Classification lecture by A. Ludwig

Taper

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Abstract

This presents notes for Ludwig's lecture in BSS-2016. It took a lot of content directly from Ludwig's handwritten notes. Those notes are available from BSS's website.

This not serves as majorly an introduction to the classification framework. It introduces the reason why the classification is done with anti-unitary symmetries and explains how the three symmetries (T,C,S) are related to the Ten-fold classes of Hamiltonians.

Contents

1 Reference suggested

sec:Reference suggested

This work is done with A. Ludwig, and his collaborators: Shinsei Ryu, Andreas Schnyder, Akira Furusaki, Joel Moore.

The most recent review (at that time) is: [Lud16]. Besides, the author also suggested many useful references: [SRFL09], [SRFL08], [RSFL10], [RML12], [Kit09], [CYR13], [Zir96], [AZ97], [HHZ05], [Ryu15], [Wit16].

sec:General Introduction

2 General Introduction

Before the classification, we should first clarify the objects that we are going to classify. There are *roughly* two types of materials that we called topological, according to Ludwig. They are:

1. Topological phase with intrinsic topological order.

   For example, the Fractional Quantum Hall state. Typically they have:

   - Ground state degeneracy on topologically non-trivial position space.
   - Anyonic excitations which may have fractional quantum numbers and non-trivial Braiding properties.

2. Symmetry Protected Topological (SPT) Phases

   (example: The (2+1)d (p+ip) superconductor has a small symmetry group $\mathbb{Z}_2$ (Fermion parity) protecting its SPT phase.)

   - which have **none** of the above properties,
   - but whose ground state cannot be continuously deformed into a direct product state (without crossing a quantum phase transition at which the gap closes), as long as the symmetry protecting the SPT is protected.

quantum phase trans?

The aim of this classification is to classify the 2nd group, that of those SPT phases.

About this classification

→ Topological insulators and topological superconductors of non-interacting fermions provide the simplest and first examples SPT phases.

They can be completely classified in any dimension of space. There are several approaches to this classification.

→ More general, interacting fermionic SPT phases is not yet fully understood. However, their classification should be built upon these "templates" of non-interacting cases.

→ This simplest and most general classification applies to the case when only anti-unitarily realized symmetries are required to protect the SPT phase.

→ (**Important Note**: all that we talked about being *realized*, are refer to the realization in 1st-quantized Hamiltonian, not in the 2nd-quantized Hamiltonian. For example, as we shall see, charge conjugation (C) is unitary in the 2nd-quantized Hamiltonian, but anti-unitary in the 1st-quantized Hamiltonian.

→ This Ten-fold way, is a framework to characterize all possible Hamiltonians (single-particle Hamiltonians). (Ref. [Zir96], [AZ97], [SRFL09].)

→ The unitarily realized symmetries are not required to protect our SPT, in the sense which will be explained immediately. The reason for not considering those unitarily realized symmetry is that their classification will depends on the specific symmetry group that implements the symmetry. Hence its classification will not be as universal as in the case of anti-unitarily realized symmetries.

Looking further: For Bosonic SPT, the group cohomology approach aims to address this issue. For fermionic SPT, there are generalizations of this approach, named group super-cohomology, which is currently under exploration.

Key Point 2.1 (Why No Unitarily realized symmetry)**.** If the Hamiltonian has a symmetry that is unitarily realized, then by Quantum Mechanics this Hamiltonian can be block diagonalized, and its blocks will not possess memory of the unitary symmetry, i.e. they do not commute with the unitary matrix representation of that symmetry. Also, those blocks are responsible for the topological properties of the system.

really? How they are responsible?

Therefore, our classification of symmetry is in fact a classification of these block Hamiltonians. And it turns out that the classification in terms of anti-unitarily realized symmetry does not depend on the specific matrix that represents the symmetry. In contrast, the classification in terms of unitarily realized symmetry depends on the specific group that implements the symmetry.

What contrast? Specific matrix v.s. Specific group?

