

# Complex Geometry - Index of Notations and ideas

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**Part I**

**Indices of Notation**



# Chapter 1

## Book

### 1.1 1. Local Theory

#### 1.1.1 1.1 Holomorphic Functions of Several Variables

**Note:** the content covered by this section is geared for accompanying my personal notes of lecture 1.

[holomorphic](#): pp.1. pp.4. Def 1.1.1. pp.10(Def.1.1.8).

[Cauchy-Riemann equations](#): pp.2

$\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$ :  $\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ ,  $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$

[Maximum principle](#): pp.3.

[Identity theorem](#): pp.3.

[Riemann extension theorem](#): pp.3. pp.9 (Prop. 1.1.7).

[Riemann mapping theorem](#): pp.3.

[Liouville theorem](#): pp.4.

[Residue theorem](#): pp.4.

[polydiscs](#)  $B_\epsilon(\omega)$ :  $\{z \mid |z_i - \omega_i| < \epsilon\}$ . pp.4.

[Hartogs' theorem](#): Prop. 1.1.4. pp.6.

[Weierstrass preparation theorem \(WPT\)](#): Prop. 1.1.6. pp.8.

[Weierstrass polynomial](#): Def. 1.1.5. pp.7.

$Z(f)$ : zero set of  $f$ . pp.9.

[biholomorphic](#): pp.10.

[\(complex\) Jacobian, regular, regular value](#): Def. 1.1.9. pp.10.

[IFT. Inverse function theorem](#): Prop 1.1.10 pp.11.

[IFT. Implicit function theorem](#): Prop 1.1.11. pp.10.

$\mathcal{O}_{\mathbb{C}^n}$ : sheaf of holomorphic functions on  $\mathbb{C}^n$ . Def. 1.1.14. pp.14.

$\mathcal{O}_{\mathbb{C}^n, z}$ : Def. 1.1.14. pp.14.

$\mathcal{O}_{\mathbb{C}^n, 0}^*$ : units of  $\mathcal{O}_{\mathbb{C}^n, 0}$ . pp.14.

[UFD, unique factorization domain, irreducible](#): Def. 1.1.16. pp.14.

[Gauss Lemma](#): pp.14.

[Weierstrass division theorem](#): Prop. 1.1.17. pp.15.

[germ of set](#): (**pp.18**)

$Z(f)$ : germ of zero set of  $f$ . (pp.18)  
 analytic germ:  $Z(f_1, \dots, f_k)$ . (pp.18)  
 analytic subset: locally are zero sets. (pp.18)  
 $I(X)$ : the set of all  $f \in \mathcal{O}_{\mathbb{C}^n, 0}$  with  $X \subset Z(f)$ . (pp.18)

### 1.1.2 1.2 Complex and Hermitian Structures

almost complex structure,  $I$ :  $I^2 = -\text{id}$ . (pp.25)  
 $V^{1,0}$  and  $V^{0,1}$ : the  $\pm i$  eigenspaces of  $I$ . (pp.25)  
 $\bigwedge^{p,q} V := \bigwedge^p V^{1,0} \oplus_{\mathbb{C}} \bigwedge^q V^{0,1}$ . (pp.27)  
 $\prod^k, \prod^{p,q}$ : natural projections. (pp.28)  
 $\mathbf{I} := \sum_{p,q} i^{p-q} \cdot \prod^{q,p}$ . (pp.28)  
 compatible: an almost complex structure  $I$  is compatible with the scalar product  $\langle, \rangle$ , if  $\langle I(v), I(w) \rangle = \langle v, w \rangle$ . (pp.28)  
 Conformal equivalence (between scalar product): (pp.29)  
 fundamental form,  $\omega := \langle I(\cdot), \cdot \rangle$ . (pp.29)

$$\omega = \frac{i}{2} \sum_i z^i \wedge \bar{z}^i = \sum_i x^i \wedge y^i.$$

(Local calculation could be found on pp.31)  
 hermitian form  $(\cdot, \cdot) := \langle, \rangle - i \cdot \omega$ . (pp.30)

## 1.2 2.Complex Manifolds

### 1.2.1 2.1 Complex and Hermitian Structures

almost complex structure: (pp.25)

### 1.2.2 2.2 Holomorphic Vector Bundles

$\tau_X$ : holomorphic tangent bundle of a complex manifold  $X$  (Def 2.2.14 at pp. 71).

