Employability and Competitiveness in Efficient Labor Markets

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Abstract

In this article, we propose a microeconomic framework to evaluate individuals' employability in efficiently organized labor markets as a function of their capacity on measureable professional attributes. We demonstrate this value coincides precisely with the normalized total duration of occupations' tasks their skill set allows them to accomplish, or what we term "inverse operational output". We further demonstrate labor market competitiveness can be defined as a complement of this employability metric, given by the percentage of an occupation's job posts subject to competition with incumbent workers (i.e. competitiveness is the employability of job seekers from other fields).

Keywords: Employability; Competitiveness; Career choice; Career development; Vocational choice; Occupational Information Network; BLS.

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Definition 1 (Skill)**.** A professional attribute, competency, or skill, of a person k can be conceptualized as a cummulative sum of successes on binary outcome variables representing tasks of progressive difficulty which require only that skill:

$$a_i^k = \sum_{l=0}^{l_i} T_{i_l}^k, \tag{1}$$

where

$$T_{i_l}^k = \begin{cases} 1, & \text{if } k \text{ succedes in a task } T_i^l \text{ of difficulty level } l; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Or, more rigorously,

$$a_i^k = \sum_{l=0}^{l_i} T(l, l_i^k), \tag{3}$$

where

$$T(l, l_i^k) = T_{i_l}^k = [l \leq l_i^k] = \begin{cases} 1, & l \leq l_i^k; \\ 0, & l > l_i^k. \end{cases} \tag{4}$$

and $l_i^k \in [0, l_i]$ is the maximum difficulty level on which k still succeedes. Thus, we can define a person k's skill level in an attribute i as the sum of their successful trials on a $T_i = \{T_i^0, \ldots, T_i^{l_i}\}$ set of tasks of increasing difficulty.

Furthermore, as we assume scales are truncated (i.e. there is a maximum difficulty level l_i, and a trivial difficulty level, which has to be zero), we can also interpret a_i^k as the *portion* of tasks one is able to accomplish out of all difficulty levels for that skill. By normalizing l_i to 100, for example, we have:

$a_i^k = 0 \iff k$ cannot perform even the most basic of attribute i's tasks; $a_i^k = 10 \iff k$ can perform only the bottom 10% of attribute i's tasks, but nothing more; $a_i^k = 50 \iff k$ can perform the easiest half of attribute i's tasks, but not the most difficult half; $a_i^k = 100 \iff k$ can perform all of attribute i's tasks.

Finally, we can define a_i^k for a continuum of task difficulty $l \in [0, 1]$:

$$a_i^k = \int_0^1 T(l, l_i^k) dl, \tag{5}$$

where $T(l, l_i^k)$ is defined as before.

Definition 2 (Skill)**.** Alternatively, we can think of a person k's professional attribute, competency, or skill, as the difficulty of the most difficult task they can accomplish, normalized by the difficulty of the most objectively difficult task of that particular skill:

$$a_i^k = \frac{l_i^k}{l_i}, \tag{6}$$

which we normalize by setting $l_i = 1$, so that

$$a_i^k = \frac{l_i^k}{1} = l_i^k, \tag{7}$$

and $l_i^k \in [0, 1]$. With this normalization, example interpretations of a_i^k are:

$a_i^k = 0 \iff k$ cannot perform even the most basic of attribute i's tasks; $a_i^k = 0.10 \iff k$ can only perform tasks of up to 10% the difficulty of attribute i's most difficult task, but nothing more; $a_i^k = 0.50 \iff k$ can perform tasks of up to half the difficulty of attribute i's most difficult task, but nothing more; $a_i^k = 1 \iff k$ can perform all of attribute i's tasks.

This is, perhaps, the most natural conceptual model for understanding competencies, as, generally, it is more intuitive to think of skill as the maximum of one's capacity, rather than the portion of tasks one could ly accomplish.

But, again, [because we assume scales to be truncated], this latter interpretation actually implies and is implied by the former. For if a task is of the same difficulty as another, then they are just as difficult in relation to that skill's most difficult task (i.e. they require the same percentage of the scale's upper limit to be performed), and, likewise, are also included in the same difficulty "bracket" (i.e. they are equivalent to the same skill test in the aggregate binary outcome interpretation), and, therefore, pressupose the same a_i^k skill level.

Of course, this equivalence is quite trivial, given that

$$\int_0^1 T(l, l_i^k) dl = 1 \times \int_0^{l_i^k} dl + 0 \times \int_{l_i^k}^1 dl = l_i^k - 0 = \frac{l_i^k}{1} = a_i^k. \tag{8}$$

This means the percentage of a skill's tasks one can accomplish is also the difficulty of the most difficult task one can accomplish relative to that skill's most difficult task.

So, however one decides to interpret skill levels, the conclusion remains the same: to be skilled in an attribute is to be able to perform the activities associated with that attribute. Put simply, the capacity to act follows virtue, for virtue is, itself, the capacity to act.

P.S.: SSL

$$a_i^k := \frac{l_i^k}{l_i} \in [0, 1] \tag{9}$$

$$\tilde{T}_i^k = \int_0^{l_i} T(l, l_i^k) dl \left(\int_0^{l_i} T(l, l_i) dl \right)^{-1} \tag{10}$$

$$= \left(\int_0^{l_i^k} 1 \times dl + \int_{l_i^k}^{l_i} 0 \times dl\right) \times \left(\int_0^{l_i} 1 \times dl\right)^{-1} \tag{11}$$

$$= \frac{l_i^k - 0}{l_i - 0} \tag{12}$$

$$= \frac{l_i^k}{l_i} \tag{13}$$

$$\therefore a_i^k = \tilde{T}_i^k \tag{14}$$

P.S.: SCL

$$\boldsymbol{a_k} := (a_1^k, \ldots, a_m^k), \boldsymbol{a_q} := (a_1^q, \ldots, a_m^q) \in [0,1]^m \ \forall\ k, q \in \{1, \ldots, n\} \tag{15}$$

$$l_q^k \leq l_q^q \ \forall\ k, q \in \{1, \ldots, n\} \tag{16}$$

$$\tilde{T}_q^k = \int_0^{l_q^q} T(l, l_q^k) dl \left(\int_0^{l_q^q} T(l, l_q^q) dl\right)^{-1} \tag{17}$$

$$= \left(\int_0^{l_q^k} 1 \times dl + \int_{l_q^k}^{l_q^q} 0 \times dl\right) \times \left(\int_0^{l_q^q} 1 \times dl\right)^{-1} \tag{18}$$

$$= \frac{l_q^k - 0}{l_q^q - 0} \tag{19}$$

$$= \frac{l_q^k}{l_q^q} \in [0,1] \tag{20}$$

$$\because \neg\ l_q^k > l_q^q \tag{21}$$

$$\because l_q^k = f(\boldsymbol{l_k}, \boldsymbol{l_q}) = f(\boldsymbol{a_k}, \boldsymbol{a_q})? \in [0,1] \ \forall\ k, q \in \{1, \ldots, n\} \tag{22}$$

$$\therefore \tilde{T}_q^k = f(\boldsymbol{a_k}, \boldsymbol{a_q}) \tag{23}$$

Now, even though these results are basically tautological, they are still important to guide our intuition. In fact, our first insight towards the Employability Theorem, namely the Skill Sufficiency Lemma (SSL), follows directly from the definitions above.

Lemma 1 (Skill Sufficiency Lemma)**.** According to the SSL, skills are necessary and sufficient to accomplish tasks. In particular, to have a skill level of $a_i^k \in [0,1]$ in attribute i is a necessary and sufficient condition for one to be capable of accomplishing the easier a_i^k portion of that attribute's tasks.

Proof. By definition,

$$T(l, l_i^k) := \begin{cases} 1, & l \leq l_i^k; \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

is a binary indicator of person k's ability to accomplish a task of difficulty $l \in [0,1]$ which requires only attribute i.

With this,

$$\tilde{T}_i^k = \int_0^1 T(l, l_i^k)dl \tag{25}$$

is the percentage of tasks requiring only attribute i that k can accomplish.

But both equivalent definitions of k's skill level in attribute i, are

$$a_i^k = \int_0^1 T(l, l_i^k)dl = l_i^k, \tag{26}$$

which is precisely the $\tilde{T}_i^k$ aggregation of $T(l, l_i^k)$ in the $[0, 1]$ interval.

Therefore, having a skill level of a_i^k is a necessary and sufficient condition to be capable of accomplishing the easier a_i^k portion of attribute i's tasks:

$$a_i^k = \tilde{T}_i^k \iff \int_0^1 T(l, l_i^k)dl = \int_0^1 T(l, l_i^k)dl. \tag{27}$$

□

Definition 3 (Complex Task)**.** A task is said to be complex if it relies on more than one attribute to be accomplished. More precisely, T_{ij}^l is a complex task of attributes i and j, if its binary outcome indicator is of the form

$$T(l, l_{ij}^k) := [l \leq l_{ij}^k], \tag{28}$$

where

$$l_{ij}^k := f(l_i^k, l_j^k) \tag{29}$$

is a strictly increasing aggregation function that returns the maximum difficulty level of the complex task T_{ij}^l a person k can accomplish based on each attribute T_{ij}^l requires. Or, generalizing for any complex task T_q^l of m attributes, requiring an entire skill set $\boldsymbol{a_q} := (a_1^q, \ldots, a_m^q)$ to be accomplished,

$$T(l, l_q^k) := [l \leq l_q^k], \tag{30}$$

where

$$l_q^k := f(\boldsymbol{l_q^k}) := f(l_1^k, \ldots, l_m^k) \tag{31}$$

and

$$\frac{\partial f(\boldsymbol{l_q^k})}{\partial l_i^k} > 0 \ \forall \ i \in \{1, \ldots, m\}. \tag{32}$$

This means none of the attributes required by the complex task are completely disposable (i.e. they are all helpful in some way). For instance, the task T_i^l, previously defined, with binary outcome $T(l, l_i^k)$ is not complex, because

$$\frac{\partial l_i^k}{\partial l_i^k} = 1, \tag{33}$$

but

$$\frac{\partial l_i^k}{\partial l_j^k} = 0, \tag{34}$$

where $i \neq j$ and $i, j \in \{1, \ldots, m\}$. Or, say the aggregation function is given by

$$f(l_i^k, l_j^k) := l_i^k - l_j^k, \tag{35}$$

so that attribute j actually hinders productivity:

$$\frac{\partial l_i^k}{\partial l_j^k} = -1. \tag{36}$$

None of these are complex tasks, for they do not coherently mobilize multiple attributes towards a unified goal.

Definition 3.1 (Weak Complexity)**.** Now, beyond these most basic rules, we can define stricter versions of "task complexity" with additional assumptions. The first version, of weak complexity, requires that

$$\frac{\partial^2 f(\boldsymbol{l_q^k})}{\partial l_i^k \partial l_j^k} > 0 \;\forall\; i \neq j \in \{1, \ldots, m\}, \tag{37}$$

meaning attributes are all complementary.

Definition 3.2 (Moderate Complexity)**.** A task is of moderate complexity if its aggregation function also meets the following criteria:

$$\lim_{l_i^k \to 0} f(\boldsymbol{l_q^k}) = 0 \;\forall\; i \in \{1, \ldots, m\}, \tag{38}$$

so that a person k's capacity to perform the complex task is weakly increasing on their capacity to perform the simple tasks of its required attributes, and goes to zero when they are unskilled in at least one of these. Thus, a moderately complex task is not reducible to any proper subset of its attributes.

For instance, a task of the form

$$T(l, l_{ij}^k) := [l \leq (1 + l_i^k) \times (1 + l_j^k) - 1] \tag{39}$$

is not moderately complex, as person k does not need every attribute to accomplish the task. Indeed, if k has precisely zero capacity in either skill i or j, then T_{ij}^l collapses to unidimensional, or simple, tasks T_i^l when

$$\begin{aligned} T(l, l_{ij}^k) &= [l \leq (1 + l_i^k) \times (1 + 0) - 1] && (40) \\ &= [l \leq l_i^k] && (41) \\ &= T(l, l_i^k), && (42) \end{aligned}$$

or T_j^l when

$$
\begin{aligned}
T(l, l_{ij}^k) &= [l \leq (1+0) \times (1 + l_j^k) - 1] && (43) \\
&= [l \leq l_j^k] && (44) \\
&= T(l, l_j^k), && (45)
\end{aligned}
$$

in which case T_{ij}^l is not *really* (moderately) complex, but rather a convolution of simple tasks. Notice, however, this does not imply there cannot be a degree of substitution between attributes. That is, moderate task complexity only means a task must require all of its attributes in *some* level, even if its functional form allows for substitution.

Definition 3.3 (Strong Complexity)**.** The strictest definition of task complexity adds the constraint that skills are aggregated by the Leontief function:

$$f(\boldsymbol{l_q^k}) := \min(\boldsymbol{l_q^k}). \tag{46}$$

Here, attributes are assumed to be perfect complements, which need to be combined in exactly the same quantities for maximum efficacy. In other words, having additional skills does not help to accomplish the task, but being unskilled in even a single attribute can undermine the whole effort. Hence, productivity is limited by the lowest competency.

Lemma 2 (Skill Composition Lemma)**.** The Skill Composition Lemma (SCL) is a generalization of the SSL and states that skills are composable to accomplish complex tasks. More precisely, let T_q^l be an activity of difficulty level l that requires the $\boldsymbol{a_q} := (a_1^q, \ldots, a_m^q)$ skill set (i.e. T_q^l is a complex task). With this, we demonstrate that any rational and sufficiently qualified economic agent can naturally "piece together", that is *compose*, attributes $\{1, \ldots, m\}$ to accomplish the T_q^l complex task.

Proof. Given

$$\tilde{T}_q^k = \int_0^1 T(l, l_q^k) dl \tag{47}$$

□

In addition to the above, we shall also specifiy tasks' duration in terms of a time allocation function.

Definition 4 (Time Allocation)**.** Time allocation is a continuous function associating tasks' difficulty with their duration, both normalized to the unit interval:

$$\mathrm{ta}(l) := \mathrm{ttc}(l) \times \left(\int_0^1 \mathrm{ttc}(l) dl \right)^{-1}, \tag{48}$$

$$\mathrm{ttc}(l) \geq 0 \; \forall \; l \in [0, 1] \wedge \int_0^1 \mathrm{ttc}(l) dl > 0, \tag{49}$$

where $\mathrm{ttc}(l)$ is the number of hours to complete a task given its difficulty level.

