HW3

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1. $\mathrm{P}(A)=0.5, \mathrm{P}(B)=0.3, \mathrm{P}(A\cap B)=0.1 \implies \mathrm{P}(A\cup B)=0.7$
   (a) $\mathrm{P}(A \mid B)=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}=\frac{0.1}{0.3}=\frac{1}{3}=0.\overline{3}$
   (b) $\mathrm{P}(B \mid A)=\frac{\mathrm{P}(B\cap A)}{\mathrm{P}(A)}=\frac{0.3}{0.5}=\frac{3}{5}=0.6$
   (c) $\mathrm{P}(A \mid A\cup B)=\frac{\mathrm{P}(A\cap(A\cup B))}{\mathrm{P}(A\cup B)}=\frac{\mathrm{P}(A)+\mathrm{P}(A\cup B)-\mathrm{P}(A\cup(A\cup B))}{\mathrm{P}(A\cup B)}=\frac{\mathrm{P}(A)+\mathrm{P}(A\cup B)-\mathrm{P}(A\cup B)}{\mathrm{P}(A\cup B)}=\frac{\mathrm{P}(A)}{\mathrm{P}(A\cup B)}=\frac{1}{2}$
   (d) $\mathrm{P}(A \mid A\cap B)=\frac{\mathrm{P}(A\cap(A\cap B))}{\mathrm{P}(A\cap B)}=1$
   (e) $\mathrm{P}(A\cap B \mid A\cup B)=\frac{\mathrm{P}((A\cap B)\cap(A\cup B))}{\mathrm{P}(A\cup B)}=\frac{\mathrm{P}(((A\cap B)\cap A)\cup((A\cap B)\cap B))}{\mathrm{P}(A\cup B)}=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(A\cup B)}=\frac{1}{7}$
2. $\mathrm{P}(A)=0.2, \mathrm{P}(B)=0.3, \mathrm{P}(A\cup B)=0.4$
   (a) $\mathrm{P}(A\cap B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cup B)=0.1$
   (b) $\mathrm{P}((A\cap B)^c)=1-\mathrm{P}(A\cap B)=0.9$
   (c) $\mathrm{P}((A\cup B)^c)=1-\mathrm{P}(A\cup B)=0.6$
   (d) $\mathrm{P}(A^c \mid B)=\frac{\mathrm{P}(A^c\cap B)}{\mathrm{P}(B)}=\frac{\mathrm{P}(B)-\mathrm{P}(A\cap B)}{\mathrm{P}(B)}=\frac{2}{3}$ [I]
3. $\mathrm{P}(E_1)=0.9, \mathrm{P}(E_2 \mid E_1)=0.8, \mathrm{P}(E_3 \mid E_1\cap E_2)=0.7$
   $\mathrm{P}(E_2)=\mathrm{P}(E_2\cap E_1)+\mathrm{P}(E_2\cap E_1^c) \implies \mathrm{P}(E_2)=\mathrm{P}(E_2\cap E_1)+\mathrm{P}(\emptyset)$ (No chance of passing E_2 if E_1 was failed.)
   $\mathrm{P}(E_2 \mid E_1)=\frac{\mathrm{P}(E_2\cap E_1)}{\mathrm{P}(E_1)} \implies \mathrm{P}(E_2)=(0.8)(0.9)$
   $\mathrm{P}(E_3 \mid E_2\cap E_1)=\frac{\mathrm{P}(E_3\cap E_2\cap E_1)}{\mathrm{P}(E_1\cap E_2)} \implies \mathrm{P}(E_3\cap E_2\cap E_1)=(0.7)(0.8)(0.9)$
   $\therefore \mathrm{P}(E_3\cap E_2\cap E_1)=\frac{63}{125}=0.504$
4. $\mathrm{P}(6 \mid i)=\begin{cases} i\le 6 \implies 0 \\ i=7 \implies \frac{1}{6} \\ i=8 \implies \frac{1}{5} \\ i=9 \implies \frac{1}{4} \\ i=10 \implies \frac{1}{3} \\ i=11 \implies \frac{1}{2} \\ i=12 \implies 1 \end{cases}$
5. $|S|=\frac{\binom{6}{1}^2}{\binom{2}{1}}=18, S_5=\{\{1,4\},\{2,3\}\}, S_7=\{\{1,6\},\{2,5\},\{3,4\}\}$
   $\mathrm{P}(5)=\frac{1}{9}, \mathrm{P}(7)=\frac{1}{6}$
   $\mathrm{P}(5 \text{ before } 7)=1\cdot\mathrm{P}(5)+0\cdot\mathrm{P}(7)+(1-\mathrm{P}(5)-\mathrm{P}(7))=\frac{2}{5}$ [II]
6. (a) **Assumptions:** $0<k\le n$; $\mathrm{P}(1^{\text{st}})=\frac{1}{n}, \mathrm{P}(2^{\text{nd}})=(1-\mathrm{P}(1^{\text{st}}))\cdot\frac{1}{n-1}=\frac{n-1}{n}\cdot\frac{1}{n-1}=\frac{1}{n}$
   $\implies \mathrm{P}(k_d^{\text{th}})=(1-\mathrm{P}(1^{\text{st}}))(1-\mathrm{P}(2^{\text{nd}}))\dots(1-\mathrm{P}(k-2))(1-\mathrm{P}(k-1))\frac{1}{n-k+1}=\frac{1}{n}$ [III]
   (b) **Assumptions:** Let w_k denote the wrong k^{th} key. $\mathrm{P}(w_1)=\frac{n-1}{n}, \mathrm{P}(w_2)=\mathrm{P}(w_1)^2 \implies \mathrm{P}(w_k)=\mathrm{P}(w_1)^k$
   The probability of selecting the correct key is $\mathrm{P}(k_c)=\frac{1}{n}$.
   $\therefore \mathrm{P}(k^{th})=\mathrm{P}(w_k)\cdot\mathrm{P}(k_c)=(\frac{n-1}{n})^k\cdot\frac{1}{n}$

