

# PROBLEMS OF QUANTUM FIELD THEORIES IN CURVED SPACETIMES

A MASTER THESIS

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# 1 First Chapter

FLRW metric

$$dl^2 = c^2 dt^2 - a^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right] \quad (1.1)$$

Weyl tensor =0 therefore the metric is conformally flat, i.e. independently of the curvature  $\kappa$  there must exist a coordinate system where

$$dl^2 = a(t) \eta_{\mu\nu} dx^\mu dx^\nu = a(t) [c^2 dt^2 - d\mathbf{x}^2] \quad (1.2)$$

the standard action describing the dynamics of a (non-minimally coupled to gravity) real scalar field is

$$s = \int \frac{1}{2} \left[ \nabla_\nu \phi \nabla^\nu \phi - \mu^2 \phi^2 - \xi R \phi^2 \right] \sqrt{-g} d^4x \quad (1.3)$$

$$\sqrt{-g} = a^4 \quad \chi = a\phi$$

$$s = \int \frac{1}{2} \left[ \partial_\nu \chi \partial^\nu \chi - \left( \mu^2 a^2 + \xi R a^2 - c^2 \frac{a''}{a} \right) \chi^2 - \partial_t \left( c^2 \chi^2 \frac{a'}{a} \right) \right] d^4x \quad (1.4)$$

dropping the time drivative

$$s = \int \frac{1}{2} \left[ \partial_\nu \chi \partial^\nu \chi - \left( \mu^2 a^2 + \xi R a^2 - c^2 \frac{a''}{a} \right) \chi^2 \right] d^4x \quad (1.5)$$

by Euler-Lagrange

$$[\partial_\nu \partial^\nu + \mu_{\text{eff}}^2(t)] \chi = 0 \quad (1.6)$$

where

$$\mu_{\text{eff}}^2(t) = (\mu^2 + \xi R) a^2 - c^2 \frac{a''}{a} \quad (1.7)$$

solutions of previous equation have the form

$$\chi = a v(t) e^{\pm i \mathbf{k} \mathbf{x} \hbar^{-1}} \quad (1.8)$$

meaning that, the dispersion relation is

$$v'' \hbar^2 + \omega^2(t) v = 0 \quad (1.9)$$

where  $\omega(t)$  is defined as

$$\omega^2(t) = \mathbf{k}^2 + \hbar^2 \mu_{\text{eff}}^2(t) = \mathbf{k}^2 + (m^2 c^2 + \xi \hbar^2 R) a(t) - \hbar^2 c^2 \frac{a''}{a} \quad (1.10)$$

now, proof that  $\text{Im}(vv'^*)$  is constant through time

$$\frac{\partial}{\partial t} \text{Im}(vv'^*) = \frac{\partial}{\partial t} \left( \frac{vv'^* - v^*v'}{2i} \right) = \frac{vv''^* - v^*v''}{2i} = 0 \quad (1.11)$$

last step is result from dispersion relation. Since dispersion relation is scalable by a time independent function,  $\text{Im}(v'v^*)$  can be determined to be a chosen value, a particular useful choice is to consider it momentum independent.

The most general solution to the main equation is

$$\chi = \int \frac{d^3 \mathbf{k}}{(2\pi \hbar)^3} \left[ a_{\mathbf{k}} v_{\mathbf{k}}(t) e^{i \mathbf{k} \mathbf{x} \hbar^{-1}} + a_{\mathbf{k}}^* v_{\mathbf{k}}^*(t) e^{-i \mathbf{k} \mathbf{x} \hbar^{-1}} \right] \quad (1.12)$$

## 1 First Chapter

The field  $\chi$  and its conjugate momentum  $\Pi = \partial_{ct}\chi$  are promoted to operators on the quantum Hilbert space, with the standar canonical conmutation relations

$$[\hat{\chi}(t, \mathbf{x}), \hat{\Pi}(t, \mathbf{y})] = i\hbar \delta^3(\mathbf{x} - \mathbf{y}) \quad (1.13)$$

$$[\hat{\chi}(t, \mathbf{x}), \hat{\chi}(t, \mathbf{y})] = [\hat{\Pi}(t, \mathbf{x}), \hat{\Pi}(t, \mathbf{y})] = 0 \quad (1.14)$$

where the operational nature of the fields arrise from the promotion of the mode amplitudes, i.e.

$$a_{\mathbf{k}} \longrightarrow \hat{a}_{\mathbf{k}} \quad a_{\mathbf{k}}^* \longrightarrow \hat{a}_{\mathbf{k}}^\dagger \quad (1.15)$$

this operators fulfill the following conmutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}^\dagger] = \frac{(2\pi\hbar)^3 \hbar c}{2\text{Im}(v'v^*)} \delta^3(\mathbf{k} - \mathbf{q}), \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{q}}^\dagger] = 0 \quad (1.16)$$

