

# PROBLEMS OF QUANTUM FIELD THEORIES IN CURVED SPACETIMES

A MASTER THESIS

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# Preface

Preface

# 1 Introduction to QFT in curved spacetimes

$$dl^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (1.1)$$

$$S = \int \left[ \frac{1}{2\kappa} (R - 2\Lambda) + \mathcal{L}_M \right] \sqrt{-g} d^4x \quad (1.2)$$

$$\kappa \equiv \frac{8\pi G}{c^4}$$

Variation of  $S$  with respect to the inverse metric ( $g^{\mu\nu}$ ) gives

$$\delta S = \int \left[ \frac{\sqrt{-g}}{2\kappa} \frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{2\kappa} \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} - \frac{\Lambda}{\kappa} \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} + \frac{\mathcal{L}_M}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} \sqrt{-g} d^4x \quad (1.3)$$

$\delta S = 0$  and

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu} \quad \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2} g_{\mu\nu} \quad (1.4)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -2 \frac{8\pi G}{c^4} \left( \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} - \frac{1}{2} \mathcal{L}_M g_{\mu\nu} \right) \quad (1.5)$$

$$T_{\mu\nu} \equiv \mathcal{L}_M g_{\mu\nu} - \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} = \frac{-2}{\sqrt{-g}} \frac{\delta (\mathcal{L}_M \sqrt{-g})}{\delta g^{\mu\nu}} \quad (1.6)$$

[1]

$$\nabla_\mu T^{\mu\nu} = 0 \quad (1.7)$$

(and its symmetric)

## 1.1 Construction of covariant actions

The equivalence principle says that field equations must be invariant with respect to local Lorentz transformations  $\Lambda(x)$ . To work on a local flat spacetime, we use the tetrad formalism

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab} \quad (1.8)$$

partial derivatives transform like

$$\partial_\mu \rightarrow \frac{\partial x^\nu}{\partial y^\mu} \partial_\nu \quad (1.9)$$

but, we see that  $e_a^\mu \partial_\mu$  is not covariant, since

$$e_a^\mu \partial^\alpha \phi(x) \rightarrow \Lambda_a^b e_b^\mu \partial_\mu [\rho(\Lambda) \phi(x)] = \Lambda_a^b e_b^\mu [\rho(\Lambda) \partial_\mu \phi + \partial_\mu \rho(\Lambda) \phi] \quad (1.10)$$

a covariant derivative  $D_\mu$  should transform as

$$D_a \phi \rightarrow \Lambda_a^b \rho(\Lambda) D_b \phi \quad (1.11)$$

therefore we need to define a better derivative, a common option is

$$D_a \equiv e_a^\mu (\partial_\mu + \Gamma_\mu) \quad (1.12)$$

where, for the derivative to be covariant, the connection  $\Gamma_\mu$  must transform as

$$\Gamma_\mu \rightarrow \rho(\Lambda) \Gamma_\mu \rho^{-1}(\Lambda) - [\partial_\mu \rho(\Lambda)] \rho^{-1}(\Lambda) \quad (1.13)$$

such connection can be written as

$$\Gamma_\mu = \frac{1}{2} \Sigma^{ab} e_a^\nu \nabla_\mu e_{b\nu} \quad (1.14)$$

where  $\Sigma^{ab}$  are the Lorentz generators and  $e_a^\nu \nabla_\mu e_{b\nu} \equiv \omega_{ab\mu}$  is the torsion free spin connection.

Under the presence of a gauge field, we would also want the derivative to be gauge invariant, and thus, the gauge four-potential  $A$  must be considered in the definition of the derivative

$$D_a \equiv e_a^\mu \left( \partial_\mu + \Gamma_\mu - \frac{i}{\hbar} e A_\mu \right) \quad (1.15)$$

where  $e$  is the coupling constant.

This derivative of course will

Now, in order to obtain the equations of motion for the matter fields, consider the lagrangian

$$\mathcal{L}_M = \mathcal{L}_M[\phi^\alpha(x), D_\mu \phi^\alpha(x)] \quad (1.16)$$

variations of  $S$  with respect of  $\phi^\alpha$  result in

$$\delta S = \int \left[ \frac{\partial \mathcal{L}_M}{\partial \phi^\alpha} \delta \phi^\alpha + \frac{\partial \mathcal{L}_M}{\partial (D_\mu \phi^\alpha)} D_\mu (\delta \phi^\alpha) \right] \sqrt{-g} d^4x \quad (1.17)$$

and thus, after applying the generalized Gauss Theorem on a curved background, and considering that field variations vanish at the boundaries, one obtains

$$\frac{\partial \mathcal{L}_M}{\partial \phi^\alpha} - D_\mu \left[ \frac{\partial \mathcal{L}_M}{\partial (D_\mu \phi^\alpha)} \right] = 0 \quad (1.18)$$

$$\Pi_\alpha \equiv \frac{\partial \mathcal{L}_M}{\partial (D_0 \phi^\alpha)} \quad (1.19)$$

## 1.2 Scalar field

$$S[\phi] = \int \frac{1}{2} \left[ \nabla_\nu \phi \nabla^\nu \phi - \mu^2 \phi^2 - \xi R \phi^2 \right] \sqrt{-g} d^4x \quad (1.20)$$

where  $\mu \equiv mc/\hbar$ ,  $R$  is the curvature scalar of the manifold. For  $\xi = 0$  the scalar field is said to be minimally coupled to gravity (nonminimally coupled otherwise); this is the simplest interactive term that couples the scalar field with the manifold; in curved spacetimes [2] a self interactive  $\lambda \phi^4$  theory needs a term proportional to  $R \phi^2$  to be renormalizable. For  $\xi = 1/6$  its said to be conformally coupled, and, in the case of a massless field ( $\mu = 0$ ) the action will be invariant under conformal transformations, i.e.

