The Metropolis-Hastings algorithm

Do HM for $N(0,1)$ with three differents sigma $(0.1,0.5,10)$ and with two different initial values $(-10,0)$

The Metropolis-Hastings algorithm process:

1. Initialize $x^{(0)}$
2. For i = 1 to k

   Sample $u \sim U(0,1)$

   Sample $x^* \sim q(x^*|x^{(i)})$

   If $u < \alpha(x^{(i)}, x^*) = \min\{ \frac{p(x^*)p(x^{(i)}|x^*)}{p(x^{(i)})p(x^*|x^{(i)})} ,1 \}$

   $x^{(i+1)} = x^*$ else $x^{(i+1)} = x^{(i)}$

因為此題$q(x^*|x^{(i)})$為常態分配為對稱分配，故$p(x^*|x^{(i)}) = p(x^{(i)}|x^*)$，

其 $\alpha(x^{(i)}, x^*) = \min\{ \frac{p(x^*)}{p(x^{(i)})} ,1 \}$

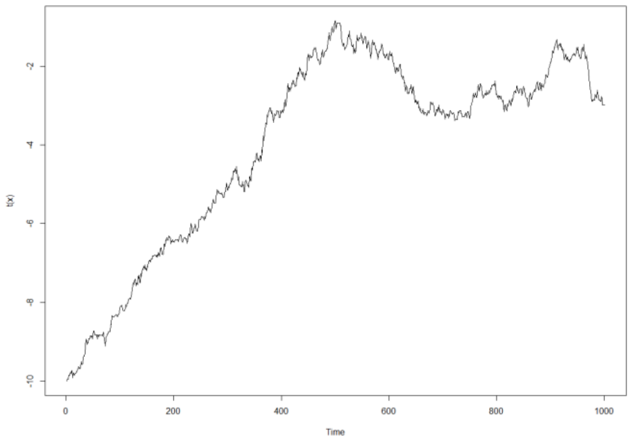
- Target : $\pi(x) = \phi(x; 0,1)$ $(N(0,1)$
- Proposal : $q(y|x) = \phi(y; x, \sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{1}{2\sigma^2}(x-y)^2)$

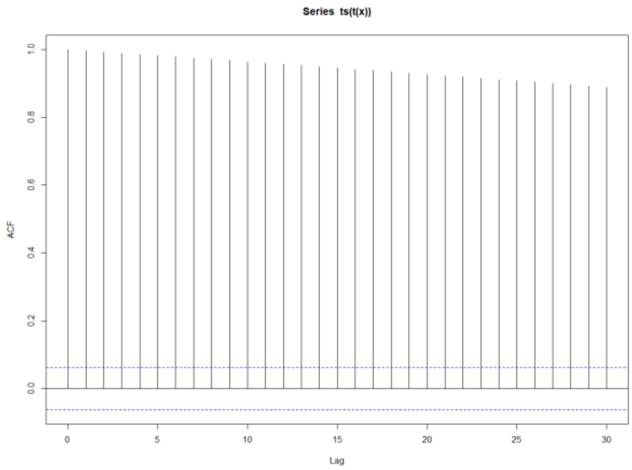
  Generate y , where $y|x^{(i)} \sim q(y|x^{(i)}) = \frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{1}{2\sigma^2}(y-x^{(i)})^2)$

  $y|x^{(i)} = x^{(i)} + \sigma * N(0,1)$
- Obtain the MCMC sequence with $x_0 = 0, -10$ and $\sigma = 0.1, 0.5, 10,$ respectively
- Here $\phi(x; \mu, \sigma)$ is pdf of $X \sim N(\mu, \sigma^2)$
- $\alpha(x, y) = \min\{\exp\left\{-\frac{y^2-x^2}{2}\right\}, 1\}$

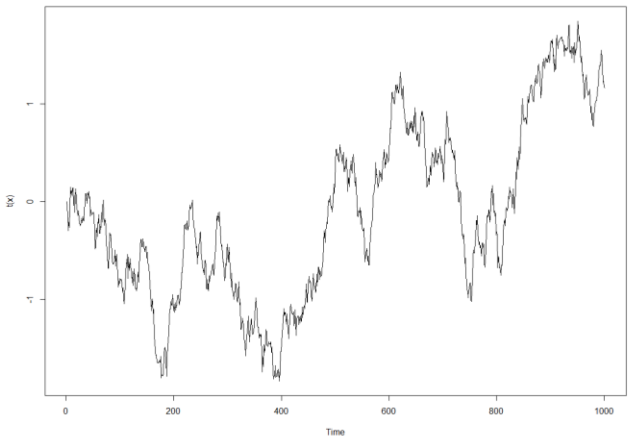
首先給定樣本起始值，並生成 Proposal distribution 的觀察值及均勻分配的亂數，比較均勻分配的亂數是否小於我們計算出的接受機率α，如果小於則更新樣本值，大於則保留原值，重複以上步驟來獲得目標分配的樣本。

