

The Employability Theorem

Cao Bittencourt

^a*B. Sc. in Economics from EPGE (FGV), RJ, Brazil.*

^b*Statistician at Atlas Career Guide Inc., FL, USA.*

Abstract

In this document, the Employability Theorem is demonstrated from a set of fairly tautological axioms, which are presupposed in quantitative career choice and career development methods.

Keywords: Employability theorem; Career choice; Career development; Vocational choice; Occupational Information Network; O*NET.

1. Proof Plan

1. basic presuppositions
2. basic lemmas
3. complex tasks
4. occupations are but tasks
5. occupations' tasks are complex
6. occupations' tasks are holistic (operation)
 - 6.1. more difficult tasks presuppose the easier tasks have been accomplished
 - 6.2. i.e. $l \in [0, 1]$ is a “progress bar” of an occupation's operation
 - 6.3. strongly holistic: each task $l \geq \bar{l}$ requires all the previous $l \in [0, \bar{l}]$, $\bar{l} \in [0, 1]$ difficulty levels to be accomplished. in addition, if all $l \in [0, 1]$ levels are not all accomplished, the whole effort is vain and the operation is not completed (i.e. round down \mathcal{U}_q when calculating operational output). furthermore, each and every $l \in [0, 1]$ difficulty level cannot be outsourced (i.e. only a perfectly qualified worker can output a unit of the occupation's operation).
 - 6.3.1. individual's time constraint is spent entirely on trying to accomplish the complex holistic task by themselves. therefore, there is no optimization to be done.

6.3.2.

$$\mathcal{U}_q = \sum_{k=1}^n [k \in \Lambda_q] \times \mathcal{U}_q^k = \sum_{k=1}^n \left[[k \in \Lambda_q] \times \int_0^1 T_q(l, l_q^k) dl \right]$$

6.4. moderately holistic: each task $l \geq \bar{l}$ requires all the previous $l \in [0, \bar{l}]$, $\bar{l} \in [0, 1]$ difficulty levels to be accomplished. in addition, if all $l \in [0, 1]$ levels are not all accomplished, the whole effort is vain and the operation is not completed (i.e. round down \mathcal{U}_q when calculating operational output). however, each and every $l \in [0, 1]$ difficulty level can be outsourced (i.e. workers can output partial units of the occupation's operation, which contribute to the operation's completion).

6.4.1. because of outsourcing, individual's time constraint is spent working from where another worker "left off", so that even if a worker cannot accomplish the entire operation by themselves, they can still contribute to the operation's completion by reducing the time highly skilled workers will have to spend on relatively more trivial tasks.

The first worker spends their entire unitary time allowance trying their hardest to accomplish the highest amount of tasks they can. When they hit their skill cap, they restart their efforts, so as to spend their entire time allowance helping out the next worker:

$$\begin{aligned} \int_0^{\tilde{T}_q^k} T_q(l, l_q^k) \times \text{ta}_q(l) dl + \int_0^{\bar{l}} T_q(l, l_q^k) \times \text{ta}_q(l) dl &= 1 \\ \int_0^{\bar{l}} 1 \times \text{ta}_q(l) dl &= 1 - \int_0^{\tilde{T}_q^k} 1 \times \text{ta}_q(l) dl \\ \int_0^{\bar{l}} \text{ta}_q(l) dl &= \int_{\tilde{T}_q^k}^1 \text{ta}_q(l) dl \\ \text{TA}_q(\bar{l}) - \text{TA}_q(0) &= \text{TA}_q(1) - \text{TA}_q(\tilde{T}_q^k) \\ \text{TA}_q(\bar{l}) &= \text{TA}_q(1) - \text{TA}_q(\tilde{T}_q^k) \\ \bar{l} &= \text{TA}_q^{-1} \left(\text{TA}_q(1) - \text{TA}_q(\tilde{T}_q^k) \right), \end{aligned}$$

so that k accomplishes tasks of difficulty levels 0 through \tilde{T}_q^k on their "first run", and restarts their effort to provide additional $l \in \left[0, \text{TA}_q^{-1} \left(\text{TA}_q(1) - \text{TA}_q(\tilde{T}_q^k) \right) \right]$ levels worth of complex tasks. Thus, the next worker does not need to start from zero, but rather from where k "left off": either \tilde{T}_q^k , \bar{l} , or some $l \in [0, \tilde{T}_q^k]$.

6.4.2.

$$\mathcal{U}_q = \left[\sum_{k=1}^n [k \in \Lambda_q] \times \mathcal{U}_q^k \right]$$

$$= \left\lfloor \sum_{k=1}^n [k \in \Lambda_q] \times \int_0^1 T_q(l, l_q^k) dl \right\rfloor$$

6.5. weakly holistic: each task $l \geq \bar{l}$ requires all the previous $l \in [0, \bar{l}]$, $\bar{l} \in [0, 1]$ difficulty levels to be accomplished. however, if not all $l \in [0, 1]$ levels are accomplished, the whole effort is not vain and the operation is partially completed (i.e. do not round \mathcal{U}_q when calculating operational output). furthermore, each and every $l \in [0, 1]$ difficulty level can be outsourced (i.e. workers can output partial units of the occupation's operation, which contribute to the operation's completion).

7. assume weak occupational complexity axiom (the other versions are too strict)

8. perhaps posit an even weaker version of occupational complexity:

8.1.

$$\frac{\partial \mathcal{U}_q}{\partial l} > 0, \tag{1}$$

$$\frac{\partial^2 \mathcal{U}_q}{\partial l^2} < 0, \tag{2}$$

so that even though tasks of a particular level are not required for the operation to “count” (i.e. partial delivery), it is still detrimental to focus too much on one subset of tasks, that is, employers are incentivised to produce the entire spectrum of difficulty levels, because marginal productivity increases when a tasks of a particular difficulty level have not been accomplished yet.

(actually, we need a indicator variable for the amount of tasks accomplished for a difficulty level, something analogous to $T_q(l)$)

9. now, because of weak occupational complexity, employers will maximize operational output by attempting to produce the entire spectrum of difficulty levels for the complex tasks of an occupation.

10. this can be done either by having only perfectly qualified employees work on the operation individually from beginning to end, or by splitting responsibilities into two, or more, types of jobs, thus allowing for less qualified, “junior” employees, to work alongside more qualified and perfectly qualified, “senior” employees towards the common goal of accomplishing the entire occupational operation.

11. additionally, because there are skill differences among workers in the labor market, any rational employer will always, and rightly, expect their employees to be of varying skill levels, rather than all perfectly qualified, so that splitting responsibilities into separate positions will not only be an alternative mode of hiring and producing, but in fact the optimal one.

