MODELING BYU MATH MIGRATION

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Abstract. This project delves into modeling the migration patterns of BYU Math students using a three-patch system to represent their presence in the Math Lab, classrooms, or elsewhere on campus. The focus is on creating a model specifically targeting the Math Lab to look at the dynamics of student migration towards it. Our goal is to provide Math Lab administrators with insights to optimize tutor scheduling during peak influx periods.

While the model captures a few trends, we acknowlege its limitations. The model found the largest influx of students into the Math Lab occurs during the break period between classes suggesting a strategic adjustment in the scheduling of Math Lab employees' shifts. We propose shifts to begin at the 30-minute mark rather than the start of the hour to optimize the Math Lab's ability to manage student influxes efficiently ensuring tutors are ready to assist when needed.

1. Background/Motivation

2. Modeling

2.1. **Building the Model.** There were six key transitions showing the migration from each patch to another over time ($L \to C, L \to E$, etc...), hence it made sense to define a matrix of transition coefficient functions, A. We defined A to be a 3×3 matrix of functions in $C([0,T];\mathbb{R})$, indexed by letters to indicate which transition each entry represents. For example, $\mathrm{A}_{LC}(t)$ represented the rate of students transitioning from Lab to Class over time see Figure 1.

We now define our conservative system of ODEs to be:

$$\begin{aligned}\dot{L} &= \mathrm{A}_{EL}(t)E + \mathrm{A}_{CL}(t)C - \mathrm{A}_{LC}(t)L - \mathrm{A}_{LE}(t)L \\ \dot{E} &= \mathrm{A}_{LE}(t)L + \mathrm{A}_{CE}(t)C - \mathrm{A}_{EL}(t)E - \mathrm{A}_{EC}(t)E \\ \dot{C} &= \mathrm{A}_{LC}(t)L + \mathrm{A}_{EC}(t)E - \mathrm{A}_{CE}(t)C - \mathrm{A}_{CL}(t)C\end{aligned} \tag{1}$$

With initial conditions

$$L(0) = 0, E(0) = 1, C(0) = 0$$

Notice that this ODE system follows the general pattern of conserved quantities transitioning between groupings, where the net derivatives all

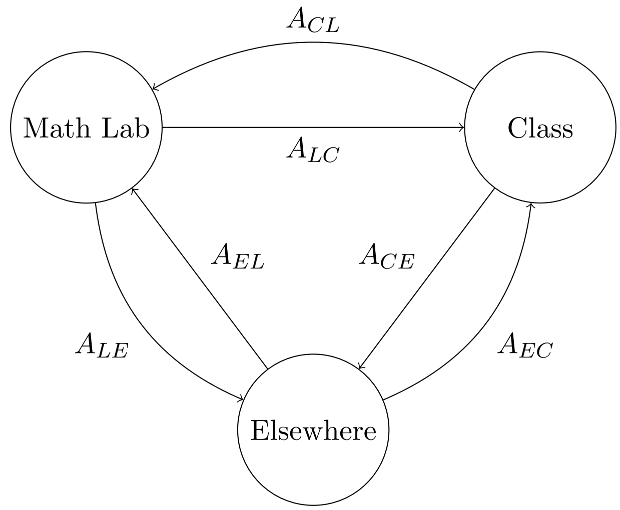


FIGURE 1. Model of Student Flow

cancel such that $\dot{P} = 0$. The nuance here is in the transition functions, which are not merely constants, but time-dependent functions.

