The Employability Theorem

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Abstract

In this document, the Employability Theorem is demonstrated from a set of fairly tautological axioms, which are presupposed in quantitative career choice and career development methods.

Keywords: Employability theorem; Career choice; Career development; Vocational choice; Occupational Information Network; O*NET.

1. Proof Plan

1. basic presuppositions
2. basic lemmas
3. complex tasks
4. occupations are but tasks
5. occupations' tasks are complex
6. occupations' tasks are holistic (operation)

   6.1. more difficult tasks presuppose the easier tasks have been accomplished

   6.2. i.e. $l \in [0, 1]$ is a "progress bar" of an occupation's operation

   6.3. strongly holistic: each task $l \geq \bar{l}$ requires all the previous $l \in [0, \bar{l}], \bar{l} \in [0, 1]$ difficulty levels to be accomplished. in addition, if all $l \in [0, 1]$ levels are not all accomplished, the whole effort is vain and the operation is not completed (i.e. round down $\mho_q$ when calculating operational output). furthermore, each and every $l \in [0, 1]$ difficulty level cannot be outsourced (i.e. only a perfectly qualified worker can output a unit of the occupation's operation).

      6.3.1. individual's time constraint is spent entirely on trying to accomplish the complex holistic task by themselves. therefore, there is no optimization to be done.

6.3.2.

$$\mho_q = \sum_{k=1}^{n} \left\lfloor [k \in \Lambda_q] \times \mho_q^k \right\rfloor = \sum_{k=1}^{n} \left\lfloor [k \in \Lambda_q] \times \int_0^1 T_q(l, l_q^k) dl \right\rfloor$$

6.4. moderately holistic: each task $l \geq \bar{l}$ requires all the previous $l \in [0, \bar{l}], \bar{l} \in [0,1]$ difficulty levels to be accomplished. in addition, if all $l \in [0,1]$ levels are not all accomplished, the whole effort is vain and the operation is not completed (i.e. round down $\mho_q$ when calculating operational output). however, each and every $l \in [0,1]$ difficulty level can be outsourced (i.e. workers can output partial units of the occupation's operation, which contribute to the operation's completion).

6.4.1. because of outsourcing, individual's time constraint is spent working from where another worker "left off", so that even if a worker cannot accomplish the entire operation by themselves, they can still contribute to the operation's completion by reducing the time highly skilled workers will have to spend on relatively more trivial tasks.
The first worker spends their entire unitary time allowance trying their hardest to accomplish the highest amount of tasks they can. When they hit their skill cap, they restart their efforts, so as to spend their entire time allowance helping out the next worker:

$$\int_0^{\tilde{T}_q^k} T_q(l, l_q^k) \times \mathrm{ta}_q(l) dl + \int_0^{\bar{l}} T_q(l, l_q^k) \times \mathrm{ta}_q(l) dl = 1$$
$$\int_0^{\bar{l}} 1 \times \mathrm{ta}_q(l) dl = 1 - \int_0^{\tilde{T}_q^k} 1 \times \mathrm{ta}_q(l) dl$$
$$\int_0^{\bar{l}} \mathrm{ta}_q(l) dl = \int_{\tilde{T}_q^k}^1 \mathrm{ta}_q(l) dl$$
$$\mathrm{TA}_q(\bar{l}) - \mathrm{TA}_q(0) = \mathrm{TA}_q(1) - \mathrm{TA}_q(\tilde{T}_q^k)$$
$$\mathrm{TA}_q(\bar{l}) = \mathrm{TA}_q(1) - \mathrm{TA}_q(\tilde{T}_q^k)$$
$$\bar{l} = \mathrm{TA}_q^{-1}\left(\mathrm{TA}_q(1) - \mathrm{TA}_q(\tilde{T}_q^k)\right),$$

so that k accomplishes tasks of difficulty levels 0 through $\tilde{T}_q^k$ on their "first run", and restarts their effort to provide additional $l \in \left[0, \mathrm{TA}_q^{-1}\left(\mathrm{TA}_q(1) - \mathrm{TA}_q(\tilde{T}_q^k)\right)\right]$ levels worth of complex tasks. Thus, the next worker does not need to start from zero, but rather from where k "left off": either $\tilde{T}_q^k$, $\bar{l}$, or some $l \in [0, \tilde{T}_q^k]$.

