

Complex Geometry - Index of Notations and ideas

Taper

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Part I

Indices of Notation

Chapter 1

Book

1.1 1. Local Theory

1.1.1 1.1 Holomorphic Functions of Several Variables

Note: the content covered by this section is geared for accompanying my personal notes of lecture 1.

[holomorphic](#): pp.1. pp.4. Def 1.1.1. pp.10(Def.1.1.8).

[Cauchy-Riemann equations](#): pp.2

$\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$: $\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$

[Maximum principle](#): pp.3.

[Identity theorem](#): pp.3.

[Riemann extension theorem](#): pp.3. pp.9 (Prop. 1.1.7).

[Riemann mapping theorem](#): pp.3.

[Liouville theorem](#): pp.4.

[Residue theorem](#): pp.4.

[polydiscs](#) $B_\epsilon(\omega)$: $\{z \mid |z_i - \omega_i| < \epsilon\}$. pp.4.

[Hartogs' theorem](#): Prop. 1.1.4. pp.6.

[Weierstrass preparation theorem \(WPT\)](#): Prop. 1.1.6. pp.8.

[Weierstrass polynomial](#): Def. 1.1.5. pp.7.

$Z(f)$: zero set of f . pp.9.

[biholomorphic](#): pp.10.

[\(complex\) Jacobian, regular, regular value](#): Def. 1.1.9. pp.10.

[IFT. Inverse function theorem](#): Prop 1.1.10 pp.11.

[IFT. Implicit function theorem](#): Prop 1.1.11. pp.10.

$\mathcal{O}_{\mathbb{C}^n}$: sheaf of holomorphic functions on \mathbb{C}^n . Def. 1.1.14. pp.14.

$\mathcal{O}_{\mathbb{C}^n, z}$: Def. 1.1.14. pp.14.

$\mathcal{O}_{\mathbb{C}^n, 0}^*$: units of $\mathcal{O}_{\mathbb{C}^n, 0}$. pp.14.

[UFD, unique factorization domain, irreducible](#): Def. 1.1.16. pp.14.

[Gauss Lemma](#): pp.14.

[Weierstrass division theorem](#): Prop. 1.1.17. pp.15.

[germ of set](#): (**pp.18**)

$Z(f)$: germ of zero set of f . (pp.18)
 analytic germ: $Z(f_1, \dots, f_k)$. (pp.18)
 analytic subset: locally are zero sets. (pp.18)
 $I(X)$: the set of all $f \in \mathcal{O}_{\mathbb{C}^n, 0}$ with $X \subset Z(f)$. (pp.18)

1.1.2 1.2 Complex and Hermitian Structures

almost complex structure, I : $I^2 = -\text{id}$. (pp.25)
 $V^{1,0}$ and $V^{0,1}$: the $\pm i$ eigenspaces of I . (pp.25)
 $\bigwedge^{p,q} V := \bigwedge^p V^{1,0} \oplus \bigwedge^q V^{0,1}$. (pp.27)
 $\prod^k, \prod^{p,q}$: natural projections. (pp.28)
 $I := \sum_{p,q} i^{p-q} \cdot \prod^{q,p}$. (pp.28)
 compatible: an almost complex structure I is compatible with the scalar product \langle, \rangle , if $\langle I(v), I(w) \rangle = \langle v, w \rangle$. (pp.28)
 Conformal equivalence (between scalar product): (pp.29)
 fundamental form, $\omega := \langle I(\cdot), \cdot \rangle$. (pp.29)

$$\omega = \frac{i}{2} \sum_i z^i \wedge \bar{z}^i = \sum_i x^i \wedge y^i.$$

(Local calculation could be found on pp.31)
 hermitian form $(\cdot, \cdot) := \langle, \rangle - i \cdot \omega$. (pp.30)
 Lefschetz operator: $L : \bigwedge^* V_{\mathbb{C}}^* \rightarrow \bigwedge^* V_{\mathbb{C}}^*$, given by $\alpha \rightarrow \omega \wedge \alpha$.

1.2 2.Complex Manifolds

1.2.1 2.1 Complex and Hermitian Structures

almost complex structure: (pp.25)

1.2.2 2.2 Holomorphic Vector Bundles

τ_X : holomorphic tangent bundle of a complex manifold X (Def 2.2.14 at pp. 71).

