Representation theory of Finite Groups

Taper

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Abstract

A note for correspoding chapter in S. Lang's book.

Contents

1 Representationas and Semisimplicity

ationas-and-Semisimplicity

Now I digress to read chapter 18 of the book [3].

Let R be a commutative ring and G a group. Let E be a R-module.

For convenience, given a representation $\rho : G \to \mathrm{Aut}_R(E)$, let $\sigma \ni G$, $x \in E$, we write σx instead of $\rho(\sigma)x$.

An R-module E, together with a representation ρ, will be called a G-**module** or G-**space**, or also a (G, R)**-module**.

If E, F are G-modules, we define a G-**homomorphism** $f : E \to F$ as a R-linear map such that $f(\sigma x) = \sigma f(x)$ for all $x \in G$ and $\sigma \in G$.

$$\begin{tikzcd} E \arrow[rr, "f"] & & F \\ & G \arrow[ul, "\rho"] \arrow[ur, "\rho'"'] \arrow[u, phantom, "\#"] & \end{tikzcd} \tag{1.0.1}$$

We make R into a G-module by making G act trivially on R, i.e. we have a trivial representation on R, itself regarded as a R-module.

Key Point 1.1. Now the we discuss a systematically way to construct a new representation from a given one. More specifically, we want to make $\mathrm{Hom}_R(E, F)$ into a G-module, i.e. construct a representation from G to $\mathrm{Hom}_R(E, F)$.

First, we define the action of G on $\mathrm{Hom}_R(E, F)$ by (let $f \in \mathrm{Hom}_R(E, F)$, $f : E \to F$).

$$([\sigma]f)(x) = \sigma(f(\sigma^{-1}x)) \tag{1.0.2}$$

We verify that this is an operation/action of G on $\mathrm{Hom}_R(E,F)$:

$$\begin{aligned}([e]f)(x) &= e(f(e^{-1}x)) = f(x)\\ ([a][b]f)(x) &= [a]bf(b^{-1}x) = abf(b^{-1}a^{1}x) = abf((ab)^{-1}x)\\ &= ([ab]f)(x)\end{aligned}$$

fact:g-hom-sg-g **Fact 1.1.** For convenience we write $([\sigma]f)(x)$ simply as $[\sigma]f(x)$. Also,

$$f(\sigma x) = \sigma f(x) \Leftrightarrow [\sigma]f(x) = \sigma f(\sigma^{-1}x) = \sigma\sigma^{-1}f(x) = f(x)$$

So f is a G-homomorphism if and only if $[\sigma]f = f$ for all $\sigma \in G$.

Remark 1.1. This is the first time I understand why σ^{-1} is so important. Without it, we don't have the beautiful fact above.

Definition 1.1 (Dual representation $\rho^\vee$)**.** When $F = R$ (so with trivial action), then $\mathrm{Hom}_R(E,F) = E^\vee$ is the dual module. If we have $\rho : G \to \mathrm{Aut}_R(E)$, a representation of G on E, then the action we just defined gives the dual representation (also called **contragredient** in the literature).

More specifically, σ acts trivially on R, so

$$\begin{aligned}\rho^\vee : &G \to \mathrm{Aut}_R(E^\vee)\\ &\sigma \mapsto [\sigma]f = f(\sigma^{-1}x)\end{aligned} \tag{1.0.3}$$

Suppose now that the modules E,F are free and finite dimensional over R (recall that a module is free when it has a basis). Let ρ be a representation of G on E, M be the matrix of $\rho(\sigma)$ with respect to a basis, $M^\vee$ be the matrix of $\rho^\vee(\sigma)$ with respect to the dual basis. Then

$$\begin{aligned}[\sigma]f : x &\mapsto f(\sigma^{-1}x)\\ x^i &\mapsto \sum_{ij} f_i (M^{-1})^i_j x^j\\ \Longrightarrow [\sigma] : f &\mapsto f\sigma^{-1}\\ f_j &\mapsto \sum_i f_i (M^{-1})^i_j\\ \begin{pmatrix} f_1\\ f_2\\ \cdots \end{pmatrix} &\mapsto (M^{-1})^T \begin{pmatrix} f_1\\ f_2\\ \cdots \end{pmatrix}\end{aligned}$$