$\Omega_X, \Omega_X^p$ : holomorphic cotangent bundle and holomorphic  $p$ -forms. (Def 2.2.14 at pp. 71)

$K_X := \det(\Omega_X) = \Omega_X^n$ , the canonical bundle of  $X$ . (Def 2.2.14 at pp. 71)

### 1.2.3 2.6 Differential Calculus on Complex Manifolds

$\bigwedge_{\mathbb{C}}^k X := \bigwedge^k (T_{\mathbb{C}} X)^*$ . (Def 2.6.7 at pp. 105)

$\bigwedge^{p,q} X := \bigwedge^p (T^{1,0} X)^* \oplus_{\mathbb{C}} \bigwedge^q (T^{0,1} X)^*$ . (Def 2.6.7 at pp. 105)

$\mathcal{A}_{X, \mathbb{C}}^k, \mathcal{A}_X^{p,q}$ : sheaves of section of the above correspond items. (Def 2.6.7 at pp. 105)

$\mathcal{A}^{p,q}(E)$ : the sheaf of  $p, q$ -forms with values in  $E$ , a complex vector bundle. (Def 2.6.22 at pp.109). Note that in particular,  $\mathcal{A}^0(E)$  is the sheaf of sections of  $E$ .



### 1.2.4 Appendix B: Sheaf Cohomology

- [illegible]



## Chapter 2

# My lecture Notes

### 2.1 Lecture 2016 Lecture 1

The first few lectures are not well noted, hence I delegate the task of recording the theorems and notations to the book's corresponding section:section 1.1.1 on page 7.

### 2.2 Lecture 4 (20160307) Complex Manifold

**Note:** we use abbreviation *mnfd* for *manifold*.

pp. A:

- [Holomorphic Atlas](#)
- [Holomorphic chart](#)
- [Complex mnfd](#)

pp. B:

- [Holomorphic function](#)
- $\mathcal{O}_X$ : sheaf of holomorphic functions on a complex mnfd  $X$ .

pp. C:

- [Hartdags' theorem](#): on complex mnfd.
- [Holomorphic functions on complex mnfd](#):

pp. D:

- [Complex Lie group](#)
- [Complex Projective Space,  \$\mathbb{CP}^n\$ , or just  \$\mathbb{P}^n\$ .](#)

pp. E:

- [Topology in  \$\mathbb{P}^n\$](#)
- [Mnfd structure on  \$\mathbb{P}^n\$ , atlas, and the \*\*canonical covering\*\*](#)

pp. F

- Grassmannian mnfd.

## 2.3 Lecture 5 Submanifolds (20160308)

pp. A:

- Affine Hypersurface (actually this is not quite different from the usual  $\mathbb{C}^n$ .)

### Part 2. **Sheaf Theory**

pp. A:

- pre-sheaf
- $\mathcal{O}_X(U)$
- $\mathcal{O}_X^*(U)$

pp. B:

- $C^\infty$
- $\underline{\mathbb{Z}}$ , sometimes simply denoted as  $\mathbb{Z}$ : sheaf of locally constant  $\mathbb{Z}$ -valued functions.
- Sheaf

pp. D:

- sheaf-morphisms

pp. E:

- Section
- $\text{Ker}(\phi)$  - sheaf of kernels.

pp. F:

- $\text{Im}(\phi)$  is a presheaf, but not a sheaf.
- $\text{Im}(\phi)$ : the sheafification of  $\text{Im}(\phi)$  above. Note that we use the same notation to denote both.

