

# Miscellaneous notes for D. Huybrechts's Complex Geometry

Taper

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## Abstract

Miscellaneous notes for D. Huybrechts's book *Introduction to Complex Geometry*, include some homeworks done.

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## 1 The structure of almost complex structures on $\mathbb{R}^4$ (exercise 1.2.1)

In exercise 1.2.1, it says that the set of all compatible almost complex structures on a euclidean space of dimension  $2n$ , is two copies of  $S^2$ .

To show it, I tried first a straight calculation. Assuming the almost complex structure  $I = (a_{ij})$ . Then we have:

Section 1.2 of Intro to Complex Geometry.

Exercises:

1.2-1.

Q: Let  $(V, \langle, \rangle)$  : euclidian space of  $\dim = 4$ .

Show:  $\{ \text{all compatible almost complex structures} \}$   
 $\cong$   
 $\text{two copies of } S^2 \cong \text{two balls.}$

Recap: compatible:  $I: I^2 = -1, \langle I(v), I(w) \rangle = \langle v, w \rangle$ .

Choose an orthogonal basis:  $e_1, \dots, e_4$

Let  $I = (a_{ij}^i)$   $\boxed{I^2 = a_{ij}^i a_{kl}^j = -\delta_{ik}^i}$

also  $\langle, \rangle \approx \delta_{ij}^i$   $\otimes$

$$\langle I(v), I(w) \rangle = (a_{ij}^i v^j) \cdot \delta_{kl}^i (a_{kl}^i w^l) = v^i \delta_{kl}^i w^k$$

( $\forall \vec{v}, \vec{w}$ )

Hence  $a_{ij}^i \delta_{kl}^i a_{kl}^j = \delta_{jl}^i$  or  $\boxed{a_{ij}^i a_{kl}^j = \delta_{jk}^i}$

For example:

$$\sum_j a_{1j}^1 a_{2j}^2 = 0 = \sum_j a_{1j}^2 a_{2j}^1 \Rightarrow \sum_{j=2}^4 a_{1j}^1 a_{2j}^2 = \sum_{j=2}^4 a_{1j}^2 a_{2j}^1$$

This can be generalized:

$$\sum_{\substack{j=1 \\ j \neq k}}^4 a_{1j}^k a_{2j}^j = \sum_{\substack{j=1 \\ j \neq k}}^4 a_{1j}^j a_{2j}^k \quad (k \neq l) \quad \left. \vphantom{\sum_{j=1}^4} \right\} \text{16 sets of eq.}$$

also:  $a_{1j}^i \sum_{j=1}^4 a_{ij}^j a_{2j}^j = - \sum_{j=1}^4 a_{1j}^i a_{2j}^j a_{ij}^j$

Figure 1: Draft

Then I discover this too hard to work, because too many equations are involved, and none of them could be eliminated by other. Meanwhile, I found a post in Math.SE about this [2]. Here are several important concepts for understanding that post.

## 1.1 Understand $\frac{GL(2n, \mathbb{R})}{GL(n, \mathbb{C})}$

### 1.1.1 Why $M_n = \frac{GL(2n, \mathbb{R})}{GL(n, \mathbb{C})}$

This is a note of my question on Math.SE [5], which explains that we can identify the set of almost complex structures with  $\frac{GL(2n, \mathbb{R})}{GL(n, \mathbb{C})}$ .

First, I try to do it when  $n = 1$ . I inject a complex number  $a + bi$  by identify it with  $a\mathbb{I} + b\mathbb{J}$ , where  $\mathbb{I}$  is the identity matrix and  $\mathbb{J}$  is  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . I take one set of basis of  $GL(2, \mathbb{R})$  as:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(I think this is a basis because the following matrix is non-singular:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

) Then the  $\frac{GL(2n, \mathbb{R})}{GL(n, \mathbb{C})}$  becomes equivalent classes represented by

$$\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$$

However, I don't know how to link this with an almost complex structure.

I have a feeling that I might have been in the wrong direction. It was pointed out that  $GL(2n, \mathbb{R})$  is not even a vector space. So what I did is in fact nonsense.