12. therefore, given expected and actual skill differences among workers, employers will split job posts based on the required skill level. thus, there will be “junior” job posts and “senior” job posts, each dedicated to accomplishing a particular subset of complex tasks with difficulty levels appropriate for employees’ respective capacity.
13. notice this does not mean all people working on “junior” positions will, necessarily, be “junior” employees themselves, that is, less qualified. indeed, if talent is abundant in the labor market, these “junior” positions will have to be filled by more qualified, or even perfectly qualified, “senior” employees. for if there were only one type of job, spanning the entire difficulty level spectrum, highly qualified workers would already have to accomplish these “junior” tasks themselves, in order to maximize operational output. however, by having two, or more, types of jobs, split by minimum required competence, highly qualified workers may specialize to the measure that there are less qualified workers available to accomplish the easier tasks. but, if there are none, they will, again, have to work on these themselves.
14. analogously, from the employers’ perspective, it does not matter who accomplishes “junior” tasks, so long as they are accomplished. thus, if highly qualified workers are abundant in a particular time period of a labor market, production is not hindered when allocating “seniors” to “junior” positions, for in these circumstances talent is not wasted. that is, because only highly qualified workers can accomplish highly demanding tasks, rational employers will generally not hire them to work on “junior” tasks, thus “saving” their talent for more difficult tasks, which a “junior” would not be able to accomplish. but, if there is enough talent to output the optimal quantity of “senior” tasks, it can actually be more productive to employ the remaining “seniors” to “junior” positions.
15. furthermore, in a continuous setting, rational employers will maximize their hiring pool by offering more than only two types of jobs. thus, there will not only be “senior” and “junior” positions, but several levels in a production hierarchy, each responsible for a particular subinterval of task difficulty, which will approximate a continuum of “seniority” as the number of workers becomes large enough.
16. now, as for employees’ work routine, rational employers will have them work over their responsibility spectrum in a proportional and optimal matter, thus avoiding wasting production (i.e. uncompleted “loops” over the responsibility spectrum). [this means each employee will spend their entire time allowance producing a partial operational output, that is a multiple of the difficulty subinterval they were hired to accomplish, which will, in turn, contribute, alongside the partial outputs of other employees, to accomplish the entire occupational operation.]
17. the reason this avoids wasting production is because [...].

18. Weak Skill Differences Axiom (WSDA)

18.1. There are, or there could be, skill differences among people in the workforce (i.e. workers are not all “clones” of one another or equally competent). Thus, the expected value of productivity is:

$$\mathbb{E}[\tilde{T}_q^k] \in [0, 1], \quad (3)$$

instead of

$$\mathbb{E}[\tilde{T}_q^k] = \tilde{T}_q^k = 1, \quad (4)$$

for all $k, q \in \{1, \dots, n\}$. This means employers do not expected every worker to be perfectly qualified and will adjust their hiring and production strategies accordingly.

19. (Binary Employability Theorem) thus, in the binary case, “junior” productive output will be given by:

$$\mathcal{U}_q^{\text{Jr}} = \frac{1}{\int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}_q(l) dl} = \left(\int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}_q(l) dl \right)^{-1}, \quad (5)$$

where $\text{ta}_q(l)$ is the time allocation function of occupation q ’s complex tasks, and time allowance (the numerator) is set to one.

20. analogously, “senior” productive output is:

$$\mathcal{U}_q^{\text{Sr}} = \frac{1}{\int_{\tilde{T}_q^{\text{Jr}}}^1 \text{ta}_q(l) dl} = \left(\int_{\tilde{T}_q^{\text{Jr}}}^1 \text{ta}_q(l) dl \right)^{-1}. \quad (6)$$

21. finally, as a mismatch in productive output due to time allocation differences between “junior” and “senior” tasks would result in wasted production, a rational employer will optimally “orchestrate” the productive effort by offering just enough “senior” job posts in the labor market to meet “junior” productivity. thus, by setting “junior” job posts to $w_q^{\text{Jr}} > 0$ and “senior” job posts to $w_q^{\text{Sr}} > 0$, we get the ratio between “junior” and “senior” positions required to output any level of occupation q ’s operation:

$$w_q^{\text{Sr}} \times \mathcal{U}_q^{\text{Sr}} = w_q^{\text{Jr}} \times \mathcal{U}_q^{\text{Jr}} \quad (7)$$

$$\therefore w_q^{\text{Sr}} \times \left(\int_{\tilde{T}_q^{\text{Jr}}}^1 \text{ta}_q(l) dl \right)^{-1} = w_q^{\text{Jr}} \times \left(\int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}_q(l) dl \right)^{-1} \quad (8)$$

$$\therefore w_q^{\text{Sr}} = w_q^{\text{Jr}} \times \left(\frac{\int_{\tilde{T}_q^{\text{Jr}}}^1 \text{ta}_q(l) dl}{\int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}_q(l) dl} \right). \quad (9)$$

with this, “senior” employability (i.e. the percentage of job posts for which they could be hired) is

$$\tilde{w}_q^{\text{Sr}} = \frac{w_q^{\text{Jr}} + w_q^{\text{Sr}}}{w_q^{\text{Jr}} + w_q^{\text{Sr}}} = 1 \quad (10)$$

and “junior” employability is

$$\tilde{w}_q^{\text{Jr}} = \frac{w_q^{\text{Jr}}}{w_q^{\text{Jr}} + w_q^{\text{Sr}}} \quad (11)$$

$$= \frac{w_q^{\text{Jr}}}{w_q^{\text{Jr}} + w_q^{\text{Jr}} \times \left(\frac{\int_0^1 \tilde{T}_q^{\text{Jr}} \text{ta}_q(l) dl}{\int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}_q(l) dl} \right)} \quad (12)$$

$$= \left(1 + \frac{\int_0^1 \tilde{T}_q^{\text{Jr}} \text{ta}_q(l) dl}{\int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}_q(l) dl} \right)^{-1} \quad (13)$$

$$= \left(1 + \frac{\int_0^1 \text{ta}_q(l) dl - \int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}_q(l) dl}{\int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}_q(l) dl} \right)^{-1} \quad (14)$$

$$= \left(1 + \frac{1 - \int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}_q(l) dl}{\int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}_q(l) dl} \right)^{-1} \quad (15)$$

$$= \left(1 + \frac{1}{\int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}_q(l) dl} - \frac{\int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}_q(l) dl}{\int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}_q(l) dl} \right)^{-1} \quad (16)$$

$$= \left(1 + \frac{1}{\int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}_q(l) dl} - 1 \right)^{-1} \quad (17)$$

$$= \left(\frac{1}{\int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}_q(l) dl} \right)^{-1} \quad (18)$$

$$= \int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}_q(l) dl. \quad (19)$$

thus, the employability of a partially qualified worker, that is a “junior”, is precisely the percentage of an operation’s total time duration their skill set allows them to accomplish (i.e. the inverse of their operational output).

