PROBLEMS OF QUANTUM FIELD THEORIES IN CURVED SPACETIMES

A MASTER THESIS

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Abstract

Quantum Field Theory is the fundamental theoretical framework of the Standard Model of elementary particles. This theory is formulated in a Minkowski space-time. However, the actual space-time metric is never of that type. On Earth, even at short distances, the metric is affected by both the force of the Earth's gravity and the solar force, but especially by the acceleration of the Earth's motion. At large distances the cosmological data lead one to think that, on average, the metric is of the Friedmann-Lemaitre-Roberson-Walker type.

The analysis of the quantization of fields in the presence of gravitational fields involves a number of theoretical issues that we intend to explore in this thesis; meaning how QFT can be adapted to introduce a gravitational background. The key aspects treated throughout this thesis include the mathematical background that will be used; scalar fields in expanding universes; thermal aspects of non inertial observers; and the renormalization problems regarding gravitational dynamics.

Contents

Conventions

The chosen convention for the metric signature will be $(+, -, -, -)$ as in [6] and most literature on Particle Physics. Common conventions and nomenclatures in Mathematics and Physics are used throughout the text, some of which are considered to be relevant:

x^μ, x	four-vector
$\mathbf{x}$	spacial vector
$g_{\mu\nu}$	general spacetime metric
$\eta_{\mu\nu}$	Minkowski spacetime metric
g	determinant of $g_{\mu\nu}$
∇_μ	covariant derivative
$S[\phi]$	action functional of a field ϕ and its derivatives
$\bar{z}$	complex conjugate of z
$A^\dagger$	hermitian conjugate of A
$R^\alpha_{\ \beta\gamma\delta}$	Riemann tensor $\quad \equiv \nabla_\delta \Gamma^\alpha_{\beta\gamma} - \nabla_\delta \Gamma^\gamma_{\ \beta\alpha} + \ldots$
$f \overleftrightarrow{\nabla}_\mu g$	$\equiv f \nabla_\nu g - (\nabla_\mu f)\, g$
γ^μ	covariant Gamma matrices, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$
G, c, $\hbar$	standard universal constants, not necessarily in natural units

Other notation will be introduced as needed.

Preface

Over the last century, Physics has evolved into two separate fundamental theories that cover two widely different energy ranges. On one side of the spectrum, Quantum Field Theory has been able to model small scale phenomena and the nature of the fundamental forces that bind matter together; and on the other, General Relativity sheds light on the proper nature of gravity and the stage in which particles move. Although these two seemingly different frameworks appear to pass all observational and experimental tests, one can find a plethora of arguments to consider that a more fundamental theory of Quantum Gravity must exist. Such unifying theory has been pursued for decades, with no definitive answer; and thus, we are left with what is expected to be its low energy limit: the theory of Quantum Fields in Curved Spacetimes, which is the focus of this thesis.

This framework is the natural extension of Quantum Field Theory, which is constructed over a Minkowski background. Such a premise may be acceptable in current experimental research such as particle accelerators, but not for astrophysical and cosmological models, where the effects of gravity do not allow for such approximation. Therefore, classical dynamical backgrounds are considered, meaning that the matter fields are to be quantized but the gravitational field itself is not. It is expected that said transition will result in new effects to be observationally tested (although experiments might be out of our technological scope).

After a formal introduction in the mathematical models that will be used throughout this thesis, we will present in each of the following chapters some new phenomenology particular to this area of Physics. The second chapter deals with the theory of scalar fields in expanding backgrounds; this cosmologically relevant scenario results in the breakage of what we understand as a particle, confronting Rabelais' *Natura abhorret vacuum* with the realization of an infinite number of possible vacuum states, all equally valid. The third chapter explores the difference of vacuum measures between two non inertial observers, which yields a radiation capable of evaporating black holes. Finally, the last chapter will dive into the semiclassical approach of gravitational dynamics, explaining how quantum fields might affect gravity over cosmological scales.

1 Introduction to QFT in Curved Spacetimes

Before diving into the thrilling phenomenology of quantum fields over a classical background, one needs to have a proper understanding of the actors in play and the basic set of equations governing their dynamics. In the present chapter we focus on that matter, presenting the general theory of classical fields, , which will be used later on, alongside the newly constructed covariant actions, to present the quantization of the free scalar field.

1.1 Matter-Gravity Action

Let us consider a dynamic universe consisting of dark energy characterized by a cosmological constant Λ and some material content described by a Lagrangian density $\mathcal{L}_{\mathrm{M}}$. The action associated with such a system would be

$$S = \int \left[\frac{1}{2\kappa}\left(R - 2\Lambda\right) + \mathcal{L}_{\mathrm{M}}\right]\sqrt{-g}\,\mathrm{d}^4x, \tag{1.1}$$

where $\kappa \equiv \frac{8\pi G}{c^4}$ is known as the Einstein gravitational constant.

The equations that would describe the classical dynamics of the system can be obtained by variations of the action presented in equation 1.1 alongside the stationary-action principle, which states that the path taken by the system is the one for which the action is stationary, meaning that $\delta S = 0$, δS being a small variation of the action. The field dynamics are given by the Euler-Lagrange equations, obtained from the Lagrangian density of said fields, depending on some set of fields $\{\phi^\alpha(x)\}$ and their covariant derivatives, i.e.

$$\mathcal{L}_{\mathrm{M}} = \mathcal{L}_{\mathrm{M}}\big[\phi^\alpha(x),\, \nabla_\mu\phi^\alpha(x)\big]. \tag{1.2}$$

Said equations can be derived from the variations of the action S with respect of ϕ^α

$$\delta S = \int \left[\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \phi^\alpha}\delta\phi^\alpha + \frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial\left(\nabla_\mu\phi^\alpha\right)}\nabla_\mu\left(\delta\phi^\alpha\right)\right]\sqrt{-g}\,\mathrm{d}^4x. \tag{1.3}$$

After applying the generalized Gauss Theorem, the stationary-action principle leads to the aforementioned Euler-Lagrange equations

$$\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \phi^\alpha} - \nabla_\mu\left[\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial\left(\nabla_\mu\phi^\alpha\right)}\right] = 0. \tag{1.4}$$

On the other hand, variations of the action S with respect to the inverse metric $(g^{\mu\nu})$ results in

$$\delta S = \int \left[\frac{1}{2\kappa}\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{2\kappa}\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} - \frac{\Lambda}{\kappa}\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta\mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}} + \frac{\mathcal{L}_{\mathrm{M}}}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x; \tag{1.5}$$

and again, by imposing the stationary principle $\delta S = 0$ and considering that (up to pure derivative terms)

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu}, \qquad \frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}g_{\mu\nu}, \tag{1.6 a,b}$$

one obtains the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -2\frac{8\pi G}{c^4}\left(\frac{\delta\mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}} - \frac{1}{2}\mathcal{L}_{\mathrm{M}}g_{\mu\nu}\right) \tag{1.7}$$

which are most commonly written in terms of the Hilbert energy-momentum tensor

$$T_{\mu\nu} \equiv \mathcal{L}_{\mathrm{M}} g_{\mu\nu} - 2\frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}} = \frac{-2}{\sqrt{-g}} \frac{\delta \left(\mathcal{L}_{\mathrm{M}}\sqrt{-g}\right)}{\delta g^{\mu\nu}}. \tag{1.8}$$

This tensor is the source of the spacetime curvature, and must not be confused with Noether's energy-momentum tensor since the two are not, in general, equivalent [5], but upon integration of the corresponding conserved currents, results are the same [14]. In addition of being symmetric, the Hilbert energy-momentum tensor is covariantly conserved, i.e.

$$\nabla_\mu T^{\mu\nu} = 0; \tag{1.9}$$

this fact is of great use once the material Hamiltonian $\mathcal{H}_{\mathrm{M}}$ is defined:

$$\mathcal{H}_{\mathrm{M}} \equiv \int T^{00} c\sqrt{-g}\, \mathrm{d}^3\mathbf{x}, \tag{1.10}$$

since it will later be used to spawn the Fock space after the quantization procedure; and thus assure no energy loses will be present on the theory.

1.2 Construction of Covariant Actions

Standard QFT alongside the Standard Model of particle physics is one of (if not the) best tested theories of Physics, which is why it is not necessary to reinvent the actions that are used on it, only a small tweak is needed to make the theory general covariant; it has been previously taken for granted that the volume element will be $\sqrt{-g}\, \mathrm{d}^4x$; but there is another consideration, the derivatives cannot be simply ∂_μ since that is not (in general) covariant. To correctly define a covariant derivative ∇_μ, one must introduce two elements: the first one is the known Christoffel symbols $\Gamma^\sigma_{\alpha\beta}$ which will contract the tensorial nature of the field; and the second one is the spin connection, given by

$$\Gamma_\mu \equiv \frac{1}{2}\Sigma^{AB}\omega_{AB\mu}; \tag{1.11}$$

where Σ^{AB} are to be understood as the Lorentz generators (the uppercase Latin indexes represent sums over a Minkowski background), and $\omega_{AB\mu}$ is the so called torsion free spin connection, defined as

$$\omega_{AB\mu} \equiv e^\nu_A \left(\partial_\mu e_{B\nu} - \Gamma^\sigma_{\nu\mu} e_{B\sigma}\right). \tag{1.12}$$

The new vector fields e^μ_A are known as the tetrad formalism coefficients, defined to transform general tensors to a local flat manifold, i.e.

$$g_{\mu\nu} = e^A_\mu e^B_\nu \eta_{ab}. \tag{1.13}$$

As a last note, if the field is coupled to a vector field A_μ the covariant derivative must be redefined as $\nabla'_\mu \equiv \nabla_\mu - \frac{i}{\hbar} e A_\mu$, where e would be the coupling constant.

1.2.1 Some Basic Examples

Scalar Field

The very first example given for a classical field is usually a (real) free scalar field $\phi(x)$ with some mass m; whose dynamics are given by the following action

$$S[\phi] = \int \frac{1}{2}\Big[\partial_\nu\phi\, \partial^\nu\phi - \mu^2\phi^2 - \xi R\phi^2\Big]\sqrt{-g}\, \mathrm{d}^4x. \tag{1.14}$$

The construction of such action arises from its Minkowskian counterpart (a primary study of the standard QFT can be found in the appendix); since the field in question is scalar, the covariant derivative is simply ∂_ν, the massive term of the action is dependant on a parameter $\mu \equiv {}^{mc}/_{\hbar}$, and a term is added as a coupling to gravity (through the Ricci scalar R) with a coupling constant ξ [1].

The inclusion of such coupling is not a mere curiosity, since it's been proven [8] that a self interactive $\lambda\phi^4$ theory needs a term proportional to $R\phi^2$ to be renormalizable. Besides this, the addition of a term proportional to $R\phi^2$ adds a new symmetry to the action, since for a massless field $\mu = 0$ with a coupling constant $\xi = {}^1/_6$, the action is invariant under Weyl transformations, i.e.

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} \equiv \Omega^2(x) g_{\mu\nu}, \tag{1.15}$$

to prove this, lets first obtain the equations of motion using the Euler-Lagrange equation 1.4, resulting in the generalized Klein-Gordon equation

$$\left[\partial_\nu \partial^\nu + \mu^2 + \xi R\right]\phi = 0, \tag{1.16}$$

and then, a Weyl transformation can be made to them, considering that the field will transform as $\phi \to \tilde{\phi} = \Omega^\beta \phi$, resulting in the following expression:

$$\begin{aligned} 0 = \mu^2 \Omega^{\beta-2}\left(\Omega^2 - 1\right)\phi + 2\left(1+\beta\right)\Omega^{\beta-3}\partial^\nu \Omega \partial_\nu \phi + \\ + \left(6\xi + \beta\right)\Omega^{\beta-3}\left(\partial_\nu \partial^\nu \Omega\right)\phi + \beta(1+\beta)\Omega^{\beta-4}\partial_\nu \Omega \partial^\nu \Omega\, \phi. \end{aligned} \tag{1.17}$$

Considering a massless field, a solution of this equation corresponds to the following values:

$$\beta = -1, \qquad \xi = \frac{1}{6}, \tag{1.18 a,b}$$

proving the Weyl invariance for such a scenario.

From its definition in equation 1.8 and the action 1.14, one can obtain the expression for the energy momentum tensor[2]

$$T_{\mu\nu} = \partial_\mu \phi\, \partial_\nu \phi - \frac{1}{2} g_{\mu\nu}\left[\partial^\sigma \phi \partial_\sigma \phi - \mu^2 \phi^2\right] + \xi\left[-R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu}\partial^\sigma \partial_\sigma + \partial_\mu \partial_\nu\right]\phi^2, \tag{1.19}$$

which has an interesting property in relation to its trace

$$T^\nu_\nu = \frac{1}{2}\left(6\xi - 1\right)\partial_\sigma \partial^\sigma \phi^2 + \mu^2 \phi^2, \tag{1.20}$$

as it is zero, for a conformal theory (meaning that it is invariant under Weyl transformations given by equation 1.15); which will not be true upon quantization, as will be proven in the last chapter.

Dirac Field

For spin ${}^1/_2$ particles, the Lorentz generators are

$$\Sigma^{AB} = -\frac{i}{2}\sigma^{AB} = \frac{1}{4}\left[\gamma^A, \gamma^B\right], \tag{1.21}$$

where γ^A are the flat gamma matrices. Therefore the covariant derivative and the connection can be written as follows

$$\nabla_\mu \equiv \partial_\mu + \Gamma_\mu, \quad \Gamma_\mu = \frac{1}{8}\omega_{AB\mu}\left[\gamma^A, \gamma^B\right]. \tag{1.22a}$$

[1] The field is said to be minimally coupled to gravity if $\xi = 0$ and nonminimally coupled otherwise.

[2] Note that in a Minkowski background, the term $R\phi^2$ present in the action written in the equation 1.14 vanishes, but the energy momentum tensor $T_{\mu\nu}$ differs from the standard expression by a pure derivative term. The new tensor is known as the improved energy-momentum tensor.

