PROBLEMS OF QUANTUM FIELD THEORIES IN CURVED SPACETIMES

A MASTER THESIS

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Abrstract

Quantum Field Theory is the fundamental theoretical structure of the Standard Model of elementary particles. This theory is formulated in a Minkowski space-time. However, the actual space-time metric is never of that type. On Earth, even at short distances, the metric is affected by both the force of the Earth's gravity and the solar force, but especially by the acceleration of the Earth's motion. At large distances the cosmological data lead one to think that, on average, the metric is of the Friedmann-Lemaitre-Roberson-Walker type.

The analysis of the quantization of fields in the presence of gravitational fields involves a number of theoretical connotations that we intend to explore in this paper. How Quantum Field Theory can be adapted when considering this gravitational background is the object of the proposal. In particular, how the structure of the quantum vacuum is affected when space-time is neither asymptotically Minkowskian, as is the case in the current Cosmological Model LCDM. There are a number of attempts to address the problem but to date none provides a satisfactory solution; the aim of the present work is to find the one that best fits the experimental observations.

Contents

Conventions

The chosen convention for the metric signature will be $(+, -, -, -)$ as in [2] and most literature on particle physics. Common conventions and nomenclatures in mathematics and physics are used throughout the text, some of which are considered to be relevant:

x^μ, x	four-vector
x^i, $\mathbf{x}$	spacial vector
$g_{\mu\nu}$	general spacetime metric
$\eta_{\mu\nu}$	Minkowski spacetime metric
g	determinant of $g_{\mu\nu}$
∇_μ	covariant derivative
$S[\phi]$	action functional of a field ϕ and it's derivatives
$\bar{z}$	complex conjugate of z
$A^\dagger$	hermitian conjugate of A
$R^\alpha_{\ \beta\gamma\delta}$	Riemann tensor $\equiv \nabla_\delta \Gamma^\alpha_{\ \beta\gamma} - \nabla_\delta \Gamma^\gamma_{\ \beta\alpha} + \dots$
$f \overleftrightarrow{\nabla}_\mu g$	$\equiv f \nabla_\nu g - (\nabla_\mu f)\, g$
γ^μ	covariant Gamma matrices $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$
G, c, $\hbar$	standard universal constants, not necessarily in natural units

Other notation will be introduced as needed.

Preface

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1 Introduction to QFT in Curved Spacetimes

1.1 The Universal Action

Consider a dynamic universe consisting of dark energy characterized by a cosmological constant Λ and some material content described by a Lagrangian density $\mathcal{L}_{\mathrm{M}}$. The action associated with such system would be

$$S = \int \left[\frac{1}{2\kappa}(R - 2\Lambda) + \mathcal{L}_{\mathrm{M}}\right]\sqrt{-g}\,\mathrm{d}^4x, \tag{1.1}$$

where $\kappa \equiv \frac{8\pi G}{c^4}$ is known as the Einstein gravitational constant.

The equations that would describe the dynamics of the system can be obtained by variations of the action presented in eq. (1.1) and the stationary-action principle, which states that the path taken by the system will result in $\delta S = 0$. The equations of motion of the matter content are dependant on its formulation; let the Lagrangian density of matter be described by some set $\{\phi^\alpha(x)\}$ of fields and their first covariant derivatives, i.e.

$$\mathcal{L}_{\mathrm{M}} = \mathcal{L}_{\mathrm{M}}\big[\phi^\alpha(x),\, \nabla_\mu\phi^\alpha(x)\big], \tag{1.2}$$

then, variations of the action S with respect of ϕ^α will result in

$$\delta S = \int \left[\frac{\partial\mathcal{L}_{\mathrm{M}}}{\partial\phi^\alpha}\delta\phi^\alpha + \frac{\partial\mathcal{L}_{\mathrm{M}}}{\partial(\nabla_\mu\phi^\alpha)}\nabla_\mu(\delta\phi^\alpha)\right]\sqrt{-g}\,\mathrm{d}^4x, \tag{1.3}$$

and thus, after applying the generalized Gauss Theorem on a general manifold, alongside the stationary-action principle, one obtains the well known Euler-Lagrange equations

$$\frac{\partial\mathcal{L}_{\mathrm{M}}}{\partial\phi^\alpha} - \nabla_\mu\left[\frac{\partial\mathcal{L}_{\mathrm{M}}}{\partial(\nabla_\mu\phi^\alpha)}\right] = 0. \tag{1.4}$$

On the other hand, variations of S with respect to the inverse metric ($g^{\mu\nu}$) results in

$$\delta S = \int \left[\frac{\sqrt{-g}}{2\kappa}\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{2\kappa}\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} - \frac{\Lambda}{\kappa}\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta\mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}} + \frac{\mathcal{L}_{\mathrm{M}}}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x; \tag{1.5}$$

and again, by imposing $\delta S = 0$ and considering that

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu}, \qquad \frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}g_{\mu\nu}, \tag{1.6 a,b}$$

one obtains the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -2\frac{8\pi G}{c^4}\left(\frac{\delta\mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}} - \frac{1}{2}\mathcal{L}_{\mathrm{M}}g_{\mu\nu}\right) \tag{1.7}$$

which are most commonly written in terms of the Hilbert energy-momentum tensor

$$T_{\mu\nu} \equiv \mathcal{L}_{\mathrm{M}}g_{\mu\nu} - \frac{\delta\mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}} = \frac{-2}{\sqrt{-g}}\frac{\delta\left(\mathcal{L}_{\mathrm{M}}\sqrt{-g}\right)}{\delta g^{\mu\nu}}. \tag{1.8}$$

This tensor is the source of the spacetime curvature, and must not be confused with the Noether's energy-momentum tensor since the two are not, in general, equivalent [1], but upon integration of the corresponding conserved currents, results are the same [5]. In addition of being symmetric, the Hilbert energy-momentum tensor is covariantly conserved, i.e.

$$\nabla_\mu T^{\mu\nu} = 0; \tag{1.9}$$

this fact is of great use once the material Hamiltonian $\mathcal{H}_\mathrm{M}$ is defined:

$$\mathcal{H}_\mathrm{M} \equiv \int T^{00} c \sqrt{-g}\, \mathrm{d}^3\mathbf{x}, \tag{1.10}$$

since it will later be used to spawn the Fock space after the quantization procedure; and thus assure no energy loses will be present on the theory.

