Exercise 2.26

刘新宇
2240502058

November 2, 2025

Notation

(1) If $0 \leq k \leq n$, then any ordered set of k orthonormal vectors in $\mathbb{R}^n$ is called a $k-$frame in $\mathbb{R}^n$. The real Stiefel manifold $V_k(\mathbb{R}^n)$ is the set of all $k-$frames. The basepoint of $V_k(\mathbb{R}^n)$ is the last k vectors $e_{n-k+1}, \ldots, e_n$ of the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. A $k-$frame $v_1, \ldots, v_k$ can be regarded as an $n \times k$ matrix by taking the ith column to be the vector v_i. In this way we identify a $k-$frame with a point in $\mathbb{R}^{nk}$ and give $V_k(\mathbb{R}^n)$ the subspace topology. We note that if $l^{'} \leq l$, then there is a map $q : V_l(\mathbb{R}^n) \longrightarrow V_{l'}(\mathbb{R}^n)$ which assigns to an $l-$frame the last l' vectors. Written $V_k(\mathbb{R}^n) = V_{k,n}$.

(2) If $0 \leq k \leq n$, then the real Grassmann manifold $G_k(\mathbb{R}^n)$ consists of all $k-$dimensional subspaces of $\mathbb{R}^n$. There is a surjection $\pi : V_k(\mathbb{R}^n) \longrightarrow G_k(\mathbb{R}^n)$ defined by $\pi(v_1, \ldots, v_k) = \langle v_1, \ldots, v_k \rangle$, where $\langle v_1, \ldots, v_k \rangle$ is the subspace of $\mathbb{R}^n$ spanned by $v_1, \ldots, v_k$. We give $G_k(\mathbb{R}^n)$ the quotient topology induced by π. Written $G_k(\mathbb{R}^n) = G_{k,n}$.

(3) Let $O^{'}(k)$ be the subgroup of all matrices

$$\begin{pmatrix} I_{n-k} & 0 \\ 0 & D \end{pmatrix}$$

for $D \in O(k)$. We form the subgroup $O(n-k)O^{'}(k)$ of $O(n)$ and this subgroup consists of all matrices of the form

$$\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$$

where $C \in O(n-k)$ and $D \in O(k)$. Then $O(n-k) \subset O(n-k)O^{'}(k) \subset O(n)$.

lemma

Lemma 1. *There is a homeomorphism $\phi_{l,n} : O(n)/O(n-l) \longrightarrow V_{l,n}$ such that the following diagram commutes*

$$\begin{tikzcd} O(n)/O(n-l) \arrow[r, "\phi_{l,n}"] \arrow[d, "p^{'}"] & V_{l,n} \arrow[d, "q"] \\ O(n)/O(n-k) \arrow[r, "\phi_{k,n}"] & V_{k,n} \end{tikzcd}$$

for $k \leq l$.

Proof. Define $\phi : O(n) \longrightarrow V_{l,n}$ by $\phi(A) = Ae_{n-l+1}, \ldots, Ae_n$ for $A \in O(n)$. Thus ϕ assigns to a matrix A the $l-$frame consisting of the last l columns of A. Using Gram-Schmidt orthogonalization, we know that ϕ is surjective. If C is an $(n-l) \times (n-l)$ matrix, then multiplication of A on the right with a matrix of the form

$$\begin{pmatrix} C & 0 \\ 0 & I_l \end{pmatrix}$$

does not change the last l columns of A, and so it follows that ϕ is constant on cosets $AO(n-l)$ in $O(n)/O(n-l)$. Thus ϕ induces a continuous surjection $\phi_{l,n} : O(n)/O(n-l) \longrightarrow V_{l,n}$. If $A, B \in O(n)$ each have the same last l columns, then $A^{-1}B = A^T B$ has the form

$$\begin{pmatrix} C & 0 \\ 0 & I_l \end{pmatrix}$$

where $C^T = C^{-1}$. This shows that $\phi_{l,n}$ is injective. Since $O(n)$ is compact and $O(n)/O(n-l)$ is Hausdorff. Thus $\phi_{l,n}$ is a continuous bijection from a compact space to a Hausdorff space, hence is a homeomorphism. The commutativity of the diagram is obviously. ☐