Maximum-Monotonic Labor Stratification Definition

Now, to generalize this conclusion for other economic configurations, we shall define notation in terms of maximum labor stratification, a productive arrangement where there are several job subtypes, indeed as many as there are jobs themselves, each with a limited spectrum of responsibilities.

Hence, mathematically,

$$l \in [\ell_{v-1}, \ell_v], \quad (20)$$

with

$$\ell_v \in [0, 1] \quad \forall v \in \{1, \dots, w_q\}, \quad (21)$$

$$l_{w_q} = 1, \quad (22)$$

$$l_0 = 0 \quad (23)$$

and

$$\sum_{v=1}^{w_q} \ell_v = 1 \quad (24)$$

is one of w_q responsibility spectra in a maximally stratified labor market, in which employment levels are unitary, or given by

$$\sum_{v=1}^{w_q} 1 = w_q, \quad (25)$$

so that any available position is its own job subtype and covers only a restrictive range of task difficulty, accounting for

$$\Omega_q^v = \frac{1}{\mathcal{U}_q^v} = \int_{\ell_{v-1}}^{\ell_v} \text{ta}(l) dl \in [0, 1] \quad (26)$$

of an operation's total time duration,

$$\sum_{v=1}^{w_q} \Omega_q^v = \sum_{v=1}^{w_q} \int_{\ell_{v-1}}^{\ell_v} \text{ta}(l) dl = \int_0^1 \text{ta}(l) dl = 1. \quad (27)$$

Intuitively speaking, we would say production in a maximally and monotonically stratified labor market is not “independent”, in the sense that employees do not work on an occupation's operation from beginning to end. This means each of them will spend all their time allowance producing a partial operational output, that is a multiple of a difficulty subinterval of complex tasks, which will, in turn, contribute, alongside the partial outputs of other employees, to accomplish the occupational operation in its entirety.

However, in a maximum labor stratification setting, these partial operational outputs will not be produced merely via “senior” and “junior” positions, as previously, but rather within a myriad of levels in a production hierarchy, approximating a continuum of “seniority” as the workforce becomes large enough.

Again, this does not mean employees are, actually, more or less competent, only that the available positions are preemptively stratified with respect to task difficulty, in order to maximize employers' hiring pool and safeguard production in the case workers are not sufficiently qualified to produce the whole responsibility spectrum by themselves (see "Maximum-Monotonic Labor Stratification Lemma" below).

[end definition of maximum-monotonic labor stratification]

Maximally Stratified Production Lemma (MSPL)

Because in a maximally and monotonically stratified labor market every position is its own job subtype (for, again, employment levels are unitary), optimal production is, then, obtained not by choosing how many workers to allocate to tasks of varying difficulty levels, but instead by setting appropriate responsibility ranges for each position (i.e. which tasks to allocate *to* workers).

Furthermore, we have just demonstrated that the maximum operational output in any labor market is exactly the number of employees in its workforce, or w_q .

[Thus, optimal responsibility ranges can be estimated by equating partial operational outputs in a stratified setting with maximum production. So, for the first job subtype,]

$$1 \times \left(\int_{\ell_0}^{\ell_1} \text{ta}(l) dl \right)^{-1} = 1 \times \left(\int_0^{\ell_1} \text{ta}(l) dl \right)^{-1} = w_q, \quad (28)$$

which means the partial operational output of the first worker, whose tasks range from $\ell_0 = 0$ to $\ell_1 \in [0, 1]$ exclusively, should produce the same amount of the $l \in [0, \ell_1]$ responsibility spectrum as would be produced in an economic configuration with maximum operational output (e.g. with w_q perfectly qualified employees working independently).

Thus, solving for ℓ_1 , we get:

$$1 \times \left(\int_0^{\ell_1} \text{ta}(l) dl \right)^{-1} = w_q \quad (29)$$

$$\therefore \int_0^{\ell_1} \text{ta}(l) dl = \frac{1}{w_q} \quad (30)$$

$$\therefore \text{TA}(l)|_0^{\ell_1} = \frac{1}{w_q} \quad (31)$$

$$\therefore \text{TA}(\ell_1) - \text{TA}(0) = \frac{1}{w_q} \quad (32)$$

$$\therefore \text{TA}^{-1}(\text{TA}(\ell_1)) = \text{TA}^{-1} \left(\frac{1}{w_q} + \text{TA}(0) \right) \quad (33)$$

$$\therefore \ell_1 = \text{TA}^{-1} \left(\frac{1}{w_q} + \text{TA}(0) \right). \quad (34)$$

Likewise, for the second worker,

$$1 \times \left(\int_{\ell_1}^{\ell_2} \text{ta}(l) dl \right)^{-1} = w_q \quad (35)$$

$$\therefore \int_{\ell_1}^{\ell_2} \text{ta}(l) dl = \frac{1}{w_q} \quad (36)$$

$$\therefore \text{TA}(l) \Big|_{\ell_1}^{\ell_2} = \frac{1}{w_q} \quad (37)$$

$$\therefore \text{TA}(\ell_2) - \text{TA}(\ell_1) = \frac{1}{w_q} \quad (38)$$

$$\therefore \text{TA}^{-1}(\text{TA}(\ell_2)) = \text{TA}^{-1} \left(\frac{1}{w_q} + \text{TA}(\ell_1) \right) \quad (39)$$

$$\therefore \text{TA}^{-1}(\text{TA}(\ell_2)) = \text{TA}^{-1} \left(\frac{1}{w_q} + \frac{1}{w_q} + \text{TA}(0) \right) \quad (40)$$

$$\therefore \ell_2 = \text{TA}^{-1} \left(\frac{2}{w_q} + \text{TA}(0) \right). \quad (41)$$

