Assignment 5

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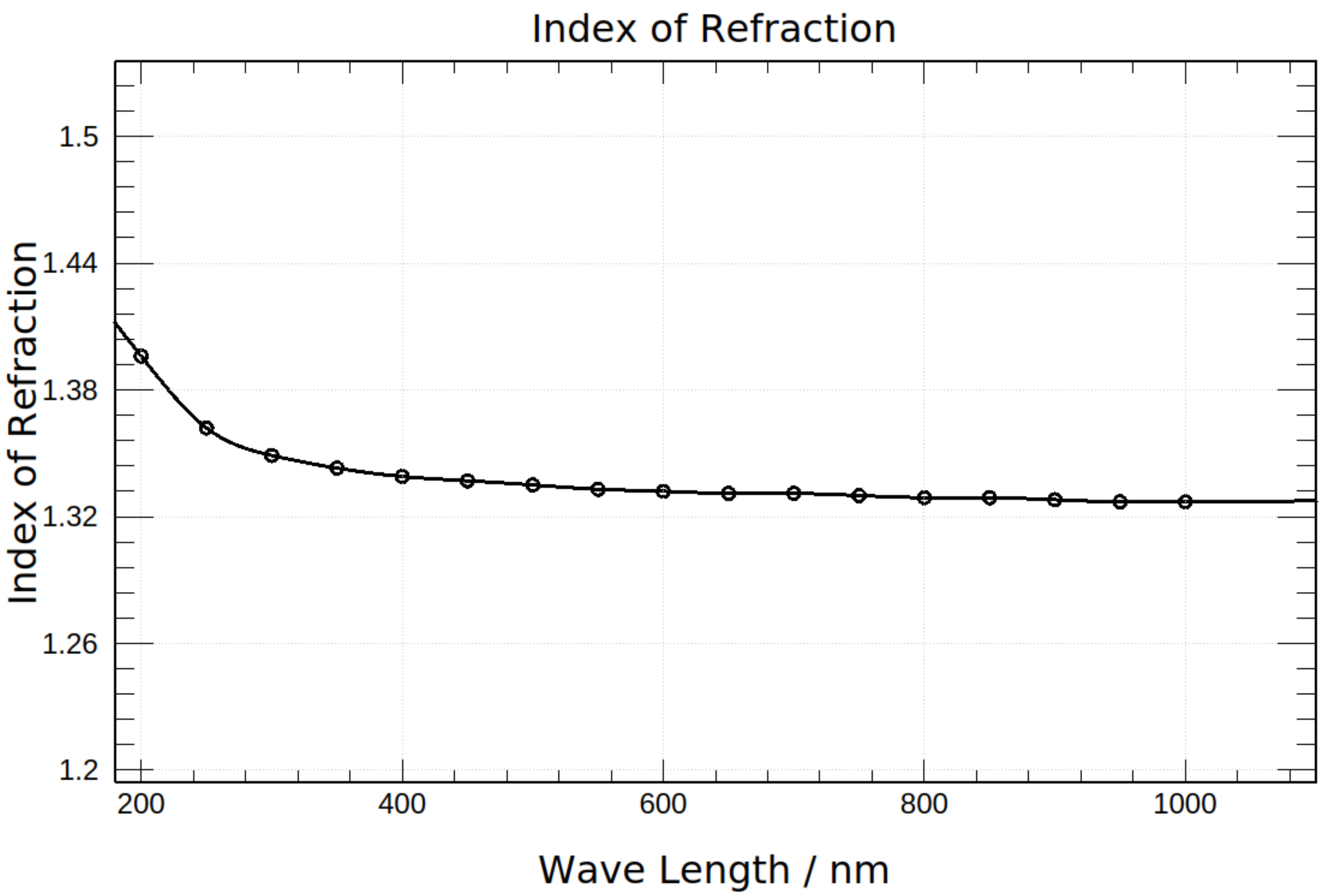
Chapter 4 Exercise 7

Cubic spline interpolation is used.

```
cat ./data.txt | ./dipersion
```



Chapter 5 Exercise 7

Use $M(\hat{A})$ to denote the matrix of operator $\hat{A}$ and $\hat{F}(\hat{A})$ to denote the function of operator $\hat{A}$. Then, we have

$$M(\hat{F}(\hat{A})) = F(M(\hat{A}))$$

So, for $\hat{H}, \hat{x}, \hat{D}$, we have

$$\begin{array}{rcl} M(\hat{x}^2) & = & M(\hat{x})^2 \\ M(\hat{D}^2) & = & M(\hat{D})^2 \\ M(\hat{H}) & = & -\frac{1}{2}M(\hat{D})^2 + \frac{1}{2}M(\hat{x})^2 \end{array}$$

