PROBLEMS OF QUANTUM FIELD THEORIES IN CURVED SPACETIMES

A MASTER THESIS

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Contents

Preface

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1 Introduction to QFT in curved spacetimes

$$\mathrm{d}l^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{1.1}$$

$$S = \int \left[\frac{1}{2\kappa}(R - 2\Lambda) + \mathcal{L}_\mathrm{M}\right]\sqrt{-g}\,\mathrm{d}^4x \tag{1.2}$$

$\kappa \equiv \frac{8\pi G}{c^4}$

Variation of S with respect to the inverse metric ($g^{\mu\nu}$) gives

$$\delta S = \int \left[\frac{\sqrt{-g}}{2\kappa}\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{2\kappa}\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} - \frac{\Lambda}{\kappa}\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta\mathcal{L}_\mathrm{M}}{\delta g^{\mu\nu}} + \frac{\mathcal{L}_\mathrm{M}}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x \tag{1.3}$$

$\delta S = 0$ and

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu} \qquad \frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}g_{\mu\nu} \tag{1.4}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -2\frac{8\pi G}{c^4}\left(\frac{\delta\mathcal{L}_\mathrm{M}}{\delta g^{\mu\nu}} - \frac{1}{2}\mathcal{L}_\mathrm{M}g_{\mu\nu}\right) \tag{1.5}$$

$$T_{\mu\nu} \equiv \mathcal{L}_\mathrm{M}g_{\mu\nu} - \frac{\delta\mathcal{L}_\mathrm{M}}{\delta g^{\mu\nu}} = \frac{-2}{\sqrt{-g}}\frac{\delta\left(\mathcal{L}_\mathrm{M}\sqrt{-g}\right)}{\delta g^{\mu\nu}} \tag{1.6}$$

[1]

$$\nabla_\mu T^{\mu\nu} = 0 \tag{1.7}$$

(and its symmetric)

1.1 Construction of covariant actions

The equivalence principle says that field equations must be invariant with respect to local Lorentz transformations $\Lambda(x)$. To work on a local flat spacetime, we use the tetrad formalism

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \tag{1.8}$$

partial derivatives transform like

$$\partial_\mu \to \frac{\partial x^\nu}{\partial y^\mu}\partial_\nu \tag{1.9}$$

but, we see that $e^\mu_a\partial_\mu$ is not covariant, since

$$e^\mu_a\partial^a\phi(x) \to \Lambda^b_a e^\mu_b\partial_\mu\left[\rho(\Lambda)\phi(x)\right] = \Lambda^b_a e^\mu_b\left[\rho(\Lambda)\partial_\mu\phi + \partial_\mu\rho(\Lambda)\,\phi\right] \tag{1.10}$$

a covariant derivative D_μ should transform as

$$D_a\phi \to \Lambda^b_a\rho(\Lambda)\,D_b\phi \tag{1.11}$$

therefore we need to define a better derivative, a common option is

$$D_a \equiv e^\mu_a\left(\partial_\mu + \Gamma_\mu\right) \tag{1.12}$$

where, for the derivative to be covariant, the connection Γ_μ must transform as

$$\Gamma_\mu \to \rho(\Lambda)\,\Gamma_\mu\rho^{-1}(\Gamma) - \left[\partial_\mu\rho(\Lambda)\right]\rho^{-1}(\Lambda) \tag{1.13}$$

such connection can be written as

$$\Gamma_\mu = \frac{1}{2}\Sigma^{ab} e_a^\nu \nabla_\mu e_{b\nu} \tag{1.14}$$

where Σ^{ab} are the Lorentz generators and $e_a^\nu \nabla_\mu e_{b\nu} \equiv \omega_{ab\mu}$ is the torsion free spin connection.

Under the presence of a gauge field, we would also want the derivative to be gauge invariant, and thus, the gauge four-potential A must be considered in the definition of the derivative

$$D_a \equiv e_a^\mu \left(\partial_\mu + \Gamma_\mu - \frac{i}{\hbar} e A_\mu\right) \tag{1.15}$$

where e is the coupling constant.

This derivative of course will

Now, in order to obtain the equations of motion for the matter fields, consider the lagrangian

$$\mathcal{L}_\mathrm{M} = \mathcal{L}_\mathrm{M}\left[\phi^\alpha(x),\, D_\mu\phi^\alpha(x)\right] \tag{1.16}$$

variations of S with respect of ϕ^α result in

$$\delta S = \int \left[\frac{\partial \mathcal{L}_\mathrm{M}}{\partial \phi^\alpha}\delta\phi^\alpha + \frac{\partial \mathcal{L}_\mathrm{M}}{\partial\left(D_\mu \phi^\alpha\right)} D_a\left(\delta\phi^\alpha\right)\right]\sqrt{-g}\,\mathrm{d}^4x \tag{1.17}$$

and thus, after applying the generalized Gauss Theorem on a curved background, and considering that field variations vanish at the boundaries, one obtains

$$\frac{\partial \mathcal{L}_\mathrm{M}}{\partial \phi^\alpha} - D_\mu\left[\frac{\partial \mathcal{L}_\mathrm{M}}{\partial\left(D_\mu \phi^\alpha\right)}\right] = 0 \tag{1.18}$$

$$\Pi_\alpha \equiv \frac{\partial \mathcal{L}_\mathrm{M}}{\partial\left(D_0 \phi^\alpha\right)} \tag{1.19}$$

