The Employability Theorem

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Abstract

In this document, the Employability Theorem is demonstrated from a set of fairly tautological axioms, which are presupposed in quantitative career choice and career development methods.

Keywords: Employability theorem; Career choice; Career development; Vocational choice; Occupational Information Network; O*NET.

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2. The Issue of Occupational Complexity

Now, because this series of articles has been of a more applied nature, so far, we did not have a need to expound upon our most basic assumptions. This is not a problem, for these are all quite reasonable, even tautological. However, to demonstrate the Employability Theorem, it is useful to make at least one these assumptions more explicit, as a valuable intuition towards the Theorem follows directly from a simple definition. Of course, we do not need to define things this way, as this first "stepping stone" is not actually used in the demonstration itself. [But, being a reasonable and insightful assumption, which can help us understand the issue of occupational complexity and, therefore, of employability,]. To minimize digressions, the fundamental axioms behind this definition were moved to the Appendix (for these, again, are practically tautological). With this said, we proceed to our definition.

Definition 1 (Skill)**.** A professional attribute, competency, or skill, of a person k can be conceptualized as a cummulative sum of successes on binary outcome variables representing tasks of progressive difficulty which require only that skill:

$$a_i^k = \sum_{l=0}^{l_i} T_{i_l}^k, \tag{1}$$

where

$$T_{i_l}^k = \begin{cases} 1, & \text{if } k \text{ succedes in a task } T_i^l \text{ of difficulty level } l; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Or, more rigorously,

$$a_i^k = \sum_{l=0}^{l_i} T(l, l_i^k), \tag{3}$$

where

$$T(l, l_i^k) = T_{i_l}^k = [l \leq l_i^k] = \begin{cases} 1, & l \leq l_i^k; \\ 0, & l > l_i^k. \end{cases} \tag{4}$$

and $l_i^k \in [0, l_i]$ is the maximum difficulty level on which k still succeedes. Thus, we can define a person k's skill level in an attribute i as the sum of their successful trials on a $T_i = \{T_i^0, \ldots, T_i^{l_i}\}$ set of tasks of increasing difficulty.

Furthermore, as we assume scales are truncated (i.e. there is a maximum difficulty level l_i, and a trivial difficulty level, which has to be zero), we can also interpret a_i^k as the *portion* of tasks one is able to accomplish out of all difficulty levels for that skill. By normalizing l_i to 100, for example, we have:

1. $a_i^k = 0 \iff k$ cannot perform even the most basic of attribute i's tasks;

2. $a_i^k = 10 \iff k$ can perform only the bottom 10% of attribute i's tasks, but nothing more;

3. $a_i^k = 50 \iff k$ can perform the easiest half of attribute i's tasks, but not the most difficult half;

4. $a_i^k = 100 \iff k$ can perform all of attribute i's tasks.

Finally, we can define a_i^k for a continuum of task difficulty $l \in [0, 1]$:

$$a_i^k = \int_0^1 T(l, l_i^k) dl, \tag{5}$$

where $T(l, l_i^k)$ is defined as before.

Definition 2 (Skill)**.** Alternatively, we can think of a person k's professional attribute, competency, or skill, as the difficulty of the most difficult task they can accomplish, normalized by the difficulty of the most objectively difficult task of that particular skill:

$$a_i^k = \frac{l_i^k}{l_i}, \tag{6}$$

which we normalize by setting $l_i = 1$, so that

$$a_i^k = \frac{l_i^k}{1} = l_i^k, \tag{7}$$

and $l_i^k \in [0, 1]$. With this normalization, example interpretations of a_i^k are:

1. $a_i^k = 0 \iff k$ cannot perform even the most basic of attribute i's tasks;

2. $a_i^k = 0.10 \iff k$ can only perform tasks of up to 10% the difficulty of attribute i's most difficult task, but nothing more;

3. $a_i^k = 0.50 \iff k$ can perform tasks of up to half the difficulty of attribute i's most difficult task, but nothing more;

4. $a_i^k = 1 \iff k$ can perform all of attribute i's tasks.

This is, perhaps, the most natural conceptual model for understanding competencies, as, generally, it is more intuitive to think of skill as the maximum of one's capacity, rather than the portion of tasks one could ly accomplish.