For the third worker,

$$1 \times \left(\int_{\ell_2}^{\ell_3} \text{ta}(l) dl \right)^{-1} = w_q \quad (42)$$

$$\therefore \int_{\ell_2}^{\ell_3} \text{ta}(l) dl = \frac{1}{w_q} \quad (43)$$

$$\therefore \text{TA}(l) \Big|_{\ell_2}^{\ell_3} = \frac{1}{w_q} \quad (44)$$

$$\therefore \text{TA}(\ell_3) - \text{TA}(\ell_2) = \frac{1}{w_q} \quad (45)$$

$$\therefore \text{TA}^{-1}(\text{TA}(\ell_3)) = \text{TA}^{-1} \left(\frac{1}{w_q} + \text{TA}(\ell_2) \right) \quad (46)$$

$$\therefore \text{TA}^{-1}(\text{TA}(\ell_3)) = \text{TA}^{-1} \left(\frac{1}{w_q} + \frac{1}{w_q} + \frac{1}{w_q} + \text{TA}(0) \right) \quad (47)$$

$$\therefore \ell_3 = \text{TA}^{-1} \left(\frac{3}{w_q} + \text{TA}(0) \right). \quad (48)$$

And so on and so forth, for every worker in the stratified labor market:

$$\ell_v = \text{TA}^{-1} \left(\frac{v}{w_q} + \text{TA}(0) \right). \quad (49)$$

Having understood what maximum-monotonic labor stratification is, we shall demonstrate that, given our axioms, such an economic configuration is, in fact, the only optimal production strategy and, so, holds in the labor market. To do so, let us first derive the maximum operational output, irrespective of productive arrangement.

Maximum Operational Output Lemma (MOOL)

In any labor market, the maximum operational output is exactly the number of employees in its workforce:

$$\mathcal{U}(\mathbf{w}_q^*, \mathcal{U}_q) = \mathbf{1}^\top \cdot \mathbf{w}_q^* = w_q, \quad (50)$$

where \mathbf{w}_q^* is the vector of optimal employment levels in a labor market with w_q employees; and \mathcal{U}_q , the vector of partial operational outputs.

Moreover, this maximum production can only be attained when the percentage of each position relative to the entire workforce respects the Proportional Employment Condition (PEC):

$$\tilde{\mathbf{w}}_q^* = \frac{\mathbf{w}_q^*}{w_q} = \mathbf{\Omega}_q, \quad (51)$$

which determines the ratio, or proportion, of a particular job subtype in a stratified labor market is the percentage of an operation's total time duration,

$$\mathbf{1}^\top \cdot \mathbf{\Omega}_q = 1, \quad (52)$$

accounted by it.

[Proof] [Let us begin with a binary setting.]

$$\mathcal{U}(\tilde{w}_q^k) = \min \left(\frac{\tilde{w}_q^k}{\int_0^{\tilde{T}_q^k} \text{ta}(l) dl}, \frac{1 - \tilde{w}_q^k}{\int_{\tilde{T}_q^k}^1 \text{ta}(l) dl} \right) \quad (53)$$

$$= \min \left(\frac{\tilde{w}_q^k}{\int_0^{\tilde{T}_q^k} \text{ta}(l) dl}, \frac{1 - \tilde{w}_q^k}{\int_0^1 \text{ta}(l) dl - \int_0^{\tilde{T}_q^k} \text{ta}(l) dl} \right) \quad (54)$$

$$= \min \left(\frac{\tilde{w}_q^k}{\Omega_q^k}, \frac{1 - \tilde{w}_q^k}{1 - \Omega_q^k} \right), \quad (55)$$

whereas the operational output of $\Omega_q^k \in [0, 1]$ is

$$\mathcal{U}(\Omega_q^k) = \min \left(\frac{\Omega_q^k}{\Omega_q^k}, \frac{1 - \Omega_q^k}{1 - \Omega_q^k} \right) = \frac{\Omega_q^k}{\Omega_q^k} = \frac{1 - \Omega_q^k}{1 - \Omega_q^k} = 1. \quad (56)$$

With this, if $\tilde{w}_q^k \in [0, 1]$ is set to $\tilde{w}_q^k > \Omega_q^k$, then

$$1 - \tilde{w}_q^k < 1 - \Omega_q^k \quad (57)$$

$$\therefore \frac{\tilde{w}_q^k}{\Omega_q^k} > 1 > \frac{1 - \tilde{w}_q^k}{1 - \Omega_q^k} \quad (58)$$

$$\therefore \mathcal{U}(\tilde{w}_q^k) = \min\left(\frac{\tilde{w}_q^k}{\Omega_q^k}, \frac{1 - \tilde{w}_q^k}{1 - \Omega_q^k}\right) = \frac{1 - \tilde{w}_q^k}{1 - \Omega_q^k} < 1 \quad (59)$$

$$\implies \mathcal{U}(\tilde{w}_q^k) < \mathcal{U}(\Omega_q^k); \quad (60)$$

and, likewise, if $\tilde{w}_q^k < \Omega_q^k$,

$$1 - \tilde{w}_q^k > 1 - \Omega_q^k \quad (61)$$

$$\therefore \frac{\tilde{w}_q^k}{\Omega_q^k} < 1 < \frac{1 - \tilde{w}_q^k}{1 - \Omega_q^k} \quad (62)$$

$$\therefore \mathcal{U}(\tilde{w}_q^k) = \min\left(\frac{\tilde{w}_q^k}{\Omega_q^k}, \frac{1 - \tilde{w}_q^k}{1 - \Omega_q^k}\right) = \frac{\tilde{w}_q^k}{\Omega_q^k} < 1 \quad (63)$$

$$\implies \mathcal{U}(\tilde{w}_q^k) < \mathcal{U}(\Omega_q^k). \quad (64)$$

Therefore,

$$\mathcal{U}(\tilde{w}_q^k) < \mathcal{U}(\Omega_q^k) = 1 \quad \forall \quad \tilde{w}_q^k \neq \Omega_q^k \in [0, 1]. \quad (65)$$

Analogously, in a setting with multiple job subtypes, optimal operational output is also trivial:

$$\mathcal{U}(\mathbf{\Omega}_q) = \min(\mathbf{\Omega}_q \mathcal{U}_q) = \frac{\Omega_q^\ell}{\Omega_q^\ell} = 1. \quad (66)$$

