PROBLEMS OF QUANTUM FIELD THEORIES IN CURVED SPACETIMES

A MASTER THESIS

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Abrstract

Quantum Field Theory is the fundamental theoretical structure of the Standard Model of elementary particles. This theory is formulated in a Minkowski space-time. However, the actual space-time metric is never of that type. On Earth, even at short distances, the metric is affected by both the force of the Earth's gravity and the solar force, but especially by the acceleration of the Earth's motion. At large distances the cosmological data lead one to think that, on average, the metric is of the Friedmann-Lemaitre-Roberson-Walker type.

The analysis of the quantization of fields in the presence of gravitational fields involves a number of theoretical connotations that we intend to explore in this paper. How Quantum Field Theory can be adapted when considering this gravitational background is the object of the proposal. In particular, how the structure of the quantum vacuum is affected when space-time is neither asymptotically Minkowskian, as is the case in the current Cosmological Model LCDM. There are a number of attempts to address the problem but to date none provides a satisfactory solution; the aim of the present work is to find the one that best fits the experimental observations.

Contents

Conventions

The chosen convention for the metric signature will be $(+, -, -, -)$ as in [2] and most literature on particle physics. Common conventions and nomenclatures in mathematics and physics are used throughout the text, some of which are considered to be relevant:

x^μ, x	four-vector
x^i, $\mathbf{x}$	spacial vector
$g_{\mu\nu}$	general spacetime metric
$\eta_{\mu\nu}$	Minkowski spacetime metric
g	determinant of $g_{\mu\nu}$
∇_μ	covariant derivative
$S[\phi]$	action functional of a field ϕ and it's derivatives
$\bar{z}$	complex conjugate of z
$A^\dagger$	hermitian conjugate of A
$R^\alpha_{\ \beta\gamma\delta}$	Riemann tensor $\quad \equiv \nabla_\delta \Gamma^\alpha_{\ \beta\gamma} - \nabla_\delta \Gamma^\gamma_{\ \beta\alpha} + \dots$
$f \overleftrightarrow{\nabla}_\mu g$	$\equiv f \nabla_\nu g - (\nabla_\mu f)\, g$
γ^μ	covariant Gamma matrices $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$
G, c, $\hbar$	standard universal constants, not necessarily in natural units

Other notation will be introduced as needed.

Preface

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1 Introduction to QFT in Curved Spacetimes

1.1 The Universal Action

Consider a dynamic universe consisting of dark energy characterized by a cosmological constant Λ and some material content described by a Lagrangian density $\mathcal{L}_{\mathrm{M}}$. The action associated with such system would be

$$S = \int \left[\frac{1}{2\kappa} (R - 2\Lambda) + \mathcal{L}_{\mathrm{M}} \right] \sqrt{-g}\, \mathrm{d}^4 x, \tag{1.1}$$

where $\kappa \equiv \frac{8\pi G}{c^4}$ is known as the Einstein gravitational constant.

The equations that would describe the dynamics of the system can be obtained by variations of the action presented in eq. (1.1) and the stationary-action principle, which states that the path taken by the system will result in $\delta S = 0$. The equations of motion of the matter content are dependant on its formulation; let the Lagrangian density of matter be described by some set $\{\phi^\alpha(x)\}$ of fields and their first covariant derivatives, i.e.

$$\mathcal{L}_{\mathrm{M}} = \mathcal{L}_{\mathrm{M}} \left[\phi^\alpha(x),\, \nabla_\mu \phi^\alpha(x) \right], \tag{1.2}$$

then, variations of the action S with respect of ϕ^α will result in

$$\delta S = \int \left[\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \phi^\alpha} \delta \phi^\alpha + \frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \left(\nabla_\mu \phi^\alpha \right)} \nabla_\mu \left(\delta \phi^\alpha \right) \right] \sqrt{-g}\, \mathrm{d}^4 x, \tag{1.3}$$

and thus, after applying the generalized Gauss Theorem on a general manifold, alongside the stationary-action principle, one obtains the well known Euler-Lagrange equations

$$\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \phi^\alpha} - \nabla_\mu \left[\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \left(\nabla_\mu \phi^\alpha \right)} \right] = 0. \tag{1.4}$$

On the other hand, variations of S with respect to the inverse metric ($g^{\mu\nu}$) results in

$$\delta S \;=\; \int \left[\frac{\sqrt{-g}}{2\kappa} \frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{2\kappa} \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} - \frac{\Lambda}{\kappa} \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}} + \frac{\mathcal{L}_{\mathrm{M}}}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} \sqrt{-g}\, \mathrm{d}^4 x; \tag{1.5}$$

and again, by imposing $\delta S = 0$ and considering that

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu}, \qquad \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2} g_{\mu\nu}, \tag{1.6 a,b}$$

one obtains the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -2 \frac{8\pi G}{c^4} \left(\frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}} - \frac{1}{2} \mathcal{L}_{\mathrm{M}} g_{\mu\nu} \right) \tag{1.7}$$

which are most commonly written in terms of the Hilbert energy-momentum tensor

$$T_{\mu\nu} \equiv \mathcal{L}_{\mathrm{M}} g_{\mu\nu} - \frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}} = \frac{-2}{\sqrt{-g}} \frac{\delta \left(\mathcal{L}_{\mathrm{M}} \sqrt{-g} \right)}{\delta g^{\mu\nu}}. \tag{1.8}$$

This tensor is the source of the spacetime curvature, and must not be confused with the Noether's energy-momentum tensor since the two are not, in general, equivalent [1], but upon integration of the corresponding conserved currents, results are the same [5]. In addition of being symmetric, the Hilbert energy-momentum tensor is covariantly conserved, i.e.

$$\nabla_\mu T^{\mu\nu} = 0; \tag{1.9}$$

this fact is of great use once the material Hamiltonian $\mathcal{H}_\mathrm{M}$ is defined:

$$\mathcal{H}_\mathrm{M} \equiv \int T^{00} c \sqrt{-g}\, \mathrm{d}^3\mathbf{x}, \tag{1.10}$$

since it will later be used to spawn the Fock space after the quantization procedure; and thus assure no energy loses will be present on the theory.

1.2 Construction of Covariant Actions

Standard quantum field theory alongside the standard model of particle physics is one of (if not the) best tested theories of physics, reason which, one doesn't need to reinvent the actions that are used on it, only a small tweak is needed to make the theory general covariant; it has been previously given for granted that the volume element will be $\sqrt{-g}\mathrm{d}^4x$, but there is another consideration, the derivatives, since in a general spacetime, they are not covariant, and thus, the inclusion of a covariant derivative ∇_μ is needed.

Let $\Xi(x)$ be a tensorial field such that under a local Lorentzian transformation $\Lambda(x)$ transforms as $\tilde{\Xi} \equiv \rho_\Lambda \Xi$. The usual covariant derivative of the transformed field will be

$$\begin{aligned}\nabla_\mu \tilde{\Xi}^{a_1,a_2,\dots}{}_{b_1,b_2,\dots} &\equiv \partial_\mu \tilde{\Xi}^{a_1,a_2,\dots}{}_{b_1,b_2,\dots} + \Gamma^{a_1}_{\sigma\mu} \tilde{\Xi}^{\sigma,a_2,\dots}{}_{b_1,b_2,\dots} + \dots - \Gamma^{\sigma}_{b_1\mu} \tilde{\Xi}^{a_1,a_2,\dots}{}_{\sigma,b_2,\dots} - \dots = \\ &= \left(\partial_\mu \rho_\Lambda\right) \Xi^{a_1,a_2,\dots}{}_{b_1,b_2,\dots} + \rho_\Lambda \partial_\mu \Xi^{a_1,a_2,\dots}{}_{b_1,b_2,\dots} + \Gamma^{a_1}_{\sigma\mu} \tilde{\Xi}^{\sigma,a_2,\dots}{}_{b_1,b_2,\dots} + \dots - \Gamma^{\sigma}_{b_1\mu} \tilde{\Xi}^{a_1,a_2,\dots}{}_{\sigma,b_2,\dots} - \dots .\end{aligned} \tag{1.11}$$

