

# Answer to homological algebra homework

刘新宇

2240502058

July 1, 2025

## Question 0.1

(1) *Proof.* • First: Let  $i : B' \rightarrow B$  and  $p : B \rightarrow B''$  be  $R$ -module homomorphisms. If for every left  $R$ -module  $N$ ,  $0 \rightarrow \text{Hom}_R(B'', N) \xrightarrow{p^*} \text{Hom}_R(B, N) \xrightarrow{i^*} \text{Hom}_R(B', N)$  is an exact sequence, then  $B' \xrightarrow{i} B \xrightarrow{p} B'' \rightarrow 0$  is an exact sequence.

$p$  is surjective: Let  $N = B''/\text{im } p$  and let  $f : B'' \rightarrow B''/\text{im } p$  be the natural map, so that  $f \in \text{Hom}_R(B'', N)$ . Then  $p^*(f) = f \circ p = 0$ , so that  $f = 0$ , because  $p^*$  is injective. Therefore  $B''/\text{im } p = 0$  and  $p$  is surjective.

$\text{im } i \subset \ker p : (p \circ i)^* = i^* \circ p^* = 0$ . Hence, if  $N = B''$  and  $g = 1_{B''}$  so that  $g \in \text{Hom}_R(B'', N)$ , then  $0 = (p \circ i)^*(g) = g \circ p \circ i = p \circ i$ , and so  $\text{im } i \subset \ker p$ .

$\ker p \subset \text{im } i$ : Choose  $N = B/\text{im } i$  and let  $h : B \rightarrow N$  be the natural map, so that  $h \in \text{Hom}_R(B, N)$ . Since  $i^*(h) = h \circ i = 0$ , so  $h \in \ker i^* = \text{im } p^*$ . Thus  $\exists h' \in \text{Hom}_R(B'', N)$ . Such that  $h = p^*(h') = h' \circ p$ . We have  $\text{im } i \subset \ker p$ , hence if  $\text{im } i \neq \ker p$ , there is an element  $b \in B$  with  $b \notin \text{im } i$  and  $b \in \ker p$ . Thus  $h(b) \neq 0$  and  $p(b) = 0$ , but  $h(b) = h' \circ p(b) = 0$ , a contradiction.

• Second: Let  $B' \xrightarrow{i} B \xrightarrow{p} B'' \rightarrow 0$  be an exact sequence of  $R$ -modules,  $C$  is a  $R$ -module. Since  $(M \otimes_R -, \text{Hom}_R(-, C))$  is an adjoint pair, there is a natural isomorphism :

$$\tau_{M,C} : \text{Hom}_R(M \otimes_R -, C) \rightarrow \text{Hom}_R(M, \text{Hom}_R(-, C)).$$

We have following commutative diagram, which vertical maps are isomorphisms.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(M \otimes_R B'', C) & \xrightarrow{(1_M \otimes p)^*} & \text{Hom}_R(M \otimes_R B, C) & \xrightarrow{(1_M \otimes i)^*} & \text{Hom}_R(M \otimes_R B', C) \\ & & \downarrow \tau''_{M,C} & & \downarrow \tau_{M,C} & & \downarrow \tau'_{M,C} \\ 0 & \longrightarrow & \text{Hom}_R(M, \text{Hom}_R(B'', C)) & \xrightarrow{(p^*)_*} & \text{Hom}_R(M, \text{Hom}_R(B, C)) & \xrightarrow{(i^*)_*} & \text{Hom}_R(M, \text{Hom}_R(B', C)) \end{array}$$

By the left exactness of  $\text{Hom}_R(-, C)$  and  $\text{Hom}_R(M, -)$ , the bottom row is exact. Thus the top row is also exact. From above, we have

$$M \otimes_R B' \longrightarrow M \otimes_R B \longrightarrow M \otimes_R B'' \longrightarrow 0$$

is an exact sequence. Hence  $M \otimes_R -$  is right exact.

□

(2) *Proof.* Let  $i : I \hookrightarrow R$  be the inclusion;  $\psi : M \otimes_R I \longrightarrow M$ , given by  $m \otimes a \mapsto am$ ; and  $\phi : M \otimes_R R \longrightarrow M$ , given by  $m \otimes r \mapsto rm$ , then  $\phi$  is an isomorphism. Since  $\psi = \phi \circ (1_M \otimes i)$ , so  $\psi$  is injective iff  $(1_M \otimes i)$  is injective.

\* (a)  $\Rightarrow$  (c) : If  $M$  is flat over  $R$ . Let  $P_\bullet$  be a projective resolution of  $N$ . Then the functor  $M \otimes_R -$  is exact, and so the complex

$$\cdots \longrightarrow M \otimes_R P_2 \longrightarrow M \otimes_R P_1 \longrightarrow M \otimes_R P_0 \longrightarrow 0$$

is exact for all  $n \geq 1$ . Therefore,  $\text{Tor}_n^R(M, N) = 0$ , for all  $n \geq 1$ .

\* (c)  $\Rightarrow$  (d) : By (c), for any  $R$ -module  $N$ ,  $\text{Tor}_1^R(M, N) = 0$ . Since  $R/I$  is an  $R$ -module, thus  $\text{Tor}_1^R(M, R/I) = 0$ , for any finitely generated ideal  $I \subset R$ .

\* (d)  $\Rightarrow$  (b) : Consider exact sequence:  $0 \longrightarrow I \xrightarrow{i} R \longrightarrow R/I \longrightarrow 0$ . We have exact sequence:  $0 = \text{Tor}_1^R(M, R/I) \longrightarrow M \otimes_R I \xrightarrow{1 \otimes i} M \otimes_R R$ . So  $1 \otimes i$  is injective, hence  $\psi$  is injective.

\* (b)  $\Rightarrow$  (a) : From (b), we have for any finitely generated ideal  $J \subset R$ ,  $1_M \otimes j : M \otimes_R J \longrightarrow M \otimes_R R$  is injective.

• First: Let  $0 \longrightarrow A \xrightarrow{i} B$  be an exact sequence of left  $R$ -modules, and let  $M$  be a right  $R$ -module. If  $u \in \ker(1_M \otimes i)$ , then there are a finitely generated submodule  $N \subset M$  and an element  $u' \in N \otimes_R A$  such that  $u' \in \ker(1_N \otimes i)$ , and  $u = (\kappa \otimes 1_A)(u')$ , where  $\kappa : N \longrightarrow M$  is the inclusion.

