If $\mathcal{U}$ is the topology on X, then $\mathcal{U}$ is a set, and so $\mathcal{U}$ is a small category; as in Example 1.19(iv), all presheaves form a category $\mathbf{pSh}(X, \mathbf{Ab})$, with morphisms $\mathrm{Hom}(\mathcal{P}, \mathcal{Q}) = \mathrm{Nat}(\mathcal{P}, \mathcal{Q})$. We call morphisms $\mathcal{P} \to \mathcal{Q}$ ***presheaf maps***. It follows that if $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then every presheaf map $\mathcal{F} \to \mathcal{G}$ is a sheaf map.

Notation. Define $\mathbf{Sh}(X, \mathbf{Ab})$ to be the full subcategory of $\mathbf{pSh}(X, \mathbf{Ab})$ generated by all sheaves over a space X. We denote the Hom sets by

$$\mathrm{Hom}_{\mathrm{sh}}(\mathcal{F}, \mathcal{F}') = \mathrm{Nat}(\mathcal{F}, \mathcal{F}').$$

Example 5.67. For each open set U of a topological space X, define

$$\mathcal{F}(U) = \{\text{continuous } f : U \to \mathbb{R}\}.$$

It is routine to see that $\mathcal{F}(U)$ is an abelian group under pointwise addition: $f + g : x \mapsto f(x) + g(x)$, and that $\mathcal{F}$ is a presheaf over X. For each $x \in X$, define an equivalence relation on $\bigcup_{U \ni x} \mathcal{F}(U)$ by $f \sim g$ if there is some open set W containing x with $f|W = g|W$. The equivalence class of f, denoted by $[x, f]$, is called a ***germ*** at x. Define E_x to be the family of all germs at x, define $E = \bigcup_{x \in X} E_x$, and define $p : E \to X$ by $p : [x, f] \mapsto x$. In our coming discussion of *associated etale-sheaves*, we will see how to topologize E so that (E, p, X) is an etale-sheaf (called the ***sheaf of germs of continuous functions over*** X). The stalks E_x of this etale-sheaf can be viewed as direct limits: the family of all open sets U containing x is a directed partially ordered set and, by Corollary 5.31, a germ $[x, f]$ is just an element of the direct limit $\varinjlim_{U \ni x} \mathcal{F}(U)$. Variations of this construction are the sheaves of ***germs of differentiable functions*** and of ***germs of holomorphic functions***. ◀

Example 5.67 generalizes; we shall see, in Theorem 5.68, that the stalks of every etale-sheaf are direct limits.

We now construct an etale-sheaf from any presheaf $\mathcal{P}$ (we do not assume that $\mathcal{P}$ is the sheaf of sections of an etale-sheaf). The next result shows that there is no essential difference between sheaves and etale-sheaves.

Theorem 5.68.

(i) *The sheaf of sections defines a functor* $\Gamma : \mathbf{Sh}_{\mathrm{et}}(X, \mathbf{Ab}) \to \mathbf{pSh}(X, \mathbf{Ab})$, *and* $\operatorname{im}\Gamma \subseteq \mathbf{Sh}(X, \mathbf{Ab})$.

(ii) *There are a functor* $\Phi : \mathbf{pSh}(X, \mathbf{Ab}) \to \mathbf{Sh}_{\mathrm{et}}(X, \mathbf{Ab})$ (*which is injective on objects*) *and a natural transformation* $\nu : 1_{\mathbf{pSh}(X,\mathbf{Ab})} \to \Gamma\Phi$ *such that* $\nu_{\mathcal{F}} : \mathcal{F} \to \Gamma\Phi(\mathcal{F})$ *is an isomorphism whenever* $\mathcal{F}$ *is a sheaf.*

(iii) *The restriction* $\Phi|\,\mathbf{Sh}(X, \mathbf{Ab})$ *is an isomorphism of categories*:

$$\mathbf{Sh}(X, \mathbf{Ab}) \cong \mathbf{Sh}_{\text{et}}(X, \mathbf{Ab}).$$

Proof.

(i) If $\varphi\colon \mathcal{S} \to \mathcal{S}'$, define $\Gamma(\varphi)\colon \Gamma(U, \mathcal{S}) \to \Gamma(U, \mathcal{S}')$ by $\sigma \mapsto \varphi\sigma$. The reader may check that Γ is a functor. Proposition 5.63 says that $\Gamma(\Box, \mathcal{S})$ is a sheaf.