From this one can conclude that, for $\nabla_\mu \tilde{\Xi}(x)$ to transform as a tensor, one should redefine the covariant derivative ∇_μ to add a term Γ_μ such that it transforms as

$$\tilde{\Gamma}_\mu \equiv \rho_\Lambda \Gamma_\mu \rho_\Lambda^{-1} - \left(\partial_\mu \rho_\Lambda\right) \rho_\Lambda^{-1}, \tag{1.12}$$

such term (that we should call connection), can be written as

$$\Gamma_\mu \equiv \frac{1}{2} \Sigma^{AB} \omega_{AB\mu}; \tag{1.13}$$

where Σ^{AB} are to be understood as the Lorentz generators (the uppercase Latin indexes represent sums over a Minkowski background), and $\omega_{AB\mu}$ is the so called torsion free spin connection, defined as

$$\omega_{AB\mu} \equiv e^\nu_A \left(\partial_\mu e_{B\nu} - \Gamma^\sigma_{\nu\mu} e_{B\sigma}\right). \tag{1.14}$$

The new vector fields e^μ_A are known as the tetrad formalism coefficients, defined to transform general tensors to a local flat manifold, i.e.

$$g_{\mu\nu} = e^A_\mu e^B_\nu \eta_{ab}. \tag{1.15}$$

Therefore, we define the covariant derivative as

$$\nabla_\mu \Xi^{a_1,a_2,\dots}{}_{b_1,b_2,\dots} \equiv \partial_\mu \Xi^{a_1,a_2,\dots}{}_{b_1,b_2,\dots} + \Gamma_\mu + \Gamma^{a_1}_{\sigma\mu} \tilde{\Xi}^{\sigma,a_2,\dots}{}_{b_1,b_2,\dots} + \dots - \Gamma^{\sigma}_{b_1\mu} \tilde{\Xi}^{a_1,a_2,\dots}{}_{\sigma,b_2,\dots} - \dots . \tag{1.16}$$

If the field $\Xi^{a_1,a_2,\dots}{}_{b_1,b_2,\dots}$ is coupled to a vector field A_μ the covariant derivative must be redefined as $\nabla'_\mu \equiv \nabla_\mu - \frac{i}{\hbar} e A_\mu$, where e would be the coupling constant.

1.2.1 Some Basic Examples

Scalar Field

The very first example given for a classical field is usually a (real) free scalar field $\phi(x)$ with some mass m; whose dynamics are given by the following action

$$S[\phi] = \int \frac{1}{2}\Big[\partial_\nu\phi\,\partial^\nu\phi - \mu^2\phi^2 - \xi R\phi^2\Big]\sqrt{-g}\,\mathrm{d}^4x. \tag{1.17}$$

The construction of such action arises from its Minkowskian counterpart (a primary study of the standard quantum field theory can be found in the appendix 4.1); since the field in question is a scalar one, the covariant derivative 1.16 its simply ∂_ν, the massive term of the action is dependant on a parameter $\mu \equiv {}^{mc}/_{\hbar}$, and an term is added as a coupling to gravity (through the Ricci scalar R) with a coupling constant ξ [1].

The inclusion of such coupling is not a mere curiosity, since it's been proven [3] that a self interactive $\lambda\phi^4$ theory needs a term proportional to $R\phi^2$ to be renormalizable. Besides this, de addition of a term proportional to $R\phi^2$ adds a new symmetry to de action, since for a massless field $\mu = 0$ with a coupling constant $\xi = {}^1/_6$, the action is invariant under conformal transformations, i.e.

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} \equiv \Omega^2(x) g_{\mu\nu}, \tag{1.18}$$

to prove this, lets first obtain the equations of motion using the Euler-Lagrange equation 1.4, resulting in the generalized Klein-Gordon equation

$$\big[\partial_\nu\partial^\nu - \mu^2 - \xi R\big]\phi = 0, \tag{1.19}$$

and then, a conformal transformation can be made to then, considering that the field will transform as $\phi \to \tilde{\phi} = \Omega^\beta\phi$, resulting on the following expression:

$$\begin{aligned} 0 = \mu^2\Omega^{\beta-2}\left(\Omega^2 - 1\right)\phi + 2\left(1+\beta\right)\Omega^{\beta-3}\partial^\nu\Omega\partial_\nu\phi + \\ + (6\xi+\beta)\Omega^{\beta-3}\left(\partial_\nu\partial^\nu\Omega\right)\phi + \beta(1+\beta)\Omega^{\beta-4}\partial_\nu\Omega\partial^\nu\Omega\,\phi. \end{aligned} \tag{1.20}$$

Considering a massless field, a solution of this equation corresponds to the following values:

$$\beta = -1, \qquad \xi = \frac{1}{6}, \tag{1.21 a,b}$$

proving the conformal invariance for such scenario.

From its definition in equation 1.8 and the action 1.17, one can obtain the expression for the energy momentum tensor

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\left[\partial^\sigma\phi\partial_\sigma\phi - \mu^2\phi^2\right] + \xi\left[-R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\partial^\sigma\partial_\sigma + \partial_\mu\partial_\nu\right]\phi^2, \tag{1.22}$$

which has an interesting property of its trace,

$$T^\nu_\nu = \frac{1}{2}\left(6\xi - 1\right)\partial_\sigma\partial^\sigma\phi^2 + \mu^2\phi^2, \tag{1.23}$$

as it is zero, for a conformal theory.

[1]The field is said to be minimally coupled to gravity if $\xi = 0$ and nonminimally coupled otherwise.

Dirac Field

For spin 1/2 particles, the Lorentz generators are

$$\Sigma^{AB} = -\frac{i}{2}\sigma^{AB} = \frac{1}{4}\left[\gamma^{A}, \gamma^{B}\right], \tag{1.24}$$

where Γ^{A} are the flat gamma matrices. Therefore the covariant derivative 1.16 and the connection 1.13 can be written as

$$\nabla_{\mu} \equiv \partial_{\mu} + \Gamma_{\mu}, \quad \Gamma_{\mu} = \frac{1}{8}\omega_{AB\mu}\left[\gamma^{A}, \gamma^{B}\right]. \tag{1.25a}$$

Taking the Dirac theory as inspiration, one could define de Dirac action in curved spacetimes as

$$S[\psi] = \int \bar{\psi}\left[i\gamma^{\nu}\nabla_{\nu} - \mu\right]\psi\sqrt{-g}\mathrm{d}^{4}x, \tag{1.26}$$

where $\Gamma^{\nu} \equiv \gamma^{A}e^{\nu}_{A}$ are the general gamma functions, which follow the next relation similar to their flat counterparts,

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}. \tag{1.27}$$

From the Euler-Lagrange equation 1.4, its obtained the generalized Dirac equation

$$\left[i\gamma^{\mu}\left(\partial_{\mu} + \Gamma_{\mu}\right) - \mu\right]\psi = 0. \tag{1.28}$$

And from its definition in equation 1.8, the energy momentum tensor will have the following expression

$$T_{\mu\nu} = \frac{1}{4}i\left\{\bar{\psi}\left(\gamma_{\mu}\nabla_{\nu} - \gamma_{\nu}\nabla_{\mu}\right) - \left[\left(\nabla_{\mu}\bar{\psi}\right)\gamma_{\nu} - \left(\nabla_{\nu}\bar{\psi}\right)\gamma_{\mu}\right]\right\}\psi. \tag{1.29}$$

A particularly interesting outcome of this field is the so called Schrödinger-Dirac equation, result of squaring the generalized Dirac operator $\left[i\gamma^{\mu}\left(\partial_{\mu} + \Gamma_{\mu}\right) - \mu\right]$ just as its done in a Minkowskian background to recover the Klein-Gordon equation,

$$\left[\nabla_{\nu}\nabla^{\nu} - \mu^{2} - \frac{1}{4}R\right]\psi = 0, \tag{1.30}$$

this expression gives another "natural" choice for the scalar field coupling to gravity ξ, comparing it with the generalized Klein-Gordon equation 2.6 finding $\xi = 1/4$.

Electromagnetic Field

Having previously studied the Dirac field, it is expected to also include the electromagnetic field, which is described by the same action as in the Minkowskian background, that is,

$$S[A_{\mu}] = \int \left(-\frac{1}{4\mu_{0}c}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_{\mathrm{Gauge}}\right)\sqrt{-g}\mathrm{d}^{4}x, \tag{1.31}$$

where $\mathcal{L}_{\mathrm{Gauge}}$ is a gauge fixing term of the form

$$\mathcal{L}_{\mathrm{Gauge}} = -\frac{1}{2\alpha}\left(\nabla_{\nu}A^{\nu}\right)^{2}. \tag{1.32}$$

The Faraday tensor is defined as

$$F_{\mu\nu} = \nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}, \tag{1.33}$$

where the last equality is a result of the symmetry of the lower indices on the Christoffel symbols.