Taking the Dirac theory as inspiration, one could define the Dirac action in curved spacetimes as

$$S[\psi] = \int \bar{\psi} \left[i\gamma^{\nu}\nabla_{\nu} - \mu \right] \psi \sqrt{-g} \mathrm{d}^4x, \tag{1.23}$$

where $\Gamma^{\nu} \equiv \gamma^{A} e_{A}^{\nu}$ are the general Gamma matrices, which follow the next relation similar to their flat counterparts,

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}. \tag{1.24}$$

From the Euler-Lagrange equation 1.4, one obtains the generalized Dirac equation

$$\left[i\gamma^{\mu} \left(\partial_{\mu} + \Gamma_{\mu} \right) - \mu \right] \psi = 0. \tag{1.25}$$

And from its definition in equation 1.8, the energy momentum tensor will have [6, sec. 3.8] the following expression

$$T_{\mu\nu} = \frac{1}{4} i \left\{ \bar{\psi} \left(\gamma_{\mu}\nabla_{\nu} + \gamma_{\nu}\nabla_{\mu} \right) - \left[\left(\nabla_{\mu}\bar{\psi} \right) \gamma_{\nu} + \left(\nabla_{\nu}\bar{\psi} \right) \gamma_{\mu} \right] \right\} \psi = T_{\nu\mu}, \tag{1.26}$$

with a trace of the form $T_{\mu}^{\mu} = m\bar{\psi}\psi$, which again, is traceless for a massless field.

A particularly interesting outcome of this field is the result of squaring the generalized Dirac operator $\left[i\gamma^{\mu} \left(\partial_{\mu} + \Gamma_{\mu} \right) - \mu \right]$ just as it is done in a Minkowskian background to recover the Klein-Gordon equation,

$$\left[\nabla_{\nu}\nabla^{\nu} - \mu^2 - \frac{1}{4} R \right] \psi = 0, \tag{1.27}$$

this expression (known as the Weitzenböck formula) gives another "natural" choice for the scalar field coupling to gravity ξ; to obtain said value, one can compare it with the generalized Klein-Gordon equation 2.6 finding $\xi = 1/4$.

Electromagnetic Field

Having previously studied the Dirac field, it is expected to also include the electromagnetic field, which is described by the same action as in the Minkowskian background, that is,

$$S[A_{\mu}] = \int \left(-\frac{1}{4c} F_{\mu\nu} F^{\mu\nu} \right) \sqrt{-g} \, \mathrm{d}^4x; \tag{1.28}$$

where the Faraday tensor is defined as

$$F_{\mu\nu} = \nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}; \tag{1.29}$$

the last equality is a result of the symmetry of the lower indices on the Christoffel symbols.

The equations of motion resulting from the action 1.28 and Euler-Lagrange 1.4 are

$$\nabla^{\nu}\nabla_{\nu}A_{\mu} + R_{\mu}^{\sigma} A_{\sigma} = 0. \tag{1.30}$$

And finally, as for completeness, the energy-momentum tensor [6, sec. 3.8] given by 1.8 is

$$T_{\mu\nu} = \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F_{\mu\alpha} F^{\alpha\nu}; \tag{1.31}$$

which is traceless.

1.3 Scalar Field Quantization

Thanks to its simplicity, the scalar field is a great field to work with, with the intent of showing some properties of a theory. For that reason, in what follows, all work will be done considering a real scalar field described by the action 1.14.

Now, let $v(x)$ be a solution of the generalized Klein-Gordon equation 2.6, then its complex conjugated $\bar{v}$ will also be an independent solution. Now consider $\{i\}$ to be some set of parameters that univocally describe a pair of solutions v_i, $\bar{v}_i$ in such a way that the most general solution of 2.6 will be

$$\phi(x) = \sum_i \left[a_i v_i(x) + \bar{a}_i \bar{v}_i(x)\right], \tag{1.32}$$

where a_i and $\bar{a}_i$ are constant factors, determined by the following external binary operation

$$\langle \phi_1(x),\, \phi_2(x)\rangle \equiv \frac{i}{\hbar}\int g^{0\nu}\left(\phi_1 \overleftrightarrow{\partial}_\nu \bar{\phi}_2\right)\sqrt{-g}\,\mathrm{d}^3\mathbf{x}, \tag{1.33}$$

such that

$$a_i = \langle v_i(x),\, \phi(x)\rangle, \quad \bar{a}_i = \langle \bar{v}_i(x),\, \phi(x)\rangle. \tag{1.34 a,b}$$

The quantization procedure is done as usual by promoting the field χ and its conjugate momentum $\Pi \equiv \partial_{ct}\phi$ to operators

$$\phi(x) \longrightarrow \hat{\phi}(x), \qquad \Pi(x) \longrightarrow \hat{\Pi}(x), \tag{1.35}$$

by promoting the constant factors to operators as well, that is

$$a_i \longrightarrow \hat{a}_i, \qquad \bar{a}_i \longrightarrow \hat{a}_i^\dagger, \tag{1.36}$$

and therefore the field operator will be written as

$$\hat{\phi}(x) = \sum_i \left[\hat{a}_i v_i(x) + \hat{a}_i^\dagger \bar{v}_i(x)\right]. \tag{1.37}$$

Once the promotion of the field to operators has been done, commutation relations between those operators must be imposed; the easiest choice would be to assume canonical quantization relations, that is,

$$\left[\hat{\phi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = i\hbar\,\delta^3\left(\mathbf{x}-\mathbf{y}\right) \qquad \left[\hat{\phi}(\mathbf{x}),\, \hat{\phi}(\mathbf{y})\right] = \left[\hat{\Pi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = 0. \tag{1.38 a-c}$$

It would be desirable to obtain a formulation similar to the well-known scalar field in a Minkoswskian background, where the Fock space is generated from a vacuum state and a set of creation and annihilation operators that follow some commutation rules. To do so, we will force the $\hat{a}_i$, $\hat{a}_i^\dagger$ operators to assume this roll, in such a way that

$$\left[\hat{a}_i,\, \hat{a}_j^\dagger\right] \propto \delta_{ij}, \qquad \left[\hat{a}_i,\, \hat{a}_j\right] = \left[\hat{a}_i^\dagger,\, \hat{a}_j^\dagger\right] = 0. \tag{1.39 a-c}$$

Thanks to the relation between the constant factors a_i and the operation $\langle v_i,\, \phi\rangle$, one can obtain the following relation

$$\begin{aligned}\left[\hat{a}_i,\, \hat{a}_j^\dagger\right] &= -\frac{1}{\hbar^2}\int\left[\left(v_i\hat{\Pi} - g^{0\nu}\left(\partial_\nu v_i\right)\hat{\phi}\sqrt{-g}\right)\Big|_{\mathbf{x}},\, \left(\bar{v}_j\hat{\Pi} - g^{0\nu}\left(\partial_\nu v_j\right)\hat{\phi}\sqrt{-g}\right)\Big|_{\mathbf{y}}\right]\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{y} = \\ &= \frac{i}{\hbar}\int g^{0\nu}\left(v_i\overleftrightarrow{\partial}_\nu \bar{v}_j\right)\sqrt{-g}\,\mathrm{d}^3\mathbf{x} = \langle v_i,\, v_j\rangle,\end{aligned} \tag{1.40}$$

where the field commutators were used. Equivalently

$$[\hat{a}_i,\, \hat{a}_j] = -\langle v_i,\, \bar{v}_j\rangle, \quad \left[\hat{a}_i^\dagger,\, \hat{a}_j^\dagger\right] = -\langle \bar{v}_i,\, v_j\rangle. \tag{1.41 a,b}$$

Therefore we must find a set of solutions $\{v_i(x),\, \bar{v}_i(x)\}$ such that

$$\langle v_i,\, v_j\rangle \propto \delta_{ij}, \quad \langle v_i,\, \bar{v}_j\rangle = \langle \bar{v}_i,\, v_j\rangle = 0. \tag{1.42 a-c}$$

With this, we can define the Fock space the usual way, starting with a vacuum state $|0\rangle$ such that the action of the annihilation operation fulfils

$$\hat{a}_i\,|0\rangle = 0 \qquad \forall i, \tag{1.43}$$

where single particle states are formed from the creation operator

$$|i\rangle \equiv \hat{a}_i^\dagger\,|0\rangle, \tag{1.44}$$

and multiparticle states as the repetitive application of said operator,

$$|i,\, j,\, \ldots\rangle = \ldots \hat{a}_j^\dagger\, \hat{a}_i^\dagger\,|0\rangle. \tag{1.45}$$

Since this is a scalar field, one might assume that the states are symmetric (describing boson particles), and this is easily confirmed, since

$$|i,\, j\rangle = \hat{a}_j^\dagger\, \hat{a}_i^\dagger\,|0\rangle = \left[\hat{a}_i^\dagger,\, \hat{a}_j^\dagger\right]|0\rangle + \hat{a}_i^\dagger\, \hat{a}_j^\dagger|0\rangle = |j,\, i\rangle. \tag{1.46}$$

1.3.1 Bogoliubov Transformations

Consider now a second set $\{u_i(x),\, \bar{u}_i(x)\}$ of solutions to the Klein-Gordon equation 2.6 such that they meet the operational rules 1.42; the field would then be expressed as

$$\phi(x) = \sum_j \left[b_j u_j(x) + \bar{b}_j \bar{u}_j(x)\right]. \tag{1.47}$$

Quantization of the field is straightforward and equivalent to the method previously presented. The relation between the v and u solutions (mode functions) will be

$$v_i(x) \equiv \sum_j \left[\alpha_{ij} u_j(x) + \beta_{ij} \bar{u}_j(x)\right], \tag{1.48}$$

where α_{ij} and β_{ij} are known as Bogoliubov coefficients, that can be obtained as

$$\alpha_{ij} \propto \langle v_i,\, u_j\rangle \qquad \beta_{ij} \propto -\langle v_i,\, \bar{u}_j\rangle. \tag{1.49 a,b}$$

Since the field is the same independently of the mode set chosen, the following expression must hold

$$\sum_i \left[\hat{a}_i v_i(x) + \hat{a}_i^\dagger v_i^*(x)\right] = \sum_j \left[\hat{b}_j u_j(x) + \hat{b}_j^\dagger u_j^*(x)\right], \tag{1.50}$$

and, as a result of the orthogonality of the mode functions, the relation between the creation and annihilation operators will be

$$\hat{a}_i = \sum_j \left(\bar{\alpha}_{ij}\hat{b}_j - \bar{\beta}_{ij}\hat{b}_j^\dagger\right), \quad \hat{a}_i^\dagger = \sum_j \left(-\beta_{ij}\hat{b}_j + \alpha_{ij}\hat{b}_j^\dagger\right). \tag{1.51 a,b}$$

Applying the commutator rules present in equation 10, one obtains new restrictions to the Bogoliubov coefficients

$$\left[\hat{a}_i,\, \hat{a}_j^\dagger\right] \propto \delta_{ij} \implies \sum_k \left(\bar{\alpha}_{ik}\alpha_{jk} - \bar{\beta}_{ik}\beta_{jk}\right) \propto \delta_{ij}, \tag{1.52}$$

$$[\hat{a}_i,\, \hat{a}_j] = 0 \implies \sum_k \left(\bar{\alpha}_{jk}\bar{\beta}_{ik} - \bar{\alpha}_{ik}\bar{\beta}_{jk}\right) = 0. \tag{1.53}$$

One might ask what the relevance of this transformation is, and would be within their right to do so, since it is not a mere mathematical result. To see the reason of this transformation, one could compute the number of v particles that are present in the u vacuum; the computation is given by

$$\langle{}_u0|\hat{N}_v|{}_u0\rangle = \sum_i \langle{}_u0|\hat{a}_i^\dagger \hat{a}_i|{}_u0\rangle = \sum_i \left[\sum_{jk} \beta_{ij}\bar{\beta}_{ik}\langle{}_u0|\hat{b}_j\hat{b}_k^\dagger|{}_u0\rangle\right] \propto \sum_{ij} |\beta_{ij}|^2. \tag{1.54}$$

The usual expectation value of a term of the form $\langle 0|\hat{N}|0\rangle$ is to be zero, and yet, it has been proven that this is not the case (in general) for the current scenario. The interpretation of such a result is that the notion of "particle" is dependent on the choice of solutions of the Klein-Gorton equation; and thus, one could define different vacuum states for different situations.

2 Scalar Fields in Expanding Universes

The methodology for the analysis of scalar fields in a general manifold was presented in the previous chapter as a preliminary for the rest of this thesis, and in particular of the present chapter. It is clear that the presence of symmetries of the theory will simplify computations, and thus, a helpful starting point might be an isotropic and homogeneous expanding universe, which is described by the so-called Friedmann–Lemaître–Robertson–Walker metric. Using reduced-circumference polar coordinates, the line element associated with such metric is written as

$$\mathrm{d}l^2 = c^2\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\mathrm{d}\Omega^2\right], \qquad \mathrm{d}\Omega \equiv \mathrm{d}\theta^2 + \sin^2\varphi\,\mathrm{d}\varphi^2, \tag{2.1 a,b}$$

where κ is the curvature of the space and $a(t)$ is the scale factor determining the expansion. The associated curvature scalar R is given by

$$R = \frac{6}{c^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right], \tag{2.2}$$

needed for the coupling of the field with gravity as previously stated.

2.1 Expanding Scalar Field Action

The Weyl tensor associated to the metric presented in equation 2.1 is identically zero, meaning that the metric is conformally flat, i.e. independent of the space curvature κ, and therefore there must exist a coordinate system where

$$\mathrm{d}l^2 = a^2(\eta)\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = a^2(\eta)\left[c^2\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2\right]; \tag{2.3}$$

working in such coordinate system will give the opportunity to use some results of standard scalar field theory. To do so, the action presented in eq. (1.14) will be rewritten in terms of a new field $\chi(x) \equiv a(\eta)\,\phi(x)$ using the fact that $\sqrt{-g} = a^4$; this results in

$$S[\chi] = \int \frac{1}{2}\left[\partial_\nu\chi\,\partial^\nu\chi - \left(\mu^2a^2 + \xi R a^2 - c^2\frac{a''}{a}\right)\chi^2 - \partial_\eta\left(c^2\chi^2\frac{a'}{a}\right)\right]\mathrm{d}^4x, \tag{2.4}$$

where $a' \equiv \partial_\eta a(\eta)$ and equivalently with a''.