Let  $u = \sum_j m_j \otimes a_j \in \ker(1_M \otimes i)$ , where  $m_j \in M$  and  $a_j \in A$ . There is an equation in  $M \otimes_R B$ ,

$$0 = (1_M \otimes i)(u) = \sum_{j=1}^n m_j \otimes ia_j.$$

Let  $F$  be the free abelian group with basis  $M \times B$ , and let  $S$  be the subgroup of  $F$  consisting of the relations of  $F/S \cong M \otimes_R B$ ; thus,  $S$  is generated by all elements in  $F$  of the form

$$\begin{aligned} & (m, b + b') - (m, b) - (m, b'), \\ & (m + m', b) - (m, b) - (m', b), \\ & (mr, b) - (m, rb). \end{aligned}$$

Let  $0 \longrightarrow S \longrightarrow F \xrightarrow{v} M \otimes_R B \longrightarrow 0$ , where  $v : (m, b) \mapsto m \otimes b$ . Since  $(1_M \otimes i)(u) = \sum_j m_j \otimes ia_j = 0$  in  $M \otimes_R B$ , we have  $\sum_j m_j \otimes ia_j = \sum_k v(m'_k, b'_k) \in v(S)$ , where  $m'_k \in M$  and  $b'_k \in B$ . Define  $N$  to be the submodule of  $M$  generated by  $m_1, \dots, m_n$  together with the (finite number of) first coordinates  $m'_k$ . Of course,  $N$  is a finitely generated submodule of  $M$ . If we define  $u' = \sum_j m_j \otimes a_j$  in  $N \otimes_R A$ , then  $(\kappa \otimes 1_A)(u') = \sum \kappa(m_j) \otimes a_j = u$ . Finally,  $(1_N \otimes i)(u') = 0$ , for we have taken care that all the relations making  $(1_N \otimes i)(u') = 0$  are present in  $N \otimes_R B$ .

• Second: If  $B$  is a right  $R$ -module, define its character module  $B^*$  as the left  $R$ -module

$$B^* = \text{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}).$$

We show that a sequence of right  $R$ -modules  $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$  is exact if and only if the sequence of character modules  $C^* \xrightarrow{\beta^*} B^* \xrightarrow{\alpha^*} A^*$  is exact.

If the original sequence is exact, then so is the sequence of character modules, for the contravariant functor  $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$  is exact, because  $\mathbb{Q}/\mathbb{Z}$  is an injective  $\mathbb{Z}$ -module.

For the converse:  $\text{im } \alpha \subset \ker \beta$ . If  $x \in A$  and  $\alpha x \notin \ker \beta$ , then  $\beta \alpha(x) \neq 0$ . there is a map  $f : C \rightarrow \mathbb{Q}/\mathbb{Z}$  with  $f \beta \alpha(x) \neq 0$ . Thus  $f \in C^*$  and  $f \beta \alpha(x) \neq 0$ , which contradicts the hypothesis that  $\alpha^* \beta^* = 0$ .

$\ker \beta \subset \text{im } \alpha$ . If  $y \in \ker \beta$  and  $y \notin \text{im } \alpha$ , than  $y + \text{im } \alpha$  is a nonzero element of  $B/\text{im } \alpha$ . Therefore, there is a map  $g : B/\text{im } \alpha \rightarrow \mathbb{Q}/\mathbb{Z}$  with  $g(y + \text{im } \alpha) \neq 0$ . If  $v : B \rightarrow B/\text{im } \alpha$  is the natural map, define  $g' = gv \in B^*$ ; note that  $g'(y) \neq 0$ . Now  $g'(\text{im } \alpha) = 0$ , so that  $0 = g' \alpha = \alpha^*(g')$  and  $g' \in \ker \alpha^* = \text{im } \beta^*$ . Thus,  $g' = \beta^*(h)$  for some  $h \in C^*$ ; that is,  $g'(y) = h\beta(y) = 0$ , which is a contradiction.

- Third: We show that a right  $R$ -module  $B$  is flat if and only if its character module  $B^*$  is an injective left  $R$ -module.

The functors  $\text{Hom}_R(-, \text{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})) = \text{Hom}_R(-, B^*)$  and  $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) \circ (B \otimes_R -)$  are naturally isomorphic. If  $B$  is flat, then each of the functors in the composite is exact, for  $\mathbb{Q}/\mathbb{Z}$  is  $\mathbb{Z}$ -injective; hence,  $\text{Hom}_R(-, B^*)$  is exact and  $B^*$  is injective.

Convvrsely, assumme  $B^*$  is an injective left  $R$ -module and  $A' \rightarrow A$  is an injection between left  $R$ -modules  $A'$  and  $A$ . Since  $\text{Hom}_R(A, B^*) = \text{Hom}_R(A, \text{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}))$ , the adjoint isomorphism gives a commutative diagram in which the vertical maps are isomorphisms.

$$\begin{array}{ccccccc}
 \text{Hom}_R(A, B^*) & \longrightarrow & \text{Hom}_R(A', B^*) & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \\
 \text{Hom}_{\mathbb{Z}}(B \otimes_R A, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \text{Hom}_{\mathbb{Z}}(B \otimes_R A', \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \\
 \downarrow = & & \downarrow = & & \\
 (B \otimes_R A)^* & \longrightarrow & (B \otimes_R A')^* & \longrightarrow & 0
 \end{array}$$

Exactness of the top row now gives exactness of the bottom row. Hence the sequence

$$0 \longrightarrow B \otimes_R A' \longrightarrow B \otimes A$$

is exact, and this gives  $B$  flat.

- Finally: Consider  $1_M \otimes l : M \otimes_R I \rightarrow M \otimes_R R$ , where  $I$  is an ideal of  $R$ . For any  $u \in \ker(1_M \otimes l)$ , use the first argument to  $0 \rightarrow I \hookrightarrow R$  and  $M$ . There is a finitely generated submodule  $N \subset M$  and  $w \in N \otimes_R I$  such that  $w \in \ker(1_N \otimes l)$  and  $u = (\kappa \otimes 1_I)(w)$ , where  $\kappa : N \hookrightarrow M$  is the inclusion. Suppose  $w = \sum_{k=1}^m n_k \otimes i_k$ ,  $n_k \in N$ ,  $i_k \in I$ . Let  $J = \langle i_1, \dots, i_m \rangle \subset I$ . Then  $w \in N \otimes_R J$ . Let  $j : J \hookrightarrow R$  be the inclusion, then  $(1_N \otimes j)(w) = (1_N \otimes l)(w) = 0$ . Consider the following commutative diagram:

$$\begin{array}{ccc}
 N \otimes_R J & \xrightarrow{1_N \otimes j} & N \otimes_R R \\
 \kappa \otimes 1_J \downarrow & & \downarrow \kappa \otimes 1_R \\
 M \otimes_R J & \xrightarrow{1_M \otimes j} & M \otimes_R R
 \end{array}$$

