frac{1}{\hbar^2}\omega^2_{\mathbf{k}}(t) v^*_{\mathbf{k}} v_{\mathbf{k}}\right] + \\ &+ \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}^{\dagger}_{-\mathbf{k}} \left[\frac{1}{c^2} v'^*_{\mathbf{k}} v'^*_{\mathbf{k}} + \frac{1}{\hbar^2}\omega^2_{\mathbf{k}}(t) v^*_{\mathbf{k}} v^*_{\mathbf{k}}\right] \Big\}\end{aligned} \tag{2.26}$$

$$\hat{\mathcal{H}} = \frac{c}{2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3} \left[\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} F_{\mathbf{k}} + \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}^{\dagger}_{-\mathbf{k}} F^*_{\mathbf{k}} + \left(2\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v'v^*)}\delta^3(\mathbf{0})\right) E_{\mathbf{k}}\right] \tag{2.27}$$

where

$$F_{\mathbf{k}}(t) = \left(\frac{1}{\hbar c}\right)^2 \left[\hbar^2 v'^2_{\mathbf{k}} + \omega^2_{\mathbf{k}}(t)\, c^2 v^2_{\mathbf{k}}\right] \tag{2.28}$$

$$E_{\mathbf{k}}(t) = \left(\frac{1}{\hbar c}\right)^2 \left[\hbar^2 |v'_{\mathbf{k}}|^2 + \omega^2_{\mathbf{k}}(t)\, c^2 |v_{\mathbf{k}}|^2\right] \tag{2.29}$$

Now, the expectation value of the hamiltonian at time t_0 in the state $|_{(v)}0\rangle$

$$\langle_{(v)}0|\hat{\mathcal{H}}(t_0)|_{(v)}0\rangle = \rho(t_0)\delta^3(\mathbf{0}) = \frac{\hbar c^2\,\delta^3(\mathbf{0})}{4\mathrm{Im}(v'v^*)}\int \mathrm{d}^3\mathbf{k}\, E_{\mathbf{k}} \tag{2.30}$$

To minimise the energy density of the vacuum state is to find the set of functions $v_{\mathbf{k}}$ that minimise $E_{\mathbf{k}}$. Suppose that $v_{\mathbf{k}}$ can be written as

$$v_{\mathbf{k}} = r_{\mathbf{k}} e^{i\alpha_{\mathbf{k}}} \tag{2.31}$$

since $\mathrm{Im}(vv'^*)$ was constant through time

$$\mathrm{Im}(v_{\mathbf{k}} v'^*_{\mathbf{k}}) = -r^2_{\mathbf{k}}\alpha'_{\mathbf{k}} \tag{2.32}$$

this means

$$E_{\mathbf{k}} = \left(\frac{1}{\hbar c}\right)^2 \left\{\hbar^2\left[r'^2_{\mathbf{k}} + \mathrm{Im}^2\left(v_{\mathbf{k}} v'^*_{\mathbf{k}}\right)\frac{1}{r^2_{\mathbf{k}}}\right] + \omega^2_{\mathbf{k}}\, c^2 r^2_{\mathbf{k}}\right\} \tag{2.33}$$

the minimum of this function must fulfil $r'_{\mathbf{k}}(t_0) = 0$. Now, if $\omega^2_{\mathbf{k}}(t_0)$ and $\mathrm{Im}(v_{\mathbf{k}} v'^*_{\mathbf{k}})$ have the same sign, the minimum of $E_{\mathbf{k}}$ happens when $r_{\mathbf{k}}(t_0) = \left[\frac{\hbar \mathrm{Im}(v_{\mathbf{k}} v'^*_{\mathbf{k}})}{\omega_{\mathbf{k}}(t_0)\, c}\right]^{1/2}$.

