HW2

Justin Nguyen

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1. $9 \cdot 10^6 = 9,000,000$ seven-digit phone numbers.
2. (a) $8! = 40320$ seatings
   (b) $5! \cdot 4 \cdot 3! = 2880$ consecutive male seatings.
   (c) $4! \cdot 2^4 = 384$ consecutive couples seatings.
3. $6m, 7s, 4e \to 2$ books
   (a) $\binom{6}{2} + \binom{7}{2} + \binom{4}{2} = 42$ choices of the two books who share the same subject.
   (b) $\binom{6}{1}\binom{7}{1} + \binom{6}{1}\binom{4}{1} + \binom{7}{1}\binom{4}{1} = 94$ choices of two books who don't share the same subject.
4. $8w, 6m \to 3w, 3m$
   (a) $\binom{8}{3}[\binom{6}{3} - \binom{4}{1}] = 896$ committees where m_1, m_2 don't work together.
   (b) $\binom{6}{3}[\binom{8}{3} - \binom{6}{1}] = 1000$ committees where w_1, w_2 don't work together.
   (c) $\binom{8}{3}\binom{6}{3} - P_3^7 = 910$ committees where w_1, m_1 don't work together.[1]
5. Let $D_6 = \{1, 2, 3, 4, 5, 6\}$, $A =$ rolling $1 \ldots 6$ in any order, and $|S| = |D_6|^6$.
   If the point $r_1 = (1, 2, 3, 4, 5, 6) \in A \implies$ other points in A must be arrangements of $r_1 \implies |A| = |D_6|!$
   $\therefore \mathrm{P}(A) = \frac{6!}{6^6} \approx 0.015$
6. $|S| = \binom{10}{5}, |A| = \binom{6}{5} \implies \mathrm{P}(A) = \frac{|A|}{|S|} \approx 0.02$
7.
8. 52 cards $\to$ 5 cards; $|S| = \binom{52}{5}$
   (a) $\mathrm{P}(3A, 2K) = \frac{\binom{4}{3}\binom{4}{2}}{\binom{52}{5}} \approx 0.000009$
   (b) $\mathrm{P}(3R_n, 2R_k) =$ TODO
9. $2w, 4h, 7a \to 1w, 2h, 3a$
   $|S| = P_6^{13}$
10. (a) Fluke $= \binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1} = 120$ arrangements.
    (b) Propose $= \binom{7}{2}\binom{5}{1}\binom{4}{2}\binom{2}{1}\binom{1}{1} = 1260$ arrangements.
    (c) Mississippi $= \binom{11}{1}\binom{10}{4}\binom{6}{4}\binom{2}{2} = 34650$ arrangements.
11. 11
12. 12

[1] P_3^7 is a simplification of the number of committees where w_1, m_1 work together.

13. **Proof:** $\sum_{k=0}^{n}\binom{n}{k}=2^n$
The Binomial Theorem states that $(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$.
Then, let $x=y=1 \implies (1+1)^n=\sum_{k=0}^{n}\binom{n}{k}1^{n-k}1^k=\sum_{k=0}^{n}\binom{n}{k}1^n=\sum_{k=0}^{n}\binom{n}{k}$.
$\therefore 2^n=\sum_{k=0}^{n}\binom{n}{k}$.

14. **Proof:** $\sum_{k=0}^{n>0}(-1)^k\binom{n}{k}=0$
Case $n=1$:

$$\sum_{k=0}^{1}(-1)^k\binom{1}{k}=(-1)^0\binom{1}{0}+(-1)^1\binom{1}{1}=0$$

Case $n=n+1$:
Note that $(\forall \alpha,\beta\in\mathbb{N})$ and $\alpha\geq\beta$

- $\binom{\alpha}{0}=\binom{\alpha}{\alpha}=1$
- $\binom{\alpha}{1}=\binom{\alpha}{\alpha-1}=\alpha$
- $\binom{\alpha}{\beta}=\binom{\alpha}{\alpha-\beta}$

Assume that $\sum_{k=0}^{n>0}(-1)^k\binom{n}{k}=0$. Then $n+1$ is:

$$\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}=$$

$$(-1)^0\binom{n+1}{0}+(-1)^1\binom{n+1}{1}+\ldots+(-1)^{n+1-k}\binom{n+1}{n+1-k}+\ldots+(-1)^k\binom{n+1}{k}+\ldots+(-1)^n\binom{n+1}{n}+(-1)^{n+1}\binom{n+1}{n+1}$$

We rearrange the terms.

$$(-1)^0\binom{n+1}{0}+(-1)^{n+1}\binom{n+1}{n+1}+\ldots+(-1)^{n+1-k}\binom{n+1}{n+1-k}+(-1)^k\binom{n+1}{k}+\ldots+(-1)^1\binom{n+1}{1}+(-1)^n\binom{n+1}{n}$$