Draft for Quantum Field Theory in a Nutshell by A. Zee. 2ed.

$q \updownarrow$

$$\langle q_F | e^{-iHT} | q_I \rangle = \langle q_F | e^{-iH\delta t} \cdots e^{-iH\delta t} | q_I \rangle$$

$$\int dq \, |q\rangle\langle q| = 1.$$

$$\int dq \, \langle q'' | q \rangle \langle q | q' \rangle = \int dq \, \delta(q-q'') \delta(q-q') = \int dq \cdot \left(\int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(q''-q)} \right) \left(\int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(q-q')} \right)$$

$$\int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ipq} = \quad q \neq 0, \; 0$$

$$q = 0 \quad \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \cdot \to \infty$$

$$\int dq \cdot \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ipq}$$

Appendix 1. (in I.1)

$$\cancel{d_K(x) = \int_{-\frac{K}{2}}^{\frac{K}{2}} \frac{dk}{2\pi} e^{ikx} = \int \frac{1}{2\pi} \cdot \frac{1}{ix} e^{ikx} \Big|_{-\frac{K}{2}}^{\frac{K}{2}} = \frac{1}{2\pi} \frac{1}{ix} \left(e^{i\frac{K}{2}x} - e^{-i\frac{K}{2}x} \right) = \frac{1}{\pi i x} \cdot \sin \frac{Kx}{2}}$$

$$d_K(x) = \int_{-\frac{K}{2}}^{\frac{K}{2}} \frac{dk}{2\pi} e^{ikx} = \frac{1}{2\pi} \cdot \frac{1}{ix} \cdot e^{ikx} \Big|_{-\frac{K}{2}}^{\frac{K}{2}} = \cancel{\frac{1}{2\pi} \frac{1}{ix} \cdot i \cdot 2 \sin\left(\frac{K}{2}x\right)}$$

$$= \frac{1}{2\pi} \frac{1}{ix} \, i \cdot 2 \cdot \sin\left(\frac{K}{2}x\right) = \frac{1}{\pi x} \cdot \sin\left(\frac{K}{2}x\right)$$

$$\int_{-\infty}^{+\infty} dx \cdot d_K(x) = \int_{-\infty}^{+\infty} dx \cdot \frac{1}{\pi x} \cdot \sin\left(\frac{K}{2}x\right) \underset{dy = \frac{K}{2} dx}{\overset{y = \frac{K}{2}x}{=\!=}} \int_{-\infty}^{+\infty} dy \cdot \frac{2}{K\pi \frac{2}{K} y} \cdot \sin y = \int_{-\infty}^{+\infty} \frac{1}{\pi} \cdot \frac{\sin y}{y}$$

$$= \cancel{\frac{1}{\pi} \cdot \mathrm{Im} \lim \oint \frac{e^{ix}}{x}} = 1$$

So $\delta(x) = \lim_{K \to \infty} d_K(x)$

$$\int_{-\infty}^{+\infty} dx \cdot \delta(x-a)\, S(x) = \int_{-\infty}^{+\infty} dx \cdot \lim_{K\to\infty}\left(\frac{1}{\pi x}\cdot \sin\frac{K(x-a)}{2}\right)\cdot S(x)$$

$$\int_{-\infty}^{+\infty} dx \frac{1}{x+i\varepsilon} = \int_{-\infty}^{+\infty} dx\, \mathcal{P}\frac{1}{x} - i\pi\delta(x)$$

$$\int_{-\infty}^{+\infty} dx \frac{1}{x+i\varepsilon} = \int_{-\infty}^{+\infty} dx \frac{x-i\varepsilon}{x^2+\varepsilon^2} = \int_{-\infty}^{+\infty} \frac{x}{x^2+\varepsilon^2} - i\int_{-\infty}^{+\infty} \frac{dx\,\varepsilon}{x^2+\varepsilon^2}$$

$$\langle q_F | e^{-iHT} | q_I\rangle = \langle q_F | e^{-iH\delta t} \cdots e^{-iH\delta t} | q_I\rangle.$$

$$= \left(\prod_{j=1}^{N-1} \int dq_j\right) \langle q_F | e^{-iH\delta t} | q_{N-1}\rangle\langle q_{N-1}| e^{-iH\delta t} | q_{N-2}\rangle \cdots \langle q_1 | e^{-iH\delta t} | q_I\rangle.$$