1.2 Construction of Covariant Actions

The equivalence principle says that field equations must be invariant with respect to local Lorentz transformations $\Lambda(x)$. To work on a local flat spacetime, we use the tetrad formalism

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \tag{1.11}$$

partial derivatives transform like

$$\partial_\mu \to \frac{\partial x^\nu}{\partial y^\mu}\partial_\nu \tag{1.12}$$

but, we see that $e^\mu_a \partial_\mu$ is not covariant, since

$$e^\mu_a \partial^a \phi(x) \to \Lambda^b_a e^\mu_b \partial_\mu \left[\rho\left(\Lambda\right)\phi(x)\right] = \Lambda^b_a e^\mu_b \left[\rho\left(\Lambda\right)\partial_\mu\phi + \partial_\mu\rho\left(\Lambda\right)\,\phi\right] \tag{1.13}$$

a covariant derivative ∇_μ of some field φ should transform as

$$\nabla_a\varphi \to \Lambda^b_a \rho\left(\Lambda\right)\nabla_b\varphi \tag{1.14}$$

therefore we need to define a better derivative, a common option is

$$\nabla_a \equiv e^\mu_a\left(\partial_\mu + \Gamma_\mu\right) \tag{1.15}$$

where, for the derivative to be covariant, the connection Γ_μ must transform as

$$\Gamma_\mu \to \rho\left(\Lambda\right)\Gamma_\mu\rho^{-1}\left(\Gamma\right) - \left[\partial_\mu\rho\left(\Lambda\right)\right]\rho^{-1}\left(\Lambda\right) \tag{1.16}$$

such connection can be written as

$$\Gamma_\mu = \frac{1}{2}\Sigma^{ab}e^\nu_a\left(\partial_\mu e_{b\nu} - \Gamma^\sigma_{\nu\mu}e_{b\sigma}\right) \equiv \frac{1}{2}\Sigma^{ab}\omega_{ab\mu} \tag{1.17}$$

where Σ^{ab} are the Lorentz generators and $\omega_{ab\mu} \equiv e^\nu_a\left(\partial_\mu e_{b\nu} - \Gamma^\sigma_{\nu\mu}e_{b\sigma}\right)$ is the torsion free spin connection.

Under the presence of a gauge field, we would also want the derivative to be gauge invariant, and thus, the gauge four-potential A must be considered in the definition of the derivative

$$\nabla_a \equiv e^\mu_a\left(\partial_\mu + \Gamma_\mu - \frac{i}{\hbar}eA_\mu\right) \tag{1.18}$$

where e is the coupling constant.

1.2.1 Some Basic Examples

Scalar Field

$$S[\phi] = \int \frac{1}{2}\Big[\partial_\nu\phi\,\partial^\nu\phi - \mu^2\phi^2 - \xi R\phi^2\Big]\sqrt{-g}\;\mathrm{d}^4x \tag{1.19}$$

where $\mu \equiv {}^{mc}/_{\hbar}$, R is the curvature scalar of the manifold. For $\xi = 0$ the scalar field is said to be minimally coupled to gravity (nonminimally coupled otherwise); this is the simplest interactive term that couples the scalar field with the manifold; in curved spacetimes [3] a self interactive $\lambda\phi^4$ theory needs a term proportional to $R\phi^2$ to be renormalizable. For $\xi = {}^1/_6$ its said to be conformally coupled, and, in the case of a massless field ($\mu = 0$) the action will be invariant under conformal transformations, i.e.

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} \equiv \Omega^2(x)g_{\mu\nu}, \tag{1.20}$$

This can be seen from the equations of motion (Klein-Gordon)

$$\big[\partial_\nu\partial^\nu - \mu^2 - \xi R\big]\phi = 0, \tag{1.21}$$

considering $\phi \to \tilde{\phi} = \Omega^\beta\phi$ one gets that

$$\begin{aligned} 0 = \mu^2\Omega^{\beta-2}\left(\Omega^2 - 1\right)\phi + 2\left(1+\beta\right)\Omega^{\beta-3}\partial^\nu\Omega\partial_\nu\phi + \\ + (6\xi+\beta)\Omega^{\beta-3}\left(\partial_\nu\partial^\nu\Omega\right)\phi + \beta(1+\beta)\Omega^{\beta-4}\partial_\nu\Omega\partial^\nu\Omega\,\phi \end{aligned} \tag{1.22}$$

this equation has a solution for massless fields, corresponding to

$$\beta = -1, \qquad \xi = \frac{1}{6} \tag{1.23 a,b}$$

The energy momentum tensor has the following form

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\left[\partial^\sigma\phi\partial_\sigma\phi - \mu^2\phi^2\right] + \xi\left[-R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\partial^\sigma\partial_\sigma + \partial_\mu\partial_\nu\right]\phi^2, \tag{1.24}$$

and its trace equals

$$T^\nu_\nu = \frac{1}{2}\left(6\xi - 1\right)\partial_\sigma\partial^\sigma\phi^2 + \mu^2\phi^2, \tag{1.25}$$

meaning that, for a free scalar theory with conformal symmetry, the energy-momentum tensor 1.24 is traceless.

