Notes of Topological Transition in a Non-Hermitian Quantum Walk

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Abstract

As suggested by the title.

Contents

1 General

sec:General

This paper discusses in general a model charaterised by the follwoing picture

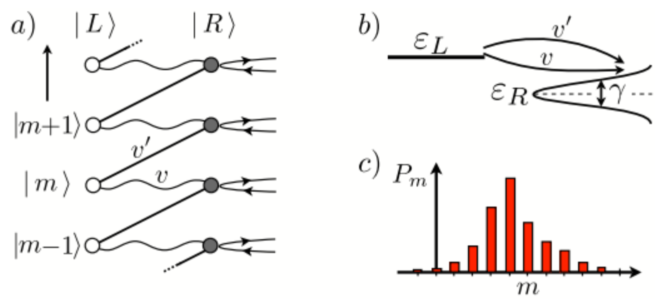

FIG. 1: Setup of the model. a) Each unit cell m contains two sites L (open circles) and R (filled circles), with each R-site connected to an external decay channel. Intracell (wavy lines) and intercell (straight lines) tunneling occur with amplitudes v and v', respectively. b) Energies of the L and R sites. Due to decay, the R-site energy obtains an imaginary part, $\tilde{\varepsilon}_R = \varepsilon_R - i\hbar\gamma/2$. c) Schematic distribution of local decay probabilities $\{P_m\}$ used to calculate the displacement (3).

Figure 1: Setup

The equation of motion is

$$\begin{aligned} i\hbar\dot{\psi}^L_m &= \varepsilon_L \psi^L_m + v\psi^R_m + v'\psi^R_{m+1} \\ i\hbar\dot{\psi}^R_m &= \tilde{\varepsilon}_R \psi^R_m + v\psi^L_m + v'\psi^L_{m-1} \end{aligned} \tag{1.0.1}$$

eq:eq-of-motion

where

$$\tilde{\varepsilon}_R = \varepsilon_R - i\hbar\gamma/2 \tag{1.0.2}$$

Important Note: Through out the text we use m as an abbreviation of R_m. So ψ_m means $\psi(R_m)$, e^{ikm} means e^{ikR_m}, and quite decivingly, e^{ik} actually is e^{ikR_1}.

sec:I-can-get

1.1 I can get

$$|\psi\rangle = \sum_m |\psi^L_m\rangle + |\psi^R_m\rangle \tag{1.1.1}$$

$$\frac{\partial}{\partial t}\langle\psi|\psi\rangle = -\sum_m \gamma \langle\psi^R_m|\psi^R_m\rangle \tag{1.1.2}$$

(All leakage from "R" site!) (draft page 1 to 2)

Strange:

$$\sum_m P_m = 1 \tag{1.1.3}$$

(draft page 3)

sec:Result

1.2 Result

" we find that the average displacement of the particle during the course of its decay, $\Delta m = \sum_m mP_m$, is exactly quantized as an integer (0 or 1 unit cells), where P_m is the probability distribution for decay from different sites.
As in the case of the quantum Hall conductance, this quantization results from an underlying topological structure; in this case it is the winding number of the relative phase between two components of the Bloch wave function.
Using the topological origin of this phenomenon, we are able to show that the quantization is insensitive to parameters and is robust against certain types of noise and decoherence.
The topological transition, which is accompanied by the formation of a non-decaying dark state, leads to a prediction of threshold-like pumping of nuclear polarization, along with strong suppression of current due to the divergence of dwell time at the threshold "

" our motivation is to provide a simple model of nuclear spin pumping in spin-blockaded double quantum dots [9, 10, 11] in the presence of competing effects of the hyperfine and spin-orbital interactions, as in Ref.[11]. "

2 Calculation

sec:Calculation

The system starts with the initial state:

$$\psi_m^L = \delta_{m,0}, \quad \psi_m^R = 0 \tag{2.0.1}$$

It evolves according to the following equation of motion 1.0.1:

$$\begin{aligned} i\hbar\dot{\psi}_m^L &= \varepsilon_L \psi_m^L + v\psi_m^R + v'\psi_{m+1}^R \\ i\hbar\dot{\psi}_m^R &= \tilde{\varepsilon}_R \psi_m^R + v\psi_m^L + v'\psi_{m-1}^L \end{aligned}$$

The paper seeks to find the following:

$$\langle \Delta m \rangle := \sum_m mP_m \tag{2.0.2}$$

$$P_m := \sum_0^\infty \gamma \left|\psi_m^R(t)\right|^2 \mathrm{d}t \tag{2.0.3}$$

eq:pm-def

The result can be obtained both numerically and analytically. First, the author provides an analytical solution.

