International Baccalaureate

MATHEMATICS

Analysis and Approaches (SL and HL)

Lecture Notes

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TOPIC 2

FUNCTIONS

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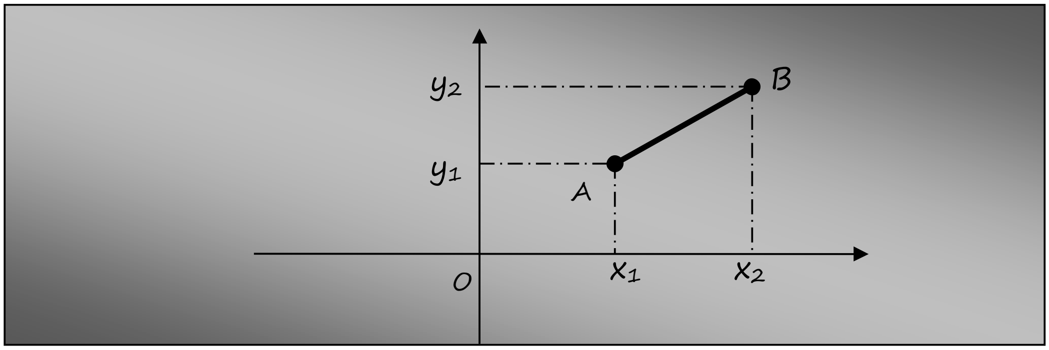
2.1 LINES (or LINEAR FUNCTIONS)

♦ BASIC NOTIONS ON COORDINATE GEOMETRY

Given two points $A(x_1,y_1)$ and $B(x_2,y_2)$



- The **gradient** or **slope** of line segment AB is given by

$$m=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}$$

This indicates the inclination of the line segment AB. As we are moving along the positive direction of the x-axis, if the line segment is

increasing	(/)	then	**m>0**
decreasing	(\ )	then	**m<0**
horizontal	(—)	then	**m=0**
vertical	(│)	then	**m is not defined**

- The distance between A and B is given by

$$d_{AB}=\sqrt{(\Delta x)^2+(\Delta y)^2}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

- The coordinates of the **midpoint** $M(x,y)$ of the line segment AB are given by

$$x=\frac{x_1+x_2}{2} \qquad y=\frac{y_1+y_2}{2}$$

EXAMPLE 1

a) Given two points A(1,4) and B(7,12)

The slope of the line segment AB is $m=\frac{\Delta y}{\Delta x}=\frac{12-4}{7-1}=\frac{4}{3}$

The distance between them is $d=\sqrt{(7-1)^2+(12-4)^2}=10$

The midpoint is $M(\frac{1+7}{2},\frac{4+12}{2})$ that is M(4,8)

b) Given two points A(1,8) and B(5,8)

It is not necessary to use the formulas. Since A and B have the same y-coordinate:

The slope of the line segment AB is m=0 (horizontal)

The distance between them is d=5-1=4

The midpoint is M(3,8)

c) Given two points A(1,5) and B(1,7)

It is not necessary to use the formulas. Since A and B have the same x-coordinate:

The slope m of the line segment AB is not defined (vertical)

The distance between them is d=7-5=2

The midpoint is M(1,6)

The notion of the **function** will be formally introduced later on, in paragraph 2.3. However, we will start by presenting two families of already known functions

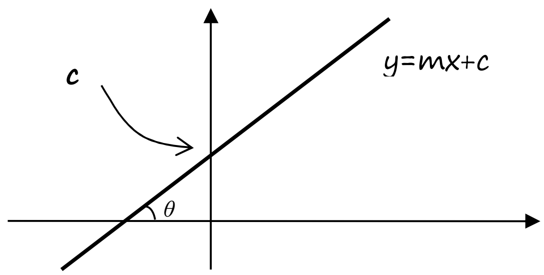
Linear functions: $y=mx+c$ or $f(x) = mx+c$

Quadratic functions: $y=ax^2+bx+c$ or $f(x) = ax^2+bx+c$

- THE EQUATION OF A LINE

Equation of a (straight) line: $y=mx+c$

m = **gradient** or **slope** c = **y-intercept**



NOTICE:

- A horizontal line has equation **$y=c$** (slope $m=0$)
- A vertical line has equation **$x=c$** (there is no slope)

  (in fact, a vertical line is not a function, that is why the equation $x=0$ is not a particular case of $y=mx+c$)

- $m=\tan\theta$, where θ is the angle between the line and x-axis

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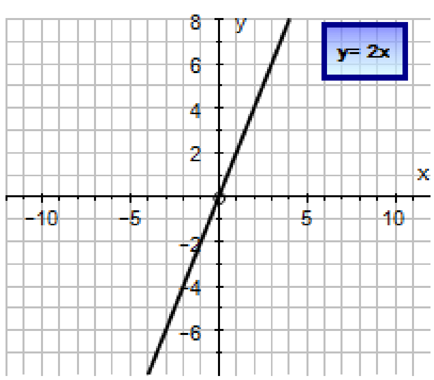
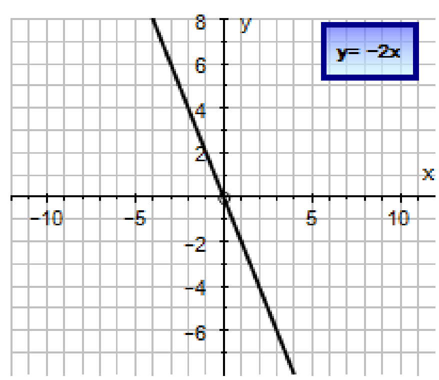
EXAMPLE 2

Look at the graphs of two lines: L_1: **$y=2x$** and L_2: **$y=-2x$**





In fact, the slope shows the rise of the line per each unit

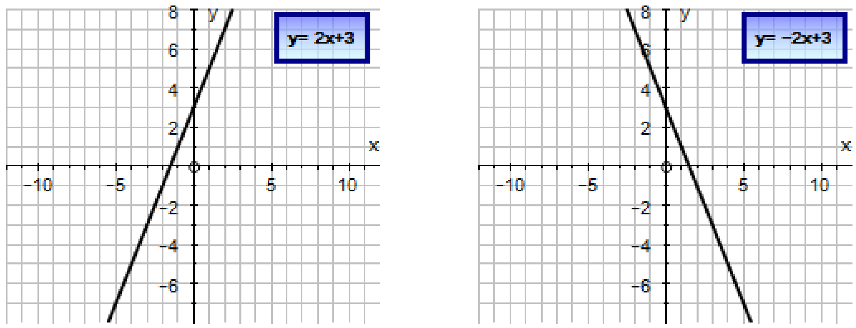
Line L_1: slope is 2 (y increases 2 units per each x-unit)

Line L_2: slope is -2 (y decreases 2 units per each x-unit)

In both cases $c=0$ (since the function passes through the origin)

EXAMPLE 3

Look at the graphs of two lines: L_1: $y=2x+3$ and L_2: $y=-2x+3$

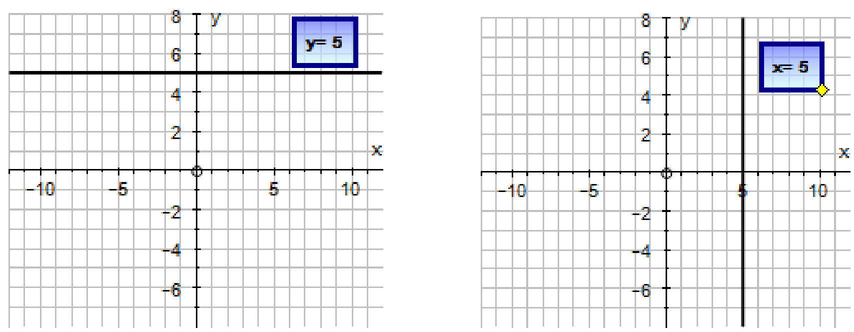


Line L_1: slope is 2 Line L_2: slope is -2

In both cases the y-intercept is 3

EXAMPLE 4

Look at the graphs of two lines: L_1: $y=5$ and L_2: $x=5$



♦ PARALLEL AND PERPENDICULAR LINES

Consider two lines: L_1: $y=m_1x+c_1$ and L_2: $y=m_2x+c_2$

Parallel lines:	$L_1 // L_2$	if	$m_1 = m_2$
Perpendicular lines:	$L_1 \perp L_2$	if	$m_2=-1/m_1$

For example,

The lines $y=3x+5$ and $y=3x+8$ are parallel

The lines $y=3x+5$ and $y=-\frac{1}{3}x+8$ are perpendicular

♦ AN ALTERNATIVE FORMULA FOR A LINE

A more general formula for a line is

$$\boxed{\text{Equation of a line:} \quad \mathbf{Ax+By=C}}$$

If $B\neq 0$, we can solve for y and obtain the form $y=mx+c$
If $B=0$, we obtain a vertical line of the form $x=c$
If $A=0$, we obtain a horizontal line of the form $y=c$

EXAMPLE 5

- From $Ax+By=C$ into the usual form
  The line $2x+3y=5$ may be expressed as $3y=-2x+5$ and finally
  $$y=-\frac{2}{3}x+\frac{5}{3}$$
- From the usual form into $Ax+By=C$
  a) The line $y=-3x+7$ may be expressed as
  $$3x+y=7$$
  b) The line $y=\frac{1}{2}x+\frac{2}{3}$ may be expressed as
  $$-\frac{1}{2}x+y=\frac{2}{3}$$
  We usually require the coefficients A,B,C to be integers.
  Multiplying by 6 we obtain
  $$-3x+6y=4$$
  c) The line $y=5$ may be expressed as $0x+y=5$
  d) The line $x=5$ may be expressed as $x+0y=5$

♦ GIVEN: A POINT AND A SLOPE

The line which

- passes through point $\mathbf{P(x_0,y_0)}$
- has slope $\mathbf{m}$

is given by

$$\boxed{\mathbf{y-y_0 = m(x-x_0)}}$$

EXAMPLE 6

The line which passes through point P(1,2), with slope m=3 is

$$y-2 = 3(x-1)$$

- Express in the form y=mx+c

  $y-2 = 3(x-1) \Leftrightarrow y=3x-3+2 \Leftrightarrow \underline{\mathbf{y=3x-1}}$

- Express in the form ax+by=c or ax+by+c=0

  $y=3x-1 \Leftrightarrow \underline{\mathbf{3x-y=1}}$ or $\underline{\mathbf{3x-y-1=0}}$

♦ GIVEN: TWO POINTS

The line which passes through the points **$P(x_1,y_1)$ and $Q(x_2,y_2)$** has slope

$$\boxed{m=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}}$$

and its equation is again given by the formula

$$\boxed{y-y_1 = m(x-x_1)}$$

EXAMPLE 7

Find the line which passes through the points P(1,2) and Q(4,7).

Express your answer in the form ax+by=c where a,b,c∈Z (integers).

Solution

The slope is $m=\frac{\Delta y}{\Delta x}=\frac{7-2}{4-1}=\frac{5}{3}$

The equation of the line is

$$y-2 = \frac{5}{3}(x-1)$$

$$\Leftrightarrow 3y-6 = 5(x-1)$$

$$\Leftrightarrow 3y-6 = 5x-5$$

and finally

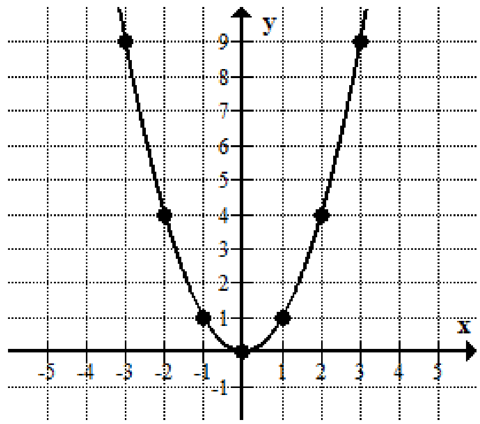
$$\mathbf{-5x+3y = 1}$$

2.2 QUADRATICS (or QUADRATIC FUNCTIONS)

♦ THE SIMPLEST QUADRATIC: $y=x^2$

Consider the function $y=x^2$. Let us find some values

x	…	−3	−2	−1	0	1	2	3	…
$y=x^2$	…	9	4	1	0	1	4	9	…



Notice that x can take any value in R. We say that

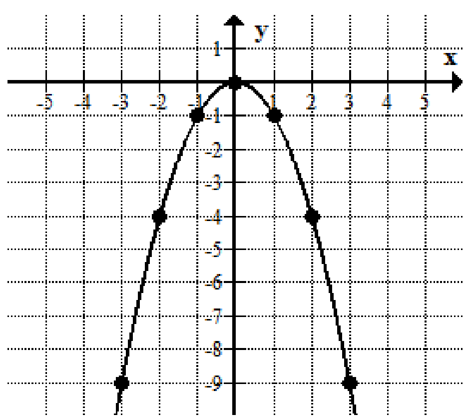
the **domain** of the function is $x\in R$

The result, i.e. the value of y, is always positive or 0. We say that

the **range** of the function is $[0,+\infty)$ (or simply $y\geq 0$).

The curve of this function is known as **parabola**.

We can easily see that the graph of the function $y=-x^2$ is

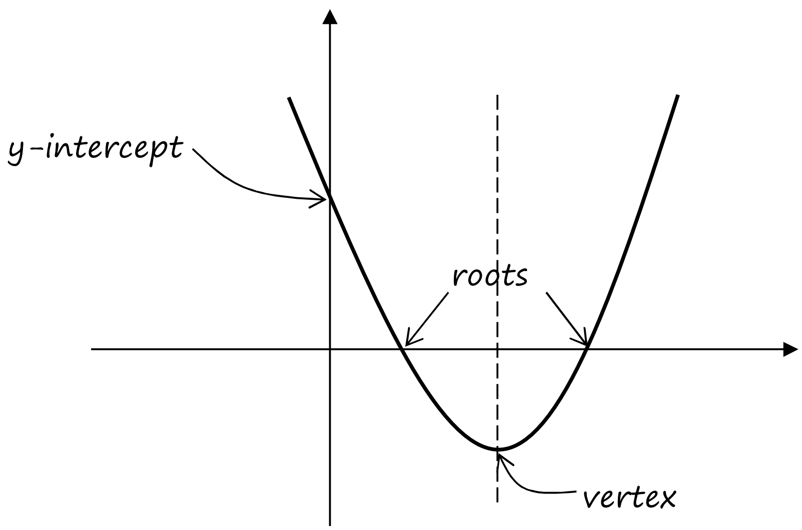


♦ THE QUADRATIC FUNCTION

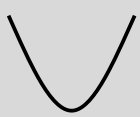
A **quadratic function** has the form

$$y=ax^2+bx+c$$

The graph of a quadratic is always a parabola. The basic characteristics of its graph as shown below:



1) $a\neq 0$. The sign of a shows the concavity of the function:

If $a>0$ the graph looks like  (concave up)

If $a<0$ the graph looks like  (concave down)

2) **Discriminant:** $\Delta=b^2-4ac$. It determines the number of roots

Δ>0: 2 roots	**Δ=0**: 1 root	**Δ<0**: No real roots

3) **x-intercepts** (or **roots**): $x_{1,2}=\frac{-b\pm\sqrt{\Delta}}{2a}$, (only if $\Delta\geq 0$)

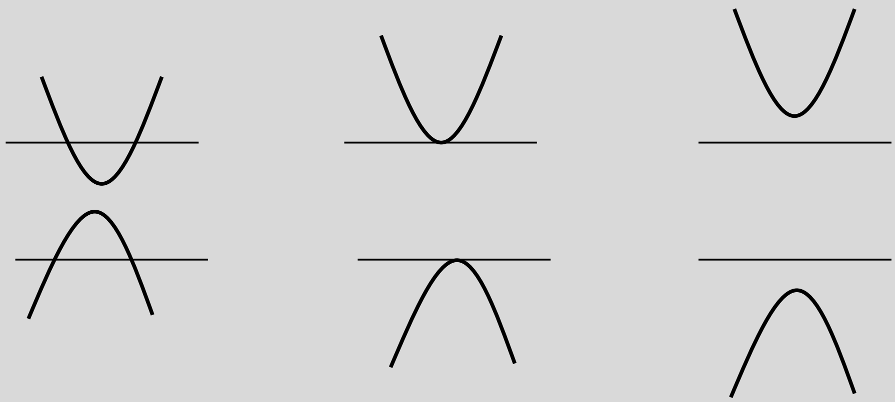
4) **y-intercept**: for $x=0$ we obtain $y=c$

5) **axis of symmetry:** $x=\frac{-b}{2a}$ (it's also the x-coordinate of the vertex)

If we know the two roots x_1, x_2 the vertex is at $x=\frac{x_1+x_2}{2}$

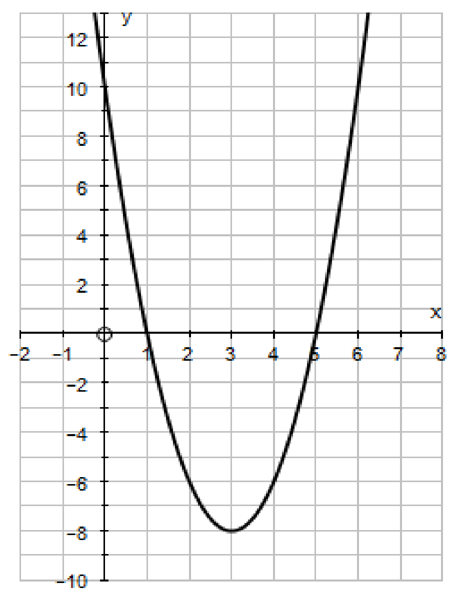
6) According to Δ, the graph looks like

Δ>0: 2 roots	**Δ=0**: 1 root	**Δ<0**: No real roots



EXAMPLE 1

Consider **$y=2x^2-12x+10$**



- $a=2$ (+tive), so the graph looks like **U** (concave up)
- $\Delta = b^2-4ac = 64>0$, thus two roots: $x_{1,2}=\dfrac{-b\pm\sqrt{\Delta}}{2a}$ = 1 and 5
- y-intercept: $y=10$
- Axis of symmetry: $x=\dfrac{-b}{2a}$ i.e. $x=3$. (Or otherwise $x=\dfrac{1+5}{2}=3$)

  For $x=3$, we obtain $y=-8$. Hence, the vertex is $V(3,-8)$

NOTICE FOR THE GDC (Casio)

We can find the roots 1 and 5 in

Equation – Polynomial (degree 2)

We can find more characteristics in **Graph mode: G-Solv (F5)**

Options	in our example
F1 (ROOT): for the roots	1 and 5
F2 (MAX) or F3 (MIN): for the vertex	(3,-8)
F4 (YCEPT): for y-intercept	10

♦ QUADRATIC INEQUALITIES

They have the form

$$ax^2+bx+c>0 \quad \text{or} \quad ax^2+bx+c\geq 0$$

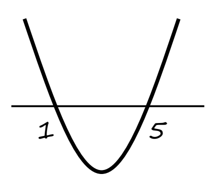
$$ax^2+bx+c<0 \quad \text{or} \quad ax^2+bx+c\leq 0$$

If we find the roots, the graph of the function gives a clear picture of the solutions.

For example, for

$$2x^2-12x+10 > 0$$

the roots are 1 and 5, the function is concave up, so it looks like



So it's positive for $x<1$ or $x>5$. We can also write $x\in]-\infty,1[\cup]5,+\infty[$

The inequality

$$2x^2-12x+10 \leq 0$$

has solutions $x\in[1,5]$.

NOTICE:

If we are given that

$$ax^2+bx+c > 0 \quad \textbf{for any } x \in R$$
$$\text{or} \quad ax^2+bx+c < 0 \quad \textbf{for any } x \in R$$

the graph does not intersect the x-axis, that is the quadratic has no real roots. Thus, $\Delta<0$

EXAMPLE 2

Let $\mathbf{f(x)=2x^2-4x+k}$. Find the values of k in each case below:

a) $f(x)=0$ has exactly one root (or two equal roots)
b) $f(x)=0$ has exactly two roots
c) $f(x)=0$ has no real roots
d) $f(x)=0$ has real roots
e) $f(x)>0$ for any $x \in R$
f) $f(x)\geq 0$ for any $x \in R$

Solution

All cases depend on the discriminant $\Delta=16-8k$

a) $\Delta=0$.

Hence, $16-8k=0 \Leftrightarrow 8k=16 \Leftrightarrow \mathbf{k=2}$

b) $\Delta>0$.

Hence, $16-8k>0 \Leftrightarrow 16>8k \Leftrightarrow \mathbf{k<2}$

c) $\Delta<0$.

Hence, $16-8k<0 \Leftrightarrow 16<8k \Leftrightarrow \mathbf{k>2}$

d) $\Delta\geq 0$. [in this case we have either one or two roots]

Hence, $16-8k\geq 0 \Leftrightarrow 16\geq 8k \Leftrightarrow \mathbf{k\leq 2}$

e) Since f(x) is always positive, it has no real roots. Thus, $\Delta<0$.

Hence, $16-8k<0 \Leftrightarrow 16<8k \Leftrightarrow \mathbf{k>2}$

f) Since f(x) is always positive or zero, it has either exactly one root or no real roots at all. Thus, $\Delta\leq 0$.

Hence, $16-8k\leq 0 \Leftrightarrow 8k\geq 16 \Leftrightarrow \mathbf{k\geq 2}$

♦ FORMS OF A QUADRATIC FUNCTION

1) Traditional form:	$y=ax^2+bx+c$	
2) Factorization form:	$y=a(x-r_1)(x-r_2)$	[r_1,r_2 roots]
3) Vertex-form:	$y=a(x-h)^2+k$	[(h,k) vertex]

NOTICE

- If we know the form $y=ax^2+bx+c$ the vertex is at
$$x=\frac{-b}{2a}$$
- If we know the form $y=a(x-r_1)(x-r_2)$, that is the roots r_1,r_2 the vertex is at their mid-point, that is
$$x=\frac{r_1+r_2}{2}$$

Since we know the x-coordinate of the vertex, that is **h**, we can also find the y-coordinate of the vertex, that is **k**. Thus we can derive the vertex form $y=a(x-h)^2+k$.

EXAMPLE 3

We consider again

$$y=2x^2-12x+10 \qquad (1)$$

We find the roots: 1 and 5. Therefore, the factorization is

$$y=2(x-1)(x-5) \qquad (2)$$

The vertex is at $x=\frac{-b}{2a}=\frac{12}{4}=3$ (or otherwise at $x=\frac{r_1+r_2}{2}=\frac{1+5}{2}=3$)

For x=3, it is y=-8, hence the vertex is (3,-8)

Therefore, the vertex-form of the quadratic is

$$y=2(x-3)^2-8 \qquad (3)$$

We may easily verify that forms (2) and (3) give (1).

Indeed,

$$y=2(x-1)(x-5) = 2(x^2-x-5x+5) = 2(x^2-6x+5) = 2x^2-12x+10$$

and

$$y=2(x-3)^2-8 = 2(x^2-6x+9)-8 = 2x^2-12x+18-8 = 2x^2-12x+10$$

♦ JUSTIFICATION OF THE VERTEX-FORM $\mathbf{y=a(x-h)^2+k}$

1) The point **(h,k)** is the vertex, i.e. a minimum or a maximum:

- If $a>0$, then

$$\begin{aligned} & a(x-h)^2 \geq 0 \qquad \text{(equality holds when } x=h\text{)} \\ \Rightarrow\quad & a(x-h)^2+k \geq k \\ \Rightarrow\quad & y \geq k \end{aligned}$$

  Therefore, at $x=h$ we obtain the minimum value $y=k$.

- If $a<0$, then

$$\begin{aligned} & a(x-h)^2 \leq 0 \qquad \text{(equality holds when } x=h\text{)} \\ \Rightarrow\quad & a(x-h)^2+k \leq k \\ \Rightarrow\quad & y \leq k \end{aligned}$$

  Therefore, at $x=h$ we obtain the maximum value $y=k$.

2) Any quadratic can be expressed in the vertex form, by the "completing the square" method.

For example, for the quadratic in EXAMPLE 3 above, we can work as follows

$$\begin{aligned} y = \underline{2x^2-12x}+10 &= 2(x^2-6x)+10 && \text{[only the first 2 terms]} \\ &= 2(x^2-6x\underline{+9-9})+10 && \text{[complete the square]} \\ &= 2(x-3)^2-18+10 \\ &= 2(x-3)^2-8 \end{aligned}$$

However, it is preferable to obtain the vertex-form as in example 3 above, that is by finding the vertex (h,k) and then expressing the quadratic as $y=a(x-h)^2+k$.

EXAMPLE 4

Let

$$y=-3x^2-15x+42 \quad (1)$$

By using the GDC,

we find the roots: -7 and 2. Thus the factorization is

$$y=-3(x+7)(x-2) \quad (2)$$

we find the vertex: V(-2.5, 60.75). Thus the vertex form is

$$y=-3(x+2.5)^2+60.75 \quad (3)$$

Notice: if you expand (2) or (3) you will obtain (1)

EXAMPLE 5

Consider $f(x)=3x^2+12x$. Find both **analytically** and **by GDC**

a) the roots and the factorization.

b) the equation of the axis of symmetry

c) the minimum value of y and the coordinates of the vertex.

d) the vertex form of f(x).

Solution

a) Analytically:

The factorization is $y = 3x^2+12x = 3x(x+4)$

So the roots are $x=0$, $x=-4$

By using GDC – Graph mode

The roots are $x=0$ and $x=-4$

So the factorization is $y=3(x-0)(x+4)$, that is $y=3x(x+4)$

b) $x=\frac{-b}{2a}=\frac{-12}{6}=-2$. That is $x=-2$.

c) Analytically:

For $x=-2$, it is $y=3(-2)^2+12(-2)=-12$. Thus $y_{min}=-12$

Thus the vertex is V(-2,-12)

By using GDC - mode: $y_{min}=-12$ and V(-2,-12).

d) $f(x) = 3(x+2)^2-12$

♦ VIETA FORMULAS

Consider the quadratic

$$y=ax^2+bx+c$$

Given that the real roots are r_1 and r_2, we define

S = the sum of the roots = r_1+r_2

P = the product of the roots = r_1r_2

Then, the Vieta formulas hold:

$$S = -\frac{b}{a}$$

$$P = \frac{c}{a}$$

Conversely, if we know the sum and the product of the roots, we can find a corresponding quadratic.

$$\boxed{y = x^2-Sx+P}$$

EXAMPLE 6

Consider again the quadratic function

$$y=2x^2-12x+10$$

The roots are 1 and 5 and indeed

their sum is $S = -\frac{b}{a} = \frac{12}{2} = 6$

their product is $P = \frac{c}{a} = \frac{10}{2} = 5$

Conversely, if we know that S = 6, and P = 5, the corresponding quadratic is

$$x^2-Sx+P$$

that is

$$x^2-6x+5$$

or any multiple of this, for example $\mathbf{2x^2-12x+10}$.