Note that the basis we use the the eigen basis of $\hat{H}$. Thus, the matrix of $\hat{H}$ is diagonal. A value whose magnitude is less than 1e-12 is printed as 0.

```
$ ./operator
x
           0    0.707107           0           0           0
    0.707107           0           1           0           0
           0           1           0     1.22474           0
           0           0     1.22474           0     1.41421
           0           0           0     1.41421           0

x^2
         0.5           0    0.707107           0           0
           0         1.5           0     1.22474           0
    0.707107           0         2.5           0     1.73205
           0     1.22474           0         3.5           0
           0           0     1.73205           0         4.5

D
           0    0.707107           0           0           0
   -0.707107           0           1           0           0
           0          -1           0     1.22474           0
           0           0    -1.22474           0     1.41421
           0           0           0    -1.41421           0

D2
        -0.5           0    0.707107           0           0
           0        -1.5           0     1.22474           0
    0.707107           0        -2.5           0     1.73205
           0     1.22474           0        -3.5           0
           0           0     1.73205           0        -4.5

Hamiltonian
         0.5           0           0           0           0
           0         1.5           0           0           0
           0           0         2.5           0           0
           0           0           0         3.5           0
           0           0           0           0         4.5
```

Chapter 5 Exercise 9

Effective potential is

$$V(r) = -\alpha\frac{4\hbar c}{3r} + \frac{r}{\hbar c a^2} + \frac{l(l+1)\hbar^2}{2\mu r^2}$$

where $\mu = m_c/2$ is the reduced mass. So, the quantization rule is

$$
\int_{r_{\min}}^{r_{\max}} \sqrt{2\mu\left(E-V(r)\right.}dr=(n+\eta)\,\pi\hbar
$$

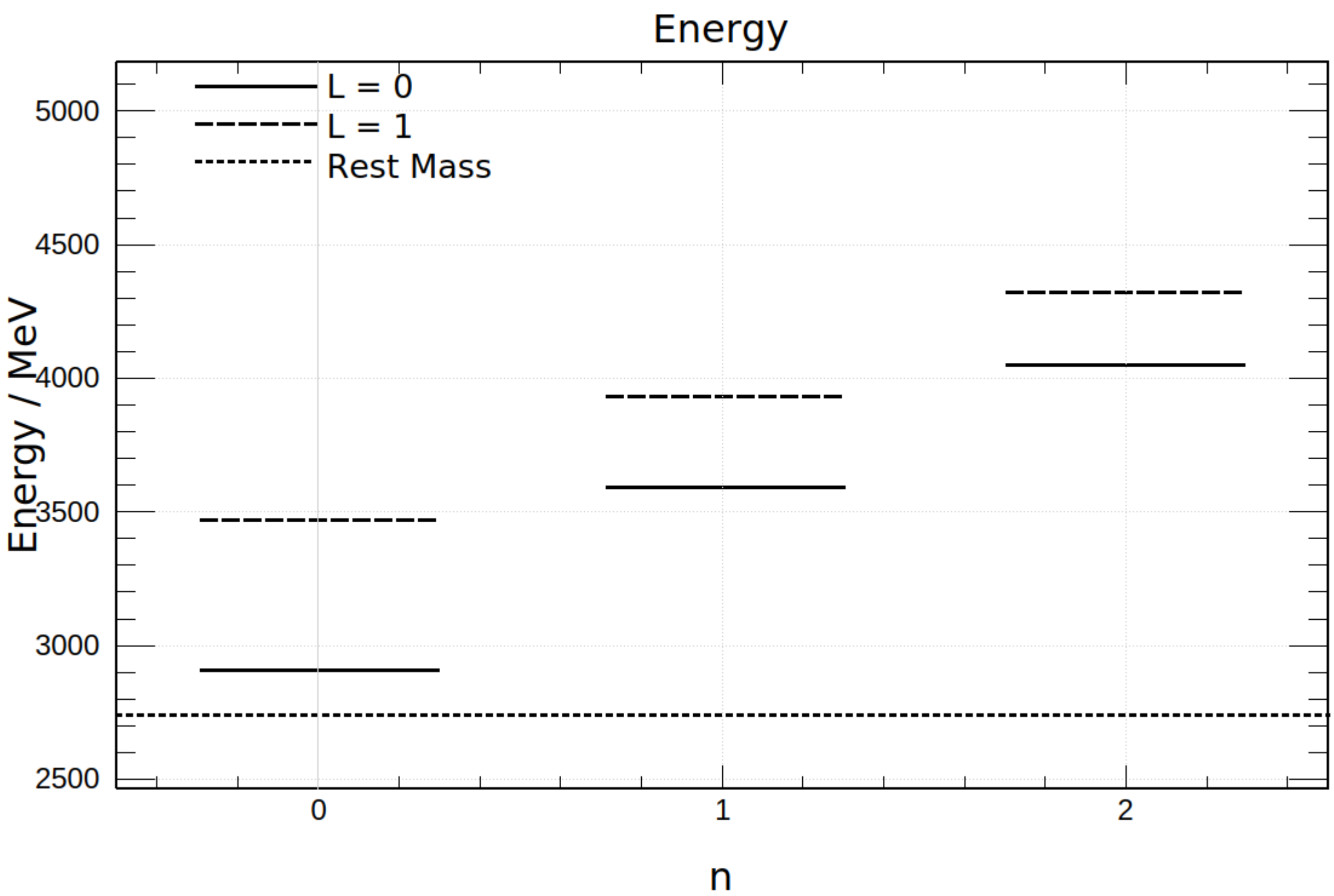
$$
\Rightarrow \int_{r_{\min}}^{r_{\max}} \sqrt{\frac{2\mu E}{\hbar^2}+\frac{8\alpha\mu c}{3\hbar r}-\frac{2\mu r}{\hbar^3 c a^2}-\frac{l(l+1)}{r^2}}dr=(n+\eta)\,\pi
$$