2.1 Analytical Solution

sec:Analytical-Solution

2.1.1 First observation

sec:First-observation

The first thing the author points out is that:

$$\frac{\partial}{\partial t}|\psi|^2 = \frac{\partial}{\partial t}\langle\psi|\psi\rangle \neq 0 \tag{2.1.1}$$

This is caused by one fundamentabl rule of inner product:

$$\frac{\partial}{\partial t}\langle f|g\rangle = \langle\frac{\partial f}{\partial t}|g\rangle + \langle g|\frac{\partial f}{\partial t}\rangle \tag{2.1.2}$$

The evolution of kets is determined by Schordinger's equation, the evolution of bras is determined by conjugating the Schordinger's equation. Thus, it is easy to see:

$$i\hbar\frac{\partial}{\partial t}|\psi|^2 = i\hbar\frac{\partial}{\partial t}\langle\psi|\psi\rangle = \langle\psi|\, H - H^\dagger\,|\psi\rangle \tag{2.1.3}$$

eq:pt-norm-1

Applying the above rules here, we easily find (notice the only non-hermitian term in the equation of motion 1.0.1 is $\tilde{\varepsilon}_R = \varepsilon_R - i\hbar\gamma/2$, so $H - H^\dagger$ is $\tilde{\varepsilon}_R - \tilde{\varepsilon}_R^* = -i\hbar\gamma$. This also explains why author choose $\gamma/2$ in $\tilde{\varepsilon}_R$, instead of γ. Thus we have

$$\frac{\partial}{\partial t}\langle\psi_m^R|\psi_m^R\rangle = -\gamma\langle\psi_m^R|\psi_m^R\rangle \tag{2.1.4}$$

eq:pt-norm-2

For the L sites, the Hamiltonian is hermitian and thus

$$\frac{\partial}{\partial t}\langle\psi_m^L|\psi_m^L\rangle = 0 \tag{2.1.5}$$

Counting them together, we let

$$|\psi\rangle = \sum_m |\psi_m^R\rangle + |\psi_m^L\rangle \tag{2.1.6}$$

and find

$$\frac{\partial}{\partial t}|\psi|^2 = \frac{\partial}{\partial t}\langle\psi|\psi\rangle = -\sum_m \gamma\langle\psi_m^R|\psi_m^R\rangle \tag{2.1.7}$$

But he the author asserts

$$\sum_m P_m = 1 \tag{2.1.8}$$

If we perceive that P_m are probabilities, then the above equation seems logical. But if I write down the formula (by eq 2.0.3):

$$\sum_m P_m = \sum_m \int_0^\infty \gamma\langle\psi_m^R|\psi_m^R\rangle \stackrel{?}{=} 1 \tag{2.1.9}$$

confusion:1

Then the result is not so obvious.

2.1.2 Pass to momentum space

sec:Pass-to-momentum-space

The basic technique is fourier expansion

$$\psi = \frac{1}{2\pi}\oint \mathrm{d}k\, e^{ikm}\psi_m^R \tag{2.1.10}$$

So the equation of motion 1.0.1 become:

$$i\hbar\frac{\partial}{\partial t}\frac{1}{2\pi}\oint \mathrm{d}k\, e^{ikm}\psi_k^L = \frac{1}{2\pi}\oint \mathrm{d}k\left(\varepsilon_L e^{ikm}\psi_k^L + ve^{ikm}\psi_k^R + v' e^{ik(m+1)}\psi_k^R\right)$$

similar to the other for ψ_m^R. The two equations can be simplified to (comparing the terms inside integration):

$$\begin{aligned} i\hbar\dot{\psi}_k^L &= \varepsilon_L\psi_k^L + v\psi_k^R + v' e^{ik}\psi_k^R \\ i\hbar\dot{\psi}_k^R &= \tilde{\varepsilon}_R\psi_k^R + v\psi_k^L + v' e^{-ik}\psi_k^L \end{aligned} \tag{2.1.11}$$

eq:eq-of-motion-in-k

Or in matirx form:

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_k^L\\ \psi_k^R\end{pmatrix} = \begin{pmatrix}\varepsilon_L & A_k\\ A_k^* & \tilde{\varepsilon}_R\end{pmatrix}\begin{pmatrix}\psi_k^L\\ \psi_k^R\end{pmatrix} \tag{2.1.12}$$

eq:eq-of-motion-in-k-mat

where $A_k := v+v'e^{ik}$, and we have asumed $v, v' \in \mathbb{R}$. This is Eq.(4) in the paper, and can be viewed as an Schordinger's equation in the momentum space, with $H(k)$ the matrix on the right-hand-side.

By exactly the same technique described from Eq. 2.1.3 to Eq. 2.1.4, we can get:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\langle\psi_k^L|\psi_k^L\rangle &= 0 \\ i\hbar\frac{\partial}{\partial t}\langle\psi_k^R|\psi_k^R\rangle &= -\gamma\langle\psi_k^R|\psi_k^R\rangle \end{aligned}$$

So (let $p_k(t) := \left|\psi_k^L(t)\right|^2 + \left|\psi_k^R(t)\right|^2$):

$$\frac{\partial}{\partial t}p_k = -\gamma\left|\psi_k^R(t)\right|^2 \tag{2.1.13}$$

eq:pt-pk

Notice

$$-\frac{i}{2\pi}\oint \mathrm{d}k\,\frac{\partial}{\partial k}(e^{ikm})\,|\psi_k^R\rangle = -i(im)\frac{1}{2\pi}\oint \mathrm{d}k\, e^{ikm}\,|\psi_k^R\rangle = m\,|\psi_m^R\rangle$$

Then

$$\begin{aligned} m\,|\psi_m^R\rangle &= -i\frac{1}{2\pi}\oint \mathrm{d}k\,\frac{\partial}{\partial k}(e^{ikm})\,|\psi_k^R\rangle \\ &= -i\frac{1}{2\pi}\left(e^{ikm}\,|\psi_k^R\rangle\Big|_{\partial\mathrm{BZ}} - \oint \mathrm{d}k\, e^{ikm}\frac{\partial}{\partial k}\,|\psi_k^R\rangle\right) \\ &= i\frac{1}{2\pi}\oint \mathrm{d}k\, e^{ikm}\frac{\partial}{\partial k}\,|\psi_k^R\rangle \end{aligned}$$

Hence

$$\langle \Delta m \rangle = \sum_m m \int_0^\infty \gamma \langle \psi_m^R | \psi_m^R \rangle = \sum_m \int_0^\infty \gamma \langle \psi_m^R | \, m \, | \psi_m^R \rangle \, \mathrm{d}t \tag{2.1.14}$$

$$= \sum_m \int_0^\infty \gamma \left(\frac{1}{2\pi} \oint \mathrm{d}k \, e^{-ikm} \langle \psi_k^R | \right) i \frac{1}{2\pi} \oint \mathrm{d}k' \, e^{ik'm} \frac{\partial}{\partial k} | \psi_k^R \rangle \tag{2.1.15}$$

$$= \sum_m \int_0^\infty \gamma \frac{i}{(2\pi)^2} \oint \mathrm{d}k \, \mathrm{d}k' \, e^{i(k'-k)m} \langle \psi_k^R | \frac{\partial}{\partial k} | \psi_k^R \rangle \tag{2.1.16}$$

Noticing

$$\sum_m e^{i(k'-k)m} = \delta(k'-k) \frac{(2\pi)^d}{v} \tag{2.1.17}$$

where v is the volumn of the primitive cell in real lattice and d is the dimention of the real lattice. So

$$\langle \Delta m \rangle = \int_0^\infty \frac{i\gamma}{2\pi v} \oint \mathrm{d}k \, \langle \psi_k^R | \frac{\partial}{\partial k} | \psi_k^R \rangle \tag{2.1.18}$$

`confusion:2` This same of Eq.(5) in the paper except a constant v. .