1.2 Scalar field

$$S[\phi] = \int \frac{1}{2}\left[\nabla_\nu\phi\,\nabla^\nu\phi - \mu^2\phi^2 - \xi R\phi^2\right]\sqrt{-g}\;\mathrm{d}^4x \tag{1.20}$$

equations of motion (Klein-Gordon)

$$\left[\nabla_\nu\nabla^\nu - \mu^2 - \xi R\right]\phi = 0 \tag{1.21}$$

$$T_{\mu\nu} = \nabla_\mu\phi\,\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left[\nabla^\sigma\phi\nabla_\sigma\phi - \mu^2\phi^2\right] + \xi\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} + g_{\mu\nu}\nabla^\sigma\nabla_\sigma - \nabla_\mu\nabla_\nu\right]\phi^2 \tag{1.22}$$

note that for minimally coupled field ($\xi = 0$) the energy-momentum tensor is equivalent to the Noether energy-momentum tensor.

Scalar product

$$\langle\phi_1(x),\,\phi_2(x)\rangle \equiv i\int g^{0\nu}\left(\phi_1\overset{\leftrightarrow}{\nabla}_\nu\phi_2^*\right)\sqrt{-g}\,\mathrm{d}^3\mathbf{x} \tag{1.23}$$

Let $v(x)$ be a solution of the Klein-Gordon equation, then $v^*(x)$ will also be an (linearly independent) solution. Let i represent the set of parameters that univocally describe a par of solutions $v_i(x)$, $v^*(x)$, therefore, the general solution of the Klein-Gordon equation will be of the form

$$\phi(x) = \sum_i \left[a_i v_i(x) + a_i^* v_i^*(x)\right] \tag{1.24}$$

where a_i, a_i^* are constant factors that can be written as

$$a_i = \langle v_i(x),\,\phi(x)\rangle \qquad a_i^* = \langle v_i^*(x),\,\phi(x)\rangle \tag{1.25}$$

Quantization of the field is done by promoting the fields to operators

$$\phi(x) \longrightarrow \hat{\phi}(x) \qquad \Pi(x) \longrightarrow \hat{\Pi}(x) \tag{1.26}$$

this is done by promoting the constant factors to operators as well, that is

$$a_i \longrightarrow \hat{a}_i \qquad a_i^* \longrightarrow \hat{a}_i^\dagger \tag{1.27}$$

and therefore

$$\hat{\phi}(x) = \sum_i \left[\hat{a}_i v_i(x) + \hat{a}_i^\dagger v_i^*(x)\right] \tag{1.28}$$

after the promotion of the fields to operators, commutation relations are imposed; the easiest choice would be to assume canonical quantization relations,

$$\left[\hat{\phi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = i\hbar\,\delta^3\left(\mathbf{x}-\mathbf{y}\right) \qquad \left[\hat{\phi}(\mathbf{x}),\, \hat{\phi}(\mathbf{y})\right] = \left[\hat{\Pi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = 0 \tag{1.29}$$

It would be desirable to obtain a formulation similar to the well known scalar field in a flat background, where the Fock space is generated from a vacuum state and a set of creation and annihilation operators that follow some commutation rules. To do so, we will force the $\hat{a}_i$, $\hat{a}_i^\dagger$ operators to assume this roll, in such a way that

$$\left[\hat{a}_i,\, \hat{a}_j^\dagger\right] \propto \delta_{ij} \qquad \left[\hat{a}_i,\, \hat{a}_j\right] = \left[\hat{a}_i^\dagger,\, \hat{a}_j^\dagger\right] = 0 \tag{1.30}$$

Thanks to the relation between the constant factors a_i and the scalar product $\langle v_i,\, \phi\rangle$, one can obtain

$$\begin{aligned}\left[\hat{a}_i,\, \hat{a}_j^\dagger\right] &= -\int \left[\left(v_i\hat{\Pi} - g^{0\nu}\left(\nabla_\nu v_i\right)\hat{\phi}\sqrt{-g}\right)\Big|_{\mathbf{x}},\, \left(v_j^*\hat{\Pi} - g^{0\nu}\left(\nabla_\nu v_j\right)\hat{\phi}\sqrt{-g}\right)\Big|_{\mathbf{y}}\right] \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{y} = \\ &= i\hbar\int g^{0\nu}\left(v_i \overleftrightarrow{\nabla}_\nu v_j^*\right)\sqrt{-g}\,\mathrm{d}^3\mathbf{x} = \hbar\langle v_i,\, v_j\rangle\end{aligned} \tag{1.31}$$

where the field commutators where used. Equivalently

$$\left[\hat{a}_i,\, \hat{a}_j\right] = -\hbar\left\langle v_i,\, v_j^*\right\rangle \qquad \left[\hat{a}_i^\dagger,\, \hat{a}_j^\dagger\right] = -\hbar\left\langle v_i^*,\, v_j\right\rangle \tag{1.32}$$