2.2. **Defining the Alpha matrix.** For a control model, we have defined a simple transition scheme where every student had an equal chance of coming to and going from each category:

$$A_{constant} = \begin{bmatrix} \text{n/a} & \mathrm{A}_{LE}(t) & \mathrm{A}_{LC}(t) \\ \mathrm{A}_{EL}(t) & \text{n/a} & \mathrm{A}_{EC}(t) \\ \mathrm{A}_{CL}(t) & \mathrm{A}_{CE}(t) & \text{n/a} \end{bmatrix} = \begin{bmatrix} & 0.1 & 0.1 \\ 0.1 & & 0.1 \\ 0.1 & 0.1 & \end{bmatrix} \tag{2}$$

With a control matrix in place, we turned to our more involved model. We changed the transition functions according to our modeling of students' classtimes (e.g. students usually enter/exit class between 10:50am to 10:55am). To be concise, we defined a helper step function:

$$\text{Let } S_{t_0,t_1,p}(t,a,b) = \begin{cases} a & \text{if } t_0 \leq t \pmod p \leq t_1 \\ b & \text{otherwise} \end{cases}$$

By then defining the shorthands $S = S_{50,55,60}$ and $S_\star = S_{0,50,60}$ (favoring transitions between class or during class respectively) we could finally write the following discontinuous A function:

$$A_{\text{discontinuous}} = \begin{bmatrix} \text{n/a} & 1/4 & S(t,\frac{1}{10},\frac{1}{60}) \\ \frac{1}{6} & \text{n/a} & S_\star(t,\frac{1}{5},\frac{1}{60}) \\ S(t,\frac{1}{20},\frac{1}{100}) & S(t,\frac{1}{20},\frac{1}{100}) & \text{n/a} \end{bmatrix} \tag{3}$$

We justified these choices of constants in (3) with our intuition of relative movement between class, lab, and elsewhere. For example, students who were elsewhere during class would quickly move to class, so we made A_{EC} relatively large for most of the hour. Also, we made A_{CL} usually larger than A_{LC}, figuring more students would stay in classes than would ever go to lab.

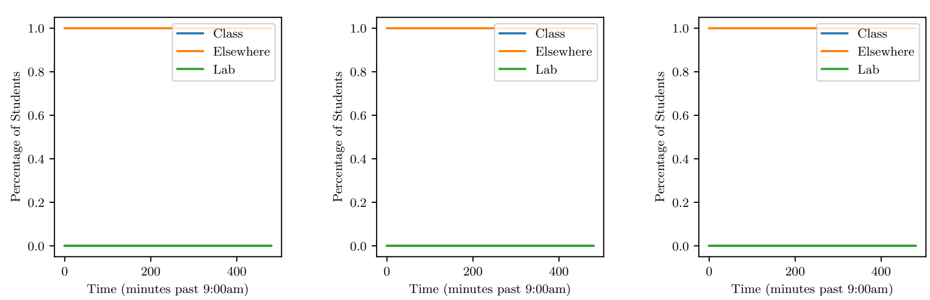
2.3. **Boundary Value Problem.** We wanted to also simulate our system ensuring that students all went elsewhere at the end of class. For this, we used a BVP solver, but we needed to add 3 parameters to add enough free variables for the BVP solver. The interest in using a boundary value problem is because all the math students are elsewhere at the beginning of the day and end of the day. Using this model will ensure that we get the proportions of math students accurate at the beginning and end of the day. The altered system was as follows:

$$\begin{aligned} \dot{L} &= p_1 \mathrm{A}_{EL}(t)E + \mathrm{A}_{CL}(t)C - p_3\mathrm{A}_{LC}(t)L - p_2\mathrm{A}_{LE}(t)L \\ \dot{E} &= p_2\mathrm{A}_{LE}(t)L + \mathrm{A}_{CE}(t)C - p_1\mathrm{A}_{EL}(t)E - \mathrm{A}_{EC}(t)E \\ \dot{C} &= p_3\mathrm{A}_{LC}(t)L + \mathrm{A}_{EC}(t)E - \mathrm{A}_{CE}(t)C - \mathrm{A}_{CL}(t)C \end{aligned} \tag{4}$$

With boundary conditions

$$L(0) = L(T) = 0, C(0) = C(T) = 0, E(0) = E(T) = 1$$

2.4. **Continuizing the Alpha Functions.** One interest we had was making $A_{discontinuous}$ continuous. We needed to have some sort of "continuizer" function. Eventually, we had success using Barycentric Lagrange Interpolation on the Chebyshev points to get a high-degree polynomial approximation. Different choices of degree led to different tradeoffs in approximating continuous coefficient functions – higher-degree polynomials were more accurate, but their derivatives were less stable. See Figure 2.