It may happen that the Hamiltonian possesses some unitarily realized symmetries. But these symmetries do not affect our classification since when the Hamiltonian becomes a block-diagonal matrix, the blocks has no memory of the unitarily realized symmetries, i.e. *unitary symmetries do not give more constraints to any of the blocks.*

Specification of above statements Suppose we have a non-interacting Hamiltonian in the 2nd-quantized Fock space:

$$\hat{H} = \sum_{A,B} \hat{\psi}^\dagger_A H_{AB} \hat{\psi}_B \tag{2.0.1}$$

eq:H-2nd

Where A, B label lattice sites (goes from 1 to N), and possibly spin indices (then goes from 1 to $2N$). H_{AB} is the 1st quantized Hamiltonian..

Is it single-particle? Make it precise!

Suppose the Hamiltonian is invariant under a symmetry transformation $\hat{U}$:

$$\hat{U}\hat{H}\hat{U}^{-1} = \hat{H} \tag{2.0.2}$$

eq:sym-in-2nd-1

And it acts on creation and annihilation operators by a matrix u:

$$\hat{\psi}'_A = \hat{U}\hat{\psi}_A\hat{U}^{-1} = \sum_B (u^\dagger)_{AB}\hat{\psi}_B \tag{2.0.3}$$

eq:sym-in-2nd-2

$$\hat{\psi}'^\dagger_A = \hat{U}\hat{\psi}^\dagger_A\hat{U}^{-1} = \sum_B \hat{\psi}^\dagger_B u_{AB} \tag{2.0.4}$$

eq:sym-in-2nd-3

(we could think about $\hat{\psi}^\dagger_A$ as a row vector, and $\hat{\psi}_B$ as a column vector. This helps us remember this rules.)

It is easy to find that preserving of canonical commutation relation ($\{\hat{\psi}'^\dagger_A, \hat{\psi}'_B\} = \delta_{A,B}$), implies that the matrix u is unitary.

Next, when we want to pass from the 2nd quantized picture to the first quantized picture, we immediately realized one thing.

The linear problem Passing from 2nd quantized to the 1st quantized (i.e. plugging Eq.2.0.2, Eq.2.0.3, Eq.2.0.4 into Eq.2.0.1), one is faced with $\hat{U}H_{AB}\hat{U}^{-1}$. If $\hat{U}$ is unitary, then H_{AB} is unchanged. Otherwise, H_{AB} becomes H^*_{AB}.

The unitary (linear) case and the block form of Hamiltonian Suppose we choose $\hat{U}$ to be unitary, then on the 1st quantized Hamiltonian will satisfy the usual condition:

$$uHu^{-1} = H \tag{2.0.5}$$

In this situation, the Hamiltonian will have a block form. More precisely, suppose all our symmetry transformations form a group G_0. Denote $\mathcal{V}$ as the N-dimensional single-particle Hilbert space spanned by the single-particle states:

$$|A\rangle = \hat{\psi}^\dagger_A |0\rangle\,, \quad A = 1, \cdots, N \tag{2.0.6}$$

($|0\rangle$ is the Fock vacuum.)

We have

Theorem 2.1 (Diagonalization of Hamiltonian in unitary representation)**.** *This space $\mathcal{V}$ decomposes into a direct sum of vector spaces $\mathcal{V}_\lambda$ associated with the irrep (irreducible representations, labeled by λ) of G_0.*

$$\mathcal{V} = \oplus_\lambda m_\lambda \mathcal{V}_\lambda \tag{2.0.7}$$

where m_λ denotes the multiplicity of λth irrep. Denote the dimension of each irrep as d_λ.

In each vector space $\mathcal{V}_\lambda$, one can choose a (orthogonal) basis of the form:

$$|v^{(\lambda)}_\alpha\rangle \otimes |w^{(\lambda)}_k\rangle \tag{2.0.8}$$

where

- *G_0 acts only only $|w^{(\lambda)}\rangle_k$, $k = 1, \cdots, d_\lambda$,*
- *H acts only on $|v^{(\lambda)}\rangle_\alpha$, $\alpha = 1, \cdots, m_\lambda$.*

Thus, each irrep λ defines a block-Hamiltonian matrix of size $m_\lambda \times m_\lambda$:

$$H^{(\lambda)}_{\alpha\beta} \equiv \langle v^{(\lambda)}_\alpha | H | v^{(\lambda)}_\beta \rangle \tag{2.0.9}$$

The exact aim of this classification This classification aims to classify sets of all block Hamiltonians $H^{(\lambda)}$ when we fix a symmetry group G_0 and let the Hamiltonian runs through all possible single-particle Hamiltonians that commute with operations (this operation should be unitarily realized on the single-particle Hamiltonian) of G_0.

