Notes about the Minimal Dimensional Method to Classify Topological Phases (preliminary)

Taper

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Abstract

Haha, leave this until later.

Contents

1 Outline

sec:Outline

The classification of topological insulators in non-interacting system is in effect a classification of $N \times N$ Hamiltonian matrix. The simple case considered here is the classification done by considering how the Hamiltonian matrix responses to those anti-unitary symmetries, when we ignored all unitary symmetries. Specifically, when we module out any unitary symmetries of the Hamiltonian, the number of meaningful anti-unitary symmetries are limited, and we only classify topological insulators based on how our Hamiltonian matrix responses to these anti-unitary symmetries.

There are two classical approaches towards this classification. The first [SRFL08] is to examine the existence of Anderson delocalization at the boundary of the insulator. The existence of robust surface conducting state is the signature of topological insulators. When some random impurity potentials could be present in these surfaces, the Anderson localization is the obstruction that surface conducting states must overcome. This approach uses the Nonlinear Sigma Models (NLσMs) to describe the surface Hamiltonian and consider when can a localization-breaking term can be added to NLσMs.

The second approach [Kit09] is to use see the classification as an extension problem of Clifford Algebras. The anti-unitary symmetries already possess some behavior similar to generators of Clifford Algebras. And sometimes Hamiltonians could be regarded as yet another generator of Clifford Algebra [MF13]. The second approach is to put the symmetries in the Clifford Algebra first, and consider adding the Hamiltonian that preserve such symmetries to the Clifford Algebra. So the problem of classification of Hamiltonians becomes a problem of extending the Clifford Algebra.

But the key to understand topological insulators, are not necessarily the Hamiltonian of the whole bulk, but the physics happening when the bulk gap closes. For example, the Chern number is the integration on a compact manifold T^d without boundary. Therefore, it should always be zero because of the Stokes theorem, unless inside this manifold there is somewhere where the formula for Chern number got "blown up", and it is just this locally "blown up" point (caused by the crossing bands) that is responsible for the nonzero Chern number, and for the non-triviality of topological insulators. More generally, we believe that there is a bulk-boundary correspondence that the properties of the bulk is reflected by the properties in the boundary.

Therefore, we can detect the different types of topological insulators when we are focused in only the crossing point of the band, or, if we shifted the energy zero level to the crossing point, in only the low energy physics. We expect a locally linear crossing energy spectrum around this point, so we expect $E = \pm\sqrt{\vec{k}^2 + m^2}$, with m being the parameter controlling the closing or opening of this spectrum. One simple Hamiltonian with only first-order derivatives to describe such a low energy physics is the Dirac Hamiltonian, as shown by the traditional discussion of Klein-Gordon equation all tells us to look at Dirac Hamiltonian.

Using the Dirac Hamiltonian, it is discovered by Dr. Chiu [Chi13] that the classification of topological insulators could be brought into very simple applications of classification and representation theory of Clifford Algebras, which have been well studied with excellent resources available online. In his classification, he established an isomorphism from $G_{\text{symmetries class}}$, the algebra generated by symmetry operators and Dirac Hamiltonians, to some Clifford Algebras. Consequently, the Hamiltonian and symmetry operators, all become some specific representations of their corresponding Clifford Algebras. Also, it will be argued later that the minimal matrix dimension of such representation, i.e. the minimal dimension of the Dirac Hamiltonian, can tell the specific type of this topological insulator. Therefore, as a whole all the classification work is reduced to the consideration of minimal matrix dimension of the representation of Clifford Algebra, in different spatial dimension.

This work may be considered as a note to Chiu's dissertation [Chi13]. However, I do not guarantee that my interpretation is the same as his as I have modified and amended some parts to make the work more, in my opinion, congruous and systematic.

Structure of this work This work are divided into several parts.

1. **Ten-fold Way** Introduces the result of Ten-fold way: why we care only about anti-unitary symmetries, why there are ten different classes.
2. **Classification by Dirac Mass Hamiltonians** gives comprehensive account of how the Minimal Dirac Hamiltonian approach works, and how the ten-fold way is derived in this approach.
3. **Looking Further** outlines the possible direction in classification under unitary symmetries, and some other results.