$$\cancel{\langle q_{j+1} | e^{-i\frac{\hat{p}^2}{2m}\delta t} | q_j\rangle = \int}$$

$$\left(\int \frac{dp}{2\pi}\langle q'|p\rangle\langle p|q''\rangle = \int \frac{dp}{2\pi} e^{ipq'}\cdot e^{-ipq''} = \frac{1}{2\pi}\int dp\, e^{ip(q'-q'')} = \delta(q'-q'')\right)$$

$$\int \frac{dp}{2\pi}\langle q_{j+1}| e^{-i\frac{\hat{p}}{2m}\delta t}\cdot |p\rangle\langle p|q_j\rangle = \int \frac{dp}{2\pi}\cdot\langle q_{j+1}| e^{-i\delta t\frac{p^2}{2\pi}} |p\rangle\langle p|q_j\rangle$$

$$= \int \frac{dp}{2\pi}\cdot e^{-i\delta t\frac{p^2}{2m}}\cdot e^{ipq_{j+1}}\cdot e^{-ipq_j} = \int \frac{dp}{2\pi}\cdot e^{-\frac{i\delta t}{2m}\cdot p^2 + i(q_{j+1}-q_j)p}$$

$$\cancel{= e^{-\frac{-(q_{j+1}-q_j)}{4(-i\frac{\delta t}{2m})}}\sqrt{\frac{\pi}{i\frac{\delta t}{2m}}} = e^{i\frac{(q_{j+1}-q_j)m}{2\delta t}}\cdot\sqrt{-i\frac{2\pi m}{\delta t}}}$$ (Calculated Using Mathematica®.).

$$\cancel{= e^{i\frac{m}{2}\delta t\frac{(q_{j+1}-q_j)}{\delta t^2}}}$$

$$= \frac{1}{2\pi} e^{-\frac{-(q_{j+1}-q_j)^2}{4(-i\frac{\delta t}{2m})}}\cdot\sqrt{\frac{\pi}{i\frac{\delta t}{2m}}} = \frac{1}{2\pi} e^{i\frac{(q_{j+1}-q_j)^2}{\frac{2\delta t}{m}}}\cdot\sqrt{\cdots} = e^{\frac{im(q_{j+1}-q_j)^2}{2\delta t}}\cdot\sqrt{\frac{-i2\pi m}{\delta t}}\,\frac{1}{2\pi}$$

$$= e^{im\frac{(q_{j+1}-q_j)^2}{2\delta t}}\sqrt{\frac{-im}{2\pi\delta t}}$$

So

$$\cancel{\langle q_F | e^{-iHt} | q_I \rangle = \prod_{j=1}^{N-1} \int e^{im\frac{(q_F - q_{N-1})^2}{2\delta t}} \sqrt{\frac{-im}{2\pi\delta t}} \cdots}$$

$$= \cancel{\left(\frac{-im}{2\pi\delta t}\right)^{\frac{N}{2}} \cdot e^{\frac{im}{2\delta t}\left((q_F - q_{N-1})^2 + \cdots (q_1 - q_I)^2\right)}}$$

$$\cancel{\int dq_1 \langle q_F | e^{-iH\delta t} | q_1 \rangle \langle q_1 | e^{-iH\delta t} | q_I \rangle =}$$

So when $\hat{H} = \frac{\hat{p}^2}{2m}$

$$\langle q_F | e^{-iH\delta t} | q_I \rangle = \prod_{j=1}^{N-1} \int dq_j \cdot \left(\frac{-im}{2\pi\delta t}\right)^{\frac{N}{2}} \cdot e^{\frac{im}{2\delta t}\left((q_F - q_{N-1})^2 + \cdots + (q_1 - q_I)^2\right)}.$$

$$= \prod_{j=1}^{N-1} \int dq_j \left(\frac{-im}{2\pi\delta t}\right)^{\frac{N}{2}} \cdot e^{\frac{im}{2}\delta t \sum_{j=0}^{N-1} \left(\frac{q_{j+1} - q_j}{\delta t}\right)^2}$$

$$\to \underbrace{\prod_{j=1}^{N-1} \int dq_j \left(\frac{-im}{2\pi\delta t}\right)^{\frac{N}{2}}}_{\downarrow} \cdot e^{\frac{im}{2} \cdot \int dt \cdot \left(\frac{dq(t)}{dt}\right)^2}$$

$$\int Dq(t) := \lim_{N\to\infty} \left(\frac{-im}{2\pi\delta t}\right)^{\frac{N}{2}} \int \prod_{j=1}^{N-1} dq_j$$

so

$$\langle q_F | e^{-iHt} | q_I \rangle = \int Dq(t) \cdot e^{\frac{im}{2}\int dt\, \dot{q}^2} = \int Dq(t)\, e^{i\int dt \cdot \frac{1}{2} m \dot{q}^2}$$