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} \equiv \Omega^2(x) g_{\mu\nu}, \quad (1.21)$$

This can be seen from the equations of motion (Klein-Gordon)

$$[\nabla_\nu \nabla^\nu - \mu^2 - \xi R] \phi = 0 \quad (1.22)$$

considering  $\phi \rightarrow \tilde{\phi} = \Omega^\beta \phi$  one gets that

$$\begin{aligned} 0 = & \mu^2 \Omega^{\beta-2} (\Omega^2 - 1) \phi + 2(1 + \beta) \Omega^{\beta-3} \nabla^\nu \Omega \nabla_\nu \phi + \\ & + (6\xi + \beta) \Omega^{\beta-3} (\nabla_\nu \nabla^\nu \Omega) \phi + \beta(1 + \beta) \Omega^{\beta-4} \nabla_\nu \Omega \nabla^\nu \Omega \phi \end{aligned} \quad (1.23)$$

this equation has a solution for massless fields, corresponding to

$$\beta = -1, \quad \xi = \frac{1}{6} \quad (1.24 \text{ a,b})$$

The energy momentum tensor has the following form

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} [\nabla^\sigma \phi \nabla_\sigma \phi - \mu^2 \phi^2] + \xi \left[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} + g_{\mu\nu} \nabla^\sigma \nabla_\sigma - \nabla_\mu \nabla_\nu \right] \phi^2 \quad (1.25)$$

note that for minimally coupled field ( $\xi = 0$ ) the energy-momentum tensor is equivalent to the Noether energy-momentum tensor.

Scalar product

$$\langle \phi_1(x), \phi_2(x) \rangle \equiv i \int g^{0\nu} \left( \phi_1 \overset{\leftrightarrow}{\nabla}_\nu \phi_2^* \right) \sqrt{-g} d^3 \mathbf{x} \quad (1.26)$$

Let  $v(x)$  be a solution of the Klein-Gordon equation, then  $v^*(x)$  will also be an (linearly independent) solution. Let  $i$  represent the set of parameters that univocally describe a par of solutions  $v_i(x)$ ,  $v^*(x)$ , therefore, the general solution of the Klein-Gordon equation will be of the form

$$\phi(x) = \sum_i [a_i v_i(x) + a_i^* v_i^*(x)] \quad (1.27)$$

where  $a_i$ ,  $a_i^*$  are constant factors that can be written as

$$a_i = \langle v_i(x), \phi(x) \rangle \quad a_i^* = \langle v_i^*(x), \phi(x) \rangle \quad (1.28)$$

Quantization of the field is done by promoting the fields to operators

$$\phi(x) \longrightarrow \hat{\phi}(x) \quad \Pi(x) \longrightarrow \hat{\Pi}(x) \quad (1.29)$$

this is done by promoting the constant factors to operators as well, that is

$$a_i \longrightarrow \hat{a}_i \quad a_i^* \longrightarrow \hat{a}_i^\dagger \quad (1.30)$$

and therefore

$$\hat{\phi}(x) = \sum_i [\hat{a}_i v_i(x) + \hat{a}_i^\dagger v_i^*(x)] \quad (1.31)$$

after the promotion of the fields to operators, commutation relations are imposed; the easiest choice would be to assume canonical quantization relations,

$$[\hat{\phi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})] = i\hbar \delta^3(\mathbf{x} - \mathbf{y}) \quad [\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{y})] = [\hat{\Pi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})] = 0 \quad (1.32)$$

It would be desirable to obtain a formulation similar to the well known scalar field in a flat background, where the Fock space is generated from a vacuum state and a set of creation and annihilation operators that follow some commutation rules. To do so, we will force the  $\hat{a}_i$ ,  $\hat{a}_i^\dagger$  operators to assume this roll, in such a way that

$$[\hat{a}_i, \hat{a}_j^\dagger] \propto \delta_{ij} \quad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \quad (1.33)$$

Thanks to the relation between the constant factors  $a_i$  and the scalar product  $\langle v_i, \phi \rangle$ , one can obtain

$$\begin{aligned} [\hat{a}_i, \hat{a}_j^\dagger] &= - \int \left[ \left( v_i \hat{\Pi} - g^{0\nu} (\nabla_\nu v_i) \hat{\phi} \sqrt{-g} \right) \Big|_{\mathbf{x}}, \left( v_j^* \hat{\Pi} - g^{0\nu} (\nabla_\nu v_j) \hat{\phi} \sqrt{-g} \right) \Big|_{\mathbf{y}} \right] d^3 \mathbf{x} d^3 \mathbf{y} = \\ &= i\hbar \int g^{0\nu} \left( v_i \overset{\leftrightarrow}{\nabla}_\nu v_j^* \right) \sqrt{-g} d^3 \mathbf{x} = \hbar \langle v_i, v_j \rangle \end{aligned} \quad (1.34)$$

where the field commutators where used. Equivalently

$$[\hat{a}_i, \hat{a}_j] = -\hbar \langle v_i, v_j^* \rangle \quad [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = -\hbar \langle v_i^*, v_j \rangle \quad (1.35)$$