2 Ten-fold Way

We begin our discussion with a background for this classification. The classification of topological phases with different non-unitary symmetries in non-interacting picture, are in essence, the classification of $N \times N$ matrices by its response to three discrete symmetries: time reversal symmetry (T), charge conjugation/particle hole symmetry (C), and chiral/sub-lattice symmetry (S). As for those unitary symmetries, one can in principle, block diagonalize the Hamiltonian $H = \mathrm{diag}(H^{\lambda_1}, H^{\lambda_2}, \cdots)$, such that each block H^{λ_i} is labeled by irreducible representations λ_i, and it has no memory of the unitary symmetries. Then, our classification is applied to those blocks and still holds.

It will be found that, those T, C are the only two meaningful non-unitary symmetries of the Single-particle system, and there are in total 10 different ways in which the Single-particle Hamiltonian could respect the three symmetries T, C, and $S = TC$. Now we explain in detail, about the reason why we care about, and only care about these three symmetries T, C, and S.

ngle-particle Hamiltonian

2.1 The symmetries of Single-particle Hamiltonian

According to Wigner's theorem, symmetries of physical system can either be unitary represented, or anti-unitarily represented. Here we show that in the case of anti-unitary symmetries, there are only 2 different kinds of them that a Single-particle Hamiltonian in first-quantized space can have. Actually, there can be more anti-unitary symmetries than this. But under reasonable assumptions, we can limit our discussion in only this two types.

The Single-particle Hamiltonian acting on Fock space is

$$\hat{H} = \sum_{A,B} \hat{\psi}_A^\dagger H_{AB} \hat{\psi}_B \tag{2.1.1}$$

eq:H-2nd

where H_{AB} are just complex numbers, $\hat{\psi}_A^\dagger$ and $\hat{\psi}_B$ are creation and annihilation operators acting on Fock space.

Here and henceforth, we will add a hat ˆ to all operators in Fock space to stress that it acts on Second-quantized Fock space. A symmetry of the Hamiltonian, represented as an operator $\hat{U}$, must have:

$$\hat{U}\hat{H}\hat{U}^{-1} = \hat{H} \tag{2.1.2}$$

eq:sym-in-2nd-1

eq:sym-in-2nd-permute

We expect $\hat{U}$ to change the creation/annihilation operators in two different ways. First, it may only permute the creation/annihilation operators:

$$\hat{\psi}'_A = \hat{U}\hat{\psi}_A\hat{U}^{-1} = \sum_B (u^\dagger)_{AB}\hat{\psi}_B \tag{2.1.3a}$$

eq:sym-in-2nd-permute-1

$$\hat{\psi}'^\dagger_A = \hat{U}\hat{\psi}^\dagger_A\hat{U}^{-1} = \sum_B \hat{\psi}^\dagger_B u_{AB} \tag{2.1.3b}$$

eq:sym-in-2nd-permute-2

eq:sym-cc

Where u is some matrix implementing the permutation. Or, it may interchange the role of creation/annihilation operators:

$$\hat{\psi}'_A = \hat{U}\hat{\psi}_A\hat{U}^{-1} = \sum_B (u^*)^\dagger_{AB}\hat{\psi}^\dagger_B \tag{2.1.4a}$$

eq:sym-cc-1

$$\hat{\psi}'^\dagger_A = \hat{U}\hat{\psi}^\dagger_A\hat{U}^{-1} = \sum_B \hat{\psi}_B u^*_{BA} \tag{2.1.4b}$$

eq:sym-cc-2

Where u^* is some other matrix implementing the interchange. We write complex conjugation u^* instead of u for convenience. In both cases (permute or interchange), to conserve the anticommutation relation between $\hat{\psi}^\dagger_A$ and $\hat{\psi}_B$ operators, one can easily show that u should be a unitary matrix.

Also, $\hat{U}$ may be linear or anti-linear. The different combinations of these conditions give us 1 unitary symmetry, 2 anti-unitary symmetries, and 1 special symmetry, to be explained below.

Case 1: Unitary Symmetry Assume the symmetry just permutes the creation/annihilation operators, as in equation 2.1.3, and assume it is linear in Second-quantized Hamiltonian, i.e. $\hat{U}i\hat{U}^{-1} = i$. The permutation relation 2.1.3 plugged into equation 2.1.2, gives

$$uHu^{-1} = H \tag{2.1.5}$$

Therefore, in this case the symmetry is unitarily realized in First-quantized Hamiltonian H_{AB}.