Ω_X, Ω_X^p : holomorphic cotangent bundle and holomorphic p -forms. (Def 2.2.14 at pp. 71)

$K_X := \det(\Omega_X) = \Omega_X^n$, the canonical bundle of X . (Def 2.2.14 at pp. 71)

1.2.3 2.6 Differential Calculus on Complex Manifolds

$\bigwedge_{\mathbb{C}}^k X := \bigwedge^k (T_{\mathbb{C}} X)^*$. (Def 2.6.7 at pp. 105)

$\bigwedge^{p,q} X := \bigwedge^p (T^{1,0} X)^* \oplus \bigwedge^q (T^{0,1} X)^*$. (Def 2.6.7 at pp. 105)

$\mathcal{A}_{X, \mathbb{C}}^k, \mathcal{A}_X^{p,q}$: sheaves of section of the above correspond items. (Def 2.6.7 at pp. 105)

$\mathcal{A}^{p,q}(E)$: the sheaf of p, q -forms with values in E , a complex vector bundle. (Def 2.6.22 at pp.109). Note that in particular, $\mathcal{A}^0(E)$ is the sheaf of sections of E .

1.2.4 Appendix B: Sheaf Cohomology

- [illegible]

Chapter 2

My lecture Notes

2.1 Lecture 2016 Lecture 1

The first few lectures are not well noted, hence I delegate the task of recording the theorems and notations to the book's corresponding section: section 1.1.1 on page 7.

2.2 Lecture 4 (20160307) Complex Manifold

Note: we use abbreviation *mnfd* for *manifold*.

pp. A:

- [Holomorphic Atlas](#)
- [Holomorphic chart](#)
- [Complex mnfd](#)

pp. B:

- [Holomorphic function](#)
- \mathcal{O}_X : sheaf of holomorphic functions on a complex mnfd X .

pp. C:

- [Hartogs' theorem](#): on complex mnfd.
- [Holomorphic functions on complex mnfd](#):

pp. D:

- [Complex Lie group](#)
- [Complex Projective Space, \$\mathbb{CP}^n\$, or just \$\mathbb{P}^n\$.](#)

pp. E:

- [Topology in \$\mathbb{P}^n\$](#)
- [Mnfd structure on \$\mathbb{P}^n\$, atlas, and the **canonical covering**](#)

pp. F

- Grassmannian mnfd.

2.3 Lecture 5 Submanifolds (20160308)

pp. A:

- Affine Hypersurface (actually this is not quite different from the usual \mathbb{C}^n .)

Part 2. **Sheaf Theory**

pp. A:

- pre-sheaf
- $\mathcal{O}_X(U)$
- $\mathcal{O}_X^*(U)$

pp. B:

- C^∞
- $\underline{\mathbb{Z}}$, sometimes simply denoted as \mathbb{Z} : sheaf of locally constant \mathbb{Z} -valued functions.
- Sheaf

pp. D:

- sheaf-morphisms

pp. E:

- Section
- $\text{Ker}(\phi)$ - sheaf of kernels.

pp. F:

- $\text{Im}(\phi)$ is a presheaf, but not a sheaf.
- $\text{Im}(\phi)$: the sheafification of $\text{Im}(\phi)$ above. Note that we use the same notation to denote both.

2.4 Lecture 6 Sheaf & Cohomology (20160315)

pp. A:

- Stalk \mathcal{F}_x .
- germ
- Directed partial order set
- Directed System

pp. B:

- Directed limit

pp. C:

- Exact Complex/ Exact Sequence.
- Exponential sequence (*mentioned under the definition of exact sequence*).
-

pp. D:

- Čech cohomology

pp. E:

- q-cochain
- coboundary operator. δ .
- $Z^p(U, \mathcal{F}) = \text{Ker.}$
- $B^p(U, \mathcal{F}) = \text{Im.}$
- $\check{H}^p(U, \mathcal{F}) = \frac{\text{Ker}}{\text{Im.}}$.