Therefore we must find a set of solutions  $\{v_i(x), v_i^*(x)\}$  such that

$$\langle v_i, v_j \rangle \propto \delta_{ij} \quad \langle v_i, v_j^* \rangle = \langle v_i^*, v_j \rangle = 0 \quad (1.36)$$

With this, we can define the Fock space the usual way, starting with a vacuum state  $|0\rangle$  such that the action of the annihilation operation fulfils

$$\hat{a}_i |0\rangle = 0 \quad \forall i \quad (1.37)$$

where single particle states are formed from the creation operator

$$|i\rangle \equiv \hat{a}_i^\dagger |0\rangle \quad (1.38)$$

and multiparticle states like

$$|i, j, \dots\rangle = \dots \hat{a}_i^\dagger \hat{a}_j^\dagger |0\rangle \quad (1.39)$$

Since this is a scalar field, one might assume that the states are symmetric (describing boson particles), and this is easily confirmed, since

$$|i, j\rangle = \hat{a}_j^\dagger \hat{a}_i^\dagger |0\rangle = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] |0\rangle + \hat{a}_i^\dagger \hat{a}_j^\dagger |0\rangle = |j, i\rangle \quad (1.40)$$

### 1.3 Bogoliubov transformations

Consider now a second set  $\{u_i(x), u_i^*(x)\}$  of solutions to the Klein-Gordon equation such that they meet the inner product rule; the field would then be written as

$$\phi(x) = \sum_j [b_j u_j(x) + b_j^* u_j^*(x)] \quad (1.41)$$

quantization of the field and creation and annihilation is straightforward. The relation between the  $v$  and  $u$  solutions would be

$$v_i(x) \equiv \sum_j [\alpha_{ij} u_j(x) + \beta_{ij} u_j^*(x)] \quad (1.42)$$

where  $\alpha_{ij}$  and  $\beta_{ij}$  are known as Bogoliubov coefficients, that can be obtained as

$$\alpha_{ij} \propto \langle v_i, u_j \rangle \quad \beta_{ij} \propto -\langle v_i, u_j^* \rangle \quad (1.43)$$

Since the field is the same independently of the mode set chosen

$$\sum_i [\hat{a}_i v_i(x) + \hat{a}_i^\dagger v_i^*(x)] = \sum_j [\hat{b}_j u_j(x) + \hat{b}_j^\dagger u_j^*(x)] \quad (1.44)$$

and, as a result of the orthogonality of the mode functions

$$\hat{a}_i = \sum_j (\alpha_{ij}^* \hat{b}_j - \beta_{ij}^* \hat{b}_j^\dagger) \quad \hat{a}_i^\dagger = \sum_j (-\beta_{ij} \hat{b}_j + \alpha_{ij} \hat{b}_j^\dagger) \quad (1.45)$$

creation and annihilation commutation relations give new restrictions to the Bogoliubov coefficients

$$[\hat{a}_i, \hat{a}_j^\dagger] \propto \delta_{ij} \implies \sum_k (\alpha_{ik}^* \alpha_{jk} - \beta_{ik}^* \beta_{jk}) \propto \delta_{ij} \quad (1.46)$$

$$[\hat{a}_i, \hat{a}_j] = 0 \implies \sum_k (\alpha_{jk}^* \beta_{ik}^* - \alpha_{ik}^* \beta_{jk}^*) = 0 \quad (1.47)$$

Now, the relevance of the Bogoliubov transformations comes from the fact that the vacuum in the  $u$  solutions, have (in general)  $v$  particles,

$$\langle u0 | \hat{N}_v | u0 \rangle = \sum_i \langle u0 | \hat{a}_i^\dagger \hat{a}_i | u0 \rangle = \sum_i \left[ \sum_{jk} \beta_{ij} \beta_{ik}^* \langle u0 | \hat{b}_j \hat{b}_k^\dagger | u0 \rangle \right] \propto \sum_{ij} |\beta_{ij}|^2 \quad (1.48)$$

therefore, there is not a unique vacuum.

## 1.4 A leap towards a continuum

Until now, it has been considered that the set of Klein-Gordon solutions could be categorised by a discrete set of parameters  $i$ , from a standard course in QFT, one of the main results is the fact that the solutions of the flat Klein-Gordon equations can be parametrised by a continuous 3-dimensional vector  $\mathbf{k}$  (which is interpreted to be the momentum of the particle). Since all computations in this section were made by considering a discrete set of parameters, it is relevant to consider the continuum case.

