Notes for Classification of topological quantum matter with symmetries

Taper

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Abstract

As title suggests.

Contents

1 Doubts

Asks

1. Page 6. Why the scalar in Schur's lemma becomes of unit lenght.
2. page 9. What does it mean by:

   > Note that unitary symmetries, which commute with the Hamiltonian, allow us to bring the Hamiltonian into a block diagonal form.

3. page 9, table I. When he talks about *codimension*, what is the dimension of the whole space? (3? 3+1?). Similar problem also exists in page 10. Note, he mentions codimension of gaples modes on page 11. He asserts that codimension 1 means 1 dimension less than the

bulk. But it should be strange to compare the dimension of defects with the dimension of the bulk.

Possibly related resources:Online notes about Imperfection:

- 0D (zero dimension) – point defects: vacancies and interstitials. Impurities.
- 1D – linear defects: dislocations (edge, screw, mixed)
- 2D – grain boundaries, surfaces.
- 3D – extended defects: pores, cracks.

Note: it is finally defined on page 12. that is: codimension of defect $d_c := d_{\text{bulk}} - d_{\text{defect}}$.

4. page 6, what does it mean by saying *"unitary symmetry"*.
5. page 10, what is a *"quantum phase diagram"*.
6. page 11, what does he says, *"Topological properties of adiabatic cycles can also be discussed in a similar manner."*. Does this mean that all previous classification does not concern the adiabatic cycles? What is *"adiabatic cycles"* exactly in his language?

   Note: "adiabatic cycle" may refer to a cycle in phase space (most likely argumented by time t parameter). "Adiabatic" describes the process to be adiabatical, i.e. vary very slowly. The detailed criterion is on page 12, just above equation 3.6:

   $$\xi|\Delta_r H(k,r)| << \varepsilon_g \tag{1.0.1}$$

7. page 12, "disinclination" is what kind of defect? Any books on crystall defects?
8. page 12, is the *"mass gap"* a massive gap or a gap composed of mass?
9. page 12, about the D: if $d_c = 1$ (line defect in a $2d$-bulk), then $D = 0$. So a line is wrapped by a point? also, on fig. 2, the $(D = 2, d = 1)$ gives a $d_{\text{defect}} = -1$! Judging from this graph, a $d_{\text{defect}} = -1$ means a temporal defect. Is this true?
10. page 42, what is a nodal system, what are nodal points?

Ask friends

1. page 13. What is a homotopy type?

Doubts

1. page 10. Amazingly, he says, *"all TIs and TSCs in the ten AZ symmetry classes are stable against disorder, and hence the assumption of translation invariance is not at all necessary"*. How can translational invariance be ignored?
2. page 12, he mentions:

   > As in the case of gapped TIs and TSCs, we are interested in the highest dimension strong topologies of the defect that do not involve lower dimensional cycles

I don't get what "strong topologies" and "lower dimensional cycles" mean.

3. page 13 right column, again he mentioned the strong topology and compactify the space into a S^{d+D}. I don't get why:

   > Physically this means the defect band theory are assumed to have trivial winding around those low-dimensional cycles.

4. page 13, What does this mean:

   > It deformation retracts from the defect complement of spacetime.

Revisit

1. page 12, bottom. How this procedure of relating real and complex classification is done?

2 Summary

2.1 About topological matter

Topological states:

- Topological insulators cannot be distinguished from ordinary, topologically trivial insulators in terms of their symmetries.
- topological insulators' topological nontriviality cannot be detected by a local order parameter.
- insulating electronic band structures can be categorized in terms of topology.
  - time-reversal are crucial in makeing a spin-orbit induced topological insulators. $\Rightarrow$ symmetry-protected topological phases (**SPT**).
  - symmetry-protected topological phase can also arise from spatial symmetry.
- There is a direct analogy between **TI** s and **TSC** s (topological insulators and topological superconductors).
- **bulk-boundary correspondence** $\Leftarrow$ topological phase transition. (pp. 11).
- **nodal systems** can exhibit band topology even though their bulk gap closes at certain points in BZ.
- All of the above are understood in terms of non-interacting or mean-field Hamiltonians. Less is known about strongly correlated systems.