2.4.1 Notes of Čech Cohomology with Coefficients in a Sheaf

pp.1:

- q-simplex σ .
- support $|\sigma|$.
- q-cocain
- $C^q(U, \mathcal{F})$
- Coboundary Operator δ .

pp.2,3,4:

- Cochain Complex
- Čech cohomology
- cocycle
- cochain
- $\check{H}^p(U, \mathcal{F}), Z^p(U, \mathcal{F}), B^p(U, \mathbb{F})$.
- $\check{H}^0(\{u_i\}, \mathcal{F}) = \mathcal{F}(X)$.

2.5 Lecture 7 Vector Bundle (20160321)

pp.1,2:

- Vector Bundle
- Trivializing covering, $\{(U_i, \tau_i)\}$.
- trivializing maps, trivializes.
- VB-equivalent of trivializing maps.
- E: total space, X: base space.

pp. 3,5:

- transition maps.
- fibre.
- $\mathcal{O}(-1)$
- cocycle condition.
- \mathcal{T}_X , Holomorphic tangent bundle.

pp. 8:

- s: section of a holomorphic vector bundle.
- \mathcal{E} : sheaf of sections of holomorphic vector bundle. $\mathcal{E}(U)$.

2.6 Lecture 8 Almost Complex Structures (20160322)

pp. 1,2:

- I : Almost Complex Structure. $I^2 = -1$. Sometime J is used in place of I .
- $V_{\mathbb{C}} := V \otimes \mathbb{C}$.
- $I_{\mathbb{C}}$: I extending to $V_{\mathbb{C}}$. Usually abbreviated simply as I .
- $V^{1,0} := \ker(I + i)$.
- $V^{0,1} := \ker(I - i)$.

2.7 Lecture 9 Exterior Algebra on Complex Manifold (20160329)

pp.1,2:

- V^* : dual of V .
- $\{dx^i, dy^i\}$.
- J^* : J extending to dual space.
- $dz^i, d\bar{z}^i$.

pp. 3:

- $S^k(V), \Lambda^k(V)$.
- s and a , symmetrization and anti-symmetrization of a tensor.
- $\Lambda^* V$.

pp. 4:

- $\Lambda^n T_{\mathcal{C}}^* X$.
- $\Lambda^* T_{\mathcal{C}}^* X$.
- $\Lambda^{p,q} T_{\mathcal{C}}^* X$.

pp. 5,6:

- \mathcal{A} : sheaf of section of cotangent bundle.
- $\mathcal{A}^n(U), \mathcal{A}^{p,q}(U)$.
- Λ on \mathcal{A} .
- d : de Rham differential.
- $\partial, \bar{\partial}$.

2.8 Lecture 10 Debeault Cohomology (20160406)

pp. 1:

- $\mathcal{H}^{p,q}(X)$.
- f^* : pull-back. Various definition from pp.1 to pp.4.

pp. 5,6,7:

- $\mathcal{A}^{p,q}(U, E) := \Gamma(U, \Lambda^{p,q} T_{\mathbb{C}}^* X \otimes E)$.
- $\bar{\partial}_E$
- $\mathcal{H}^{p,q}(X, E)$.
- $\bar{\partial}$ -Poincaré lemma in one variable.

2.9 Lecture 11 (20160412)

pp.1,2,3:

- $\bar{\partial}$ -Poincaré lemma in n-dimension
- Ω_X^p : holomorphic p-forms. On pp.2.
- $\check{H}^q(X, \Omega^p)(\check{\text{Cech}}) \cong \mathcal{H}_{\bar{\partial}}^{p,q}(X)(\text{Dolbeault})$. On pp.3.

pp. 6,7:

- Analytic Subvarity.
- Analytic Hybersurface.
- Cousin's Problem.

2.10 Lecture 12 Hermitian Structure on Manifold Manifold (20160418)

pp. 1,2,3:

- I compatible with $\langle -, - \rangle$.
- ω : Fundamental form associated with \langle, \rangle and I . $\omega(v, w) := \langle I(v), w \rangle$.
- Conformal Equivalence.
- \langle, \rangle : Hermitian Inner Product.

pp. 4:

- $(,): \text{ s.t. } (v, w) := \langle v, w \rangle - i\omega(v, w) = \langle v, w \rangle - i \langle I(v), w \rangle$

pp. 5:

- $\langle, \rangle_{\mathbb{C}}$ be s.t. $\langle v \otimes \alpha, w \otimes \beta \rangle := \alpha \bar{\beta} \langle v, w \rangle$.

pp. 6:

- $\frac{1}{2}(\langle, \rangle) = \langle, \rangle_{\mathbb{C}}|_{V^{1,0}}$

pp. 7,8:

- Local computations: z_i, h_{ij} ,
- $\omega = (\dots dx^i \dots dy^i)$
- ω , Fundamental form on Riemannian Mnfd.
- Kähler mnfd: $d\omega \equiv 0$.

