Notes of Basic Topolgy

Taper

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Abstract

A note of Basic Topology, based on *Basic Topology* by M.A. Armstrong.

Contents

There are several parts that I will skipped for convenience. Those include chapter 1 - Introduction, chapter 2 - Continuity, chapter 3 - Compactness and Connectedness, and chapter 4 - Identification Spaces. Below is some especially confusing part that I would like to note:

1 Special Notes

sec:Special-Notes

About map In book [1], a map is defined as a continuous function (page 32), which is confusing. In this note, I will not use this convention and will always states continuity clearly.

Basic facts about maps Assuming domain $f = X$, codomain $f = Y$.

$$f(U \cup V) = f(U) \cup f(V) \tag{1.0.1}$$
$$f(U \cap V) \subseteq f(U) \cap f(V) \tag{1.0.2}$$
$$f(U^c) \supseteq f(U)^c, \text{ i.e. } f(U)^c \subseteq f(U^c) \tag{1.0.3}$$
$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V) \tag{1.0.4}$$
$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V) \tag{1.0.5}$$
$$f^{-1}(U^c) = [f^{-1}(U)]^c \tag{1.0.6}$$

Smallest the Largest Topolgy The set of all possible topolgies on X is partially ordered by inclusion. For a certain characteristics $\mathcal{C}$, it is possible to have the smallest or the largest one.

The **smallest topolgy** $\mathcal{T}_{\min}$ is the one such that, for any $\mathcal{T}'$ satisfying $\mathcal{C}$, $\mathcal{T}_{\min} \subseteq \mathcal{T}'$. The **largest topolgy** $\mathcal{T}_{\max}$ is the one such that, for any $\mathcal{T}'$ satisfying $\mathcal{C}$, $\mathcal{T}' \subseteq \mathcal{T}_{\max}$. Synonyms of these two words are:

- Larger: stronger, finer.
- Smaller: weaker, coarser.

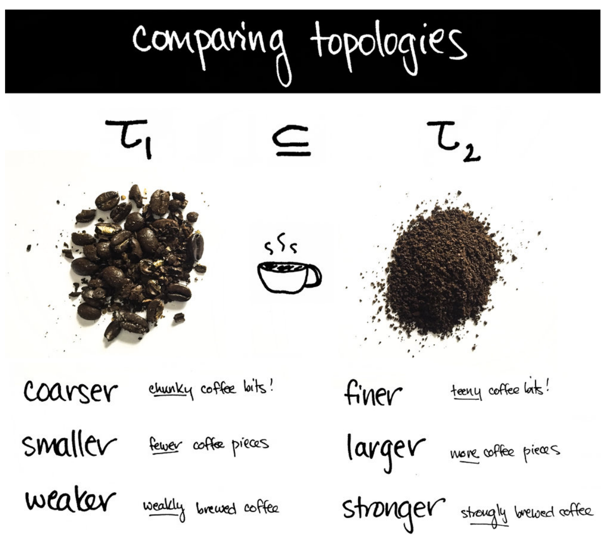


Figure 1: Comparing topologies and coffee (Credit: math3ma)

For example, assuming we have

$$f : X \to Y \tag{1.0.7}$$

where f is any function.

If X has topolgy $\mathcal{T}_X$, we ask then what kind of topolgy on Y will make f a continuous function. First, all $f^{-1}(V)$, with $V \in \mathcal{T}_Y$ should be open in

X. So, the easiest choice is to make $\mathcal{T}_{Y,\min} = \{\varnothing, Y\}$, this is the smallest topolgy. Also, any set $V \in Y$ such that $f^{-1}(V) \notin \mathcal{T}_X$ should not be in $\mathcal{T}_Y$. Then the largest topolgy is $\mathcal{T}_{Y,\max} = \{V \subset Y | f^{-1}(V) \in \mathcal{T}_X\}$.

If Y has topolgy $\mathcal{T}_Y$, we also ask what kind of topolgy on X will make f a continuous function. First, all $V \in \mathcal{T}_Y$, their preimage $f^{-1}(V)$ must be in $\mathcal{T}_X$. So the smallest topolgy is $\mathcal{T}_{X,\min} = \{f^{-1}(V) | V \in \mathcal{T}_Y\}$. Than what about the largest topolgy? We consider, what kind of sets cannot be inside $\mathcal{T}_X$. First, can $(f^{-1}(V))^c = f^{-1}(V^c)$ be in $\mathcal{T}_X$? Yes. Since unless the space is connected, there can be sets being both open and closed (other than X and $\varnothing$). Any other restrictions? No that I can think of. So, the largest topolgy $\mathcal{T}_{X,\max} = 2^X$, the set of all subsets of X. (The notation 2^X is taken from the page 4 of book [2].

A summary:

Table 1: Largest and Smallest Topolgies

$X \xrightarrow{f} Y$	Smallest	Largest
Given $\mathcal{T}_X$	$\mathcal{T}_{Y,\min} = \{\varnothing, Y\}$	$\mathcal{T}_{Y,\max} = \{V \subset Y \| f^{-1}(V) \in \mathcal{T}_X\}$
Given $\mathcal{T}_Y$	$\mathcal{T}_{X,\min} = \{f^{-1}(V) \| V \in \mathcal{T}_Y\}$	$\mathcal{T}_{X,\max} = 2^X$
No constraint	$\{\varnothing, X\}$	2^X

Facts about subspace/induced topolgy Let Y be a subspace of a topological space X wit induced topolgy.

Fact 1.1. A set $H \subseteq Y$ is open in Y if and only if $H = F \cap Y$ for some open set F in X.

Fact 1.2. A set $H \subseteq Y$ is closed in Y if and only if $H = F \cap Y$ for some closed set F in X.

Fact 1.3. A set H is open/closed in $X \Rightarrow H$ is open/closed in Y. But the converse may not be true. The converse statement depends on whether Y is open or closed in X.

1.1 Lebesgue lamme

sec:Lebesgue lamme

This is a very important lemma, which is why I gave it a seperate section. It is labeled (3.11) in book [1].

thm:lebesgue-lemma

Theorem 1.1 (Lebesgue Lemma). *Let X be a compact metric space and let $\mathscr{F}$ be an open cover of X. Then there exists a real number $\delta > 0$ (called the **Lebesgue number** of $\mathscr{F}$) such that any subset of X of diameter less than δ is contained in some member of $\mathscr{F}$.*

2 A Brief Note of Chapter 4 - Identification Spaces

sec:Brief-Note-Chapter-4

2.1 Identification topology

ec:Identification topology

Definition 2.1 (Identification Topology). Let X be a topological space and let $\mathscr{P}$ be a family of disjoint nonempty subsets of X such that $\cup\mathscr{P} =$

X. Such a family is usually called a partition of X. Let Y be a new space whose points are the members of $\mathscr{P}$. Let $\pi : X \to Y$ sends each point of X to the subset of $\mathscr{P}$. Define a topolgy $\mathcal{T}_Y$ on Y to be the largest topolgy such that the π is continuous. This $\mathcal{T}_Y$ is called the idetification topolgy. And Y is called the **identification space** .

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d, equal] \\ & \mathscr{P} \end{tikzcd}$$

Theorem 2.1. *Let Y be an idetification space defined as above and let Z be an arbitrary topological space. A function $f : Y \to Z$ is continuous if and ony if the composition $f \circ \pi : X \to Z$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[rr, bend left, "f\circ\pi"] \arrow[dr] & Y \arrow[r, "f"] \arrow[d, equal] & Z \\ & \mathscr{P} & \end{tikzcd}$$

Definition 2.2 (Identification Map)**.** Let $f : X \to Y$ be an onto continuous map and suppose that the topolgy on Y is the largest for which f is continuous. Then we call f an identification map.

The naming "identification map" is because:

Theorem 2.2. *Any function $f : X \to Y$ gives rise to a partition of X whose members are the subsets $\{f^{-1}(y)\}$, where $y \in Y$. Let Y_* denote the identification space associated with this partition, and $\pi : X \to Y_*$ the usual continuous map.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] & Y \\ \{f^{-1}(y)\} \arrow[r, equal] & Y_* \end{tikzcd}$$

If f is an identification map, then:

1. *the spaces Y and Y_* are homeomorphic;*
2. *a function $g : Y \to Z$ is continuous if and only if the composition $g \circ f : X \to Z$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] \arrow[d, "\pi"] & Y \arrow[r, "g"] \arrow[d, equal] & Z \\ \{f^{-1}(y)\} \arrow[r, equal] & Y_* & \end{tikzcd}$$

Theorem 2.3. *Let $f : X \to Y$ be an onto continuous map. If f maps open sets of X to open sets of Y, or closed sets to closed sets, then f is an identification map, i.e. $\mathcal{T}_y$ is the largest topolgy such that f is continuous.*

coro:idmap-coro

Corollary 2.1. *Let $f : X \to Y$ be an onto continuous map. If X is compact and Y is Hausdorff, then f is an identification map.*

Definition 2.3 (Torus)**.** Torus is the unit square $[0, 1] \times [0, 1]$, with 1. opposite edge identified; 2. four edge points identified.