Therefore we must find a set of solutions $\{v_i(x),\, v_i^*(x)\}$ such that

$$\langle v_i,\, v_j\rangle \propto \delta_{ij} \qquad \langle v_i,\, v_j^*\rangle = \langle v_i^*,\, v_j\rangle = 0 \tag{1.33}$$

With this, we can define the Fock space the usual way, starting with a vacuum state $|0\rangle$ such that the action of the annihilation operation fulfils

$$\hat{a}_i\,|0\rangle = 0 \qquad \forall i \tag{1.34}$$

where single particle states are formed from the creation operator

$$|i\rangle \equiv \hat{a}_i^\dagger\,|0\rangle \tag{1.35}$$

and multiparticle states like

$$|i,\, j,\, \ldots\rangle = \ldots\hat{a}_j^\dagger\,\hat{a}_i^\dagger\,|0\rangle \tag{1.36}$$

Since this is a scalar field, one might assume that the states are symmetric (describing boson particles), and this is easily confirmed, since

$$|i,\, j\rangle = \hat{a}_j^\dagger\,\hat{a}_i^\dagger\,|0\rangle = \left[\hat{a}_i^\dagger,\, \hat{a}_j^\dagger\right]|0\rangle + \hat{a}_i^\dagger\,\hat{a}_j^\dagger|0\rangle = |j,\, i\rangle \tag{1.37}$$

1.3 Bogoliubov transformations

Consider now a second set $\{u_i(x),\, u_i^*(x)\}$ of solutions to the Klein-Gordon equation such that they meet the inner product rule; the field would then be written as

$$\phi(x) = \sum_j \left[b_j u_j(x) + b_j^* u_j^*(x)\right] \tag{1.38}$$

quantization of the field and creation and annihilation is straightforward. The relation between the v and u solutions would be

$$v_i(x) \equiv \sum_j \left[\alpha_{ij} u_j(x) + \beta_{ij} u_j^*(x)\right] \tag{1.39}$$

where α_{ij} and β_{ij} are known as Bogoliubov coefficients, that can be obtained as

$$\alpha_{ij} \propto \langle v_i,\, u_j\rangle \qquad \beta_{ij} \propto -\langle v_i,\, u_j^*\rangle \tag{1.40}$$

Since the field is the same independently of the mode set chosen

$$\sum_i \left[\hat{a}_i v_i(x) + \hat{a}_i^\dagger v_i^*(x)\right] = \sum_j \left[\hat{b}_j u_j(x) + \hat{b}_j^\dagger u_j^*(x)\right] \tag{1.41}$$

and, as a result of the orthogonality of the mode functions

$$\hat{a}_i = \sum_j \left(\alpha_{ij}^* \hat{b}_j - \beta_{ij}^* \hat{b}_j^\dagger\right) \qquad \hat{a}_i^\dagger = \sum_j \left(-\beta_{ij} \hat{b}_j + \alpha_{ij} \hat{b}_j^\dagger\right) \tag{1.42}$$

creation and annihilation commutation relations give new restrictions to the Bogoliubov coefficients

$$\left[\hat{a}_i,\, \hat{a}_j^\dagger\right] \propto \delta_{ij} \implies \sum_k \left(\alpha_{ik}^* \alpha_{jk} - \beta_{ik}^* \beta_{jk}\right) \propto \delta_{ij} \tag{1.43}$$

$$\left[\hat{a}_i,\, \hat{a}_j\right] = 0 \implies \sum_k \left(\alpha_{jk}^* \beta_{ik}^* - \alpha_{ik}^* \beta_{jk}^*\right) = 0 \tag{1.44}$$

Now, the relevance of the Bogoliubov transformations comes from the fact that the vacuum in the u solutions, have (in general) v particles,

$$\langle {}_u 0|\hat{N}_v|{}_u 0\rangle = \sum_i \langle {}_u 0|\hat{a}_i^\dagger \hat{a}_i|{}_u 0\rangle = \sum_i \left[\sum_{jk} \beta_{ij}\beta_{ik}^* \langle {}_u 0|\hat{b}_j \hat{b}_k^\dagger|{}_u 0\rangle\right] \propto \sum_{ij} |\beta_{ij}|^2 \tag{1.45}$$

therefore, there is not a unique vacuum.

1.4 A leap towards a continuum

Until now, it has been considered that the set of Klein-Gordon solutions could be categorised by a discrete set of parameters i, from a standard course in QFT, one of the main results is the fact that the solutions of the flat Klein-Gordon equations can be parametrised by a continuous 3-dimensional vector $\mathbf{k}$ (which is interpreted to be the momentum of the particle). Since all computations in this section where made by considering a discrete set of parameters, it is relevant to consider the continuum case.

