HW6

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1. $Y \sim \text{Poisson}(3.5) \implies \mathrm{P}(Y = y) = \frac{3.5^y e^{-3.5}}{y!}$
   (a) $\mathrm{P}(Y \geq 2) = 1 - \mathrm{P}(Y = 0) - \mathrm{P}(Y = 1) \approx 0.8641$
   (b) $\mathrm{P}(Y \leq 1) = \mathrm{P}(Y = 0) + \mathrm{P}(Y = 1) \approx 0.1359$
2. $Y \sim \text{Poisson}(3) \implies \mathrm{P}(Y = y) = \frac{3^y e^{-3}}{y!}$
   (a) $\mathrm{P}(Y \geq 3) = 1 - \sum_{y=0}^{2} \mathrm{P}(Y = y) \approx 0.5768$
   (b)
   $$\mathrm{P}(Y \geq 3 \mid Y \geq 1) = \frac{\mathrm{P}(Y \geq 3 \cap Y \geq 1)}{\mathrm{P}(Y \geq 1)} = \frac{\mathrm{P}(Y \geq 3)}{\mathrm{P}(Y \geq 1)} \approx 0.6070$$
3. $\mathrm{P}(Y = 4) = 3\,\mathrm{P}(Y = 2)$
   Let $Y \sim \text{Poisson}(\lambda)$ be the number of claims filed $\implies \mathrm{P}(Y = y) = \frac{\lambda^y e^{-\lambda}}{y!}$.
   We solve for λ, the mean of the number of claims
   $$\frac{\lambda^4 e^{-\lambda}}{4!} = 3\frac{\lambda^2 e^{-\lambda}}{2!} \implies \lambda = \sqrt{\frac{3 \cdot 4!}{2!}} = 6$$
   Then, $\mathrm{Var}(Y) = 6$.
4. Let $\mathrm{I} \sim \text{Poisson}(3)$ and $\mathrm{II} \sim \text{Poisson}(4)$ represent the number of cars arriving at entrance I and II respectively.
   The probability that a total of three cars will arrive at the parking lot in a given hour is
   $$\mathrm{P}(\mathrm{I} + \mathrm{II} = 3) = \sum_{y=0}^{3} \mathrm{P}(\mathrm{I} = y)\,\mathrm{P}(\mathrm{II} = 3 - y) \approx 0.0521$$
   Note: $Y \sim \text{Poisson}(7)$ with $\mathrm{P}(Y = 3)$ is equivalent.
5. Let the number of customers who purchase the item during the day be $Y \sim \text{Poisson}(2) \implies \mathrm{P}(Y = y) = \frac{2^y e^{-2}}{y!}$
   The cost of the item is given by $C(Y) = 100(2^{-Y})$.
   The expected cost of the item shall be
   $$\begin{aligned}\mathrm{E}[C(Y)] &= \sum_{y=0}^{\infty} C(y)\,\mathrm{P}(Y = y) = \sum_{y=0}^{\infty} 100 \cdot 2^{-y} \frac{2^y e^{-2}}{y!} \\ &= 100e^{-2} \sum_{y=0}^{\infty} \frac{1}{y!} \\ &= 100e^{-2}e = 100e^{-1} \approx \$36.79\end{aligned}$$
6.

y	$Y \sim \text{Binomial}(20, 0.05)$	$Y \sim \text{Poisson}(1)$	Difference
0	0.35848592	0.36787944	0.0093935188
1	0.3773536	0.36787944	0.0094741614
2	0.1886768	0.18393972	0.0047370807

The binomial distribution approximates the Poisson distribution decently well in this case. The largest error being smaller than 0.01

All decimals are rounded to 4 places.

7. Let $Y \sim \text{Poisson}(800000 \cdot \binom{30}{6}^{-1})$ represent the number of winners with the approximation for $\lambda = n \cdot p$. The probability that the state loses money is given by

$$\mathrm{P}(Y \geq 2) = 1 - \mathrm{P}(Y = 0) - \mathrm{P}(Y = 1) \approx 0.3898$$

which only differs, before rounding, from the binomial distribution's value by 1.0244×10^{-7}.

8. $\mathrm{P}(Y = y) = \binom{n}{y} p^y (1-p)^{n-y}$
Using the Binomial Theorem, it follows that

$$\begin{aligned} M_Y(t) = \mathrm{E}[e^{tY}] &= \sum_{y=0}^{n} e^{ty} \binom{n}{y} p^y (1-p)^{n-y} \\ &= \sum_{y=0}^{n} \binom{n}{y} (pe^t)^y (1-p)^{n-y} = [pe^t + (1-p)]^n \end{aligned}$$

9. $M_Y(t) = \frac{1}{6}e^t + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}$

$$\begin{aligned} \mathrm{E}[Y] &= \frac{\mathrm{d}}{\mathrm{d}t} M_Y(t)\Big|_{t=0} = \frac{1}{6}e^0 + \frac{4}{6}e^0 + \frac{9}{6}e^0 = \frac{14}{6} \\ \mathrm{E}[Y^2] &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} M_Y(t)\Big|_{t=0} = \frac{1}{6}e^0 + \frac{8}{6}e^0 + \frac{27}{6}e^0 = 6 \\ \mathrm{Var}(Y) &= \mathrm{E}[Y^2] - \mathrm{E}[Y]^2 = 6 - \left(\frac{14}{6}\right)^2 = \frac{5}{9} \end{aligned}$$