To prove this, consider that

$$\begin{aligned} [\hat{\chi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})] &= \frac{1}{c} \int \frac{d^3\mathbf{k} d^3\mathbf{q}}{(2\pi\hbar)^6} \left\{ [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] v_{\mathbf{k}} v'_{\mathbf{q}} e^{i(\mathbf{k}\mathbf{x} + \mathbf{q}\mathbf{y})\hbar^{-1}} + [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{q}}^\dagger] v_{\mathbf{k}}^* v'_{\mathbf{q}}^* e^{i(\mathbf{k}\mathbf{x} - \mathbf{q}\mathbf{y})\hbar^{-1}} + \right. \\ &\quad \left. + [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}^\dagger] v_{\mathbf{k}} v_{\mathbf{q}}^* e^{i(\mathbf{k}\mathbf{x} - \mathbf{q}\mathbf{y})\hbar^{-1}} - [\hat{a}_{\mathbf{q}}, \hat{a}_{\mathbf{k}}^\dagger] v_{\mathbf{k}}^* v'_{\mathbf{q}} e^{-i(\mathbf{k}\mathbf{x} - \mathbf{q}\mathbf{y})\hbar^{-1}} \right\} \end{aligned} \quad (1.17)$$

if the operators  $\hat{a}$  and  $\hat{a}^\dagger$  are to be understood as creation and annihilation operators, they must fulfill

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}^\dagger] = \alpha \delta^3(\mathbf{k} - \mathbf{q}), \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{q}}^\dagger] = 0 \quad (1.18)$$

where  $\alpha \in \mathbb{C}$ , and thus

$$[\hat{\chi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})] = \frac{\alpha}{c} \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^6} 2i\text{Im}(v_{\mathbf{k}} v_{\mathbf{k}}^*) e^{i(\mathbf{k}\mathbf{x} - \mathbf{q}\mathbf{y})\hbar^{-1}} \quad (1.19)$$

considering  $\text{Im}(v'v^*)$  momentum independent, and remembering the canonical conmutation relations, one finds that

$$\alpha \text{Im}(vv^*) = \frac{1}{2} \hbar c (2\pi\hbar)^3 \quad (1.20)$$

The hamiltonian

$$\hat{\mathcal{H}}(t) = \int \frac{c}{2} \left[ \hat{\Pi}^2 + (\nabla \hat{\chi})^2 + \mu_{\text{eff}}^2(t) \hat{\chi}^2 \right] d^3\mathbf{x} \quad (1.21)$$

$$\begin{aligned} \hat{\Pi}^2 &= \frac{1}{c^2} \int \frac{d^3\mathbf{k} d^3\mathbf{q}}{(2\pi\hbar)^6} \left[ \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}} v'_{\mathbf{k}} v'_{\mathbf{q}} e^{i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}}^\dagger v'_{\mathbf{k}} v_{\mathbf{q}}^* e^{i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} + \right. \\ &\quad \left. + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}} v_{\mathbf{k}}^* v'_{\mathbf{q}} e^{-i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}}^\dagger v_{\mathbf{k}}^* v_{\mathbf{q}}^* e^{-i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} \right] \end{aligned} \quad (1.22)$$

$$\begin{aligned} (\nabla \hat{\chi})^2 &= -\frac{1}{\hbar^2} \int \frac{d^3\mathbf{k} d^3\mathbf{q}}{(2\pi\hbar)^6} \mathbf{k}\mathbf{q} \left[ \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}} v_{\mathbf{k}} v_{\mathbf{q}} e^{i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} - \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}}^\dagger v_{\mathbf{k}} v_{\mathbf{q}}^* e^{i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} - \right. \\ &\quad \left. - \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}} v_{\mathbf{k}}^* v_{\mathbf{q}} e^{-i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}}^\dagger v_{\mathbf{k}}^* v_{\mathbf{q}}^* e^{-i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} \right] \end{aligned} \quad (1.23)$$

$$\begin{aligned} \hat{\chi}^2 &= \int \frac{d^3\mathbf{k} d^3\mathbf{q}}{(2\pi\hbar)^6} \left[ \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}} v_{\mathbf{k}} v_{\mathbf{q}} e^{i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}}^\dagger v_{\mathbf{k}} v_{\mathbf{q}}^* e^{i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} + \right. \\ &\quad \left. + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}} v_{\mathbf{k}}^* v_{\mathbf{q}} e^{-i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}}^\dagger v_{\mathbf{k}}^* v_{\mathbf{q}}^* e^{-i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} \right] \end{aligned} \quad (1.24)$$