A common computation in many fields of physics is the determination of the density of states $D(\mathbf{k})$ describing the number of modes with momentum between $\mathbf{k}$ and $\mathbf{k}+\mathrm{d}\mathbf{k}$. Consider a system with volume V, where the field goes to zero at its boundary; in this case, the permitted values of momenta must meet

$$k^i = n^i \frac{\pi\hbar}{V^{1/3}}, \qquad n^i \in \mathbb{Z} \tag{1.46}$$

Let $N(k)$ be the number of states with momentum modulus less than k, that is, the states such that

$$n = \sqrt{(n^1)^2 + (n^2)^2 + (n^3)^2} < k\frac{V^{1/3}}{\pi\hbar} \tag{1.47}$$

considering a flat momentum space[1] and a large enough volume, $N(k)$ will be essentially equal to an eight of the volume of a sphere with radius $kV^{1/3}/\pi\hbar$, that is

$$N(k) \approx \frac{1}{8}\frac{4}{3}\pi\left(k\frac{V^{1/3}}{\pi\hbar}\right)^3 = \frac{V}{6\pi^2\hbar^3}k^3 \tag{1.48}$$

meaning, that the density of states will be

$$D(\mathbf{k}) \equiv D(k) = \frac{\mathrm{d}N(k)}{\mathrm{d}k} \approx \frac{V}{2\pi^2\hbar^3}k^2 \tag{1.49}$$

With this, one could approximate a discrete sum over a parameter i to an integral over a continuum $\mathbf{k}$

$$\sum_i f_i = \int_0^\infty D(k) f_k \mathrm{d}k \approx \int_0^\infty \frac{V}{2\pi^2\hbar^3} f_k k^2 \mathrm{d}k \equiv \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3} f_{\mathbf{k}} \tag{1.50}$$

where it has been defined.

$$4\pi V f_k k^2 \equiv \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} f_{\mathbf{k}} \sin\varphi \mathrm{d}\theta \mathrm{d}\varphi \tag{1.51}$$

therefore $\mathrm{d}^3\mathbf{k}/(2\pi\hbar)^3$ is to be understood as the volume element of the momentum space.

2 Scalar field in an expanding universe

FLRW metric

$$\mathrm{d}l^2 = c^2\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\mathrm{d}\Omega^2\right] \tag{2.1}$$

Weyl tensor =0 therefore the metric is conformally flat, i.e. independently of the curvature κ there must exist a coordinate system where

$$\mathrm{d}l^2 = a(t)\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = a(t)\left[c^2\mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2\right] \tag{2.2}$$

the standard action describing the dynamics of a (non-minimally coupled to gravity) real scalar field is

$$s = \int \frac{1}{2}\Big[\nabla_\nu\phi\,\nabla^\nu\phi - \mu^2\phi^2 - \xi R\phi^2\Big]\sqrt{-g}\,\mathrm{d}^4x \tag{2.3}$$

$\sqrt{-g} = a^4$ $\chi = a\phi$

$$s = \int \frac{1}{2}\Big[\partial_\nu\chi\,\partial^\nu\chi - \left(\mu^2a^2 + \xi Ra^2 - c^2\frac{a''}{a}\right)\chi^2 - \partial_t\left(c^2\chi^2\frac{a'}{a}\right)\Big]\mathrm{d}^4x \tag{2.4}$$

dropping the time derivative

$$s = \int \frac{1}{2}\Big[\partial_\nu\chi\,\partial^\nu\chi - \left(\mu^2a^2 + \xi Ra^2 - c^2\frac{a''}{a}\right)\chi^2\Big]\mathrm{d}^4x \tag{2.5}$$

[1]In contrast to modified theories of relativity in which this is not the case, like the κ-Poincaré relativity.

by Euler-Lagrange

$$\left[\partial_\nu \partial^\nu + \mu^2_{\text{eff}}(t)\right] \chi = 0 \tag{2.6}$$

where

$$\mu^2_{\text{eff}}(t) = \left(\mu^2 + \xi R\right) a^2 - \frac{a''}{ac^2} \tag{2.7}$$

solutions of previous equation have the form

$$\chi = a\, v(t) e^{\pm i\mathbf{kx}\hbar^{-1}} \tag{2.8}$$

meaning that, the dispersion relation is

$$v''\hbar^2 + \omega^2(t)\, v = 0 \tag{2.9}$$

where $\omega(t)$ is defined as

$$\omega^2_{\mathbf{k}}(t) = \mathbf{k}^2 + \hbar^2 \mu^2_{\text{eff}}(t) = \mathbf{k}^2 + \left(m^2c^2 + \xi\hbar^2 R\right) a^2(t) - \hbar^2 \frac{a''}{ac^2} \tag{2.10}$$

now, proof that $\text{Im}(vv'^*)$ is constant through time

$$\frac{\partial}{\partial t}\text{Im}(vv'^*) = \frac{\partial}{\partial t}\left(\frac{vv'^* - v^*v'}{2i}\right) = \frac{vv''^* - v^*v''}{2i} = 0 \tag{2.11}$$

last step is result from dispersion relation. Since dispersion relation is scalable by a time independent function, $\text{Im}(v'v^*)$ can be determined to be a chosen value, a particular useful choice is to consider it momentum independent. $\text{Im}(v'v^*) = W[v, v^*]$ therefore, if its not equal to 0, they are linearly independent solutions to dispersion relation.