10. (a)

$$\mathrm{E}[e^{tY}] = \left[\frac{1}{3}e^t + \frac{2}{3}\right]^5 \implies Y \sim \text{Binomial}(5, 1/3)$$

(b)

$$\mathrm{E}[e^{tY}] = \frac{e^t}{2 - e^t} = \frac{pe^t}{1 - (1-p)e^t} \implies p = \frac{1}{2} \implies Y \sim \text{Geometric}(1/2)$$

(c)

$$E[e^{tY}] = e^{2(e^t - 1)} \implies Y \sim \text{Poisson}(2)$$

11. (a) $M_Y(0) = \mathrm{E}[e^0] = \mathrm{E}[1] = 1$
(b) If $W = 3Y \implies M_W(t) = M_Y(3t)$

Proof. It immediately follows from the definitions

$$M_W(t) = \mathrm{E}[e^{t3Y}] = M_Y(3t) = \mathrm{E}[e^{3tY}]$$

□

(c) If $U = aY + b \implies M_U(t) = e^{bt} M_Y(at)$

Proof.

$$M_U(t) = \mathrm{E}[e^{t(aY+b)}] = \mathrm{E}[e^{aYt}e^{bt}] = e^{bt}\,\mathrm{E}[e^{aYt}] = e^{bt} M_Y(3t)$$

□

12. If $r(t) = \ln[M_Y(t)] \implies \frac{\mathrm{d}^1}{\mathrm{d}t^1} r(t)\Big|_{t=0} = \mathrm{E}[Y]$ and $\frac{\mathrm{d}^2}{\mathrm{d}t^2} r(t)\Big|_{t=0} = \mathrm{Var}(Y)$

Proof. First, we prove that $r^{(1)}(0) = \mathrm{E}[Y]$

$$\begin{aligned} M_Y(t) &= \mathrm{E}[e^{tY}] = \sum_y e^{ty}\,\mathrm{P}(Y = y) \\ r(t) &= \ln(M_Y(t)) = \ln\left[e^{0t}\,\mathrm{P}(Y=0) + e^t\,\mathrm{P}(Y=1) + e^{2t}\,\mathrm{P}(Y=2) + \cdots\right] \\ \frac{\mathrm{d}^1}{\mathrm{d}t^1} r(t) &= \frac{0 \cdot \mathrm{P}(Y=0) + e^t\,\mathrm{P}(Y=1) + 2e^{2t}\,\mathrm{P}(Y=2) + \cdots}{\mathrm{P}(Y=0) + e^t\,\mathrm{P}(Y=1) + e^{2t}\,\mathrm{P}(Y=2) + \cdots} \\ \frac{\mathrm{d}^1}{\mathrm{d}t^1} r(t)\Big|_{t=0} &= \frac{0 \cdot \mathrm{P}(Y=0) + 1e^0\,\mathrm{P}(Y=1) + 2e^0\,\mathrm{P}(Y=2) + \cdots}{\mathrm{P}(Y=0) + e^0\,\mathrm{P}(Y=1) + e^0\,\mathrm{P}(Y=2) + \cdots} = \frac{\mathrm{E}[Y]}{1} = \mathrm{E}[Y] \end{aligned}$$

We see that the numerator expands to the definition of $\mathrm{E}[Y]$, and the denominator is the sum of the sample space. Next, we show the latter, that $r^{(2)}(0) = \mathrm{Var}(Y)$

Let $f(t) = 0 \cdot \mathrm{P}(Y=0) + e^t\,\mathrm{P}(Y=1) + 2e^{2t}\,\mathrm{P}(Y=2) + \cdots \implies \dfrac{\mathrm{d}}{\mathrm{d}t} f(t) = 0 \cdot P(Y=0) + e^t\,\mathrm{P}(Y=1) + 4e^{2t}\,\mathrm{P}(Y=2) + \cdots$

Let $g(t) = \mathrm{P}(Y=0) + e^t\,\mathrm{P}(Y=1) + e^{2t}\,\mathrm{P}(Y=2) + \cdots \implies \dfrac{\mathrm{d}}{\mathrm{d}t} g(t) = 0 \cdot \mathrm{P}(Y=0) + e^t\,\mathrm{P}(Y=1) + 2e^{2t}\,\mathrm{P}(Y=2) + \cdots$

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2} r(t)\right|_{t=0} = \frac{\frac{\mathrm{d}}{\mathrm{d}t} f(0) g(0) - f(0) \frac{\mathrm{d}}{\mathrm{d}t} g(0)}{g(0)^2} = \frac{\mathrm{E}[Y^2] \cdot 1 - \mathrm{E}[Y] \cdot \mathrm{E}[Y]}{1} = \mathrm{E}[Y^2] - \mathrm{E}[Y]^2 = \mathrm{Var}(Y)$$

$\square$