A common computation in many fields of physics is the determination of the density of states  $D(\mathbf{k})$  describing the number of modes with momentum between  $\mathbf{k}$  and  $\mathbf{k}+d\mathbf{k}$ . Consider a system with volume  $V$ , where the field goes to zero at its boundary; in this case, the permitted values of momenta must meet

$$k^i = n^i \frac{\pi \hbar}{V^{1/3}}, \quad n^i \in \mathbb{Z} \quad (1.49)$$

Let  $N(k)$  be the number of states with momentum modulus less than  $k$ , that is, the states such that

$$n = \sqrt{(n^1)^2 + (n^2)^2 + (n^3)^2} < k \frac{V^{1/3}}{\pi \hbar} \quad (1.50)$$

considering a flat momentum space<sup>1</sup> and a large enough volume,  $N(k)$  will be essentially equal to an eighth of the volume of a sphere with radius  $kV^{1/3}/\pi\hbar$ , that is

$$N(k) \approx \frac{1}{8} \frac{4}{3} \pi \left( k \frac{V^{1/3}}{\pi \hbar} \right)^3 = \frac{V}{6\pi^2 \hbar^3} k^3 \quad (1.51)$$

meaning, that the density of states will be

$$D(\mathbf{k}) \equiv D(k) = \frac{dN(k)}{dk} \approx \frac{V}{2\pi^2 \hbar^3} k^2 \quad (1.52)$$

With this, one could approximate a discrete sum over a parameter  $i$  to an integral over a continuum  $\mathbf{k}$

$$\sum_i f_i = \int_0^\infty D(k) f_k dk \approx \int_0^\infty \frac{V}{2\pi^2 \hbar^3} f_k k^2 dk \equiv \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3} f_{\mathbf{k}} \quad (1.53)$$

where it has been defined.

$$4\pi V f_k k^2 \equiv \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} f_{\mathbf{k}} \sin \varphi d\theta d\varphi \quad (1.54)$$

therefore  $d^3\mathbf{k}/(2\pi\hbar)^3$  is to be understood as the volume element of the momentum space.

## 2 Scalar field in an expanding universe

FLRW metric

$$dl^2 = c^2 dt^2 - a^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right] \quad (2.1)$$

Weyl tensor = 0 therefore the metric is conformally flat, i.e. independently of the curvature  $\kappa$  there must exist a coordinate system where

$$dl^2 = a(t) \eta_{\mu\nu} dx^\mu dx^\nu = a(t) [c^2 dt^2 - d\mathbf{x}^2] \quad (2.2)$$

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<sup>1</sup>In contrast to modified theories of relativity in which this is not the case, like the  $\kappa$ -Poincaré relativity.



## 2 Scalar field in an expanding universe

the standard action describing the dynamics of a (non-minimally coupled to gravity) real scalar field is

$$s = \int \frac{1}{2} \left[ \nabla_\nu \phi \nabla^\nu \phi - \mu^2 \phi^2 - \xi R \phi^2 \right] \sqrt{-g} \, d^4x \quad (2.3)$$

$$\sqrt{-g} = a^4 \quad \chi = a\phi$$

$$s = \int \frac{1}{2} \left[ \partial_\nu \chi \partial^\nu \chi - \left( \mu^2 a^2 + \xi R a^2 - c^2 \frac{a''}{a} \right) \chi^2 - \partial_t \left( c^2 \chi^2 \frac{a'}{a} \right) \right] d^4x \quad (2.4)$$

dropping the time derivative

$$s = \int \frac{1}{2} \left[ \partial_\nu \chi \partial^\nu \chi - \left( \mu^2 a^2 + \xi R a^2 - c^2 \frac{a''}{a} \right) \chi^2 \right] d^4x \quad (2.5)$$

by Euler-Lagrange

$$[\partial_\nu \partial^\nu + \mu_{\text{eff}}^2(t)] \chi = 0 \quad (2.6)$$

where

$$\mu_{\text{eff}}^2(t) = (\mu^2 + \xi R) a^2 - \frac{a''}{ac^2} \quad (2.7)$$

solutions of previous equation have the form

$$\chi = a v(t) e^{\pm i \mathbf{k} \mathbf{x} \hbar^{-1}} \quad (2.8)$$

meaning that, the dispersion relation is

$$v'' \hbar^2 + \omega^2(t) v = 0 \quad (2.9)$$

where  $\omega(t)$  is defined as

$$\omega_{\mathbf{k}}^2(t) = \mathbf{k}^2 + \hbar^2 \mu_{\text{eff}}^2(t) = \mathbf{k}^2 + (m^2 c^2 + \xi \hbar^2 R) a^2(t) - \hbar^2 \frac{a''}{ac^2} \quad (2.10)$$

now, proof that  $\text{Im}(vv'^*)$  is constant through time

$$\frac{\partial}{\partial t} \text{Im}(vv'^*) = \frac{\partial}{\partial t} \left( \frac{vv'^* - v^*v'}{2i} \right) = \frac{vv''^* - v^*v''}{2i} = 0 \quad (2.11)$$

last step is result from dispersion relation. Since dispersion relation is scalable by a time independent function,  $\text{Im}(v'v^*)$  can be determined to be a chosen value, a particular useful choice is to consider it momentum independent.  $\text{Im}(v'v^*) = W[v, v^*]$  therefore, if its not equal to 0, they are linearly independent solutions to dispersion relation.