The most general solution to the main equation is

$$\chi = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\left[a_{\mathbf{k}} v_{\mathbf{k}}(t) e^{i\mathbf{kx}\hbar^{-1}} + a^*_{\mathbf{k}} v^*_{\mathbf{k}}(t) e^{-i\mathbf{kx}\hbar^{-1}}\right] \tag{2.12}$$

The field χ and its conjugate momentum $\Pi = \partial_{ct}\chi$ are promoted to operators on the quantum Hilbert space, with the standar canonical conmutation relations

$$\left[\hat{\chi}(t,\mathbf{x}), \hat{\Pi}(t,\mathbf{y})\right] = i\hbar\, \delta^3(\mathbf{x} - \mathbf{y}) \tag{2.13}$$

$$\left[\hat{\chi}(t,\mathbf{x}), \hat{\chi}(t,\mathbf{y})\right] = \left[\hat{\Pi}(t,\mathbf{x}), \hat{\Pi}(t,\mathbf{y})\right] = 0 \tag{2.14}$$

where the operational nature of the fields arrise from the promotion of the mode amplitudes, i.e.

$$a_{\mathbf{k}} \longrightarrow \hat{a}_{\mathbf{k}} \qquad a^*_{\mathbf{k}} \longrightarrow \hat{a}^\dagger_{\mathbf{k}} \tag{2.15}$$

this operators fulfill the following conmutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = \frac{(2\pi\hbar)^3\hbar c}{2\text{Im}(v'v^*)}\delta^3(\mathbf{k} - \mathbf{q})\,, \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = 0 \tag{2.16}$$

(note that $\hat{a}_{\mathbf{k}} \neq \hat{a}_{-\mathbf{k}}$)

To prove this, consider that

$$\begin{aligned}\left[\hat{\chi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = \frac{1}{c}\int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^6}\Big\{&[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}]\, v_{\mathbf{k}} v'_{\mathbf{q}} e^{i(\mathbf{kx}+\mathbf{qy})\hbar^{-1}} + \left[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}\right] v^*_{\mathbf{k}} v^{*\prime}_{\mathbf{q}} e^{i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}} + \\ &+ \left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}\right] v_{\mathbf{k}} v^{*\prime}_{\mathbf{q}} e^{i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}} - \left[\hat{a}_{\mathbf{q}}, \hat{a}^\dagger_{\mathbf{k}}\right] v^*_{\mathbf{k}} v'_{\mathbf{q}} e^{-i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}}\Big\}\end{aligned} \tag{2.17}$$

if the operators $\hat{a}$ and $\hat{a}^\dagger$ are to be understood as creation and annihilation operators, they must fulfill

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = \alpha\delta^3(\mathbf{k}-\mathbf{q}), \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = 0 \tag{2.18}$$

where $\alpha \in \mathbb{C}$, and thus

$$\left[\hat{\chi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = \frac{\alpha}{c}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^6} 2i\mathrm{Im}(v_{\mathbf{k}}v^{*'}_{\mathbf{k}})e^{i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}} \tag{2.19}$$

considering $\mathrm{Im}(v'v^*)$ momentum independent, and remembering the canonical conmutation relations, one finds that

$$\alpha\mathrm{Im}(vv^{*'}) = \frac{1}{2}\hbar c(2\pi\hbar)^3 \tag{2.20}$$

Let $\xi = 0$, i.e. work in the nonminimally coupled scalar field; then the Hamiltonian will be

$$\hat{\mathcal{H}}(t) = \int \frac{c}{2}\left[\hat{\Pi}^2 + \left(\nabla\hat{\chi}\right)^2 + \mu^2_{\text{eff}}(t)\hat{\chi}^2\right]\mathrm{d}^3\mathbf{x} \tag{2.21}$$

$$\hat{\Pi}^2 = \frac{1}{c^2}\int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^6}\Big[\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{q}}v'_{\mathbf{k}}v'_{\mathbf{q}}e^{i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{q}}v'_{\mathbf{k}}v'^*_{\mathbf{q}}e^{i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} + \\ + \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{q}}v'^*_{\mathbf{k}}v'_{\mathbf{q}}e^{-i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{q}}v'^*_{\mathbf{k}}v'^*_{\mathbf{q}}e^{-i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}}\Big] \tag{2.22}$$

