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Irreducible Triangulations of the Klein Bottle

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We determine the complete list of the irreducible triangulations of the Klein bottle, up to equivalence, analyzing their structures. © 1997 Academic Press

1. INTRODUCTION

A *triangulation* of a closed surface is a simple graph embedded on the surface so that each face is triangular and that any two faces have at most one edge in common. (The latter is needed only for the sphere to exclude K_3 from the spherical triangulations.) It is often regarded as a 2-simplicial complex together with its triangular faces. Two triangulations G and G' of a closed surface F^2 are said to be *equivalent* if there is a homeomorphism $h: F^2 \to F^2$ with $h(G) = G'$. In the combinatorial sense, such a homeomorphism can be thought of as an isomorphism between two graphs which induces a bijection between their faces. We shall say that two triangulations are *isomorphic* to each other when they are isomorphic as graphs neglecting their embeddings.

Let *abc* and *acd* be two faces which share an edge *ac* in a triangulation G. The *contraction* of *ac* is to delete the edge *ac* and to identify the path *bad* with *bcd*, shrinking the quadrilateral region bounded by the cycle *abcd*, as shown in Fig. 1. An edge e of G is said to be *contractible* if the contraction of e yields another triangulation of the surface where G is embedded.

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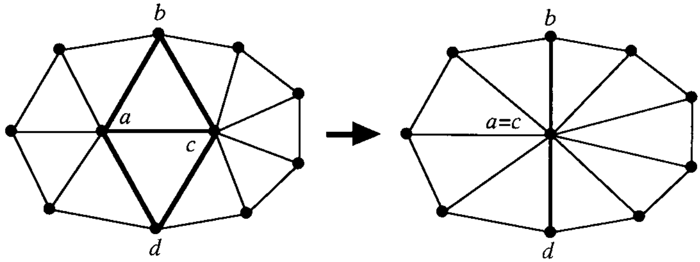


FIG. 1. Contraction of an edge in a triangulation.

Thus, when the surface is not the sphere, then an edge e of G is not contractible if and only if e is contained in at least three cycles of length 3, two of which bound the two faces incident to e. In this case, the graph obtained from G by contracting e would not be simple.

A triangulation is said to be *irreducible* if it has no contractible edge. It is not so difficult to see that the only irreducible triangulation of the sphere is the unique embedding of K_4, that is, the tetrahedron [10]. Barnette [1] has already shown that there are precisely two irreducible triangulations of the projective plane and they are equivalent to ones given in Fig. 2. Lawrencenko [4] has determined the complete list of the irreducible triangulations of the torus, which are 21 in number. In this paper, we shall classify the irreducible triangulations of the Klein bottle, discussing their structures.

THEOREM 1. *There are precisely* 25 *irreducible triangulations of the Klein bottle, up to equivalence.*

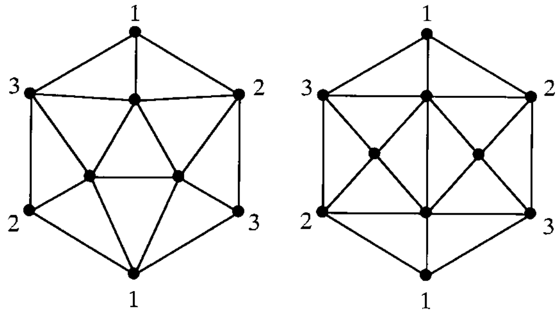


FIG. 2. Irreducible triangulations of the projective plane.

Figures 13, 14, and 15 at the end of this paper present their complete list. In the first two figures, namely Figs. 13 and 14, the pair of horizontal sides of each rectangle should be identified in parallel and vertical ones in antiparallel to get an actual triangulation on the Klein bottle. Such a triangulation is called a *handle type*, which is characterized precisely in Section 3.

On the other hand, each triangulation in Fig. 15 is called a *crosscap type* and can be obtained from two copies of irreducible triangulations of the projective plane, given in Fig. 2, by pasting them together along one face. It is easy to see that any triangulation of this type is actually irreducible. Identify each antipodal pair of vertices on the boundary of each of the two hexagons which form one picture.

To distinguish their types, handle type or crosscap type, will lead us to a systematic method to generate all the irreducible triangulations of the Klein bottle. It seems, however, so difficult to classify the irreducible triangulations of a given closed surface, in general, although any closed surface has only finitely many irreducible triangulations [2]. Recently, Nakamoto and Ota [6] has shown that any irreducible triangulation of a closed surface F^2 with Euler characteristic $\chi(F^2) \leqslant 0$ has at most $171(2-\chi(F^2))-72$ vertices. Their linear bound is the best one at the present, but it is 270 for the Klein bottle while the largest irreducible triangulations of the Klein bottle has only 11 vertices.

The classification of irreducible triangulations involves many applications. For example, Negami [9] has shown that any two triangulations of each closed surface with the same and sufficiently large number of vertices can be transformed into each other by a sequence of operations called *diagonal flips*, connecting this phenomenon to the finiteness of irreducible triangulations in number. Although his proof is so theoretical, we will be able to prove it more concretely for a fixed closed surface if we have the complete list of the irreducible triangulations of the surface.

Also, Lawrencenko and Negami [5] have classified those graphs that triangulate both the torus and the Klein bottle. The essence of their classification is to identify the graph which can be embedded on the torus and the Klein bottle as their irreducible triangulations, that is, one which belongs to the intersection of the set of irreducible triangulations of the torus and that of the Klein bottle. Section 4 will give an argument to show that such a graph is isomorphic to Kh1 in our list.

In Section 2, we shall describe the topology on the Klein bottle, which will be useful for the reader unfamiliar to the Klein bottle. The proof of Theorem 1 will be given in Section 3, consisting of three lemmas, namely Lemmas 4, 5, and 6. In particular, Lemma 4 is the key for our classification. Section 4 includes some observations on irreducible triangulations of the Klein bottle.

2. TOPOLOGY ON THE KLEIN BOTTLE

A simple closed curve on a closed surface is said to be *trivial* if it bounds a 2-cell region on the surface, and to be *essential* otherwise. The Klein bottle includes precisely three types of essential simple closed curves, up to homeomorphism. A simple closed curve is called a *meridian* if it cuts open the Klein bottle into an annulus (or a cylinder). Given a meridian, there is a simple closed curve crossing it at a point, which corresponds to an arc joining the two boundary components of the annulus. Such a curve is called a *longitude* and is characterized as one which lies along the center line of a Möbius band.

In terms of topology, a meridian can be defined as an orientation-preserving nonseparating simple closed curve while a longitude is an orientation-reversing nonseparating simple closed curve. There is another type of a simple closed curve, called an *equator*, which is a separating essential simple closed curve. This is orientation-preserving necessarily. We shall visualize these essential simple closed curves as follows (see Fig. 3).

Choose a meridian and a longitude so that they cross each other only once. (We say that two simple closed curves *cross* each other when they intersect transversely.) If one cuts the Klein bottle along both the meridian and the longitude, then a rectangle will be obtained. Conversely, identify the horizontal pair of its sides in parallel and the vertical one in antiparallel. Then the Klein bottle will be obtained and the horizontal and vertical sides will become a longitude and a meridian, respectively.

Divide each of vertical sides into four segments of equal length and label the three dividing points on the left side with 1, 2, and 3 and those of the right side with 3, 2, 1 downwards. Then each pair of points with the same label should be identified to a single point in the Klein bottle. The arc

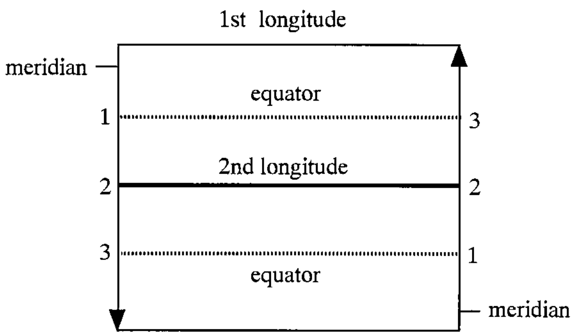


FIG. 3. Essential simple closed curves in the Klein bottle.

between the two middle points of vertical sides with label 2 forms an essential simple closed curve in the Klein bottle. This is also called a longitude. Actually, this longitude crosses the meridian only once and its tubular neighborhood is homeomorphic to the Möbius band. Thus, we can choose two disjoint longitudes in the Klein bottle.

The union of the two arcs between 1 and 3 forms an essential simple closed curve which separates the Klein bottle into two Möbius bands, each of which contains one of the two longitudes. This curve is called an *equator* in this paper. Prepare two copies of the projective plane and remove a disk from each of them. Each of the resulting objects is often called a *crosscap* and is homeomorphic to the Möbius band. The Klein bottle will be obtained from the union of those crosscaps by pasting them along their boundary curves. In this case, the boundary curves of the crosscaps will be an equator in the Klein bottle. (A crosscap will correspond to a longitude rather than the Möbus band in our later context.)