Remark 2.1. The identification map and corollary 2.1 can be used to show that torus is homeomorphic to two copies of circles:$S^1 \times S^1$. This is mentioned in page 68 of [1].

Definition 2.4 (Cone CX)**.** The cone of any space CX is formed from $X \times I$, where I is the unit interval $[0, 1]$, with certain identification. The identification shrinks all points in one surface into one point. This is discussed in page 68 of [1].

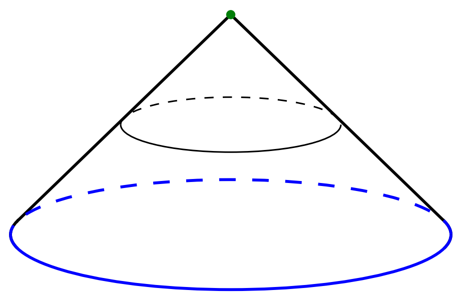


Figure 2: Cone of a Circle (Wikipedia)

Remark 2.2. There is another definition of cone CX when X in imbeded into $\mathbb{E}^n$, may be found on page 68 of [1]. Cone constructed in this way is called a geometric cone. It is made up of all straight line segments that join $v = (0, 0, \cdots, 1) \in \mathbb{E}^{n+1}$ to some point of X.

Lemma 2.1. *The geometric cone on X is homeomorphic to CX.*

Definition 2.5 (Quotient Space)**.** Let X be a topological space, A be its subspace. Then X/A menas the X with subspace A identified to a point.

1. the set A.
2. the individual points of $X \setminus A$.

Remark 2.3. In this notation, CX becomes $(X \times I)/(X \times \{1\})$.

Fact 2.1.

$$B^n/S^{n-1} \cong S^n \tag{2.1.1}$$

where $\cong$ menas homeomorphic. This is proved on page 69. Intuitively, this is like wrap a lower dimension ball surround the higher dimension ball.

Definition 2.6 ($f \cup g$)**.** Let X, Y $f \cup g$ subsets of a topological space and give each of X, Y, and$X \cup Y$ the induced topolgy. If $f : X \to Z$ and $g : Y \to Z$ are functions which agree on the intersection of X and Y, we can define

$$\begin{gathered} f \cup g : X \cup Y \to Z \qquad (2.1.2) \\ (f \cup g)(x) = f(x),\, x \in X \\ (f \cup g)(x) = g(x),\, x \in Y \end{gathered}$$

We say that $f \cup g$ are formed by 'glueing together' the functions f and g.

Lemma 2.2 (Glueing lemma (closed))**.** *If X and Y are closed in $X \cup Y$, and if both f and g are continuous, then $f \cup g$ are continuous.*

Similarly,

Lemma 2.3 (Glueing lemma (open))**.** *If X and Y are open in $X \cup Y$, and if both f and g are continuous, then $f \cup g$ are continuous.*

These two lemmas are seen as a special case of the following theorem, explained in page 70.

Define $X + Y$ to be the disjoint union of spaces X, Y. Define $j : X + Y \to X \cup Y$ which restrict to either X or Y is just the inclusion in $X \cup Y$.

Theorem 2.4. *If j is an identification map, and if both $f : X \to Z$ and $g : X \to Z$ are continuous, then $f \cup g : X \cup Y \to Z$ is continuous.*

$$\begin{tikzcd}
X+Y \arrow[r, "j"] & X \cup Y \arrow[r, "f \cup g"] & Z \\
X \arrow[urr, "f"] & Y \arrow[ur, bend right, "g"'] &
\end{tikzcd}$$

This can be generalized as follows. Let $X_\alpha, \alpha \in A$ be a family of subsets of a topological space and give each X_α and the union $\cup X_\alpha$, the induced topolgy. Let Z be a space and suppose we are given maps $f_\alpha : X_\alpha \to Z$, one for each α in A, such that if $\alpha, \beta \in A$,

$$f_\alpha \Big|_{X_\alpha \cap X_\beta} = f_\beta \Big|_{X_\alpha \cap X_\beta}$$

Define function $F : \cup X_\alpha \to Z$ by glueing together f_α. Let $\oplus X_\alpha$ be the disjoint unin of spaces X_α. Let $j : \oplus X_\alpha \to \cup X_\alpha$ be similarly defined.

Theorem 2.5. *If j is an identification map, and if each f_α is continuous, then F is continuous.*

Note: When j is the identification map, then $\cup X_\alpha$ has the identification topolgy instead of the subspace topology. The two will be quite different, as discussed on page 70 to 71 of [1].

Definition 2.7 (Projective space P^n)**.** A discussion of real P^n may be found on page 71.

Attaching maps and $X \cup_f Y$ Let:

$$Y \supseteq A \xrightarrow{f} X \tag{2.1.3}$$

where X, Y are topological spaces, f is continuous. We identify the disjoint union $X + Y$ using f, partitioning them into:

1. pairs of points $\{a, f(a)\}$ where $a \in A$;
2. individual points of $Y \setminus A$;
3. individual points of $X \setminus \mathrm{Im}(f)$.

The result identification space is denoted $X \cup_f Y$, and f is called the attaching map. This process can also be viewed as:

$$X \cup_f Y = (X \amalg Y)/\{f(A) \sim A\} \tag{2.1.4}$$

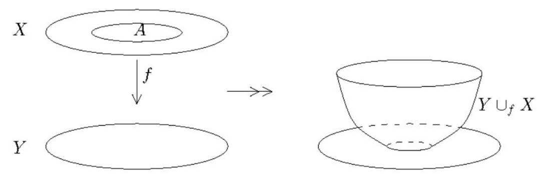


Figure 3: Attaching Space (credit: nLab

Example 2.1. P^2 can be seen as attaching a closed disc D to the boundary of M, a Mobius strip, as discussed in page 72 of [1]. Geometrically, this simply shrinks the boundary of M into a point. And an ant travelling around this point can point out the direction just as in P^2.

Remark 2.4. It is remarked that properties such as compactness, connectedness, and path-connectedness is inherited in identification. However, Hausdorff-ness is not. An counter example can be found in page 72 of [1].

sec:Topological-Groups

2.2 Topological Groups

In simple words, **topological groups** are objects that has both a topolgy on it and a group structure in it. And the two structures must be compatible. Specifically, the multiplication map $a \cdot b$ and the inverse map $a \to a^{-1}$ are continuous. Homomorphisms between are both group-homomorphisms and topological-homomorphisms (continuous maps). Isomorphisms are both group-isomorphisms and topolgy-isomorphisms (homeomorphisms). A sub-(topological group) is both a subgroup and has subspace topolgy. For convenience of language, use $\mathcal{TPG}$ denotes the category of topological groups. [1]

Example 2.2. The $\mathbb{R}$ is a topological group. The $\mathbb{Z}$ with discrete topology form the sub-(topological group) of $\mathbb{R}$. The quotient $\mathbb{R}/\mathbb{Z}$ forms a topological group. The map $f : \mathbb{R} \to S^1$ induces a homeomorphism $\mathbb{R}/\mathbb{Z} \cong S^1$, which is also a group isomorphisms, i.e. it is a $\mathcal{TPG}$-isomorphism.

Example 2.3. Similarly, R^n.

Example 2.4. The circle is also one. The group structure is combination of degrees.

Example 2.5. Any group with discrete topology.

Example 2.6. The torus considered as the product of two circles. (Take the producttopology and the product group structure.

[1]This notation is nowhere popular or accepted. I use it to only to save space and time.

Example 2.7. Three sphere S^3 considered as the unit sphere in the space of quaterions $\mathbb{H}$.

Remember this? :

$$\begin{tikzcd}
 & i \arrow[dr] \arrow[d, phantom, "\#"] & \\
k \arrow[ur] & & j \arrow[ll]
\end{tikzcd}$$

The unit sphere are unit quaterions, see more Versor.

