HW1

Justin Nguyen

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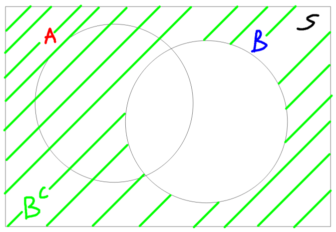
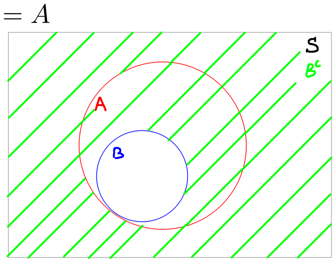
1. $A = A \cap S, S = B \cup B^c$

   (a) **Proof:** $A = (A \cap B) \cup (A \cap B^c)$
   $(A \cap B) \cup (A \cap B^c)$
   $= A \cap (B \cup B^c)$ (Distributive Law)
   $= A \cap S$ (Definition of a complement)
   $= A$

   (b) **Proof:** $B \subseteq A \implies A = B \cup (A \cap B^c)$
   $B \cup (A \cap B^c)$
   $= (B \cup A) \cap (B \cup B^c)$ (Distributive)
   $= (B \cup A) \cap S$ (Definition of a complement)
   $= A \cap S$
   $= A$

2. (a) **Proof:** $P(A) = P(A \cap B) + P(A \cap B^c)$
   We know $A = A \cap (B \cup B^c) = A \cap S = A \implies P(A) = P(A \cap (B \cup B^c))$. (Axiom)
   Thus $P(A) = P((A \cap B) \cup (A \cap B^c))$. (distributive)
   B, B^c are disjoint $\implies A \cap B, A \cap B^c$ are also disjoint.
   $\therefore P(A) = P(A \cap B) + P(A \cap B^c)$.

   (b) **Proof:** $P(A \cap B) = P(B) - P(A^c \cap B)$
   $P(B) = P(A \cap B) + P(A^c \cap B)$ (From 2a)
   $P(A \cap B) = P(A \cap B) + P(A^c \cap B) - P(A^c \cap B) \iff P(A \cap B) = P(A \cap B)$

   (c) If $B \subseteq A \implies P(A) = P(B) + P(A \cap B^c)$
   Counterexample: Let $S = \{1, 2 \cdots 10\}, A = \{1, 2, 3\}, B = \{1\}$
   $P(B) + P(A \cap B^c) = \frac{3}{10} + \frac{2}{10} \neq P(A) = \frac{3}{10}$

3. (a) $A \cap B$
   (b) $A \cup B$
   (c) $(A \cap B)^c$
   (d) $(A \cap B^c) \cup (A^c \cap B)$

4. (a) $\sum_{i=1}^{5} E_i = 1 \implies P(E_4) = 0.3, P(E_5) = 0.15$
   (b) $P(E_1) = 0.3 \implies P(E_2) = 0.1 \implies P(E_{3 \ldots 5}) = 0.2$

5. $P(s) = 8\%, P(b) = 6\%, P(s \cap b) = 2\%$
   (a) $P(b) = 6\%$
   (b) $P(b \cup s) = P(b) + P(s) - P(s \cap b) = 12\%$
   (c) $P(b \cap s^c) + P(s \cap b^c)$
   $= [P(b) - P(b^c \cap s^c)] + [P(s) - P(s^c \cap b^c)]$ (from 2a)
   $= P(b) + P(s) - 2P(s^c \cap b^c)$
   $= P(b) + P(s) - 2P(s \cup b) = 10\%$

6. $P(H) = 70\%, P(D) = 30\%, P(H - D) = P(H \cap D^c) = 60\%$
   $P(D - H) = P(D \cap H^c)$
   $= P(D) + P(H^c) - P(D \cup H^c)$
   $= 60\% - (100\% - P(H \cap D^c)) = 20\%$

7. (a) $S = \{\text{HH, HT, TH, TT}\}$
   (b) Yes, all points are equally likely. Each one has a 25% probability.
   (c) $A = \{\text{HT, TH}\}, B = \{\text{HH, HT, TH}\}$
   (d) i. $P(A) = \frac{2}{4}$
   ii. $P(B) = \frac{3}{4}$
   iii. $P(A \cap B) = \frac{2}{4}$
   iv. $P(A \cup B) = \frac{3}{4}$
   v. $P(A^c \cup B) = \frac{1}{4}$

8. (a) $S = \{$
   $(V_1, V_1), (V_1, V_2), (V_1, V_3),$
   $(V_2, V_1), (V_2, V_2), (V_2, V_3),$
   $(V_3, V_1), (V_3, V_2), (V_3, V_3)$
   $\}$
   (b) All points are equally likely. Thus each is $\frac{1}{9}$.
   (c) $A = \{(V_1, V_1), (V_2, V_2), (V_3, V_3)\}, B = \{(V_1, V_2), (V_2, V_1), (V_2, V_2), (V_2, V_3), (V_3, V_2)\}$
   i. $P(A) = \frac{3}{9}$
   ii. $P(B) = \frac{5}{9}$
   iii. $P(A \cup B) = \frac{7}{9}$
   iv. $P(A \cap B) = \frac{1}{9}$

9. (a) Let a be the member of the minority group.
   $S = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$
   (b) All points are equally likely, thus each event has probability $\frac{1}{6}$.
   (c) $P(a) = \frac{3}{6}$

10. **Proof:** $A_1, A_2, \ldots$ is a partition of S, and $B \subseteq S \implies P(B) = \sum_{i=1}^{\infty} P(B \cap A_i)$
   $(\forall i, j) A_i, A_j$ are disjoint $\implies B \cap A_i, B \cap A_j$ are disjoint and thus have no overlap.
   If A_i, A_j were not disjoint, then it would not form a partition of S, which would contradict the given information.
   $\therefore P(B) = \sum_{i=1}^{\infty} P(B \cap A_i)$.