# Condensed Matter Field Theory notes

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**Abstract**

Notes of book [AS10], and another book [GR96] for information about path integral.

## Contents



# 1 Todo

1. Understand in what case can the Gaussian integral formula can be applied. In another word, understand the analytical continuation of the Gaussian integral. See for example, buttom of pp.343 of [GR96].

`todo:analytical-c`

2. Understand the Wick Rotation, cf. pp.356(ch11.5) of [GR96]. This is related to todo 1
3. Understand how the constant term in the path integral of a Feynman Kernal will(or will not) affect the physics. Understand the mathematical rationale to support this. (cf. buttom of pp.344 of [GR96].
4. I have doubt about the correctness of pp.110 eq 3.28 till pp.111 (Construction recipe of the path integral), especially about his argument, the size of the *Planck cell*.

Bonus objectives:

1. Find abou the similarity between Path Integral of a free particle and the solution to a classical diffusion equation (cf. pp.112, footnote 1 of [AS10]).
2. Those marked *todo* in [AS10].

## 2 pp.33 eq.1.43

sec:Table

In page 33 of [AS10], the author derives a difference of action, when we have a symmetry transformation paraterized by $\omega_a$:

$$x_\mu \to x'_\mu = x_\mu + \frac{\partial x_\mu}{\omega_a}|_{\omega=0}\omega_a(x) \tag{2.0.1}$$

$$\phi^i(x) \to \phi'(x') = \phi^i(x) + \omega_a(x)F^i_a[\phi] \tag{2.0.2}$$

We have:

$$\mathcal{L} = \mathcal{L}(\phi^i(x), \partial_{x_\mu}\phi^i(x)) \tag{2.0.3}$$

$$\mathcal{L}' = \mathcal{L}'(\phi'^i(x'), \partial_{x'_\mu}\phi'^i(x')) \tag{2.0.4}$$

$$= \mathcal{L}\left(\phi^i + F^i_a\omega_a, \left(\delta_{\mu\nu} - \partial_{x_\mu}(\omega_a\partial_{\omega_a}x_\mu)\right)\partial_{x_\nu}(\phi^i + F^i_a\omega_a)\right) \tag{2.0.5}$$

And

$$\Delta S = \int \mathrm{d}^m x'\,\mathcal{L}' - \int \mathrm{d}^m x\,\mathcal{L} \tag{2.0.6}$$

eq:dS-integrand

$$\begin{aligned} &= \int \mathrm{d}^m x\left(1 + \partial_{x_\mu}\left(\omega_a\partial_{\omega_a}x_\mu\right)\right) \\ &\times \mathcal{L}\left(\phi^i + F^i_a\omega_a, \left(\delta_{\mu\nu} - \partial_{x_\mu}\left(\omega_a\partial_{\omega_a}x_\mu\right)\right)\partial_{x_\nu}(\phi^i + F^i_a\omega_a)\right) \\ &- \int \mathrm{d}^m x\,\mathcal{L}(\phi^i(x), \partial_{x_\mu}\phi^i(x)) \end{aligned} \tag{2.0.7}$$

Then he argues that, "for constant parameters $\omega_a$ the action difference $\Delta a$ vanishes". Therefore "the leading contribution to the action difference of a symmetry transformation must be linear in the derivative $\partial_{x_\mu}\omega_a$".

Then he writes that "A straightforward expansion of the formula above for $\Delta S$ shows that these terms are given by"

$$\Delta S = -\int \mathrm{d}^m x\, j^a_\mu(x)\partial_{x_\mu}\omega_a \tag{2.0.8}$$

where $j^a_\mu$ is:

$$j^a_\mu = \left( \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \partial_{x_\nu}\phi^i - \mathcal{L}\delta_{\mu\nu} \right) \frac{\partial x_\nu}{\partial \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} F^i_a \tag{2.0.9}$$

I am partically confused about how to do the "straightforward expansion". I guess I should do $\frac{\partial}{\partial(\partial_{x_\mu}\omega_a)}$ to the integrand inside expression for $\Delta S$, though I don't really understand the reason. Even so, the integrand contains terms like $\partial_{x_\mu}\partial_{\omega_a}x_\mu$, which I don't know how to deal with.