2.3 FUNCTIONS, DOMAIN, RANGE, GRAPH

♦ DEFINITION

Let us formally introduce the notion of the **function:**

$$f: X \to Y$$

A **<u>function</u>** f from a set X to a set Y assigns
to **each** element x of X
a **unique** element y of Y

We write:

$f(x)=y$

$f: x \mapsto y$

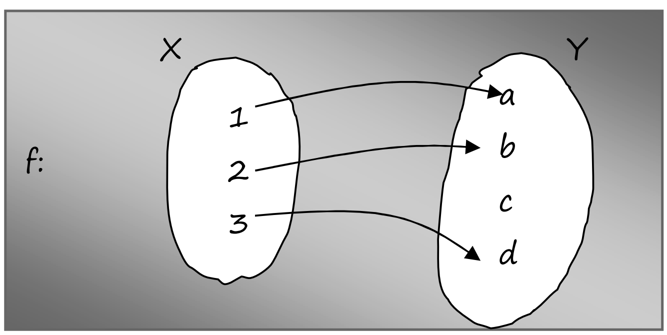
We say:

f maps x to y

y is the image of x

<u>EXAMPLE 1</u>

Let X={1,2,3} and Y={a,b,c,d}. The following is a function $f: X \to Y$



Indeed, **each** element of X has a **unique** image in Y.

We say

f	maps	1 to a	or	a is the image of 1
		2 to b		b is the image of 2
		3 to d		d is the image of 3

We write

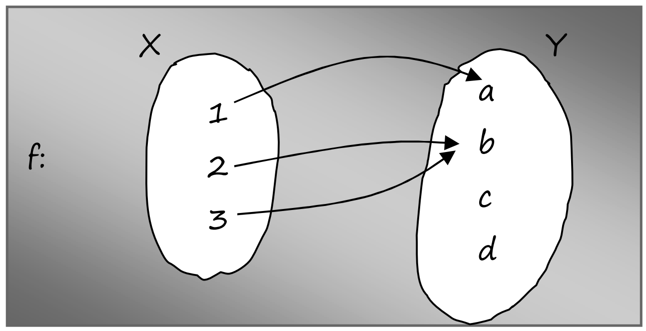
$f(1)=a$, $f(2)=b$, $f(3)=d$

or $f: 1 \mapsto a$ $f: 2 \mapsto b$ $f: 3 \mapsto d$

EXAMPLE 2

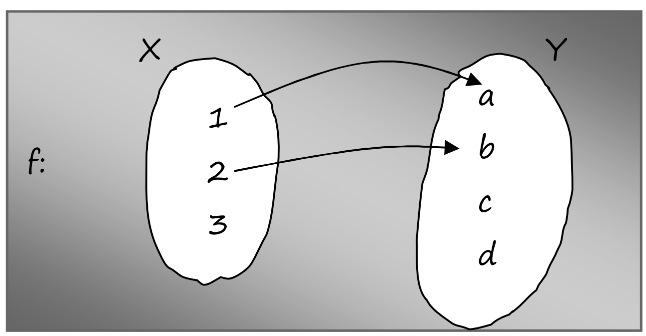
Let X={1,2,3} and Y={a,b,c,d}

- The following is a function



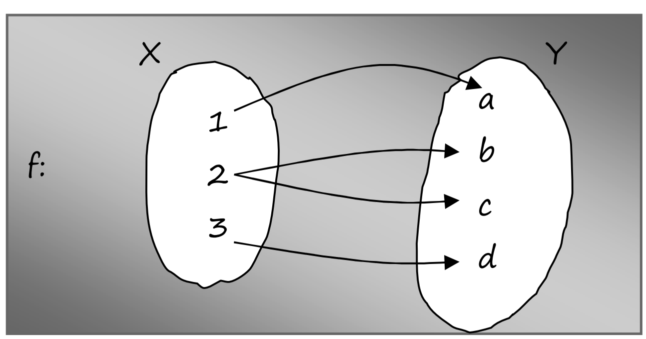
(we do not mind if two elements of X have the same image)

- Notice though that the following is not a function



(we said "**each** x of X", but here 3 has no image)

- Finally, the following is not a function



(we said "**unique** y of Y", but 2 has two images)

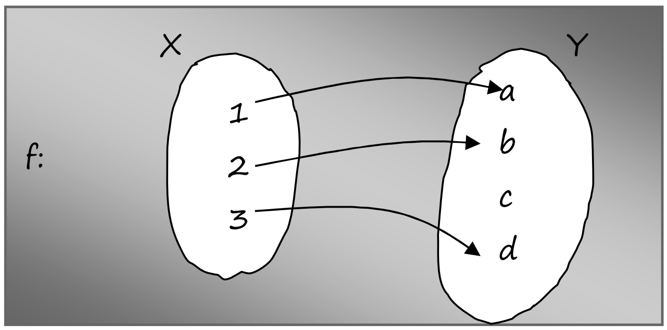
♦ DOMAIN AND RANGE

For a function $f: X \to Y$,

The set of all x's involved is called **DOMAIN**

The set of all y's involved (only the images) is called **RANGE**

Consider again the function $f: X \to Y$ given by



Then DOMAIN : $x \in X = \{1,2,3\}$

RANGE : $y \in \{a,b,d\}$

We usually denote the domain by D_f and the range by R_f.

The range is not necessarily the whole set Y, it may be part of Y.

Here, the sets X and Y are subsets of R, the set of real numbers.

Our functions usually have a specific pattern. For example, consider the function f which maps

$1 \mapsto 2 \qquad 2 \mapsto 4 \qquad 3 \mapsto 6 \qquad 4 \mapsto 8$ and so on

in other words f maps each value x to its double 2x.

We say that the function $f: R \to R$, is given by

$$f: x \mapsto 2x$$

or $f(x)=2x$

or $y=2x$

Thus the formula of the function gives any possible result, e.g.

$f(15)=30, \qquad f(2.4) = 4.8 \qquad$ etc

If we restrict the function f from R to the interval X=[0,10], we still have the function f: X→ R, given by

$$f(x)=2x, \quad 0\le x\le 10$$

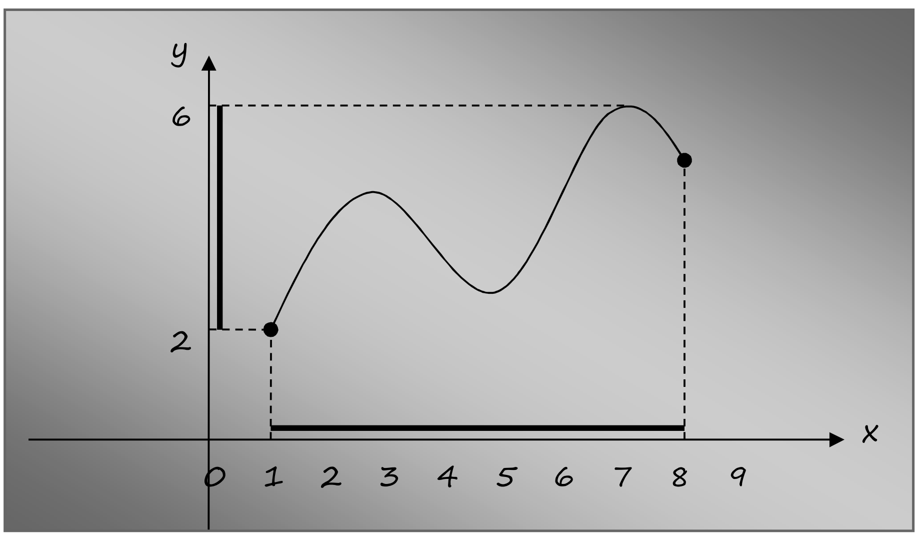
but now

DOMAIN : $x\in[0,10]$
RANGE : $y\in[0,20]$ (why?)

- GRAPH

We know that the pairs (x,y) that satisfy the equation of the function y=f(x) can be represented as points (x,y) on the Cartesian plane and form the **graph** of the function.

The graph clearly shows the DOMAIN and the RANGE of the function. For example,



DOMAIN:	Projection on the x-axis,	i.e. D_f: $x\in[1,8]$
RANGE:	Projection on the y-axis,	i.e. R_f: $y\in[2,6]$

We may observe, for example, that the points

(1,2), (5,3), (7,6), (8,5) lie on the curve.

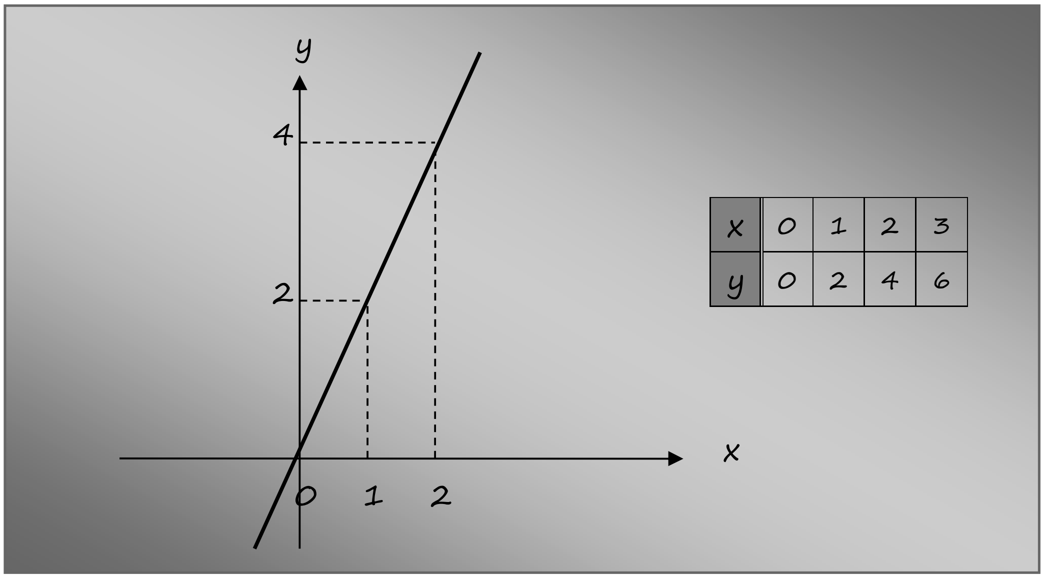
That implies

$$f(1)=2 \qquad f(5)=3 \qquad f(7)=6 \qquad f(8)=5$$

We have already studied the graphs of two families of functions; linear and quadratic functions. The graphs are straight lines and parabolas respectively.

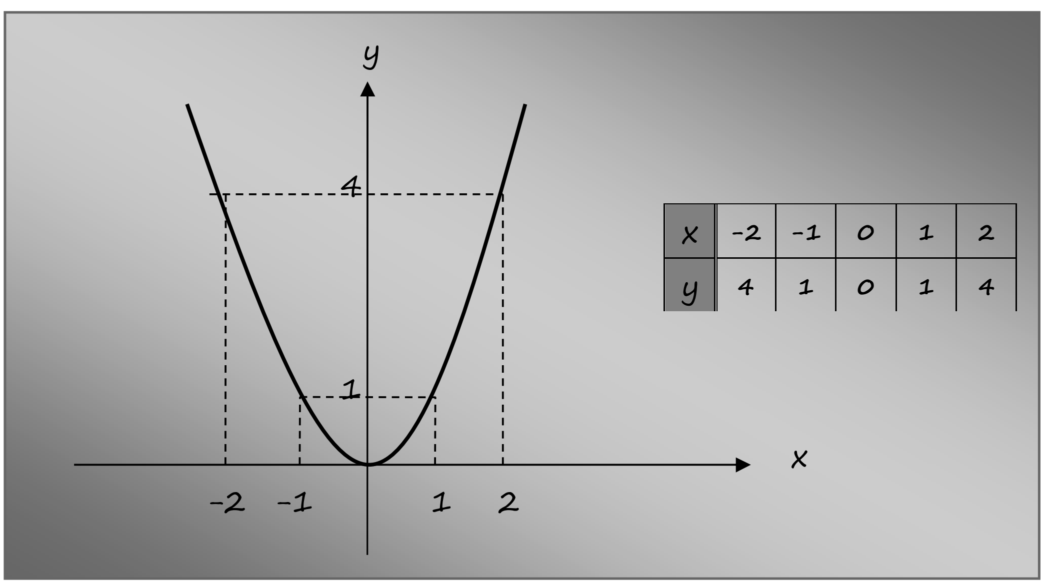
EXAMPLE 3

- $f(x)=2x$, or otherwise $y=2x$ is represented by the graph



Here D_f: $x\in R$ R_f: $y\in R$

- $f(x)=x^2$, or otherwise $y= x^2$ is represented by the graph

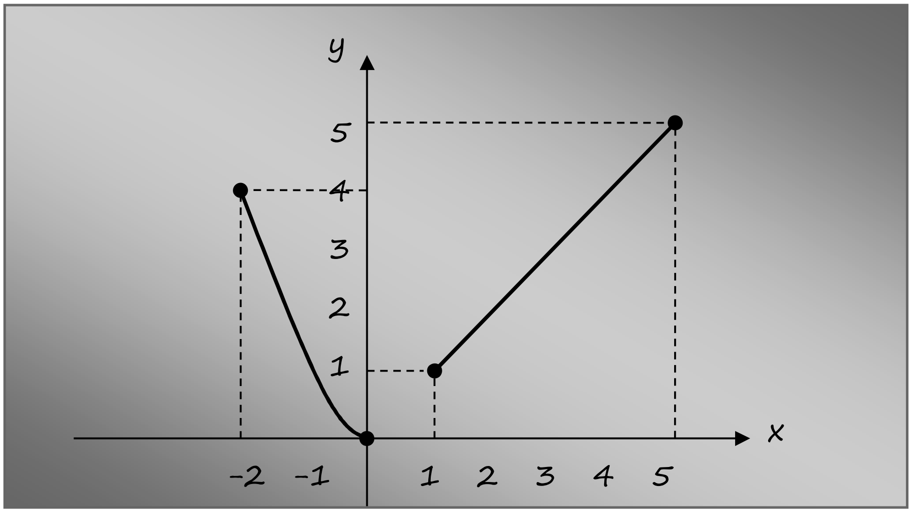


Here D_f: $x\in R$ R_f: $y\in[0,+\infty)$

EXAMPLE 4

Consider the function $f(x)=\begin{cases} x^2, & -2 \le x \le 0 \\ x, & 1 \le x \le 5 \end{cases}$

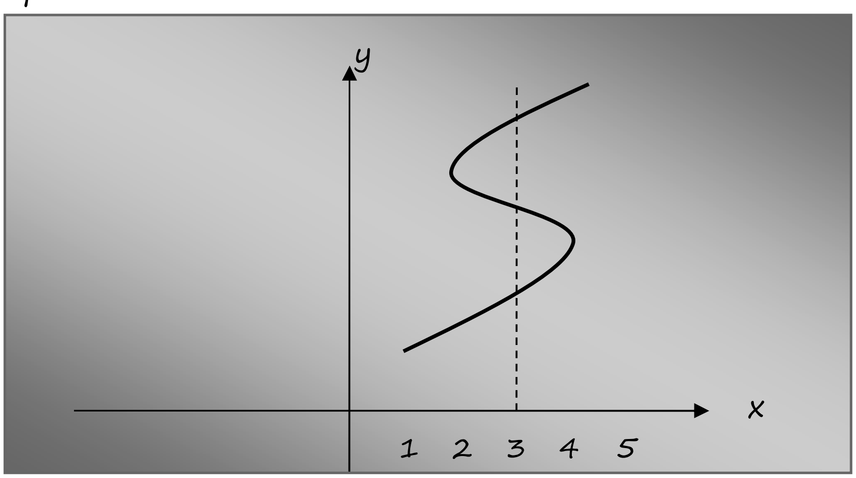
The graph is given below



Clearly, $D_f: x \in [-2,0] \cup [1,5]$ and $R_f: y \in [0,5]$

NOTICE:

The graph also shows if we have a function or not



This is not a function, since f(3) for example is not unique!

Vertical line test:

Any vertical line intersects the graph at most once.

♦ AN "AGGREEMENT" FOR THE DOMAIN

Usually, a function is simply given as a formula of the form $y=f(x)$, where x and y are real variables.

If the domain of the function is not given, we agree that

D_f **is R**

or D_f **is the largest possible subset of R**

For example,

- if f is given by $f(x)=2x$, we assume that $x\in R$
- if f is given by $f(x)=\frac{2}{x}$, we assume that $x\in R-\{0\}=R^*$

  (we may also write D_f: $x\neq 0$)

We mainly deal with the following cases

1. $f(x)$ is a function with no restrictions on x, for example a polynomial [say $f(x)=2x^3+3x^2+1$], then

   $\boxed{D_f = R}$

2. $f(x) = \frac{A}{B}$, then B cannot be 0, thus

   $\boxed{D_f = R - \{\text{roots of the equation } B=0\}}$

3. $f(x) = \sqrt{A}$, then $A\geq 0$.

   $\boxed{D_f = \text{the solution set of the inequality } A\geq 0}$

4. $f(x) = \log A$ or $f(x)=\ln A$, then $A>0$. [1]

   $\boxed{D_f = \text{the solution set of the inequality } A>0}$

5. $f(x)$ = is a combination of all the above.

   We find the subset of R where all our restrictions hold.

---

[1] The functions $f(x)=\log x$ and $f(x)=\ln x$ are not known yet. They will be introduced later on within this topic.

EXAMPLE 5

a) $f(x) = 3x-9$. Clearly, D_f: $x\in R$

b) $f(x) = \dfrac{5}{3x-9}$. Restriction: $3x-9 \neq 0$

Solve: $3x-9=0 \Leftrightarrow 3x=9 \Leftrightarrow x = 3$

Thus, D_f: $x\in R-\{3\}$. We may also write D_f: $x\neq3$

c) $f(x) = \sqrt{3x-9}$. Restriction: $3x-9\geq0$

Solve: $3x-9\geq0 \Leftrightarrow 3x\geq9 \Leftrightarrow x \geq 3$

Thus, D_f: $x\in[3,+\infty)$. We may also write D_f: $x\geq3$

d) $f(x) = \ln(3x-9)$. Restriction: $3x-9>0$

Solve: $3x-9>0 \Leftrightarrow 3x>9 \Leftrightarrow x > 3$

Thus, D_f: $x\in(3,+\infty)$. We may also write D_f: $x>3$

e) $f(x) = \dfrac{x+2}{x^2-3x+2}$ Restriction: $x^2-3x+2\neq0$

Solve: $x^2-3x+2=0 \Leftrightarrow x=1$ or $x=2$

Thus, D_f: $x\in R-\{1,2\}$

f) $f(x) = \sqrt{x-1}+\sqrt{2-x}$ Restrictions: $x-1 \geq0$ and $2-x \geq0$

Solve: $x-1 \geq0 \Leftrightarrow x \geq 1$

$2-x \geq0 \Leftrightarrow x \leq 2$

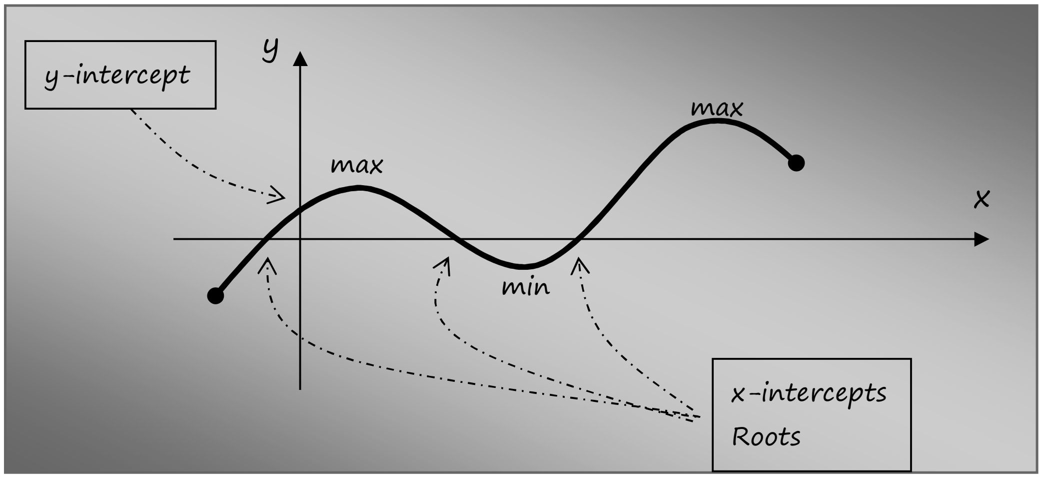
Thus, D_f: $x\in[1,2]$ We may also write D_f: $1\leq x\leq2$

g) $f(x) = \dfrac{\sqrt{1-x^2}}{x}$ Restrictions: $1-x^2 \geq0$ and $x\neq0$

Solve: $1-x^2 \geq0 \Leftrightarrow x^2 \leq 1 \Leftrightarrow -1\leq x\leq1$

Thus, D_f: $x\in[-1,0)\cup(0,1]$

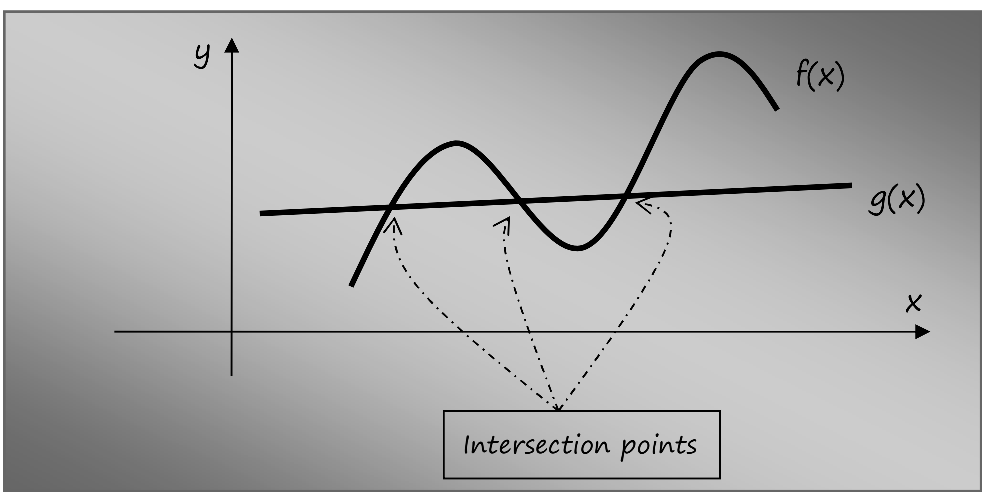
♦ SPECIFIC POINTS ON A GRAPH



For y=f(x)

- ***y*-intercept**: We set x=0 and find y
- ***x*-intercepts (roots)**: We solve the equation f(x)=0
- **local max-min:** (as shown above)

When we have two graphs y=f(x) and y=g(x), it also useful to know the **intersection points** of the two graphs



These points (x,y) can be found by solving the equation **f(x)=g(x)** to obtain x and then using either y=f(x) or y=g(x) to obtain y.

All notions above, namely **y-intercept**, **x-intercepts** (or **roots**), **max**, **min**, **intersection points** can be easily found in **GDC – Graph mode**.

EXAMPLE 6

Consider the functions $f(x)=(x-3)^2-4$ and $g(x)=x-5$.

For f :

y-intercept: for $x=0$, we obtain $\boxed{y=5}$

x-intercepts or roots: We solve $(x-3)^2-4=0$

$(x-3)^2-4=0 \Leftrightarrow (x-3)^2=4 \Leftrightarrow x-3=\pm 2 \Leftrightarrow x=2+3$ or $x=-2+3$

Hence $\boxed{x=5 \text{ or } x=1}$

max-min: for this particular function (quadratic), we know that there is only a minimum.

We have a min at the vertex, i.e. at point $\boxed{(3,-4)}$

We say: We have a <u>min at x=3.</u> The <u>min value is y=-4</u>

For intersection points of f and g :

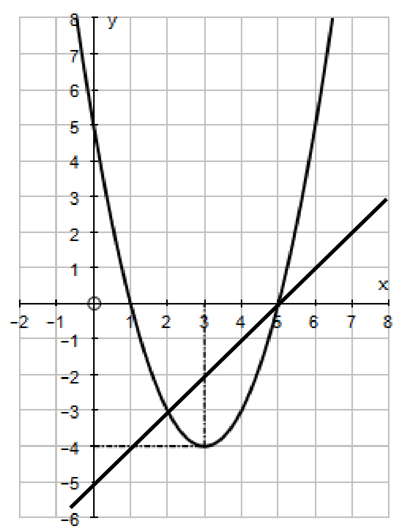
$f(x)=g(x) \Leftrightarrow (x-3)^2-4=x-5 \Leftrightarrow x^2-6x+9-4=x-5 \Leftrightarrow x^2-7x+10=0$

$\Leftrightarrow x=2$ or $x=5$

By using either $f(x)$ or $g(x)$ we find $y=-3$, $y=0$ respectively.

Hence, the curves intersect at points $\boxed{(2,-3) \text{ and } (5,0)}$

Indeed, the graphs of $f(x)$ and $g(x)$ are as follows



Remark: Confirm all the results by using **GDC – Graph mode.**

♦ SOVING EQUATIONS AND INEQUALITIES BY USING GRAPHS

We can solve

- equations of the form **$f(x)=g(x)$**
- inequalities of the form **$f(x)>g(x)$** or **$f(x)\geq g(x)$**

by using **GDC – graph mode**

METHOD A: we find the intersection points of the graphs

$y_1 = f(x)$

$y_2 = g(x)$

Solutions of **$f(x)=g(x)$**: x-coordinates of intersection points

Solutions of **$f(x)>g(x)$**: intervals where $y_1=f(x)$ is above $y_2=g(x)$

METHOD B: we find the roots of the graph

$y_1 = f(x)-g(x)$

Solutions of **$f(x)-g(x)=0$**: the roots of the graph

Solutions of **$f(x)-g(x)>0$**: intervals where $y_1=f(x)-g(x)$ is positive

---

EXAMPLE 7

Consider again the functions of Example 6

$$f(x)=(x-3)^2-4 \quad \text{and} \quad g(x)=x-5.$$

a) Solve the equation $f(x)=g(x)$.

