Group Theory in Physics (Course Note)

Taper

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Abstract

This is our course note for the course about group theoy, with its application in physics.

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1. Combine group theory with you research.
2. Mid-term and final.
3. No homework, cause it is already graduate level.

4. Professor Fei Ye. (Phone: 88018229, 228 in Research building 2), T.A. Zhe Zhang. (110 Research building 2).

1 20160919

He first introduces several common examples of symmetries in our life and physics. Omitted, with one exception:

He mentions that there is one more symmetry in the Hydrogen Hamiltonian: the Laplace-Runge-Lenz symmetry. (So its symmetry group is not just $SO(3)$, but two copies of $SO(3)$ that forms a $SO(4)$. And using the representation of $SO(4)$, the complete spectrum of Hydrogen Hamiltonian is solved. Hence this $SO(4)$ is the largest symmetry of Hydrogen Hamiltonian.

1.1 Digression about Lenz vector

Since the class is too boring, I checked about the Lenz vector via Google and found this Math.SE question [2]

The first answer to that post is:

---

1) **Problem**. The Kepler Problem has Hamiltonian

$$H \ := \ \frac{p^2}{2m} - \frac{k}{q},$$

where m is the 2-body reduced mass. The [Laplace–Runge–Lenz vector](http://en.wikipedia.org/wiki/Laplace

$$A^j \ := \ a^j + km\frac{q^j}{q}, \qquad a^j \ := \ (\mathbf{L}\times\mathbf{p})^j \ = \ \mathbf{q}\cdot\mathbf{p}\, p^j - p^2\, q^j, \qquad \mathbf{L} \ := \ \mathbf{q}\times\mathbf{p}.$$

2) **Action**. The Hamiltonian Lagrangian is

$$L_H \ := \ \dot{\mathbf{q}}\cdot\mathbf{p} - H,$$

and the action is

$$S[\mathbf{q},\mathbf{p}] \ = \ \int \mathrm{d}t \ L_H.$$

The non-zero fundamental canonical Poisson brackets are

$$\{q^i, p^j\} \ = \ \delta^{ij}.$$

3) **Inverse Noether's Theorem**. Quite generally in the Hamiltonian formulation, given a constant of motion Q, then the infinitesimal variation

$$\delta \ = \ \varepsilon\{Q, \cdot\}$$

is a global off-shell symmetry of the action S (modulo boundary terms). Here ε is an infinitesimal global parameter, and $X_Q = \{Q, \cdot\}$ is a Hamiltonian vector field with Hamiltonian generator Q. The full Noether current is (minus) Q, see e.g. my answer to [this question](http://physics.stackexchange.com/q/8626/2451). (The words **on-shell**. and **off-shell**. refer to whether the equations of motion are satisfied or not.)

4) **Variation**. Let us check that the three Laplace–Runge–Lenz components A^j are Hamiltonian generators of three continuous global off-shell symmetries of the action S. In detail, the infinitesimal variations $\delta = \varepsilon_j\{A^j, \cdot\}$ read

$$
\begin{aligned}
\delta q^i &= \varepsilon_j\{A^j, q^i\}, \qquad \{A^j, q^i\} = 2p^i q^j - q^i p^j - \mathbf{q}\cdot\mathbf{p}\,\delta^{ij}, \\
\delta p^i &= \varepsilon_j\{A^j, p^i\}, \qquad \{A^j, p^i\} = p^i p^j - p^2\,\delta^{ij} + km\left(\frac{\delta^{ij}}{q} - \frac{q^i q^j}{q^3}\right), \\
\delta t &= 0,
\end{aligned}
$$

where ε_j are three infinitesimal parameters.

5) Notice for later that

$$
\begin{aligned}
\mathbf{q}\cdot\delta\mathbf{q} &= \varepsilon_j(\mathbf{q}\cdot\mathbf{p}\,q^j - q^2\,p^j), \\
\mathbf{p}\cdot\delta\mathbf{p} &= \varepsilon_j km\left(\frac{p^j}{q} - \frac{\mathbf{q}\cdot\mathbf{p}\,q^j}{q^3}\right) = -\frac{km}{q^3}\mathbf{q}\cdot\delta\mathbf{q}, \\
\mathbf{q}\cdot\delta\mathbf{p} &= \varepsilon_j(\mathbf{q}\cdot\mathbf{p}\,p^j - p^2\,q^j) = \varepsilon_j a^j, \\
\mathbf{p}\cdot\delta\mathbf{q} &= 2\varepsilon_j(p^2\,q^j - \mathbf{q}\cdot\mathbf{p}\,p^j) = -2\varepsilon_j a^j .
\end{aligned}
$$

6) The Hamiltonian is invariant

$$
\delta H = \frac{1}{m}\mathbf{p}\cdot\delta\mathbf{p} + \frac{k}{q^3}\mathbf{q}\cdot\delta\mathbf{q} = 0,
$$

showing that the Laplace–Runge–Lenz vector A^j is classically a constant of motion

$$
\frac{dA^j}{dt} \approx \{A^j, H\} + \frac{\partial A^j}{\partial t} = 0.
$$

(We will use the $\approx$ sign to stress that an equation is an on-shell equation.)