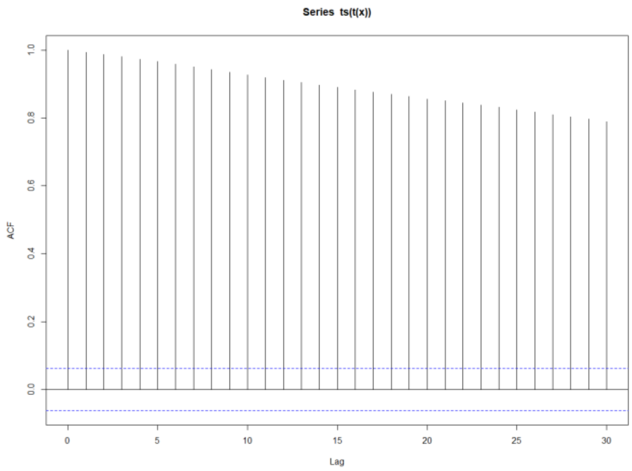
情形一（$x_0 = -10$ ； $\sigma = 0.1$）



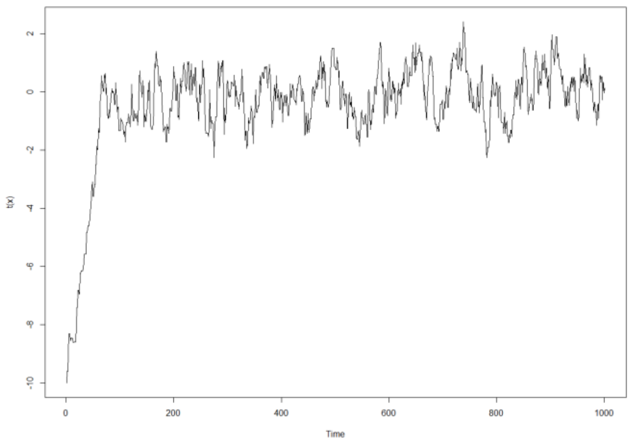


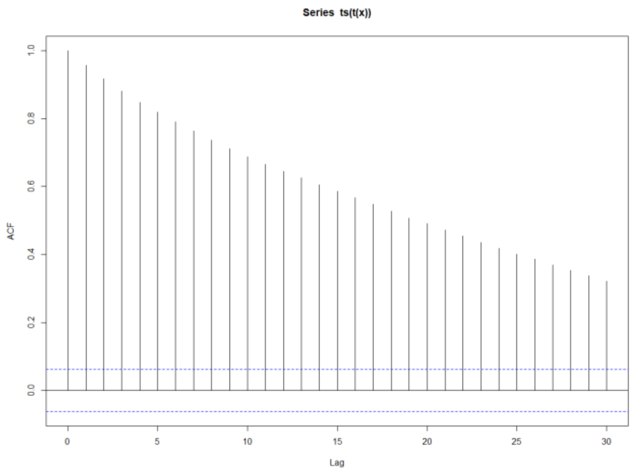
情形二（$x_0 = 0$ ； $\sigma = 0.1$）



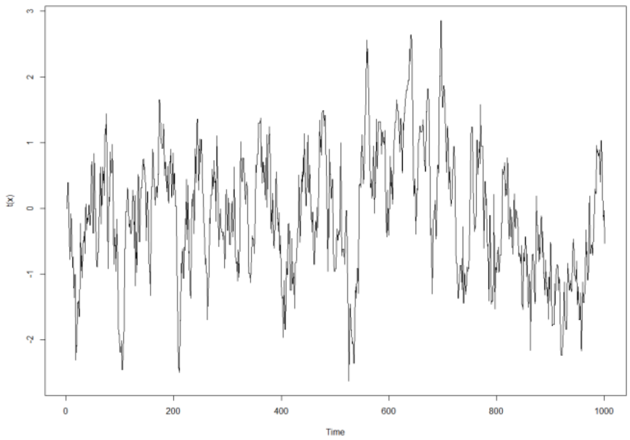


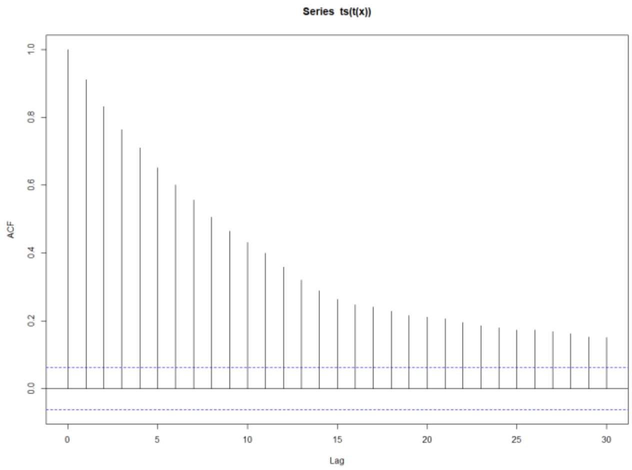