Appendix 2

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}ax^2}\,dx = \sqrt{\frac{2\pi}{a}}$$

$$\int_{-\infty}^{+\infty} x^{2n} e^{-\frac{1}{2}ax^2}\,dx = \int_{-\infty}^{+\infty} x^{2n-2}\left(\frac{d}{da} e^{-\frac{1}{2}ax^2}\right)\cdot(-2)\,dx$$

$$= \int_{-\infty}^{+\infty} (-2)^n \left(\frac{d}{da}\right)^n e^{-\frac{1}{2}ax^2}\,dx = (-2)^n \cdot \frac{d^n}{da^n}\cdot\sqrt{\frac{2\pi}{a}} \qquad (a^{-\frac{1}{2}})$$

$$-\frac{3}{2}\ \frac{5}{\ }\cdots -\frac{\ }{2}$$

$$= (-2)^n \sqrt{2\pi}\cdot\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)\cdot a^{-\frac{2n+1}{2}}$$

$$= \sqrt{2\pi}\cdot 1\cdot 3\cdot 5\cdots(2n-1)\cdot a^{-\frac{1}{2}}\cdot\frac{1}{a^n} = \frac{(2n-1)!!}{a^n}\cdot\sqrt{\frac{2\pi}{a}}$$

~~$\langle xxxxxx \rangle$~~ ~~$\langle xx\ xx\ xx \rangle$~~ $\langle x^{2n} \rangle$

~~Wrong~~

Wrong ↗ ~~$C_6^2 = \frac{6\times 5}{2} = 30\ 15$~~ ~~$C_{2n}^2 = \frac{(2n)(2n-1)}{2}$~~ $= n(2n-1)$

$$e^{-\frac{1}{2}x^T A\cdot x + J^T\cdot x}, \quad A = O^{-1}\cdot D\cdot O \Rightarrow OA = DO$$

$$= O^T D O \qquad (O^{-1} = O^T)$$

$$e^{-\frac{1}{2}x^T O^T D O x + J^T O^T O x}$$

$$\prod_i \left(\frac{2\pi}{D_{ii}}\right)^{\frac{1}{2}}\cdot e^{\sum_i J_i'^2/2D_{ii}}$$

$$e^{-\frac{1}{2}(Ox)^T D\, Ox + (OJ)^T Ox} = e^{-\frac{1}{2}y^T D y + J' y}.$$

$$= e^{-\frac{1}{2}y_i D_{ii} y_i + (J')_i y_i}. \xrightarrow{\text{integration}} \left(\frac{2\pi}{a}\right)^{\frac{N}{2}} \cancel{e^{J^2/2a\,N}} \left(\prod_i \sqrt{\frac{2\pi}{D_{ii}}}\right) e^{J_i'^2 y_i}$$

Changing variables : $y = OX \Rightarrow \frac{\partial\vec{y}}{\partial\vec{x}} = O \Rightarrow dy = \left|\frac{\partial\vec{y}}{\partial\vec{x}}\right|\cdot dx$

$$\Rightarrow dx_1\cdots dx_n = dx = \frac{1}{|\det O|}\,dy$$

Also $\prod_i \left(\frac{2\pi}{D_{ii}}\right)^{\frac{1}{2}} = \frac{(2\pi)^{\frac{N}{2}}}{\left(\prod_i D_{ii}\right)^{\frac{1}{2}}} = \left(\frac{(2\pi)^N}{\det[D]}\right)^{\frac{1}{2}} = \sqrt{\frac{(2\pi)^N}{\det(A)}}$

$$\sum_i J_i'^2 / 2D_{ii} = \frac{1}{2}\sum_i (OJ)_i \frac{1}{D_{ii}} (OJ_i) = \frac{1}{2}(OJ)^T \frac{1}{D}(OJ)$$

$$= \frac{1}{2} J^T A^{-1} J$$

$\det(O) = 1$ since O is orthogonal!