Example 2.8. The **orthogonal group** $\mathrm{O}(n)$, of $n \times n$ orthogonal real matrices. It is easy to check that $\mathrm{O}(n-1)$ is a sub-$\mathcal{TPG}$ of $\mathrm{O}(n)$.

Definition 2.8 (Left translation L_x)**.** For $x \in G$, the function

$$L_x : G \to G \tag{2.2.1}$$

$$g \mapsto xg \tag{2.2.2}$$

is called a left translation by x. Similarly we have **right translation** R_x .

Fact 2.2. L_x and R_x are homeomorphisms (But not group-isomorphisms).

Remark 2.5. This shows that a topological group has a certain homogeneity as a topological space. For if $x, y \in G$, then $L_{yx^{-1}}$ maps x to y and is a homeomorphism. Therefore G *exhibits the same topological structure locally near each point.*

Theorem 2.6. *Let G is a topological group, let K be a connected component of G which contains the identity element. Then K is a closed normal subgroup of G.*

Fact 2.3. If $G = \mathrm{O}(n)$, then $K = \mathrm{SO}(n)$.

Theorem 2.7. *In a connected topological group, any neighbourhood of the identity element is a set generates the whole group.*

The two theorems above is summarised as

Table 2: caption

topology		group/topology
e + connected	$\Rightarrow$	closed & normal subgroup
e + neighbourhood	$\Rightarrow$	generator

A bit more examples about matrices:

Example 2.9. $\mathbb{M}(n)$ the $n \times n$ matrices, is not a topological group. But its subspace GL(n) , specifically, $\mathrm{GL}(\mathrm{n}, \mathbb{R})$ or $\mathrm{GL}(\mathrm{n}, \mathbb{C})$, is a topological group. This is demonstrated in page 76, theorem 4.12.

Fact 2.4. GL(n) is not compact. It has two idsjoint nonempty open sets: those with positive and those with negative determinants.

Theorem 2.8. $\mathrm{O}(n)$ *and* $\mathrm{SO}(n)$ *are closed and compact.* $\mathrm{SO}(n)$ *is a sub-$\mathcal{TPG}$ of* $\mathrm{O}(n)$*.*

Fact 2.5. $\mathrm{SO}(2) \cong S^1$ and $\mathrm{SO}(3) \cong P^3$. Here $\cong$ means isomorphisms of topological groups.

Remark 2.6. These two facts established on page 77. The first one can be eaily guess. Since a rotation is obviously determined by a rotation degree on S^1. Mathematically we have

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cong e^{i\theta} \tag{2.2.3}$$

The second one is proved mathematical in book [1]. But it has a physical argument. Remember we have the the homogeneous coordinates for P^3, such as $[1, \theta_x, \theta_y, \theta_z]$. As indicated in my labels, the three free coordinates θ_i can be regarded as rotation in 3-dimensional space. This rotation preserves the orientation, so it is in SO, not in O.

`sec:Orbit-Space`

2.3 Orbit Space

Definition 2.9 (Group Action on Topology Space)**.** A topological group G is said to act as a group of homeomorphisms on a space X if each group element (let $g, h \in G$) induces a homeomorphism of the space in such a way that:

1. $(hg)(x) = h(g(x))$, $\forall x \in X$;
2. $e(x) = x$, $\forall x \in X$, where $e = gg^{-1}$;
3. the function $G \times X \to X$,$(g, x) \mapsto g(x)$ is continuous.

The subset of X, consisting of $g(x)$ for all $g \in G$, is called an **orbit** of $x \in X$, written $O(x)$. Thought, it more convenient to write it just as Gx , as in textbooks of abstract algebra.

Fact 2.6. A common fact in abstract algebra here is: each orbit Gx is disjoint. If two $Gx \cap Gy \neq \varnothing$, then $Gx = Gy$.

By above fact, orbits partitions X, hence we can form the Identification space, with every elements in X identified with their brothers in the same orbit. The result is **orbit space** X/G .

`ex:R-over-Z-T`

Example 2.10. $\mathbb{Z}$ acts on $\mathbb{R}$ by addition $x \mapsto x + n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$. It partitioned $\mathbb{R}$ into intervals, for each $x \in X$, $x \sim x + n, \forall n \in \mathbb{Z}$. The orbit space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1.

An action G on X is called **transitive** , if and only if the orbit space X/G is the trivial point $\{1\}$. Or equivalently, the only orbit is the whole space, i.e. $Gx = G$, $\forall x \in G$.

Example 2.11. The orthogonal action $O(n)$ on S^{n-1} is transitive. Physically, this is saying that $\forall x \in S^{n-1}$, it can be rotated into $\forall y \in S^{n-1}$. A mathematical proof is on page 79 of [1]

A lot of examples from book [1]

Example 2.12. Extending example 2.10:

$$\mathbb{E}^2/(\mathbb{Z} \times \mathbb{Z}) = T \text{ (torus)} \tag{2.3.1}$$

Here = means homeomorphism.

Example 2.13.

$$S^n/\mathbb{Z}_2 = P^n \tag{2.3.2}$$

Here = means homeomorphism.

Example 2.14 (Three ways of $\mathbb{Z}_2$ acting on T)**.** The detailed procedure is to be found on page 91 of [1]. Here's a picture to visualize the action:

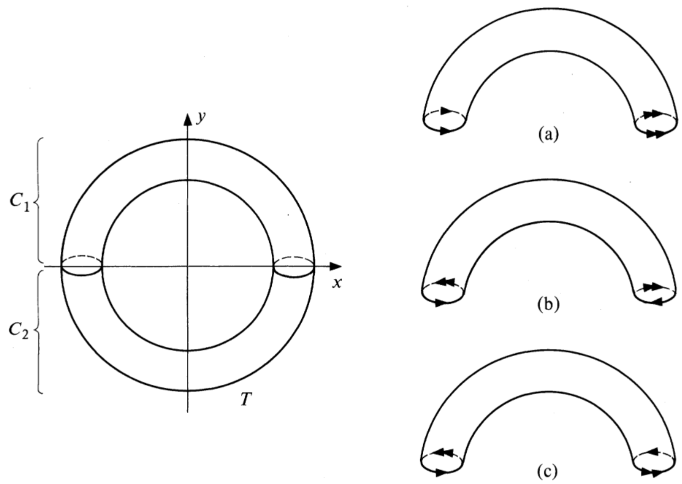


Figure 4:

The results are (a) a sphere; (b) a torus; (c) a Klein bottle.

Example 2.15. If G is a topological group, and H is $\mathcal{TPG}$-subgroup. Then, the left cosets of right cosets can be canonically seen as orbits. See more on page 81, example 4.

Example 2.16.

$$\mathrm{O}(n)/\mathrm{O}(n-1) = S^{n-1} \tag{2.3.3}$$

$$\mathrm{SO}(n)/\mathrm{SO}(n-1) = S^{n-1} \tag{2.3.4}$$

Here = means homeomorphism. The first is established mathematically in page 82 of [1]. The second is mentioned there, indicating a similar proof.

Here I give an argument. Consider a unit vector y in S^{n-1}, if we want to rotate another unit vector e_1 to y, since the action is transitive, we can easily find a $A \in \mathrm{O}(n)$ to do this. But in addition, we can also find that $A \cdot B$, where $B \in \mathrm{O}(n-1)$ rotates the space around e_1 (thus leaving e_1 un-affected) also do our job. So there is an $\mathrm{O}(n-1)$ redundancy in $\mathrm{O}(n) \to S^{n-1}$. Similar for the second relation.

Theorem 2.9. *Let G acts on X and suppose that both G and X/G are connected, then X is connected.*

Fact 2.7. Using the theorem above, one can deduce that: $\mathrm{SO}(1)$ is connected, S^{n-1} is connected, so $\mathrm{SO}(n)$ is connected.