(A) $A_{interp.7}$ (B) $A_{interp.12}$ (C) $A_{interp.25}$

FIGURE 2. Continuizing $A_{discontinuous}$ with various degrees of Barycentric Lagrange Interpolation at the Chebyshev points. Our ensuing models use the 12-degree interpolation (bottom-left).

2.5. **Incorporating Mid-Day Busy Peaks.** Since the math lab traffic peaks mid-day in what resembles resembles a normal distribution, we simply added (or convolved) gaussian curves to A_{EL} and A_{CL} in $A_{interp.12}$. Note that this modified A isn't purely hour-wise periodic, but still works the same.

3. Results

The following are the results of differing alpha matrices. All of these simulations use a sensible domain (9am–5pm). The initial values are given below Equation (1), and BVP used these as initial *and* final boundary conditions.

3.1. **Constant Alphas.** Simulation our model on the constant alpha matrix as given in Equation (2) gave the following results. See Figure 3. As seen below, this timeplot doesn't have anything interesting to show.

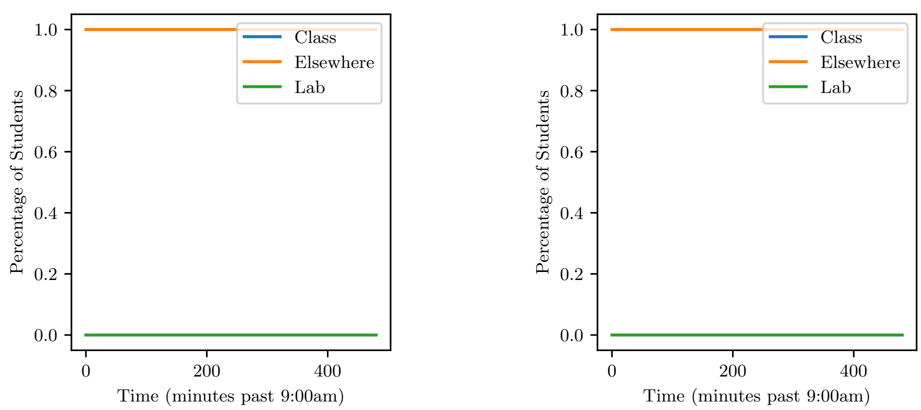


Figure 3. The constant alpha functions (left) along with the timeplot using IVP (right).

3.2. **Discontinuous Alphas.** Simulating our model with $A_{discontinuous}$ in Equation (3) resulted in Figure 4.

The timeplot shows peaks during the transition period. We also see that the model shows majority of math students are in class throughout the day.

3.3. **Boundary Value Problem.** Simulating our model with Equation (4) and the given boundary conditions resulted in Figure 5. In this one we notice the numerical software fitting boundary conditions, which in turn makes the functions almost fit those endpoints every hour. We also see some of the plot going above 1 and below 0. This doesn't accurately represent the population of math students since we can't have more than 100% of students and less than 0%. The peaks as seen in the normal IVP are present here, but they are more stark.