The equations of motion resulting from the action 1.31 and Euler-Lagrange 1.4 are

$$\nabla^{\nu}\nabla_{\nu}A_{\mu} + R^{\sigma}_{\mu}A_{\sigma} - (1-\alpha)\nabla_{\mu}\nabla^{\nu}A_{\nu} = 0. \tag{1.34}$$

And finally, as for completeness, the energy-momentum tensor given by 1.8 is

$$\begin{aligned} T_{\mu\nu} = -\frac{1}{\mu_0}\left(F_{\mu\alpha}F^{\alpha\nu} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) + \\ + \alpha\left\{A_{\mu}\left(\nabla_{\nu}\nabla_{\rho}A^{\rho}\right) + \left(\nabla_{\mu}\nabla_{\rho}A^{\rho}\right)A_{\nu} - g_{\mu\nu}\left[A^{\rho}\left(\nabla\rho\nabla_{\sigma}A^{\sigma}\right) + \frac{1}{2}\left(\nabla_{\rho}A^{\rho}\right)^{2}\right]\right\}. \end{aligned} \tag{1.35}$$

1.3 Scalar field Quantization

Thanks to its simplicity, the scalar field is a great field to work with, with the intent of showing some properties of a theory. For that reason, for what follows, all work will be done considering a real scalar field described by the action 1.17.

Now, let $v(x)$ be a solution of the generalized Klein-Gordon equation 2.6, then its complex conjugated $\bar{v}$ will also be an independent solution. Now consider i to be a set of parameters that univocally describe a par of solutions v_i, $\bar{v}_i$ in such a way that the most general solution of 2.6 will be

$$\phi(x) = \sum_i \left[a_i v_i(x) + \bar{a}_i\bar{v}_i(x)\right], \tag{1.36}$$

where a_i and $\bar{a}_i$ are constant factors, determined by the following external binary operation

$$\langle\phi_1(x),\, \phi_2(x)\rangle \equiv \frac{i}{\hbar}\int g^{0\nu}\left(\phi_1\overleftrightarrow{\partial}_{\nu}\bar{\phi}_2\right)\sqrt{-g}\,\mathrm{d}^3\mathbf{x}, \tag{1.37}$$

such that

$$a_i = \langle v_i(x),\, \phi(x)\rangle, \quad \bar{a}_i = \langle \bar{v}_i(x),\, \phi(x)\rangle. \tag{1.38 a,b}$$

The quantization procedure is done by promoting the field χ and its conjugate momentum $\Pi \equiv \partial_{ct}\chi$ to operators

$$\phi(x) \longrightarrow \hat{\phi}(x), \qquad \Pi(x) \longrightarrow \hat{\Pi}(x), \tag{1.39}$$

by promoting the constant factors to operators as well, that is

$$a_i \longrightarrow \hat{a}_i, \qquad \bar{a}_i \longrightarrow \hat{a}_i^{\dagger}, \tag{1.40}$$

and therefore

$$\hat{\phi}(x) = \sum_i \left[\hat{a}_i v_i(x) + \hat{a}_i^{\dagger}\bar{v}_i(x)\right]. \tag{1.41}$$

One the promotion of the field to operators have been done, commutation relations between those operators must be imposed; the easiest choice would be to assume canonical quantization relations, that is,

$$\left[\hat{\phi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = i\hbar\,\delta^3\left(\mathbf{x}-\mathbf{y}\right) \qquad \left[\hat{\phi}(\mathbf{x}),\, \hat{\phi}(\mathbf{y})\right] = \left[\hat{\Pi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = 0. \tag{1.42 a-c}$$

It would be desirable to obtain a formulation similar to the well known scalar field in a Minkoswskian background, where the Fock space is generated from a vacuum state and a set of creation and annihilation operators that follow some commutation rules. To do so, we will force the $\hat{a}_i$, $\hat{a}_i^{\dagger}$ operators to assume this roll, in such a way that

$$\left[\hat{a}_i,\, \hat{a}_j^{\dagger}\right] \propto \delta_{ij}, \qquad \left[\hat{a}_i,\, \hat{a}_j\right] = \left[\hat{a}_i^{\dagger},\, \hat{a}_j^{\dagger}\right] = 0. \tag{1.43 a-c}$$

Thanks to the relation between the constant factors a_i and the operation $\langle v_i,\, \phi\rangle$, one can obtain the following relation

$$\left[\hat{a}_i,\, \hat{a}_j^\dagger\right] = -\frac{1}{\hbar^2}\int\left[\left(v_i\hat{\Pi} - g^{0\nu}\left(\partial_\nu v_i\right)\hat{\phi}\sqrt{-g}\right)\Big|_{\mathbf{x}},\, \left(\bar{v}_j\hat{\Pi} - g^{0\nu}\left(\partial_\nu v_j\right)\hat{\phi}\sqrt{-g}\right)\Big|_{\mathbf{y}}\right]\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{y} =$$
$$= \frac{i}{\hbar}\int g^{0\nu}\left(v_i\overleftrightarrow{\partial}_\nu\bar{v}_j\right)\sqrt{-g}\,\mathrm{d}^3\mathbf{x} = \langle v_i,\, v_j\rangle, \quad (1.44)$$

where the field commutators where used. Equivalently

$$[\hat{a}_i,\, \hat{a}_j] = -\langle v_i,\, \bar{v}_j\rangle, \quad \left[\hat{a}_i^\dagger,\, \hat{a}_j^\dagger\right] = -\langle \bar{v}_i,\, v_j\rangle. \quad (1.45\text{ a,b})$$

Therefore we must find a set of solutions $\{v_i(x),\, \bar{v}_i(x)\}$ such that

$$\langle v_i,\, v_j\rangle \propto \delta_{ij}, \quad \langle v_i,\, \bar{v}_j\rangle = \langle \bar{v}_i,\, v_j\rangle = 0. \quad (1.46\text{ a-c})$$

With this, we can define the Fock space the usual way, starting with a vacuum state $|0\rangle$ such that the action of the annihilation operation fulfils

$$\hat{a}_i\,|0\rangle = 0 \qquad \forall i \quad (1.47)$$

where single particle states are formed from the creation operator

$$|i\rangle \equiv \hat{a}_i^\dagger\,|0\rangle \quad (1.48)$$

and multiparticle states like

$$|i,\, j,\, \ldots\rangle = \ldots\hat{a}_j^\dagger\,\hat{a}_i^\dagger\,|0\rangle \quad (1.49)$$

Since this is a scalar field, one might assume that the states are symmetric (describing boson particles), and this is easily confirmed, since

$$|i,\, j\rangle = \hat{a}_j^\dagger\,\hat{a}_i^\dagger\,|0\rangle = \left[\hat{a}_i^\dagger,\, \hat{a}_j^\dagger\right]|0\rangle + \hat{a}_i^\dagger\,\hat{a}_j^\dagger|0\rangle = |j,\, i\rangle \quad (1.50)$$

1.3.1 Bogoliubov Transformations

Consider now a second set $\{u_i(x),\, \bar{u}_i(x)\}$ of solutions to the Klein-Gordon equation 2.6 such that they meet the operational rules 1.46; the field would then be written as

$$\phi(x) = \sum_j\left[b_j u_j(x) + \bar{b}_j\bar{u}_j(x)\right]. \quad (1.51)$$

Quantization of the field is straightforward and equivalent to the method previously presented. The relation between the v and u solutions (mode functions) will be

$$v_i(x) \equiv \sum_j\left[\alpha_{ij}u_j(x) + \beta_{ij}\bar{u}_j(x)\right], \quad (1.52)$$

where α_{ij} and β_{ij} are known as Bogoliubov coefficients, that can be obtained as

$$\alpha_{ij} \propto \langle v_i,\, u_j\rangle \qquad \beta_{ij} \propto -\langle v_i,\, \bar{u}_j\rangle. \quad (1.53)$$