Next, the book [1] (page 82 to 85) introduces several three spaces (**Lens space** , **irrational flow** on T torus, **fundamental region** or in my word *space filling shapes*) and two group **Euclidean group** (page 84) and **plane-crystallographic group** (page 85). To save time, I leave here only some pictures:

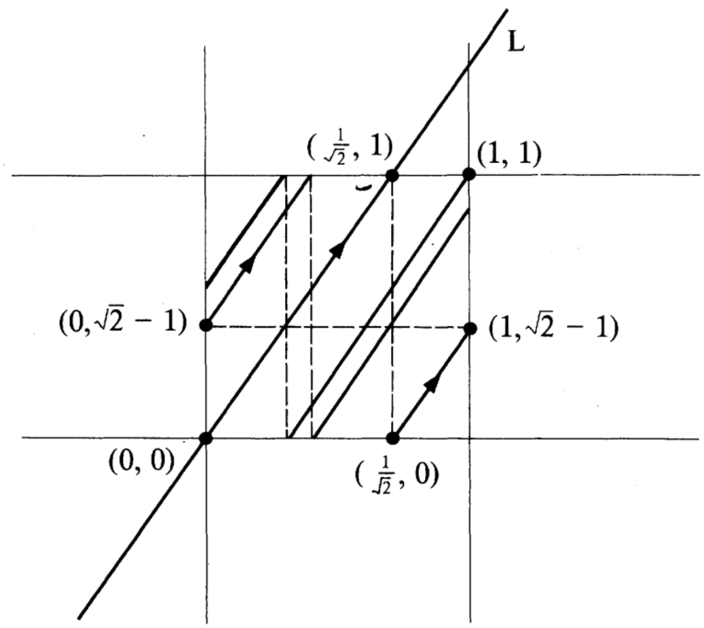


Figure 5: Irrational Flow on T

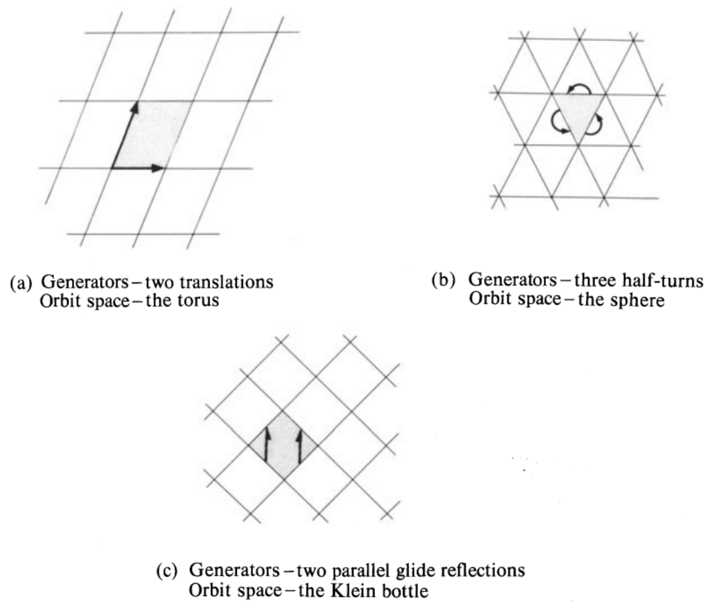


Figure 6: Space-filling Shapes

3 Chapter 5 - Fundamental Groups

apter-5-Fundamental-Groups

3.1 Homotopic maps

sec:Homotopic-maps

By a **loop** we mena a continuous map $\alpha : I \to X$ such that $\alpha(0) = \alpha(1)$. We can view is also as a continuous map $\alpha : S^1 \to X$. It is said to be based at the point $\alpha(0)$. Two loops α and β with the same **base point** can be multiplied, and their product is defined on page 87. A visualization of $\alpha \cdot \beta$ is here:

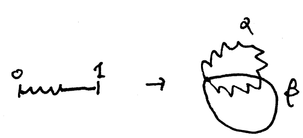


Figure 7: Multiply loops

But this product is not sufficient to become a group. At least, the multiplication is not associative:

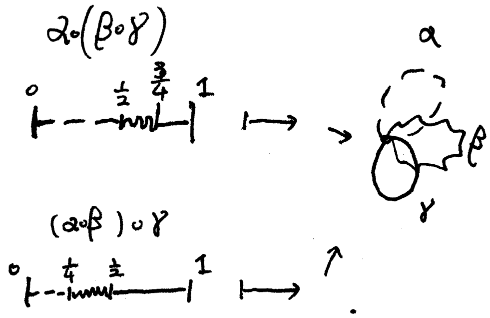


Figure 8: Multiplication of loop is not associative

But clearly the two result are exactly if we do care how long they occupy on the interval I, if the interval I is considered as a time parameter. So we define the following homotopy relation between loops. If we can find a family $\{f_r\}$ of maps, one for each $r \in [0,1]$, such that $f_0 = \alpha$, $f_1 = \beta$, then we say that the loops α and β are homotopic. Schematically,

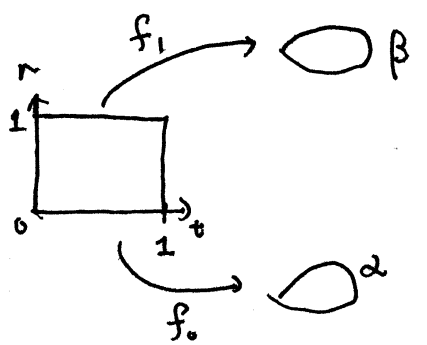


Figure 9: Homotopy for loops

This relation can be generalized to any continuous maps:

Definition 3.1 (Homotopic)**.** Let $f, g : X \to Y$ be continuous maps. Then f is homotopic to g if there exists a map $F : X \times I \to Y$ such that $F(x, O) = f(x)$ and $F(x, 1) = g(x)$ for alt points $x \in X$.

The map F is called a **homotopy** from f to g, and we write $f \simeq_F g$. In addition, if f and g agree on some $A \subset X$, we may wish to deform f to g without altering the values of f on A. In this case we ask for a homotopy F from f to g with the additional property that

$$F(a, t) = f(a) \text{ for all } a \in A, \text{ for all } t \in I \tag{3.1.1}$$

when such a homotopy exists, we say the f **is homotopic to** g **relative to** A and write $f \simeq_F g$ **rel** A .

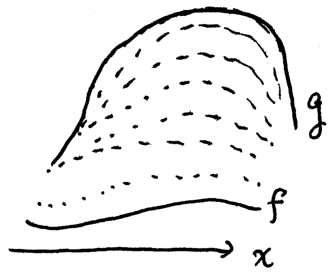


Figure 10: Homotopy Scheme

When f and g are loops, then the homotopic relation for loops are just saying that $f \simeq g$ rel $\{0, 1\}$.

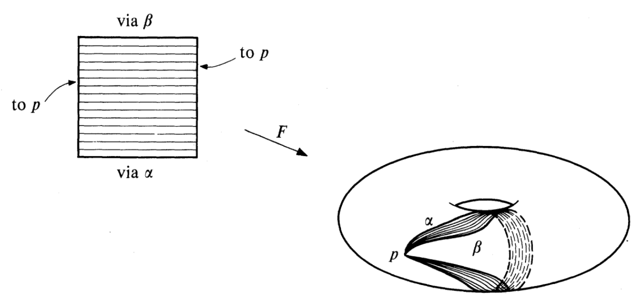


Figure 11: Homotopy for loops 2 (pp. 88 of [1])

Example 3.1. The author shows on page 88 of [1] that: when C is a convex subset of a euclidean space, let $f, g : X \to C$ be continuous maps, then $f \simeq_F g$, where F is $F(x, t) = (1 - t)f(x) + tg(x)$. Note that if f and g agree on a subset A of X, then this homotopy is a homotopy relative to A. This F is called a **straight-line homotopy** .

Example 3.2. Let $f, g : X \to S^n$ be continuous maps. We can take S^n to be the unit sphere in $\mathbb{E}^{n+1}$, and think of f, g as continuous maps into $\mathbb{E}^{n+1}$, then we may form a similar "straight-line homotopy" from f to g by:

$$F(x, t) = \frac{(1 - t)f(x) + tg(x)}{||(1 - t)f(x) + tg(x)||} \tag{3.1.2}$$

Notice that the ball B^{n+1} is a convex set, so the numerator lies inside the ball. When normalized (as in $F(x, t)$), the numerator is a point on the sphere.