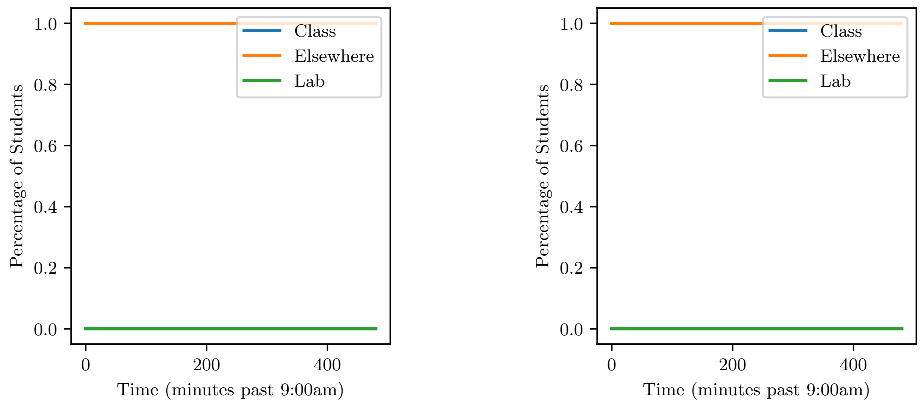


FIGURE 4. The discontinuous alpha functions (left) along with the timeplot using IVP (right).

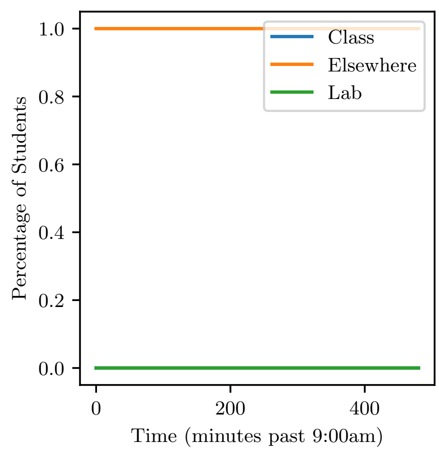


FIGURE 5. Timeplot of our ODE system with $A_{discontinuous}$, but using a BVP solver.

3.4. **Interpolation of Alphas.** Simulating our model using the $A_{interp.12}$ polynomial, since these ones barely dipped below zeros, yeilded the results in Figure 6. The peaks once again occur, but the trend seems to match the polynomial in middle right of the transition matrix which is $A_{E\to C}$. This is probably the case because in reality the majority of students are going from elsewhere to class, but it is interesting that the lab population also takes that form because $A_{E\to C}$ isn't in the $\dot{L}$ equation.

Finally, simulating our model on the convolved polynomial showed the results in Figure 7. The IVP timeplot shows that the peaks will rise towards the middle of the day and then drop as the day continues. On the other

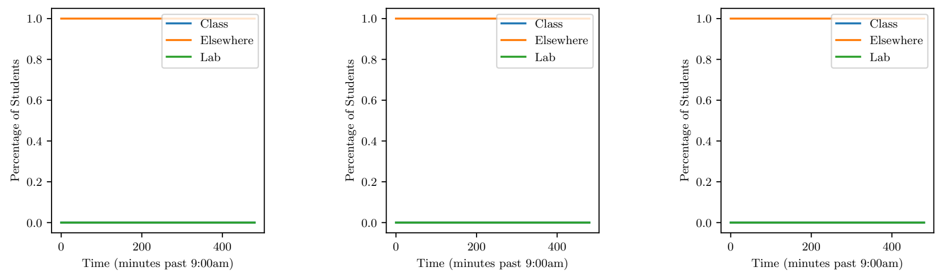


FIGURE 6. Simulating with $A_{interp.12}$ (left), giving the timeplots using both IVP (center) and BVP (right).

hand the BVP timeplot is unaccurate because the scale of the percentage of students is not realisitc. It is seeming to get a lot of numerical noise and doesn't provide useful information.