7) The variation of the Hamiltonian Lagrangian L_H is a total time derivative

$$
\begin{aligned}
\delta L_H &= \delta(\dot{\mathbf{q}}\cdot\mathbf{p}) = \dot{\mathbf{q}}\cdot\delta\mathbf{p} - \dot{\mathbf{p}}\cdot\delta\mathbf{q} + \frac{d(\mathbf{p}\cdot\delta\mathbf{q})}{dt} \\
&= \varepsilon_j\left(\dot{\mathbf{q}}\cdot\mathbf{p}\,p^j - p^2\,\dot{q}^j + km\left(\frac{\dot{q}^j}{q} - \frac{\mathbf{q}\cdot\dot{\mathbf{q}}\,q^j}{q^3}\right)\right)
\end{aligned}
$$

$$-\varepsilon_j \left(2\dot{\mathbf{p}} \cdot \mathbf{p}\, q^j - \dot{\mathbf{p}} \cdot \mathbf{q}\, p^j - \mathbf{p} \cdot \mathbf{q}\, \dot{p}^j\right) - 2\varepsilon_j \frac{da^j}{dt}$$
$$= \varepsilon_j \frac{df^j}{dt}, \qquad f^j := A^j - 2a^j,$$

and hence the action S is invariant off-shell up to boundary terms.

8) **Noether current**. The bare Noether current j^k is

$$j^k := \frac{\partial L_H}{\partial \dot{q}^i}\{A^k, q^i\} + \frac{\partial L_H}{\partial \dot{p}^i}\{A^k, p^i\} = p^i\{A^k, q^i\} = -2a^k.$$

The full Noether current J^k (which takes the total time-derivative into account) becomes (minus) the Laplace–Runge–Lenz vector

$$J^k := j^k - f^k = -2a^k - (A^k - 2a^k) = -A^k.$$

J^k is conserved on-shell

$$\frac{dJ^k}{dt} \approx 0,$$

due to Noether's first Theorem. Here k is an index that labels the three symmetries.

However, I don't really understand the content inside. I asked professor Ye whether we can find some physics about this conserved quantity, and he answered with no.

The next answer is also interesting:

While Kepler second law is simply a statement of the conservation of angular momentum (and as such it holds for all systems described by central forces), the first and the third laws are special and are linked with the unique form of the newtonian potential $-k/r$. In particular, Bertrand theorem assures that *only* the newtonian potential and the harmonic potential kr^2 give rise to closed orbits (no precession). It is natural to think that this must be due to some kind of symmetry of the problem. In fact, the particular symmetry of the newtonian potential is described exactly by the conservation of the RL vector (it can be shown that the RL vector is conserved iff the potential is central and newtonian). This, in turn, is due to a more general symmetry: if conservation of angular momentum is linked to the group of special orthogonal transformations in 3-dimensional space $SO(3)$, conservation of the RL vector must be linked to a 6-dimensional group of symmetries, since in this case there are apparently six conserved quantities (3 components of L and 3 components of $\mathcal{A}$). In the case of bound orbits, this group is $SO(4)$, the group of rotations in 4-dimensional space.

Just to fix the notation, the RL vector is:

$$\mathcal{A} = \mathbf{p} \times \mathbf{L} - \frac{km}{r}\mathbf{x} \tag{1.1.1}$$

Calculate its total derivative:

$$\frac{d\mathcal{A}}{dt} = -\nabla U \times (\mathbf{x} \times \mathbf{p}) + \mathbf{p} \times \frac{d\mathbf{L}}{dt} - \frac{k\mathbf{p}}{r} + \frac{km(\mathbf{p}\cdot\mathbf{x})}{r^3}\mathbf{x} \tag{1.1.2}$$

Make use of Levi-Civita symbol to develop the cross terms:

$$\epsilon_{sjk}\epsilon_{sil} = \delta_{ji}\delta_{kl} - \delta_{jl}\delta_{ki} \tag{1.1.3}$$

Finally:

$$\frac{d\mathcal{A}}{dt} = \left(\mathbf{x}\cdot\nabla U - \frac{k}{r}\right)\mathbf{p} + \left[(\mathbf{p}\cdot\mathbf{x})\frac{k}{r^3} - 2\mathbf{p}\cdot\nabla U\right]\mathbf{x} + (\mathbf{p}\cdot\mathbf{x})\nabla U \tag{1.1.4}$$

Now, if the potential $U = U(r)$ is central:

$$(\nabla U)_j = \frac{\partial U}{\partial x_j} = \frac{dU}{dr}\frac{\partial r}{\partial x_j} = \frac{dU}{dr}\frac{x_j}{r} \tag{1.1.5}$$

so

$$\nabla U = \frac{dU}{dr}\frac{\mathbf{x}}{r} \tag{1.1.6}$$

Substituting back:

$$\frac{d\mathcal{A}}{dt} = \frac{1}{r}\left(\frac{dU}{dr} - \frac{k}{r^2}\right)\left[r^2\mathbf{p} - (\mathbf{x}\cdot\mathbf{p})\mathbf{x}\right] \tag{1.1.7}$$

Now, you see that if U has *exactly* the newtonian form then the first parenthesis is zero and so the RL vector is conserved. Maybe there's some slicker way to see it (Poisson brackets?), but this works anyway.

1.2 Coming back to the course

After mentioning the Poincáre group, he produces to review some concepts about linear algebra:

1. The axioms of linear space, using quantum mechanics as basic example (Omitted).
2. Some common concepts of linear space: linear-independence, subspace, direct sum, linear operators, its matrix representation. (Omitted)

3. Introducing the complete antisymmetric tensor $\epsilon^{a_1,\cdots,a_n}$. Some properties:

$$\frac{1}{(m-n)!}\sum_{a_{n+1},\cdots,a_m}\epsilon_{a_1,\cdots,a_n,a_{n+1},a_m}\epsilon_{b_1,\cdots,b_n,a_{n+1},a_m}$$
$$=\sum_{p_1,\cdots,p_n}\epsilon_{p_1,\cdots,p_n}\delta_{a_1,b_{p_1}}\cdots\delta_{a_n,b_{p_n}} \qquad (1.2.1)$$
$$\epsilon_{ab}\epsilon_{rs}=\delta_{ar}\delta_{bs}-\delta_{as}\delta_{br} \qquad (1.2.2)$$
$$\sum_d \epsilon_{abd}\epsilon_{rsd}=\delta_{ar}\delta_{bs}+\delta_{as}\delta_{br} \qquad (1.2.3)$$