Any two meridians are *isotopic* to each other on the Klein bottle. That is, they can be transformed into each other by a continuous deformation, called an *isotopy*. Thus, we may say that the meridian is unique on the Klein bottle, up to isotopy, while a disjoint system of two longitudes and an equator is not unique.

The following statements will work as the axioms on the topology of the Klein bottle and will be useful to recognize which a given simple closed curve is, a meridian, a longitude, or an equator.

- If an essential simple closed curve does not meet a meridan, then it is another meridian.
- If a simple closed curve crosses a meridian only once, then it is a logitude.
- If a simple closed curve crosses a meridian twice and if such a pair of crossing points cannot be eliminated by any isotopy, then it is an equator.
- If a simple closed curve crosses each of a disjoint pair of longitudes once, then it is a meridian.
- If an essential simple closed curve is disjoint from one of a disjoint pair of longitudes, then it is isotopic to the other.
- If an essential simple closed curve is disjoint from an equator, then it is a longitude.
- If an essential simple closed curve is disjoint from a disjoint pair of longitudes, then it is an equator.

3. PROOF OF THEOREM

The following lemma is an easy criterion for a triangulation to be irreducible and is found as Lemma 3 in [9].

LEMMA 2. *A triangulation of a closed surface except the sphere is irreducible if and only if each edge of it lies on an essential cycle of length* 3.

A cycle C in a triangulation is called a *polygon* if it bounds a 2-cell region on the surface and an edge lying in such a 2-cell is called a *diagonal* of C if it joins two vertices on C. The following lemma is an immediate consequence of Lemma 2.

LEMMA 3. *Let C be a polygon in an irreducible triangulation and A the 2-cell region bounded by C. Then*:

(i) *If there is a vertex u inside A, then all the neighbors $u_1, ..., u_n$ of u are contained in C and each edge uu_i is contained in an essential cycle uu_iu_j through an edge u_iu_j outside A for some u_j.*

(ii) *The two ends of each diagonal of C are joined by a path of length* 2 *outside A.*

This lemma is useful to decide the partial structure of irreducible triangulations. For example, any polygon of length 3 bounds a face and the quadrilateral region bounded by a polygon of length 4 either is divided into two triangles by a diagonal or contains only one vertex of degree 4.

The following lemma, presents a fundamental fact on our classification of the irreducible triangulations of the Klein bottle. An irreducible triangulation of the Klein bottle is of *handle type* or of *crosscap type* if it contains a cycle of length 3 which is a meridian or an equator, respectively.

LEMMA 4. *The irreducible triangulations of the Klein bottle can be classified into two disjoint classes, handle types and crosscap types.*

Proof. Let T be an irreducible triangulation of the Klein bottle. Suppose that T contains a meridian C of length 3 and another essential cycle C' which separates the Klein bottle. Then, C cuts C' into at least two segments and each of those segments has to have length at least 2 since C induces a complete graph in T. Thus, C' has length at least 4. This implies that any irreducible triangulation of handle type cannot be of crosscap type.

Now we shall find an equator of length 3 in T, assuming that T is not of handle type, that is, does not include any meridian of length 3 in turn.

Note that a cycle is a meridian if it crosses each of two disjoint longitudes once.

Let u_0 be a vertex of T which attains the minimum degree of T and hence $\deg u_0 = 4$, 5 or 6 since the mean degree of any triangulation on the Klein bottle is equal to 6. Let $u_1, ..., u_n$ be the neighbors of u_0 with $n = \deg u_0 \leqslant 6$ which lie along the link of u_0 in T cyclically in this order. Consider essential cycles of length 3 through u_0 and suppose that none of them separates the Klein bottle. That is, they are not equators. Then, we can find a pair of those, say $Q_1 = u_0 u_1 u_k$ and $u_0 u_i u_j$, with $1 < i < k < j$ and regard Q_1 as a longitude.

To visualize this situation, we cut open the Klein bottle along Q_1 into a hexagonal disk with boundary $u_0 u_1 u_k u_0 u_1 u_k$ and with one crosscap Q_2 inside. (The crosscap or a longitude Q_2 is not assumed to be a cycle in T at this stage.) In this hexagon, the link of u_0 in T splits into two horizontal paths $u_1 u_2 \cdots u_i \cdots u_k$ and $u_k \cdots u_j \cdots u_n u_1$. If $u_0 u_i u_j$ crossed Q_2, then it would be a meridian. Thus, the edge $u_i u_j$ lying vertically separates the rectangular region in the middle of the hexagon into two regions with boundaries $u_1 \cdots u_i u_j \cdots u_k$ and $u_i \cdots u_k u_1 \cdots u_j$. We may suppose that the former L is a 2-cell region and the latter R contains the crosscap Q_2. See Fig. 4, where each u_t is denoted simply by t.

First, we shall show that $\deg u_0 = n < 6$. Assume that the minimum degree of T is $\deg u_0 = 6$ and reselect u_i and u_j so that the 2-cell region L is as wide as possible. The boundary $u_1 \cdots u_i u_j \cdots u_k$ of L is a polygon in T and its chords lying outside L, if any, have ends in $\{u_1, u_i, u_k, u_j\}$. (A *chord* of a cycle is an edge which joins two vertices of the cycle and which does not belong to it.) By Lemma 3, if L contained a vertex inside, it would have degree at most 4, contrary to our assumption on $\deg u_0$. On the other hand, if L is divided by only diagonals, then its boundary cycle will contain a vertex of degree at most 5, except the case when L is a quadrilateral. Thus, we have $i = 2$ and $j = k + 1$ and the quadrilateral L is divided into two triangles with either an edge $u_1 u_j$ or $u_k u_2$.

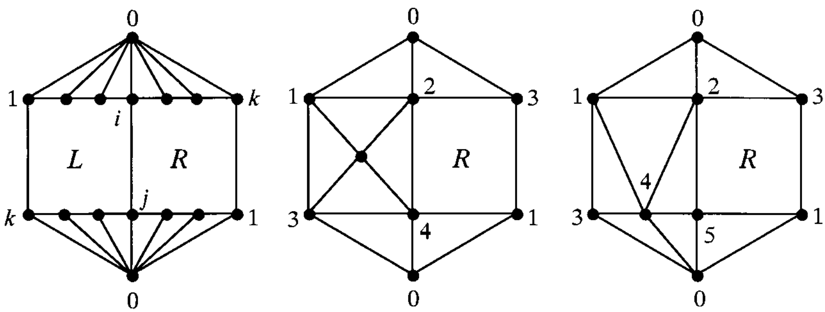


FIG. 4. The Klein bottle cut open along $u_0 u_1 u_k$.

We may suppose the second case up to symmetry. Then $k \geqslant 4$. For there would be multiple edges between u_2 and $u_k = u_3$ otherwise. Consider an essential cycle $u_0u_3u_h$ of length 3 which has to contain u_0u_3. If $h \leqslant k$, that is, if u_h lies on the top of R, then $h = k = 5$, $j = 6$ and $u_3u_h = u_3u_5$ must cross the crosscap Q_2 in R. In this case, u_4 must be incident to an edge u_4u_l which also crosses Q_2 and such that $u_0u_4u_l$ forms an essential cycle of length 3. If u_4u_l joined the top and the bottom of R, then the essential cycle $u_0u_4u_l$ would be an meridian since it crosses each of the disjoint pair of longitudes Q_1 and Q_2. Otherwise, $u_l = u_2$ and $u_2u_4u_5 = u_2u_4u_k$ would be an meridian. In either case, it is contrary to the assumtion of T being not of handle type.

Thus, u_3u_h must join the top and the bottom of R. If it crossed the longitude Q_2, then $u_0u_3u_h$ would be a meridian, a contradiction. So the edge u_3u_h splits R into two regions R' and R'', which are bounded by $u_2u_3u_h \cdots u_j$ and $u_3 \cdots u_ku_1 \cdots u_h$, possibly $u_h = u_1$, One of them includes the crosscap Q_2 and the other is a 2-cell region. If R' is a 2-cell region, then it should be divided with only diagonals by Lemma 3. In this case, if $u_h \neq u_1$, then $L \cup R'$ would be a 2-cell region wider than L, contrary to our choice of u_i and u_j. Otherwise, there would be a diagonal u_3u_l incident to u_3 in R' and the region with boundary $u_1u_2u_3u_l \cdots u_k$ would be a 2-cell one wider than L. On the other hand, if R'' were a 2-cell region, then either u_k or u_1 would have degree less than 6; if there is a diagonal incident to one of u_k and u_1 in R'', then the diagonal disturbs the other to have degree 6.

All of cases with $n = 6$ include contradictions. Therefore, $n \leqslant 5$ and we have more concrete pictures (the middle and right in Fig. 4), with renumbering if needed, by Lemma 3. Then the cycle $u_2u_3u_1u_n$, with $n = 4$ or 5, is an equator of length 4 which separates the two longitudes Q_1 and Q_2. Let S_1 and S_2 be the subsurfaces bounded by $u_2u_3u_1u_n$ at each side in the Klein bottle, containing and not containing u_0, respectively, each of which is homeomorphic to the Möbius band. That is, S_1 is the outside of $u_2u_3u_1u_n$ while S_2 is the rectangular region with a crosscap Q_2 in our picture.