METHOD A: Look at the graphs of $y_1=f(x)$ and $y_2=g(x)$ (see Example 6). The intersection points occur at $\boxed{x=2,\ x=5}$

METHOD B: The equation can be written

$$f(x)-g(x) = (x-3)^2 - 4 - (x-5) = 0$$

Look at the graph of $y_1=f(x)-g(x)$ (see GDC). Roots: $\boxed{x=2,\ x=5}$

b) Solve the inequality $f(x)>g(x)$.

METHOD A: the graph of $y_1=f(x)$ is above $y_2=g(x)$ (see Example 6) when $\boxed{x<2 \text{ or } x>5}$

METHOD B: the graph of $y_1=f(x)-g(x)$ (see GDC) is positive outside the roots, that is when $\boxed{x<2 \text{ or } x>5}$

EXAMPLE 8

Solve the equation $2^x = 2x+3$.

(a) by using the function SolveN of your GDC

(b) by considering the graphs of

$$y_1=2^x$$

$$y_2=2x+3.$$

(c) by considering the graph

$$y=2^x - (2x+3)$$

Solution

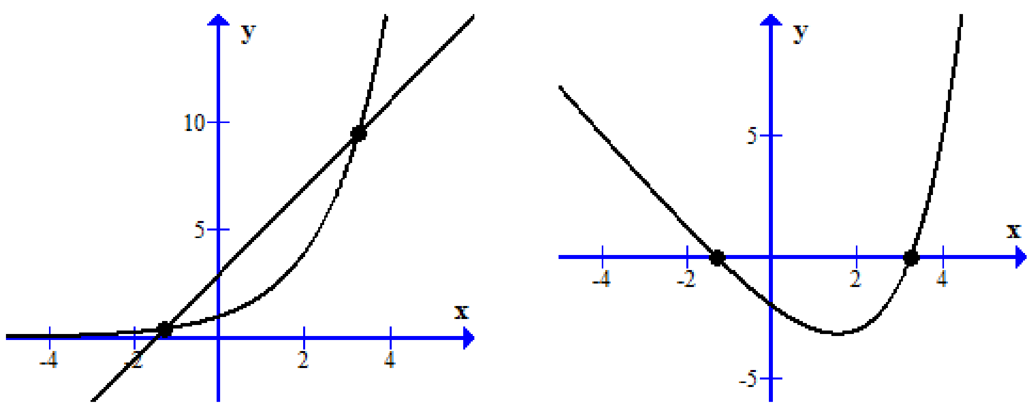
(a) SolveN gives two roots:

$x = -1.29643 \cong -1.30$

$x = 3.24702 \cong 3.25$

For the following we need the diagrams

diagram 1 (for (b)) diagram 2 (for (c))



(b) Intersection points in diagram 1: $x \cong -1.30$ and $x \cong 3.25$

(c) Roots of the function in diagram 2: $x \cong -1.30$ and $x \cong 3.25$

Further question: (d) Solve the inequality $2^x < 2x+3$

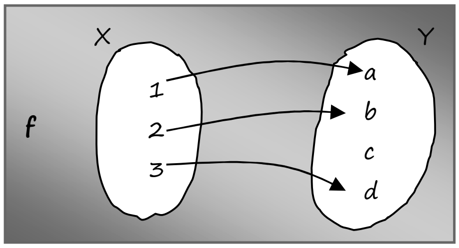
Solution

According to either diagram 1, or diagram 2

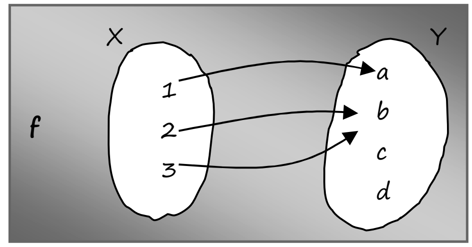
$$-1.30 < x < 3.25$$

♦ ONE-TO-ONE vs MANY-TO-ONE FUNCTIONS **(mainly for HL)**

Consider again the two functions below.



this function is **one-to-one**



this function is **many-to-one**

The formal definition for a one-to-one function says that different elements of X map to different elements of Y, that is

> A function $f: X \to Y$ is **one-to-one** if for any x_1, x_2 in X
> $$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$
> or equivalently (the contrapositive statement)
> $$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

(the contrapositive definition is more practical for exercises).

Graphically, it is easy to confirm that the function is one-to-one:

> **Horizontal line test:**
> Any horizontal line intersects the graph at most once.

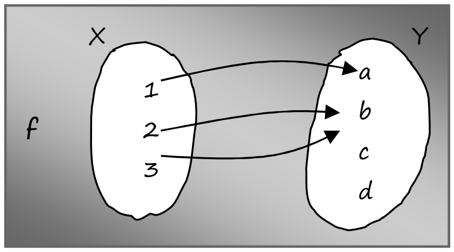
EXAMPLE 9

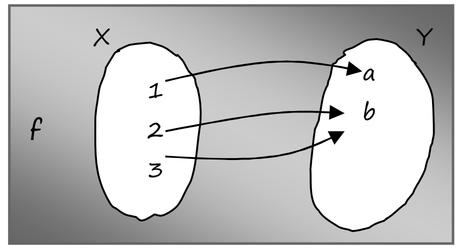
Look at the functions of Example 3.

- the function $f(x)=2x$ is one-to-one, since
$$f(x_1)=f(x_2) \Rightarrow 2x_1=2x_2 \Rightarrow x_1=x_2$$
OR, since any horizontal line intersects the graph at most once.
- the function $f(x)=x^2$ is many-to-one, since different elements may map to the same image, e.g. $f(2)=4$ but also $f(-2)=4$.
OR, since a horizontal line may intersect the graph twice.

♦ ONTO FUNCTIONS **(only for HL – optional but good to know)**

Consider the following two functions





As you see, in the second example the range of f is coincides with Y. In other words, any element of Y is an image of some element of X.

We say that

f maps X **onto** Y or simply f is **onto**

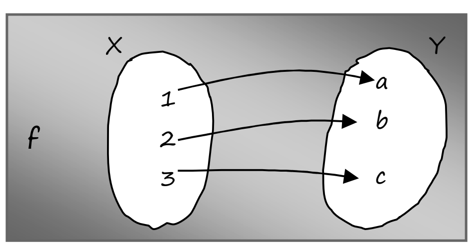
Notice though, that this property is "recoverable". Just ignore the elements of Y that are not images and the function becomes onto.

EXAMPLE 10

- the function f:R → R, given by f(x)=2x is **onto**, since the range of this function is R.
- the function f:R → R, given by $f(x)=x^2$ is **not onto**, since the range of this function is $[0,+\infty)$, which is a proper subset of R. However, if the function is given as f:R → $[0,+\infty)$, it is onto.

♦ 1-1 AND ONTO FUNCTIONS **(only for HL - optional)**

Consider the function



This is **one-to-one and onto.**

The function f:R→R, given by f(x)=2x, as well as any linear function, is one-to-one and onto.

2.4 COMPOSITION OF FUNCTIONS: $f \circ g$

- DISCUSSION

Consider the function $f(x)=x^2$

Notice that

$$f(5) = 5^2$$
$$f(a) = a^2$$
$$f(3a+5) = (3a+5)^2$$
$$f(3x+5) = (3x+5)^2$$

In the last case the input value for f is another function of x.

In this way, we combine two functions,

$$f(x)=x^2 \qquad \text{and} \qquad g(x)= 3x+5$$

and create a new function $y=(3x+5)^2$.

This new function is denoted by $\boldsymbol{f \circ g}$.

- DEFINITION

> For two functions f and g, the **composite function $f \circ g$** is a new function defined by
>
> $$(f \circ g)(x)=f(g(x))$$

The operation is called **composition**.
We say that $f \circ g$ is the **composite function** of f and g.

For the functions $f(x)=x^2$ and $g(x)=3x+5$ given above, we find $(f \circ g)(x)$ as follows

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(3x+5) \\ &= (3x+5)^2\end{aligned}$$

In the same way we define the composite function $(g \circ f)(x)$. It is given by

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^2) \\ &= 3x^2+5\end{aligned}$$

That is

$$(f \circ g)(x)= (3x+5)^2 \qquad \text{while} \qquad (g \circ f)(x)= 3x^2+5$$

NOTICE:

- In general

$$\mathbf{f \circ g \neq g \circ f}$$

- It is not necessary to write the answer so analytically. You can answer directly. Look again:

$$f(x)=x^2 \qquad \text{and} \qquad g(x)=3x+5$$

For $f \circ g$ you just plug g into f.

$$(f \circ g)(x)= (3x+5)^2$$

For $g \circ f$ you just plug f into g.

$$(g \circ f)(x)= 3x^2+5$$

- For three functions

$$f(x)=x^2, \quad g(x)=3x+5, \quad h(x)=\sqrt{x}$$

we can define $(f \circ g \circ h)(x)$.

We just plug h into g, to obtain

$$(g \circ h)(x)= 3\sqrt{x}+5$$

and the result into f to obtain

$$(f \circ g \circ h)(x)= (3\sqrt{x}+5)^2$$

We can easily verify that

$$\mathbf{f \circ (g \circ h)=(f \circ g) \circ h}$$

EXAMPLE 1

Let $f(x)=2x^2-1$ and $g(x)=x+1$. Find

(a) $(f\circ g)(x)$ (b) $(g\circ f)(x)$ (c) $(f\circ g)(1)$ (d) $(g\circ f)(1)$

Solution

(a) $(f\circ g)(x) = 2(x+1)^2-1$

(b) $(g\circ f)(x) = (2x^2-1)+1 = 2x^2$

(c) From (a), we have

$$(f\circ g)(1)=7$$

(d) From (b), we have

$$(g\circ f)(1) = 2$$

Notice for questions (c) and (d)

For $(f\circ g)(1)$ and $(g\circ f)(1)$, it is not necessary to find $(f\circ g)(x)$ and $(g\circ f)(x)$ first. Alternatively, we can directly apply the definition as follows

(c) $(f\circ g)(1) = f(g(1)) = f(2) = 7$ [since $g(1)=2$]

(d) $(g\circ f)(1) = g(f(1)) = g(1) = 2$ [since $f(1)=1$]

We may also define the function $f\circ f$ in the obvious way:

$$(f\circ f)(x) = f(f(x))$$

That is, we plug f into itself.

For example, if $f(x)=2x-1$, then

$$(f\circ f)(x) = f(2x-1) = 2(2x-1)-1 = 4x-3$$

EXAMPLE 2

Let $f(x)=\frac{x+1}{2}$ and $g(x)=\sqrt{x}$

Find (a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

(c) $(f \circ f)(x)$ (d) $(g \circ g)(x)$

(e) $(f \circ f \circ f)(x)$ in two ways: as $f \circ (f \circ f)$ and as $(f \circ f) \circ f$

Solution

(a) $(f \circ g)(x) = \frac{\sqrt{x}+1}{2}$ (b) $(g \circ f)(x) = \sqrt{\frac{x+1}{2}}$

(c) $(f \circ f)(x) = \frac{\frac{x+1}{2}+1}{2} = \frac{\frac{x+3}{2}}{2} = \frac{x+3}{4}$

(d) $(g \circ g)(x) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$

(e) $(f \circ f \circ f)(x) = [f \circ (f \circ f)](x) = \frac{\frac{x+3}{4}+1}{2} = \frac{\frac{x+7}{4}}{2} = \frac{x+7}{8}$

Or $= [(f \circ f) \circ f](x) = \frac{\frac{x+1}{2}+3}{4} = \frac{\frac{x+7}{2}}{4} = \frac{x+7}{8}$

♦ THE IDENTITY FUNCTION $i(x)$

It is the simple function that maps x to itself

$$i(x)=x \qquad \text{or} \qquad i: x \mapsto x$$

Notice that

$(f \circ i)(x) = f(i(x)) = f(x)$

$(i \circ f)(x) = i(f(x)) = f(x)$

That is

$$f \circ i = f \qquad \text{and} \qquad i \circ f = f$$

- PRESUPPOSITION FOR $f \circ g$ AND $g \circ f$ (**Mainly for HL**)

Let $f: A \to B$ and $g: B \to C$.

Then

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \\ x \arrow[r, mapsto] \arrow[rr, bend right, mapsto, "g \circ f"'] & f(x) \arrow[r, mapsto] & g(f(x)) \end{tikzcd}$$

That is in $g \circ f$, f is applied first and then g

Notice also that $g \circ f$ can be defined only if the Range of f is inside the Domain of g.

Similar observations may be done for $f \circ g$. Thus,

Function	**Observation**	**Presupposition**
$f \circ g$	g is applied first and then f	$R_g \subseteq D_f$
$g \circ f$	f is applied first and then g	$R_f \subseteq D_g$

2.5 THE INVERSE FUNCTION: f^{-1}

♦ DISCUSSION

Consider the function $f(x)=x+10$. It maps

$$0 \mapsto 10$$
$$1 \mapsto 11$$
$$2 \mapsto 12 \qquad \text{etc.}$$

The "inverse" procedure is also a function:

$$10 \mapsto 0$$
$$11 \mapsto 1$$
$$12 \mapsto 2 \qquad \text{etc.}$$

It is called the **inverse function** of f and it is denoted by $\mathbf{f^{-1}}$.

Obviously

$$f^{-1}(x)=x-10$$

In fact, f and f^{-1} are **inverse** to each other.

♦ FORMAL DEFINITION

Let $f:R\to R$

The **inverse function** $\mathbf{f^{-1}}$ is a new function such that

$$f(x)=y \quad \Leftrightarrow \quad f^{-1}(y)=x.$$

♦ HOW DO WE FIND $\mathbf{f^{-1}}$?

Steps f is given	**Example** $f(x) = x+10$
1. Set $f(x)=y$	$x+10 = y$
2. Solve for x	$x = y-10$
3. Keep the solution but replace y by x	$f^{-1}(x)=x-10$

NOTICE:

1. The inverse function of f^{-1} is f itself. That is

$$\boxed{(f^{-1})^{-1} = f}$$

2. The domain of f becomes range of f^{-1} and vice-versa:

$$\boxed{D_{f^{-1}} = R_f} \qquad \boxed{R_{f^{-1}} = D_f}$$

3. It holds

$$\boxed{(f^{-1}\circ f)(x) = x = (f\circ f^{-1})(x)}$$

For example, for $f(x) = x+10$ and $f^{-1}(x)=x-10$:

$(f\circ f^{-1})(x)=f(f^{-1}(x))=f(x-10)=(x-10)+10=x$

$(f^{-1}\circ f)(x)= f^{-1}(f(x))=f(x+10)=(x+10)-10=x$

In other words, $\mathbf{f^{-1}\circ f=i}$ and $\mathbf{f\circ f^{-1}=i}$, where $\mathbf{i}$ is the identity function.

EXAMPLE 1

Let $f(x)=3x+5$. Find (a) $f^{-1}(x)$ (b) $f^{-1}(11)$

Solution

(a) We follow the three steps:

- Set $3x+5=y$
- $3x+5=y \Leftrightarrow 3x = y-5 \Leftrightarrow x = \dfrac{y-5}{3}$
- $f^{-1}(x)= \dfrac{x-5}{3}$

(b) Since we know $f^{-1}(x)= \dfrac{x-5}{3}$, it is $f^{-1}(11) = 2$

Alternatively: It is not necessary to find $f^{-1}(x)$.

If $f^{-1}(11)=x$ then $f(x)=11$. Hence

$$3x+5 = 11 \Leftrightarrow 3x = 6 \Leftrightarrow x=2.$$

Thus, $f^{-1}(11) =2$

Remark:

Verify that

the inverse function of $f^{-1}(x)=\dfrac{x-5}{3}$ is $f(x)=3x+5$.

- Set $\dfrac{x-5}{3}=y$
- $\dfrac{x-5}{3}=y \Leftrightarrow x-5=3y \Leftrightarrow x=3y+5$
- The inverse function is $y=3x+5$

In other words f and f^{-1} are inverse to each other.

EXAMPLE 2

Let $f(x)=2x^2-1$ where $x\geq 0$. Find (a) $f^{-1}(x)$ (b) $f^{-1}(49)$

Solution

(a) We follow the three steps:

- Set $2x^2-1=y$
- $2x^2-1=y \Leftrightarrow 2x^2=y+1 \Leftrightarrow x^2=\dfrac{y+1}{2} \Leftrightarrow x=\sqrt{\dfrac{y+1}{2}}$
- $f^{-1}(x)=\sqrt{\dfrac{x+1}{2}}$

(b) Since we know $f^{-1}(x)=\sqrt{\dfrac{x+1}{2}}$, it is

$f^{-1}(49)=\sqrt{25}=5$

or again

$f^{-1}(49)=x$ implies $f(x)=49$

$\Leftrightarrow 2x^2-1=49 \Leftrightarrow x^2=25 \Leftrightarrow x=5$

So $f^{-1}(49)=5$

EXAMPLE 3

Let $f(x)=\frac{x+1}{x-2}$

(a) Show that $f^{-1}(x)=\frac{2x+1}{x-1}$

(b) Verify that $f \circ f^{-1}$ is the identity function [that is $(f \circ f^{-1})(x)=x$]

(c) Find the domain and the range of the functions f and f^{-1}

Solution

(a) $\frac{x+1}{x-2} = y \Leftrightarrow y(x-2) = x+1$

$\Leftrightarrow yx-2y = x+1$

$\Leftrightarrow yx-x = 2y+1$

$\Leftrightarrow x(y-1)=2y+1$

$\Leftrightarrow x=\frac{2y+1}{y-1}$

Hence, $\boxed{f^{-1}(x)=\frac{2x+1}{x-1}}$

(b) $$(f \circ f^{-1})(x) = \frac{\frac{2x+1}{x-1}+1}{\frac{2x+1}{x-1}-2} = \frac{\frac{2x+1+x-1}{x-1}}{\frac{2x+1-2x+2}{x-1}} = \frac{\frac{3x}{x-1}}{\frac{3}{x-1}} = \frac{3x}{3} = x$$

That is $(f \circ f^{-1})(x)=x$ (identity function)

[In a similar way we can show that $(f^{-1} \circ f)(x)=x$]

(c) The domain of f corresponds to the range of f^{-1} and vice versa:

D_f	R_f
$x \neq 1$	$y \neq 2$
$D_{f^{-1}}$	$R_{f^{-1}}$
$x \neq 2$	$y \neq 1$

EXAMPLE 4

Let $f(x)=1-2x$ and $g(x)=\frac{1}{x}$. Find

(a) $(fog)(x)$ **(b)** $(gof)(x)$ **(c)** $(gof^{-1})(x)$

(d) $(fog^{-1})(x)$ **(e)** $(fog)^{-1}(x)$ **(f)** $(f^{-1}og^{-1})(x)$

Solution

(a) $(fog)(x) = f(g(x)) = f(\frac{1}{x}) = 1-2\frac{1}{x} = 1-\frac{2}{x}$

(b) $(gof)(x) = g(f(x)) = g(1-2x) = \frac{1}{1-2x}$

(c) We firstly need f^{-1}. Since $f(x)=1-2x$

$$1-2x = y \Leftrightarrow 1-y = 2x \Leftrightarrow x = \frac{1-y}{2}. \qquad \text{Hence } f^{-1}(x)= \frac{1-x}{2}$$

Now $(gof^{-1})(x) = \frac{2}{1-x}$

(d) We firstly need g^{-1}. Since $g(x)=\frac{1}{x}$

$$\frac{1}{x}= y \Leftrightarrow x = \frac{1}{y}. \qquad \text{Hence } g^{-1}(x)= \frac{1}{x} \quad [\text{ that is } g^{-1} = g\ ]$$

Then, $(fog^{-1})(x) = 1-\frac{2}{x}$

(e) We are looking for the inverse function of $(fog)(x) = 1-\frac{2}{x}$

$$1-\frac{2}{x}=y \Leftrightarrow 1-y = \frac{2}{x} \Leftrightarrow x = \frac{2}{1-y}. \quad \text{Thus , } (fog)^{-1}(x) = \frac{2}{1-x}$$

(f) $(f^{-1}og^{-1})(x) = \dfrac{1-\frac{1}{x}}{2} = \dfrac{x-1}{2x}$

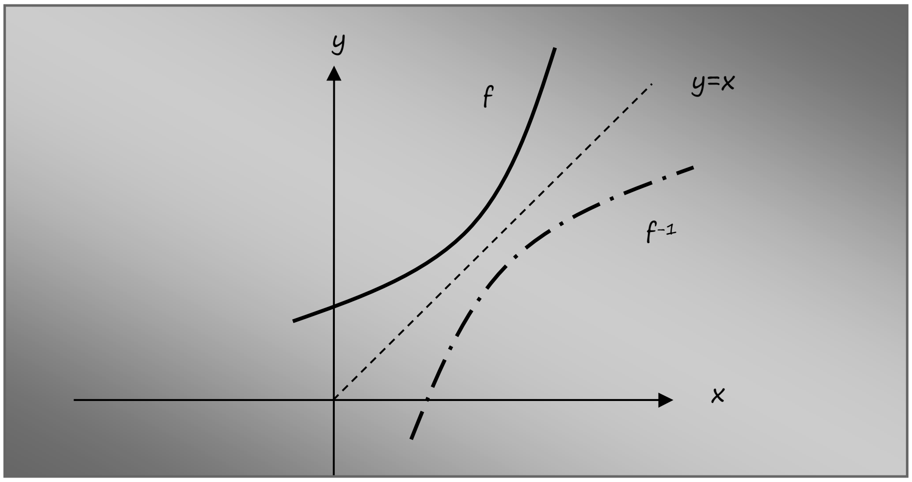
NOTICE:

Notice that $(fog)^{-1} \neq f^{-1}og^{-1}$. In fact it holds

$$\boxed{(fog)^{-1} = g^{-1}of^{-1}}$$

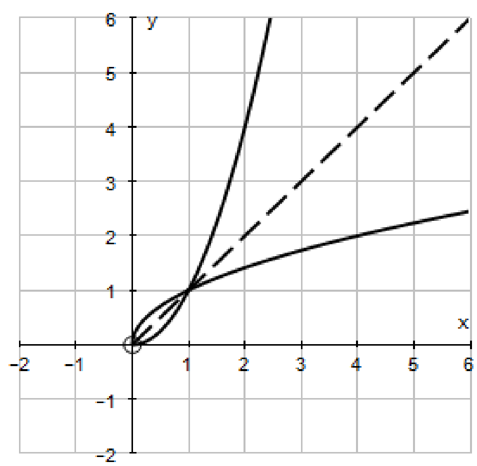
- GRAPH OF f^{-1}

The graph of f^{-1} is a reflection of f about the line y=x



EXAMPLE 5

If $f(x)=x^2$, for $x\geq 0$, then $f^{-1}(x)=\sqrt{x}$. Their graphs are



Notice: if f is increasing then f and f^{-1} may intersect **only on the line y=x.** Thus, in order to find the intersection points, instead of

$$f(x) = f^{-1}(x)$$

we can solve

$$f(x) = x$$

Here, $f(x)=x \Leftrightarrow x^2 = x \Leftrightarrow x^2-x = 0 \Leftrightarrow x(x-1)=0 \Leftrightarrow x=0 \text{ or } x=1$

The intersection points are (0,0) and (1,1).

NOTICE:

We say that the function f is **self-inverse** if $f^{-1}=f$.

Then it also holds

$$(f\circ f)(x)=x$$

i.e. $f\circ f$ is the identity function I.

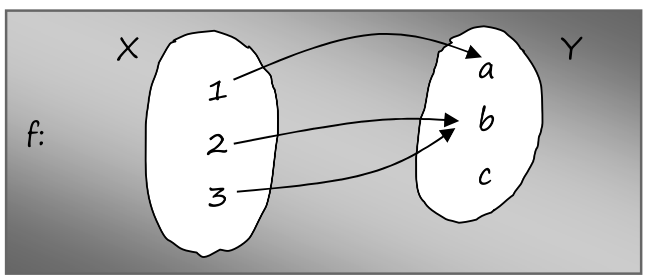
The graph of a self-inverse function is symmetric about y=x.

The simplest example is $f(x)=\frac{1}{x}$, since $f^{-1}(x)=\frac{1}{x}$.

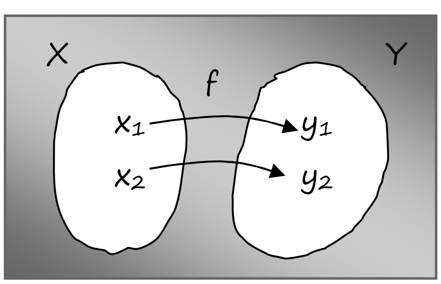
Another example is $f(x)=\frac{2x-6}{x-2}$ (please confirm!)

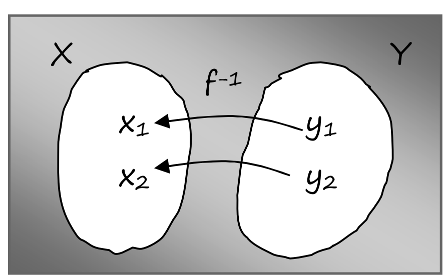
♦ PRESUPPOSITION FOR f^{-1} (**Mainly for HL**)

Consider the function



The inverse function f^{-1} doesn't exist, since $f^{-1}(b)$ is not uniquely determined (is it 2 or 3?). Hence, for f^{-1} to exist, different values of x should map to different values of y:





In other words, the function has to be **one-to-one**

(in fact, it has to be one-to-one and onto!)

NOTICE: Remember that

a function must satisfy the **vertical line test**.

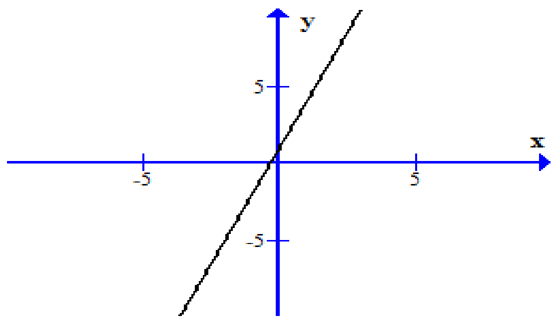
a "1-1" function must also satisfy the **horizontal line test**

Horizontal line test

Any horizontal line intersects the graph at most once

EXAMPLE 6

(a) The function f(x)=3x+1 is "1-1" since it is a straight line and satisfies the horizontal line test.