Since the field is the same independently of the mode set chosen:

$$\sum_i\left[\hat{a}_i v_i(x) + \hat{a}_i^\dagger v_i^*(x)\right] = \sum_j\left[\hat{b}_j u_j(x) + \hat{b}_j^\dagger u_j^*(x)\right] \quad (1.54)$$

and, as a result of the orthogonality of the mode functions, the relation between the creation and annihilation operators will be

$$\hat{a}_i = \sum_j \left(\bar{\alpha}_{ij}\hat{b}_j - \bar{\beta}_{ij}\hat{b}_j^\dagger\right), \quad \hat{a}_i^\dagger = \sum_j \left(-\beta_{ij}\hat{b}_j + \alpha_{ij}\hat{b}_j^\dagger\right). \tag{1.55 a,b}$$

Applying commutator rules 10 give new restrictions to the Bogoliubov coefficients

$$\left[\hat{a}_i,\, \hat{a}_j^\dagger\right] \propto \delta_{ij} \implies \sum_k \left(\bar{\alpha}_{ik}\alpha_{jk} - \bar{\beta}_{ik}\beta_{jk}\right) \propto \delta_{ij}, \tag{1.56}$$

$$[\hat{a}_i,\, \hat{a}_j] = 0 \implies \sum_k \left(\bar{\alpha}_{jk}\bar{\beta}_{ik} - \bar{\alpha}_{ik}\bar{\beta}_{jk}\right) = 0. \tag{1.57}$$

The reader might ask what the relevance of this transformation is, and it would be in its right to do so, since it is not a mere mathematical result. To see the reason of this transformation, one could compute the number of v particles that are present in the u vacuum; the computation is given by

$$\langle {}_u0|\hat{N}_v|{}_u0\rangle = \sum_i \langle {}_u0|\hat{a}_i^\dagger \hat{a}_i|{}_u0\rangle = \sum_i \left[\sum_{jk} \beta_{ij}\bar{\beta}_{ik}\langle {}_u0|\hat{b}_j\hat{b}_k^\dagger|{}_u0\rangle\right] \propto \sum_{ij} |\beta_{ij}|^2. \tag{1.58}$$

The usual expectation value of a term of the form $\langle 0|\hat{N}|0\rangle$ is to be zero, and yet, it has been proven that this is not the case (in general) for the current scenario. The interpretation of such result is that the notion of "particle" is dependent on the choice of solutions of the Klein-Gorton equation; and thus, one could define different vacuum states for different situations.

1.3.2 A Leap Towards a Continuum

Until now, it has been considered that the set of Klein-Gordon solutions could be categorised by a discrete set of parameters i, from a standard course in QFT, one of the main results is the fact that the solutions of the flat Klein-Gordon equations can be parametrised by a continuous 3-dimensional vector $\mathbf{k}$ (which is interpreted to be the momentum of the particle). Since all computations in this section where made by considering a discrete set of parameters, it is relevant to consider the continuum case.

A common computation in many fields of physics is the determination of the density of states $D(\mathbf{k})$ describing the number of modes with momentum between $\mathbf{k}$ and $\mathbf{k} + \mathrm{d}\mathbf{k}$. Consider a system with volume V, where the field goes to zero at its boundary; in this case, the permitted values of momenta must meet

$$k^i = n^i \frac{\pi\hbar}{V^{1/3}}, \qquad n^i \in \mathbb{N} \tag{1.59}$$

Let $N(k)$ be the number of states with momentum modulus less than k, that is, the states such that

$$n = \sqrt{(n^1)^2 + (n^2)^2 + (n^3)^2} < k\frac{V^{1/3}}{\pi\hbar} \tag{1.60}$$

considering a flat momentum space[2] and a large enough volume, $N(k)$ will be essentially equal to an eight of the volume of a sphere with radius $kV^{1/3}/\pi\hbar$, that is

$$N(k) \approx \frac{1}{8}\frac{4}{3}\pi\left(k\frac{V^{1/3}}{\pi\hbar}\right)^3 = \frac{V}{6\pi^2\hbar^3}k^3 \tag{1.61}$$

meaning, that the density of states will be

$$D(\mathbf{k}) \equiv D(k) = \frac{\mathrm{d}N(k)}{\mathrm{d}k} \approx \frac{V}{2\pi^2\hbar^3}k^2 \tag{1.62}$$

[2]In contrast to modified theories of relativity in which this is not the case, like the κ-Poincaré relativity.

With this, one could approximate a discrete sum over a parameter i to an integral over a continuum $\mathbf{k}$

$$\sum_i f_i = \int_0^\infty D(k) f_k \mathrm{d}k \approx \int_0^\infty \frac{V}{2\pi^2\hbar^3} f_k k^2 \mathrm{d}k \equiv \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3} f_\mathbf{k} \tag{1.63}$$

where it has been defined.

$$4\pi V f_k k^2 \equiv \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} f_\mathbf{k} \sin\varphi \mathrm{d}\theta \mathrm{d}\varphi \tag{1.64}$$

therefore ${}^{\mathrm{d}^3\mathbf{k}}/_{(2\pi\hbar)^3}$ is to be understood as the volume element of the momentum space.

2 Scalar Fields in Expanding Universes

The methodology for the analysis of scalar fields in a general manifold was presented in the previous chapter as a preliminary for the rest of this work, and in particular of the present chapter. It is clear that the presence of symmetries of the theory will simplify computations, and thus, a great start might be an isotropic and homogeneous expanding universe, which is described by the so called Friedmann–Lemaître–Robertson–Walker metric. Using reduced-circumference polar coordinates, the line element associated with such metric is written as

$$\mathrm{d}l^2 = c^2\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\mathrm{d}\Omega^2\right], \quad \mathrm{d}\Omega \equiv \mathrm{d}\theta^2 + \sin^2\varphi\mathrm{d}\varphi^2, \tag{2.1 a,b}$$

where κ is the curvature of the space and $a(t)$ is the scale factor determining the expansion. The associated curvature scalar R is given by

$$R = \frac{6}{c^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right], \tag{2.2}$$

needed to the coupling of the field with gravity as previously stated.

2.1 Expanding scalar field action

The Weyl tensor associated to the metric presented in eq. (2.1) is identically zero, meaning that the metric is conformally flat, i.e. independently of the space curvature κ, and therefore there must exist a coordinate system where

$$\mathrm{d}l^2 = a^2(\eta)\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = a^2(\eta)\left[c^2\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2\right], \tag{2.3}$$

working in such coordinate system will give the opportunity to use some results of standard scalar field theory. To do so, the action presented in eq. (1.17) will be rewritten in terms of a new field $\chi(x) \equiv a(\eta)\,\phi(x)$ using the fact that $\sqrt{-g} = a^4$

$$S[\chi] = \int \frac{1}{2}\left[\partial_\nu\chi\,\partial^\nu\chi - \left(\mu^2 a^2 + \xi R a^2 - c^2\frac{a''}{a}\right)\chi^2 - \partial_\eta\left(c^2\chi^2\frac{a'}{a}\right)\right]\mathrm{d}^4x, \tag{2.4}$$

where $a' \equiv \partial_\eta a(\eta)$ and equivalently with a''.

Dropping the time derivative will result on the following action for the sacalar χ field

$$S[\chi] = \int \frac{1}{2}\Big[\partial_\nu\chi\,\partial^\nu\chi - \left(\mu^2 a^2 + \xi R a^2 - c^2\frac{a''}{a}\right)\chi^2\Big]\mathrm{d}^4x, \tag{2.5}$$

being the main source for the current study. In order to obtain the expressions describing the dynamics of this field, the Euler-Lagrange equations in eq. (1.4) will be used, resulting in the generalized Klein-Gordon equation

$$\left[\partial_\nu\partial^\nu + \mu^2_{\text{eff}}(t)\right]\chi = 0, \qquad \mu^2_{\text{eff}}(t) = \left(\mu^2 + \xi R\right)a^2 - \frac{a''}{ac^2}. \tag{2.6 a,b}$$

Solutions of the eq. (2.6) are dependent on an integration constant related to the momentum $\mathbf{k}$, and are of the form

$$\chi_{\mathbf{k}}(x) = \alpha_{\mathbf{k}}v_{\mathbf{k}}(\eta)e^{-i\mathbf{k}\mathbf{x}\hbar^{-1}} + \bar{\alpha}_{\mathbf{k}}\bar{v}_{\mathbf{k}}(\eta)e^{i\mathbf{k}\mathbf{x}\hbar^{-1}}, \tag{2.7}$$

and upon substitution in (2.6), one gets the following differential equation

$$v''_{\mathbf{k}}\hbar^2 + \omega^2_{\mathbf{k}}(\eta)\,v_{\mathbf{k}} = 0 \tag{2.8}$$

where the dispersion relation $\omega_{\mathbf{k}}(\eta)$ is defined as,

$$\omega_{\mathbf{k}}^2(\eta) = \mathbf{k}^2 + \hbar^2\mu_{\text{eff}}^2(\eta) = \mathbf{k}^2 + \left(m^2c^2 + \xi\hbar^2 R\right) a^2 - \hbar^2 \frac{a''}{ac^2}. \tag{2.9}$$