Example 3.3. This is example is best illustrated by pictures:

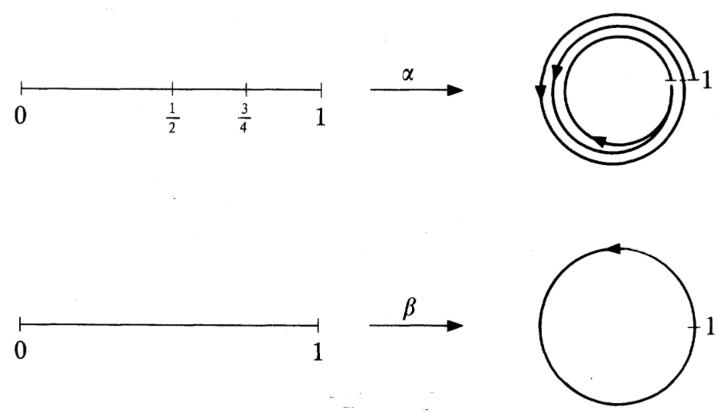


Figure 12: Loops to loops (pp. 89 of [1])

Geometrieally, α winds eaeh of the segments $[O, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{4}]$, $[\frac{3}{4}, 1]$ once round the eirc1e, the first two being wound in an anticlockwise direetion, and the third clockwise. The loop β simply winds the whole interval $[0, 1]$ once round the circle anticlockwise.

The book [1] gives a homotopy F between α and β on page 89. But it is best to imagine α and β being metal coils, and this F just describes the process when one magically strach and unfold the coil from α to β.

Notice that this coil is connected head to tail, so it is essential that there is not pole inside the coil in order that one can unfold the coil from α to β.

I think we already feel this, but the book proves it on page 90, that

Lemma 3.1. *The relation of 'homotopy' is an equivalence relation on the set of all maps from X to Y.*

Also

Lemma 3.2. *The relation of 'homotopy relative to a subset A of X' is an equivalence relation on the set of all maps from X to Y which agree with some give map on A.*

The book also mentions that

Lemma 3.3. *Homotopy behaves well with respect to composition of maps*

which means precisely that:

- If $f \simeq_F g$ rel A, then $hf \simeq_{hF} hg$ rel A.

$$\begin{tikzcd} A \subset X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \arrow[r, "h"] & Z \end{tikzcd}$$

- If $g \simeq_G h$ rel B, then $gf \simeq_F hf$ rel $f^{-1}B$ via the homotopy $F(x,t) = G(f(x),t)$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & Z \\ \cup & \cup & \\ f^{-1}B & B & \end{tikzcd}$$

n-of-the-fundamental-group

3.2 Construction of the fundamental group

Theorem 3.1. *The set of homotopy classes of loops in X based at $p \in X$ forms a group under the multiplication $\langle\alpha\rangle \cdot \langle\beta\rangle = \langle\alpha \cdot \beta\rangle$. The identity and inverse elements are defined on page 92 and 93 of [1].*

Remark 3.1. Notice that the fundamental group dependes heavily on the base point. Especially when the space X is disconnected. A natural intuition is that when the space is path-connected, then any two paths can be connected. If two points can be connected, then two loops based on different points can be connected. That why we have the following.

Theorem 3.2. *If X is a path-connected then $\pi_1(X,p)$ $\pi_1(X,q)$ are isomorphic for any two points $p,q \in X$.*

Proof. The proof is on page 94 of [1]. □

Definition 3.2 (f_*)**.** Suppose we have a continuous map $f : X \to Y$, f can induce a map ($p \in X, q \in Y$ and $q = f(p)$).

$$f_* : \pi_1(X,p) \to \pi_1(Y,q) \tag{3.2.1}$$
$$\langle\alpha\rangle \mapsto \langle f \circ \alpha\rangle \tag{3.2.2}$$

This map is actually a homomorphism.

Fact 3.1. By construction, we have for

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$(g \circ f)_* = g_* \circ f_* \tag{3.2.3}$$

Fact 3.2. With a homeomorphism $h : X \to Y$ and the above fact, we see that homeomorphic spaces have isomorphic fundamental groups.

sec:Sectin-5.3

3.3 Sectin 5.3

This section calculates the following facts:

#	Space	Fundamental group
1	Conves subset of $\mathbb{E}^n$	{e}
2	Circle	$\mathbb{Z}$
3	S^1	$\mathbb{Z}$
4	S^n, $n \geq 2$	{e}
5	Torus $S^1 \times S^1$	$\mathbb{Z} \times \mathbb{Z}$
6	$P^n, , n \geq 2$	$\mathbb{Z}_2$
7	Klein bottle	$\{a, b \mid a^2 = b^2\}$
8	Len space $L(p, q)$	$\mathbb{Z}_p$

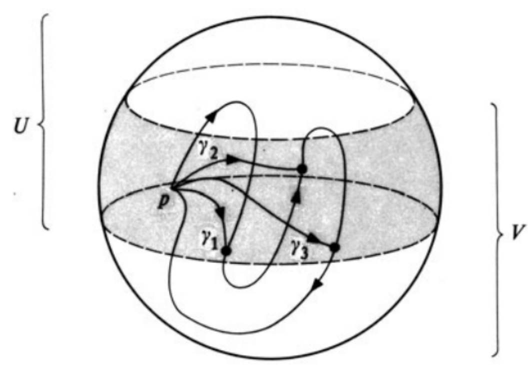
Convex subset of $\mathbb{E}^n$ is on page 96 of [1].

Definition 3.3 (simply connected)**.** A path-connected space whose fundamental group is trivial is said to be simply connected.

The n-sphere S^n is on page 99. To prove it, we require a theorem:

Theorem 3.3. *Let X be a space which can be written as the union of two simply connected open sets U, V in such a way that $U \cap V$ is path-connected. Then X is simply connected.*

Proof. Proof is on page 99 of [1]. It requires the Lebesgue lemma 1.1. A picture illustrating the proof is provided here:



□

Then, the two cover of S^n are simply $U = S^n - \{x\}$ and $V = S^n - \{y\}$, where x, y are two distinct points on S^n. Since U and V are homeomorphic to $\mathbb{E}^n$, S^n is simply-connected.

sec:Continued-Section-5.3

3.4 Continued Section 5.3

In this section, I will follow the lecture by professor Li Qin. We show that

Theorem 3.4.

$$\pi_1(S^1) = \mathbb{Z}$$

Proof. Given any integer $n \in \mathbb{Z}$, we give a loop by associatng each n

$$\begin{aligned} \pi : \mathbb{R} &\to S^1 \\ t &\mapsto e^{2\pi i t} \end{aligned}$$

Define

$$\gamma_n : [0,1] \to \mathbb{R} \tag{3.4.1}$$

$$s \mapsto ns \tag{3.4.2}$$

Then $\phi_n := \pi \circ \gamma_n$ has the property that $\phi_n(0) = 1$, $\phi_n(1) = 1$. So we obtain

$$\phi : \mathbb{Z} \to \pi_1(S^1) \tag{3.4.3}$$

Geometrically, we ϕ_n loops around S^1 in n turns.

We need to prove that ϕ is a isomorphism. This is done by:

1. Prove that ϕ is a homomorphism;
2. Prove that ϕ is bijective.

First,

$$\begin{aligned} \gamma_n &: s \mapsto ns \\ \gamma_m &: s \mapsto ms \end{aligned}$$

We need

$$\langle \pi \circ \gamma_{m+n} \rangle = \langle \pi \circ \gamma_m \rangle \langle \pi \circ \gamma_n \rangle$$

Define $\sigma : [0,1] \to \mathbb{R}$, $s \mapsto \gamma_n(s) + m$, this is a translation of real line. Then $\pi \circ \sigma = \pi \circ \gamma_n$. Then

$$\langle \pi \circ \gamma_m \rangle \langle \pi \circ \gamma_n \rangle = \langle \pi \circ \gamma_m \rangle \langle \pi \circ \sigma \rangle = \langle \pi \circ (\gamma_m \circ \sigma) \rangle$$

γ_{m+n} has the same domain and codomain of $\gamma_m \circ \sigma$, and they obviously share the same start and the same end point. Therefore these two path are homotopic relative to $\{0, 1\}$. Therefore

$$\langle \pi \circ \gamma_{m+n} \rangle = \langle \pi \circ (\gamma_m \circ \sigma) \rangle \tag{3.4.4}$$

Or

$$\phi_{m+n} = \phi_m \phi_n \tag{3.4.5}$$

Second, we need to show that this map is surjective. Notice that $\pi : \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$, is like a projection of a circulatory path onto a circle S^1. This map is locally homeomorphic. We can find a cover of S^1 as the combination of

$$\begin{aligned} U &= S^1 \setminus \{-1\} \\ V &= S^1 \setminus \{1\} \end{aligned}$$

Then $\pi^{-1}(V)$ are the intervals on $\mathbb{R}$ excluding the whole integer points. Similarly, $\pi^{-1}(U)$ are those intervals on $\mathbb{R}$ excluding those half-integer points. In each of those intervals the map π is bijective. Now we need a lemma:

thm:path-lifting-lemma-1

Lemma 3.4 (Path-lifting lemma).