More mathematically:

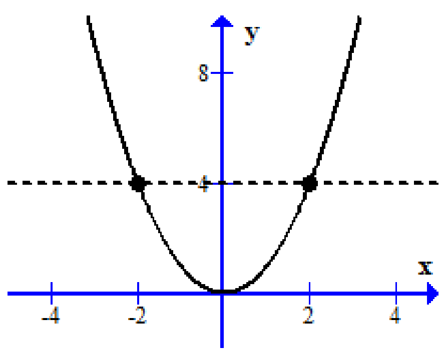
$$f(x_1)=f(x_2) \Rightarrow 3x_1+1=3x_2+1 \Rightarrow 3x_1=3x_2 \Rightarrow x_1=x_2$$

Hence f is "1-1" and f^{-1} exists.

We can easily find $f^{-1}(x)=\dfrac{x-1}{3}$

(b) The function $f(x)=x^2$ is not "1-1"

Indeed, f does not satisfies the horizontal line test, as two different values may map to the same image, for example f(-2)=4=f(2).



However,

- if we consider

$$f(x)=x^2, \quad x\geq 0$$

then f is "1-1" (horizontal line test) and f^{-1} exists.

$$f^{-1}(x)=\sqrt{x} \quad \text{(look at example 5)}$$

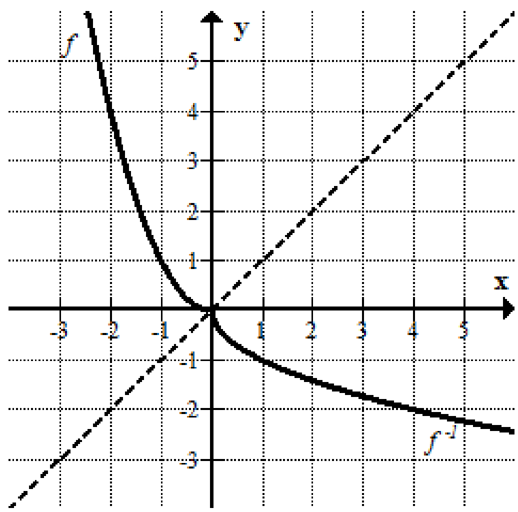
- Similarly, if we consider the restriction

$$f(x)=x^2, \quad x\leq 0$$

then f is "1-1" (horizontal line test) and f^{-1} exists. then

$$f^{-1}(x)=-\sqrt{x}$$

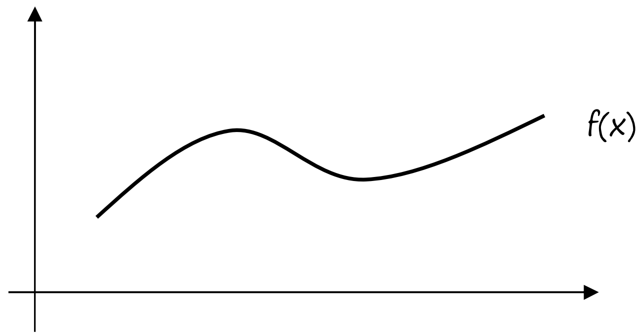
In this case the graphs of f and f inverse are as follows



2.6 TRANSFORMATIONS OF FUNCTIONS

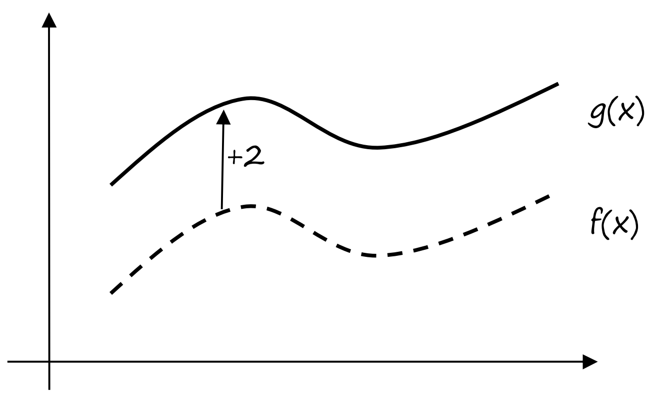
♦ DISCUSSION

Consider a function f(x).



Let's think of the new function $g(x)=f(x)+2$

In fact, we add 2 units to any value of $y=f(x)$, thus the whole graph of f(x) moves 2 units up.



We say that this is a **vertical translation** of the graph.

In a similar way we can describe other transformations of f(x), not only in a vertical direction (applied on y) but also in a horizontal direction (applied on x).

Let us present the most important transformations in a concise way!

♦ THE BASIC TRANSFORMATIONS

Consider the original function $y=f(x)$.
(In the following tables we assume $a>0$ and $b>1$)

VERTICAL TRANSFORMATIONS				
Function	**Transformation**		**Example: $f(x)=x^2$**	
$f(x)+a$	vertical **translation** a units up	↑ a	$g(x)=x^2+2$	↑ 2
$f(x)-a$	vertical **translation** a units down	↓ a	$g(x)=x^2-2$	↓ 2
$bf(x)$	vertical **stretch** with **scale factor b**	↕ $\times b$	$g(x)=2x^2$	↕ $\times 2$
$f(x)/b$	vertical **stretch** with **scale factor 1/b** (shrink)	$\div b$	$g(x)=x^2/2$	$\div 2$
$-f(x)$	**reflection** in the x-axis		$g(x)=-x^2$	

Now, as far as the horizontal transformations below are concerned, we obtain, perhaps, the opposite of what we expect!

HORIZONTAL TRANSFORMATIONS				
Function	**Transformation**		**Example: $f(x)=x^2$**	
$f(x+a)$	horizontal **translation** a units to the left	← a	$g(x)=(x+2)^2$	← 2
$f(x-a)$	horizontal **translation** a units to the right	→ a	$g(x)=(x-2)^2$	→ 2
$f(bx)$	horizontal **stretch** with **scale factor 1/b** (shrink)	$\div b$	$g(x)=(2x)^2$	$\div 2$
$f(x/b)$	horizontal **stretch** with **scale factor b**	↔ $\times b$	$g(x)=(x/2)^2$	↔ $\times 2$
$f(-x)$	**reflection** in the y-axis		$g(x)=(-x)^2$	

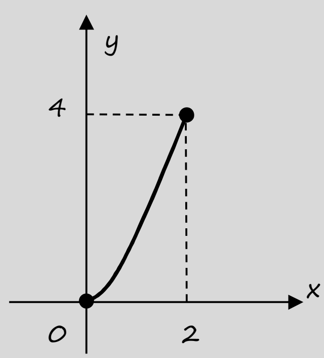
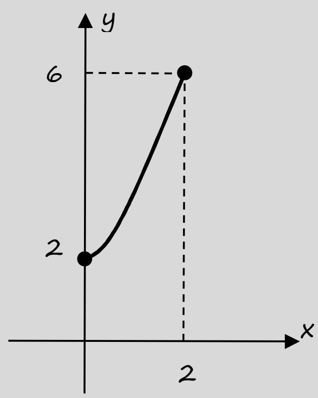
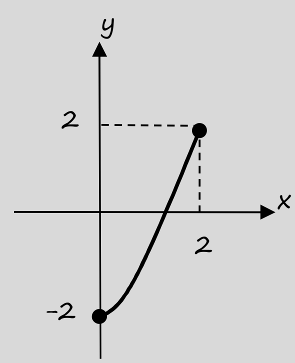
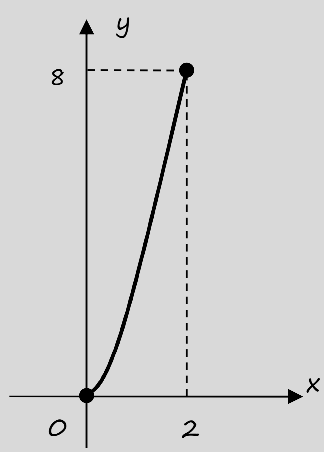
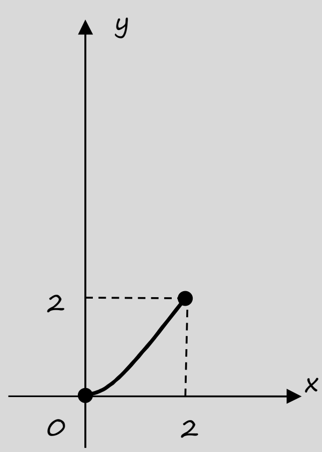
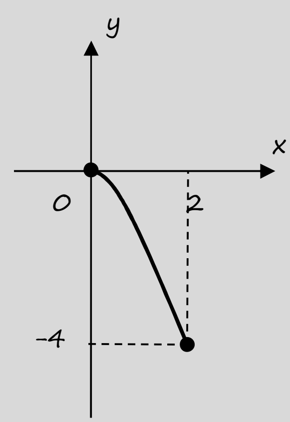
EXAMPLE 1

Let us observe the basic transformations of the function

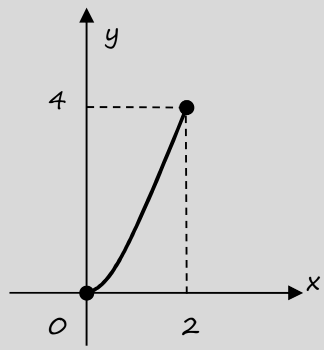
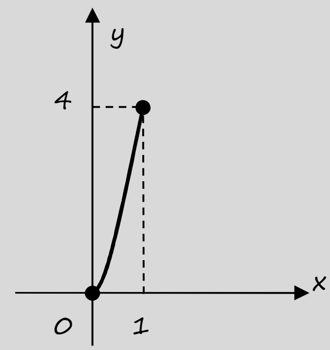
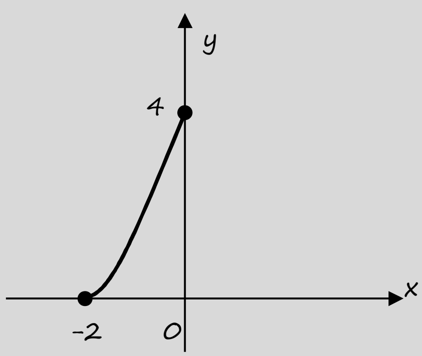
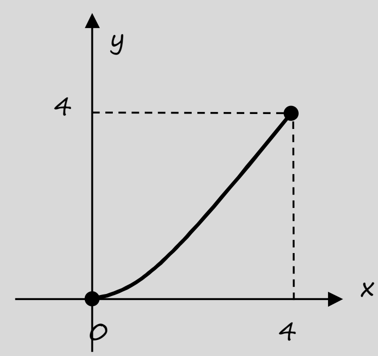
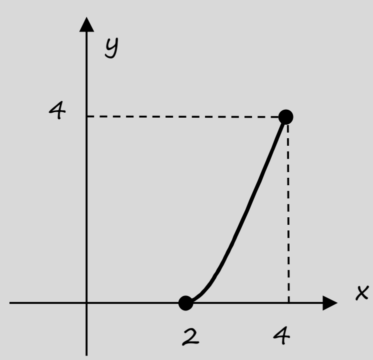
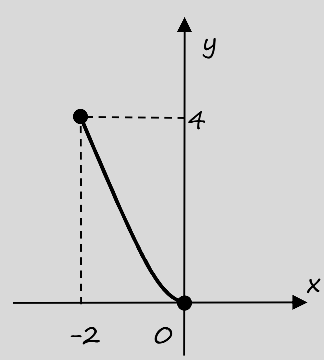
$$f(x) = x^2, \quad 0 \le x \le 2$$

in connection with the two tables above.

Let us see the vertical transformations first

VERTICAL TRANSFORMATIONS		
$f(x)=x^2$ [original function] 	$f(x)=x^2+2$ [2 units up] 	$f(x)=x^2-2$ [2 units down] 
$f(x)=2x^2$ [vertical stretch, s.f. 2] 	$f(x)=x^2/2$ [vertical stretch s.f. ½ That is shrink (÷2)] 	$f(x)=-x^2$ [reflection in x-axis] 

Next, we observe the horizontal transformations

HORIZONTAL TRANSFORMATIONS	
$f(x)=x^2$ [original function] 	$f(x)=(2x)^2$ [horizontal stretch, s.f. ½ That is shrink (÷2)] 
$f(x)=(x+2)^2$ [2 units to the left] 	$f(x)=(x/2)^2$ [horizontal stretch, s.f. 2] 
$f(x)=(x-2)^2$ [2 units to the right] 	$f(x)=(-x)^2$ [reflection in y-axis] 

EXAMPLE 2

Let **A(6,10)** be a point on the curve of $y=f(x)$.

Let us present some basic transformations as well as the corresponding images of the point A.

VERTICAL TRANSFORMATIONS			
Function	**Transformation**		**Image of A**
$f(x)+2$	vertical **translation** 2 units up	↑ 2	A′(6,12)
$f(x)-2$	vertical **translation** 2 units down	↓ 2	A′(6,8)
$2f(x)$	vertical **stretch** with **scale factor 2**	×2	A′(6,20)
$f(x)/2$	vertical **stretch** with **scale factor 1/2** (shrink)	÷2	A′(6,5)
$-f(x)$	**reflection** in the x-axis		A′(6,-10)

HORIZONTAL TRANSFORMATIONS			
Function	**Transformation**		**Example: $f(x)=x^2$**
$f(x+2)$	horizontal **translation** 2 units to the left	2 ←	A′(4,10)
$f(x-2)$	horizontal **translation** 2 units to the right	2 →	A′(8,10)
$f(2x)$	horizontal **stretch** with **scale factor 1/2** (shrink)	÷2	A′(3,10)
$f(x/2)$	horizontal **stretch** with **scale factor 2**	×2	A′(12,10)
$f(-x)$	**reflection** in the y-axis		A′(-6,10)

NOTICE:

The horizontal translation by a units (to the right or to the left)

is also denoted by the translation vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$

A vertical translation by b units (up or down)

is also denoted by the translation vector $\begin{pmatrix} 0 \\ b \end{pmatrix}$

The combination of those two translations is denoted by $\begin{pmatrix} a \\ b \end{pmatrix}$

Of course we may have a combination of several simple transformations.

For example, $2f(x-3)+5$ implies

a vertical stretch with scale factor 2, followed by

a horizontal translation 3 units to the right, followed by

a vertical translation 5 units up

NOTICE:

Remember the vertex form of a quadratic function

$$y=a(x-h)^2+k$$

This is a combination of transformations of the simple quadratic function $y=x^2$

Indeed, If $a>0$

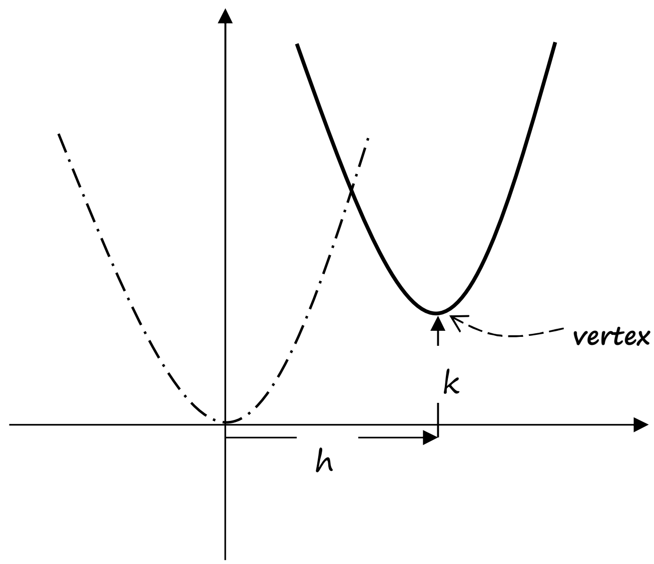
x^2	original function
ax^2	<u>vertical stretch</u> by scale factor a
$a(x-h)^2$	<u>horizontal translation</u> by **h units**
$a(x-h)^2+k$	<u>vertical translation</u> by **k units**

(if $a<0$, we also have a reflection about x-axis)

The two translations by $\begin{pmatrix} h \\ k \end{pmatrix}$ imply that the initial vertex (0,0) of the function x^2 moves

h units horizontally, and
k units vertically,

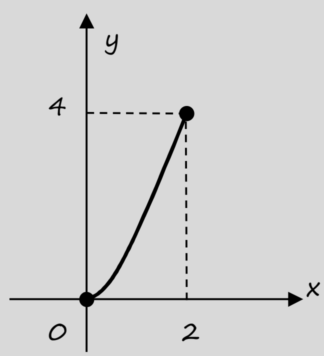
thus its new position is (h,k)



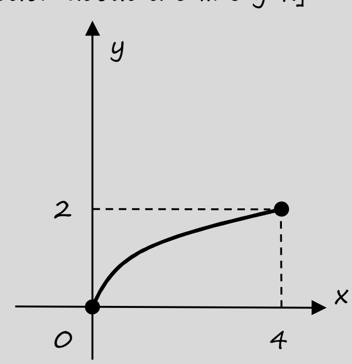
- THE INVERSE FUNCTION TRANSFORMATION

We have already seen that $f^{-1}(x)$ causes a reflection in the line y=x.

$f(x)=x^2$
[initial function]



$f^{-1}(x)=\sqrt{x}$
[reflection about the line y=x]



The image of the point A(2,4) is A′(4,2).

♦ CORRECT ORDER OF TRANSFORMATIONS

Suppose that we are looking for the image of point A(8,8) under a sequence of transformations. Pay attention to the order!
For two consecutive transformations, we distinguish three cases.

- ***Case A: vertical – horizontal***

One does not affect the other, so any order is correct!

Example
2f(x+3)

order	image
f(x)	(8,8)
2f(x)	(8,16)
2f(x+3)	**(5,16)**

or

order	image
f(x)	(8,8)
f(x+3)	(5,8)
2f(x+3)	**(5,16)**

- ***Case B: vertical – vertical***

The order matters! Changes affect, the **whole expression**

Example 1	order	description	image
2f(x)+6	f(x)		(8,8)
	2f(x)	we multiply the whole expression by 2	(8,16)
	2f(x)+6	we add 6 to the whole expression	**(8,22)**

Example 2	order	description	image
2[f(x)+3]	f(x)		(8,8)
	f(x)+3	we add 3 to the whole expression	(8,11)
	2[f(x)+3]	we multiply the whole expression by 2	**(8,22)**

- ***Case C: horizontal – horizontal***

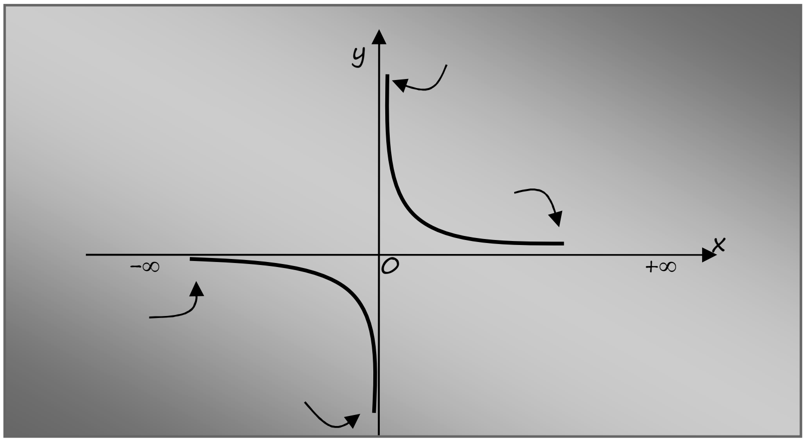
This is the most confusing case! At each step, **only x changes**

Example 1	order	description	image
f(2x+6)	f(x)		(8,8)
	f(x+6)	x changes to x+6	(2,8)
	f(2x+6)	x changes to 2x	**(1,8)**

Example 2	order	description	image
f(2(x+3))	f(x)		(8,8)
	f(2x)	x changes to 2x	(4,8)
	f(2(x+3))	x changes to x+3	**(1,8)**

2.7 ASYMPTOTES

Look at the graph of the function $f(x)=\frac{1}{x}$



Notice: as x tends to $+\infty$ the value of y tends to 0 **(the x-axis)**

Also as x tends to $-\infty$ the value of y approaches 0 **(the x-axis)**

We say that

the x-axis (that is the line **y=0**) is a **horizontal asymptote**

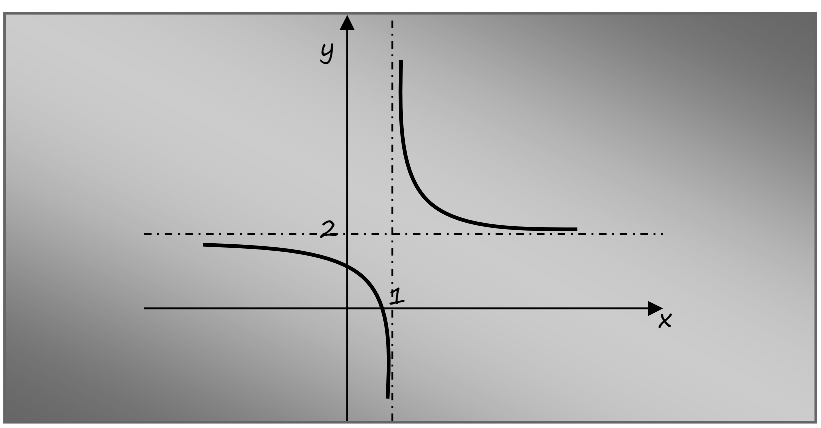
Moreover,

for values of x near 0 **(y-axis),** the value of y tends to $+\infty$ or $+\infty$

We say that

the y-axis (that is the line **x=0**) is a **vertical asymptote**

Similarly, for $g(x)=\frac{1}{x-1}+2$ (f moved 1 unit right and 2 units up).



Now the line y=2 is a horizontal asymptote

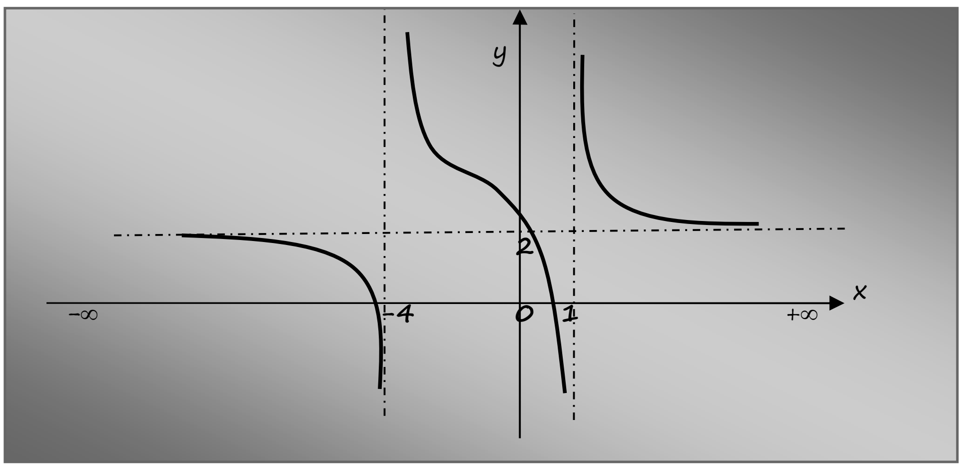
the line x=1 is a vertical asymptote

In general,

For Vertical Asymptotes: we are looking at points x=a where the function is not defined

For Horizontal Asymptotes: we observe what happens if x tends to +∞ or −∞. If the function approaches the line y=b we say that y=b is a horizontal asymptote!

In the following graph:



The function is not defined at x=−4 and x=1, so

the lines x=−4 and x=1 are vertical asymptotes

As x tends to +∞ or −∞ the graph approaches the line y=2, so

the line y=2 is a horizontal asymptote

In this section we concentrate on rational functions of the form

$$f(x)=\frac{Ax+B}{Cx+D}$$

and their asymptotes. It can be shown that such a function can be derived from original function

$$f(x)=\frac{1}{x}$$

by a sequence of transformations.

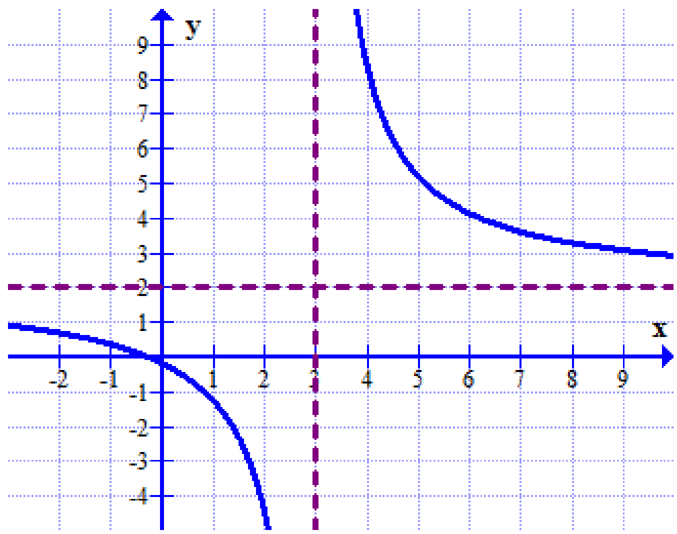
♦ RATIONAL FUNCTIONS OF THE FORM $f(x)=\dfrac{Ax+B}{Cx+D}$,

These functions possess one vertical and one horizontal asymptote.

For example, the function

$$f(x)=\frac{4x+1}{2x-6}$$

looks like



1) Vertical Asymptotes: x=a

At points where the function is not defined.

We solve

$$2x-6=0 \Leftrightarrow x=3$$

Hence

$$\boxed{\text{The line } x=3 \text{ is a vertical asymptote}}$$

2) Horizontal Asymptotes: y=b

The line

$$y=\frac{A}{C} \text{ is a horizontal asymptote}$$

(we consider only the leading coefficients!)

For our example,

$$y=\frac{4}{2}=2,$$

Hence

$$\boxed{\text{The line } y=2 \text{ is a horizontal asymptote}}$$

Notice

The domain is $x \neq 3$ while the vertical asymptote is $x=3$.

The range is $y \neq 2$ while the vertical asymptote is $y=2$.

Two short explanations for the horizontal asymptote:

- The function can be written as follows:

$$f(x)=\frac{4x+1}{2x-6}=\frac{2(2x-6)+13}{2x-6}=\frac{2(2x-6)}{2x-6}+\frac{13}{2x-6}=2+\frac{13}{2x-6}$$

As x tends to $+\infty$ or $-\infty$ the fraction $\frac{13}{2x-6}$ approaches 0.

- If we divide everything by x we obtain:

$$f(x)=\frac{4x+1}{2x-6}=\frac{4+\frac{1}{x}}{2-\frac{6}{x}}$$

As x tends to $+\infty$ or $-\infty$ the fractions $\frac{1}{x}$ and $\frac{6}{x}$ approach 0.

In both cases f(x), that is the value of y, approaches 2.