情形三（$x_0 = -10$ ； $\sigma = 0.5$）



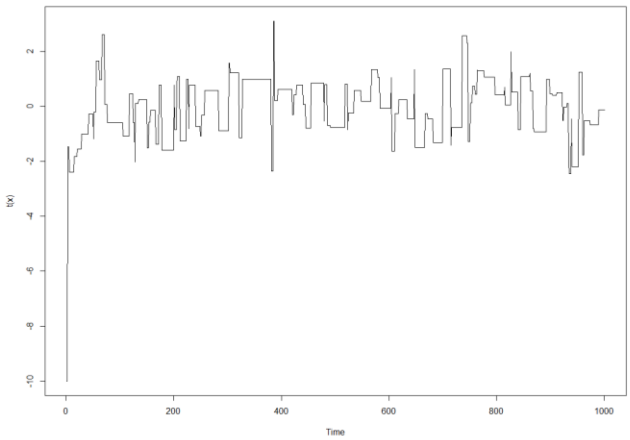


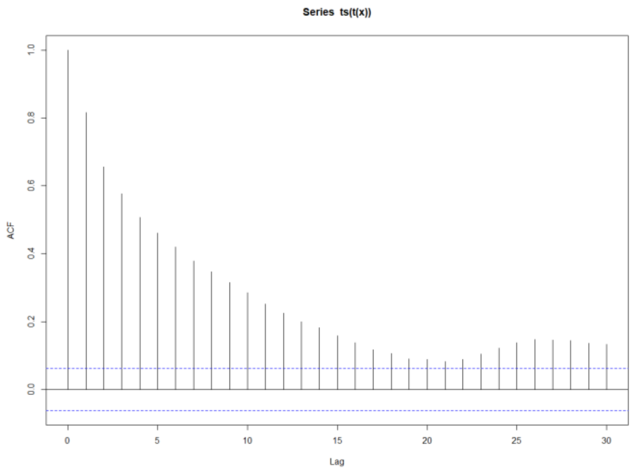
情形四 ($x_0 = 0$; $\sigma = 0.5$)



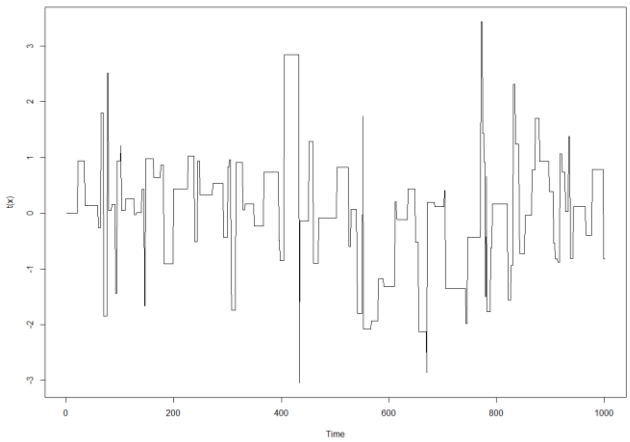


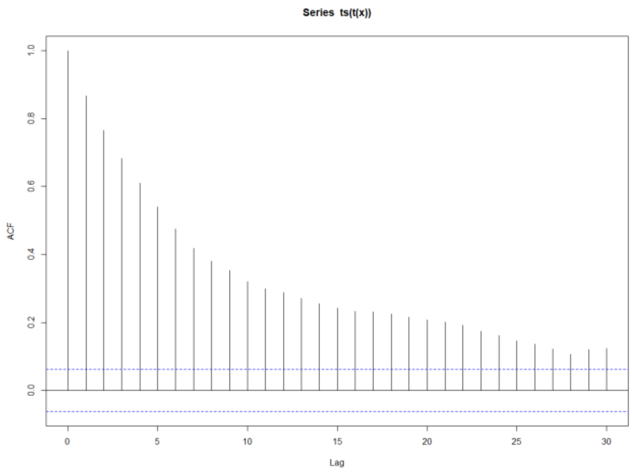
情形五 ($x_0 = -10$; $\sigma = 10$)





