HW5

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1. $R = \{00, 0, 1, 2, \ldots, 36\}$; $P(W) = \frac{12}{38}$
   Let Y represent y trials until r wins $\implies Y \sim \text{NB}(r, P(W)) \wedge \text{P}(Y = y) = \binom{y-1}{r-1} \text{P}(W)^r \, \text{P}(W^c)^{y-r}$

   (a) In this case, $Y \sim \text{NB}(5, \text{P}(W^c))$ and $y = 5$.
   $\text{P}(Y = 5) = \binom{5-1}{5-1} \text{P}(W^c)^5 \, \text{P}(W^c)^0 = (\frac{26}{38})^5 \approx 0.149951$

   (b) $Y \sim \text{NB}(1, \text{P}(W)) \implies \text{P}(Y = 4) = \binom{4-1}{1-1} \text{P}(W^c)^3 \, \text{P}(W)^1 = (\frac{26}{38})^3 \cdot \frac{12}{38} \approx 0.10115\ldots$

2. $\text{P}(C) = 41\%$
   $Y \sim \text{Geometric}(\text{P}(C^c)) \implies \text{P}(Y = y) = \text{P}(C)^{y-1} \, \text{P}(C^c) = (41\%)^{y-1}(59\%)$ for $y = 1, 2, \ldots$
   This is geometric because we want only the first success.

3. $Y \sim \text{Geometric}(p)$

   (a)

   Theorem. $\text{P}(Y > a) = (1 - p)^a$

   Proof.

$$P(Y > a) = \sum_{y=a+1}^{\infty} (1-p)^{y-1} p = \frac{p}{1-p} \sum_{y=a+1}^{\infty} (1-p)^y$$

$$\text{Let } S = \sum_{y=a+1}^{\infty} (1-p)^y = (1-p)^{a+1} + (1-p)^{a+2} + \ldots$$

$$(1-p)S = (1-p)^{a+2} + (1-p)^{a+3} + \ldots \implies S - (1-p)S = S - S + Sp \implies S = \frac{(1-p)^{a+1}}{p}$$

$$P(Y > a) = \frac{p}{1-p} \sum_{y=a+1}^{\infty} (1-p)^y = \frac{p}{1-p} \cdot \frac{(1-p)^{a+1}}{p} = (1-p)^a$$

$$\therefore P(Y > a) = (1-p)^a$$

   □

   (b)

   Theorem. $\text{P}(Y = a + k \mid Y > a) = \text{P}(Y = k)$

   Proof.

$$\text{P}(Y = k) = (1-p)^{k-1} p$$

$$\text{P}(Y = a + k \mid Y > a) = \frac{\text{P}(Y = a + k \cap Y > a)}{\text{P}(Y > a)} = \frac{\text{P}(Y = a + k \cap (Y = a + 1 \cup Y = a + 2 \ldots \cup \ldots Y = a + k))}{\text{P}(Y > a)}$$

$$= \frac{\text{P}(Y = a + k)}{\text{P}(Y > a)} = \frac{(1-p)^{a+k-1} p}{(1-p)^a} = (1-p)^{k-1} p$$

$$\therefore \text{P}(Y = a + k \mid Y > a) = \text{P}(Y = k) = (1-p)^{k-1} p$$

   □

4. $Y \sim \text{Geometric}(0.3) \implies \text{P}(Y = y) = (0.7)^{y-1}(0.3)$
   $\text{P}(Y > a) = (0.7)^a \geq 0.1 \iff a \ln(0.7) \geq \ln(0.1) \implies a \geq \frac{\ln(0.1)}{\ln(0.7)}$
   Pick $a \geq \frac{\ln(0.1)}{\ln(0.7)}$ so that $\text{P}(Y > a) \geq 0.1$

5. $\mathrm{P}(A) = 40\%$
$Y \sim \mathrm{NB}(3, \mathrm{P}(A)) \implies \mathrm{P}(Y = y) = \binom{y-1}{3-1} \mathrm{P}(A)^{3-1} \mathrm{P}(A^c)^{y-3}$
$\mathrm{P}(Y = 10) = \binom{10-1}{3-1} \mathrm{P}(A)^{3-1} \mathrm{P}(A^c)^{10-3} = \binom{9}{2}(40\%)^2(60\%)^7 \approx 0.1612\ldots$