## 2.4 Lecture 6 Sheaf & Cohomology (20160315)

pp. A:

- Stalk  $\mathcal{F}_x$ .
- germ
- Directed partial order set
- Directed System

pp. B:

- Directed limit

pp. C:

- Exact Complex/ Exact Sequence.
- Exponential sequence (*mentioned under the definition of exact sequence*).
- 

pp. D:

- Čech cohomology

pp. E:

- q-cochain
- coboundary operator.  $\delta$ .
- $Z^p(U, \mathcal{F}) = \text{Ker.}$
- $B^p(U, \mathcal{F}) = \text{Im.}$
- $\check{H}^p(U, \mathcal{F}) = \frac{\text{Ker}}{\text{Im.}}$ .

### 2.4.1 Notes of Čech Cohomology with Coefficients in a Sheaf

pp.1:

- q-simplex  $\sigma$ .
- support  $|\sigma|$ .
- q-cocain
- $C^q(U, \mathcal{F})$
- Coboundary Operator  $\delta$ .

pp.2,3,4:

- Cochain Complex
- Čech cohomology
- cocycle
- cochain
- $\check{H}^p(U, \mathcal{F}), Z^p(U, \mathcal{F}), B^p(U, \mathbb{F})$ .
- $\check{H}^0(\{u_i\}, \mathcal{F}) = \mathcal{F}(X)$ .

## 2.5 Lecture 7 Vector Bundle (20160321)

pp.1,2:

- Vector Bundle
- Trivializing covering,  $\{(U_i, \tau_i)\}$ .
- trivializing maps, trivializes.
- VB-equivalent of trivializing maps.
- E: total space, X: base space.

pp. 3,5:

- transition maps.
- fibre.
- $\mathcal{O}(-1)$
- cocycle condition.
- $\mathcal{T}_X$ , Holomorphic tangent bundle.

pp. 8:

- s: section of a holomorphic vector bundle.
- $\mathcal{E}$ : sheaf of sections of holomorphic vector bundle.  $\mathcal{E}(U)$ .

## 2.6 Lecture 8 Almost Complex Structures (20160322)

pp. 1,2:

- $I$ : Almost Complex Structure.  $I^2 = -1$ . Sometime  $J$  is used in place of  $I$ .
- $V_{\mathbb{C}} := V \otimes \mathbb{C}$ .
- $I_{\mathbb{C}}$ :  $I$  extending to  $V_{\mathbb{C}}$ . Usually abbreviated simply as  $I$ .
- $V^{1,0} := \ker(I + i)$ .
- $V^{0,1} := \ker(I - i)$ .

## 2.7 Lecture 9 Exterior Algebra on Complex Manifold (20160329)

pp.1,2:

- $V^*$ : dual of  $V$ .
- $\{dx^i, dy^i\}$ .
- $J^*$ :  $J$  extending to dual space.
- $dz^i, d\bar{z}^i$ .

pp. 3:

- $S^k(V), \Lambda^k(V)$ .
- $s$  and  $a$ , symmetrization and anti-symmetrization of a tensor.
- $\Lambda^* V$ .

pp. 4:

- $\Lambda^n T_{\mathcal{C}}^* X$ .
- $\Lambda^* T_{\mathcal{C}}^* X$ .
- $\Lambda^{p,q} T_{\mathcal{C}}^* X$ .

pp. 5,6:

- $\mathcal{A}$ : sheaf of section of cotangent bundle.
- $\mathcal{A}^n(U), \mathcal{A}^{p,q}(U)$ .
- $\Lambda$  on  $\mathcal{A}$ .
- $d$ : de Rham differential.
- $\partial, \bar{\partial}$ .

