

Answer to homological algebra homework

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Question 0.1

(1) *Proof.* • First: Let $i : B' \rightarrow B$ and $p : B \rightarrow B''$ be R -module homomorphisms. If for every left R -module N , $0 \rightarrow \text{Hom}_R(B'', N) \xrightarrow{p^*} \text{Hom}_R(B, N) \xrightarrow{i^*} \text{Hom}_R(B', N)$ is an exact sequence, then $B' \xrightarrow{i} B \xrightarrow{p} B'' \rightarrow 0$ is an exact sequence.

p is surjective: Let $N = B''/\text{im } p$ and let $f : B'' \rightarrow B''/\text{im } p$ be the natural map, so that $f \in \text{Hom}_R(B'', N)$. Then $p^*(f) = f \circ p = 0$, so that $f = 0$, because p^* is injective. Therefore $B''/\text{im } p = 0$ and p is surjective.

$\text{im } i \subset \ker p$: $(p \circ i)^* = i^* \circ p^* = 0$. Hence, if $N = B''$ and $g = 1_{B''}$ so that $g \in \text{Hom}_R(B'', N)$, then $0 = (p \circ i)^*(g) = g \circ p \circ i = p \circ i$, and so $\text{im } i \subset \ker p$.

$\ker p \subset \text{im } i$: Choose $N = B/\text{im } i$ and let $h : B \rightarrow N$ be the natural map, so that $h \in \text{Hom}_R(B, N)$. Since $i^*(h) = h \circ i = 0$, so $h \in \ker i^* = \text{im } p^*$. Thus $\exists h' \in \text{Hom}_R(B'', N)$. Such that $h = p^*(h') = h' \circ p$. We have $\text{im } i \subset \ker p$, hence if $\text{im } i \neq \ker p$, there is an element $b \in B$ with $b \notin \text{im } i$ and $b \in \ker p$. Thus $h(b) \neq 0$ and $p(b) = 0$, but $h(b) = h' \circ p(b) = 0$, a contradiction.

• Second: Let $B' \xrightarrow{i} B \xrightarrow{p} B'' \rightarrow 0$ be an exact sequence of R -modules, C is a R -module. Since $(M \otimes_R -, \text{Hom}_R(-, C))$ is an adjoint pair, there is a natural isomorphism :

$$\tau_{M,C} : \text{Hom}_R(M \otimes_R -, C) \rightarrow \text{Hom}_R(M, \text{Hom}_R(-, C)).$$

We have following commutative diagram, which vertical maps are isomorphisms.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(M \otimes_R B'', C) & \xrightarrow{(1_M \otimes p)^*} & \text{Hom}_R(M \otimes_R B, C) & \xrightarrow{(1_M \otimes i)^*} & \text{Hom}_R(M \otimes_R B', C) \\ & & \downarrow \tau_{M,C}'' & & \downarrow \tau_{M,C} & & \downarrow \tau_{M,C}' \\ 0 & \longrightarrow & \text{Hom}_R(M, \text{Hom}_R(B'', C)) & \xrightarrow{(p^*)_*} & \text{Hom}_R(M, \text{Hom}_R(B, C)) & \xrightarrow{(i^*)_*} & \text{Hom}_R(M, \text{Hom}_R(B', C)) \end{array}$$

By the left exactness of $\text{Hom}_R(-, C)$ and $\text{Hom}_R(M, -)$, the bottom row is exact. Thus the top row is also exact. From above, we have

$$M \otimes_R B' \longrightarrow M \otimes_R B \longrightarrow M \otimes_R B'' \longrightarrow 0$$

is an exact sequence. Hence $M \otimes_R -$ is right exact.

□

(2) *Proof.* Let $i : I \hookrightarrow R$ be the inclusion; $\psi : M \otimes_R I \longrightarrow M$, given by $m \otimes a \mapsto am$; and $\phi : M \otimes_R R \longrightarrow M$, given by $m \otimes r \mapsto rm$, then ϕ is an isomorphism. Since $\psi = \phi \circ (1_M \otimes i)$, so ψ is injective iff $(1_M \otimes i)$ is injective.

★ (a) \Rightarrow (c) : If M is flat over R . Let P_\bullet be a projective resolution of N . Then the functor $M \otimes_R -$ is exact, and so the complex

$$\cdots \longrightarrow M \otimes_R P_2 \longrightarrow M \otimes_R P_1 \longrightarrow M \otimes_R P_0 \longrightarrow 0$$

is exact for all $n \geq 1$. Therefore, $\text{Tor}_n^R(M, N) = 0$, for all $n \geq 1$.

★ (c) \Rightarrow (d) : By (c), for any R -module N , $\text{Tor}_1^R(M, N) = 0$. Since R/I is an R -module, thus $\text{Tor}_1^R(M, R/I) = 0$, for any finitely generated ideal $I \subset R$.

★ (d) \Rightarrow (b) : Consider exact sequence: $0 \longrightarrow I \xrightarrow{i} R \longrightarrow R/I \longrightarrow 0$. We have exact sequence: $0 = \text{Tor}_1^R(M, R/I) \longrightarrow M \otimes_R I \xrightarrow{1 \otimes i} M \otimes_R R$. So $1 \otimes i$ is injective, hence ψ is injective.

★ (b) \Rightarrow (a) : From (b), we have for any finitely generated ideal $J \subset R$, $1_M \otimes j : M \otimes_R J \longrightarrow M \otimes_R R$ is injective.

