# HW11

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1. $\{Y_1, Y_2, \ldots, Y_6\}$ i. i. d. $\sim$ Exponential$(\beta)$
   $\mathrm{P}\left(\min\{Y_1, Y_2, \ldots, Y_6\} \geq a\right) = \mathrm{P}\left(Y_1 \geq \alpha\right)\mathrm{P}\left(Y_2 \geq \alpha\right)\ldots\mathrm{P}\left(Y_6 \geq \alpha\right) = F_{Y_1}(a)^6 =$
   $$\left(\int_a^\infty \beta^{-1}\exp\left[-\beta^{-1}y\right]\mathrm{d}y\right)^6 = \exp\left(-6a/\beta\right)$$

2. $\{Y_1, Y_2, \ldots Y_n\}$ i. i. d. with pdf $f_Y(y) = e^\theta e^{-y}, y > \theta$
   $f_{Y_{(1)}} = n\left[1 - F_Y(y)\right]^{n-1} \cdot f_Y(y) =$
   $$n\left(1 - e^\theta \int_\theta^\infty e^{-y}\,\mathrm{d}y\right)e^\theta e^{-y} = n\left(1 - e^\theta\left[-e^{-y}\right]_\theta^\infty\right)e^\theta e^{-y} = n(1 - e^{2\theta})e^\theta e^{-y}$$

3. $\{Y_1, \ldots, Y_n\}$ i. i. d. $\sim$ Beta$(2, 2)$
   (a) $F_{Y_{(n)}}(y) = F_Y(y)^n =$
   $$\left(\frac{\Gamma(2+2)}{\Gamma(2)\Gamma(2)}\int y^{2-1}(1-y)^{2-1}\,\mathrm{d}y\right)^n = \left(\frac{\Gamma(4)}{\Gamma(2)^2}\int y(1-y)\,\mathrm{d}y\right)^n = \left(3y^2 - 2y^3\right)^n = y^{2n}\sum_{k=0}^n \binom{n}{k}(3)^k(-2y)^{n-k}$$
   (b) $f_{Y_{(n)}}(y) = \frac{\mathrm{d}}{\mathrm{d}y}F_{Y_{(n)}}(y) = 6n\left(y - y^2\right)\left(3y^2 - 2y^3\right)^{n-1}$

4. $Y = \{Y_1, Y_2, \ldots, Y_5\}$ i. i. d. $\sim$ Uniform$(0, 1)$
   $\mathrm{P}\left(1/4 < Y_{\phi_{0.5}} < 3/4\right) = \mathrm{P}\left(1/4 < Y_3 < \frac{3}{4}\right) = F_{Y(3)}(3/4) - F_{Y(3)}(1/4) =$
   $$\frac{5!}{(3-1)!(5-3)!}\int_{1/4}^{3/4} F_Y(y)^{3-1}\left[1 - F_Y(y)\right]^{5-3} f_Y(y)\,\mathrm{d}y = 30\int_{1/4}^{3/4} F_Y(y)^2\left[1 - F_Y(y)\right]^2 f_Y(y)\,\mathrm{d}y$$
   $$= 30\int_{1/4}^{3/4} y^2(1-y)^2 \cdot 1\,\mathrm{d}y = 30\int_{1/4}^{3/4} y^2 - 2y^3 + y^4\,\mathrm{d}y = 30\left[\frac{1}{3}y^3 - \frac{2}{4}y^4 + \frac{1}{5}y^5\right]_{1/4}^{3/4} = \frac{203}{256}$$

5.

   **Theorem.** *If* $\{Y_1, \ldots, Y_n\} \sim$ Uniform$(0, \theta = 1) \implies Y_{(k)} \sim$ Beta$(\alpha, \beta)$

   *Proof.* We first find the pdf of the $k^{\text{th}}$ order statistic. We know that
   $$f_Y(y) = 1 \text{ and } F_Y(y) = y$$
   then,
   $$f_{Y(k)}(y) = \frac{n!}{(k-1)!(n-k)!}\left[F_Y(y)\right]^{k-1}\left[1 - F_Y(y)\right]^{n-k} = \frac{n!}{(k-1)!(n-k)!}y^{k-1}(1-y)^{n-k}$$
   A random variable $Y \sim$ Beta$(\alpha, \beta)$ iff its pdf is
   $$f_Y(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}y^{\alpha-1}(1-y)^{\beta-1}$$
   Then, we must have that $\alpha = k$ and $\beta = n - k + 1 \implies Y_{(k)} \sim$ Beta$(k, n-k+1)$ $\square$

All decimals are rounded to 4 places.