## 2.8 Lecture 10 Debeault Cohomology (20160406)

pp. 1:

- $\mathcal{H}^{p,q}(X)$ .
- $f^*$ : pull-back. Various definition from pp.1 to pp.4.

pp. 5,6,7:

- $\mathcal{A}^{p,q}(U, E) := \Gamma(U, \Lambda^{p,q} T_{\mathbb{C}}^* X \otimes E)$ .
- $\bar{\partial}_E$
- $\mathcal{H}^{p,q}(X, E)$ .
- $\bar{\partial}$ -Poincaré lemma in one variable.

## 2.9 Lecture 11 (20160412)

pp.1,2,3:

- $\bar{\partial}$ -Poincaré lemma in n-dimension
- $\Omega_X^p$ : holomorphic p-forms. On pp.2.
- $\check{H}^q(X, \Omega^p)(\check{\text{Cech}}) \cong \mathcal{H}_{\bar{\partial}}^{p,q}(X)(\text{Dolbeault})$ . On pp.3.

pp. 6,7:

- Analytic Subvarity.
- Analytic Hybersurface.
- Cousin's Problem.

## 2.10 Lecture 12 Hermitian Structure on Manifold Manifold (20160418)

pp. 1,2,3:

- $I$  compatible with  $\langle -, - \rangle$ .
- $\omega$ : Fundamental form associated with  $\langle, \rangle$  and  $I$ .  $\omega(v, w) := \langle I(v), w \rangle$ .
- Conformal Equivalence.
- $\langle, \rangle$ : Hermitian Inner Product.

pp. 4:



- $(,):$  s.t.  $(v, w) := \langle v, w \rangle - i\omega(v, w) = \langle v, w \rangle - i \langle I(v), w \rangle$

pp. 5:

- $\langle, \rangle_{\mathbb{C}}$  be s.t.  $\langle v \otimes \alpha, w \otimes \beta \rangle := \alpha \bar{\beta} \langle v, w \rangle$ .

pp. 6:

- $\frac{1}{2}(\langle, \rangle) = \langle, \rangle_{\mathbb{C}}|_{V^{1,0}}$

pp. 7,8:

- Local computations:  $z_i, h_{ij}$ ,
- $\omega = (\dots dx^i \dots dy^i)$
- $\omega$ , Fundamental form on Riemannian Mnfd.
- Kähler mnfd:  $d\omega \equiv 0$ .

## 2.11 Lecture 13 Kähler Manifold (20160419)

pp.1:

- Local computation:  $\omega = (\dots dz^i \dots d\bar{z}^i)$

pp.4:

- Fubini-Study Metric on  $\mathbb{CP}^n$ .

## 2.12 Lecture 14 Hodge Theory (20160425)

pp.1:

- $\langle, \rangle$  on  $\Lambda^k V$
- vol: volume element.
- $*$ : Hodge Star Operator.

pp.4:

- $L$ : Lefschetz Operator
- $\Lambda$ : adjoint of  $L$ .  $\Lambda = *^{-1} \circ L \circ *$ .

pp.5 :

- $*, L, \Lambda$  on Kähler mnfd.
- $d^* := (-1)^{m*(k+1)+1} * \circ d \circ *$ , adjoint of  $d$ . On a Kähler mnfd,  $d^* = -* \circ d \circ *$
- $\Delta := d^* \circ d + d \circ d^*$ .

pp. 6:

- $\bar{\partial}^*, \partial^*$ : Similar to the above for  $d$ .
- $\Delta_{\partial}, \Delta_{\bar{\partial}}$ : Similar to the above for  $d$ .