[I] Alternatively, $\mathrm{P}(A^c \mid B)=1-\mathrm{P}(A \mid B)$

[II] Alternatively, I was shown: $\mathrm{P}(5 \text{ before } 7)=\frac{\mathrm{P}(5)}{\mathrm{P}(5)+\mathrm{P}(7)}=\frac{2}{5}$, and this looks like Bayes' Thm.

[III] This is amazing. Why does it work out this way?

7. $\mathrm{P}(R_E \cup R_O) = 85\%, \mathrm{P}(R_E) = 75\%, \mathrm{P}(R_E \cap R_O) = \mathrm{P}(R_E)\,\mathrm{P}(R_O)$
   $75\%\,\mathrm{P}(R_O) = 75\% + P(R_O) - 85\%$
   $\therefore \mathrm{P}(R_O) = 40\%$

8. (i) $\mathrm{P}(C_C) = 2\,\mathrm{P}(C_D)$ (ii) $\mathrm{P}(C_C \cap C_D) = \mathrm{P}(C_C)\,\mathrm{P}(C_D)$ (iii) $\mathrm{P}(C_C \cap C_D) = 0.15$
   $\mathrm{P}(C_C)\,\mathrm{P}(C_D) = 2\,\mathrm{P}(C_D)^2 \implies \mathrm{P}(C_D) = \frac{\sqrt{30}}{20}, \mathrm{P}(C_C) = \frac{\sqrt{30}}{10}$
   $\mathrm{P}((C_C \cup C_D)^c) = 1 - \mathrm{P}(C_C \cup C_D) = 1 - (\mathrm{P}(C_C) + \mathrm{P}(C_D) - \mathrm{P}(C_C \cap C_D)) = \frac{23-3\sqrt{30}}{20} \approx 0.328\ldots$

9. $\mathrm{P}(-|B) = 30\%, \mathrm{P}(-|A) = 60\%;\ \mathrm{P}(A) = \frac{1}{4}, \mathrm{P}(B) = \frac{3}{4}$
   $\mathrm{P}(A \mid -) = \frac{\mathrm{P}(A)\,\mathrm{P}(-|A)}{\mathrm{P}(A)\,\mathrm{P}(-|A)+\mathrm{P}(B)\,\mathrm{P}(-|B)} = \frac{2}{5} = 0.4$

10. $\mathrm{P}(I) = 46\%, \mathrm{P}(L) = 30\%, \mathrm{P}(C) = 24\%$
    $\mathrm{P}(V \mid I) = 35\%, \mathrm{P}(V \mid L) = 62\%, \mathrm{P}(V \mid C) = 58\%$

