Assignment 2
Task 1a)

Taylor expansions of $f(x-\beta\Delta x)$ and $f(x+\beta\Delta x)$

$$f(x-\beta\Delta x) = f(x) - \beta\Delta x \cdot f'(x) + \frac{(\beta\Delta x)^2}{2} \cdot f''(x) - \frac{(\beta\Delta x)^3}{6} \cdot f'''(x) + O(\Delta x^4)$$

$$f(x+\beta\Delta x) = f(x) + \beta\Delta x \cdot f'(x) + \frac{(\beta\Delta x)^2}{2} \cdot f''(x) + \frac{(\beta\Delta x)^3}{6} \cdot f'''(x) + O(\Delta x^4)$$

Substitute into $D(x)$

$$D(x) = \frac{1}{\Delta x}\Big(\alpha_1 f(x-\beta\Delta x) + \alpha_2 f(x) + \alpha_3 f(x+\beta\Delta x)\Big)$$

$$= \frac{1}{\Delta x}\Big[\alpha_1 \Big(f(x) - \beta\Delta x \cdot f'(x) + \frac{(\beta\Delta x)^2}{2} \cdot f''(x) - \frac{(\beta\Delta x)^3}{6} \cdot f'''(x) + O(\Delta x^4)\Big)$$
$$+ \alpha_2 f(x)$$
$$+ \alpha_3 \Big(f(x) + \beta\Delta x \cdot f'(x) + \frac{(\beta\Delta x)^2}{2} \cdot f''(x) + \frac{(\beta\Delta x)^3}{6} \cdot f'''(x) + O(\Delta x^4)\Big)\Big]$$

$$= \frac{1}{\Delta x}\Big[(\alpha_1+\alpha_2+\alpha_3) f(x) + (\alpha_3-\alpha_1)\beta\Delta x\, f'(x) + (\alpha_1+\alpha_3)\frac{(\beta\Delta x)^2}{2} \cdot f''(x)$$
$$+ (\alpha_3-\alpha_1)\frac{(\beta\Delta x)^3}{6} f'''(x) + 2O(\Delta x^4)\Big]$$

$$= \frac{1}{\Delta x}(\alpha_1+\alpha_2+\alpha_3) f(x) + (\alpha_3-\alpha_1)\beta f'(x) + (\alpha_1+\alpha_3)\frac{\beta^2\Delta x}{2} f''(x) + \underbrace{(\alpha_3-\alpha_1)\frac{\beta^3\Delta x^2}{6} + O(\Delta x^3)}_{O(\Delta x^2)}$$

Rearrange:

$$D(x) - (\alpha_3-\alpha_1)\beta f'(x) = \frac{1}{\Delta x}(\alpha_1+\alpha_2+\alpha_3) f(x) + (\alpha_1+\alpha_3)\frac{\beta^2\Delta x}{2} f''(x) + O(\Delta x^2)$$

For $D(x)$ to have second order accuracy

$$D(x) - f'(x) = O(\Delta x^2)$$

so the coefficient of $f'(x)$ must be 1 and the coefficients of $f(x)$ and $f''(x)$ must be 0.

So $(\alpha_3 - \alpha_1)\beta = 1$ (1)

$\alpha_1 + \alpha_2 + \alpha_3 = 0$ (2)

$(\alpha_1 + \alpha_3)\beta^2 = 0$ (3)

Starting with (3), since (1) = 1 $\beta \neq 0$ so

$$\alpha_1 + \alpha_3 = 0$$
$$\alpha_1 = -\alpha_3$$

Substituting into (1)

$$(\alpha_3 - (-\alpha_3))\beta = 1$$

$$2\alpha_3\beta = 1$$
$$\alpha_3 = \frac{1}{2\beta}$$

so $\alpha_1 = -\alpha_3 = -\frac{1}{2\beta}$

(2) $\alpha_1 + \alpha_2 + \alpha_3 = 0$
$$\alpha_2 + 0 = 0$$
$$\alpha_2 = 0$$

Therefore the only possible values of $\alpha_1, \alpha_2, \alpha_3$ which guaruntee second order accuracy for $D(x)$ are

$$\alpha_1 = -\alpha_3 = \frac{1}{2\beta} \text{ and } \alpha_2 = 0$$