Dirac Field

$$\Gamma_\mu = \frac{1}{8}\omega^{ab}_\mu\left[\gamma_a,\,\gamma_b\right] \tag{1.26}$$

$$S[\psi] = \int \bar{\psi}\left[i\gamma^\mu\left(\partial_\mu + \Gamma_\mu\right) - \mu\right]\psi\sqrt{-g}\mathrm{d}^4x \tag{1.27}$$

$$\{\gamma^\mu,\,\gamma^\nu\} = 2g^{\mu\nu} \tag{1.28}$$

$$\left[i\gamma^\mu\left(\partial_\mu + \Gamma_\mu\right) - \mu\right]\psi = 0 \tag{1.29}$$

$$T_{\mu\nu} = \frac{1}{4}i\left\{\bar{\psi}\left(\gamma_\mu\nabla_\nu - \gamma_\nu\nabla_\mu\right) - \left[\left(\nabla_\mu\bar{\psi}\right)\gamma_\nu - \left(\nabla_\nu\bar{\psi}\right)\gamma_\mu\right]\right\}\psi \tag{1.30}$$

Schrödinger-Dirac equation

$$\left[\nabla_\nu\nabla^\nu - \mu^2 - \frac{1}{4}R\right]\psi = 0 \tag{1.31}$$

$\xi = \frac{1}{4}$

Electromagnetic Field

$$S[A_\mu] = \int \left(-\frac{1}{4\mu_0 c} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_{\text{Gauge}} \right) \sqrt{-g} \mathrm{d}^4 x \tag{1.32}$$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.33}$$

where the last equality is a result of the symmetry of the lower indices on the Christoffel symbols.

$$\mathcal{L}_{\text{Gauge}} = -\frac{1}{2\alpha} \left(\nabla_\nu A^\nu \right)^2 \tag{1.34}$$

$$\nabla^\nu \nabla_\nu A_\mu + R^\alpha_\mu A_\alpha - (1-\alpha) \nabla_\mu \nabla^\nu A_\nu = 0 \tag{1.35}$$

$$\begin{aligned} T_{\mu\nu} = -\frac{1}{\mu_0} \left(F_{\mu\alpha} F^{\alpha\nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) + & \\ + \alpha \left\{ A_\mu \left(\nabla_\nu \nabla_\rho A^\rho \right) + \left(\nabla_\mu \nabla_\rho A^\rho \right) A_\nu - g_{\mu\nu} \left[A^\rho \left(\nabla \rho \nabla_\sigma A^\sigma \right) + \frac{1}{2} \left(\nabla_\rho A^\rho \right)^2 \right] \right\} & \end{aligned} \tag{1.36}$$

1.3 Scalar field Quantization

Scalar product

$$\langle \phi_1(x),\, \phi_2(x) \rangle \equiv i \int g^{0\nu} \left(\phi_1 \overleftrightarrow{\partial}_\nu \phi_2^* \right) \sqrt{-g}\, \mathrm{d}^3\mathbf{x} \tag{1.37}$$

Let $v(x)$ be a solution of the Klein-Gordon equation, then $v^*(x)$ will also be an (linearly independent) solution. Let i represent the set of parameters that univocally describe a par of solutions $v_i(x)$, $v^*(x)$, therefore, the general solution of the Klein-Gordon equation will be of the form

$$\phi(x) = \sum_i \left[a_i v_i(x) + a_i^* v_i^*(x) \right] \tag{1.38}$$

where a_i, a_i^* are constant factors that can be written as

$$a_i = \langle v_i(x),\, \phi(x) \rangle \qquad a_i^* = \langle v_i^*(x),\, \phi(x) \rangle \tag{1.39}$$

Quantization of the field is done by promoting the field χ and its conjugate momentum $\Pi \equiv \partial_{ct}\chi$ to operators

$$\phi(x) \longrightarrow \hat{\phi}(x) \qquad \Pi(x) \longrightarrow \hat{\Pi}(x) \tag{1.40}$$

this is done by promoting the constant factors to operators as well, that is

$$a_i \longrightarrow \hat{a}_i \qquad a_i^* \longrightarrow \hat{a}_i^\dagger \tag{1.41}$$

and therefore

$$\hat{\phi}(x) = \sum_i \left[\hat{a}_i v_i(x) + \hat{a}_i^\dagger v_i^*(x) \right] \tag{1.42}$$

after the promotion of the fields to operators, commutation relations are imposed; the easiest choice would be to assume canonical quantization relations,

$$\left[\hat{\phi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y}) \right] = i\hbar\, \delta^3 \left(\mathbf{x} - \mathbf{y} \right) \qquad \left[\hat{\phi}(\mathbf{x}),\, \hat{\phi}(\mathbf{y}) \right] = \left[\hat{\Pi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y}) \right] = 0. \tag{1.43 a-c}$$

It would be desirable to obtain a formulation similar to the well known scalar field in a flat background, where the Fock space is generated from a vacuum state and a set of creation and annihilation operators that follow some commutation rules. To do so, we will force the $\hat{a}_i$, $\hat{a}_i^\dagger$ operators to assume this roll, in such a way that

$$\left[\hat{a}_i,\, \hat{a}_j^\dagger \right] \propto \delta_{ij} \qquad \left[\hat{a}_i,\, \hat{a}_j \right] = \left[\hat{a}_i^\dagger,\, \hat{a}_j^\dagger \right] = 0 \tag{1.44}$$

Thanks to the relation between the constant factors a_i and the scalar product $\langle v_i, \phi\rangle$, one can obtain