where $\eta=3/4$ if $l=0$, $\eta=1/3$ if $l=1$ because classically particles cannot reach the cener of the system if the angular momentum is nonzero. Choose $b=10^{-15}m$ as the length unit. The reduced position $\xi=x/a$. The quantization rule can be rewritten as

$$
\int_{\xi_{\min}}^{\xi_{\max}} \sqrt{\epsilon+\frac{\beta}{\xi}-\gamma\xi-\frac{l(l+1)}{\xi^2}}d\xi=(n+\eta)\,\pi
$$

where

$$
\begin{cases}
\epsilon=\dfrac{E}{E_0} \\
E_0=\dfrac{\hbar^2}{2\mu b^2}\approx 28.4219\ \ MeV \\
\beta=\dfrac{8\alpha\mu c b}{3\hbar}\approx 3.51768 \\
\gamma=\dfrac{2\mu b^3}{\hbar^3 c a^2}\approx 30.1959
\end{cases}
$$



Chapter 5 Exercise 12

Suppose we wait for a long enough time so that all N atoms decay.

Part a

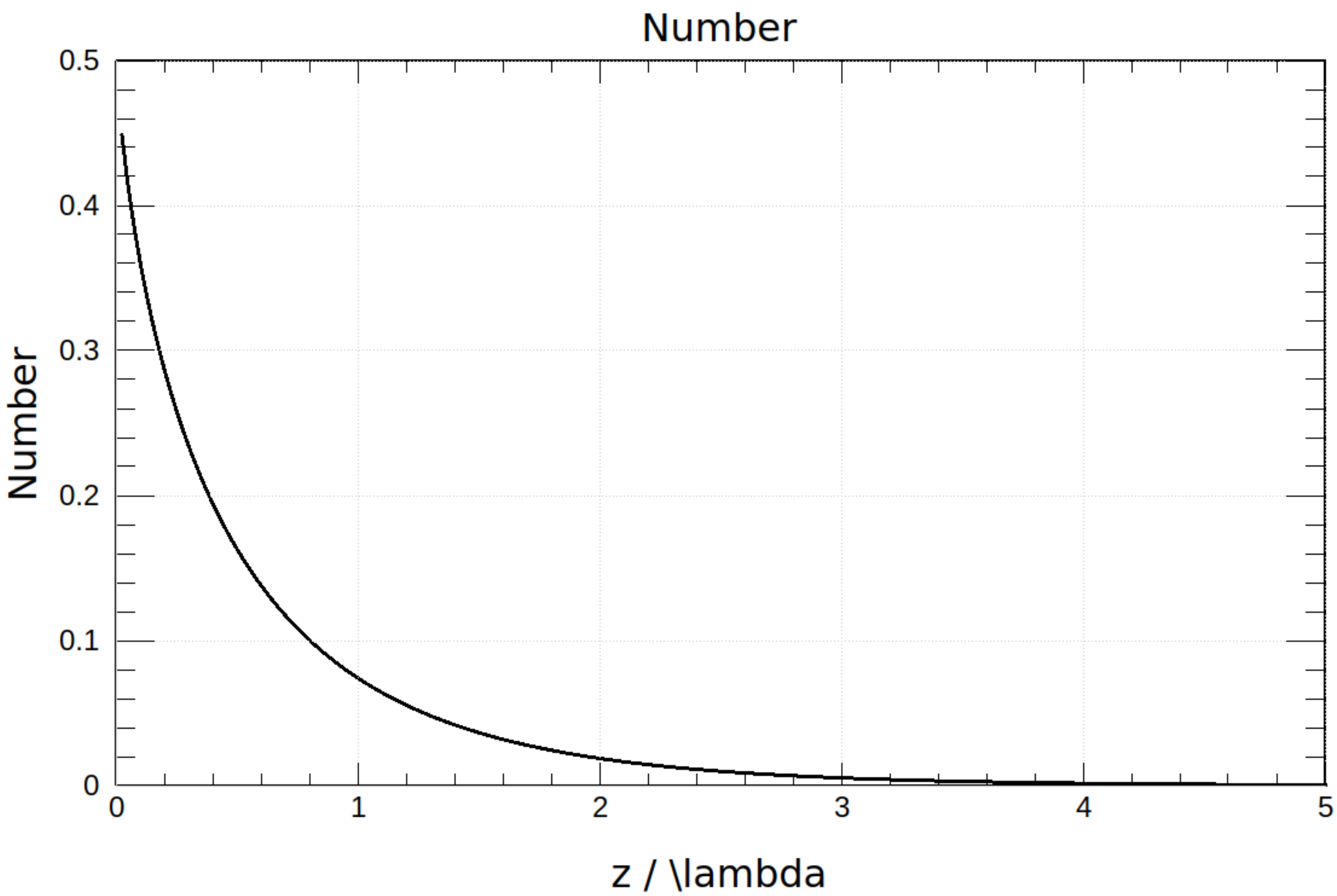
$$
\begin{aligned}
F\left(\frac{z}{\lambda}\right) &= \frac{N}{4\pi}\int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} \sin\theta d\theta \left(1 - \int_0^{\frac{z}{\cos\theta}} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} dx\right) \\
&= \frac{N}{2}\left(e^{-\frac{z}{\lambda}} - \frac{z}{\lambda}\Gamma\left(0, \frac{z}{\lambda}\right)\right)
\end{aligned}
$$

where $\Gamma(a, z)$ is the upper incomplete gamma function. Check ssymptotic behaviours of $\Gamma(\frac{z}{\lambda})$

$$
\begin{aligned}
\lim_{z \to 0} F\left(\frac{z}{\lambda}\right) &= \frac{N}{2} \\
\lim_{z \to \infty} F\left(\frac{z}{\lambda}\right) &= 0
\end{aligned}