(ii) Given a presheaf $\mathcal{P}$ of abelian groups over a space X, we first construct its ***associated etale-sheaf*** $\mathcal{P}^{\text{et}} = (E^{\text{et}}, p^{\text{et}}, X)$. For each $x \in X$, the index set consisting of all open neighborhoods $U \ni x$, partially ordered by reverse inclusion, is a directed set. Define $E_x^{\text{et}} = \varinjlim_{U \ni x} \mathcal{P}(U)$ (generalizing the stalks of the sheaf of germs in Example 5.67).

$$\begin{tikzcd}
E_x^{\text{et}} = \varinjlim_{U \ni x} \mathcal{P}(U) & \\
& \mathcal{P}(V) \arrow[ul, "\rho_x^V"'] \arrow[d, "\rho_U^V"] \\
& \mathcal{P}(U) \arrow[uul, "\rho_x^U"]
\end{tikzcd}$$

Since the index set is directed, Corollary 5.31(iii) says that the elements of $E_x^{\text{et}} = \varinjlim \mathcal{P}(U)$ are equivalence classes $[\rho_x^U(\sigma)]$, where $U \ni x$, $\sigma \in \mathcal{P}(U)$, and $\rho_x^U : \mathcal{P}(U) \to E_x^{\text{et}}$ is an insertion morphism of the direct limit; moreover, $[\rho_x^U(\sigma)] + [\rho_x^{U'}(\sigma')] = [\rho_x^W \rho_W^U(\sigma) + \rho_x^W \rho_W^{U'}(\sigma')]$, where $W \subseteq U \cap U'$ (thus, $[\rho_x^U(\sigma)]$ generalizes $[x, f]$ in Example 5.67). Define $E^{\text{et}} = \bigcup_{x \in X} E_x^{\text{et}}$, and define a surjection $p^{\text{et}}\colon E^{\text{et}} \to X$ by $[\rho_x^U(\sigma)] \mapsto x$.

If $U \subseteq X$ is a nonempty open set and $\sigma \in \mathcal{P}(U)$, define

$$\langle \sigma, U \rangle = \{[\rho_x^U(\sigma)] : x \in U\}.$$

We claim that $\langle \sigma, U \rangle \cap \langle \sigma', U' \rangle$ either is empty or contains a subset of the same form. If $e \in \langle \sigma, U \rangle \cap \langle \sigma', U' \rangle$, then $e = [\rho_x^U(\sigma)] = [\rho_y^{U'}(\sigma')]$, where $x \in U$, $\sigma \in \mathcal{P}(U)$, and $y \in U'$, $\sigma' \in \mathcal{P}(U')$. But $x = p^{\text{et}}[\rho_x^U(\sigma)] = p^{\text{et}}[\rho_y^{U'}(\sigma')] = y$, so that $x \in U \cap U'$. By Lemma 5.30(ii), there is an open $W \subseteq U \cap U'$ with $W \ni x$ and $[\rho_W^U \rho_x^W(\sigma)] = [\rho_W^{U'} \rho_x^W(\sigma')]$; call this element $[\tau]$; note that $\langle \tau, W \rangle \subseteq \langle \sigma, U \rangle \cap \langle \sigma', U' \rangle$, as desired. Equip E^{et} with the topology[7] generated

[7]This is the coarsest topology on E that makes all sections continuous.

by all $\langle\sigma, U\rangle$; it follows that these sets form a base for the topology; that is, every open set is a union of $\langle\sigma, U\rangle$s.

To see that $(E^{\text{et}}, p^{\text{et}}, X)$ is a protosheaf, we must show that the surjection p^{et} is a local homeomorphism. If $e \in E^{\text{et}}$, then $e = [\rho_x^U(\sigma)]$ for some $x \in X$, where U is an open neighborhood of x and $\sigma \in \mathcal{P}(U)$. If $S = \langle\sigma, U\rangle$, then S is an open neighborhood of e, and it is routine to see that $p^{\text{et}}|S \colon S \to U$ is a homeomorphism.