6.4.2.

$$\mho_q = \left\lfloor \sum_{k=1}^{n} [k \in \Lambda_q] \times \mho_q^k \right\rfloor$$

$$= \left\lfloor \sum_{k=1}^{n} [k \in \Lambda_q] \times \int_0^1 T_q(l, l_q^k) dl \right\rfloor$$

6.5. weakly holistic: each task $l \geq \bar{l}$ requires all the previous $l \in [0, \bar{l}], \bar{l} \in [0, 1]$ difficulty levels to be accomplished. however, if not all $l \in [0, 1]$ levels are accomplished, the whole effort is not vain and the operation is partially completed (i.e. do not round $\mho_q$ when calculating operational output). furthermore, each and every $l \in [0, 1]$ difficulty level can be outsourced (i.e. workers can output partial units of the occupation's operation, which contribute to the operation's completion).

7. assume weak occupational complexity axiom (the other versions are too strict)

8. perhaps posit an even weaker version of occupational complexity:

8.1.

$$\frac{\partial \mho_q}{\partial l} > 0, \tag{1}$$

$$\frac{\partial^2 \mho_q}{\partial l^2} < 0, \tag{2}$$

so that even though tasks of a particular level are not required for the operation to "count" (i.e. partial delivery), it is still detrimental to focus too much on one subset of tasks, that is, employers are incentivised to produce the entire spectrum of difficulty levels, because marginal productivity increases when a tasks of a particular difficulty level have not been accomplished yet.

(actually, we need a indicator variable for the amount of tasks accomplished for a difficulty level, something analogous to $T_q(l)$)

9. now, because of weak occupational complexity, employers will maximize operational output by attempting to produce the entire spectrum of difficulty levels for the complex tasks of an occupation.

10. this can be done either by having only perfectly qualified employees work on the operation individually from beginning to end, or by splitting responsabilities into two, or more, types of jobs, thus allowing for less qualified, "junior" employees, to work alongside more qualified and perfectly qualified, "senior" employees towards the common goal of accomplishing the entire occupational operation.

11. additionally, because there are skill differences among workers in the labor market, any rational employer will always, and rightly, expect their employees to be of varying skill levels, rather than all perfectly qualified, so that splitting responsabilities into separate positions will not only be an alternative mode of hiring and producing, but in fact the optimal one.

12. therefore, given expected and actual skill differences among workers, employers will split job posts based on the required skill level. thus, there will be "junior" job posts and "senior" job posts, each dedicated to accomplishing a particular subset of complex tasks with difficulty levels appropriate for employees' respective capacity.

13. notice this does not mean all people working on "junior" positions will, necessarily, be "junior" employees themselves, that is, less qualified. indeed, if talent is abundant in the labor market, these "junior" positions will have to be filled by more qualified, or even perfectly qualified, "senior" employees. for if there were only one type of job, spanning the entire difficulty level spectrum, highly qualified workers would already have to accomplish these "junior" tasks themselves, in order to maximize operational output. however, by having two, or more, types of jobs, split by minimum required competence, highly qualified workers may specialize to the measure that there are less qualified workers available to accomplish the easier tasks. but, if there are none, they will, again, have to work on these themselves.

14. analogously, from the employers' perspective, it does not matter who accomplishes "junior" tasks, so long as they are accomplished. thus, if highly qualified workers are abundant in a particular time period of a labor market, production is not hindered when allocating "seniors" to "junior" positions, for in these circumstances talent is not wasted. that is, because only highly qualified workers can accomplish highly demanding tasks, rational employers will generally not hire them to work on "junior" tasks, thus "saving" their talent for more difficult tasks, which a "junior" would not be able to accomplish. but, if there is enough talent to output the optimal quantity of "senior" tasks, it can actually be more productive to employ the remaining "seniors" to "junior" positions.

15. furthermore, in a continuous setting, rational employers will maximize their hiring pool by offering more than only two types of jobs. thus, there will not only be "senior" and "junior" positions, but several levels in a production hierarchy, each responsible for a particular subinterval of task difficulty, which will approximate a continuum of "seniority" as the number of workers becomes large enough.

16. now, as for employees' work routine, rational employers will have them work over their responsability spectrum in a proportional and optimal matter, thus avoiding wasting production (i.e. uncompleted "loops" over the responsability spectrum). this means each employee will spend their entire time allowance producing a partial operational output, that is a multiple of the difficulty subinterval they were hired to accomplish, which will, in turn, contribute, alonside the partial outputs of other employees, to accomplish the entire occupational operation.

17. the reason this avoids wasting production is because [...].