Notice, as well, we termed normalized task duration "time allocation". This is on purpose, as time constraints are assumed to coincide with aggregate normalized duration. Or, in other words, all employees have the same unitary time allowance, while tasks' normalized duration, likewise integrates to a dimensionless time unit, so that any worker, if sufficiently qualified, can output every task by themselves. Therefore, an independent employee producing the entire $l \in [0, 1]$ responsability spectrum must spend their time allowance in accordance with normalized duration, or the *percentage* of a single worker's time that has to be *allocated* to complete a task; hence the name "time allocation".

Having understood what "tasks" are mathematically, we can proceed with our goal of quantifying employability. To this end, we note the following.

Observation 1 (Occupational Reducibility)**.** From a practical standpoint, occupations are reducible to their activities: a "job" is nothing but a collection of tasks which have to be executed in a particular time frame.

Thus, we conclude one is employable to the measure of one's productivity: if a person can perform an occupation's tasks, they can, thereby, be employed in its labor market.

However, how exactly should we assess productivity when speaking of whole occupations? For, though a job is just a collection of tasks, we do not know *which* tasks – whether simple, or complex, and if so, how complex – constitute "an occupation". Morever, these activities of varying complexity and productivity requirements usually vary from position to position. So, it appears our situation here is somewhat nebulous. In fact, we could even go so far as to replace our initial observation and say: an occupation is a "black box of tasks".

Again, this is what we refer to as the issue of "occupational complexity" and it is our main obstacle towards an economicly precise concept of employability and labor market competitiveness. But we have a good solution for this problem.

Observation 2 (Attribute Complementary and Occupational Atomicity)**.** Professional attributes serve different purposes in different occupations, and each occupation's attributes complement one another to output a mostly homogenous "product", which is the essence of its activity and makes it unique among all occupations. Hence, what a particular job produces is an indivisible, "atomic", set of tasks, essentially other than what is produced elsewhere in the labor market.

For example, an artist, an airline pilot and a surgeon have *manual dexterity* as part of their skill set. Nevertheless, one's dexterity is combined with their creativity and applied to producing art, while the others' are combined with specific, technical, knowledge and applied, respectively, to maneuvering airbuses and performing surgery. And, for this reason, each of these's skills are not necessarily transferable to the other's activities: thus, indeed, to be talented with the scalpel does not mean one is any good with a paint brush; for the painter's ability is not really "manual dexterity" itself, but "artistic-manual-dexterity", whereas a surgeon's is "surgical-manual-dexterity" (the hyphens emphasize occupational atomicity, or indivisibility, as the same competency, combined to another set of skills yields quite different results).

And we could further state all occupations' simple tasks (i.e. those requiring only one attribute) amount to very little, and perhaps nothing, in terms of time allocation, so that employees in each labor market spend their time allowances on occupations' complex, and essential, tasks (i.e. those that differentiate them). For, if this was not the case, it would not imply an occupation is not complex, only that it was incorrectly categorized: it would not be an issue with the theory here proposed, but an empirical, classification error. Therefore, in such cases, the occupation should be split into however many suboccupations are needed until each of them is indivisible (i.e. "atomic").

Finally, as with attribute atomicity (see [ref]), occupational atomicity too does not rule out some level of specialization and skill substitution within a job: we do not assume every single position is identical, or even deals with exactly the same subject matter; what we do suppose is that an occupation's job posts are sufficiently and essentially equivalent, despite irrelevant differences in difficulty and specialization, to the point where a person's productivity and employability remains constant across them (i.e. occupations are "well-defined").

And, because of these two insightful, yet fairly uncontroversial observations (viz. of reducibility and atomicity), it seems we have good logical grounds to "sidestep" the issue of occupational complexity by an axiom.

Axiom 1 (Occupational Complexity Axiom)**.** Any occupation can be thought of as one indivisible activity that mobilizes workers' entire skill set. We call this "holistic task" an occupation's *operation*.

Mathematically, an occupational operation is just a series of complex tasks on a continuum of difficulty levels normalized to the unit interval, all of which are indispensable for the whole operation to be accomplished.

Axiom 1.1 (Strong Occupational Complexity Axiom, SOCA)**.** Let us denote, then, "operational output" (abbreviated to "o.o.") with the standard IPA (International Phonetic Alphabet, 1919) symbol for the near-close near-back rounded vowel (i.e. the "double o" sound in words such as "boot"):

$$\mho_q := \mho_q^{\text{IP}} := \sum_{v=1}^{w_q} \left[\tilde{T}_q^v = 1\right] \times \left(\int_0^1 \text{ta}(l)dl\right)^{-1} = \sum_{v=1}^{w_q} \left[\tilde{T}_q^v = 1\right] \tag{50}$$

$$\because \text{ta}(l) := \text{ttc}(l) \times \left(\int_0^1 \text{ttc}(l)dl\right)^{-1} \tag{51}$$

$$\implies \int_0^1 \text{ta}(l)dl = \left(\int_0^1 \text{ttc}(l)dl\right)^{-1} \times \int_0^1 \text{ttc}(l)dl = 1, \tag{52}$$

where the upperscript "IP" indicates production is organized independently, as each worker outputs the $l \in [0, 1]$ responsability spectrum by themselves. So, aggregate output in a labor market is exactly the number of perfectly qualified employees in it (i.e. those capable of producing all tasks without outsourcing).

Intuitively, what this formulation entails is a scenario of several individuals working in parallel entirely disconnected from one another. And, of course, this is hardly the case in a real economy. Thus, we should define occupational complexity in weaker terms, allowing for at least a degree of outsourcing.

Axiom 1.2 (Moderate Occupational Complexity Axiom, MOCA)**.** With moderate occupational complexity, we assume production can be split into $p_q \in \{1, 2, 3, \dots\}$ positions, or job subtypes, each responsible for their own subset of tasks with increasing difficulty levels.

Therefore, though the operation, in itself, remains "indivisible", employers may split it apart and outsource it, so long as every subtask is completed (i.e. if this "stratified" operation is, then, "pieced back together" with all its parts).

More precisely,

$$\mho_q := \min\left(\boldsymbol{w_q}(\tilde{\boldsymbol{T}}_{\boldsymbol{q}}, \boldsymbol{\ell_q}) \times \mho_{\boldsymbol{q}}(\boldsymbol{\ell_q})\right) + [p_q > 1] \times \boldsymbol{w_q}^\top \cdot \boldsymbol{\varsigma_q}(\boldsymbol{\ell_q}), \tag{53}$$

where

$$\mho_{\boldsymbol{q}}(\boldsymbol{\ell_q}) := \left(\mho_q^1, \dots, \mho_q^{p_q}\right) := \left(\frac{1}{\int_0^{\ell_1^q} \mathrm{ta}(l)dl}, \dots, \frac{1}{\int_{\ell_{p_q}^q}^1 \mathrm{ta}(l)dl}\right) \tag{54}$$

is the vector of partial operational outputs, as a function of

$$\boldsymbol{\ell_q} := \left(\ell_0^q, \dots, \ell_{p_q}^q\right) := (0, \dots, 1) \in [0,1]^{p_q}, \tag{55}$$

$$\sum_{v=1}^{p_q} \int_{\ell_{v-1}^q}^{\ell_v^q} \mathrm{ta}(l)dl := 1 \tag{56}$$

responsability bounds; while

$$\boldsymbol{w_q}(\tilde{\boldsymbol{T}}_{\boldsymbol{q}}, \boldsymbol{\ell_q}) := \left(\sum_{r=1}^{w_q^1}\left[\tilde{T}_1^r \geq \ell_1^q\right], \dots, \sum_{r=1}^{w_q^{p_q}}\left[\tilde{T}_{p_q}^r \geq \ell_{p_q}^q\right]\right) \geq \boldsymbol{0}, \tag{57}$$

$$0 \leq \boldsymbol{1}^\top \cdot \boldsymbol{w_q}(\tilde{\boldsymbol{T}}_{\boldsymbol{q}}, \boldsymbol{\ell_q}) \leq \boldsymbol{1}^\top \cdot \boldsymbol{w_q}(\boldsymbol{1}, \boldsymbol{\ell_q}) := w_q \tag{58}$$

are effective (i.e. sufficiently qualified) workers per position; and

$$\boldsymbol{\varsigma_q}(\boldsymbol{\ell_q}) := \left(\varsigma_1^q, \dots, \varsigma_{p_q}^q\right) \in \mathbb{R}^{p_q}, \tag{59}$$

$$\varsigma_v^q := \int_{\ell_{v-1}^q}^{\ell_v^q} \mathrm{sg}(l)dl - \int_{\ell_{v-1}^q}^{\ell_v^q} \mathrm{sc}(l)dl \tag{60}$$

is the net stratification effect, which measures whether the gains in efficiency due to splitting job posts into separate positions,

$$\sum_{v=1}^{p_q} \int_{\ell_{v-1}^q}^{\ell_v^q} \mathrm{sg}(l)dl \geq 0 \tag{61}$$

outweigh its cost,

$$\sum_{v=1}^{p_q} \int_{\ell_{v-1}^q}^{\ell_v^q} \mathrm{sc}(l)dl \geq 0. \tag{62}$$

Note the Leontief function here signifies, again, moderately complex operations are "indivisible", so that aggregate production is set to the lowest partial operational output.

Axiom 1.3 (Weak Occupational Complexity Axiom, WOCA)**.** Weakly complex operations are just the same as in MOCA, with an additional assumption about the net effects of labor stratification:

$$\boldsymbol{w_q}^\top \cdot \boldsymbol{\varsigma_q}(\boldsymbol{\ell_q}) = 0 \implies \mho_q = \min\Big(\boldsymbol{w_q}(\boldsymbol{\tilde{T}_q}, \boldsymbol{\ell_q}) \times \mho_{\boldsymbol{q}}(\boldsymbol{\ell_q})\Big). \tag{63}$$

In other words, weak occupational complexity asserts the gains and costs of splitting job posts are both negligeable or cancel each other out. This means employers may stratify positions without either gain or loss to production.

And we could further specifiy even weaker versions of the axiom, with any

$$\mho_q := \mho\Big(\boldsymbol{w_q}(\boldsymbol{\tilde{T}_q}, \boldsymbol{\ell_q}) \times \mho_{\boldsymbol{q}}(\boldsymbol{\ell_q})\Big) + [p_q > 1] \times \boldsymbol{w_q}^\top \cdot \boldsymbol{\varsigma_q}(\boldsymbol{\ell_q}) \tag{64}$$

aggregation function, yielding the very same conclusions we demonstrate in this paper, if it is "well-behaved" enough (e.g. if it satisfies Inada conditions). But, for simplicity's sake and mathematical convenience, we shall assume weak occupational complexity (WOCA) going forward.

Lemma 3 (Occupational Composition Lemma)**.** Skill sets are composable to accomplish occupations' operations.

Proof. □

To conclude this section, let us summarize what we have so far defined. With the aim of estimating individuals' employability in efficiently organized labor markets, we analyzed occupations in terms of their tasks and the required competence to complete them. We started from a fairly tautological notion of "skill" as one's capacity to accomplish tasks in a given domain and generalized, from this, another notion, that of complex tasks. Finally, we observed occupations are, in practice, reducible to their tasks and, though very much complex and hard to quantify in this regard, with an important axiom, we derived a theoretically sound method to measure overall productivity in any occupation's labor market. Thus, we have effectively "sidestepped" the issue of occupational complexity by assuming complexity.

3. Market Conditions and Employer Behavior

Axiom 2 (Employer Rationality Axiom, ERA)**.** Employers are rational and only hire individuals to work on tasks for which they are qualified. Additionaly, if labor stratification is allowed, rational employers will split job posts and outsource activities if they expect workers cannot accomplish the whole operation.

Mathematically, a rational employer's optimization problem is to choose vectors of employment levels $\boldsymbol{w_q}$ and responsability bounds $\boldsymbol{\ell_q}$ for each of p_q positions and implement a production strategy that maximizes operational output, given the available talent $\mathbb{E}[\tilde{\boldsymbol{T}}_{\boldsymbol{q}}]$, or expected productivity in the workforce:

$$\max_{p_q, \boldsymbol{w_q}, \boldsymbol{\ell_q}} \mathbb{E}\left[\mho\left(\boldsymbol{w_q}(\tilde{\boldsymbol{T}}_{\boldsymbol{q}}, \boldsymbol{\ell_q}), \mho_{\boldsymbol{q}}(\boldsymbol{\ell_q})\right) \mid \mathbb{E}[\tilde{\boldsymbol{T}}_{\boldsymbol{q}}]\right] \tag{65}$$

$$\text{s.t.} \quad p_q \in \{1, 2, 3, \dots\}, \tag{66}$$

$$\boldsymbol{\ell_q} \in [0,1]^{p_q}, \tag{67}$$

$$\sum_{v=1}^{p_q} \int_{\ell_{v-1}^q}^{\ell_v^q} \mathrm{ta}(l) dl = 1, \tag{68}$$

$$\boldsymbol{w_q} \geq \boldsymbol{0}, \tag{69}$$

$$\boldsymbol{1}^\top \cdot \boldsymbol{w_q} = w_q. \tag{70}$$

Note

Axiom 3 (Hireability Axiom)**.** Any rational employer hires employees by evaluating a hireability statistic, which quantifies potential employees' expected productivity, their educational attainment, and years of experience.

Axiom 3.1 (Weak Hireability Axiom, WHA)**.**

$$\mathbb{E}\left|h_q^k - \mathbb{E}(h_q^k)\right| \in [0,1] \tag{71}$$

Axiom 3.2 (Moderate Hireability Axiom, MHA)**.**

$$\mathbb{E}\left|h_q^k - \mathbb{E}(h_q^k)\right| = 0 \tag{72}$$

Axiom 3.3 (Strong Hireability Axiom, SHA)**.**

$$\mathbb{E}(h_q^k) = h_q^k \tag{73}$$

Axiom 4 (Productivity Differentia Axiom, PDA)**.** There are, or there could be, skill differences in the workforce (i.e. employees are, likely, not all equally competent "clones" of one another). So, the expected value of productivity is:

$$\mathbb{E}[\tilde{T}_q^v] \in [0,1], \tag{74}$$

instead of

$$\mathbb{E}[\tilde{T}_q^v] = \tilde{T}_q^v = 1, \tag{75}$$

for all $v \in \{1, \dots, w_q\}, q \in \{1, \dots, n\}$. This means employers do not expected every worker to be perfectly qualified and will adjust their hiring and production strategies accordingly.

