1 Continuous Random Variable

Cumulative Distribution Function $F_Y(y)$: The probability that Y will take on a value less than y.
Properties: • $\lim_{y \to -\infty} F_Y(Y) = 0$ • $\lim_{y \to \infty} F_Y(Y) = 1$ • $F_Y(b) - F_Y(a) = \mathrm{P}(a \leq Y \leq b)$
Probability Density Function $f_Y(y)$: The probability per unit length.
Quantile ϕ_p: $F_Y(y) = \mathrm{P}(y \leq \phi_p) = p$

1.1 Multivariate Probability Distributions

Joint CDF: $F(y_1, y_2) = \mathrm{P}(Y_1 \leq y_1, Y_2 \leq y_2) =$

$$\int_{-\infty}^{y_2} \int_{-\infty}^{y_1} f(t_1, t_2) \, \mathrm{d}t_1 \, \mathrm{d}t_2$$

Joint PDF: $f(y_1, y_2)$
Marginal PDF:

$$(1) f_{y_1}(y_1) = \int f(y_1, y_2) \, \mathrm{d}y_2 \qquad (2) f_{y_2}(y_2) = \int f(y_1, y_2) \, \mathrm{d}y_1$$

Conditional Density:

$$f(y_1 \mid y_2) = \frac{f(y_1, y_2)}{f_{y_2}(y_2)} = \frac{f(y_1, y_2)}{\int f(y_1, y_2) \, \mathrm{d}y_1}$$

1.2 Independence

Y_1, Y_2 are independent

- $\iff F(y_1, y_2) = F_{y_1}(y_2) F_{y_2}(y_2)$
- $\iff f(y_1, y_2) = f_{y_1}(y_1) f_{y_2}(y_2)$
- $\iff f(y_1 \mid y_2) = f_{y_1}(y_1)$
- $\iff \mathrm{E}[Y_1, Y_2] = \mathrm{E}[Y_1] \, \mathrm{E}[Y_2]$
- $\implies f(y_1, y_2) = g(y_1) h(y_2)$

1.3 Expectation

Expectation and Joint Expectation:

$$(1) \, \mathrm{E}[Y] = \int y f_Y(y) \, \mathrm{d}y \qquad (2) \, \mathrm{E}[Y_1 Y_2] = \int \int y_1 y_2 f(y_1, y_2) \, \mathrm{d}y_1 \, \mathrm{d}y_2$$

Variance and Joint Variance:

$$(1) \operatorname{Var}[Y_1] = \mathrm{E}[Y^2] - \mathrm{E}[Y]^2 \qquad (2) \operatorname{Var}[aY_1 \pm bY_2] = a^2 \operatorname{Var}[Y_1] + b^2 \operatorname{Var}[Y_2] \pm 2ab \operatorname{Cov}[Y_1, Y_2]$$

Covariance:

$$\operatorname{Cov}[Y_1, Y_2] = \mathrm{E}[Y_1 Y_2] - \mathrm{E}[Y_1] \, \mathrm{E}[Y_2]$$

Properties:

- $\mathrm{E}[a \cdot f(y) + b \cdot g(y) + \cdots] = a \, \mathrm{E}[f(y)] + b \, \mathrm{E}[g(y)] + \cdots$
- $\operatorname{Var}[af(y) + bg(y) + c] = a^2 \operatorname{Var}[f(y)] + b^2 \operatorname{Var}[g(y)]$
- If Y_1, Y_2 independent $\implies \operatorname{Cov}[Y_1, Y_2] = 0$

1.3.1 Conditional Expectation

$$\mathrm{E}[Y_1 \mid Y_2 = y_2] = \int y_1 f(y_1 \mid y_2) \, \mathrm{d}y_1 = \int y_1 \frac{f(y_1, y_2)}{\int f(y_1, y_2) \, \mathrm{d}y_1} \, \mathrm{d}y_1$$

2 Continuous Distributions Dictionary

- $Y \sim \text{Uniform}(a, b)$:

$$f(y) = \frac{1}{b-a};\ \ y \in [a, b] \implies \mu = \frac{a+b}{2} \text{ and } \sigma^2 = \frac{(b-a)^2}{12}$$

$$M_Y(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}$$

- $Y \sim \text{Normal}(\mu, \sigma^2)$:[I]