4. Analysis

4.1. Stability.

4.2. Pitfalls/Errors.

5. Conclusion

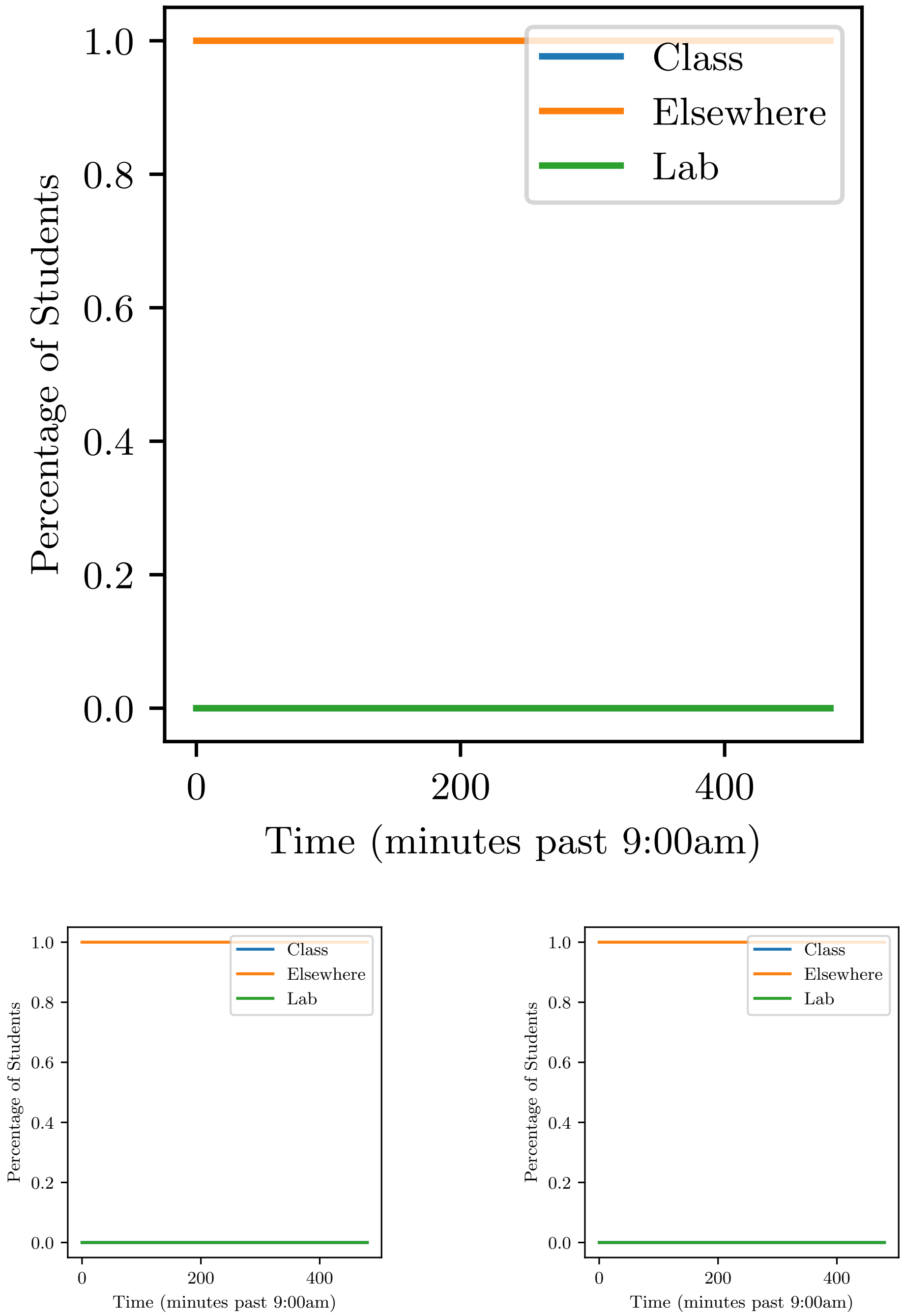


FIGURE 7. Simulating our IVP with $A_{interp.12}$ plus some gaussian terms for daily traffic (top), along with the time-plot using IVP (bottom-left) and BVP (bottom-right). our BVP spikes far outside our usual bounds at transitions with daily traffic.

References

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