$$

In the program, N is set to 1.

```
$ ./ex12
```



Part b

Ignore the limit of the speed of light and assume the gamma ray travels through the material instantly. This should be a resonable approximation in practice. The surface flux density is

$$
\begin{aligned}
f &= \int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left(nr^2 \sin\theta dr d\theta d\phi\right) \left(\frac{1}{\tau} e^{-\frac{t}{\tau}}\right) \left(\frac{\cos\theta}{4\pi r^2}\right) \left(1 - \int_0^r \frac{1}{\lambda} e^{-\frac{x}{\lambda}} dx\right) \\
&= \frac{\lambda n}{4} \frac{e^{-\frac{t}{\tau}}}{\tau}
\end{aligned}
$$

Part c

Define

$$
\begin{cases}
Density\ \ of\ \ Granite & \rho = 2.7g/cm^3 \\
Weight\ \ fraction\ \ of\ \ K_2O & \alpha = 4\% \\
Natural\ \ Abundance\ \ of\ \ K^{40} & \beta = 0.012\% \\
Gamma\ \ ray\ \ production\ \ efficiency & \eta = 11\% \\
Half\ \ of\ \ K^{40} & t_0 = 10^9 years \\
Mass of K_2O & m = 1.56415 \times 10^{-22} g
\end{cases}
$$

So,

$$
\tau = t_0 / \ln(2)
$$
$$
n = \frac{2\rho\alpha}{m}\beta
$$

Surface flux density

$$
f = \eta \frac{\lambda n}{4} \frac{e^{-\frac{t}{\tau}}}{\tau}
$$
$$
\approx 0.62602 \left(\frac{1}{2}\right)^{-\frac{t}{1.2\times10^9 yr}} cm^{-2} s^{-1}
$$

If we wait long enough time so that all K^{40} decay. Gamma ray surface number density is

$$
N = \int_0^\infty f dt = \frac{n\eta\lambda}{4} \approx 3.41782 \times 10^{16} \ \ cm^{-2}
$$