Note we do not assign any specific probability distribution to workers' productivity. Hence, this axiom is as general as it can be.

4. The Employability Theorem

4.1. What is Employability?

When we speak of "employability" what we generally mean is rather trivial: the capacity to find employment. Thus, we say someone is "employable" if they are easily *hireable* and could get a job in a large portion of the labor market.

Definition 5 (Employability)**.** Mathematically, then, the employability of a person k is the percentage of jobs posts for which they are sufficiently qualified:

$$\tilde{W}_q^k := \left[h_q^k \geq \frac{1}{2}\right] \sum_{v=1}^{p_q} \left[\tilde{T}_q^k \geq \ell_q^v\right] \tilde{w}_q^v \in [0,1], \tag{76}$$

$$\sum_{v=1}^{p_q} \tilde{w}_q^v := \left(\frac{1}{w_q}\right) \sum_{v=1}^{p_q} w_q^v := 1, \tag{77}$$

where $\ell_q^v \in [0,1]$ is the upper responsability bound, or the minimum productivity to be hired in one of p_q types of positions in a labor market with a w_q workforce; while $h_q^k \in [0,1]$ is the hireability statistic accounting for other selection criteria, such as years of education, experience, etc.

And we can further aggregate employability for n occupations to assess how many of all available w jobs in the economy are suitable for one's skill set:

$$\tilde{W}_k := \sum_{q=1}^{n} \tilde{W}_q^k := \sum_{q=1}^{n} \left[h_q^k \geq \frac{1}{2}\right] \sum_{v=1}^{p_q} \left[\tilde{T}_q^k \geq \ell_q^v\right] \tilde{w}_q^v \in [0,1], \tag{78}$$

$$\sum_{q=1}^{n} \tilde{w}_q := \left(\frac{1}{w}\right) \sum_{q=1}^{n} w_q := 1, \tag{79}$$

$$w := \sum_{q=1}^{n} w_q. \tag{80}$$

4.2. Introductory Example: Employability with Two Types of Workers

With these basic axioms in place, we can attempt to derive an employability coefficient as presented in Definition 5 above. For ease of understanding, though, let us begin with a simple example and, then, proceed with a more complete, and robust, theorem.

In this subsection, we shall estimate employability in an occupation q's labor market where there are two types of workers with varying productivity. The first type – call them “juniors” – have lower skill and cannot accomplish tasks with difficulty levels $l > \tilde{T}_q^{\mathrm{Jr}} \in [0, 1)$. And the other type of employee are perfectly qualified “seniors”, with $\tilde{T}_q^{\mathrm{Sr}} = 1$ productive capacity.

Now, because of weak occupational complexity (WOCA), employers will maximize operational output by producing the entire $l \in [0, 1]$ spectrum of occupation q's complex tasks, subject to each task's duration. This can be done either by having only perfectly qualified employees work on these independently, from beginning to end, or by splitting responsabilities into two, or more, types of jobs, thus allowing for less qualified, “junior” employees, to work alongside “seniors” towards the common goal of accomplishing the whole operation.

Additionally, because we assume there to be skill differences among workers in the labor market, any rational employer will always, and rightly, expect potential employees to be of varying skill levels, rather than all perfectly qualified, so that splitting responsabilities into separate positions will not only be an alternative mode of hiring and producing, but in fact the optimal one.

Therefore, employers will stratify job offers based on required competence, providing “junior” and “senior” positions, both dedicated to their own subset of complex tasks with difficulty levels appropriate for employees' capacity.

Notice this does not mean those working on “junior” positions will, necessarily, be “juniors” themselves, that is, less qualified. Indeed, if talent is abundant in the labor market, these positions will have to be filled by more qualified, or even perfectly qualified, “senior” employees. For if there were only one type of job, spanning the entire responsability spectrum, these highly qualified workers would already have to accomplish “junior” tasks, in order to maximize operational output. However, by having two, or more, types of jobs, requiring more, or less, productivity, they may specialize to the measure there are sufficient employees allocated to easier tasks.

Either way, if the available talent is enough to output occupation q's operation, employability in such a market will be determined by the ratio of junior and senior job posts, as we demonstrate below.

Theorem 1 (Binary Employability Theorem, BET)**.** In a labor market with two types of workers with varying productivity, each worker's employability is the inverse of their maximum operational output.

Proof. In the binary case, “junior” productive output will be given by:

$$\mho_q^{\mathrm{Jr}} := \frac{1}{\int_0^{\tilde{T}_q^{\mathrm{Jr}}} \mathrm{ta}(l) dl} = \left(\int_0^{\tilde{T}_q^{\mathrm{Jr}}} \mathrm{ta}(l) dl \right)^{-1}, \tag{81}$$

where $\mathrm{ta}(l)$ is the time allocation function of occupation q's complex tasks, and time allowance (the numerator) is set to one. Analogously, “senior” output is:

$$\mho_q^{\mathrm{Sr}} := \frac{1}{\int_{\tilde{T}_q^{\mathrm{Jr}}}^1 \mathrm{ta}(l) dl} = \left(\int_{\tilde{T}_q^{\mathrm{Jr}}}^1 \mathrm{ta}(l) dl \right)^{-1}. \tag{82}$$

Furthermore, as a mismatch in operational output due to time allocation differences between "junior" and "senior" tasks would result in wasted production, a rational employer will optimally "orchestrate" the productive effort by offering just enough "senior" job posts in the labor market to meet "junior" productivity. So, by setting "junior" job posts to $w_q^{\mathrm{Jr}} > 0$ and "senior" job posts to $w_q^{\mathrm{Sr}} > 0$, we get the ratio between "junior" and "senior" positions required to output any level of occupation q's operation:

$$\begin{aligned} w_q^{\mathrm{Sr}} \times \mho_q^{\mathrm{Sr}} &= w_q^{\mathrm{Jr}} \times \mho_q^{\mathrm{Jr}} && (83) \\ \therefore w_q^{\mathrm{Sr}} \times \left(\int_{\tilde{T}_q^{\mathrm{Jr}}}^{1} \mathrm{ta}(l) dl \right)^{-1} &= w_q^{\mathrm{Jr}} \times \left(\int_0^{\tilde{T}_q^{\mathrm{Jr}}} \mathrm{ta}(l) dl \right)^{-1} && (84) \\ \therefore w_q^{\mathrm{Sr}} &= w_q^{\mathrm{Jr}} \times \left(\frac{\int_{\tilde{T}_q^{\mathrm{Jr}}}^{1} \mathrm{ta}(l) dl}{\int_0^{\tilde{T}_q^{\mathrm{Jr}}} \mathrm{ta}(l) dl} \right). && (85) \end{aligned}$$

With this, "senior" employability (i.e. the percentage of job posts for which they could be hired) is

$$\tilde{w}_q^{\mathrm{Sr}} := \frac{w_q^{\mathrm{Jr}} + w_q^{\mathrm{Sr}}}{w_q^{\mathrm{Jr}} + w_q^{\mathrm{Sr}}} = 1, \tag{86}$$

while "junior" employability is

$$\begin{aligned} \tilde{w}_q^{\mathrm{Jr}} &:= \frac{w_q^{\mathrm{Jr}}}{w_q^{\mathrm{Jr}} + w_q^{\mathrm{Sr}}} && (87) \\ &= \frac{w_q^{\mathrm{Jr}}}{w_q^{\mathrm{Jr}} + w_q^{\mathrm{Jr}} \times \left(\frac{\int_{\tilde{T}_q^{\mathrm{Jr}}}^{1} \mathrm{ta}(l) dl}{\int_0^{\tilde{T}_q^{\mathrm{Jr}}} \mathrm{ta}(l) dl} \right)} && (88) \\ &= \left(1 + \frac{\int_{\tilde{T}_q^{\mathrm{Jr}}}^{1} \mathrm{ta}(l) dl}{\int_0^{\tilde{T}_q^{\mathrm{Jr}}} \mathrm{ta}(l) dl} \right)^{-1} && (89) \\ &= \left(1 + \frac{\int_0^{1} \mathrm{ta}(l) dl - \int_0^{\tilde{T}_q^{\mathrm{Jr}}} \mathrm{ta}(l) dl}{\int_0^{\tilde{T}_q^{\mathrm{Jr}}} \mathrm{ta}(l) dl} \right)^{-1} && (90) \\ &= \left(1 + \frac{1 - \int_0^{\tilde{T}_q^{\mathrm{Jr}}} \mathrm{ta}(l) dl}{\int_0^{\tilde{T}_q^{\mathrm{Jr}}} \mathrm{ta}(l) dl} \right)^{-1} && (91) \\ &= \left(1 + \frac{1}{\int_0^{\tilde{T}_q^{\mathrm{Jr}}} \mathrm{ta}(l) dl} - \frac{\int_0^{\tilde{T}_q^{\mathrm{Jr}}} \mathrm{ta}(l) dl}{\int_0^{\tilde{T}_q^{\mathrm{Jr}}} \mathrm{ta}(l) dl} \right)^{-1} && (92) \\ &= \left(1 + \frac{1}{\int_0^{\tilde{T}_q^{\mathrm{Jr}}} \mathrm{ta}(l) dl} - 1 \right)^{-1} && (93) \end{aligned}$$

$$= \left(\frac{1}{\int_0^{\tilde{T}_q^{\mathrm{Jr}}} \mathrm{ta}(l)dl} \right)^{-1} \tag{94}$$

$$= \int_0^{\tilde{T}_q^{\mathrm{Jr}}} \mathrm{ta}(l)dl. \tag{95}$$

Thus, the employability of a partially qualified worker, that is a "junior", is precisely the percentage of an operation's total time duration their skill set allows them to accomplish (i.e. the inverse of their operational output). □

4.3. General Employability Theorem

Now, to generalize this conclusion, we shall define notation in terms of maximum labor stratification, a productive arrangement where there are several job subtypes, indeed as many as there are jobs themselves, each with a limited spectrum of responsabilities.

Definition 6 (Maximum Labor Stratification)**.** Hence, mathematically,

$$l \in [\ell_{v-1}, \ell_v], \tag{96}$$

with

$$\ell_v \in [0,1] \;\forall\; v \in \{1, \ldots, w_q\}, \tag{97}$$

$$\ell_{w_q} := 1, \tag{98}$$

$$\ell_0 := 0 \tag{99}$$

is one of w_q responsability spectra in a maximally stratified labor market, in which employment levels are unitary, or given by

$$\sum_{v=1}^{w_q} 1 = w_q, \tag{100}$$

so that any available position is its own job subtype and covers only a restrictive range of task difficulty, accounting for

$$\Omega_q^v := \frac{1}{\mho_q^v} = \int_{\ell_{v-1}}^{\ell_v} \mathrm{ta}(l)dl \in [0,1] \tag{101}$$

of an operation's total time duration,

$$\sum_{v=1}^{w_q} \Omega_q^v := \sum_{v=1}^{w_q} \int_{\ell_{v-1}}^{\ell_v} \mathrm{ta}(l)dl = \int_0^1 \mathrm{ta}(l)dl := 1. \tag{102}$$

Intuitively speaking, we would say production in a maximally (and monotonically) stratified labor market is not "independent", in the sense that employees do not work on an occupation's operation from beginning to end. This means each of them will spend all their time allowance producing a partial operational output, that is a multiple of a difficulty subinterval of complex tasks, which will, in turn, contribute, alonside the partial outputs of other employees, to accomplish[ing?] the occupational operation in its entirety.

However, in a maximum labor stratification setting, these partial operational outputs will not be produced merely via "senior" and "junior" positions, as previously, but rather within a myriad of levels in a production hierarchy, approximating a continuum of "seniority" as the workforce becomes large enough.

Again, this does not mean employees are, themselves, more or less competent, only that available job posts are preemptively stratified with respect to task difficulty, in order to maximize employers' hiring pool and safeguard production in the case workers are not sufficiently qualified to produce the whole responsability spectrum independently (see "Maximum Labor Stratification Lemma").

Having understood what maximum labor stratification is, one may wonder whether there could be more than w_q job subtypes in a labor market. For though it is intuitive to think of w_q, the workforce size, as the upper bound for stratification, if we allow for partial hiring, with "fractional jobs",

$$w_q^v > 0 \ \forall \ v \in \{1, \dots, p\}, \tag{103}$$

$$\sum_{v=1}^{p} w_q^v := w_q, \tag{104}$$

where $p \in \{1, 2, 3, \dots\}$ is the number of positions in a labor market, then workers can allocate fractions of their time allowance to multiple responsability spectra, and the productive arrangement we have just defined, may not, technically speaking, be "maximally stratified".

Indeed, if it were possible to stratify beyond w_q, rational employers would readly do so, for, again, labor stratification reduces the uncertainty around production and serves as an insurance policy to guarantee the available talent is sufficient to output an occupation's operation.

But, because of this, if p can be greater than w_q, the optimal production strategy would, logically, be to offer as many types of jobs as possible, even infinitely many.

Hence, infinite labor stratification is defined as an economic configuration where labor markets are subdivided into infinitesimal jobs, each contributing very little to production. In fact, in such a market, "job posts" are so small as to be indistinguishable from tasks themselves[1]

$$\because \lim_{p \to \infty} (\ell_v - \ell_{v-1}) = 0 \implies \Omega_q^v := \int_{\ell_{v-1}}^{\ell_v} \mathrm{ta}(l) dl = \mathrm{ta}(l) \tag{105}$$

$$\wedge \lim_{p \to \infty} \tilde{w}_q^v := \lim_{p \to \infty} \left(\frac{w_q^v}{w_q} \right) =: \tilde{w}_q(l) = \mathrm{ta}(l) \in [0, 1] \ \forall \ v \in \{1, \dots, p\} \tag{106}$$

[1]See the Proportional Employment Condition in "Maximum Operational Output Lemma".

$$\therefore w_q(l) = w_q \times \tilde{w}_q(l) = w_q \times \mathrm{ta}(l) \wedge \int_0^1 w_q(l)dl = w_q. \tag{107}$$

Therefore, employers are guaranteed maximum insurance against workers' potential underqualification; and employability is simply

$$\tilde{W}_q^k = \left[h_q^k \geq \frac{1}{2}\right] \int_0^1 T(l, l_q^k)\tilde{w}_q(l)dl = \left[h_q^k \geq \frac{1}{2}\right] \int_0^{\tilde{T}_q^k} \mathrm{ta}(l)dl, \tag{108}$$

where the hireability statistic $h_q^k \in [0, 1]$ accounts for hiring requirements other than productivity; and $\tilde{w}_q(l)$ is the proportion of fractional positions for a particular job subtype, which coincides with its time allocation when there are infinite "jobs", each dedicated to a single, infinitely narrow task. We note, as well, this formula is the same as it was in binary labor stratification (with "junior" and "senior" positions). Thus, again, employability is the percentage of an operation's total duration one can accomplish.