So $\iint \dots = \left(\frac{(2\pi)^N}{\det A}\right)^{\frac{1}{2}} e^{\frac{1}{2} J^T \cdot A^{-1} \cdot J}$

$\boxed{\begin{array}{l} \cancel{A = A^{-1}} \\ A = A^T \\ \text{so} \\ A^{-1} = (A^{-1})^T \end{array}}$

$$\frac{d}{dJ_j}\frac{d}{dJ_i}\left(e^{\frac{1}{2}J^T A^{-1} J}\right) = \frac{d}{dJ_j}\left(e^{\frac{1}{2}J^T A^{-1} J} \cdot \frac{1}{2} \cdot 2\left(J_i (A^{-1})_{ii} + \sum_{k \neq i} J_k (A^{-1})_{ik}\right)\right)$$

$$= \frac{d}{dJ_j}\left(e^{\frac{1}{2}J^T (A^{-1}) J} \cdot \sum_k J_k (A^{-1})_{ik}\right)$$

$$= e^{\frac{1}{2}J^T A^{-1} J} \cdot \frac{d}{dJ_j}\left(\frac{1}{2}J^T A^{-1} J\right) \sum_k J_k (A^{-1})_{ik}$$ (this term vanishes when $J = 0$)

$$+ e^{\frac{1}{2}J^T A^{-1} J} (A^{-1})_{ij}$$

$$\overset{J=0}{=} (A^{-1})_{ij}$$

$$\frac{d}{dJ_k}\frac{d}{dJ_j}\frac{d}{dJ_i}\left(e^{\frac{1}{2}J^T \underset{\sim}{A}^{-1} J}\right) = e^{\frac{1}{2}J^T A^{-1} J} \frac{d}{dJ_k}\left(\frac{1}{2}J^T A^{-1} J\right) \frac{d}{dJ_j}\left(\frac{1}{2}J^T A^{-1} J\right) \sum_k J_k (A^{-1})_{ik}$$

$$+ e^{\frac{1}{2}J^T A^{-1} J} \cdot (A^{-1})_{jk} \cdot \sum_k J_k (A^{-1})_{ik} + e^{\frac{1}{2}J^T A^{-1} J} \frac{d}{dJ_j}\left(\frac{1}{2}J^T A^{-1} J\right)(A^{-1})_{ik}$$

$$+ (A^{-1})_{ij}\, e^{\frac{1}{2}J^T A^{-1} J} \cdot \sum_l J_l (A^{-1})_{kl}$$

$$\frac{d}{dJ_l}\frac{d}{dJ_k}\frac{d}{dJ_j}\frac{d}{dJ_i}\left(e^{\frac{1}{2}J^TA^{-1}J}\right)$$

$$= \frac{dA}{dJ_l} + \frac{d}{dJ_l}\left(e^{\frac{1}{2}J^TA^{-1}J}\right)(A^{-1})_{jk}\sum_m J_m (A^{-1})_{im} + e^{\frac{1}{2}J^TA^{-1}J}\cdot(A^{-1})_{jk}(A^{-1})_{il}$$

$$+ \frac{d}{dJ_l}\left(e^{\frac{1}{2}J^TA^{-1}J}\right)\cdot(A^{-1})_{ik}\cdot\frac{d}{dJ_j}\left(\frac{1}{2}J^TA^{-1}J\right) + e^{\frac{1}{2}J^TA^{-1}J}(A^{-1})_{ik}\cdot(A^{-1})_{jl}$$

$$+ \frac{d}{dJ_l}\left(e^{\frac{1}{2}J^TA^{-1}J}\right)(A^{-1})_{ij}\cdot\sum_m J_m (A^{-1})_{km} + e^{\frac{1}{2}J^TA^{-1}J}(A^{-1})_{ij}(A^{-1})_{kl}$$

$$\overset{J=0}{=} (A^{-1})_{jk}(A^{-1})_{il} + (A^{-1})_{ik}(A^{-1})_{jl} + (A^{-1})_{ij}(A^{-1})_{kl}.$$