EXAMPLE 1

Look at some rational functions and their asymptotes:

Function	**Vertical Asymptotes** (denominator = 0)	**Horizontal Asymptote** (divide leading coefficients)
$f(x)=\frac{3x-7}{x-5}$	$x=5$	$y=3$
$f(x)=\frac{3x-7}{2x-5}$	$x=\frac{5}{2}$	$y=\frac{3}{2}$
$f(x)=\frac{8x-7}{2x+4}$	$x=-2$	$y=4$
$f(x)=\frac{7}{x-5}$	$x=5$	$y=0$
$f(x)=\frac{7}{x-5}+3$	$x=5$	$y=3$

EXAMPLE 2

Let $f(x)=\dfrac{3x+2}{x-4}$

We can easily find that the inverse function is $f^{-1}(x)=\dfrac{4x+2}{x-3}$

Notice what happens with the asymptotes:

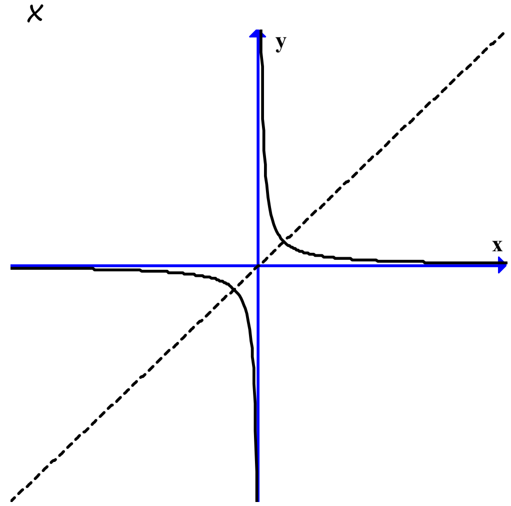
	Domain	Range	V.A.	H.A
f(x)	$x\neq4$	$y\neq3$	$x=4$	$y=3$
$f^{-1}(x)$	$x\neq3$	$y\neq4$	$x=3$	$y=4$

♦ SELF-INVERSE FUNCTIONS

A function is said to be **self-inverse** if $f^{-1}(x)=f(x)$

Such a function is **symmetric in the line y=x**.

For example $f(x)=\dfrac{1}{x}$ is a self-inverse function.



Indeed, $y=\dfrac{1}{x}\Leftrightarrow x=\dfrac{1}{y}$ hence, $f^{-1}(x)=\dfrac{1}{x}$

Several rational functions are self-inverse. For example

$$f(x)=\frac{2x+3}{x-2}=f^{-1}(x)$$

The asymptotes for those two functions are x=2 and y=2.

2.8 EXPONENTS – THE EXPONENTIAL FUNCTION a^x

♦ THE EXPONENTIAL 2^x

Let us define the power 2^x, as x moves along the sets

$N = \{0,1,2,3,\ldots\}$	**Natural numbers**
$Z = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$	**Integers**
$Q = \{\text{fractions } \frac{m}{n} \mid m,n\in Z,\ n\neq 0\}$	**Rational numbers**
$R = Q$ + irrational numbers[†]	**Real numbers**

1) If $x=n\in N$, then

$$2^0 = 1$$

$$2^n = 2\cdot 2\cdot 2 \cdots 2 \text{ (n times)}$$

For example $2^3=8$

2) If $x=-n$, where $n\in N$, then

$$2^{-n} = \frac{1}{2^n}$$

Thus we know 2^x for any $x\in Z$.

For example $2^{-3}=\frac{1}{2^3}=\frac{1}{8}$

3) If $x=\frac{m}{n}$, where $m,n\in Z$, $n\neq 0$, then

$$2^{\frac{m}{n}} = \sqrt[n]{2^m}$$

Thus we know 2^x for any $x\in Q$

For example, $2^{\frac{2}{3}}=\sqrt[3]{2^2}=\sqrt[3]{4}$, $\quad 2^{\frac{2}{3}}=\sqrt{2^3}=\sqrt{8}$, $\quad 2^{\frac{1}{2}}=\sqrt{2}$

[†] That is numbers that cannot be expressed as fractions, eg π, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$

4) If x=irrational, then

2^x = given by a calculator!

The definition is beyond our scope, thus we trust technology!

Thus we know 2^x for any $x\in R$

For example, $2^{\pi} = 8.8249779$

In general, if $a>0$ we define

$a^0 = 1$

$a^n = a \cdot a \cdot a \cdots a$ (n times)

$a^{-n} = \dfrac{1}{a^n}$

$a^{\frac{m}{n}} = \sqrt[n]{a^m}$

a^x = given by a calculator!

NOTICE

- If $a<0$, a^x is defined only for $x=n\in Z$
- $0^x=0$ only if $x\neq 0$
- 0^0 is not defined

- PROPERTIES

All known properties of powers are still valid for exponents $x\in R$

(1) $a^x a^y = a^{x+y}$ (3) $(ab)^x = a^x b^x$ (5) $(a^x)^y = a^{xy}$

(2) $\dfrac{a^x}{a^y} = a^{x-y}$ (4) $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Here $a,b>0$ and $x,y\in R$

EXAMPLE 1

- $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$
- $\left(\frac{1}{5}\right)^{-2} = \frac{1}{5^{-2}} = 5^2 = 25$
- $\left(\frac{3}{5}\right)^{-2} = \left(\frac{5}{3}\right)^{2} = \frac{25}{9}$
- $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$ or $8^{2/3} = (2^3)^{2/3} = 2^{3\cdot(2/3)} = 2^2 = 4$
- $27^{-4/3} = \sqrt[3]{27^{-4}} = \sqrt[3]{\frac{1}{27^4}} = \sqrt[3]{\left(\frac{1}{27}\right)^4} = \sqrt[3]{\left(\frac{1}{27}\right)}^{4} = \left(\frac{1}{3}\right)^4 = \frac{1}{81}$

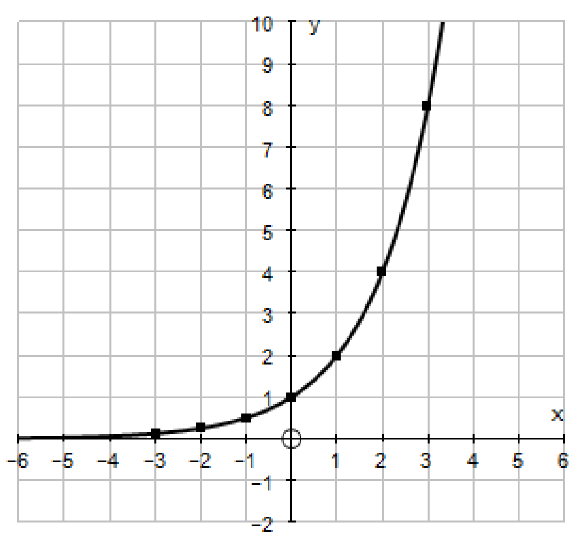
♦ THE EXPONENTIAL FUNCTION $f(x)=a^x$ (where $a>0$)

Consider $f: R \to R$

$f(x)=2^x$

Let us estimate some values

x	...	−3	−2	−1	0	1	2	3	...
$y=2^x$	...	1/8	1/4	1/2	0	1	4	8	...



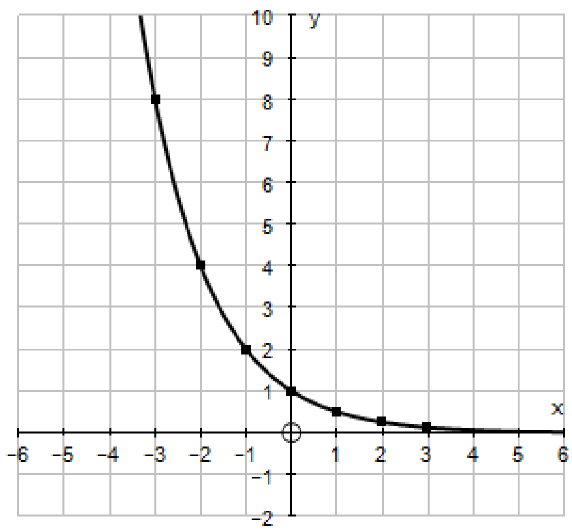
Domain: $x \in R$
Range: $y>0$

Consider now $g: R \to R$

$g(x)=0.5^x$ [that is $g(x)=\left(\frac{1}{2}\right)^x=\frac{1}{2^x}$]

Let us estimate some values

x	…	−3	−2	−1	0	1	2	3	…
$y=2^x$	…	8	4	2	0	1/2	1/4	1/8	…



Domain: $x \in R$

Range: $y>0$

NOTICE

1) $f(x)=a^x$ is <u>always</u> positive (even if $x<0$)

2) $g(x)=\left(\frac{1}{a}\right)^x=\frac{1}{a^x}=a^{-x}$. Thus, $g(x)$ is a reflection of $f(x)=a^x$ about the y-axis [look at the graphs of $f(x)$ and $g(x)$ above]

3) if $a>1$, then $f(x)=a^x$ <u>**increases**</u> (the graph looks like that of 2^x)
if $a<1$, then $f(x)=a^x$ <u>**decreases**</u> (the graph looks like that of 0.5^x)
if $a=1$, then $f(x)=1^x=1$ is constant

4) if $a\neq1$, function $f(x)=a^x$ is "one-one", i.e.

$$\boxed{a^x=a^y \Rightarrow x=y}$$

This property helps us to solve exponential equations!

EXAMPLE 2

Solve the following equations

(a) $2^{3x-1} = 2^{x+2}$ (b) $2^{3x-1} = 4^{x+2}$ (c) $4^{3x-1} = 8^{x+2}$

(d) $\frac{1}{2^{3x-1}} = 4^{x+2}$ (e) $\sqrt{2}^{\,3x-1} = 4^{x+2}$

Solution

Our attempt will be to induce a common base in both sides

(a) We have already a common base. Thus

$$2^{3x-1} = 2^{x+2} \Leftrightarrow 3x-1 = x+2 \Leftrightarrow 2x = 3$$

$$\Leftrightarrow \boxed{x=3/2}$$

(b) We can write $4=2^2$. Thus

$$2^{3x-1} = 4^{x+2} \Leftrightarrow 2^{3x-1} = 2^{2x+4} \Leftrightarrow 3x-1 = 2x+4$$

$$\Leftrightarrow \boxed{x = 5}$$

(c) We can write $4=2^2$ and $8=2^3$. Thus

$$4^{3x-1} = 8^{x+2} \Leftrightarrow 2^{6x-2} = 2^{3x+6} \Leftrightarrow 6x-2 = 3x+6$$

$$\Leftrightarrow 3x=8 \Leftrightarrow \boxed{x = 8/3}$$

(d) We apply the property $\frac{1}{2^n}=2^{-n}$. Thus

$$\frac{1}{2^{3x-1}} = 4^{x+2} \Leftrightarrow 2^{-3x+1} = 2^{2x+4} \Leftrightarrow -3x+1 = 2x+4$$

$$\Leftrightarrow 5x = -3 \Leftrightarrow \boxed{x=-3/5}$$

(e) We apply the property $\sqrt{2} = 2^{1/2}$. Thus

$$\sqrt{2}^{\,3x-1} = 4^{x+2} \Leftrightarrow 2^{\frac{3x-1}{2}} = 2^{2x+4} \Leftrightarrow \frac{3x-1}{2} = 2x+4$$

$$\Leftrightarrow 3x-1= 4x+8 \Leftrightarrow \boxed{x=-9}$$

♦ THE NUMBER e

There is a specific irrational number

$$e=2.7182818...$$

which plays an important role in mathematics. The number e is almost as popular as the irrational number $\pi=3.14...$

An approximation of e is given below. Consider the expression

$$\left(1+\frac{1}{n}\right)^n$$

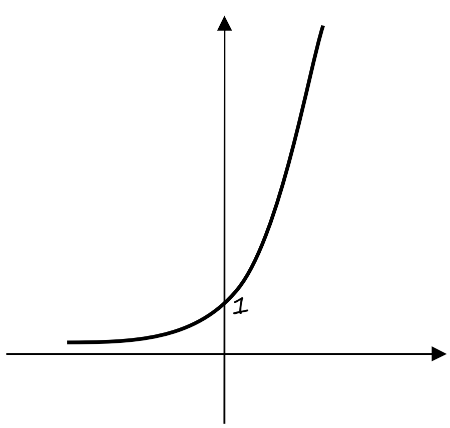
For $n=1$	the result is	2
For $n=2$	the result is	2.25
For $n=10$	the result is	2.5937424...
For $n=100$	the result is	2.7048138...
For $n=1000$	the result is	2.7169239...
For $n=10^6$	the result is	2.7182804...

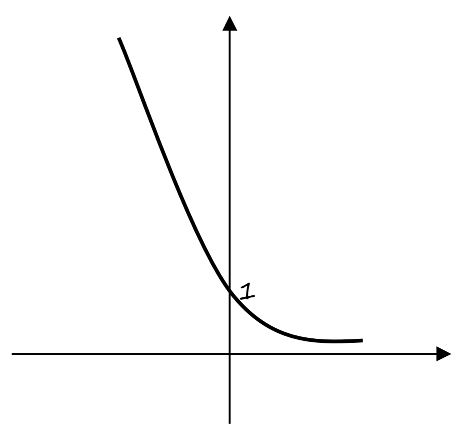
As n tends to $+\infty$ this expression tends to $e=2.7182818...$

♦ THE EXPONENTIAL e^x

The exponential function $f(x)=e^x$ appears in many applications. The graph looks like any function of the form $f(x)=a^x$. We present the graphs of

$f(x) = e^x$ $f(x)=e^{-x}$

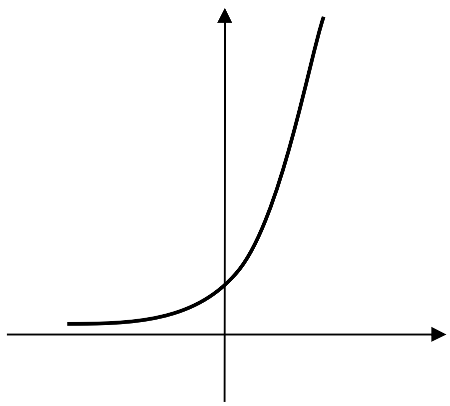




♦ ASYMPTOTES OF EXPONENTIAL FUNCTIONS

Observe the exponential functions ($a>0$, $a\neq 1$)

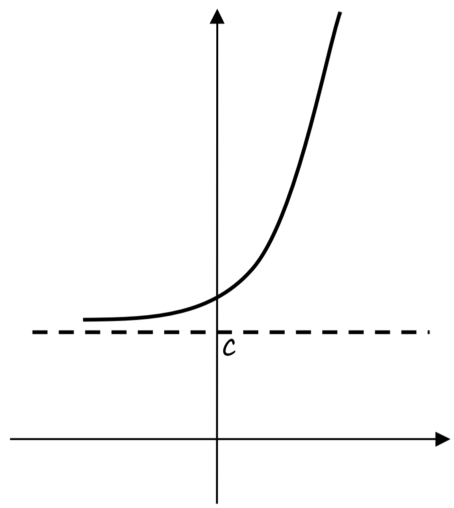
$f(x)=a^x$



horizontal asymptote: $y=0$

y-intercept: $y=1$

$f(x)=a^x+c$



horizontal asymptote: $y=c$

y-intercept: $y=c+1$

EXAMPLE 3

Function	Horizontal Asymptote	y-intercept
$f(x)=2^x$	line $y=0$	$y=1$
$f(x)=2^{-x}$	line $y=0$	$y=1$
$f(x)=e^x$	line $y=0$	$y=1$
$f(x)=e^{3x}$	line $y=0$	$y=1$
$f(x)=3e^x$	line $y=0$	$y=3$
$f(x)=-3e^x$	line $y=0$	$y=-3$
$f(x)= e^x+5$	line $y=5$	$y=6$
$f(x)=3e^x+5$	line $y=5$	$y=8$
$f(x)=e^{x-2}$	line $y=0$	$y=e^{-2}$

2.9 LOGARITHMS – THE LOGARITHMIC FUNCTION $y=\log_a x$

♦ DISCUSSION

Let us compare the equations of the form $x^2=a$ and $2^x=a$

Quadratic equation $x^2=a$	Exponential equation $2^x=a$
Some equations are straightforward $x^2=4 \Leftrightarrow x=\pm 2$ $x^2=9 \Leftrightarrow x=\pm 3$ $x^2=16 \Leftrightarrow x=\pm 4$	Some equations are straightforward $2^x=4 \Leftrightarrow x=2$ $2^x=8 \Leftrightarrow x=3$ $2^x=16 \Leftrightarrow x=4$
but what about $x^2=7$? We don't know an exact answer	but what about $2^x=7$? We don't know an exact answer!
Mathematicians invented the symbol $\sqrt{\ }$ and called the solutions $\pm\sqrt{7}$	Mathematicians invented the symbol $\log_2$ and called the solution $\log_2 7$
Similarly, the solution of $x^3=7$ is denoted by $\sqrt[3]{7}$	Similarly, the solution of $3^x=7$ is denoted by $\log_3 7$

Let us introduce this new notion more formally.

♦ THE LOGARITHM $\log_2 x$

This number is called **logarithm of x to the base 2**. It is connected to the exponential 2^x. The definition is given by

$$\log_2 x = b \Leftrightarrow 2^b = x$$

For example,

$\log_2 8 = 3$, since $2^3=8$

$\log_2 16 = 4$, since $2^4=16$

etc.

For example, for the value of $\log_2 8$, we think in the following way:

$2^{\text{what exponent}}$ gives 8? The answer is 3

Hence $\log_2 8=3$

Working in the same way, let us find $\log_2 32$

It is $\log_2 32=5$ [since $2^5=32$]

However, for $\log_2 7=?$, we should think:

$2^{\text{what exponent}}$ gives 7? Well, this is not easy to answer!

Our GDC gives $\log_2 7=2.80735\ldots$

This implies that

$$2^{2.80735\ldots\ldots}=7$$

EXAMPLE 1

Look at the following results:

- $\log_2 64=6$
- $\log_2 2=1$ in general, $\log_a a=1$ for any base a.
- $\log_2 1=0$ in general, $\log_a 1=0$, for any base a
- $\log_2 2^5=5$
- $\log_2 2^{100}=100$ In general $\boldsymbol{\log_2 2^x=x}$
- $\log_2 2^{1453}=1453$

♦ THE LOGARITHM $\log_a x$

In the same way, for any base $a>0$, $a\neq 1$ we define

$$\boxed{\log_a x = b \Leftrightarrow a^b=x}$$

For example, $\log_3 9 = 2$ (since $3^2=9$)

NOTICE

Once upon a time $\log_{10}x$ has been the most popular logarithm!!!

Due to its popularity, for this particular logarithm the base 10 is usually omitted

We write **$\log x$** instead of **$\log_{10}x$**

For example,

$\log 100 = 2$, since $10^2=100$

$\log 1000 = 3$, since $10^3=1000$

$\log 10000 = 4$, since $10^4=10000$

Use your GDC to confirm these results!
Also notice that

$$\log 10=1 \quad \text{and} \quad \log 1=0$$

EXAMPLE 2

- $\log 1000000 = 6$,
- $\log 10^7=7$ In general $\log 10^x=x$
- $\log 10^{1453}=1453$

But also, for very small numbers

- $\log 0.001 = -3$,
- $\log 0.000001 = -6$.

♦ THE NATURAL LOGARITHM **$\ln x$**

We will frequently use the special logarithm to the base

$$e=2.7182818\ldots$$

Instead of $\log_e x$, we denote it by **$\ln x$.**
Hence,

$$\boxed{\ln x=b \Leftrightarrow e^b=x}$$

♦ THE LOGARITHMIC FUNCTION $f(x)=\log_a x$

A new function is defined

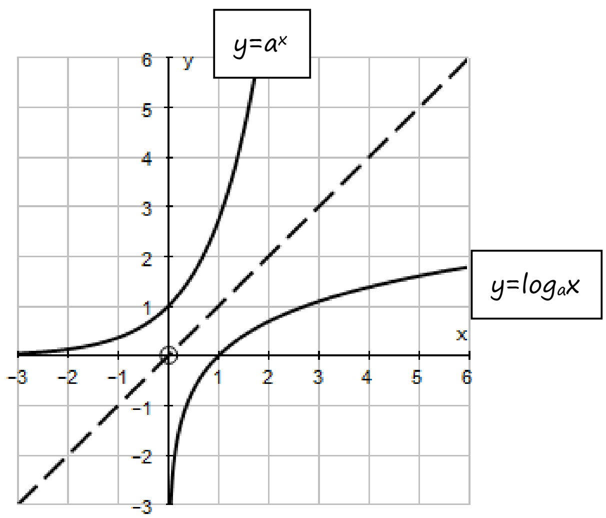
$$f(x)=\log_a x$$

In fact, this is the inverse of the exponential function $y=a^x$

$$\boxed{\text{If } f(x)=a^x \text{ then } f^{-1}(x)=\log_a x}$$

Indeed, $a^x=y \Leftrightarrow x=\log_a y$, hence $f^{-1}(x)=\log_2 x$

If $a>1$ (for example if $a=2$), the graphs of these two functions look like



Observations:

- For $y=a^x$: **Domain**: $x\in R$ **Range**: $y>0$
- For $y=\log_a x$: **Domain**: $x>0$ **Range**: $y\in R$
- The x-axis is a horizontal asymptote of $y=a^x$
- The y-axis is a vertical asymptote of $y=\log_a x$
- $y=a^x$ always passes through (0,1)
- $y=\log_a x$ always passes through (1,0)

♦ BASIC PROPERTIES OF LOGARITHMS

For any base a ($a>0$, $a\neq1$)

- $\log_a 1 = 0$
- $\log_a a = 1$

- $\log_a a^x = x$
- $a^{\log_a x} = x$

These results can be obtained by the definition of logarithm!

♦ THREE ALGEBRAIC LAWS

For simplicity reasons, we use log instead of $\log_a$.

1) $\log xy = \log x + \log y$
2) $\log \frac{x}{y} = \log x - \log y$
3) $\log x^n = n\log x$

or

- $\log x + \log y = \log xy$
- $\log x - \log y = \log \frac{x}{y}$
- $n\log x = \log x^n$

These laws hold for any base a.

Remark: The 3 lows are respectively derived from the properties

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^m)^n = a^{nm}$$

since for all of them $a^{LHS} = a^{RHS}$ and thus LHS=RHS.

NOTICE

The first two laws can be combined to obtain:

$$\log A + \log B - \log C + \log D = \log \frac{ABD}{C}$$

If we also have coefficients, we obtain

$$2\log A + 3\log B - 4\log C + 5\log D = \log \frac{A^2B^3D^5}{C^4}$$

In this way, **we collect many logs into one log**.

For example,

$$\mathbf{2}\log 3 + \mathbf{3}\log 4 - \mathbf{4}\log 2 = \log \frac{3^2 4^3}{2^4} = \log 36$$

Apparently, this works for ln as well:

$$\mathbf{2}\ln 3 + \mathbf{3}\ln 4 - \mathbf{4}\ln 2 = \ln \frac{3^2 4^3}{2^4} = \ln 36$$

Look at also the opposite direction

$$\boxed{\log \frac{A^2 B^3 D^5}{C^4} = \mathbf{2}\log A + \mathbf{3}\log B - \mathbf{4}\log C + \mathbf{5}\log D}$$

In this way, **we split one log into many logs**.

For example,

$$\log 15 = \log(3\times 5) = \log 3 + \mathbf{2}\log 5$$

$$\log_2 \frac{9}{5} = \log_2 \frac{3^2}{5} = 2\log_2 3 - \log_2 5$$

$$\ln 72 = \ln(8\times 9) = \ln 2^3 3^2 = \mathbf{3}\ln 2 + \mathbf{2}\ln 3$$

EXAMPLE 3

Suppose $\ln x = \mathbf{a}$, $\ln y = \mathbf{b}$, $\ln z = \mathbf{c}$.

Express the following logarithms in terms of **a, b, c.**

$\ln xy$	$= \ln x + \ln y = \mathbf{a+b}$
$\ln x^2$	$= 2\ln x = \mathbf{2a}$
$\ln \frac{y}{z}$	$= \ln y - \ln z = \mathbf{b-c}$
$\ln \frac{x^3 y}{z^2}$	$= 3\ln x + \ln y - 2\ln z = \mathbf{3a+b-2c}$
$\ln \frac{1}{x}$	$= \ln 1 - \ln x = 0 - a = \mathbf{-a}$ or $\ln \frac{1}{x} = \ln x^{-1} = -\ln x = \mathbf{-a}$
$\ln \sqrt{x}$	$= \ln x^{\frac{1}{2}} = \frac{1}{2}\ln x = \frac{\mathbf{a}}{\mathbf{2}}$

EXAMPLE 4

Suppose $\log_3 2=$**m**, $\log_3 5=$**n**. Express the following in terms of **m, n**.

$$\log_3 10, \quad \log_3 50, \quad \log_3 \frac{5}{2}, \quad \log_3 2.5$$

Solution

- $\log_3 10 = \log_3(2\times5) = \log_3 2 + \log_3 5 = \mathbf{m+n}$
- $\log_3 50 = \log_3(2\times5^2) = \log_3 2 + 2\log_3 5 = \mathbf{m+2n}$
- $\log_3 \frac{2}{5} = \log_3 2 - \log_3 5 = \mathbf{m-n}$
- $\log_3 2.5 = \log_3 \frac{5}{2} = \log_3 5 - \log_3 2 = \mathbf{n-m}$

NOTICE (Just for information, you may skip it!)

The logarithm to the base 10 indicates the size of a number!

Any n-digit number has a logarithm between n-1 and n.

Indeed, since $\log 100=2$ and $\log 1000=3$

any 3-digit integer, (that is between 100 and 999 inclusive)

has a logarithm within the interval $[2,3)$

For example, $\log 173 = 2.238 \ldots$

Similarly, any 10-digit number has a logarithm within $[9,10)$

For example, $\log 1234567890 = 9.091 \ldots$

Question: how many digits does the number 2^{100} have?

This is a huge number! The GDC cannot show all its digits.

However, $\log 2^{100} = 100\log 2 = 30.1$

which implies that the number 2^{100} has 31 digits!

♦ SIMPLE LOGARITHMIC EQUATIONS

They have the form

$$log_a x = b$$

We use the definition to solve them:

$$x = a^b$$

EXAMPLE 5

Solve the logarithmic equations

(a) $log_2(x+2)=3$ (a) $log(x+2)=3$ (c) $ln(x+2)=3$

Solution

(a) $x+2=2^3 \Leftrightarrow x+2=8 \Leftrightarrow x=6$

(b) $x+2=10^3 \Leftrightarrow x+2=1000 \Leftrightarrow x=998$

(c) $x+2=e^3 \Leftrightarrow x=e^3-2$

Notice

Of course the solutions may be obtained by a GDC.