We may formalize this conclusion as follows.

Lemma 4 (Infinite Stratification Lemma, ISL)**.** If fractional job posts are allowed, with

$$w_q^v > 0 \ \forall \ v \in \{1, \ldots, p\}, p \in \{1, 2, 3, \ldots\}, \tag{109}$$

$$\sum_{v=1}^{p} w_q^v := w_q, \tag{110}$$

employers' optimal choice is to infinitely split positions as infinitesimal tasks,

$$\lim_{p \to \infty} \tilde{w}_q^v =: \tilde{w}_q(l) = \mathrm{ta}(l), \tag{111}$$

so that employability becomes:

$$\tilde{W}_q^k = \left[h_q^k \geq \frac{1}{2}\right] \int_0^{\tilde{T}_q^k} \mathrm{ta}(l)dl. \tag{112}$$

Proof. See above. □

All this said, infinitely stratified markets are rather abstract, and it is not realistic to think of actual job posts as infinitesimal tasks; for, then, the very concept of a "job" itself disappears. Fractional positions do not make much sense in reality, where jobs usually deal with a set of multiple responsabilities. Furthermore, a maximally – though not infinitely – stratified labor market with sufficient positions, will, in practice, yield the same results when w_q is large enough, so that we do not even need to consider infinite labor stratification as a production strategy.

Axiom 5 (Maximum Stratification Axiom, MSA)**.** Therefore, let us assume

$$p \in \{1, \ldots, w_q\}, \tag{113}$$

$$\sum_{v=1}^{p} w_q^v := w_q, \tag{114}$$

and

$$w_q^v \geq 1 \; \forall \; v \in \{1, \ldots, p\}, \tag{115}$$

as it is somewhat arbitrary setting minimum employment levels to any value other than one; for then it would always be optimal to choose an even smaller value than that, in which case we would converge back to an infinitely stratified labor market. Thus, we define there has to be at least one worker per position.

With this, we can now demonstrate that, given the above, maximum labor stratification is, in fact, the most efficient production strategy and, so, holds in the labor market. But, to do so, we must first derive an upper limit for aggregate operational output, irrespective of productive arrangement, to serve as our "benchmark" and show other strategies cannot yield higher production.

Lemma 5 (Maximum Operational Output Lemma, MOOL)**.** The maximum operational output of any labor market is exactly the number of employees in its workforce:

$$\mho_q^* = \mho(\boldsymbol{w_q^*}, \boldsymbol{\mho_q}) = \min(\boldsymbol{w_q^*} \times \boldsymbol{\mho_q}) = w_q, \tag{116}$$

where $\boldsymbol{w_q^*}$ is the vector of optimal employment levels in a labor market with w_q employees; and $\boldsymbol{\mho_q}$, the vector of partial operational outputs. Or, assuming maximum labor stratification with unitary employment levels,

$$\mho_q^* = \mho(\boldsymbol{1}, \boldsymbol{\mho_q}(\boldsymbol{\ell_q^*})) = \min(\boldsymbol{1} \times \boldsymbol{\mho_q}(\boldsymbol{\ell_q^*})) = w_q, \tag{117}$$

where $\boldsymbol{\ell_q^*}$ are optimal stratification bounds for the responsability spectra of occupation q's job posts (see "Optimal Stratification Lemma" below).

Moreover, when optimizing employment levels, this maximum production can only be attained when the percentage of each position relative to the entire workforce respects the Proportional Employment Condition (PEC):

$$\boldsymbol{\tilde{w}_q^*} := \frac{\boldsymbol{w_q^*}}{w_q} = \boldsymbol{\Omega_q}, \tag{118}$$

which determines the ratio, or proportion, of a particular job subtype in a labor market is the percentage of an operation's total time duration,

$$\boldsymbol{1}^\top \cdot \boldsymbol{\Omega_q} := 1, \tag{119}$$

accounted by it.

Proof. As we want to derive maximum operational output, throughout this proof we assume

$$\tilde{T}_q^v \geq \ell_v^q \; \forall \; v \in \{1, \ldots, w_q\} \iff \boldsymbol{w_q}(\boldsymbol{\tilde{T}_q}, \boldsymbol{\ell_q}) = \boldsymbol{w_q}, \tag{120}$$

that is all employees are sufficiently qualified for their responsabilities.

With this, we begin with the most trivial of economic configurations, that of independent production with perfectly qualified workers. In this scenario, each employee devotes their unitary time allowance, which coincides with the total time duration of occupation q's operation,

$$\int_0^1 \mathrm{ta}(l)dl := 1, \tag{121}$$

to output exactly one productive unit:

$$1 \times \left(\int_0^1 \mathrm{ta}(l)dl\right)^{-1} = 1; \tag{122}$$

while w_q of such employees working in parallel, yield an output of

$$w_q \times \left(\int_0^1 \mathrm{ta}(l)dl\right)^{-1} = w_q. \tag{123}$$

Here, we have taken occupation q's responsability spectrum $l \in [0, 1]$ as a whole, or as a single, "holistic", task, covering all its activities; and we have found the maximum amount that can be produced of it is one unit per worker, or w_q aggregate units.

However, it can be easier to comprehend this result if we analyze responsability spectra individually, as if a perfectly qualified, independent, employee worked on a series of tasks, which sum to their time allowance,

$$\mathbf{1}^\top \cdot \boldsymbol{\Omega_q} := 1. \tag{124}$$

Note that, as each worker's time allowance is the same as operations' total duration, failing to output any single task by overemphasizing another would nullify the whole productive effort. Hence, the optimal choice of hours to allocate to any responsability spectrum has to be the minimum time required to complete it, or

$$\Omega_q^\ell \in [0, 1]. \tag{125}$$

Furthermore, by the definition of partial operational output (ref) above, one outputs $\mho_q^\ell$ when spending their unitary time allowance to produce a responsability spectrum. So, the output, with only Ω_q^ℓ time units, is:

$$\Omega_q^\ell \mho_q^\ell := \left(\frac{1}{\mho_q^\ell}\right) \times \mho_q^\ell = 1. \tag{126}$$

Finally, as weak occupational complexity implies the production function is homothetic, the aggregate operational output of w_q perfectly qualified employees working independently is:

$$\mho_q^* = \min(\boldsymbol{\Omega_q} \times \boldsymbol{\mho_q}) \times w_q = \Omega_q^\ell \mho_q^\ell \times w_q = 1 \times w_q = w_q. \tag{127}$$

Therefore, a perfectly qualified employee working full-time and independently can output one unit of an occupation's complex tasks with one unit of their time (i.e. their entire time allowance). And, likewise, a workforce with w_q employees identical to this one produces w_q units of operational output. Or, to put it simply, a maximally productive person acchieves maximum production.

We, now, proceed with the binary setting presented above, where employers choose a $\tilde{w}_q^{\text{Jr}} \in (0,1)$ percentage of less qualified (i.e. "junior") job posts to offer, which determines the remaining $\tilde{w}_q^{\text{Sr}} := 1 - \tilde{w}_q^{\text{Jr}} \in (0,1)$ percentage of perfectly qualified (or "senior") positions.

In this case,

$$
\begin{aligned}
\mho(\tilde{w}_q^{\text{Jr}}) &= \min\left(\tilde{w}_q^{\text{Jr}} \times \mho_q^{\text{Jr}}, \tilde{w}_q^{\text{Sr}} \times \mho_q^{\text{Sr}}\right) && (128)\\
&= \min\left(\frac{\tilde{w}_q^{\text{Jr}}}{\int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}(l)dl}, \frac{1-\tilde{w}_q^{\text{Jr}}}{\int_{\tilde{T}_q^{\text{Jr}}}^1 \text{ta}(l)dl}\right) && (129)\\
&= \min\left(\frac{\tilde{w}_q^{\text{Jr}}}{\int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}(l)dl}, \frac{1-\tilde{w}_q^{\text{Jr}}}{\int_0^1 \text{ta}(l)dl - \int_0^{\tilde{T}_q^{\text{Jr}}} \text{ta}(l)dl}\right) && (130)\\
&= \min\left(\frac{\tilde{w}_q^{\text{Jr}}}{\Omega_q^{\text{Jr}}}, \frac{1-\tilde{w}_q^{\text{Jr}}}{1-\Omega_q^{\text{Jr}}}\right), && (131)
\end{aligned}
$$

whereas the operational output of employing $\Omega_q^{\text{Jr}} \in (0,1)$ is

$$
\mho(\Omega_q^{\text{Jr}}) = \min\left(\frac{\Omega_q^{\text{Jr}}}{\Omega_q^{\text{Jr}}}, \frac{1-\Omega_q^{\text{Jr}}}{1-\Omega_q^{\text{Jr}}}\right) = \frac{\Omega_q^{\text{Jr}}}{\Omega_q^{\text{Jr}}} = \frac{1-\Omega_q^{\text{Jr}}}{1-\Omega_q^{\text{Jr}}} = 1. \quad (132)
$$

With this, if $\tilde{w}_q^{\text{Jr}}$ is set to $\tilde{w}_q^{\text{Jr}} > \Omega_q^{\text{Jr}}$, then

$$
\begin{aligned}
&1-\tilde{w}_q^{\text{Jr}} < 1-\Omega_q^{\text{Jr}} && (133)\\
&\therefore \frac{\tilde{w}_q^{\text{Jr}}}{\Omega_q^{\text{Jr}}} > 1 > \frac{1-\tilde{w}_q^{\text{Jr}}}{1-\Omega_q^{\text{Jr}}} && (134)\\
&\therefore \mho(\tilde{w}_q^{\text{Jr}}) = \min\left(\frac{\tilde{w}_q^{\text{Jr}}}{\Omega_q^{\text{Jr}}}, \frac{1-\tilde{w}_q^{\text{Jr}}}{1-\Omega_q^{\text{Jr}}}\right) = \frac{1-\tilde{w}_q^{\text{Jr}}}{1-\Omega_q^{\text{Jr}}} < 1 && (135)\\
&\implies \mho(\tilde{w}_q^{\text{Jr}}) < \mho(\Omega_q^{\text{Jr}}) = 1; && (136)
\end{aligned}
$$

and, if $\tilde{w}_q^{\text{Jr}} < \Omega_q^{\text{Jr}}$,

$$
\begin{aligned}
&1-\tilde{w}_q^{\text{Jr}} > 1-\Omega_q^{\text{Jr}} && (137)\\
&\therefore \frac{\tilde{w}_q^{\text{Jr}}}{\Omega_q^{\text{Jr}}} < 1 < \frac{1-\tilde{w}_q^{\text{Jr}}}{1-\Omega_q^{\text{Jr}}} && (138)\\
&\therefore \mho(\tilde{w}_q^{\text{Jr}}) = \min\left(\frac{\tilde{w}_q^{\text{Jr}}}{\Omega_q^{\text{Jr}}}, \frac{1-\tilde{w}_q^{\text{Jr}}}{1-\Omega_q^{\text{Jr}}}\right) = \frac{\tilde{w}_q^{\text{Jr}}}{\Omega_q^{\text{Jr}}} < 1 && (139)
\end{aligned}
$$

$$\implies \mho(\tilde{w}_q^{\mathrm{Jr}}) < \mho(\Omega_q^{\mathrm{Jr}}) = 1. \tag{140}$$

Hence,

$$\mho(\tilde{w}_q^{\mathrm{Jr}}) < \mho(\Omega_q^{\mathrm{Jr}}) = 1 \;\forall\; \tilde{w}_q^{\mathrm{Jr}} \neq \Omega_q^{\mathrm{Jr}} \in (0,1). \tag{141}$$

Analogously, with multiple job subtypes, optimal operational output is:

$$\mho(\boldsymbol{\Omega_q}) = \min(\boldsymbol{\Omega_q} \times \boldsymbol{\mho_q}) = \frac{\Omega_q^\ell}{\Omega_q^\ell} = 1, \tag{142}$$

for, again, since

$$1 =: \mathbf{1}^\top \cdot \boldsymbol{\tilde{w}_q} = \mathbf{1}^\top \cdot \boldsymbol{\Omega_q} := 1, \tag{143}$$

choosing any $\tilde{w}_q^\ell \neq \Omega_q^\ell$ implies the proportion of at least one position, say $\tilde{w}_q^r$, is impacted, and aggregate output along with it, either because

$$\tilde{w}_q^\ell > \Omega_q^\ell \tag{144}$$

$$\therefore \frac{\tilde{w}_q^\ell}{\Omega_q^\ell} > 1 > \frac{\tilde{w}_q^r}{\Omega_q^r} \tag{145}$$

$$\therefore \mho(\boldsymbol{\tilde{w}_q}) = \min(\boldsymbol{\tilde{w}_q} \times \boldsymbol{\mho_q}) = \frac{\tilde{w}_q^r}{\Omega_q^r} < 1 \tag{146}$$

$$\implies \mho(\boldsymbol{\tilde{w}_q}) < \mho(\boldsymbol{\Omega_q}) = 1; \tag{147}$$

or, alternatively, because

$$\tilde{w}_q^\ell < \Omega_q^\ell \tag{148}$$

$$\therefore \frac{\tilde{w}_q^\ell}{\Omega_q^\ell} < 1 < \frac{\tilde{w}_q^r}{\Omega_q^r} \tag{149}$$

$$\therefore \mho(\boldsymbol{\tilde{w}_q}) = \min(\boldsymbol{\tilde{w}_q} \times \boldsymbol{\mho_q}) = \frac{\tilde{w}_q^\ell}{\Omega_q^\ell} < 1 \tag{150}$$

$$\implies \mho(\boldsymbol{\tilde{w}_q}) < \mho(\boldsymbol{\Omega_q}) = 1. \tag{151}$$

Thus,

$$\mho(\boldsymbol{\tilde{w}_q}, \boldsymbol{\mho_q}) < \mho(\boldsymbol{\Omega_q}, \boldsymbol{\mho_q}) = 1 \tag{152}$$

$$\therefore \mho(\boldsymbol{w_q}, \boldsymbol{\mho_q}) < \mho(w_q \times \boldsymbol{\Omega_q}, \boldsymbol{\mho_q}) = w_q \tag{153}$$

$$\forall\; \boldsymbol{\tilde{w}_q} \neq \boldsymbol{\Omega_q} \in [0,1]^p, p \in \{1, \ldots, w_q\}, \tag{154}$$

$$1 =: \mathbf{1}^\top \cdot \boldsymbol{\tilde{w}_q} = \mathbf{1}^\top \cdot \boldsymbol{\Omega_q} := 1. \tag{155}$$

We can derive the same conclusion for a maximally stratified labor market, as well. But here, instead of determining the proportion of job subtypes with a $\boldsymbol{w_q^*}$ vector of employment levels, employers maximize production selecting optimal $\boldsymbol{\ell_q^*}$ responsability bounds for w_q unique job posts.