$$\langle x_i x_j\rangle = \frac{\frac{d}{dJ_i}\frac{d}{dJ_j}\left(e^{\frac{1}{2}J^TA^{-1}J}\right)\Big|_{J=0}\cdot\left(\frac{(2\pi)^N}{\det A}\right)^{\frac{1}{2}}}{\left(\frac{(2\pi)^N}{\det A}\right)^{\frac{1}{2}}} = (A^{-1})_{ij}.$$

$$L = \sum_a \frac{1}{2} m_a \dot{q}_a^2 - \sum_{ab} \frac{1}{2} k_{ab} (q_a - q_b)^2 - \sum_a W(q_a)$$

$$q_a(t) \to \varphi(\vec{x}, t) \qquad \sum_a m_a \to \int d^2x\, \sigma \qquad \dot{q}_a^2 = \cancel{\frac{\partial\varphi}{\partial x}} \left(\frac{\partial \varphi}{\partial t}\right)^2$$

$$\cancel{\int k_{ab}} (q_a - q_b)^2 \to (ds)^2 \to \ell^2 \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right)$$

$$\sum_{ab} k_{ab} \frac{1}{2} (q_a - q_b)^2 \to \int d^2x\, \rho \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right)$$

$$S \to \int_0^T dt \int dx^2 \frac{1}{2} \left\{ \sigma \left(\frac{\partial \varphi}{\partial t}\right)^2 - \rho \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) \underbrace{- \tau \varphi^2 - \varsigma \varphi^4}_{} \right\}$$

Why no φ, φ^3, etc. ?

0+1 :

$$S = \int_0^T dt \; ? \quad \to \quad S = \int_0^T dt\, L.$$

In 0 dimension, we have only a point.

$\cancel{\delta L(\varphi) = 0}$ $\quad L = L(\partial\varphi, \varphi)$, $\partial\varphi = \partial_\mu \varphi$

$$\delta L = \frac{\partial L}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu\varphi) + \frac{\partial L}{\partial \varphi} \delta\varphi$$

$$\delta \partial_\mu \varphi = \partial_\mu (\delta\varphi)$$

so $$\delta \int d^Dx\, L(\varphi) = \int d^Dx \left(\frac{\partial L}{\partial(\partial_\mu\varphi)} \delta(\partial_\mu\varphi) + \frac{\partial L}{\partial\varphi} \delta\varphi \right)$$

$$= \int d^Dx \frac{\partial L}{\partial(\partial_\mu \varphi)} \partial_\mu(\delta\varphi) + \frac{\partial L}{\partial\varphi}\delta\varphi = \frac{\partial L}{\partial(\partial_\mu\varphi)} \delta\varphi \Big|_{\text{boundary}} + \int d^Dx \left[-\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu\varphi)} \right) + \frac{\partial L}{\partial\varphi} \right] \delta\varphi$$

$$= 0 \quad \Rightarrow \quad \partial_\mu \frac{\partial L}{\partial(\partial_\mu\varphi)} - \frac{\partial L}{\partial\varphi} = 0$$

, or using the ~~correct~~ correct symbol for variation

$$\partial_\mu \frac{\delta L}{\delta(\partial_\mu\varphi)} - \frac{\delta L}{\delta\varphi} = 0.$$

Free field:

$$\partial_\mu\left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)}\right) - \frac{\delta \mathcal{L}}{\delta \varphi} = 0 \qquad \cancel{\partial_\nu} \quad \mathcal{L} = \frac{1}{2}\left((\partial\varphi)^2 - m^2\varphi^2\right)$$

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} = \partial^\mu \varphi \quad , \quad \frac{\delta \mathcal{L}}{\delta \varphi} = -m^2 \varphi .$$

so $\partial_\mu \partial^\mu \varphi + m^2 \varphi = 0 \Rightarrow \cancel{\partial^2\varphi + m} \ (\partial^2 + m^2)\varphi = 0$

Explicitly: $\cancel{\frac{\partial^2\varphi}{\partial x^2}} \ \nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial t^2} + m^2 \varphi = 0 .$

or $(\nabla^2 + m^2)\varphi = \frac{\partial^2 \varphi}{\partial t^2}$

assume $\varphi = e^{i(\omega t - \vec{k}\cdot\vec{x})}$. $\cancel{\nabla\varphi}$