Hence

$$M^\vee = (M^{-1})^T \tag{1.0.4}$$ eq:M-dual-M-1T

Next we discuss the tensor product instead of Hom. Let E,E' be (G,R)-modules. We form their tensor product $E \otimes E'$, taken over R. then we define an action of G on $E \otimes E'$ by (let $\sigma \in G$):

$$\sigma(x \otimes x') := \sigma x \otimes \sigma x' \tag{1.0.5}$$

fact:Ev-F--Hom-G **Fact 1.2.** Suppose that E, F are free and finite dimensional over R. Then the R-isomorphism

$$E^\vee \otimes F \overset{g}{\cong} \mathrm{Hom}_R(E,F) \tag{1.0.6}$$

$$f_i \otimes x^j \overset{g}{\cong} A^j_i = (f_i x^j) \tag{1.0.7}$$ eq:fx-g-aji

is a G-isomorphism.

Proof. In this proof, we will use Einstein summation convention. We see that

$$\sigma(f_i \otimes x^j) = (\sigma^\vee f_i) \otimes (\sigma x^j) = f_k (M_E^{-1})_i^k \otimes (M_F)_l^j x^l \tag{1.0.8}$$

So

$$g(\sigma(f_i \otimes x^j)) = g(f_k (M_E^{-1})_i^k \otimes (M_F)_l^j x^l) = (M_E^{-1})_i^k A_k^l (M_F)_l^j \tag{1.0.9}$$

Meanwhile, for $h \in \mathrm{Hom}_R(E, F)$,

$$\begin{align} [\sigma]h : x &\mapsto \sigma h(\sigma^{-1}x) \tag{1.0.10}\\ x^i &\mapsto \sigma h_k^i (M_E^{-1})_j^k x^j \tag{1.0.11}\\ &= (M_F)_l^i h_k^l (M_E^{-1})_j^k x^j \tag{1.0.12}\end{align}$$

So G acts on $h \in \mathrm{Hom}_R(E, F)$ is

$$[\sigma] : h_j^i \mapsto (M_F)_l^i h_k^l (M_E^{-1})_j^k \tag{1.0.13}$$

So, consider $g(f_i \otimes x^j) \in \mathrm{Hom}_R(E, F)$, we have

$$[\sigma][g(f_i \otimes x^j)] = [\sigma]A_i^j = (M_F)_l^j A_k^l (M_E^{-1})_i^k$$

Hence

$$g(\sigma(f_i \otimes x^j)) = [\sigma][g(f_i \otimes x^j)] \tag{1.0.14}$$

eq:g-iso-sggs

□

Whether E is free or not, we define the G**-invariant submodule of** E to be $\mathrm{inv}_G(E) = R$-submodule of elements $x \in E$ such that $\sigma x = x$ for all $\sigma \in G$. If E, F are free, we have an R-isomorphism

$$\mathrm{inv}_G(E^\vee \otimes F) \cong \mathrm{Hom}_G(E, F) \tag{1.0.15}$$

Proof. I give only a finite dimensional proof (for no proof is provided in Lang's book [3]. In finite dimensional case, this fact is closely related to the fact 1.2:

$$E^\vee \otimes F \overset{g}{\cong} \mathrm{Hom}_R(E, F)$$

Let me use the notation used before. The isomorphism is actually the g defnied for equation 1.0.7.

On one hand, for $f_i \otimes x^j \in \mathrm{inv}_G(E^\vee \otimes F)$, denote $g(f_i \otimes x^j)$ by A_i^j. Then $g(\sigma(f_i \otimes x^j)) = g(f_i \otimes x^j) = A_i^j$ for any $\sigma \in G$. By equation 1.0.14, we have $[\sigma]A_i^j = A_i^j$, hence by fact 1.1 (notice that $A_i^j \in \mathrm{Hom}_R(E, F)$), $A_i^j \in \mathrm{Hom}_G(E, F)$. The converse is similar.

