HW10

Justin Nguyen

November 20, 2024

1. $f(y_1, y_2) = y_1 + y_2$, $0 < y_1, y_2 < 1$
   (a) $\mathrm{E}[Y_1] = \int_0^1 y_1(y_1 + y_2)\,\mathrm{d}y_1 = \frac{1}{3} + \frac{y^2}{2}$
   (b) $\mathrm{E}[Y_2] = \int_0^1 y_2(y_1 + y_2)\,\mathrm{d}y_2 = \frac{y_1}{2} + \frac{1}{3}$
2. $f(y_1, y_2) = 4y_1y_2, 0 < y_1, y_2 < 1 \implies y_1, y_2$ independent.
   (a) $\mathrm{E}[Y_1] = 4y_2 \int_0^1 y_1^2\,\mathrm{d}y_1 = \frac{4y_2}{3}$
   (b) $\mathrm{Var}(Y_1) = \mathrm{E}[Y_1^2] - \mathrm{E}[Y_1]^2 = y_2 - \left(\frac{4y_2}{3}\right)^2$
   (c) $\mathrm{E}[Y_1 - Y_2] = \mathrm{E}[Y_1] - \mathrm{E}[Y_2] = \frac{4}{3}y_2 - 4y_1 \int_0^1 y_2^2\,\mathrm{d}y_2 = \frac{4}{3}(y_2 - y_1)$
3. $f(y_1, y_2) = 1/y_2, 0 < y_1 < y_2 < 1$
   (a) $\mathrm{E}[Y_1Y_2] =$
   $$\int_0^1 \int_0^{y_2} y_1\,\mathrm{d}y_1\,\mathrm{d}y_2 = \frac{1}{2}\int_0^1 y_2^2\,\mathrm{d}y_2 = \left[\frac{1}{2\cdot 3}y_2^3\right]_0^1 = \frac{1}{6}$$
   (b) $\mathrm{E}[Y_1] =$
   $$\int_0^1 \int_0^{y_2} \frac{y_1}{y_2}\,\mathrm{d}y_1\,\mathrm{d}y_2 = \int_0^1 \left[\frac{y_1^2}{2y_2}\right]_0^{y_2}\mathrm{d}y_2 = \int_0^1 \frac{y_2}{2}\,\mathrm{d}y_2 = \left[y_2^2/4\right]_0^1 = \frac{1}{4}$$
   (c) $\mathrm{E}[Y_2] =$
   $$\int_0^1 \int_0^{y_2} 1\,\mathrm{d}y_1\,\mathrm{d}y_2 = \frac{1}{2}$$
   (d) $\mathrm{Cov}(Y_1, Y_2) = \mathrm{E}[Y_1Y_2] - \mathrm{E}[Y_1]\,\mathrm{E}[Y_2] = \frac{1}{6} - \frac{1}{2\cdot 4} = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$
4. $f(y_1, y_2) = 1$, $0 < y_1 < 2$, $0 < y_2 < 1$, $2y_2 < y_1$
   $\mathrm{E}[Y_1 - Y_2] =$
   $$\int_0^2 \int_0^{y_1/2} y_1 - y_2\,\mathrm{d}y_2\,\mathrm{d}y_1 = \int_0^2 \left[y_1y_2 - \frac{1}{2}y_2^2\right]_0^{y_1/2}\mathrm{d}y_1 = \frac{3}{8}\int_0^2 y_1^2\,\mathrm{d}y_1 = 1$$
5. $f(y_1, y_2) = 6(1 - y_2), 0 \le y_1 \le y_2 \le 1$
   (a) $\mathrm{Cov}(Y_1, Y_2) = \mathrm{E}[Y_1Y_2] - \mathrm{E}[Y_1]\,\mathrm{E}[Y_2] = \frac{3}{20} - \frac{1}{8} = \frac{1}{40}$
      i. $\mathrm{E}[Y_1Y_2] =$
      $$6\int_0^1 (1 - y_2)y_2 \int_0^{y_2} y_1\,\mathrm{d}y_1\,\mathrm{d}y_2 = 3\int_0^1 y_2^3 - y_2^4\,\mathrm{d}y_2 = 3\left[\frac{1}{4}y_2^4 - \frac{1}{5}y_2^5\right]_0^1 = \frac{3}{20}$$
      ii. $\mathrm{E}[Y_1] =$
      $$6\int_0^1 (1 - y_2) \int_0^{y_2} y_1\,\mathrm{d}y_1\,\mathrm{d}y_2 = 3\int_0^1 y_2^2 - y_2^3\,\mathrm{d}y_2 = \left[y_2^3 - \frac{3}{4}y_2^4\right]_0^1 = \frac{1}{4}$$
      iii. $\mathrm{E}[Y_2] =$
      $$6\int_0^1 (1 - y_2)y_2 \int_0^{y_2} \mathrm{d}y_1\,\mathrm{d}y_2 = 6\int_0^1 y_2^2 - y_2^3\,\mathrm{d}y_2 = \left[2y_2^3 - \frac{3}{2}y_2^4\right]_0^1 = \frac{1}{2}$$
   (b) $\mathrm{Var}(Y_1 - 3Y_2) = \mathrm{Var}(Y_1) + 3^2\,\mathrm{Var}(Y_2) - 2(3)\,\mathrm{Cov}(Y_1, Y_2) = \frac{27}{80}$
      i. $\mathrm{Var}(Y_1) = \mathrm{E}[Y_1^2] - \mathrm{E}[Y_1]^2 = \frac{1}{10} - \frac{1}{16} = \frac{3}{80}$
      ii. $\mathrm{Var}(Y_2) = \mathrm{E}[Y_2^2] - \mathrm{E}[Y_2]^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$