If u_2 has degree 2 in $T \cap S_2$, then there is a face with boundary $u_2u_3u_n$ in S_2 and the cycle $u_3u_1u_n$ is essential and separates the Klein bottle, which concludes that T is of crosscap type. Thus, we may assume symmetrically that not only u_2 but also u_3, u_1 and u_n have degree at least 3 in S_2. We shall show that u_2 and u_1 have a common neighbor inside S_2 and that u_3 and u_n do.

It suffices to discuss it for the pair of u_2 and u_1. First, suppose that u_2 has degree 3 in $T \cap S_2$ with the third neighbor x. Then the essential cycle through u_2x must be u_2xu_1 with the edge u_1u_2 going across S_1, and hence x is adjacent to both u_2 and u_1.

Suppose that both u_2 and u_1 have degree at least 4. If they have no common neighbors in S_2 other than u_3 and u_n, then we can choose two disjoint longitudes of length 3 in S_2 one of which passes through an edge incident to u_2 and the other through one incident to u_1. However, this is not the case since any two orientation-reversing simple closed curves cross each other in the Möbius band. Thus, u_2 and u_1 have a third common neighbors, say x.

Now we turn back to the previous picture of the hexagon and add a path $u_2 x u_1$ of length 2 to the region R with a crosscap Q_2 bounded by $u_2 u_3 u_1 u_n$. Then there is an essential cycle $u_1 u_2 x$ of length 3, which crosses Q_1 once. If this cycle crossed Q_2, then it would be a meridian, contrary to our assumption on T. Thus, $u_1 u_2 x$ must be homotopic to the longitude $u_1 u_0 u_3$. This implies that either (i) $u_2 u_3 u_1 x$ or (ii) $u_2 u_n u_1 x$ bounds a 2-cell and x has to be a common neighbor of u_3 and u_n which we have found above. However, either $u_3 x$ or $u_5 x$, corresponding to (i) or (ii), would be contractible when $\deg u_0 = n = 5$, a contradiction. When $\deg u_0 = 4$, then the cycle $u_3 u_4 x$ must be homotopic to the longitude $u_3 u_0 u_1$ and either (iii) $u_3 u_2 u_4 x$ or (iv) $u_3 u_1 u_4 x$ bounds a 2-cell. Combining cases (i), (ii), (iii), and (iv), we conclude that one of four cycles $u_2 u_3 x$, $u_3 u_1 x$, $u_1 u_4 x$, and $u_4 u_2 x$ of length 3 separates the Klein bottle into two Möbius bands and hence T is of crosscap type. ∎

Negami has already classified the 6-regular triangulations of the Klein bottle, up to equivalence, with two types of their standard forms called the handle type and the crosscap type, in his thesis [8] and also in [7]. From his classification, it follows easily that if any 6-regular triangulation of the Klein bottle is irreducible, then it is of handle type and is equivalent to Kh14 in our classification. Using this fact, we can shorten the above proof slightly.

LEMMA 5. *There exist precisely* 4 *irreducible triangulations of crosscap type, up to equivalence.*

Proof. Let T be an irreducible triangulation on the Klein bottle and C an essential cycle of length 3 in T which is an equator of the Klein bottle. Then T splits naturally into two triangulations of the projective plane, say T_1 and T_2, so that C bounds a face in both T_1 and T_2.

Suppose that T_1 is not irreducible and let ab be any contractible edge of T_1. Since T is irreducible, there is an essential cycle C' of length 3 in T containing ab. If C' is not contained in T_1, then it must be contained in T_2 and ab lines on C. Otherwise, C' is not essential and bounds a 2-cell region including the face which C bounds. Suppose that C' does not coincide with C in addition and that C' is the innermost one among such cycles. Then we can find an edge lying in the region bounded by $C \cup C'$ such that it could not be contained in an essential cycle of length 3 in T on the Klein bottle. Thus, $C' = C$ and hence ab lies on C. This implies that there are at most three contractible edges of T_1, which are contained in C. Moreover we shall show that none of those three edges on C is contractible in T_1.

Let B_1 and B_2 be the left and right triangulations of the projective plane given in Fig. 2, respectively, which are irreducible. Then B_1 is isomorphic to K_6 while B_2 is isomorphic to $K_4 + \overline{K_3}$ as just graphs. Let B' be another trianglation of the projective plane and suppose that either B_1 or B_2, say B, can be obtained from B' by contracting one edge xy.

If $\deg x = 3$, then all the three edges incident to x, including xy, are contractible. If $\deg x = 4$, then the edge xy' not lying on the two triangles bounding faces incident to xy will be contractible. For contraction of xy' yields the same triangulation as that of xy does. If $\deg x \geqslant 5$ and if $\deg y \geqslant 5$, then the vertex obtained as the result of contracting xy has degree at least 6. In this case, B must be B_2 since B_1 is 5-regular, and B' is equivalent to the one given in Fig. 5, up to symmetry. Then B' has five contractible edges, which are marked with $\times$ in the figure.

In all cases, B' has a pair of two contractible edges which cannot be contained together in any face. It follows that any triangulation of the projective plane has such a pair of edges unless it is either B_1 or B_2. Now recall that a contractible edge in T_1, if any, lies on the cycle C bounding a face. This and what we have just concluded imply that T_1 is either B_1 or B_2 and so is T_2.

Consider the symmetry of the two irreducible triangulations B_1 and B_2 to classify the ones obtained by gluing two copies of them. In B_1, any pair of faces with ordered triples of vertices can be transformed into each other by an automorphism. In B_2, any face is incident to three vertices of degrees 4, 6, 6 and any pair of faces can be translated into each other by an automorphism, too, but the correspondence between vertices incident to those faces should preserve their degrees.

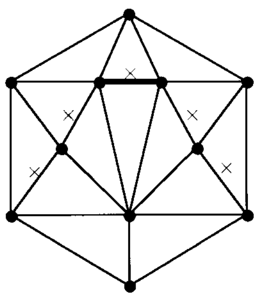


FIG. 5. Splitting a vertex of degree 6 in B_2.

The above observation implies the following. The irreducible triangulation obatined from two B_1's and that from B_1 and B_2 are unique up to equivalence, and there are precisely two equivalence types of irreducible triangulations obatined from two B_2's. These four irreducible triangulations of crosscap type, Kc1 to Kc4, are not equivalent to one another since they have different degree sequences, as shown in Fig. 15. ∎

LEMMA 6. *There exist precisely* 21 *irreducible triangulations of handle type, up to equivalence.*

Proof. Let T be an irreducible triangulation on the Klein bottle and abc a cycle of length 3 in T which is a meridian of the Klein bottle. Cut the Klein bottle along abc into an annulus and put the annulus on the plane. Then the cycle abc splits into the outermost cycle $a'b'c'$ and the innermost one $a''b''c''$ in the triangulation of this annulus. Add two extra vertices x and y temporarily and join them to $a'b'c'$ and $a''b''c''$, respectively. By Menger's theorem, there are three inner disjoint paths joining x and y since any triangulation is 3-connected.

Let P_0, P_1, P_2 be the segments of these paths across the annulus. We may suppose that P_0 joins a' to a'', P_1 joins b' to c'' and P_2 joins c' to b'' after relabeling if necessary. Thus, P_0 is a longitude and $P_1 \cup P_2$ is an equator in the Klein bottle. In this situation, the Klein bottle separates into three rectangular regions bounded by $P_0 \cup P_1 \cup a'b' \cup a''c''$, $P_1 \cup P_2 \cup b'c' \cup c''b''$, and $P_2 \cup P_0 \cup c'a' \cup b''a''$, and denoted by R_{01}, R_{12}, and R_{02}, respectively. Futhermore, we choose these paths P_0, P_1, P_2 with the meridian abc so as to minimize their lengths $|P_i|$.

Paying attention to the meridian abc, the longitude P_0 and the equator $P_1 \cup P_2$, we shall find and classify the standard forms of irreducible triangulations of the Klein bottle in the following five steps. The headings of those steps will show their local targets. The local conclusion of Step 1 is that $|P_0| = 3$ and $|P_1|, |P_2| \leqslant 3$. Figures 6, 7, and 8 present those in Steps 2, 3 and 4.