4. Some special matrices.
5. Fact: If $R\Gamma=\Gamma R$, and Γ is diagonal. (let $\mu\neq\nu$) Then if $\Gamma_{\mu\mu}\neq\Gamma_{\nu\nu}$, we have: $R_{\mu\nu}=R_{\nu\mu}=0$. On the other hand, if $R_{\mu\nu}\neq 0$, then $\Gamma_{\mu\mu}=\Gamma_{\nu\nu}$. This is obviously from:

$$\sum_j R^i_j\Gamma^j_k=\sum_j\Gamma^i_jR^j_k \Longrightarrow R^i_k\Gamma^k_k=\Gamma^i_iR^i_k$$

   where the first is automatically summed, and the second is not.
6. A linear functional is closed w.r.t. a vector space. (Omitted)
7. ... then this linear functional can be expressed as a matrix w.r.t to a basis of this vector space. (Omitted)
8. Invariant subspace. (Omitted)
9. Transformation of basis. (Omitted)
10. Direct sum of operators:
   Let vector spaces $L=L_1\oplus L_2$, with $L=\langle e_i\rangle$, $L_1=\langle e'_1,\cdots e'_n\rangle$,$L_2=\langle e'_{n+1},\cdots,e'_m\rangle$, $e'_\nu=\sum_\mu e_\mu S_{\mu\nu}$. Assume that L_1,L_2 are invariant w.r.t A, an linear operator. If:

$$Ae'_\mu=\sum_{\nu=1}^{m}e'_\nu R'_{\nu\mu} \qquad (1.2.4)$$

   we have obviously:

$$Ae'_\mu=\sum_{\nu=1}^{n}e'_\nu R'_{\nu\mu}\text{for }\mu\in\{1\cdots n\} \qquad (1.2.5)$$
$$Ae'_\mu=\sum_{\nu=n}^{m}e'_\nu R'_{\nu\mu}\text{for }\mu\in\{n\cdots m\} \qquad (1.2.6)$$

   i.e., A's matrix representation has two diagonal blocks. Using this fact, A after a linear transformation (by S), could be written as $R_1\oplus R_2$, where the meaning of R_1/R_2 is obvious.
11. Eigenvalues and the characteristic equation. (Omitted) Some properties:
   (a) Trace $=\sum_i\lambda_i$

(b) Determinant $= \prod_i \lambda_i$

(c) Geometric multiplicity $\leq$ Algebraic multiplicity, or

$$\text{dim} V_{\lambda_1} \leq n_1$$

.

12. Inner product and orthonormal basis. (Omitted) Here we define matrix Ω to be, when a basis $\{e_i\}$is given:

    Definition 1.1.

    $$\Omega_{ij} \equiv \langle e_i, e_j \rangle \tag{1.2.7}$$

13. Adjoint operator:

    Let A be a linear operator represented by matrix A^i_j. Let its adjoint $A^\dagger$ be represented by R^i_J. Then using $\langle A^\dagger e_j, e_i \rangle = \langle e_j, A e_i \rangle$, we will get $(R^k_j)^* \Omega_{ki} = \Omega_{jk} A^k_i$, i.e. $(R^T)^* \Omega = \Omega A$, so:

    $$R = \Omega^{-1} A^\dagger \Omega \tag{1.2.8}$$

    where we have used the fact that $\Omega^\dagger = \Omega$.

    Note that $(R^k_j)^* \Omega_{ki}$ is not $\Omega^T R^*$. (Be careful and you will find out why.)

    This is very different from my previous naive concept when Ω is not identity matrix, i.e. when the basis is not orthonormal.

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He first introduces some important matrices:

2.1 Some Important Matrices

Unitary matrix Eigenvalues of Unitary matrices has modulus 1, i.e. $|\lambda| = 1$. This can be proved directly. Also, Unitary matrices are unitarily diagonalizable. This is a result of the following Spectral Theorem:

Theorem 2.1 (Spectral Theorem)**.** *One matrix A is normal (i.e. $A^\dagger A = AA^\dagger$), if and only if it is unitarily diagonalizable.*

Proof. If A is normal, then by Schur decomposition, we can write $A = UTU^\dagger$, here U is unitary and T is upper-triangular. Using the condition of being normal, one can show directly that T is in fact also normal. Now we show that any triangular matrix that is normal must be diagonal. Observe that we have $\langle e_i, T^\dagger T e_i \rangle = \langle e_i, TT^\dagger e_i \rangle$, i.e. $\langle T^\dagger e_i, T^\dagger e_i \rangle = \langle Te_i, Te_i \rangle$. This is saying that the norm of the first column of $A^\dagger$ is equal to the norm of the first column of A. Obviously A has to be diagonal.

The converse is obvious. $\square$

Also, unitary matrix's eigenvector corresponding to different eigenvalues are orthogonal. This is a direct result of fact mentioned above.

Hermitian matrices They have real eigenvalues and orthogonal eigenvectors (proof omitted). Also, if $\det(R^\dagger R) \neq 0$, then $R^\dagger R > 0$, i.e. it is positive-definite.

This is wong: An example is that the matrix Ω introduced in the previous lecture has $\det(\Omega^\dagger \Omega) = \det(\Omega)$, hence $\det(\Omega) = 1$ (it cannot be 0), hence it is positive definite.