If there is a minimum, then

$$v_{\mathbf{k}}(t_0) = \left[\frac{\hbar \mathrm{Im}(v_{\mathbf{k}} v_{\mathbf{k}}^{\prime *})}{\omega_{\mathbf{k}}(t_0)\, c}\right]^{1/2} e^{i\alpha_{\mathbf{k}}(t_0)} \qquad v_{\mathbf{k}}^{\prime}(t_0) = -c\frac{\omega_{\mathbf{k}}(t_0)}{ih}\, v_{\mathbf{k}}(t_0) \tag{2.34}$$

under these functions,

$$E_{\mathbf{k}}(t_0) = 2\frac{\mathrm{Im}(v_{\mathbf{k}} v_{\mathbf{k}}^{\prime *})}{\hbar c}\omega_{\mathbf{k}}(t_0) \qquad F_{\mathbf{k}}(t_0) = 0 \tag{2.35}$$

meaning

$$\hat{\mathcal{H}}(t_0) = \mathrm{Im}(v v^{\prime *})\frac{1}{\hbar}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\left(2\hat{a}_{\mathbf{k}}^{\dagger}\hat{a}_{\mathbf{k}} + \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v^{\prime} v^{*})}\delta^3(\mathbf{0})\right)\omega_{\mathbf{k}}(t_0) \tag{2.36}$$

which is equivalent to the standard Hamiltonian for a scalar field without the presence of gravity.
Bogolyubov Transformation The expression of the field χ at two different times must be related to a Bogoliubov transformation, with coefficients

$$\alpha_{\mathbf{kp}} = \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v^{\prime} v^{*})}\langle \chi_{\mathbf{k}}(t_0),\, \chi_{\mathbf{p}}(t)\rangle \qquad \beta_{\mathbf{kp}} = -\frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v^{\prime} v^{*})}\langle \chi_{\mathbf{k}}(t_0),\, \chi_{\mathbf{p}}^{*}(t)\rangle \tag{2.37}$$

since the field can be written as $\chi_{\mathbf{k}} = v_{\mathbf{k}} e^{-i\mathbf{kx}/h}$ from the expression of the inner product one can see that

$$\alpha_{\mathbf{kp}} \propto \delta^3(\mathbf{k}-\mathbf{p}) \qquad \beta_{\mathbf{kp}} \propto \delta^3(\mathbf{k}+\mathbf{p}) \tag{2.38}$$

therefore it is possible to write

$$v_{\mathbf{k}}(t) = \alpha_{\mathbf{k}} v_{\mathbf{k}}(t_0) + \beta_{\mathbf{k}} v_{\mathbf{k}}^{*}(t_0) \tag{2.39}$$

where, recalling that $\mathrm{Im}(v_{\mathbf{k}}^{\prime} v_{\mathbf{k}}^{*})$ is constant through time,

$$|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1 \tag{2.40}$$

To obtain the value of $\langle_{(t_0)}0|\hat{\mathcal{H}}(t)|_{(t_0)}0\rangle$ lets first compute

$$\langle_{(t_0)}0|\hat{\mathcal{N}}_{\mathbf{k}}^{(a)}(t)|_{(t_0)}0\rangle = \langle_{(t_0)}0|\hat{a}_{\mathbf{k}}^{\dagger}(t)\hat{a}_{\mathbf{k}}(t)|_{(t_0)}0\rangle = \left|\beta_{\mathbf{k}}\right|^2 \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v v^{\prime *})}\delta^3(\mathbf{0}) \tag{2.41}$$

therefore

$$\langle_{(t_0)}0|\hat{\mathcal{H}}(t)|_{(t_0)}0\rangle = \delta^3(\mathbf{0})\int \mathrm{d}^3\mathbf{k}\left(\frac{1}{2} + \left|\beta_{\mathbf{k}}\right|^2\right) c\,\omega_{\mathbf{k}}(t) \geq \langle_{(t_0)}0|\hat{\mathcal{H}}(t_0)|_{(t_0)}0\rangle \tag{2.42}$$

meaning, if $\beta_{\mathbf{k}} \neq 0$ for all $\mathbf{k}$ then, at a time $t > t_0$ the energy density will be different in relation to the original vacuum.