2.11 Lecture 13 Kähler Manifold (20160419)

pp.1:

- Local computation: $\omega = (\dots dz^i \dots d\bar{z}^i)$

pp.4:

- Fubini-Study Metric on \mathbb{CP}^n .

2.12 Lecture 14 Hodge Theory (20160425)

pp.1:

- \langle, \rangle on $\Lambda^k V$
- vol: volume element.
- $*$: Hodge Star Operator.

pp.4:

- L : Lefschetz Operator
- Λ : adjoint of L . $\Lambda = *^{-1} \circ L \circ *$.

pp.5 :

- $*$, L , Λ on Kähler mnfd.
- $d^* := (-1)^{m*(k+1)+1} * \circ d \circ *$, adjoint of d . On a Kähler mnfd, $d^* = -* \circ d \circ *$
- $\Delta := d^* \circ d + d \circ d^*$.

pp. 6:

- $\bar{\partial}^*, \partial^*$: Similar to the above for d .
- $\Delta_{\partial}, \Delta_{\bar{\partial}}$: Similar to the above for d .

2.13 Lecture 15 Hodge Theory on Manifold (20160426)

pp.1:

- $(,)$ on $\mathcal{A}^*(X)$. $(\alpha, \beta) := \int_X g_{\mathbb{C}}(\alpha, \beta) \text{vol}$

pp.3:

- $\mathcal{H}^k(X, g)$: d-harmonic forms. Sometimes we replace \mathcal{H} with \mathcal{H} for harmonic forms, so is for symbols below.
- $\mathcal{H}_{\bar{\partial}}^k(X, g)$: $\bar{\partial}$ -harmonic forms. (Be careful to distinguish this with Dolbeault Cohomology groups).
- $\mathcal{H}_{\partial}^k(X, g)$: ∂ -harmonic forms.

pp. 5:

- $\mathcal{H}_d^k(X, g) \cong \mathcal{H}_d^{2n-k}(X, g)$, Poincaré duality
- $\mathcal{H}_{\bar{\partial}}^{p,q}(X, g) \cong (\mathcal{H}_{\bar{\partial}}^{n-p, n-q}(X, g))^*$, both are harmonic forms, called Serre Duality.

pp. 6,7:

- $\mathcal{A}^{p,q} = \bar{\partial} \mathcal{A}^{p,q-1}(X) \oplus \bar{\partial}^* \mathcal{A}^{p,q+1}(X) \oplus \mathcal{H}_{\bar{\partial}}^{p,q}(X, g)$:
Hodge decomposition
- $\mathcal{H}_{\bar{\partial}}^{p,q}$ (harmonic forms) $\cong \mathcal{H}_{\bar{\partial}}^{p,q}(X)$ (Dolbeault Cohomology group)
- $\mathcal{H}_d^{p,q}$ (harmonic forms) $\cong \mathcal{H}_{dR}^{p,q}(X)$ (de Rham Cohomology group)

pp. 8:

- A lot of isomorphisms between *de Rham*, *Dolbeault* and *harmonic forms*.

2.14 Lecture 16 Harmonic forms on Kähler Manifold (20160503)

pp.1:

- $\Delta_{\partial} = \Delta_{\bar{\partial}} = \frac{1}{2} \Delta_d$, for Kähler mnfd.

2.15 Lecture 17 Hermitian Vector Bundle (20160510)

pp.1,3:

- Hermitian Vector Bundle. pp.1
- Antilinear map. pp.3.

- Hermitian Inner Product on $\mathcal{A}^{p,q}(X, E)$. pp.4
- $\bar{*}_E$ Hodge Operator on Hermitian vector bundle. pp.5.
- $\bar{\partial}_E^*$

pp. 8:

- Kodaira-Serre Duality.