Solving eq. (2.8) will in turn give the form of the set of solutions $\{\chi_{\mathbf{k}}\}$ needed to describe the general expression of $\chi(x)$; since we are currently considering general expansion parameters and curvature scalars, the following computations will be made using a general set of $v_{\mathbf{k}}$ functions. This functions nevertheless have some interesting properties, such as being capable of a choice of normalization, and a constant of motion: the imaginary part of $v_{\mathbf{k}}\bar{v}'_{\mathbf{k}}$. Lets check the last statement

$$\frac{\partial}{\partial\eta}\text{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}}) = \frac{\partial}{\partial\eta}\left(\frac{v_{\mathbf{k}}\bar{v}'_{\mathbf{k}} - \bar{v}_{\mathbf{k}}v'_{\mathbf{k}}}{2i}\right) = \frac{v_{\mathbf{k}}\bar{v}'_{\mathbf{k}} - \bar{v}_{\mathbf{k}}v''_{\mathbf{k}}}{2i} = 0 \tag{2.10}$$

last step is result from dispersion relation. Since the functions $v_{\mathbf{k}}$ are capable to a choice in normalization, we will choose a set of solutions of eq. (2.8) such that $\text{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}})$ its independent of the momentum $\mathbf{k}$, and equal for all modes, this constant of motion will simply be defined as

$$\text{Im}(v\bar{v}') \equiv \text{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}}), \quad \forall\,\mathbf{k}. \tag{2.11}$$

The most general solution $\chi(x)$ of equation eq. (2.6) can be written as a Fourier mode expansion

$$\chi(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\left[a_{\mathbf{k}}v_{\mathbf{k}}(\eta)e^{-i\mathbf{k}\mathbf{x}\hbar^{-1}} + \bar{a}_{\mathbf{k}}\bar{v}_{\mathbf{k}}(\eta)e^{i\mathbf{k}\mathbf{x}\hbar^{-1}}\right]. \tag{2.12}$$

2.2 Quantization

After promoting the fields to operators, and imposing commutation relations, one gets the following rules for the creation and annihilation operators from equations 10

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = \frac{(2\pi\hbar)^3\hbar c}{2\text{Im}(v\bar{v}')}\delta^3(\mathbf{k}-\mathbf{q}), \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = 0, \tag{2.13 a-c}$$

to be able to verify the value of the proportional factor in 2.13.a, lets compute the commutator between $\hat{\chi}$ and $\hat{\Pi}$, i.e.

$$\begin{aligned}\left[\hat{\chi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})\right] = \frac{1}{c}\int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^6}\Big\{[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}]\, v_{\mathbf{k}}v'_{\mathbf{q}}e^{-i(\mathbf{kx}+\mathbf{qy})\hbar^{-1}} + \left[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}\right]\bar{v}_{\mathbf{k}}\bar{v}'_{\mathbf{q}}e^{-i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}} + \\ + \left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}\right] v_{\mathbf{k}}\bar{v}'_{\mathbf{q}}e^{-i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}} - \left[\hat{a}_{\mathbf{q}}, \hat{a}^\dagger_{\mathbf{k}}\right]\bar{v}_{\mathbf{k}}v'_{\mathbf{q}}e^{i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}}\Big\}\end{aligned} \tag{2.14}$$

using expresions 2.13 and considering that the proportional factor of 2.13.a to be α, previous expression simplify to the following one,

$$\left[\hat{\chi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})\right] = \frac{\alpha}{c}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^6} 2i\text{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}})e^{-i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}}; \tag{2.15}$$

since $\text{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}})$ was considered to be momentum independent, this implies that

$$\left[\hat{\chi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})\right] = i\frac{2\alpha\text{Im}(v\bar{v}')}{c(2\pi\hbar)^3}\delta^3(\mathbf{x}-\mathbf{y}), \tag{2.16}$$

and, from equation 1.42.a one can solve for α, resulting in the value present in equation 2.13.

The next step in the quantization procedure is to obtain the Hamiltonian $\hat{\mathcal{H}}$ that spans the Fock space; to do so, we use the definition in equation 1.10 alongside the energy momentum tensor 1.22.

As a simplification, lets consider a minimally coupled theory, i.e. $\xi = 0$; then the Hamiltonian will be

$$\hat{\mathcal{H}}(t) = \int \frac{c}{2}\left[\hat{\Pi}^2 + \left(\nabla\hat{\chi}\right)^2 + \mu_{\text{eff}}^2(t)\hat{\chi}^2\right] \mathrm{d}^3\mathbf{x}. \tag{2.17}$$

Substitution of the general expression of $\hat{\chi}$ (from equation 2.12) and $\hat{\Pi}$ (remembering that $\Pi \equiv \partial_0\chi$) one will get the following expansion,

$$\hat{\mathcal{H}} = \frac{c}{2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\left[\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}F_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\bar{F}_{\mathbf{k}} + \left(2\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\delta^3(\mathbf{0})\right)E_{\mathbf{k}}\right], \tag{2.18}$$

where the functions $F_{\mathbf{k}}(t)$ and $E_{\mathbf{k}}(t)$ are defined as

$$F_{\mathbf{k}}(t) = \left(\frac{1}{\hbar c}\right)^2\left[\hbar^2 v'^2_{\mathbf{k}} + \omega^2_{\mathbf{k}}(t)\,c^2 v^2_{\mathbf{k}}\right], \quad E_{\mathbf{k}}(t) = \left(\frac{1}{\hbar c}\right)^2\left[\hbar^2\left|v'_{\mathbf{k}}\right|^2 + \omega^2_{\mathbf{k}}(t)\,c^2\left|v_{\mathbf{k}}\right|^2\right]. \tag{2.19 a,b}$$

2.3 Instantaneous Vacuum State

Note that the only way a vacuum state $|0\rangle$ could remain an eigenstate of the Hamiltonian 2.18 at all times, would be if $F_{\mathbf{k}}(t) = 0$, at all times, i.e.

$$F_{\mathbf{k}}(t) = \left(\frac{1}{\hbar c}\right)^2\left[\hbar^2 v'^2_{\mathbf{k}} + \omega^2_{\mathbf{k}}(t)\,c^2 v^2_{\mathbf{k}}\right] = 0, \tag{2.20}$$

solving for $v_{\mathbf{k}}$ gives the following expression

$$v_{\mathbf{k}}(t) = \mathrm{C}\exp\left[\pm\frac{c}{i\hbar}\int \omega_{\mathbf{k}}(\eta)\,\mathrm{d}\eta\right], \tag{2.21}$$

which is not compatible with 2.8 except for a time independent dispersion relation $\omega_{\mathbf{k}}$.

The last result implies that, at different times, one can (and should) define different vacuum states; and thus, we will define the *instantaneous vacuum state* $|_{(t_0)}0\rangle$ as the one that at some time t_0 will minimize the energy density. Since all possible states are related by Bogolyubov transformations, finding the instantaneous vacumm state is the same as finding the set of functions $v_{\mathbf{f}}$ that are simultaneously solution of 2.8 and minimize

$$\langle_{(t_0)}0|\hat{\mathcal{H}}(t_0)|_{(t_0)}0\rangle = \rho(t_0)\delta^3(\mathbf{0}) = \frac{\hbar c^2\,\delta^3(\mathbf{0})}{4\mathrm{Im}(v\bar{v}')}\int \mathrm{d}^3\mathbf{k}\,E_{\mathbf{k}} \tag{2.22}$$

To minimise the energy density of the vacuum state is to find the set of functions $v_{\mathbf{k}}$ that minimise $E_{\mathbf{k}}$. Suppose that $v_{\mathbf{k}}$ can be written as

$$v_{\mathbf{k}} = r_{\mathbf{k}}e^{i\alpha_{\mathbf{k}}} \tag{2.23}$$

since $\mathrm{Im}(v\bar{v}')$ was constant through time

$$r^2_{\mathbf{k}}\alpha'_{\mathbf{k}} = -\mathrm{Im}(v\bar{v}') \tag{2.24}$$

this means

$$E_{\mathbf{k}} = \left(\frac{1}{\hbar c}\right)^2\left\{\hbar^2\left[r'^2_{\mathbf{k}} + \mathrm{Im}^2\left(v\bar{v}'\right)\frac{1}{r^2_{\mathbf{k}}}\right] + \omega^2_{\mathbf{k}}\,c^2 r^2_{\mathbf{k}}\right\} \tag{2.25}$$

the minimum of this function must fulfil $r'_{\mathbf{k}}(t_0) = 0$. Now, if $\omega^2_{\mathbf{k}}(t_0)$ and $\mathrm{Im}(v\bar{v}')$ have the same sign, the minimum of $E_{\mathbf{k}}$ happens when $r_{\mathbf{k}}(t_0) = \left[\frac{\hbar\mathrm{Im}(v\bar{v}')}{\omega_{\mathbf{k}}(t_0)\,c}\right]^{1/2}$.