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right];\ \ y \in (\pm\infty) \implies \mu = \mu \text{ and } \sigma^2 = \sigma^2$$

$$M_Y(t) = \exp\left[\mu t + \frac{t^2\sigma^2}{2}\right]$$

- $Y \sim \text{Gamma}(\alpha, \beta)$:

$$f(y) = \frac{y^{\alpha-1}\exp\left[-y\beta^{-1}\right]}{\Gamma(\alpha)\beta^\alpha};\ \ y \in (0, \infty) \implies \mu = \alpha\beta \text{ and } \sigma^2 = \alpha\beta^2$$

$$M_Y(t) = (1-\beta t)^{-\alpha}$$

- $Y \sim \text{Exponential}(\beta) = Y \sim \text{Gamma}(1, \beta)$:

$$f(y) = \beta^{-1}\exp\left[-y\beta^{-1}\right];\ \ y \in (0, \infty) \implies \mu = \beta \text{ and } \sigma^2 = \beta^2$$

$$M_Y(t) = (1-\beta t)^{-1}$$

- $Y \sim \chi^2(\nu) = Y \sim \text{Gamma}(\nu/2, 2)$:

$$f(y) = \frac{y^{\nu/2-1}\exp\left[-y/2\right]}{\Gamma(\nu/2)2^{\nu/2}};\ \ y^2 > 0 \implies \mu = \nu \text{ and } \sigma^2 = 2\nu$$

- $Y \sim \text{Beta}(\alpha, \beta)$:

$$f(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1};\ \ y \in (0,1) \implies \mu = \frac{\alpha}{\alpha+\beta} \text{ and } \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

2.1 Special Cases

- $Y \sim \chi^2(1) = [Y \sim \text{Normal}(0,1)]^2$
- $Y \sim \text{Beta}(1,1) = Y \sim \text{Uniform}(0,1)$

2.2 Gamma Function

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}\,\mathrm{d}y$$

$$\Gamma(k \in \mathbb{N}) = (k-1)!$$

2.3 Standardizing the Normal Distribution

If $Y \sim \text{Normal}(\mu, \sigma^2) \implies$

$$Z = \frac{Y-\mu}{\sigma} \sim \text{Normal}(0,1)$$

[I] CDF is not in closed form.

3 Discrete Distribution Dictionary

- $Y \sim \text{Binomial}(n, p)$: Observing $y \in [0, n]$ successes in fixed n trials.

$$\text{P}(Y = y) = \binom{n}{y} p^y (1-p)^{n-y} \implies \mu = n \cdot p \text{ and } \sigma^2 = np(1-p)$$

$$M_Y(t) = \left[pe^t + (1-p)\right]^n$$

- $Y \sim \text{NB}(r, p)$: Observing the fixed r^{th} success in $y \in [r, \infty)$ trials.

$$\text{P}(Y = y) = \binom{y-1}{r-1} p^r (1-p)^{y-r} \implies \mu = \frac{r}{p} \text{ and } \sigma^2 = \frac{r(1-p)}{p^2}$$

$$M_Y(t) = \left[\frac{pe^t}{1-(1-p)e^t}\right]^r$$

- $Y \sim \text{Geometric}(p) = \text{NB}(1, p)$: Observing 1 success in $y \in [1, \infty)$ trials.

$$\text{P}(Y = y) = (1-p)^{y-1} p \implies \mu = \frac{1}{p} \text{ and } \sigma^2 = \frac{1-p}{p^2}$$

$$M_Y(t) = \frac{pe^t}{1-(1-p)e^t}$$

- $Y \sim \text{Hypergeometric}(N, r, n)$: Observing $y \in \begin{cases} [0, n] \text{ if } n \leq r, \\ [0, r] \text{ if } n > r \end{cases}$ successes in n draws, without replacement, from a population of N that contains r success states.

$$\text{P}(Y = y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}} \implies \mu = \frac{r \cdot n}{N} \text{ and } \sigma^2 = n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$$

- $Y \sim \text{Poisson}(\lambda)$: Observing $y \in [0, \infty)$ indepedent events that occur with a constant mean rate of λ in a fixed interval of time or area. Note: $Y \sim \text{Poisson}(n \cdot p) = Y \sim \text{Binomial}(n, p)$ if n is very large, or p is small. That is to say, it may be used to approximate the binomial distribution.

$$\text{P}(Y = y) = \frac{\lambda^y e^{-\lambda}}{y!} \implies \mu = \lambda \text{ and } \sigma^2 = \lambda$$

$$M_Y(t) = e^{\lambda(e^t - 1)}$$

3.1 Moment Generating Functions

MGF of Y:

$$M_Y(t) = \mathrm{E}[e^{tY}] = \int e^{ty} f(y) \, \mathrm{d}y$$

$k^{\mathbf{th}}$ **Moment of Y**: $\mathrm{E}[Y^k]$

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} M_Y(t) \Big|_{t=0} = \begin{cases} k = 1 & \Longrightarrow \mathrm{E}[Y] \\ k = 2 & \Longrightarrow \mathrm{E}[Y^2] \\ \vdots & \\ k = n & \Longrightarrow \mathrm{E}[Y^n] \end{cases}$$

4 Calculus

Chain Rule:

$$(f \circ g)' = (f \circ g)' g'$$

Product Rule:

$$(f \cdot g)' = f'g + fg'$$

Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Integration by parts:

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u$$

Power Rule:

$$\int x^n \, \mathrm{d}x = \frac{1}{n+1} x^{n+1}$$