~~the~~ then $\nabla \varphi = -i\vec{k}\varphi \quad \nabla^2 = -\vec{k}^2 \varphi$

$\frac{\partial \varphi}{\partial t} = i\omega\varphi \quad \frac{\partial^2 \varphi}{\partial t^2} = -\omega^2 \varphi$

so $(m^2 - \vec{k}^2)\varphi = -\omega^2 \varphi \Rightarrow \omega^2 = \vec{k}^2 - m^2$

But if the signature of metric is changed:

$$-\nabla^2 \varphi + \frac{\partial^2 \varphi}{\partial t^2} + m^2 \varphi = 0$$

then $\cancel{(m^2 - \nabla^2)\varphi = \frac{\partial^2 \varphi}{\partial t^2}} \quad (\nabla^2 - m^2)\varphi = \frac{\partial^2 \varphi}{\partial t^2}$

or $\cancel{m^2 + k^2 = \omega^2}$ or $-k^2 - m^2 = -\omega^2$

so $\omega^2 = k^2 + m^2$.

Then, why a change in signature would alter the dispersion relationship $\omega(k)$?

$$Z = \int D\varphi \cdot e^{i\int d^4x \left\{\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] + J\varphi\right\}}$$

$$\int d^4x \, (\partial\varphi)^2 = \int d^4x \, \partial^\mu\varphi \cdot \partial_\mu\varphi = \cancel{\int d^4x} \, 0 - \int d^4x \, \varphi \, (\partial^\mu\partial_\mu\varphi)$$

$$= 0 - \int d^4x \, \varphi \, (\partial^2\varphi).$$

so $\quad \frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right] \to -\frac{1}{2}\varphi(\partial^2 + m^2)\varphi$

$$\int D\varphi \, e^{i \, a^4 \cdot \sum_i -\frac{1}{2}\varphi_i \cdot (M_{ij}M_{jk}\varphi_k + m^2\varphi) + J\varphi}$$

$$= \cancel{\int D\varphi \, e^{i \cdot a^4 \cdot -\frac{1}{2}\varphi^T \cdot M^2 \cdot \varphi + m^2 I}} \int D\varphi \, e^{i a^4 (-\frac{1}{2}\varphi^T (M^2 + m^2)\varphi + J\varphi}$$

$$-(\partial^2 + m^2) D(x-y) = -(\partial^2 + m^2)\int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - m^2 + i\varepsilon}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{k^2 - m^2}{k^2 - m^2 + i\varepsilon} e^{ik(x-y)} = \cancel{\int \frac{d^4k}{(2\pi)^4} (k^2 - m^2) \, P\frac{1}{x} - i\pi\delta(k^2 - m^2) \, e^{ik(x-y)}}$$

$$\frac{1}{x + i\varepsilon} = P\frac{1}{x} - i\pi\delta(x)$$

$$= \int \frac{d^4k}{(2\pi)^4} (k^2 - m^2)\left(P\frac{1}{(k^2 - m^2)} - i\pi\delta(k^2 - m^2)\right) e^{ik(x-y)}$$

$$\delta(k^2 - m^2) = \frac{1}{|2k|} \cdot \delta(k = m) + \frac{1}{|2k|}\delta(k = -m) \qquad d(k^2 - m^2) = 2k\,dk$$

$$\cancel{P\frac{1}{k^2 - m^2} = \int \frac{d^4k}{(2\pi)^4} (k^2 - m^2) \, e^{ik(x-y)} \cdot \frac{k^2 - m^2}{(k^2 - m^2)^2 + \varepsilon^2}}$$

$$\cancel{\frac{k^2 - m^2}{k^2 - m^2 + i\varepsilon} = \frac{(k+m)(k-m)}{(k+m)(k-m) + i\varepsilon}}$$

$$\int \frac{d^4\vec{k}}{(2\pi)^4}\left(P\frac{1}{\vec{k}^2-m^2}\right)\frac{\vec{k}^2-m^2}{\cancel{k^2-m^2+i\varepsilon}}e^{i\vec{k}(x-y)}$$

