HW8

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1. $Y \sim \text{Normal}(10, 36) \equiv Z \sim \text{Normal}(0,1) = \frac{Y-10}{6}$
   (a) $\mathrm{P}(Y > 5) = \mathrm{P}\left(Z > \frac{-5}{6}\right) = \mathrm{P}\left(Z < \frac{5}{6}\right) = 1 - \mathrm{P}\left(Z > \frac{5}{6}\right) = 0.7967$
   (b) $\mathrm{P}(4 < Y < 16) = \mathrm{P}(-1 < Z < 1) = 1 - 2\,\mathrm{P}(Z > 1) = 0.6826$

2. Let $R \sim \text{Normal}(40, 4^2)$ represent the annual rainfall.
   The probability of at most 50 inches of rainfall is

$$\mathrm{P}(R \leq 50) = \mathrm{P}\left(Z \leq \frac{50-40}{4}\right) = 1 - \mathrm{P}(Z > 2.5) = 1 - 0.0062 = 0.9938$$

   Because years are discrete, and we want 10 years in a row with a rainfall of at most 50 inches, the most fitting discrete distribution would be $Y \sim \text{Binomial}(10, \mathrm{P}(Z \leq 2.5))$. It must be that the probability of 10 years in a row of at most 50 inches of rainfall is

$$\mathrm{P}(Y = 10) = \binom{10}{10} \mathrm{P}(Z \leq 2.5)^{10} \approx 0.9397$$

3. (a) $\mathrm{P}(Z > z_0) = 0.5 \implies z_0 = 0$
   (b)

$$\mathrm{P}(-z_0 < Z < z_0) = \int_{-z_0}^{z_0} Z \,\mathrm{d}z = 0.9$$

   Because the normal distribution is symmetric, this integral is equivalent to

$$2\int_0^{z_0} Z \,\mathrm{d}z = 0.9 \implies \int_0^{z_0} Z \,\mathrm{d}z = 0.45 \equiv \mathrm{P}(0 < Z < z_0)$$

   Then we convert

$$\mathrm{P}(0 < Z < z_0) = \mathrm{P}(Z > 0 \cap Z < z_0) = 1 - \mathrm{P}(Z < 0) - \mathrm{P}(Z > z_0) = 0.45 \implies \mathrm{P}(Z > z_0) = 0.05$$

   $\therefore z_0 = 1.65.$

4. Let $Y \sim \text{Normal}(10, 4)$ represent the working lifetime in years of a client. The 14^{th} percentile is a value such that $\mathrm{P}(Y \leq \phi_{0.14}) = 0.14$. Equivalently,

$$\mathrm{P}\left(Z \leq \frac{\phi_{0.14} - 10}{2}\right) = 0.14$$

   We can invert the inside and get

$$\mathrm{P}\left(Z \geq \frac{10 - \phi_{0.14}}{2}\right) = 0.14 \implies \frac{10 - \phi_{0.14}}{2} = 1.08 \implies \phi_{0.14} = 10 - 2(1.08) = 7.84$$

   So, $\phi_{0.14} = 7.84$ years.

5. Let $Y \sim \text{Exponential}(2)$ represent the repair time.
   (a) $\mathrm{P}(Y \geq 2) = \frac{1}{2}\int_2^\infty e^{-y/2}\,\mathrm{d}y = \frac{1}{e}$
   (b) By the memoryless property, $\mathrm{P}(Y \geq 9 + 1 \mid Y \geq 9) = \mathrm{P}(Y \geq 1) = e^{-1/2} \approx 0.6065$

6. (a) Let $Y \sim \text{Exponential}(20)$ represent the thousands of miles before a car is junked.
   By the memoryless property, $\mathrm{P}(Y \geq 10 + 20 \mid Y \geq 10) = \mathrm{P}(Y \geq 20) = e^{-1} \approx 0.3679$

All decimals are rounded to 4 places.

(b) $Y \sim \text{Uniform}(0, 40)$, so now we want $\mathrm{P}\left(Y \geq 30 \mid Y \geq 10\right) = \frac{\mathrm{P}(Y \geq 30)}{\mathrm{P}(Y \geq 10)} = 1/3$

7. $Y \sim \text{Exponential}(\beta) \implies f_Y(y) = \frac{e^{-y/\beta}}{\beta}$
Median of $Y = \mathrm{P}\left(Y \leq \phi_{0.5}\right) = 0.5$, thus

$$\int_0^{\phi_{0.5}} \frac{e^{-y/\beta}}{\beta} \,\mathrm{d}y = -e^{-\phi_{0.5}/\beta} - (-e^0) \implies e^{-\phi_{0.5}/\beta} = 0.5$$

Solving for $\phi_{0.5}$, we get

$$\phi_{0.5} = \ln 0.5^{-\beta} = \ln 2^\beta$$

8. Let $Y \sim \text{Exponential}(\beta)$ represent the number of days that elapse between the beginning of the calendar year before an an accident occurs.
First, we solve for β,

$$\mathrm{P}\left(Y \leq 50\right) = \int_0^{50} \frac{e^{-y/\beta}}{\beta} \,\mathrm{d}y = -e^{-50/\beta} + e^0 = 0.3 \implies \beta = \frac{50}{\ln(10/7)}$$

Then,

$$\mathrm{P}\left(Y \leq 80\right) = F_Y(80) - F_Y(0) = 1 - e^{-80/\beta} \approx 0.4349$$