$$\begin{aligned}
\hat{\mathcal{H}} = \frac{c}{2} \int \frac{d^3\mathbf{k}d^3\mathbf{q}}{(2\pi\hbar)^3} \left\{ \hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{q}} \left[ \frac{1}{c^2}v'_{\mathbf{k}}v'_{\mathbf{q}} - \left( \frac{1}{\hbar^2}\mathbf{k}\mathbf{q} - \mu_{\text{eff}}^2 \right) v_{\mathbf{k}}v_{\mathbf{q}} \right] \delta^3(\mathbf{k} + \mathbf{q}) + \right. \\
+ \hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{q}}^\dagger \left[ \frac{1}{c^2}v'_{\mathbf{k}}v'_{\mathbf{q}}^* + \left( \frac{1}{\hbar^2}\mathbf{k}\mathbf{q} + \mu_{\text{eff}}^2 \right) v_{\mathbf{k}}v_{\mathbf{q}}^* \right] \delta^3(\mathbf{k} - \mathbf{q}) + \\
+ \hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{q}} \left[ \frac{1}{c^2}v_{\mathbf{k}}^*v'_{\mathbf{q}} + \left( \frac{1}{\hbar^2}\mathbf{k}\mathbf{q} + \mu_{\text{eff}}^2 \right) v_{\mathbf{k}}^*v_{\mathbf{q}} \right] \delta^3(\mathbf{k} - \mathbf{q}) + \\
\left. + \hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{q}}^\dagger \left[ \frac{1}{c^2}v_{\mathbf{k}}^*v_{\mathbf{q}}^* - \left( \frac{1}{\hbar^2}\mathbf{k}\mathbf{q} - \mu_{\text{eff}}^2 \right) v_{\mathbf{k}}^*v_{\mathbf{q}}^* \right] \delta^3(\mathbf{k} + \mathbf{q}) \right\} \quad (1.25)
\end{aligned}$$

$$\begin{aligned}
\hat{\mathcal{H}} = \frac{c}{2} \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3} \left\{ \hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k}} \left[ \frac{1}{c^2}v'_{\mathbf{k}}v'_{\mathbf{k}} + \frac{1}{\hbar^2}\omega^2(t)v_{\mathbf{k}}v_{\mathbf{k}} \right] + \right. \\
+ \hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k}}^\dagger \left[ \frac{1}{c^2}v'_{\mathbf{k}}v'_{\mathbf{k}}^* + \frac{1}{\hbar^2}\omega^2(t)v_{\mathbf{k}}v_{\mathbf{k}}^* \right] + \\
+ \hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}} \left[ \frac{1}{c^2}v_{\mathbf{k}}^*v'_{\mathbf{k}} + \frac{1}{\hbar^2}\omega^2(t)v_{\mathbf{k}}^*v_{\mathbf{k}} \right] + \\
\left. + \hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}}^\dagger \left[ \frac{1}{c^2}v_{\mathbf{k}}^*v_{\mathbf{k}}^* + \frac{1}{\hbar^2}\omega^2(t)v_{\mathbf{k}}^*v_{\mathbf{k}}^* \right] \right\} \quad (1.26)
\end{aligned}$$

$$\hat{\mathcal{H}} = \frac{c}{2} \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3} \left[ \hat{a}_{\mathbf{k}}^2 F_{\mathbf{k}} + \hat{a}_{\mathbf{k}}^{\dagger 2} F_{\mathbf{k}}^* + \left( 2\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \frac{(2\pi\hbar)^3 \hbar c}{2\text{Im}(v'v^*)} \delta^3(\mathbf{0}) \right) E_{\mathbf{k}} \right] \quad (1.27)$$

where

$$F_{\mathbf{k}}(t) = \left( \frac{1}{\hbar c} \right)^2 \left[ \hbar^2 v_{\mathbf{k}}'^2 + \omega^2(t) c^2 v_{\mathbf{k}}^2 \right] \quad (1.28)$$

$$E_{\mathbf{k}}(t) = \left( \frac{1}{\hbar c} \right)^2 \left[ \hbar^2 |v'_{\mathbf{k}}|^2 + \omega^2(t) c^2 |v_{\mathbf{k}}|^2 \right] \quad (1.29)$$

Now, the expectation value of the hamiltonian at time  $t_0$  in the state  $|_{(v)}0\rangle$

$$\langle_{(v)}0|\hat{\mathcal{H}}(t_0)|_{(v)}0\rangle = \rho(t_0)\delta^3(\mathbf{0}) = \frac{\hbar c^2 \delta^3(\mathbf{0})}{4\text{Im}(v'v^*)} \int d^3\mathbf{k} E_{\mathbf{k}} \quad (1.30)$$

# Notas sobre unidades

- $[s] = [\hbar]$
- $[a] = [\xi] = 1$
- $[\mu] = [L]^{-1}$
- $[R] = [L]^{-2}$
- $[\phi] = [\chi] = [\hbar]^{1/2}[L]^{-1}$
- $[\Pi] = [\hbar]^{1/2}[L]^{-2}$
- $[a_{\mathbf{k}}] = [\hbar]^{1/2}[L]^2$

# Bibliography

- [1] Viatcheslav Mukhanov and Sergei Winitzki. *Introduction to Quantum Effects in Gravity*. Cambridge University Press, 2007.