18. (Binary Employability Theorem) thus, in the binary case, “junior” productive output will be given by:

$$\mho_q^{\mathrm{Jr}} = \frac{1}{\int_0^{\tilde{Y}_q^{\mathrm{Jr}}} \mathrm{ta}_q(l)dl} = \left(\int_0^{\tilde{Y}_q^{\mathrm{Jr}}} \mathrm{ta}_q(l)dl\right)^{-1}, \tag{3}$$

where $\mathrm{ta}_q(l)$ is the time allocation function of occupation q’s complex tasks, and time allowance (the numerator) is set to one.

19. analogously, “senior” productive output is:

$$\mho_q^{\mathrm{Sr}} = \frac{1}{\int_{\tilde{Y}_q^{\mathrm{Jr}}}^1 \mathrm{ta}_q(l)dl} = \left(\int_{\tilde{Y}_q^{\mathrm{Jr}}}^1 \mathrm{ta}_q(l)dl\right)^{-1}. \tag{4}$$

20. finally, as a mismatch in productive output due to time allocation differences between “junior” and “senior” tasks would result in wasted production, a rational employer will optimally “orchestrate” the productive effort by offering just enough “senior” job posts in the labor market to meet “junior” productivity. thus, by setting “junior” job posts to $w_q^{\mathrm{Jr}} > 0$ and “senior” job posts to $w_q^{\mathrm{Sr}} > 0$, we get the ratio between “junior” and “senior” positions required to output any level of occupation q’s operation:

$$w_q^{\mathrm{Sr}} \times \mho_q^{\mathrm{Sr}} = w_q^{\mathrm{Jr}} \times \mho_q^{\mathrm{Jr}} \therefore \tag{5}$$

$$w_q^{\mathrm{Sr}} \times \left(\int_{\tilde{Y}_q^{\mathrm{Jr}}}^1 \mathrm{ta}_q(l)dl\right)^{-1} = w_q^{\mathrm{Jr}} \times \left(\int_0^{\tilde{Y}_q^{\mathrm{Jr}}} \mathrm{ta}_q(l)dl\right)^{-1} \therefore \tag{6}$$

$$w_q^{\mathrm{Sr}} = w_q^{\mathrm{Jr}} \times \left(\frac{\int_{\tilde{Y}_q^{\mathrm{Jr}}}^1 \mathrm{ta}_q(l)dl}{\int_0^{\tilde{Y}_q^{\mathrm{Jr}}} \mathrm{ta}_q(l)dl}\right). \tag{7}$$

with this, “senior” employability (the percentage of job posts for which they could be hired) is

$$\tilde{w}_q^{\mathrm{Sr}} = \frac{w_q^{\mathrm{Jr}} + w_q^{\mathrm{Sr}}}{w_q^{\mathrm{Jr}} + w_q^{\mathrm{Sr}}} = 1 \tag{8}$$

and “junior” employability is

$$\tilde{w}_q^{\mathrm{Jr}} = \frac{w_q^{\mathrm{Jr}}}{w_q^{\mathrm{Jr}} + w_q^{\mathrm{Sr}}} \tag{9}$$

$$= \frac{w_q^{\mathrm{Jr}}}{w_q^{\mathrm{Jr}} + w_q^{\mathrm{Jr}} \times \left(\frac{\int_{\tilde{Y}_q^{\mathrm{Jr}}}^1 \mathrm{ta}_q(l)dl}{\int_0^{\tilde{Y}_q^{\mathrm{Jr}}} \mathrm{ta}_q(l)dl}\right)} \tag{10}$$

$$= \left(1 + \frac{\int_{\tilde{Y}_q^{\mathrm{Jr}}}^{1} \mathrm{ta}_q(l)dl}{\int_0^{\tilde{Y}_q^{\mathrm{Jr}}} \mathrm{ta}_q(l)dl} \right)^{-1} \tag{11}$$

$$= \left(1 + \frac{\int_0^1 \mathrm{ta}_q(l)dl - \int_0^{\tilde{Y}_q^{\mathrm{Jr}}} \mathrm{ta}_q(l)dl}{\int_0^{\tilde{Y}_q^{\mathrm{Jr}}} \mathrm{ta}_q(l)dl} \right)^{-1} \tag{12}$$

$$= \left(1 + \frac{1 - \int_0^{\tilde{Y}_q^{\mathrm{Jr}}} \mathrm{ta}_q(l)dl}{\int_0^{\tilde{Y}_q^{\mathrm{Jr}}} \mathrm{ta}_q(l)dl} \right)^{-1} \tag{13}$$