**Solution**. The reality is a bit more complicated. We first do a first order expasion to get the infinitesimal difference:

$$\mathcal{L}' - \mathcal{L} \tag{2.0.10}$$

$$\approx \frac{\partial \mathcal{L}}{\partial \phi^i} F^i_a \omega_a + \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left[ \partial_\mu \left( F^i_a \omega_a \right) - \partial_\mu \left( \omega_a \frac{\partial x_\nu}{\partial \omega_a} \right) \partial_\nu \left( \phi^i + F^i_a \omega_a \right) \right]$$

$$= \quad \omega_a \left[ \frac{\partial \mathcal{L}}{\partial \phi^i} F^i_a + \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left( \partial_\mu F^i_a - \partial_\mu (\frac{\partial x_\nu}{\partial \omega_a}) \partial_\nu (\phi^i + F^i_a \omega_a) \right) \right] \tag{2.0.11}$$

eq:l-l-omega

$$+ \partial_\mu \omega_a \left[ \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left( F^i_a - \frac{\partial x_\nu}{\partial \omega_a} \partial_\nu (\phi^i + F^i_a \omega_a) \right) \right] \tag{2.0.12}$$

eq:l-l-pmu-omega

We also discover the integrand in Eq.2.0.6 to be

$$\left( 1 + \partial_\mu (\omega_a \frac{\partial x_\mu}{\partial \omega_a}) \right) \mathcal{L}' - \mathcal{L} \tag{2.0.13}$$

$$= \left( 1 + \partial_\mu (\omega_a \frac{\partial x_\mu}{\partial \omega_a}) \right) (\mathcal{L}' - \mathcal{L}) + \left( \partial_\mu (\omega_a \frac{\partial x_\mu}{\partial \omega_a}) \right) \mathcal{L} \tag{2.0.14}$$

eq:integrand-l-density

For the first term $\left(1 + \partial_\mu(\omega_a \frac{\partial x_\mu}{\partial \omega_a})\right)(\mathcal{L}' - \mathcal{L})$, the $(\mathcal{L}' - \mathcal{L})$ already has terms of first order of $\omega_a$ and of first order of $\partial_\nu \omega_a$. For our purpose, the second order terms $(\partial_\nu(F^i_a \omega_a))$ from item 2.0.11 and item 2.0.12 can be ignored. Also, the item $(\partial_\mu(\omega_a \frac{\partial x_\mu}{\partial \omega_a}))(\mathcal{L}' - \mathcal{L})$ in eq.2.0.14 can also be ignored.

Therefore the integrand in Eq.2.0.6 becomes

$$(\mathcal{L}' - \mathcal{L}) + \left( \partial_\mu (\omega_a \frac{\partial x_\mu}{\partial \omega_a}) \right) \mathcal{L} \tag{2.0.15}$$

$$= \omega_a \left[ \frac{\partial \mathcal{L}}{\partial \phi^i} F^i_a + \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left( \partial_\mu F^i_a - (\partial_\mu \frac{\partial x_\nu}{\partial \omega_a}) \partial_\nu (\phi^i + F^i_a \omega_a) \right) + (\partial_\nu \frac{\partial x_\mu}{\partial \omega_a}) \mathcal{L} \right] \tag{2.0.16}$$

$$+ \partial_\mu \omega_a \left[ \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left( F^i_a - \frac{\partial x_\nu}{\partial \omega_a} \partial_\nu (\phi^i + F^i_a \omega_a) \right) + \frac{\partial x_\mu}{\partial \omega_a} \mathcal{L} \right] \tag{2.0.17}$$

Therefore, the term we seek, i.e. the coefficient of $\partial_\mu \omega_a$ is

$$\frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left( F^i_a - \frac{\partial x_\nu}{\partial \omega_a} \partial_\nu (\phi^i + F^i_a \omega_a) \right) + \frac{\partial x_\mu}{\partial \omega_a} \mathcal{L} \tag{2.0.18}$$

$$= \left( \mathcal{L}\delta_{\mu\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \partial_\nu \phi^i \right) \frac{\partial x_\nu}{\partial \omega_a} + \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} F^i_a \tag{2.0.19}$$

which is what we expect in equation 1.43 of [AS10].