Now each stalk E_x^{et} is an abelian group. To see that addition is continuous, take $(e, e') \in E^{\text{et}} + E^{\text{et}}$; that is, $e = [\rho_x^U(\sigma)]$ and $e' = [\rho_x^{U'}(\sigma')]$. We may assume the representatives have been chosen so that $\sigma, \sigma' \in \mathcal{P}(U)$ for some U, so that $e+e' = [\rho_x^U(\sigma+\sigma')]$. Let $V^{\text{et}} = \langle\sigma+\sigma', V\rangle$ be a basic open neighborhood of $e+e'$. If $\alpha\colon E^{\text{et}} + E^{\text{et}} \to E^{\text{et}}$ is addition, then it is easy to see that if $U^{\text{et}} = [\langle\tau, W\rangle \times \langle\tau', W\rangle] \cap (E^{\text{et}}+E^{\text{et}})$, then $\alpha(U^{\text{et}}) \subseteq V^{\text{et}}$. Thus, α is continuous. As inversion $E^{\text{et}} \to E^{\text{et}}$ is also continuous, $\mathcal{P}^{\text{et}} = (E^{\text{et}}, p^{\text{et}}, X)$ is an etale-sheaf.

Define $\Phi\colon \mathbf{pSh}(X, \mathbf{Ab}) \to \mathbf{Sh}_{\text{et}}(X, \mathbf{Ab})$ on objects by $\Phi(\mathcal{P}) = \mathcal{P}^{\text{et}} = (E^{\text{et}}, p^{\text{et}}, X)$. Note that Φ is injective on objects, for if $\mathcal{P} \neq \mathcal{P}'$, then $\{\varinjlim_{U\ni x} \mathcal{P}(U)\} \neq \{\varinjlim_{U\ni x} \mathcal{P}'(U)\}$, and so their direct limits are distinct (of course, they may be isomorphic). Hence, $\mathcal{P}^{\text{et}} \neq \mathcal{P}'^{\text{et}}$ and $\Phi\mathcal{P} \neq \Phi\mathcal{P}'$. To define Φ on morphisms, let $\varphi\colon \mathcal{P}_1 \to \mathcal{P}_2$ be a presheaf map, and let $\mathcal{P}_i^{\text{et}} = (E_i^{\text{et}}, p_i^{\text{et}}, X)$ for $i = 1, 2$. For each $x \in X$, φ induces a morphism of direct systems $\{\mathcal{P}_1(U) : U \ni x\} \to \{\mathcal{P}_2(U) : U \ni x\}$ and, hence, a homomorphism $\varphi_x\colon \varinjlim_{U\ni x} \mathcal{P}_1(U) \to \varinjlim_{U\ni x} \mathcal{P}_1(U)$; that is, $\varphi_x\colon (E_1^{\text{et}})_x \to (E_2^{\text{et}})_x$. Finally, define $\Phi(\varphi)\colon E_1^{\text{et}} \to E_2^{\text{et}}$ by $e_x \mapsto \varphi_x(e_x)$ for all $e_x \in (E_1^{\text{et}})_x$. We let the reader prove that $\Phi(\varphi)$ is an etale-map and that Φ is a functor.

Given a presheaf $\{\mathcal{P}, \rho_U^V\}$ and an open subset $U \subseteq X$ (that is, $U \in \mathcal{U}$), a base for the topology of E^{et} consists of all $\langle\sigma, U\rangle = \{[\rho_x^U(\sigma)] : x \in U\}$. Define $\sigma^{\text{et}}\colon U \to E^{\text{et}}$ by $\sigma^{\text{et}}(x) = [\rho_x^U(\sigma)]$; Exercise 5.39(i) on page 301 now says that $\sigma^{\text{et}} \in \Gamma(U, \mathcal{P}^{\text{et}})$. Define $\nu_U\colon \mathcal{P}(U) \to \Gamma(U, \mathcal{P}^{\text{et}})$ by $\sigma \mapsto \sigma^{\text{et}}$. If V is an open set containing U, then it easy to see that $\nu_V = \nu_U \rho_U^V$, so that the family $\{\nu_U : U \in \mathcal{U}\}$ gives a presheaf map $\nu_{\mathcal{P}}\colon \mathcal{P} \to \Gamma(\square, \mathcal{P}^{\text{et}})$. We let the reader check that $\nu = (\nu_U)$ is a natural transformation $1_{\mathbf{pSh}(X,\mathbf{Ab})} \to \Gamma\Phi$.

