HW4

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1. Given that $y \in [0,3] \implies$

$$P(Y=0) = P(Y=3) = \frac{1}{3!}$$
$$P(Y=1) = P(Y=2) = \frac{1}{3}$$

Which is valid, because

$$\sum_{y=0}^{3} P(Y=y) = \frac{1}{6} + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} = 1$$

2. (a) $y \in \{2, 3, \ldots, 12\}$

(b) $y \in \{-5, -4, \ldots, 0, \ldots 4, 5\} = [-5, 5]$

(c) Note that $\sum_{y=-5}^{5} P(Y=y) = 1$

y	$P(Y=y)$
-5	$1/36$
-4	$2/36$
-3	$3/36$
-2	$4/36$
-1	$5/36$
0	$6/36$
1	$5/36$
2	$4/36$
3	$3/36$
4	$2/36$
5	$1/36$

3. $P(Y=y) = \frac{1}{(y+1)(y+2)}$ for $y \in \mathbb{N}$

$$P(Y=1) = \frac{1}{6}$$
$$P(Y \le 4) = \frac{5}{6}$$
$$P(Y=1 \mid Y \le 4) = \frac{P(Y=1 \cap Y \le 4)}{P(Y \le 4)}$$
$$P(Y \le 4) = \sum_{y=0}^{4} P(Y=y) = P(\bigcup_{y=0}^{4} Y=y) \implies (Y=y_1) \cap (Y=y_2) = \emptyset : y_1, y_2 \in \mathbb{N}$$
$$\therefore P(Y=1 \mid Y \le 4) = \frac{P(Y=1 \cap (Y=1 \cup \ldots \cup Y=4))}{P(Y \le 4)} = \frac{P(Y=1)}{P(Y \le 4)} = \frac{1}{5}$$

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4. Let $y \in m_y = \{-4, 5, 15\}$ $P(Y=\$15) = P(J) + P(Q) = \frac{8}{52}$
$P(Y=\$5) = P(K) + P(A) = \frac{8}{52}$
$P(Y=-\$4) = \frac{36}{52}$
$\therefore E(Y) = \sum_{y \in m_y} y\, P(Y=y) = \frac{4}{13} \approx \$0.31$

5. $S = \{(R), (R^c, R), (R^c, R^c, R), (R^c, R^c, R^c)\}$; $\mathrm{P}(R) = \frac{18}{38}, \mathrm{P}(R^c) = \frac{20}{38}$

   (a) $y = \{-3, -1, 0, 1\}$

   (b) $\mathrm{P}(Y > 0) = \mathrm{P}(Y \geq 1) = \mathrm{P}(Y = 1) = \mathrm{P}(R)$

   (c) $E(Y) = \sum_y y\,\mathrm{P}(Y = y) = -\frac{651}{6859} \approx -0.095$

6. $\mathrm{P}(Y = 1) = 0, \mathrm{P}(Y = 2) = \frac{1}{6}, \mathrm{P}(Y = 3) = \frac{2}{6}, \mathrm{P}(Y = 4) = \frac{3}{6}$
   $E(Y) = \sum_y y\,\mathrm{P}(Y = y) = 0 \cdot 1 + 2 \cdot \frac{1}{6} + 3 \cdot \frac{2}{6} + 4 \cdot \frac{3}{6} = \frac{10}{3}$

7. Let $i \in \{1, 2, 3, 4\}$, $|B_1| = 40, |B_2| = 33, |B_3| = 25, |B_4| = 50$ with drivers D_1, D_2, D_3, D_4

   (a) $\mathrm{P}(X = B_i) = \frac{|B_i|}{148}$
   $\mathrm{P}(Y = B_i) = \frac{1}{4}$

   (b) i. $\mu_X = E(X) = \sum_i i\,\mathrm{P}(X = i) = \frac{5914}{148} \approx 39.28\ldots$
   ii. $\mu_Y = E(Y) = \sum_i i\,\mathrm{P}(Y = i) = 37$