Dropping the total time derivative will result in the following action for scalar field χ

$$S[\chi] = \int \frac{1}{2}\left[\partial_\nu\chi\,\partial^\nu\chi - \left(\mu^2a^2 + \xi R a^2 - c^2\frac{a''}{a}\right)\chi^2\right]\mathrm{d}^4x, \tag{2.5}$$

which will be the focus of this chapter. In order to obtain the expressions describing the dynamics of this field, the Euler-Lagrange equations in eq. (1.4) will be used, resulting in the generalized Klein-Gordon equation

$$\left[\partial_\nu\partial^\nu + \mu^2_{\text{eff}}(\eta)\right]\chi = 0, \qquad \mu^2_{\text{eff}}(\eta) = \left(\mu^2 + \xi R\right)a^2 - \frac{a''}{ac^2}. \tag{2.6 a,b}$$

Solutions of the equation 2.6 are dependent on an integration constant related to the momentum $\mathbf{k}$, and are of the form

$$\chi_{\mathbf{k}}(x) = \alpha_{\mathbf{k}}v_{\mathbf{k}}(\eta)e^{-i\mathbf{k}\mathbf{x}\hbar^{-1}} + \bar{\alpha}_{\mathbf{k}}\bar{v}_{\mathbf{k}}(\eta)e^{i\mathbf{k}\mathbf{x}\hbar^{-1}}, \tag{2.7}$$

and upon substitution in (2.6), one obtains the following differential equation

$$v''_{\mathbf{k}}\,\hbar^2 + \omega^2_{\mathbf{k}}(\eta)\, v_{\mathbf{k}} = 0 \tag{2.8}$$

where the dispersion relation $\omega_{\mathbf{k}}(\eta)$ is defined as,

$$\omega^2_{\mathbf{k}}(\eta) = \mathbf{k}^2 + \hbar^2\mu^2_{\text{eff}}(\eta) = \mathbf{k}^2 + \left(m^2c^2 + \xi\hbar^2 R\right) a^2 - \hbar^2 \frac{a''}{ac^2}. \tag{2.9}$$

Solving eq. (2.8) will in turn give the form of the set of solutions $\{\chi_{\mathbf{k}}\}$ needed to describe the general expression of $\chi(x)$; since we are currently considering general expansion parameters and curvature scalars, the following computations will be carried out using a general set of $v_{\mathbf{k}}$ functions. These functions nevertheless have some interesting properties, such as being capable of a choice of normalization, and a constant of motion: the imaginary part of $v_{\mathbf{k}}\bar{v}'_{\mathbf{k}}$. The last statement can be verified by the following computation

$$\frac{\partial}{\partial\eta}\text{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}}) = \frac{\partial}{\partial\eta}\left(\frac{v_{\mathbf{k}}\bar{v}'_{\mathbf{k}} - \bar{v}_{\mathbf{k}}v'_{\mathbf{k}}}{2i}\right) = \frac{v_{\mathbf{k}}\bar{v}'_{\mathbf{k}} - \bar{v}_{\mathbf{k}}v''_{\mathbf{k}}}{2i} = 0; \tag{2.10}$$

where the last step is result from dispersion relation. Since the functions $v_{\mathbf{k}}$ are capable of a choice in normalization, we will choose a set of solutions of eq. (2.8) such that $\text{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}})$ is independent of the momentum $\mathbf{k}$, and equal for all modes, this constant of motion will simply be defined as

$$\text{Im}(v\bar{v}') \equiv \text{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}}), \quad \forall\,\mathbf{k}. \tag{2.11}$$

The most general solution $\chi(x)$ of equation eq. (2.6) can be written as a Fourier mode expansion

$$\chi(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\left[a_{\mathbf{k}}v_{\mathbf{k}}(\eta)e^{-i\mathbf{k}\mathbf{x}\hbar^{-1}} + \bar{a}_{\mathbf{k}}\bar{v}_{\mathbf{k}}(\eta)e^{i\mathbf{k}\mathbf{x}\hbar^{-1}}\right]. \tag{2.12}$$

2.2 Quantization

After promoting the fields to operators, and imposing commutation relations, one obtains the following rules for the creation and annihilation operators from equations 10.a,b,c

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = \frac{(2\pi\hbar)^3\hbar c}{2\text{Im}(v\bar{v}')}\delta^3(\mathbf{k}-\mathbf{q})\,, \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = 0. \tag{2.13 a-c}$$

To be able to verify the value of the proportional factor in 2.13.a, lets compute the commutator between $\hat{\chi}$ and $\hat{\Pi}$, i.e.

$$\begin{aligned}\left[\hat{\chi}(\mathbf{x}),\,\hat{\Pi}(\mathbf{y})\right] = \frac{1}{c}\int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^6}\Big\{&[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}]\, v_{\mathbf{k}}v'_{\mathbf{q}}e^{-i(\mathbf{k}\mathbf{x}+\mathbf{q}\mathbf{y})\hbar^{-1}} + \left[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}\right]\bar{v}_{\mathbf{k}}\bar{v}'_{\mathbf{q}}e^{-i(\mathbf{k}\mathbf{x}-\mathbf{q}\mathbf{y})\hbar^{-1}} + \\ &+ \left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}\right] v_{\mathbf{k}}\bar{v}'_{\mathbf{q}}e^{-i(\mathbf{k}\mathbf{x}-\mathbf{q}\mathbf{y})\hbar^{-1}} - \left[\hat{a}_{\mathbf{q}}, \hat{a}^\dagger_{\mathbf{k}}\right]\bar{v}_{\mathbf{k}}v'_{\mathbf{q}}e^{i(\mathbf{k}\mathbf{x}-\mathbf{q}\mathbf{y})\hbar^{-1}}\Big\}\end{aligned} \tag{2.14}$$

using expresions 2.13.a-c and considering that the proportional factor of 2.13.a to be α, the previous expression simplify to the following one,

$$\left[\hat{\chi}(\mathbf{x}),\,\hat{\Pi}(\mathbf{y})\right] = \frac{\alpha}{c}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^6}2i\text{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}})e^{-i(\mathbf{k}\mathbf{x}-\mathbf{q}\mathbf{y})\hbar^{-1}}; \tag{2.15}$$

since $\text{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}})$ was considered to be momentum independent, this implies that

$$\left[\hat{\chi}(\mathbf{x}),\,\hat{\Pi}(\mathbf{y})\right] = i\frac{2\alpha\text{Im}(v\bar{v}')}{c(2\pi\hbar)^3}\delta^3(\mathbf{x}-\mathbf{y}), \tag{2.16}$$

and, from equation 1.38.a one can solve for α, resulting in the value present in equation 2.13.

The next step in the quantization procedure is to obtain the Hamiltonian $\hat{\mathcal{H}}$ that spans the Fock space; to do so, we use the definition in equation 1.10 alongside the energy momentum tensor 1.19. As a simplification, lets consider a minimally coupled theory, i.e. $\xi = 0$; then the Hamiltonian will be

$$\hat{\mathcal{H}}(t) = \int \frac{c}{2}\left[\hat{\Pi}^2 + \left(\nabla\hat{\chi}\right)^2 + \mu_{\text{eff}}^2(t)\hat{\chi}^2\right] \mathrm{d}^3\mathbf{x}. \tag{2.17}$$

Substitution of the general expression of $\hat{\chi}$ (from equation 2.12) and $\hat{\Pi}$ (remembering that $\Pi \equiv \partial_0\chi$) one will get the following expansion,

$$\hat{\mathcal{H}}(\eta) = \frac{c}{2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\left[\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}F_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\bar{F}_{\mathbf{k}} + \left(2\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\delta^3(\mathbf{0})\right)E_{\mathbf{k}}\right], \tag{2.18}$$

where the functions $F_{\mathbf{k}}(t)$ and $E_{\mathbf{k}}(t)$ are defined as

$$F_{\mathbf{k}}(\eta) = \left(\frac{1}{\hbar c}\right)^2\left[\hbar^2 v_{\mathbf{k}}'^2 + \omega_{\mathbf{k}}^2(t)\,c^2 v_{\mathbf{k}}^2\right], \quad E_{\mathbf{k}}(\eta) = \left(\frac{1}{\hbar c}\right)^2\left[\hbar^2|v_{\mathbf{k}}'|^2 + \omega_{\mathbf{k}}^2(t)\,c^2|v_{\mathbf{k}}|^2\right]. \tag{2.19 a,b}$$

2.3 Instantaneous Vacuum State

Note that the only way a vacuum state $|0\rangle$ could remain an eigenstate of the Hamiltonian 2.18 at all times, would be if $F_{\mathbf{k}}(\eta) = 0$, at all times, i.e.

$$F_{\mathbf{k}}(\eta) = \left(\frac{1}{\hbar c}\right)^2\left[\hbar^2 v_{\mathbf{k}}'^2 + \omega_{\mathbf{k}}^2(\eta)\,c^2 v_{\mathbf{k}}^2\right] = 0, \tag{2.20}$$

solving for $v_{\mathbf{k}}$ gives the following expression

$$v_{\mathbf{k}}(\eta) = \mathrm{C}\exp\left[\pm\frac{c}{i\hbar}\int \omega_{\mathbf{k}}\left(\eta\right)\mathrm{d}\eta\right], \tag{2.21}$$

which is not compatible with 2.8 except for a time-independent dispersion relation $\omega_{\mathbf{k}}$.

The last result implies that, at different times, one can (and should) define different vacuum states; and thus, we will define the *instantaneous vacuum state* $|_{(\eta_0)}0\rangle$ as the one that at some time η_0 will minimize the energy density. Since all possible states are related by Bogolyubov transformations, finding the instantaneous vacumm state is the same as finding the set of functions $v_{\mathbf{f}}$ that are simultaneously solution of equation 2.8 and minimize

$$\langle_{(\eta_0)}0|\hat{\mathcal{H}}(\eta_0)|_{(\eta_0)}0\rangle = \rho(\eta_0)\delta^3(\mathbf{0}) = \frac{\hbar c^2\,\delta^3(\mathbf{0})}{4\mathrm{Im}(v\bar{v}')}\int \mathrm{d}^3\mathbf{k}\,E_{\mathbf{k}} \tag{2.22}$$

To minimize the energy density of the vacuum state is to find the set of functions $v_{\mathbf{k}}$ that minimize $E_{\mathbf{k}}$. Suppose that $v_{\mathbf{k}}$ can be written as

$$v_{\mathbf{k}} = r_{\mathbf{k}}e^{i\alpha_{\mathbf{k}}} \tag{2.23}$$

since $\mathrm{Im}(v\bar{v}')$ was constant through time,

$$r_{\mathbf{k}}^2\alpha_{\mathbf{k}}' = -\mathrm{Im}(v\bar{v}'). \tag{2.24}$$

This means that

$$E_{\mathbf{k}} = \left(\frac{1}{\hbar c}\right)^2\left\{\hbar^2\left[r_{\mathbf{k}}'^2 + \mathrm{Im}^2\left(v\bar{v}'\right)\frac{1}{r_{\mathbf{k}}^2}\right] + \omega_{\mathbf{k}}^2\,c^2 r_{\mathbf{k}}^2\right\} \tag{2.25}$$

the minimum of this function must fulfil $r_{\mathbf{k}}'(\eta_0) = 0$. Now, if $\omega_{\mathbf{k}}^2(\eta_0)$ and $\mathrm{Im}(v\bar{v}')$ have the same sign, the minimum of $E_{\mathbf{k}}$ happens when $r_{\mathbf{k}}(\eta_0) = \left[\frac{\hbar\mathrm{Im}(v\bar{v}')}{\omega_{\mathbf{k}}(\eta_0)\,c}\right]^{1/2}$.

If there is a minimum, then

$$v_{\mathbf{k}}(\eta_0) = \left[\frac{\hbar \mathrm{Im}(v\bar{v}')}{\omega_{\mathbf{k}}(\eta_0)\, c}\right]^{1/2} e^{i\alpha_{\mathbf{k}}(\eta_0)}, \qquad v'_{\mathbf{k}}(\eta_0) = -c\frac{\omega_{\mathbf{k}}(\eta_0)}{ih}\, v_{\mathbf{k}}(\eta_0) \tag{2.26 a,b}$$

under these functions,

$$E_{\mathbf{k}}(\eta_0) = 2\frac{\mathrm{Im}(v\bar{v}')}{\hbar c}\omega_{\mathbf{k}}(\eta_0), \quad F_{\mathbf{k}}(\eta_0) = 0; \tag{2.27a}$$

meaning that

$$\hat{\mathcal{H}}(\eta_0) = \mathrm{Im}(v\bar{v}')\frac{1}{\hbar}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\left(2\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\delta^3(\mathbf{0})\right)\omega_{\mathbf{k}}(t_0), \tag{2.28}$$

which is equivalent to the standard Hamiltonian for a scalar field without the presence of gravity.

But what is the energy of the instantaneous vacuum at a different time? The relation of such energies can be computed considering that the field at some time η can be described as some Bogoliubov transformation of the same field at a time η_0.