$$= \left(1 + \frac{1}{\int_0^{\tilde{Y}_q^{\mathrm{Jr}}} \mathrm{ta}_q(l)dl} - \frac{\int_0^{\tilde{Y}_q^{\mathrm{Jr}}} \mathrm{ta}_q(l)dl}{\int_0^{\tilde{Y}_q^{\mathrm{Jr}}} \mathrm{ta}_q(l)dl} \right)^{-1} \tag{14}$$

$$= \left(1 + \frac{1}{\int_0^{\tilde{Y}_q^{\mathrm{Jr}}} \mathrm{ta}_q(l)dl} - 1 \right)^{-1} \tag{15}$$

$$= \left(\frac{1}{\int_0^{\tilde{Y}_q^{\mathrm{Jr}}} \mathrm{ta}_q(l)dl} \right)^{-1} \tag{16}$$

$$= \int_0^{\tilde{Y}_q^{\mathrm{Jr}}} \mathrm{ta}_q(l)dl. \tag{17}$$

thus, the employability of a partially qualified worker, that is a “junior”, is precisely the percentage of an operation’s total time duration their skill set allows them to accomplish (i.e. the inverse of their operational output).

21. (Multiple Employability Theorem) Let us now work towards a generalized employability theorem by considering an occupation q whose operation can be broken down into multiple types of jobs in a hierarchy: one requiring perfect qualification (a senior position); one requiring moderate skill levels (a mid-junior / mid-senior position); the other requiring somewhat less skill (a junior position); and the last one requiring very little skill (an intern position).

    Each of these job types is indispensable and equally contributes to occupation q’s operation, so that operational output is given by the Leontief function:

$$\mho_q = \min\left(w_q^{\mathrm{Ir}} \mho_q^{\mathrm{Ir}}, w_q^{\mathrm{Jr}} \mho_q^{\mathrm{Jr}}, w_q^{\mathrm{Mr}} \mho_q^{\mathrm{Mr}}, w_q^{\mathrm{Sr}} \mho_q^{\mathrm{Sr}} \right). \tag{18}$$

    With this, a rational employer will avoid wasting production by setting:

$$w_q^{\mathrm{Ir}} \mho_q^{\mathrm{Ir}} = w_q^{\mathrm{Jr}} \mho_q^{\mathrm{Jr}} = w_q^{\mathrm{Mr}} \mho_q^{\mathrm{Mr}} = w_q^{\mathrm{Sr}} \mho_q^{\mathrm{Sr}}, \tag{19}$$

    which yields

$$w_q^{\mathrm{Jr}} = w_q^{\mathrm{Ir}} (\mho_q^{\mathrm{Ir}} / \mho_q^{\mathrm{Jr}}); \tag{20}$$

$$w_q^{\mathrm{Mr}} = w_q^{\mathrm{Ir}}(\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Mr}}); \tag{21}$$

$$w_q^{\mathrm{Sr}} = w_q^{\mathrm{Ir}}(\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Sr}}). \tag{22}$$

Finally, employability will be calculated by the job types a person is qualified for based on their productivity. For instance, a person with senior-level productivity (i.e. a perfectly qualified worker) will be employable on every role:

$$\tilde{w}_q^{\mathrm{Sr}} = \frac{w_q^{\mathrm{Ir}} + w_q^{\mathrm{Jr}} + w_q^{\mathrm{Mr}} + w_q^{\mathrm{Sr}}}{w_q^{\mathrm{Ir}} + w_q^{\mathrm{Jr}} + w_q^{\mathrm{Mr}} + w_q^{\mathrm{Sr}}} = 1. \tag{23}$$

However, a person less qualified than a senior, say with moderate productivity, will be eligible for every position but senior-level ones:

$$\begin{aligned}
\tilde{w}_q^{\mathrm{Mr}} &= \frac{w_q^{\mathrm{Ir}} + w_q^{\mathrm{Jr}} + w_q^{\mathrm{Mr}}}{w_q^{\mathrm{Ir}} + w_q^{\mathrm{Jr}} + w_q^{\mathrm{Mr}} + w_q^{\mathrm{Sr}}} && (24)\\
&= \frac{w_q^{\mathrm{Ir}} + w_q^{\mathrm{Ir}}(\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Jr}}) + w_q^{\mathrm{Ir}}(\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Mr}})}{w_q^{\mathrm{Ir}} + w_q^{\mathrm{Ir}}(\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Jr}}) + w_q^{\mathrm{Ir}}(\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Mr}}) + w_q^{\mathrm{Ir}}(\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Sr}})} && (25)\\
&= \frac{1 + (\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Jr}}) + (\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Mr}})}{1 + (\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Jr}}) + (\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Mr}}) + (\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Sr}})}. && (26)
\end{aligned}$$