9. Let $Y \sim \text{Exponential}(1/20)$ represent the monthly commission for an agent. Our STD is 20, and our mean is 20.
The CDF of Y is $F_Y(y) = 1 - e^{-0.05y}$. We want

$$\mathrm{P}\left(10 < Y < 30\right) = F_Y(30) - F_Y(10) \approx 0.3834$$

10. If $Y \sim \text{Gamma}(\alpha, \beta) \implies \mathrm{E}[Y^k] = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\beta^k$ where $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y} \,\mathrm{d}y$

Proof. The MGF of Y is $M_Y(t) = (1 - \beta t)^{-\alpha}$, and $\frac{\mathrm{d}^k}{\mathrm{d}t^k} M_Y(t)|_{t=0} = \mathrm{E}[Y^k]$

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}t^k} M_Y(t)\right|_{t=0} = \left.\frac{\mathrm{d}^k}{\mathrm{d}t^k} (1 - \beta t)^{-a}\right|_{t=0} = (-1)^k(-(\alpha + k - 1))(-(\alpha + k - 2))\ldots\alpha\beta^k(1 - \beta t)^{-(a+k)}\Big|_{t=0}$$

If k is odd, then we get $(-1)^k$ times odd negative terms which is positive.
If k is even, we get $(-1)^n$ times even negative terms which is positive.
Then, evaluated at $t = 0$

$$\mathrm{E}[Y^k] = (\alpha + k - 1)(\alpha + k - 2)\ldots\alpha\beta^k = \frac{\Gamma(a + k)}{\Gamma(a)}\beta^k$$

□

11. $f_Y(y) = k\left(y^3 e^{-y/2}\right), y > 0 \implies Y \sim \text{Gamma}(4, 2)$

$$k\int_0^\infty y^{4-1}e^{-y/2} \,\mathrm{d}y = 1 \implies k = \frac{1}{2^4\Gamma(4)}$$

$Y \sim \chi^2(8) = Y \sim \text{Gamma}(8/2, 2)$

12. If $Y \sim \text{Beta}(4, 3) \implies k = \frac{\Gamma(4+3)}{\Gamma(4)\Gamma(3)} = 60$

13. $Y \sim \text{Beta}(3, 2)$

(a) The probability that a batch can't be sold is

$$\mathrm{P}\left(Y > 0.4\right) = 1 - \int_0^{0.4} 12y^2(1 - y) \,\mathrm{d}y = 1 - \int_0^{0.4} 12y^2 - 12y^3 \,\mathrm{d}y = 1 - \left[4y^3 - 3y^4\right]_0^{0.4} = 1 - \frac{112}{625} = 0.8208$$

(b)

$$\mathrm{E}[Y] = \int_0^1 12y^3 - 12y^4 \,\mathrm{d}y = 60\%$$

14. Integrate by parts

$$\int_0^\infty e^{ty}\left(ye^{-y}\right)\mathrm{d}y = \int_0^\infty ye^{(t-1)y}\,\mathrm{d}y = \left[\frac{ye^{(t-1)y}}{t-1} - \frac{e^{(t-1)y}}{(t-1)^2}\right]_0^\infty$$

Assume $|t| < 1$ so

$$M_Y(t) = \lim_{y\to\infty}\left[\frac{y}{e^{(t-1)y}(t-1)} - \frac{1}{e^{(t-1)y}(t-1)^2}\right] - \left(-\frac{1}{(t-1)^2}\right) = 0 + \frac{1}{(t-1)^2}$$

15.

$$M_Y(t) = \mathrm{E}[e^{tY}] = \int_{-1}^0 e^{ty}(1+y)\,\mathrm{d}y + \int_0^1 e^{ty}(1-y)\,\mathrm{d}y$$

After integrating by parts,

$$= \left[(1+y)t^{-1}e^{ty} - t^{-2}e^{ty}\right]_{-1}^0 + \left[(1-y)t^{-1}e^{ty} + t^{-2}e^{ty}\right]_0^1$$

Evaluating gives us

$$\left[(t^{-1}-t^{-2}) + (t^{-2}e^{-t})\right] + \left[(t^{-2}e^t) - (t^{-1}+t^{-2})\right] = -2t^{-2} + t^{-2}e^{-t} + t^{-2}e^t$$

So,

$$M_Y(t) = t^{-2}\left(e^t + e^{-t} - 2\right)$$

16.

$$\left.\frac{\mathrm{d}^3}{\mathrm{d}t^3}e^{t^2/2}\right|_{t=0} = \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}te^{t^2/2}\right|_{t=0} = \left.\frac{\mathrm{d}^1}{\mathrm{d}t^1}(e^{t^2/2} + t^2e^{t^2/2})\right|_{t=0} = \left.te^{t^2/2} + 2te^{t^2/2} + t^3e^{t^2/2}\right|_{t=0} = 0$$

17. $\sigma = \sqrt{\mathrm{E}[Y^2] - \mathrm{E}[Y]^2}$
We calculate the first and second MGF

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(1-500t)^{-4} = \frac{\mathrm{d}^1}{\mathrm{d}t^1}(-4)(-500)(1-500t)^{-5} = (-5)(-4)(-500)^2(1-500t)^{-6}$$

thus

$$\mathrm{E}[Y^2] = 20\cdot 500^2 \text{ and } \mathrm{E}[Y] = 4\cdot 500 \implies \sigma = \sqrt{20\cdot 500^2 - 2000^2} = 1000$$