# Solution for HW6 20161109

Taper

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**Abstract**

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## 1 Validate that $\mathrm{Tr} = \sum \lambda_i$ with $\sigma_x$

The characteristic equations is

$$\begin{aligned}
&\det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0 \\
&\Rightarrow \lambda^2 - 1 = 0 \\
&\Rightarrow \lambda = \pm 1 \\
&\Rightarrow \lambda_1 + \lambda_2 = 0
\end{aligned}$$

Meanwhile, we have $\mathrm{Tr}(\sigma_x) = 0 + 0 = 0$, confirming theorem.

## 2 Prove $[X, Y] = [Y, Z] = [X, Z] = 0$

$$\begin{aligned}
[X, Y] &= \int \mathrm{d}x \left|x\right\rangle \left\langle x\right| x \int \mathrm{d}y \left|y\right\rangle \left\langle y\right| y - \int \mathrm{d}y \left|y\right\rangle \left\langle y\right| y \int \mathrm{d}x \left|x\right\rangle \left\langle x\right| x \\
&= \iint \mathrm{d}x\mathrm{d}y (xy - yx) \left|x\right\rangle \left\langle x\right| \left|y\right\rangle \left\langle y\right| \\
&= \iint \mathrm{d}x\mathrm{d}y \, 0 \left|x\right\rangle \left\langle x\right| \left|y\right\rangle \left\langle y\right| \\
&= 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
[Y, Z] &= \int \mathrm{d}y \left|y\right\rangle \left\langle y\right| y \int \mathrm{d}z \left|z\right\rangle \left\langle z\right| z - \int \mathrm{d}z \left|z\right\rangle \left\langle z\right| z \int \mathrm{d}y \left|y\right\rangle \left\langle y\right| y \\
&= \iint \mathrm{d}y\mathrm{d}z (yz - zy) \left|y\right\rangle \left\langle y\right| \left|z\right\rangle \left\langle z\right| \\
&= \iint \mathrm{d}y\mathrm{d}z \, 0 \left|y\right\rangle \left\langle y\right| \left|z\right\rangle \left\langle z\right| \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
[X, Z] &= \int \mathrm{d}x \left|x\right\rangle \left\langle x\right| x \int \mathrm{d}z \left|z\right\rangle \left\langle z\right| z - \int \mathrm{d}z \left|z\right\rangle \left\langle z\right| z \int \mathrm{d}x \left|x\right\rangle \left\langle x\right| x \\
&= \iint \mathrm{d}x\mathrm{d}z (xz - zx) \left|x\right\rangle \left\langle x\right| \left|z\right\rangle \left\langle z\right| \\
&= \iint \mathrm{d}x\mathrm{d}z \, 0 \left|x\right\rangle \left\langle x\right| \left|z\right\rangle \left\langle z\right| \\
&= 0
\end{aligned}
$$

# 3 Prove

## 3.1 Translation operator is unitary

*Proof.* Since

$$
\begin{aligned}
\left|T_a\beta\right\rangle \equiv T_a \left|\beta\right\rangle &= T_a \int \mathrm{d}^3x \left|x\right\rangle \left\langle x|\beta\right\rangle \\
&= \int \mathrm{d}^3x \left|x+b\right\rangle \left\langle x|\beta\right\rangle
\end{aligned}
$$

So

$$
\left\langle T_a\beta\right| = \int \mathrm{d}^3x \left\langle x|\beta\right\rangle^* \left\langle x+b\right|
$$

Then

$$
\begin{aligned}
\left\langle T_a\beta|\gamma\right\rangle &= \int \mathrm{d}^3x \left\langle x|\beta\right\rangle^* \left\langle x+b\right| \int \mathrm{d}^3x' \left\langle x'|\gamma\right\rangle \left|x'\right\rangle \\
&= \iint \mathrm{d}^3x \, \mathrm{d}^3x' \, \delta(x+a-x') \left\langle x|\beta\right\rangle^* \left\langle x'|\gamma\right\rangle \\
&= \iint \mathrm{d}^3x \, \mathrm{d}^3x' \, \delta(x-(x'-a)) \left\langle x|\beta\right\rangle^* \left\langle x'|\gamma\right\rangle \\
&= \int \mathrm{d}^3x \left\langle x|\beta\right\rangle^* \left\langle x\right| \int \mathrm{d}^3x' \left\langle x'|\gamma\right\rangle \left|x'-a\right\rangle \\
&= \int \mathrm{d}^3x \left\langle x|\beta\right\rangle^* \left\langle x\right| \left( T_{-a} \int \mathrm{d}^3x' \left\langle x'|\gamma\right\rangle \left|x'\right\rangle \right) \\
&= \left\langle \beta|T_{-a}\gamma\right\rangle
\end{aligned}
$$

So

$$
T_a^\dagger = T_{-a} \tag{3.1.1}
$$

So, with the result of section 3.3, we have

$$
T_a T_a^\dagger = T_a T_{-a} = T_{a-a} = T_0 = \mathbb{1}
$$

□

Hence it is unitary.

## 3.2 $[T_a, T_b] = 0$

*Proof.* Notice that

$$\begin{aligned} T_{a+b}\left|\alpha\right\rangle = T_{a+b}\int \mathrm{d}^3x\left|x\right\rangle\left\langle x|\alpha\right\rangle &= \int \mathrm{d}^3x\left|x+(a+b)\right\rangle\left\langle x|\alpha\right\rangle \\ &= \int \mathrm{d}^3x\left|x+(b+a)\right\rangle\left\langle x|\alpha\right\rangle \\ &= T_{b+a}\int \mathrm{d}^3x\left|x\right\rangle\left\langle x|\alpha\right\rangle \\ &= T_{b+a}\left|\alpha\right\rangle \end{aligned}$$

So $T_{a+b} = T_{b+a}$. Then use the result of section 3.3, we have

$$[T_a, T_b] = T_aT_b - T_bT_a = T_{a+b} - T_{b+a} = T_{a+b} - T_{a+b} = 0$$

□

## 3.3 $T_aT_b = T_{a+b}$

*Proof.*

$$\begin{aligned} T_aT_b\left|\alpha\right\rangle &= T_aT_b\int \mathrm{d}^3x\left|x\right\rangle\left\langle x|\alpha\right\rangle \\ &= T_a\int \mathrm{d}^3x\left|x+b\right\rangle\left\langle x|\alpha\right\rangle \\ &= \int \mathrm{d}^3x\left|x+b+a\right\rangle\left\langle x|\alpha\right\rangle \\ &= \int \mathrm{d}^3x\left|x+(a+b)\right\rangle\left\langle x|\alpha\right\rangle \\ &= T_{a+b}\int \mathrm{d}^3x\left|x\right\rangle\left\langle x|\alpha\right\rangle \end{aligned}$$

So $T_aT_b = T_{a+b}$. □