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = -\int \left[\left(v_i\hat{\Pi} - g^{0\nu}\left(\partial_\nu v_i\right)\hat{\phi}\sqrt{-g}\right)\Big|_{\mathbf{x}}, \left(v_j^*\hat{\Pi} - g^{0\nu}\left(\partial_\nu v_j\right)\hat{\phi}\sqrt{-g}\right)\Big|_{\mathbf{y}}\right] \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{y} = \\ = i\hbar\int g^{0\nu}\left(v_i\overleftrightarrow{\partial}_\nu v_j^*\right)\sqrt{-g}\,\mathrm{d}^3\mathbf{x} = \hbar\langle v_i, v_j\rangle \quad (1.45)$$

where the field commutators where used. Equivalently

$$[\hat{a}_i, \hat{a}_j] = -\hbar\,\langle v_i, v_j^*\rangle \qquad \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = -\hbar\,\langle v_i^*, v_j\rangle \quad (1.46)$$

Therefore we must find a set of solutions $\{v_i(x), v_i^*(x)\}$ such that

$$\langle v_i, v_j\rangle \propto \delta_{ij} \qquad \langle v_i, v_j^*\rangle = \langle v_i^*, v_j\rangle = 0 \quad (1.47)$$

With this, we can define the Fock space the usual way, starting with a vacuum state $|0\rangle$ such that the action of the annihilation operation fulfils

$$\hat{a}_i\,|0\rangle = 0 \qquad \forall i \quad (1.48)$$

where single particle states are formed from the creation operator

$$|i\rangle \equiv \hat{a}_i^\dagger\,|0\rangle \quad (1.49)$$

and multiparticle states like

$$|i, j, \ldots\rangle = \ldots\hat{a}_j^\dagger\,\hat{a}_i^\dagger\,|0\rangle \quad (1.50)$$

Since this is a scalar field, one might assume that the states are symmetric (describing boson particles), and this is easily confirmed, since

$$|i, j\rangle = \hat{a}_j^\dagger\,\hat{a}_i^\dagger\,|0\rangle = \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right]|0\rangle + \hat{a}_i^\dagger\,\hat{a}_j^\dagger|0\rangle = |j, i\rangle \quad (1.51)$$

1.3.1 Bogoliubov Transformations

Consider now a second set $\{u_i(x), u_i^*(x)\}$ of solutions to the Klein-Gordon equation such that they meet the inner product rule; the field would then be written as

$$\phi(x) = \sum_j \left[b_j u_j(x) + b_j^* u_j^*(x)\right] \quad (1.52)$$

quantization of the field and creation and annihilation is straightforward. The relation between the v and u solutions would be

$$v_i(x) \equiv \sum_j \left[\alpha_{ij}u_j(x) + \beta_{ij}u_j^*(x)\right] \quad (1.53)$$

where α_{ij} and β_{ij} are known as Bogoliubov coefficients, that can be obtained as

$$\alpha_{ij} \propto \langle v_i, u_j\rangle \qquad \beta_{ij} \propto -\langle v_i, u_j^*\rangle \quad (1.54)$$

Since the field is the same independently of the mode set chosen

$$\sum_i \left[\hat{a}_i v_i(x) + \hat{a}_i^\dagger v_i^*(x)\right] = \sum_j \left[\hat{b}_j u_j(x) + \hat{b}_j^\dagger u_j^*(x)\right] \quad (1.55)$$

and, as a result of the orthogonality of the mode functions

$$\hat{a}_i = \sum_j \left(\alpha_{ij}^* \hat{b}_j - \beta_{ij}^* \hat{b}_j^\dagger\right) \qquad \hat{a}_i^\dagger = \sum_j \left(-\beta_{ij} \hat{b}_j + \alpha_{ij} \hat{b}_j^\dagger\right) \tag{1.56}$$

creation and annihilation commutation relations give new restrictions to the Bogoliubov coefficients

$$\left[\hat{a}_i,\, \hat{a}_j^\dagger\right] \propto \delta_{ij} \implies \sum_k \left(\alpha_{ik}^* \alpha_{jk} - \beta_{ik}^* \beta_{jk}\right) \propto \delta_{ij} \tag{1.57}$$

$$[\hat{a}_i,\, \hat{a}_j] = 0 \implies \sum_k \left(\alpha_{jk}^* \beta_{ik}^* - \alpha_{ik}^* \beta_{jk}^*\right) = 0 \tag{1.58}$$

Now, the relevance of the Bogoliubov transformations comes from the fact that the vacuum in the u solutions, have (in general) v particles,

$$\langle_u 0|\hat{N}_v|_u 0\rangle = \sum_i \langle_u 0|\hat{a}_i^\dagger \hat{a}_i|_u 0\rangle = \sum_i \left[\sum_{jk} \beta_{ij}\beta_{ik}^* \langle_u 0|\hat{b}_j \hat{b}_k^\dagger|_u 0\rangle\right] \propto \sum_{ij} |\beta_{ij}|^2 \tag{1.59}$$

therefore, there is not a unique vacuum.

1.3.2 A Leap Towards a Continuum

Until now, it has been considered that the set of Klein-Gordon solutions could be categorised by a discrete set of parameters i, from a standard course in QFT, one of the main results is the fact that the solutions of the flat Klein-Gordon equations can be parametrised by a continuous 3-dimensional vector $\mathbf{k}$ (which is interpreted to be the momentum of the particle). Since all computations in this section where made by considering a discrete set of parameters, it is relevant to consider the continuum case.