Step 1. *Bounding* $|P_0|, |P_1|, |P_2|$. First, we shall show that P_0 has length 3. Let $a' = x_0, x_1, x_2, ..., x_n = a''$ be the vertices lying along P_0 in this order, with $n = |P_0|$. Then x_i and x_j are not adjacent for each i and j with $i+2 \leqslant j$ by our choice of P_0. Suppose that $n \geqslant 4$. To put each edge $x_i x_{i+1}$ on an essential cycle of length 3 with $1 \leqslant i \leqslant n-2$, both x_i and x_{i+1} must be adjacent to one of b or c. If $x_1 x_2 b$ and $x_2 x_3 b$ are such essential cycles, then $x_1 x_2 x_3 b'$ (or $x_1 x_2 x_3 b''$) forms a cycle of length 4 which bounds a quadrilateral region within R_{01} (or R_{02}) and an edge $x_2 b''$ (or $x_2 b'$) lies in R_{02} (or R_{01}). By Lemma 3, this quadrilateral has to contain either a diagonal or a vertex of degree 4 adjacent to x_1, x_2, x_3 and b'. The first case is however contrary to the simpleness of T or the minimality of $|P_0|$ while

x_1 and x_3 would be adjacent in the second case, a contradiction. Thus, we may assume that there exist edges $x_1 b'$ and $x_3 c''$ in R_{01} and edges $x_2 b''$ and $x_2 c'$ in R_{02}, up to symmetry. In this case, no essential cycle of length 3 containing $a' x_1$ can be found under our assumption. Therefore, $|P_0| = n = 3$.

Remark. Our arguments in the previous paragraph works to conclude the following: Let P_0, P_1, and P_2 be three disjoint paths joining $\{a', b', c'\}$ to $\{a'', b'', c''\}$ in the annulus with boundary cycles $a'b'c'$ and $a''b''c''$. If P_0 forms a longitude and if $P_1 \cup P_2$ forms an equator in the Klein bottle, then there is a path P'_0 of length 3 which has the same ends as P_0 and is disjoint from $P_1 \cup P_2$.

Second, we shall show that $|P_1|$ and $|P_2|$ can be assumed to be less than 4. Let $b' = y_0, y_1, ..., y_m = c''$ be the vertices along P_1 and suppose that $|P_1| = m \geqslant 4$. To illustrate our arguments, cut open the Klein bottle into a rectangle by the meridian abc and the longitude P_0 and put the rectangle so that the vertical pair of sides correspond to the meridian and that the left side is labeled with $abca$ (or $a'b'c'a'$) and the right side with $acba$ (or $a''c''b''a''$) downwards. Thus, the two copies of $P_0 = a' x_1 x_2 a''$ are placed on the top and the bottom of this rectrangle.

For each edge $y_i y_{i+1}$ with $1 \leqslant i \leqslant m-2$, we have the following three possibilities on the essentail cycle of length 3 which contains $y_i y_{i+1}$:

(a) There is a path $a' y_i y_{i+1} a''$ of length 3 in R_{01} joining the two ends of P_0.

(b) $i = 1$ and there is an edge $y_{i+1} b'' (= y_2 b'')$ in R_{12}.

(c) $i = m-2$ and there is an edge $y_i c' (= y_{m-2} c')$ in R_{12}.

It is obvious that the same case does not happen for consequtive two edgs $y_i y_{i+1}$ and $y_{i+1} y_{i+2}$. It follows that $|P_1| = m = 4$ and we may assume that either (b) and (c), or (b) and (a) happen for $y_1 y_2$ and $y_2 y_3$, up to symmetry.

Let $c' = z_0, z_1, ..., z_l = b''$ be the vertices on P_2 and suppose the first case. That is, the middle point y_2 of P_1 is adjacent to c' and b'' in R_{12}. In this case, we can conclude that $|P_2| = 2$ or 3 since the corresponding cases to (b) and (c) cannot happen for P_2. If $|P_2| = 2$, then the quadrilateral region $c' y_2 b'' z_1$ in R_{01} must be divided by the diagonal $y_2 z_1$. The edge $y_2 z_1$ may be assumed to be contained in an essential cycle given as a path $x_j y_2 z_1 x_j$ for some $j \in \{0, 1, 2\}$ or $a' y_2 z_1 a''$. In the former case, we can use the meridian $x_j y_2 z_1 x_j$ instead of abc. Cut the rectangle along this new meridian and paste the left and right sides. (One of the two halves should be turned over.) Then the new P_0, P_1 and P_2 will have length 3. In the latter case, consider edges incident to z_1 in R_{02} different from $z_1 y_2$, $z_1 c'$, $z_1 b''$ and $z_1 a''$. Then, one of them will be contained in an essential cycle given as $x_j y_2 z_1 x_j$

and we can reselect the meridian abc and paths P_0, P_1 and P_2 so that their lengths are equal to 3, as well as in the first case.

Suppose that $|P_2| = l = 3$ with (b) and (c). Then the edge $z_1 z_2$ must be contained in an essencial cycle $az_1 z_2$ given as $a'z_1 z_2 a''$ in R_{02} and the edges $c'z_1$ and $b''z_2$ must be contained in essentail cycles $c'z_1 x_j c''$ and $b''z_2 x_j b'$, respectively, for some $j \in \{1, 2\}$ in common. In this case, we can choose $a'z_1 z_2 a''$ (in R_{02}), $c'y_2 b''$ (in R_{12}) and $b'x_j c''$ (in R_{01}) as new paths P_0, P_1 and P_2 respectively, each of which has length at most 3.

Now suppose that (b) and (a) happen. That is, there exist edges $a'y_2$, $y_3 a''$ in R_{01} and $y_2 b''$ in R_{12}. In this situation, the essential cycle of length 3 containing $c''y_3$ must be given as $c''y_3 x_j c'$ for some $j \in \{1, 2\}$ with $y_3 x_j$ in R_{01} and $x_j c'$ in R_{02}. Then there exist two paths $P'_1 = a'y_2 y_3 c''$ and $P'_2 = c'x_1 x_2 a''$ if $j = 1$ (or $= c'x_2 a''$ if $j = 2$). Consider a path between b' and b'' in the region bounded by $P_2 \cup b''y_2 y_1 b'c'$. If any such path passes through either y_2 or c', then $\{y_2, c'\}$ will be a cut of the region and there will be the edge $y_2 c'$, which reduces to the previous case. Otherwise, we can find a path P'_0 of length 3 between b' and b'' in R_{12}, disjoint from $P'_1 \cup P'_2$. Thus, we use P'_0, P'_1 and P'_2 instead of P_0, P_1 and P_2 since $|P'_i| \leqslant 3$.

Step 2. *Recognizing the inside of* R_{01} (Fig. 6). Now we shall determine the partial structures inside each of the rectangles R_{01}, R_{12}, and R_{02}. We may assume that the triple $\{P_0, P_1, P_2\}$ satisfies the following conditions:

(I) $|P_0| = 3$.

(II) $|P_1| \leqslant 3$, $|P_2| \leqslant 3$.

(III) $|P_1| + |P_2|$ is the smallest among those triples with (I) and (II).

(IV) The number of faces of T contained in $R_{01} \cup R_{02}$ is the smallest among those with (I), (II) and (III).

Consider R_{01}, assuming that $|P_1| = 3$. First suppose that there are two disjoint paths W_1 and W_2 joining x_i to y_i for $i = 1, 2$ inside R_{01}. Then the cycle passing through W_1 and going along $y_1 y_2 c'' a'' x_2 x_1$ will be a polygon. Since y_2 is not adjacent to any vertex, except y_1 and c'', on the polygon outside it, W_2 has to be an edge $x_2 y_2$ by Lemma 3. Similarly, such a polygon through W_2 forces W_1 to be an edge $x_1 y_1$. So we have the structure given as R_{01}-0 in Fig. 6 in this case. By Lemma 3, each of three rectangular regions in R_{01}-0 contains no vertex and is divided into two triangles by a diagonal.

Add two extra vertices x and y temporarily so that x is adjacent to a', x_1, x_2, a'' and y to b', y_1, y_2, b''. By Menger's theorem, if there are not four inner disjoint paths between x and y, including $xa'b'y$ and $xa''c''y$, then there is a cut with (at most) three vertices which separates x and y.

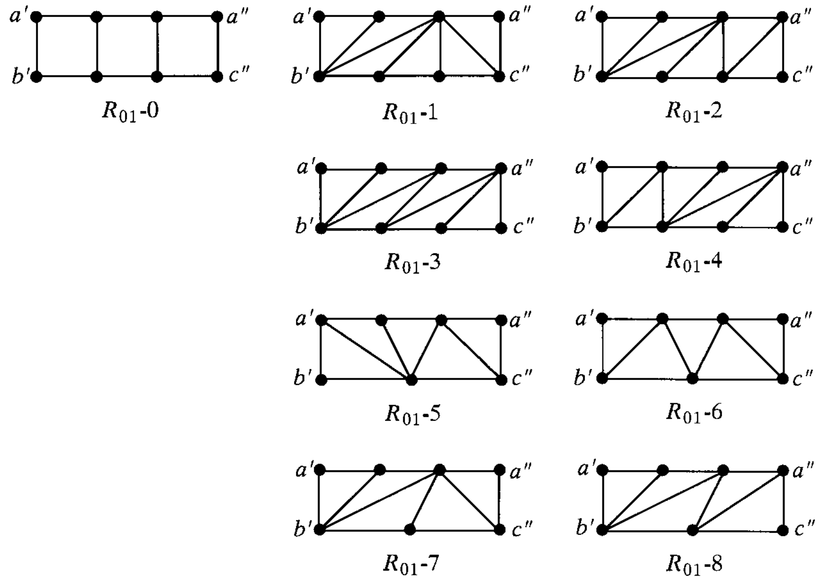


FIG. 6. Structures inside R_{01}.

