HW7

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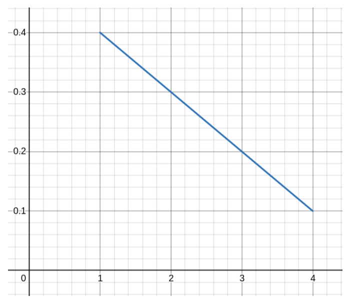
1. $F_Y(y) = \frac{1}{2} - \frac{y}{10}, 1 \leq y \leq 4$



2. (a) $\mathrm{P}(Y = 1) = \mathrm{P}(Y = 2) = \ldots = \mathrm{P}(Y = 5) = \frac{1}{5}$

   (b) $F_Y(y) = \begin{cases} y = 0 & \implies 0 \\ 1 \leq y \leq 5 & \implies \frac{y}{5} \end{cases}$

   (c) i. $\mathrm{P}(Y < 3) = 1 - \mathrm{P}(2) - \mathrm{P}(1) - \mathrm{P}(0) = \frac{2}{5}$

   ii. $\mathrm{P}(Y \leq 3) = F(3) = \frac{3}{5}$

   iii. $\mathrm{P}(Y = 3) = \frac{1}{5}$

3. $f_Y(y) = c(1 - y^2)$ for $y \in (-1, 1)$

   (a)

$$1 = \int_{-1}^{1} f_Y(y)\,\mathrm{d}y = c\int_{-1}^{1} 1 - y^2\,\mathrm{d}y = c\left[y - \frac{y^3}{3}\right]_{y=-1}^{y=1} = c \cdot \frac{4}{3} \implies c = \frac{3}{4}$$

   (b) $F_Y(y) = \frac{3}{4}(y - \frac{y^3}{3})$

4. $F_Y(y) = 1 - e^{-y^2}, y \geq 0$

   (a) $F_Y(y) = 1 - e^{-y^2} = 0.3 \implies y = \sqrt{\ln\left(\frac{10}{7}\right)}$

   $\phi_{0.3} = F_Y^{-1}(0.3) = \sqrt{\ln\left(\frac{10}{7}\right)} \approx 0.5972$

   (b) $\frac{\mathrm{d}}{\mathrm{d}y}F_y(y) = f_y(y) = 2ye^{-y^2}, y \geq 0$

   (c) $\mathrm{P}(Y \geq 2) = 1 - F_y(2) \approx 0.0183$

   (d) $\mathrm{P}(Y > 1 \mid Y > 2) = \frac{\mathrm{P}(Y>1|Y>2)}{\mathrm{P}(Y>2)} = 1$

5.

$$\int_0^1 a + by^2\,\mathrm{d}y = ay + \frac{b}{3}y^3\Big|_0^1 = 1 \implies a + \frac{b}{3} = 1$$

$$\mathrm{E}[Y] = \frac{3}{5} = \int_0^1 y(a + by^2)\,\mathrm{d}y$$

$$= \int_0^1 ay + by^3\,\mathrm{d}y = \frac{a}{2}y^2 + \frac{b}{4}y^4\Big|_0^1 \implies \frac{a}{2} + \frac{b}{4} = \frac{3}{5}$$

$$\implies a = \frac{3}{5}, b = \frac{6}{5}$$

All decimals are rounded to 4 places.

6. $f_Y(y) = ye^-y, y \geq 0$

$$\mathrm{E}[Y] = \int_0^\infty y(ye^{-y})\,\mathrm{d}y = -e^{-y}\left(y^2+2y+2\right)\Big|_0^\infty$$
$$= \lim_{y\to\infty} \frac{y^2+2y+2}{-e^y} + 2 = 0 + 2 \text{ hours}$$

7. $f_Y(y) = \frac{3}{64}y^2(4-y), 0 < y < 4$

   (a)

$$\mathrm{E}[Y] = \frac{3}{64}\int_0^4 y\cdot y^2(4-y)\,\mathrm{d}y = \frac{3}{64}(y^4 - \frac{1}{5}y^5)\Big|_0^4 = \frac{12}{5}$$
$$\mathrm{Var}(Y) = \mathrm{E}[Y^2] - \mathrm{E}[Y]^2 = \frac{3}{64}\int_0^4 y^2\cdot y^2(4-y)\,\mathrm{d}y - \left(\frac{12}{5}\right)^2$$
$$= \frac{3}{64}\left(\frac{4}{5}y^5 - \frac{1}{6}y^6\right)\Big|_0^4 - \frac{144}{25} = \frac{16}{25}$$

   (b) Let the weekly cost be $C = 200Y$.

$$\mathrm{E}[C] = 200\,\mathrm{E}[Y] = \$480$$
$$\mathrm{Var}(C) = \mathrm{E}[(200Y)^2] - (200\,\mathrm{E}[Y])^2 = 25600 \implies \sigma = \$160$$

   (c) The probability of it exceeding \$600 is given by

$$\mathrm{P}\left(C > 600\right) = \mathrm{P}\left(Y > 3\right) = 1 - \int_0^3 f_Y(y)\,\mathrm{d}y \approx 0.2617$$

   So, we should expect the operating costs to exceed \$600 about every 1 in 3 weeks. This is often enough that it could be considered a concern.