• First: Let $0 \longrightarrow A \xrightarrow{i} B$ be an exact sequence of left R -modules, and let M be a right R -module. If $u \in \ker(1_M \otimes i)$, then there are a finitely generated submodule $N \subset M$ and an element $u' \in N \otimes_R A$ such that $u' \in \ker(1_N \otimes i)$, and $u = (\kappa \otimes 1_A)(u')$, where $\kappa : N \longrightarrow M$ is the inclusion. Let $u = \sum_j m_j \otimes a_j \in \ker(1_M \otimes i)$, where $m_j \in M$ and $a_j \in A$. There is an equation in $M \otimes_R B$,

$$0 = (1_M \otimes i)(u) = \sum_{j=1}^n m_j \otimes ia_j.$$

Let F be the free abelian group with basis $M \times B$, and let S be the subgroup of F consisting of the relations of $F/S \cong M \otimes_R B$; thus, S is generated by all elements in F of the form

$$\begin{aligned} (m, b + b') - (m, b) - (m, b'), \\ (m + m', b) - (m, b) - (m', b), \\ (mr, b) - (m, rb). \end{aligned}$$

Let $0 \longrightarrow S \longrightarrow F \xrightarrow{v} M \otimes_R B \longrightarrow 0$, where $v : (m, b) \mapsto m \otimes b$. Since $(1_M \otimes i)(u) = \sum_j m_j \otimes ia_j = 0$ in $M \otimes_R B$, we have $\sum_j m_j \otimes ia_j = \sum_k v(m'_k, b'_k) \in v(S)$, where $m'_k \in M$ and $b'_k \in B$. Define N to be the submodule of M generated by m_1, \dots, m_n together with the (finite number of) first coordinates m'_k . Of course, N is a finitely generated submodule of M . If we define $u' = \sum_j m_j \otimes a_j$ in $N \otimes_R A$, then $(\kappa \otimes 1_A)(u') = \sum \kappa(m_j) \otimes a_j = u$. Finally, $(1_N \otimes i)(u') = 0$, for we have taken care that all the relations making $(1_N \otimes i)(u') = 0$ are present in $N \otimes_R B$.

• Second: If B is a right R -module, define its character module B^* as the left R -module

$$B^* = \text{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}).$$

We show that a sequence of right R -modules $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is exact if and only if the sequence of character modules $C^* \xrightarrow{\beta^*} B^* \xrightarrow{\alpha^*} A^*$ is exact.

If the original sequence is exact, then so is the sequence of character modules, for the contravariant functor $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ is exact, because \mathbb{Q}/\mathbb{Z} is an injective \mathbb{Z} -module.

For the converse: $\text{im } \alpha \subset \ker \beta$. If $x \in A$ and $\alpha x \notin \ker \beta$, then $\beta\alpha(x) \neq 0$. there is a map $f : C \rightarrow \mathbb{Q}/\mathbb{Z}$ with $f\beta\alpha(x) \neq 0$. Thus $f \in C^*$ and $f\beta\alpha(x) \neq 0$, which contradicts the hypothesis that $\alpha^*\beta^* = 0$.

$\ker \beta \subset \text{im } \alpha$. If $y \in \ker \beta$ and $y \notin \text{im } \alpha$, then $y + \text{im } \alpha$ is a nonzero element of $B/\text{im } \alpha$. Therefore, there is a map $g : B/\text{im } \alpha \rightarrow \mathbb{Q}/\mathbb{Z}$ with $g(y + \text{im } \alpha) \neq 0$. If $v : B \rightarrow B/\text{im } \alpha$ is the natural map, define $g' = gv \in B^*$; note that $g'(y) \neq 0$. Now $g'(\text{im } \alpha) = 0$, so that $0 = g'\alpha = \alpha^*(g')$ and $g' \in \ker \alpha^* = \text{im } \beta^*$. Thus, $g' = \beta^*(h)$ for some $h \in C^*$; that is, $g'(y) = h\beta(y) = 0$, which is a contradiction.

• Third: We show that a right R -module B is flat if and only if its character module B^* is an injective left R -module.

The functors $\text{Hom}_R(-, \text{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})) = \text{Hom}_R(-, B^*)$ and $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) \circ (B \otimes_R -)$ are naturally isomorphic. If B is flat, then each of the functors in the composite is exact, for \mathbb{Q}/\mathbb{Z} is \mathbb{Z} -injective; hence, $\text{Hom}_R(-, B^*)$ is exact and B^* is injective.

Conversely, assume B^* is an injective left R -module and $A' \rightarrow A$ is an injection between left R -modules A' and A . Since $\text{Hom}_R(A, B^*) = \text{Hom}_R(A, \text{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}))$, the adjoint isomorphism gives a commutative diagram in which the vertical maps are isomorphisms.

$$\begin{array}{ccccc} \text{Hom}_R(A, B^*) & \longrightarrow & \text{Hom}_R(A', B^*) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \text{Hom}_{\mathbb{Z}}(B \otimes_R A, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \text{Hom}_{\mathbb{Z}}(B \otimes_R A', \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \\ \downarrow = & & \downarrow = & & \\ (B \otimes_R A)^* & \longrightarrow & (B \otimes_R A')^* & \longrightarrow & 0 \end{array}$$

Exactness of the top row now gives exactness of the bottom row. Hence the sequence

$$0 \longrightarrow B \otimes_R A' \longrightarrow B \otimes_R A$$

is exact, and this gives B flat.

• Finally: Consider $1_M \otimes l : M \otimes_R I \rightarrow M \otimes_R R$, where I is an ideal of R . For any $u \in \ker(1_M \otimes l)$, use the first argument to $0 \rightarrow I \hookrightarrow R$ and M . There is a finitely generated submodule $N \subset M$ and $w \in N \otimes_R I$ such that $w \in \ker(1_N \otimes l)$ and $u = (\kappa \otimes 1_I)(w)$, where $\kappa : N \hookrightarrow M$ is the inclusion. Suppose $w = \sum_{k=1}^m n_k \otimes i_k$, $n_k \in N$, $i_k \in I$. Let $J = \langle i_1, \dots, i_m \rangle \subset I$. Then $w \in N \otimes_R J$. Let $j : J \hookrightarrow R$ be the inclusion, then $(1_N \otimes j)(w) = (1_N \otimes l)(w) = 0$. Consider the following commutative diagram:

$$\begin{array}{ccc} N \otimes_R J & \xrightarrow{1_N \otimes j} & N \otimes_R R \\ \kappa \otimes 1_J \downarrow & & \downarrow \kappa \otimes 1_R \\ M \otimes_R J & \xrightarrow{1_M \otimes j} & M \otimes_R R \end{array}$$