Case 2: Anti-unitary T Symmetry Assume the symmetry just permutes the creation/annihilation operators, as in equation 2.1.3, but assume it is anti-linear in Second-quantized Hamiltonian, i.e. $\hat{U}i\hat{U}^{-1} = -i$. The permutation relation 2.1.3 plugged into equation 2.1.2, gives a different result, since $\hat{U}H_{AB}\hat{U}^{-1} = H^*_{AB}$ now.

$$uH^*u^\dagger = H \tag{2.1.6}$$

Or:

$$uKHu^tK = uKH(uK)^{-1} = H \tag{2.1.7}$$

where K is complex conjugation. This symmetry is called Time-reversal symmetry, and is realized in First-quantized Hamiltonian as an anti-unitary operator $T = uK$.

Case 3: Anti-unitary C Symmetry Assume the symmetry just interchange the creation/annihilation operators, as in equation 2.1.4, but assume it is linear in Second-quantized Hamiltonian, i.e. $\hat{U}i\hat{U}^{-1} = i$. The interchange relation 2.1.4 plugged into equation 2.1.2, gives: [1]

$$u(H - \frac{1}{2}\operatorname{tr}(H))^t u^\dagger = -(H - \frac{1}{2}\operatorname{tr}(H)) \tag{2.1.8}$$

eq:sym-C-cond

Taking the trace of above equality will give $2\operatorname{tr}(H) = N\operatorname{tr}(H)$, since in solids $N >> 2$, we must have $\operatorname{tr}(H) = 0$. Then the above equality simplifies into [2]:

$$uH^* u^\dagger = uKH(uK)^{-1} = -H \tag{2.1.9}$$

This type of symmetry is called charge-conjugation symmetry. It is also called particle-hole symmetry in condensed matter physics. It is realized as $C = uK$ with $CHC^{-1} = -H$ for 1st-quantized single-particle Hamiltonian.

Case 4: Unitary S Symmetry Assume the symmetry now interchange the creation/annihilation operators, as in equation 2.1.4, and assume it is anti-linear in Second-quantized Hamiltonian, i.e. $\hat{U}i\hat{U}^{-1} = -i$. The interchange relation 2.1.4 plugged into equation 2.1.2, gives:

$$u(H - \frac{1}{2}\operatorname{tr}(H))u^\dagger = -(H - \frac{1}{2}\operatorname{tr}(H)) \tag{2.1.10}$$

eq:sym-S-cond

and since $N >> 2$, we have $\operatorname{tr}(H) = 0$. Then

$$uHu^\dagger = -H \tag{2.1.11}$$

This symmetry will be called the chiral symmetry, denoted $\hat{S} = u$. It is unitarily realized in First-quantized Hamiltonian, but since $\{S, H\} = 0$ instead of $[S, H] = 0$, it is not a traditional symmetry that we are used to. Also, it is easy to see that S is a combination of T and C,[3] and the symmetry property of Hamiltonian under T or C uniquely defined the symmetry property of Hamiltonian under S. But there is one exception. When the Hamiltonian does not obey T and C, it may or may not obey $S = TC$ as a whole.

Squaring of T, C, S Squaring of T/C symmetry operators should be proportional to identity $\mathbb{1}$, hence it is only a phase as they are unitary. This can be viewed from two perspectives. First, we expect the system, after applying twice of symmetry operation T/C, should come back to the same state, except a possible phase difference. Second, T^2/C^2 commutes with all T/C symmetric Hamiltonians (easy to derive) in all irreducible representations of unitary symmetries (to be explained later), therefore by Schur's lemma, they must be proportional to a constant, which is a phase.

It is easy to find that this phase $e^{i\delta}$should be ± 1. For example, let $T = uK$, then $T^2 = uu^*$, and $(uu^*)u = u(u^*u)$ gives us $e^{2i\delta} = 1$, hence $e^{i\delta} = \pm 1$.