The result turns out to be independent of the group G_0, and of course independent of its irrep λ.

3 The Anti-Unitary Symmetries

he Anti-Unitary Symmetries

It turns out that under reasonable assumptions, there are only a finite number of anti-unitary symmetries. Let us revisit our definition of symmetry transformation:

$$\hat{U}\hat{H}\hat{U}^{-1} = \hat{H} \tag{2.0.2, revisit}$$

If it acts on creation/annihilation operators by:

$$\hat{\psi}'_A = \hat{U}\hat{\psi}_A\hat{U}^{-1} = \sum_B (u^\dagger)_{AB}\hat{\psi}_B \tag{2.0.3, revisit}$$

$$\hat{\psi}'^\dagger_A = \hat{U}\hat{\psi}^\dagger_A\hat{U}^{-1} = \sum_B \hat{\psi}^\dagger_B u_{BA} \tag{2.0.4, revisit}$$

Case 1 Instead of requiring $\hat{U}$ to be linear (which would give a unitarily realized symmetry on single-particle Hamiltonian), we require it to be anti-linear:

$$\hat{U}i\hat{U}^{-1} = -i \tag{3.0.10}$$

Then, calculation shows:

$$uH^*u^\dagger = H \tag{3.0.11}$$

where H is 1st quantized single-particle Hamiltonian. This symmetry is called time-reversal symmetry, denoted $\hat{\tau}$, its action on 1st-quantized single-particle Hamiltonian H can be denoted using

$$T \equiv \hat{\tau}\Big|_{\text{1st quantized Hilbert space}} \tag{3.0.12}$$

and by above formula, we have:

$$T = uK \tag{3.0.13}$$

where K is complex conjugation. And we have the usual property $THT^{-1} = H$, where $T^{-1} = u^tK$. (u^t is the transpose of u).

Case 2 If instead, we change eq.2.0.3 and eq.2.0.4 to such that

$$\hat{\psi}'_A = \hat{U}\hat{\psi}_A\hat{U}^{-1} = \sum_B (u^*)^\dagger_{AB}\hat{\psi}^\dagger_B \tag{3.0.14}$$

eq:sym-cc-1

$$\hat{\psi}'^\dagger_A = \hat{U}\hat{\psi}^\dagger_A\hat{U}^{-1} = \sum_B \hat{\psi}_B u^*_{BA} \tag{3.0.15}$$

eq:sym-cc-2

(Again, a tip is to remember that $\hat{\psi}^\dagger_A$ is a row vector, and $\hat{\psi}_A$ is a column vector)

In words, we require creation operators transform into annihilation operators, and annihilation operators transform into creation operators.

Now proceed again in the same way, we have the choice of linear or antilinear $\hat{U}$. If $\hat{U}$ is linear, then the condition $\hat{U}\hat{H}\hat{U}^{-1} = \hat{H}$ leads to the equation:

$$u(H - \frac{1}{2}\,\mathrm{tr}(H))^t u^\dagger = -(H - \frac{1}{2}\,\mathrm{tr}(H)) \tag{3.0.16}$$

eq:sym-C-cond

(notice in calculation that $\sum_i \hat{\psi}_i^\dagger \hat{\psi}_i = \mathbb{1}$ on 1st-quantized single-particle Hilbert space.)

Taking the trace of above equality will give $2\,\mathrm{tr}(H) = N\,\mathrm{tr}(H)$, since in solids $N >> 2$, we must have $\mathrm{tr}(H) = 0$. Then the above equality simplifies into (note that $H^t = H^*$ for Hermitian H):

$$uH^* u^\dagger = -H \tag{3.0.17}$$

This type of symmetry is called charge-conjugation symmetry. It is also called particle-hole symmetry in condensed matter physics. Notice that if we denote this symmetry in 2nd-quantized Hilbert space as $\hat{C}$, then in 1st-quantized single-particle Hilbert space, we have

$$C \equiv \hat{C}\Big|_{\text{1st-quantized}}, \quad \text{and}\, C = uK \tag{3.0.18}$$

with

$$CHC^{-1} = -H \tag{3.0.19}$$

for 1st-quantized single-particle Hamiltonian.