From equations 1.49 one deduces that the Bogoliubov coefficients will be given by

$$\alpha_{\mathbf{kp}} = \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\langle \chi_{\mathbf{k}}(\eta_0),\, \chi_{\mathbf{p}}(\eta)\rangle, \qquad \beta_{\mathbf{kp}} = -\frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\langle \chi_{\mathbf{k}}(\eta_0),\, \bar{\chi}_{\mathbf{p}}(\eta)\rangle, \tag{2.29 a,b}$$

and, since the field can be expressed as $\chi_{\mathbf{k}} = v_{\mathbf{k}}e^{i\mathbf{kx}/\hbar}$; from the definition of the binary operation $\langle\cdot,\cdot\rangle$ in expression 1.33, one can see that

$$\alpha_{\mathbf{kp}} \propto \delta^3(\mathbf{k}-\mathbf{p}), \qquad \beta_{\mathbf{kp}} \propto \delta^3(\mathbf{k}+\mathbf{p}), \tag{2.30 a,b}$$

and thus, it is possible to write $v_{\mathbf{k}}$ at an arbitrary time t as

$$v_{\mathbf{k}}(\eta) = \alpha_{\mathbf{k}}v_{\mathbf{k}}(\eta_0) + \beta_{\mathbf{k}}\bar{v}_{\mathbf{k}}(\eta_0); \tag{2.31}$$

where, recalling that $\mathrm{Im}(v\bar{v}')$ is constant through time, the relation between $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ must be

$$|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1. \tag{2.32}$$

The energy of the instantaneous vacuum state $|_{(\eta_0)}0\rangle$ at a time η is given by $\langle_{(\eta_0)}0|\hat{\mathcal{H}}(\eta)|_{(\eta_0)}0\rangle$, were the hamiltonian $\hat{\mathcal{H}}(\eta)$ is given by expression 2.18; to compute that, lets first obtain

$$\langle_{(\eta_0)}0|\hat{a}^{\dagger}_{\mathbf{k}}(\eta)\hat{a}_{\mathbf{k}}(\eta)|_{(\eta_0)}0\rangle = \left|\beta_{\mathbf{k}}\right|^2\frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\delta^3(\mathbf{0}); \tag{2.33}$$

therefore the energy of the instantaneous vacuum state $|_{(\eta_0)}0\rangle$ at a time η is given by

$$\langle_{(\eta_0)}0|\hat{\mathcal{H}}(\eta)|_{(\eta_0)}0\rangle = \delta^3(\mathbf{0})\int \mathrm{d}^3\mathbf{k}\left(\frac{1}{2} + \left|\beta_{\mathbf{k}}\right|^2\right)c\,\omega_{\mathbf{k}}(\eta) \neq \langle_{(\eta_0)}0|\hat{\mathcal{H}}(\eta_0)|_{(\eta_0)}0\rangle. \tag{2.34}$$

As expected, the energy density at different times will not be the same, since the definition of $|_{(\eta_0)}0\rangle$ is the state that minimizes the energy density at some particular time, meaning that other state must have a lower energy density at a different time.

2.4 Case Study: de Sitter Universe

The de Sitter Universe is a flat FLRW metric with no matter or radiation, but it does have a a positive cosmological constant Λ. Per the Friedmann equations,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho + \Lambda c^2}{3} - \frac{\kappa c^2}{a^2}, \quad (\rho = \kappa = 0) \tag{2.35}$$

the expansion parameter $a(t)$ will be equal to

$$a(t) = a_1 e^{H_\Lambda t} + a_2 e^{-H_\Lambda t}\,, \tag{2.36}$$

where $H_\Lambda = \sqrt{\Lambda c^2/3}$ is the Hubble-Lemaître constant. The most common choice is to set $a_2 = 0$, and thus, consider an always expanding universe.

The line element describing the motion of particles through this universe is given by

$$\mathrm{d}l^2 = c^2\mathrm{d}t^2 - a^2(t)\mathrm{d}\mathbf{x}^2, \tag{2.37}$$

more commonly expressed as a function of the conformal time η defined as

$$\eta \equiv -\int_t^\infty \frac{\mathrm{d}t'}{a(t')} = -\frac{1}{a_1 H_\Lambda} e^{-H_\Lambda t} = -\frac{1}{a(t) H_\Lambda}; \tag{2.38}$$

and thus, the line element will be given by

$$\mathrm{d}l^2 = \frac{1}{H_\Lambda \eta^2}\left[c^2\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2\right], \tag{2.39}$$

which has the same form as 2.3.

Now, from equation 2.2 one can compute the curvature scalar $R = {}^{12}/_{c^2} H_\Lambda^2$, and thus, the dispersion relation 2.9 will be given by the following expression

$$\omega_{\mathbf{k}}^2(\eta) = \mathbf{k}^2 + \left[\left(\frac{mc^2}{H_\Lambda}\right)^2 + 2\,(6\xi - 1)\,\hbar^2\right]\frac{1}{c^2\eta^2}, \tag{2.40}$$

from which one might obtain the solutions of differential equation 2.8; to do so it is best to use the following change of variables,

$$s \equiv -k\eta\frac{c}{\hbar}, \qquad v_{\mathbf{k}} \equiv \sqrt{s} f(s), \tag{2.41 a,b}$$

obtaining Bessel's differential equation

$$s^2\frac{\mathrm{d}^2 f}{\mathrm{d}s^2} + s\frac{\mathrm{d}f}{\mathrm{d}s} + \left(s^2 - \nu^2\right) f(s) = 0, \tag{2.42}$$

with a parameter

$$\nu^2 \equiv (3 - 16\xi)\,\frac{3}{4} - \left(\frac{mc^2}{H_\Lambda \hbar}\right)^2. \tag{2.43}$$

The solutions of said differential equation are given by the so called Bessel functions of the first kind $J_\nu(s)$ and $Y_\nu(s)$; therefore the $v_{\mathbf{k}}$ functions can be deduced as

$$f(s) = AJ_\nu(s) + BY_\nu(s) \implies v_{\mathbf{k}}(\eta) = \sqrt{k|\eta|\frac{c}{\hbar}}\left[A_{\mathbf{k}} J_\nu(k|\eta|c/\hbar) + B_{\mathbf{k}} Y_\nu(k|\eta|c/\hbar)\right] \tag{2.44}$$

For $\nu^2 \geq 0$, both J_ν and Y_ν will be real functions, but for $\nu^2 < 0$ they will be complex functions [12]; for simplicity, we will focus on the $\nu^2 \geq 0$ case. In addition, consider that the choice of $\mathrm{Im}(v\bar{v}')$ will translate into a restriction on the relation between the $A_{\mathbf{k}}$, $B_{\mathbf{k}}$ parameters;

$$\mathrm{Im}\left(v\bar{v}'\right) = ik^2|\eta|\frac{c}{\hbar}\left(A_{\mathbf{k}}\bar{B}_{\mathbf{k}} - \bar{A}_{\mathbf{k}} B_{\mathbf{k}}\right) W\left[J_\nu\left(k|\eta|\right), Y_\nu\left(k|\eta|\right)\right], \tag{2.45}$$

where $W[f,g] \equiv fg' - f'g$ is the Wronskian, particularly [1, 10.5.2] for the Bessel functions, one obtains $W\left[J_\nu\left(s\right), Y_\nu\left(s\right)\right] = \frac{2}{\pi s}$ and thus, the relation between the $A_{\mathbf{k}}$, $B_{\mathbf{k}}$ parameters will be

$$\left(A_{\mathbf{k}}\bar{B}_{\mathbf{k}} - \bar{A}_{\mathbf{k}}B_{\mathbf{k}}\right) = -i\frac{\pi}{2k}\mathrm{Im}\left(v\bar{v}'\right). \tag{2.46}$$

As previously seen, the choice of a set of solutions of the Klein-Gordon equation defines the choice of a vacuum; an example is the instantaneous vacuum, but for a de Sitter background there is a special choice, the Bunch-Davies vacuum[1].

2.4.1 Bunch-Davies Vacuum

Consider the behaviour of the field in the distant past, that is, $k|\eta| \to \infty$; at this moment, the dispersion relation 2.40 is given by $\omega_{\mathbf{k}} \approx k$, therefore, the solutions of 2.8 are given by

$$v_{\mathbf{k}}(\eta) \approx \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}e^{i\omega_{\mathbf{k}}\eta}, \tag{2.47}$$

which is the usual flat temporal mode function. The same limit can be taken over the solution 2.44, considering the asymptotic expressions for the Bessel functions [1, 10.7.8], one obtains

$$v_{\mathbf{k}}(\eta) \approx \sqrt{\frac{2}{\pi}}\left[A_{\mathbf{k}}\cos\lambda_\nu + B_{\mathbf{k}}\sin\lambda_\nu\right], \quad \lambda_\nu \equiv k|\eta|\frac{c}{\hbar} - \frac{\pi}{2}\nu - \frac{\pi}{4}. \tag{2.48 a,b}$$

One could easily recover the flat temporal mode function by imposing the relation $B_{\mathbf{k}} = iA_{\mathbf{k}}$; this choice, alongside the relation 2.46, meaning that $|A_{\mathbf{k}}| = \sqrt{\pi/4k\mathrm{Im}\left(v\bar{v}'\right)}$; and therefore, the Bunch-Davies temporal mode functions are given (except for an irrelevant phase) by

$$v_{\mathbf{k}}(\eta) = \sqrt{\frac{\pi}{4}|\eta|\mathrm{Im}\left(v\bar{v}'\right)}\,H_\nu^{(1)}\left(k|\eta|\right), \tag{2.49}$$

where $H_\nu^{(1)}(s) \equiv J_\nu(s) + iY_\nu(s)$ is known as the Hankel function of the first kind.

This vacuum state is of particular interest [3] since it is not time dependent, it is invariant under the de Sitter symmetry group and, for a conformal theory (i.e. $m = 0$ and $\xi = 1/6$) the temporal mode function is given by[2]

$$v_{\mathbf{k}}(\eta) = -i\sqrt{\frac{\hbar\mathrm{Im}\left(v\bar{v}'\right)}{2\omega_{\mathbf{k}}c}}e^{i\omega_{\mathbf{k}}|\eta|c/\hbar} \tag{2.50}$$

which is (up to an irrelevant phase) the standard plane wave temporal mode function.

Any other vacuum state can be obtained via a Bogoliubov transformation of the Bunch-Davies vacuum; from the relation of the Bogoliubov parameters in expression 2.32, one could define a new set of temporal mode functions $\{u_{\mathbf{k}}\}$ as

$$u_{\mathbf{k}}(\eta) \equiv \cosh\alpha\, v_{\mathbf{k}}(\eta) + e^{i\beta}\sinh\alpha\,\bar{v}_{\mathbf{k}}(\eta), \tag{2.51}$$

where $0 \leq \alpha < \infty$ and $-\pi \leq \beta < \pi$ are constant parameters defining the transformation. This new vacuum state denoted by $|_{(\alpha,\beta)}0\rangle$ is known as a Mottola-Allen vacuum.

[1] Also known as euclidean vacuum.

[2] For such a theory, $\omega_{\mathbf{k}} = k$ and $\nu = 1/2$; and for such value, the Hankel function can be written as

$$H_{1/2}^{(1)}(s) = -i\sqrt{\frac{2}{\pi s}}e^{is}.$$

3 The Unruh Effect

3.1 Accelerated Observers and Unruh Temperature

In contrast to previous chapters, in this section there will be no effects of gravitation, and a flat $1+1$ spacetime (for simplicity) will be considered; unlike standard flat QFT, here a non inertial (accelerated) observer will be considered. Let an observer measure a self constant two-acceleration $\alpha^\mu \equiv {}^{\mathrm{d}^2 x^\mu}/_{\mathrm{d}\tau^2}$, this quantity must meet the condition

$$\alpha^2 = \left(c\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2}\right)^2 - \left(\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2}\right)^2. \tag{3.1}$$

Solutions of such differential equation can be written as

$$t(\tau) = t_0 + t_1\tau \pm \frac{c}{\alpha}\sinh\left(\frac{\alpha\tau}{c}\right), \quad x(\tau) = x_0 + x_1\tau \pm \frac{c^2}{\alpha}\cosh\left(\frac{\alpha\tau}{c}\right), \tag{3.2 a,b}$$

where t_0, t_1, x_0, x_1 are constant real parameters. Since the trajectory must ensure that $c^2\mathrm{d}\tau^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2$, then the "velocities" must be such that $ct_1 = x_1 = 0$; furthermore, for simplicity we consider some coordinates in which $ct_0 = x_0 = 0$. Said trajectory can be described as the hyperbola

$$x^2 - c^2t^2 = \frac{c^2}{\alpha^2}. \tag{3.3}$$

A coordinate system known as Rindler coordinates is specifically used in the description of accelerated observers; such coordinates do not map the whole spacetime, and must be divided into two maps: depending on if $x > c|t|$ or otherwise. The coordinates used will be named (η, ξ) ($(\tilde\eta, \tilde\xi)$ if working in the second chart); the first could be understood as some sort of temporal coordinate, while the second can be understood as a parameter determining the acceleration of the observer. Said coordinate system can be written[1] as:

- Chart $x > c|t|$ (3.1 blue section),

$$t(\eta,\,\xi) \equiv \frac{c}{\alpha}\sinh\left(\frac{\alpha\eta}{c}\right)e^{\alpha\xi/c^2}, \tag{3.4 a}$$

$$x(\eta,\,\xi) \equiv \frac{c^2}{\alpha}\cosh\left(\frac{\alpha\eta}{c}\right)e^{\alpha\xi/c^2}, \tag{3.4 b}$$

- Chart $x < c|t|$ (3.1 purple section),

$$t(\tilde\eta,\,\tilde\xi) \equiv -\frac{c}{\alpha}\sinh\left(\frac{\alpha\tilde\eta}{c}\right)e^{\alpha\tilde\xi/c^2}, \tag{3.5 a}$$

$$x(\tilde\eta,\,\tilde\xi) \equiv -\frac{c^2}{\alpha}\cosh\left(\frac{\alpha\tilde\eta}{c}\right)e^{\alpha\tilde\xi/c^2}, \tag{3.5 b}$$

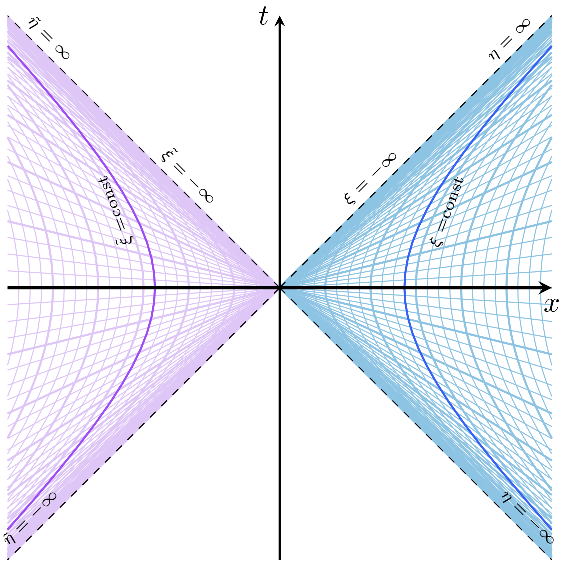

Figure 3.1: Rindler Coordinate System.