Likewise, juniors and interns will be employable on an even smaller portion of jobs, respectively:

$$\begin{aligned}
\tilde{w}_q^{\mathrm{Jr}} &= \frac{w_q^{\mathrm{Ir}} + w_q^{\mathrm{Jr}}}{w_q^{\mathrm{Ir}} + w_q^{\mathrm{Jr}} + w_q^{\mathrm{Mr}} + w_q^{\mathrm{Sr}}} && (27)\\
&= \frac{w_q^{\mathrm{Ir}} + w_q^{\mathrm{Ir}}(\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Jr}})}{w_q^{\mathrm{Ir}} + w_q^{\mathrm{Ir}}(\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Jr}}) + w_q^{\mathrm{Ir}}(\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Mr}}) + w_q^{\mathrm{Ir}}(\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Sr}})} && (28)\\
&= \frac{1 + (\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Jr}})}{1 + (\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Jr}}) + (\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Mr}}) + (\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Sr}})} && (29)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{w}_q^{\mathrm{Ir}} &= \frac{w_q^{\mathrm{Ir}}}{w_q^{\mathrm{Ir}} + w_q^{\mathrm{Jr}} + w_q^{\mathrm{Mr}} + w_q^{\mathrm{Sr}}} && (30)\\
&= \frac{w_q^{\mathrm{Ir}}}{w_q^{\mathrm{Ir}} + w_q^{\mathrm{Ir}}(\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Jr}}) + w_q^{\mathrm{Ir}}(\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Mr}}) + w_q^{\mathrm{Ir}}(\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Sr}})} && (31)\\
&= \left(1 + (\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Jr}}) + (\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Mr}}) + (\mho_q^{\mathrm{Ir}}/\mho_q^{\mathrm{Sr}})\right)^{-1}. && (32)
\end{aligned}$$

22. (Generalized Employability Theorem) And, generalizing for any w_q number of job subtypes, in accordance with rational employers' hiring and production stratification strategy,

$$\tilde{w}_q^k = \frac{\sum_{\ell=1}^{w_q}\left[\tilde{Y}_q^k \geq \frac{\ell}{w_q}\right]\left(\mho_q^1/\mho_q^\ell\right)}{\sum_{\ell=1}^{w_q}\left(\mho_q^1/\mho_q^\ell\right)}. \tag{33}$$

As partial operational output is given by

$$\mho_q^\ell = \left(\int_{\frac{\ell-1}{w_q}}^{\frac{\ell}{w_q}} \mathrm{ta}_q(l) dl \right)^{-1}, \tag{34}$$

the employability of person k at occupation q's w_q positions is:

$$\begin{aligned}
\tilde{w}_q^k &= \frac{\sum_{\ell=1}^{w_q} \left[\tilde{Y}_q^k \geq \frac{\ell}{w_q} \right] \frac{\left(\int_{\frac{1-1}{w_q}}^{\frac{1}{w_q}} \mathrm{ta}_q(l) dl \right)^{-1}}{\left(\int_{\frac{\ell-1}{w_q}}^{\frac{\ell}{w_q}} \mathrm{ta}_q(l) dl \right)^{-1}}}{\sum_{\ell=1}^{w_q} \frac{\left(\int_{\frac{1-1}{w_q}}^{\frac{1}{w_q}} \mathrm{ta}_q(l) dl \right)^{-1}}{\left(\int_{\frac{\ell-1}{w_q}}^{\frac{\ell}{w_q}} \mathrm{ta}_q(l) dl \right)^{-1}}} && (35) \\
&= \frac{\sum_{\ell=1}^{w_q} \left[\tilde{Y}_q^k \geq \frac{\ell}{w_q} \right] \left(\frac{\int_{\frac{\ell-1}{w_q}}^{\frac{\ell}{w_q}} \mathrm{ta}_q(l) dl}{\int_0^{\frac{1}{w_q}} \mathrm{ta}_q(l) dl} \right)}{\sum_{\ell=1}^{w_q} \left(\frac{\int_{\frac{\ell-1}{w_q}}^{\frac{\ell}{w_q}} \mathrm{ta}_q(l) dl}{\int_0^{\frac{1}{w_q}} \mathrm{ta}_q(l) dl} \right)}. && (36)
\end{aligned}$$