Case 3 If we now require again $\hat{U}$ being anti-linear, and proceed with similar arguments, we would have, very similar to eq.3.0.16, the equality:

$$u(H - \frac{1}{2}\,\mathrm{tr}(H))u^\dagger = -(H - \frac{1}{2}\,\mathrm{tr}(H)) \tag{3.0.20}$$

eq:sym-S-cond

and since $N >> 2$, we have $\mathrm{tr}(H) = 0$. Then

$$uHu^\dagger = -H \tag{3.0.21}$$

This symmetry will be called the chiral symmetry, denoted $\hat{S}$. And we have a simple relation for its action on 2nd-quantized Hamiltonian and the 1st-quantized Hamiltonian:

$$S \equiv \hat{S}\Big|_{\text{1st-quantized}}, \quad \text{and } S = u \tag{3.0.22}$$

with

$$SHS^{-1} = -H \tag{3.0.23}$$

It is easy to see that S is just a combination of T and C symmetry. This will be noted in the coming section.

3.1 Summary

sec:Summary

There are 3 different symmetries that do not lie in our usual sense of symmetry, which is unitarily realized and commutes with H. They are summarized here. Also, there is a some other property listed below. They are proved in this lecture, but are either commonly available in most textbooks, or easy to prove. So I did not give details of proof below.

Time-reversal :

- 2nd-quantized: $\hat{\tau}\hat{H}\hat{\tau}^{-1} = \hat{H}$
- Relation:

$$\hat{\tau}\hat{\psi}_A\hat{\tau}^{-1} = \sum_B (u_T^\dagger)_{AB}\hat{\psi}_B \tag{3.1.1}$$

$$\hat{\tau}\hat{\psi}_A^\dagger\hat{\tau}^{-1} = \sum_B \hat{\psi}_B^\dagger (u_T)_{BA} \tag{3.1.2}$$

$$\hat{\tau}i\hat{\tau}^{-1} = -i \tag{3.1.3}$$

- 1st-quantized: $T = u_T K$, $THT^{-1} = H$
- Squared term

  If T commutes with H, then T^2 also commutes with H. Calculation shows $T^2 = u_T u_T^* = \pm 1$. For fermion system, $T^2 = (-1)$. Hence in 2nd-quantized Hilbert Space, $\hat{T}^2 = (-1)^{\hat{Q}}$, where $\hat{Q} = \sum_A \hat{\psi}_A^\dagger \hat{\psi}_A$. This is exactly the Fermion number parity operator.

Charge-conjugation :

- 2nd-quantized: $\hat{C}\hat{H}\hat{C}^{-1} = \hat{H}$
- Relation:

$$\hat{C}\hat{\psi}_A\hat{C}^{-1} = \sum_B (u_C^*)_{AB}^\dagger \hat{\psi}_B^\dagger \tag{3.1.4}$$

$$\hat{C}\hat{\psi}_A^\dagger\hat{C}^{-1} = \sum_B \hat{\psi}_B (u_C^*)_{BA} \tag{3.1.5}$$

$$\hat{C}i\hat{C}^{-1} = i \tag{3.1.6}$$

- 1st-quantized: $C = u_C K$, $CHC^{-1} = -H$, and $\mathrm{tr}(H) = 0$.
- Squared term

  If C commutes with H, then C^2 also commutes with H. Calculation shows $C^2 = u_C u_C^* = \pm 1$. For fermion system, $C^2 = (-1)$. Hence in 2nd-quantized Hilbert Space, $\hat{C}^2 = (-1)^{\hat{Q}}$, where $\hat{Q} = \sum_A \hat{\psi}_A^\dagger \hat{\psi}_A$. This is exactly the Fermion number parity operator.
- Action on $\mathcal{F}_q$:

  Consider the $\mathcal{F}_q$, the eigenspace of $\hat{Q} = \sum_{A=1}^N \hat{\psi}_A^\dagger \hat{\psi}_A$, with eigenvalue q. Then, direct calculation shows:

$$\hat{C}\hat{Q}\hat{C}^{-1} = N - \hat{Q} \tag{3.1.7}$$

  Therefore, $\hat{C}$ links $\mathcal{F}_q$ and $\mathcal{F}_{N-q}$.