2.16 Lecture 18 Connection (20160516)

- ∇ : connection. pp.1.
- Trivial connections. pp.2
- $\mathcal{A}^1(M, \text{End}(E)) := \Gamma(M, \Lambda^1 M \otimes \text{End}(E))$. pp.3. Also, one may find how elements in this sheaf act on $\mathcal{A}^0(M)$ on pp.173, inside proof of proposition 4.2.3.
- $s \in \mathcal{A}^0(E)$ is Parallel/flat/constant $\Leftrightarrow \Delta(s) = 0$. pp.4.
- $\Delta = d + A$. pp.4.
- Δ be compatible with hermitian structure on E . pp.5.
- Δ be compatible with holomorphic vector bundle. pp.6.
- $A = \bar{H}^{-1} \partial H$. Chern connection. pp.6.

2.17 Lecture 19 Holomorphic Connection & Curvature(20160517)

- Holomorphic Connection. pp.1.
- $At(E)$: Atiyah class of E . pp.2.
- Δ^k . pp.4.
- F_Δ : curvature associated with Δ . pp.5.
- $F_\Delta = dA + A \wedge A$: Cartan structure equation. pp.6.
- First Chern class of complex line bundle.

2.18 Lecture 20 Divisors & (Holomorphic) Line Bundles (20160524)

- Analytic Subvariety. pp.1.
- Regular/Smooth Point. pp.1.
- Singular Point. pp.2.
- Irreducible analytic subvariety. pp.2.
- $\dim(Y)$: dimension of analytic subvariety. pp.2. Also pp.4.
- Affine algebraic varieties. pp.3.
- Projective algebraic varieties. pp.3.
- Hypersurface. pp.4.
- Divisor, $Div(X)$:=group of all divisors. pp.5.
- Effective divisor. pp.6.
- $Ord_Y(f)$: order of function. pp.6. Also pp.8.
- Meromorphic function on complex mnfd.
- (f) : divisor given by a global meromorphic function.
- Principal divisor. pp.8.

2.19 Lecture 21 Divisors & (Holomorphic) Line Bundles (20160530)

- $H^0(X, K_X^*/\mathcal{O}_X^*) \cong Div(X)$. pp.1.
- $Pic(X)$: Picard group, all holomorphic line bundles. pp.3.
- $Pic(X) \cong \check{H}^1(X, \mathcal{O}_X^*)$. pp.3.
- $\mathcal{O}(D)$: line bundle given by divisor D . pp.5.
- Linear equivalent of divisors.
- *: used only in this section to denote the map:

$$(Div(X)/Pic(X)) \hookrightarrow Pic(X)$$

pp.6.

- $Z(s)$: divisor constructed from nonzero section $s \in H^0(X, L)$ for a line bundle L .

2.20 Lecture 22 Divisors & (Holomorphic) Line Bundles (20160606)

- Base point of a line bundle. pp.4.
- $Bs(L) :=$ set of all base points of line bundle L . pp.4.
- $\mathcal{O}(1), \mathcal{O}(k)$. pp.6.

Part II

Indices of Results

Theorems, Remarks, etc.

Chapter 3

Local Theory

3.1 1.1 Holomorphic Functions of Several Variables

Proposition 3.1.1. *The local ring $\mathcal{O}_{\mathbb{C}^n,0}$ is a UFD.*

(pp.14 of [1])

Proposition 3.1.2. *Weierstrass division theorem Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ and $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ be a Weierstrass polynomial of degree d . Then there exist $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ of degree $< d$ and $h \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $f = g \cdot h + r$. The functions h and r are uniquely determined.*

(pp.15 of [1])

Proposition 3.1.3. *The local UFT $\mathcal{O}_{\mathbb{C}^n,0}$ is Noetherian.*

(pp.16 of [1])

Corollary 3.1.1. *Let $g \in \mathcal{O}_{\mathbb{C}^n,0}$ be an irreducible function. If $f \in \mathcal{O}_{\mathbb{C}^n,0}$ vanishes on $Z(g)$, then g divides f .*

(pp.16 of [1])