A common computation in many fields of physics is the determination of the density of states $D(\mathbf{k})$ describing the number of modes with momentum between $\mathbf{k}$ and $\mathbf{k}+\mathrm{d}\mathbf{k}$. Consider a system with volume V, where the field goes to zero at its boundary; in this case, the permitted values of momenta must meet

$$k^i = n^i \frac{\pi\hbar}{V^{1/3}}\,, \qquad n^i \in \mathbb{N} \tag{1.60}$$

Let $N(k)$ be the number of states with momentum modulus less than k, that is, the states such that

$$n = \sqrt{(n^1)^2 + (n^2)^2 + (n^3)^2} < k\frac{V^{1/3}}{\pi\hbar} \tag{1.61}$$

considering a flat momentum space[1] and a large enough volume, $N(k)$ will be essentially equal to an eight of the volume of a sphere with radius $kV^{1/3}/\pi\hbar$, that is

$$N(k) \approx \frac{1}{8}\frac{4}{3}\pi \left(k\frac{V^{1/3}}{\pi\hbar}\right)^3 = \frac{V}{6\pi^2\hbar^3}k^3 \tag{1.62}$$

meaning, that the density of states will be

$$D(\mathbf{k}) \equiv D(k) = \frac{\mathrm{d}N(k)}{\mathrm{d}k} \approx \frac{V}{2\pi^2\hbar^3}k^2 \tag{1.63}$$

With this, one could approximate a discrete sum over a parameter i to an integral over a continuum $\mathbf{k}$

$$\sum_i f_i = \int_0^\infty D(k) f_k \mathrm{d}k \approx \int_0^\infty \frac{V}{2\pi^2\hbar^3} f_k k^2 \mathrm{d}k \equiv \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3} f_\mathbf{k} \tag{1.64}$$

[1] In contrast to modified theories of relativity in which this is not the case, like the κ-Poincaré relativity.

where it has been defined.

$$4\pi V f_k k^2 \equiv \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} f_{\mathbf{k}} \sin\varphi \mathrm{d}\theta \mathrm{d}\varphi \tag{1.65}$$

therefore ${}^{\mathrm{d}^3\mathbf{k}}/_{(2\pi\hbar)^3}$ is to be understood as the volume element of the momentum space.

2 Scalar Field in an Expanding Eniverse

The methodology for the analysis of scalar fields in a general manifold was presented in the previous chapter as a preliminary for the rest of this work, and in particular of the present chapter. It is clear that the presence of symmetries of the theory will simplify computations, and thus, a great start might be an isotropic and homogeneous expanding universe, which is described by the so called Friedmann–Lemaître–Robertson–Walker metric. Using reduced-circumference polar coordinates, the line element associated with such metric is written as

$$\mathrm{d}l^2 = c^2\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\mathrm{d}\Omega^2\right], \quad \mathrm{d}\Omega \equiv \mathrm{d}\theta^2 + \sin^2\varphi\mathrm{d}\varphi^2, \tag{2.1 a,b}$$

where κ is the curvature of the space and $a(t)$ is the scale factor determining the expansion.

2.1 Expanding scalar field action

The Weyl tensor associated to the metric presented in eq. (2.1) is identically zero, meaning that the metric is conformally flat, i.e. independently of the space curvature κ, and therefore there must exist a coordinate system where

$$\mathrm{d}l^2 = a^2(\eta)\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = a^2(\eta)\left[c^2\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2\right], \tag{2.2}$$

working in such coordinate system will give the opportunity to use some results of standard scalar field theory. To do so, the action presented in eq. (1.19) will be rewritten in terms of a new field $\chi(x) \equiv a(\eta)\,\phi(x)$ using the fact that $\sqrt{-g} = a^4$

$$S[\chi] = \int \frac{1}{2}\left[\partial_\nu\chi\,\partial^\nu\chi - \left(\mu^2a^2 + \xi Ra^2 - c^2\frac{a''}{a}\right)\chi^2 - \partial_\eta\left(c^2\chi^2\frac{a'}{a}\right)\right]\mathrm{d}^4x, \tag{2.3}$$

where $a' \equiv \partial_\eta a(\eta)$ and equivalently with a''.

Dropping the time derivative will result on the following action for the sacalar χ field

$$S[\chi] = \int \frac{1}{2}\left[\partial_\nu\chi\,\partial^\nu\chi - \left(\mu^2a^2 + \xi Ra^2 - c^2\frac{a''}{a}\right)\chi^2\right]\mathrm{d}^4x, \tag{2.4}$$

being the main source for the current study. In order to obtain the expressions describing the dynamics of this field, the Euler-Lagrange equations in eq. (1.4) will be used, resulting in the generalized Klein-Gordon equation

$$\left[\partial_\nu\partial^\nu + \mu^2_{\text{eff}}(t)\right]\chi = 0, \qquad \mu^2_{\text{eff}}(t) = \left(\mu^2 + \xi R\right)a^2 - \frac{a''}{ac^2}. \tag{2.5 a,b}$$

Solutions of the eq. (2.5) are dependent on an integration constant related to the momentum $\mathbf{k}$, and are of the form

$$\chi_{\mathbf{k}}(x) = \alpha_{\mathbf{k}}v_{\mathbf{k}}(\eta)e^{-i\mathbf{kx}\hbar^{-1}} + \bar{\alpha}_{\mathbf{k}}\bar{v}_{\mathbf{k}}(\eta)e^{i\mathbf{kx}\hbar^{-1}}, \tag{2.6}$$

and upon substitution in (2.5), one gets the following differential equation

$$v''_{\mathbf{k}}\hbar^2 + \omega^2_{\mathbf{k}}(\eta)\,v_{\mathbf{k}} = 0 \tag{2.7}$$

where the dispersion relation $\omega_{\mathbf{k}}(\eta)$ is defined as,

$$\omega^2_{\mathbf{k}}(\eta) = \mathbf{k}^2 + \hbar^2\mu^2_{\text{eff}}(\eta) = \mathbf{k}^2 + \left(m^2c^2 + \xi\hbar^2R\right)a^2 - \hbar^2\frac{a''}{ac^2}. \tag{2.8}$$