But the square of S is tricky. Since $S = TC = u_T u_C^*$, and each unitary matrix u_T and u_C has a phase freedom (as they acts on creation/annihilation operators),

[1] Note that in calculating this, $\sum_i \hat{\psi}_i^\dagger \hat{\psi}_i = \mathbb{1}$ on 1st-quantized single-particle Hilbert space.
[2] note that $H^t = H^*$ for Hermitian H
[3] In fact, we could have defined $S = TC$ or $S = CT$, and obtained the same result.

we can always pick a phase such that S^2 is some phase we want. Sometimes we pick $S^2 = 1$. Sometime we want $\{T, C\} = 0$, and pick another specific phase for S^2.[4]

Why Ten Classes For the three unitary symmetries under consideration, as will be mentioned, S symmetry is a combination of T and C, $S = TC$. It will be obtained that $T^2 = \pm 1$, $C^2 = \pm 1$, and S^2 is undetermined (depending on the phase choice[5] we give for T and C). Therefore, there are in total 10 different possible ways that this 3 symmetries could be combined together: We denote $T^2 = 0$ ($C^2 = 0$) to symbolize that the system does not follow $T(C)$ symmetry. Then $T^2 = 0, \pm 1$ and $C^2 = 0, \pm 1$ gives us $3 \times 3 = 9$ different ways. But if $T^2 = C^2 = 0$, S may or may not conserve. Therefore, we have $9 - 1 + 2 = 10$ different possible ways. In each way the topological invariants of the Hamiltonian respecting these symmetries are tabulated in Figure.1 in [Lud16]. Each class is given a name (A, AII, etc.), and their properties in different spatial dimensions respect a periodic structure. For the class A and class AII, they are named complex classes and respect a 2-fold period. For the other eight classes are named real classes and respect a 8-fold period, which is why we only listed the first $0 \to 7$ spatial dimensions in Table 7.

2.2 Dealing with Unitary Symmetries

g with Unitary Symmetries

As mentioned, we could module out those unitary symmetries. This statement is made precise by the following theorem about unitary symmetries of the Hamiltonian.

Theorem 2.1 (Diagonalization of Hamiltonian in unitary representation)**.** *This space $\mathcal{V}$ decomposes into a direct sum of vector spaces $\mathcal{V}_\lambda$ associated with the irrep (irreducible representations, labeled by λ) of G_0.*

$$\mathcal{V} = \oplus_\lambda m_\lambda \mathcal{V}_\lambda \tag{2.2.1}$$

where m_λ denotes the multiplicity of λth irrep. Denote the dimension of each irrep as d_λ.

In each vector space $\mathcal{V}_\lambda$, one can choose a (orthogonal) basis of the form:

$$|v_\alpha^{(\lambda)}\rangle \otimes |w_k^{(\lambda)}\rangle \tag{2.2.2}$$

where

- *G_0 acts only only $|w^{(\lambda)}\rangle_k$, $k = 1, \cdots, d_\lambda$,*
- *H acts only on $|v^{(\lambda)}\rangle_\alpha$, $\alpha = 1, \cdots, m_\lambda$.*

Therefore, with all unitary symmetries ignored, we are classifying how Hamiltonian will be like when it respects, in 10 different ways, the combinations of T, C, S symmetries.

[4] This is a bit troublesome. First we denote $T^2 = \varepsilon_T$, $C^2 = \varepsilon_C$, then it can be found that $(u_T)^t = \varepsilon_T u_T$, $(u_C)^t = \varepsilon_C u_C$. Then, with $S = u_T u_C^*$, we have $S^\dagger = \varepsilon_C \varepsilon_T u_C u_T^* = \varepsilon_C \varepsilon_T CT$, or $CT = \varepsilon_C \varepsilon_T S^\dagger$. Since $TC + CT = 0$, we have $S + \varepsilon_C \varepsilon_T S^\dagger = 0$, or $S^2 = -\varepsilon_C \varepsilon_T$.

[5] To be explained later.

Differently Realized Anti-unitary Symmetries It should be noted that there is still a freedom of unitary matrix that implements the T or C symmetry. Therefore, we could have, for example, two different time-reversal symmetries $T_1 = u_1 K$ and $T_2 = u_2 K$. They are obviously related by a unitary matrix, say $T_1 = u_{12} T_2$. And we could easily get that u_{12} is also a unitary symmetry of the Hamiltonian (if the Hamiltonian respect both T_1 and T_2). Therefore, upon enlarging the symmetry group G_0 to include the element u_{12} and repeat the process described in the theorem above, we only have one time-reversal symmetry. Similar analysis could be done for the C symmetry as well.