Chiral/sub-lattice :

- 2nd-quantized: $\hat{S}\hat{H}\hat{S}^{-1} = \hat{H}$

- Relation:

$$\hat{S}\hat{\psi}_A\hat{S}^{-1} = \sum_B (u_S^*)^\dagger_{AB}\hat{\psi}^\dagger_B \tag{3.1.8}$$

$$\hat{S}\hat{\psi}^\dagger_A\hat{S}^{-1} = \sum_B \hat{\psi}_B(u_S^*)_{BA} \tag{3.1.9}$$

$$\hat{S}i\hat{S}^{-1} = -i \tag{3.1.10}$$

- 1st-quantized: $S = u_S$, $SHS^{-1} = -H$, and $\mathrm{tr}(H) = 0$.
- **Relation between Chiral, Time-reversal, and Charge-conjugation**
  We have $\hat{S} = \hat{\tau}\hat{C}$, $u_S = u_T u_C^*$.
- Squared term
  If S commutes with H, then S^2 also commutes with H. Calculation shows $S^2 = u_S^2$. But since $u_S = u_T u_C^*$, we can always make a phase choice of u_T and u_C such that $S^2 = u_S^2 = 1$.
- Let $S' \equiv CT$. If $T^2 = \epsilon_T \mathbb{1}$, $C^2 = \epsilon_C \mathbb{1}$, and we adopt the phase choice such that $S^2 = 1$, then it can be shown that $S = \epsilon_C\epsilon_T S'$, or

$$TC = \epsilon_C\epsilon_T CT \tag{3.1.11}$$

3.2 How many anti-unitary symmetries are there?

ary symmetries are there?

As we have seen, there are three classes of anti-unitary symmetry transformations. Within each class, there is actually one unique anti-unitary symmetry transformation, module those of unitary ones. To be more specific, suppose we have two realizations of charge-conjugation symmetry:

$$C_1 = u_{C,1}K, \quad C_2 = u_{C,2}K \tag{3.2.1}$$

They are clearly related by a unitary matrix $u_{12} = u_{C,1}u^\dagger_{C,2}$, and $C_1 = u_{12}C_2$. Then, if we blindly assume the 1st-quantized Hamiltonian has symmetry C_1:

$$C_1HC_1^{-1} = -H \tag{3.2.2}$$

then, plugging C_2 inside gives

$$C_2HC_2^{-1} = -u^\dagger_{12}Hu_{12} \tag{3.2.3}$$

Therefore, if we further enlarge the group G_0 to include $u^\dagger_{12}$ (i.e. by requiring that H commutes with $u^\dagger_{12}$), then with $G_0' = G_0 \cup \{u^\dagger_{12}\}$, the classification will be done and nothing new is actually introduced. Therefore, there is only one C symmetry to be considered, module the unitary matrices.

Similar argument applies for T. But for S, Ludwig specifically point out that S's expression should always be kept explicit. I am confused about the reason.

Why S so special?

4 Incorporating Superconductors

orporating Superconductors

We can consider the Bogoliubov-de Gennes (BdG) Hamiltonians of fermionic excitations inside superconductors in the same way. Except that

- The particle-number $\hat{Q}$ is not conserved
- There is a built-in charge-conjugation symmetry, to be explained later.