I think that for infinite dimensional cases, we might replace $\sum$ with $\int$ and use the same logic here. □

Definition 1.2 (Sum $\rho \oplus \rho'$)**.** If $\rho : G \to \mathrm{Aut}_R(E)$, $\rho' : G \to \mathrm{Aut}_R(E')$ are two representation of G, we define their sum $\rho \oplus \rho'$ to be the representation on the direct sum $E \oplus E'$, with $\sigma \in G$ acting componentwise.

Give G, observe the G-isomorphism classes of representations have an additive monoid structure under the above direct sum, and also have an associative multiplicative structure under the tensor product. With the notation of representations, we denote this product by $\rho \otimes \rho'$. This product is distributive with respect to the addition (i.e. direct sum), by their definition.

Definition 1.3 (Trace Tr_G)**.** If G is a finite group, and E is a G-module, then we can define the trace $\mathrm{Tr}_G : E \to E$ which is an R-homomorphism, namely

$$\mathrm{Tr}_G(x) = \sum_{\sigma \in G} \sigma x \tag{1.0.16}$$

Fact 1.3. Tr_G belongs to $\mathrm{inv}_G(E)$, i.e. is fixed under the operation of any $\sigma \in G$:

$$\tau \,\mathrm{Tr}_G(x) = \sum_{\sigma \in G} \tau\sigma x = \sum_{\sigma' \in G} \sigma' x$$

Let E, F be twn G-modules, and $f : E \to F$ is an R-homomorphism of G-modules, then we can easily extend this definition to define $\mathrm{Tr}_G(f)$ by

$$\begin{aligned} &\mathrm{Tr}_G(f) : E \to F \\ &\mathrm{Tr}_G(f) = \sum_{\sigma \in G} [\sigma] f \end{aligned} \tag{1.0.17}$$

Also, $\mathrm{Tr}_G(f)$ is invariant under any $\sigma \in G$, so it is a G-homomorphism, i.e. $\mathrm{Tr}_G(f) \in \mathrm{Hom}_G(E, F)$.

Proposition 1.1. *Let G be a finite group and let E, E', F, F' be G-modules, let*

$$E' \xrightarrow{\phi} E \xrightarrow{f} F \xrightarrow{\psi} F'$$

be R-homomorphism, and asume that ϕ, ψ are G-homomorphisms. Then

$$\mathrm{Tr}_G(\psi \circ f \circ \phi) = \psi \circ \mathrm{Tr}_G(f) \circ \phi \tag{1.0.18}$$

Proof. Proof is provided on the book [3]. The essential point is that

$$[\sigma](\psi \circ f \circ \phi) = \sigma\psi\sigma^{-1}\sigma f \sigma^{-1}\sigma\phi\sigma^{-1} = [\sigma]\psi \circ [\sigma] f \circ [\sigma]\phi \tag{1.0.19}$$

□

Theorem 1.1 (Maschke)**.** *Let G be a finite group of order n, and let k be a field whose characteristic does not divide n. then the group ring $k[G]$ is semisimple.*

Recall the an object is called **semisimple** if it is the direct sum of simple objects. A **simple** object is some ireducible building blocks, without being too simple to be simple. [1] A **simple ring** is a nonzero ring who has no two-sided ideals other than the zero ideal and itself.[2]

The proof of this theorem is provided in page 666, section 18.1 of [3]. I do not understand it. So I do not put it down here.

[1] As remarked in nLab (link), There is a general principle in mathematics that "A trivial object is too simple to be simple". For example, 1 is not a prime number.

[2] I did not find the definition for a simple ring on Lang's book [3]. Rather, I found it online.

sec:Characters

2 Characters

Assuming in this section that, G: a finite group. k: a field of characteristic not dividing $|G|$. E, F: finite dimensional k-spaces. $n = |G|$.