$$\left(\nabla\hat{\chi}\right)^2 = -\frac{1}{\hbar^2}\int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^6}\mathbf{kq}\Big[\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{q}}v_{\mathbf{k}}v_{\mathbf{q}}e^{i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}} - \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{q}}v_{\mathbf{k}}v^*_{\mathbf{q}}e^{i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} - \\ - \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{q}}v^*_{\mathbf{k}}v_{\mathbf{q}}e^{-i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{q}}v^*_{\mathbf{k}}v^*_{\mathbf{q}}e^{-i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}}\Big] \tag{2.23}$$

$$\hat{\chi}^2 = \int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^6}\Big[\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{q}}v_{\mathbf{k}}v_{\mathbf{q}}e^{i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{q}}v_{\mathbf{k}}v^*_{\mathbf{q}}e^{i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} + \\ + \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{q}}v^*_{\mathbf{k}}v_{\mathbf{q}}e^{-i(\mathbf{k}-\mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{q}}v^*_{\mathbf{k}}v^*_{\mathbf{q}}e^{-i(\mathbf{k}+\mathbf{q})\mathbf{x}\hbar^{-1}}\Big] \tag{2.24}$$

$$\hat{\mathcal{H}} = \frac{c}{2}\int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^3}\Bigg\{\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{q}}\left[\frac{1}{c^2}v'_{\mathbf{k}}v'_{\mathbf{q}} - \left(\frac{1}{\hbar^2}\mathbf{kq} - \mu^2_{\text{eff}}\right)v_{\mathbf{k}}v_{\mathbf{q}}\right]\delta^3(\mathbf{k}+\mathbf{q}) + \\ + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{q}}\left[\frac{1}{c^2}v'_{\mathbf{k}}v'^*_{\mathbf{q}} + \left(\frac{1}{\hbar^2}\mathbf{kq} + \mu^2_{\text{eff}}\right)v_{\mathbf{k}}v^*_{\mathbf{q}}\right]\delta^3(\mathbf{k}-\mathbf{q}) + \\ + \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{q}}\left[\frac{1}{c^2}v'^*_{\mathbf{k}}v'_{\mathbf{q}} + \left(\frac{1}{\hbar^2}\mathbf{kq} + \mu^2_{\text{eff}}\right)v^*_{\mathbf{k}}v_{\mathbf{q}}\right]\delta^3(\mathbf{k}-\mathbf{q}) + \\ + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{q}}\left[\frac{1}{c^2}v'^*_{\mathbf{k}}v'^*_{\mathbf{q}} - \left(\frac{1}{\hbar^2}\mathbf{kq} - \mu^2_{\text{eff}}\right)v^*_{\mathbf{k}}v^*_{\mathbf{q}}\right]\delta^3(\mathbf{k}+\mathbf{q})\Bigg\} \tag{2.25}$$

$$\hat{\mathcal{H}} = \frac{c}{2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\Bigg\{\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}\left[\frac{1}{c^2}v'_{\mathbf{k}}v'_{\mathbf{k}} + \frac{1}{\hbar^2}\omega^2_{\mathbf{k}}(t)v_{\mathbf{k}}v_{\mathbf{k}}\right] + \\ + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\left[\frac{1}{c^2}v'_{\mathbf{k}}v'^*_{\mathbf{k}} + \frac{1}{\hbar^2}\omega^2_{\mathbf{k}}(t)v_{\mathbf{k}}v^*_{\mathbf{k}}\right] + \\ + \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\left[\frac{1}{c^2}v'^*_{\mathbf{k}}v'_{\mathbf{k}} + \frac{1}{\hbar^2}\omega^2_{\mathbf{k}}(t)v^*_{\mathbf{k}}v_{\mathbf{k}}\right] + \\ + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\left[\frac{1}{c^2}v'^*_{\mathbf{k}}v'^*_{\mathbf{k}} + \frac{1}{\hbar^2}\omega^2_{\mathbf{k}}(t)v^*_{\mathbf{k}}v^*_{\mathbf{k}}\right]\Bigg\} \tag{2.26}$$

$$\hat{\mathcal{H}} = \frac{c}{2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\left[\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}F_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}F^*_{\mathbf{k}} + \left(2\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v'v^*)}\delta^3(\mathbf{0})\right)E_{\mathbf{k}}\right] \tag{2.27}$$

where

$$F_{\mathbf{k}}(t) = \left(\frac{1}{\hbar c}\right)^2\left[\hbar^2 v'^2_{\mathbf{k}} + \omega^2_{\mathbf{k}}(t)\,c^2 v^2_{\mathbf{k}}\right] \tag{2.28}$$

$$E_{\mathbf{k}}(t) = \left(\frac{1}{\hbar c}\right)^2\left[\hbar^2|v'_{\mathbf{k}}|^2 + \omega^2_{\mathbf{k}}(t)\,c^2|v_{\mathbf{k}}|^2\right] \tag{2.29}$$