The most general solution to the main equation is

$$\chi = \int \frac{d^3 \mathbf{k}}{(2\pi \hbar)^3} \left[ a_{\mathbf{k}} v_{\mathbf{k}}(t) e^{i \mathbf{k} \mathbf{x} \hbar^{-1}} + a_{\mathbf{k}}^* v_{\mathbf{k}}^*(t) e^{-i \mathbf{k} \mathbf{x} \hbar^{-1}} \right] \quad (2.12)$$

The field  $\chi$  and its conjugate momentum  $\Pi = \partial_c \chi$  are promoted to operators on the quantum Hilbert space, with the standar canonical conmutation relations

$$[\hat{\chi}(t, \mathbf{x}), \hat{\Pi}(t, \mathbf{y})] = i \hbar \delta^3(\mathbf{x} - \mathbf{y}) \quad (2.13)$$

$$[\hat{\chi}(t, \mathbf{x}), \hat{\chi}(t, \mathbf{y})] = [\hat{\Pi}(t, \mathbf{x}), \hat{\Pi}(t, \mathbf{y})] = 0 \quad (2.14)$$

where the operational nature of the fields arrise from the promotion of the mode amplitudes, i.e.

$$a_{\mathbf{k}} \longrightarrow \hat{a}_{\mathbf{k}} \quad a_{\mathbf{k}}^* \longrightarrow \hat{a}_{\mathbf{k}}^\dagger \quad (2.15)$$

this operators fulfill the following commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}^\dagger] = \frac{(2\pi\hbar)^3 \hbar c}{2\text{Im}(v'v^*)} \delta^3(\mathbf{k} - \mathbf{q}), \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{q}}^\dagger] = 0 \quad (2.16)$$

(note that  $\hat{a}_{\mathbf{k}} \neq \hat{a}_{-\mathbf{k}}$ )

To prove this, consider that

$$\begin{aligned} [\hat{\chi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})] &= \frac{1}{c} \int \frac{d^3\mathbf{k} d^3\mathbf{q}}{(2\pi\hbar)^6} \left\{ [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] v_{\mathbf{k}} v'_{\mathbf{q}} e^{i(\mathbf{k}\mathbf{x} + \mathbf{q}\mathbf{y})\hbar^{-1}} + [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{q}}^\dagger] v_{\mathbf{k}}^* v_{\mathbf{q}}'^* e^{i(\mathbf{k}\mathbf{x} - \mathbf{q}\mathbf{y})\hbar^{-1}} + \right. \\ &\quad \left. + [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}^\dagger] v_{\mathbf{k}} v_{\mathbf{q}}'^* e^{i(\mathbf{k}\mathbf{x} - \mathbf{q}\mathbf{y})\hbar^{-1}} - [\hat{a}_{\mathbf{q}}, \hat{a}_{\mathbf{k}}^\dagger] v_{\mathbf{k}}^* v_{\mathbf{q}}' e^{-i(\mathbf{k}\mathbf{x} - \mathbf{q}\mathbf{y})\hbar^{-1}} \right\} \end{aligned} \quad (2.17)$$

if the operators  $\hat{a}$  and  $\hat{a}^\dagger$  are to be understood as creation and annihilation operators, they must fulfill

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}^\dagger] = \alpha \delta^3(\mathbf{k} - \mathbf{q}), \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{q}}^\dagger] = 0 \quad (2.18)$$

where  $\alpha \in \mathbb{C}$ , and thus

$$[\hat{\chi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})] = \frac{\alpha}{c} \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^6} 2i\text{Im}(v_{\mathbf{k}} v_{\mathbf{k}}'^*) e^{i(\mathbf{k}\mathbf{x} - \mathbf{q}\mathbf{y})\hbar^{-1}} \quad (2.19)$$

considering  $\text{Im}(v'v^*)$  momentum independent, and remembering the canonical commutation relations, one finds that

$$\alpha \text{Im}(v v'^*) = \frac{1}{2} \hbar c (2\pi\hbar)^3 \quad (2.20)$$

Let  $\xi = 0$ , i.e. work in the nonminimally coupled scalar field; then the Hamiltonian will be

$$\hat{\mathcal{H}}(t) = \int \frac{c}{2} \left[ \hat{\Pi}^2 + (\nabla \hat{\chi})^2 + \mu_{\text{eff}}^2(t) \hat{\chi}^2 \right] d^3\mathbf{x} \quad (2.21)$$

$$\begin{aligned} \hat{\Pi}^2 &= \frac{1}{c^2} \int \frac{d^3\mathbf{k} d^3\mathbf{q}}{(2\pi\hbar)^6} \left[ \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}} v'_{\mathbf{k}} v'_{\mathbf{q}} e^{i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}}^\dagger v'_{\mathbf{k}} v_{\mathbf{q}}'^* e^{i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} + \right. \\ &\quad \left. + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}} v_{\mathbf{k}}'^* v'_{\mathbf{q}} e^{-i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}}^\dagger v_{\mathbf{k}}'^* v_{\mathbf{q}}'^* e^{-i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} \right] \end{aligned} \quad (2.22)$$

$$\begin{aligned} (\nabla \hat{\chi})^2 &= -\frac{1}{\hbar^2} \int \frac{d^3\mathbf{k} d^3\mathbf{q}}{(2\pi\hbar)^6} \mathbf{k}\mathbf{q} \left[ \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}} v_{\mathbf{k}} v_{\mathbf{q}} e^{i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} - \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}}^\dagger v_{\mathbf{k}} v_{\mathbf{q}}'^* e^{i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} - \right. \\ &\quad \left. - \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}} v_{\mathbf{k}}'^* v_{\mathbf{q}} e^{-i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}}^\dagger v_{\mathbf{k}}'^* v_{\mathbf{q}}'^* e^{-i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} \right] \end{aligned} \quad (2.23)$$