$d^4\vec{k} = \cancel{k}\cdot dk\cdot d\Omega$
length, direction

$$= \int \frac{d\Omega}{(2\pi)^4}\cdot dk\left(P\frac{1}{k^2-m^2}\right)\frac{k^2-m^2}{\cancel{k^2-m^2+i\varepsilon}}e^{i\vec{k}(x-y)}$$

$$= \int \frac{d\Omega}{(2\pi)^4}\cdot\frac{d(k^2-m^2)}{2k}\frac{k^2-m^2}{(k^2-m^2)^2+\varepsilon^2}\cdot\cancel{\frac{k^2-m^2}{k^2-m^2}}(k^2-m^2)\,e^{ik(x-y)} \quad (\varepsilon\to 0)$$

$$\to \int \frac{d\Omega}{(2\pi)^4}\frac{d(k^2-m^2)}{2k}e^{ik(x-y)} = \int\frac{d\Omega\, dk}{(2\pi)^4}e^{ik(x-y)} = \delta^4(x-y)$$

Hence,

$$\int \frac{d^4k}{(2\pi)^4}\frac{k^2-m^2}{k^2-m^2+i\varepsilon}e^{ik(x-y)} \cancel{= \int\frac{d^4k}{(2\pi)^4}\left(P\frac{1}{k^2-m^2}\right)(k^2-m^2)e^{ik(x-y)} - i\pi\delta(k^2-m^2)(k^2-m^2)e^{ik(x-y)}}$$

$$\cancel{= \int\frac{d^4k}{(2\pi)^4}} = \int\frac{d^4k}{(2\pi)^4}\left[P\frac{1}{k^2-m^2}\underbrace{-i\pi\delta(k^2-m^2)}_{\text{integrate into zero}}\right](k^2-m^2)\,e^{ik(x-y)}$$

$$= \int\frac{d^4k}{(2\pi)^4}\left(P\frac{1}{k^2-m^2}\right)(k^2-m^2)\,e^{ik(x-y)}$$

$$= \delta^4(x-y)$$

Getting $D(x)$.:

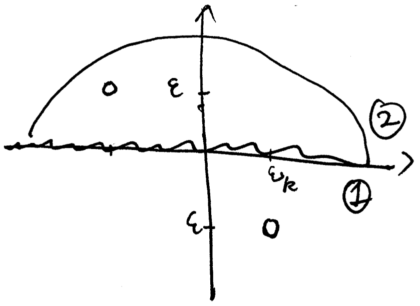
$\omega_k = \sqrt{\vec{k}^2+m^2}$ ~~let $k^2-m^2+i\varepsilon=0 \Rightarrow k = \pm\sqrt{m^2-i\varepsilon}$~~

$$D(x) = \int\frac{d^4k}{(2\pi)^4}\frac{e^{ikx}}{k^2-m^2+i\varepsilon}$$

(Note. in this part $\vec{k}$ is just the space component of the four-vector k^μ)

Let $k^2-m^2+i\varepsilon=0$ i.e. $k^{0\,2}-\vec{k}^2-m^2+i\varepsilon=0$, $\Rightarrow k^0 = \pm\sqrt{\omega_k^2-i\varepsilon} = \pm\omega_k\sqrt{1-i\frac{\varepsilon}{\omega_k^2}}$
$\overset{\varepsilon\to 0}{\approx} \pm\omega_k\left(1-\frac{1}{2}i\frac{\varepsilon}{\omega_k^2}\right)$

But ε is infinitesimal small, so $k^0 \approx \pm\left(\omega_k - \frac{1}{2}i\frac{\varepsilon}{\omega_k}\right) \approx \pm(\omega_k - i\varepsilon)$



$$\int_{-\infty}^{+\infty}\frac{dk^0}{(2\pi)^4}\,\cancel{e^{ik^0(x-y)}}\cdot\frac{e^{i[k^0(x-y)^0-\vec{k}(\vec{x}-\vec{y})]}}{(k^0+\omega_k\mp i\varepsilon)(k^0-\omega_k+i\varepsilon)}.$$

When $y^\mu=0$.
$$\int_{-\infty}^{+\infty}\frac{dk^0}{(2\pi)^4}\frac{e^{i(k^0x^0-\vec{k}\vec{x})}}{(k^0+\omega_k\mp i\varepsilon)(k^0-\omega_k+i\varepsilon)} =$$