    (a) $\mathrm{P}(I \mid V) = \frac{\mathrm{P}(V|I)\,\mathrm{P}(I)}{\mathrm{P}(V|I)\,\mathrm{P}(I)+\mathrm{P}(V|L)\,\mathrm{P}(L)+\mathrm{P}(V|C)\,\mathrm{P}(C)} = \frac{805}{2431} \approx 0.331\ldots$
    (b) $\mathrm{P}(L \mid V) = \frac{\mathrm{P}(V|L)\,\mathrm{P}(L)}{\mathrm{P}(V|I)\,\mathrm{P}(I)+\mathrm{P}(V|L)\,\mathrm{P}(L)+\mathrm{P}(V|C)\,\mathrm{P}(C)} = \frac{930}{2431} \approx 0.382\ldots$
    (c) $\mathrm{P}(C \mid V) = \frac{\mathrm{P}(V|C)\,\mathrm{P}(C)}{\mathrm{P}(V|I)\,\mathrm{P}(I)+\mathrm{P}(V|L)\,\mathrm{P}(L)+\mathrm{P}(V|C)\,\mathrm{P}(C)} = \frac{696}{2431} \approx 0.286\ldots$
    (d) $\frac{\text{2431 participating voters}}{\text{5000 voters}} = \frac{(35)(42)+(62)(30)+(58)(24)}{100^2} = \mathrm{P}(V \mid I)\,\mathrm{P}(I) + \mathrm{P}(V \mid L)\,\mathrm{P}(L) + \mathrm{P}(V \mid C)\,\mathrm{P}(C)$[IV]

11. $B_1 = (b, w), B_2 = (2b, w);\ \mathrm{P}(B_1) = \mathrm{P}(B_2) = \frac{1}{2}$

    (a) $\mathrm{P}(b) = \mathrm{P}(b \mid B_1)\,\mathrm{P}(B_1) + \mathrm{P}(b \mid B_2)\,\mathrm{P}(B_2) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{2}{3} = \frac{7}{12}$
    (b) $\mathrm{P}(w) = \frac{5}{12}$
    $\mathrm{P}(B_1 \mid w) = \frac{\mathrm{P}(w|B_1)\,\mathrm{P}(B_1)}{\mathrm{P}(w)} = \frac{3}{5}$

12. $D_6 = \{1\ldots6\};\ 5w + 10b = |D_6|$
    $|S_{n\in D_6}| = \binom{15}{n}$ and $\mathrm{P}(\text{any roll}) = \frac{1}{6}$

    (a) Let $W_{k\in D_6}$ denote the events of getting k white marbles with a k dice roll. Thus, $|W_{k\in D_6}| = \binom{5}{k}$. Note $|W_6| = 0$.
    Then, $\mathrm{P}(W) = \sum_{k\in D_6} \frac{|W_k|}{|S_k|} \cdot P(\text{roll } k) = \frac{5}{66} = 0.0\overline{75}$.
    (b) $\mathrm{P}(3) = \frac{1}{6}, \mathrm{P}(W_3 \mid 3) = \frac{2}{91} \implies \mathrm{P}(3 \mid W_3) = \frac{\mathrm{P}(W_3|3)\,\mathrm{P}(3)}{\mathrm{P}(W_3)} = \frac{1}{6} = 0.1\overline{6}$ [V]

13. Let $A, B : A \cap B = \emptyset$ and $\mathrm{P}(B) > 0$.
    Proof: $\mathrm{P}(A \mid A \cup B) = \frac{\mathrm{P}(A)}{\mathrm{P}(A)+\mathrm{P}(B)}$
    This immediately follows from $\mathrm{P}(A \mid A \cup B) = \frac{\mathrm{P}(A\cap(A\cup B))}{\mathrm{P}(A\cup B)} = \frac{\mathrm{P}(A)}{\mathrm{P}(A)+\mathrm{P}(B)}$
    ■

14. **Proof:** $\mathrm{P}(A \cap B) \geq 1 - \mathrm{P}(A^c) - \mathrm{P}(B^c)$

    $$\mathrm{P}(A \cap B) = \mathrm{P}(A) - \mathrm{P}(A \cap B^c)$$

    $$\mathrm{P}(A \cap B) = \mathrm{P}(A) - [\mathrm{P}(A) + \mathrm{P}(B^c) - \mathrm{P}(A \cup B^c)] = \mathrm{P}(A \cup B^c) - \mathrm{P}(B^c)$$

    $$\mathrm{P}(A \cup B^c) - \mathrm{P}(B^c) \geq 1 - \mathrm{P}(A^c) - \mathrm{P}(B^c) \iff \mathrm{P}(A \cup B^c) \geq 1 - \mathrm{P}(A^c) \iff \mathrm{P}(A \cup B^c) \geq \mathrm{P}(A)$$

    $$\therefore \mathrm{P}(A \cap B) \geq 1 - \mathrm{P}(A^c) - \mathrm{P}(B^c)$$

[IV] This is weird. Why does this make sense?
[V] This makes sense, and is also funny, because the probability of getting a 3 on a dice roll doesn't depend on anything. This is akin to proving $x = x$ in a proof.