**Question**: as for why we should ignore the term with $\omega_a$, there are two posts ( [1], [2] ) might be useful for a thought.

confusion

I had great doubt about this problem. Though I have posted an answer on [1], I don't think that answer is satisfactory.

# 3 About $\dot{q}$ and imaginary time

The $\dot{q}$ in all the path integrals, especially eq.3.6 and eq.3.8, is in fact a shorthand for the divided difference $\frac{q_{n+1}-q_n}{\Delta t}$ as in pp.99 (the buttom). It is not exactly the same as $\dot{q}$. pp.343 of [GR96] also mentioned that in this sense, the Lagrangian in all path integrals is not identical with the ordinary Lagrange function. Though, I still do not know if this matters at all.

In the most common imaginary time transformation, such as those mentioned in pp.106 of [AS10], and pp.358 of [GR96], we have the transformation $t \to -i\tau$. This in effect change all the $\Delta t$ in, e.g. eq 3.5 (pp.99) of [AS10], to $-i\Delta\tau$. Therefore, the divided difference

$$\frac{q_{n+1}-q_n}{\Delta t} \to \frac{q_{n+1}-q_n}{-i\Delta\tau}$$

Therefore,

$$\dot{q} \to i\partial_\tau q$$
$$\dot{q}^2 \to -\partial_\tau^2 q$$

# 4 Eq. 3.5

It is not so obvious to get Eq.3.5 in pp.99 of [AS10]. Here is my notes.

According to the book, Eq.3.3 is turned into (I set $\hbar = 1$ occasionally, though sometimes I forgot that I have set $\hbar = 1$, orz):

$$\begin{aligned}\langle q_f| &\int \mathrm{d}q_N\,\mathrm{d}p_N\,|q_N\rangle\langle q_N|p_N\rangle\langle p_N|\,e^{-i\hat{T}\Delta t}e^{-i\hat{V}\Delta t}\times\\ &\int \mathrm{d}q_{N-1}\,\mathrm{d}p_{N-1}\,|q_{N-1}\rangle\langle q_{N-1}|p_{N-1}\rangle\langle p_{N-1}|\,e^{-i\hat{T}\Delta t}e^{-i\hat{V}\Delta t}\times\dots\\ &\int \mathrm{d}q_1\,\mathrm{d}p_1\,|q_1\rangle\langle q_1|p_1\rangle\langle p_1|\,e^{-i\hat{T}\Delta t}e^{-i\hat{V}\Delta t}\,|q_i\rangle \qquad (4.0.20)\end{aligned}$$

Notice that

T has only p, V has only q

$$\langle q|p\rangle = \frac{\exp(iqp/\hbar)}{\sqrt{2\pi\hbar}} \qquad (4.0.21)$$

$$\langle p_N|\,e^{-i\hat{T}\Delta t} = \langle p_N|\,e^{-iT(p_N)\Delta t} \qquad (4.0.22)$$

$$e^{-i\hat{V}\Delta t}\,|q_{N-1}\rangle = e^{-iV(q_{N-1})\Delta t}\,|q_{N-1}\rangle \qquad (4.0.23)$$

$$(4.0.24)$$

Also,

$$\langle q_N|p_N\rangle \langle p_N| e^{-i\hat{T}\Delta t}e^{-i\hat{V}\Delta t}|q_{N-1}\rangle = \frac{e^{iq_Np_N/\hbar}}{\sqrt{2\pi\hbar}}\langle p_N| e^{-iT(p_N)\Delta t}e^{-iV(q_{N-1})\Delta t}|q_{N-1}\rangle$$

$$=\frac{e^{iq_Np_N/\hbar}}{\sqrt{2\pi\hbar}}\langle p_N|q_{N-1}\rangle e^{-iT(p_N)\Delta t}e^{-iV(q_{N-1})\Delta t} = \frac{e^{ip_N(q_N-q_{N-1})/\hbar}}{2\pi\hbar}e^{-i[T(p_N)+V(q_{N-1})]\Delta t} \tag{4.0.25}$$

etc. Now we have to pay special attentiont to the start and end. For the start, we have a

$$\int \mathrm{d}q_N \langle q_f|q_N\rangle = \int \mathrm{d}q_N\, \delta(q_N - q_f)$$