Or, to keep notation concise,

$$\tilde{w}_q^k = \sum_{\ell=1}^{w_q} \left[\tilde{Y}_q^k \geq \frac{\ell}{w_q} \right] \tilde{\Omega}_q^{\ell 1}, \tag{37}$$

where

$$\Omega_q^\ell = \frac{1}{\mho_q^\ell} = \int_{\frac{\ell-1}{w_q}}^{\frac{\ell}{w_q}} \mathrm{ta}_q(l) dl \tag{38}$$

is the inverse of operational output $\mho_q^\ell$, or the percentage of an operation's total time duration accounted by position ℓ's tasks, and

$$\begin{aligned}
\tilde{\Omega}_q^{\ell 1} &= \left(\frac{\mho_q^1}{\mho_q^\ell} \right) \times \left(\sum_{\ell=1}^{w_q} \frac{\mho_q^1}{\mho_q^\ell} \right)^{-1} && (39) \\
&= \left(\frac{\Omega_q^\ell}{\Omega_q^1} \right) \times \left(\sum_{\ell=1}^{w_q} \frac{\Omega_q^\ell}{\Omega_q^1} \right)^{-1} && (40)
\end{aligned}$$

is normalized inverse operational output.

23. (Aggregate Employability Theorem) With this, the aggregate employability of a person k on the labor market is:

$$\tilde{w}_k = \sum_{q=1}^{n} \left[h_q^k \geq \frac{1}{2} \right] \sum_{\ell=1}^{w_q} \left[\tilde{Y}_q^k \geq \frac{\ell}{w_q} \right] \left(\tilde{\Omega}_q^\ell / \tilde{\Omega}_q^1 \right), \tag{41}$$

where h_q^k is the hireability statistic and the other variables are defined as above. And, for a particular Λ_L subset of the labor market,

$$\tilde{w}_{\Lambda_L}^k = \sum_{q=1}^{n} [q \in \Lambda_L] \left[h_q^k \geq \frac{1}{2} \right] \sum_{\ell=1}^{w_q} \left[\tilde{Y}_q^k \geq \frac{\ell}{w_q} \right] \tilde{\Omega}_q^{\ell 1}. \tag{42}$$

actually, aggregate employability is:

$$\begin{aligned} \tilde{W}_k &= \sum_{q=1}^{n} \sum_{\ell=1}^{w_q} \left[h_q^k \geq \frac{1}{2} \right] \left[\tilde{T}_q^k \geq \frac{\ell}{w_q} \right] \tilde{w}_q^\ell & (43) \\ &= \sum_{q=1}^{n} \sum_{\ell=1}^{w_q} \left[h_q^k \geq \frac{1}{2} \right] \left[\tilde{T}_q^k \geq \frac{\ell}{w_q} \right] \frac{w_q^\ell}{w_q} & (44) \\ &= \sum_{q=1}^{n} \sum_{\ell=1}^{w_q} \left[h_q^k \geq \frac{1}{2} \right] \left[\tilde{T}_q^k \geq \frac{\ell}{w_q} \right] \Omega_q^\ell & (45) \\ &= \sum_{q=1}^{n} \sum_{\ell=1}^{w_q} \left[h_q^k \geq \frac{1}{2} \right] \left[\tilde{T}_q^k \geq \frac{\ell}{w_q} \right] \int_{\frac{\ell-1}{w_q}}^{\frac{\ell}{w_q}} \mathrm{ta}(l) dl & (46) \end{aligned}$$

24. under which assumptions is it optimal to output the entire responsability spectrum? Leontief functional form? Weakly holistic operation:

$$\frac{\partial \mho_q}{\partial w_q^\ell \mho_q^\ell} > 0 \tag{47}$$

$$\frac{\partial^2 \mho_q}{\partial \left(w_q^\ell \mho_q^\ell \right)^2} < 0 \tag{48}$$

$$\frac{\partial^2 \mho_q}{\partial \left(w_q^\ell \mho_q^\ell \right) \partial \left(w_q^r \mho_q^r \right)} > 0 \tag{49}$$

25. Independent Operational Output Lemma (IOOL)

    25.1. From Weak Occupational Complexity Axiom (WOCA)

    25.2. The optimal operational output of a perfectly qualified employee working independently is:

$$w_q^\ell \mho_q^\ell = 1 \iff w_q^\ell = \Omega_q^\ell \in [0, 1], \tag{50}$$

as even the weakest version of occupational complexity implies all difficulty levels are complementary and have to be accomplished for optimal operational output. In this formulation, though, employment $w_q^\ell \in [0,1]$ is partial, that is: the same worker is hired to accomplish every responsability spectrum. This is because such an independent labor market is not stratified at all.