The Tenfold Way The Tenfold Way, is originally the ten different ways to write down a Hamiltonian when there are ten different combinations of symmetries that the system respect. The classes of Hamiltonian (or more precisely, the evolution operators e^{itH}) are tabulated in Figure.1 of [Lud16]. Ludwig classified the topological insulators in each spatial dimension by computing the homotopy groups for each dimension. On the other hand, this classification can also be done using K-theory.

Translational Symmetry Since we are classifying in the solids, it is natural to expect translational symmetry to present. Also, the addition of translational symmetry not only

produces the same classification[6], but also generalize to arbitrary dimensions.

The Hamiltonian in k-space that preserve these three discrete symmetries should follow[7]:

$$TH(k)T^{-1} = H(-k) \tag{2.2.3a}$$
eq:T-sym-Hk

$$CH(k)C^{-1} = -H(-k) \tag{2.2.3b}$$
eq:C-sym-Hk

$$SH(k)S^{-1} = -S(k) \tag{2.2.3c}$$
eq:S-sym-Hk

3 Classification by Dirac Mass Terms

imal Hamiltonian Approach

Dirac Hamiltonian and Topological Property of the Boundary We first note that different topological phases are distinguished by the closing and then opening of the gap, i.e. a quantum phase transition. If we adiabatically flatten the energy spectrum to two bands of energy $\pm E$, the transition (closing of gap) will happen at some point where locally the energy is of a Dirac cone type. Therefore, the Dirac Hamiltonian captures this transition behavior. Then, a class of topological phases, or more specifically, a class of Hamiltonian matrix that shares the same topological invariant, can be *represented* by the Dirac Hamiltonian in this class.

3.1 Crossing Spectrum and Edge Mode

ng Spectrum and Edge Mode

Here we show that the crossing spectrum characteristic of topological insulators can be characterized by a Dirac type Hamiltonian, and with this Dirac Hamiltonian

[6] This is still a mystery to me which this is true. The exact reason is related to some isomorphism between certain homotopy groups. Interested reader could see Ludwig's lecture here for a clue.

[7] It is easy to anticipate the change $k \to -k$ for anti-unitary symmetries, since they change the phase factor when one Fourier transform the Hamiltonian.

we could find an Edge mode solution built into it.

Crossing Spectrum and Dirac Hamiltonian In some local region around the crossing point, the energy spectrum should look like:

$$E = \pm\sqrt{\mathbf{k}^2 + M^2} \tag{3.1.1}$$

where **k** is the crystal momentum and M is some constant responsible for the opening and closing of the gap, or in some cases, a parameter in which we can control to open or close the gap.

As has been analysed in most Quantum Mechanics textbook[8], this energy-momentum relationship requires a Dirac type Hamiltonian. Then, the Dirac Hamiltonian we use is:

$$H_{\text{Dirac}} = M\gamma_0 + \sum_{i=1}^{d} k_i\gamma_i \tag{3.1.2}$$

eq:dirac-H

where d is the spatial dimension of the physical system, γ_i obeys the Clifford Algebra anti-commutation relation:

$$\{\gamma_i, \gamma_j\} = 2\delta_{i,j}\mathbb{1} \tag{3.1.3}$$

Edge Mode and M Now we explain why the Dirac Hamiltonian eq. 3.1.2 is useful for capturing the topological phase. First, it has a gapless state. Due to the anti-commutation relation of γ_i, we can show[9] that the energies of Dirac Hamiltonian are:

$$E_{\pm} = \pm\sqrt{M^2 + \sum_{i=1}^{d} k_i^2} \tag{3.1.4}$$

Therefore, as M varies from negative to 0 to positive, the system goes from gaped to gapless and again to gaped state. This shows that the M is parameter controlling the quantum phase transition, and the system with $M < 0$ or $M > 0$ possesses two different quantum phase.

In addition, we can analyse its edge mode. Assuming the two material of different quantum phases touches each other in the dth dimension. Then we should replace k_d by $-i\partial_{x_d}$, since the translational invariance is broken in that direction. Now the eigenvalue equation is:

$$(M\gamma_0 + \sum_{i=1}^{d-1} k_i\gamma_i - i\partial_{x_d}\gamma_d)\Phi = E\Phi \tag{3.1.5}$$

Past experience teaches us the ansatz (assuming $M(x_d)$ goes to $\pm\infty$ as $x_d \to \pm\infty$, so that the leading term below does not blow up the wavefunction):

$$\Phi = e^{-\int_0^{x_d} M(x'_d)\mathrm{d}x'_d}\phi(x_1, \cdots, x_{d-1}) \tag{3.1.6}$$

[8]See for example, p.99 of [Gre97]. Although their discussion is in dimension $d = 3$, it is easy to generalize it to any dimension.

