HW2

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1. $9 \cdot 10^6 = 9,000,000$ seven-digit phone numbers.
2. (a) $8! = 40320$ seatings
   (b) $5! \cdot 4 \cdot 3! = 2880$ consecutive male seatings.
   (c) $4! \cdot 2^4 = 384$ consecutive couples seatings.
3. $6m, 7s, 4e \to 2$ books
   (a) $\binom{6}{2} + \binom{7}{2} + \binom{4}{2} = 42$ choices of the two books who share the same subject.
   (b) $\binom{6}{1}\binom{7}{1} + \binom{6}{1}\binom{4}{1} + \binom{7}{1}\binom{4}{1} = 94$ choices of two books who don't share the same subject.
4. $8w, 6m \to 3w, 3m$
   (a) $\binom{8}{3}[\binom{6}{3} - \binom{4}{1}] = 896$ committees where m_1, m_2 don't work together.
   (b) $\binom{6}{3}[\binom{8}{3} - \binom{6}{1}] = 1000$ committees where w_1, w_2 don't work together.
   (c) $\binom{8}{3}\binom{6}{3} - P_3^7 = 910$ committees where w_1, m_1 don't work together.[1]
5. Let $D_6 = \{1,2,3,4,5,6\}$, $A =$ rolling $1 \ldots 6$ in any order, and $|S| = |D_6|^6$.
   If the point $r_1 = (1,2,3,4,5,6) \in A \implies$ other points in A must be arrangements of $r_1 \implies |A| = |D_6|!$
   $\therefore \mathrm{P}(A) = \frac{6!}{6^6} \approx 0.015$
6. $|S| = \binom{10}{5}, |A| = \binom{6}{5} \implies \mathrm{P}(A) = \frac{|A|}{|S|} \approx 0.02\ldots$
7. $4s_w, 2s_b, 6s_r, 3s_g \to 4s$
   $|S| = \binom{15}{4}$
   (a) $P(2s_1, 2s_2) = \frac{\binom{4}{2}[\binom{2}{2}+\binom{6}{2}+\binom{3}{2}]+\binom{2}{2}[\binom{6}{2}+\binom{3}{2}]+\binom{6}{2}\binom{3}{2}}{\binom{15}{4}} = \frac{177}{1365} \approx 0.13$
   (b) $P(1s_r) = 1 - P(\text{no reds}) = 1 - \frac{P_4^9}{P_4^{15}} \approx 0.91$
8. 52 cards $\to$ 5 cards; $|S| = \binom{52}{5}$
   (a) $\mathrm{P}(3A, 2K) = \frac{\binom{4}{3}\binom{4}{2}}{\binom{52}{5}} \approx 0.000009\ldots$
   (b) $\mathrm{P}(\text{full house}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} \approx 0.001\ldots$
9. $2w, 4h, 7a \to 1w, 2h, 3a$
   If every claim is different, and the process order doesn't matter, then $|S| = \binom{13}{6}$
   There is only one way to select $\{w_1, h_1, h_2, a_1, a_2, a_3\} \implies |A| = 1$
   $\therefore \mathrm{P}(A) = \frac{1}{\binom{13}{6}} \approx 0.0006\ldots$
10. (a) Fluke $= \binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1} = 120$ arrangements
    (b) Propose $= \binom{7}{2}\binom{5}{1}\binom{4}{2}\binom{2}{1}\binom{1}{1} = 1260$ arrangements
    (c) Mississippi $= \binom{11}{1}\binom{10}{4}\binom{6}{4}\binom{2}{2} = 34650$ arrangements
11. $3u, 4r, 2z, 1c$; $|S| = \binom{10}{3,4,3} = 4200$ rankings
    $P(1 \text{ winner, } 2 \text{ losers}) = \frac{\binom{7}{2,4,1}}{4200} = 0.025$
12. $9m = 2m_\alpha + 7m_x \to 3p_1, 3p_2, 3p_3$; $|S| = \binom{9}{3,3,3} = 1680$ outcomes
    $P(2m_\alpha \to 3p_1) = \frac{\binom{7}{1,3,3}}{1680} = 0.08\overline{3}$

[1] P_3^7 is a simplification of the number of committees where w_1, m_1 work together.

13. **Proof:** $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
The Binomial Theorem states that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.
Then, let $x = y = 1 \implies (1+1)^n = (2)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k} 1^n = \sum_{k=0}^{n} \binom{n}{k}$.
$\therefore \sum_{k=0}^{n} \binom{n}{k} = 2^n$.

14. **Proof:** $\sum_{k=0}^{n>0} (-1)^k \binom{n}{k} = 0$
If $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \implies (1-1)^n = 0 = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$
$\therefore \sum_{k=0}^{n>0} (-1)^k \binom{n}{k} = 0$.

15. **Proof:** $\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$
Proven earlier, $2^n = \sum_{k=0}^{n} \binom{n}{k} \implies \frac{n}{2} \cdot 2^n = n2^{n-1} = \frac{n}{2} \sum_{k=0}^{n} \binom{n}{k}$