And, again, since

$$\mathbf{1}^\top \cdot \tilde{\mathbf{w}}_q = \mathbf{1}^\top \cdot \mathbf{\Omega}_q = 1, \quad (67)$$

choosing any $\tilde{w}_q^\ell \neq \Omega_q^\ell \in [0, 1]$ implies the proportion of at least one position, say \tilde{w}_q^r , is impacted, and thus aggregate output along with it, either because

$$\tilde{w}_q^\ell > \Omega_q^\ell \quad (68)$$

$$\therefore \frac{\tilde{w}_q^\ell}{\Omega_q^\ell} > 1 > \frac{\tilde{w}_q^r}{\Omega_q^r} \quad (69)$$

$$\therefore \mathcal{U}(\mathbf{w}_q) = \min(\mathbf{w}_q \mathcal{U}_q) = \frac{\tilde{w}_q^r}{\Omega_q^r} < 1 \quad (70)$$

$$\implies \mathcal{U}(\mathbf{w}_q) < \mathcal{U}(\mathbf{\Omega}_q); \quad (71)$$

or, alternatively, because

$$\tilde{w}_q^\ell < \Omega_q^\ell \quad (72)$$

$$\therefore \frac{\tilde{w}_q^\ell}{\Omega_q^\ell} < 1 < \frac{\tilde{w}_q^r}{\Omega_q^r} \quad (73)$$

$$\therefore \mathcal{U}(\mathbf{w}_q) = \min(\mathbf{w}_q \mathcal{U}_q) = \frac{\tilde{w}_q^\ell}{\Omega_q^\ell} < 1 \quad (74)$$

$$\implies \mathcal{U}(\mathbf{w}_q) < \mathcal{U}(\mathbf{\Omega}_q). \quad (75)$$

Thus,

$$\mathcal{U}(\tilde{\mathbf{w}}_q, \mathcal{U}_q) < \mathcal{U}(\mathbf{\Omega}_q, \mathcal{U}_q) = 1 \quad (76)$$

$$\iff \mathcal{U}(\mathbf{w}_q, \mathcal{U}_q) < \mathcal{U}(w_q \mathbf{\Omega}_q, \mathcal{U}_q) = w_q \quad (77)$$

$$\forall \tilde{\mathbf{w}}_q \neq \mathbf{\Omega}_q \quad (78)$$

As every worker in the labor market is assumed to have the same unitary time allowance, which coincides with operations' total time duration,

$$\int_0^1 \text{ta}(l) dl = 1, \quad (79)$$

there cannot be, even in theory, any productive arrangement yielding operational output higher than that obtained when perfectly qualified employees work independently. For, indeed, if employers establish other modes of production, workers' time allowances and tasks' time requirements will not change. In addition, splitting responsibilities across multiple job subtypes does not affect operational output (WOCA). Therefore, perfectly qualified independent production is the “ceiling” or “benchmark” for all production strategies.

$$\int_0^1 \text{ta}(l) dl = 1 \quad (80)$$

$$\therefore \mathcal{U}_q^* = 1 \times \left(\int_0^1 \text{ta}(l) dl \right)^{-1} = 1 \quad (81)$$

$$\therefore w_q \times \mathcal{U}_q^* = w_q \times \left(\int_0^1 \text{ta}(l) dl \right)^{-1} = w_q \quad (82)$$

$$\therefore \sum_{v=1}^{w_q} \int_0^1 \text{ta}(l) dl = \sum_{v=1}^{w_q} 1 = w_q \times 1 = w_q \quad (83)$$

The mathematical proof for this is quite straightforward. Because the aggregate production function is of the Leontief functional form (for simplicity's sake, here considering only two factors of production, namely an easier responsibility spectrum and a harder responsibility spectrum), that is

$$\mathcal{U}_q(\tilde{w}_q^k) = \min \left[\tilde{w}_q^k \left(\int_0^{\tilde{T}_q^k} \text{ta}(l) dl \right)^{-1}, (1 - \tilde{w}_q^k) \left(\int_{\tilde{T}_q^k}^1 \text{ta}(l) dl \right)^{-1} \right], \quad (84)$$

and because time allowances are unitary and coincide with total time requirements, optimal relative employment levels are trivial and have to be

$$\tilde{w}_q^k = \int_0^{\tilde{T}_q^k} \text{ta}(l) dl \in [0, 1] \quad (85)$$

and

$$1 - \tilde{w}_q^k = \int_{\tilde{T}_q^k}^1 \text{ta}(l) dl \quad (86)$$

$$= \int_0^1 \text{ta}(l) dl - \int_0^{\tilde{T}_q^k} \text{ta}(l) dl \quad (87)$$

$$= 1 - \int_0^{\tilde{T}_q^k} \text{ta}(l) dl \in [0, 1] \quad (88)$$

to avoid hindering production, so that

$$\mathcal{U}_q(\tilde{w}_q^k) = \min \left[\tilde{w}_q^k \left(\int_0^{\tilde{T}_q^k} \text{ta}(l) dl \right)^{-1}, (1 - \tilde{w}_q^k) \left(\int_{\tilde{T}_q^k}^1 \text{ta}(l) dl \right)^{-1} \right] \quad (89)$$

$$= \tilde{w}_q^k \left(\int_0^{\tilde{T}_q^k} \text{ta}(l) dl \right)^{-1} \quad (90)$$

$$= (1 - \tilde{w}_q^k) \left(\int_{\tilde{T}_q^k}^1 \text{ta}(l) dl \right)^{-1} \quad (91)$$

$$= \left(\int_0^{\tilde{T}_q^k} \text{ta}(l) dl \right) \left(\int_0^{\tilde{T}_q^k} \text{ta}(l) dl \right)^{-1} \quad (92)$$

$$= \left(\int_{\tilde{T}_q^k}^1 \text{ta}(l) dl \right) \left(\int_{\tilde{T}_q^k}^1 \text{ta}(l) dl \right)^{-1} \quad (93)$$

$$= 1 \quad (94)$$

it, then, follows that, if we set any relative employment level above the optimal one $\tilde{v}_q^k > \tilde{w}_q^k, \tilde{v}_q^k, \tilde{w}_q^k \in [0, 1]$, operational output will be lower, for

$$\mathcal{U}_q(\tilde{v}_q^k) = \min \left[\tilde{v}_q^k \left(\int_0^{\tilde{T}_q^k} \text{ta}(l) dl \right)^{-1}, (1 - \tilde{v}_q^k) \left(\int_{\tilde{T}_q^k}^1 \text{ta}(l) dl \right)^{-1} \right] \quad (95)$$

$$= \min \left[\frac{\tilde{v}_q^k}{\int_0^{\tilde{T}_q^k} \text{ta}(l) dl}, \frac{1 - \tilde{v}_q^k}{\int_{\tilde{T}_q^k}^1 \text{ta}(l) dl} \right] \quad (96)$$

$$= \min \left[\frac{\tilde{v}_q^k}{\int_0^{\tilde{T}_q^k} \text{ta}(l) dl}, \frac{1 - \tilde{v}_q^k}{1 - \int_0^{\tilde{T}_q^k} \text{ta}(l) dl} \right] \quad (97)$$