2.2 Features of Tenfold for Gapped System

Topological equivalence . l : adiabatic path or continuous path in phase diagram. Then topological class are characterized by invariance of gap under l.

Trivial class : The topological classes of state characterized by atomic insulators.

Weak topological phase : nontrivial topological phase relies on lattice-translational symmetry. **Strong topological phase** : ... that does not rely on translational symmetry.

Classfication of Gapless

- 2 complex symmetry classes, 8 real symmetry classes.
- The presence or absence of weak TIs or TSCs in a given symmetry classes can be deduced from that of strong TIs or TSCs in lower dimension in the same symmetry class. So the classification aims for the strong ones.
- **dimensional shift** (pp. 11).
- **primary series** , **first/second decendants** , **even series** : class of items in the periodic table. (pp. 12).

Classfication with defects

- **defect Hamiltonian** $H(k,r)$,$r = (x,t)$. Key: r varies slowly ($\xi|\nabla_r H(r,k)| \ll \varepsilon_g$). page 13.
- Characterization (wrapped by S^D)
  1. s: AZ symmetry class
  2. d: bulk dimension
  3. d_c: defect codimension
     - real AZ class: $s-\delta \pmod 8$ ($\delta = d - D$, topological dimension)
     - complex AZ classes, similarly
- Relate real and complex: forgetful functor.
- **bulk-defect correspondence** : guarantees gapless defect excitation.

Topological Invariants

Base space :

$$(\mathbf{k},\mathbf{r}) \in \mathrm{BZ}^d \times \mathcal{M}^D \xrightarrow{\text{compactify}} S^{d+D} \tag{2.2.1}$$

Table 3 in page 13, is a summary of invariants: **s** := AZ symmetry class. **Ch** : Chern number. **CS** :Chern-Simons invariant. ν : Winding number. **FK** : Fu-Kane.

Table 1: caption

	s even	s odd
$\mathbb{Z}$	Ch	ν
$\mathbb{Z}_2^{(1)}$	CS	FK
$\mathbb{Z}_2^{(2)}$	FK	CS ($\widetilde{\text{CS}}$)

2.3 Chern Number

It detects an obstruction in defining a set of bloch wave functions smooothly over the base space.

Fibres The fibres of the fibre-bundle on the base space are wave functions under adiabatic changes. They may be twisted.

Twist : Think of the Möbius strip as a twisted fibre bundle. A good resource: Cylinder and Möbius strip as fiber bundles.

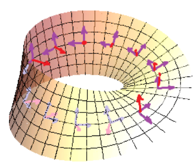

Figure 1: Möbius Strip, Credit: Quora

Theorem 2.1. *The set of Bloch functions (i.e. projectors) defines a map from the base space to $U(N_+ + N_-)/U(N_+) \times U(N_-)$. Topological distinct maps of this type can be classified by their homotopy group*

$$\pi_{d+D}[U(N_+ + N_-)/U(N_+) \times U(N_-)] \tag{2.3.1}$$

When $N_\pm$ is large enough, and when $d + D$ is even, this group is $\mathbb{Z}$, the integer topological invariant is the Chern number.

2.4 Winding number

- Can only be defined in the presence of chiral symmetry.
- Relavant space: $U(N), \Rightarrow$
- Q matrix (projector) Can be classified by $\pi_{d+D}[U(N)]$, which is nontrivial when $d+D$ is odd. When $d+D$ is odd, $\pi_{d+D}[U(N)] = \mathbb{Z}$, with maps characterized by the winding number.

2.5 Chern-Simons invariant

- Can be defined when $d + D$ is odd.
- Symmetry $\to$ Quantized/Discrete CS invariant. e.g.
  - chiral symmetry quantizes CS
- Not gauge-invariant.
  - But $W := \exp\{2\pi i\, \text{CS}\}$ is gauge-invariant.

2.6 Fu-Kane Invariant

- Construction requires special attention in the choice of basis

2.7 K-theory Approach

- Topological nontrivial phases $\Rightarrow$ have massive Dirac Hamiltonian representatives

3 Anchor

References

[1] http://journals.aps.org/rmp/abstract/10.1103/RevModPhys.88.035005

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