Condensed Matter Field Theory notes

Taper

February 21, 2017

Abstract

Notes of book [AS10].

Contents

1 pp.33 eq.1.43 **1**

2 License **3**

1 pp.33 eq.1.43

sec:Table

In page 33 of [AS10], the author derives a difference of action, when we have a symmetry transformation paraterized by ω_a:

$$x_\mu \to x'_\mu = x_\mu + \frac{\partial x_\mu}{\omega_a}|_{\omega=0}\omega_a(x) \tag{1.0.1}$$

$$\phi^i(x) \to \phi'(x') = \phi^i(x) + \omega_a(x) F^i_a[\phi] \tag{1.0.2}$$

We have:

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}(\phi^i(x), \partial_{x_\mu}\phi^i(x)) && (1.0.3)\\
\mathcal{L}' &= \mathcal{L}'(\phi'^i(x'), \partial_{x'_\mu}\phi'^i(x')) && (1.0.4)\\
&= \mathcal{L}\left(\phi^i + F^i_a\omega_a, \left(\delta_{\mu\nu} - \partial_{x_\mu}(\omega_a\partial_{\omega_a}x_\mu)\right)\partial_{x_\nu}(\phi^i + F^i_a\omega_a)\right) && (1.0.5)
\end{aligned}$$

And

$$\begin{aligned}
\Delta S &= \int \mathrm{d}^m x'\,\mathcal{L}' - \int \mathrm{d}^m x\,\mathcal{L} && (1.0.6)\quad \text{eq:dS-integrand}\\
&= \int \mathrm{d}^m x \left(1 + \partial_{x_\mu}\left(\omega_a\partial_{\omega_a}x_\mu\right)\right)\\
&\quad\times \mathcal{L}\left(\phi^i + F^i_a\omega_a, \left(\delta_{\mu\nu} - \partial_{x_\mu}\left(\omega_a\partial_{\omega_a}x_\mu\right)\right)\partial_{x_\nu}(\phi^i + F^i_a\omega_a)\right)\\
&\quad - \int \mathrm{d}^m x\,\mathcal{L}(\phi^i(x), \partial_{x_\mu}\phi^i(x)) && (1.0.7)
\end{aligned}$$

Then he argues that, "for constant parameters ω_a the action difference Δa vanishes". Therefore "the leading contribution to the action difference of a symmetry transformation must be linear in the derivative $\partial_{x_\mu}\omega_a$".

Then he writes that "A straightforward expansion of the formula above for ΔS shows that these terms are given by"

$$\Delta S = -\int \mathrm{d}^m x\, j^a_\mu(x)\partial_{x_\mu}\omega_a \tag{1.0.8}$$

where j^a_μ is:

$$j^a_\mu = \left(\frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)}\partial_{x_\nu}\phi^i - \mathcal{L}\delta_{\mu\nu}\right)\frac{\partial x_\nu}{\partial \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)}F^i_a \tag{1.0.9}$$

I am partically confused about how to do the "straightforward expansion". I guess I should do $\frac{\partial}{\partial(\partial_{x_\mu}\omega_a)}$ to the integrand inside expression for ΔS, though I don't really understand the reason. Even so, the integrand contains terms like $\partial_{x_\mu}\partial_{\omega_a}x_\mu$, which I don't know how to deal with.

Solution. The reality is a bit more complicated. We first do a first order expasion to get the infinitesimal difference:

$$\begin{aligned}
&\mathcal{L}' - \mathcal{L} && (1.0.10)\\
\approx&\frac{\partial \mathcal{L}}{\partial \phi^i}F^i_a\omega_a + \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)}\left[\partial_\mu\left(F^i_a\omega_a\right) - \partial_\mu\left(\omega_a\frac{\partial x_\nu}{\partial\omega_a}\right)\partial_\nu\left(\phi^i + F^i_a\omega_a\right)\right]\\
=\ \ &\omega_a\left[\frac{\partial \mathcal{L}}{\partial \phi^i}F^i_a + \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)}\left(\partial_\mu F^i_a - \partial_\mu(\frac{\partial x_\nu}{\partial\omega_a})\partial_\nu(\phi^i + F^i_a\omega_a)\right)\right] && (1.0.11)\ \boxed{\texttt{eq:l-l-omega}}\\
+&\partial_\mu\omega_a\left[\frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)}\left(F^i_a - \frac{\partial x_\nu}{\partial\omega_a}\partial_\nu(\phi^i + F^i_a\omega_a)\right)\right] && (1.0.12)\ \boxed{\texttt{eq:l-l-pmu-omega}}
\end{aligned}$$

We also discover the integrand in Eq.1.0.6 to be

$$\begin{aligned}
&\left(1 + \partial_\mu(\omega_a\frac{\partial x_\mu}{\partial\omega_a})\right)\mathcal{L}' - \mathcal{L} && (1.0.13)\\
=&\left(1 + \partial_\mu(\omega_a\frac{\partial x_\mu}{\partial\omega_a})\right)(\mathcal{L}' - \mathcal{L}) + \left(\partial_\mu(\omega_a\frac{\partial x_\mu}{\partial\omega_a})\right)\mathcal{L} && (1.0.14)\ \boxed{\texttt{eq:integrand-l-density}}
\end{aligned}$$

For the first term $\left(1 + \partial_\mu(\omega_a\frac{\partial x_\mu}{\partial\omega_a})\right)(\mathcal{L}' - \mathcal{L})$, the $(\mathcal{L}' - \mathcal{L})$ already has terms of first order of ω_a and of first order of $\partial_\nu\omega_a$. For our purpose, the second order terms $(\partial_\nu(F^i_a\omega_a))$ from item 1.0.11 and item 1.0.12 can be ignored. Also, the item $(\partial_\mu(\omega_a\frac{\partial x_\mu}{\partial\omega_a}))(\mathcal{L}' - \mathcal{L})$ in eq.1.0.14 can also be ignored.

Therefore the integrand in Eq.1.0.6 becomes

$$\begin{aligned}
&(\mathcal{L}' - \mathcal{L}) + \left(\partial_\mu(\omega_a\frac{\partial x_\mu}{\partial\omega_a})\right)\mathcal{L} && (1.0.15)\\
=&\omega_a\left[\frac{\partial \mathcal{L}}{\partial \phi^i}F^i_a + \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)}\left(\partial_\mu F^i_a - (\partial_\mu\frac{\partial x_\nu}{\partial\omega_a})\partial_\nu(\phi^i + F^i_a\omega_a)\right) + (\partial_\nu\frac{\partial x_\mu}{\partial\omega_a})\mathcal{L}\right] && (1.0.16)\\
+&\partial_\mu\omega_a\left[\frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)}\left(F^i_a - \frac{\partial x_\nu}{\partial\omega_a}\partial_\nu(\phi^i + F^i_a\omega_a)\right) + \frac{\partial x_\mu}{\partial\omega_a}\mathcal{L}\right] && (1.0.17)
\end{aligned}$$

Therefore, the term we seek, i.e. the coefficient of $\partial_\mu \omega_a$ is

$$\frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu} \phi^i)} \left(F_a^i - \frac{\partial x_\nu}{\partial \omega_a} \partial_\nu (\phi^i + F_a^i \omega_a) \right) + \frac{\partial x_\mu}{\partial \omega_a} \mathcal{L} \tag{1.0.18}$$

$$= \left(\mathcal{L} \delta_{\mu\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu} \phi^i)} \partial_\nu \phi^i \right) \frac{\partial x_\nu}{\partial \omega_a} + \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu} \phi^i)} F_a^i \tag{1.0.19}$$

which is what we expect in equation 1.43 of [AS10].

Question: as for why we should ignore the term with ω_a, there are two posts ([1], [2]) might be useful for a thought.

References

[AS10] Alexander. Altland and Ben BD Ben Simons. *Condensed matter field theory*. Cambridge University Press, 2010.

2 License