Then  $(1_M \otimes j)(\kappa \otimes 1_J)(w) = (\kappa \otimes 1_R)(1_N \otimes j)(w) = (\kappa \otimes 1_R)(0) = 0$ . Since  $J$  is finitely generated, so  $1_M \otimes j$  is injective, thus  $(\kappa \otimes 1_J)(w) = 0$ . But  $u = (\kappa \otimes 1_I)(w) = (\kappa \otimes 1_I)(\sum_{k=1}^m n_k \otimes i_k) =$

$\sum_{k=1}^m \kappa(n_k) \otimes i_k = (\kappa \otimes 1_J)(w)$ . Hence  $u = 0$  and  $1_M \otimes l$  is injective. Therefore, for any ideal  $I$  of  $R$ , the sequence  $0 \longrightarrow A \otimes_R I \xrightarrow{1_M \otimes l} A \otimes_R R$  is exact. Use the second argument, we have  $(A \otimes_R R)^* \longrightarrow (A \otimes_R I)^* \longrightarrow 0$  is exact, and use the third argument we have

$$\text{Hom}_R(R, A^*) \longrightarrow \text{Hom}_R(I, A^*) \longrightarrow 0$$

is exact. This says that every map from any ideal  $I$  to  $A^*$  extends to a map  $R \longrightarrow A^*$ . Hence  $A^*$  is injective, by the Baer Criterion and so  $A$  is flat.

□

## Question 0.2

(1) *Proof.*  $\star(c) \Rightarrow (a)$  : Since  $M$  is projective, so  $\text{Tor}_1^R(M, N) = 0$ , for any  $R$ -module  $N$ . Thus  $M$  is flat. Since  $M$  is also finitely generated, suppose  $M = \langle a_1, \dots, a_n \rangle$ , and let  $F$  be the free  $R$ -module with basis  $\{x_1, \dots, x_n\}$ . Define  $\phi : F \longrightarrow M$  by  $\phi : x_j \mapsto a_j$ , so there is an exact sequence  $0 \longrightarrow \ker \phi \longrightarrow F \xrightarrow{\phi} M \longrightarrow 0$ . This sequence split, because  $M$  is projective, so that  $F \cong M \oplus \ker \phi$ . Now  $\ker \phi$  is finitely generated, for it is a direct summand, hence an image, of the finitely generated module  $F$ . Therefore,  $M$  is finitely presented.

$\star(a) \Rightarrow (c)$  : • First: We show that if  $A$  is a flat right  $R$ -module and  $I$  is a left ideal, then the  $\mathbb{Z}$ -map  $\theta_A : A \otimes_R I \longrightarrow AI$ , given by  $a \otimes i \mapsto ai$ , is an isomorphism.

Let  $\kappa : I \longrightarrow R$  be the inclusion, and let  $\phi_A : A \otimes_R R \longrightarrow A$  be the isomorphism  $a \otimes r \mapsto ar$ . The composite

$$\phi_A(1_A \otimes \kappa) : A \otimes_R I \longrightarrow A \otimes_R R \longrightarrow A$$

is given by  $a \otimes i \mapsto ai \in R$ , and its image is  $AI$ . Now  $1_A \otimes \kappa$  is an injection, because  $A$  is flat, and so composing it with the isomorphism  $\phi_A$  is an injection. Therefore, the composite  $\phi_A(1_A \otimes \kappa)$  is an injection, so that  $\theta_A : a \otimes i \mapsto ai$ , is an isomorphism.

• Second: Let  $0 \longrightarrow K \longrightarrow F \xrightarrow{\phi} A \longrightarrow 0$  be an exact sequence of right  $R$ -module in which  $F$  is flat. Then  $A$  is a flat module if and only if  $K \cap FI = KI$  for every finitely generated left ideal  $I$ .

We give a preliminary discussion before proving this. For every left ideal  $I$ , right exactness of  $\square \otimes_R I$  gives exactness of

$$K \otimes_R I \longrightarrow F \otimes_R I \xrightarrow{\varphi \otimes 1} A \otimes_R I \rightarrow 0.$$

By the first argument, there is an isomorphism  $\theta_F : F \otimes_R I \rightarrow FI$  with  $f \otimes i \mapsto fi$ ; of course,  $\theta_K : K \otimes_R I \rightarrow KI$  is a surjection. The following diagram commutes, where inc is the inclusion and nat is the natural map.

$$\begin{array}{ccccccc} K \otimes_R I & \longrightarrow & F \otimes_R I & \xrightarrow{\varphi \otimes 1} & A \otimes_R I & \longrightarrow & 0 \\ \downarrow \theta_K & & \downarrow \theta_F & & \downarrow \gamma & & \\ KI & \xrightarrow{\text{inc}} & FI & \xrightarrow{\text{nat}} & FI/KI & \longrightarrow & 0 \end{array}$$

It is easy to see that there exists a map  $\gamma : A \otimes_R I \rightarrow FI/KI$ , given by  $\varphi f \otimes i \mapsto fi + KI$ , where  $f \in F$  and  $i \in I$ ; since  $\theta_K$  is a surjection and  $\theta_F$  is an isomorphism, the map  $\gamma$  is an isomorphism.

Now

$$\varphi(FI) = \left\{ \varphi \left( \sum_j f_j i_j \right) : f_j \in F, i_j \in I \right\} = \left\{ \sum_j (\varphi f_j) i_j \right\} = AI.$$

Therefore, the first isomorphism theorem provides an isomorphism

$$\delta : FI/(FI \cap K) \rightarrow \varphi(FI) = AI,$$

namely,  $fi + (FI \cap K) \mapsto \varphi(fi)$ . We assemble these maps to obtain the composite  $\sigma$ :

$$FI/KI \xrightarrow{\gamma^{-1}} A \otimes_R I \xrightarrow{\theta_A} AI \xrightarrow{\delta^{-1}} FI/(FI \cap K).$$

Explicitly,  $\sigma : fi + KI \mapsto fi + (FI \cap K)$ . But  $KI \subseteq FI \cap K$ , so that  $\sigma$  is the enlargement of coset map of the third isomorphism theorem and, hence,  $\ker \sigma = (FI \cap K)/KI$ . Therefore,  $\sigma$  is an isomorphism if and only if  $KI = FI \cap K$ . Moreover, since the flanking maps  $\gamma^{-1}$  and  $\delta^{-1}$  are isomorphisms,  $\sigma$  is an isomorphism if and only if  $\theta_A$  is an isomorphism.

If  $A$  is flat, then the first argument says that  $\theta_A$  is an isomorphism. Therefore,  $\sigma$  is an isomorphism and  $KI = FI \cap K$ . Conversely, if  $KI = FI \cap K$  for every finitely generated left ideal  $I$ , then  $\theta_A$  is an isomorphism, and thus  $A$  is flat.