So every $q_N$ is replaced by $q_f$. For the end, we have

$$\langle q_1|p_1\rangle \langle p_1| e^{-i\hat{T}\Delta t}e^{-i\hat{V}\Delta t}|q_i\rangle = e^{-i[T(p_1)+V(q_i)]}\frac{e^{ip_1(q_1-q_i)}}{2\pi\hbar}$$

Together we have the whole thing into:

$$\begin{aligned}&\int \mathrm{d}q_1\cdots \mathrm{d}q_{N-1}\,\mathrm{d}p_1\,\mathrm{d}p_N\,\frac{1}{(2\pi\hbar)^N}\times\\ &e^{i\left[p_1(q_1-q_i)+\cdots p_N(q_N-q_{N-1})\right]}\times\\ &e^{-i\left[T(p_1)+\cdots+T(p_N)+V(q_i)+V(q_1)+\cdots+V(q_{N-1})\right]}\end{aligned} \tag{4.0.26}$$

which is exactly eq.(3.5) in book.

# 5 Eq 9.4

The Hamiltonian for particle on a ring is claimed to be (Eq. 9.1 of [AS10], pp. 498):

$$H = \frac{1}{2}(-i\partial_\phi - A)^2 = \frac{1}{2}(p-A)^2 \tag{5.0.27}$$

The book [AS10] claims that

$$L = \frac{1}{2}\dot{\phi}^2 - iA\dot{\phi} \tag{5.0.28}$$

I am quite confused, especially about the appearance of $\dot{\phi}$. Can any explain a bit?

How I tried: Since the inverse of a Legendre transformation is Legendre transformation itself,

$$\text{Denote } x \equiv \frac{\partial H}{\partial p} = p - A, \text{ so,} \tag{5.0.29}$$

$$p = x + A, \quad H = \frac{1}{2}x^2, \text{ so,} \tag{5.0.30}$$

$$L = xp - H = x(x+A) - \frac{1}{2}x^2 = \frac{1}{2}x^2 + xA \tag{5.0.31}$$

So my calculation found that the Lagrangian of above Hamiltonian is:

$$L = \frac{1}{2}x^2 + xA \tag{5.0.32}$$

where

$$x = \frac{\partial H}{\partial p} \tag{5.0.33}$$

## References


[AS10] Alexander. Altland and Ben BD Ben Simons. *Condensed Matter Field Theory (Second Edition)*. Cambridge University Press, 2010. URL: `http://www.cambridge.org/us/academic/subjects/physics/condensed-matter-physics-nanoscience-and-mesoscopic-physics/condensed-matter-field-theory-2nd-edition?format=HB{&}isbn=9780521769754`.

[GR96] Walter Greiner and Joachim Reinhardt. *Field Quantization*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1996. URL: `http://link.springer.com/10.1007/978-3-642-61485-9`, `doi:10.1007/978-3-642-61485-9`.


# 6 A brief Summary of Quantum double well

The Quantum double well:

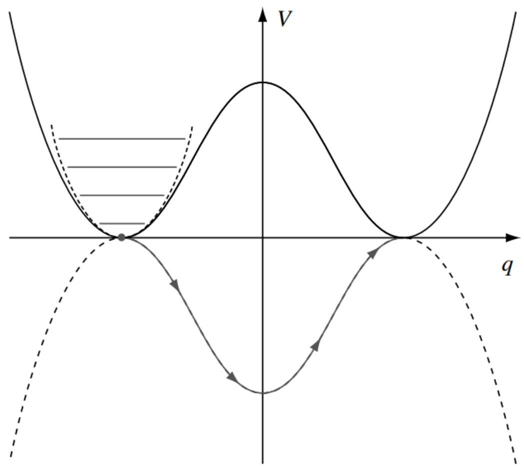


Figure 1: Quantum double well
The solid line is the double well potential. The dotted line in the second quarter is to remind us of the single well potential and its sets of eigenvalues. The inverted potential in Euclidean time is shown below the $q$-axis.

