實驗設計 HW4

統碩一 106354012 曹立諭

2018/04/10

Exercise

Write down the pdf form of a multivariate normaldistribution.

The pdf :

$$f(\mathbb{X}) = \frac{1}{(2\pi)^{\frac{k}{2}}\ |\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\mathbb{X}-\mu)'\Sigma^{-1}(\mathbb{X}-\mu)\right\}, \text{where} \quad \mathbb{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}, \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{bmatrix}$$

實驗設計 HW4
統碩一 106354012 曹立諭

2018/04/10

Question A.1

Let y be an N by 1 random vector with $E(y)=X\beta$, and $Var(y)=\sigma^2 I_N$, where X is N by p and β is p by 1. Let $Y=Py$, where P is a projection (not necessarily orthogonal) onto the range of X.

(a) Find the mean and (co)variance of Y and $y-Y$.

We know $y\in\mathbb{R}^{N\times 1}$ $,E(y)=X\beta$ $,Var(y)=\sigma^2 I$

Let $Y=Py$,

$$E(Y)=E(Py)=E\big(P(X\beta+\epsilon)\big)=E(PX\beta+P\epsilon)=PX\beta=X(X'X)^{-1}X'X\beta=X\beta$$

$$\begin{aligned}Var(Y)&=Var(Py)=Var\big(P(X\beta+\epsilon)\big)=Var(PX\beta+P\epsilon)=Var(P\epsilon)=PVar(\epsilon)P'\\&=P*\sigma^2 I*P'=\sigma^2 PP'=\sigma^2 I\end{aligned}$$

$$E(y-Y)=E(y-Py)=E\big((I-P)y\big)=(1-P)X\beta=X\beta-PX\beta=X\beta-X\beta=0$$

$$\begin{aligned}Var(y-Y)&=Var(y-Py)=Var\big((I-P)y\big)=(I-P)Var(y)(I-P)'\\&=(I-P)(I-P')\sigma^2=(I-P'-P+PP')\sigma^2\end{aligned}$$

$$Cov(Y,y-Y)=Cov(Py,(I-P)y)=P(I-P)'Var(y)=P(I-P')\sigma^2$$

(b) Prove that $Cov(Y\ ,y-Y\ )$ is 0 if and only if P is an orthogonal projection.

$\Rightarrow$

$$Cov(Y,y-Y)=Cov(Py,(I-P)y)=P(I-P)'Var(y)=P(I-P')\sigma^2=0$$

$\because$ $\sigma^2\neq 0$, $P(I-P')=P-PP'=0$, $P=PP'$

$P'=(PP')'=PP'=P$

$\therefore$ $we\ get\ P=P'\ ,P\ is\ an\ orthogonal\ projection$

$\Leftarrow$

P is an orthogonal projection ,then we know $P^2=P=P'$

$$\begin{aligned}Cov(Y,y-Y)&=Cov(Py,(I-P)y)=P(I-P)'Var(y)=P(I'-P')\sigma^2\\&=P(I-P)\sigma^2=(P-PP)\sigma^2=0\end{aligned}$$

By $\Rightarrow$ & $\Leftarrow$, we proof $Cov(Y,y-Y)$ is 0 iff P is an orthogonal projection.

實驗設計 HW4

統碩一 106354012 曹立諭

2018/04/10

Question A.2

Let $y = X\beta + \epsilon$, where ϵ is iid $N(0, \sigma^2)$; y is N by 1, X is N by p, and β is p by 1. Let g be any N by 1 vector. What is the distribution of $(g'y)^2$? What, if anything, changes when $g'X$ is zero?

$y = X\beta + \epsilon$, $\epsilon \sim N(o, \sigma^2 I)$

$$E(y) = E(X\beta + \epsilon) = X\beta \ , Var(y) = Var(X\beta + \epsilon) = \sigma^2 I$$

so $y \sim N(X\beta, \sigma^2 I)$

$$E(g'y) = E\big(g'(X\beta + \epsilon)\big) = g'X\beta$$

$$Var(g'y) = Var\big(g'(X\beta + \epsilon)\big) = Var(g'\epsilon) = g'g * \sigma^2$$

Then we get $g'y \sim N(g'X\beta, g'g * \sigma^2)$

$\left(\frac{g'y}{g'g*\sigma^2}\right)^2$ has a (non-central) chi-square distribution with v=1 degree of freedom and its non-centrality parameter is $\Omega = \frac{(g'X\beta)^2}{2g'g*\sigma^2}$

So $(g'y)^2$ has a (non-central) gamma distribution with $\alpha = \frac{1}{2}$ & $\beta = 2g'g * \sigma^2$

and its non-centrality parameter is $\Omega = \frac{(g'X\beta)^2}{2g'g*\sigma^2}$

$$E((g'y)^2) = g'g * \sigma^2 + (g'X\beta)^2$$

When $g'X = 0$

$$\left(\frac{g'y - 0}{\sqrt{g'g \ \ \sigma^2}}\right)^2 \sim \chi_{(1)} \quad => \quad (g'y)^2 \sim g'g * \sigma^2 * \chi_{(1)}$$