But, again, [because we assume scales to be truncated], this latter interpretation actually implies and is implied by the former. For if a task is of the same difficulty as another, then they are just as difficult in relation to that skill's most difficult task (i.e. they require the same percentage of the scale's upper limit to be performed), and, likewise, are also included in the same difficulty "bracket" (i.e. they are equivalent to the same skill test in the aggregate binary outcome interpretation), and, therefore, pressupose the same a_i^k skill level.

Of course, this equivalence is quite trivial, given that

$$\int_0^1 T(l, l_i^k) dl = 1 \times \int_0^{l_i^k} dl + 0 \times \int_{l_i^k}^1 dl = l_i^k - 0 = \frac{l_i^k}{1} = a_i^k. \tag{8}$$

This means the percentage of a skill's tasks one can accomplish is also the difficulty of the most difficult task one can accomplish relative to that skill's most difficult task.

So, however one decides to interpret skill levels, the conclusion remains the same: to be skilled in an attribute is to be able to perform the activities associated with that attribute. Put simply, the capacity to act follows virtue, for virtue is, itself, the capacity to act.

Now, even though these results are basically tautological, they are still important to guide our intuition. In fact, our first insight towards the Employability Theorem, namely the Skill Sufficiency Lemma (SSL), follows directly from the definitions above.

Lemma 1 (Skill Sufficiency Lemma)**.** *According to the SSL, skills are necessary and sufficient to accomplish tasks. In particular, to have a skill level of $a_i^k \in [0, 1]$ in attribute i is a necessary and sufficient condition for one to be capable of accomplishing the easier a_i^k portion of that attribute's tasks.*

Proof. By definition,

$$T(l, l_i^k) = \begin{cases} 1, & l \leq l_i^k; \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

is a binary indicator of person k's ability to accomplish a task of difficulty $l \in [0, 1]$ which requires only attribute i.

With this,

$$\tilde{T}_i^k = \int_0^1 T(l, l_i^k) dl \tag{10}$$

is the percentage of tasks requiring only attribute i that k can accomplish.

But both equivalent definitions of k's skill level in attribute i, are

$$a_i^k = \int_0^1 T(l, l_i^k) dl = l_i^k, \tag{11}$$

which is precisely the $\tilde{T}_i^k$ aggregation of $T(l, l_i^k)$ in the $[0, 1]$ interval.

Therefore, having a skill level of a_i^k is a necessary and sufficient condition to be capable of accomplishing the easier a_i^k portion of attribute i's tasks:

$$a_i^k = \tilde{T}_i^k \iff \int_0^1 T(l, l_i^k) dl = \int_0^1 T(l, l_i^k) dl. \tag{12}$$

$\square$

Definition 3 (Complex Task)**.** A task is said to be complex if it relies on more than one attribute to be accomplished. More precisely, T_{ij}^l is a complex task of attributes i and j, if its binary outcome indicator is of the form

$$T(l, l_{ij}^k) = [l \leq l_{ij}^k], \tag{13}$$

where

$$l_{ij}^k = f(l_i^k, l_j^k) \tag{14}$$

is a strictly increasing aggregation function that returns the maximum difficulty level of the complex task T_{ij}^l a person k can accomplish based on each attribute T_{ij}^l requires. Or, generalizing for any complex task T_q^l of m attributes, requiring an entire skill set $\boldsymbol{a_q} = (a_1^q, \ldots, a_m^q)$ to be accomplished,

$$T(l, l_q^k) = [l \leq l_q^k], \tag{15}$$

where

$$l_q^k = f(\boldsymbol{l_q^k}) = f(l_1^k, \ldots, l_m^k) \tag{16}$$

and

$$\frac{\partial f(\boldsymbol{l_q^k})}{\partial l_i^k} > 0 \; \forall \; i \in \{1, \ldots, m\}. \tag{17}$$

This means none of the attributes required by the complex task are completely disposable (i.e. they are all helpful in some way). For instance, the task T_i^l, previously defined, with binary outcome $T(l, l_i^k)$ is not complex, because

$$\frac{\partial l_i^k}{\partial l_i^k} = 1, \tag{18}$$

but

$$\frac{\partial l_i^k}{\partial l_j^k} = 0, \tag{19}$$

where $i \neq j$ and $i, j \in \{1, \ldots, m\}$. Or, say the aggregation function is given by

$$f(l_i^k, l_j^k) = l_i^k - l_j^k, \tag{20}$$

so that attribute j actually hinders productivity:

$$\frac{\partial l_i^k}{\partial l_j^k} = -1. \tag{21}$$

None of these are complex tasks, for they do not coherently mobilize multiple attributes towards a unified goal.