## 2.13 Lecture 15 Hodge Theory on Manifold (20160426)

pp.1:

- $(,)$  on  $\mathcal{A}^*(X)$ .  $(\alpha, \beta) := \int_X g_{\mathbb{C}}(\alpha, \beta) \text{vol}$

pp.3:

- $\mathcal{H}^k(X, g)$ : d-harmonic forms. Sometimes we replace  $\mathcal{H}$  with  $\mathcal{H}$  for harmonic forms, so is for symbols below.
- $\mathcal{H}_{\bar{\partial}}^k(X, g)$ :  $\bar{\partial}$ -harmonic forms. (Be careful to distinguish this with Dolbeault Cohomology groups).
- $\mathcal{H}_{\partial}^k(X, g)$ :  $\partial$ -harmonic forms.

pp. 5:

- $\mathcal{H}_d^k(X, g) \cong \mathcal{H}_d^{2n-k}(X, g)$ , Poincaré duality
- $\mathcal{H}_{\bar{\partial}}^{p,q}(X, g) \cong (\mathcal{H}_{\partial}^{n-p, n-q}(X, g))^*$ , both are harmonic forms, called Serre Duality.

pp. 6,7:

- $\mathcal{A}^{p,q} = \bar{\partial}^* \mathcal{A}^{p,q-1}(X) \oplus \bar{\partial}^* \mathcal{A}^{p,q+1}(X) \oplus \mathcal{H}_{\bar{\partial}}^{p,q}(X, g)$ :  
Hodge decomposition
- $\mathcal{H}_{\bar{\partial}}^{p,q}$ (harmonic forms)  $\cong \mathcal{H}_{\bar{\partial}}^{p,q}(X)$ (Dolbeault Cohomology group)
- $\mathcal{H}_d^{p,q}$ (harmonic forms)  $\cong \mathcal{H}_{dR}^{p,q}(X)$ (de Rham Cohomology group)

pp. 8:

- A lot of isomorphisms between *de Rham*, *Dolbeault* and *harmonic forms*.

## 2.14 Lecture 16 Harmonic forms on Kähler Manifold (20160503)

pp.1:

- $\Delta_{\partial} = \Delta_{\bar{\partial}} = \frac{1}{2} \Delta_d$ , for Kähler mnfd.

## 2.15 Lecture 17 Hermitian Vector Bundle (20160510)

pp.1,3:

- Hermitian Vector Bundle. pp.1
- Antilinear map. pp.3.

- Hermitian Inner Product on  $\mathcal{A}^{p,q}(X, E)$ . pp.4
- $\bar{*}_E$  Hodge Operator on Hermitian vector bundle. pp.5.
- $\bar{\partial}_E^*$

pp. 8:

- Kadaira-Serre Duality.

## 2.16 Lecture 18 Connection (20160516)

- $\nabla$ : connection. pp.1.
- Trivial connections. pp.2
- $\mathcal{A}^1(M, \text{End}(E)) := \Gamma(M, \Lambda^1 M \otimes \text{End}(E))$ . pp.3. Also, one may find how elements in this sheaf act on  $\mathcal{A}^0(M)$  on pp.173, inside proof of proposition 4.2.3.
- $s \in \mathcal{A}^0(E)$  is Parallel/flat/constant  $\Leftrightarrow \Delta(s) = 0$ . pp.4.
- $\Delta = d + A$ . pp.4.
- $\Delta$  be compatible with hermitian structure on  $E$ . pp.5.
- $\Delta$  be compatible with holomorphic vector bundle. pp.6.
- $A = \bar{H}^{-1} \partial H$ . Chern connection. pp.6.

## 2.17 Lecture 19 Holomorphic Connection & Curvature(20160517)

- Holomorphic Connection. pp.1.
- $At(E)$ : Atiyah class of  $E$ . pp.2.
- $\Delta^k$ . pp.4.
- $F_\Delta$ : curvature associated with  $\Delta$ . pp.5.
- $F_\Delta = dA + A \wedge A$ : Cartan structure equation. pp.6.
- First Chern class of complex line bundle.