Such a cut may be assumed to be contained in either a path $b'wa''$ or $b'wc''$, up to symmetry, where w is a vertex in R_{01}. In the second case, if w lies inside R_{01}, not on its boundary, then P_1 should be replaced with the path $b'wc''$ of length 2 by the assumption (III) for $\{P_0, P_1, P_2\}$. On the other hand, if w lies on the boundary of R_{01}, then w must be x_1 or x_2 and it reduces to the first case.

When there is a path $b'wa''$, then we have three cases: (a) w is inside R_{01}, (b) $w = x_2$, or (c) $w = y_1$.

In case (a), consider any edge ws incident to w in the region bounded by the polygon $b'wa''c''y_2y_1$. By Lemma 3, ws should be a diagonal, that is, s should be one of y_1, y_2, and c''. If $s = y_2$ or $s = c''$, then it is contrary to the assumption (IV) or (III) for $\{P_0, P_1, P_2\}$. If $s = y_1$, then the essential cycle of length 3 including $ws = wy_1$ has to be wsc which is presented as $c''wsc'$ and there is a path $b'wc''$ of length 2, contrary to (III) again.

Suppose case (b). Then $a'b'x_2x_1$ and $b'x_2a''c''y_2y_1$ will be polygons in T. If the polygon $a'b'x_2x_1$ contains a vertex v, then v is adjacent to all of a', b', x_2, and x_1 and there is an edge x_1b'' in R_{02}. In this case, we find two paths $P_1' = b'x_2a''$ and $P_2' = a'x_1b''$ of length 2 and a path P_0' joining c' to c'' in R_{12}, through neither b' nor b''. (If any path between c' and c'' in R_{12} passed through b' or b'', then $\{b', b''\}$ would be a cut of R_{12} and T would include a self-loop $b'b''$.) By the previous remark on $|P_0|$, we can assume that $|P_0'| = 3$. Thus, $\{P_0', P_1', P_2'\}$ can be used as a new triple, after relabeling on $\{a, b, c\}$, with $|P'_1| = 2$ and $|P'_2| = 2$. This is contrary to the assumption (III) for $\{P_0, P_1, P_2\}$. Therefore, the polygon $a'b'x_2x_1$ has to be divided by only the diagonal $b'x_1$.

Consider the other polygon $b'x_2a''c''y_2y_1$ and suppose that there is a vertex v inside it. Since v has degree at least 4, it is adjacent to at least two of the vertices on P_1, say y_i and $y_j (i < j)$. If $j - i \geqslant 2$, then there would be a path between b' and c'' which passes through v and which has length 3 or 2, contrary to (IV) or (III). Otherwise, v has degree 4 and $j = i + 1$. In this case, y_i must be adjacent to a'' and y_{i+1} to x_2 by Lemma 3, but these two cannot hold together, a contradiction. Thus, the polygon $b'x_2a''c''y_2y_1$ contains no vertex and is divided by three diagonals as given as R_{01}-1, R_{01}-2, and R_{01}-3 in Fig. 6.

Suppose case (c) in turn. Then $y_1 y_2 c'' a''$ will be a polygon in T and it has to have the diagonal $a''y_2$ by Lemma 3. For $y_1 y_2 c'' a''$ admits no chord outside. On the other hand, $a'b'y_1a''x_2x_1$ is not a polygon. To find a suitable polygon, consider the triangular face $a''y_1w$ incident to $a''y_1$ in the region bounded by $a'b'y_1a''x_2x_1$. If w did not lie on its boundary, then wy_1 would not be contained in any essential cycle of length 3, contrary to T being irreducible. So w has to coincide with x_2 and we have the polygon $a'b'y_1x_2x_1$; otherwise, the simpleness of T would be broken. If there is a vertex v inside the polygon, v is adjacent to a', x_1, x_2, and b'' and there is an edge x_1b'' in R_{02}. In this case, we can find another triple $\{P''_0, P''_1, P''_2\}$ such that $P''_1 = a'x_1b''$, $P''_2 = b'y_1a''$, and P''_0 joins c' to c'', missing b', y_1, and b''. (If such a P''_0 could not exist, then there would be either a self-loop $b'b''$ or a pair of multiple edges $b'y_1$ and $b''y_1$.) This is contrary to (III). Therefore, the region bounded by $a'b'y_1x_2x_1$ has to be divided by two diagonals and we have R_{01}-3 and R_{01}-4 in case (c).

Now assume that $|P_1| = 2$ and hence $P_1 = b'y_1c''$. Consider inner disjoint paths between y_1 and an extra vertex x adjacent to a', x_1, x_2, and a''. If there are four such paths, then two of them contain two inner disjoint paths W_1 and W_2 which join x_1 and x_2 to y_1 in R_{01}. Choose W_1 and W_2 so as to minimize their length. Then the cycle $W_2 \cup y_1b'a'x_1x_2$ will be a polygon in T. Since there is no chord incident to y_1 outside the polygon, W_1 has to have length 1, that is, $W_1 = x_1 y_1$ by Lemma 3. Similarly, $W_2 = x_2 y_1$ and each of two rectangular regions bounded by $a'b'y_1x_1$ and $a''c''y_1x_2$ contains only one diagonal by Lemma 3 again. Thus, we have R_{01}-5 and R_{01}-6 up to symmetry, in this case.

If there are not four inner disjoint paths between y_1 and x, then we may assume that there is either a path $b'wa''$ or $b'wc''$ of length 2, up to symmetry, whose vertices form a cut of R_{01}. In the latter case, w has to be either x_1 or x_2 by the assumption (IV) and this reduces to the former case. So suppose that there is a path $b'wa''$ in R_{01}. Then a cycle $b'wa''c''y_1$ will be a polygon in T. If w lies inside R_{01}, then any edge inside $b'wa''c''y_1$ cannot be incident to w by Lemma 3. This requires the diagonal $b'a''$ of the polygon, which breaks the simpleness of T, a contradiction. Thus, $w = x_2$.

If there is a vertex v inside the polygon $b'x_2a''c''y_1$, then v has to be adjacent to b', x_2, a'', and c'' by Lemma 3. In this case, there is a path $b'vc''$ of length 2, contrary to the assumption (IV). Thus, the inside of the polygon $b'x_2a''c''y_1$ should be divided by two diagonals and we have R_{01}-7 and R_{01}-8 after seeing that the quadrilateral region bounded by $a'x_1x_2b'$ contains no vertex. If the polygon $a'x_1x_2b'$ contains a vertex v inside, then v is adjacent to all of a', x_1, x_2, and b' and there is an edge x_1b'' in R_{02}. In this case, the diagonal x_2y_1 cannot lie on any essential cycle of length 3, a contradiction.

Step 3. *Recognizing the inside of* R_{12} (Fig. 7). Now we shall consider R_{12} in turn. First suppose that $|P_1| = |P_2| = 3$ and add two extra vertices y and z adjacent to $\{b', y_1, y_2, c''\}$ and $\{c', z_1, z_2, b''\}$, respectively. In this case, the existence of a cut with three vertices will be contrary to either the assumtion (III) for $\{P_0, P_1, P_2\}$ or the simpleness or the triangulation. So there exist four inner disjoint paths between y and z, including $yb'c'z$ and $yc''b''z$, by Menger's theorem. Let W_1 and W_2 be the segments of the other two in R_{12}, which join y_1 to z_1 and y_2 to z_2, respectively. Then $W_1 \cup z_1z_2b''c''y_2y_1$ and $W_2 \cup z_2z_1c'b'y_1y_2$ will be polygons in T. These polygons force W_1 and W_2 to be edges by Lemma 3 and we have R_{12}-0 in Fig. 7. Each square in R_{12}-0 should contain one diagonal.

When $|P_1| = 3$ and $|P_2| = 2$, we consider four inner disjoint paths between y and z_1 with Menger's theorem again. If there is a cut of three vertices separating y and z_1, then those vertices form a path $c'wb''$ of length 2 and w is inside R_{12}. The polygon $c'wb''z_1$, however, adimts neither (i) nor (ii) in Lemma 3. Thus, there are two inner disjoint paths W_1 and W_2 joining y_1 and y_2 to z_1 in R_{12} and each of them has to be a single edges by the argument similar to the previous paragraph. The regions bounded

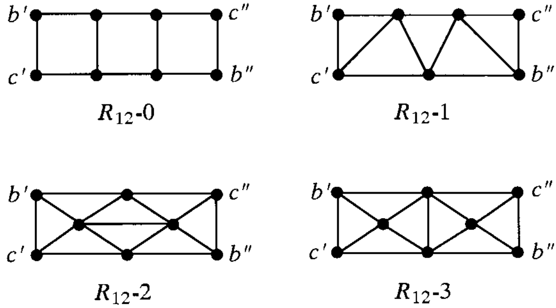


FIG. 7. Structures inside R_{12}.

by $b'c'z_1y_1$ and $c''b''z_1y_2$ should contain the diagonals y_1c' and y_2b'' by the simmpleness of T, and hence, we have R_{12}-1 in this case.