If $\mathcal{F}$ is a sheaf, we show that $\nu_{\mathcal{F}}\colon \mathcal{F} \to \Gamma(\square, \mathcal{F}^{\text{et}})$ is an isomorphism using Exercise 5.41 on page 301. It suffices to prove, for each open U, that $\nu_U\colon \mathcal{F}(U) \to \Gamma(U, \mathcal{F}^{\text{et}})$, given by $\sigma \to \sigma^{\text{et}}$, is a bijection. To see that ν_U is injective, suppose that $\sigma, \tau \in \mathcal{F}(U)$ and $\sigma^{\text{et}} = \tau^{\text{et}}$. For each $x \in U$, we have $\rho_x^U(\sigma) = \rho_x^U(\tau)$; that is, there is an open neighborhood W_x of x with $\sigma|W_x = \tau|W_x$. The family of all such W_x is an open cover of U, and so Proposition 5.58(iv) gives $\sigma = \tau$. To see that ν_U is surjective, let $\beta \in \Gamma(U, \mathcal{F}^{\text{et}})$. For each $x \in U$, there is a basic open set $\langle U, \sigma_x \rangle$ containing $\beta(x)$, where $\sigma_x \in \mathcal{F}(U_x)$. The gluing condition, Proposition 5.58(v), shows that there is $\sigma \in \mathcal{F}(U)$ with $\sigma|U_x = \sigma_x$ for all $x \in U$, and another application of Proposition 5.58(iv) gives $\beta = \sigma^{\text{et}}$. Thus, ν_U is a bijection.

(iii) This follows easily from parts (i) and (ii). •

The stalks of the etale-sheaf of germs in Example 5.67 are direct limits, as are the stalks of $\mathcal{P}^{\text{et}}$; we now define the stalks of an arbitrary presheaf.

Definition. If $\mathcal{P}$ is a presheaf on a space X, then the ***stalk*** at $x \in X$ is

$$\mathcal{P}_x = \varinjlim_{U \ni x} \mathcal{P}(U).$$

For each $x \in X$, the presheaf map $\varphi \colon \mathcal{P} \to \mathcal{Q}$ induces a morphism of direct systems $\{\mathcal{P}(U) : U \ni x\} \to \{\mathcal{Q}(U) : U \ni x\}$, which, in turn, gives the homomorphism $\varphi_x \colon \varinjlim_{U \ni x} \mathcal{P}(U) \to \varinjlim_{U \ni x} \mathcal{Q}(U)$ defined by $\varphi_x \colon [\sigma] \mapsto [\varphi\sigma]$, where $\sigma \in \mathcal{P}(U)$ and $x \in U$. Exercise 5.33 on page 272 shows that $\varinjlim$ is a functor $\mathbf{Dir}(I, \mathbf{Ab}) \to \mathbf{Ab}$, where $\mathbf{Dir}(I, \mathbf{Ab})$ is the category of direct systems of abelian groups over $I = \{U \ni x\}$. Hence, if $\mathcal{P} \xrightarrow{\varphi} \mathcal{Q} \xrightarrow{\psi} \mathcal{R}$ are presheaf maps, then $(\psi\varphi)_x = \psi_x\varphi_x$. See Exercise 5.45 on page 302 for a description of ν_x, where $\nu \colon \mathcal{P} \to \Gamma(\square, \mathcal{P}^{\text{et}})$ is the natural map in Theorem 5.68.

Lemma 5.69. *Let $\varphi, \psi \colon \mathcal{P} \to \mathcal{F}$ be presheaf maps, where $\mathcal{P}$ is a presheaf and $\mathcal{F}$ is a sheaf. If φ, ψ agree on stalks, that is, $\varphi_x = \psi_x$ for all $x \in X$, then $\varphi = \psi$.*

Proof. We must show that $\varphi_U = \psi_U$ for all open U. Given U, choose $x \in U$ and $e_x = [\sigma_x] \in \mathcal{P}_x$, where $\sigma_x \in \mathcal{P}(U_x)$ for some open $U_x \ni x$ with $U_x \subseteq U$. By hypothesis,

$$[\varphi\sigma_x] = \varphi_x([\sigma_x]) = \psi_x([\sigma_x]) = [\psi\sigma_x] \text{ in } \varinjlim_{U \ni x} \mathcal{F}(U).$$

By the definition of equality in direct limits, there are open neighborhoods W_x of x with $\varphi\sigma_x|W_x = \psi\sigma_x|W_x$, and $(W_x)_{x \in U}$ is an open cover of U. Since the equalizer condition holds for the sheaf $\mathcal{F}$, the restrictions determine a unique section; that is, $\varphi\sigma_x = \psi\sigma_x$. Hence, $\varphi_U = \psi_U$ and $\varphi = \psi$. •

Theorem 5.70. *Let $\mathcal{P} = \{\mathcal{P}(U), \rho_U^V\}$ be a presheaf of abelian groups over a space X, let $\mathcal{P}^{\text{et}} = (E^{\text{et}}, p^{\text{et}}, X)$ be its associated etale-sheaf, and let*