   (c) i. $\mathrm{Var}(X) = E(X^2) - \mu_X^2 \approx 82.2032\ldots$
   ii. $\mathrm{Var}(Y) = E(Y^2) - \mu_Y^2 = 84.5$

8. Given $E(Y) = 1, \mathrm{Var}(Y) = E(Y^2) - E(Y)^2 = 5 \implies E(Y^2) = 6$

   (a) $E((Y + 2)^2) = E(Y^2 + 4Y + 4) = E(Y^2) + 4E(Y) + E(4) = 14$

   (b) $\mathrm{Var}(4 + 3Y) = E((3Y + 4)^2) - E(3Y + 4)^2 = 9E(Y^2) - 9E(Y)^2 = 9 \cdot 6 - 9 = 45$

9. Let Y be the random variable representing the number of recoveries, and given $\mathrm{P}(R) = 0.7$
   We can say that $Y \sim \mathrm{Binomial}(10, \mathrm{P}(R)) \implies$ Let $\mathrm{P}(Y = y) = \binom{10}{y} \mathrm{P}(R)^y\,\mathrm{P}(R^c)^{10-y}$.

   (a) $\mathrm{P}(Y = 4) \approx 0.036756909$

   (b) $\mathrm{P}(Y \geq 3) = \sum_{y=3}^{10} \mathrm{P}(Y = y) \approx 0.9999940951$

   (c) $\mathrm{P}(5 \leq Y \leq 7) = \sum_{y=5}^{7} \mathrm{P}(Y = y) \approx 0.5698682262$

   (d) $\mathrm{P}(Y \leq 8) = \sum_{y=0}^{8} \mathrm{P}(Y = y) \approx 0.8506916541$

10. Given $\mathrm{P}(B_p) = 0.8$
    Let Y be the random variable representing the number of correct bits in the encoding message M for a bit b. Because a wrong bit is the opposite bit $\bar{b}$, and the machine transmits bits one at a time, which means the bits are independent of one another, therefore we can assume

$$Y \sim \mathrm{Binomial}(5, \mathrm{P}(B_p)) \implies \mathrm{P}(Y = y) = \binom{5}{y} \mathrm{P}(B_p)^y\,\mathrm{P}(B_p^C)^{5-y}$$

    It is assumed that b was sent correctly if M contains more of b than $\bar{b}$.
    In the case of 5 digits, at least 3 or more b bits means the message was properly sent $\implies$

$$\mathrm{P}(Y \geq 3) = \sum_{y=3}^{5} \mathrm{P}(Y = y) = \frac{2944}{3125} = 0.94208$$

    Then, the probability that b was sent incorrectly is

$$\mathrm{P}((Y \geq 3)^C) = 1 - \mathrm{P}(Y \geq 3) = 0.05792 = \mathrm{P}(Y < 3)$$

11. Given $\mathrm{P}(F) = 0.2 \implies \mathrm{P}(F^c) = 0.8$,
    Let Y be the random variable of the number of operating components that can operate past 1000 hours.
    Then $Y \sim \mathrm{Binomial}(4, \mathrm{P}(F^c)) \implies \mathrm{P}(Y = y) = \binom{4}{y} \mathrm{P}(F^c)^y\,\mathrm{P}(F)^{4-y}$

    (a) $\mathrm{P}(Y = 2) = \frac{96}{625} = 0.1536$

    (b) If the system has operated for more than 1000 hours, this implies that $\mathrm{P}(Y \geq 2)$ is given.

$$\therefore \mathrm{P}(Y = 2 \mid Y \geq 2) = \frac{\mathrm{P}(Y = 2 \cap Y \geq 2)}{\mathrm{P}(Y \geq 2)} = \frac{\mathrm{P}(Y = 2)}{\mathrm{P}(Y \geq 2)} = \frac{3}{19} \approx 0.157894736842$$