Finally suppose that $|P_1|=|P_2|=2$, and hence, $P_1=b'y_1c''$ and $P_2=c'z_1b''$. Then there exist three inner disjoint paths between y_1 and z_1, including $y_1b'c'z_1$ and $y_1c''b''z_1$. Let $W=w_0w_1...w_r$ be the third path between $y_1=w_0$ and $z_1=w_r$ across R_{12}.

Suppose that at least one of two polygons $W\cup z_1c'b'y_1$ and $W\cup z_1b''c''y_1$, say the former, contains a vertex v inside. Then v has to be adjacent to b', c', and two w_i's, say w_j and w_k with $0\leqslant j<k\leqslant r$, which are adjacent to c'' and b'', respectively. If $j>0$, then the path $w_0w_1\cdots w_j$ should be a diagonal inside the polygon $b'vw_jc''w_0$ and $w_j=w_1$ has to be adjacent to b'' by Lemma 3. By the simpleness of T, the polygon $b'vw_1w_0$ contains the diagonal $vw_0=vy_1$. So we may assume that $j=0$. Similarly, $k=r$ and there is an edge $vw_r=vz_1$ in R_{12}. Now we have the polygon $vz_1b''c''y_1$ and W has length at most 2 by Lemma 3. If $|W|=2$, then w_1 is adjacent to y_1, z_1, b'', and c'', and the polygon $vz_1w_1y_1$ contains either the diagonal vw_1 or y_1z_1. If $|W|=1$, then the polygon $y_1z_1b''c''$ contains a vertex of degree 4. Thus, we have R_{12}-2 and R_{12}-3 in this case.

Suppose that neither the polygon $W\cup z_1c'b'y_1$ nor $W\cup z_1b''c''y_1$ contains any vertex inside. By Lemma 3, only diagonals divide their regions into triangles. We reselect here W so as to minimize its length and hence there is no such diagonal w_iw_j joining two vertices on W. Consider the triangular face $b'c'w_i$ incident to $b'c'$ inside $W\cup z_1c'b'y_1$. Then there is an edge incident to w_i inside $W\cup z_1b''c''y_1$, which joins w_i to b'' or c''. In either case, there would be multiple edges incident to w_i in T, a contradiction. Thus, this is not the case.

Step 4. *Composing partial structures in triangulations* (Fig. 8). Now we have prepared the parts to construct the irreducible triangulations of the Klein bottle. Let R_{02}-i be the picture obtained from the vertical reflexion of R_{01}-i $(0\leqslant i\leqslant 8)$ by relabeling b' and c'' with c' and b'', respectively. Let R_{02}-$(-i)$ be the horizontal reflexion of R_{02}-i. From our arguments above, each irreducible triangulation of the Klein bottle can be constructed as the union of R_{01}-i, R_{12}-j, and R_{02}-k with some diagonals added in the quadrilateral regions $(0\leqslant i\leqslant 8, 0\leqslant j\leqslant 3, -8\leqslant k\leqslant 8)$. Let $[i, j, k]$ denote such a configuration. It remains to classify those configulations $[i, j, k]$ with tedious routine. Here we shall show only our guide line to complete the list of Figs. 13 and 14.

First omit ones which give nonsimple graphs and ones which can be recognized immediately to be equivalent to another up to symmetry. Then we have:

(i) $[0,0,0]$, $[0,1,-5]$, $[0,1,6]$, $[1,0,0]$, $[1,1,-5]$, $[2,0,0]$, $[5,2,-5]$, $[5,3,-5]$, $[5,3,6]$, $[6,2,6]$, $[6,3,6]$

(ii) $[2, 0, 2]$, $[2, 0, 3]$, $[2, 0, 4]$, $[3, 0, 4]$

(iii) $[3, 0, 0]$, $[4, 0, 0]$

(iv) $[0, 1, 1]$, $[0, 1, 8]$, $[1, 0, 4]$, $[2, 1, -5]$, $[2, 1, 8]$, $[3, 1, -5]$, $[3, 1, 8]$, $[4, 1, -5]$, $[4, 1, 6]$, $[4, 1, 7]$, $[4, 1, 8]$

(v) $[5, 3, 5]$, $[5, 3, -7]$, $[5, 2, -8]$, $[5, 3, -8]$, $[8, 2, 8]$, $[8, 3, 8]$

(vi) $[4, 0, -4]$, $[4, 1, 5]$

(vii) $[5, 2, 5]$, $[5, 2, 6]$, $[5, 2, -7]$.

Each member of the first group (i) remains in Figs. 13 and 14 and has one of the six partial structures PS1 to PS6 given in Fig. 8 unless it is $[1, 1, -5]$ which is equivalent to Kh3 in Fig. 13.

Trying to find those structures in the others, we can omit groups (ii) to (vii). Each member of group (ii) includes three disjoint meridians of length 3 and reduces to $[0, 0, 0]$. Group (iii) consists of ones which include PS2 or PS3 and reduce to $[1, 0, 0]$ or $[2, 0, 0]$ while (iv) contains ones which include PS4 and reduce to $[0, 1, -5]$ or $[0, 1, 6]$. The members of group (v) include PS5 or PS6 and can be recognized to be equivalent to ones given as $[5, x, -5]$, $[5, x, 6]$, or $[6, x, 6]$ with $x = 2, 3$. The configuration $[4, 0, -4]$ in group (vi) should be omitted since it is contrary to the assumption (III) for the triple $\{P_0, P_1, P_2\}$ after adding diagonals to be an irreducible triangulation. On the other hand, $[4, 1, 5]$ should be omitted since it is equivalent to $[1, 1, -5]$. Group (vii) includes only ones which are not irreducible.

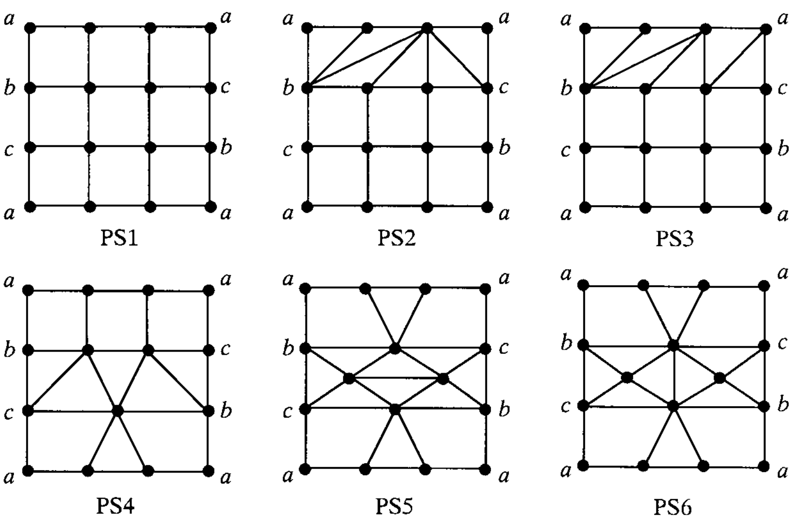


FIG. 8. Partial structures of irreducible triangulations.

Step 5. *Classifying triangulations up to equivalence*. There remains only a tedious routine. Add diagonals in each quadrilateral regions and classify them up to equivalence. Then we obtain the irreducible triangulations of handle type, Kh1 to Kh21, given in Figs. 13 and 14. Most of them will be distinguished from one another by their degree sequences. ∎

A given irreducible triangulation of handle type might be given by a picture which can be found in neither Figs. 13, 14, nor 15. To recognize its standard form in the three lists, find a suitable meridian of length 3 and the triple $\{P_0, P_1, P_2\}$ associated with the meridian so that they satisfy the conditions (I) to (IV) in Step 2 of the above proof.

4. OBSERVATIONS

In this section, we shall show some observations on the irreducible triangulations of the Klein bottle, Kh1 to Kh21 and Kc1 to Kc4. Recall that we have classified them up to equivalence.

In fact, our list includes a pair of inequivalent triangulations which are isomorphic as graphs, namely Kh2 and Kh5. Their degree sequences, (7, 7, 7, 6, 6, 5, 5, 5), suggest an isomorphism between them. Any isomorphism carries three vertices of degree 7 in Kh2 onto those in Kh5, but it does not extend to any homeomorphism over the Klein bottle. For, those vertices form a triangle bounding a face in Kh2 while they form a longitude of length 3. That is the reason why Kh2 and Kh5 are not equivalent as triangulations on the Klein bottle.

Also, our list includes one toroidal graph, which is Kh1, and the others cannot be embedded on the torus, as shown in Theorem 10. To prove this, we need some criteria, Lemmas 7, 8, and 9, to decide whether an irreducible triangulation of the Klein bottle is embeddable in the torus or not. Recall that a triangulation is irreducible if and only if each of its edges is contained in at least three cycles of length 3. This condition is purely combinatorial and hence if an irreducible triangulation of the Klein bottle is embeddable in the torus, then such an embedding on the torus gives an irreducible triangulation of the torus, too.