Solving eq. (2.7) will in turn give the form of the set of solutions $\{\chi_{\mathbf{k}}\}$ needed to describe the general expression of $\chi(x)$; since we are currently considering general expansion parameters and curvature scalars, the following computations will be made using a general set of $v_{\mathbf{k}}$ functions. This functions nevertheless have some interesting properties, such as being capable of a choice of normalization, and a constant of motion: the imaginary part of $v_{\mathbf{k}}\bar{v}'_{\mathbf{k}}$. Lets check the last statement

$$\frac{\partial}{\partial\eta}\mathrm{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}}) = \frac{\partial}{\partial\eta}\left(\frac{v_{\mathbf{k}}\bar{v}'_{\mathbf{k}} - \bar{v}_{\mathbf{k}}v'_{\mathbf{k}}}{2i}\right) = \frac{v_{\mathbf{k}}\bar{v}'_{\mathbf{k}} * - \bar{v}_{\mathbf{k}}v''_{\mathbf{k}}}{2i} = 0 \tag{2.9}$$

last step is result from dispersion relation. Since the functions $v_{\mathbf{k}}$ are capable to a choice in normalization, we will choose a set of solutions of eq. (2.7) such that $\mathrm{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}})$ its independent of the momentum $\mathbf{k}$, and equal for all modes, this constant of motion will simply be defined as

$$\mathrm{Im}(v\bar{v}') \equiv \mathrm{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}}), \quad \forall\, \mathbf{k}. \tag{2.10}$$

The most general solution $\chi(x)$ of equation eq. (2.5) can be written as a Fourier mode expansion

$$\chi(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\left[a_{\mathbf{k}}v_{\mathbf{k}}(\eta)e^{-i\mathbf{k}\mathbf{x}\hbar^{-1}} + \bar{a}_{\mathbf{k}}\bar{v}_{\mathbf{k}}(\eta)e^{i\mathbf{k}\mathbf{x}\hbar^{-1}}\right] \tag{2.11}$$

2.2 Quantization

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\delta^3(\mathbf{k}-\mathbf{q})\,, \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = 0 \tag{2.12 a-c}$$

$$\begin{aligned}\left[\hat{\chi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = \frac{1}{c}\int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^6}\Big\{&[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}]\, v_{\mathbf{k}}v'_{\mathbf{q}}e^{-i(\mathbf{k}\mathbf{x}+\mathbf{q}\mathbf{y})\hbar^{-1}} + \left[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}\right]\bar{v}_{\mathbf{k}}\bar{v}'_{\mathbf{q}}e^{-i(\mathbf{k}\mathbf{x}-\mathbf{q}\mathbf{y})\hbar^{-1}} + \\ &+ \left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}\right]v_{\mathbf{k}}\bar{v}'_{\mathbf{q}}e^{-i(\mathbf{k}\mathbf{x}-\mathbf{q}\mathbf{y})\hbar^{-1}} - \left[\hat{a}_{\mathbf{q}}, \hat{a}^\dagger_{\mathbf{k}}\right]\bar{v}_{\mathbf{k}}v'_{\mathbf{q}}e^{i(\mathbf{k}\mathbf{x}-\mathbf{q}\mathbf{y})\hbar^{-1}}\Big\}\end{aligned} \tag{2.13}$$

using eq. (2.12) and considering that the proportional factor of 2.12.a to be α

$$\left[\hat{\chi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = \frac{\alpha}{c}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^6}2i\mathrm{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}})e^{-i(\mathbf{k}\mathbf{x}-\mathbf{q}\mathbf{y})\hbar^{-1}} \tag{2.14}$$

since $\mathrm{Im}(v_{\mathbf{k}}\bar{v}'_{\mathbf{k}})$ was considered to be momentum independent,

$$\left[\hat{\chi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = i\frac{2\alpha\mathrm{Im}(v\bar{v}')}{c(2\pi\hbar)^3}\delta^3(\mathbf{x}-\mathbf{y}) \tag{2.15}$$

and, from equation 1.43.a one can solve for α, resulting in the value present in equation 2.12.

Let $\xi = 0$, i.e. work in the nonminimally coupled scalar field; then the Hamiltonian will be

$$\hat{\mathcal{H}}(t) = \int \frac{c}{2}\left[\hat{\Pi}^2 + \left(\nabla\hat{\chi}\right)^2 + \mu^2_{\text{eff}}(t)\hat{\chi}^2\right]\mathrm{d}^3\mathbf{x} \tag{2.16}$$

$$\hat{\mathcal{H}} = \frac{c}{2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\left[\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}F_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\bar{F}_{\mathbf{k}} + \left(2\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\delta^3(\mathbf{0})\right)E_{\mathbf{k}}\right] \tag{2.17}$$

where

$$F_{\mathbf{k}}(t) = \left(\frac{1}{\hbar c}\right)^2\left[\hbar^2 v'^2_{\mathbf{k}} + \omega^2_{\mathbf{k}}(t)\, c^2 v^2_{\mathbf{k}}\right], \quad E_{\mathbf{k}}(t) = \left(\frac{1}{\hbar c}\right)^2\left[\hbar^2\left|v'_{\mathbf{k}}\right|^2 + \omega^2_{\mathbf{k}}(t)\, c^2\left|v_{\mathbf{k}}\right|^2\right]. \tag{2.18 a,b}$$

2.3 Instantaneous Vacuum State

Note that the only way a vacuum state $|0\rangle$ could remain an eigenstate of the Hamiltonian at all times, would be if $F_{\mathbf{k}} = 0$, which would mean

$$F_{\mathbf{k}}(t) = \left(\frac{1}{\hbar c}\right)^2 \left[\hbar^2 v_{\mathbf{k}}'^2 + \omega_{\mathbf{k}}^2(t)\, c^2 v_{\mathbf{k}}^2\right] = 0, \tag{2.19}$$

solving for $v_{\mathbf{k}}$ gives the following expression

$$v_{\mathbf{k}}(t) = \mathrm{C}\exp\left[\pm\frac{c}{i\hbar}\int \omega_{\mathbf{k}}(\eta)\,\mathrm{d}\eta\right], \tag{2.20}$$

which is not compatible with 2.7 except for a time independent dispersion relation $\omega_{\mathbf{k}}$.