So, let

$$\boldsymbol{\ell^*_q} := (\ell^*_0, \dots, \ell^*_{w_q}) := (0, \dots, 1) \in [0,1]^{w_q}, \tag{156}$$

with

$$\sum_{v=1}^{w_q} \int_{\ell^*_{v-1}}^{\ell^*_v} \mathrm{ta}(l)dl = \int_0^1 \mathrm{ta}(l)dl := 1 \tag{157}$$

be the vector of optimal responsability bounds that maximizes operational output, such that

$$\mho(\boldsymbol{\ell^*_q}) = \min(\mathbf{1} \times \mho_{\boldsymbol{q}}(\boldsymbol{\ell^*_q})) = 1 \times \left(\int_{\ell^*_{v-1}}^{\ell^*_v} \mathrm{ta}(l)dl \right)^{-1} := w_q, \tag{158}$$

as with the previous economic configurations.

Note employers could, again, attempt to increase production beyond this level if they, now, reduced the responsabilities of a particular job subtype by setting

$$\ell_v < \ell^*_v \implies \left(\int_{\ell^*_{v-1}}^{\ell_v} \mathrm{ta}(l)dl \right)^{-1} > \left(\int_{\ell^*_{v-1}}^{\ell^*_v} \mathrm{ta}(l)dl \right)^{-1} := w_q. \tag{159}$$

Nevertheless, because every worker has the same unitary time allowance, this would also entail the complementary subinterval of complex tasks $l \in [\ell_v, \ell^*_v]$ would either not be produced at all, in which case

$$\mho(\boldsymbol{\ell_q}) = 0 \times \left(\int_{\ell_v}^{\ell^*_v} \mathrm{ta}(l)dl \right)^{-1} = 0, \tag{160}$$

or that it would be produced with a $1 - \omega^v_q \in [0,1]$ fraction of a time unit, yielding some quantity

$$\mho(\boldsymbol{\ell_q}, \boldsymbol{\omega_q}) = (1 - \omega^v_q) \times \left(\int_{\ell_v}^{\ell^*_v} \mathrm{ta}(l)dl \right)^{-1}, \tag{161}$$

where $\omega^v_q \in [0,1]$ is the percentage of worker v's time allowance dedicated to the emphasized $l \in [\ell^*_{v-1}, \ell_v]$ responsability spectrum.

Furthermore, because aggregate operational output is given by the Leontief production function,

$$\mho_q := \mho(\mathbf{1}, \mho_{\boldsymbol{q}}(\boldsymbol{\ell_q}, \boldsymbol{\omega_q})) := \min(\mathbf{1} \times \mho_{\boldsymbol{q}}(\boldsymbol{\ell_q}, \boldsymbol{\omega_q})), \tag{162}$$

$$\mho_{\boldsymbol{q}}(\boldsymbol{\ell_q}, \boldsymbol{\omega_q}) := (\mho^1_q, \dots, \mho^{w_q}_q), \tag{163}$$

$$\mho^v_q := \min\left(\frac{\omega^v_q}{\int_{\ell^*_{v-1}}^{\ell_v} \mathrm{ta}(l)dl}, \frac{1 - \omega^v_q}{\int_{\ell_v}^{\ell^*_v} \mathrm{ta}(l)dl} \right), \tag{164}$$

it would be pointless if only a subset of employees were to increase their operational output by themselves; for an occupation's complex tasks are all complementary: they work together to acchieve its operation. Hence, for $\mho_q(\boldsymbol{\ell_q}, \boldsymbol{\omega_q})$ to be greater than $\mho_q(\boldsymbol{\ell_q^*}) = w_q$,

$$\mho_q^v > w_q \ \forall \ v \in \{1, \ldots, w_q\}, \tag{165}$$

which requires all partial operational outputs to surpass the following point of equilibrium:

$$\mho(\boldsymbol{\ell_q}, \boldsymbol{\omega_q}) = \mho(\boldsymbol{\ell_q^*}) = \min(\mathbf{1} \times \mho_{\boldsymbol{q}}(\boldsymbol{\ell_q^*})) = w_q \tag{166}$$

$$\iff \frac{\omega_q^v}{\int_{\ell_{v-1}^*}^{\ell_v} \mathrm{ta}(l)dl} = \frac{1-\omega_q^v}{\int_{\ell_v}^{\ell_v^*} \mathrm{ta}(l)dl} = w_q \ \forall \ v \in \{1, \ldots, w_q\} \tag{167}$$

$$\iff \omega_q^v = w_q \int_{\ell_{v-1}^*}^{\ell_v} \mathrm{ta}(l)dl \wedge 1 - \omega_q^v = w_q \int_{\ell_v}^{\ell_v^*} \mathrm{ta}(l)dl. \tag{168}$$

Now, if any single $\omega_q^v \in [0, 1]$ is set to

$$\omega_q^v > w_q \int_{\ell_{v-1}^*}^{\ell_v} \mathrm{ta}(l)dl, \tag{169}$$

then, indeed,

$$\frac{\omega_q^v}{\int_{\ell_{v-1}^*}^{\ell_v} \mathrm{ta}(l)dl} > w_q, \tag{170}$$

but also

$$\frac{1-\omega_q^v}{\int_{\ell_v}^{\ell_v^*} \mathrm{ta}(l)dl} < w_q \tag{171}$$

$$\implies \mho_q^v = \min\left(\frac{\omega_q^v}{\int_{\ell_{v-1}^*}^{\ell_v} \mathrm{ta}(l)dl}, \frac{1-\omega_q^v}{\int_{\ell_v}^{\ell_v^*} \mathrm{ta}(l)dl}\right) = \frac{1-\omega_q^v}{\int_{\ell_v}^{\ell_v^*} \mathrm{ta}(l)dl} < w_q \tag{172}$$

$$\therefore \mho_q(\boldsymbol{\ell_q}, \boldsymbol{\omega_q}) < \mho_q(\boldsymbol{\ell_q^*}) = w_q; \tag{173}$$

and, conversely,

$$\omega_q^v < w_q \int_{\ell_{v-1}^*}^{\ell_v} \mathrm{ta}(l)dl \tag{174}$$

$$\implies \frac{\omega_q^v}{\int_{\ell_{v-1}^*}^{\ell_v} \mathrm{ta}(l)dl} < w_q < \frac{1-\omega_q^v}{\int_{\ell_v}^{\ell_v^*} \mathrm{ta}(l)dl} \tag{175}$$

$$\implies \mho_q^v = \min\left(\frac{\omega_q^v}{\int_{\ell_{v-1}^*}^{\ell_v} \mathrm{ta}(l)dl}, \frac{1-\omega_q^v}{\int_{\ell_v}^{\ell_v^*} \mathrm{ta}(l)dl}\right) = \frac{\omega_q^v}{\int_{\ell_{v-1}^*}^{\ell_v} \mathrm{ta}(l)dl} < w_q \tag{176}$$

$$\therefore \mho_q(\boldsymbol{\ell_q}, \boldsymbol{\omega_q}) < \mho_q(\boldsymbol{\ell^*_q}) = w_q; \tag{177}$$

so that

$$\nexists\ \boldsymbol{\ell_q}, \boldsymbol{\omega_q} \in [0,1]^{w_q} \mid \mho_q(\boldsymbol{\ell_q}, \boldsymbol{\omega_q}) > \mho_q(\boldsymbol{\ell^*_q}) = w_q, \tag{178}$$

$$\sum_{v=1}^{w_q} \left(\int_{\ell^*_{v-1}}^{\ell_v} \mathrm{ta}(l)dl + \int_{\ell_v}^{\ell^*_v} \mathrm{ta}(l)dl \right) = \int_0^1 \mathrm{ta}(l)dl := 1. \tag{179}$$

Finally, even with an $\boldsymbol{\omega^v_q}$ vector of partial time allocations for each worker, at least one difficulty subinterval would have to be neglected to emphasize another,

$$\because 1 =: \mathbf{1}^\top \cdot \boldsymbol{\omega^v_q} = \mathbf{1}^\top \cdot \boldsymbol{\Omega^v_q} := 1 \implies \min \left(\boldsymbol{\omega^v_q} \times \mho_{\boldsymbol{q}}(\boldsymbol{\ell_q}) \right) < w_q \tag{180}$$

$$\therefore \mho_q(\boldsymbol{\ell^v_q}, \boldsymbol{\omega^v_q}) < \mho_q(\boldsymbol{\ell^*_q}) = w_q, \tag{181}$$

as before.

Thus, we have demonstrated there cannot be, in any productive arrangement, a higher aggregate operational output than w_q, that is the number of employees in the workforce, as all attempts to increase production, actually, end up hindering it.

The intuition for this is quite simple. Production strategies can merely distribute the available talent across an occupation's responsability spectrum: they are but ways of splitting and organizing tasks conveniently (via independent production, or any level of labor stratification); they do not, however, change activities' time requirements, nor the time allowances of employees, both of which are, by definition, equivalent. So, these economic configurations only serve to "safeguard" operational output against worker's potential underqualification. The main limiting factors to production, then, are workers' capacity and time itself. Hence, we may say, somewhat tautologically, the most one can produce in a day is a "day's work". □

In the lemma above, we have assumed there to be an optimal $\boldsymbol{\ell^*_q}$ vector of responsability bounds maximizing operational output in stratified production. We shall, now, devote out attention to describing what such a vector would have to be like and, thus, how production is optimally stratified.

Lemma 6 (Optimal Stratification Lemma, OSL)**.** Because in a maximally and monotonically stratified labor market every position is its own job subtype (for, again, employment levels are unitary), optimal production is, then, obtained not by choosing how many workers to allocate to tasks of varying difficulty levels, but instead by setting appropriate responsability ranges for each position (i.e. which tasks to allocate *to* workers). The bounds for these ranges are:

$$\lambda^q_v = \mathrm{TA}^{-1} \left(\frac{v}{w_q} + \mathrm{TA}(0) \right) \ \forall\ v \in \{1, \ldots, w_q\}, \tag{182}$$

where $\mathrm{TA}(l)$ is the anti-derivative of the time allocation function $\mathrm{ta}(l)$, and $\mathrm{TA}^{-1}(l)$, its inverse.

Proof. We have just demonstrated the maximum operational output in any labor market, with or without unique, unitary, positions, is exactly

$$\mho_q^* = \min(\boldsymbol{w_q^*} \times \mho_{\boldsymbol{q}}) = \min(\mathbf{1} \times \mho_{\boldsymbol{q}}(\boldsymbol{\ell_q^*})) = w_q, \tag{183}$$

or the number of employees in its workforce.

Therefore, optimal bounds for responsability spectra can be calculated by equating partial operational outputs with maximum production; for if maximum-monotonic labor stratification is to be optimal, it must yield the same partial outputs as any efficient production strategy.

So, for the first job subtype,

$$1 \times \left(\int_{\ell_0}^{\ell_1} \mathrm{ta}(l) dl \right)^{-1} = 1 \times \left(\int_{0}^{\ell_1} \mathrm{ta}(l) dl \right)^{-1} = w_q, \tag{184}$$

which means the partial operational output of the first worker, whose tasks range from $\ell_0 = 0$ to $\ell_1 \in [0, 1]$ exclusively, should produce the same amount of the $l \in [0, \ell_1]$ responsability spectrum as would be produced in an economic configuration with maximum operational output (e.g. with w_q perfectly qualified employees working independently).