$$\begin{aligned} \hat{\chi}^2 &= \int \frac{d^3\mathbf{k} d^3\mathbf{q}}{(2\pi\hbar)^6} \left[ \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}} v_{\mathbf{k}} v_{\mathbf{q}} e^{i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}}^\dagger v_{\mathbf{k}} v_{\mathbf{q}}'^* e^{i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} + \right. \\ &\quad \left. + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}} v_{\mathbf{k}}'^* v_{\mathbf{q}} e^{-i(\mathbf{k} - \mathbf{q})\mathbf{x}\hbar^{-1}} + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}}^\dagger v_{\mathbf{k}}'^* v_{\mathbf{q}}'^* e^{-i(\mathbf{k} + \mathbf{q})\mathbf{x}\hbar^{-1}} \right] \end{aligned} \quad (2.24)$$

$$\begin{aligned} \hat{\mathcal{H}} &= \frac{c}{2} \int \frac{d^3\mathbf{k} d^3\mathbf{q}}{(2\pi\hbar)^3} \left\{ \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}} \left[ \frac{1}{c^2} v'_{\mathbf{k}} v'_{\mathbf{q}} - \left( \frac{1}{\hbar^2} \mathbf{k}\mathbf{q} - \mu_{\text{eff}}^2 \right) v_{\mathbf{k}} v_{\mathbf{q}} \right] \delta^3(\mathbf{k} + \mathbf{q}) + \right. \\ &\quad + \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{q}}^\dagger \left[ \frac{1}{c^2} v'_{\mathbf{k}} v_{\mathbf{q}}'^* + \left( \frac{1}{\hbar^2} \mathbf{k}\mathbf{q} + \mu_{\text{eff}}^2 \right) v_{\mathbf{k}} v_{\mathbf{q}}^* \right] \delta^3(\mathbf{k} - \mathbf{q}) + \\ &\quad + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}} \left[ \frac{1}{c^2} v_{\mathbf{k}}'^* v'_{\mathbf{q}} + \left( \frac{1}{\hbar^2} \mathbf{k}\mathbf{q} + \mu_{\text{eff}}^2 \right) v_{\mathbf{k}}^* v_{\mathbf{q}} \right] \delta^3(\mathbf{k} - \mathbf{q}) + \\ &\quad \left. + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{q}}^\dagger \left[ \frac{1}{c^2} v_{\mathbf{k}}'^* v_{\mathbf{q}}'^* - \left( \frac{1}{\hbar^2} \mathbf{k}\mathbf{q} - \mu_{\text{eff}}^2 \right) v_{\mathbf{k}}^* v_{\mathbf{q}}^* \right] \delta^3(\mathbf{k} + \mathbf{q}) \right\} \end{aligned} \quad (2.25)$$

$$\begin{aligned} \hat{\mathcal{H}} = \frac{c}{2} \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3} \left\{ \hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} \left[ \frac{1}{c^2}v'_{\mathbf{k}}v'_{\mathbf{k}} + \frac{1}{\hbar^2}\omega_{\mathbf{k}}^2(t)v_{\mathbf{k}}v_{\mathbf{k}} \right] + \right. \\ \left. + \hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k}}^\dagger \left[ \frac{1}{c^2}v'_{\mathbf{k}}v_{\mathbf{k}}'^* + \frac{1}{\hbar^2}\omega_{\mathbf{k}}^2(t)v_{\mathbf{k}}v_{\mathbf{k}}'^* \right] + \right. \\ \left. + \hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}} \left[ \frac{1}{c^2}v_{\mathbf{k}}'^*v'_{\mathbf{k}} + \frac{1}{\hbar^2}\omega_{\mathbf{k}}^2(t)v_{\mathbf{k}}'^*v_{\mathbf{k}} \right] + \right. \\ \left. + \hat{a}_{\mathbf{k}}^\dagger\hat{a}_{-\mathbf{k}}^\dagger \left[ \frac{1}{c^2}v_{\mathbf{k}}'^*v_{\mathbf{k}}'^* + \frac{1}{\hbar^2}\omega_{\mathbf{k}}^2(t)v_{\mathbf{k}}'^*v_{\mathbf{k}}'^* \right] \right\} \quad (2.26) \end{aligned}$$

$$\hat{\mathcal{H}} = \frac{c}{2} \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3} \left[ \hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}F_{\mathbf{k}} + \hat{a}_{\mathbf{k}}^\dagger\hat{a}_{-\mathbf{k}}^\dagger F_{\mathbf{k}}^* + \left( 2\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}} + \frac{(2\pi\hbar)^3\hbar c}{2\text{Im}(v'v^*)}\delta^3(\mathbf{0}) \right) E_{\mathbf{k}} \right] \quad (2.27)$$

where

$$F_{\mathbf{k}}(t) = \left( \frac{1}{\hbar c} \right)^2 \left[ \hbar^2 v_{\mathbf{k}}'^2 + \omega_{\mathbf{k}}^2(t) c^2 v_{\mathbf{k}}^2 \right] \quad (2.28)$$

$$E_{\mathbf{k}}(t) = \left( \frac{1}{\hbar c} \right)^2 \left[ \hbar^2 |v_{\mathbf{k}}'|^2 + \omega_{\mathbf{k}}^2(t) c^2 |v_{\mathbf{k}}|^2 \right] \quad (2.29)$$

Now, the expectation value of the hamiltonian at time  $t_0$  in the state  $|_{(v)}0\rangle$

$$\langle_{(v)}0|\hat{\mathcal{H}}(t_0)|_{(v)}0\rangle = \rho(t_0)\delta^3(\mathbf{0}) = \frac{\hbar c^2 \delta^3(\mathbf{0})}{4\text{Im}(v'v^*)} \int d^3\mathbf{k} E_{\mathbf{k}} \quad (2.30)$$