Next he sets $|\psi_k^R\rangle = u_k(t) e^{i\theta_k(t)}$, where u is the length and θ is the angle, of $|\psi_k^R\rangle$.

The author notes in [18] that the $u_k(t) > 0$ for $t > 0$, but this is obvious from Eq.2.1.13, which is translated into

$$\frac{\partial}{\partial t}(u_k)^2 = -\gamma (u_k)^2 \tag{2.1.19}$$

Thus $(u_k)^2$ is an exponential function, which is always positive unless the initial condition is 0.

Note

$$\oint \mathrm{d}k \, u_k \partial_k u_k = \frac{1}{2} \oint \mathrm{d}k \, \partial_k (u_k)^2 = \frac{1}{2} \left. (u_k)^2 \right|_{\text{two points of the same value}} = 0 \tag{2.1.20}$$

Then

$$\langle \Delta m \rangle = i\gamma \int_0^\infty dt \oint \frac{dk}{2\pi} \cdot \overset{u_k(t) e^{-i\theta_k(t)}}{\overset{\uparrow}{\underline{\psi_k^{R*}}}} \, \partial_k \left(u_k(t) e^{i\theta_k(t)} \right)$$

$$= i\gamma \int_0^\infty dt \oint \frac{dk}{2\pi} \; u_k(t) e^{-i\theta_k(t)} \cdot \left[\left(\partial_k u_k(t) \right) e^{i\theta_k(t)} + u_k(t) e^{i\theta_k(t)} \cdot i \left(\partial_k \theta_k(t) \right) \right]$$

$$\oint dk \left(u_k \partial_k u_k \right) = 0$$

$$= i\gamma \int_0^\infty dt \oint \frac{dk}{2\pi} \; u_k^2(t) \cdot i \, \partial_k \theta_k(t) = -\gamma \int_0^\infty dt \oint \frac{dk}{2\pi} \, u_k^2(t) \, \partial_k \theta_k(t).$$

$$= \oint \frac{dk}{2\pi} \int_0^\infty \left(\partial_t P_k \right) \left(\partial_k \theta_k(t) \right).$$

so

$$\langle \Delta m\rangle = \oint \frac{\mathrm{d}k}{2\pi}\int_0^\infty \mathrm{d}t\,(\frac{\partial p_k}{\partial t})(\frac{\partial \theta_k(t)}{\partial k}) \tag{2.1.21}$$

This is Eq.7 in the paper, except that I have ignored the $\frac{1}{v}$ difference between my result and the paper's. Then it is straight forward to obtain

$$\langle \Delta m\rangle = \mathcal{I}_0 - \int_0^\infty \mathrm{d}t \oint \frac{\mathrm{d}k}{2\pi} p_k \partial_t(\partial_k\theta_k) \tag{2.1.22}$$

eq:m-I0-int

where

$$\mathcal{I}_0 = \oint \frac{\mathrm{d}k}{2\pi}\left(p_k \left.\frac{\partial\theta_k}{\partial k}\right|_{t=0}^{\infty}\right) \tag{2.1.23}$$

Now the second term in eq.2.1.22 can be turned into (assuming $\theta_k(t)$ is smooth so that order of differentiation does not matter).

$$-\int_0^\infty \mathrm{d}t \oint \frac{\mathrm{d}k}{2\pi} p_k \partial_t(\partial_k\theta_k) = \int_0^\infty \mathrm{d}t \oint \frac{\mathrm{d}k}{2\pi}\frac{\partial p_k}{\partial k}\frac{\partial \theta_k}{\partial t} \tag{2.1.24}$$

eq:to-be-0

This is eq.(9) in the paper.

