# HW9

Justin Nguyen

November 13, 2024

1. $f(y_1, y_2) = e^{-(y_1+y_2)}, 0 < y_1, y_2 < \infty \implies f(y_1, y_2) = g(y_1)h(y_2) = e^{-y_1} \cdot e^{-y_2} \implies Y_1, Y_2$ independent.
   (a) $\mathrm{P}\left(Y_1 < 1, Y_2 > 5\right) = \int_0^1 e^{-y_1}\,\mathrm{d}y_1 \cdot \int_5^\infty e^{-y_2}\,\mathrm{d}y_2 = [-e^{-y_1}]_0^1 \cdot [-e^{-y_2}]_5^\infty = (1 - e^{-1})e^{-5} \approx 0.0043$
   (b) $\mathrm{P}\left(Y_1 + Y_2 < 3\right) =$
   $$\int_0^3 \int_0^{3-y_2} e^{-(y_1+y_2)}\,\mathrm{d}y_1\,\mathrm{d}y_2 = \int_0^3 e^{-y_2}\left[\int_0^{3-y_2} e^{-y_1}\,\mathrm{d}y_1\right]\mathrm{d}y_2 = \int_0^3 e^{-y_2}\left[-e^{-y_1}\right]_0^{3-y_2}\mathrm{d}y_2 = \int_0^3 e^{-y_2} - e^{-3}\,\mathrm{d}y_2$$
   Finally, we arrive at
   $$\left[-e^{-y_2} - y_2 e^{-3}\right]_0^3 = 1 - 4e^{-3} \approx 0.8009$$
   (c) $\mathrm{P}\left(Y_1 < Y_2\right) =$
   $$\int_0^\infty \int_0^{y_2} e^{-(y_1+y_2)}\,\mathrm{d}y_1\,\mathrm{d}y_2 = 0.5$$
   I am too lazy to re-type the work, but it is similar to 1(b).
2. $f(y_1, y_2) = y_1^{-1}, 0 < y_2 < y_1 < 1$
   $$\int_0^1 \int_0^{y_1} y_1^{-1}\,\mathrm{d}y_2\,\mathrm{d}y_1 = \int_0^1 y_1^{-1} \int_0^{y_1} 1\,\mathrm{d}y_2\,\mathrm{d}y_1 = \int_0^1 y_1^{-1} y_1\,\mathrm{d}y_1 = \int_0^1 1\,\mathrm{d}y_1 = 1$$
   $\therefore f(y_1, y_2)$ is a JDF of $Y_1, Y_2$.
3. $f(y_1, y_2) = y_1 \cdot e^{-y_1 y_2} \cdot e^{-y_1}$; $y_1 > 0, y_2 > 0$
   $f(y_1 \mid y_2) = \frac{f(y_1,y_2)}{f_{y_2}(y_2)}$
   $$f_{y_2}(y_2) = \int_0^\infty y_1 e^{-y_1(y_2+1)}\,\mathrm{d}y_1 = \left[\frac{-y_1}{y_2+1}e^{-y_1(y_2+1)} + \frac{1}{y_2+1}\int e^{-y_1(y_2+1)}\,\mathrm{d}y_1\right]_0^\infty = \left[e^{-y_1(y_2+1)}\left(\frac{1 - y_1 y_2 - y_1}{(y_2+1)^2}\right)\right]_0^\infty$$
   After evaluating,
   $$f_{y_2}(y_2) = \frac{1}{(y_2+1)^2} \implies f(y_1 \mid y_2) = (y_2+1)^2 y_1 e^{-y_1(y_2+1)}$$
4. $f(y_1, y_2) = c(y_1^2 - y_2^2)e^{-y_1}$, $0 \le y_1 < \infty$, $-y_1 \le y_2 \le y_1$
   $f(y_2 \mid y_1) = \frac{f(y_1,y_2)}{f_{y_1}(y_1)}$
   $$f_{y_1}(y_1) = ce^{-y_1}\int_{-y_1}^{y_1} (y_1^2 - y_2^2)\,\mathrm{d}y_2 = ce^{-y_1}\left[y_1^2 y_2 - \frac{1}{3}y_2^3\right]_{-y_1}^{y_1} = \frac{4c}{3}y_1^3 e^{-y_1}$$
   So,
   $$f(y_2 \mid y_1) = \frac{3(y_1^2 - y_2^2)}{4y_1^3}$$

All decimals are rounded to 4 places.