6. $G_k \sim \text{Normal}(14, 2^2)$

   (a) Let $G = \{G_1, \ldots, G_{100}\}$. We want to know

$$\mathrm{P}\left(\overline{G} > 14.5\right) = \mathrm{P}\left(Z > \frac{14.5 - 14}{2/\sqrt{100}}\right) = .0062$$

   (b) $\mathrm{P}\left(g_1 < \overline{G} < g_2\right) = 0.95$
   We have that

$$\mathrm{P}\left(g_1 < \overline{G} < g_2\right) = \mathrm{P}\left(\frac{g_1 - 14}{2/\sqrt{100}} < Z < \frac{g_2 - 14}{2/\sqrt{100}}\right) = 0.95$$

   An interval corresponding to 0.95 means that at either side, we have 2.5% is excluded. Then,

$$\mathrm{P}\left(Z > z_2\right) = 0.025 \implies z_2 = 1.96 \implies \mathrm{P}\left(-1.96 < Z < 1.96\right) = 0.95$$

   So, $\frac{g_1 - 14}{2/\sqrt{100}} = -1.96 \implies g_1 = -1.96 \cdot (2/\sqrt{100}) + 14 = 13.608$, and
   $\frac{g_2 - 14}{2/\sqrt{100}} = 1.96 \implies g_2 = 1.96 \cdot (2/\sqrt{100}) + 14 = 14.392$.

7. $n = 100, \mu = \mu, \sigma = 2.5 \implies H = \{H_k : k \in [1, 100]\}$ i. i. d. $\sim \text{Normal}(\mu, 2.5^2)$
   $\mathrm{P}\left(|\overline{H} - \mu| < 0.5\right) =$

$$\mathrm{P}\left(\frac{-0.5}{\sigma/\sqrt{n}} < \frac{\overline{H} - \mu}{\sigma/\sqrt{n}} < \frac{0.5}{\sigma/\sqrt{n}}\right) = \mathrm{P}\left(-2 < Z < 2\right) = 2(\mathrm{P}\left(Z > 0\right) - \mathrm{P}\left(Z > 2\right)) = 0.9544$$

8. $C = \{C_1, \ldots, C_{36}\}$ i. i. d. $\sim \text{Normal}(19400, 6000^2)$
   $\mathrm{P}\left(\overline{C} > 20,000\right) =$

$$\mathrm{P}\left(Z > \frac{20000 - 19400}{6000/\sqrt{36}}\right) = 0.2743$$

9. $C = \{C_1, \ldots, C_{2025}\} \sim \text{Normal}(3125, 540^2)$
   $\mathrm{P}\left(c_1 < \overline{C} < c_2\right) = 0.95 =$

$$\mathrm{P}\left(\frac{c_1 - 3125}{540/\sqrt{2025}} < Z < \frac{c_2 - 3125}{540/\sqrt{2025}}\right) = 0.95 \implies \frac{c_1 - 3125}{540/\sqrt{2025}} = -1.96 \text{ and } \frac{c_2 - 3125}{540/\sqrt{2025}} = 1.96$$

   Then, $c_1 = 3148.52, c_2 = 3101.48$.

10. $\mathrm{P}\left(I\right) = 1/7$; $P = \sum_{k=1}^{612} P_k \sim \text{Binomial}(612, \mathrm{P}\left(I\right))$
    Let $np = 612\,\mathrm{P}\left(I\right)$, then $\mathrm{P}\left(90 < P < 150\right) =$

$$\mathrm{P}\left(\frac{90 - np}{\sqrt{np(1 - \mathrm{P}\left(I\right))}} < Z < \frac{150 - np}{\sqrt{np(1 - \mathrm{P}\left(I\right))}}\right) = \mathrm{P}\left(0.2970 - \frac{1}{2} < Z < 7.2281 + \frac{1}{2}\right) = 0.5804$$

11. $n = 160, p = 0.05$

    (a) $Y \sim \text{Binomial}(160, 0.05) \implies \mathrm{P}\left(Y \leq 5\right) = \sum_{y=0}^{5} \binom{160}{y} p^y (1-p)^{160-y} = 0.184224295731$

    (b) $Y \sim \text{Poisson}(160 \cdot 0.05) \implies \mathrm{P}\left(Y \leq 5\right) = \sum_{y=0}^{5} \frac{\lambda^y e^{-\lambda}}{y!} = 0.19123606208$

    (c) $\mathrm{P}\left(Y \leq 5\right) = \mathrm{P}\left(Z \leq \frac{5 - np}{\sqrt{np(1-p)}} + \frac{1}{2}\right) = 0.2810$

$$e = \begin{cases} d_{1h} := D_1(h) + K_1 h + K_2 h^2 \\ d_{2h} := D_1(2h) + 2K_1 h + 4K_2 h^2 \\ d_{3h} := D_1(3h) + 3K_1 h + 9K_2 h^2 \end{cases} \implies e = \alpha d_{1h} + \beta d_{2h} + \gamma d_{3h} \implies \begin{cases} \alpha + \beta + \gamma = 1 \\ \alpha + 2\beta + 3\gamma = 0 \\ \alpha + 4\beta + 9\gamma = 0 \end{cases}$$

$\therefore \alpha = 3, \beta = -3, \gamma = 1$