Definition 3.1 (Weak Complexity)**.** Now, beyond these most basic rules, we can define stricter versions of "task complexity" with additional assumptions. The first version, of weak complexity, requires that

$$\frac{\partial^2 f(\boldsymbol{l_q^k})}{\partial l_i^k \partial l_j^k} > 0 \;\forall\; i, j \in \{1, \ldots, m\}, i \neq j, \tag{22}$$

meaning attributes are complementary.

Definition 3.2 (Moderate Complexity)**.** A task is of moderate complexity if its aggregation function also meets the following criteria:

$$\lim_{l_i^k \to 0} f(\boldsymbol{l_q^k}) = 0 \;\forall\; i \in \{1, \ldots, m\}, \tag{23}$$

so that a person k's capacity to perform the complex task is weakly increasing on their capacity to perform the simple tasks of its required attributes, and goes to zero when they are unskilled in at least one of these. Thus, a moderately complex task is not reducible to any proper subset of its attributes.

For instance, a task of the form

$$T(l, l_{ij}^k) = [l \leq (1 + l_i^k) \times (1 + l_j^k) - 1] \tag{24}$$

is not moderately complex, as person k does not need every attribute to accomplish the task. Indeed, if k has precisely zero capacity in either skill i or j, then T_{ij}^l collapses to unidimensional, or simple, tasks T_i^l when

$$\begin{aligned} T(l, l_{ij}^k) &= [l \leq (1 + l_i^k) \times (1 + 0) - 1] &(25)\\ &= [l \leq l_i^k] &(26)\\ &= T(l, l_i^k), &(27) \end{aligned}$$

or T_j^l when

$$\begin{aligned} T(l, l_{ij}^k) &= [l \leq (1 + 0) \times (1 + l_j^k) - 1] &(28)\\ &= [l \leq l_j^k] &(29)\\ &= T(l, l_j^k), &(30) \end{aligned}$$

in which case T_{ij}^l is not *really* (moderately) complex, but rather a convolution of simple tasks. Notice, however, this does not imply there cannot be a degree of substitution between attributes. That is, moderate task complexity only means a task must require all of its attributes in *some* level, even if its functional form allows for substitution.

Definition 3.3 (Strong Complexity)**.** The strictest definition of task complexity adds the constraint that skills are aggregated by the Leontief function:

$$f(\boldsymbol{l_q^k}) = \min(\boldsymbol{l_q^k}). \tag{31}$$

Here, attributes are assumed to be perfect complements, which need to be combined in exactly the same quantities for maximum efficacy. In other words, having additional skills does not help to accomplish the task, but being unskilled in even a single attribute can undermine the whole effort. Hence, productivity is limited by the lowest competency.

Lemma 2 (Skill Composition Lemma)**.** *The Skill Composition Lemma (SCL) is a generalization of the SSL and states that skills are composable to accomplish complex tasks. More precisely, let T_q^l be an activity of difficulty level l that requires the $\boldsymbol{a_q} = (a_1^q, \ldots, a_m^q)$ skill set (i.e. T_q^l is a complex task). With this, we demonstrate that any rational and sufficiently qualified economic agent can naturally "piece together", that is compose, attributes $\{1, \ldots, m\}$ to accomplish the T_q^l complex task.*

Proof. Given

$$\tilde{T}_q^k = \int_0^1 T(l, l_q^k) dl \tag{32}$$

$\square$

Axiom 1 (Occupational Reducibility Axiom)**.** Occupations can be reduced to their tasks.

Axiom 2 (Occupational Complexity Axiom)**.** All of an occupation's tasks can be thought of as one indivisible task, which mobilizes their entire skill set. We call this "holistic task" an occupation's *operation.*

Mathematically, an occupational operation, is just a complex task that has a continuum of difficulty levels in the unit interval, all of which are indispensable for the task to be accomplished. We denote "operational output" (OO) with the standard IPA symbol for the near-close near-back rounded vowel (i.e. the "double o" sound in words such as "boot"):

$$\mho_q^k = \left\lfloor \int_0^1 T(l, l_q^k) dl \right\rfloor . \tag{33}$$

By this formulation, the amount of an occupations's operation a person k can output is the floor of what they accomplish of the operation's complex tasks. Of course, this value is zero if they are not perfectly qualified, which at first can seem too strong of a pressupposition, but is, in fact, an effective strategy to "side-step" the issue of occupational complexity and is, also, quite reasonable, as we shall demonstrate below.