$$= \min \left[\frac{\tilde{v}_q^k}{\tilde{w}_q^k}, \frac{1 - \tilde{v}_q^k}{1 - \tilde{w}_q^k} \right] \therefore \quad (98)$$

$$\tilde{v}_q^k > \tilde{w}_q^k \wedge \tilde{v}_q^k, \tilde{w}_q^k \in [0, 1] \implies \frac{\tilde{v}_q^k}{\tilde{w}_q^k} > \frac{1 - \tilde{v}_q^k}{1 - \tilde{w}_q^k} \quad (99)$$

$$\therefore \min \left[\frac{\tilde{v}_q^k}{\tilde{w}_q^k}, \frac{1 - \tilde{v}_q^k}{1 - \tilde{w}_q^k} \right] = \frac{1 - \tilde{v}_q^k}{1 - \tilde{w}_q^k} \quad (100)$$

$$= \frac{1 - \tilde{v}_q^k}{\int_{\tilde{T}_q^k}^1 \text{ta}(l) dl} \quad (101)$$

$$\therefore \frac{1 - \tilde{v}_q^k}{\int_{\tilde{T}_q^k}^1 \text{ta}(l) dl} < \frac{1 - \tilde{w}_q^k}{\int_{\tilde{T}_q^k}^1 \text{ta}(l) dl} \quad (102)$$

$$= \frac{\tilde{w}_q^k}{\int_0^{\tilde{T}_q^k} \text{ta}(l) dl} \quad (103)$$

$$= 1 \quad (104)$$

In other words, one day is sufficient to produce a “day’s work”. Or, put another way, the most one can produce in a day is a “day’s work”.

22. Independent Operational Output Lemma (IOOL)

22.1. From Weak Occupational Complexity Axiom (WOCA)

Let us, then, first consider an independent production strategy with maximum worker productivity (i.e. the “ceiling” or “benchmark” for all other production strategies). Furthermore, because we want to compare this production strategy to that of maximum-monotonic labor stratification, we shall analyze operational output in a stratified fashion as well.

In this scenario, the optimal choice of hours to allocate to a responsibility spectrum is trivial:

$$w_q^\ell = \Omega_q^\ell \in [0, 1]. \quad (105)$$

Now, by the definition of partial operational output (ref) above, one produces \mathcal{U}_q^ℓ when devoting their entire unitary time allowance to the

$$l \in \left[\frac{\ell - 1}{w_q}, \frac{\ell}{w_q} \right], \ell \in \{1, \dots, w_q\} \quad (106)$$

responsibility spectrum. Therefore, with only $\Omega_q^\ell \in [0, 1]$ time units, operational output is:

$$\Omega_q^\ell \mathcal{U}_q^\ell = \left(\frac{1}{\mathcal{U}_q^\ell} \right) \times \mathcal{U}_q^\ell = 1. \quad (107)$$

And, as Weak Occupational Complexity implies operational output is homothetic, the aggregate production of w_q perfectly qualified employees working independently is:

$$w_q^\ell \mathcal{U}_q^\ell \times w_q = \Omega_q^\ell \mathcal{U}_q^\ell \times w_q = 1 \times w_q = w_q. \quad (108)$$

Hence, a perfectly qualified employee working full-time and independently can output one unit of an occupation's complex tasks with one unit of their time (i.e. their entire time allowance). And, likewise, a workforce with w_q employees identical to this one produces w_q units of operational output. Or, to put it simply, a maximally productive person achieves maximum production.

23. Productivity Sufficiency Lemma (PSL)

23.1. dsds

24. Maximum-Monotonic Labor Stratification Lemma (MLSL)

24.1. From Employer Rationality Axiom, Weak Skill Difference Axiom, and Weak Occupational Complexity Axiom.

24.1.1. Now, to generalize this conclusion for other economic configurations, we shall define notation in terms of maximum labor stratification, that is a productive arrangement in which there are not one (“homogeneous” or “independent”), nor two (“juniors” and “seniors”), but rather several job subtypes, indeed as many as there are jobs themselves, each with a limited spectrum of responsibilities. Furthermore, we shall demonstrate that, given our axioms, such an economic configuration is, in fact, the only optimal production strategy and, so, holds in the labor market. Hence, the Maximum-Monotonic Labor Stratification Lemma (MLSL) states that a perfectly rational employer (ERA), which expects there could be skill differences in the workforce (WSDA), and can split operational output without either gain or loss to production (WOCA), will, therefore, strategically stratify their job offers monotonically, and even maximally, so that, if indeed there happens to be skill differences in the labor market, they can, then, allocate less competent workers to easier roles, and avoid wasting talent, thus “saving their best” for the most demanding tasks.

Mathematically,

$$l \in \left[\frac{\ell - 1}{w_q}, \frac{\ell}{w_q} \right], \ell \in \{1, \dots, w_q\} \quad (109)$$

is a responsibility spectrum in a maximally stratified labor market, in which employment levels are given by

$$\sum_{\ell=1}^{w_q} w_q^\ell = w_q, \quad (110)$$

so that any available position is its own job subtype and covers only a restrictive range of task difficulty, accounting for

$$\Omega_q^\ell = \frac{1}{\bar{U}_q^\ell} = \int_{\frac{\ell-1}{w_q}}^{\frac{\ell}{w_q}} \text{ta}(l) dl \quad (111)$$

of an operation's total time duration

$$\sum_{\ell=1}^{w_q} \Omega_q^\ell = \sum_{\ell=1}^{w_q} \int_{\frac{\ell-1}{w_q}}^{\frac{\ell}{w_q}} \text{ta}(l) dl = \int_0^1 \text{ta}(l) dl = 1. \quad (112)$$

Intuitively speaking, we would say production in a maximally and monotonically stratified labor market is not “independent”, [in the sense that each employee does not work on the entire operation from beginning to end. This means each employee] will spend all their time allowance producing a partial operational output, that is a multiple of the difficulty subinterval they were hired to accomplish, which will, in turn, contribute, alongside the partial outputs of other employees, to accomplish the complete occupational operation.

However, in a maximally stratified labor market, these partial operational outputs, will not be produced merely via “senior” and “junior” positions, as previously, but rather within a myriad of levels in a production hierarchy, each responsible for a particular subinterval of task difficulty, approximating a continuum of “seniority” as the workforce becomes large enough.

Again, this does not mean employees are, actually, more or less competent, only that the available positions are [preemptively] stratified with respect to task difficulty.