[9]Note that $H^2_{\text{Dirac}} = E^2\mathbb{1}$, this tells us it can only have eigenvalues $E_{\pm}$. Note that this also explains why we choose not $\gamma_i^2 = -1$, which would give non-sensible result $H^2_{\text{Dirac}} = -E^2\mathbb{1}$.

which after plugging inside the eigenvalue equation, leads to

$$\left(M(x_d)(\gamma_0 + i\gamma_d) + \sum_{i=1}^{d-1} k_i \gamma_i \right) \phi = E\phi \tag{3.1.7}$$

After left multiplication of γ_0, we have:

$$\left(M(\mathbb{1} + i\gamma_0\gamma_d) + \sum_{i=1}^{d-1} k_i \gamma_i \right) \phi = E\gamma_0\phi \tag{3.1.8}$$

Since $i\gamma_0\gamma_d$ squares to 1, it has eigenvalues ± 1. Also, $i\gamma_0\gamma_d$ and γ_0 anticommute, so they share the same eigenspaces with γ_0 mapping all the $+1$ eigenvectors of $i\gamma_0\gamma_d$ to the -1 eigenvectors and vice versa. Now we want this state to be a surface state, so we choose the -1 eigenvectors (denoted ϕ_-) of $i\gamma_0\gamma_d$ to kill the bulk term. This leads to:

$$\gamma_0 \sum_{i=1}^{d-1} k_i \gamma_i \phi_- = \gamma_0 E \phi_- \tag{3.1.9}$$

As mentioned, γ_0 switches the two eigenspaces, so we must necessarily have a surface Dirac Hamiltonian, by projecting all those matrices to the -1 eigenspaces:

$$H_{\text{surf}}\phi_- = \sum_{i=1}^{d-1} \Gamma_{i-}\phi_- = E\phi_- \tag{3.1.10}$$

eq:H-Dirac-surf

3.2 From Homotopy Classification to Minimal Dirac Hamiltonian Method

Homotopy to Minimal Dirac

Spectral Flattening The classification of topological insulators concerns different classes of Hamiltonian that cannot be adiabatically changed to each without closing or opening of a gap. Hence, we could in general adiabatically change our parameters such that the spectrum of Hamiltonian is simplified into two bands of energy ± 1. Under this condition, we have $H^2 = 1$, which already hints that the Hamiltonian itself is a viable candidate for generating Clifford Algebras.

Extension Problem with General Examples The general theme of homotopy classification of classifying topological insulators are done by the considering the *extension problem*, i.e. different ways to extend from a Clifford containing only symmetry operators, to a Clifford Algebra containing Hamiltonians. For example, for class A in 0-dimension, the empty space is nothing, hence $C\ell_0(\mathbb{C})$[10]. And $H^2 = 1$ as mentioned earlier, hence the extension problem is from $C\ell_0(\mathbb{C}) \to C\ell_1(\mathbb{C})$, whose possible representations form some mathematical structure called *classifying space* C_0 [MF13]. And since we are in 0-dimension, all operators are maps starting from $T^0 = S^0$. So our classified objects are different classes of maps from $S^0 \to C_0$, which is mathematically captured by the 0-th homotopy group $\pi_0(C_0) = \mathbb{Z}$. For class AIII which has only chiral symmetry operator S, we choose a phase such that $S^2 = 1$, i.e. making it a candidate for generators of Clifford Algebra. The symmetry condition $\{H, S\} = 0$ tells us now we have a

[10] The reason for using different Clifford Algebra ($C\ell(\mathbb{R})$ or $C\ell(\mathbb{C})$ for different symmetry classes will be explained in section 3.3.

extension problem of $C\ell_1(\mathbb{C}) \to C\ell_2(\mathbb{C})$, whose classifying space is C_1, and has $\pi_0(C_1) = 0$. Therefore, in 0-dimension, the class A has $\mathbb{Z}$ topological insulators and class AIII has trivial topological insulators. The case for real symmetry classes are not too complicated and is concisely mentioned in [MF13].