The Euclidean path integral is

$$G_E(a, \pm a : \tau) = \int_{q(0)=\pm a, q(\tau)=a} Dq \exp\left(-\frac{1}{\hbar}\int_0^\tau \mathrm{d}\tau' \left(\frac{m}{2}\dot{q}^2 + V(q)\right)\right) \tag{6.0.34}$$

with the imaginary time saddle point equation

$$-m\ddot{q} + V'(q) = 0 \tag{6.0.35}$$

It is turned into several contributions:

$$A = A_{cl} \times A_{qu} \tag{6.0.36}$$

where $A$ denotes transition amplitude, "cl" for classical, "qu" for quantum.

The classical contribution includes a stationary not moving part $A_{cl,st}$, a moving but bouncing back and forth in the inverted potential part $A_{cl,inst}$. The moving and bouncing back and forth motion is termed **instanton** , see p.117( [AS10]) for details.

The $A_{cl,st}$ is $e^0 = 1$ since the stationary path $q \equiv 0$. The $A_{cl,inst}$ is calculated in the book (pp.116-117), and the result is:

$$A_{cl,inst,one-trip} = \exp\left(-\frac{1}{\hbar}S_{inst}\right) \tag{6.0.37}$$

for one bouncing trip. Here $S_{inst}$ is given by eq.3.36 in p.117. Adding them together gives:

$$A_{cl,inst} = \sum_{n \text{ even/odd}}^{\infty} \frac{1}{n!} \left(\tau K e^{-S_{inst}/\hbar}\right)^n = \cosh\left(\tau K e^{-S_{inst}/\hbar}\right) \text{ or } \sinh\left(\tau K e^{-S_{inst}/\hbar}\right) \tag{6.0.38}$$

where $K$ is just some prefactor explained in p.118 and calculated in pp.122-123.

The quantum contribution is due to the fluctuation of path around the classical path. This part is also divided into two categories. The $A_{qu,st}$ for fluctuation around the stationary path is calculated in the section about single well green function, spcifically eq.3.31 in p.114. It is approximated in p.119 to be

$$A_{qu,st,n} = e^{-\omega(\tau_{i+1}-\tau_i)/2} \tag{6.0.39}$$

and adding together gives

$$A_{qu,st} = \prod_i e^{-\omega(\tau_{i+1}-\tau_i)/2} = e^{-\tau\omega} \tag{6.0.40}$$

The rest $A_{qu,inst}$ from fluctuation around instanton is assumed to be negligible (p.119).

In summary:

| Path | Classical (0th order) | Fluctuation (2nd order) |
|---|---|---|
| Stationary | 1 | $\approx e^{-\omega\tau}$ |
| Instanton | $e^{-\frac{1}{\hbar}S_{\text{inst}}}$, or $\cosh\left(\tau K e^{-S_{\text{inst}}/\hbar}\right)$ $\sinh\left(\tau K e^{-S_{\text{inst}}/\hbar}\right)$ | $\approx 1$ |

# 7 Tunneling of Quantum field

This part is particular hard for me. There are majorly three problems. The first and most obvious one, is why do we require a "thin wall" approximation? Why is this approximation physical?

The next problem is how is expansion of first order in $f$ will make the following equation:

$$m\partial_r^2\phi = -\frac{m}{r}\partial_r\phi + \partial_\phi\left(V(\phi) - f\phi\right) \tag{7.0.41}$$

into a simplified form:

$$m\partial_r^2\phi = \partial_\phi(\phi) \tag{7.0.42}$$

The last problem is that, the action in the new coordinate is:

$$\int_0^{r_0} \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \left[m(\partial_r\phi)^2 + V(\phi) - f\phi\right] r \tag{7.0.43}$$

where I have ignored several constants, and have used the formula:

$$\mathrm{d}\tau\,\mathrm{d}x = r\,\mathrm{d}r\,\mathrm{d}\theta \tag{7.0.44}$$

For the moment ignores $-f\phi r$, the first two terms:

$$\int_0^{2\pi} \mathrm{d}\theta \int_0^{r_0} \mathrm{d}r \left[m(\partial_r\phi)^2 + V(\phi)\right] r \tag{7.0.45}$$

is still one step away from producing a $S_{\text{inst}}$ term:

$$S_{\text{inst}} := \int_0^T \mathrm{d}\tau \left(\frac{m}{2}(\partial_\tau\phi)^2 + V(\phi)\right) \tag{7.0.46}$$

# 8 Eq. 3.50

This equation is actually a lemma in combinatorics, see

Wikipedia Baker–Campbell–Hausdorff formula.