5. $f(y_1,y_2)=k(1-y_2), 0\le y_1\le y_2\le 1$

   (a) Solving for k

$$k\int_0^1\int_0^{y_2} 1-y_2\,\mathrm{d}y_1\,\mathrm{d}y_2 = k\int_0^1 [(1-y_2)y_1]_0^{y_2}\,\mathrm{d}y_2 = k\int_0^1 y_2-y_2^2\,\mathrm{d}y_2 = k\left[\frac{1}{2}y_2^2-\frac{1}{3}y_2^3\right]_0^1 = \frac{k}{6} \implies k=6$$

   (b) $\mathrm{P}\left(Y_1\le 3/4, Y_2\ge 1/2\right)=$

$$6\int_{1/2}^{3/4}\int_0^{1/2} 1-y_2\,\mathrm{d}y_1\,\mathrm{d}y_2 = 6\int_{1/2}^{3/4}[(1-y_2)y_1]_0^{1/2}\,\mathrm{d}y_2 = 6/2\int_{1/2}^{3/4}1-y_2\,\mathrm{d}y_2 = 3\left[y_2-\frac{1}{2}y_2^2\right]_{1/2}^{3/4} = \frac{9}{32}$$

   (c) i. $f_{y_1}(y_1)=$

$$6\int_{y_1}^1 1-y_2\,\mathrm{d}y_2 = 6\left[y_2-\frac{1}{2}y_2^2\right]_{y_1}^1 = 3-6y_1+3y_1^2$$

   ii. $f_{y_2}(y_2)=$

$$6\int_0^{y_2} 1-y_2\,\mathrm{d}y_1 = 6\int_0^{y_2}1-y_2\,\mathrm{d}y_1 = 6y_2(1-y_2)$$

   (d) $f(y_1\mid y_2)=\frac{6(1-y_2)}{6y_2(1-y_2)}=\frac{1}{y_2}$

   (e) $f(y_2\mid y_1)=\frac{6(1-y_2)}{3(y_1-1)^2}=\frac{2(1-y_2)}{(y_1-1)^2}$

   (f) $\mathrm{P}\left(Y_2\ge 3/4\mid Y_1=1/2\right)=$

$$\int_{3/4}^1 f(y_2\mid y_1=1/2)\,\mathrm{d}y_2 = 8\int_{3/4}^1 1-y_2\,\mathrm{d}y_2 = 8\left[y_2-\frac{1}{2}y_2^2\right]_{3/4}^1 = 1/4$$

6. (a) $f(y_1,y_2)=y_1e^{-(y_1+y_2)}=y_1e^{-y_1}e^{-y_2}=g(y_1)h(y_2)\implies Y_1,Y_2$ independent.

   (b) No, $Y_1$ depends on $Y_2$ as part of its domain.

7. $f(y_1,y_2)=y_1+y_2, 0<y_1,y_2<1$

   (a) It is impossible to separate $Y_1,Y_2$'s JDF into products of separable single variable functions, therefore it is not independent.

   (b) i. $f_{y_1}(y_1)=\int_0^1 y_1+y_2\,\mathrm{d}y_2=\left[y_1y_2+\frac{1}{2}y_2^2\right]_0^1=y_1+1/2$

   ii. $f_{y_2}(y_2)=\int_0^1 y_1+y_2\,\mathrm{d}y_1=\left[\frac{1}{2}y_1^2+y_2y_1\right]_0^1=y_2+1/2$

   (c) $\int_0^1\int_0^{1-y_2} y_1+y_2\,\mathrm{d}y_1\,\mathrm{d}y_2=\frac{1}{3}$

8. $Y_1\sim\text{Exponential}(\lambda_1^{-1}), Y_2\sim\text{Exponential}(\lambda_2^{-1})\implies \mathrm{P}\left(Y_1\le Y_2\right)=\frac{\lambda_1}{\lambda_1+\lambda_2}$

*Proof.* $f_{Y_1}(y_1)=\lambda_1e^{-\lambda_1y_1}$ and $f_{Y_2}(y_2)=\lambda_2e^{-\lambda_2y_2}\implies f(y_1,y_2)=f_{Y_1}(y_1)f_{Y_2}(y_2)$.
So, $\mathrm{P}\left(Y_1<Y_2\right)$

$$=\int_0^\infty \lambda_2e^{-\lambda_2y_2}\int_0^{y_2}\lambda_1e^{-\lambda_1y_1}\,\mathrm{d}y_1\,\mathrm{d}y_2=\int_0^\infty\lambda_2e^{-\lambda_2y_2}\left[-e^{-\lambda_1y_1}\right]_0^{y_2}\mathrm{d}y_2=\int_0^\infty\lambda_2e^{-\lambda_2y_2}\left[1-e^{-\lambda_1y_2}\right]\mathrm{d}y_2$$

$$=\lambda_2\int_0^\infty e^{-\lambda_2y_2}-e^{-y_2(\lambda_2+\lambda_1)}\,\mathrm{d}y_2=\left[-e^{-\lambda_2y_2}+\frac{\lambda_2}{\lambda_1+\lambda_2}e^{-y_2(\lambda_1+\lambda_2)}\right]_0^\infty=0-\left[-1+\frac{\lambda_2}{\lambda_1+\lambda_2}\right]=1-\frac{\lambda_2}{\lambda_1+\lambda_2}$$

$$=\frac{\lambda_1}{\lambda_1+\lambda_2}$$

$\square$