If there is a minimum, then

$$v_{\mathbf{k}}(t_0) = \left[\frac{\hbar\mathrm{Im}(v\bar{v}')}{\omega_{\mathbf{k}}(t_0)\,c}\right]^{1/2} e^{i\alpha_{\mathbf{k}}(t_0)} \qquad v'_{\mathbf{k}}(t_0) = -c\frac{\omega_{\mathbf{k}}(t_0)}{ih}\,v_{\mathbf{k}}(t_0) \tag{2.26}$$

under these functions,

$$E_{\mathbf{k}}(t_0) = 2\frac{\mathrm{Im}(v\bar{v}')}{\hbar c}\omega_{\mathbf{k}}(t_0) \qquad F_{\mathbf{k}}(t_0) = 0 \tag{2.27}$$

meaning

$$\hat{\mathcal{H}}(t_0) = \mathrm{Im}(v\bar{v}')\frac{1}{\hbar}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\left(2\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\delta^3(\mathbf{0})\right)\omega_{\mathbf{k}}(t_0) \tag{2.28}$$

which is equivalent to the standard Hamiltonian for a scalar field without the presence of gravity.

Bogolyubov Transformation The expression of the field χ at two different times must be related to a Bogoliubov transformation, with coefficients

$$\alpha_{\mathbf{kp}} = \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\langle \chi_{\mathbf{k}}(t_0),\, \chi_{\mathbf{p}}(t)\rangle \qquad \beta_{\mathbf{kp}} = -\frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\langle \chi_{\mathbf{k}}(t_0),\, \bar{\chi}_{\mathbf{p}}(t)\rangle \tag{2.29}$$

since the field can be written as $\chi_{\mathbf{k}} = v_{\mathbf{k}}e^{i\mathbf{kx}/\hbar}$ from the expression of the inner product one can see that

$$\alpha_{\mathbf{kp}} \propto \delta^3(\mathbf{k}-\mathbf{p}) \qquad \beta_{\mathbf{kp}} \propto \delta^3(\mathbf{k}+\mathbf{p}) \tag{2.30}$$

therefore it is possible to write

$$v_{\mathbf{k}}(t) = \alpha_{\mathbf{k}}v_{\mathbf{k}}(t_0) + \beta_{\mathbf{k}}\bar{v}_{\mathbf{k}}(t_0) \tag{2.31}$$

where, recalling that $\mathrm{Im}(v\bar{v}')$ is constant through time,

$$|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1 \tag{2.32}$$

To obtain the value of $\langle_{(t_0)}0|\hat{\mathcal{H}}(t)|_{(t_0)}0\rangle$ lets first compute

$$\langle_{(t_0)}0|\hat{\mathcal{N}}^{(a)}_{\mathbf{k}}(t)|_{(t_0)}0\rangle = \langle_{(t_0)}0|\hat{a}^\dagger_{\mathbf{k}}(t)\hat{a}_{\mathbf{k}}(t)|_{(t_0)}0\rangle = \left|\beta_{\mathbf{k}}\right|^2\frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\delta^3(\mathbf{0}) \tag{2.33}$$

therefore

$$\langle_{(t_0)}0|\hat{\mathcal{H}}(t)|_{(t_0)}0\rangle = \delta^3(\mathbf{0})\int \mathrm{d}^3\mathbf{k}\left(\frac{1}{2} + \left|\beta_{\mathbf{k}}\right|^2\right)c\,\omega_{\mathbf{k}}(t) \geq \langle_{(t_0)}0|\hat{\mathcal{H}}(t_0)|_{(t_0)}0\rangle \tag{2.34}$$

meaning, if $\beta_{\mathbf{k}} \neq 0$ for all $\mathbf{k}$ then, at a time $t > t_0$ the energy density will be different in relation to the original vacuum.

2.4 Case of Study: de Sitter Universe

The de Sitter Universe is a flat FLRW metric with no matter or radiation, but it does have a a positive cosmological constant Λ. Per the Friedmann equations,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho + \Lambda c^2}{3} - \frac{\kappa c^2}{a^2}, \quad (\rho = \kappa = 0) \tag{2.35}$$

the expansion parameter $a(t)$ will be equal to

$$a(t) = a_1 e^{H_\Lambda t} + a_2 e^{-H_\Lambda t}\,, \tag{2.36}$$

where $H_\Lambda = \sqrt{\Lambda c^2/3}$ it's the Hubble-Lemaître constant. The most common choice is to set $a_2 = 0$, and thus, consider an always expanding universe.

The line element describing the motion of particles through this universe is given by

$$\mathrm{d}l^2 = c^2\mathrm{d}t^2 - a^2(t)\mathrm{d}\mathbf{x}^2, \tag{2.37}$$

more commonly expressed as a function of the conformal time η defined as [1]

$$\eta \equiv -\int_t^\infty \frac{\mathrm{d}t'}{a(t')} = -\frac{1}{a_1\,H_\Lambda}e^{-H_\Lambda t} = -\frac{1}{a(t)H_\Lambda}; \tag{2.38}$$

and thus, the line element will be given by

$$\mathrm{d}l^2 = \frac{1}{H_\Lambda \eta^2}\left[c^2\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2\right], \tag{2.39}$$

which has the same form as 2.3.

Now, from equation 2.2 one can compute the curvature scalar $= H_\Lambda^2 12/c^2$, and thus, the dispersion relation 2.9 will be given by the following expression

$$\omega_{\mathbf{k}}^2(\eta) = \mathbf{k}^2 + \left[\left(\frac{mc^2}{H_\Lambda}\right)^2 + 2\,(6\xi - 1)\,\hbar^2\right]\frac{1}{c^2\eta^2}, \tag{2.40}$$

from which one might obtain the solutions of the differential equation 2.8; to do so it is best to use the following change of variables,

$$s \equiv -k\eta, \qquad\qquad v_{\mathbf{k}} \equiv \sqrt{|s|}f(s), \tag{2.41 a,b}$$

obtaining the Bessel's differential equation

$$s^2\frac{\mathrm{d}^2 f}{\mathrm{d}s^2} + s\frac{\mathrm{d}f}{\mathrm{d}s} + \left(s^2 - \nu^2\right) f(s) = 0, \tag{2.42}$$

with a parameter

$$\nu^2 \equiv (3 - 16\xi)\,\frac{3}{4}\frac{\hbar^2}{c^2} - \left(\frac{mc}{H_\Lambda}\right)^2. \tag{2.43}$$

The solutions of such differential equation are given by the so called Bessel functions of the first kind $J_\nu(s)$ and $Y_\nu(s)$; therefore the $v_{\mathbf{k}}$ functions can be deduced as

$$f(s) = AJ_\nu(s) + BY_\nu(s) \implies v_{\mathbf{k}}(\eta) = \sqrt{k|\eta|}\left[A_{\mathbf{k}}J_\nu(k|\eta|) + B_{\mathbf{k}}Y_\nu(k|\eta|)\right] \tag{2.44}$$

For $\nu^2 \geq 0$, both J_ν and Y_ν will be real functions, but for $\nu < 0$ they will be complex functions [4]; for simplicity, we will focus on the $\nu \geq 0$ case. In addition, consider that the choice of $\mathrm{Im}(v\bar{v}')$ will translate in a restriction on the relation $B_{\mathbf{k}} = B_{\mathbf{k}}(A_{\mathbf{k}})$.

