HW10

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1. $f(y_1, y_2) = y_1 + y_2, \, 0 < y_1, y_2 < 1$

   (a) $E[Y_1] = \int_0^1 y_1(y_1 + y_2) \, dy_1 = \frac{1}{3} + \frac{y_2}{2}$

   (b) $E[Y_2] = \int_0^1 y_2(y_1 + y_2) \, dy_2 = \frac{y_1}{2} + \frac{1}{3}$

2. $f(y_1, y_2) = 4y_1 y_2, \, 0 < y_1, y_2 < 1 \implies y_1, y_2$ independent.

   (a) $E[Y_1] = 4y_2 \int_0^1 y_1^2 \, dy_1 = \frac{4y_2}{3}$

   (b) $\text{Var}(Y_1) = E[Y_1^2] - E[Y_1]^2 = y_2 - \left(\frac{4y_2}{3}\right)^2$

   (c) $E[Y_1 - Y_2] = E[Y_1] - E[Y_2] = \frac{4}{3}y_2 - 4y_1 \int_0^1 y_2^2 \, dy_2 = \frac{4}{3}(y_2 - y_1)$

3. $f(y_1, y_2) = 1/y_2, \, 0 < y_1 < y_2 < 1$

   (a) $E[Y_1 Y_2] =$
   $$\int_0^1 \int_0^{y_2} y_1 \, dy_1 \, dy_2 = \frac{1}{2} \int_0^1 y_2^2 \, dy_2 = \left[\frac{1}{2 \cdot 3} y_2^3\right]_0^1 = \frac{1}{6}$$

   (b) $E[Y_1] =$
   $$\int_0^1 \int_0^{y_2} \frac{y_1}{y_2} \, dy_1 \, dy_2 = \int_0^1 \left[\frac{y_1^2}{2y_2}\right]_0^{y_2} dy_2 = \int_0^1 \frac{y_2}{2} \, dy_2 = [y_2^2/4]_0^1 = \frac{1}{4}$$

   (c) $E[Y_2] =$
   $$\int_0^1 \int_0^{y_2} 1 \, dy_1 \, dy_2 = \frac{1}{2}$$

   (d) $\text{Cov}(Y_1, Y_2) = E[Y_1 Y_2] - E[Y_1]E[Y_2] = \frac{1}{6} - \frac{1}{2 \cdot 4} = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$

4. $f(y_1, y_2) = 1, \, 0 < y_1 < 2, \, 0 < y_2 < 1, \, 2y_2 < y_1$
   $E[Y_1 - Y_2] =$
   $$\int_0^2 \int_0^{y_1/2} y_1 - y_2 \, dy_2 \, dy_1 = \int_0^2 \left[y_1 y_2 - \frac{1}{2}y_2^2\right]_0^{y_1/2} dy_1 = \frac{3}{8} \int_0^2 y_1^2 \, dy_1 = 1$$

5. $f(y_1, y_2) = 6(1 - y_2), \, 0 \le y_1 \le y_2 \le 1$

   (a) $\text{Cov}(Y_1, Y_2) = E[Y_1 Y_2] - E[Y_1]E[Y_2] = \frac{3}{20} - \frac{1}{8} = \frac{1}{40}$

      i. $E[Y_1 Y_2] =$
   $$6 \int_0^1 (1 - y_2) y_2 \int_0^{y_2} y_1 \, dy_1 \, dy_2 = 3 \int_0^1 y_2^3 - y_2^4 \, dy_2 = 3\left[\frac{1}{4}y_2^4 - \frac{1}{5}y_2^5\right]_0^1 = \frac{3}{20}$$

      ii. $E[Y_1] =$
   $$6 \int_0^1 (1 - y_2) \int_0^{y_2} y_1 \, dy_1 \, dy_2 = 3 \int_0^1 y_2^2 - y_2^3 \, dy_2 = \left[y_2^3 - \frac{3}{4}y_2^4\right]_0^1 = \frac{1}{4}$$

      iii. $E[Y_2] =$
   $$6 \int_0^1 (1 - y_2) y_2 \int_0^{y_2} dy_1 \, dy_2 = 6 \int_0^1 y_2^2 - y_2^3 \, dy_2 = \left[2y_2^3 - \frac{3}{2}y_2^4\right]_0^1 = \frac{1}{2}$$

   (b) $\text{Var}(Y_1 - 3Y_2) = \text{Var}(Y_1) + 3^2 \text{Var}(Y_2) - 2(3) \text{Cov}(Y_1, Y_2) = \frac{27}{80}$

      i. $\text{Var}(Y_1) = E[Y_1^2] - E[Y_1]^2 = \frac{1}{10} - \frac{1}{16} = \frac{3}{80}$

      ii. $\text{Var}(Y_2) = E[Y_2^2] - E[Y_2]^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$

All decimals are rounded to 4 places.

6. X is the benefit to the surgeon, and Y to the hospital. $\text{Var}(X) = 5000, \text{Var}(Y) = 10000, \text{Var}(X+Y) = 17000$. From the given information,

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\,\text{Cov}(X,Y) = 17000 \iff \text{Cov}(X,Y) = \text{E}[XY] - \text{E}[X]\,\text{E}[Y] = 1000$$

So we can interpret the question as asking

$$\text{Var}(X + 100 + 1.1Y) = \text{Var}(X) + 1.1^2\,\text{Var}(Y) + 2(1.1)\,\text{Cov}(Y_1, Y_2) = 19300$$

7. $f(y_1, y_2) = y_2^{-1}, 0 < y_1 < y_2 < 1$
$\text{E}[Y_1 \mid Y_2 = y_2] =$

$$\int_0^{y_2} y_1 \frac{y_2^{-1}}{\int_{y_1}^1 y_2^{-1}\,dy_2}\,dy_1 = \int_0^{y_2} y_1 \frac{y_2^{-1}}{\ln(y_2^{-1})}\,dy_1 = \frac{y_2}{\ln(y_2^{-1})}$$

8. $f(y_1, y_2) = 6(1 - y_2), 0 \le y_1 \le y_2 \le 1$
$\text{E}[Y_2 \mid Y_1 = y_1] =$

$$\int_{y_1}^1 y_2 f(y_2 \mid y_1)\,dy_2 = \int_{y_1}^1 y_1 \frac{6(1 - y_2)}{\int_0^{y_2} 6(1 - y_2)\,dy_1}\,dy_2 = \int_{y_1}^1 y_1 \frac{1}{y_2}\,dy_2 = \frac{y_1}{\ln(y_1^{-1})}$$

9. $f(y_1, y_2) = \frac{e^{-y_2}}{y_2}, 0 < y_1 < y_2 < \infty$
$\text{E}[Y_1^3 \mid Y_2 = y_2] =$

$$\int_0^{y_2} y_1^3 \frac{e^{-y_2}/y_2}{}\,dy_1$$

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