For (a) and (b), **SolveN** gives the exact solutions x=6 and x=998
For (c) it gives an approximation $x \cong 18.1$
(this is not the **exact** solution, it is the approximate value of e^3-2).

Furthermore, if the equation contains a parameter, for example

$$log_2(x+a)=3$$

we cannot use GDC. The solution must be expressed in terms of a:

$$x+a=2^3 \Leftrightarrow x=8-a$$

In paper 2 (GDC allowed) we can use our calculator to solve more complicated logarithmic equations'

In paper 1 (GDC not allowed) we have to present the analytical solution for equations which involve more than one logarithms.

For an equation that involves more than one logarithms, our target will be to bring it in one of the forms

- **$\log A = \log B$** so that **$A = B$**
- **$\log_b A=c$** so that **$A=b^c$** by definition

The resulting equations will be easier to deal with.

EXAMPLE 6

Solve the equations

(a) $\log_2 x+\log_2(x+2)= \log_2 3$

(b) $\log_2 x+\log_2(x+2)= 3$

(c) $\log_2 x+\log_2(x-4)-\log_2(2x-3)=\log_2 3$

Solutions

(a) $\log_2 x+\log_2(x+2)= \log_2 3 \Leftrightarrow \log_2 x(x+2)= \log_2 3$

$\Leftrightarrow x(x+2)=3$

$\Leftrightarrow x^2+2x-3=0$

The solutions are $x=1$ and $x=-3$

The second solution is rejected since $x>0$ by the original equation. Therefore $\boxed{x=1}$.

(b) $\log_2 x+\log_2(x+2)= 3 \Leftrightarrow \log_2 x(x+2)= 3$

$\Leftrightarrow x(x+2)=2^3$

$\Leftrightarrow x^2+2x-8=0$

The solutions are $x=2$ and $x=-4$ (rejected)

Therefore $\boxed{x=2}$.

(c) $\log_2 x+\log_2(x-4)-\log_2(2x-3)=\log_2 3 \Leftrightarrow \log_2\dfrac{x(x-4)}{2x-3} = \log_2 3$

$$\Leftrightarrow \frac{x(x-4)}{2x-3}=3 \Leftrightarrow x^2-4x=6x-9 \Leftrightarrow x^2-10x+9=0$$

The solutions are $x=1$ or $x=9$

The first solution is rejected since $x>4$.

Therefore, $\boxed{x=9}$

♦ CHANGE OF BASE

Using our GDC we find that

$$\log_2 3 \cong 1.58496...$$

Look at also the values of $\frac{\log 3}{\log 2}$ and $\frac{\ln 3}{\ln 2}$. We find the same result!

In fact, we can express $\log_2 3$ in the form $\frac{\log_* 3}{\log_* 2}$ using any base we like, that is

$$\log_2 3 = \frac{\log 3}{\log 2} = \frac{\ln 3}{\ln 2} = \frac{\log_7 3}{\log_7 2} = ...$$

In general, we can change $\log_a b$ into $\frac{\log_* b}{\log_* a}$, using any base we like.

The formula

$$\boxed{\log_a b = \frac{\log_c b}{\log_c a}}$$

is known as "**change of base**".

EXAMPLE 7

Suppose $\log_2 x = \mathbf{a}$, $\log_2 y = \mathbf{b}$, $\log_2 z == \mathbf{c}$. Express the following in terms of **a,b,c**.

$$\log_4 x, \qquad \log_y x, \qquad \log_x yz, \qquad \log_y 2xz$$

Solution

- $\log_4 x = \frac{\log_2 x}{\log_2 4} = \frac{a}{2}$
- $\log_y x = \frac{\log_2 x}{\log_2 y} = \frac{a}{b}$
- $\log_x yz = \frac{\log_2 yz}{\log_2 x} = \frac{\log_2 y + \log_2 z}{\log_2 x} = \frac{b+c}{a}$
- $\log_y 2xz = \frac{\log_2 2xz}{\log_2 y} = \frac{1+a+c}{b}$

EXAMPLE 8

Suppose $\ln 2=\mathbf{m}$, $\ln 5=\mathbf{n}$. Express the following in terms of **m, n**.

$$\log_5 e, \quad \log_4 5^3$$

Solution

- $\log_5 e = \dfrac{\ln e}{\ln 5} = \dfrac{\mathbf{1}}{\mathbf{n}}$
- $\log_4 5 = \dfrac{\ln 5}{\ln 4} = \dfrac{\ln 5}{\ln 2^2} = \dfrac{\mathbf{n}}{\mathbf{2m}}$

We may need the change base formula for an equation as well.

EXAMPLE 9

Solve the equation

$$\log_4(x+12) = 1 + \frac{1}{2}\log_2 x$$

Solution

We have to change base 4 to 2.

$$\frac{\log_2(x+12)}{\log_2 4} = 1 + \frac{1}{2}\log_2 x$$

$$\Leftrightarrow \frac{\log_2(x+12)}{2} = 1 + \frac{1}{2}\log_2 x$$

$$\Leftrightarrow \log_2(x+12) = 2 + \log_2 x$$

$$\Leftrightarrow \log_2(x+12) - \log_2 x = 2$$

$$\Leftrightarrow \log_2 \frac{x+12}{x} = 2$$

$$\Leftrightarrow \frac{x+12}{x} = 4$$

$$\Leftrightarrow x+12 = 4x$$

$$\Leftrightarrow 3x = 12$$

$$\Leftrightarrow x = 4$$

2.10 EXPONENTIAL EQUATIONS

In these equations the unknown x is in the exponent. The simplest exponential equation has the form

$$a^x = b \qquad (*)$$

The definition of the logarithm gives the solution: $x=\log_a b$.

Another technique is to apply on both sides of **(*)** the same logarithm:

apply $\log_a$	$a^x = b \Rightarrow \log_a a^x = \log_a b \Rightarrow x = \log_a b$ (in fact, the definition)
apply log	$a^x = b \Rightarrow \log a^x = \log b \Rightarrow x\log a = \log b \Rightarrow x = \dfrac{\log b}{\log a}$
apply ln	$a^x = b \Rightarrow \ln a^x = \ln b \Rightarrow x\ln a = \ln b \Rightarrow x = \dfrac{\ln b}{\ln a}$
apply $\log_c$	$a^x = b \Rightarrow \log_c a^x = \log_c b \Rightarrow x\log_c a = \log_c b \Rightarrow x = \dfrac{\log_c b}{\log_c a}$

<u>Notice</u>: the first row gives the result of the definition; the other rows give alternative expressions.
In fact, the results justify the change of base formula!

Let us keep in mind the following two expressions for the solution:

$a^x = b$	$x = \log_a b$
	$x = \dfrac{\ln b}{\ln a}$

<u>EXAMPLE 1</u>

Solve the equation $2(5^x) = 9$.

<u>Solution</u>

$$2(5^x) = 9 \Leftrightarrow 5^x = 4.5 \Leftrightarrow \boxed{x = \log_5 4.5} \text{ or otherwise } \boxed{x = \frac{\ln 4.5}{\ln 5}}$$

For exponential equations of base e, we anyway use ln( ).

EXAMPLE 2

Solve the equation $10e^{2x} = 85$

Solution

We first divide by 10:

$$10e^{2x} = 85 \Leftrightarrow e^{2x} = 8.5 \Leftrightarrow \ln e^{2x} = \ln 8.5 \Leftrightarrow 2x = \ln 8.5 \Leftrightarrow \boxed{x = \frac{\ln 8.5}{2}}$$

EXAMPLE 3

Solve the equation $5^x = 2^{x+1}$. Express the result in the form $\frac{\ln a}{\ln b}$.

Solution

Method A: Let us apply ln on both sides

$$\begin{aligned} 5^x = 2^{x+1} &\Leftrightarrow \ln 5^x = \ln 2^{x+1} \\ &\Leftrightarrow x\ln 5 = (x+1)\ln 2 \\ &\Leftrightarrow x\ln 5 = x\ln 2 + \ln 2 \\ &\Leftrightarrow x\ln 5 - x\ln 2 = \ln 2 \\ &\Leftrightarrow x(\ln 5 - \ln 2) = \ln 2 \\ &\Leftrightarrow x = \frac{\ln 2}{\ln 5 - \ln 2} \Leftrightarrow \boxed{x = \frac{\ln 2}{\ln \frac{5}{2}}} \end{aligned}$$

Method B: Simplify the equation to the form $a^x=b$; then apply ln

$$\begin{aligned} 5^x = 2^{x+1} &\Leftrightarrow 5^x = 2^x 2 \\ &\Leftrightarrow \frac{5^x}{2^x} = 2 \\ &\Leftrightarrow \left(\frac{5}{2}\right)^x = 2 \\ &\Leftrightarrow \boxed{x = \frac{\ln 2}{\ln \frac{5}{2}}} \end{aligned}$$

Remarks

- This is the exact answer. If we are looking for an answer to 3sf, the calculator gives x=0.756.
- We can use any logarithm instead of ln( ), for example log( ).
- If an expression in the form $\log_a b$ is required, the answer is

$$\left(\frac{5}{2}\right)^x = 2 \Leftrightarrow x = \log_{\frac{5}{2}} 2$$

♦ EXPONENTIAL MODELLING (GROWTH OR DECAY)

In many applications a quantity increases or decreases exponentially according to time.

Suppose that a population **P** at time **t** (years after a certain time) is given by the formula

$$\boxed{P=P_o e^{kt}}$$

- If k>0, the population increases (e.g. $P=1000e^{0.2t}$)
- If k<0, the population decreases (e.g. $P=1000e^{-0..2t}$)

Question 1: What is the **initial** population (at starting time)?

Initial means t=0. Since $e^0=1$

$$P=P_o$$

Thus, the coefficient P_o is always the initial value of P.

Let us consider the case where the initial population is 1000 and P is given by

$$P=1000e^{0.2t}$$

Question 2: What is the population after 3 years?

For t=3

$$P=1000e^{0.2\times3}=1822$$

Question 3: The population after t years is 2500. Find t.

$$1000e^{0.2t}=2500 \Leftrightarrow e^{0.2t}=2.5$$
$$\Leftrightarrow \ln e^{0.2t} = \ln 2.5$$
$$\Leftrightarrow 0.2t = \ln 2.5$$
$$\Leftrightarrow t = \frac{\ln 2.5}{0.2} = 4.58 \text{ years}$$

(OR directly by GDC, $t = 4.58$ years)

Question 4: The population doubles after t years. Find t.

It's the same as in Question 3. We set $P=2000$, or in general $P=2P_o$

Sometimes the constant k is not known. We are given some information to find its value. Suppose

$$P=P_oe^{kt}$$

Question 5: Given that the population doubles every 4 years, find k.

For $t=4$, $P=2P_o$,

$$P_oe^{k4}=2P_o \Leftrightarrow e^{4k}=2$$
$$\Leftrightarrow 4k = \ln 2$$
$$\Leftrightarrow k = \frac{\ln 2}{4} = 0.173$$

Let us see an example of decay.

EXAMPLE 4

The mass m of a radio-active substance at time t hours is given by

$$\mathbf{m = 4e^{-kt}}$$

(a) The mass is 1 kg after 5 hours. Find k.
(b) What is the mass after 3 hours?
(c) The mass reduces to a half after t hours. Find t.

Solution

(a) For $t=5$, $m=1$, thus

$$4e^{-k5}=1$$

$$\Leftrightarrow e^{-5k} = \frac{1}{4} \Leftrightarrow e^{5k} = 4 \Leftrightarrow 5k=\ln 4 \Leftrightarrow \mathbf{k} = \frac{\ln 4}{5} \ (\cong \mathbf{0.277})$$

Therefore,

$$\mathbf{m = 4e^{-0.277t}}$$

(b) For t=3,

$$m = 4e^{-0.277 \times 3} = \mathbf{1.74}$$

(c) For m=2,

$$4e^{-0.277t} = 2$$

$$\Leftrightarrow e^{-0.277t} = 0.5 \;\Leftrightarrow\; -0.277t = \ln 0.5 \Leftrightarrow t = \frac{\ln 0.5}{-0.277} = \mathbf{2.50\ hours}$$

This time (the quantity reduces to a half) is known as **half-life time.**

♦ MORE EXPONENTIAL EQUATIONS **(mainly for HL)**

Let us look at some additional examples

<u>EXAMPLE 5</u>

Solve the equation

$$6^x 7^{x-1} = 3^{x-2}$$

Express the result in the form $\frac{\ln a}{\ln b}$

<u>Solution</u>

Although we can apply ln( ) on both sides and obtain

$$x\ln 6 + (x-1)\ln 7 = (x-2)\ln 3$$

which is a linear equation and can be solved as usual, I will recommend the quicker method: to simplify first the equation to the form $a^x=b$;

$$6^x 7^{x-1} = 3^{x-2} \Leftrightarrow \frac{6^x 7^x}{7} = \frac{3^x}{3^2}$$

$$\Leftrightarrow \frac{6^x 7^x}{3^x} = \frac{7}{3^2}$$

$$\Leftrightarrow 14^x = \frac{7}{9}$$

$$\text{(now apply ln)} \quad \Leftrightarrow \boxed{x = \frac{\ln(7/9)}{\ln 14}}$$

Notice:

Mind the following (common mistake)

$$A \pm B = C \quad \textbf{does not imply} \quad \ln A \pm \ln B = \ln C$$

$$\textbf{it only implies} \quad \ln(A \pm B) = \ln C$$

If an equation that ivolves a sum of exponentials, it doesn't help to apply a logarithm, as $\log(a^x \pm b^x)$ cannot be simplified.

For example, the equation: $2^x + 3^x = 7$ cannot be solved analytically. Only by using GDC-SolveN, we can obtain the solution x=1.356…

If you see such an equation in paper 1, it is highly likely to be a hidden quadratic!

EXAMPLE 6

Solve the equations:

(a) $6e^x + \frac{12}{e^x} = 17$ (b) $6(10^{2x})+12=17(10^x)$

Solution

(a) Let $y=e^x$. Then

$$6y + \frac{12}{y} = 17 \Leftrightarrow 6y^2-17y+12=0$$

There are two solutions: $y=\frac{3}{2}$, $y=\frac{4}{3}$

- For $y=\frac{3}{2}$, $e^x = \frac{3}{2} \Leftrightarrow \boxed{x=\ln\frac{3}{2}}$
- For $y=\frac{4}{3}$, $e^x = \frac{4}{3} \Leftrightarrow \boxed{x=\ln\frac{4}{3}}$

(b) Let $y=10^x$. Then

$$6y^2-17y+12=0$$

There are two solutions: $y=\frac{3}{2}$, $y=\frac{4}{3}$

- For $y=\frac{3}{2}$, $10^x = \frac{3}{2} \Leftrightarrow \boxed{x=\log\frac{3}{2}}$
- For $y=\frac{4}{3}$, $10^x = \frac{4}{3} \Leftrightarrow \boxed{x=\log\frac{4}{3}}$

EXAMPLE 7

Solve the system of equations

$$2(3^x)-3(2^y)=-22 \quad \text{and} \quad 5(3^x)+\frac{1}{2}(2^y)=9$$

Solution

Let $A=3^x$ and $B=2^y$. Then

$$2A-3B=-22 \quad \text{and} \quad 5A+\frac{1}{2}B=9$$

The solution is A=1, B=8. Hence,

$$3^x=1 \Leftrightarrow x=\log_3 1 \Leftrightarrow \boxed{x=0} \quad \text{and} \quad 2^y=8 \Leftrightarrow y=\log_2 8 \Leftrightarrow \boxed{y=3}$$

ONLY FOR

HL

2.11 POLYNOMIAL FUNCTIONS (for HL)

♦ DEFINITION

A **polynomial function**, or simply a **polynomial** is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, all $a_i \in R$ and $n \in N$.

The highest power of x is called **degree** of the polynomial. We write

$$\deg f(x) = n$$

For example

$f(x) = 5x^4+3x^2-7x+2$	$\deg f(x)=4$
$g(x) = x^5-2x^3+5x-7$	$\deg g(x)=5$

We also use the following terminology for polynomials of a particular degree:

$\deg f(x) = 0$	$f(x)=a$	(constant function)
$\deg f(x) = 1$	$f(x)=ax+b$	(linear function)
$\deg f(x) = 2$	$f(x)=ax^2+bx+c$	(quadratic function)
$\deg f(x) = 3$	$f(x)=ax^3+bx^2+cx+d$	(cubic function)
$\deg f(x) = 4$	$f(x)=ax^4+bx^3+cx^2+dx+e$	(quartic function)

Notice though that the degree of the zero polynomial $f(x)=0$ is undefined[‡]

[‡] In some books the degree of the zero polynomial is defined to be -1 or $-\infty$.

♦ ADDITION AND MULTIPLICATION OF POLYNOMIALS

When we add or multiply polynomials the result is also a polynomial. We perform these operations in the obvious way!.

EXAMPLE 1

Let $f(x) = 3x^2-2x+5$ and $g(x)= 2x^3-7x+1$

Then

$$f(x)+g(x) = (3x^2-2x+5)+(2x^3-7x+1) = 2x^3+3x^2-9x+6$$

$$f(x)g(x) = (3x^2-2x+5)(2x^3-7x+1)$$

$$= 6x^5-21x^3+3x^2-4x^4+14x^2-2x+10x^3-35x+5$$

$$= 6x^5-4x^4-11x^3+17x^2-37x+5$$

Here, $\deg f(x)=2$, $\deg g(x)=3$ while

$$\deg[f(x)+g(x)]=3 \qquad \deg[f(x)g(x)]=5$$

In general

If $\deg f(x)=n$, $\operatorname{def} g(x)=m$ with $n>m$ (i.e max degree = n)

$$\deg[f(x)+g(x)] = n \qquad \deg[f(x)g(x)] = n+m$$

If $\deg f(x)=n$, $\operatorname{def} g(x)=n$ (equal degrees)

$$\deg[f(x)+g(x)] \le n \qquad \deg[f(x)g(x)] = 2n$$

NOTICE

Look at the last line: it is not $\deg[f(x)+g(x)] = n$ since $f(x)$ and $g(x)$ may have opposite leading coefficients; for example

$$f(x)=3x^2+7x, \qquad g(x)= -3x^2+2 \qquad (n=2)$$

Then

$$f(x)+g(x)= (3x^2+7x)+(-3x^2+2)=-7x+2 \qquad \deg=1<2$$

$$f(x)g(x)= (3x^2+7x)(-3x^2+2)=-9x^4-21x^3+6x^2+14x \qquad \deg=4$$

♦ DIVISION OF POLYNOMIALS

Since $(2x)(3x+1)=6x^2+2x$, we can derive that

$$\frac{6x^2+2x}{2x}=3x+1$$

But how can we divide polynomials in general?

REMEMBER When we divide two integers, say $a:b$ or $\frac{a}{b}$, we obtain

$$a=bq+r$$

where q=quotient and r=remainder ($0\le r<b$)

For example 23:5 gives quotient=4 and remainder=2, so

$$23=5\cdot 4+2$$

The same applies for polynomials

If we divide two polynomials, $f(x)$ by $g(x)$, we obtain two polynomials

the **quotient** $q(x)$

the **remainder** $r(x)$

such that

$$f(x)=g(x)q(x)+r(x)$$

where

$r(x)=0$ or $\deg r(x) < \deg g(x)$

Let us describe the process of the so-called **long division** by using an example.

EXAMPLE 2

We will divide $f(x)=2x^3-4x^2+5x-1$ by $g(x)= x^2+3x+1$

[As the way of dividing varies in different countries we present two methods: the left to the right and the right to the left division]

left to the right method	instructions	right to the left method
$\begin{array}{l\|l} 2x^3-4x^2+5x-1 & x^2+3x+1 \\ \cline{2-2} & \end{array}$	**step 1**	$x^2+3x+1\overline{)2x^3-4x^2+5x-1}$
$\begin{array}{l\|l} 2x^3-4x^2+5x-1 & x^2+3x+1 \\ \cline{2-2} & 2x \end{array}$	**step 2** divide $2x^3:x^2 = 2x$	$\begin{array}{r} 2x \\ x^2+3x+1\overline{)2x^3-4x^2+5x-1} \end{array}$
$\begin{array}{l\|l} 2x^3-4x^2+5x-1 & x^2+3x+1 \\ \cline{2-2} 2x^3+6x^2+2x & 2x \end{array}$	**step 3** multiply $2x$ by $g(x)$	$\begin{array}{r} 2x \\ x^2+3x+1\overline{)2x^3-4x^2+5x-1} \\ 2x^3+6x^2+2x \end{array}$
$\begin{array}{l\|l} 2x^3-4x^2+5x-1 & x^2+3x+1 \\ \cline{2-2} -\ 2x^3+6x^2+2x & 2x \\ \cline{1-1} -10x^2+3x-1 & \end{array}$	**step 4** subtract	$\begin{array}{r} 2x \\ x^2+3x+1\overline{)2x^3-4x^2+5x-1} \\ -\ 2x^3+6x^2+2x \\ \hline -10x^2+3x-1 \end{array}$

repeat with $-10x^2+3x-1$ and x^2+3x+1

$\begin{array}{l\|l} 2x^3-4x^2+5x-1 & x^2+3x+1 \\ \cline{2-2} -\ 2x^3+6x^2+2x & 2x-10 \\ \cline{1-1} -10x^2+3x-1 & \end{array}$	**step 5** divide $-10x^2:x^2=-10$	$\begin{array}{r} 2x-10 \\ x^2+3x+1\overline{)2x^3-4x^2+5x-1} \\ -\ 2x^3+6x^2+2x \\ \hline -10x^2+3x-1 \end{array}$
$\begin{array}{l\|l} 2x^3-4x^2+5x-1 & x^2+3x+1 \\ \cline{2-2} -\ 2x^3+6x^2+2x & 2x-10 \\ \cline{1-1} -10x^2+3x-1 & \\ -10x^2-30x-10 & \end{array}$	**step 6** multiply -10 by $g(x)$	$\begin{array}{r} 2x-10 \\ x^2+3x+1\overline{)2x^3-4x^2+5x-1} \\ -\ 2x^3+6x^2+2x \\ \hline -10x^2+3x-1 \\ -10x^2-30x-10 \end{array}$
$\begin{array}{l\|l} 2x^3-4x^2+5x-1 & x^2+3x+1 \\ \cline{2-2} -\ 2x^3+6x^2+2x & 2x-10 \\ \cline{1-1} -10x^2+3x-1 & \\ -\ -10x^2-30x-10 & \\ \cline{1-1} 33x+9 & \end{array}$	**step 7** subtract	$\begin{array}{r} 2x-10 \\ x^2+3x+1\overline{)2x^3-4x^2+5x-1} \\ -\ 2x^3+6x^2+2x \\ \hline -10x^2+3x-1 \\ -\ -10x^2-30x-10 \\ \hline 33x+9 \end{array}$

Hence, $q(x)=2x-10$, $r(x)=33x+9$ and

$$2x^3-4x^2+5x-1 = (x^2+3x+1)(2x-10) + (33x+9)$$

NOTICE

In number theory, the division a=bq+r also gives $\frac{a}{b}=q+\frac{r}{b}$

For example $\frac{23}{5}=4+\frac{3}{5}$.

Similarly, the division of polynomials gives

$$\frac{f(x)}{g(x)}=q(x)+\frac{r(x)}{g(x)}$$

In our example

$$\frac{2x^3-4x^2+5x-1}{x^2+3x+1} = 2x-10 + \frac{33x+9}{x^2+3x+1}$$

If r(x)=0, then f(x)=g(x)q(x). Then we say that

f(x) is divisible by g(x)
or g(x) divides exactly f(x)
or g(x) is a factor of f(x)

EXAMPLE 3

Let us divide $f(x)=2x^3+2x^2-x-1$ by $g(x)=2x^2-1$

We present the long division in one step:

left to the right method		right to the left method
$\begin{array}{r\|l} 2x^3+2x^2-x-1 & 2x^2-1 \\ \cline{2-2} -\ 2x^3 \qquad -x & x+1 \\ \cline{1-1} 2x^2 \quad -1 & \\ -\ 2x^2 \quad -1 & \\ \cline{1-1} 0 & \end{array}$	notice that the remainder r(x) is 0	$\begin{array}{r} x+1 \\ 2x^2-1 \overline{)\,2x^3+2x^2-x-1} \\ -\ 2x^3 \qquad -x \\ \hline 2x^2 \quad -1 \\ -\ 2x^2 \quad -1 \\ \hline 0 \end{array}$

Therefore,

$$2x^3+2x^2-x-1 = (2x^2-1)(x+1)$$

or otherwise

$$\frac{2x^3+2x^2-x-1}{2x^2-1} = x+1$$

♦ THE FACTOR THEOREM

f(x) is divisible by $(x-\mathbf{a})$ $\Leftrightarrow$ $f(\mathbf{a}) = 0$

or otherwise

$(x-\mathbf{a})$ is a factor of f(x) $\Leftrightarrow$ **a** is a root of f(x)

Proof

($\Rightarrow$) If f(x) is divisible by $(x-\mathbf{a})$ then $f(x)=(x-\mathbf{a})q(x)$ for some q(x)

then $f(\mathbf{a})=0$

($\Leftarrow$) Let $f(\mathbf{a})=0$. We divide f(x) by $(x-\mathbf{a})$ and obtain

$$f(x)=(x-\mathbf{a})q(x)+r \qquad (r \text{ must be constant})$$

But then, $f(\mathbf{a})=0 \Rightarrow r=0$. That is

$$f(x)=(x-\mathbf{a})q(x)$$

ie f(x) is divisible by $(x-\mathbf{a})$

♦ THE REMAINDER THEOREM

When f(x) is divided be $(x-\mathbf{a})$ the remainder is $f(\mathbf{a})$

Proof

We divide f(x) by $(x-\mathbf{a})$. Suppose $f(x)=(x-\mathbf{a})q(x)+r$. Then $f(\mathbf{a}) = r$

EXAMPLE 4

Let $f(x)=x^3+x^2-x+2$. Find the remainder when f(x) is divided by

$$(x-1),\ (x+1),\ (x-2),\ (x+2)$$

$f(1)=3$, hence the remainder when f(x) is divided by $(x-1)$ is 3

$f(-1)=3$, hence the remainder when f(x) is divided by $(x+1)$ is 3

$f(2)=12$, hence the remainder when f(x) is divided by $(x-2)$ is 12

$f(-2)=0$, hence f(x) is divisible by $(x+2)$, ie $(x+2)$ is a factor of f(x)

EXAMPLE 5 (the factor theorem for quadratics)

Let $f(x)=ax^2+bx+c$ be a quadratic with two roots p and q, that is

$$f(p)=0 \text{ and } f(q)=0$$

Then $f(x)$ is divisible by $(x-p)$ and $(x-q)$. Indeed, we know that

$$f(x) = a(x-p)(x-q)$$

EXAMPLE 6

Solve the equation $x^3+x^2-x+2=0$.