Below is one correct answer I got:

An almost-complex structure is a matrix  $J$  such that  $J^2 = -I$  is the negative identity. As you said, one example of such a matrix  $J$  is

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Interpreting  $GL(n, \mathbb{C})$  as a subgroup of  $GL(2n; \mathbb{R})$  depends on having fixed such an almost-complex structure. Once we have a matrix  $J$ , we can call a matrix  $A \in GL(2n; \mathbb{R})$  complex-linear if it commutes with  $J$ , i.e.  $AJA^{-1} = J$ .

(The idea is that  $\mathbb{C}$ -linear maps  $T$  are just real linear maps with the additional property that  $T(iv) = iT(v)$  for all vectors  $v$ )

Given any matrix  $A \in GL(2n; \mathbb{R})$ , we get another almost-complex structure  $AJA^{-1}$ . This is the same almost-complex structure  $J$  if and

only if  $A \in GL(n; \mathbb{C})$ . On the other hand, all almost-complex structures are similar (although it may take some work to be convincing that they are similar over  $\mathbb{R}$  and not only  $\mathbb{C}$ ) since they are diagonalizable with the same eigenvalues  $\pm i$ . That gives you a bijection

$$GL(2n; \mathbb{R})/GL(n; \mathbb{C}) \longrightarrow \{\text{almost - complex structures}\}$$

under which a class  $A \cdot GL(n; \mathbb{C})$  corresponds to the almost-complex structure  $AJA^{-1}$ .

I questioned him:

1. Why  $AJA^{-1}$  is the same almost-complex structure  $J$  if and only if  $A \in GL(n; \mathbb{C})$ .
2. Why all almost-complex structures are similar over  $\mathbb{R}$ .

He responded that:

1. is the definition of  $GL(n; \mathbb{C})$  as matrices  $A$  with  $AJA^{-1} = J$ .
2. comes from the fact that any real matrices that are similar over  $\mathbb{C}$  are already similar over  $\mathbb{R}$ . This isn't trivial but it has been asked and answered many times on this site: here is one reference [6].

Inside that reference, the following theorem is proved:

**Theorem 1.1.** *Let  $E$  be a field, let  $F$  be a subfield, and let  $A$  and  $B$  be  $n \times n$  matrices with coefficients in  $F$ . If  $A$  and  $B$  are similar over  $E$ , they are similar over  $F$ .*

However, I still have doubts about the following question: For  $A \in GL(2n, \mathbb{R})$ , if  $AJA^{-1} = J$ , can we conclude that  $A$  is inside  $GL(n, \mathbb{C})$ ?

### 1.1.2 Why $GL(n, \mathbb{C}) \hookrightarrow GL^+(2n, \mathbb{R})$

To understand that post [2], I also read this [3]. In it, it asks how to prove that

$$GL(n, \mathbb{C}) \hookrightarrow GL^+(2n, \mathbb{R}) \tag{1.1.1}$$

for any  $n$ . The questioner gives the intuition for this fact:

how about since as Lie groups,  $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$  and  $GL(n, \mathbb{C})$  is connected but  $GL(2n, \mathbb{R})$  has two connected components, one for positive determinant and one for negative determinant? And the identity has positive determinant, so it must lie in that component.

Someone answered that question:

The claim is: If  $V$  is an  $n$ -dimensional complex vector space with underlying  $2n$ -dimensional real vector space  $W$ , then the canonical group monomorphism  $GL(V) \rightarrow GL(W)$  lands inside  $GL^+(W) = \{f \in GL(W) : \det(f) > 0\}$ . The purpose of this abstract reformulation is that we may use operations on vector spaces in order to simplify the problem: If  $V'$  is another finite-dimensional complex vector space with underlying real vector space  $W'$ , the diagram

$$\begin{array}{ccc}
\mathrm{GL}(V) \times \mathrm{GL}(V') & \rightarrow & \mathrm{GL}(W) \times \mathrm{GL}(W') \\
\downarrow & & \downarrow \\
\mathrm{GL}(V \oplus V') & \rightarrow & \mathrm{GL}(W \oplus W')
\end{array} \tag{1.1.2}$$