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\pi"] \\ {[0,1]} \arrow[ur, "f"] \arrow[r, "\sigma"] & S^1 \end{tikzcd} \tag{3.4.6}$$

Assuming we have π and σ, both are continuous maps. More specifically, σ is a path in S^1 which begins at the point $1 \in S^1$. Then there is a unique path $\tilde{\sigma}$ in $\mathbb{R}$ which begins at $0 \in \mathbb{R}$ and satisfies $\pi \circ \tilde{\sigma} = \sigma$.

Proof. The proof is on page 97 to 98 of [1]. The class gives me enough intuition to understand the proof.

The intuition is that, by Lebesgue lemma, we can divide the interval $[0,1]$ fine enough such that each divided part is maped to only one of the cover U or V. We thus break a path σ into small paths σ_i. Each σ_i can be lifted into a path $\tilde{\sigma}_i$ in $\mathbb{R}$. But such lifting can be arbitrary because the inverse of π is not a good function. To resolve this ambiguity, one requires the first path should starts with $0 \in \mathbb{R}$, and the second should be continuously connected to the first, and so is the third, fourth, etc. This fixes the ambiguity and the paths $\tilde{\sigma}_i$ when connected give the required path $\tilde{\sigma}$. □

Note that $\tilde{\sigma}(0) = 0$, $\tilde{\sigma}(1)$ is an integer. Now for any loop $\gamma : [0,1] \to S^1$ based at 1, we can find a lifting $\tilde{\gamma} : [0,1] \to \mathbb{R}$ such that $\tilde{\gamma}(0) = 0$,$\tilde{\gamma}(1) = n$, and $\gamma = \pi \circ \tilde{\gamma}$. Then $\tilde{\gamma} \cong \gamma_n$ rel $\{0,1\}$, also $\langle \pi \circ \tilde{\gamma} \rangle = \langle \pi \circ \gamma_n \rangle$. Hence for any path γ we find a n such that $\gamma = \phi(n)$. So *the map is surjective.*

We need another lemma to prove that it is injective.

Lemma 3.5 (Homotopy-lifting lemma). *If $F : [0,1] \times [0,1] \to S^1$ is a map such that $F(0,t) = F(1,t) = 1$ for $0 \le t \le 1$, then there exists a unique $\widetilde{F} : [0,1] \times [0,1] \to \mathbb{R}$ such that*

$$\pi \circ \widetilde{F} = F \tag{3.4.7}$$

$$\widetilde{F}(0,t) = 0,\ 0 \le t \le 1 \tag{3.4.8}$$

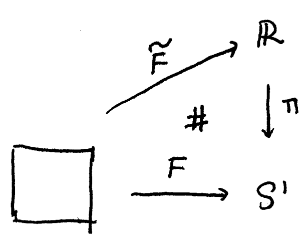


Figure 13: Homotopy lifting lemma-1

Proof. The proof is on page 98 of [1]. The idea is exactly the same as in lemma 3.4. □

Now we proof that the map ϕ is injective. Suffice to prove that $\mathrm{Ker}(\phi)$ is trivial. Suppose $\phi(n) = \pi \circ \gamma_n$ is homotopic to the constant loop. Then choose a homotopy F from $\pi \circ \gamma_n$ to the constant loop. By the homotopy-lifting lemma we can find $\widetilde{F} : [0,1] \times [0,1] \to \mathbb{R}$ such that $\pi \circ \widetilde{F} = F$. Also $\widetilde{F}(0,t) = 0$. We can find the vertical bottom is 0 and verticl top is γ_n. Right line is integers and can only be 0.

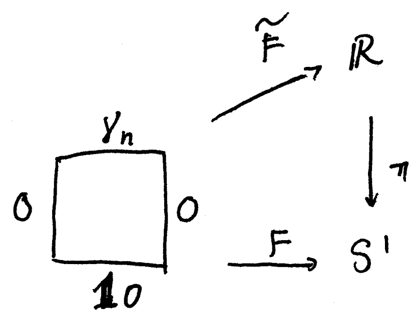


Figure 14: Schematic Draft

Hence γ_n starts at 0 and ends at 0. One can find that $\gamma_n \cong 0$. This completes the proof of injectivity. Hence completes the whole proof. □

We have an application,

Theorem 3.5 (Brouwer Fixed Point theorem). *A contiuous map f : $B^2 \to B^2$ (B^2 : 2D-closed dicks) must have a fixed point. That is, $\exists x \in B^2$ such that $f(x) = x$.*

Proof. Assuming that this theorem is false, that is $\forall x \in B^2$, $x \neq f(x)$, then we have a straight path from $f(x)$ to x. We can extends this path to cuts the boundary of B^2 at $h(x)$. This is for all $x \in B^2$, hence we have a map $h : B^2 \to S^1$. Also, $h|_{S^1}$ is obvious an identity map. But S^1 can be included inside B^2, so we have:

$$S^1 \to B^2 \to S^1 \tag{3.4.9}$$

Hence we have a series of homomorphism of fundamental groups:

$$\pi_1(S^1) \to \pi_1(B^2) \to \pi_1(S^1) \tag{3.4.10}$$

and the composite is identity map. But observe that B^2 is a convex set and hence its fundamental group is trivial. But S^1 has non-trivial fundamental group. It is then impossible to form such a chain of homomorphism whose product is identity map. Contradiction! □

Remark 3.2. This theorem can be extended to higher dimensional case. But the proof cannot be the same because for higher dimension $\pi_1(S^n)$ is no longer non-trivial.

Another application, which we need a theorem to help:

Theorem 3.6.

$$\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \otimes \pi_1(X, y_0) \tag{3.4.11}$$

Proof. We use the projection maps: P_1 and P_2. Then, the map

$$\begin{aligned} (P_1)_* &: \pi_1(X \times Y) \to \pi_1(X) \\ (P_2)_* &: \pi_1(X \times Y) \to \pi_1(Y) \end{aligned}$$

and their composition formed into

$$\langle \alpha \rangle \mapsto (\langle P_1 \circ \alpha \rangle, \langle P_2 \circ \alpha \rangle)$$

this map is surjective, injective, and is homomorphism. The detail can be found on page page 101 of [1]. □

Fact 3.3. By this theorem, the two objects S^2 and $S^1 \times S^1$ is not homeomorphic, since their fundamental groups are not the same (the former is trivial and the later is $\mathbb{Z} \times \mathbb{Z}$).

sec:Homotopy-Type

3.5 Homotopy Type

Notice that a homeomorphic map will make two space have the same fundamental group, but two space having the same fundamental group may not be homeomorphic. For example, the plane and the S^2 both have the same $\pi_1 = \{e\}$, but the plane is not compact while the 2-sphere is, i.e. they are not homeomorphic.

Notice also that homeomorphic requires $fg^{-1} = \mathbb{1}$, whereas under homotopic relation, we may require $fg^{-1} \cong \mathbb{1}$. Therefore we may also define a new type of relationship between topological spaces:

Definition 3.4 (Homotopy type)**.** Two spaces X and Y have the same homotopy type, or they are homotopy equivalent, if there exist maps:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

such that $g \circ f \cong \mathbb{1}_X$, and $f \circ g \cong \mathbb{1}_Y$.

Fact 3.4. This relationship must be an equivalence relation on topological spaces, as confirmed by lemma (5.16) on page 103 of [1].

Remark 3.3. The topological spaces X and Y are not necessarily path-connected! But we have:

Fact 3.5. If X and Y have the same homotopy type then X is path-connected if and only if Y is path-connected. (This is noted in a footnote on page 107 of [1].)

Definition 3.5 (Deformation retraction)**.** Let A be a subspace of X. Let a homotopy $G : X \times I \to X$ which is relative to A and for which

$$\begin{cases} G(x,0) = x \\ G(x,1) \in A \end{cases}$$

for all $x \in X$. Then G will be called a deformation retraction of X onto A.

Remark 3.4. If there is a deformation retraction of X onto A, then of course X and A have the same homotopy type.