The BdG Hamiltonian uses Nambu Spinor:

$$\hat{\chi} \equiv \begin{pmatrix} \hat{\chi}_1 \\ \cdots \\ \hat{\chi}_N \\ \hat{\chi}_{N+1} \\ \cdots \\ \hat{\chi}_{2N} \end{pmatrix} = \begin{pmatrix} \hat{\psi}_1 \\ \cdots \\ \hat{\psi}_N \\ \hat{\psi}_1^\dagger \\ \cdots \\ \hat{\psi}_N^\dagger \end{pmatrix} = \begin{pmatrix} \hat{\psi} \\ (\hat{\psi}^\dagger)^t \end{pmatrix} \tag{4.0.4}$$

(It seems that the author is ignoring spin here.) Here, we regard $\hat{\psi}^\dagger$ as a row vector, and $\hat{\psi}$ as a column vector. The BdG Hamiltonian is written as

Check the spin indices later

$$\hat{H} = \frac{1}{2}\hat{\chi}^\dagger H \hat{\chi} = \frac{1}{2}\sum_{A,B=1}^{2N} \hat{\chi}^\dagger{}_A H_{AB} \hat{\chi}_B = \frac{1}{2}\begin{pmatrix} \hat{\psi}^\dagger & \hat{\psi}^t \end{pmatrix} H \begin{pmatrix} \hat{\psi} \\ (\hat{\psi}^\dagger)^t \end{pmatrix} \tag{4.0.5}$$

eq:super-H

The 1st-quantized Hamiltonian H has the of 4 $N \times N$ blocks:

$$H = \begin{bmatrix} \Xi & \Delta \\ \Delta^* & -\Xi^t \end{bmatrix} \tag{4.0.6}$$

eq:super-H-def

where Ξ is Hermitian, and $\Delta = -\Delta^t$ due to fermion statistics.

Why fermion statistics leads to this?

Writing explicitly, we have:

$$\hat{H} = \sum_{a,b=1}^{N} \hat{\psi}_a^\dagger \Xi_{ab} \hat{\psi}_b + \frac{1}{2}\sum_{a,b=1}^{N} \left(\hat{\psi}_a^\dagger \Delta_{ab} \hat{\psi}_b^\dagger + \hat{\psi}_a \Delta_{ab}^* \hat{\psi}_b \right) \tag{4.0.7}$$

A striking difference is that $\hat{\chi}$ and $\hat{\chi}^\dagger$ are related by a linear transformation, whereas in non-superconducting case, $\hat{\psi}^\dagger$ and $\hat{\psi}$ are linear-independent. Explicitly, since $\left(\hat{\chi}^\dagger\right)^t = \begin{bmatrix} \left(\hat{\psi}^\dagger\right)^t \\ \hat{\psi} \end{bmatrix}$, then clearly:

$$\left(\hat{\chi}^\dagger\right)^t = \tau_1 \hat{\chi} \tag{4.0.8}$$

where τ_1 interchange the components:

$$\tau_1 = \begin{bmatrix} 0 & \mathbb{1}_{N\times N} \\ \mathbb{1}_{N\times N} & 0 \end{bmatrix} \tag{4.0.9}$$

Taking the transpose again gives

$$\hat{\chi}^\dagger = \hat{\chi}^t \tau_1 \tag{4.0.10}$$

eq:super-relat-1

and then,

$$\hat{\chi} = \tau_1 (\hat{\chi}^\dagger)^t \tag{4.0.11}$$

eq:super-relat-2

The Charge-conjugation Symmetry of Superconductors There is one charge-conjugation symmetry associated with the matrix τ_1. If one plugs eq.4.0.10 and eq.4.0.11 into the Hamiltonian (eq.4.0.5), one will find (note that $\mathrm{tr}(H) = 0$ by eq.4.0.6, and be careful about summation index):

$$\frac{1}{2}\hat{\chi}^\dagger H \hat{\chi} = \frac{1}{2}\hat{\chi}^\dagger (-\tau_1 H \tau_1)^t \hat{\chi} \tag{4.0.12}$$

Therefore (note that $H^t = H^*, \tau_1^t = \tau_1, \tau_1^2 = \mathbb{1}$):

$$\tau_1 H^* \tau_1 = -H \tag{4.0.13}$$

Therefore, the BdG Hamiltonian has built-in charge-conjugation symmetry, realized by $u_C = \tau_1$.

Encompass non-interacting problem and superconducting problem The way to put the two different kind of problems together, is to work with operator $\hat{\Psi}/\hat{\Psi}^\dagger$ which has first few components for ordinary annihilation/creation operators, and the last few components for superconductor Nambu spinors, and with H consisting of two diagonal blocks for each respective.

This sounds silly, perhaps I do not get some more deep reason?

5 TO BE CONTINUED

6 License

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