6. $\mathrm{P}(B) = 60\%$
   (a) Let $Y \sim \mathrm{Geometric}(40\%)$ represent the dropped calls until pick-up $\implies \mathrm{P}(Y = y) = (60\%)^{y-1}(40\%)$
   i. $\mathrm{P}(Y = 1) = 40\%$ ii. $\mathrm{P}(Y = 2) = \frac{6}{25}$ iii. $\mathrm{P}(Y = 3) = \frac{18}{125}$
   (b) Let $Y \sim \mathrm{NB}(2, 40\%)$ represent the dropped calls until both my friend and I are received.
   Then the probability that four tries will be necessary is
   $$\mathrm{P}(Y = 4) = \binom{4-1}{2-1} \mathrm{P}(B^c)^{4-2} \mathrm{P}(B)^2 = \frac{1323}{10000} = 0.1323$$

7. $\mathrm{P}(D) = 0.4$
Let $Y \sim \mathrm{Geometric}(\mathrm{P}(D))$ represent the number of hurricanes until damage occurs $\implies \mathrm{P}(Y = y) = (0.6)^{y-1}(0.4)$

y	$\mathrm{P}(Y = y)$
2	0.24
3	0.144
4	0.0864
5	0.05184

8. $Y \sim \mathrm{Hypergeometric}(100, 6, 10) \implies \mathrm{P}(Y = y) = \frac{\binom{6}{y}\binom{94}{10-y}}{\binom{100}{10}}$
   (a) $\mathrm{P}(Y = 0) = \frac{\binom{6}{0}\binom{94}{10}}{\binom{100}{10}} \approx 0.5223\ldots$
   (b) $\mathrm{P}(Y > 2) = \mathrm{P}(Y \geq 3) = \sum_{y=3}^{6} \mathrm{P}(Y = y) \approx 0.0126$

9. (a) Let $Y \sim \mathrm{Hypergeometric}(20, 2, 4)$ represent the number of defects $\implies \mathrm{P}(Y = y) = \frac{\binom{2}{y}\binom{18}{4-y}}{\binom{20}{4}}$
   We want zero defects, this probability is $\mathrm{P}(Y = 0) = \frac{12}{19}$, but because we want the probability of rejection this is simply $1 - \mathrm{P}(Y = 0) = \frac{7}{19} \approx 0.3684\ldots$
   (b) Y changes to $Y \sim \mathrm{NB}(4, 0.9) \implies \mathrm{P}(Y = y) = \binom{y-1}{4-1}(0.9)^4(0.1)^{y-4}$
   Then, the probability of acceptance is $\mathrm{P}(Y = 4) = (0.9)^4$, but we want to know the rejection which is then $1 - \mathrm{P}(Y = 4) = 0.3439$

10. Let $Y \sim \mathrm{Hypergeometric}(20, 8, 6) \implies \mathrm{P}(Y = y) = \frac{\binom{8}{y}\binom{12}{6-y}}{\binom{20}{6}}$ represent the number of jurors who are black.
$E(Y) = \frac{6 \cdot 8}{20} = 2.4$

y	$\mathrm{P}(Y = y)$
0	0.0238
1	0.1635
2	0.3576
3	0.3179
4	0.1192
5	0.0173
6	0.0007

On average, we should expect about 2 or 3 black jurors. There is reason to doubt the randomness of this selection. In particular, the probability of choosing just one black juror is 16.35%, while the most likely configurations involve either 2 or 3 at 35.67% and 31.79% respectively.

11. Let $Y \sim \mathrm{Hypergeometric}(20, 15, 4)$ represent the number of sampled cocaine packets
$\implies \mathrm{P}(Y = y) = \frac{\binom{15}{y}\binom{5}{4-y}}{\binom{20}{4}}$

Let X represent the number of noncocaine packets. Because X depends on how many noncocaine packets were taken in Y, there would be $5 - (4 - y) = 1 + y$ noncocaine packets left.
Then $(X|Y) \sim \mathrm{Hypergeometric}(16, y + 1, 2) \implies \mathrm{P}(X = x \mid Y = y) = \frac{\binom{y+1}{x}\binom{16-(y+1)}{2-x}}{\binom{16}{2}}$

The probability of six packets with 4 containing cocaine and 2 not is
$$\mathrm{P}(Y = 4)\,\mathrm{P}(X = 2 \mid Y = 4) = \frac{91}{3876} \approx 0.0235$$

12. $\mathrm{P}(W) = \frac{1}{\binom{30}{6}}$

If each ticket is \$1 then the state makes \$800,000. The state will lose money if there are at least 2 winners.
Thus, let $Y \sim \text{Binomial}(800000, \mathrm{P}(W))$ represent the number of winners out of 800,000 participants
$\implies \mathrm{P}(Y = y) = \binom{800000}{y} \mathrm{P}(W)^y \, \mathrm{P}(W^c)^{800000-y}$ Then, the probability that the state will lose money is given by

$$\mathrm{P}(Y \geq 2) = 1 - \mathrm{P}(Y = 0) - \mathrm{P}(Y = 1) \approx 0.3898\ldots$$