情形六 ($x_0 = 0$; $\sigma = 10$)



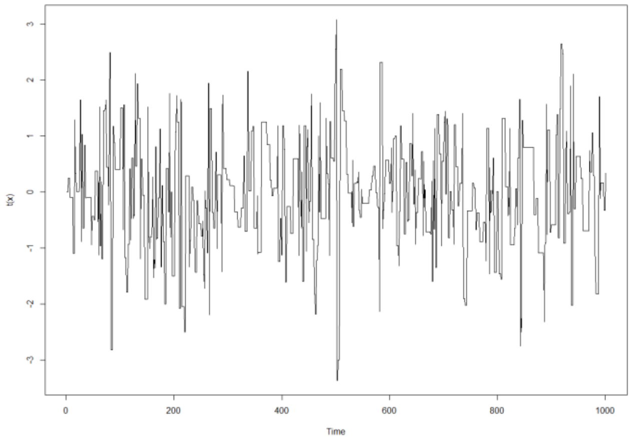


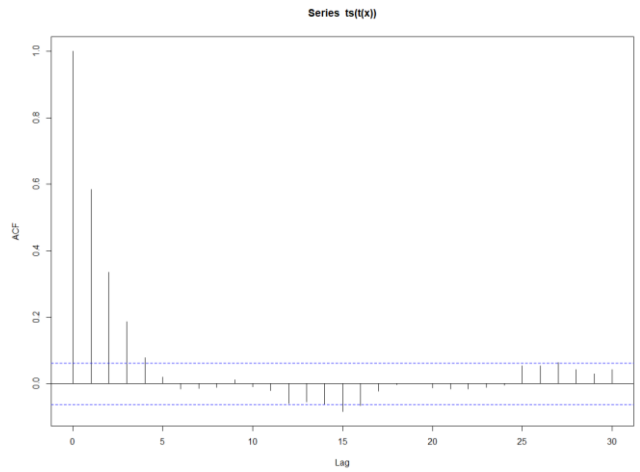
接著我們計算各情形的接受機率

情形	一	二	三	四	五	六
Acc_rate	0.893	0.980	0.829	0.833	0.114	0.100

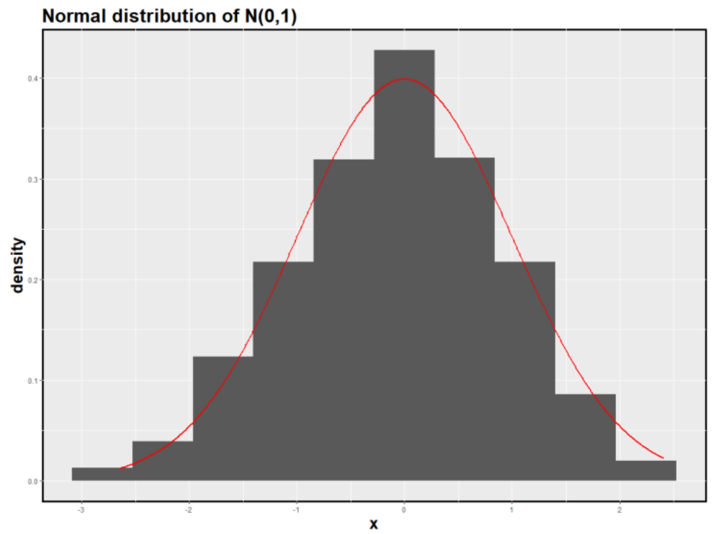
Question : small σ ($\sigma = 0.1$) gives (higher, lower) acceptance rate?

我們從上面表格發現當我們的變異數越大時，其樣本的震盪幅度較大，接受機率會逐漸下降，我們的樣本之間的相關程度也會隨著變異數的增加有下降的趨勢，最適合的接受機率約在 30%到 40%之間，所以我們尋找接受機率為 30%到 40%之間，並將此情形為 $(x_0 = 0\ ;\ \sigma = 3)$接受機率為 34.9%，將其圖表示在下方





接著我們將這組樣本繪製成直方圖與真實分配比較，與實際分配非常相似。



總結:

不論我們選取的起始點為 0 或-10，我們都可以從圖形中發現其樣本最後震盪的幅度會逐漸縮小至$(-3,3)$之間，而且其觀察值間的相關程度隨著變異數的增加，有逐漸降低的傾向，而從表格紀錄各情形的接受機率之中，發現當我們的變異數增加時，其接受機率也有逐漸下降的傾向。