Actually $\det(\Omega^\dagger\Omega) \neq \det(\Omega)$, because

$$\sum_\rho |e_\rho\rangle \langle e_\rho| \neq 1 (\text{unless the basis is orthonormal}) \tag{2.1.1}$$

Therefore we need anthoer argument for Ω being positive-definite. It is provided in page 11 of [1].

Orthogonal matrix For an orthogonal matrix over $\mathbb{C}$, it is quite troublesome. For example, if $Ra = \lambda a$ and $\lambda \neq \pm 1$, then we have $a^T a = 0$, which is quite bad because this force a to have complex components.

Orthogonal matrix over $\mathbb{R}$ In this case, we have similar result. But it is easy to show that for an orthogonal matrix R having only real elements, then its eigenvalues $\lambda = \pm 1$.

Then he proceeds to direct product.

Direct product and also the Kronecker Product of two matrices. Properties (let $T = R \otimes S$):

1. $\dim T = \dim R \times \dim S$
2. $\mathrm{tr}(T) = \mathrm{tr}(R)\mathrm{tr}(S)$
3. $\otimes$ commutes with the operation of inverse, transpose, and transpose conjugation.
4. $$\frac{d}{d\alpha}(R(\alpha) \otimes S(\alpha)) = R'(\alpha) \otimes S(\alpha) + R(\alpha) \otimes S'(\alpha) \tag{2.1.2}$$
5. when the dimentions are the same:
   (a) $(R_1 \otimes S_1)(R_2 \otimes S_2) = (R_1 R_2) \otimes (S_1 S_2)$

2.2 Symmetry and Group

Finally we arrived in the group theory.

Symmetry examples Dipole transition. $\langle \phi_f | \hat{P} | \phi_i \rangle$, must happen when the parity of ϕ_i and ϕ_f is of opposite parity. (pp.18 of [1])

Group

Definition 2.1 (Group)**.** Omitted.

Some basic properties (Omitted).

Definition 2.2 (Abel Group)**.** Omitted.

Definition 2.3 (Cardinality of group $\#A$)**.** Omitted.

Multiplication table Facts: group of order $1, 2$, and 3 are unique up to an isomorphism.

Definition 2.4 (Cyclic group, generators)**.** Omitted.

固有转动是指的那些 $det(M) > 0$ 的转动. 用C_n来表示他们.
Also, 周期 of R is just $\langle R \rangle$.
Let σ for spatial reflection.

Definition 2.5 ($C_N, \bar{C}_N$)**.** $\bar{C}_N = C_N * \sigma$

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3.1 Common Concepts in Group

Introducing to various groups:S_4, V_4, D_3, all omited. (**pp.22-23 of [1]**)

D_n group. See pp. 25-26 of the book [1]. Note that here the n refers to the n-polygon, not that the group is of order n. For the mathematicians, they might be comfortable with D_n means the dihedral group of order n, but is actually the group of symmetries of $n/2$-polygon.

Definition 3.1 (Subgroup)**.** Omitted.

Fact 3.1. One only has to check the closeness for determining a subgroup, if it is of finite order.

However, for group of infinite order, one has to check the existance of unit and inverse elements.

Examples of subgroup (**pp.26 of [1]**)

Noteworth:C_6 has three copies of D_2, this can be intuitively guessed by the fact that a hexago has three rectanle in it.

Definition 3.2 (Coset)**.** Omitted.

Properties of coset (omitted).

Definition 3.3 (Index of subgroup)**.** Omitted.

Proposition 3.1. *Two elements R and T belongs to the same coset kH, if and only if $R^{-1}T \in H$.*

Proof. Omitted. □

Definition 3.4 (Normal/Invariant subgroup)**.** A subgroup is normal or nnvariant subgrouproup(also invariant), if and only if for any $x \in G$, we have $xH = Hx$.

Fact 3.2. If H has index 2, then it must be normal/invariant. This is obvious.

Definition 3.5 (Quotient)**.** Omitted.

Note that quotient group (a.k.a. factor group) is only defined for a normal subgroup.

Example 3.1. D_3 (Using the multiplication table). Omitted because it is too complex to be typed down here.

3.2 Conjugacy classes

Definition 3.6 (Conjugate)**.** If exists $S \in G$, s.t. $R' = S^{-1}RS$, then we say R' is conjugate to R.

see (**pp.28-30 of [1]**) This is clearly an equivalence relationship. By this we can define conjugate class, denoted by $C = \{R_1, \cdots\}$, then we have the characterization $C = \{s^{-1}R_i s | \forall s \in G\}$, for any R_i. We then have the following facts (all are obvious):

Fact 3.3.

1. The unit class formed just by the unit element.
2. The inverse class formed by just all the inverse element. $C^{-1} = \{R_i^{-1}\}$. If $C = C^{-1}$, then C is called self-inverse.
3. The order of elements in a class is just the same.
4. For $\forall T, S \in G$, TS and ST are conjugate to each other. This means that all elements symmetric on the multiplication table is conjugate to each other.
5. For two elements R, R' conjugate to each other, both can be expressed by the products of two elements in two different way. (Isn't this too obvious to mention.)
6. Let G be a rotation group. Suppose it has an axis of the order of n, with its operation denoted as R,, we can get a new axis by the following steps. (Supose we have another rotation S),
   (a) S^{-1}, rotate m back to n.
   (b) R, rotate about n around $2\pi/n$,
   (c) S rotate n to m

   Result: $S^{-1}RS$ rotate around a new axis m about $2\pi/n$. So $R' = S^{-1}RS$ and R is calle the equivalent axis.

   Also, if $m = -n$, then they are called polar axis to each other.
7. C_n, which is an abel group, every element form a conjugate class by itself. Specifically, e and $R^{n/2}$ are self inverse, if n is even.