Now, the expectation value of the hamiltonian at time t_0 in the state $|_{(v)}0\rangle$

$$\langle_{(v)}0|\hat{\mathcal{H}}(t_0)|_{(v)}0\rangle = \rho(t_0)\delta^3(\mathbf{0}) = \frac{\hbar c^2\,\delta^3(\mathbf{0})}{4\mathrm{Im}(v'v^*)}\int \mathrm{d}^3\mathbf{k}\,E_{\mathbf{k}} \tag{2.30}$$

To minimise the energy density of the vacuum state is to find the set of functions $v_{\mathbf{k}}$ that minimise $E_{\mathbf{k}}$. Suppose that $v_{\mathbf{k}}$ can be written as

$$v_{\mathbf{k}} = r_{\mathbf{k}}e^{i\alpha_{\mathbf{k}}} \tag{2.31}$$

since $\mathrm{Im}(vv'^*)$ was constant through time

$$\mathrm{Im}(v_{\mathbf{k}}v'^*_{\mathbf{k}}) = -r^2_{\mathbf{k}}\alpha'_{\mathbf{k}} \tag{2.32}$$

this means

$$E_{\mathbf{k}} = \left(\frac{1}{\hbar c}\right)^2\left\{\hbar^2\left[r'^2_{\mathbf{k}} + \mathrm{Im}^2\left(v_{\mathbf{k}}v'^*_{\mathbf{k}}\right)\frac{1}{r^2_{\mathbf{k}}}\right] + \omega^2_{\mathbf{k}}\,c^2 r^2_{\mathbf{k}}\right\} \tag{2.33}$$

the minimum of this function must fulfil $r'_{\mathbf{k}}(t_0) = 0$. Now, if $\omega^2_{\mathbf{k}}(t_0)$ and $\mathrm{Im}(v_{\mathbf{k}}v'^*_{\mathbf{k}})$ have the same sign, the minimum of $E_{\mathbf{k}}$ happens when $r_{\mathbf{k}}(t_0) = \left[\frac{\hbar\mathrm{Im}(v_{\mathbf{k}}v'^*_{\mathbf{k}})}{\omega_{\mathbf{k}}(t_0)\,c}\right]^{1/2}$.

If there is a minimum, then

$$v_{\mathbf{k}}(t_0) = \left[\frac{\hbar\mathrm{Im}(v_{\mathbf{k}}v'^*_{\mathbf{k}})}{\omega_{\mathbf{k}}(t_0)\,c}\right]^{1/2} e^{i\alpha_{\mathbf{k}}(t_0)} \qquad v'_{\mathbf{k}}(t_0) = -c\frac{\omega_{\mathbf{k}}(t_0)}{ih}\,v_{\mathbf{k}}(t_0) \tag{2.34}$$

under these functions,

$$E_{\mathbf{k}}(t_0) = 2\frac{\mathrm{Im}(v_{\mathbf{k}}v'^*_{\mathbf{k}})}{\hbar c}\omega_{\mathbf{k}}(t_0) \qquad F_{\mathbf{k}}(t_0) = 0 \tag{2.35}$$

meaning

$$\hat{\mathcal{H}}(t_0) = \mathrm{Im}(vv'^*)\frac{1}{\hbar}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3}\left(2\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v'v^*)}\delta^3(\mathbf{0})\right)\omega_{\mathbf{k}}(t_0) \tag{2.36}$$

which is equivalent to the standard Hamiltonian for a scalar field without the presence of gravity.
Bogolyubov Transformation The expression of the field χ at two different times must be related to a Bogoliubov transformation, with coefficients

$$\alpha_{\mathbf{kp}} = \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v'v^*)}\langle\chi_{\mathbf{k}}(t_0),\,\chi_{\mathbf{p}}(t)\rangle \qquad \beta_{\mathbf{kp}} = -\frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v'v^*)}\langle\chi_{\mathbf{k}}(t_0),\,\chi^*_{\mathbf{p}}(t)\rangle \tag{2.37}$$

since the field can be written as $\chi_{\mathbf{k}} = v_{\mathbf{k}}e^{-i\mathbf{kx}/h}$ from the expression of the inner product one can see that

$$\alpha_{\mathbf{kp}} \propto \delta^3(\mathbf{k}-\mathbf{p}) \qquad \beta_{\mathbf{kp}} \propto \delta^3(\mathbf{k}+\mathbf{p}) \tag{2.38}$$

therefore it is possible to write

$$v_{\mathbf{k}}(t) = \alpha_{\mathbf{k}}v_{\mathbf{k}}(t_0) + \beta_{\mathbf{k}}v^*_{\mathbf{k}}(t_0) \tag{2.39}$$

where, recalling that $\mathrm{Im}(v'_{\mathbf{k}}v^*_{\mathbf{k}})$ is constant through time,

$$|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1 \tag{2.40}$$

To obtain the value of $\langle_{(t_0)}0|\hat{\mathcal{H}}(t)|_{(t_0)}0\rangle$ lets first compute