Then $(1_M \otimes j)(\kappa \otimes 1_J)(w) = (\kappa \otimes 1_R)(1_N \otimes j)(w) = (\kappa \otimes 1_R)(0) = 0$. Since J is finitely generated, so $1_M \otimes j$ is injective, thus $(\kappa \otimes 1_J)(w) = 0$. But $u = (\kappa \otimes 1_I)(w) = (\kappa \otimes 1_I)(\sum_{k=1}^m n_k \otimes i_k) =$

$\sum_{k=1}^m \kappa(n_k) \otimes i_k = (\kappa \otimes 1_J)(w)$. Hence $u = 0$ and $1_M \otimes l$ is injective. Therefore, for any ideal I of R , the sequence $0 \longrightarrow A \otimes_R I \xrightarrow{1_M \otimes l} A \otimes_R R$ is exact. Use the second argument, we have $(A \otimes_R R)^* \longrightarrow (A \otimes_R I)^* \longrightarrow 0$ is exact, and use the third argument we have

$$\text{Hom}_R(R, A^*) \longrightarrow \text{Hom}_R(I, A^*) \longrightarrow 0$$

is exact. This says that every map from any ideal I to A^* extends to a map $R \longrightarrow A^*$. Hence A^* is injective, by the Baer Criterion and so A is flat.

□

Question 0.2

(1) *Proof.* $\star (c) \Rightarrow (a)$: Since M is projective, so $\text{Tor}_1^R(M, N) = 0$, for any R -module N . Thus M is flat. Since M is also finitely generated, suppose $M = \langle a_1, \dots, a_n \rangle$, and let F be the free R -module with basis $\{x_1, \dots, x_n\}$. Define $\phi : F \longrightarrow M$ by $\phi : x_j \mapsto a_j$, so there is an exact sequence $0 \longrightarrow \ker \phi \longrightarrow F \xrightarrow{\phi} M \longrightarrow 0$. This sequence split, because M is projective, so that $F \cong M \oplus \ker \phi$. Now $\ker \phi$ is finitely generated, for it is a direct summand, hence an image, of the finitely generated module F . Therefore, M is finitely presented.

$\star (a) \Rightarrow (c)$: • First: We show that if A is a flat right R -module and I is a left ideal, then the \mathbb{Z} -map $\theta_A : A \otimes_R I \longrightarrow AI$, given by $a \otimes i \mapsto ai$, is an isomorphism.

Let $\kappa : I \longrightarrow R$ be the inclusion, and let $\phi_A : A \otimes_R R \longrightarrow A$ be the isomorphism $a \otimes r \mapsto ar$. The composite

$$\phi_A(1_A \otimes \kappa) : A \otimes_R I \longrightarrow A \otimes_R R \longrightarrow A$$

is given by $a \otimes i \mapsto ai \in R$, and its image is AI . Now $1_A \otimes \kappa$ is an injection, because A is flat, and so composing it with the isomorphism ϕ_A is an injection. Therefore, the composite $\phi_A(1_A \otimes \kappa)$ is an injection, so that $\theta_A : a \otimes i \mapsto ai$, is an isomorphism.

• Second: Let $0 \longrightarrow K \longrightarrow F \xrightarrow{\phi} A \longrightarrow 0$ be an exact sequence of right R -module in which F is flat. Then A is a flat module if and only if $K \cap FI = KI$ for every finitely generated left ideal I .

We give a preliminary discussion before proving this. For every left ideal I , right exactness of $\square \otimes_R I$ gives exactness of

$$K \otimes_R I \longrightarrow F \otimes_R I \xrightarrow{\varphi \otimes 1} A \otimes_R I \longrightarrow 0.$$

By the first argument, there is an isomorphism $\theta_F : F \otimes_R I \rightarrow FI$ with $f \otimes i \mapsto fi$; of course, $\theta_K : K \otimes_R I \rightarrow KI$ is a surjection. The following diagram commutes, where inc is the inclusion and nat is the natural map.

$$\begin{array}{ccccccc} K \otimes_R I & \longrightarrow & F \otimes_R I & \xrightarrow{\varphi \otimes 1} & A \otimes_R I & \longrightarrow & 0 \\ \downarrow \theta_K & & \downarrow \theta_F & & \downarrow \gamma & & \\ KI & \xrightarrow{\text{inc}} & FI & \xrightarrow{\text{nat}} & FI/KI & \longrightarrow & 0 \end{array}$$

It is easy to see that there exists a map $\gamma : A \otimes_R I \rightarrow FI/KI$, given by $\varphi f \otimes i \mapsto fi + KI$, where $f \in F$ and $i \in I$; since θ_K is a surjection and θ_F is an isomorphism, the map γ is an isomorphism. Now

$$\varphi(FI) = \left\{ \varphi \left(\sum_j f_j i_j \right) : f_j \in F, i_j \in I \right\} = \left\{ \sum_j (\varphi f_j) i_j \right\} = AI.$$

Therefore, the first isomorphism theorem provides an isomorphism

$$\delta : FI/(FI \cap K) \rightarrow \varphi(FI) = AI,$$

namely, $fi + (FI \cap K) \mapsto \varphi(fi)$. We assemble these maps to obtain the composite σ :

$$FI/KI \xrightarrow{\gamma^{-1}} A \otimes_R I \xrightarrow{\theta_A} AI \xrightarrow{\delta^{-1}} FI/(FI \cap K).$$

Explicitly, $\sigma : fi + KI \mapsto fi + (FI \cap K)$. But $KI \subseteq FI \cap K$, so that σ is the enlargement of coset map of the third isomorphism theorem and, hence, $\ker \sigma = (FI \cap K)/KI$. Therefore, σ is an isomorphism if and only if $KI = FI \cap K$. Moreover, since the flanking maps γ^{-1} and δ^{-1} are isomorphisms, σ is an isomorphism if and only if θ_A is an isomorphism.