If we know one root then we may use division to find the remaining roots.

In Example 4, we saw that -2 is a root. Hence $(x+2)$ is a factor.

We divide x^3+x^2-x+2 by $(x+2)$ and get $q(x)= x^2-x+1$ (left as exercise)

The equation takes the form

$$(x+2)(x^2-x+1)=0$$

However, the quadratic x^2-x+1 has no real roots, so our equation has only one root, ie $x=-2$.

EXAMPLE 7

Let $f(x) = x^3-6x^2+11x-6$. Solve the equation $f(x) = 0$.

Solution

We can easily observe that $x=1$ is a solution since $f(1)=0$.

We divide $f(x)$ by the factor $(x-1)$ and find the quotient (x^2-5x+6).

(it is left as exercise!)

The equation takes the form

$$(x-1)(x^2-5x+6)=0$$

But the quadratic (x^2-5x+6) has two roots, $x=2$ and $x=3$. Thus the equation has three solutions, namely 1,2 and 3.

Notice also that the full factorization of the cubic equation gives

$$x^3-6x^2+11x-6=0$$

$$\Leftrightarrow (x-1)(x-2)(x-3)=0$$

$$\Leftrightarrow x=1 \quad or\ x=2 \quad or\ x=3$$

REMARK (useful for guessing roots)

Consider the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad \text{where } a_i \in Z$$

We may look for roots among the following

Potential integer roots: $\pm$ factors of a_0

Potential rational roots: $\pm \dfrac{\text{factor of } a_0}{\text{factor of } a_n}$

EXAMPLE 8

Let $f(x) = 2x^3-7x^2-17x+10$.

Potential integer roots: $\pm1, \pm2, \pm5, \pm10$

Potential rational roots: $\pm\dfrac{1}{2}, \pm\dfrac{5}{2}$ ($\pm\dfrac{2}{2}$ and $\pm\dfrac{10}{2}$ are integers)

Among those, we can verify that

$$f(-2)=0, \quad f(5)=0, \quad f(1/2)=0.$$

We could also find the first root, say $x=-2$, and then divide $f(x)$ by the factor $(x+2)$ to obtain the remaining quadratic factor.

Indeed, the long division will give

$$2x^3-7x^2-17x+10 = (x+2)(2x^2-11x+5)$$

and the quadratic factor has two roots, $x=5$ and $x=1/2$.

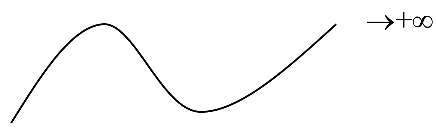
♦ THE GRAPH OF A CUBIC FUNCTION

Consider a cubic function

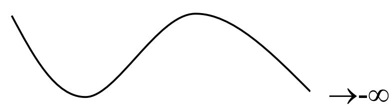
$$f(x) = ax^3+bx^2+cx+d$$

The leading coefficient a determines the behavior of the graph towards the right end:

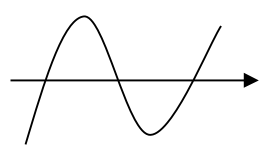
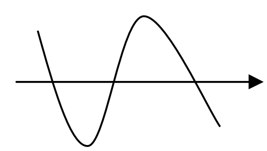
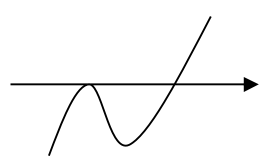
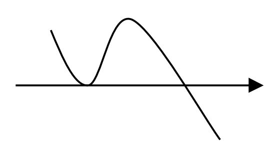
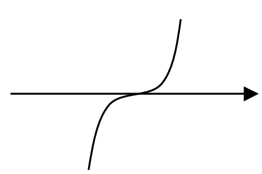
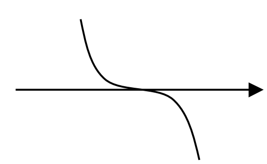
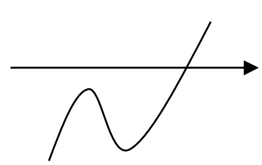
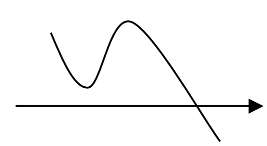
- If $a>0$, for large values of x, $f(x)\to+\infty$ and the graph looks like



- If $a<0$, for large values of x, $f(x)\to-\infty$ and the graph looks like



The factorization of the cubic function determines the position of the graph in relation to the x-axis:

$f(x)$	$a>0$	$a<0$
$a(x-r_1)(x-r_2)(x-r_3)$		
$a(x-r_1)^2(x-r_2)$		
$a(x-r_1)^3$		
$a(x-r_1)(x^2-px+qx)$ *irreducible*		

2.12 SUM AND PRODUCT OF ROOTS (for HL)

The fundamental theorem of algebra states that a polynomial of degree n has n roots (real or complex[§]). Here, we denote by

$S = r_1+r_2+\cdots+r_n$ the sum of the roots

$P = r_1r_2\cdots r_n$ the product of the roots

♦ QUADRATIC FUNCTIONS

We have seen that for a quadratic function

$$f(x) = ax^2+bx+c \qquad (1)$$

there are always two roots r_1 and r_2 (real or complex).

We may have

- r_1,r_2 real, $r_1 \neq r_2$
- r_1,r_2 real, $r_1 = r_2$
- r_1,r_2 conjugate complex roots

In any case, the factorization over C is

$$f(x) = a(x-r_1)(x-r_2)$$

Thus

$$f(x) = a(x^2-r_1x-r_2x+r_1r_2)$$
$$= ax^2-a(r_1+r_2)x+ar_1r_2x \qquad (2)$$

By comparing (1) and (2) we obtain

$$b=-a(r_1+r_2) \qquad \text{and} \qquad c= ar_1r_2$$

and finally

$$\boxed{S= r_1+r_2=-\frac{b}{a}} \qquad \boxed{P = r_1r_2=\frac{c}{a}}$$

These relations are known as **Vieta formulae.**

[§] Don't worry if you haven't seen complex numbers yet. Just accept that there are n roots!

- CUBIC FUNCTIONS

Consider now the cubic function

$$f(x) = ax^3+bx^2+cx+d \qquad (1)$$

According to the fundamental theorem of algebra the factorization of f(x) over C is

$$f(x) = a(x-r_1)(x-r_2)(x-r_3)$$

The constant term is

$$-ar_1r_2r_3$$

Thus, by (1)

$$d= -ar_1r_2r_3 \Rightarrow r_1r_2r_3 = -\frac{d}{a}$$

The coefficient of x^2 is

$$-ar_3-ar_2-ar_1 = -a(r_1+r_2+r_3)$$

Thus, by (1)

$$b= -a(r_1+r_2+r_3) \Rightarrow r_1+r_2+r_3 = -\frac{b}{a}$$

Hence,

$$\boxed{S= r_1+r_2+r_3=-\frac{b}{a}} \qquad \boxed{P = r_1r_2r_3=-\frac{d}{a}}$$

Notice

Usually a cubic function is expressed in the form

$$f(x) = a_3x^3+a_2x^2+a_1x+a_0$$

The Vieta formulae take the form

$$\boxed{S= r_1+r_2+r_3=-\frac{a_2}{a_3}} \qquad \boxed{P = r_1r_2r_3=-\frac{a_0}{a_3}}$$

♦ THE GENERAL CASE

Consider the general form of a polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \qquad (1)$$

According to the fundamental theorem of algebra the factorization of f(x) over C is

$$f(x) = a_n(x-r_1)(x-r_2)\dots(x-r_n)$$

The constant term is

$$(-1)^n a_n r_1 r_2 \cdots r_n$$

Thus, by (1)

$$a_0 = (-1)^n a_n r_1 r_2 \cdots r_n \Rightarrow r_1 r_2 \cdots r_n = (-1)^n \frac{a_0}{a_n}$$

The coefficient of x^{n-1} is

$$-a_n r_1 - a_n r_2 - \cdots - a_n r_n = -a_n(r_1 + r_2 + \cdots + r_n)$$

Thus, by (1)

$$a_{n-1} = -a_n(r_1 + r_2 + \cdots + r_n) \Rightarrow r_1 + r_2 + \cdots + r_n = -\frac{a_{n-1}}{a_n}$$

Hence,

$$\boxed{S = r_1 + r_2 + \cdots + r_n = -\frac{a_{n-1}}{a_n}} \qquad \boxed{P = r_1 r_2 \cdots r_n = (-1)^n \frac{a_0}{a_n}}$$

NOTICE (just for information!)

By considering the coefficients of x^{n-2}, x^{n-3} etc we similarly obtain

The sum S_2 of all possible pairs $r_i r_j$ is $\frac{a_{n-2}}{a_n}$

The sum S_3 of all possible triples $r_i r_j r_k$ is $-\frac{a_{n-3}}{a_n}$

and so on.

EXAMPLE 1

Let $f(x)=2x^3+ax^2+bx+c$

The sum of the roots is 3.5, the product of the roots is -5 and the polynomial is divided by (x+2). Find the values of a,b and c.

Solution

$$S=-\frac{a_2}{a_3} \Rightarrow -\frac{a}{2}=3.5 \Rightarrow a = -7$$

$$P=(-1)^3\frac{a_0}{a_3} \Rightarrow -\frac{c}{2}=-5 \Rightarrow c = 10$$

By the factor theorem

$$f(-2)=0 \quad \Rightarrow -16+4a-2b+c = 0$$
$$\Rightarrow -16-28-2b+10 = 0$$
$$\Rightarrow b=-17$$

EXAMPLE 2

Let $f(x)=ax^4-10x^3+bx+c$

The sum of the roots is 2, the product of the roots is -5. and the polynomial is divided by (x-1). Find the values of a,b and c.

Solution

$$S=-\frac{a_3}{a_4} \Rightarrow \frac{10}{a}=2 \Rightarrow a = 5$$

$$P=(-1)^4\frac{a_0}{a_4} \Rightarrow \frac{c}{a}=-5 \Rightarrow c = -25$$

By the factor theorem

$$f(1)=0 \quad \Rightarrow a-10+b+c = 0$$
$$\Rightarrow 5-10+b-25 = 0$$
$$\Rightarrow b=30$$

2.13 RATIONAL FUNCTIONS – PARTIAL FRACTIONS (for HL)

♦ RATIONAL FUNCTIONS

A **rational** function has the form

$$\boxed{f(x)=\frac{p(x)}{q(x)}}$$

where p(x) and q(x) are polynomials.

For example

$$f(x)=\frac{2x-5}{x^2-4x+3}$$

We have already seen rational functions of the form

$$f(x)=\frac{Ax+B}{Cx+D}$$

and their asymptotes.

Again, for the asymptotes of a rational function in general, we work as follows:

1) Vertical Asymptotes: **x=a**

At points where the function is not defined. Thus, we solve the equation q(x)=0.
For example,

$$f(x)=\frac{2x-5}{x^2-4x+3}$$

we solve

$$x^2-4x+3=0 \Leftrightarrow x=1 \text{ or } x=3$$

Hence

$$\boxed{\text{The lines } x=1 \text{ and } x=3 \text{ are vertical asymptotes}}$$

2) Horizontal Asymptotes: y=b

We only consider the leading coefficients of p(x) and q(x).

We distinguish three cases:

- $deg p(x) = deg q(x)$, $y = \dfrac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}$
- $deg p(x) < deg q(x)$, $y=0$
- $deg p(x) > deg q(x)$, there is no horizontal asymptote

For example,

$f(x)=\dfrac{4x^2-3x+1}{2x^2+7x-6}$ The line $y=2$ is a horizontal asymptote

$f(x)=\dfrac{3x+1}{2x^2+7x-6}$ The line $y=0$ is a horizontal asymptote

$f(x)=\dfrac{4x^2-3x+1}{2x-6}$ There is no horizontal asymptote

Notice also that,

if $f(x)=\dfrac{p(x)}{q(x)}$ has a horizontal asymptote $y=b$,

then $g(x)=\dfrac{p(x)}{q(x)}+c$ has a horizontal asymptote $y=b+c$

as $g(x)$ is the function $f(x)$ shifted c units up.

EXAMPLE 1

Function	Vertical Asymptotes (denominator = 0)	Horizontal Asymptote (divide leading coefficients)
$f(x)=\dfrac{7x^2-5x+1}{x^2-3x+2}$	$x=1$, $x=2$	$y=7$
$f(x)=\dfrac{7x^2-5x+1}{2x^2-6x+4}$	$x=1$, $x=2$	$y=7/2$
$f(x)=\dfrac{-5x+1}{x^2-3x+2}$	$x=1$, $x=2$	$y=0$
$f(x)=\dfrac{-5x+1}{x^2-3x+2}+8$	$x=1$, $x=2$	$y=8$
$f(x)=\dfrac{7x^2-5x+1}{-3x+6}$	$x=2$	none

EXAMPLE 2

Find the intercepts the domain and the asymptotes of the function

$$f(x)=\frac{x^2-6x+8}{x^2-4x+3}$$

Use your GDC to sketch the graph of f(x) and hence find its range.

Solution

It is

$$f(x)=\frac{x^2-6x+8}{x^2-4x+3}=\frac{(x-2)(x-4)}{(x-1)(x-3)}$$

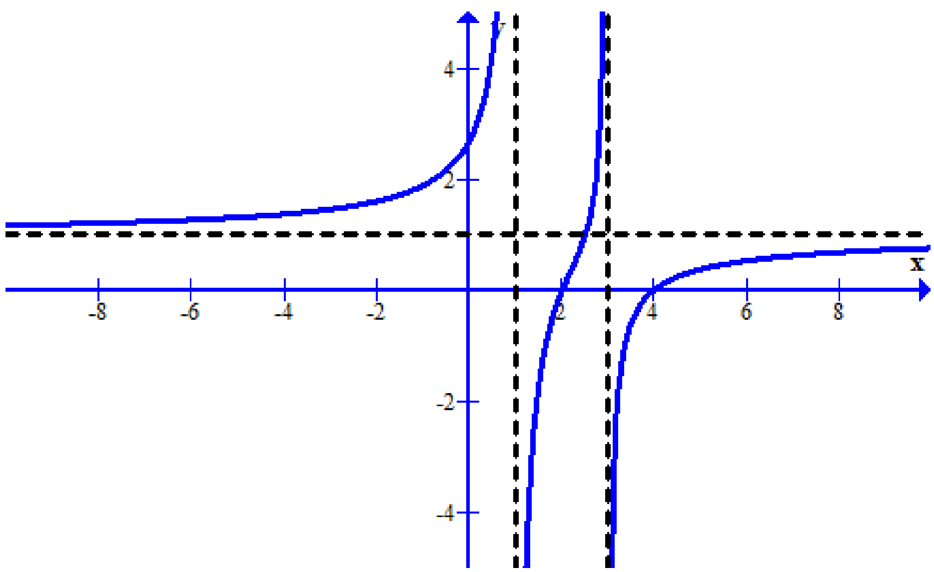
x-intercepts (or roots): x=2, x=4

y-intercept: For x=0, y=8/3

Domain: x≠1, x≠3

VA: x=1, x=3

HA: y=1



According to the graph the **range** is $y\in R$

Notice: that the value of the asymptote y=1 is not excluded from the range.

EXAMPLE 3

Find the intercepts the domain and the asymptotes of the function

$$f(x)=\frac{x^2-3x-4}{x^2-4x+3}$$

Use your GDC to sketch the graph of f(x) and hence find its range.

Solution

It is

$$f(x)=\frac{x^2-3x-4}{x^2-4x+3}=\frac{(x+1)(x-4)}{(x-1)(x-3)}$$

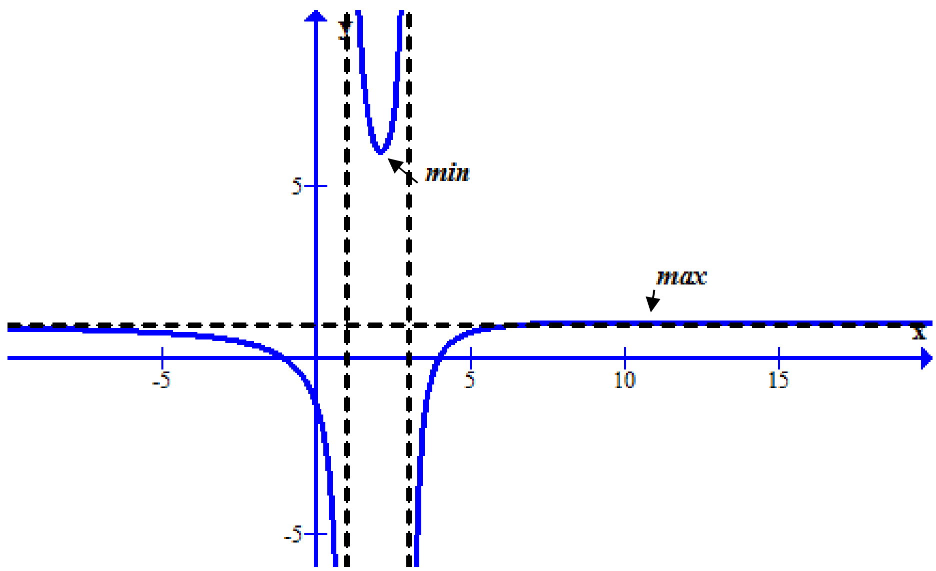
x-intercepts (or roots): $x=-1$, $x=4$

y-intercept: For $x=0$, $y=-4/3$

Domain: $x\neq1$, $x\neq3$

VA: $x=1$, $x=3$

HA: $y=1$



By using the GDC, we find that:

there is a **local min** at (2.1,5.95) and a **local max** at (11.9,1.05)

According to the graph the **range** is $y\in\,]-\infty,1.05]\cup[5.95,+\infty[$

Later on we will be able to find the local min and the local max without a GDC, by using derivatives!

♦ OBLIQUE ASYMPTOTES

We have seen that for a rational function of the form

$$f(x)=\frac{ax^2+bx+c}{dx+e}$$

there is no horizontal asymptote. But there is an **oblique asymptote.** If the division of the two polynomials gives the quotient $q(x)=Ax+B$ and the remainder r, then

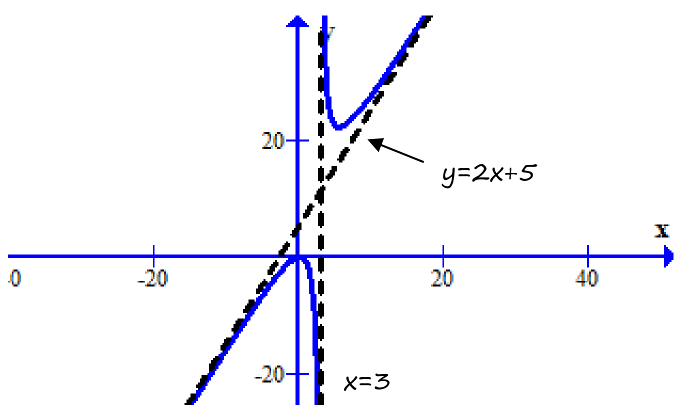
$$f(x)=\frac{ax^2+bx+c}{dx+e}=(Ax+B)+\frac{r}{dx+e}$$

As $x\to\pm\infty$, the last fraction tends to 0 and thus $f(x) \to Ax+B$.
That is the graph of y=f(x) approaches the oblique line **y=Ax+B**.

EXAMPLE 4

$$f(x)=\frac{4x^2-2x+1}{2x-6}$$

- The vertical asymptote is **x=3**.
- There is no horizontal asymptote.
- As $4x^2-2x+1$ divided by $2x-6$ gives q(x)=2x+5 (and r=31) the oblique asymptote is **y=2x+5**.



Justification: $f(x)=\dfrac{4x^2-2x+1}{2x-6}=2x+5+\dfrac{31}{2x-6}\to 2x+5$ as $x\to\pm\infty$

<u>**Notice**</u>. The same situation occurs for a rational function $f(x)=\dfrac{p(x)}{q(x)}$ where deg q(x) is **one less** than deg p(x).

♦ PARTIAL FRACTIONS (only the easiest case)

We only consider rational functions of the form

$$f(x)=\frac{a'}{ax^2+bx+c} \qquad \text{and} \qquad f(x)=\frac{a'x+b'}{ax^2+bx+c}$$

If the denominator ax^2+bx+c has two roots, say $x=r_1$ and $x=r_2$, we can express the functions in the form of **partial fractions**:

$$f(x)=\frac{A}{x-r_1}+\frac{B}{x-r_2}$$

We will demonstrate the way by using an example.

EXAMPLE 5

$$f(x)=\frac{3x-5}{x^2-4x+3}.$$

The denominator has two roots: x=1, x=3. Thus

$$f(x)=\frac{A}{x-1}+\frac{B}{x-3}$$

Method 1

$$\frac{A}{x-1}+\frac{B}{x-3}=\frac{A(x-3)+B(x-1)}{(x-1)(x-3)}=\frac{(A+B)-(3A+B)}{(x-1)(x-3)}$$

Comparing with the numerator of the original function

$$A+B = 3$$

$$3A+B = 5$$

The solution of the system gives **A=1** and **B=2**.

Method 2

$$\frac{3x-5}{x^2-4x+3}=\frac{A}{x-1}+\frac{B}{x-3}$$

Multiply by (x-1)(x-3):

$$A(x-3)+B(x-1) = 3x-5$$

For x=3 we obtain: $2B=4\Rightarrow \mathbf{B=2}$

For x=1 we obtain: $-2A=-2\Rightarrow \mathbf{A=1}$

Therefore,

$$f(x)=\frac{1}{x-1}+\frac{2}{x-3}$$

2.14 POLYNOMIAL AND RATIONAL INEQUALITIES (for HL)

Let f(x) be a polynomial. By factorizing f(x) we can easily sketch a graph and thus solve the **polynomial inequalities**

$f(x)>0 \qquad f(x)<0 \qquad f(x)\geq 0 \qquad f(x)\leq 0$

When we factorize f(x) we may find

- linear factors of the form $(x-a)$
- irreducible quadratic factors of the form (x^2+bx+c) [with $\Delta<0$]

Only the roots of the linear factors affect the inequality. We can sketch a graph of the polynomial, having in mind that

in a **single** root the graph **crosses** the x-axis

in a **double** root the graph just **touches** the x-axis

In general, for a root which is repeated **n times**

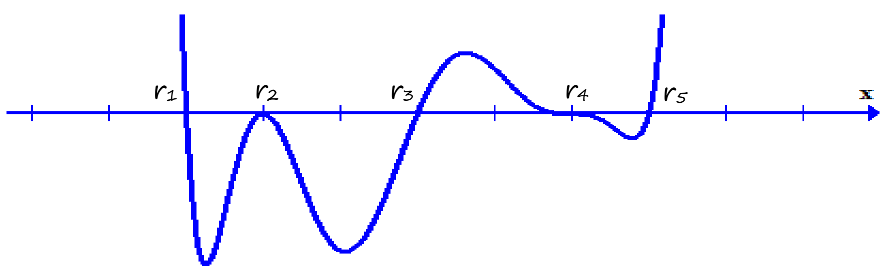
if n is **odd** it behaves as a single root (change of sign)

if n is **even** it behaves as a double root (no change of sign)

For example, if

$$f(x) = a(x-r_1)(x-r_2)^2(x-r_3)(x-r_4)^3(x-r_5)$$

and a>0 the graph looks like



The sign of a shows the behavior of the curve towards $+\infty$.

Now the signs of the function are shown in the table below

x	$-\infty$	r_1		r_2		r_3		r_4		r_5		$+\infty$
f(x)	+	0	−	0	−	0	+	0	−	0	+	

EXAMPLE 1

Solve the inequality

$$2x^3-7x^2-17x+10 > 0$$

Solution

We have seen earlier that this cubic function has three single roots, -2, 0.5 and 5. Thus the inequality becomes

$$2(x+2)(x-0.5)(x-5) > 0$$

We obtain

x	$-\infty$		-2		0.5		5		$+\infty$
f(x)		−	0	+	0	−	0	+	

Hence, the solution is $x\in\ ]-2,0.5[\ \cup\ ]5,+\infty[$

EXAMPLE 2

Solve the inequalities

(a) $3(x-1)^2(x-5) > 0$

(b) $3(x-1)^2(x-5) \geq 0$

(c) $3(x-1)^2(x-5)(x^2+1) \geq 0$

Solution

The quadratic factor x^2+1 in (c) has no real roots (irreducible). It is always positive so it doesn't affect the sign of the polynomial.

We obtain

x	$-\infty$		1		5		$+\infty$
f(x)		−	0	−	0	+	

Hence, the solutions are

(a) $x>5$

(b) $x=1$ or $x\geq5$

(c) $x=1$ or $x\geq5$

For a rational function of the form $\frac{f(x)}{g(x)}$ remember that

$$\frac{f(x)}{g(x)} > 0 \Leftrightarrow f(x)g(x) > 0$$

Therefore, by factorizing f(x) and g(x) we can think of the polynomial f(x)g(x) and thus solve the **rational inequalities**

$$\frac{f(x)}{g(x)}>0 \qquad \frac{f(x)}{g(x)}<0 \qquad \frac{f(x)}{g(x)}\geq 0 \qquad \frac{f(x)}{g(x)}\leq 0$$

In case the inequality is either ≥ or ≤, remember to include the roots of the numerator f(x) and exclude the roots of the denominator g(x).

EXAMPLE 3

Solve the inequalities

$$(a)\ \frac{(x-1)(x-3)^2}{(x-2)(x^2+x+1)} \leq 0, \qquad (b)\ \frac{(x-1)(x^2+x+1)}{(x-3)^2(x-2)} \geq 0$$

(factorization is already given).

Solution

Notice that the same factors appear in both inequalities. If we multiply all factors we obtain the polynomial

$$(x-1)(x-2)(x-3)^2(x^2+x+1)$$

We obtain

x	$-\infty$		1		2		3		$+\infty$
f(x)		+	0	−	0	+	0	+	

Hence, the solutions are

(a) $x\in [1,2[\cup \{3\}$

[we exclude the root x=2 of the denominator]

(b) $x\in]-\infty,1] \cup]2,3[\cup]3,+\infty[$.