The last result implies that, at different times one can (and should) define different vacuum states; and thus, we will define the *instantaneous vacuum state* $|_{(v)}0\rangle$ as the one that at some time t_0 will minimize the energy density. Since all possible states are related by Bogolyubov transformations, finding the instantaneous vacumm state is the same as finding the set of functions $v_{\mathbf{f}}$ that are simultaneously solution of 2.7 and minimize

$$\langle_{(v)}0|\hat{\mathcal{H}}(t_0)|_{(v)}0\rangle = \rho(t_0)\delta^3(\mathbf{0}) = \frac{\hbar c^2\,\delta^3(\mathbf{0})}{4\mathrm{Im}(v\bar{v}')}\int \mathrm{d}^3\mathbf{k}\,E_{\mathbf{k}} \tag{2.21}$$

To minimise the energy density of the vacuum state is to find the set of functions $v_{\mathbf{k}}$ that minimise $E_{\mathbf{k}}$. Suppose that $v_{\mathbf{k}}$ can be written as

$$v_{\mathbf{k}} = r_{\mathbf{k}} e^{i\alpha_{\mathbf{k}}} \tag{2.22}$$

since $\mathrm{Im}(v\bar{v}')$ was constant through time

$$r_{\mathbf{k}}^2 \alpha_{\mathbf{k}}' = -\mathrm{Im}(v\bar{v}') \tag{2.23}$$

this means

$$E_{\mathbf{k}} = \left(\frac{1}{\hbar c}\right)^2 \left\{\hbar^2\left[r_{\mathbf{k}}'^2 + \mathrm{Im}^2\left(v\bar{v}'\right)\frac{1}{r_{\mathbf{k}}^2}\right] + \omega_{\mathbf{k}}^2\, c^2 r_{\mathbf{k}}^2\right\} \tag{2.24}$$

the minimum of this function must fulfil $r_{\mathbf{k}}'(t_0) = 0$. Now, if $\omega_{\mathbf{k}}^2(t_0)$ and $\mathrm{Im}(v\bar{v}')$ have the same sign, the minimum of $E_{\mathbf{k}}$ happens when $r_{\mathbf{k}}(t_0) = \left[\frac{\hbar \mathrm{Im}(v\bar{v}')}{\omega_{\mathbf{k}}(t_0)\,c}\right]^{1/2}$.

If there is a minimum, then

$$v_{\mathbf{k}}(t_0) = \left[\frac{\hbar\mathrm{Im}(v\bar{v}')}{\omega_{\mathbf{k}}(t_0)\,c}\right]^{1/2} e^{i\alpha_{\mathbf{k}}(t_0)} \qquad v_{\mathbf{k}}'(t_0) = -c\frac{\omega_{\mathbf{k}}(t_0)}{ih}\,v_{\mathbf{k}}(t_0) \tag{2.25}$$

under these functions,

$$E_{\mathbf{k}}(t_0) = 2\frac{\mathrm{Im}(v\bar{v}')}{\hbar c}\omega_{\mathbf{k}}(t_0) \qquad F_{\mathbf{k}}(t_0) = 0 \tag{2.26}$$

meaning

$$\hat{\mathcal{H}}(t_0) = \mathrm{Im}(v\bar{v}')\frac{1}{\hbar}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\left(2\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\delta^3(\mathbf{0})\right)\omega_{\mathbf{k}}(t_0) \tag{2.27}$$

which is equivalent to the standard Hamiltonian for a scalar field without the presence of gravity.

Bogolyubov Transformation The expression of the field χ at two different times must be related to a Bogoliubov transformation, with coefficients

$$\alpha_{\mathbf{kp}} = \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\langle\chi_{\mathbf{k}}(t_0),\, \chi_{\mathbf{p}}(t)\rangle \qquad \beta_{\mathbf{kp}} = -\frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\langle\chi_{\mathbf{k}}(t_0),\, \bar{\chi}_{\mathbf{p}}(t)\rangle \tag{2.28}$$

since the field can be written as $\chi_{\mathbf{k}} = v_{\mathbf{k}}e^{i\mathbf{kx}/\hbar}$ from the expression of the inner product one can see that

$$\alpha_{\mathbf{kp}} \propto \delta^3(\mathbf{k}-\mathbf{p}) \qquad \beta_{\mathbf{kp}} \propto \delta^3(\mathbf{k}+\mathbf{p}) \tag{2.29}$$

therefore it is possible to write

$$v_{\mathbf{k}}(t) = \alpha_{\mathbf{k}}v_{\mathbf{k}}(t_0) + \beta_{\mathbf{k}}\bar{v}_{\mathbf{k}}(t_0) \tag{2.30}$$

where, recalling that $\mathrm{Im}(v\bar{v}')$ is constant through time,

$$|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1 \tag{2.31}$$

To obtain the value of $\langle_{(t_0)}0|\hat{\mathcal{H}}(t)|_{(t_0)}0\rangle$ lets first compute

$$\langle_{(t_0)}0|\hat{\mathcal{N}}^{(a)}_{\mathbf{k}}(t)|_{(t_0)}0\rangle = \langle_{(t_0)}0|\hat{a}^\dagger_{\mathbf{k}}(t)\hat{a}_{\mathbf{k}}(t)|_{(t_0)}0\rangle = \left|\beta_{\mathbf{k}}\right|^2\frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v\bar{v}')}\delta^3(\mathbf{0}) \tag{2.32}$$

therefore

$$\langle_{(t_0)}0|\hat{\mathcal{H}}(t)|_{(t_0)}0\rangle = \delta^3(\mathbf{0})\int \mathrm{d}^3\mathbf{k}\left(\frac{1}{2}+\left|\beta_{\mathbf{k}}\right|^2\right)c\,\omega_{\mathbf{k}}(t) \geq \langle_{(t_0)}0|\hat{\mathcal{H}}(t_0)|_{(t_0)}0\rangle \tag{2.33}$$

meaning, if $\beta_{\mathbf{k}} \neq 0$ for all $\mathbf{k}$ then, at a time $t > t_0$ the energy density will be different in relation to the original vacuum.