Thus, solving for ℓ_1, we get:

$$1 \times \left(\int_{0}^{\ell_1} \mathrm{ta}(l) dl \right)^{-1} = w_q \tag{185}$$

$$\therefore \int_{0}^{\ell_1} \mathrm{ta}(l) dl = \frac{1}{w_q} \tag{186}$$

$$\therefore \mathrm{TA}(l)\Big|_0^{\ell_1} = \mathrm{TA}(\ell_1) - \mathrm{TA}(0) = \frac{1}{w_q} \tag{187}$$

$$\therefore \mathrm{TA}^{-1}(\mathrm{TA}(\ell_1)) = \mathrm{TA}^{-1}\left(\frac{1}{w_q} + \mathrm{TA}(0) \right) \tag{188}$$

$$\therefore \ell_1 = \mathrm{TA}^{-1}\left(\frac{1}{w_q} + \mathrm{TA}(0) \right). \tag{189}$$

Likewise, for the second worker,

$$1 \times \left(\int_{\ell_1}^{\ell_2} \mathrm{ta}(l) dl \right)^{-1} = w_q \tag{190}$$

$$\therefore \int_{\ell_1}^{\ell_2} \mathrm{ta}(l) dl = \frac{1}{w_q} \tag{191}$$

$$\therefore \mathrm{TA}(l)\Big|_{\ell_1}^{\ell_2} = \mathrm{TA}(\ell_2) - \mathrm{TA}(\ell_1) = \frac{1}{w_q} \tag{192}$$

$$\therefore \mathrm{TA}^{-1}(\mathrm{TA}(\ell_2)) = \mathrm{TA}^{-1}\left(\frac{1}{w_q} + \mathrm{TA}(\ell_1) \right) \tag{193}$$

$$\therefore \mathrm{TA}^{-1}(\mathrm{TA}(\ell_2)) = \mathrm{TA}^{-1}\left(\frac{1}{w_q} + \frac{1}{w_q} + \mathrm{TA}(0)\right) \tag{194}$$

$$\therefore \ell_2 = \mathrm{TA}^{-1}\left(\frac{2}{w_q} + \mathrm{TA}(0)\right). \tag{195}$$

For the third worker,

$$1 \times \left(\int_{\ell_2}^{\ell_3} \mathrm{ta}(l)dl\right)^{-1} = w_q \tag{196}$$

$$\therefore \int_{\ell_2}^{\ell_3} \mathrm{ta}(l)dl = \frac{1}{w_q} \tag{197}$$

$$\therefore \mathrm{TA}(l)\Big|_{\ell_2}^{\ell_3} = \mathrm{TA}(\ell_3) - \mathrm{TA}(\ell_2) = \frac{1}{w_q} \tag{198}$$

$$\therefore \mathrm{TA}^{-1}(\mathrm{TA}(\ell_3)) = \mathrm{TA}^{-1}\left(\frac{1}{w_q} + \mathrm{TA}(\ell_2)\right) \tag{199}$$

$$\therefore \mathrm{TA}^{-1}(\mathrm{TA}(\ell_3)) = \mathrm{TA}^{-1}\left(\frac{1}{w_q} + \frac{1}{w_q} + \frac{1}{w_q} + \mathrm{TA}(0)\right) \tag{200}$$

$$\therefore \ell_3 = \mathrm{TA}^{-1}\left(\frac{3}{w_q} + \mathrm{TA}(0)\right). \tag{201}$$

And so on and so forth, up to the very last worker:

$$1 \times \left(\int_{\ell_{w_q-1}}^{\ell_{w_q}} \mathrm{ta}(l)dl\right)^{-1} = w_q \tag{202}$$

$$\therefore \int_{\ell_{w_q-1}}^{\ell_{w_q}} \mathrm{ta}(l)dl = \frac{1}{w_q} \tag{203}$$

$$\therefore \mathrm{TA}(l)\Big|_{\ell_{w_q-1}}^{\ell_{w_q}} = \mathrm{TA}(\ell_{w_q}) - \mathrm{TA}(\ell_{w_q-1}) = \frac{1}{w_q} \tag{204}$$

$$\therefore \mathrm{TA}^{-1}(\mathrm{TA}(\ell_{w_q})) = \mathrm{TA}^{-1}\left(\frac{1}{w_q} + \mathrm{TA}(\ell_{w_q-1})\right) \tag{205}$$

$$\therefore \mathrm{TA}^{-1}(\mathrm{TA}(\ell_{w_q})) = \mathrm{TA}^{-1}\left(\frac{1}{w_q} + \cdots + \frac{1}{w_q} + \mathrm{TA}(0)\right) \tag{206}$$

$$\therefore \ell_{w_q} = \mathrm{TA}^{-1}\left(\frac{w_q}{w_q} + \mathrm{TA}(0)\right) := 1 \tag{207}$$

$$\iff \mathrm{TA}^{-1}\left(\frac{w_q}{w_q} + \mathrm{TA}(0)\right) = \mathrm{TA}^{-1}\left(1 + \mathrm{TA}(0)\right) = 1 \tag{208}$$

$$\iff \mathrm{TA}\left(\mathrm{TA}^{-1}\left(1 + \mathrm{TA}(0)\right)\right) = \mathrm{TA}(1) \tag{209}$$

$$\iff \mathrm{TA}(1) - \mathrm{TA}(0) = \int_0^1 \mathrm{ta}(l)dl = 1, \tag{210}$$

which is true, by definition,

$$\because \mathrm{ta}(l) := \mathrm{ttc}(l) \times \left(\int_0^1 \mathrm{ttc}(l) dl\right)^{-1} \tag{211}$$

$$\implies \int_0^1 \mathrm{ta}(l) dl = \left(\int_0^1 \mathrm{ttc}(l) dl\right)^{-1} \times \int_0^1 \mathrm{ttc}(l) dl = 1. \tag{212}$$

And, with this condition met, we can finally arrive, by the induction above, to a general form of optimal responsability ranges:

$$\ell_v = \mathrm{TA}^{-1}\left(\frac{v}{w_q} + \mathrm{TA}(0)\right) \ \forall \ v \in \{1, \ldots, w_q\}. \tag{213}$$

□

Having derived the optimal responsability ranges for a maximally stratified labor market with unitary employment levels, we shall demonstrate other types of labor stratification cannot attain in an efficient economy because: 1) no other strategy has lower productivity requirements (MPL); 2) and, given MSA, there is only one set of optimal responsability bounds and employment levels (ESL).

Lemma 7 (Minimum Productivity Lemma, MPL)**.** Maximally stratified markets have the lowest barrier of entry out of all valid productive arrangements.

Proof. To show maximally stratified markets pose the lowest barrier of entry, let us consider the minimum required productivity for each economic configuration.

If production is organized independently,

$$\mho_q = w_q \iff \tilde{T}_q^v = 1 \ \forall \ v \in \{1, \ldots, w_q\}, \tag{214}$$

that is, either all w_q employees are perfectly qualified, or maximum operational output (see MOOL) is not achieved. Moreover, in a binary setting,

$$\mho_q = w_q \iff \tilde{T}_q^v \geq \tilde{T}_q^{\mathrm{Jr}} \in [0, 1) \ \forall \ v \in \{1, \ldots, w_q^{\mathrm{Jr}}\} \wedge \tilde{T}_q^v = 1, \tag{215}$$

for the rest of the workforce (i.e. all junior employess have at least junior productivity, and all senior employees are perfectly qualified), which means productivity requirements are lower in this economic configuration, with a weighted productivity of, at least,

$$\tilde{w}_q^{\mathrm{Jr}} \times \tilde{T}_q^{\mathrm{Jr}} + (1 - \tilde{w}_q^{\mathrm{Jr}}) \times 1 < 1 \because \tilde{T}_q^{\mathrm{Jr}} \in [0, 1) \wedge \tilde{w}_q^{\mathrm{Jr}} \in (0, 1), \tag{216}$$

rather than $1 \times 1 = 1$, with independent production. And, if there were three levels of seniority, with w_q^{Ir} interns (less qualified than juniors and seniors), productivity requirements would be even lower:

$$\tilde{w}_q^{\mathrm{Ir}} \times \tilde{T}_q^{\mathrm{Ir}} + \tilde{w}_q^{\mathrm{Jr}} \times \tilde{T}_q^{\mathrm{Jr}} + (1 - \tilde{w}_q^{\mathrm{Jr}} - \tilde{w}_q^{\mathrm{Ir}}) \times 1 \tag{217}$$

$$< \tilde{w}_q^{\mathrm{Jr}} \times \tilde{T}_q^{\mathrm{Jr}} + (1 - \tilde{w}_q^{\mathrm{Jr}}) \times 1 \tag{218}$$

$$< 1 \tag{219}$$

$$\because \tilde{T}_q^{\mathrm{Ir}} < \tilde{T}_q^{\mathrm{Jr}} \in [0,1) \wedge \tilde{w}_q^{\mathrm{Ir}}, \tilde{w}_q^{\mathrm{Jr}} \in (0,1) \wedge \tilde{w}_q^{\mathrm{Ir}} + \tilde{w}_q^{\mathrm{Jr}} \in (0,1). \tag{220}$$

And the pattern continues for all levels of labor stratification, up to the limit of w_q unique positions in a maximally and monotonically stratified labor market, where minimum required productivity is the lowest it can be when infinite stratification is ruled out (MSA), with an average of

$$\sum_{v=1}^{w_q} \tilde{w}_q^v \times \tilde{T}_q^v = \left(\frac{1}{w_q}\right) \sum_{v=1}^{w_q} \ell_q^v = \left(\frac{1}{w_q}\right) \sum_{v=1}^{w_q} \mathrm{TA}^{-1} \left(\frac{v}{w_q} + \mathrm{TA}(0)\right). \tag{221}$$

More thoroughly, though,

$$p_q = 1 \implies \underline{\boldsymbol{\tilde{T}_q^{\mathrm{IP}}}} = \underbrace{(1, \ldots, 1)}_{w_q^1(1)=w_q}, \tag{222}$$

$$p_q = 2 \implies \underline{\boldsymbol{\tilde{T}_q^{\mathrm{BS}}}} = (\underbrace{\tilde{T}_q^{\mathrm{Jr}}, \ldots, \tilde{T}_q^{\mathrm{Jr}}}_{w_q^1(2)}, \underbrace{1, \ldots, 1}_{w_q^2(2)}), \tag{223}$$

$$p_q = 3 \implies \underline{\boldsymbol{\tilde{T}_q^{\mathrm{3S}}}} = (\underbrace{\tilde{T}_q^{\mathrm{Ir}}, \ldots, \tilde{T}_q^{\mathrm{Ir}}}_{w_q^1(3)}, \underbrace{\tilde{T}_q^{\mathrm{Jr}}, \ldots, \tilde{T}_q^{\mathrm{Jr}}}_{w_q^2(3)}, \underbrace{1, \ldots, 1}_{w_q^3(3)}), \tag{224}$$

$$\vdots \tag{225}$$

$$p_q = w_q \implies \underline{\boldsymbol{\tilde{T}_q^{\mathrm{MS}}}} = (\underbrace{\lambda_1^q}_{1}, \ldots, \underbrace{\lambda_{w_q}^q}_{1}) := (\underbrace{\lambda_1^q}_{1}, \ldots, \underbrace{1}_{1}) \tag{226}$$

[does $\tilde{T}_q^r$ have to be constant across differently stratified markets?]

are the vectors of minimum required productivity (hence, the underline) to implement each strategy, where the superscripts stand for "independent production" (IP), "binary stratification" (BS), "three-fold stratification" (3S), and "maximum stratification" (MS), respectively, while

$$w_q^v(p_q) \geq 1 \;\forall\; v \in \{1, \ldots, p_q\}, p_q \in \{1, \ldots, w_q\} \tag{227}$$

are employment levels per job subtype as a function of labor stratification.

Or, in matrix form,

$$\underline{\boldsymbol{\tilde{T}_q}} = \begin{bmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ \tilde{T}_q^{\mathrm{Jr}} & \cdots & \tilde{T}_q^{\mathrm{Jr}} & \tilde{T}_q^{\mathrm{Jr}} & \cdots & 1 \\ \tilde{T}_q^{\mathrm{Ir}} & \cdots & \tilde{T}_q^{\mathrm{Ir}} & \tilde{T}_q^{\mathrm{Jr}} & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^q & \cdots & \lambda_{w_q^1(3)}^q & \lambda_{1+w_q^1(3)}^q & \cdots & 1 \end{bmatrix}. \tag{228}$$

It is clear in $\underline{\boldsymbol{\tilde{T}_q}}$ elements are weakly decreasing on their row number (i.e. those higher up in the matrix are greater or equal to those below them), as

$$\lambda_1^q < \cdots < \tilde{T}_q^{\mathrm{Ir}} < \tilde{T}_q^{\mathrm{Jr}} < \cdots < \tilde{T}_q^{\mathrm{Sr}} := 1, \tag{229}$$

$$\lambda^q_{w^1_q(3)} < \cdots < \tilde{T}^{\mathrm{Ir}}_q < \tilde{T}^{\mathrm{Jr}}_q < \cdots < \tilde{T}^{\mathrm{Sr}}_q := 1, \tag{230}$$

$$\lambda^q_{1+w^1_q(3)} < \cdots < \tilde{T}^{\mathrm{Jr}}_q = \tilde{T}^{\mathrm{Jr}}_q < \cdots < \tilde{T}^{\mathrm{Sr}}_q := 1, \tag{231}$$

and similarly for every other row in every column but the last one, which is the unit vector. So, we generalize this as follows:

$$\lambda^q_v \leq \tilde{\underline{T}}^r_q \ \forall\ v \in \left\{1, \ldots, \sum_{z=1}^{r} w^z_q(p_q)\right\}, r \in \{1, \ldots, p_q\}, p_q \in \{1, \ldots, w_q\}; \tag{232}$$

that is, a maximally stratified market minimizes productivity requirements. □

Now that we have proven maximum labor stratification with unitary employment is the production strategy with the lowest barrier of entry, let us also show it is the only feasible mode of maximum stratification.

Lemma 8 (General Stratification Lemma, GSL)**.** A generalization of the Optimal Stratification Lemma, the GSL states that, for any degree of labor stratification – whether binary, three-fold, etc –, rational employers' will always choose to split more difficult positions up to the (predefined) limit of one worker per job subtype, leaving the remaining responsabilities to the first, and easiest, role.

Lemma 9 (Efficient Stratification Lemma, ESL)**.** Any efficient labor market where employers choose both $\boldsymbol{w_q}$ and $\boldsymbol{\ell_q}$ with $p \in \{1, \ldots, w_q\}$ types of job posts converges to maximum labor stratification with $p = w_q$ unique positions;

$$\boldsymbol{\ell_q} = \boldsymbol{\ell^*_q} := (\ell^*_0, \ldots, \ell^*_p) := (0, \ldots, 1), \tag{233}$$

$$\ell^*_v = \mathrm{TA}^{-1}\left(\frac{v}{w_q} + \mathrm{TA}(0)\right) \ \forall\ v \in \{1, \ldots, w_q\} \tag{234}$$

optimal responsability bounds; and unitary employment levels, $\boldsymbol{w_q} = \boldsymbol{1}$.

Proof 1. In the MPL above, we have shown maximum stratification minimizes productivity requirements and, thus, the chance workers may not be sufficiently qualified for their responsabilities (i.e. it is the safest production strategy). So,

$$p := w_q \tag{235}$$

is the optimal number of positions in a labor market where employers can split workers' activities without gain or loss to production (WOCA), while attaining the same w_q maximum operational output as all economic strategies (MOOL).