3 Scalar field in a de Sitter Universe

The de Sitter Universe is a flat FLRW metric with no matter or radiation, but it does have a a positive cosmological constant Λ. Per the Friedmann equations,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho + \Lambda c^2}{3} - \frac{\kappa c^2}{a^2} \tag{3.1}$$

the expansion parameter $a(t)$ will be equal to

$$a(t) = a_1 e^{H_\Lambda t} + a_2 e^{-H_\Lambda t}\ , \qquad H_\Lambda = \sqrt{\frac{\Lambda c^2}{3}} \tag{3.2}$$

$a_2 = 0$

$$\mathrm{d}l^2 = c^2\mathrm{d}t^2 - a^2(t)\mathrm{d}\mathbf{x}^2 \tag{3.3}$$

$$\eta \equiv -\int_t^\infty \frac{\mathrm{d}t'}{a(t')} = -\frac{1}{a_1\, H_\Lambda} e^{-H_\Lambda t} = -\frac{1}{a(t) H_\Lambda} \tag{3.4}$$

[1]

$$\mathrm{d}l^2 = \frac{1}{H_\Lambda \eta^2}\left[c^2\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2\right] \tag{3.5}$$

$$R = \frac{6}{c^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] = \frac{12}{c^2} H_\Lambda^2 \tag{3.6}$$

$$\omega_{\mathbf{k}}^2(\eta) = \mathbf{k}^2 + \left[\left(\frac{mc^2}{H_\Lambda}\right)^2 + 2\left(6\xi - 1\right)\hbar^2\right]\frac{1}{c^2\eta^2} \tag{3.7}$$

$$v_{\mathbf{k}}''\hbar^2 + \omega_{\mathbf{k}}^2(\eta)\, v_{\mathbf{k}} = 0 \tag{3.8}$$

change of variables

$$s \equiv -k\eta \qquad v_{\mathbf{k}} \equiv \sqrt{s} f(s) \tag{3.9}$$

$$s^2\frac{\mathrm{d}^2 f}{\mathrm{d}s^2} + s\frac{\mathrm{d}f}{\mathrm{d}s} + \left(s^2 - \nu^2\right) f(s) = 0 \tag{3.10}$$

$$\nu^2 \equiv (3 - 16\xi)\,\frac{3}{4}\frac{\hbar^2}{c^2} - \left(\frac{mc}{H_\Lambda}\right)^2 \tag{3.11}$$

$$f(s) = AJ_\nu(s) + BY_\nu(s) \implies v_{\mathbf{k}}(\eta) = \sqrt{k|\eta|}\left[A_{\mathbf{k}} J_\nu(k|\eta|) + B_{\mathbf{k}} Y_\nu(k|\eta|)\right] \tag{3.12}$$

$J_\nu(x)$, $Y_\nu(x)$ are the Bessel functions of the firs kind. $\mathrm{Im}(vv'^*)$ creates a restriction on the relation $B_{\mathbf{k}} = B_{\mathbf{k}}(A_{\mathbf{k}})$

[1]Considering $a_2 \neq 0$ one would obtain that

$$\eta = \frac{\arctan\left(\sqrt{\frac{a_2}{a_1}} e^{-H_\Lambda t}\right)}{H_\Lambda \sqrt{a_1 a_2}}$$

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Scalar field in Minkowski background

$$\eta_{\mu\nu} = \mathrm{diag}\,(+,\ -,\ -,\ -) \tag{1}$$

$$S[\phi] = \int \frac{1}{2}\Big[\partial_\nu\phi\,\partial^\nu\phi - \mu^2\phi^2\Big]\ \mathrm{d}^4x \tag{2}$$

Units

- $[s] = [\hbar]$
- $[a] = [\xi] = 1$
- $[\mu] = [L]^{-1}$
- $[R] = [L]^{-2}$
- $[\phi] = [\chi] = [\hbar]^{1/2}[L]^{-1}$
- $[\Pi] = [\hbar]^{1/2}[L]^{-2}$
- $[a_{\mathbf{k}}] = [\hbar]^{1/2}[L]^2$

Questions & To-Do

1 Dudas

- How do you know that there is a set of solutions of Klein Gordon such that the inner product fulfils the given results?
- It's the Hamiltonian well defined?
- Can you always write a FLRW metric as a flat one with a coordinate change?
- Covariant derivatives, spin connection.

2 To-Do

- Chapter 2: equations suppose that a_k are associated with $e^{-Et\hbar^{-1}}$, they should be associates with **positive** exponential....
- Minkowski scalar field.

Computations