Let G be a graph and H a subgraph in G. A *bridge* for H in G is a subgraph induced by one of components of $G - H$, say B, and those edges joining B to H. An edge also is called a bridge if it does not belong to H and if its two ends do, and is said to be *singular* in particular. A singular bridge is often called a *chord* of H. Note that any two distinct bridges contain no edge in common and that if they meet each other, then their intersection consists of only some vertices of H. Thus, H and bridges for H give an edge decomposition of G.

LEMMA 7. *Let G be an irreducible triangulation on the torus and K an induced subgraph of G with at most 4 vertices. Then either $G-K$ is connected, or $G-K$ has exactly two components and one of them consists of only one vertex having degree 4 in G.*

Proof. Let abc be the boundary cycle of any face of G with vertices a, b, and c. Suppose that two edges ab and bc belong to two distinct bridges for K, say B_i and B_j, respectively. Then the vertex b and ac must belong to K since b belongs to both B_i and B_j and ac joins two vertices in the different bridges. It follows that B_i and B_j are singular bridges consisting of ab and bc, respectively, which is contrary to K being an induced subgraph in G. Thus, we can assign a unique label x to each face of G so that $x=i$ if its boundary contains an edge of a bridge B_i and that $x=0$ if the three edges on its boundary belong to K. This implies that the number of bridges for K coincides with the number of nontriangular faces of K on the torus; any triangular face of K does not contain any vertex of G since G is irreducible.
On the other hand, it is easy to observe that K has at most one nontriangular face on the torus unless K is isomorphic to K_4. If K has two nontriangular faces, then K is equivalent to the unique 2-cell embedding of K_4 on the torus with one quadrilateral face and one octagonal face. The quadrilateral one must contain precisely one vertex, which has degree 4 in G, by Lemma 3. So the lemma follows. ∎

Here we shall say that G *splits* into H and K if both H and K are disjoint induced subgraphs in G with $V(G)=V(H)\cup V(K)$ and if each vertex is incident to an edge belonging to neither H nor K.

LEMMA 8. *Let G be an irreducible triangulation on the torus which splits into K_5 and K_3. Then G is isomorphic to* Kh1.

Proof. Suppose that G splits into H and a triangle Δ and that H is isomorphic to K_5. Then H has precisely four triangular faces and one octagonal face on the torus since H has only one bridge and is contained in the irreducible triangulation G. Such an embedding of H can be given as in Fig. 9. (Identify each pair of parallel sides of the rectangle to get the torus.) Put Δ inside the octagonal face and add edges to triangulate the torus. Then only one of the resulting triangulations is irreducible and is given as in Fig. 10. The graph is isomorpic to Kh1. ∎

LEMMA 9. *Let G be a triangulation on the torus which splits into H and K_3. If H can be obtained from $K_{3,3}$ by adding three edges, then G is not irreducible.*

Proof. Let Δ be the subgraph isomorphic to K_3 with vertices x, y, and z. We call the three edges added to $K_{3,3}$ the chords of H here. By

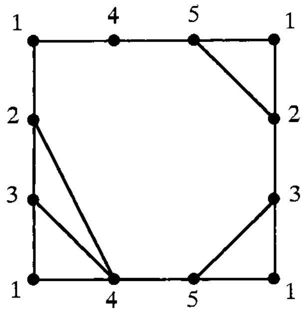


FIG. 9. An embedding of K_5 with one octagonal face.

Euler's formula, H has precisely five triangular faces and one nonagonal face which should contain Δ.

It is well known that $K_{3,3}$ has precisely two inequivalent embeddings on the torus, shown in Fig. 11. To construct the embedding of H, we have to add three chords to these embeddings. However, the left one cannot be used to do it since only three chords cannot divide two of three hexagonal regions into triangular faces. On the other hand, two of three chords of H should be added to the two quadrilateral regions in the right embedding of $K_{3,3}$ and the other one cuts off a triangle from the decagonal region, which looks like the union of two hexagons. There are two ways to cut off the triangle from it, up to symmetry. Add a chord 23 (or bc). Then we have the nonagon $32c1a2b1c$ (or $bc1a2b1c3$). If bc cuts off the triangle $bc3$, then $a3$ will be a contractible edge in G.

Put $\Delta = xyz$ inside the nonagon, which contains all the neighbors of x, y and z. Let u, v and w be the three vertices on the nonagon such that there are triangular faces xyu, yzv and zxw. Consider an edge joining a to Δ, say ax. If a coincides with neither u nor w, then ax will be contractible since

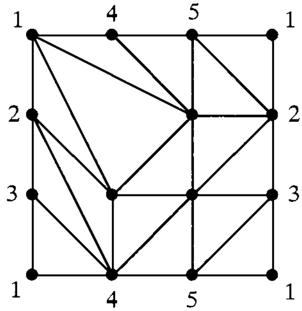


FIG. 10. The toroidal triangulation isomorphic to Kh1.

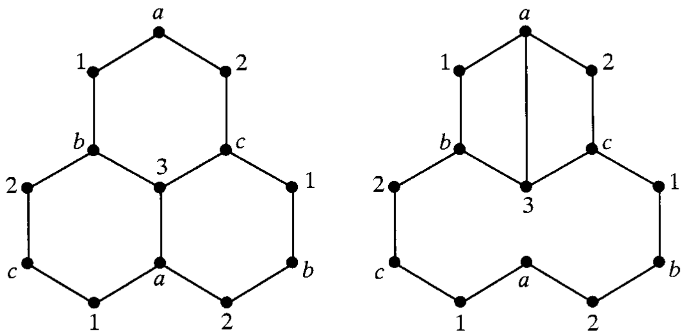


FIG. 11. Two inequivalent embeddings of $K_{3,3}$ on the torus.

a appears only once on the nonagon. Similarly, if none of edges incident to a, 3 and b (or 2) is contractible, then $\{a, 3, b\}$ (or $\{a, 3, 2\}$) will coincide with $\{u, v, w\}$. Adding edges between Δ and the nonagon, we will obtain the pictures in Fig. 12. In either case, Δ includes a contractible edge. ∎

THEOREM 10. *If an irreducible triangulation of the Klein bottle can be embedded in the torus, then it is equivalent to* Kh1.

Proof. Let G be an irrducible triangulation of the Klein bottle. First suppose that G is of crosscap type. Then G can be obtained as a union of two irreducible triangulations B' and B'' of the projective plane. If $G = B' \cup B''$ were embedded in the torus, then B' would be contained in a face of B'', which is contrary to the nonplanarity of B' and B''.

Suppose that G is of handle type. If G is equivalent to one of Kh1 to Kh6, then the middle part R_{12} of each figure induces K_5 and, hence, G splits into K_5 and K_3. By Lemma 8, if G is embeddable in the torus, then G is equivalent to Kh1 since Kh1 is not isomorphic to any other. In case of Kh7 to Kh11, the boundary of R_{12} contains only four vertices of G. If K is the subgraph induced by the four vertices in G, then $G - K$ has two or three components and does not satisfy the condition in Lemma 7. Thus,

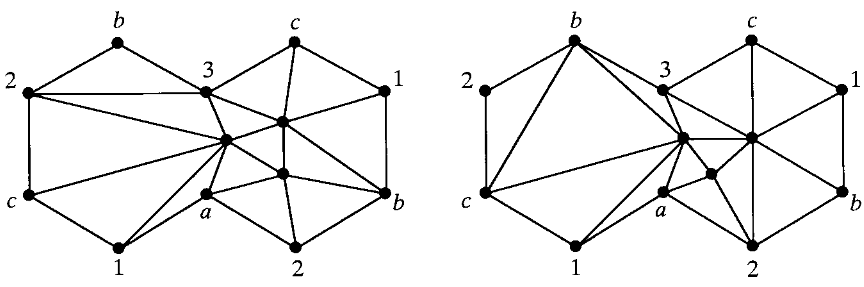


FIG. 12. Dividing the decagonal region.