In addition, by the Maximum Stratification Axiom,

$$\sum_{v=1}^{p} w^v_q := w_q \wedge w^v_q \geq 1 \ \forall\ v \in \{1, \ldots, p\}, p \in \{1, \ldots, w_q\} \tag{236}$$

$$\therefore p := w_q \wedge \sum_{v=1}^{w_q} 1 = w_q \implies \sum_{v=1}^{p} w^v_q = \sum_{v=1}^{w_q} w^v_q = w_q \tag{237}$$

$$\iff w_q^v = 1 \;\forall\; v \in \{1, \ldots, w_q\} \tag{238}$$

$$\iff \boldsymbol{w_q} = \mathbf{1}. \tag{239}$$

Therefore, the above implies, given the PEC and OSL,

$$\int_{\ell_{v-1}}^{\ell_v} \mathrm{ta}(l)dl = \tilde{w}_q^v = \frac{1}{w_q} \wedge \left(\int_{\ell_{v-1}^*}^{\ell_v^*} \mathrm{ta}(l)dl \right)^{-1} := w_q \tag{240}$$

$$\implies \int_{\ell_{v-1}}^{\ell_v} \mathrm{ta}(l)dl = \frac{1}{\left(\int_{\ell_{v-1}^*}^{\ell_v^*} \mathrm{ta}(l)dl \right)^{-1}} = \int_{\ell_{v-1}^*}^{\ell_v^*} \mathrm{ta}(l)dl \tag{241}$$

$$\iff \ell_v = \ell_v^* = \mathrm{TA}^{-1}\left(\frac{v}{w_q} + \mathrm{TA}(0) \right) \;\forall\; v \in \{1, \ldots, w_q\}. \tag{242}$$

□

Proof 2. Though simple in itself, it is easier to understand this proof iteratively. Hence, as before, we can define the number of job subtypes to be

$$p := w_q, \tag{243}$$

as this is the upper limit for labor stratification (MSA), which maximally safeguards production (MPL, MOOL); then, proceed by setting

$$w_q^p := w_q^{w_q} = 1, \tag{244}$$

as MSA requires $w_q^v \geq 1 \;\forall\; v \in \{1, \ldots, p\}$. Thus, by the Proportional Employment Condition,

$$\int_{\ell_{p-1}}^{\ell_p} \mathrm{ta}(l)dl = \int_{\ell_{w_q-1}}^{\ell_{w_q}} \mathrm{ta}(l)dl := \int_{\ell_{w_q-1}}^{1} \mathrm{ta}(l)dl = \frac{1}{w_q}, \tag{245}$$

we estimate

$$\ell_{p-1} = \mathrm{TA}^{-1}\left(\mathrm{TA}(1) - \frac{1}{w_q} \right) \tag{246}$$

$$= \mathrm{TA}^{-1}\left(1 + \mathrm{TA}(0) - \frac{1}{w_q} \right) \tag{247}$$

$$= \mathrm{TA}^{-1}\left(\frac{w_q}{w_q} - \frac{1}{w_q} + \mathrm{TA}(0) \right) \tag{248}$$

$$= \mathrm{TA}^{-1}\left(\frac{w_q - 1}{w_q} + \mathrm{TA}(0) \right) \tag{249}$$

$$= \ell_{w_q-1}^* \tag{250}$$

$$\because \mathrm{TA}(1) - \mathrm{TA}(0) = \int_0^1 \mathrm{ta}(l)dl = 1 \tag{251}$$

by minimizing the amount of perfectly qualified workers for efficient production. And, similarly, for the $l \in [\ell_{p-2}, \ell_{p-1}]$ responsability spectrum, if $w_{p-1} \geq 1$ is

$$w_{p-1} := w_{w_q-1} = 1, \tag{252}$$

we minimize productivity requirements:

$$\int_{\ell_{p-2}}^{\ell_{p-1}} \mathrm{ta}(l)dl = \int_{\ell_{w_q-2}}^{\ell_{w_q-1}} \mathrm{ta}(l)dl = \frac{1}{w_q} \tag{253}$$

$$\implies \ell_{p-2} = \ell^*_{w_q-2} = \mathrm{TA}^{-1}\left(\frac{w_q-2}{w_q} + \mathrm{TA}(0)\right). \tag{254}$$

Again, ℓ_{p-3} is derived by minimizing the number of highly productive workers:

$$\int_{\ell_{p-3}}^{\ell_{p-2}} \mathrm{ta}(l)dl = \int_{\ell_{w_q-3}}^{\ell_{w_q-2}} \mathrm{ta}(l)dl = \frac{1}{w_q} \tag{255}$$

$$\implies \ell_{p-3} = \ell^*_{w_q-3} = \mathrm{TA}^{-1}\left(\frac{w_q-3}{w_q} + \mathrm{TA}(0)\right); \tag{256}$$

and, so on and so forth, always minimizing productivity requirements with $w_q^v := 1 \; \forall \, v \in \{1, \ldots, w_q\}$. Notice the pattern here is the same as in the Optimal Stratification Lemma. Therefore, responsability bounds, and also employment levels, are the same as well. Hence,

$$p = w_q, \tag{257}$$

$$\boldsymbol{\ell_q} = \boldsymbol{\ell_q^*} := (\ell_0^*, \ldots, \ell_p^*) := (0, \ldots, 1), \tag{258}$$

$$\ell_v^* = \mathrm{TA}^{-1}\left(\frac{v}{w_q} + \mathrm{TA}(0)\right) \; \forall \, v \in \{1, \ldots, w_q\}, \tag{259}$$

$$\boldsymbol{w_q} = \mathbf{1}. \tag{260}$$

□

Finally, let us show the economy's productivity has to be capable of supporting maximum-monotonic labor stratification.

Lemma 10 (Productivity Sufficiency Lemma, PSL)**.** The available talent in a labor market is, at least, sufficient to allow for maximally stratified production.

Proof 1. If talent were not sufficient to produce occupation q's entire $l \in [0, 1]$ responsability spectrum, then, as aggregate operation output is given by the Leontief function (see WOCA, MOOL),

$$\mho_q := \mho(\boldsymbol{w_q}(\tilde{\boldsymbol{T}}_{\boldsymbol{q}}, \boldsymbol{\ell_q}), \mho_{\boldsymbol{q}}(\boldsymbol{\ell_q})) := \min\left(\boldsymbol{w_q}(\tilde{\boldsymbol{T}}_{\boldsymbol{q}}, \boldsymbol{\ell_q}) \times \mho_{\boldsymbol{q}}(\boldsymbol{\ell_q})\right), \tag{261}$$

employers' optimal choice would be to save their resources and completely shutdown the productive effort. Therefore,

$$\neg \, \mho_q > 0 \iff \boldsymbol{w_q^*} = \mathbf{0} \iff \mathbf{1}^\top \cdot \boldsymbol{w_q^*} = 0 \tag{262}$$

$$\therefore w_q > 0 \iff \mho_q > 0 \iff \boldsymbol{w_q}(\tilde{\boldsymbol{T}}_{\boldsymbol{q}}, \boldsymbol{\ell_q}) \geq \mathbf{1}, \tag{263}$$

$$0 \leq \mathbf{1}^\top \cdot \boldsymbol{w_q}(\tilde{\boldsymbol{T}}_{\boldsymbol{q}}, \boldsymbol{\ell_q}) \leq \mathbf{1}^\top \cdot \boldsymbol{w_q}(\mathbf{1}, \boldsymbol{\ell_q}) := w_q. \tag{264}$$

In other words, simply because this occupation's labor market exists we know the talent employed is sufficient to output all its responsability spectra.

Furthermore, as rational employers will not overhire (ERA), for this would reduce their profit, we also know not a single position in the labor market violates the Proportional Employment Condition (see MOOL).

Otherwise, employers would lay off excess workers to downscale the workforce from a suboptimal $w_q > 0$ to some $w_q^* \leq w_q$, again, to save resources. Thus, the current workforce, necessarily, has to be of the optimal

$$\sum_{v=1}^{p} w_q^v = w_q = w_q^* \tag{265}$$

size and respect the PEC,

$$\tilde{w_q^v} = \Omega_q^v \in [0, 1], \tag{266}$$

$$\sum_{v=1}^{p} \Omega_q^v := 1 \tag{267}$$

at every level. Hence, we must have precisely $w_q^v = w_q \times \Omega_q^v \geq 1 \ \forall \ v \in \{1, \ldots, p\}, p \in \{1, \ldots, w_q\}$ employees in each position.

In addition, we have ruled out infinite labor stratification (see MSA), and demonstrated any efficiently stratified labor market is characterized by the very same responsability spectra, with w_q unique positions, and unitary employment (OSL, ESL). So, the labor market cannot be more than maximally stratified in accordance with Definition 5.

At last, from all valid production strategies we have considered, maximum labor stratification is that which has the lowest barrier of entry, minimizing productivity requirements (MPL).

Therefore, if a labor market has any employees at all, the available talent in it has to be, at least, sufficient for maximally stratified production:

$$\tilde{T}_q^v \geq \mathrm{TA}^{-1}\left(\frac{v}{w_q} + \mathrm{TA}(0)\right) \ \forall \ v \in \{1, \ldots, w_q\}. \tag{268}$$

$\square$

Proof 2. Another proof for this lemma is to start with the minimum productivity condition above,

$$\boldsymbol{w_q}(\tilde{\boldsymbol{T}}_{\boldsymbol{q}}, \boldsymbol{\ell_q}) \geq \mathbf{1} \iff \sum_{r=1}^{w_q^v} [\tilde{T}_q^r \geq \ell_v^q] \geq 1 \ \forall \ v \in \{1, \ldots, p_q\}, \tag{269}$$

and write, for every production strategy, what it implies.

With independent production (IP),

$$\boldsymbol{w_q}(\tilde{\boldsymbol{T}}_{\boldsymbol{q}}, \boldsymbol{\ell_q}) \geq \mathbf{1} \iff \sum_{v=1}^{w_q} [\tilde{T}_q^v = 1] \geq 1 \tag{270}$$

$$\therefore w_q > 0 \iff \mho_q^{\mathrm{IP}} \in [1, w_q]. \tag{271}$$

So, if occupation q's labor market exists at all (i.e. $w_q > 0$), it must have at least one perfectly qualified employee, yielding at least one productive unit.

Now, binary labor stratification (BS) demands a perfectly qualified "senior" and a sufficiently qualified "junior" for a minimum aggregate output $\mho_q^{\mathrm{BS}} > 1$, greater than that of independent production:

$$\boldsymbol{w_q}(\tilde{\boldsymbol{T}}_{\boldsymbol{q}}, \boldsymbol{\ell_q}) \geq \mathbf{1} \iff \sum_{r=1}^{w_q^{\mathrm{Jr}}} [\tilde{T}_q^r \geq \ell_{\mathrm{Jr}}^q] \geq 1 \wedge \sum_{r=1}^{w_q^{\mathrm{Sr}}} [\tilde{T}_q^r = 1] \geq 1 \tag{272}$$

$$\therefore w_q > 0 \iff \mho_q^{\mathrm{BS}} \in \left[\min\left(\frac{1}{\int_0^{\ell_{\mathrm{Jr}}^q} \mathrm{ta}(l)dl}, \frac{1}{\int_{\ell_{\mathrm{Jr}}^q}^1 \mathrm{ta}(l)dl} \right), w_q \right]. \tag{273}$$

And, again, the pattern repeats for all production strategies, up to maximum labor stratification, where

$$\boldsymbol{w_q}(\tilde{\boldsymbol{T}}_{\boldsymbol{q}}, \boldsymbol{\ell_q}) \geq \mathbf{1} \iff \sum_{r=1}^{w_q^1} [\tilde{T}_q^r \geq \ell_1^q] \geq 1 \wedge \cdots \wedge \sum_{r=1}^{w_q^{w_q}} [\tilde{T}_q^r = 1] \geq 1. \tag{274}$$

But

$$\because p_q = w_q \iff \boldsymbol{w_q} = \mathbf{1} \wedge \boldsymbol{\ell_q} = \boldsymbol{\lambda_q} \tag{275}$$

$$\therefore \sum_{r=1}^{1} [\tilde{T}_q^r \geq \lambda_1^q] = 1 \wedge \cdots \wedge \sum_{r=1}^{1} [\tilde{T}_q^r = 1] = 1 \tag{276}$$

$$\iff \tilde{T}_q^v \geq \lambda_v^q \ \forall \ v \in \{1, \ldots, w_q\}, \tag{277}$$

which means all employees in a maximally stratified labor market necessarily meet the minimum productivity requirements for their responsabilities (i.e. the available talent has to be at least sufficient for maximally stratified production).

Therefore,

$$w_q > 0 \iff \mho_q^{\mathrm{MS}} \in \left[\min\left(\frac{1}{\int_0^{\lambda_1^q} \mathrm{ta}(l)dl}, \ldots, \frac{1}{\int_{\lambda_{w_q-1}^q}^1 \mathrm{ta}(l)dl} \right), w_q \right] \tag{278}$$

$$\iff \mho_q^{\mathrm{MS}} \in \left[\left(\int_{\lambda_{v-1}^q}^{\lambda_v^q} \mathrm{ta}(l)dl \right)^{-1}, w_q \right] \tag{279}$$

$$\iff \mho_q^{\mathrm{MS}} \in [w_q, w_q] \tag{280}$$

$$\iff \mho_q^{\mathrm{MS}} = w_q, \tag{281}$$

which is also the maximum operational output of any production strategy (MOOL). Hence, productivity requirements are minimized, while aggregate output is maximized. Or, put another way, in this arrangement the minimum guarantees the maximum. □

From the above, it follows logically that maximum-monotonic labor stratification is the optimal production strategy and, so, holds in the labor market.

Lemma 11 (Maximum Labor Stratification Lemma, MLSL)**.** The Maximum Labor Stratification Lemma (MLSL) states that a perfectly rational employer (ERA), which expects there could be skill differences in the workforce (PDA), and can split operational output without gain or loss to production (WOCA), will, therefore, strategically stratify their job offers monotonically, and even maximally, so that, if indeed there happens to be skill differences in the labor market, they can, then, allocate less competent workers to easier roles, and avoid wasting talent, thus "saving their best" for the most demanding tasks.

Proof. We have demonstrated any productive arrangement can only yield, at most, w_q units of an occupation q's operation, and this if the talent employed is sufficiently qualified (MOOL).

We have also demonstrated that, simply because a labor market exists at all, its workers' productivity has to be, at least, sufficient for maximally stratified production (PSL) when infinite stratification is ruled out (ISL, MSA).