One can check that these coordinates meet the following hyperbolic relation,

$$x^2 - c^2t^2 = \frac{c^2}{\alpha^2}e^{2\alpha\xi/c^2}, \tag{3.6}$$

[1]The choice is not unique, depending on the definition of ξ the exponential factor can be substituted by a different form, such as $(1+\xi)$ or simply by ξ.

and thus, an observer with coordinates (η, ξ) can be said to experience an acceleration $\alpha e^{-\alpha\xi/c^2}$. Using said coordinate system, the line element will be[2],

$$c^2 \mathrm{d}\tau^2 = e^{\alpha\xi/c^2} \left[c^2 \mathrm{d}\eta^2 - \mathrm{d}\xi^2\right]. \tag{3.7}$$

The fact that a non-inertial reference system will experience some deviations of a theory in relation to an inertial one should not surprise any reader, since its a basic result of elementary physics, and thus, it is also expected in a relativistic quantum theory to be true. To be able to demonstrate this, we will consider the simplest possible case, a massless and minimally coupled scalar field ϕ; whose equation of motion given by the Klein-Gordon equation 1.16 will be

$$e^{-2\alpha\xi/c^2} \left[\partial_{c\eta}^2 - \partial_\xi^2\right] \phi = 0. \tag{3.8}$$

Solutions of the previous equation are the usual plane waves[3], just as it would be for an inertial observer. Since the solutions of an accelerated reference frame and an inertial observer are the same, it would seem that all phenomena will be described equally by both, but it is in fact not the same, since their metric description will be different, and thus, the field will be related by a Bogoliubov transformation; that is, the two observers might disagree on their definition of the vacuum state.

In order to explicitly compute this, it will be useful to use the so-called null coordinates $u \equiv c\eta - \xi$ and $v \equiv c\eta + \xi$ for the accelerated observer, and $U \equiv ct - x$ and $V \equiv ct + x$, through which the solution of the equation of motion will be (depending on the needed chart)

$$\phi_\omega^u \equiv e^{i\omega u\,\hbar^{-1}}, \quad \phi_\omega^v \equiv e^{i\omega v\,\hbar^{-1}}, \quad \phi_\omega^{\tilde{u}} \equiv e^{i\omega \tilde{u}\,\hbar^{-1}}, \quad \phi_\omega^{\tilde{v}} \equiv e^{i\omega \tilde{v}\,\hbar^{-1}}. \tag{3.9 a-d}$$

In addition, null coordinates will also be used for the inertial observer, those being $U \equiv ct - x$ and $V \equiv ct + x$; and the relation to the accelerated null coordinates will be

$$U(u,\, v) = -\frac{c^2}{\alpha} e^{-\alpha u/c^2}, \qquad V(u,\, v) = \frac{c^2}{\alpha} e^{\alpha v/c^2}. \tag{3.10 a,b}$$

Using null coordinates, the field can be described as two independent non-interactive fields (since there is no autointeraction term), that is, $\phi \equiv \phi_u + \phi_v$; in what follows, we will only consider the ϕ_u field (defined below), but the same can be done for the ϕ_v field. Now, lets expand the ϕ_u field as a set of Rindler modes 3.9.a,c, meaning

$$\phi_u \equiv \int_0^\infty \frac{\mathrm{d}\omega}{(2\pi\hbar)\sqrt{2\omega}} \left\{\Theta(-U)\left[a_\omega^u \phi_\omega^u + \bar{a}_\omega^u \bar{\phi}_\omega^u\right] + \Theta(U)\left[a_\omega^{\tilde{u}} \phi_\omega^{\tilde{u}} + \bar{a}_\omega^{\tilde{u}} \bar{\phi}_\omega^{\tilde{u}}\right]\right\}. \tag{3.11}$$

Clearly, it can also be written in terms of planar waves in U coordinates for the inertial observer.

Since we were interested in the difference in vacuum states, we would need to obtain the Bogoliubov coefficients connecting the Rindler modes with the inertial ones; to do so, we compute the coefficient $\beta_{\Omega,\omega}$ using 1.49.b, obtaining

$$\beta_{\Omega\omega}^u \propto \langle \phi_\Omega^u, \bar{\phi}_\omega^U \rangle = \frac{1}{2\hbar^2}\sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^\infty \exp\left\{i\left[\omega U(u) + \Omega u\right]\hbar^{-1}\right\} \mathrm{d}u, \tag{3.12}$$

where the expression for $U(u)$ is given by 3.10.a. This integral is solvable using the change of variables $z \equiv i\left(\omega\hbar c^2/\alpha\right)\exp(-au/c^2)$ which will result in

$$\int_{-\infty}^\infty \exp\left\{i\left[\omega U(u) + \Omega u\right]\hbar^{-1}\right\} \mathrm{d}u = \frac{1}{\alpha}\left(-i\frac{\alpha\hbar}{\omega c^2}\right)^{ic^2\Omega/\alpha\hbar} \int_0^\infty z^{ic^2\Omega/\alpha\hbar - 1} e^{-z} \mathrm{d}z; \tag{3.13}$$

[2]Note that as it would be expected, the metric is conformally flat.

[3]This is a direct result of the fact that, for a $1+1$ scalar theory, Weyl invariance is obtained for $m = \xi = 0$.

where the integral is the Gamma function $\Gamma\left(ic^2\Omega/\alpha\hbar\right)$. This means that the Bogoliubov coefficient $\beta^u_{\Omega\omega}$ can be written (up to some proportionality constant) as

$$\beta^u_{\Omega\omega} \propto \Gamma\left(i\frac{\Omega c^2}{\alpha\hbar}\right)\sqrt{\frac{\Omega}{\omega}}\left(\frac{c^2\omega}{\alpha\hbar}\right)^{-i\frac{\omega c^2}{\alpha\hbar}} e^{-\frac{\pi\Omega c^2}{2\alpha\hbar}}. \tag{3.14}$$

Now, from 1.54 we know how to compute the number of particles of a given momentum Ω that an accelerated observer would measure in the inertial vacuum, that is

$$N_\Omega \equiv \langle_U 0|\left(\hat{a}^u_\Omega\right)^\dagger \hat{a}^u_\Omega|_U 0\rangle \propto \int_0^\infty \frac{\mathrm{d}\Omega}{(2\pi\hbar)\sqrt{2\Omega}}|\beta^u_{\Omega\Omega}| \propto \frac{1}{e^{\frac{2\pi\Omega c}{\alpha\hbar}}-1}\delta(0). \tag{3.15}$$

This expression has some resemblance with the expected value of particles with energy Ω for Bose-Einstein statistics; if we were to interpret it as such (which is not an overreach, considering the bosonic nature of the scalar field particles), a temperature might be defined as

$$T_0 \equiv \frac{\alpha\hbar}{2\pi c k_B}. \tag{3.16}$$

This expression corresponds to the temperature that a stationary observer would predict for an accelerated one, which is different than the temperature that the accelerated observer himself would measure. The relation between both expressions is given by Tolman's law[4], stating that the proper temperature T_{Unruh} measured by the accelerating observer, is given by

$$T_{\text{Unruh}} = \sqrt{g_{00}}\,T_0 = \left(\alpha e^{-\alpha/c^2\xi}\right)\frac{\hbar}{2\pi c k_B} \equiv \frac{a\hbar}{2\pi c k_B}; \tag{3.17}$$

where a is the acceleration measured by the non-inertial observer, as stated in equation 3.6.

3.2 Application to Black Holes: Hawking Radiation

According with the no hair conjecture, black holes can be univocally described only by three parameters: its mass M, its angular momentum J, and its electric charge Q. For simplicity, lets consider a Schwarzschild black hole[5], which considers that $J = Q = 0$. The line element describing the spacetime of such black hole [10] is given (using spherical coordinates) by

$$c^2\mathrm{d}\tau^2 = \left(1-\frac{2GM}{c^2 r}\right)c^2\mathrm{d}t^2 - \left(1-\frac{2GM}{c^2 r}\right)^{-1}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2\right), \tag{3.18}$$

where $R_S \equiv 2GM/c^2$ is the so called *Schwarzschild radius*; the apparent singularity at $r = R_S$ is (as we will show shortly) not a physical one, but a by-product of the coordinate system.

In order to further simplify the problem (and to relate it back to the previous sections of the present thesis), lets consider a $1+1$ Schwarzschild black hole; to obtain such a solution, simply take the limit $\mathrm{d}\theta = \mathrm{d}\varphi = 0$ at the solution 3.18. There are two interesting coordinate systems to describe this spacetime; the "tortoise" coordinates and the Kruskal–Szekeres coordinates; each of which are interesting for different reasons. The first of the two comes from the use of the so-called tortoise coordinate, given by the expression

$$\mathrm{d}r^* \equiv \left(1-\frac{R_S}{r}\right)^{-1}\mathrm{d}r, \tag{3.19}$$

[4]A sketch of a proof goes as follows: consider a conserved energy E_0 measured by an observer in a stationary gravitational field, said energy relates to the energy measured by another observer by $E_0 = \sqrt{g_{00}}E$; since the thermodynamic relation between energy and temperature comes from the entropy S as $T_0 = \partial S/\partial E_0$, then the proper temperature must follow the relation given by Tolman's law.

[5]This is the first solution of the Einstein field equations 1.7, found by Karl Schwarzschild in 1916; Einstein himself was quoted to be amazed by the simplicity and the speed of Schwarzschild's derivation.

meaning that the line element will be

$$c^2\mathrm{d}\tau^2 = \left[1 - \frac{R_S}{r\,(r^*)}\right]\left[c^2\mathrm{d}t^2 - \mathrm{d}r^*\right]; \tag{3.20}$$

similarly to the previous section, it is useful to write it using null coordinates $u \equiv ct - r^*$ and $v \equiv ct + r^*$,

$$c^2\mathrm{d}\tau^2 = \left[1 - \frac{R_S}{r\,(u,v)}\right]\left[\mathrm{d}^2u - \mathrm{d}^2v\right]. \tag{3.21}$$

Last expression can be rewritten considering that[6]

$$1 - \frac{R_S}{r\,(u,v)} = \frac{R_S}{r}\exp\left(1 - \frac{r}{R_S}\right)\exp\left(\frac{v-u}{2R_S}\right) \tag{3.22}$$

and, defining the Kruskal–Szekeres null coordinates $(U,\, V)$ as

$$U \equiv -2R_S e^{-u/2R_S}, \qquad V \equiv 2R_S e^{v/2R_S}, \tag{3.23 a,b}$$

the line element can be expressed as

$$c^2\tau^2 = \frac{R_S}{r(U,V)} e^{1-r(U,\,V)/R_S}\mathrm{d}U\mathrm{d}V. \tag{3.24}$$

Note that both coordinate systems make the metric conformally flat, and the relation between the null tortoise and null Kruskal–Szekeres coordinates given by equation 3.23 is the same as in the Unruh effect in equation 3.10 (with the redefinition $\alpha = {c^2}/{2R_S}$); therefore the procedure of the previous section can be used here, and thus, according to expression 3.16 and Tolman's law, a tortoise observer located at $r \to \infty$ will measure that the black hole emits a thermal bath of massless boson particles at a temperature

$$T_{\text{Hawking}} = \frac{\hbar c^3}{8\pi G M k_B}. \tag{3.25}$$

If one were to consider a $3+1$ black hole [16, sec. 9.1.4] described by the line element given by 3.18, the Klein-Gordon equation would not be $\partial^\mu\partial_\nu\phi = 0$, but would have an effective potential

$$\mathcal{V}_{\text{eff}} \equiv \left(1 - \frac{R_S}{r}\right)\left[\frac{R_S}{r^3} + \frac{l(l+1)}{r^2}\right], \tag{3.26}$$

with l being the quantum orbital angular momentum of the state. The effect of said potential is such that an escaping wave, upon reaching $r \to \infty$, would have changed its frequency Ω, which is why equation 3.15 would be affected by a greybody factor $\Gamma(\Omega)$

$$N_\Omega \propto \frac{\Gamma(\Omega)}{e^{\Omega/k_B T_{\text{Hawking}}} - 1}\delta(\mathbf{0}). \tag{3.27}$$

[6] To check this, one must use the integrated form of the tortoise coordinate 3.19 given by

$$r^*(r) = r - R_S + R_S \ln\left(\frac{r}{R_S} - 1\right).$$

3.2.1 Black Hole Thermodynamics

Once it has been proven that an observer can detect an emission of particles from a black hole, the question of the origin of the required energy for the creation of such particles arises. One can consider that the energy required to create Unruh particles can come from the energy used for continuous acceleration, but this cannot be the answer for the Hawking radiation, since both observers are free-falling. Therefore the only possible answer must be that the particles extract their energy directly from the black hole, meaning that a black hole must be in thermodynamic equilibrium with the field. This reasoning can be used to deduce the evolution of black holes; as stated before, black holes can be described univocally by three parameters, and thus, its fundamental entropy relation can be written as

$$\mathrm{d}S = \left(\frac{\partial S}{\partial M}\right)\mathrm{d}M + \left(\frac{\partial S}{\partial J}\right)\mathrm{d}J + \left(\frac{\partial S}{\partial Q}\right)\mathrm{d}Q. \tag{3.28}$$

For a Schwarzschild black hole, this relation simplifies (since $Q = J = 0$), and considering that the energy must equal Mc^2, then one can deduce the so-called Bekenstein-Hawking [7] entropy

$$\mathrm{d}S = \frac{c^2}{T_{\text{Hawking}}}\mathrm{d}M \implies S_{\text{BH}} = \frac{4\pi G k_B}{\hbar c}M^2. \tag{3.29}$$

Since we are considering that the field extracts energy from the black hole, and the energy of the black hole is proportional to its mass, it is then clear that the black hole must be losing mass. One can easily deduce the temporal mass expression considering the black hole as a black body, and thus following the Stefan-Boltzmann law for the luminosity L, i.e.

$$L \equiv -c^2\frac{\mathrm{d}M}{\mathrm{d}t} = \epsilon A\sigma T^4_{\text{Hawking}} \tag{3.30}$$

where σ is the Stefan-Boltzmann constant and ϵ is a factor of correction for possible greybody effects and deviations from the use of the electromagnetic field (mostly loss of degrees of freedom). Solutions of such differential equation are

$$M(t) = M_0\left(1 - \frac{t}{t_{BH}}\right)^{1/3}, \quad t_{BH} \equiv 5120\frac{\pi G^2}{\epsilon\hbar c^4}M_0^3. \tag{3.31 a,b}$$

To get an idea of the strength of said radiation, lets consider an average stellar black hole (the most numerous type), which masses around $100M_\odot$; considering that at $t = 0$, a black hole has that mass, it would have lost all of it (it is said to have "evaporated") in about $\sim 2.1\cdot 10^{73}$ years, an unfathomable magnitude comparable to the age of the universe ($13.7\cdot 10^9$ years).

