Miscellaneous notes for D. Huybrechts's Complex Geometry

Taper

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Abstract

Miscellaneous notes for D. Huybrechts's book *Introduction to Complex Geometry*, include some homeworks done.

Contents

1 The structure of almost complex structures on $\mathbb{R}^{2n}$ 1

2 License 4

1 The structure of almost complex structures on $\mathbb{R}^{2n}$

In exercise 1.2.1, it says that the set of all compatible almost complex structures on a euclidean space of dimension $2n$, is two copies of S^2.

To show it, I tried first a straight calculation. Assuming the almost complex structure $I = (a_{ij})$. Then we have:

Section 1.2 of Intro → Complex Geometry.

Exercises:

1.2-1.

Q: Let $(V, \langle , \rangle)$: euclidian space of dim $= 4$.

Show: $\{$ all compatible almost complex structures $\}$

$\wr\wr$

two copies of S^2 ~~$\neq$~~ $=$ ~~a~~ two balls.

Recap: compatible I: $I^2 = -1$, $\langle I(v), I(w) \rangle = \langle v, w \rangle$

Choose an orthogonal basis: $e_1 \cdots e_4$

Let $I = (a^i{}_j)$ $\quad \boxed{I^2 = a^i{}_j a^j{}_k = -\delta^i{}_k}$

also $\langle , \rangle \approx \delta^i{}_j$

$\langle I(v), I(w) \rangle = (a^i{}_j v^j) \cdot \delta^i{}_k (a^k{}_l w^l) = v^i \delta^i{}_k w^k$

$(\forall \vec{v}, \vec{w})$

Hence $\quad a^i{}_j \delta^i{}_k a^k{}_l = \delta_{jl}$ or $\boxed{a^i{}_j a^i{}_k = \delta_{jk}}$

For example:

$\sum_j a^1{}_j a^j{}_2 = 0 = \sum_j a^j{}_1 a^j{}_2 \Rightarrow \sum_{j=2}^{4} a^1{}_j a^j{}_2 = \sum_{j=2}^{4} a^j{}_1 a^j{}_2$

This can be generalized: $\sum_{\substack{j=1 \\ j \neq k}}^{4} a^k{}_j a^j{}_l = \sum_{\substack{j=1 \\ j \neq k}}^{4} a^j{}_k a^j{}_l \quad (k \neq l)$ $\Big\}$ 16 sets of eq.

Also: $\sum_{j=1}^{4} a^i{}_j a^j{}_i = -\sum_{j=1}^{4} a^j{}_i a^j{}_i$

Figure 1: Draft

Then I discover this too hard to work, because too many equations are involed, and none of them could be eliminiated by other. Meanwhile, I found a post in Math.SE about this: set of almost complex structures on $\mathbb{R}^4$ as two disjoint spheres.

To understand that post, I read this: Does $GL(n,\mathbb{C})$ inject into $GL^+(2n,\mathbb{R})$ for all n?. However, the proof inside is not perfect:

> The claim is: If V is an n-dimensional complex vector space with underlying $2n$-dimensional real vector space W, then the canonical group monomorphism $\mathrm{GL}(V) \to \mathrm{GL}(W)$ lands inside $\mathrm{GL}^+(W) = \{f \in \mathrm{GL}(W) : \det(f) > 0\}$. The purpose of this abstract reformulation is that we may use operations on vector spaces in order to simplify the problem: If V' is another finite-dimensional complex vector space with underlying real vector space W', the diagram
>
> $$\begin{array}{ccc} \mathrm{GL}(V) \times \mathrm{GL}(V') & \to & \mathrm{GL}(W) \times \mathrm{GL}(W') \\ \downarrow & & \downarrow \\ \mathrm{GL}(V \oplus V') & \to & \mathrm{GL}(W \oplus W') \end{array} \tag{1.0.1}$$
>
> commutes, and the image of $\mathrm{GL}^+(W) \times \mathrm{GL}^+(W')$ is contained in $\mathrm{GL}^+(W \oplus W')$. Therefore, if some element in $\mathrm{GL}(V \oplus V')$ lies in the image of $\mathrm{GL}(V) \times \mathrm{GL}(V')$, it suffices to consider the components. Combining this with the fact that $\mathrm{GL}(V)$ is generated by elementary matrices (after chosing a basis of V), we may reduce the whole problem to the following three types of matrices:
>
> - the 1×1-matrices (λ),
> - the 2×2-matrices $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$,
> - and the 2×2-matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
>
> Write $\lambda = a + ib$ with $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then, the complex 1×1-matrix (λ) becomes the real 2×2-matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, which has determinant $a^2 + b^2 > 0$. The complex 2×2-matrix $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$ becomes the real 4×4-matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & -b & 1 & 0 \\ b & a & 0 & 1 \end{pmatrix}$, which has determinant 1. Finally, the complex 2×2-matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ becomes the real 4×4-matrix $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, which has determinant 1.

This proof is not complete because, to build the proof from $\mathbb{R}^2$ to $\mathbb{R}^{2n}$, it requries, in his argument, that any element in $\mathrm{GL}(V \oplus V)$ is in the image of $\mathrm{GL}(V) \times \mathrm{GL}(V')$, which is not the case.

On the other hand, it seems that this property can be proved directly by calculation.

References

[1] D Huybrechts's Introduction to Complex Geometry. Springer.

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