PROBLEMS OF QUANTUM FIELD THEORIES IN CURVED SPACETIMES

A MASTER THESIS

Submitted in partial fulfillment of the requirements for the award of

Master's Degree in Physics of the Universe: Cosmology, Astrophysics, Particles and Astroparticles

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August 19, 2023

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1 First Chapter

FLRW metric

$$\mathrm{d}l^2 = c^2\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\mathrm{d}\Omega^2\right] \tag{1.1}$$

Weyl tensor =0 therefore the metric is conformally flat, i.e. independently of the curvature κ there must exist a coordinate system where

$$\mathrm{d}l^2 = a(t)\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = a(t)\left[c^2\mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2\right] \tag{1.2}$$

the standard action describing the dynamics of a (non-minimally coupled to gravity) real scalar field is

$$s = \int \frac{1}{2}\left[\nabla_\nu\phi\,\nabla^\nu\phi - \mu^2\phi^2 - \xi R\phi^2\right]\sqrt{-g}\,\mathrm{d}^4x \tag{1.3}$$

$\sqrt{-g} = a^4$ $\chi = a\phi$

$$s = \int \frac{1}{2}\left[\partial_\nu\chi\,\partial^\nu\chi - \left(\mu^2a^2 + \xi Ra^2 - c^2\frac{a''}{a}\right)\chi^2 - \partial_t\left(c^2\chi^2\frac{a'}{a}\right)\right]\mathrm{d}^4x \tag{1.4}$$

dropping the time drivative

$$s = \int \frac{1}{2}\left[\partial_\nu\chi\,\partial^\nu\chi - \left(\mu^2a^2 + \xi Ra^2 - c^2\frac{a''}{a}\right)\chi^2\right]\mathrm{d}^4x \tag{1.5}$$

by Euler-Lagrange

$$\left[\partial_\nu\partial^\nu + \mu^2_{\text{eff}}(t)\right]\chi = 0 \tag{1.6}$$

where

$$\mu^2_{\text{eff}}(t) = \left(\mu^2 + \xi R\right)a^2 - c^2\frac{a''}{a} \tag{1.7}$$

solutions of previous equation have the form

$$\chi = a\,v(t)e^{\pm i\mathbf{kx}\hbar^{-1}} \tag{1.8}$$

meaning that, the dispersion relation is

$$v''\hbar^2 + \omega^2(t)\,v = 0 \tag{1.9}$$

where $\omega(t)$ is defined as

$$\omega^2(t) = \mathbf{k}^2 + \hbar^2\mu^2_{\text{eff}}(t) = \mathbf{k}^2 + \left(m^2c^2 + \xi\hbar^2R\right)a(t) - \hbar^2c^2\frac{a''}{a} \tag{1.10}$$

now, proof that $\mathrm{Im}(v'v^*)$ is constant through time

$$\frac{\partial}{\partial t}\mathrm{Im}(v'v^*) = \frac{\partial}{\partial t}\left(\frac{v'v^* - {v^*}'v}{2i}\right) = \frac{v''v^* - {v^*}''v}{2i} = 0 \tag{1.11}$$

last step is result from dispersion relation. Since dispersion relation is scalable by a time independent function, $\mathrm{Im}(v'v^*)$ can be determined to be a chosen value, a particular useful choice is to consider it momentum independent.

The most general solution to the main equation is

$$\chi = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3} \left[a_{\mathbf{k}} v_{\mathbf{k}}(t) e^{i\mathbf{kx}\hbar^{-1}} + a^*_{\mathbf{k}} v^*_{\mathbf{k}}(t) e^{-i\mathbf{kx}\hbar^{-1}}\right] \tag{1.12}$$

The field χ and its conjugate momentum $\Pi = \partial_{ct}\chi$ are promoted to operators on the quantum Hilbert space, with the standar canonical conmutation relations

$$\left[\hat{\chi}(t, \mathbf{x}), \hat{\Pi}(t, \mathbf{y})\right] = i\hbar\, \delta^3(\mathbf{x} - \mathbf{y}) \tag{1.13}$$

$$\left[\hat{\chi}(t, \mathbf{x}), \hat{\chi}(t, \mathbf{y})\right] = \left[\hat{\Pi}(t, \mathbf{x}), \hat{\Pi}(t, \mathbf{y})\right] = 0 \tag{1.14}$$

where the operational nature of the fields arrise from the promotion of the mode amplitudes, i.e.

$$a_{\mathbf{k}} \longrightarrow \hat{a}_{\mathbf{k}} \qquad a^*_{\mathbf{k}} \longrightarrow \hat{a}^\dagger_{\mathbf{k}} \tag{1.15}$$

this operators fulfill the following conmutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = \frac{(2\pi\hbar)^3\hbar c}{2\mathrm{Im}(v'v^*)}\delta^3(\mathbf{k} - \mathbf{q})\,, \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = 0 \tag{1.16}$$

To prove this, consider that

$$\begin{aligned}\left[\hat{\chi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = \frac{1}{c}\int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{q}}{(2\pi\hbar)^6} \Big\{ [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}]\, v_{\mathbf{k}} v'_{\mathbf{q}} e^{i(\mathbf{kx}+\mathbf{qy})\hbar^{-1}} + \left[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}\right] v^*_{\mathbf{k}} v^{*\prime}_{\mathbf{q}} e^{i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}} + \\ + \left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}\right] v_{\mathbf{k}} v^{*\prime}_{\mathbf{q}} e^{i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}} - \left[\hat{a}_{\mathbf{q}}, \hat{a}^\dagger_{\mathbf{k}}\right] v^*_{\mathbf{k}} v'_{\mathbf{q}} e^{-i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}} \Big\}\end{aligned} \tag{1.17}$$

if the operators $\hat{a}$ and $\hat{a}^\dagger$ are to be understood as creation and annihilation operators, they must fulfill

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = \alpha\delta^3(\mathbf{k} - \mathbf{q})\,, \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{q}}] = 0 \tag{1.18}$$

where $\alpha \in \mathbb{C}$, and thus

$$\left[\hat{\chi}(\mathbf{x}),\, \hat{\Pi}(\mathbf{y})\right] = \frac{\alpha}{c}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^6} 2i\mathrm{Im}(v_{\mathbf{k}} v^{*\prime}_{\mathbf{k}}) e^{i(\mathbf{kx}-\mathbf{qy})\hbar^{-1}} \tag{1.19}$$

considering $\mathrm{Im}(v'v^*)$ momentum independent, and remembering the canonical conmutation relations, one finds that

$$\alpha\mathrm{Im}(vv^{*\prime}) = \frac{1}{2}\hbar c(2\pi\hbar)^3 \tag{1.20}$$

The hamiltonian

$$\hat{\mathcal{H}}(t) = \int \frac{1}{2}\left[\hat{\Pi}^2 + \left(\nabla\hat{\chi}\right)^2 + \mu^2_{\text{eff}}(t)\hat{\chi}^2\right] \mathrm{d}^3\mathbf{x} \tag{1.21}$$

Notas sobre unidades

- $[s] = [\hbar]$
- $[a] = [\xi] = 1$
- $[R] = [\mu] = [L]^{-2}$
- $[\phi] = [\chi] = [\hbar]^{1/2}[L]^{-1}$
- $[\Pi] = [\hbar]^{1/2}[L]^{-2}$
- $[a_{\mathbf{k}}] = [\hbar]^{1/2}[L]^{2}$

Bibliography

[1] Viatcheslav Mukhanov and Sergei Winitzki. *Introduction to Quantum Effects in Gravity*. Cambridge University Press, 2007.