- Third: Let  $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$  be an exact sequence of right  $R$ -modules, where  $F$  is free with basis  $\{x_j : j \in J\}$ . For each  $v \in F$ , define  $I(v)$  to be the left ideal in  $R$  generated by the “coordinates”  $r_1, \dots, r_t \in R$  of  $v$ , where  $v = x_{j_1}r_1 + \dots + x_{j_t}r_t$ . Then  $A$  is flat if and only if  $v \in KI(v)$  for every  $v \in K$ .

If  $A$  is flat and  $v \in K$ , then  $v \in K \cap FI(v) = KI(v)$ , by the second argument.

Conversely, let  $I$  be any left ideal, and let  $v \in K \cap FI$ . Then  $I(v) \subset I$ , so the hypothesis gives  $v \in KI(v) \subset KI$ . Hence,  $K \cap FI \subset KI$ . As the reverse inclusion always holds, the second argument says that  $A$  is flat.

- Fourth: Let  $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$  be an exact sequence of right  $R$ -modules, where  $F$  is free. The following statements are equivalent.

- (i)  $A$  is flat.
- (ii) For every  $v \in K$ , there is an  $R$ -map  $\theta : F \rightarrow K$  with  $\theta(v) = v$ .
- (iii) For every  $v_1, \dots, v_n \in K$ , there is an  $R$ -map  $\theta : F \rightarrow K$  with  $\theta(v_i) = v_i$  for all  $i$ .

(i)  $\Rightarrow$  (ii): Assume that  $A$  is flat. Choose a basis  $\{x_j : j \in J\}$  of  $F$ . If  $v \in K$ , then  $I(v)$  is the left ideal generated by  $r_1, \dots, r_l$ , where  $v = x_{j_1}r_1 + \dots + x_{j_l}r_l$ . By the third argument,  $v \in KI(v)$ , and so  $v = \sum k_p s_p$ , where  $k_p \in K$  and  $s_p \in I(v)$ . Hence,  $s_p = \sum u_{pi} r_i$ , where  $u_{pi} \in R$ . Rewrite:  $v = \sum k'_i r_i$ , where  $k'_i = \sum k_p u_{pi} \in K$ , and define  $\theta : F \rightarrow K$  by  $\theta(x_{j_i}) = k'_i$  and  $\theta(x_j) = 0$  for all other basis elements  $x_j$ . Clearly,  $\theta(v) = v$ .

(ii)  $\Rightarrow$  (i): Let  $v \in K$ , and let  $\theta : F \rightarrow K$  be a map with  $\theta(v) = v$ . Choose a basis  $\{x_j : j \in J\}$  of  $F$ , and write  $v = x_{j_1}r_1 + \dots + x_{j_l}r_l$ . Then  $v = \theta(v) = \theta(x_{j_1})r_1 + \dots + \theta(x_{j_l})r_l \in KI(v)$ . Hence,  $A$  is flat, by the third argument.

Since (iii) obviously implies (ii), it only remains to prove (ii)  $\Rightarrow$  (iii). The proof is by induction on  $n$ . The base step is our hypothesis (ii). Let  $v_1, \dots, v_n \in K$ , where  $n \geq 2$ . There is a map  $\theta_n : F \rightarrow K$  with  $\theta_n(v_n) = v_n$ . By induction, there is a map  $\theta' : F \rightarrow K$  with  $\theta'[v_i - \theta_n(v_i)] = v_i - \theta_n(v_i)$  for all  $i = 1, \dots, n-1$ . Now define  $\theta : F \rightarrow K$  by

$$\theta(u) = \theta_n(u) + \theta'[u - \theta_n(u)]$$

for all  $u \in F$ . It is routine to see that  $\theta(v_i) = v_i$  for all  $i$ .

- Finally: If  $M$  is flat and of finite presentation, we show that  $M$  is projective.

Let  $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$  be an exact sequence of  $R$ -modules, where  $K, F$  are finitely generated and  $F$  is free. If  $K = \langle v_1, \dots, v_n \rangle$ , then the fourth argument gives  $\theta : F \rightarrow K$ , with  $\theta(v_i) = v_i$  for all  $i$  (because  $M$  is flat). Therefore,  $K$  is a retraction of  $F$ , and hence it is a direct summand:  $F \cong K \oplus M$ . Therefore,  $M$  is projective.

$\star (b) \Rightarrow (c)$  : Assume (b) holds: There exist  $f_1, \dots, f_s \in R$  with  $(f_1, \dots, f_s) = R$  such that  $M_{f_i} = M \otimes_R R_{f_i}$  is a free  $R_{f_i}$ -module for each  $i$ . Since  $M$  is finitely presented (hence finitely generated), it suffices to show that  $M_{\mathfrak{m}}$  is a free  $R_{\mathfrak{m}}$ -module for every maximal ideal  $\mathfrak{m} \subset R$ .

Fix a maximal ideal  $\mathfrak{m} \subset R$ . As  $(f_1, \dots, f_s) = R$ , there exists some  $f_i \notin \mathfrak{m}$  (otherwise  $\mathfrak{m}$  would contain all  $f_i$  and thus 1, a contradiction). For this  $f_i$ , we have:

$$M_{f_i} \cong R_{f_i}^{k_i} \quad (\text{free module over } R_{f_i})$$

Localize further at  $\mathfrak{m}$ . Since  $f_i \notin \mathfrak{m}$ , the ideal  $\mathfrak{m}R_{f_i}$  is prime in  $R_{f_i}$  and:

$$(M_{f_i})_{\mathfrak{m}R_{f_i}} \cong M \otimes_R (R_{f_i})_{\mathfrak{m}R_{f_i}} \cong M \otimes_R R_{\mathfrak{m}} = M_{\mathfrak{m}}$$

As  $M_{f_i}$  is free over  $R_{f_i}$ , its localization  $(M_{f_i})_{\mathfrak{m}R_{f_i}}$  is free over  $(R_{f_i})_{\mathfrak{m}R_{f_i}} \cong R_{\mathfrak{m}}$ . Thus  $M_{\mathfrak{m}}$  is free over  $R_{\mathfrak{m}}$ .

Since  $M_{\mathfrak{m}}$  is free (hence projective) over  $R_{\mathfrak{m}}$  for every maximal ideal  $\mathfrak{m}$  and  $M$  is finitely generated,  $M$  is projective over  $R$ .

$\star (c) \Rightarrow (b)$  : Assume (c) holds:  $M$  is projective. We construct elements  $f_1, \dots, f_s \in R$  generating the unit ideal such that  $M_{f_i}$  is free over  $R_{f_i}$  for each  $i$ .