12. $\mathrm{P}(-) = 2\,\mathrm{P}(+) \implies \mathrm{P}(-) = \frac{2}{3}, \mathrm{P}(+) = \frac{1}{3}$
$\mathrm{P}(p \mid +) = 0.8, \mathrm{P}(p \mid -) = 0.4$
Let Y be the random variable represent the number of passing examiners, p the event of passing an examiner, and $\mathrm{P}(P_{n\in\{3,5\}})$ be the probabilities of passing the exam for n examiners.
If it happens to be an on-day, then

$$Y_n^+ \sim \text{Binomial}(n, \mathrm{P}(p \mid +)) = \begin{cases} n = 3 \implies \mathrm{P}(Y_3^+ = y) = \binom{3}{y}\,\mathrm{P}(p \mid +)^y\,\mathrm{P}(p^c \mid +)^{3-y} \\ n = 5 \implies \mathrm{P}(Y_5^+ = y) = \binom{5}{y}\,\mathrm{P}(p \mid +)^y\,\mathrm{P}(p^c \mid +)^{5-y} \end{cases}$$

Because passing the majority of the examinations means passing, then we should examine when Y_n has a majority.

$$\mathrm{P}(Y_3^+ \geq 2) = \sum_{y=2}^{3} \mathrm{P}(Y_3^+ = y) = \frac{112}{125} = 0.896$$
$$\mathrm{P}(Y_5^+ \geq 3) = \sum_{y=3}^{5} \mathrm{P}(Y_5^+ = y) = \frac{2944}{3125} = 0.94208$$

Then, in the case of an off-day,

$$Y_n^- \sim \text{Binomial}(n, \mathrm{P}(p \mid -))$$

And, again, we examine when Y_n is the majority,

$$\mathrm{P}(Y_3^- \geq 2) = \sum_{y=2}^{3} \mathrm{P}(Y_3^- = y) = \frac{44}{125} = 0.352$$
$$\mathrm{P}(Y_5^- \geq 3) = \sum_{y=3}^{5} \mathrm{P}(Y_5^- = y) = \frac{992}{3125} = 0.31744$$

Thus, in general,

$$\mathrm{P}(P_3) = \mathrm{P}(Y_3^+ \geq 2)\,\mathrm{P}(+) + \mathrm{P}(Y_3^- \geq 2)\,\mathrm{P}(-) = \frac{8}{15} = 0.5\overline{3}$$
$$\mathrm{P}(P_5) = \mathrm{P}(Y_5^+ \geq 3)\,\mathrm{P}(+) + \mathrm{P}(Y_5^- \geq 3)\,\mathrm{P}(-) = \frac{4928}{9375} = 0.525653\overline{3}$$

$\therefore$ The student should choose 3 tests.

13. Let $Y \sim \text{Binomial}(100, \mathrm{P}(d)) \implies \mathrm{P}(Y = y) = \binom{100}{y}\,\mathrm{P}(d)^y\,\mathrm{P}(d^c)^{100-y}$

$$\mathrm{P}(Y = 3) = 2\,\mathrm{P}(Y = 2) \iff \binom{100}{3}\,\mathrm{P}(d)^3(1 - \mathrm{P}(d))^{97} = 2\binom{100}{2}\,\mathrm{P}(d)^2(1 - \mathrm{P}(d))^{98} \implies \mathrm{P}(d) = \frac{3}{52}$$

14. $Y \sim \text{Binomial}(4, 10\%) \implies E(Y) = \sum_{y=0}^{4} y\binom{4}{y}(10\%)^y(90\%)^{4-y} = \frac{2}{5}$
$E(Y^2) = \frac{13}{25} \implies E(C) = 3E(Y^2) + E(Y) + E(2) = \$3.96$

15. Let Y represent the number of heads observed $\implies Y \sim \text{Binomial}(10, p)$
$\mathrm{P}((HTT\ldots) \mid Y = 6) = \frac{\mathrm{P}((HTT\ldots) \cap Y=6)}{\mathrm{P}(Y=6)} = \frac{p^6(1-p)^4\binom{7}{2}\binom{5}{5}}{\binom{10}{6}p^6(1-p)^4} = \frac{1}{10}$

16. **Proof:** $\mathrm{P}(Y > 1 \mid Y \geq 1) = \frac{1-(1-p)^n - n(1-p)^{n-1}p}{1-(1-p)^n}$