Lemma 3.1.1. *For any germ $X \subset \mathbb{C}^n$ the set $I(X) \subset \mathcal{O}_{\mathbb{C}^n,0}$ is an ideal. If $(A) \subset \mathcal{O}_{\mathbb{C}^n,0}$ denotes the ideal generated by the subset $A \subset \mathcal{O}_{\mathbb{C}^n,0}$, then $Z(A) = Z((A))$ and $Z(A)$ is analytic.*

(pp.18 of [1])

Lemma 3.1.2. *If $X_1 \subset X_2$, then $I(X_2) \subset I(X_1)$. If $I_1 \subset I_2$, then $Z(I_2) \subset Z(I_1)$. For any analytic germ X one has $Z(I(X)) = X$. For any ideal $I \subset \mathcal{O}_{\mathbb{C}^n,0}$, one has $I \subset I(Z(I))$.*

(pp.18 of [1])

3.2 1.2 Complex and Hermitian Structures

Lemma 3.2.1. *If I is an almost complex structure on a real vector space V , then V admits in a natural way the structure of a complex vector space*

(pp.25 of [1])

Remark 3.2.1. *An almost complex structure can only exist on an even dimensional real vector space.*

Corollary 3.2.1. *Any almost complex structure on V induces a natural orientation on V .*

(pp.25 of [1])

Lemma 3.2.2. *Let V be a real vector space endowed with an almost complex structure I . Then*

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

Complex conjugation on $V_{\mathbb{C}}$ induces an \mathbb{R} -linear isomorphism $V^{1,0} \cong V^{0,1}$.

(pp.26 of [1])

Remark 3.2.2. *Two almost complex structures on $V_{\mathbb{C}}$: I and i , coincide on the subspace $V^{1,0}$ but differ by a sign on $V^{0,1}$.*

Lemma 3.2.3. *Let V be a real vector space endowed with an almost complex structure I . Then the dual space $V^* = \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$ has a natural almost complex structure given by $I(f)(v) = f(I(v))$. The induced decomposition on $(V^*)_{\mathbb{C}} = \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) = (V_{\mathbb{C}})^*$ is given by*

$$(V^*)^{1,0} = \{f \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(I(v)) = if(v)\} = (V^{1,0})^*$$

$$(V^*)^{0,1} = \{f \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(I(v)) = -if(v)\} = (V^{0,1})^*$$

Also note that $(V^)^{1,0} = \text{Hom}_{\mathbb{C}}((V, I), \mathbb{C})$.*

Proposition 3.2.1. *For a real vector space V endowed with an almost complex structure I , one has:*

1. $\bigwedge^{p,q} V$ is in a canonical way a subspace of $\bigwedge^{p+q} V_{\mathbb{C}}$.
2. $\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V$.
3. Complex conjugation on $\bigwedge^* V_{\mathbb{C}}$ defines a (\mathbb{C} -linear) isomorphism $\bigwedge^{p,q} V \cong \bigwedge^{q,p} V$, i.e. $\bigwedge^{p,q} V = \bigwedge^{q,p} V$.
4. The exterior product is of bidegree $(0,0)$.

(pp.27 of [1])

Remark 3.2.3. *Local calculation of $V^{1,0}, (V^*)^{1,0}$*

$$z_i = \frac{1}{2}(x_i - y_i), \bar{z}_i = \frac{1}{2}(x_i + iy_i)$$

$$z^i = x^i + iy^i, \bar{z}^i = x^i - iy^i$$

$$I(z_i) = iz_i, I(z^i) = iz^i$$

(pp.27 to 28 of [1])

Lemma 3.2.4. *For any $m \leq \dim_{\mathbb{C}} V^{1,0}$, one has*

$$(-2i)^m (z_1 \wedge \bar{z}_1) \wedge \cdots \wedge (z_m \wedge \bar{z}_m) = (x_1 \wedge y_1) \wedge \cdots \wedge (x_m \wedge y_m).$$

For $m = \dim_{\mathbb{C}} V^{1,0}$, this defines a positive oriented volume form for the natural orientation of V .

Also

$$\left(\frac{i}{2}\right)^m (z^1 \wedge \bar{z}^1) \wedge \cdots \wedge (z^m \wedge \bar{z}^m) = (x^1 \wedge y^1) \wedge \cdots \wedge (x^m \wedge y^m).$$

Bibliography

- [1] Complex Geometry

Part III

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