Since  $M$  is finitely presented and projective, for any prime ideal  $\mathfrak{p} \subset R$ , the localization  $M_{\mathfrak{p}}$  is projective over  $R_{\mathfrak{p}}$ . As  $R_{\mathfrak{p}}$  is local, a finitely generated projective module over a local ring is free. Thus  $M_{\mathfrak{p}}$  is free over  $R_{\mathfrak{p}}$ .

Define the rank function:

$$r : \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}, \quad \mathfrak{p} \mapsto \text{rank}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$$

We show  $r$  is locally constant. Fix  $\mathfrak{p} \in \text{Spec}(R)$  and let  $r(\mathfrak{p}) = n$ . Since  $M$  is finitely presented, there is an exact sequence:

$$R^a \rightarrow R^b \rightarrow M \rightarrow 0$$

Localizing at  $\mathfrak{p}$  gives:

$$R_{\mathfrak{p}}^a \rightarrow R_{\mathfrak{p}}^b \rightarrow M_{\mathfrak{p}} \rightarrow 0, \quad \text{with } M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^n$$

Let  $K = \ker(R^b \rightarrow M)$ . As  $M$  is finitely presented,  $K$  is finitely generated. Localizing at  $\mathfrak{p}$ :

$$K_{\mathfrak{p}} = \ker(R_{\mathfrak{p}}^b \rightarrow M_{\mathfrak{p}}) \cong R_{\mathfrak{p}}^{b-n}$$

Since  $K$  is finitely generated, there exists  $f \notin \mathfrak{p}$  such that  $K_f$  is free over  $R_f$  of rank  $b - n$ . Over  $R_f$ , the sequence:

$$R_f^a \rightarrow R_f^b \rightarrow M_f \rightarrow 0$$

is exact with  $\ker(R_f^b \rightarrow M_f) = K_f$  free. Thus the sequence splits, and  $M_f$  is free over  $R_f$  of rank  $n$ . Hence  $r(\mathfrak{q}) = n$  for all  $\mathfrak{q} \in D(f) = \{\mathfrak{q}' \mid f \notin \mathfrak{q}'\}$ , proving  $r$  is locally constant.

The sets  $U_n = \{\mathfrak{p} \mid r(\mathfrak{p}) = n\}$  form an open cover of  $\text{Spec}(R)$ . Since  $\text{Spec}(R)$  is quasi-compact, there exists a finite subcover corresponding to ranks  $n_1, \dots, n_k$ . For each  $\mathfrak{p} \in \text{Spec}(R)$ , we have an  $f_{\mathfrak{p}} \notin \mathfrak{p}$  such that  $M_{f_{\mathfrak{p}}}$  is free over  $R_{f_{\mathfrak{p}}}$ . The open sets  $D(f_{\mathfrak{p}})$  cover  $\text{Spec}(R)$ . By quasi-compactness, there is a finite subcover:

$$\text{Spec}(R) = D(f_1) \cup \dots \cup D(f_s)$$

for some  $f_1, \dots, f_s \in R$ . This implies  $(f_1, \dots, f_s) = R$ . By construction,  $M_{f_i}$  is free over  $R_{f_i}$  for each  $i$ , proving (b).

□

(2) *Proof.* • First: We use Question 0.1 (2) (b)  $\Rightarrow$  (a) to show that  $S/I$  is a flat  $S$ -module. For any finitely generated ideal  $J \subset S$ , suppose  $J = \langle b_1, \dots, b_m \rangle$ , where  $b_i \in S$  for all  $i$ . We need to show the map  $\psi : (S/I) \otimes_S J \rightarrow S/I$ , given by  $\bar{s} \otimes j \mapsto j\bar{s}$  is injective. Since  $(S/I) \otimes_S J \cong J/(IJ)$  and the corresponding map is  $\phi : J/(IJ) \rightarrow S/I$ , given by  $j + IJ \mapsto j + I$  with kernel  $I \cap J$ . Thus  $\phi$  is injective iff  $I \cap J = IJ$ . But  $IJ \subset I \cap J$ , so we only need to show  $I \cap J \subset IJ$ . For any  $x \in I \cap J$ ,  $x$  has finite support, thus there are  $s_1, \dots, s_m \in S$  such that  $x = \sum_{k=1}^m s_k b_k$ . Let  $F = \text{supp}(x)$ , then  $F$  is finite. For every  $k$ , define  $a_k \in S$  by: if  $i \in F$ , then  $\pi_i(a_k) = \pi_i(s_k)$ ; if  $i \notin F$ , then  $\pi_i(a_k) = 0$ . Since  $F$  is finite, so  $a_k$  has finite support, thus  $a_k \in I$ . Let  $y = \sum_{k=1}^m a_k b_k$ , then  $y \in IJ$ . Consider  $x - y = \sum_{k=1}^m (s_k - a_k) b_k$ . For any  $i \in F$ , we have  $\pi_i(s_k - a_k) = 0$ . Therefore, if  $i \in F$ , then  $\pi_i(x) = \sum_k \pi_i(s_k) \pi_i(b_k) = \sum_k \pi_i(a_k) \pi_i(b_k) = \pi_i(y)$ , so  $\pi_i(x - y) = 0$ ; if  $i \notin F$ , then  $\pi_i(x) = 0$  and  $\pi_i(a_k) = 0$ , thus  $\pi_i(y) = 0$ , so  $\pi_i(x - y) = 0$ . Therefore  $x - y = 0$ , and  $x = y \in IJ$ . Hence  $I \cap J \subset IJ \Rightarrow I \cap J = IJ$ , which implies  $\phi$  is injective, and so is  $\psi$ . Thus  $S/I$  is flat.

• Second: If  $S/I$  is projective, then the exact sequence  $0 \longrightarrow I \longrightarrow S \xrightarrow{\pi} S/I \longrightarrow 0$  is split. So there is a  $S$ -module homomorphism  $\sigma : S/I \rightarrow S$ , such that  $\pi \sigma = 1_{S/I}$ . Take  $a = (1, 1, 1, \dots) \in S$ , then  $\bar{a} \in S/I$ . Let  $d = \sigma(\bar{a}) - a \in S$ , since  $\pi \sigma(\bar{a}) = \bar{a}$ , so  $\sigma(\bar{a}) - a \in I$ . Thus  $d$  has finite support  $F$ . Take  $j \notin F$ , define  $e_j \in S$  by  $\pi_i(e_j) = \delta_{ij}$ , then  $e_j \in I$ , so  $\bar{e}_j = 0$  in  $S/I$ . Since  $\sigma$  is a  $S$ -map and  $e_j \bar{a} = \bar{e}_j \bar{a}$ . We have  $\pi_i(e_j a) = \pi_i(e_j) \pi_i(a)$ . Thus, if  $i = j$ , then  $\pi_i(e_j a) = 1$ , otherwise  $\pi_i(e_j a) = 0$ , i.e.  $e_j a = e_j$ , and  $\bar{e}_j \bar{a} = \bar{e}_j = 0$ . Therefore:

$$e_j \sigma(\bar{a}) = \sigma(e_j \bar{a}) = \sigma(\bar{e}_j \bar{a}) = \sigma(0) = 0.$$

In particular,  $\pi_j(\sigma(\bar{a})) = \pi_j(e_j \sigma(\bar{a})) = 0$ . Since  $j \notin F$ , so  $\pi_j(d) = 0$ . That is  $\pi_j(\sigma(\bar{a})) = \pi_j(a) = 1$ , a contradiction. Hence  $S/I$  is not projective.