[7] In reality, they presented their result not as a function of the mass M, but as a function of the surface area A, and thus, the proper Bekenstein-Hawking entropy would be $S_{BH} = {}^{c^3 k_B}/_{\hbar G}A$.

4 Problems Related to the Energy-Momentum Tensor

Up to this moment, we have focused on the phenomenology of the quantum fields on classical backgrounds; but we must not forget the dynamics of the gravitational field $g_{\mu\nu}$, given by equations of motion 1.28. These equations show that the source of gravitational dynamics is the energy-momentum tensor given by equation 1.8; but then a conundrum arises: since we are considering quantum fields, the tensor must be a quantum operator, even though the metric remains a classical quantity. The unification of both models is one of the most sought after theories of Physics, and well out of scope of this thesis; therefore for the remaining sections, the so-called semiclassical approximation will be used. This model considers both classical gravity and quantum fields, but imposes the hypothesis that the source of the gravity dynamics is given by the expected value of the energy-momentum tensor $\langle T_{\mu\nu}\rangle$, i.e.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}\langle T_{\mu\nu}\rangle\,. \tag{4.1}$$

Here we encounter another problem, which is to correctly define this expected value from a classical perspective, which would entail defining what we know as the effective action W; to be used as the matter action on the definition given by equation 1.8. This ad hoc hypothesis will result in other problems that will be solved in the following sections.

4.1 The Effective Action W

To demonstrate the methodology, it is sufficient to consider the vacuum expectation of a scalar field, which is described by the action presented in equation 1.14. To obtain the vacuum expectation value of $T_{\mu\nu}$, we use the path integral formulation of QFT, where it can be computed as

$$\langle T_{\mu\nu}\rangle = \frac{\int \mathcal{D}[\phi]\, T_{\mu\nu}\, e^{iS_{\mathrm{M}}[\phi]\hbar^{-1}}}{\int \mathcal{D}[\phi]\, e^{iS_{\mathrm{M}}[\phi]\hbar^{-1}}} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta W}{\delta g^{\mu\nu}}, \tag{4.2}$$

the last equality will be the proper definition of the effective action W, action W, chosen so that it mimics the energy momentum tensor described by equation 1.8.

It is possible to compute W as a closed expression, to do this, consider the generating functional $Z[J]$ defined as

$$Z[J] \equiv \int \mathcal{D}[\phi]\, \exp\left\{iS_{\mathrm{M}}[\phi]\hbar^{-1} + i\int \mathrm{d}^4 J(x)\phi(x)\right\}; \tag{4.3}$$

where $J(x)$ is a possible external current that will be considered zero for the following treatment. Substituting the definition of $T_{\mu\nu}$ (again, expression 1.8) on the previous equation, one finds the following relation

$$\langle T_{\mu\nu}\rangle = \frac{-2}{Z[0]\sqrt{-g}}\int \mathcal{D}[\phi]\,\frac{\delta S_{\mathrm{M}}[\phi]}{\delta g^{\mu\nu}}\, e^{iS_{\mathrm{M}}[\phi]\hbar^{-1}} = \frac{2i\hbar}{Z[0]\sqrt{-g}}\frac{\delta Z[0]}{\delta g^{\mu\nu}}, \tag{4.4}$$

comparing this expression with the definition in equation 4.2, it its easily solvable that

$$W = -i\hbar \ln\left(\frac{Z[0]}{\mu_0\hbar^{1/2}}\right) + \mathrm{C}; \tag{4.5}$$

where we have introduced an integral constant C, and a constant μ_0 with the same units as μ (that is, inverse of length), in order to ensure that the argument of the logarithm remains dimensionless

Since we are considering a particular field action, W can be simplified even further, given that the action $S_{\mathrm{M}}[\phi]$ given by equation 1.14 can be written as

$$S_{\mathrm{M}}[\phi] = \frac{1}{2}\int \partial_\nu\left[\sqrt{-g}\,\phi\,\partial^\nu\phi\right]\mathrm{d}^4x - \int \frac{1}{2}\phi\left[\partial_\nu\partial^\nu + \mu^2 + \xi R\right]\phi\sqrt{-g}\,\mathrm{d}^4x, \tag{4.6}$$

where the first term is a total derivative and thus can be dropped. Then we can use the Dirac delta function to write

$$\phi(x) = \int \phi(y)\frac{\delta^4(x-y)}{\sqrt{-g(x)}}\sqrt{-g(y)}\,\mathrm{d}^4y; \tag{4.7}$$

meaning that the action might be expressed as

$$S_{\mathrm{M}}[\phi] = -\frac{1}{2}\int \sqrt{-g(x)}\,\mathrm{d}^4x\int \phi(x)K(x,y)\phi(y)\sqrt{-g(y)}\,\mathrm{d}^4y; \tag{4.8}$$

where we defined the function $K(x,y) \equiv \left[\partial_\nu\partial^\nu + \mu^2 + \xi R(x)\right]\delta^4(x-y)/\sqrt{-g(x)}$. But what exactly does $K(x,y)$ represent? Considering the definition of an inverse matrix

$$\int K(x,y)K^{-1}(y,z)\sqrt{-g(y)}\,\mathrm{d}^4y = \frac{\delta^4(x-z)}{\sqrt{-g(z)}}, \tag{4.9}$$

one ends up with the following relation

$$\left[\partial_\nu\partial^\nu + \mu^2 + \xi R(x)\right]K^{-1}(x,z) = \frac{\delta^4(x-z)}{\sqrt{-g(z)}}. \tag{4.10}$$

This expression can be seen (provided that the appropriate boundary conditions at the poles of its Fourier transform are met) as the action of the (linear) differential operator $\left[\partial_\nu\partial^\nu + \mu^2 + \xi R(x)\right]$ on a Green function $-K^{-1}(x,z)$; in particular, it represents the relation between said operator and the Feynman propagator $G_F(x,z) \equiv -i\langle 0|\mathrm{T}\left(\phi(x)\phi(z)\right)|0\rangle$, meaning that

$$K(x,y) = -G_F^{-1}(x,y). \tag{4.11}$$

Using this relation, we can express the action as

$$S_{\mathrm{M}}[\phi] = \frac{1}{2}\int \sqrt{-g(x)}\,\mathrm{d}^4x\int \phi(x)G_F^{-1}(x,y)\phi(y)\sqrt{-g(y)}\,\mathrm{d}^4y, \tag{4.12}$$

which can be interpreted as the product of matrices of continuous index $\phi^\dagger G_F\,\phi$ (where G_F is the operator related to the propagator $G_F(x,y)$ through $G_F(x,y) = \langle x|G_F|y\rangle$), and thus, one can compute the value of $Z[0]$ as

$$Z[0] = \int \mathcal{D}[\phi]\,\exp\left\{i\frac{\phi^\dagger G_F^{-1}\phi}{2\hbar}\right\} \propto \left[\hbar^{-1}\det\left(-G_F\right)\right]^{-1/2}. \tag{4.13}$$

Substituting this value in equation 4.5, and by appropriately choosing the value of the constant C to compensate for the proportionality factor [1], one can deduce the following expression for the effective action

$$W = -\frac{i\hbar}{2}\ln\left[\mu_0^{-2}\det\left(-G_F\right)\right] = -\frac{i\hbar}{2}\mathrm{Tr}\left[\ln\left(-\mu_0^{-2}G_F\right)\right]. \tag{4.14}$$

Here we introduced the trace of the operator $\ln\left(-\mu_0^{-2}G_F\right)$, this can be computed using the definition of the trace,

$$\mathrm{Tr}[A] \equiv \int A(x,x)\sqrt{-g(x)}\,\mathrm{d}^4x = \int \langle x|A|x\rangle\sqrt{-g}\,\mathrm{d}^4x; \tag{4.15}$$

which will be relevant in what follows.

[1] Nevertheless, if another choice were to be made, the sum of both the constant C and the logarithm of the proportionality factor is not a function of the metric, and thus, it is irrelevant under variations of $g_{\mu\nu}$.

4.1.1 Renormalization

Even for a flat background, it is well known that the energy-momentum tensor is divergent in nature; the main representative of said property is the vacuum energy (for example, one might see the Hamiltonian given by 2.18). The solution to this issue is obtained by a renormalization program, in particular we use the dimensional regularization procedure in what follows, for which we would need to use the DeWitt-Schwinger representation of the propagator, given by

$$G_F^{\mathrm{DS}}(x,y,\mu) \equiv -i\frac{\Delta^{1/2}(x,y)}{(4\pi)^{(d+1)/2}}\int_0^\infty F(x,y;is)\exp\left[-is\mu^2+\frac{\sigma(x,y)}{2is}\right](is)^{-(d+1)/2}\,\mathrm{d}(is); \tag{4.16}$$

more information on this representation can be found in the appendix; as for now, the following definitions might be needed:

- d indicates the number of spacial dimensions.
- $\Delta(x,y) \equiv -\det\left[\partial_\mu\partial^\nu\sigma(x,y)\right]\left[g(x)g(y)\right]^{-1/2}$ is the so-called Van Vleck determinant.
- $\sigma(x,y)$ is one half of the squared geodesic distance between two events x and y.
- $F(x,y;is)$ is a geometry-dependent function, which will be expanded as a power series of $(is)^n$.

The usefulness of said representation is apparent once the propagator G_F is written in the following integral form

$$G_F = -K^{-1} = -\int_0^\infty e^{-isK}\mathrm{d}(is), \tag{4.17}$$

which can be compared to the DeWitt-Schwinger representation to deduce the following expression

$$\left\langle x\left|e^{-isK}\right|y\right\rangle = i\frac{\Delta^{1/2}(x,y)}{(4\pi)^{(d+1)/2}}F(x,y;is)\exp\left[-is\mu^2+\frac{\sigma(x,y)}{2is}\right](is)^{-(d+1)/2}; \tag{4.18}$$

which in turn implies that

$$\left\langle x\left|(is)^{-1}e^{-isK}\right|y\right\rangle = i\int_{\mu^2}^\infty \frac{\Delta^{1/2}(x,y)}{(4\pi)^{(d+1)/2}}F(x,y;is)\exp\left[-is\,m^2+\frac{\sigma(x,y)}{2is}\right](is)^{-(d+1)/2}\,\mathrm{d}m^2. \tag{4.19}$$

The connection with the effective action W given by equation 4.14 comes from the exponential integral function, since

$$\int_0^\infty (is)^{-1}e^{-isK}\mathrm{d}(is) = -\ln\left(\mu_0^2\,K\right)+\mathrm{C}' = \ln\left(-\mu_0^{-2}\,G_F\right)+\mathrm{C}'; \tag{4.20}$$

therefore, with the trace definition given in equation 4.15, one deduces that the effective action can be expressed as the following function of the DeWitt-Schwinger representation of the propagator,

$$W = \frac{i\hbar}{2}\int_{\mu^2}^\infty \mathrm{d}m^2\int G_F(x,x,m)\,\sqrt{-g(x)}\,\mathrm{d}^4x; \tag{4.21}$$

from which one can define its Lagrangian,

$$\mathcal{L}_{\mathrm{eff}} \equiv \lim_{y\to x}\frac{i\hbar c}{2}\int_{\mu^2}^\infty G_F^{\mathrm{DS}}(x,y,m)\,\mathrm{d}m^2; \tag{4.22}$$

from which we will base the renormalization procedure.