8. $f_Y(y) = \begin{cases} 1 < y \leq 10 & \implies 2y^{-3} \\ y > 10 & \implies 0 \end{cases}$

$$\mathrm{E}[Y] = \int_1^{10} y\cdot 2y^{-3}\,\mathrm{d}y + \int_{10}^\infty y\cdot 0\,\mathrm{d}y = \int_1^{10} 2y^{-2}\,\mathrm{d}y + 0 = 1.8$$

9. Let $Y \sim \mathrm{Uniform}(0, 60)$ be the time at which the passenger arrives
   $\implies f_Y(y) = \frac{1}{60}, y \in (0, 60)$ and $F_Y(y) = \frac{y}{60}, y \in (0, 60)$
   Assumptions:

   - Both trains cannot be at the station.
   - The train departs the same time it arrives.

   The chance to catch B lies in intervals: $(0, 5), (15, 20), (30, 35), (45, 50)$, which are the spaces in between the departures of A. Then, the total chance to catch B from 7:00 to 8:00 AM is

$$\mathrm{P}\left(B\right) = \sum_{y=0}^{3} F_Y(15y+5) - F_Y(15y) = \frac{1}{3}$$
$$\mathrm{P}\left(A\right) = 1 - \mathrm{P}\left(B\right) = \frac{2}{3}$$

   Thus, the passenger gets on train A two thirds of the time.

10. Let $Y \sim \mathrm{Uniform}(0, L)$ be the point chosen $\implies f_Y(y) = \frac{1}{L}, F_Y(y) = \frac{y}{L}$.
    We want to know when

$$\frac{\min(F_Y(L-\alpha), F_Y(\alpha))}{\max(F_Y(L-\alpha), F_Y(\alpha))} = \frac{\min(L-\alpha, \alpha)}{\max(L-\alpha, \alpha)} < \frac{1}{4}, 0 < \alpha < L$$

This must mean that in the case of $\alpha < L - \alpha$

$$\frac{\alpha}{L-\alpha} < \frac{1}{4} \implies \alpha < \frac{1}{5}$$

And in the other case,

$$\frac{L-\alpha}{\alpha} < \frac{1}{4} \implies \alpha > \frac{4L}{5}$$

Thus, the probability of the ratio of the short to long segment being less than $\frac{1}{4}$ is

$$1 - \mathrm{P}\left(\frac{L}{5} \leq Y \leq \frac{4L}{5}\right) = 1 - \left(F_Y\left(\frac{4L}{5}\right) - F_Y\left(\frac{L}{5}\right)\right) = \frac{2}{5}$$

11. Let $Y \sim \text{Uniform}(0, 30)$ represent the time the bus gets here $\implies f_Y(y) = \frac{1}{30}, F_Y(y) = \frac{y}{30}$.
    (a) $\mathrm{P}(Y > 10) = 1 - F_Y(10) = \frac{2}{3}$
    (b) $\mathrm{P}(Y > 25 \mid Y > 15) = \frac{\mathrm{P}(Y>25)}{Y>15} = \frac{1-F_Y(25)}{1-F_Y(15)} = \frac{1}{3}$
12. $Y \sim \text{Uniform}(\theta_1, \theta_2), \mathrm{P}(Y \leq m) = 0.5 \implies m = \phi_{0.5} = \frac{\theta_2+\theta_1}{2}$
13. $X \sim \text{Uniform}(a, 100) \implies f_X(x) = \frac{1}{100-a}, F_X(x) = \frac{x}{100-a}$
    $Y \sim \text{Uniform}(1.25a, 100) \implies f_Y(y) = \frac{1}{100-1.25a}, F_Y(y) = \frac{y}{100-1.25a}$

$$\mathrm{E}[X^2] = \frac{19600}{3} = \frac{1}{100-a}\int_a^{100} x^2 \, \mathrm{d}x = \frac{x^3}{3(100-a)}\bigg|_a^{100} = \frac{100^3 - a^3}{3(100-a)} = \frac{(100-a)(100^2 + 100a + a^2)}{3(100-a)}$$
$$19600 = 100^2 + 100a + a^2 \implies a = 60$$

Now, $F_Y(y) = \frac{y}{25} \implies F_Y^{-1}(y) = 25y$. So $\phi_{0.8} = F_Y^{-1}(0.8) = 25(0.8) = 20$