If A is flat, then the first argument says that θ_A is an isomorphism. Therefore, σ is an isomorphism and $KI = FI \cap K$. Conversely, if $KI = FI \cap K$ for every finitely generated left ideal I , then θ_A is an isomorphism, and thus A is flat.

• Third: Let $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ be an exact sequence of right R -modules, where F is free with basis $\{x_j : j \in J\}$. For each $v \in F$, define $I(v)$ to be the left ideal in R generated by the “coordinates” $r_1, \dots, r_t \in R$ of v , where $v = x_{j_1} r_1 + \dots + x_{j_t} r_t$. Then A is flat if and only if $v \in KI(v)$ for every $v \in K$.

If A is flat and $v \in K$, then $v \in K \cap FI(v) = KI(v)$, by the second argument.

Conversely, let I be any left ideal, and let $v \in K \cap FI$. Then $I(v) \subset I$, so the hypothesis gives $v \in KI(v) \subset KI$. Hence, $K \cap FI \subset KI$. As the reverse inclusion always holds, the second argument says that A is flat.

• Fourth: Let $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ be an exact sequence of right R -modules, where F is free. The following statements are equivalent.

- (i) A is flat.
- (ii) For every $v \in K$, there is an R -map $\theta : F \rightarrow K$ with $\theta(v) = v$.
- (iii) For every $v_1, \dots, v_n \in K$, there is an R -map $\theta : F \rightarrow K$ with $\theta(v_i) = v_i$ for all i .

(i) \Rightarrow (ii): Assume that A is flat. Choose a basis $\{x_j : j \in J\}$ of F . If $v \in K$, then $I(v)$ is the left ideal generated by r_1, \dots, r_l , where $v = x_{j_1} r_1 + \dots + x_{j_l} r_l$. By the third argument, $v \in KI(v)$, and so $v = \sum k_p s_p$, where $k_p \in K$ and $s_p \in I(v)$. Hence, $s_p = \sum u_{pi} r_i$, where $u_{pi} \in R$. Rewrite: $v = \sum k'_i r_i$, where $k'_i = \sum k_p u_{pi} \in K$, and define $\theta : F \rightarrow K$ by $\theta(x_{j_i}) = k'_i$ and $\theta(x_j) = 0$ for all other basis elements x_j . Clearly, $\theta(v) = v$.

(ii) \Rightarrow (i): Let $v \in K$, and let $\theta : F \rightarrow K$ be a map with $\theta(v) = v$. Choose a basis $\{x_j : j \in J\}$ of F , and write $v = x_{j_1} r_1 + \dots + x_{j_l} r_l$. Then $v = \theta(v) = \theta(x_{j_1}) r_1 + \dots + \theta(x_{j_l}) r_l \in KI(v)$. Hence, A is flat, by the third argument.

Since (iii) obviously implies (ii), it only remains to prove (ii) \Rightarrow (iii). The proof is by induction on n . The base step is our hypothesis (ii). Let $v_1, \dots, v_n \in K$, where $n \geq 2$. There is a map $\theta_n : F \rightarrow K$ with $\theta_n(v_n) = v_n$. By induction, there is a map $\theta' : F \rightarrow K$ with $\theta'[v_i - \theta_n(v_i)] = v_i - \theta_n(v_i)$ for all $i = 1, \dots, n-1$. Now define $\theta : F \rightarrow K$ by

$$\theta(u) = \theta_n(u) + \theta'[u - \theta_n(u)]$$

for all $u \in F$. It is routine to see that $\theta(v_i) = v_i$ for all i .

• Finally: If M is flat and of finite presentation, we show that M is projective.

Let $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ be an exact sequence of R -modules, where K, F are finitely generated and F is free. If $K = \langle v_1, \dots, v_n \rangle$, then the fourth argument gives $\theta : F \longrightarrow K$, with $\theta(v_i) = v_i$ for all i (because M is flat). Therefore, K is a retraction of F , and hence it is a direct summand: $F \cong K \oplus M$. Therefore, M is projective.

★ (b) \Rightarrow (c) : Assume (b) holds: There exist $f_1, \dots, f_s \in R$ with $(f_1, \dots, f_s) = R$ such that $M_{f_i} = M \otimes_R R_{f_i}$ is a free R_{f_i} -module for each i . Since M is finitely presented (hence finitely generated), it suffices to show that $M_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$ -module for every maximal ideal $\mathfrak{m} \subset R$.

Fix a maximal ideal $\mathfrak{m} \subset R$. As $(f_1, \dots, f_s) = R$, there exists some $f_i \notin \mathfrak{m}$ (otherwise \mathfrak{m} would contain all f_i and thus 1, a contradiction). For this f_i , we have:

$$M_{f_i} \cong R_{f_i}^{k_i} \quad (\text{free module over } R_{f_i})$$

Localize further at \mathfrak{m} . Since $f_i \notin \mathfrak{m}$, the ideal $\mathfrak{m}R_{f_i}$ is prime in R_{f_i} and:

$$(M_{f_i})_{\mathfrak{m}R_{f_i}} \cong M \otimes_R (R_{f_i})_{\mathfrak{m}R_{f_i}} \cong M \otimes_R R_{\mathfrak{m}} = M_{\mathfrak{m}}$$

As M_{f_i} is free over R_{f_i} , its localization $(M_{f_i})_{\mathfrak{m}R_{f_i}}$ is free over $(R_{f_i})_{\mathfrak{m}R_{f_i}} \cong R_{\mathfrak{m}}$. Thus $M_{\mathfrak{m}}$ is free over $R_{\mathfrak{m}}$.

Since $M_{\mathfrak{m}}$ is free (hence projective) over $R_{\mathfrak{m}}$ for every maximal ideal \mathfrak{m} and M is finitely generated, M is projective over R .

★ (c) \Rightarrow (b) : Assume (c) holds: M is projective. We construct elements $f_1, \dots, f_s \in R$ generating the unit ideal such that M_{f_i} is free over R_{f_i} for each i .