Proposition 3.2. *For an normal subgroup, every conjugate element is also inside the same normal subgroup. This indicates that an normal subgroup can be decomposed into a series of sum of conjugate classes.*

Proof. If $R \in H$, then we show that $S^{-1}RS \in H$, this is obviously since it belongs to $S^{-1}HS$. □

Example 3.2. For $D_3 : E, D, F, A, B, C$, their orders are respetively $1, 3, 3, 2, 2, 2$. We have the following conjugate classes:

1. $\{E\}$. Is self inverse.
2. $\{D, F\}$. D is conjugate to F. We can see this physically by looking at rotation from the front or the below. This class is also self-inverse.
3. $\{A, B, C\}$, is clearly a self-inverse conjugate class.

Example 3.3 (D_6)**.** Ramiliarize one with the formulae for D_n. Hint: use the order of elements to find classes of conjugate. Then use the proposition 3.2 to find the subgroups.

4 Skipped Two lectures

Due to GRE physics preparation. Concepts that may have been covered:

- Conjugacy classes
- Representation of Groups.
  - Character of representation. (Notes for this part is delayed to be contained in the next lecture's note).
  - Equivalence between representation.
- Transformation of Fields
- Group Algebra and Regular Representation.

4.1 Conjugacy classes (continued)

One important group is the symmetric group. It is important because that every finite group of order n can be embedded inside the symmetric group S_n (Cayley's theorem) [1].

The conjugacy classes in symemtric is pretty easy to find. What we need to know is the following observation (from [3])

Key Point 4.1. Consider two permutation π and σ:

$$\pi = \begin{pmatrix} 1 & \cdots & n \\ \pi(1) & \cdots & \pi(n) \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{pmatrix} \tag{4.1.1}$$

Then we have by direct calculation:

$$\sigma\pi\sigma^{-1} = \begin{pmatrix} \sigma(1) & \cdots & \sigma(n) \\ \sigma(\pi(1)) & \cdots & \sigma(\pi(n)) \end{pmatrix} \tag{4.1.2}$$

Therefore, the cycle structure of π is unchanged under any "conjugacy transformation".

What do I mean by cycle structure? Let's see an example. Suppose we have a cycle (123):

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 2 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd}$$

Then the conjugated map $\sigma(123)\sigma^{-1}$ will have:

$$\begin{tikzcd} & \sigma(1) \arrow[dr] & \\ \sigma(2) \arrow[ur] & & \sigma(3) \arrow[ll] \end{tikzcd}$$

[1] However, it is hard sometimes to find the smallest possible symmetric group to embed into. For example, S_3 has 6 elements and can thus be embedded into S_6. But obviously it can be embedded better just into itself S_3.

So it has a cycle of $(\sigma(1)\sigma(2)\sigma(3))$. We see that in general, the cycle type is unchanged under conjugacy transformation. [2]

On the other hand, all elements of the same cycle type belong to the same conjugacy class. For example, consider two permutations: $(i_1 i_2 i_3)(i_4 i_5)$ and $(\pi_{i_1}\pi_{i_2}\pi_{i_3})\,(\pi_{i_4}\pi_{i_5})$, where π permutes the five numbers $i_1\cdots i_5$. Consider the map:

$$\sigma \equiv \begin{pmatrix} i_1 & \cdots & i_5 \\ \pi_{i_1} & \cdots & \pi_{i_5} \end{pmatrix} \tag{4.1.3}$$

Then

$$\begin{aligned}
&\sigma(i_1 i_2 i_3)(i_4 i_5)\sigma^{-1} = \\
&\begin{pmatrix} i_1 & \cdots & i_5 \\ \pi_{i_1} & \cdots & \pi_{i_5} \end{pmatrix}(i_1 i_2 i_3)(i_4 i_5)\begin{pmatrix} \pi_{i_1} & \cdots & \pi_{i_5} \\ i_1 & \cdots & i_5 \end{pmatrix} \\
&= (\pi_{i_1}\pi_{i_2}\pi_{i_3})(\pi_{i_4}\pi_{i_5})
\end{aligned}$$

The proof for the general case is similar. So we have the theorem below:

Theorem 4.1 (Cycle type determines conjugacy class)**.** *Two permutations are conjugate in the symmetric group if and only if they have the same cycle type.*

Remark 4.1. Since a cycle type is just a set of unordered integer partition of number n. The above theorem means that the set of conjugacy classes in the symmetric group on a finite set is in bijection with the set of unordered integer partitions of the size of the set.

4.2 Group Representation

Definition 4.1 (Representation)**.** A representation of a group G is a continuous homomorphism D from G to the group of automorphisms of a vector space V:

$$D : G \mapsto \mathrm{Aut}V \tag{4.2.1}$$

V is called the *represetation space*, and the *dimension of the representation* is the dimension of V.

Related concepts. (The following is copied from [3].)

- There is always the representation $D(g) = 1$ for all g. If $\dim V = 1$, this is called the *trivial representation* .
- The matrix groups, i.e. $\mathrm{GL}(n, K)$ and subgroups, naturally have the representation "by themselves", i.e. by$n \times n$ matrices acting on K_n and satisfying the defining constraints (e.g. nonzero determinant). This is loosely called the *fundamental or defining representation.*

[2] For a permutation, its cycle type is determined by decomposing it into independent cycles. For independent cycles, the cycle type is unique. For example, $(135)(24)$ has a cycle type of $(3, 2)$, or $(2, 3)$ since the order is irrelevant. But if we decompose it into non-independent cycles, we cannot determine its cycle type. For example, $(135) = (15)(13)$, which is ambiguous if we were to tell the cycle type.