Then contour must be ~~the~~ around ~~the~~ the upper half plane, because:

$$\lim_{k^0\to\frown}\frac{k^0 e^{ik^0x^0}}{(k^0+\square)(k^0-\square)}=\lim_{\substack{k^0_x\to\pm\infty\\ k^0_y\to+\infty}}\frac{k^0e^{ik^0_xx^0-k^0_yx^0}}{(k^0+\square)(k^0-\square)}=0 \qquad (x^0>0)$$

i.e. if $k^0=k^0_x+ik^0_y$, $k^0_x\to\pm\infty$, $k^0_y\to+\infty$

on the other hand

$$\lim_{k^0\to\smile}\frac{k^0e^{ik^0x^0}}{(k^0+\square)(k^0-\square)}=\lim_{\substack{k^0_x\to\pm\infty\\ k^0_y\to-\infty}}\frac{k^0\cdot e^{ik^0_xx^0-k^0_yx^0}}{(k^0+\square)(k^0-\square)}\to+\infty$$

Using the upper contour integration:

$$\int_{-\infty}^{+\infty}\frac{dk^0}{(2\pi)^4}\cdot\frac{e^{i[k^0x^0-\vec{k}\vec{x}]}}{(k^0+\omega_k-i\varepsilon)(k^0-\omega_k+i\varepsilon)}=2\pi i\cdot\frac{1}{(2\pi)^4}\left((k^0+\omega_k-i\varepsilon)\frac{e^{i(k^0x^0-\vec{k}x)}}{(k^0+\omega_k-i\varepsilon)(k^0-\omega_k+i\varepsilon)}\right)\Bigg|_{k^0=-\omega_k+i\varepsilon}$$

$$=\frac{2\pi i}{(2\pi)^4}\cdot\frac{e^{i[(-\omega_k+i\varepsilon)x^0-\vec{k}\vec{x}]}}{2(-\omega_k+i\varepsilon)}=\frac{i}{(2\pi)^3}\cdot\frac{e^{-i(\omega_kx^0+\vec{k}\vec{x})}\cdot e^{-\varepsilon x^0}}{-2\omega_k+2i\varepsilon}$$

On the other hand, the method fail to use "ε" (fail to eliminate "ε").

So I try another way:

$$\int_{-\infty}^{+\infty}\frac{dk^0}{(2\pi)^4}\frac{e^{i(k^0x^0-\vec{k}\vec{x})}}{k^{02}-\vec{k}^2-m^2+i\varepsilon}=\int_{-\infty}^{+\infty}\frac{\cancel{dk^{02}}}{(2\pi)^4}\cdot\frac{d[(k^0)^2]}{2k^0}\frac{e^{i(k^0x^0-\vec{k}\vec{x})}}{(k^0)^2-\vec{k}^2-m^2+i\varepsilon}=\cancel{\frac{1}{(2\pi)^4}\lim_{\varepsilon\to0^+}\int\frac{d(k^0)^2}{2k^0}\frac{(k^0)^2}{(k^0)^2+\varepsilon^2}}e^{i}$$

But this can't be done either. Because we can't change from $\int_{-\infty}^{+\infty}dx$ to $\int_{-\infty}^{+\infty}d(x^2)$ ✗

When x: space like, $x^0=0$ $(t=0)$.

$$D(x) = -i\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_k}\cdot\Big(\underbrace{e^{i\vec{k}\cdot\vec{x}}\tfrac{1}{2} + e^{-i\vec{k}\cdot\vec{x}}\tfrac{1}{2}}_{\cos(\vec{k}\vec{x})}\Big) = -i\int \frac{d^3k}{(2\pi)^3}\,\frac{\cos\vec{k}\vec{x}}{2\omega_k}$$

$\omega_k = \sqrt{\vec{k}^2+m^2}$, so $D(x) = -i\int \frac{d^3k}{(2\pi)^3}\,\frac{\cos(\vec{k}\cdot\vec{x})}{2\sqrt{\vec{k}^2+m}}$

I.4 From Field to Particle to Force

$$\cancel{W(J) = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,J_2^*(k)\,\frac{1}{k^2-m^2+i\varepsilon}\,J_1(k)}$$

$J_a(x) = \delta^{(3)}(\vec{x}-\vec{x}_a)$

$$W(J) = -\frac{1}{2}\iint d^4x\,d^4y\; J(x)\,D(x-y)\,J(y)$$

$\downarrow$

$$= -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,J^*(k)\,\frac{1}{k^2-m^2+i\varepsilon}\,J(k).$$