To minimise the energy density of the vacuum state is to find the set of functions  $v_{\mathbf{k}}$  that minimise  $E_{\mathbf{k}}$ . Suppose that  $v_{\mathbf{k}}$  can be written as

$$v_{\mathbf{k}} = r_{\mathbf{k}} e^{i\alpha_{\mathbf{k}}} \quad (2.31)$$

since  $\text{Im}(vv'^*)$  was constant through time

$$\text{Im}(v_{\mathbf{k}}v_{\mathbf{k}}'^*) = -r_{\mathbf{k}}^2 \alpha'_{\mathbf{k}} \quad (2.32)$$

this means

$$E_{\mathbf{k}} = \left( \frac{1}{\hbar c} \right)^2 \left\{ \hbar^2 \left[ r_{\mathbf{k}}'^2 + \text{Im}^2(v_{\mathbf{k}}v_{\mathbf{k}}'^*) \frac{1}{r_{\mathbf{k}}^2} \right] + \omega_{\mathbf{k}}^2 c^2 r_{\mathbf{k}}^2 \right\} \quad (2.33)$$

the minimum of this function must fulfil  $r_{\mathbf{k}}'(t_0) = 0$ . Now, if  $\omega_{\mathbf{k}}^2(t_0)$  and  $\text{Im}(v_{\mathbf{k}}v_{\mathbf{k}}'^*)$  have the same sign, the minimum of  $E_{\mathbf{k}}$  happens when  $r_{\mathbf{k}}(t_0) = \left[ \frac{\hbar \text{Im}(v_{\mathbf{k}}v_{\mathbf{k}}'^*)}{\omega_{\mathbf{k}}(t_0) c} \right]^{1/2}$ .

If there is a minimum, then

$$v_{\mathbf{k}}(t_0) = \left[ \frac{\hbar \text{Im}(v_{\mathbf{k}}v_{\mathbf{k}}'^*)}{\omega_{\mathbf{k}}(t_0) c} \right]^{1/2} e^{i\alpha_{\mathbf{k}}(t_0)} \quad v_{\mathbf{k}}'(t_0) = -c \frac{\omega_{\mathbf{k}}(t_0)}{i\hbar} v_{\mathbf{k}}(t_0) \quad (2.34)$$

under these functions,

$$E_{\mathbf{k}}(t_0) = 2 \frac{\text{Im}(v_{\mathbf{k}}v_{\mathbf{k}}'^*)}{\hbar c} \omega_{\mathbf{k}}(t_0) \quad F_{\mathbf{k}}(t_0) = 0 \quad (2.35)$$

meaning

$$\hat{\mathcal{H}}(t_0) = \text{Im}(vv'^*) \frac{1}{\hbar} \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3} \left( 2\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}} + \frac{(2\pi\hbar)^3\hbar c}{2\text{Im}(v'v^*)}\delta^3(\mathbf{0}) \right) \omega_{\mathbf{k}}(t_0) \quad (2.36)$$

which is equivalent to the standard Hamiltonian for a scalar field without the presence of gravity.

**Bogolyubov Transformation** The expression of the field  $\chi$  at two different times must be related to a Bogoliubov transformation, with coefficients

$$\alpha_{\mathbf{k}\mathbf{p}} = \frac{(2\pi\hbar)^3 \hbar c}{2\text{Im}(v'v^*)} \langle \chi_{\mathbf{k}}(t_0), \chi_{\mathbf{p}}(t) \rangle \quad \beta_{\mathbf{k}\mathbf{p}} = -\frac{(2\pi\hbar)^3 \hbar c}{2\text{Im}(v'v^*)} \langle \chi_{\mathbf{k}}(t_0), \chi_{\mathbf{p}}^*(t) \rangle \quad (2.37)$$

since the field can be written as  $\chi_{\mathbf{k}} = v_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}/\hbar}$  from the expression of the inner product one can see that

$$\alpha_{\mathbf{k}\mathbf{p}} \propto \delta^3(\mathbf{k} - \mathbf{p}) \quad \beta_{\mathbf{k}\mathbf{p}} \propto \delta^3(\mathbf{k} + \mathbf{p}) \quad (2.38)$$

therefore it is possible to write

$$v_{\mathbf{k}}(t) = \alpha_{\mathbf{k}} v_{\mathbf{k}}(t_0) + \beta_{\mathbf{k}} v_{\mathbf{k}}^*(t_0) \quad (2.39)$$

where, recalling that  $\text{Im}(v'_{\mathbf{k}} v_{\mathbf{k}}^*)$  is constant through time,

$$|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1 \quad (2.40)$$

To obtain the value of  $\langle_{(t_0)} 0 | \hat{\mathcal{H}}(t) |_{(t_0)} 0 \rangle$  lets first compute

$$\langle_{(t_0)} 0 | \hat{\mathcal{N}}_{\mathbf{k}}^{(a)}(t) |_{(t_0)} 0 \rangle = \langle_{(t_0)} 0 | \hat{a}_{\mathbf{k}}^\dagger(t) \hat{a}_{\mathbf{k}}(t) |_{(t_0)} 0 \rangle = |\beta_{\mathbf{k}}|^2 \frac{(2\pi\hbar)^3 \hbar c}{2\text{Im}(v'v^*)} \delta^3(\mathbf{0}) \quad (2.41)$$