- Two representations D and D' are called *equivalent* if they are related by a similarity transformation, i.e. if there is an operator S such that

$$SD(g)S^{-1} = D'(g) \tag{4.2.2}$$

  for all g. Note that S does not depend on g! Two equivalent representation can be thought of as the same representation in different bases. We will normally regard equivalent representations as being equal.

  Note also here S represents a transformation of basis. If we know the transformation of vectors X, then

$$X^{-1}D(g)X = D'(g) \tag{4.2.3}$$

- A representation is called *faithful* if it is injective, i.e. $\ker D = \{e\}$, or in other words, if $D(g_1) \neq D(g_2)$ whenever $g_1 \neq g_2$.
- If V is equipped with a (positive definite) scalar product, D is unitary if it preserves that scalar product, i.e if

$$\langle u, v\rangle = \langle D(g)u, D(g)v\rangle \tag{4.2.4}$$

  for all $g \in G$. (Here we assume that V is a complex vector space, as that is the most relevant case. Otherwise one could define orthogonal representations etc.)

4.2.1 Regular Representation

Now we construce a simple representation for all groups. It utilizes a vector space constructed from group itself called Group Algebra.

Definition 4.2 (Group algebra/Monoid algebra)**.** Let A be a commutative ring, G be a monoid, written multiplicatively. Then the monoid ring $A[G]$ consists of those finite formal linear combinations v of the form:

$$v = \sum_{g\in G} v_g g \tag{4.2.5}$$

where $v_g \in A$. The v_g are seen as coefficients and g are seen as the basis vectors that can be multiplied. Soe the addition is defined as:

$$v + w = \sum_{g\in G}(v_g + w_g)g \tag{4.2.6}$$

And the multiplication is

$$vw = \sum_{g,g'\in G}(v_g w_{g'})gg' \tag{4.2.7}$$

If G is a group then the corresponding $A[G]$ is called a *group algebra.*

A rigorious definition would use the concept of function to define the "formal linear combination". Please refer to page 105 (section II.3) of [4].

Remark 4.2. Note that both A and G can be naturally embeded into $A[G]$. This will helps us to define a representation of group G later.

Example 4.1. (From page 106 of [4].) Polynomial rings are special cases. In n variables, consider a multiplicative free abelian group of rank n. Let $X_l, \cdots X_n$ be generators. Let G be the multiplicative subset consisting of elements $X_1^{v_1}, \cdots, X_n^{v_n}$, where $v_i \leq 0$ for all i. Then G is a monoid, and it is eas to verify that $A[G]$ is just $A[X_1, \cdots, X_n]$.

Here we take A to be $\mathbb{C}$, then we have clearly $\mathbb{C}[G]$ a vector space equipped with a bilinear map (the product), i.e. an algebra over a field. Its dimension is clear equal to the order of G. We can also give it an inner product by:

$$\langle v, w\rangle = \sum_{g\in G} v_g^* w_g \tag{4.2.8}$$

Now we can define the regular representation:

Definition 4.3 (Regular Representation)**.** A regular representation of a group G is the following endomorphism of $\mathbb{C}[G]$:

$$D_{\text{reg}} : v \mapsto g \cdot v \tag{4.2.9}$$

where $v \in \mathbb{C}[G]$, $g \in G \hookrightarrow \mathbb{C}[G]$.

Remark 4.3. This representation can be just seen as a permutation of basis vectors, because:

$$g \cdot v = \sum_{h\in G} v_h(gh) = \sum_{h'\in G} v_{g^{-1}h'}\, h' \tag{4.2.10}$$

Therefore, it is unitary.

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5.1 Unitarity of Representation

Theorem 5.1 (Unitary Representation)**.** *For finite groups and for compact Lie groups, all representations are equivalent to a unitary representation.*

Remark 5.1. Before the formal proof, I remarked that since any finite group can be embedded inside the symmetric group (Cayley's theorem), and the symmetric has clearly a unitary representation (perhaps the easiest representation one could ever construct besides the trivial representation), one could naturally guess whether it is possible that any representation could be turned into a unitary one.

As for the compact Lie group case, it is just mentioned in [3] and will not be proved here. It is mentioned in page 25 of [3] that:

> For compact Lie groups, however, there exists a (unique) translationally invariant measure, so we can replace $\sum_g \to \int \mathrm{d}g$ and the integral is convergent because of compactness. Then the proof directly carries over.

Proof. For $D(g)$, we need to find a X such that $\bar{D}(g) \equiv X^{-1}D(g)X$ is unitary.

Since

$$1 = \bar{D}^\dagger \bar{D}$$

One will find

$$(XX^\dagger)^{-1} = D^\dagger (XX^\dagger)^{-1} D$$

Then let

$$H \equiv \sum_{s\in G} D^\dagger(s) D(s) \tag{5.1.1}$$

One can verify that

$$D^\dagger(g) H D(g) = H \tag{5.1.2}$$

Now we construct X from H. We have

$$H = (XX^\dagger)^{-1} \tag{5.1.3}$$

Notice we have H is Hermitian by above equation. Also, H is positive definite (easily seen from the definition of H and $a^\dagger H a \geq 0$. Also H is of full rank.).

Then we have $UHU^{-1} = \text{diag}\{\gamma_1, \gamma_2, \cdots\}$ and $\gamma_i > 0$. The rest for constructing X should be obvious. □

Remark 5.2 (Examples of Non-unitary representation)**.** It is remard in [3] that for infinite and non-compact groups, the representation are not unitary in general. He gives two examples:

- The group $\mathbb{Z}^*$ acting on $\mathbb{C}$ by mulplication is certain not equivalent to any unitary representation.
- Arugment for a representation of non-compact Lie group cannot be made unitary by a similarity transformation:

  > The group of unitary operators on a finite dimensional vector space is isomorphic to $U(n)$ (for complex vector spaces, $O(n)$ otherwise), and is hence compact. But there cannot be a bijective continuous map between a compact and a non-compact space, so faithful finite-dimensional representations of non-compact groups will be non-unitary.