commutes, and the image of  $\mathrm{GL}^+(W) \times \mathrm{GL}^+(W')$  is contained in  $\mathrm{GL}^+(W \oplus W')$ . Therefore, if some element in  $\mathrm{GL}(V \oplus V')$  lies in the image of  $\mathrm{GL}(V) \times \mathrm{GL}(V')$ , it suffices to consider the components. Combining this with the fact that  $\mathrm{GL}(V)$  is generated by elementary matrices (after choosing a basis of  $V$ ), we may reduce the whole problem to the following three types of matrices:

- the  $1 \times 1$ -matrices  $(\lambda)$ ,
- the  $2 \times 2$ -matrices  $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$ ,
- and the  $2 \times 2$ -matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

Write  $\lambda = a + ib$  with  $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ . Then, the complex  $1 \times 1$ -matrix  $(\lambda)$  becomes the real  $2 \times 2$ -matrix  $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ , which has determinant  $a^2 + b^2 > 0$ . The complex  $2 \times 2$ -matrix  $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$

becomes the real  $4 \times 4$ -matrix  $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & -b & 1 & 0 \\ b & a & 0 & 1 \end{pmatrix}$ , which has

determinant 1. Finally, the complex  $2 \times 2$ -matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  be-

comes the real  $4 \times 4$ -matrix  $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ , which has deter-

minant 1.

However, this proof is not complete because, to build the proof from  $\mathbb{R}^2$  to  $\mathbb{R}^{2n}$ , it requires, in his argument, that any element in  $\mathrm{GL}(V \oplus V)$  is in the image of  $\mathrm{GL}(V) \times \mathrm{GL}(V')$ , which is not the case.

On the other hand, it seems that this property can be proved directly by calculation. The following will be a notes of a paper [4], which one comment mentions in the Math.SE post [3].

### 1.1.3 Determinants of Block Matrices

This paper tries to prove the theorem:

**Theorem 1.2.** *Let  $R$  be a commutative subring of  ${}^n F^n$ , where  $F$  is a field (or a commutative ring), and let  $M \in {}^m R^m$ . Then*

$$\det_F \mathbf{M} = \det_F(\det_R \mathbf{M}) \tag{1.1.3}$$

In particular, we have:

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \det_F(AD - BC) \quad (1.1.4)$$

Note that, that the ring being is commutative excludes some ambiguity. For example, when the ring 4 is not commutative, then the quantity:

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \quad (1.1.5)$$

is not well-defined. It can be  $AD - BC$ , or  $DA - CB$ , etc.

Before the proof of the main theorem, it establishes several facts:

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \det_F \mathbf{A} \det_F \mathbf{D} \quad (1.1.6)$$

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} = \det_F \mathbf{A} \det_F \mathbf{D} \quad (1.1.7)$$

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{pmatrix} = \det_F -\mathbf{C} \det_F \mathbf{B} \quad (1.1.8)$$

$$\det_F \mathbf{A} \det_F \mathbf{D} = \det_F \mathbf{I}_n \det_F(\mathbf{AD}) \quad (1.1.9)$$

He first builds up a seemingly simplified, but is actually different version of the main theorem:

**Theorem 1.3.** *Let  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in {}^n F^n$ . Let  $\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$ .*

*If  $\mathbf{CD} = \mathbf{DC}$ , then,*

$$\det_F \mathbf{M} = \det_F(\mathbf{AD} - \mathbf{BC}) \quad (1.1.10)$$

and similar results:

$$\text{if } \mathbf{AC} = \mathbf{CA} \text{ then, } \det_F \mathbf{M} = \det_F(\mathbf{AD} - \mathbf{CB}) \quad (1.1.11)$$

$$\text{if } \mathbf{BD} = \mathbf{DB} \text{ then, } \det_F \mathbf{M} = \det_F(\mathbf{DA} - \mathbf{BC}) \quad (1.1.12)$$

$$\text{if } \mathbf{AB} = \mathbf{BA} \text{ then, } \det_F \mathbf{M} = \det_F(\mathbf{DA} - \mathbf{CB}) \quad (1.1.13)$$