All decimals are rounded to 4 places.

6. X is the benefit to the surgeon, and Y to the hospital. $\operatorname{Var}(X) = 5000, \operatorname{Var}(Y) = 10000, \operatorname{Var}(X+Y) = 17000$. From the given information,

$$\operatorname{Var}(X+Y) = \operatorname{Var}(X) + \operatorname{Var}(Y) + 2\operatorname{Cov}(X,Y) = 17000 \iff \operatorname{Cov}(X,Y) = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y] = 1000$$

So we can interpret the question as asking

$$\operatorname{Var}(X + 100 + 1.1Y) = \operatorname{Var}(X) + 1.1^2 \operatorname{Var}(Y) + 2(1.1)\operatorname{Cov}(Y_1, Y_2) = 19300$$

7. $f(y_1, y_2) = y_2^{-1}, 0 < y_1 < y_2 < 1$
$\mathrm{E}[Y_1 \mid Y_2 = y_2] =$

$$\int_0^{y_2} y_1 \frac{y_2^{-1}}{\int_{y_1}^1 y_2^{-1}\,\mathrm{d}y_2}\,\mathrm{d}y_1 = \int_0^{y_2} y_1 \frac{y_2^{-1}}{\ln(y_2^{-1})}\,\mathrm{d}y_1 = \frac{y_2}{\ln(y_2^{-1})}$$

8. $f(y_1, y_2) = 6(1 - y_2), 0 \le y_1 \le y_2 \le 1$
$\mathrm{E}[Y_2 \mid Y_1 = y_1] =$

$$\int_{y_1}^1 y_2 f(y_2 \mid y_1)\,\mathrm{d}y_2 = \int_{y_1}^1 y_1 \frac{6(1-y_2)}{\int_0^{y_2} 6(1-y_2)\,\mathrm{d}y_1}\,\mathrm{d}y_2 = \int_{y_1}^1 y_1 \frac{1}{y_2}\,\mathrm{d}y_2 = \frac{y_1}{\ln(y_1^{-1})}$$

9. $f(y_1, y_2) = \frac{e^{-y_2}}{y_2}, 0 < y_1 < y_2 < \infty$ $\mathrm{E}[Y_1^3 \mid Y_2 = y_2] =$

$$\int_0^{y_2} y_1^3 \frac{y_2^{-1} e^{-y_2}}{\int_0^{y_2} y_2^{-1} e^{-y_2}\,\mathrm{d}y_1}\,\mathrm{d}y_1 = \int_0^{y_2} y_1^3 y_2^{-1}\,\mathrm{d}y_1 = \left[\frac{1}{4} y_1^4 y_2^{-1}\right]_0^{y_2} = \frac{1}{4} y_2^3$$

10. If Y_1, Y_2 independent then $\mathrm{E}[Y_1 \mid Y_2 = y_2] = \mathrm{E}[Y_1]$ for all y_2

Proof.

$$\mathrm{E}[Y_1 \mid Y_2 = y_2] = \int y_1 f(y_1 \mid y_2)\,\mathrm{d}y_1 = \int y_1 \frac{f(y_1, y_2)}{f_{y_2}(y_2)}\,\mathrm{d}y_1 = \int y_1 f_{y_1}(y_1)\,\mathrm{d}y_1 = \mathrm{E}[Y_1]$$

□