Furthermore, as Weak Occupational Complexity requires operational output to be homothetic, the aggregate production of w_q perfectly qualified employees working independently is:

$$w_q^\ell \mho_q^\ell \times w_q = 1 \times w_q = w_q. \tag{51}$$

26. Equivalent Operational Output Lemma (EOOL)

26.1. Now, as WOCA states there are no gains nor losses in production due to partial operational outsourcing, the following must hold, in any labor market, even in a maximally stratified one:

$$w_q^\ell \mho_q^\ell = w_q, \tag{52}$$

where w_q is the aggregate operational output of the

$$l \in \left[\frac{\ell - 1}{w_q}, \frac{\ell}{w_q}\right] \tag{53}$$

responsability spectrum of q's complex tasks when w_q perfectly qualified employees work independently; and $w_q^\ell \mho_q^\ell$ is the production of w_q^ℓ employees with at least minimum qualification working in a maximally stratified labor market, so that

$$\sum_{\ell=1}^{w_q} w_q^\ell = w_q. \tag{54}$$

Or, in other words, operational output is equivalent in any production strategy provided the talent employed is sufficiently qualified.

Finally, we derive the Proportional Employment Condition (PEC):

$$w_q^\ell \mho_q^\ell = w_q \iff \frac{w_q^\ell}{w_q} = \Omega_q^\ell \in [0,1], \tag{55}$$

which determines the ratio, or proportion, of a particular job subtype in a stratified labor market is exactly the percentage of an operation's total time duration accounted by it. Thus, activities that require more time also require more dedicated employees working on them full-time, and vice-versa.

27. Weak Skill Differences Axiom (WSDA)

27.1. There are, or there could be, skill differences among people in the workforce (i.e. workers are not all “clones” of one another or equally competent). Thus, the expected value of productivity is:

$$\mathbb{E}[\tilde{T}_q^k] \in [0,1], \tag{56}$$

instead of

$$\mathbb{E}[\tilde{T}_q^k] = \tilde{T}_q^k = 1, \tag{57}$$

for all $k, q \in \{1, \ldots, n\}$. This means employers do not expected every worker to be perfectly qualified and will adjust their hiring and production strategies accordingly.

28. Monotonic/Maximal Labor Stratification Lemma (MLSL)

28.1. From Employer Rationality Axiom, Weak Skill Difference Axiom, and Weak Occupational Complexity Axiom.

28.1.1. The Monotonic/Maximal Labor Stratification Lemma (MLSL) states that a perfectly rational employer (ERA), which expects there could be skill differences in the workforce (WSDA), and can split operational output without either gain or loss to production (WOCA), will, therefore, strategically stratify their job offers monotonically, and even maximally, so that, if indeed there happens to be skill differences in the labor market, they can, then, allocate less competent workers to easier roles, and avoid wasting talent, thus “saving their best” for the most demanding tasks.
Mathematically,

$$l \in \left[\frac{\ell - 1}{w_q}, \frac{\ell}{w_q}\right], \ell \in \{1, \ldots, w_q\} \tag{58}$$

is a responsability spectrum in a maximally stratified labor market, in which employment levels are given by

$$\sum_{\ell=1}^{w_q} w_q^\ell = w_q, \tag{59}$$

so that any available position is its own job subtype and convers only a restrictive range of task difficulty, accounting for

$$\Omega_q^\ell = \int_{\frac{\ell-1}{w_q}}^{\frac{\ell}{w_q}} \mathrm{ta}(l)dl \tag{60}$$

of an operation's total time duration, that is

$$\sum_{\ell=1}^{w_q} \Omega_q^\ell = \sum_{\ell=1}^{w_q} \int_{\frac{\ell-1}{w_q}}^{\frac{\ell}{w_q}} \mathrm{ta}(l)dl = \int_0^1 \mathrm{ta}(l)dl = 1. \tag{61}$$