2.1.3 Arguing that the Integration 2.1.24 is zero

sec:Argument-1

confusion:3

Now the author tries to argue that p_k and $\partial_t\theta_k$ are even functions of k, so that the integration 2.1.24 is 0. I cannot follow his argument . But I find another model, which can found on Wikipedia - Rabi Cycle. The Rabi Cycle model can be applied only when $\gamma = 0$, i.e. there is no dissipation. But since the formulae above are continues, if the integration p_k and $\partial_t\theta_k$ are even, then they should be even when $\gamma = 0$, unless some very strange singular behavior exist.

$\gamma = 0$ case using Two-level system model

Suppose we have a Hermitian Hamiltonian (let $\{E_0, W_1, W_2, \Delta\} \subset \mathbb{R}$):

$$H = E_0\mathbb{1} + W_1\sigma_1 + W_2\sigma_2 + \Delta\sigma_3 = \begin{pmatrix} E_0+\Delta & W_1 - iW_2 \\ W_1 + iW_2 & E_0 - \Delta\end{pmatrix} \tag{2.1.25}$$

Supose the system starts at $|\psi(0)\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, then its time evolution will be (Set $\hbar = 1$. Calculation is done in the Wikipedia page, and is checked myself.):

$$|\psi(t)\rangle = \cos(\theta/2)e^{-iE_+t}\,|E_+\rangle - \sin(\theta/2)e^{-i\phi}e^{-iE_-t}\,|E_-\rangle \tag{2.1.26}$$

eq:time-evolve-rabi

where $|E_+\rangle = e^{i\phi/2}\begin{pmatrix}\cos(\theta/2)\\ e^{-i\phi}\sin\theta/2\end{pmatrix}$ and $|E_-\rangle = \begin{pmatrix}-e^{i\phi}\sin(\theta/2)\\ \cos(\theta/2)\end{pmatrix}$ are orthonormal eigenkets, and

$$\begin{aligned} &W := W_1 + iW_2, \quad E_\pm = E_0 \pm \sqrt{\Delta^2 + |W|^2} \\ &\sin(\theta) := \frac{|W|}{\sqrt{\Delta^2+|W|^2}},\ \cos(\theta) := \frac{\Delta}{\sqrt{\Delta^2+|W|^2}}, \end{aligned} \tag{2.1.27}$$

ϕ is such that $W = |W|e^{-i\phi}$.

Here for this paper. Let us suppose for the moment $\gamma = 0$, then we see that $W = A_k^*$, so $W_1 = v + v' \cos k$, $W_2 = -v' \sin k$. Also E_0 and Δ are linear combination of ε_R and ε_L:

$$E_0 = \frac{\varepsilon_R + \varepsilon_L}{2}, \quad \Delta = \frac{\varepsilon_R - \varepsilon_L}{2} \tag{2.1.28}$$

The system still starts with $|\psi(0)\rangle$ (since $\psi_k^L(0) = 1$, $\psi_k^R(0) = 0$). The time evolved state still follows eq.2.1.26, where the dependence on k lies in $\sin(\theta), \cos(\theta)$, and ϕ. Then by definition, $p_k = ||\psi(t)\rangle|^2$, $u_k(t)$ is the second component of $|\psi(t)\rangle$.

I was doubtful of the fact that p_k is even in k, so I did a numerical check (see file "OneD_quantum_walk_test_even_functions_pk_uk.m"), and unfortunately my guess seems to be wrong. I have given arbitrary numbers to time t, v, v', energies ε_R and ε_L, and the resulted p_k is always even in k, the resulted u_k is always constantly 0 in both time variable t and k.

This motivates me to calculate it analytically. The results is contained in files

"`Elementary Calculation of Time Evolution in Two-level model.nb`"

and

"`Elementary Calculation to get pk and uk from Two-Level model.nb`".

(their pdf version files are also included)

The result is that I can understand p_k is even in k, but I cannot see why $\partial_t u_k$ because its calculation involves getting the angle of a complex number, which has not good analytical expressions.

3 Anchor

sec:Anchor

References

1dwalk

[1] Topological Transition in a Non-Hermitian Quantum Walk, arXiv

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