  But then he mentions something about a perculiar case, the Lorentz group, which I do not understand.

Anyway, he summarised in the following:

> To summarise, for finite groups and for compact Lie groups, all representations are equivalent to a unitary one (and we will usually take them to be unitary from now on). For infinite groups and non-compact Lie groups, on the other hand, finite-dimensional faithful representations are never unitary. Finally, some non-compact groups may have representations which are unitary with respect to a non-definite scalar product, such as the Lorentz group

From class, Ye Fei proved that:

Theorem 5.2. *For any two equivalent representation, there is always a unitary matrix to relate the two, i.e. exits Y unitary, s.t.*

$$\bar{D}(g) = Y^{-1} D(g) Y$$

Proof. Suppose we have two unitary representation $D(g)$ and $\bar{D}(g)$, related by

$$\bar{D}(g) = X^{-1} D(g) X$$

where X is not necessarily unitary.

Let $H \equiv X^\dagger X$, it is easy to prove that H is Hermitian and positive definite. Direct calculation shows that

$$\bar{D}^{-1}(g) H \bar{D}(g) = H$$

That is $\bar{D}(g)$ and H commute. Then we construct Y from H. Let V be such that

$$V^{-1} H V = \Gamma \tag{5.1.4}$$

where Γ is a diagonal matrix of H's eigenvalues. Obviously V has to be unitary. Define $\bar{Y}$ to be

$$\bar{Y} \equiv V \sqrt{\Gamma} V^{-1} \tag{5.1.5}$$

By direct calculation, we have

$$(X\bar{Y})^\dagger (X\bar{Y}) = \mathbb{1}$$

One can show that

$$\bar{Y}^{-1} \bar{D} \bar{Y} = \bar{D}$$

with laborious calculation. Then one can easily verify that $Y \equiv \bar{Y} X$ is the required unitary transformation. □

5.2 Reduciblility of Representations

Definition 5.1 (Reducible/Irreducible Representation)**.** A representation D is called *reducible* if V contains an invariant subspace. Otherwise D is called irreducible.

A representation is called *fully reducible* if V can be written as the direct sum of irreducible invariant subspaces, i.e. $V = V_1 \oplus \cdots \oplus V_p$, all the V_i are invariant and the restriction of D to each V_i is irreducible.

Example 5.1. The regular representation mentioned before is reducible. For example, the vector

$$V = \sum_{g \in G} g \tag{5.2.1}$$

spans an invariant subspace of the operator D_{reg}.

Example 5.2.

- The representation of finite group is obviously fully reducible or irreducible (since a unitary matrix cannot have off diagonal blocks).

- The representation of translation group is not fully reducible. For example, for translation we have: $T_a T_b = T_{a+b}$, One can confirm that the following representation obeys the above relationship:

$$T_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

  but this is obviously not fully reducible.

Definition 5.2 (Interwiner)**.** Given two representations D_1 and D_2 acting on V_1 and V_2, an intertwiner between D_1 and D_2 is a linear operator

$$F : V_1 \mapsto V_2 \tag{5.2.2}$$

which "commutes" with G in the sense that

$$FD_1(g) = D_2(g)F \tag{5.2.3}$$

for all $g \in G$.

(From pp.30 of [3])

The existence of an intertwiner has a number of consequences. First, D_1 and D_2 are equivalent exactly if there exists an invertible intertwiner. Second, the kernel and the image of F are invariant subspaces: Assume $v \in \operatorname{Ker} F$, i.e. $Fv = 0$. Then

$$FD_1 v = D_2 F v = D_2 0 = 0 \tag{5.2.4}$$

so $D_1 v \in \operatorname{Ker} F$. On the other hand, let $w_2 = F w_1$ be an arbitrary element of the image of F. Then from the definition we have

$$D_2 w_2 = D_2 F w_1 = F D_1 w_1 \tag{5.2.5}$$

which is again in the image of F. Now if D_1 is irreducible, the only invariant subspaces, hence the only possible kernels, are $\{0\}$ and V_1 itself, so F is either injective or zero. Similarly, if D_2 is irreducible, F is either surjective or zero. Taking these statements together, we arrive at Schur's Lemma:

Lemma 5.1 (Schur's lemma I)**.** *An intertwiner between two irreducible representations is either an isomorphism, in which case the representations are equivalent, or zero*

An important special case is the one where $D_1 = D_2$. In that case, we see that F is essentially unique. More precisely, we have the following theorem, also often called Schur's Lemma:

Lemma 5.2 (Schur's lemma II)**.** *If D is an irreducible finite-dimensional representation on a complex vector space and there is an endomorphism F of V which satisfies*

$$FD(g) = D(g)F \tag{5.2.6}$$

for all $g \in G$, then F is a multiple of the identity, $F = \lambda \mathbb{1}$

Proof. Note that F has at least one eigenvector v with eigenvalue λ. (This is where we need V to be a complex vector space: A real matrix might have complex eigenvalues, and hence no real eigenvectors.) Clearly, $F - \lambda\mathbb{1}$ is also an intertwiner, and it is not an isomorphism since it annihilates v. Hence, by Schur's Lemma, it vanishes, thus $F = \lambda\mathbb{1}$. □

(From the book [1])

5.3 Orthogonality Relations and Counting Irreducible Representations

Theorem 5.3 (Orthogonality Theorem for Representations)**.** *For finite group G, let $D^i(G)$ and $D^j(G)$ be its two irreducible representation. Then, as a vector in group algebra, they have the following orthogonal relationship:*

$$\sum_{h\in G} D^i_{\mu\rho}(h^{-1})D^j_{\nu\lambda}(h) = \frac{N}{d_j}\delta_{ij}\delta_{\mu\nu}\delta_{\rho\lambda} \tag{5.3.1}$$

N is the order of the group, and d_j is the dimension of representation $D^j(G)$. If in addition, the two representations are unitary, we have

$$\sum_{h\in G} D^{i*}_{\mu\rho}(h)D^j_{\nu\lambda}(h) = \frac{N}{d_j}\delta_{ij}\delta_{\mu\nu}\delta_{\rho\lambda} \tag{5.3.2}$$

Proof. **Note**: The following proof is clumsy and only applies for the unitary case. For a good proof, please refer to page 44 of [3].