[1] Considering $a_2 \neq 0$ one would obtain that

$$\eta = \frac{\arctan\left(\sqrt{\frac{a_2}{a_1}}e^{-H_\Lambda t}\right)}{H_\Lambda\sqrt{a_1 a_2}}$$

3 The Unruh Effect

3.1 Accelerated Observers and Unruh Temperature

Flat $1+1$ spacetime,

$$\alpha^{\mu} \equiv \frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\tau^2} \tag{3.1}$$

$$\alpha^2 = \left(c\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2}\right)^2 - \left(\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2}\right)^2 \tag{3.2}$$

$$t(\tau) = t_0 + t_1\tau \pm \frac{c}{\alpha}\sinh\left(\frac{\alpha\tau}{c}\right), \quad x(\tau) = x_0 + x_1\tau \pm \frac{c^2}{\alpha}\cosh\left(\frac{\alpha\tau}{c}\right). \tag{3.3 a,b}$$

$t_0, t_1, x_0, x_1 \in \mathbb{R}$ but since $c^2\mathrm{d}\tau^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2$ one deduces that $ct_1 = x_1 = 0$, for simplicity we consider some coordinates in which $ct_0 = x_0 = 0$.

$$x^2 - c^2t^2 = \frac{c^2}{\alpha^2} \tag{3.4}$$

- $x > c|t|$

$$t(\eta,\, \xi) \equiv \frac{c}{\alpha}\sinh\left(\frac{\alpha\eta}{c}\right)e^{\alpha\xi/c^2}, \tag{3.5a}$$

$$x(\eta,\, \xi) \equiv \frac{c^2}{\alpha}\cosh\left(\frac{\alpha\eta}{c}\right)e^{\alpha\xi/c^2}, \tag{3.5a b}$$

- $x < c|t|$

$$t(\tilde\eta,\, \tilde\xi) \equiv -\frac{c}{\alpha}\sinh\left(\frac{\alpha\tilde\eta}{c}\right)e^{\alpha\tilde\xi/c^2}, \tag{3.6a}$$

$$x(\tilde\eta,\, \tilde\xi) \equiv -\frac{c^2}{\alpha}\cosh\left(\frac{\alpha\tilde\eta}{c}\right)e^{\alpha\tilde\xi/c^2}, \tag{3.6a b}$$

$x^2 - c^2t^2 = \frac{c^2}{\alpha^2}e^{2\alpha\xi/c^2}$

$$c^2\mathrm{d}\tau^2 = e^{\alpha\xi/c^2}\left[c^2\mathrm{d}\eta^2 - \mathrm{d}\xi^2\right] \tag{3.7}$$

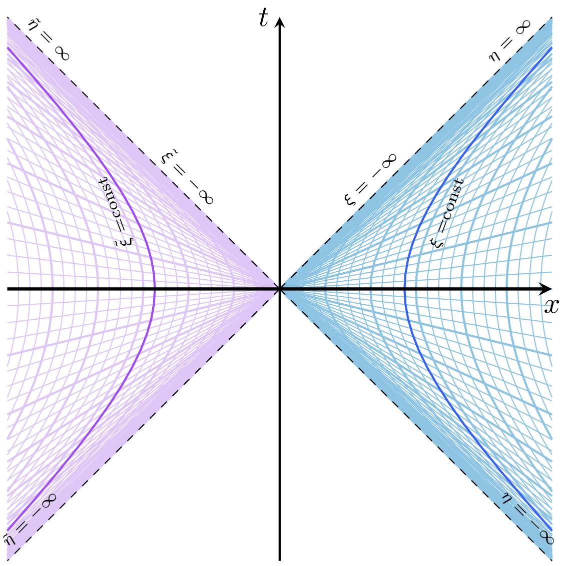

Figure 3.1: Rindler Coordinates (Left and Right charts).

Now consider a massless and minimally coupled scalar field described by the action 1.17, the equation of motion would be [1]

$$e^{-2\alpha\xi/c^2}\left[\partial_{c\eta}^2 - \partial_{\xi}^2\right]\phi = 0 \tag{3.8}$$

use of null coordinates $u \equiv c\eta - \xi$ $v \equiv c\eta + \xi$

$$\phi^u_\omega \equiv e^{i\omega u\,\hbar^{-1}}, \quad \phi^v_\omega \equiv a^v_\omega e^{i\omega v\,\hbar^{-1}}, \quad \phi^{\tilde u}_\omega \equiv a^{\tilde u}_\omega e^{i\omega\tilde u\,\hbar^{-1}}, \quad \phi^{\tilde v}_\omega \equiv a^{\tilde v}_\omega e^{i\omega\tilde v\,\hbar^{-1}}, \tag{3.9 a-d}$$

[1] As it turns out, for $1+1$ the conformal invariance is obtained by a massless minimally coupled theory.

$U \equiv ct - x$ and $V \equiv ct + x$

$$U(u,\, v) = -\frac{c^2}{\alpha} e^{-\alpha u/c^2}, \qquad V(u,\, v) = \frac{c^2}{\alpha} e^{\alpha v/c^2}, \tag{3.10a}$$

$\phi \equiv \phi_u + \phi_v$

$$\phi_u \equiv \int_0^\infty \frac{\mathrm{d}\omega}{(2\pi\hbar)\sqrt{2\omega}} \left\{\Theta(-U)\left[a_\omega^u \phi_\omega^u + \bar{a}_\omega^u \bar{\phi}_\omega^u\right] + \Theta(U)\left[a_\omega^{\tilde{u}} \phi_\omega^{\tilde{u}} + \bar{a}_\omega^{\tilde{u}} \bar{\phi}_\omega^{\tilde{u}}\right]\right\} \tag{3.11}$$

$$\beta_{\Omega\omega} \propto \langle \Theta(-U)\phi_\Omega^u + \Theta(U)\phi_\Omega^{\tilde{u}}, \bar{\phi}_\omega^U \rangle = \frac{1}{2\hbar^2}\sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{\infty} \exp\left\{i\left[\omega U(u) + \Omega u\right]\hbar^{-1}\right\} \mathrm{d}u \tag{3.12}$$

$z \equiv i\left({}^{\omega c^2}/_{\alpha}\right) \exp\left({}^{au}/_{c^2}\right)$

$$\beta_{\Omega\omega} \propto \frac{c^2}{\alpha\hbar^2}\Gamma\left(i\frac{\omega c^2}{\alpha\hbar}\right)\sqrt{\frac{\Omega}{\omega}}\left(\frac{c^2\omega}{\alpha\hbar}\right)^{-i\frac{\omega c^2}{\alpha\hbar}} e^{-\frac{\pi\omega c^2}{2\alpha\hbar}} \tag{3.13}$$

$$N_\omega \equiv \langle {}_U0|\hat{a}_\omega^\dagger \hat{a}_\omega|{}_U0\rangle = \int_0^\infty \frac{\mathrm{d}\omega}{(2\pi\hbar)\sqrt{2\omega}}|\beta_{\omega\omega}| \propto \frac{1}{e^{\frac{2\pi\omega c}{\alpha\hbar}} - 1}\delta(0) \tag{3.14}$$

$$T_0 \equiv \frac{\alpha\hbar}{2\pi c k_B} \tag{3.15}$$

This is not the temperature that would be measured by an accelerated Rindler observer; consider the *conserved* energy E_0 measured by an observed in an stationary gravitational field, which relates to the energy measured by another observer by $E_0 = \sqrt{g_{00}}\, E$, since the thermodynamical relation for the energy and the temperature comes from the entropy ${}^1/_{T_0} = {}^{\partial S}/_{E_0}$, then the proper temperature will be $T = \sqrt{g_{00}}\, T_0$; this is known as Tolman's law, and results in

$$T_{\text{Unruh}} = \sqrt{g_{00}}\, T_0 = \left(\alpha e^{-\alpha/c^2 \xi}\right)\frac{\hbar}{2\pi c k_B} \equiv \frac{a\hbar}{2\pi c k_B} \tag{3.16}$$

3.2 Application to Black Holes: Hawking Radiation

$$c^2\mathrm{d}\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2\mathrm{d}t^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right) \tag{3.17}$$