And, indeed, regardless of workers' actual capacity, this arrangement must hold, because the expected value of operational output – and, therefore, of producers' revenue – is higher and constant when applying a stratified production strategy when compared to an independent production strategy; for in such a strategy, production is more easily limited by skill differences, and so the expected value of operational output is potentially lower, but never higher. That is,

$$\mathbb{E}[\tilde{T}_q^k] \in [0, 1] \quad \forall k, q \in \{1, \dots, n\} \quad (113)$$

implies that

$$\int_0^1 T(l, \tilde{T}_q^k) \text{ta}(l) dl \quad (114)$$

$$\leq \tilde{v}_q^k \int_0^1 T(l, \tilde{T}_q^k) \text{ta}(l) dl + (1 - \tilde{v}_q^k) \int_0^1 T(l, 1) \text{ta}(l) dl \quad (115)$$

$$\leq \tilde{w}_q^k \int_0^{\tilde{T}_q^k} T(l, \tilde{T}_q^k) \text{ta}(l) dl + (1 - \tilde{w}_q^k) \int_{\tilde{T}_q^k}^1 T(l, 1) \text{ta}(l) dl \quad (116)$$

$$= \sum_{k=1}^{n?} \sum_{\ell=1}^{w_q} [k \in \Lambda_q^\ell] \left[\tilde{T}_q^k \geq \frac{\ell}{w_q} \right] \tilde{w}_q^\ell \int_{\frac{\ell-1}{w_q}}^{\frac{\ell}{w_q}} \text{ta}(l) dl \quad (117)$$

$$= \int_0^1 T(l, 1) \text{ta}(l) dl \quad (118)$$

$$= \int_0^1 \text{ta}(l) dl \quad (119)$$

$$= 1, \quad (120)$$

where $\tilde{v}_q^k > \tilde{w}_q^k$, with $\tilde{v}_q^k, \tilde{w}_q^k \in [0, 1]$, is an inefficient allocation of workers above the optimal relative employment level \tilde{w}_q^k in a semi-stratified labor market; and the double sum in equation (ref) is the output of a maximally stratified labor market, in which every $\ell \in \{1, \dots, w_q\}$ job subtype is but a fraction of available positions, with a partial workforce of $w_q \times \tilde{w}_q^\ell$ individuals, all exclusively dedicated to their own responsibility spectrum, and identified by $[k \in \Lambda_q^\ell]$ employment statuses that are evaluated to 1 if they are employed in a particular Λ_q^ℓ strata of the labor market, and to 0 for all other job subtypes; while the remaining equations are the maximum operational output of a labor market with w_q perfectly qualified employees working independently on the entire responsibility spectrum of occupation q 's operation. Or, more succinctly,

$$\mathbb{E}[\mathcal{U}_q^{\text{IP}} \mid \mathbb{E}[\tilde{T}_q^k] \in [0, 1] \forall k, q \in \{1, \dots, n\}] \quad (121)$$

$$\leq \mathbb{E}[\mathcal{U}_q^{\text{IS}} \mid \mathbb{E}[\tilde{T}_q^k] \in [0, 1] \forall k, q \in \{1, \dots, n\}] \quad (122)$$

$$\leq \mathbb{E}[\mathcal{U}_q^{\text{MS}} \mid \mathbb{E}[\tilde{T}_q^k] \in [0, 1] \forall k, q \in \{1, \dots, n\}] \quad (123)$$

$$= \mathbb{E}[\mathcal{U}_q^{\text{IP}} \mid \tilde{T}_q^k = 1 \forall k, q \in \{1, \dots, n\}], \quad (124)$$

where each of the terms above represents the expected value of aggregate operational output given the expected productivity in the workforce, for the three production strategies: maximum-monotonic labor stratification (MS), imperfect-monotonic labor stratification (IS), and independent production (IP).

In other words, splitting responsibilities in accordance with competence is always as productive as the maximum operational output (viz. that which is obtained when employing perfectly qualified workers independently), provided employees are sufficiently qualified for their responsibilities. But, again, this is, by definition, guaranteed by employers' rationality, as well as the simple fact the economy is already producing its current operational output (Operational Equilibrium Lemma, OEL).

Therefore, employing potentially underqualified workers to output the entire responsibility spectrum $l \in [0, 1]$ independently can only be as productive as the labor stratification strategy, but never more than it. Independent production, then, is a suboptimal strategy when employers expect there to be skill differences in the workforce.

Thus, maximum-monotonic labor stratification follows as an insurance policy against worker's potential underqualification: for if talent is lacking in the labor market, there is nothing to gain by employing individuals which are not sufficiently qualified for a difficult job, whereas if talent is abundant, there is nothing to lose when employing overqualified individuals to a job below their skill level.

Hence, given the same w_q workforce, operational output in a maximally stratified labor market is always greater or equal to the output of any other economic configuration. It is, therefore, always optimal to monotonically and maximally stratify responsibilities across w_q unique positions, each focused on increasingly demanding tasks.

24.2. Monotonic labor stratification is required and follows logically from employers' perfect rationality axiom.

24.3. maximum labor stratification is optional, but also follows logically from employers' perfect rationality axiom.

24.3.1. Because the General Employability Theorem (GET) holds true for imperfectly stratified labor markets as well, for less than maximum labor stratification is mathematically equivalent to just a variable change. This said, imperfect labor market stratification leads to inefficiencies in hiring, as the base requirements for each stratum are higher than they would be if labor was maximally stratified.

Finally, to identify the optimal \tilde{w}_q^ℓ relative employment levels in the labor market, and thus calculate a generalized employability coefficient, we compare the two production strategies that, as demonstrated, guarantee maximum operational output (viz. maximum-monotonic labor stratification and perfectly qualified independent production).

25. Equivalent Operational Output Lemma (EOOL)

25.1. Now, as WOCA states there are no gains nor losses in production due to partial operational outsourcing, the following must hold, in any labor market, even in a maximally stratified one:

$$w_q^\ell \mathcal{U}_q^\ell = w_q, \quad (125)$$

where w_q is the aggregate operational output of the

$$l \in \left[\frac{\ell-1}{w_q}, \frac{\ell}{w_q} \right], \ell \in \{1, \dots, w_q\} \quad (126)$$

responsability spectrum of q 's complex tasks when w_q perfectly qualified employees work independently (IOOL); and $w_q^\ell \mathcal{U}_q^\ell$ is the production of w_q^ℓ employees with at least minimum qualification working in a maximally stratified labor market, so that

$$\sum_{\ell=1}^{w_q} w_q^\ell = w_q. \quad (127)$$

Or, in other words, operational output is equivalent in any production strategy provided the talent employed is sufficiently qualified.