It says:

$$\begin{aligned}\mathrm{Ad}_{e^X} Y &\equiv e^X Y e^{-X} = e^{\mathrm{ad}_X} Y \\ &\equiv Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \cdots \end{aligned} \tag{8.0.47}$$

or just

$$\mathrm{Ad}_{e^X} = e^{\mathrm{ad}_X} \tag{8.0.48}$$

Here the author uses the notation $[X, \,]Y \equiv \mathrm{ad}_X Y \equiv [X,Y]$.

# 9 p.138 Exercise

Using eq.3.50 on p.138, it is straightforward to obtain:

$$\begin{aligned} n_1 &= \cos\phi\cos\theta\,\langle S_1\rangle + \cos\phi\sin\theta\,\langle S_3\rangle - \sin\phi\,\langle S_2\rangle \\ n_2 &= \cos\phi\,\langle S_2\rangle + \sin\phi\cos\theta\,\langle S_1\rangle + \sin\theta\sin\phi\,\langle S_3\rangle \\ n_3 &= \cos\theta\,\langle S_3\rangle + \sin\theta\,\langle S_1\rangle \end{aligned}$$

And we have

$$\begin{aligned} \langle S_1\rangle &= \langle\uparrow|S_1|\uparrow\rangle = 0 \\ \langle S_2\rangle &= \langle\uparrow|S_2|\uparrow\rangle = 0 \\ \langle S_3\rangle &= \langle\uparrow|S_3|\uparrow\rangle = S \end{aligned}$$

by the representation thoery of SU(2).

# 10 Eq. 3.52

$$\begin{aligned} \langle\partial_\tau g|g\rangle &= \langle\uparrow|\partial_\tau e^{i\phi\hat{S}_3 + i\theta\hat{S}_2} e^{-i\phi\hat{S}_3 - i\theta\hat{S}_2}|\uparrow\rangle \\ &= \langle\uparrow|e^{i\phi\hat{S}_3 + i\theta\hat{S}_2} i(\hat{S}_3\partial_\tau\phi + \hat{S}_2\partial_\tau\theta) e^{-i\phi\hat{S}_3 - i\theta\hat{S}_2}|\uparrow\rangle \\ &= i\left(\partial_\tau\phi\,\langle g|\hat{S}_3|g\rangle + \partial_\tau\theta\,\langle g|\hat{S}_2|g\rangle\right) \\ &= i(n_3\partial_\tau\phi + n_2\partial_\tau\theta) \\ &= iS(\cos\theta\partial_\tau\phi + \sin\theta\sin\phi\partial_\tau\theta) \end{aligned} \tag{10.0.49}$$

Then how is $\partial_\tau\theta$ missing in (3.52).

# 11 Boson coherent state (p.159)

The aim is to show:

$$a \exp\left(\phi a^\dagger\right) |0\rangle = \phi \exp\left(\phi a^\dagger\right) |0\rangle \tag{11.0.50}$$

Use $aa^\dagger = 1 + a^\dagger a$. And observe:

$$\begin{aligned}
a(a^\dagger)^n &= (a^\dagger)^{n-1} + a^\dagger a (a^\dagger)^{n-1} \\
&= (a^\dagger)^{n-1} + (a^\dagger)^{n-1} + (a^\dagger)^2 a (a^\dagger)^{n-2} \\
&= 3(a^\dagger)^{n-1} + (a^\dagger)^3 a (a^\dagger)^{n-3} \\
&\dots \\
&= n(a^\dagger)^{n-1} + (a^\dagger)^n a
\end{aligned}$$