The dimensional renormalization considers a theory with some spacial dimension d, which at the end will approach the value $d=3$. Here we must consider the units of each quantity, since we desire that they remain the same of the $3+1$ case; upon integration of the last expression, one obtains

$$\mathcal{L}_{\mathrm{eff}} = \lim_{y\to x}\frac{\hbar c}{2\mu_0^{d-3}}\frac{\Delta^{1/2}(x,y)}{(4\pi)^{(d+1)/2}}\int_0^\infty F(x,y;is)\exp\left[-is\mu^2+\frac{\sigma(x,y)}{2is}\right](is)^{-(d+3)/2}\,\mathrm{d}(is); \tag{4.23}$$

where the term μ_0^{d-3} was introduced so that the units of $\mathcal{L}_{\text{eff}}$ remain the same as for $d=3$ (note that for such case the term equals 1 with no units). Considering the values of d for which this expression is convergent, and taking the limit $y \to x$; we can expand the function $F(x;is)$ as a power series of the form

$$F(x;is) \approx \sum_{n=0}^{\infty} a_n(x)(is)^n; \tag{4.24}$$

and obtain the following expression of the effective Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{eff}} = \frac{\hbar c}{2\mu_0^{d-3}} \frac{1}{(4\pi)^{(d+1)/2}} \sum_{n=0}^{\infty} a_n(x) \int_0^{\infty} \exp\left(-is\mu^2\right)(is)^{n-(d+3)/2}\, \mathrm{d}(is) = \\ = \frac{\hbar c}{2(4\pi)^{(d-1)/2}} \left(\frac{\mu}{\mu_0}\right)^{(d-3)} \sum_{n=0}^{\infty} a_n(x)\mu^{2(2-n)}\Gamma\left(n - \frac{d+1}{2}\right).\end{aligned} \tag{4.25}$$

Now we can pinpoint the divergences of the Lagrangian, which come from the terms proportional to $a_0(x)$, $a_1(x)$ and $a_2(x)$ since the Gamma functions diverge at those values of n at the limit $d \to 3$ as follows:

$$\Gamma\left(-\frac{d+1}{2}\right) = \frac{4}{d^2-1}\left(\frac{2}{3-d} - \gamma\right) + \mathcal{O}(d-3), \tag{4.26a}$$

$$\Gamma\left(1-\frac{d+1}{2}\right) = \frac{2}{1-d}\left(\frac{2}{3-d} - \gamma\right) + \mathcal{O}(d-3), \tag{4.26b}$$

$$\Gamma\left(2-\frac{d+1}{2}\right) = \frac{2}{3-d} - \gamma + \mathcal{O}(d-3). \tag{4.26c}$$

From this, one can define the divergent term $\mathcal{L}_{\text{eff}}^{\infty}$ of the Lagrangian as[2]

$$\mathcal{L}_{\text{eff}}^{\infty} = -\lim_{d\to 3} \frac{\hbar c}{2(4\pi)^2}\left\{\frac{1}{d-3} + \frac{1}{2}\left[\gamma + 2\ln\left(\frac{\mu}{\mu_0}\right)\right]\right\}\left[\mu^4 a_0(x) - \mu^2 a_1(x) + 2a_2(x)\right]; \tag{4.27}$$

where $a_0(x)$, $a_1(x)$ and $a_2(x)$ are given by the following equations

$$a_0(x) = 1, \quad a_1(x) = \left(\frac{1}{6} - \xi\right) R, \tag{4.28 a,b}$$

$$a_2(x) = \frac{1}{180}\left(R_{\alpha\beta\gamma\sigma}R^{\alpha\beta\gamma\sigma} - R_{\alpha\beta}R^{\alpha\beta}\right) - \frac{1}{6}\left(\frac{1}{5} - \xi\right)\partial_\nu\partial^\nu R + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2; \tag{4.28 c}$$

which are dependent on the metric geometry.

As previously mentioned, we do not need to disregard the divergent terms of the theory, since they should be felt by the gravitational field; to solve this problem, one should pay attention to the geometry dependence of the divergence terms, finding that they could be reabsorbed into the Einstein field equations. Considering that the gravitational Lagrangian is given by $\mathcal{L}_G \equiv {}^1\!/_{2\kappa}\,(R - \Lambda)$, then we could define an effective gravitational Lagrangian as

$$\mathcal{L}'_{\text{G}} \equiv \mathcal{L}_{\text{G}} - \mathcal{L}_{\text{eff}}^{\infty} = \left(A + \frac{1}{2\kappa}\right) R - \left(B + \frac{1}{\kappa}\Lambda\right) - \lim_{d\to 3} \frac{a_2(x)\hbar c}{(4\pi)^2}\left\{\frac{1}{d-3} + \frac{1}{2}\left[\gamma + 2\ln\left(\frac{\mu}{\mu_0}\right)\right]\right\}; \tag{4.29}$$

[2] To obtain this expression, one must consider the following representation

$$\left(\frac{\mu}{\mu_0}\right)^{d-3} = 1 + (d-3)\ln\left(\frac{\mu}{\mu_0}\right) + \mathcal{O}\left[(d-3)^2\right]$$

where two new (divergent) quantities were defined,

$$A \equiv \lim_{d\to 3} \frac{\mu^2\,\hbar c}{2(4\pi)^2}\left(\frac{1}{6}-\xi\right)\left\{\frac{1}{d-3}+\frac{1}{2}\left[\gamma+2\ln\left(\frac{\mu}{\mu_0}\right)\right]\right\}, \tag{4.30 a}$$

$$B \equiv \lim_{d\to 3} \frac{\mu^4\,\hbar c}{2(4\pi)^2}\left\{\frac{1}{d-3}+\frac{1}{2}\left[\gamma+2\ln\left(\frac{\mu}{\mu_0}\right)\right]\right\}. \tag{4.30 b}$$

Doing so would allow us to absorb the divergences and renormalize both the gravitational constant G and the cosmological constant Λ as

$$G' \equiv G\,(1+2\kappa A)^{-1}\,, \quad \Lambda' \equiv (\Lambda + B\kappa)\,. \tag{4.31 a,b}$$

But, what is the meaning of such an absorption? One might ask how the renormalization can result in the known values of the constants, specially when the added terms are divergent. To answer this question, one must reconsider what is experimentally measured, the renormalized value G' or the bare constant G? Since there is no way of measuring bare constants (given that the fields are present even in the vacuum); the only measurable quantities would be the ones we called renormalized and, as a result, one finds that the bare constants could be also divergent, but never to be measured.

This absorption might work for the first two divergent terms (that is, $a_0(x)$ and $a_1(x)$), but as shown in equation 4.30, there are terms not present in the Einstein field Lagrangian, so one might ask about this extra term. From the expression of $a_2(x)$, one finds that the geometry dependence is of fourth order of derivatives of $g_{\mu\nu}$; which are not present on standard general relativity, but are considered on modified theories of gravity such as four derivatives Stelle theory [22]. In such theories [18], the Hilbert action is modified as $S[g]=\int \frac{1}{2\kappa}f(R)\sqrt{-g}\,\mathrm{d}^4x$, where the function $f(R)$ depends on a linear combination of up to four derivatives, usually described as $f(R)\equiv 2\Lambda+R+\alpha R^2-\beta R_{\mu\nu}R^{\mu\nu}$; such theories have some attractive qualities such as their renormalizable nature (at the expense of the need of the loss of unitarity as a result of an emergent ghost particle); and the absence of singularities on black hole solutions.

With this, we can finally define our renormalized theory, given by the following action

$$S[g,\phi] = \int \left[\frac{1}{2\kappa}f(R)-\mathcal{L}_{\text{eff}}^{\infty}\right]\sqrt{-g}\,\mathrm{d}^4x + W_{\text{ren}}; \tag{4.32}$$

the expression of W_{ren} is given by (see [6] eq. 6.89) the difference of the effective Lagrangian, and its divergent part

$$W_{\text{ren}} = \int \left[\mathcal{L}_{\text{eff}}-\mathcal{L}_{\text{eff}}^{\infty}\right]\sqrt{-g}\,\mathrm{d}^4x = -\frac{\hbar}{64\pi^2}\int\int_0^\infty \ln(is)\partial_{is}^3\left[F(x;is)e^{-is\mu^2}\right]\sqrt{-g}\,\mathrm{d}(is)\mathrm{d}^4x. \tag{4.33}$$

4.2 The Conformal Anomaly

As for many other quantum theories, some results resulting from the classical theory might break upon quantization, such is the case of the so-called conformal anomaly that will be discussed in this section.

Considering a conformal scalar field described by the action 1.14 with $\mu=0$ and $\xi={}^1\!/\!_6$, it is easy to check that the energy momentum tensor is traceless (from equation 1.20), that is

$$T_\nu^\nu = \frac{1}{2}\,(6\xi-1)\,\partial_\sigma\partial^\sigma\phi^2+\mu^2\phi^2 = 0; \tag{4.34}$$

but the vacuum expectation of the renormalized trace (which we will simply denote as $\langle T_\nu^\nu\rangle_{\text{ren}}$) will not be zero.

One can convince oneself that the vacuum expectation of the trace is indeed zero;

$$\langle T_\nu^\nu\rangle \equiv \langle T_\nu^\nu\rangle_{\text{div}}+\langle T_\nu^\nu\rangle_{\text{ren}} = 0; \tag{4.35}$$

from which one obtains that $\langle T^{\nu}_{\nu}\rangle_{\text{ren}} = -\langle T^{\nu}_{\nu}\rangle_{\text{div}}$. Here we used the labels *div* and *ren* not to tell their finitude; but to determine the origin of those quantities, meaning that $\langle T^{\nu}_{\nu}\rangle_{\text{ren}}$ should be computed from W_{ren}; as we will see, both quantities will be finite.

By doing this, it will be simpler to compute the expression for $\langle T^{\nu}_{\nu}\rangle_{\text{div}}$, from which we will later recover the desired value. To do so, we compute the effective divergent action W_{div} from the effective divergent Lagrangian $\mathcal{L}^{\infty}_{\text{eff}}$; from equation 4.25 one can see that the only relevant term for W_{div} will be the term proportional to $a_2(x)$, resulting in

$$W_{\text{div}} = \frac{\hbar}{2(4\pi)^2} \lim_{\substack{d\to 3\\ \mu\to 0}} \left(\frac{\mu}{\mu_0}\right)^{(d-3)} \Gamma\left(2-\frac{d-1}{2}\right) \int a_2(x)\sqrt{-g}\,\mathrm{d}^4x. \tag{4.36}$$

For a conformal theory, $a_2(x)$ can be written as

$$a_2(x) = \alpha\left[F(x) - \frac{2}{3}\partial_\sigma\partial^\sigma R\right] + \beta G(x) \tag{4.37}$$

plus a term proportional to $(d-3)^2$ which will vanish once the limit $d\to 3$ is taken. In this form, the coefficients are $\alpha \equiv {}^{1}/_{120}$ and $\beta \equiv -{}^{1}/_{360}$; and the functions are given by

$$F(x) \equiv R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 2R^{\alpha\beta}R_{\alpha\beta} + \frac{1}{3}R^2, \quad G(x) \equiv R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta}R_{\alpha\beta} + R^2. \tag{4.38 a,b}$$

To obtain the value of $\langle T^{\nu}_{\nu}\rangle_{\text{div}}$ one needs to compute

$$\langle T^{\nu}_{\nu}\rangle_{\text{div}} \equiv \frac{2g^{\mu\nu}}{\sqrt{-g}}\frac{\delta W_{\text{div}}}{\delta g^{\mu\nu}}; \tag{4.39}$$

which can be done using the following equalities [11]

$$\frac{2g^{\mu\nu}}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int F(x)\sqrt{-g}\,\mathrm{d}^{d+1}x = -(d-3)\left[F(x) - \frac{2}{3}\partial_\sigma\partial^\sigma R\right], \tag{4.40 a}$$

$$\frac{2g^{\mu\nu}}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int G(x)\sqrt{-g}\,\mathrm{d}^{d+1}x = -(d-3)G(x). \tag{4.40 b}$$

The result of said computation is given by the next expression

$$\langle T^{\nu}_{\nu}\rangle_{\text{div}} = \frac{\hbar}{2(4\pi)} \lim_{\substack{d\to 3\\ \mu\to 0}} \left(\frac{\mu}{\mu_0}\right)^{(d-3)} (d-3)\,\Gamma\left(2-\frac{d-1}{2}\right)\left[\alpha\left[F(x) - \frac{2}{3}\partial_\sigma\partial^\sigma R\right] + \beta G(x)\right], \tag{4.41}$$

which will deliver the value of $\langle T^{\nu}_{\nu}\rangle_{\text{ren}}$ once its relationship with its *div* counterpart is considered alongside the Gamma function expansion given by equation 4.26.c and the definition of $a_2(x)$ (although there are some ambiguities resulting in the presence of total derivatives [21], [4]):

$$\langle T^{\nu}_{\nu}\rangle_{\text{ren}} = -\frac{a_2(x)\hbar}{16\pi^2}. \tag{4.42}$$

As it turns out [6, sec. 6.4] this is enough information to recover $\langle T^{\nu}_{\mu}\rangle_{\text{ren}}$ (that is, the vacuum expectation value of the tensor, not just its trace) if both the background and the quantum field are conformally flat. As a relevant cosmological example, one can compute the value of said quantity for a de Sitter universe described by a metric given by equation 2.3 with a scale factor $a(t) = \exp(H_\Lambda t)$, resulting in

$$\langle T^{\nu}_{\mu}\rangle_{\text{ren}} = \frac{H_\Lambda^4\hbar}{960\pi^2c^4}\delta^{\nu}_{\mu}, \tag{4.43}$$

which (as expected from a de Sitter Universe) corresponds with a "dark energy" with negative pressure P and energy density $\rho = -P = {}^{H_\Lambda^4\hbar}/_{(960\pi^2c^4)}$.

5 Final Discussion

The theory of Quantum Fields in Curved Spacetimes might be the best probe into experimental and observational testing on new Quantum Gravity proposals, standing on the fringe of current physics knowledge. Although mostly accepted by the community, it lacks experimental and observational support; the sheer magnitude of cosmological metrics combined with the measurement precision needed to observe quantum phenomenology creates a technological barrier that has yet to be overcome. Even though no direct evidence has been found, analogue models of the Unruh effect made in condensed matter systems [17] have measured the expected radiation predicted by Hawking.

The framework presented in this thesis is considered as the best approach to date to model the interaction of matter and gravitational fields within current physical models. This is what is expected to be recovered at the infrared range of a quantum gravity unification theory; despite the fact that no definitive model of said theory has been found, there are proposals of phenomenology to be found at low energies (see [2] for a recent comprehensive report).

This thesis has looked into simple models like scalar fields or backgrounds with plenty of symmetries, alongside the phenomenology that might results from them, such as the creation of new particles, different vacua definitions and thermal baths for non-inertial observers. Further areas of interest of advanced complexity within the field might include the study of Spinor Bundles, scattering processes; gravitational perturbation theory or any of the countless theorems resulting from this framework, such as the Harrison-Zeldovich theorem, which states that conformal theories do not create massive particles. Said results could have substantial relevance for cosmological evolution and astrophysical formations. It is in this regard that physicists hold the Theory of Quantum Fields in Curved Spacetimes, since it might help to deepen our knowledge on unexpected effects that otherwise would have not been considered, such as black hole evaporation, and to add new questions to guide the research to new horizons, such as the problem with information loss as a result of black hole evaporation.

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Scalar Field in Minkowski Background

Throughout this thesis, the main field used to introduce the theory of quantum field in curved spacetimes, was the scalar field. Therefore, it would be good practice to introduce the theory of scalar fields on a Minkowski background.