Definition 2.1 (Character χ)**.** The character χ of a representation $\rho : k[G] \to \mathrm{End}_k(E)$ is defined as

$$\chi : k[G] \to k \tag{2.0.20}$$

$$\alpha \mapsto \mathrm{Tr}(\rho(\alpha)) \tag{2.0.21}$$

We sometimes write it as χ_ρ or χ_E.

The **trivial character** is sometimes denoted as χ_0 or 1_G .

Remark 2.1. By the k-linearity of trace function, we observe that characters are determined by its values on G.

Definition 2.2 (Isomorphic Representations)**.** Two representations ρ, ϕ of G on spaces E, F are isomorphic if and only if there is a G-isomorphism between E and F.

$$\begin{tikzcd} & G \arrow[dl, "\rho"'] \arrow[dr, "\phi"] \arrow[d, phantom, "\#"] & \\ E \arrow[rr, "\rho"'] & & F \end{tikzcd} \tag{2.0.22}$$

$$f(\rho(\sigma)) = \phi(\sigma)f, \forall \sigma \in G \tag{2.0.23}$$

In matrix notation,

$$F\rho_\sigma = \phi_\sigma F, \forall \sigma \in G \tag{2.0.24}$$

Note: In everything that follows, we are interested only in isomorphism classes of representations.

If E, F are G-spaces, then their direct sum $E \oplus F$ is also a G-space, the operation of G being componentwise. If $x \oplus y \in E \oplus F$ with $x \in E$, $y \in F$, then $\sigma(x \oplus y) = \sigma x \oplus \sigma y$.

Similarly, the tensor product $E \otimes_k F = E \otimes F$ is a G-space, the operation G being given by $\sigma(x \otimes y) = \sigma x \otimes \sigma y$.

prop:character-ring

Proposition 2.1. *If e, F are G-spaces, then*

$$\chi_E + \chi_F = \chi_{E \oplus F} \tag{2.0.25}$$

$$\chi_E \chi_F = \chi_{E \otimes F} \tag{2.0.26}$$

If $\chi^\vee$ denotes the character of the dual representation on $E^\vee$, then

$$\chi^\vee(\sigma) = \chi(\sigma^{-1}) \tag{2.0.27}$$

In particular, when $k = \mathbb{C}$, then

$$\chi^\vee(\sigma) = \overline{\chi(\sigma)} \tag{2.0.28}$$

Proof. The first relation is obvious.

The second relation holds from the following observation,

$$\sigma(v_i \otimes w_j) = \sigma v_i \otimes \sigma w_j = A_{i\nu} v_v \otimes B_{j\mu} w_\mu$$

So its character is

$$\chi_{E\otimes F}(\sigma) = A_{ii}B_{jj} = \chi_E(\sigma)\chi_F(\sigma)$$

The third relation follows from that the matrix of dual representation is the matrix $(M^{-1})^T$, see equation 1.0.4.

The last relation will be proved in corollary □

We now extend the definition of **character** to a function of G, which can be written in a linear combination of characters of representations with arbitrary integer coefficients. Then, the previous characters are very special. Such characters associated with representations will be called **effective characters** . Note that everything we have defined of course depends on the field k, and we shall add **over k** to ouirexpressions if we need to specify the field k.

Remark 2.2. By proposition 2.1, all the characters form a ring structure.

By a **simple** or **irreducible character** of G one means the character of a simple representation, i.e., the character associated with a simple $k[G]$-module.

3 Anchor

sec:Anchor

References

book [1] Zhongqi Ma, Group Theory in Physics

Ludeling [2] Lecture Notes for physics751: Group Theory (for Physicists), by C Ludeling. Link

lang-algebra [3] Serge Lang. Algebra. Revised 3rd. Springer.

4 License

Nomenclature

(G, R)-module , page 1

1_G , page 5