## 2.18 Lecture 20 Divisors & (Holomorphic) Line Bundles (20160524)

- Analytic Subvariety. pp.1.
- Regular/Smooth Point. pp.1.
- Singular Point. pp.2.
- Irreducible analytic subvariety. pp.2.
- $\dim(Y)$ : dimension of analytic subvariety. pp.2. Also pp.4.
- Affine algebraic varieties. pp.3.
- Projective algebraic varieties. pp.3.
- Hypersurface. pp.4.
- Divisor,  $Div(X)$ :=group of all divisors. pp.5.
- Effective divisor. pp.6.
- $Ord_Y(f)$ : order of function. pp.6. Also pp.8.
- Meromorphic function on complex mnfd.
- $(f)$ : divisor given by a global meromorphic function.
- Principal divisor. pp.8.

## 2.19 Lecture 21 Divisors & (Holomorphic) Line Bundles (20160530)

- $H^0(X, K_X^*/\mathcal{O}_X^*) \cong Div(X)$ . pp.1.
- $Pic(X)$ : Picard group, all holomorphic line bundles. pp.3.
- $Pic(X) \cong \check{H}^1(X, \mathcal{O}_X^*)$ . pp.3.
- $\mathcal{O}(D)$ : line bundle given by divisor  $D$ . pp.5.
- Linear equivalent of divisors.
- \*: used only in this section to denoted the map:

$$(Div(X)/Pic(X)) \hookrightarrow Pic(X)$$

pp.6.

- $Z(s)$ : divisor constructed from nonzero section  $s \in H^0(X, L)$  for a line bundle  $L$ .

## 2.20 Lecture 22 Divisors & (Holomorphic) Line Bundles (20160606)

- Base point of a line bundle. pp.4.
- $Bs(L) :=$  set of all base points of line bundle  $L$ . pp.4.
- $\mathcal{O}(1), \mathcal{O}(k)$ . pp.6.



**Part II**

**Indices of Results**





Theorems, Remarks, etc.



## Chapter 3

# Local Theory

### 3.1 1.1 Holomorphic Functions of Several Variables

**Proposition 3.1.1.** *The local ring  $\mathcal{O}_{\mathbb{C}^n,0}$  is a UFD.*

(pp.14 of [1])

**Proposition 3.1.2.** *Weierstrass division theorem Let  $f \in \mathcal{O}_{\mathbb{C}^n,0}$  and  $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$  be a Weierstrass polynomial of degree  $d$ . Then there exist  $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$  of degree  $< d$  and  $h \in \mathcal{O}_{\mathbb{C}^n,0}$  such that  $f = g \cdot h + r$ . The functions  $h$  and  $r$  are uniquely determined.*

(pp.15 of [1])

**Proposition 3.1.3.** *The local UFT  $\mathcal{O}_{\mathbb{C}^n,0}$  is Noetherian.*

(pp.16 of [1])

**Corollary 3.1.1.** *Let  $g \in \mathcal{O}_{\mathbb{C}^n,0}$  be an irreducible function. If  $f \in \mathcal{O}_{\mathbb{C}^n,0}$  vanishes on  $Z(g)$ , then  $g$  divides  $f$ .*

(pp.16 of [1])

**Lemma 3.1.1.** *For any germ  $X \subset \mathbb{C}^n$  the set  $I(X) \subset \mathcal{O}_{\mathbb{C}^n,0}$  is an ideal. If  $(A) \subset \mathcal{O}_{\mathbb{C}^n,0}$  denotes the ideal generated by the subset  $A \subset \mathcal{O}_{\mathbb{C}^n,0}$ , then  $Z(A) = Z((A))$  and  $Z(A)$  is analytic.*

(pp.18 of [1])

**Lemma 3.1.2.** *If  $X_1 \subset X_2$ , then  $I(X_2) \subset I(X_1)$ . If  $I_1 \subset I_2$ , then  $Z(I_2) \subset Z(I_1)$ . For any analytic germ  $X$  one has  $Z(I(X)) = X$ . For any ideal  $I \subset \mathcal{O}_{\mathbb{C}^n,0}$ , one has  $I \subset I(Z(I))$ .*