Following is an example of a deformation retraction:

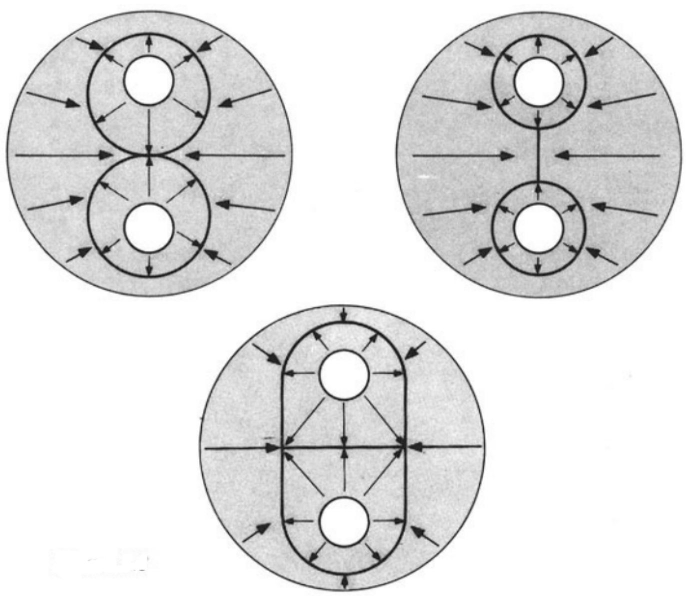


Figure 15: Three Deformation Retractions (from [1])

Exmaples from the book:

Example 3.4. Homeomorphic spaces have the same homotopy type.

Example 3.5. Any convex subset of a euclidean space is homotopy equivalent to a point.

Example 3.6. $\mathbb{E}^n \setminus \{0\}$ has the homotopy type of S^{n-1}. This is shown on page 104 of [1], and is illustrated there by:

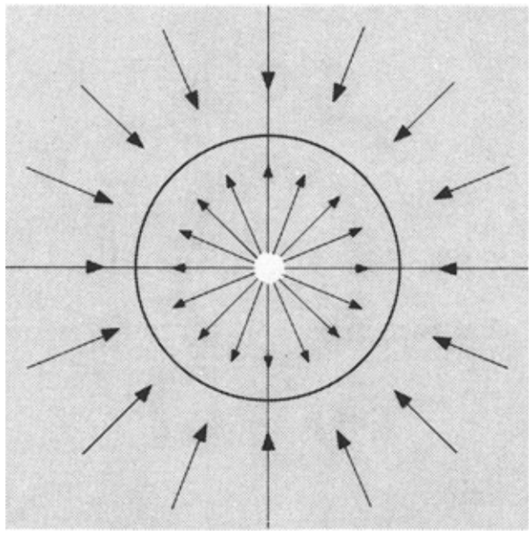


Figure 16: $E^n \setminus \{0\}$ and S^{n-1}

Theorem 3.7. *If $f \cong_F g : X \to Y$ then $g_* : \pi_1(X,p) \to \pi_1(Y, g(p))$ is equal to the composition*

$$\pi_1(X,p) \xrightarrow{f_*} \pi_1(Y, f(p)) \xrightarrow{\gamma_*} \pi_1(Y, g(p)) \tag{3.5.1}$$

where γ is the path joining $f(p)$ to $g(p)$ in Y defined by $\gamma(s) = F(p,s)$.

Proof. Proof can be found on page 105 of [1]. □

Theorem 3.8. *If two path-connected spaces are of the same homotopy type, then they have isomorphic fundamental group.*

Proof. Proof can be found on page 106 of [1]. Here's a note using the notation in that book:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

It should be noticed that the continuity allows one to identify the diagram:

$$\begin{tikzcd} \pi_1(X,p) \arrow[r, "(gf)_*"] \arrow[rr, bend right, "\mathbb{1}"'] & \pi_1(X, gf(p)) \arrow[r, "\gamma_*^{-1}"] & \pi_1(X,p) \end{tikzcd}$$

with the diagram:

$$\begin{tikzcd} \pi_1(X) \arrow[r, "(gf)_*"] \arrow[rr, bend right, "\mathbb{1}"'] & \pi_1(X) \arrow[r, "\gamma_*^{-1}"] & \pi_1(X) \end{tikzcd}$$

□

Fact 3.6. Using the above fact, we can find that the Möbius strip, the cylinder, the punctured plane $\mathbb{E}^2 \setminus \{0\}$, and the solid torus, all have the homotopy type of a circle S^1, and consequiently have $\mathbb{Z}$ as fundamental group.

Fact 3.7. Also, $\mathbb{E}^n \setminus \{0\}$ deformation-retracts onto S^{n-1} and is therefore a simply connected space when $n \geq 3$.

Definition 3.6 (Contractible)**.** A space X is called contractible if the identity map $\mathbb{1}_X$ is homotopic to the constant map at some point of X.

Theorem 3.9.

1. *A space is contractible if and only if it has the homotopy type of a point.*
2. *A contractible space is simply connected.*
3. *Any two maps into a contractible space are homotopic.*
4. *If X is contractible, then $\mathbb{1}_X$ is homotopic to the constant map at x for any $x \in X$.*

Example 3.7. Here is a contractible space that is really hard to imagine. It is called the topologist's dunce hat. It is formed by identifying the sides of a triangle in the manner indicated in the following figure:

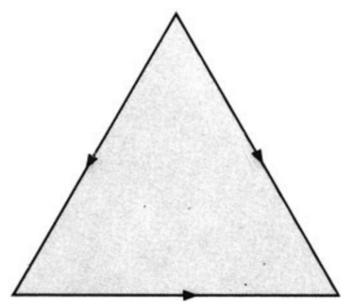


Figure 17: Topologist's dunce hat

`fig:topo-d-hat`

You may imagine identifying a pair of sides, then identify the identitied edge to the resting side. The direction for identifying is really important. This space is contractible. Despite some hard words said by the book, it is really simple to see on figure 17. Draw any loop on it and shrink it. Noticing that all edges are identified, though in some odd direction.

Remark 3.5. An identity function $\mathbb{1}_X$ on a contractible space X is homotopic to constant function $\mathbb{1}_p$ on a point $p \in X$. But this homotopy may not keep the point p fixed. That is, we do not necessarily have $\mathbb{1}_X \cong \mathbb{1}_p$ rel p.

An example on page l08 of [1]. In a space called the topologist's comb:

$$K = \{\frac{1}{n} | n \in \mathbb{N} \setminus \{0\}\}$$
$$\text{comb space} = (\{0\} \times [0,1]) \cup (K \times [0,1]) \cup ([0,1] \times \{0\})$$

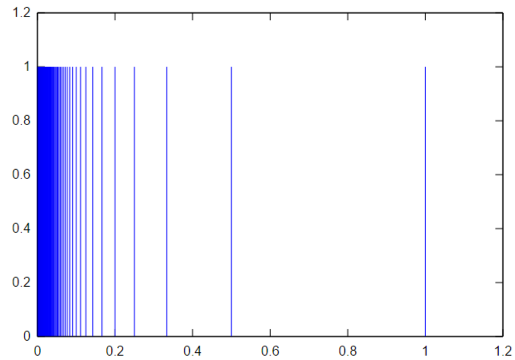


Figure 18: Topologist's Comb (from Wikipedia)

We cannot continuously defrom the identity function $\mathbb{1}_X$ to the constant function on $p = (0, 1)$. Intuitively, there are infinite points around p that are infinitesimally close to it. But the identity function must shrink from all these infinite points to the base line $[0, 1]$ without afftecting the value on p. This sounds unlikely.

Analytically, the argument is provided by Pedro in his Math.SE post:

> " **Idea:** Take the sequence $x_n = (\frac{1}{n}, 1)$, it converges to x_0. If existed such homotopy $H(x, t)$ then the sequences $H(x_n, t)$ would still converge to x_0.
> You have for each neighborhood U of x_0 a number $N_{(U,t)}$ and $\epsilon_{(U,t)}$ such that $H(x_n, s) \in U$ for $n > N_{(U,t)}$ and $|t - s| < \epsilon_{(U,t)}$.
> Covering the interval I with $(t - \epsilon_{(U,t)}, t + \epsilon_{(U,t)})$ and taking a finite subcover you obtain a number N such that $H(x_n, t) \in U$ for any $n > N$ and $t \in [0, 1]$.
> Now you can use disconnectedness of a small neighborhood to show that the homotopy can't take the elements of the sequence to x_0. "

Note: The teacher decided to temporarily switched to the next chapter, which is about a technique to patch the whole space. He will come back to the remaining sections of chapter 5 later.