Let

$$Y^{\mu\nu}_{\rho\lambda} \equiv \delta_{\rho\lambda}\delta_{\mu\nu} \tag{5.3.3}$$

Then let

$$X^{\mu\nu} = \sum_{h\in G} D^i(h^{-1})Y^{\mu\nu}D^j(h)$$

One can find by direct calculation

$$X^{\mu\nu}_{\rho\lambda} = \sum_{h\in G} D^{i*}_{\mu\rho}(h)D^j_{\nu\lambda}(h) \tag{5.3.4}$$

And also through direct calculation, one finds

$$D^i(s)X^{\mu\nu} = X^{\mu\nu}D^j(s)$$

for any $s \in G$. Then $X^{\mu\nu}$ is a interwiner. So the case for $i \neq j$ is obvious. When $i = j$, we have:

$$X = \lambda\mathbb{1}$$

Now we find the λ, i.e. the eigenvalue of $X^{\mu\nu}$.

Now since

$$X^{\mu\nu}_{\rho\lambda} = \lambda^{\mu\nu}\delta_{\rho\lambda}$$

One can find two fact by direct calculation:

$$\sum_{\rho} X^{\mu\nu}_{\rho\lambda} = d_j\lambda^{\mu\nu}$$
$$\sum_{\rho} X^{\mu\nu}_{\rho\lambda} = N\delta^{\mu\nu}$$

Hence $\lambda^{\mu\nu} = \frac{N}{d_j}\delta^{\mu\nu}$. □

The above relation gives us the first clue to the total number of irreducible representations by the following corollary.

Corollary 5.1.

$$\sum_j m_j^2 = N = |G| \tag{5.3.5}$$

where the index j runs over all possible irreducible representations.

Proof. Define the following vector in group algebra

$$v^i_{\mu\nu} = \sqrt{\frac{d_j}{N}} \sum_{h\in G} \left(D^i(h)\right)_{\mu\nu} h \tag{5.3.6}$$

Then by the orthogonality theorem, this vector is orthonomal in vector space/group algebra $\mathbb{C}[G]$. But the dimension of this vector space is N, and for each representation D^i we have d_i^2 vectors like the one above, which are all orthogonal to each other. Note that orthogonal implies linear independence, hence we have:

$$\sum_j m_j^2 \le N$$

To make the inequality an equality, it is suffice to prove that such $v^i_{\mu\nu}$ forms a basis of the group algebra $\mathbb{C}[G]$. In page 45 to 46 of [3], he shows that for any $g \in G \hookrightarrow \mathbb{C}[G]$, one has

$$g = \sum_{h\in G} \left(D_{\text{reg}}(h)\right)_{ge} h = \sum_{h\in G} \left[\sum_{j,\mu\nu} c^j_{\mu\nu} \left(D^j(h)\right)_{\mu\nu}\right] h \tag{5.3.7}$$

where e is the unit in the group. c are some hard to tell constants. To show this, [3] uses that fact that D_{reg} is unitary, so it is completely reducible into irreducible components, which have to be (some of) the D^i, i.e. $D_{\text{reg}} = U(D^{i_1} \oplus D^{i_2} \oplus \cdots)U^\dagger$. The detailed proof is not reproduced here.

Then, g is a linear combination with coefficients in $\left(D^j(h)\right)_{\mu\nu}$, hence a linear combination of $v^i_{\mu\nu}$. Since g can be any basis in the group algebra, this shows that the $\{v^i_{\mu\nu}\}$ is complete. □

Remark 5.3. The above proof shows in some sense that the regular representation contains all irreducible representation as components. Since all $v^i_{\mu\nu}$ appears in the regular representation.

5.4 Characters

Definition 5.3 (Character of Representation)**.** Given a representation D of G over vector space on field K, the character $\chi : G \to K$ is defined as the trace

$$\chi(g) = \operatorname{Tr} D(g) \tag{5.4.1}$$

Remark 5.4. Since trace is invariant under similarity transformation (remember that $\operatorname{Tr} AB = \operatorname{Tr} BA$), trace can be used to distinguishes between non-equivalent representations. Hence it is called the character of that representation.

Using the orthogonality relation in Section 5.3 it is esay to see we have

Corollary 5.2.

$$\sum_{h\in G}\chi^i(h^{-1})\chi^j(h) = N\delta_{ij} \tag{5.4.2}$$

Or when the representations are unitary:

$$\sum_{h\in G}\chi^{i*}(h)\chi^j(h) = N\delta_{ij} \tag{5.4.3}$$

Here again $N = |G|$.

6 Anchor

References

[1] Zhongqi Ma, Group Theory in Physics

[2] What symmetry causes the Runge-Lenz vector to be conserved?

[3] Lecture Notes for physics751: Group Theory (for Physicists), by C Ludeling. Link

[4] Serge Lang. Algebra. Revised 3rd. Springer.

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