And, with this, we derive the Proportional Employment Condition (PEC):

$$w_q^\ell \mathcal{U}_q^\ell = w_q \iff \frac{w_q^\ell}{w_q} = \Omega_q^\ell \in [0, 1], \quad (128)$$

which determines the ratio, or proportion, of a particular job subtype in a stratified labor market is exactly the percentage of an operation's total time duration accounted by it. Thus, activities that require more time also require more dedicated employees working on them full-time, and vice-versa.

Notice, too, this is the same result we had found in the binary stratification example above (of a labor market with only “junior” and “senior” employees):

$$\tilde{w}_q^{\text{Jr}} = \frac{w_q^{\text{Jr}}}{w_q^{\text{Jr}} + w_q^{\text{Sr}}} = \frac{w_q^{\text{Jr}}}{w_q} = \int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}_q(l) dl = \Omega_q^{\text{Jr}}. \quad (129)$$

Therefore, this condition must hold in any efficient labor market, irrespective of its stratification's “granularity”.

26. Definition of aggregate employability in a maximally and monotonically stratified labor market is:

$$\tilde{W}_k = \sum_{q=1}^n \tilde{W}_q^k, \quad (130)$$

where

$$\tilde{W}_q^k = \sum_{\ell=1}^{w_q} \left[h_q^k \geq \frac{1}{2} \right] \left[\tilde{T}_q^k \geq \frac{\ell}{w_q} \right] \tilde{w}_q^\ell \quad (131)$$

$$= \sum_{\ell=1}^{w_q} \left[h_q^k \geq \frac{1}{2} \right] \left[\tilde{T}_q^k \geq \frac{\ell}{w_q} \right] \frac{w_q^\ell}{w_q} \quad (132)$$

is partial employability, that is one's employability in a particular occupation q in the labor market.

$$\tilde{W}_k = \left(\frac{1}{W} \right) \sum_{q=1}^n \sum_{v=1}^{w_q} \left[h_q^k \geq \frac{1}{2} \right] \left[\tilde{T}_q^k \geq \ell_v \right] \quad (133)$$

$$= \left(\frac{1}{W} \right) \sum_{q=1}^n \sum_{v=1}^{w_q} \left[h_q^k \geq \frac{1}{2} \right] \left[\tilde{T}_q^k \geq \text{TA}_q^{-1} \left(\frac{v}{w_q} + \text{TA}_q(0) \right) \right] \quad (134)$$

27. Competitiveness in a maximally-monotonically stratified labor market with irregular responsibility ranges (i.e. without partial hiring):

$$\tilde{v}s_k = \left(\frac{1}{W} \right) \sum_{q=1}^n \left[\ddot{u}_k^q \geq \frac{1}{2} \right] \sum_{v=1}^{w_k} \left[h_k^q \geq \frac{1}{2} \right] \left[\tilde{T}_k^q \geq \ell_v \right] \quad (135)$$

$$= \left(\frac{1}{W} \right) \sum_{q=1}^n \left[\ddot{u}_k^q \geq \frac{1}{2} \right] \sum_{v=1}^{w_k} \left[h_k^q \geq \frac{1}{2} \right] \left[\tilde{T}_k^q \geq \text{TA}_k^{-1} \left(\frac{v}{w_k} + \text{TA}_k(0) \right) \right] \quad (136)$$

Competitiveness in a maximally-monotonically stratified labor market with irregular responsibility ranges (i.e. without partial hiring):

$$\tilde{v}s_k = \left(\frac{1}{W} \right) \sum_{q=1}^n \left[\ddot{u}_k^q \geq \frac{1}{2} \right] \sum_{v=1}^{w_k} \left[h_k^q \geq \frac{1}{2} \right] \left[\tilde{T}_k^q \geq \ell_v \right] \quad (137)$$

$$= \left(\frac{1}{W} \right) \sum_{q=1}^n \left[\ddot{u}_k^q \geq \frac{1}{2} \right] \sum_{v=1}^{w_k} \left[h_k^q \geq \frac{1}{2} \right] \left[\tilde{T}_k^q \geq \text{TA}_k^{-1} \left(\frac{v}{w_k} + \text{TA}_k(0) \right) \right] \quad (138)$$

P.S.: SSL

$$d_i^k := \frac{l_i^k}{l_i} \in [0, 1] \quad (139)$$

$$\tilde{T}_i^k = \int_0^{l_i} T(l, l_i^k) dl \left(\int_0^{l_i} T(l, l_i) dl \right)^{-1} \quad (140)$$

$$= \left(\int_0^{l_i^k} 1 \times dl + \int_{l_i^k}^{l_i} 0 \times dl \right) \times \left(\int_0^{l_i} 1 \times dl \right)^{-1} \quad (141)$$

$$= \frac{l_i^k - 0}{l_i - 0} \quad (142)$$

$$= \frac{l_i^k}{l_i} \quad (143)$$

$$\therefore a_i^k = \tilde{T}_i^k \quad (144)$$

P.S.: SCL

$$\mathbf{a}_k := (a_1^k, \dots, a_m^k), \mathbf{a}_q := (a_1^q, \dots, a_m^q) \in [0, 1]^m \quad \forall k, q \in \{1, \dots, n\} \quad (145)$$

$$l_q^k \leq l_q^q \quad \forall k, q \in \{1, \dots, n\} \quad (146)$$

$$\tilde{T}_q^k = \int_0^{l_q^q} T(l, l_q^k) dl \left(\int_0^{l_q^q} T(l, l_q^q) dl \right)^{-1} \quad (147)$$

$$= \left(\int_0^{l_q^k} 1 \times dl + \int_{l_q^k}^{l_q^q} 0 \times dl \right) \times \left(\int_0^{l_q^q} 1 \times dl \right)^{-1} \quad (148)$$

$$= \frac{l_q^k - 0}{l_q^q - 0} \quad (149)$$

$$= \frac{l_q^k}{l_q^q} \in [0, 1] \quad (150)$$

$$\because \neg l_q^k > l_q^q \quad (151)$$

$$\because l_q^k = f(\mathbf{l}_k, \mathbf{l}_q) = f(\mathbf{a}_k, \mathbf{a}_q)? \in [0, 1] \quad \forall k, q \in \{1, \dots, n\} \quad (152)$$

$$\therefore \tilde{T}_q^k = f(\mathbf{a}_k, \mathbf{a}_q) \quad (153)$$