4 Chapter 6 - Triangulation

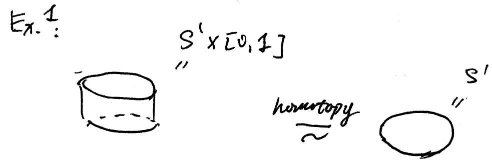
sec:Chapter6-Triangulation

4.1 Section 6.1 Triangulating Spaces

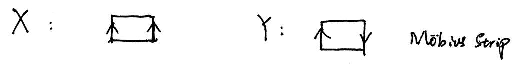
sec:Triangulating-Spaces

He gave to examples first,

Example 4.1. A trivial example:



Example 4.2. A non trivial one. That is, the Möbius Strip also has the homotopy type of a ciclr S^1.



X, Y can be both shrink to S^1. , though shrink Y, one must be careful to check that the shrinking function is compatible on the identified edges.

But for complicated spaces, its homotopy type will not be so easy to calculate in with simple imagination. Here we will introduce a new technique to actually calculate the homotopy type of topological spaces.

The technique is called Triangulation. The most visual example comes from the computer vision technology (though I cannot find a picture by directly Googling triangulation). The idea we wants to stress is that the Triangulation is like use small patches of triangles to patch and cover the surface of 3D smooth objects. Increase the overall number of patches and make each patch get smaller. In the limit of this process one might get to recover the original image.

Definition 4.1 (Simplex of $\dim k$)**.** The stanford simplex of $\dim k$ in R^{k+1} is defined as follows. Let $v_i = (0, \cdots, 0, 1, 0, \cdots 0)$, where 1 is in $i+1$ coordinates. That is:

$$v_0 = (1, 0, \cdots)$$

$$v_1 = (0, 1, 0 \cdots)$$

$$\cdots$$

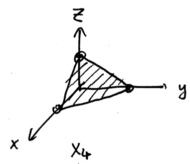
$$v_0 = (0, \cdots, 0, 1)$$

Then the simplex is the smallest convex set containing $\{v_0, \cdots, v_k\}$ in $\mathbb{R}^{k+1}$. It is also called a **k-simplex** .

Ex. $k = 0$: a point

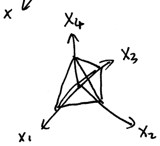
$k = 1$: a closed line segment connecting v_0, v_1.

$k = 2$:



is a triangle

$k = 3$:



is a ~~octa~~hedral
pyramid
tetra
四面体.

Example 4.3.

Fact 4.1. Any point in a k-simplex is of the form:

$$\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_k v_k \tag{4.1.1}$$

where each $\lambda_i \geq 0$, and $\sum_{i=0}^{k} \lambda_k = 1$.

Example 4.4. There is a special point in the k-simplex:

$$\frac{1}{k+1} \sum_{i=0}^{k} v_i \tag{4.1.2}$$

When $k = 2$, this is just the usual center of gravity of triangle.

We want to study those space which is the union of a finite collection of simplexes which fit together nicely in some Euclidean space. These are called "triangulable spaces". More specifically:

Definition 4.2 (faces)**.** If A and B are simplexes, and if the vertices of B form a subset of vertices of A. Then B is called a face of A, written as $B < A$.

Definition 4.3 (Simplicial Complex)**.** A **finite** collection of simplexes in some Euclidean space in $\mathbb{R}^n$ is called a simplicial complex, if

1. whenever a simplex lies in this collection, then so does its faces
2. whenever two simplexes in this collection intersect, their intersection is a common face of these two simplexes.

Example 4.5.

Definition 4.4 (Topology on simplicial complex)**.** The topology on a simplicial complex is given by the subspace topology of Euclidean space. Let K be a simplicial complex, we denote $|K|$ as the topology of K. This $|K|$ is called a **polyhedron** .

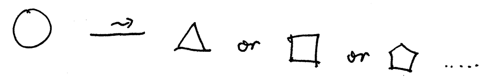
Definition 4.5 (Triangulation and Triangulable)**.** A triangulation on a topological space X consists of a simplicial complex K, and a homeomorphism h:

$$h : |K| \to X$$

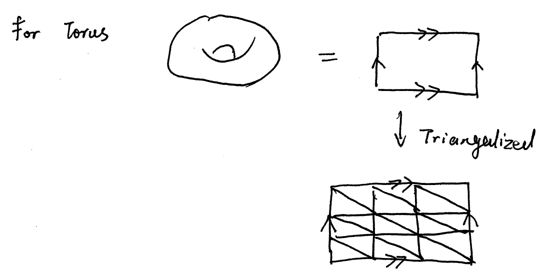
A space is called triangulable if such simplicial complex $|K|$ and homeomorphism h exists.

Fact 4.2. If X is triangulable, then X is compact and can be made into a metric space. (Notice that the simplicial complex is a finite collection.)

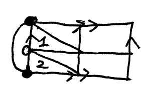
Remark 4.1. Trigulation is in general not unique. For example:



Example 4.6. Here is a triangulation on the torus:



Notice that the following diagram is not a triangulation of torus:



because when ~~△~~ △1 intersects △2, ~~it is not~~ the intersection (2 points) is not a face!

Lemma 4.1. *Let K be a simplicial complex in $\mathbb{R}^n$, then*

1. *$|K|$ is compact.*
2. *Each point of $|K|$ lies in the interior of exactly one simplex of K.*
3. *If we take the simplexes of K seperately, and give their union the identification topology, we obtain $|K|$. Thus we have two equivalent ways to view the topology on K.*

4. *If $|K|$ is connected, then $|K|$ is also path-connected.*

Proof. 1. Obviously.

2. Since K is connected by simplexes, a point p must obviously be contained in some simplex. Also, since on each simplex p must be on the interior of it, or interior of its components. We only need to prove that the containing simplex is unique. Suppose p is the interior of simplexes A and B. Notice that their intersection must be their faces. The only face of A or B containing interior points is A or B itself. Hence $A = B$.

3. By definition of identification topology, we can see that in the identified space, a subset $C \subset |K|$, C is closed $\Leftrightarrow C \cap |A|$ is closed in $|A|$ for all simplexes $A \in K$. The rest of the proof is obvious.

4. We need only to show that $|K|$ is locally path-connected. But any point $p \in K$, $p \in$ interior of some A where $A \in K$. That is, we can find a neighbourhood U of p, and U is contained in A ($U \subset A$). (More specifically, we can find ε such that $B_\varepsilon(p) \cup |K| = B_\varepsilon(p) \cup |A|$) But a simplex A is obviously locally path-connected. Hence U (or $B_\varepsilon(p)$) is path connected, hence $|K|$ is path-connected.

□

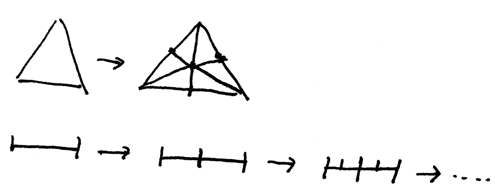
sec:Barycentric-Division

4.2 Section 6.2 Barycentric Division

Now we begin to make the triangulation more and more detailed. We start from a simplicial complex K and constructing a new simplicial complex K_1, such that

1. $|K| = |K_1|$, homeomorphically.
2. diameter $|K_1| <$ diameter $|K|$.

The general construction method is called Barycentric division, visualized as:



We first define a concept

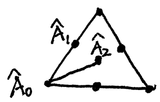
Definition 4.6 ($\hat{A}$ Bary center)**.** Let A be a simplex, then the bery center of A, denoted $\hat{A}$, is the point:

$$\hat{A} := \frac{1}{k+1}(v_0 + \cdots + v_k) \tag{4.2.1}$$

Then, the general process is: The new space K_2 is such that

1. The vertices of K_1 are bary centers of **ALL** simplexes of K. Note the the bary center of a 0-simplex is just itself. So in general, K_1 has more vortices of K.
2. A collection $\hat{A}_0, \cdots, \hat{A}_k$ of such bary centers will form a vortex of a k-simplex in K_1 if and only if there is a permutation σ of $\{0, 1, \cdots, k\}$ such that
$$A_{\sigma 0} < A_{\sigma 1} < \cdots < A_{\sigma k} \tag{4.2.2}$$

Remark 4.2. The second point above tries to do the following



we have: $A_0 < A_1$

a point

an edge of original triangle

$A_0 < A_2$ the original triangle.

Also $A_0 < A_1 < A_2$

sec:Anchor

5 Anchor

References

book [1] M.A. Armstrong. Basic Topology. 2ed.

Singer.Thorpe [2] I.M. Singer, J.A. Thorpe. Lecture Notes on Elementary Topology and Geometry. UTM.

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