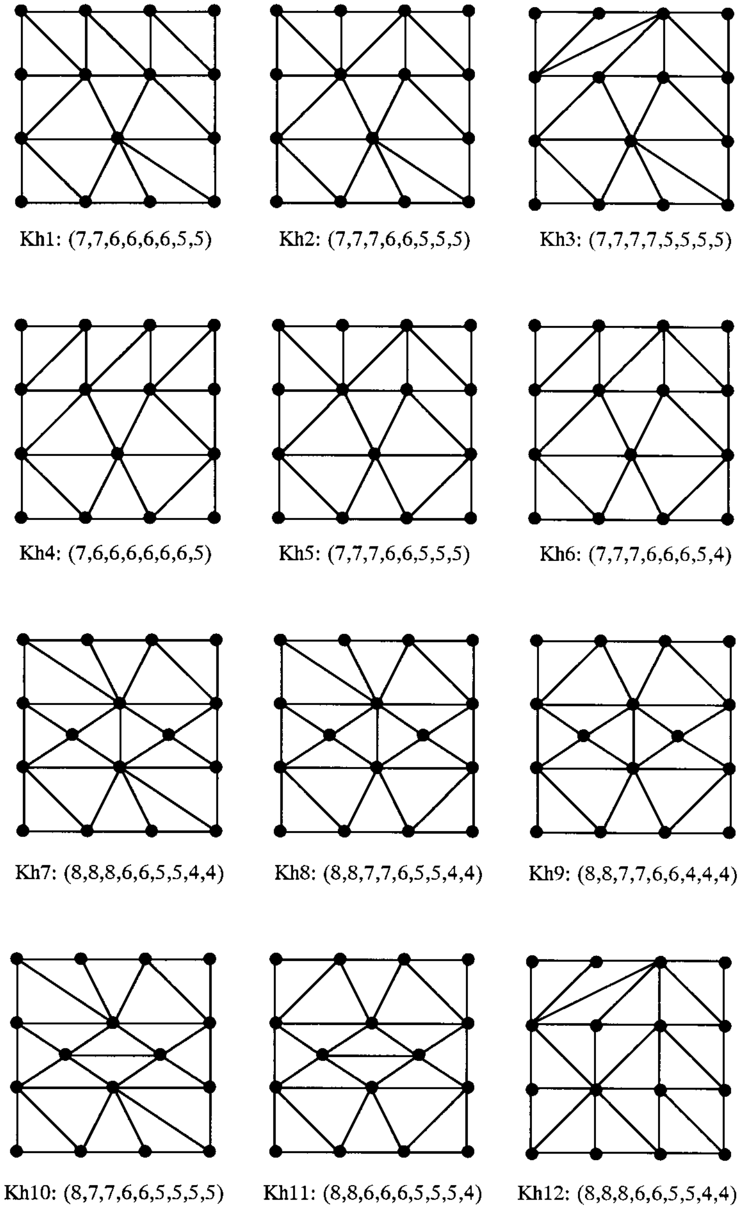


FIG. 13. Irreducible triangulations of the Klein bottle, No. 1.

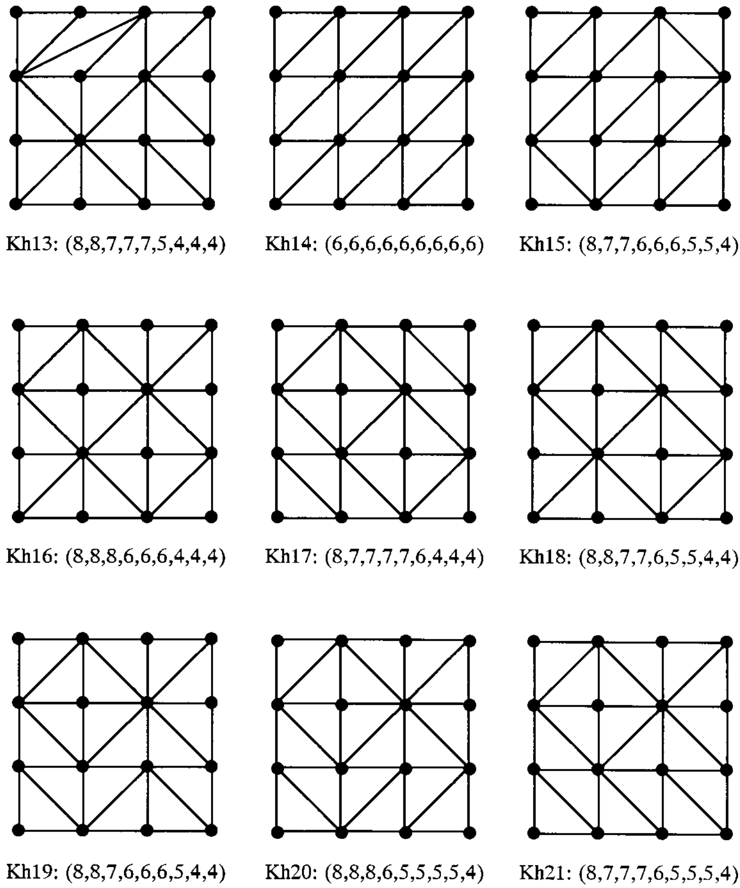


FIG. 14. Irreducible triangulations of the Klein bottle, No. 2.

G is not toroidal in this case. When G is one of Kh12 to Kh21, then R_{12} induces $K_{3,3}$ with three chords added. So we can apply Lemma 9 to this case and conclude that G is not toroidal since it is irreducible. ∎

One might expect that the irreducible triangulations of the Klein bottle work for the initial step of an induction to prove some fact on general triangulations of the Klein bottle. For example, we can find many cycles with suitable properties in them, which can be used to show the existence of such cycles in general cases. Every irreducible triangulation of the Klein bottle includes:

- A disjoint pair of longitudes and a meridian which crosses each of the longitudes only once.

- A meridian and an equator which cross each other at precisely two vertices.
- A hamilton cycle which is trivial on the Klein bottle.
- A hamilton cycle which is a meridian.
- A hamilton cycle which is a longitude.
- A hamilton cycle which is an equator.

Splitting of vertices stretches cycles in a triangulation, preserving their topological property on the surface. Thus, the following theorem is an immediate consequence from the above observations:

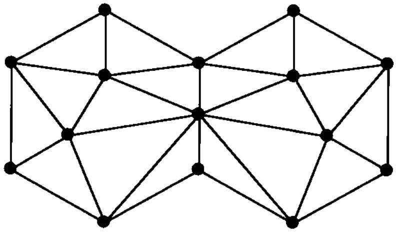
THEOREM 11. *Every triangulation of the Klein bottle includes a meridian, a longitude and an equator as its cycles.*

Recently, Brunet, Nakamoto, and Negami [3] have shown that every 5-connected triangulation of the Klein bottle includes a hamilton cycle which is trivial. This does not follow directly from our observations since the hamiltonicity of a cycle is not preserved by vertex splitting in general. However, their proof is based on the fact that every irreducible triangulation of handle type includes two disjoint meridians. More generally, we can show the following characterization for those triangulations:

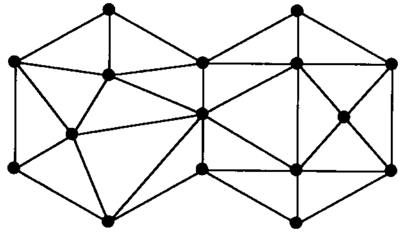
THEOREM 12. *A triangulation of the Klein bottle includes two disjoint meridians if and only if it does not include an equator of length* 3.

Proof. The necessity is clear. Let G be a triangulation of the Klein bottle which does not contain any equator as its cycle of length 3. It is obvious that any vertex splitting preserves two disjoint meridians. So it suffices to show that G includes two disjoint meridians under the assumption that any edge contraction of G yields either a nonsimple graph or a triangulation which includes an equator of length 3. For example, any irreducible triangulation of handle type satisfies this assumption and includes two disjoint meridians while any crosscap type does not. Thus, we may suppose that G is not irreducible.

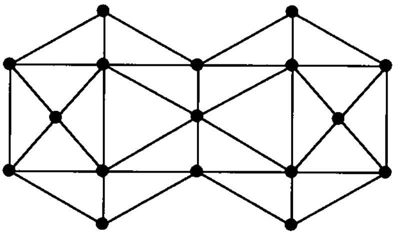
Let e be a contractible edge in G and let G/e denote the triangulation obtained from G by contracting e. By our assumption, G/e has to have an equator C' of length 3, which comes from an equator C of length 4 in G with $C' = C/e$. If G/e is irreducible, then it is of crosscap type and C' separates it into two irreducible triangulations of the projective plane. In this case, it is easy to construct the concrete picture of G from Fig. 15 and to observe that G includes two disjoint meridians actually. Otherwise, we can choose a sequence of contractible edges $e_1, ..., e_n$ in G so that $G/\{e, e_1, ..., e_n\}$ is an irreducible triangulation of crosscap type. Then G is



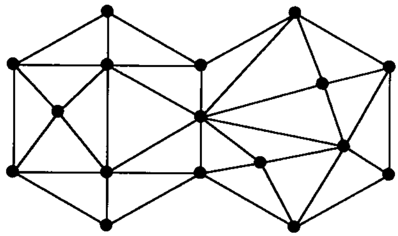
Kc1: (8,8,8,5,5,5,5,5,5)



Kc2: (9,9,7,6,6,5,5,5,4,4)



Kc3: (10,10,6,6,6,6,6,4,4,4,4)



Kc4: (10,8,8,6,6,6,6,4,4,4,4)

FIG. 15. Irreducible triangulations of the Klein bottle, No. 3.

contractible to $G/\{e_1, ..., e_n\}$ and the latter has the same picture as G in the previous case. Thus, $G/\{e_1, ..., e_n\}$ includes two disjoint meridians and so G does. ∎

Since a 4-connected triangulation cannot contain an essential separating cycle of length 3, The following corollary immediately follows from Theorem 12:

COROLLARY 13. *Every 4-connected triangulation of the Klein bottle includes two disjoint meridians as its cycles.*

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