These equalities can be proved easily by the following:

$$\begin{pmatrix} D & 0 \\ -C & i \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} AD - BC & B \\ CD - DC & D \end{pmatrix} = \begin{pmatrix} AD - BC & B \\ 0 & D \end{pmatrix} \text{ when } C, D \text{ commutes}$$

$$\begin{pmatrix} D & -B \\ 0 & i \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} DA - BC & DB - BD \\ C & D \end{pmatrix} = \begin{pmatrix} DA - BC & 0 \\ C & D \end{pmatrix} \text{ when } D, B \text{ commutes.}$$

The author also gives an illuminative explanation for why

$$(\det_F \mathbf{M} - \det_F(\mathbf{AD} - \mathbf{BC})) \det_F \mathbf{D} = 0$$

necessarily implies:

$$\det_F \mathbf{M} = \det_F(\mathbf{AD} - \mathbf{BC})$$

However, I am dubious about this conclusion, since I think it needs in addition that the polynomial ring  $F[x]$  has not nonzero zero divisor.

Having demonstrated the above simple case, the author continues to prove the main theorem. He proves by induction. He first uses:

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} \begin{pmatrix} dI & 0 \\ -c & 1 \end{pmatrix} = \begin{pmatrix} A_0 & b \\ 0 & d \end{pmatrix} \quad (1.1.14)$$

where  $A, A_0 \in {}^{m-1}R^{m-1}, b \in {}^{m-1}R, c \in R^{m-1}, d \in R$ . Therefore, (let  $M = \begin{pmatrix} A & b \\ c & d \end{pmatrix}$ ) with similar reason mentioned before, he shows if:

$$\det_F \mathbf{A}_0 = \det_F(\det_R \mathbf{A}_0) \quad (1.1.15)$$

(which is true by induction) then:

$$\det_F \mathbf{M} = \det_F(\det_R \mathbf{M}) \quad (1.1.16)$$

Proof completes.

He also mentions a corollary:

**Corollary 1.1.** *Let  $\mathbf{P} \in {}^n F^n$  and  $\mathbf{Q} \in {}^m F^m$ , then*

$$\det_F(\mathbf{P} \otimes \mathbf{Q}) = (\det_F \mathbf{P})^m (\det_F \mathbf{Q})^n \quad (1.1.17)$$

The proof is quite straightforward and is omitted.

#### 1.1.4 Why $GL(n, \mathbb{C}) \hookrightarrow GL^+(2n, \mathbb{R})$ (continued)

With above theorem, the proof of equation 1.1.1 is straight forward. Since for  $(a_{ij}) = A \in GL(n, \mathbb{C})$ , it injects into  $GL(2n, \mathbb{R})$  as matrices of the form:

$$\begin{pmatrix} \dots & \dots & \dots \\ \dots & H a_{ij} & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

where:

$$H(z \equiv x + iy) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

Since  $H(a_{ij})$  commutes with each other (proved by calculation), we can use the theorem in previous part to show that:

$$\det_{\mathbb{R}}(A) = \det_{\mathbb{R}}(\det_{\mathbb{C}}(A)) = |\det_{\mathbb{C}}(A)|^2 \geq 0$$

Notice that I have been sloopy in language, but the meaning should be clear.

## 2 Anchor

### References

- [1] D Huybrechts's Introduction to Complex Geometry.
- [2] set of almost complex structures on  $\mathbb{R}^4$  as two disjoint spheres.
- [3] Does  $GL(n, \mathbb{C})$  inject into  $GL^+(2n, \mathbb{R})$  for all  $n$ ?
- [4] John R. Silvester, Determinants of Block Matrices. Available in WebArchive link: <https://web.archive.org/web/20140505161153/http://www.mth.kcl.ac.uk/~jrs/gazette/blocks.pdf>
- [5] Making sense of  $\frac{GL(2n, \mathbb{R})}{GL(n, \mathbb{C})}$
- [6] Similar matrices and field extensions

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