Since M is finitely presented and projective, for any prime ideal $\mathfrak{p} \subset R$, the localization $M_{\mathfrak{p}}$ is projective over $R_{\mathfrak{p}}$. As $R_{\mathfrak{p}}$ is local, a finitely generated projective module over a local ring is free. Thus $M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$.

Define the rank function:

$$r : \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}, \quad \mathfrak{p} \mapsto \text{rank}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$$

We show r is locally constant. Fix $\mathfrak{p} \in \text{Spec}(R)$ and let $r(\mathfrak{p}) = n$. Since M is finitely presented, there is an exact sequence:

$$R^a \rightarrow R^b \rightarrow M \rightarrow 0$$

Localizing at \mathfrak{p} gives:

$$R_{\mathfrak{p}}^a \rightarrow R_{\mathfrak{p}}^b \rightarrow M_{\mathfrak{p}} \rightarrow 0, \quad \text{with } M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^n$$

Let $K = \ker(R^b \rightarrow M)$. As M is finitely presented, K is finitely generated. Localizing at \mathfrak{p} :

$$K_{\mathfrak{p}} = \ker(R_{\mathfrak{p}}^b \rightarrow M_{\mathfrak{p}}) \cong R_{\mathfrak{p}}^{b-n}$$

Since K is finitely generated, there exists $f \notin \mathfrak{p}$ such that K_f is free over R_f of rank $b - n$. Over R_f , the sequence:

$$R_f^a \rightarrow R_f^b \rightarrow M_f \rightarrow 0$$

is exact with $\ker(R_f^b \rightarrow M_f) = K_f$ free. Thus the sequence splits, and M_f is free over R_f of rank n . Hence $r(\mathfrak{q}) = n$ for all $\mathfrak{q} \in D(f) = \{\mathfrak{q}' \mid f \notin \mathfrak{q}'\}$, proving r is locally constant.

The sets $U_n = \{\mathfrak{p} \mid r(\mathfrak{p}) = n\}$ form an open cover of $\text{Spec}(R)$. Since $\text{Spec}(R)$ is quasi-compact, there exists a finite subcover corresponding to ranks n_1, \dots, n_k . For each $\mathfrak{p} \in \text{Spec}(R)$, we have an $f_{\mathfrak{p}} \notin \mathfrak{p}$ such that $M_{f_{\mathfrak{p}}}$ is free over $R_{f_{\mathfrak{p}}}$. The open sets $D(f_{\mathfrak{p}})$ cover $\text{Spec}(R)$. By quasi-compactness, there is a finite subcover:

$$\text{Spec}(R) = D(f_1) \cup \dots \cup D(f_s)$$

for some $f_1, \dots, f_s \in R$. This implies $(f_1, \dots, f_s) = R$. By construction, M_{f_i} is free over R_{f_i} for each i , proving (b). □

(2) *Proof.* • First: We use Question 0.1 (2) (b) \Rightarrow (a) to show that S/I is a flat S -module. For any finitely generated ideal $J \subset S$, suppose $J = \langle b_1, \dots, b_m \rangle$, where $b_i \in S$ for all i . We need to show the map $\psi : (S/I) \otimes_S J \rightarrow S/I$, given by $\bar{s} \otimes j \mapsto j\bar{s}$ is injective. Since $(S/I) \otimes_S J \cong J/(IJ)$ and the corresponding map is $\phi : J/(IJ) \rightarrow S/I$, given by $j + IJ \mapsto j + I$ with kernel $I \cap J$. Thus ϕ is injective iff $I \cap J = IJ$. But $IJ \subset I \cap J$, so we only need to show $I \cap J \subset IJ$. For any $x \in I \cap J$, x has finite support, thus there are $s_1, \dots, s_m \in S$ such that $x = \sum_{k=1}^m s_k b_k$. Let $F = \text{supp}(x)$, then F is finite. For every k , define $a_k \in S$ by: if $i \in F$, then $\pi_i(a_k) = \pi_i(s_k)$; if $i \notin F$, then $\pi_i(a_k) = 0$. Since F is finite, so a_k has finite support, thus $a_k \in I$. Let $y = \sum_{k=1}^m a_k b_k$, then $y \in IJ$. Consider $x - y = \sum_{k=1}^m (s_k - a_k) b_k$. For any $i \in F$, we have $\pi_i(s_k - a_k) = 0$. Therefore, if $i \in F$, then $\pi_i(x) = \sum_k \pi_i(s_k) \pi_i(b_k) = \sum_k \pi_i(a_k) \pi_i(b_k) = \pi_i(y)$, so $\pi_i(x - y) = 0$; if $i \notin F$, then $\pi_i(x) = 0$ and $\pi_i(a_k) = 0$, thus $\pi_i(y) = 0$, so $\pi_i(x - y) = 0$. Therefore $x - y = 0$, and $x = y \in IJ$. Hence $I \cap J \subset IJ \Rightarrow I \cap J = IJ$, which implies ϕ is injective, and so is ψ . Thus S/I is flat.

• Second: If S/I is projective, then the exact sequence $0 \rightarrow I \rightarrow S \xrightarrow{\pi} S/I \rightarrow 0$ is split. So there is a S -module homomorphism $\sigma : S/I \rightarrow S$, such that $\pi\sigma = 1_{S/I}$. Take $a = (1, 1, 1, \dots) \in S$, then $\bar{a} \in S/I$. Let $d = \sigma(\bar{a}) - a \in S$, since $\pi\sigma(\bar{a}) = \bar{a}$, so $\sigma(\bar{a}) - a \in I$. Thus d has finite support F . Take $j \notin F$, define $e_j \in S$ by $\pi_i(e_j) = \delta_{ij}$, then $e_j \in I$, so $\bar{e}_j = 0$ in S/I . Since σ is a S -map and $e_j \bar{a} = \bar{e}_j \bar{a}$. We have $\pi_i(e_j a) = \pi_i(e_j) \pi_i(a)$. Thus, if $i = j$, then $\pi_i(e_j a) = 1$, otherwise $\pi_i(e_j a) = 0$, i.e. $e_j a = e_j$, and $\bar{e}_j \bar{a} = \bar{e}_j = 0$. Therefore:

$$e_j \sigma(\bar{a}) = \sigma(e_j \bar{a}) = \sigma(\bar{e}_j \bar{a}) = \sigma(0) = 0.$$

In particular, $\pi_j(\sigma(\bar{a})) = \pi_j(e_j \sigma(\bar{a})) = 0$. Since $j \notin F$, so $\pi_j(d) = 0$. That is $\pi_j(\sigma(\bar{a})) = \pi_j(a) = 1$, a contradiction. Hence S/I is not projective. □

Question 0.3

- (1) An object I of \mathcal{A} is called injective if the left exact functor $\text{Hom}_{\mathcal{A}}(-, I)$ is exact.