Furthermore, Definition 3 implies there is no upside to employing underqualified workers, as

$$\tilde{T}_q^k < \tilde{T}_q^v \implies \left[\tilde{T}_q^k \geq \tilde{T}_q^v\right] \mho_q^v = 0, \tag{282}$$

that is, if an employee cannot fully output a responsability spectrum, their contribution to production is void, limiting aggregate operational output. This means choosing any production strategy other than maximum-monotonic labor stratification is a risk with no upside, when workers may have varying productivity (PDA). For, as we have shown, this arrangement by itself guarantees minimum productivity requirements (PSL) and maximum operational output (MOOL). So, it would be irrational of employers to organize production in another manner. Or, more succinctly, we can write for all $v \in \{1, \ldots, w_q\}$:

$$\mathbb{E}[\mho_q^{\mathrm{IP}} \mid \mathbb{E}[\tilde{T}_q^v] \in [0,1]] \tag{283}$$

$$\leq \mathbb{E}[\mho_q^{\mathrm{BS}} \mid \mathbb{E}[\tilde{T}_q^v] \in [0,1]] \tag{284}$$

$$\leq \mathbb{E}[\mho_q^{\mathrm{3S}} \mid \mathbb{E}[\tilde{T}_q^v] \in [0,1]] \tag{285}$$

$$\vdots \tag{286}$$

$$\leq \mathbb{E}[\mho_q^{\mathrm{MS}} \mid \mathbb{E}[\tilde{T}_q^v] \in [0,1]] \tag{287}$$

$$= \mathbb{E}[\mho_q^{\mathrm{IP}} \mid \tilde{T}_q^v = 1] = w_q, \tag{288}$$

where each of the terms above represents the expected value of aggregate operational output in production strategies other than infinite stratification, given the workforce's expected productivity.

In other words, splitting responsabilities in accordance with competence is always as productive as the maximum operational output (viz. that which is obtained when employing perfectly qualified workers independently), provided employees are sufficiently qualified for their responsabilities. But, again, this is, by definition, guaranteed by employers' rationality, as well as the simple fact the economy is already producing its current operational output (see PSL).

Therefore, employing potentially underqualified workers to output the entire responsability spectrum $l \in [0, 1]$ independently can only be as productive as the labor stratification strategy, but never more than it. Indepedent production, then, is a suboptimal strategy when employers expect there to be skill differences in the workforce. And the same logic also applies to less than maximally-stratified arrangements.

Thus, maximum labor stratification follows as an insurance policy against worker's potential underqualification: for if talent is lacking in the labor market, there is nothing to gain by employing individuals which are not sufficiently qualified for a difficult job, whereas if talent is abundant, there is nothing to lose when employing overqualified individuals to a job below their skill level.

Hence, given the same w_q workforce, operational output in a maximally stratified labor market is always greater or equal to the output of any other economic configuration. It is, therefore, always optimal to monotonically and maximally stratify responsabilities across w_q unique positions, each focused on increasingly demanding tasks. □

With this, we have shown maximally stratified production is the only efficient arrangement that holds in reality. So, we can, finally, derive a general employability coefficient by estimating employability in such markets.

General Employability Theorem (GET)**.** Because maximum labor stratification is the safest and most efficient production strategy, rational employers will always choose to implement it. Therefore, an individual's employability in a maximally stratified economy,

$$\tilde{W}_k = \left(\frac{1}{w}\right) \sum_{q=1}^{n} \left[h_q^k \geq \frac{1}{2}\right] \sum_{v=1}^{w_q} \left[\tilde{T}_q^k \geq \mathrm{TA}^{-1}\left(\frac{v}{w_q} + \mathrm{TA}(0)\right)\right] \tag{289}$$

is their actual employability in reality.

Proof. We have just demonstrated (MLSL) that maximum labor stratification is the only optimal productive arrangement and, given our assumptions, attains in reality. So, one's employability in this market is their actual employability.

Moreover, in accordance with Definition 5, employability is

$$\tilde{W}_k := \sum_{q=1}^{n} \tilde{W}_q^k := \sum_{q=1}^{n} \left[h_q^k \geq \frac{1}{2}\right] \sum_{v=1}^{p} \left[\tilde{T}_q^k \geq \tilde{T}_q^v\right] \tilde{w}_q^v, \tag{290}$$

which in a maximally stratified market becomes (see OSL, ESL)

$$\tilde{W}_k = \sum_{q=1}^{n} \left[h_q^k \geq \frac{1}{2}\right] \sum_{v=1}^{w_q} \left[\tilde{T}_q^k \geq \ell_q^v\right] \left(\frac{1}{w_q}\right) \tag{291}$$

$$= \left(\frac{1}{w}\right) \sum_{q=1}^{n} \left[h_q^k \geq \frac{1}{2}\right] \sum_{v=1}^{w_q} \left[\tilde{T}_q^k \geq \mathrm{TA}^{-1}\left(\frac{v}{w_q} + \mathrm{TA}(0)\right)\right]. \tag{292}$$

□

4.4. Corollaries

The General Employability Theorem has a few interesting collolaries, the first, and simplest, of which is a more compact, and intuitive, version of it.

Corollary 1 (Simplified Employability Corollary, SEC)**.** We want to show that, as with the BET and ISL, so too in a maximally and monotonically stratified labor market, employability is the percentage of an operation's total time duration one is capable of producing. Or, mathematically,

$$\tilde{W}_q^k = \int_0^{\tilde{T}_q^k} \mathrm{ta}(l)dl := \Omega_q^k \in [0,1] \;\forall\; k, q \in \{1, \ldots, n\}. \tag{293}$$

Proof. To prove this result, let us, then, first consider what would be the employability of person k if they had exactly the minimum required productivity for every job subtype. So, for instance, when $v = 1$,

$$\tilde{T}_q^k := \ell_1 = \mathrm{TA}^{-1}\left(\frac{1}{w_q} + \mathrm{TA}(0)\right) \implies \tilde{W}_q^k = \frac{1}{w_q}, \tag{294}$$

as a productivity coefficient of $\tilde{T}_q^k = \ell_1$ is just enough to be hireable on the easiest job in occupation q's labor market, but not on the second, much less on the remaining, more difficult, positions.

Likewise, for other values of v, we have

$$\tilde{T}_q^k := \ell_2 = \mathrm{TA}^{-1}\left(\frac{2}{w_q} + \mathrm{TA}(0)\right) \implies \tilde{W}_q^k = \frac{2}{w_q}, \tag{295}$$

$$\tilde{T}_q^k := \ell_3 = \mathrm{TA}^{-1}\left(\frac{3}{w_q} + \mathrm{TA}(0)\right) \implies \tilde{W}_q^k = \frac{3}{w_q}, \tag{296}$$

$$\vdots \tag{297}$$

$$\tilde{T}_q^k := \ell_v = \mathrm{TA}^{-1}\left(\frac{v}{w_q} + \mathrm{TA}(0)\right) \implies \tilde{W}_q^k = \frac{v}{w_q}, \tag{298}$$

so that we may derive the following pattern for any $v \in \{1, \ldots, w_q\}$:

$$\tilde{T}_q^k = \mathrm{TA}^{-1}\left(\tilde{W}_q^k + \mathrm{TA}(0)\right) \tag{299}$$

$$\therefore \mathrm{TA}(\tilde{T}_q^k) = \mathrm{TA}\left(\mathrm{TA}^{-1}\left(\tilde{W}_q^k + \mathrm{TA}(0)\right)\right) \tag{300}$$

$$\therefore \mathrm{TA}(\tilde{T}_q^k) = \tilde{W}_q^k + \mathrm{TA}(0) \tag{301}$$

$$\therefore \tilde{W}_q^k = \mathrm{TA}(\tilde{T}_q^k) - \mathrm{TA}(0) = \int_0^{\tilde{T}_q^k} \mathrm{ta}(l)dl := \Omega_q^k \in [0,1], \tag{302}$$

as we wanted to show.

However, because $\tilde{T}_q^k \in [0,1]$ is not as discretized as responsability ranges $l \in [\ell_{v-1}, \ell_v], v \in \{1, \ldots, w_q\}$ are, and because rational employers do not hire insufficiently qualified employees, we must approximate $\tilde{T}_q^k$ with the closest

$$\tilde{T}_q^\kappa := \left(\frac{1}{w_q}\right) \sum_{v=1}^{w_q} \left[\tilde{T}_q^k \geq \ell_v\right] \tag{303}$$

productivity estimate, such that $\tilde{T}_q^k \geq \tilde{T}_q^\kappa$, but $\tilde{T}_q^k \approx \tilde{T}_q^\kappa$, where $\tilde{T}_q^\kappa = \ell_\kappa \in \{\ell_0, \ldots, \ell_{w_q}\}$ determines the most demanding task for which k is still productive. Therefore, the adjusted coefficient is:

$$\tilde{W}_q^k = \int_0^{\tilde{T}_q^\kappa} \mathrm{ta}(l)dl := \Omega_q^\kappa \approx \int_0^{\tilde{T}_q^k} \mathrm{ta}(l)dl \in [0,1], \tag{304}$$

when w_q is large enough.

Of course, this assumes candidate k is evaluated as “employable” in accordance with the hireability statistic

$$\left[h_q^k \geq \frac{1}{2}\right], \tag{305}$$

which accounts for selection criteria besides minimum required productivity. Hence, a more complete formulation would be:

$$\tilde{W}_q^k = \left[h_q^k \geq \frac{1}{2}\right] \int_0^{\tilde{T}_q^\kappa} \mathrm{ta}(l)dl; \tag{306}$$

or, in the aggregate form,

$$\tilde{W}_k = \left(\frac{1}{w}\right) \sum_{q=1}^{n} \left[h_q^k \geq \frac{1}{2}\right] \int_0^{\tilde{T}_q^\kappa} \mathrm{ta}(l)dl := \left(\frac{1}{w}\right) \sum_{q=1}^{n} \left[h_q^k \geq \frac{1}{2}\right] \Omega_q^\kappa \tag{307}$$

$$\forall\ \tilde{T}_q^\kappa \in \{\ell_0, \ldots, \ell_{w_q}\}; k, q \in \{1, \ldots, n\}. \tag{308}$$

□

In addition, the General Employability Theorem can be used to prove the General Competitiveness Corollary (GCC) with the following definition.

Definition 7 (Competitiveness)**.** Labor market competitiveness can be defined in a variety of ways. The typical is to think of competitiveness as a ratio of job seekers to the number of available positions. Thus, we say an occupation's labor market is "competitive" if there are too many incumbents per job post.

However, this definition has two main flaws: 1) it can be somewhat cumbersome, if not impossible, to gather all necessary data, for every labor market, under shifting conditions, to accurately assess competitiveness; 2) and even if such data are available and trustworthy, incumbents per job posts, in and of itself, is not that much of an interpretable, or at least complete, statistic.

So, we may propose an additional, alternative, definition of competitiveness as the percentage of the workforce which would be *willing* and *able* to compete for jobs in a particular labor market:

$$\tilde{\mathrm{vs}}_k := \sum_{q=1}^{n} \left[\ddot{u}_k^q \geq \ddot{u}_q^q\right] \tilde{W}_k^q \tag{309}$$

$$:= \sum_{q=1}^{n} \left[\ddot{u}_k^q \geq \ddot{u}_q^q\right] \left[h_k^q \geq \frac{1}{2}\right] \sum_{v=1}^{p} \left[\tilde{T}_k^q \geq \tilde{T}_k^v\right] \tilde{w}_k^v \in [0,1], \tag{310}$$

where

$$\left[\ddot{u}_k^q \geq \ddot{u}_q^q\right], \tag{311}$$

$$\ddot{u}_k^q := \mathrm{eq}(\hat{u}_k^q) := \mathrm{eq}\left(u(\boldsymbol{y}, \boldsymbol{v_q}, \mathbf{A}, \dots)\right) \in [0,1], \tag{312}$$

or equivalent, scaled, utility[2] indicates if the relative utility of person q, with preferences for attributes $\boldsymbol{v_q} \in [0,1]^m$, working at a job with a skill set $\boldsymbol{a_q} \in [0,1]^m$, and earning a wage of y_q, would find it sufficiently more rewarding to switch to an occupation k, requiring a skill set $\boldsymbol{a_k} \in [0,1]^m$, and paying y_k; while the other terms are defined as previously.

Thus, what the $\tilde{\mathrm{vs}}_k$ ("versus") competitiveness statistic tells us is that if workers of type q are willing to compete for an occupation k's job posts, they are evaluated in terms of their employability (cf. Definition 5) and, if found sufficiently qualified, counted as viable incumbents. Or, in other words, competitiveness is the employability of willing and able workers from other labor markets (i.e. it is a complement of employability).

And, with this, we may derive the General Competitiveness Corollary.

Corollary 2 (General Competitiveness Corollary, GCC)**.** The competitiveness of an occupation's labor market is the percentage of the aggregate workforce w that is willing and able to compete for its job posts.

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[2]The *umlauts* are not derivatives, but a shorthand notation for the equivalence operator defined in another paper.

Proof. By Definition 7 and the General Employability Theorem,

$$\tilde{\mathrm{v}}\mathrm{s}_k := \sum_{q=1}^{n} \left[\ddot{u}_k^q \geq \ddot{u}_q^q\right] \left[h_k^q \geq \frac{1}{2}\right] \sum_{v=1}^{p} \left[\tilde{T}_k^q \geq \tilde{T}_k^v\right] \tilde{w}_k^v \tag{313}$$

$$= \left(\frac{1}{w}\right) \sum_{q=1}^{n} \left[\ddot{u}_k^q \geq \ddot{u}_q^q\right] \left[h_k^q \geq \frac{1}{2}\right] \sum_{v=1}^{w_k} \left[\tilde{T}_k^q \geq \ell_v\right] \tag{314}$$

$$= \left(\frac{1}{w}\right) \sum_{q=1}^{n} \left[\ddot{u}_k^q \geq \ddot{u}_q^q\right] \left[h_k^q \geq \frac{1}{2}\right] \sum_{v=1}^{w_k} \left[\tilde{T}_k^q \geq \mathrm{TA}^{-1}\left(\frac{v}{w_k} + \mathrm{TA}(0)\right)\right]. \tag{315}$$

□

5. Example Implementation

5.1. Functional Specifications

5.2. Occupational Information Network Data

5.3. Results

6. Discussion

7. Conclusion

Appendix A – Basic Definitions

Appendix B – Employability and Competitiveness Statistics

Appendix C – Proof Layout