Introduction

The simplest action for a (real) scalar field without interactions, might be

$$S[\phi] = \int \frac{1}{2}\left[\partial_\nu \phi\, \partial^\nu \phi - \mu^2 \phi^2\right] \mathrm{d}^4 x. \tag{1}$$

From this action, one can obtain the equations of motion of the field from the Euler-Lagrange equations, which would yield as a result the so-called Klein-Gordon equation,

$$\left(\partial_\nu \partial^\nu - \mu^2\right) \phi = 0, \tag{2}$$

with solutions of the form

$$\phi_{\mathbf{k}} = a_{\mathbf{k}} e^{ikx\,\hbar^{-1}} + \bar{a}_{\mathbf{k}} e^{-ikx\,\hbar^{-1}}. \tag{3}$$

Substitution of this solution in the Klein-Gordon equation results in the following dispersion relation

$$k_\nu k^\nu = \hbar^2 \mu^2, \tag{4}$$

which gives a relation between the parameter μ and the mass of the field m through $\mu = {}^{mc}/_{\hbar}$.

The most general solution of the Klein-Gordon equation can be written as a mode expansion of solutions of the form 3, which would be

$$\phi(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3 2k_0} \phi_{\mathbf{k}} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3 2k_0} \left(a_{\mathbf{k}} e^{ikx\,\hbar^{-1}} + \bar{a}_{\mathbf{k}} e^{-ikx\,\hbar^{-1}}\right); \tag{5}$$

where $\frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3 2k_0}$ is a Lorentz-invariant measure. It is convenient to redefine $a_{\mathbf{k}} \to a_{\mathbf{k}}(2k_0)^{-1/2}$, and thus, the field would be described as

$$\phi(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3 \sqrt{2k_0}} \left(a_{\mathbf{k}} e^{ikx\,\hbar^{-1}} + \bar{a}_{\mathbf{k}} e^{-ikx\,\hbar^{-1}}\right). \tag{6}$$

Quantization

The methodology to canonically quantize a field comes from the promotion of the field $\phi(x)$ and its conjugated momenta $\Pi(x) \equiv \partial_{ct}$ into quantum operators,

$$\phi(x) \longrightarrow \hat{\phi}(x), \qquad \Pi(x) \longrightarrow \hat{\Pi}(x),$$

to do so, the most common procedure is to promote the mode constant factors to quantum operators, which will be known as annihilation and creator operators,

$$a_{\mathbf{k}} \longrightarrow \hat{a}_{\mathbf{k}}, \qquad \bar{a}_{\mathbf{k}} \longrightarrow \hat{a}^\dagger_{\mathbf{k}}.$$

This will imply that the quantum field operator $\hat{\phi}(x)$ will have the form

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3 \sqrt{2k_0}} \left(\hat{a}_{\mathbf{k}} e^{ikx\,\hbar^{-1}} + \hat{a}^\dagger_{\mathbf{k}} e^{-ikx\,\hbar^{-1}}\right). \tag{7}$$

In addition to the promotion, some commutation rules must be imposed, to do so, Dirac proposes the following procedure: replacing the Poisson Brackets of the phase space with commutators, as

$$\{A, B\} \to \frac{1}{i\hbar}\left[\hat{A}, \hat{B}\right]; \tag{8}$$

the Poisson Bracket for a coordinate q_i and a conjugate momentum p_j is $\{q_i, p_j\} = \delta_{ij}$, therefore it is natural to consider as commutation rules the following,

$$\left[\hat{\phi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})\right] = i\hbar\,\delta^3(\mathbf{x}-\mathbf{y}) \qquad \left[\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{y})\right] = \left[\hat{\Pi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})\right] = 0.. \tag{9 a-c}$$

Substitution of the quantum field expression in these commutators will give the needed commutation rules for the annihilation and creation operators:

$$\left[\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{q}}\right] = (2\pi\hbar)^3\,\hbar^2\delta^3(\mathbf{k}-\mathbf{q}), \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = \left[\hat{a}^{\dagger}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{q}}\right] = 0. \tag{10 a-c}$$

Hamiltonian and Fock space

The most relevant quantum operator is without doubt, the Hamiltonian $\hat{\mathcal{H}}$, which is given by Noether's theorem as the conserved current

$$\hat{\mathcal{H}} = \int\left[\frac{\partial\hat{\mathcal{L}}}{\partial\left(\partial_0\hat{\phi}\right)} - \hat{\mathcal{L}}\right]\mathrm{d}^3\mathbf{x}; \tag{11}$$

for the given field, one obtains the following expression for the Hamiltonian

$$\hat{\mathcal{H}} = \int\left(\hat{\Pi}\partial_0\hat{\phi} - \hat{\mathcal{L}}\right)\mathrm{d}^3\mathbf{x} = \int\frac{c}{2}\left[\hat{\Pi}^2 + \left(\nabla\hat{\phi}\right)^2 + \mu^2\hat{\phi}^2\right]\mathrm{d}^3\mathbf{x}; \tag{12}$$

and once the quantum field expansion is substituted, it will be simplified as

$$\hat{\mathcal{H}} = \int E_{\mathbf{p}}\left[\hat{a}_{\mathbf{p}}\hat{a}^{\dagger}_{\mathbf{p}} + \frac{1}{2}(2\pi\hbar)^3\hbar^2\delta(\mathbf{0})\right]\frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3\hbar^2}. \tag{13}$$

Note that the constant term

$$\hat{\mathcal{H}}_{\mathrm{div}} = \frac{1}{2}(2\pi\hbar)^3\hbar^2\delta(\mathbf{0})\int E_{\mathbf{p}}\frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3\hbar^2} \tag{14}$$

is divergent, and known as vacuum energy; it is useful to define a 'normal" ordered Hamiltonian without this term, which will be zero for a vacuum state

$$:\hat{\mathcal{H}}:= \int E_{\mathbf{p}}\ \hat{a}_{\mathbf{p}}\hat{a}^{\dagger}_{\mathbf{p}}\frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3\hbar^2}. \tag{15}$$

The Fock space $\{|\mathbf{p}\rangle\}$ generated by the Hamiltonian is formed from a vacuum (no particles) state $|0\rangle$ which is annihilated by $\hat{a}_{\mathbf{p}}$, i.e.

$$\hat{a}_{\mathbf{p}}|0\rangle = 0; \tag{16}$$

other one particle states, with a given momentum $\mathbf{p}$, are formed from the vacuum state after applying the creator operator

$$|\mathbf{p}\rangle \equiv \hat{a}^{\dagger}_{\mathbf{p}}|0\rangle; \tag{17}$$

similar to a quantum oscillator. Multiparticle states are formed after the chain use of the creator operators

$$|\mathbf{p}_1, \mathbf{p}_2, \ldots\rangle \equiv \ldots\hat{a}^{\dagger}_{\mathbf{p}_2}\hat{a}^{\dagger}_{\mathbf{p}_1}|0\rangle; \tag{18}$$

note that this quantum states are bosonic, since

$$|\mathbf{p}_1, \mathbf{p}_2\rangle = \hat{a}^{\dagger}_{\mathbf{p}_2}\hat{a}^{\dagger}_{\mathbf{p}_1}|0\rangle = \left[\hat{a}^{\dagger}_{\mathbf{p}_2}, \hat{a}^{\dagger}_{\mathbf{p}_1}\right]|0\rangle + \hat{a}^{\dagger}_{\mathbf{p}_1}\hat{a}^{\dagger}_{\mathbf{p}_2}|0\rangle = |\mathbf{p}_2, \mathbf{p}_1\rangle. \tag{19}$$

DeWitt-Schwinger Representation

One of the most important objects in a theory of quantum fields are the correlation functions; in particular one of them takes on greater significance: the Feynman propagator $G_F(x,x')$, which can be defined for a free scalar field as the solution of the following homegeneous equation

$$\left[\partial_\nu \partial^\nu + \mu^2 + \xi R(x)\right] G_F(x,x') = -\left[-g(x)\right]^{-1/2} \delta^{(d+1)}(x-x'). \tag{1}$$

In what follows, we will present a particularly useful representation of the scalar field propagator following the treatment given by [9]. This representation has important applications on regularization, as was presented in the corresponding section of this thesis (sec. 4).

Before presenting the mathematical formalism, one must introduce the Riemann normal coordinates y^μ for a given given point x, whose origin is considered x'. Using these particular coordinates, one might expand

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{1}{3} R_{\mu\alpha\nu\beta}\, y^\alpha y^\beta - \frac{1}{6} \nabla\gamma R_{\mu\alpha\nu\beta}\, y^\alpha y^\beta y^\gamma + \left[\frac{1}{20} \nabla_\gamma \nabla_\delta\, R_{\mu\alpha\nu\beta} + \frac{2}{45} R_{\alpha\mu\beta\lambda} R^\lambda{}_{\gamma\nu\delta}\right] y^\alpha y^\beta y^\gamma y^\delta + \ldots, \tag{2}$$

where all coefficients are evaluated at $y=0$.

Next, a new two-point function is defined from $G_F(x,x')$ as

$$\mathcal{G}_F(x,x') \equiv \left[-g(x)\right]^{-1/4} G_F(x,x'), \tag{3}$$

alongside its Fourier transform

$$\mathcal{G}_F(x,x') = \int \frac{\mathrm{d}^{d+1}k}{(2\pi\hbar)^{d+1}} \mathcal{G}_F(k) e^{-iky\hbar^{-1}}; \tag{4}$$

where $ky \equiv \eta_\mu \nu k^\mu y^\nu$.

By using Riemann normal coordinates in the propagator equation, and converting to k-space, $\mathcal{G}_F(k)$ can be solved by iteration to any desired derivative order (also known as adiabatic order). In particular, for order 4 one obtains

$$\mathcal{G}_F(k) \approx \hbar^2 \left(k^2 - \mu^2\hbar^2\right)^{-1} - \hbar^4 \left(\frac{1}{6} - \xi\right) \left[R - \frac{i}{2} \left(\nabla_\alpha R\right) \partial^\alpha + \frac{1}{3} a_{\alpha\beta} \partial^\alpha \partial^\beta\right] \left(k^2 - \mu^2\hbar^2\right)^{-2} + \\ + \hbar^3 \left[\frac{2}{3} a^\lambda_\lambda - \left(\frac{1}{6} - \xi\right)^2 R^2\right] \left(k^2 - \mu^2\hbar^2\right)^{-3}, \tag{5}$$

with the derivative referring to k-space, that is, $\partial_\alpha \equiv \partial/\partial k^\alpha$; and the element $a_{\alpha\beta}$ defined as

$$a_{\alpha\beta} \equiv \left[\frac{1}{2}\left(\frac{1}{6} - \xi\right) + \frac{1}{120}\right] \nabla_\alpha \nabla_\beta\, R - \frac{1}{40} \nabla_\lambda \nabla^\lambda\, R_{\alpha\beta} - \frac{1}{30} R_\alpha^{\ \lambda} R_{\lambda\beta} + \frac{1}{60} R^{\kappa\ \lambda}_{\ \alpha\ \beta} R_{\kappa\lambda} + \frac{1}{60} R^{\lambda\mu\kappa}{}_\alpha R_{\lambda\mu\kappa\beta}. \tag{6}$$

If one were to write

$$\mathcal{G}_F(x,x') = \int \frac{\mathrm{d}^{d+1}k}{(2\pi\hbar)^{d+1}} e^{iky\hbar^{-1}} \left[a_0(x,x') - a_1(x,x') \frac{\partial}{\partial\mu^2} + a_2(x,x') \left(\frac{\partial}{\partial\mu^2}\right)^2\right] \hbar^2 \left(k^2 - \mu^2\hbar^2\right)^{-1}, \tag{7}$$

then (up to fourth adiabatic order), the element $a_i(x,x')$ must equal

$$a_0(x,x') = 1, \quad a_1(x,x') = \left(\frac{1}{6} - \xi\right) \left(R - \frac{1}{2} \nabla_\alpha R\, y^\alpha\right) - \frac{1}{3} a_{\alpha\beta} y^\alpha y^\beta \tag{8 a,b}$$

$$a_2(x,x') = \frac{1}{2} \left(\frac{1}{6} - \xi\right)^2 R^2 + \frac{1}{3} a^\lambda_\lambda; \tag{8 c}$$

all right hand side elements evaluated at x'.

The next step is to consider the integral representation

$$\hbar^2(k^2 - \mu^2\hbar^2 + i\epsilon)^{-1} = -\int_0^\infty e^{-is(k^2-\mu^2\hbar^2+i\epsilon)\hbar^{-2}} \mathrm{d}(is), \tag{9}$$

which, when added into equation 7 will give as a result

$$\mathcal{G}_F(x,x') = -(4\pi)^{-(d+1)/2} \int_0^\infty F(x,x';is) \exp\left[-is\mu^2 + \frac{\sigma(x,x')}{2is}\right] \mathrm{d}(is), \tag{10}$$

where $\sigma(x,x') \equiv 1/2 y_\alpha y^\alpha$ can be be understood as half of the squared geodesic distance between x and x'. Here we have introduced the function $F(x,x';is)$ representing the dependence of the $a_i(x,x')$ terms up to an adiabatic order N, such that

$$F(x,x';is) \equiv \sum_{n=0}^{N/2} a_n(x,x')(is)^n. \tag{11}$$

Finally, using the relation between $\mathcal{G}_F(x,x')$ and the propagator $G_F(x,x')$, one can obtain what is known as the DeWitt-Schinger representation of the scalar propagator; given by

$$G_F^{\mathrm{DS}}(x,x') \equiv -i\frac{\Delta^{1/2}(x,x')}{(4\pi)^{(d+1)/2}} \int_0^\infty F(x,y;is) \exp\left[-is\mu^2 + \frac{\sigma(x,x')}{2is}\right] (is)^{-(d+1)/2} \mathrm{d}(is); \tag{12}$$

where $\Delta(x,x')$ is the so called Van Vleck determinant, defined as

$$\Delta(x,x') \equiv -\det\left[\partial_\mu\partial_\nu\sigma(x,x')\right] \left[g(x)g(x')\right]^{-1/2}. \tag{13}$$

As a last note, the equality only really holds if the closed expression of the function $F(x,x';is)$ is known, otherwise it must be understood as a limiting expansion, in which each $a_i(x,x')$ term follows some recursive rules.