An object P of \mathcal{A} is called projective if the left exact functor $\text{Hom}_{\mathcal{A}}(P, -)$ is exact.

- (2) *Proof.* • First: We show that a chain complex P is a projective object in $\mathbf{Ch}(\mathcal{A})$ if and only if it is a split exact complex of projectives.

(\Rightarrow): Given an object $B \in \mathcal{A}$, let $\rho^n(B)$ be the complex with B concentrated in degree n ; given a morphism $f : P_n \rightarrow B$, define a chain map $F = (F_i) : P \rightarrow \rho^n(B)$, where $F_n = f$ and all other $F_i = 0$. Similarly, if $g : A \rightarrow B$ is an epimorphism and $\rho^n(A)$ is the complex with A concentrated in degree n , then there is a chain map $G = (G_i) : \rho^n(A) \rightarrow \rho^n(B)$, where $G_n = g$ and all other $G_i = 0$. Since P is projective in $\mathbf{Ch}(\mathcal{A})$, there is a chain map $H : P \rightarrow \rho^n(A)$ with $GH = F$. It follows that $gh = f$, and so P_n is projective.

$$\begin{array}{ccc} & P_n & \\ \swarrow h & \downarrow f & \\ A & \xrightarrow{g} & B \end{array} \qquad \begin{array}{ccc} & P & \\ \swarrow H & \downarrow F & \\ \rho^n(A) & \xrightarrow{G} & \rho^n(B) \end{array}$$

Consider $\mathbf{Cone}(1_P)_n = P_{n-1} \oplus P_n$, and $d_n^C : P_{n-1} \oplus P_n \rightarrow P_{n-2} \oplus P_{n-1}$, given by $(a, b) \mapsto (-d_{n-1}(a), d_n(b) - a)$. There exists short exact sequence

$$0 \longrightarrow P \xrightarrow{i} \mathbf{Cone}(1_P) \xrightarrow{\phi} P[-1] \longrightarrow 0,$$

where $i_n : P_n \rightarrow P_{n-1} \oplus P_n$, given by $i_n(b) = (0, b)$; and $\phi_n : P_{n-1} \oplus P_n \rightarrow P[-1]_n = P_{n-1}$, given by $\phi_n(a, b) = -a$. Since P is projective, it is easy to see that $P[-1]$ is projective. Hence, there is a chain map $r : P[-1] \rightarrow \mathbf{Cone}(1_P)$, such that $\phi \circ r = 1_{P[-1]}$. That is, $r_n : P_{n-1} \rightarrow P_{n-1} \oplus P_n$, such that $\phi_n r_n(a) = \phi_n(r_n^{(1)}(a), r_n^{(2)}(a)) = -r_n^{(1)}(a) = a$. It follows that $r_n(a) = (-a, t_n(a))$, where $t_n : P_{n-1} \rightarrow P_n$. Since r is a chain map, we have $d_n^C r_n = r_{n-1} d_n^{P[-1]}$.

$$\begin{aligned} d_n^C r_n(a) &= d_n^C(-a, t_n(a)) = (d_{n-1}(a), d_n t_n(a) + a), \\ r_{n-1} d_n^{P[-1]}(a) &= r_{n-1}(-d_{n-1}(a)) = (d_{n-1}(a), -t_{n-1} d_{n-1}(a)). \end{aligned}$$

Thus, $d_n t_n + t_{n-1} d_{n-1} = -1_{P_{n-1}}$, let $s_{n-1} = t_n : P_{n-1} \rightarrow P_n$. Then $d_n s_{n-1} + s_{n-2} d_{n-1} = -1_{P_{n-1}}$. It follows that $1_P : P \rightarrow P$ is null-homotopic, and therefore P is split exact.

(\Leftarrow): Suppose P is split exact and each P_n is projective in \mathcal{A} . Then there exists a contracting homotopy $s : P \rightarrow P[1]$, such that $d_{n+1}^P s_n + s_{n-1} d_n^P = 1_{P_n}$. We need to show that for any epi chain map $e : A \rightarrow B$ and chain map $f : P \rightarrow B$ in $\mathbf{Ch}(\mathcal{A})$, there exists chain map $g : P \rightarrow A$ with $eg = f$. Since $e_n : A_n \rightarrow B_n$ is epi and P_n is projective for each n , there exists $k_n : P_n \rightarrow A_n$ with $e_n k_n = f_n$. Define $c_n = d_n^A k_n - k_{n-1} d_n^P : P_n \rightarrow A_{n-1}$. Since e and f are chain maps, so

$$\begin{aligned} e_{n-1} c_n &= e_{n-1}(d_n^A k_n - k_{n-1} d_n^P) \\ &= e_{n-1} d_n^A k_n - e_{n-1} k_{n-1} d_n^P \\ &= d_n^B e_n k_n - f_{n-1} d_n^P \\ &= d_n^B f_n - f_{n-1} d_n^P \\ &= 0. \end{aligned}$$

