Notes of Group Theory and Physics

Taper

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Abstract

Notes of Group Theory and Physics written by Sternberg [Ste94].

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1 The Classification of the finite subgroups of SO(3) (1.8)

of-finite-subgroups-of-SO3

Previous section established one important formula for this section. Suppose a group G acts on a set M. Define the **fix point set** $\mathrm{FP}(a)$ of an element $a \in G$ as the set of all $m \in M$ such that $am = m$, i.e. left invariant under a. Denote G_m for $m \in M$ as the isotropy subgroup of m. Then we have

$$\sum_{a \neq e} \# \mathrm{FP}(a) = \sum_{\text{orbits}} \frac{\#G}{\#G_m} (\#G_m - 1) \tag{1.0.1}$$

eq:fp-orbit-formula

The proof is on section 1.7 of [Ste94].

This section classifies all possible finite subgroups of SO(3). The classification of finite subgroups of O(3) is on the next section. We basically study the action of $G = \mathrm{SO}(3)$ on $M = S^2$, the unit sphere. The first interesting discovery is

Theorem 1.1 ((Euler))**.** *When the dimension n is odd, any $a \in \mathrm{SO}(n)$ leaves at least one non-zero vector invariant, i.e. any $a \in \mathrm{SO}(n)$, $\mathrm{Ker}(a - I) \neq \varnothing$, or there is always a $\mathbf{v} \neq 0$, such that $a\mathbf{v} = \mathbf{v}$.*

This implies that any rotation in odd dimensional space is a rotation about some fixed axis (since $a\mathbf{v} = \mathbf{v}$ implies $(a\mathbf{p}) \cdot \mathbf{v} = (a\mathbf{v}) \cdot (a\mathbf{p}) = \mathbf{v} \cdot \mathbf{p} = 0$).

Then the book analyses the formula counting fix point sets and orbits 1.0.1. Use new symbols for three numbers:

three-symbols-for-3-numbers

$$\begin{aligned} n &= \#G \\ r &= \text{number of orbits of } G \\ n_i &= \#G_m, \text{ where } m \in i\text{th orbit.} \end{aligned}$$

Then we have

$$2 - \frac{2}{n} = r - \sum_1^r \frac{1}{n_i} \tag{1.0.2}$$

This equation can be simplified by considering the practical numerical values of n, r and n_i, with $n_i \leq n$. By eliminating case by case, he finally arrived at five sets of possible values:

tab:label

Table 1: Finite rotation groups

r	(n_1, n_2, n_3)	$\#G$	Schoenflies	Hermann-Mauguin	Note
2	(n,n,0)	n	C_n	n	Cyclic group
3	$(2, 2, k), k \geq 2$	$2k$	D_k	222 for D_2, $k2$ otherwise	Dihedral group
3	$(2, 3, 3)$	12	T	23	of regular tetrahedron
3	$(2, 3, 4)$	24	O	432	of regular octahedron
3	$(2, 3, 5)$	60	I	not mentioned	of icosahedron

For details about derivation please visit pp. 28 to 31 of [Ste94].

But we need to exclude some subgroups from the list of crystallographic groups. The result is that, on the first row, n is restricted to be $1, 2, 3, 4$ and 6; on the second row, k is restricted to be $2, 3, 4$ and 6. And I is excluded from the list of crystallographic groups.

The reason is the common one on Solid States classes about possible space-filling polygons (see pp.31 of [Ste94]). Also, a good proof about possible rotation angles in three-dimension for a lattice is provided on pp.31 to 32 of [Ste94]. The key is that the rotation matrix could be made of integers, and hence its characteristic (trace) should be an integer.

Next the author mentioned the **atomic hypothesis** , an interesting historical account of our view on crystals. The essence is that because only above mentioned angles occurred in rotational symmetries of crystals, we can presume that a crystal is not a continuum, but is "built up from discrete subunits in a regular repetitive pattern" (pp.32).

He also mentions the **law of rational indices** which found a basis for modern way of labeling different faces of a lattice (the (100) side, etc.). But this law is too long to be typed here.

In sum, we have only following finite subgroups of SO(3) that is interested for crystals:

$$C_1, C_2, C_3, C_4, C_6, D_2, D_3, D_4, D_6, T, O$$

References

[Ste94] Shlomo. Sternberg. *Group theory and physics*. Cambridge University Press, 1994.

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