$$\langle_{(t_0)}0|\hat{\mathcal{N}}^{(a)}_{\mathbf{k}}(t)|_{(t_0)}0\rangle = \langle_{(t_0)}0|\hat{a}^\dagger_{\mathbf{k}}(t)\hat{a}_{\mathbf{k}}(t)|_{(t_0)}0\rangle = \left|\beta_{\mathbf{k}}\right|^2 \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(vv'^*)}\delta^3(\mathbf{0}) \tag{2.41}$$

therefore

$$\langle_{(t_0)}0|\hat{\mathcal{H}}(t)|_{(t_0)}0\rangle = \delta^3(\mathbf{0})\int \mathrm{d}^3\mathbf{k}\left(\frac{1}{2} + \left|\beta_{\mathbf{k}}\right|^2\right) c\,\omega_{\mathbf{k}}(t) \geq \langle_{(t_0)}0|\hat{\mathcal{H}}(t_0)|_{(t_0)}0\rangle \tag{2.42}$$

meaning, if $\beta_{\mathbf{k}} \neq 0$ for all $\mathbf{k}$ then, at a time $t > t_0$ the energy density will be different in relation to the original vacuum.

3 de Sitter Universe

The de Sitter Universe is a flat FLRW metric with no matter or radiation, but it does have a a positive cosmological constant Λ. Per the Friedmann equations,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho + \Lambda c^2}{3} - \frac{\kappa c^2}{a^2} \tag{3.1}$$

the expansion parameter $a(t)$ will be equal to

$$a(t) = a_1 e^{H_\Lambda t} + a_2 e^{-H_\Lambda t}\,, \qquad H_\Lambda = \sqrt{\frac{\Lambda c^2}{3}} \tag{3.2}$$

$a_2 = 0$

$$\mathrm{d}l^2 = c^2\mathrm{d}t^2 - a^2(t)\mathrm{d}\mathbf{x}^2 \tag{3.3}$$

$$\eta \equiv -\int_t^\infty \frac{\mathrm{d}t'}{a(t')} = -\frac{1}{a_1\,H_\Lambda}e^{-H_\Lambda t} = -\frac{1}{a(t)H_\Lambda} \tag{3.4}$$

[1]

$$\mathrm{d}l^2 = \frac{1}{H_\Lambda \eta^2}\left[c^2\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2\right] \tag{3.5}$$

$$R = \frac{6}{c^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] = \frac{12}{c^2}H_\Lambda^2 \tag{3.6}$$

$$\omega^2_{\mathbf{k}}(\eta) = \mathbf{k}^2 + \left[\left(\frac{mc^2}{H_\Lambda}\right)^2 + 2\left(6\xi - 1\right)\hbar^2\right]\frac{1}{c^2\eta^2} \tag{3.7}$$

$$v''_{\mathbf{k}}\hbar^2 + \omega^2_{\mathbf{k}}(\eta)\,v_{\mathbf{k}} = 0 \tag{3.8}$$

change of variables

$$s \equiv -k\eta \qquad v_{\mathbf{k}} \equiv \sqrt{s}f(s) \tag{3.9}$$

$$s^2\frac{\mathrm{d}^2 f}{\mathrm{d}s^2} + s\frac{\mathrm{d}f}{\mathrm{d}s} + \left(s^2 - \nu^2\right)f(s) = 0 \tag{3.10}$$

$$\nu^2 \equiv (3 - 16\xi)\,\frac{3}{4}\frac{\hbar^2}{c^2} - \left(\frac{mc}{H_\Lambda}\right)^2 \tag{3.11}$$

$$f(s) = AJ_\nu(s) + BY_\nu(s) \implies v_{\mathbf{k}}(\eta) = \sqrt{k|\eta|}\left[A_{\mathbf{k}}J_\nu(k|\eta|) + B_{\mathbf{k}}Y_\nu(k|\eta|)\right] \tag{3.12}$$

[1] Considering $a_2 \neq 0$ one would obtain that

$$\eta = \frac{\arctan\left(\sqrt{\frac{a_2}{a_1}}e^{-H_\Lambda t}\right)}{H_\Lambda\sqrt{a_1 a_2}}$$

Scalar field in Minkowski background

$$\eta_{\mu\nu} \tag{1}$$

Units

- $[s] = [\hbar]$
- $[a] = [\xi] = 1$
- $[\mu] = [L]^{-1}$
- $[R] = [L]^{-2}$
- $[\phi] = [\chi] = [\hbar]^{1/2}[L]^{-1}$
- $[\Pi] = [\hbar]^{1/2}[L]^{-2}$
- $[a_{\mathbf{k}}] = [\hbar]^{1/2}[L]^{2}$

Questions

- How do you know that there is a set of solutions of Klein Gordon such that the inner product fulfils the given results?
- It's the Hamiltonian well defined?
- Can you always write a FLRW metric as a flat one with a coordinate change?
- Covariant derivatives, spin connection.

Computations

Bibliography

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