□

## Question 0.3

(1) An object  $I$  of  $\mathcal{A}$  is called injective if the left exact functor  $\text{Hom}_{\mathcal{A}}(-, I)$  is exact.

An object  $P$  of  $\mathcal{A}$  is called projective if the left exact functor  $\text{Hom}_{\mathcal{A}}(P, -)$  is exact.

(2) *Proof.* • First: We show that a chain complex  $P$  is a projective object in  $\mathbf{Ch}(\mathcal{A})$  if and only if it is a split exact complex of projectives.

( $\Rightarrow$ ): Given an object  $B \in \mathcal{A}$ , let  $\rho^n(B)$  be the complex with  $B$  concentrated in degree  $n$ ; given a morphism  $f : P_n \rightarrow B$ , define a chain map  $F = (F_i) : P \rightarrow \rho^n(B)$ , where  $F_n = f$  and all other  $F_i = 0$ . Similarly, if  $g : A \rightarrow B$  is an epimorphism and  $\rho^n(A)$  is the complex with  $A$  concentrated in degree  $n$ , then there is a chain map  $G = (G_i) : \rho^n(A) \rightarrow \rho^n(B)$ , where  $G_n = g$  and all other  $G_i = 0$ . Since  $P$  is projective in  $\mathbf{Ch}(\mathcal{A})$ , there is a chain map  $H : P \rightarrow \rho^n(A)$  with  $GH = F$ . It follows that  $gh = f$ , and so  $P_n$  is projective.

$$\begin{array}{ccc} & P_n & \\ h \swarrow & \downarrow f & \searrow H \\ A & \xrightarrow{g} & B \\ & \rho^n(A) & \xrightarrow{G} \rho^n(B) \end{array}$$

Consider  $\mathbf{Cone}(1_P)_n = P_{n-1} \oplus P_n$ , and  $d_n^C : P_{n-1} \oplus P_n \rightarrow P_{n-2} \oplus P_{n-1}$ , given by  $(a, b) \mapsto (-d_{n-1}(a), d_n(b) - a)$ . There exists short exact sequence

$$0 \longrightarrow P \xrightarrow{i} \mathbf{Cone}(1_P) \xrightarrow{\phi} P[-1] \longrightarrow 0,$$

where  $i_n : P_n \rightarrow P_{n-1} \oplus P_n$ , given by  $i_n(b) = (0, b)$ ; and  $\phi_n : P_{n-1} \oplus P_n \rightarrow P[-1]_n = P_{n-1}$ , given by  $\phi_n(a, b) = -a$ . Since  $P$  is projective, it is easy to see that  $P[-1]$  is projective. Hence, there is a chain map  $r : P[-1] \rightarrow \mathbf{Cone}(1_P)$ , such that  $\phi \circ r = 1_{P[-1]}$ . That is,  $r_n : P_{n-1} \rightarrow P_{n-1} \oplus P_n$ , such that  $\phi_n r_n(a) = \phi_n(r_n^{(1)}(a), r_n^{(2)}(a)) = -r_n^{(1)}(a) = a$ . It follows that  $r_n(a) = (-a, t_n(a))$ , where  $t_n : P_{n-1} \rightarrow P_n$ . Since  $r$  is a chain map, we have  $d_n^C r_n = r_{n-1} d_n^{P[-1]}$ .

$$\begin{aligned} d_n^C r_n(a) &= d_n^C(-a, t_n(a)) = (d_{n-1}(a), d_n t_n(a) + a), \\ r_{n-1} d_n^{P[-1]}(a) &= r_{n-1}(-d_{n-1}(a)) = (d_{n-1}(a), -t_{n-1} d_{n-1}(a)). \end{aligned}$$

Thus,  $d_n t_n + t_{n-1} d_{n-1} = -1_{P_{n-1}}$ , let  $s_{n-1} = t_n : P_{n-1} \rightarrow P_n$ . Then  $d_n s_{n-1} + s_{n-2} d_{n-1} = -1_{P_{n-2}}$ . It follows that  $1_P : P \rightarrow P$  is null-homotopic, and therefore  $P$  is split exact.

( $\Leftarrow$ ): Suppose  $P$  is split exact and each  $P_n$  is projective in  $\mathcal{A}$ . Then there exists a contracting homotopy  $s : P \rightarrow P[1]$ , such that  $d_{n+1}^P s_n + s_{n-1} d_n^P = 1_{P_n}$ . We need to show that for any epi chain map  $e : A \rightarrow B$  and chain map  $f : P \rightarrow B$  in  $\mathbf{Ch}(\mathcal{A})$ , there exists chain map  $g : P \rightarrow A$  with  $eg = f$ . Since  $e_n : A_n \rightarrow B_n$  is epi and  $P_n$  is projective for each  $n$ , there exists  $k_n : P_n \rightarrow A_n$  with  $e_n k_n = f_n$ . Define  $c_n = d_n^A k_n - k_{n-1} d_n^P : P_n \rightarrow A_{n-1}$ . Since  $e$  and  $f$  are chain maps, so

$$\begin{aligned} e_{n-1} c_n &= e_{n-1} (d_n^A k_n - k_{n-1} d_n^P) \\ &= e_{n-1} d_n^A k_n - e_{n-1} k_{n-1} d_n^P \\ &= d_n^B e_n k_n - f_{n-1} d_n^P \\ &= d_n^B f_n - f_{n-1} d_n^P \\ &= 0. \end{aligned}$$