Then (notice $a|0\rangle = 0$):

$$\begin{aligned}
\sum_n a \frac{\phi^n (a^\dagger)^n}{n!} |0\rangle &= \sum_n \phi\, \phi^{n-1} \frac{n(a^\dagger)^{n-1} + (a^\dagger)^n a}{n!} |0\rangle \\
&= \sum_n \phi\, \phi^{n-1} \frac{n(a^\dagger)^{n-1}}{n!} |0\rangle \\
&= \phi \sum_n \phi^n \frac{(a^\dagger)^n}{n!} |0\rangle \\
&= \phi \exp\left\{\phi a^\dagger\right\} |0\rangle
\end{aligned}$$

# 12 Something about Grassmann Algebra, Grassmann Calculus

I believe that there are several important points missing in the book, when he talks about Grassmann.

**The first** thing is the derivative rule. The differentiation with respect to Grassmann numbers should follow some version of **graded Leibniz rule** . But I am not sure about the details for the following reason. Consider the traditional graded Leibniz rule:

$$\mathrm{d}(fg) = (\mathrm{d}f)g + (-1)^{[f]} f \mathrm{d}g \tag{12.0.51}$$

where $[f]$ is something defined as the parity, or the degree of f.

Now comes the problem: how to define a grade for something like:

$$f = \eta_1 + \eta_2 + \eta_3 + \eta_1\eta_2 + \eta_1\eta_2\eta_3 \tag{12.0.52}$$

$$g = \eta_1 - \eta_2 - \eta_3 + \eta_1\eta_3 - \eta_1\eta_2\eta_3 \tag{12.0.53}$$

understand graded Leibniz rule

By calculation, I found:

$$\frac{\partial}{\partial\eta_1}(fg) = -2\eta_2 - 2\eta_3 - 2\eta_2\eta_3 \tag{12.0.54}$$

$$(\frac{\partial}{\partial\eta_1}f)g = \eta_1 - \eta_2 - \eta_3 + \eta_1\eta_3 - \eta_1\eta_2 - \eta_2\eta_3 - \eta_1\eta_2\eta_3 \tag{12.0.55}$$

$$f(\frac{\partial}{\partial\eta_1}g) = \eta_1 + \eta_2 + \eta_3 + \eta_1\eta_3 + \eta_1\eta_2 + \eta_2\eta_3 + \eta_1\eta_2\eta_3 \tag{12.0.56}$$

It is easy to see that:

$$\frac{\partial}{\partial\eta_1}(fg) \neq (\frac{\partial}{\partial\eta_1}f)g \pm f(\frac{\partial}{\partial\eta_1}g) \tag{12.0.57}$$

A more general calculation gives:

$$\frac{\partial}{\partial\eta}\left[(a+\eta b)(c+\eta d)\right] = bc + (-1)^{[a]}ad \tag{12.0.58}$$

$$\left(\frac{\partial}{\partial\eta}(a+\eta b)\right)(c+\eta d) = bc + b\eta d \tag{12.0.59}$$

$$(a+\eta b)\left(\frac{\partial}{\partial\eta}(c+\eta d)\right) = ad + \eta bd \tag{12.0.60}$$

where $(-1)^{[a]}$ is the number such that $a\eta = (-1)^{[a]}\eta a$. So, there is no way to reconcile the above calculations into something like:

$$\frac{\partial}{\partial\eta}\left[(a+\eta b)(c+\eta d)\right] = \left(\frac{\partial}{\partial\eta}(a+\eta b)\right)(c+\eta d) + (-1)^{?}(a+\eta b)\left(\frac{\partial}{\partial\eta}(c+\eta d)\right)$$

**The second** point is very important. It is about the postulations of Grassmann variables. I postulate that (in addition to the postulations in p.162 of [AS10]):

1. All $\eta_i, \bar{\eta}_i, a_i, a_i^\dagger$ anti-commute with each other. From this we also have $\eta_i^2 = 0$, etc.
2. $\{\mathrm{d}\eta_i, \mathrm{d}\eta_j\} = 0$ for $(i \neq j)$.
3. $\int \mathrm{d}\eta\,(af(\eta) + bg(\eta)) = a\int \mathrm{d}\eta\, f(\eta) + b\int \mathrm{d}\eta\, g(\eta)$
4. $\{\eta_i, \mathrm{d}\eta_j\} = 0$, for $i \neq j$.
5. $\{\bar{\eta}_i, \mathrm{d}\eta_j\} = 0$, for $\forall i, j$.
6. $\frac{\partial}{\partial\eta}(af(\eta)) = a\frac{\partial}{\partial\eta}f(\eta)$, for $a \in \mathbb{C}$.