This is advantageous because the expected value of operational output – and, therefore, of producers' revenue – is higher and constant when applying a stratified production strategy when compared to an independent production strategy (i.e. one where employees work independently on the entire operation from beginning to end); for in such a strategy, production is more easily limited by skill differences, and so the expected value of operational output is potentially lower, but never higher. That is,

$$\mathbb{E}[\tilde{T}_q^k] \in [0,1] \;\forall\; k,q \in \{1,\ldots,n\} \tag{62}$$

implies that

$$\begin{aligned}
&\int_0^1 T(l,\tilde{T}_q^k)\mathrm{ta}(l)dl && (63)\\
&\leq \tilde{v}_q^k \int_0^1 T(l,\tilde{T}_q^k)\mathrm{ta}(l)dl + (1-\tilde{v}_q^k)\int_0^1 T(l,1)\mathrm{ta}(l)dl && (64)\\
&\leq \tilde{w}_q^k \int_0^{\tilde{T}_q^k} T(l,\tilde{T}_q^k)\mathrm{ta}(l)dl + (1-\tilde{w}_q^k)\int_{\tilde{T}_q^k}^1 T(l,1)\mathrm{ta}(l)dl && (65)\\
&= \sum_{k=1}^{n}\sum_{\ell=1}^{w_q}\left[k \in \Lambda_q^\ell\right]\left[\tilde{T}_q^k \geq \frac{\ell}{w_q}\right]\tilde{w}_q^\ell \int_{\frac{\ell-1}{w_q}}^{\frac{\ell}{w_q}} \mathrm{ta}(l)dl && (66)\\
&= \int_0^1 T(l,1)\mathrm{ta}(l)dl && (67)\\
&= \int_0^1 \mathrm{ta}(l)dl && (68)\\
&= 1, && (69)
\end{aligned}$$

where $\tilde{v}_q^k > \tilde{w}_q^k$, with $\tilde{v}_q^k, \tilde{w}_q^k \in [0,1]$, is an inefficient allocation of workers above the optimal employability coefficient $\tilde{w}_q^k$ in a semi-stratified labor market; and the double sum in equation (66) is the output of a maximally stratified labor market, in which every $\ell \in \{1,\ldots,w_q\}$ job subtype is but a fraction of available positions, with a partial workforce of $w_q \times \tilde{w}_q^\ell$ individuals, all exclusively dedicated to their own responsability spectrum, and identified by $[k \in \Lambda_q^\ell]$ employment statuses that are evaluated to 1 if they are employed in a particular Λ_q^ℓ strata of the labor market, and to 0 for all other job subtypes; while the remaining equations are the maximal operational output of a labor market with w_q perfectly qualified employees working independently on the entire responsability spectrum of occupation q's operation.
In other words, splitting responsabilities in accordance with competence is always as productive as the maximum operational output (viz. that which is obtained when employing perfectly

qualified workers independently), provided employees are sufficiently qualified for their responsabilities. But, again, this is, by definition, guaranteed by employers' rationality, as well as the simple fact the economy is already producing its current operational output (Operational Equilibrium Lemma, OEL).

Therefore, employing potentially underqualified workers to output the entire responsability spectrum $l \in [0, 1]$ independently can only be as productive as the labor stratification strategy, but never more than it. Indepedent production, then, is a suboptimal strategy when employers expect there to be skill differences in the workforce.

Thus, maximum-monotonic labor stratification follows as an insurance policy against worker's potential underqualification; for if talent is lacking in the labor market, there is nothing to gain by employing individuals which are not sufficiently qualified for a difficult job, whereas if talent is abundant, there is nothing to lose when employing overqualified individuals to a job below their skill level.

Hence, given the same w_q workforce, operational output in a maximally stratified labor market is always greater or equal to the output of any other economic configuration. It is, therefore, always optimal to monotonically and maximally stratify responsabilities across w_q unique positions, each focused on increasingly demanding tasks.

28.2. Monotonic labor stratification is required and follows logically from employers' perfect rationality axiom.

28.3. Maximal labor stratification is optional, but also follows logically from employers' perfect rationality axiom.

28.3.1. Because the General Employability Theorem (GET) holds true for imperfectly stratified labor markets as well, for less than maximal labor stratification is mathematically equivalent to just a variable change. This said, imperfect labor market stratification leads to inefficiencies in hiring, as the base requirements for each stratum are higher than they would be if labor was maximally stratified.

29.