Thus  $c_n$  factor through  $\ker e_{n-1}$ , and it is easy to see that  $c = \{c_n\} : P \rightarrow \ker e[-1]$  is a chain map, so  $d_n^A c_{n+1} = -c_n d_{n+1}^P$ . Define  $g_n = k_n + c_{n+1}s_n : P_n \rightarrow A_n$ . Then  $e_n g_n = e_n(k_n + c_{n+1}s_n) = e_n k_n = f_n$ , so  $eg = f$ . Also

$$\begin{aligned} d_n^A g_n - g_{n-1} d_n^P &= d_n^A(k_n + c_{n+1}s_n) - (k_{n-1} + c_n s_{n-1})d_n^P \\ &= (d_n^A k_n - k_{n-1} d_n^P) + (d_n^A c_{n+1}s_n - c_n s_{n-1} d_n^P) \\ &= c_n + (-c_n d_{n+1}^P s_n - c_n s_{n-1} d_n^P) \\ &= c_n - c_n (d_{n+1}^P s_n + s_{n-1} d_n^P) \\ &= c_n - c_n 1_{P_n} \\ &= 0. \end{aligned}$$

Hence  $g = \{g_n\}$  is a chain map from  $P$  to  $A$  with  $eg = f$ . Therefore  $P$  is a projective object in  $\mathbf{Ch}(\mathcal{A})$ .

- Second: We prove  $(a) \iff (b)$

$(b) \Rightarrow (a)$ : Suppose  $P \cong \mathbf{Cone}(1_K)$ . Since  $\mathbf{Cone}(1_K)_n = K_{n-1} \oplus K_n$ , and each  $K_n$  is projective in  $\mathcal{A}$ , so each  $\mathbf{Cone}(1_K)_n$  is projective in  $\mathcal{A}$ . Define  $s_n : \mathbf{Cone}(1_K)_n \rightarrow \mathbf{Cone}(1_K)_{n+1}$  by  $s_n(c_{n-1}, c_n) = (-c_n, 0)$ . A routine calculation show that  $D = DsD$ , where  $D_n : \mathbf{Cone}(1_K)_n \rightarrow \mathbf{Cone}(1_K)_{n-1}$  is the differential given by  $D_n(c_{n-1}, c_n) = (0, -c_{n-1})$ , so that  $\mathbf{Cone}(1_K)$  is split. And it is easy to see that  $\ker D_n = \text{im } D_{n+1}$  for all  $n$ , hence,  $\mathbf{Cone}(1_K)$  is exact. Therefore  $P \cong \mathbf{Cone}(1_K)$  is projective in  $\mathbf{Ch}(\mathcal{A})$ .

$(a) \Rightarrow (b)$ : Suppose  $P$  is projective in  $\mathbf{Ch}(\mathcal{A})$ , then  $P$  is a split exact complex of projectives. So  $B_n = \text{im } d_{n+1}^P = \ker d_n^P$  and the sequence  $0 \rightarrow B_n \rightarrow P_n \rightarrow B_{n-1} \rightarrow 0$  is split exact. Thus  $P_n \cong B_{n-1} \oplus B_n$ , Since  $P_n$  is projective, so  $B_n$  and  $B_{n-1}$  is projective for all  $n$ . Also,  $d_n^B : B_n \rightarrow B_{n-1}$  is 0. Define chain complex  $K$  by  $K_n = B_n$ , then  $K$  is a chain complex of projective objects in  $\mathcal{A}$  with vanishing differentials. Then,  $\mathbf{Cone}(1_K)_n = K_{n-1} \oplus K_n = B_{n-1} \oplus B_n \cong P_n$ . For each  $n$ , we have an isomorphism  $\phi_n : \mathbf{Cone}(1_K)_n \rightarrow P_n$ , and all the  $\phi_n$  assemble a chain map  $\phi : (\mathbf{Cone}(1_K), D') \rightarrow (P, d^P)$ . Since each  $\phi_n$  is an isomorphism, so  $\phi$  is an isomorphism.

□

- (3) *Proof.* • First: If  $f : A \rightarrow B$  is a morphism in  $\mathcal{A}$ , then define  $\sum^k(f)$  is the complex with  $f$  concentrated in degrees  $(k, k-1)$ ; that is,  $A$  is the term of degree  $k$ ,  $B$  is the term of degree  $k-1$ , all other terms are 0, and  $f$  is the  $k$ th differential.

We show that if  $P$  is a projective object in an abelian category  $\mathcal{A}$  and  $k \in \mathbb{Z}$ , then  $\sum^k(1_P)$  is projective in  $\mathbf{Comp}(\mathcal{A})$ .

Consider the diagram in  $\mathbf{Comp}(\mathcal{A})$ . Here,  $g$  and  $g'$  are parts of an epic chain map  $\mathcal{C} \rightarrow \mathcal{C}'$ , so that each of them is epic in  $\mathcal{A}$ . Since  $P$  is projective, there is  $h : P \rightarrow C_k$  with  $gh = f$ . Define  $h' : P \rightarrow C_{k-1}$  by  $h' = dh$ . All the faces of the prism commute, with the possible exception of the triangle on the right. In particular,  $(h, h') : \sum^k(1_P) \rightarrow \mathcal{C}$  is a chain map. It remains to prove that

$g'h' = f'$ . But  $g'h' = g'dh = d'gh = d'f = f'$ .

$$\begin{array}{ccccc}
& & P & & \\
& \swarrow h & \downarrow f & \searrow 1_P & \\
C_k & \xleftarrow{g} & C'_k & \xrightarrow{h'} & P \\
\downarrow d & & \downarrow f' & & \downarrow \\
C_{k-1} & \xleftarrow{g'} & C'_{k-1} & \xrightarrow{d'} &
\end{array}$$

- Second: If an abelian category  $\mathcal{A}$  has enough projectives, then so does  $\mathbf{Comp}(\mathcal{A})$ .

Let  $\mathcal{C} = \cdots \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \longrightarrow \cdots$  be a complex in  $\mathcal{A}$ . For each  $n$ , there exists a projective  $P_n$  and an epic  $g_n : P_n \rightarrow C_n$ . Consider the following chain map  $G_n : \sum^n(1_P) \rightarrow \mathcal{C}$ :

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & P_n & \xrightarrow{1_{P_n}} & P_n \longrightarrow 0 \longrightarrow \cdots \\
& & \downarrow & & \downarrow g_n & & \downarrow d_n g_n \\
\cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \xrightarrow{d_n} & C_{n-1} \longrightarrow C_{n-2} \longrightarrow \cdots
\end{array}$$

Now  $\Sigma = \oplus_{n \in \mathbb{Z}} \sum^n(1_{P_n})$  is projective in  $\mathbf{Comp}(\mathcal{A})$ , by the first argument, and  $G = \oplus G_n : \Sigma \rightarrow \mathcal{C}$  is an epimorphism.