$R_S \equiv {}^{2GM}/_{c^2}$

consider a two dimensional black hole

$$c^2\mathrm{d}\tau^2 = \left(1 - \frac{R_S}{r}\right)c^2\mathrm{d}t^2 - \left(1 - \frac{R_S}{r}\right)^{-1}\mathrm{d}r^2 \tag{3.18}$$

tortoise coordinate

$$\mathrm{d}r^* \equiv \left(1 - \frac{R_S}{r}\right)^{-1}\mathrm{d}r \tag{3.19}$$

$$c^2\mathrm{d}\tau^2 = \left[1 - \frac{R_S}{r\left(r^*\right)}\right]\left[c^2\mathrm{d}t^2 - \mathrm{d}r^*\right] \tag{3.20}$$

[2]

$$T_{\text{Hawking}} = \frac{\hbar c^3}{8\pi G M k_B} \tag{3.21}$$

[2]From a non quantum point of view, one could consider the acceleration experienced by an observer situated at $r = R_S$ without angular momentum, which will be equal to $a = {}^{GM}/_{R_S^2} = {}^{c^4}/_{4GM}$

$$\mathrm{d}S = \left(\frac{\partial S}{\partial M}\right)\mathrm{d}M + \left(\frac{\partial S}{\partial J}\right)\mathrm{d}J + \left(\frac{\partial S}{\partial Q}\right)\mathrm{d}Q \tag{3.22}$$

$Q = J = 0$ $E = Mc^2$

$$\mathrm{d}S = \frac{c^2}{T_{\text{Hawking}}}\mathrm{d}M \implies S = \frac{4\pi G k_B}{\hbar c}M^2 \tag{3.23}$$

Bekenstein-Hawking [3]

Since black holes are the perfect example of a black body, we could use the Stefan-Boltzmann law for the luminosity L

$$L = -c^2\frac{\mathrm{d}M}{\mathrm{d}t} = \epsilon A\sigma T^4 \tag{3.24}$$

where $\sigma = {}^{\pi^2 k_B^4}/_{60\hbar^3 c^2}$ its the Stefan-Boltzmann constant, ϵ is a factor of correction for possible greybody effects

$$M(t) = M_0\left(1 - \frac{t}{t_{BH}}\right)^{1/3}, \quad t_{BH} \equiv 5120\frac{\pi G^2}{\epsilon\hbar c^4}M_0^3 \tag{3.25}$$

stellar black holes (the most numerous type) have masses around $100M_\odot$, meaning a time of evaporation of about $\sim 2.1\cdot 10^{73}$ years.

TBD

[3] In reality, they presented their result not as a function of the mass M, but as a function of the surface area A, and thus, the proper Bekenstein-Hawking entropy would be $S_{BH} = {}^{c^3 k_B}/_{\hbar G}A$.

4 Energy Momentum Tensor Renormalization

TBD

4.1 The Conformal Anomaly

TBD

5 Final Discussions

TBD

Bibliography

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Scalar field in Minkowski background

$$\eta_{\mu\nu} = \mathrm{diag}\,(+,\,-,\,-,\,-) \tag{1}$$

$$S[\phi] = \int \frac{1}{2}\Big[\partial_\nu\phi\,\partial^\nu\phi - \mu^2\phi^2\Big]\,\mathrm{d}^4x \tag{2}$$

$$\left(\partial_\nu\partial^\nu - \mu^2\right)\phi = 0 \tag{3}$$

$$\phi_{\mathbf{k}} = a_{\mathbf{k}}e^{ikx\,\hbar^{-1}} + \bar{a}_{\mathbf{k}}e^{-ikx\,\hbar^{-1}} \tag{4}$$

$$k_\nu k^\nu = \hbar^2\mu^2 \tag{5}$$

$$\phi(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3 2k_0}\phi_{\mathbf{k}} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3 2k_0}\left(a_{\mathbf{k}}e^{ikx\,\hbar^{-1}} + \bar{a}_{\mathbf{k}}e^{-ikx\,\hbar^{-1}}\right) \tag{6}$$

$\frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3 2k_0}$ is a Lorentz-invariant measure. It is convenient to redefine $a_{\mathbf{k}} \to a_{\mathbf{k}}(2k_0)^{-1/2}$

$$\phi(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3\sqrt{2k_0}}\left(a_{\mathbf{k}}e^{ikx\,\hbar^{-1}} + \bar{a}_{\mathbf{k}}e^{-ikx\,\hbar^{-1}}\right) \tag{7}$$

$$\begin{aligned} \phi(x) &\longrightarrow \hat{\phi}(x), \qquad & \Pi(x) &\longrightarrow \hat{\Pi}(x), \\ a_{\mathbf{k}} &\longrightarrow \hat{a}_{\mathbf{k}}, \qquad & \bar{a}_{\mathbf{k}} &\longrightarrow \hat{a}^\dagger_{\mathbf{k}}, \end{aligned}$$

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3\sqrt{2k_0}}\left(\hat{a}_{\mathbf{k}}e^{ikx\,\hbar^{-1}} + \hat{a}^\dagger_{\mathbf{k}}e^{-ikx\,\hbar^{-1}}\right) \tag{8}$$

$$\Big[\hat{\phi}(\mathbf{x}),\,\hat{\Pi}(\mathbf{y})\Big] = i\hbar\,\delta^3\left(\mathbf{x}-\mathbf{y}\right) \qquad \Big[\hat{\phi}(\mathbf{x}),\,\hat{\phi}(\mathbf{y})\Big] = \Big[\hat{\Pi}(\mathbf{x}),\,\hat{\Pi}(\mathbf{y})\Big] = 0. \tag{9 a-c}$$

$$\Big[\hat{a}_{\mathbf{k}},\,\hat{a}^\dagger_{\mathbf{q}}\Big] = (2\pi\hbar)^3\,\hbar^2\delta^3\left(\mathbf{k}-\mathbf{q}\right), \qquad [\hat{a}_{\mathbf{k}},\,\hat{a}_{\mathbf{q}}] = \Big[\hat{a}^\dagger_{\mathbf{k}},\,\hat{a}^\dagger_{\mathbf{q}}\Big] = 0. \tag{10 a-c}$$

$$\begin{aligned}\Big[\hat{\phi}(\mathbf{x}),\,\hat{\Pi}(\mathbf{y})\Big] = -\frac{1}{i\hbar}\int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{2(2\pi\hbar)^6}\sqrt{\frac{q_0}{k_0}}\Big\{\left[a_{\mathbf{k}},a_{\mathbf{q}}\right]e^{i(\mathbf{kx}+\mathbf{qy})\hbar^{-1}} - \Big[a_{\mathbf{k}},a^\dagger_{\mathbf{q}}\Big]e^{i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}} - \\ - \Big[a_{\mathbf{q}},a^\dagger_{\mathbf{k}}\Big]e^{-i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}} - \Big[a^\dagger_{\mathbf{k}},a^\dagger_{\mathbf{q}}\Big]e^{-i(\mathbf{kx}+\mathbf{qy})\hbar^{-1}}\Big\} = \frac{\alpha}{(2\pi\hbar)^3\hbar^2}i\hbar\delta^3\left(\mathbf{x}-\mathbf{y}\right)\end{aligned} \tag{11}$$

$$\hat{\mathcal{H}} = \int\left(\hat{\Pi}\partial_0\hat{\phi} - \hat{\mathcal{L}}\right)\mathrm{d}^3\mathbf{x} = \int \frac{c}{2}\left[\hat{\Pi}^2 + \left(\nabla\hat{\phi}\right)^2 + \mu^2\hat{\phi}^2\right]\mathrm{d}^3\mathbf{x} \tag{12}$$

Units

- $[S] = [\hbar]$
- $[a] = [\xi] = 1$
- $[\mu] = [L]^{-1}$
- $[R] = [L]^{-2}$
- $[\phi] = [\chi] = [\hbar]^{1/2}[L]^{-1}$
- $[\Pi] = [\hbar]^{1/2}[L]^{-2}$
- $[a_{\mathbf{k}}] = [\hbar]^{1/2}[L]^{2}$

Questions & To-Do

1 Questions

- How do you know that there is a set of solutions of Klein Gordon such that the inner product fulfils the given results?
- It's the Hamiltonian well defined?
- Can you always write a FLRW metric as a flat one with a coordinate change?
- Covariant derivatives, spin connection.

2 To-Do

- Minkowski scalar field.
- Move Units appendix to conventions.
- Add references! (e.g. covariant derivative).
- Particle detector and notion of particle (Unruh-DeWitt, ...). Maybe after Scalar field quantization 1.3 section.