It is not so simple to tell whether we should have

$$[\eta_i, \mathrm{d}\eta_i] = 0\,?$$

or

$$\{\eta_i, \mathrm{d}\eta_i\} = 0\,?$$

But we should have

$$\{\eta_i, \mathrm{d}\eta_j\} = 0 \quad \text{for } i \neq j$$

For if $[\eta_i, \mathrm{d}\eta_j] = 0$, then we have a contradiction:

$$\begin{aligned}\int \mathrm{d}\eta_j\, \eta_i\eta_j &= \eta_i \int \mathrm{d}\eta_j\, \eta_j = \eta_i \\ &= -\int \mathrm{d}\eta_j\, \eta_j\eta_i = -\eta_i \quad \text{contradiction!}\end{aligned} \tag{12.0.61}$$

And similar argument applies for $\{\bar\eta_i, \mathrm{d}\eta_j\} = 0$, $\forall i, j$.

There is also another point missing. The relations between $\eta$ and $|0\rangle\langle 0|$ is ambiguous. It seems that there is no harm in assuming:

$$\eta\,|0\rangle\langle 0| = |0\rangle\langle 0|\,\eta \tag{12.0.62}$$

Because, if so, we have:

$$a\,|\eta\rangle\langle\eta| = \eta\,|\eta\rangle\langle\eta| = |\eta\rangle\langle\eta|\,\eta \tag{12.0.63}$$

without conflict with:

$$\bar\eta\,|\eta\rangle\langle\eta| = |\eta\rangle\langle\eta|\,\bar\eta = |\eta\rangle\langle\eta|\,a^\dagger \tag{12.0.64}$$

With above assumptions, I have:

**Theorem 12.1.** *Assuming that $a, b, c, d$ either commute or anticommute with $\eta$.*

$$\int \mathrm{d}\eta \left(\frac{\partial}{\partial\eta}(a+\eta b)\right)(c+\eta d) = (-1)^s \int \mathrm{d}\eta\,(a+\eta b)\left(\frac{\partial}{\partial\eta}(c+\eta d)\right) \tag{12.0.65}$$

eq:inteByPart-grass

*where $s$ is such that $b\eta = (-1)^s \eta b$.*

This can also be generalized into

$$\int \mathrm{d}\eta \left(\frac{\partial}{\partial\eta} f(\eta)\right) g(\eta) = (-1)^s \int \mathrm{d}\eta\, f(\eta)\left(\frac{\partial}{\partial\eta} g(\eta)\right) \tag{12.0.66}$$

where $s$ is some appropriate constant as in eq 12.0.65. Here we have assumed that any $f(\eta)$, we can decompose it into $f = a + \eta b$, where $a$ and $b$ do not contain $\eta$. We also assume $a, b$ either commute or anticommute with $\eta$.

*Proof.* Direct calculation. $\square$

**Change of variables in integration** First we consider a simple example.

$$\int \mathrm{d}\phi\, a + b\phi = b \tag{12.0.67}$$

Let $v = \lambda\phi$, since

$$\int \mathrm{d}v\, a + b\frac{v}{\lambda} = \frac{b}{\lambda} \tag{12.0.68}$$

We naturally require

$$\mathrm{d}\phi = \lambda\,\mathrm{d}v \tag{12.0.69}$$

For multivariable case, it is easy to guess that:

$$\mathbf{v} = M\phi \tag{12.0.70}$$

$$\mathrm{d}\phi_1\,\mathrm{d}\phi_2 \cdots \mathrm{d}\phi_N = \det(M)\,\mathrm{d}v_1\,\mathrm{d}v_2 \cdots \mathrm{d}v_N \tag{12.0.71}$$

(Took from Wikipedia Berezin Integral.)

# 13 License