□

## Question 0.4

*Proof.* We show if  $\mathcal{A}$  is an abelian category, then the class  $Q$  of quasi-isomorphisms in the homotopy category  $\mathbf{K}(\mathcal{A})$  satisfies the following property: Given a quasi-isomorphism  $q \in Q$  and a morphism  $f$  in  $\mathbf{K}(\mathcal{A})$  (with same target), then there exists an object  $W$ , a morphism  $g$  and a quasi-isomorphism  $t \in Q$  such that the following diagram is commutative

$$\begin{array}{ccc}
W & \xrightarrow{g} & Z \\
t \downarrow & & \downarrow q \\
X & \xrightarrow{f} & Y
\end{array}$$

We have given a morphism  $f$  and a quasi-isomorphism  $q$ . By axiom (TR2) of triangulated category for the triangulated category  $\mathbf{K}(\mathcal{A})$ , there exists a distinguished triangle  $Z \xrightarrow{q} Y \xrightarrow{u} U \xrightarrow{v} Z[1]$ .

Similarly, considering  $uf : X \rightarrow U$ , there is a distinguished triangle  $W \xrightarrow{t} X \xrightarrow{uf} U \xrightarrow{w} W[1]$ . Applying axioms (TR4) and (TR3) we can deduce the existence of the morphism  $g$  (and  $g[1]$ ) in the following commutative diagram:

$$\begin{array}{ccccccc}
W & \xrightarrow{t} & X & \xrightarrow{uf} & U & \xrightarrow{w} & W[1] \\
g \downarrow & & \downarrow f & & \downarrow id & & \downarrow g[1] \\
Z & \xrightarrow{q} & Y & \xrightarrow{u} & U & \xrightarrow{v} & Z[1]
\end{array}$$

Since  $q$  is a quasi-isomorphism, by assumption, the long exact homology sequence applied to the bottom row yields that  $H_n(U) = 0$  for all  $n \in \mathbb{Z}$ , and then the long exact homology sequence for the top row implies that  $t$  must be a quasi-isomorphism, as desired.

In our question, we have distinguished triangles:  $K \xrightarrow{f} L \xrightarrow{u} U \xrightarrow{v} K[1]$  and  $W \xrightarrow{f'} L' \xrightarrow{ug} U \xrightarrow{w} W[1]$ . Hence we have

$$\begin{array}{ccccccc} W & \xrightarrow{f'} & L' & \xrightarrow{ug} & U & \xrightarrow{w} & W[1] \\ g' \downarrow & & \downarrow g & & \downarrow id & & \downarrow g'[1] \\ K & \xrightarrow{f} & L & \xrightarrow{u} & U & \xrightarrow{v} & K[1] \end{array}$$

Let  $W = K'$ .  $f$  is a quasi-isomorphism  $\Rightarrow H_n(U) = 0$ . Use the long exact sequence  $\Rightarrow f'$  is a quasi-isomorphism.

The class of quasi-isomorphisms does not in general satisfy the conditions of the preceding argument already in  $\mathbf{Ch}(\mathcal{A})$ ; it is crucial first to pass to the homotopy category. Hence, generally speaking, there is no such triple  $(K', f', g')$  so that  $f'$  is a quasi-isomorphism and that  $gf' = fg'$ .

□

## Question 0.7

*Proof.* Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories, we show that the nerve functor  $N$  induce a bijection:

$$\theta : \text{Fun}(\mathcal{C}, \mathcal{D}) \longrightarrow \text{Hom}(N(\mathcal{C}), N(\mathcal{D})).$$

First we have  $\theta$  is injective: a functor  $F : \mathcal{C} \longrightarrow \mathcal{D}$  is determined by its behavior on the objects and morphisms of  $\mathcal{C}$ , and therefore by the behavior of  $\theta(F)$  on the vertices and edges of simplicial set  $N(\mathcal{C})$ .

Let us prove the surjection of  $\theta$ . Let  $f : N(\mathcal{C}) \longrightarrow N(\mathcal{D})$  be a morphism of simplicial sets; we wish to show that there exists a functor  $F : \mathcal{C} \longrightarrow \mathcal{D}$  such that  $f = \theta(F)$ . For each  $n \geq 0$ , the morphism  $f$  determines a map of sets  $N_n(\mathcal{C}) \longrightarrow N_n(\mathcal{D})$ , which we also denote by  $f$ . In the case  $n = 0$ , this map carries each object  $C \in \mathcal{C}$  to an object of  $\mathcal{D}$ , which we denote by  $F(C)$ . For every pair of objects  $C, D \in \mathcal{C}$ , the map  $f$  carries each morphism  $u : C \longrightarrow D$  to a morphism  $f(u)$  in the category  $\mathcal{D}$ . Since  $f$  commutes with face operators, the morphism  $f(u)$  has source  $F(C)$  and target  $F(D)$ , and can therefore be regarded as an element of  $\text{Hom}_{\mathcal{D}}(F(C), F(D))$ ; we denote this element by  $F(u)$ . We will complete the proof by verify the following:

- (a). The preceding construction determines a functor  $F : \mathcal{C} \longrightarrow \mathcal{D}$ ,
- (b). We have an equality  $f = \theta(F)$  of maps from  $N(\mathcal{C})$  to  $N(\mathcal{D})$ .

To prove (a), we first note that the compatibility of  $f$  with degeneracy operators implies that we have  $F(id_C) = id_{F(C)}$  for each  $C \in \mathcal{C}$ . It will therefore suffice to show that for every pair of composable morphisms  $u : C \longrightarrow D$  and  $v : D \longrightarrow E$  in the category  $\mathcal{C}$ , we have  $F(v) \circ F(u) = F(v \circ u)$  as elements of the set  $\text{Hom}_{\mathcal{D}}(F(C), F(E))$ . For this, we observe that the diagram  $C \xrightarrow{u} D \xrightarrow{v} E$  can be identified with a 2-simplex  $\sigma$  of  $N(\mathcal{C})$ . Using the equality  $d_i^2(f(\sigma)) = f(d_i^2(\sigma))$  for  $i = 0, 2$ , we see that  $f(\sigma)$  corresponds to the diagram  $F(C) \xrightarrow{F(u)} F(D) \xrightarrow{F(v)} F(E)$  in  $\mathcal{D}$ . We now compute  $F(v) \circ F(u) = d_1^2(f(\sigma)) = f(d_1^2(\sigma)) = F(v \circ u)$ . This complete the proof of (a).

To prove (b), we must show that  $f(\tau) = \theta(F)(\tau)$  for each  $n$ -simplex  $\tau$  of  $N(\mathcal{C})$ . This follows by construction in the case  $n \leq 1$ , and follows in general since an  $n$ -simplex of  $N(\mathcal{D})$  is determined by its 1-dimensional faces.

□