Thus c_n factor through $\ker e_{n-1}$, and it is easy to see that $c = \{c_n\} : P \longrightarrow \ker e[-1]$ is a chain map, so $d_n^A c_{n+1} = -c_n d_{n+1}^P$. Define $g_n = k_n + c_{n+1} s_n : P_n \longrightarrow A_n$. Then $e_n g_n = e_n(k_n + c_{n+1} s_n) = e_n k_n = f_n$, so $eg = f$. Also

$$\begin{aligned} d_n^A g_n - g_{n-1} d_n^P &= d_n^A(k_n + c_{n+1} s_n) - (k_{n-1} + c_n s_{n-1}) d_n^P \\ &= (d_n^A k_n - k_{n-1} d_n^P) + (d_n^A c_{n+1} s_n - c_n s_{n-1} d_n^P) \\ &= c_n + (-c_n d_{n+1}^P s_n - c_n s_{n-1} d_n^P) \\ &= c_n - c_n (d_{n+1}^P s_n + s_{n-1} d_n^P) \\ &= c_n - c_n 1_{P_n} \\ &= 0. \end{aligned}$$

Hence $g = \{g_n\}$ is a chain map from P to A with $eg = f$. Therefore P is a projective object in $\mathbf{Ch}(\mathcal{A})$.

• Second: We prove $(a) \iff (b)$

$(b) \Rightarrow (a)$: Suppose $P \cong \mathbf{Cone}(1_K)$. Since $\mathbf{Cone}(1_K)_n = K_{n-1} \oplus K_n$, and each K_n is projective in \mathcal{A} , so each $\mathbf{Cone}(1_K)_n$ is projective in \mathcal{A} . Define $s_n : \mathbf{Cone}(1_K)_n \longrightarrow \mathbf{Cone}(1_K)_{n+1}$ by $s_n(c_{n-1}, c_n) = (-c_n, 0)$. A routine calculation show that $D = DsD$, where $D_n : \mathbf{Cone}(1_K)_n \longrightarrow \mathbf{Cone}(1_K)_{n-1}$ is the differential given by $D_n(c_{n-1}, c_n) = (0, -c_{n-1})$, so that $\mathbf{Cone}(1_K)$ is split. And it is easy to see that $\ker D_n = \text{im } D_{n+1}$ for all n , hence, $\mathbf{Cone}(1_K)$ is exact. Therefore $P \cong \mathbf{Cone}(1_K)$ is projective in $\mathbf{Ch}(\mathcal{A})$.

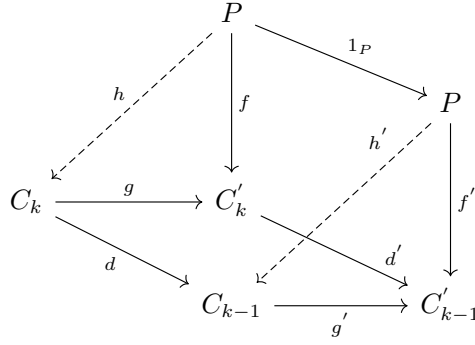
$(a) \Rightarrow (b)$: Suppose P is projective in $\mathbf{Ch}(\mathcal{A})$, then P is a split exact complex of projectives. So $B_n = \text{im } d_{n+1}^P = \ker d_n^P$ and the sequence $0 \longrightarrow B_n \longrightarrow P_n \longrightarrow B_{n-1} \longrightarrow 0$ is split exact. Thus $P_n \cong B_{n-1} \oplus B_n$. Since P_n is projective, so B_n and B_{n-1} is projective for all n . Also, $d_n^B : B_n \longrightarrow B_{n-1}$ is 0. Define chain complex K by $K_n = B_n$, then K is a chain complex of projective objects in \mathcal{A} with vanishing differentials. Then, $\mathbf{Cone}(1_K)_n = K_{n-1} \oplus K_n = B_{n-1} \oplus B_n \cong P_n$. For each n , we have an isomorphism $\phi_n : \mathbf{Cone}(1_K)_n \longrightarrow P_n$, and all the ϕ_n assemble a chain map $\phi : (\mathbf{Cone}(1_K), D') \longrightarrow (P, d^P)$. Since each ϕ_n is an isomorphism, so ϕ is an isomorphism. \square

(3) *Proof.* • First: If $f : A \longrightarrow B$ is a morphism in \mathcal{A} , then define $\sum^k(f)$ is the complex with f concentrated in degrees $(k, k-1)$; that is, A is the term of degree k , B is the term of degree $k-1$, all other terms are 0, and f is the k th differential.

We show that if P is a projective object in an abelian category \mathcal{A} and $k \in \mathbb{Z}$, then $\sum^k(1_P)$ is projective in $\mathbf{Comp}(\mathcal{A})$.

Consider the diagram in $\mathbf{Comp}(\mathcal{A})$. Here, g and g' are parts of an epic chain map $\mathcal{C} \longrightarrow \mathcal{C}'$, so that each of them is epic in \mathcal{A} . Since P is projective, there is $h : P \longrightarrow C_k$ with $gh = f$. Define $h' : P \longrightarrow C_{k-1}$ by $h' = dh$. All the faces of the prism commute, with the possible exception of the triangle on the right. In particular, $(h, h') : \sum^k(1_P) \longrightarrow \mathcal{C}$ is a chain map. It remains to prove that

$g'h' = f'$. But $g'h' = g'dh = d'gh = d'f = f'$.



- Second: If an abelian category \mathcal{A} has enough projectives, then so does $\mathbf{Comp}(\mathcal{A})$.

Let $\mathcal{C} = \cdots \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \longrightarrow \cdots$ be a complex in \mathcal{A} . For each n , there exists a projective P_n and an epic $g_n : P_n \longrightarrow C_n$. Consider the following chain map $G_n : \sum^n (1_P) \longrightarrow \mathcal{C}$:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & P_n & \xrightarrow{1_{P_n}} & P_n & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & g_n \downarrow & & \downarrow d_n g_n & & \downarrow & & \\ \cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & C_{n-2} & \longrightarrow & \cdots \end{array}$$

Now $\Sigma = \oplus_{n \in \mathbb{Z}} \Sigma^n(1_{P_n})$ is projective in $\mathbf{Comp}(\mathcal{A})$, by the first argument, and $G = \oplus G_n : \Sigma \longrightarrow \mathcal{C}$ is an epimorphism.