[we exclude the roots x=2 and x=3 of the denominator]

Mind the difference between equations and inequalities.

EXAMPLE 4

Solve (a) $\frac{x+1}{x-2}=x-3$ (b) $\frac{x+1}{x-2}\geq x-3$

(a) $\frac{x+1}{x-2}=x-3 \Leftrightarrow x+1=(x-2)(x-3)$

$\Leftrightarrow x+1=x^2-5x+6$

$\Leftrightarrow x^2-6x+5=0$

$\Leftrightarrow x=1 \text{ or } x=5$

(b) we present two solutions, one without GDC, one by GDC.

Solution without GDC (analytical)

Now we **cannot** cross multiply! We move everything to the LHS:

$$\frac{x+1}{x-2}-(x-3)\geq 0 \Leftrightarrow \frac{x+1-(x-3)(x-2)}{x-2}\geq 0$$

$$\Leftrightarrow \frac{x+1-x^2+5x-6}{x-2}\geq 0$$

$$\Leftrightarrow \frac{-x^2+6x-5}{x-2}\geq 0$$

$$\Leftrightarrow \frac{-(x-1)(x-5)}{x-2}\geq 0$$

We obtain

x	$-\infty$	1	2	5	$+\infty$
$f(x)$	+	0 −	‖ +	0 −	

Hence, the solution is $x\in\ ]-\infty,1]\cup\ ]2,5]$

Solution by GDC

We sketch the graph of $f(x)=\frac{x+1}{x-2}-(x-3)$

We construct a table as above with **all the critical values:**

- the roots of the function: $x=1$, $x=5$
- the values where the function is not defined: $x=2$

Based on the graph we complete the signs on the table as above

---

2.15 MODULUS EQUATIONS AND INEQUALITIES (for HL)

Remember that, if a is a positive constant,

$$|x| = a \Leftrightarrow x=a \text{ or } x=-a$$

$$|x| < a \Leftrightarrow -a<x<a$$

$$|x| > a \Leftrightarrow x<-a \text{ or } x>a$$

In this way we can solve easy equations or inequalities involving only one absolute value.

EXAMPLE 1

(a) $|2x-3|=5 \Leftrightarrow 2x-3=5 \text{ or } 2x-3=-5$

$\Leftrightarrow 2x=8 \text{ or } 2x=-2$

$\Leftrightarrow x=4 \text{ or } x=-1$

(b) $|2x-3|<5 \Leftrightarrow -5<2x-3<5$

$\Leftrightarrow -2<2x<8$

$\Leftrightarrow -1<x<4$

(c) $|2x-3|>5 \Leftrightarrow 2x-3<-5 \text{ or } 2x-3>5$

$\Leftrightarrow 2x<-2 \text{ or } 2x>8$

$\Leftrightarrow x<-1 \text{ or } x>4$

EXAMPLE 2

(a) $\left|\dfrac{x-1}{x-2}\right|=5 \Leftrightarrow \dfrac{x-1}{x-2}=5 \text{ or } \dfrac{x-1}{x-2}=-5$

$\Leftrightarrow x-1=5x-10 \text{ or } x-1=-5x+10$

$\Leftrightarrow 4x=9 \text{ or } 6x=11$

$\Leftrightarrow x=9/4 \text{ or } x=11/6$

The inequality here is more complicated.

(b) $\left|\dfrac{x-1}{x-2}\right|<5 \quad \Leftrightarrow \quad -5<\dfrac{x-1}{x-2}<5$

We solve separately,

$$\frac{x-1}{x-2}<5 \Leftrightarrow \frac{x-1}{x-2}-5<0 \Leftrightarrow \frac{x-1-5x+10}{x-2}<0 \Leftrightarrow \frac{-4x+9}{x-2}<0$$

We obtain

x	$-\infty$	2	9/4	$+\infty$
f(x)	$-$	0 $+$	0 $-$	

Thus x<2 or x> 9/4 (1)

Similarly

$$\frac{x-1}{x-2}>-5 \Leftrightarrow \frac{x-1}{x-2}+5>0 \Leftrightarrow \frac{x-1+5x-10}{x-2}>0 \Leftrightarrow \frac{6x-11}{x-2}>0$$

We obtain

x	$-\infty$	11/6	2	$+\infty$
f(x)	$+$	0 $-$	0 $+$	

Thus x<11/6 or x> 2 (2)

(1) and (2) together give: x<11/6 or x>9/4

Alternative solution:

Since both sides of the inequality are positive

$$\left|\frac{x-1}{x-2}\right|<5 \Leftrightarrow \left|\frac{x-1}{x-2}\right|^2<5^2 \Leftrightarrow (x-1)^2<25(x-2)^2$$

$$\Leftrightarrow \quad x^2-2x+1<25(x^2-4x+4)$$

$$\Leftrightarrow \quad 24x^2-98x+99>0$$

$$\Leftrightarrow \quad 24(x-9/4)(x-11/6)>0$$

$$\Leftrightarrow \quad x<11/6 \text{ or } x>9/4$$

Things become even more complicated when more that one absolute values are involved or there is a variable outside the absolute value. We have to find first the zeros of the absolute values and investigate the different cases.

EXAMPLE 3

(a) $|x-1|=3x+2$

We find the zeros: $x-1=0 \Leftrightarrow x=1$

CASE 1: $x<1$

$|x-1|=3x+2 \Leftrightarrow -x+1 = 3x+2 \Leftrightarrow 4x=-1 \Leftrightarrow x=-1/4$ (accepted)

CASE 2: $x>1$

$|x-1|=3x+2 \Leftrightarrow x-1 = 3x+2 \Leftrightarrow 2x=-3 \Leftrightarrow x=-3/2$ (rejected)

Final answer (the union of the two cases): $x=-1/4$

(b) $|x-1|<3x+2$

We find the zeros: $x-1=0 \Leftrightarrow x=1$

CASE 1: $x<1$

$|x-1|<3x+2 \Leftrightarrow -x+1 < 3x+2 \Leftrightarrow 4x>-1 \Leftrightarrow x>-1/4$

Thus $x>-1/4$

CASE 2: $x>1$

$|x-1|<3x+2 \Leftrightarrow x-1 < 3x+2 \Leftrightarrow 2x>-3 \Leftrightarrow x>-3/2$

Thus $x>1$

Final answer (the union of the two cases): $x>-1/4$

Alternative graphical solution for $|x-1|-3x-2<0$

We can easily sketch the graph of $f(x)=|x-1|-3x-2$

We know that the graph consists of linear parts.

For $x=1$, $f(1)=-5$
For $x=0$ (before 1): $f(0)=-1$
For $x=2$ (after 1): $f(2)=-7$

We sketch the graph and observe which part is negative.

EXAMPLE 4

$$|x-1|+|x-2|=x$$

We find the zeros of the absolute values: $x=1$ and $x=2$

CASE 1: $x<1$

$-x+1-x+2=x \Leftrightarrow 3x = 3 \Leftrightarrow x=1$ (rejected)

CASE 2: $1\leq x<2$

$x-1-x+2=x \Leftrightarrow x=1$ (accepted)

CASE 3: $x\geq 2$

$x-1+x-2=x \Leftrightarrow x=3$ (accepted)

Final answer (the union of the three cases): $x=1$ or $x=3$

EXAMPLE 5

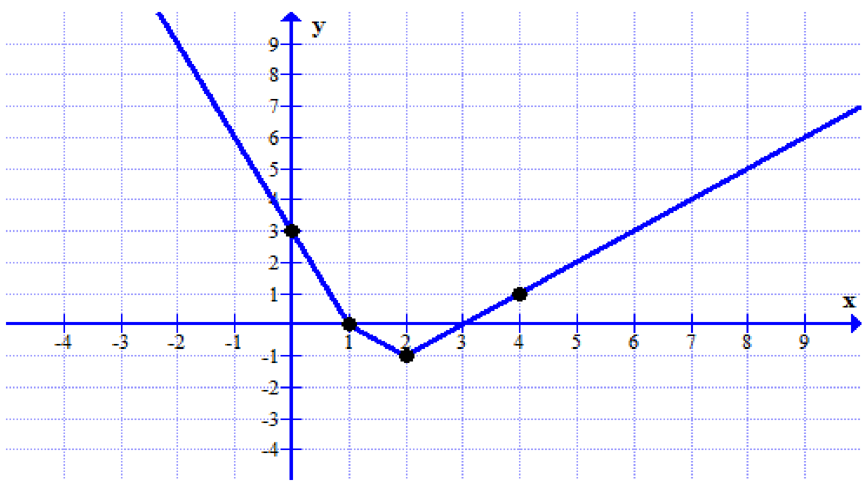
Sketch the graph of $f(x)=|x-1|+|x-2|-x$

We find the zeros of the absolute values: $x=1$ and $x=2$

We know that the graph consists of linear parts. Thus we need 4 points on the graph, the two values above, one before, one after:

$$f(1)=0, \quad f(2)=-1, \quad f(0)=3, \quad f(4)=1$$

We just connect them:



2.16 SYMMETRIES OF f(x) – MORE TRANSFORMATIONS (for HL)

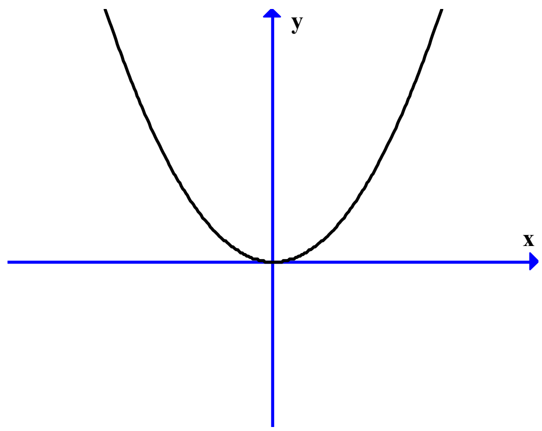
- EVEN AND ODD FUNCTIONS

A function is said to be **even** if

$$f(-x)=f(x)$$

Such a function is **symmetric in y-axis**.

For example $f(x)=x^2$ is an even function.

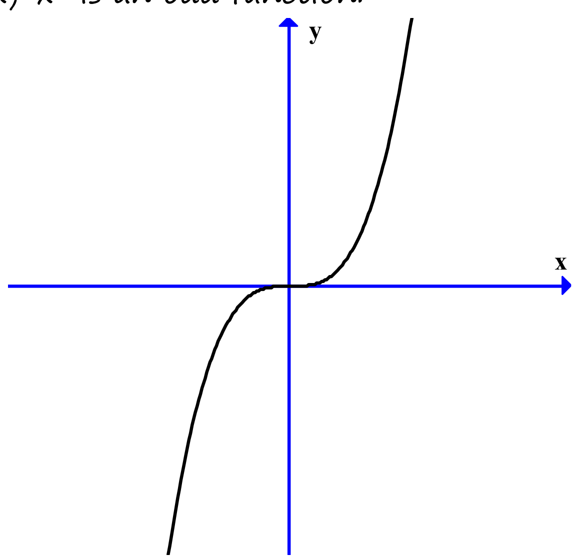


A function is said to be **odd** if

$$f(-x)=-f(x)$$

Thus a function is **symmetric about the origin**.

For example $f(x)=x^3$ is an odd function.

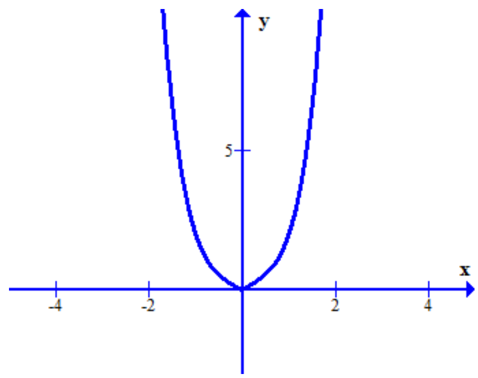
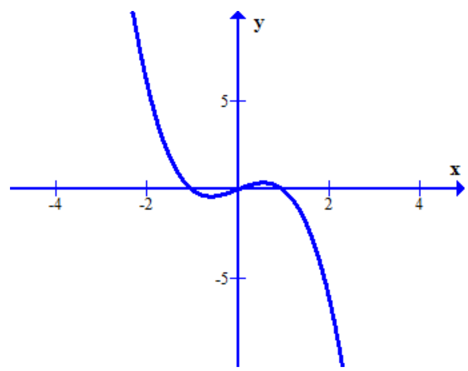
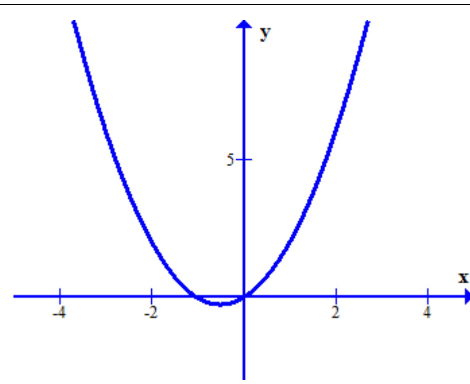


EXAMPLE 1

Investigate whether the following functions are even or odd.

a) $f(x)=x^4+|x|$ b) $g(x)=x-x^3$ c) $h(x)=x+x^2$

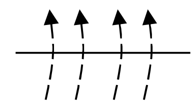
Solution

(a)	$f(-x) = (-x)^4+|-x|$ $= x^4+|x|$ $= f(x)$ hence the function is **even**.	
(b)	$g(-x) = (-x)-(-x)^3$ $=-x+x^3$ $=-(x-x^3)$ $=-g(x)$ hence the function is **odd**.	
(c)	$h(-x) = (-x)+(-x)^2$ $= -x+x^2$ hence the function is **neither even nor odd**.	

♦ THE ABSOLUTE VALUE TRANSFORMATIONS

Consider the initial function $f(x)$.

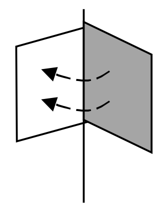
(a) The new function $|f(x)|$



- preserves any positive part of $f(x)$
- reflects any negative part of $f(x)$ in x-axis

  [this is because $f(x)<0$ implies that $|f(x)|=-f(x)$]

(b) The new function $f(|x|)$

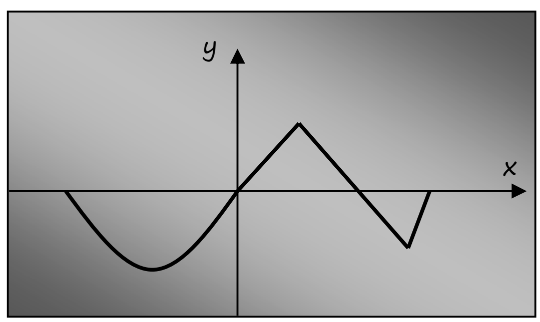


- ignores $f(x)$ for $x<0$
- reflects $f(x)$, $x\geq 0$ in y-axis

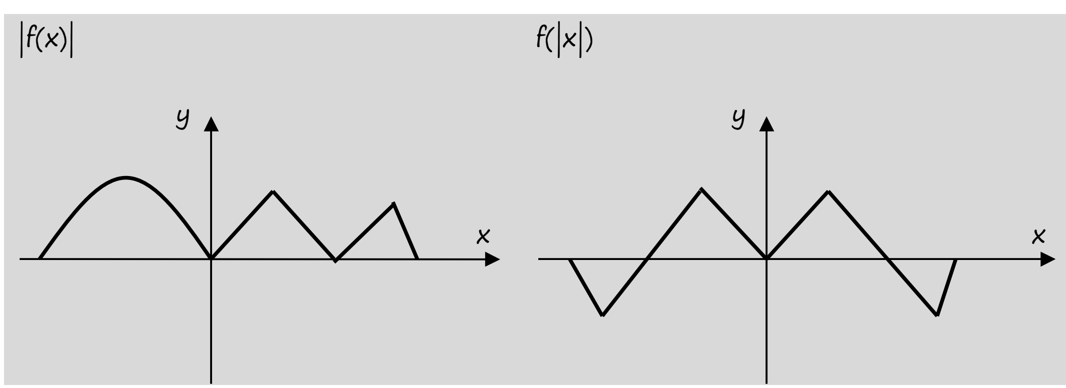
  [this is because $x<0$ implies that $f(|x|)=f(-x)$]

EXAMPLE 2

Let $f(x)$ have the graph

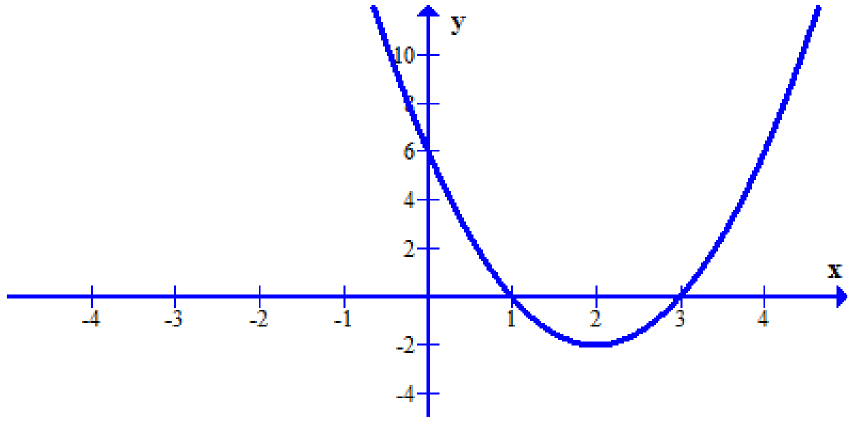


Then

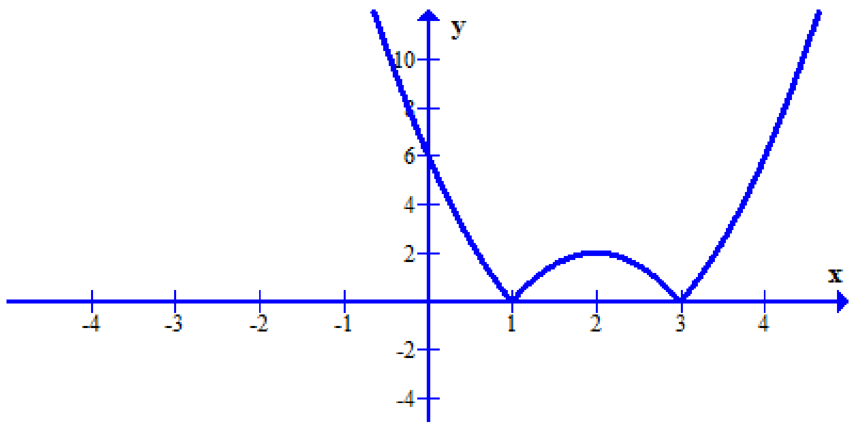


EXAMPLE 3

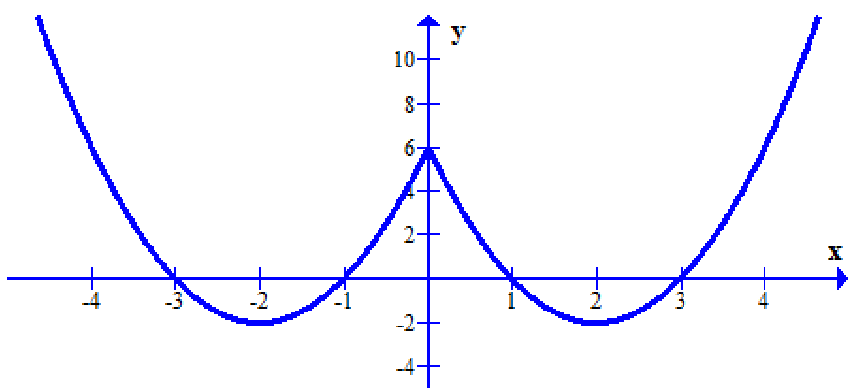
Let $f(x) = 2x^2 - 8x + 6$



Then $|f(x)| = |2x^2 - 8x + 6|$ has the graph



while $f(|x|) = 2|x|^2 - 8|x| + 6$ has the graph



♦ THE RECIPROCAL FUNCTION $\frac{1}{f(x)}$

Another transformation of the function f(x) is $\boxed{g(x)=\frac{1}{f(x)}}$

We notice the following:

- If x=a is a root of f(x) then g(x) is not defined at x=a (V.A.)
- If x=a is vertical asymptote of f(x) then x=a is a root of g(x)
- Any y=a concept (H.A., y-intercept etc) becomes $y=\frac{1}{a}$

Thus, in order to sketch the graph of the reciprocal function $\frac{1}{f(x)}$ we follow the rules:

1) V.A. become roots and roots become V.A.

2) H.A. y=a becomes H.A. $y=\frac{1}{a}$

3) Any characteristic point (x, y) becomes $(x, \frac{1}{y})$

   max at (x, y) becomes **min** at (x, 1/y) (and vice versa)

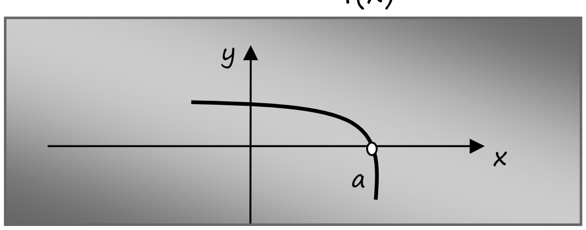
   y-intercept (0, y) becomes **y-intercept** (0, 1/y), etc.

4) If f(x) is positive/negative, g(x) is also positive/negative

5) If f(x) is increasing, g(x) is decreasing (and vice versa)

NOTICE

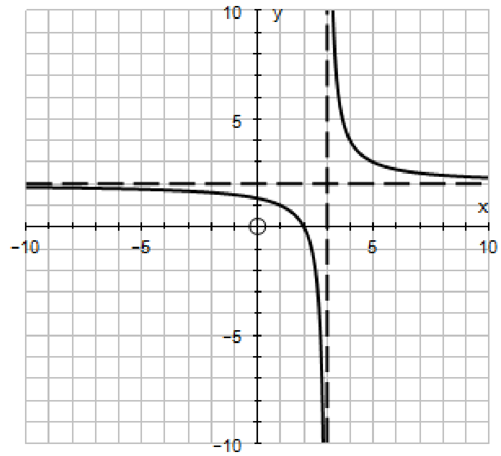
- In fact, the V.A. x=a becomes not exactly a root but a point of discontinuity on x axis, since $g(x)=\frac{1}{f(x)}\neq 0$. The graph looks like



- If y=0 is a HA, in the reciprocal y tends to +∞ or −∞ accordingly.

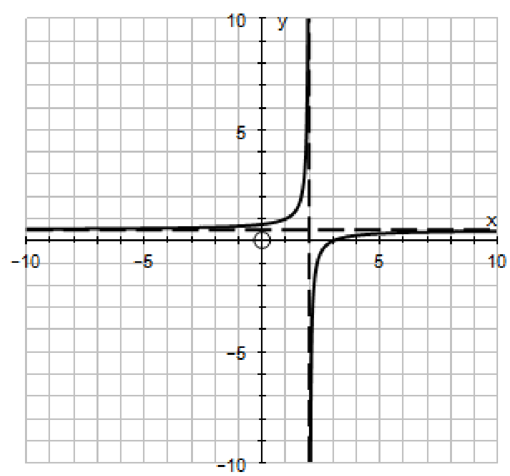
EXAMPLE 4

Consider the function $f(x)=\dfrac{2x-4}{x-3}$



Observations on f(x)	**Conclusions for $\dfrac{1}{f(x)}$**
Root: x=2	V.A: x=2
V,A,: x=3	Root: x=3
H.A.: y=2	H.A.: y=1/2
y-intercept y=4/3	y-intercept y=3/4

For $\dfrac{1}{f(x)}$(i.e. $\dfrac{x-3}{2x-4}$) we indicate roots, asymptotes and carry on



- THE TRANSFORMATION $y=[f(x)]^2$

Consider the function y=f(x)

What about the function $y=[f(x)]^2$?

(1) Whatever lies on the line y=1 remains the same

(2) Whatever lies on the line y=–1 goes to y=1.

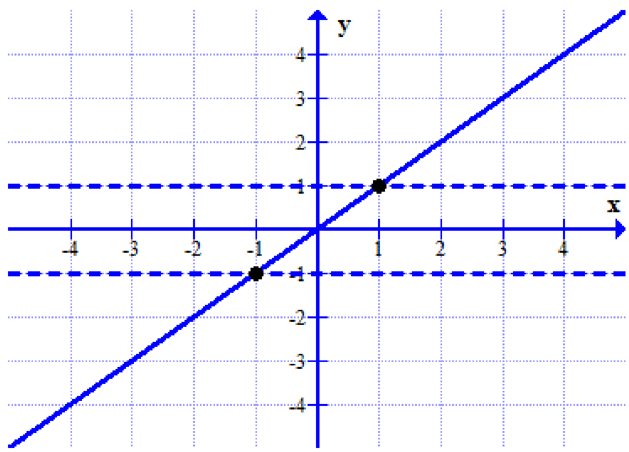
(3) For the positive part of the function:

We stretch everything above y=1: 2 becomes 4 etc

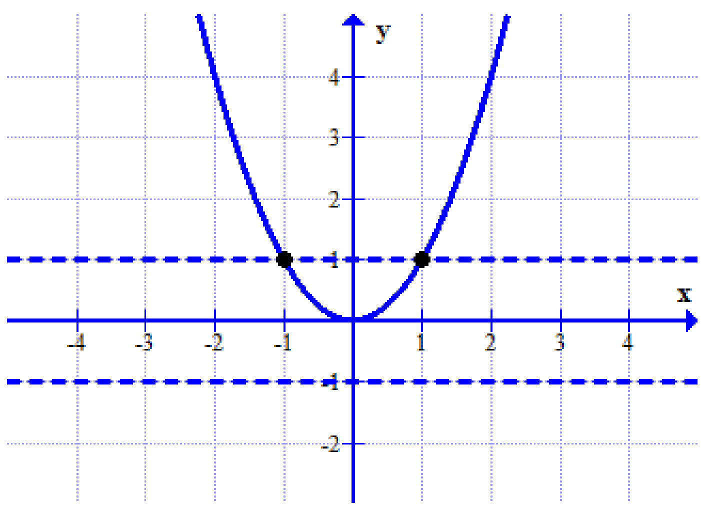
We shrink everything below y=1: 1/2 becomes 1/4 etc

(4) The negative part becomes positive and behaves as in 3

The easy example of f(x)=x is indicative.



We obtain



which is the known curve of $y=x^2$ ☺