(pp.18 of [1])

### 3.2 1.2 Complex and Hermitian Structures

**Lemma 3.2.1.** *If  $I$  is an almost complex structure on a real vector space  $V$ , then  $V$  admits in a natural way the structure of a complex vector space*

(pp.25 of [1])

**Remark 3.2.1.** *An almost complex structure can only exist on an even dimensional real vector space.*

**Corollary 3.2.1.** *Any almost complex structure on  $V$  induces a natural orientation on  $V$ .*

(pp.25 of [1])

**Lemma 3.2.2.** *Let  $V$  be a real vector space endowed with an almost complex structure  $I$ . Then*

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

*Complex conjugation on  $V_{\mathbb{C}}$  induces an  $\mathbb{R}$ -linear isomorphism  $V^{1,0} \cong V^{0,1}$ .*

(pp.26 of [1])

**Remark 3.2.2.** *Two almost complex structures on  $V_{\mathbb{C}}$ :  $I$  and  $i$ , coincide on the subspace  $V^{1,0}$  but differ by a sign on  $V^{0,1}$ .*

**Lemma 3.2.3.** *Let  $V$  be a real vector space endowed with an almost complex structure  $I$ . Then the dual space  $V^* = \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$  has a natural almost complex structure given by  $I(f)(v) = f(I(v))$ . The induced decomposition on  $(V^*)_{\mathbb{C}} = \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) = (V_{\mathbb{C}})^*$  is given by*

$$(V^*)^{1,0} = \{f \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(I(v)) = if(v)\} = (V^{1,0})^*$$

$$(V^*)^{0,1} = \{f \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(I(v)) = -if(v)\} = (V^{0,1})^*$$

*Also note that  $(V^*)^{1,0} = \text{Hom}_{\mathbb{C}}((V, I), \mathbb{C})$ .*

**Proposition 3.2.1.** *For a real vector space  $V$  endowed with an almost complex structure  $I$ , one has:*

1.  $\bigwedge^{p,q} V$  is in a canonical way a subspace of  $\bigwedge^{p+q} V_{\mathbb{C}}$ .
2.  $\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V$ .
3. Complex conjugation on  $\bigwedge^* V_{\mathbb{C}}$  defines a ( $\mathbb{C}$ -linear) isomorphism  $\bigwedge^{p,q} V \cong \bigwedge^{q,p} V$ , i.e.  $\bigwedge^{p,q} V = \bigwedge^{q,p} V$ .
4. The exterior product is of bidegree  $(0,0)$ .

(pp.27 of [1])

**Remark 3.2.3.** *Local calculation of  $V^{1,0}, (V^*)^{1,0}$*

$$z_i = \frac{1}{2}(x_i - y_i), \bar{z}_i = \frac{1}{2}(x_i + iy_i)$$

$$z^i = x^i + iy^i, \bar{z}^i = x^i - iy^i$$

$$I(z_i) = iz_i, I(z^i) = iz^i$$

(pp.27 to 28 of [1])

**Lemma 3.2.4.** *For any  $m \leq \dim_{\mathbb{C}} V^{1,0}$ , one has*

$$(-2i)^m (z_1 \wedge \bar{z}_1) \wedge \cdots \wedge (z_m \wedge \bar{z}_m) = (x_1 \wedge y_1) \wedge \cdots \wedge (x_m \wedge y_m).$$

*For  $m = \dim_{\mathbb{C}} V^{1,0}$ , this defines a positive oriented volume form for the natural orientation of  $V$ .*

*Also*

$$\left(\frac{i}{2}\right)^m (z^1 \wedge \bar{z}^1) \wedge \cdots \wedge (z^m \wedge \bar{z}^m) = (x^1 \wedge y^1) \wedge \cdots \wedge (x^m \wedge y^m).$$



# Bibliography

- [1] Complex Geometry





# **Part III**

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