□

Question 0.4

Proof. We show if \mathcal{A} is an abelian category, then the class Q of quasi-isomorphisms in the homotopy category $\mathbf{K}(\mathcal{A})$ satisfies the following property: Given a quasi-isomorphism $q \in Q$ and a morphism f in $\mathbf{K}(\mathcal{A})$ (with same target), then there exists an object W , a morphism g and a quasi-isomorphism $t \in Q$ such that the following diagram is commutative

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ t \downarrow & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

We have given a morphism f and a quasi-isomorphism q . By axiom (TR2) of triangulated category for the triangulated category $\mathbf{K}(\mathcal{A})$, there exists a distinguished triangle $Z \xrightarrow{q} Y \xrightarrow{u} U \xrightarrow{v} Z[1]$.

Similarly, considering $uf : X \longrightarrow U$, there is a distinguished triangle $W \xrightarrow{t} X \xrightarrow{uf} U \xrightarrow{w} W[1]$.

Applying axioms (TR4) and (TR3) we can deduce the existence of the morphism g (and $g[1]$) in the following commutative diagram:

$$\begin{array}{ccccccc} W & \xrightarrow{t} & X & \xrightarrow{uf} & U & \xrightarrow{w} & W[1] \\ g \downarrow & & \downarrow f & & \downarrow id & & \downarrow g[1] \\ Z & \xrightarrow{q} & Y & \xrightarrow{u} & U & \xrightarrow{v} & Z[1] \end{array}$$

Since q is a quasi-isomorphism, by assumption, the long exact homology sequence applied to the bottom row yields that $H_n(U) = 0$ for all $n \in \mathbb{Z}$, and then the long exact homology sequence for the top row implies that t must be a quasi-isomorphism, as desired.

In our question, we have distinguished triangles: $K \xrightarrow{f} L \xrightarrow{u} U \xrightarrow{v} K[1]$ and $W \xrightarrow{f'} L' \xrightarrow{ug} U \xrightarrow{w} W[1]$. Hence we have

$$\begin{array}{ccccccc} W & \xrightarrow{f'} & L' & \xrightarrow{ug} & U & \xrightarrow{w} & W[1] \\ g' \downarrow & & \downarrow g & & \downarrow id & & \downarrow g'[1] \\ K & \xrightarrow{f} & L & \xrightarrow{u} & U & \xrightarrow{v} & K[1] \end{array}$$

Let $W = K'$. f is a quasi-isomorphism $\Rightarrow H_n(U) = 0$. Use the long exact sequence $\Rightarrow f'$ is a quasi-isomorphism.

The class of quasi-isomorphisms does not in general satisfy the conditions of the preceding argument already in $\mathbf{Ch}(\mathcal{A})$; it is crucial first to pass to the homotopy category. Hence, generally speaking, there is no such triple (K', f', g') so that f' is a quasi-isomorphism and that $gf' = fg'$. □

Question 0.7

Proof. Let \mathcal{C} and \mathcal{D} be categories, we show that the nerve functor N induce a bijection:

$$\theta : \text{Fun}(\mathcal{C}, \mathcal{D}) \longrightarrow \text{Hom}(N(\mathcal{C}), N(\mathcal{D})).$$

First we have θ is injective: a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is determined by its behavior on the objects and morphisms of \mathcal{C} , and therefore by the behavior of $\theta(F)$ on the vertices and edges of simplicial set $N(\mathcal{C})$.

Let us prove the surjection of θ . Let $f : N(\mathcal{C}) \longrightarrow N(\mathcal{D})$ be a morphism of simplicial sets; we wish to show that there exists a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ such that $f = \theta(F)$. For each $n \geq 0$, the morphism f determines a map of sets $N_n(\mathcal{C}) \longrightarrow N_n(\mathcal{D})$, which we also denote by f . In the case $n = 0$, this map carries each object $C \in \mathcal{C}$ to an object of \mathcal{D} , which we denote by $F(C)$. For every pair of objects $C, D \in \mathcal{C}$, the map f carries each morphism $u : C \longrightarrow D$ to a morphism $f(u)$ in the category \mathcal{D} . Since f commutes with face operators, the morphism $f(u)$ has source $F(C)$ and target $F(D)$, and can therefore be regarded as an element of $\text{Hom}_{\mathcal{D}}(F(C), F(D))$; we denote this element by $F(u)$. We will complete the proof by verify the following:

- (a). The preceding construction determines a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$,
- (b). We have an equality $f = \theta(F)$ of maps from $N(\mathcal{C})$ to $N(\mathcal{D})$.

To prove (a), we first note that the compatibility of f with degeneracy operators implies that we have $F(id_C) = id_{F(C)}$ for each $C \in \mathcal{C}$. It will therefore suffice to show that for every pair of composable morphisms $u : C \longrightarrow D$ and $v : D \longrightarrow E$ in the category \mathcal{C} , we have $F(v) \circ F(u) = F(v \circ u)$ as elements of the set $\text{Hom}_{\mathcal{D}}(F(C), F(E))$. For this, we observe that the diagram $C \xrightarrow{u} D \xrightarrow{v} E$ can be identified with a 2-simplex σ of $N(\mathcal{C})$. Using the equality $d_i^2(f(\sigma)) = f(d_i^2(\sigma))$ for $i = 0, 2$, we see that $f(\sigma)$ corresponds to the diagram $F(C) \xrightarrow{F(u)} F(D) \xrightarrow{F(v)} F(E)$ in \mathcal{D} . We now compute $F(v) \circ F(u) = d_i^2(f(\sigma)) = f(d_i^2(\sigma)) = F(v \circ u)$. This complete the proof of (a).

To prove (b), we must show that $f(\tau) = \theta(F)(\tau)$ for each n -simplex τ of $N(\mathcal{C})$. This follows by construction in the case $n \leq 1$, and follows in general since an n -simplex of $N(\mathcal{D})$ is determined by its 1-dimensional faces.

□