therefore

$$\langle_{(t_0)} 0 | \hat{\mathcal{H}}(t) |_{(t_0)} 0 \rangle = \delta^3(\mathbf{0}) \int d^3\mathbf{k} \left( \frac{1}{2} + |\beta_{\mathbf{k}}|^2 \right) c \omega_{\mathbf{k}}(t) \geq \langle_{(t_0)} 0 | \hat{\mathcal{H}}(t_0) |_{(t_0)} 0 \rangle \quad (2.42)$$

meaning, if  $\beta_{\mathbf{k}} \neq 0$  for all  $\mathbf{k}$  then, at a time  $t > t_0$  the energy density will be different in relation to the original vacuum.

### 3 de Sitter Universe

The de Sitter Universe is a flat FLRW metric with no matter or radiation, but it does have a positive cosmological constant  $\Lambda$ . Per the Friedmann equations,

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G\rho + \Lambda c^2}{3} - \frac{\kappa c^2}{a^2} \quad (3.1)$$

the expansion parameter  $a(t)$  will be equal to

$$a(t) = a_1 e^{H_\Lambda t} + a_2 e^{-H_\Lambda t}, \quad H_\Lambda = \sqrt{\frac{\Lambda c^2}{3}} \quad (3.2)$$

$a_2 = 0$

$$dl^2 = c^2 dt^2 - a^2(t) d\mathbf{x}^2 \quad (3.3)$$

$$\eta \equiv - \int_t^\infty \frac{dt'}{a(t')} = - \frac{1}{a_1 H_\Lambda} e^{-H_\Lambda t} = - \frac{1}{a(t) H_\Lambda} \quad (3.4)$$

<sup>1</sup>

$$dl^2 = \frac{1}{H_\Lambda \eta^2} [c^2 d\eta^2 - d\mathbf{x}^2] \quad (3.5)$$

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<sup>1</sup>Considering  $a_2 \neq 0$  one would obtain that

$$\eta = \frac{\arctan\left(\sqrt{\frac{a_2}{a_1}} e^{-H_\Lambda t}\right)}{H_\Lambda \sqrt{a_1 a_2}}$$

$$R = \frac{6}{c^2} \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right] = \frac{12}{c^2} H_\Lambda^2 \quad (3.6)$$

$$\omega_{\mathbf{k}}^2(\eta) = \mathbf{k}^2 + \left[ \left( \frac{mc^2}{H_\Lambda} \right)^2 + 2(6\xi - 1)\hbar^2 \right] \frac{1}{c^2\eta^2} \quad (3.7)$$

$$v_{\mathbf{k}}''\hbar^2 + \omega_{\mathbf{k}}^2(\eta) v_{\mathbf{k}} = 0 \quad (3.8)$$

change of variables

$$s \equiv -k\eta \quad v_{\mathbf{k}} \equiv \sqrt{s}f(s) \quad (3.9)$$

$$s^2 \frac{d^2 f}{ds^2} + s \frac{df}{ds} + (s^2 - \nu^2) f(s) = 0 \quad (3.10)$$

$$\nu^2 \equiv (3 - 16\xi) \frac{3}{4} \frac{\hbar^2}{c^2} - \left( \frac{mc}{H_\Lambda} \right)^2 \quad (3.11)$$

$$f(s) = AJ_\nu(s) + BY_\nu(s) \implies v_{\mathbf{k}}(\eta) = \sqrt{k|\eta|} [A_{\mathbf{k}}J_\nu(k|\eta|) + B_{\mathbf{k}}Y_\nu(k|\eta|)] \quad (3.12)$$

$J_\nu(x)$ ,  $Y_\nu(x)$  are the Bessel functions of the first kind.  $\text{Im}(vv'^*)$  creates a restriction on the relation  $B_{\mathbf{k}} = B_{\mathbf{k}}(A_{\mathbf{k}})$

# Scalar field in Minkowski background

$$\eta_{\mu\nu} \tag{1}$$

## Units

- $[s] = [\hbar]$
- $[a] = [\xi] = 1$
- $[\mu] = [L]^{-1}$
- $[R] = [L]^{-2}$
- $[\phi] = [\chi] = [\hbar]^{1/2}[L]^{-1}$
- $[\Pi] = [\hbar]^{1/2}[L]^{-2}$
- $[a_{\mathbf{k}}] = [\hbar]^{1/2}[L]^2$

## Questions

- How do you know that there is a set of solutions of Klein Gordon such that the inner product fulfils the given results?
- It's the Hamiltonian well defined?
- Can you always write a FLRW metric as a flat one with a coordinate change?
- Covariant derivatives, spin connection.

## Computations

# Bibliography

- [1] Mark Robert Baker, Natalia Kiriushcheva, and Sergei Kuzmin. Noether and hilbert (metric) energy-momentum tensors are not, in general, equivalent. *Nuclear Physics B*, 962:115240, 2021.
- [2] T S Bunch, P Panangaden, and L Parker. On renormalisation of  $\lambda\phi^4$  field theory in curved space-time. i. *Journal of Physics A: Mathematical and General*, 13(3):901, mar 1980.
- [3] Viatcheslav Mukhanov and Sergei Winitzki. *Introduction to Quantum Effects in Gravity*. Cambridge University Press, 2007.