[intuition for operational output formulation]

[clarify the indivisibility of the "holistic task" does not imply tasks cannot be outsourced, but that the whole operation has to be accomplished in its entirety]

[examples in defense of operational output formulation]

[how this operational output helps to side-step the issue of occupational complexity]

So, in a way, a "holistic task", or operation, is a "doubly complex" task, as, in addition to being complex as already defined, it is also strictly complex in its difficulty levels, for failing to accomplish even the most basic of difficulty levels nullifies the entire operation. It is, in other words, a strongly complex task itself made up of various complex tasks that perfectly complement one another.

Aggregate occupational operation output:

$$\mho_q = \left\lfloor \sum_{k=1}^{n} [k \in \Lambda_q]? \right\rfloor , \tag{34}$$

where Λ_q is the set of people working in occupation q.

Labor market taxa (Λ):

$$\Lambda_1^1 = \Lambda(1,1) = \{1, \ldots, n\} \iff k, q \in \Lambda_1^1 \tag{35}$$

$$\Lambda = \{\Lambda_1^1, \ldots, \Lambda_n^{\bar{L}}\} \tag{36}$$

$$\Lambda^{-1}(k) = \Lambda_k^{\bar{L}} \tag{37}$$

"[...] the whole is something besides the parts" Aristotle 980a Metaphysics, Translated by W. D. Ross

"[...] the whole is not the same as the sum of its parts" Aristotle 100a, Topics, Translated by W. D. Ross

Lemma 3 (Occupational Composition Lemma)**.** *Skill sets are composable to accomplish occupations' operations.*

Proof. □

3. Labor Market Conditions and Employer Behavior

Axiom 3 (Rationality Axiom)**.** Employers are rational and will only pay for employees to work on tasks they can accomplish. Additionaly, employers will outsource parts of an occupation's operation if their employees cannot accomplish the entire operation.

Axiom 4 (Hireability Axiom)**.** Any rational employer hires employees by evaluating a hireability statistic, which quantifies potential employees' expected productivity, their educational attainment, and years of experience.

Axiom 4.1 (Weak Hireability Axiom)**.**

$$\mathbb{E}\left|h_q^k - \mathbb{E}(h_q^k)\right| \in [0,1] \tag{38}$$

Axiom 4.2 (Moderate Hireability Axiom)**.**

$$\mathbb{E}\left|h_q^k - \mathbb{E}(h_q^k)\right| = 0 \tag{39}$$

Axiom 4.3 (Strong Hireability Axiom)**.**

$$\mathbb{E}(h_q^k) = h_q^k \tag{40}$$

4. Task Difficulty and Time Allocation

Axiom 5 (Task Duration Axiom)**.** More difficult tasks and operations require more time to complete than easier tasks and operations.

Time allocation by difficulty level:

$$\mathrm{ta}_q(\bar{l}) = \frac{\mathrm{ttc}(\bar{l})}{\int_0^1 \mathrm{ttc}(l)dl} \tag{41}$$

5. The Employability Theorem

5.1. Demonstration

Employability Theorem. *The employability of a person in a particular occupation is the percentage of that occupation's operation total time duration that their skill set allows them to accomplish.*

Proof. Employment by difficulty level:

$$w_q(\bar{l}) = w_q \times \mathrm{ta}_q(\bar{l}) \tag{42}$$

Employability per occupation:

$$\begin{aligned}
w_q^k &= [\tilde{Y}_q^k \ddot{\tau}_q^k s_{kq}^\theta \geq 0.5] \times \int_0^1 T(l, l_q^k) w_q(l) dl && (43)\\
&= [\tilde{Y}_q^k \ddot{\tau}_q^k s_{kq}^\theta \geq 0.5] \times \int_0^1 [l_q^k \geq l] w_q \mathrm{ta}_q(l) dl && (44)\\
&= [\tilde{Y}_q^k \ddot{\tau}_q^k s_{kq}^\theta \geq 0.5] \times w_q \left(\int_0^{l_q^k} 1 \times \mathrm{ta}_q(l) dl + \int_{l_q^k}^1 0 \times \mathrm{ta}_q(l) dl \right) && (45)\\
&= [\tilde{Y}_q^k \ddot{\tau}_q^k s_{kq}^\theta \geq 0.5] \times w_q \int_0^{l_q^k} \mathrm{ta}_q(l) dl && (46)
\end{aligned}$$