Lemma 2. *The function $\pi : V_{k,n} \longrightarrow G_{k,n}$ is an open mapping.*

Proof. We identify elements of $V_{k,n}$ with $n \times k$ matrices of rank k, by assigning to each $v_1, \ldots, v_k \in V_{k,n}$, the matrix A whose columns are $v_1, \ldots, v_k$. If P is a nonsingular $k \times k$ matrix, then $\theta_P : V_{k,n} \longrightarrow V_{k,n}$ defined by $\theta_P(A) = AP$ is a homeomorphism since $\theta_{P^{-1}}$ is its inverse. If $x, y \in V_{k,n}$ with corresponding matrices A and B, then $\pi(x) = \pi(y)$ iff $B = AP$, for some nonsingular $k \times k$ matrix P. Hence if $U \subset V_{k,n}$, then

$$\pi^{-1}(\pi(U)) = \bigcup_P \theta_P(U),$$

for all nonsingular $k \times k$ matrices P. Therefore if U is open in $V_{k,n}$, then $\pi^{-1}(\pi(U))$ is open in $V_{k,n}$. Hence $\pi(U)$ is open in $G_{k,n}$, and so π is an open map. ☐

Exercise 2.26

We show that there is a homeomorphism $\psi_{k,n} : O(n)/(O(n-k)O^{'}(k)) \longrightarrow G_{k,n}$ such that the following diagram is commutative

$$\begin{tikzcd} O(n)/O(n-k) \arrow[r, "\phi_{k,n}"] \arrow[d, "p^{'}"] & V_{k,n} \arrow[d, "\pi"] \\ O(n)/(O(n-k)O^{'}(k)) \arrow[r, "\psi_{k,n}"] & G_{k,n} \end{tikzcd}$$

Proof. We define $\psi : O(n) \longrightarrow G_{k,n}$ by $\psi(A) = \langle Ae_{n-k+1}, \ldots, Ae_n \rangle$, for $A \in O(n)$. If $B \in O(n-k)O^{'}(k)$, then $\psi(AB) = \langle ABe_{n-k+1}, \ldots, ABe_n \rangle = \langle Ae_{n-k+1}, \ldots, Ae_n \rangle = \psi(A)$. Thus ψ is constant on cosets of $O(n-k)O^{'}(k)$. Again, using Gram-Schmidt orthogonalization, we know that ψ is surjective. And so ψ

induces a continuous surjection $\psi_{k,n} : O(n)/(O(n-k)O^{'}(k)) \longrightarrow G_{k,n}$. It follows immediately that the above diagram is commutative. To show that $\psi_{k,n}$ is injective, suppose that $\psi_{k,n}(A) = \psi_{k,n}(B) = W$, for $A, B \in O(n)$. Then $Ae_{n-k+1}, \ldots, Ae_n$ and $Be_{n-k+1}, \ldots, Be_n$ are bases for W. Let P be the change of basis matrix from the first basis to the second. Also, $\langle Ae_1, \ldots, Ae_{n-k} \rangle = \langle Be_1, \ldots, Be_{n-k} \rangle = W^{\perp}$ with bases $Ae_1, \ldots, Ae_{n-k}$ and $Be_1, \ldots, Be_{n-k}$. Let Q be the change of basis matrix from the first basis to the second one. Then

$$B = A \begin{pmatrix} Q & 0 \\ 0 & P \end{pmatrix} \quad and \quad \begin{pmatrix} Q & 0 \\ 0 & P \end{pmatrix} \in O(n-k)O^{'}(k).$$

This shows that $\psi_{k,n}$ is injective. Finally, since $\phi_{k,n}$ is a homeomorphism, and π is an open map. Suppose $U \in O(n)/(O(n-k)O^{'}(k))$ is open, then $\psi_{k,n}(U) = \pi \circ \phi_{k,n}(p^{'-1}(U))$, which is open in $G_{k,n}$. It follows that $\psi_{k,n}$ is an open map. Hence $\psi_{k,n}$ is a homeomorphism. $\square$