3 Scalar Field in a de Sitter Universe

The de Sitter Universe is a flat FLRW metric with no matter or radiation, but it does have a a positive cosmological constant Λ. Per the Friedmann equations,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho + \Lambda c^2}{3} - \frac{\kappa c^2}{a^2} \tag{3.1}$$

the expansion parameter $a(t)$ will be equal to

$$a(t) = a_1 e^{H_\Lambda t} + a_2 e^{-H_\Lambda t}\,, \qquad H_\Lambda = \sqrt{\frac{\Lambda c^2}{3}} \tag{3.2}$$

$a_2 = 0$

$$\mathrm{d}l^2 = c^2 \mathrm{d}t^2 - a^2(t)\mathrm{d}\mathbf{x}^2 \tag{3.3}$$

$$\eta \equiv -\int_t^\infty \frac{\mathrm{d}t'}{a(t')} = -\frac{1}{a_1 H_\Lambda} e^{-H_\Lambda t} = -\frac{1}{a(t) H_\Lambda} \tag{3.4}$$

[1]

$$\mathrm{d}l^2 = \frac{1}{H_\Lambda \eta^2}\left[c^2 \mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2\right] \tag{3.5}$$

$$R = \frac{6}{c^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] = \frac{12}{c^2} H_\Lambda^2 \tag{3.6}$$

$$\omega_{\mathbf{k}}^2(\eta) = \mathbf{k}^2 + \left[\left(\frac{mc^2}{H_\Lambda}\right)^2 + 2\left(6\xi - 1\right)\hbar^2\right]\frac{1}{c^2\eta^2} \tag{3.7}$$

$$v_{\mathbf{k}}''\hbar^2 + \omega_{\mathbf{k}}^2(\eta)\, v_{\mathbf{k}} = 0 \tag{3.8}$$

change of variables

$$s \equiv -k\eta \qquad v_{\mathbf{k}} \equiv \sqrt{s} f(s) \tag{3.9}$$

$$s^2 \frac{\mathrm{d}^2 f}{\mathrm{d}s^2} + s\frac{\mathrm{d}f}{\mathrm{d}s} + \left(s^2 - \nu^2\right) f(s) = 0 \tag{3.10}$$

$$\nu^2 \equiv (3 - 16\xi)\,\frac{3}{4}\frac{\hbar^2}{c^2} - \left(\frac{mc}{H_\Lambda}\right)^2 \tag{3.11}$$

$$f(s) = AJ_\nu(s) + BY_\nu(s) \implies v_{\mathbf{k}}(\eta) = \sqrt{k|\eta|}\left[A_{\mathbf{k}} J_\nu(k|\eta|) + B_{\mathbf{k}} Y_\nu(k|\eta|)\right] \tag{3.12}$$

$J_\nu(x)$, $Y_\nu(x)$ are the Bessel functions of the firs kind. For $\nu^2 \geq 0$ J_ν and Y_ν will be real functions, but for $\nu < 0$ they will be complex functions [4]. $\mathrm{Im}(v\bar{v}')$ creates a restriction on the relation $B_{\mathbf{k}} = B_{\mathbf{k}}(A_{\mathbf{k}})$

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[1] Considering $a_2 \neq 0$ one would obtain that

$$\eta = \frac{\arctan\left(\sqrt{\frac{a_2}{a_1}} e^{-H_\Lambda t}\right)}{H_\Lambda \sqrt{a_1 a_2}}$$

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Scalar field in Minkowski background

$$\eta_{\mu\nu} = \mathrm{diag}\,(+,\,-,\,-,\,-) \tag{1}$$

$$S[\phi] = \int \frac{1}{2}\Big[\partial_\nu\phi\,\partial^\nu\phi - \mu^2\phi^2\Big]\;\mathrm{d}^4x \tag{2}$$

$$\left(\partial_\nu\partial^\nu - \mu^2\right)\phi = 0 \tag{3}$$

$$\phi_{\mathbf{k}} = a_{\mathbf{k}}e^{ikx\,\hbar^{-1}} + \bar{a}_{\mathbf{k}}e^{-ikx\,\hbar^{-1}} \tag{4}$$

$$k_\nu k^\nu = \hbar^2\mu^2 \tag{5}$$

Units

- $[S] = [\hbar]$
- $[a] = [\xi] = 1$
- $[\mu] = [L]^{-1}$
- $[R] = [L]^{-2}$
- $[\phi] = [\chi] = [\hbar]^{1/2}[L]^{-1}$
- $[\Pi] = [\hbar]^{1/2}[L]^{-2}$
- $[a_{\mathbf{k}}] = [\hbar]^{1/2}[L]^2$

Questions & To-Do

1 Questions

- How do you know that there is a set of solutions of Klein Gordon such that the inner product fulfils the given results?
- It's the Hamiltonian well defined?
- Can you always write a FLRW metric as a flat one with a coordinate change?
- Covariant derivatives, spin connection.

2 To-Do

- Minkowski scalar field.
- Move Units appendix to conventions.

Computations