And with $l_q^k = \tilde{Y}_q^k = \tilde{Y}(\boldsymbol{a_k}, \boldsymbol{a_q}) = Y(\boldsymbol{a_k}, \boldsymbol{a_q}) / Y(\boldsymbol{a_q}, \boldsymbol{a_q})$,

$$w_q^k = \left\lfloor [\tilde{Y}_q^k \ddot{\tau}_q^k s_{kq}^\theta \geq 0.5] \times w_q \int_0^{\tilde{Y}_q^k} \mathrm{ta}_q(l) dl \right\rfloor \tag{47}$$

□

5.2. Corollaries

Corollary 1 (Proportionate Compensation Corollary)**.** *Because employers are rational, employees will be payed the exact portion of the market value they generate with their work.*

Proof. □

Corollary 2 (Aggregate Employability Corollary)**.** *A person's employability in a certain subset of the labor market is calculated by their average employability on that subset of the labor market, weighted by each occupation's employment levels.*

Proof. Aggregate employability (entire economy):

$$\tilde{w}_k = \frac{1}{W}\sum_{q=1}^{n} w_q^k \tag{48}$$

$$= \frac{1}{W}\sum_{q=1}^{n}\left\lfloor [\tilde{Y}_q^k \ddot{\tau}_q^k s_{kq}^{\theta} \geq 0.5] \int_0^{\tilde{Y}_q^k} w_q \mathrm{ta}_q(l)dl \right\rfloor, \tag{49}$$

$$W = \sum_{k=1}^{n} w_k, \tag{50}$$

$$\tilde{Y}_q^k = \tilde{Y}(\boldsymbol{a_k}, \boldsymbol{a_q}) = \frac{\prod_{i=1}^{m} \max(1+a_i^k, 1+a_i^q)^{\sigma_i^q}}{\prod_{i=1}^{m} (1+a_i^q)^{\sigma_i^q}}, \tag{51}$$

$$\sigma_i^q = \frac{a_i^q}{\sum_{i=1}^{m} a_i^q} \tag{52}$$

P.S.: think of a notation for economic taxa / aggregation levels.
Aggregate employability (particular subset of the economy):

$$\tilde{w}_{?!}^k = \tilde{w}_k(?,!) = \sum_{q=1}^{n} [q \in ?!][\tilde{Y}_q^k \ddot{\tau}_{kq}^{\theta} \geq 0.5] w_q^k \tag{53}$$

$$= \sum_{q=1}^{n}\left\lfloor [q \in ?!][\tilde{Y}_q^k \ddot{\tau}_{kq}^{\theta} \geq 0.5] \int_0^{\tilde{Y}_q^k} w_q \mathrm{ta}_q(l)dl \right\rfloor, \tag{54}$$

$$\tilde{Y}_q^k = \tilde{Y}(\boldsymbol{a_k}, \boldsymbol{a_q}) = \frac{\prod_{i=1}^{m} \max(1+a_i^k, 1+a_i^q)^{\sigma_i^q}}{\prod_{i=1}^{m} (1+a_i^q)^{\sigma_i^q}}, \tag{55}$$

$$\sigma_i^q = \frac{a_i^q}{\sum_{i=1}^{m} a_i^q} \tag{56}$$

□

Corollary 3 (Occupational Competitiveness Corollary)**.**

Proof. □

Corollary 4 (Aggregate Competitiveness Corollary)**.**

Proof. □

6. Example Implementation

6.1. Functional Specifications

6.2. Occupational Information Network Data

6.3. Results

7. Discussion

8. Conclusion

[In this paper, we demonstrated the Employability Theorem, and its corollaries. We also proposed a few functional specifications and analyzed the employability and competitiveness planes which they determine. In the next article, we shall implement these methods with real data from the Occupational Information Network.]

Appendix A – Basic Definitions

Appendix B – Employability and Competitiveness Statistics