However, the method to obtain such classifying spaces are mathematical daunting and requires K-theory. We try to simplify it by looking at Dirac Hamiltonians, since they should capture the essence of topological states.

Extension Problem with Dirac Hamiltonian The Dirac Hamiltonian

$$H = \vec{k} \cdot \vec{\gamma} + m\tilde{\gamma}_0 \tag{3.2.1}$$

has a gap closing and opening mass term $m\tilde{\gamma}_0(r)$, depending on some parameter r. Suppose that we have a domain wall squeezed by two bulk regions A and B. Now as the parameter r changes freely from region A to region B, the symmetry condition will force the matrix γ_0 to explorer some space having the same homotopy type of some classifying space (again!). For example, for class A in 2-dimension, the Hamiltonian without mass term $(k_1\gamma_1 + k_2\gamma_2)$ consists of two gamma matrices, generating a $C\ell_2(\mathbb{C})$, whereas adding the mass term, we have $C\ell_3(\mathbb{C})$. Since there is no symmetry in class A, the extension problem concerns different ways to extend the algebra $C\ell_2(\mathbb{C}) \to C\ell_3(\mathbb{C})$, which tells us the number of unitarily non-equivalent mass terms. Detailed examples can be found in section III.C.1 of [CTSR16].

The View from SPEMT Let us consider a modified Dirac Hamiltonian with extra mass term:

$$H = M\tilde{\gamma}_0 + \sum_{i=1}^{d} k_i\gamma_i + \sum_{j=1}^{D} m_j\tilde{\gamma}_j \tag{3.2.2}$$

eq:H-spemt

Here, the parameter M characterized the domain wall between different phases of topological insulators. The extra mass term m_j represents a perturbation caused by disorder. I argue that we can do our classification in the following way. First, we consider the case when $D = 0$, i.e. without perturbation, and we consider how many different mass terms we could have. Second, we consider whether we can add an extra mass term which respect the symmetry of the Hamiltonian ($D = 1$) or not. Tthis symmetry preserving extra mass term is denoted as SPEMT. If a SPEMT can be added, then the system is not robust against perturbation and gap may be opened by disorder. So it is not protected by symmetry and is trivial topological insulator. If a SPEMT cannot be added, then this system is topological non-trivial.

One best thing about this classification is that it can be done by considering only the minimal matrix dimension of those gamma matrices in Hamiltonian. The argument is that, complex gamma matrices are of even dimension (except the trivial cases of $C\ell_{0/1}(\mathbb{C})$)[11]. Therefore, gamma matrices of different matrix dimension are built by tensor products of Pauli matrices. [12] Now, increasing the dimension of the matrix, is equivalent to tensoring them. One can add more bands of the same type, or add more bands of a different type in the smaller dimension, or add trivial bands. In all cases, the triviality of topological insulators can be detected

[11] See this post [Phy], or the p.12 and Theorem in p.5 of this [Wes98].

[12] Or, the non-trivial irreducible representations (turns out that there are at most 2) are built by tensoing Pauli matrices.

in minimal dimension, since adding trivial bands can be ignored (insensitive to addition of trivial bands), and the possible different types of matrices are limited.

However, to distinguish between a $\mathbb{Z}$ topological insulator and a $\mathbb{Z}_2$ topological insulator, we need to increase the matrix dimension to consider multiple copies of the Dirac Hamiltonian. If the topological state is stable for an arbitrary copies, then this is a $\mathbb{Z}$ topological insulator. If the topological state is stable only of an odd number of copies, this is a $\mathbb{Z}_2$ topological insulator.

Another way to view this way of classification, is to look at the surface Dirac Hamiltonian 3.1.10. Then the above mentioned approach is to see if the surface mode can be gapped, so as to detect the topological properties of the bulk.

3.3 Isomorphism between Symmetry Classes and Clifford Algebras

sec:Isomorphism

As with other classifications, we construct isomorphisms between Dirac matrices and Clifford Algebras, after which we are faced with an extension problem. The isomorphism is constructed now.

A bit of notation note. We use $\mathrm{C}\ell_{p,q}(\mathbb{R})$ to denote a Clifford Algebra over $\mathbb{R}$ with p for $+$ signature and q for $-$ signature. We use $\mathrm{C}\ell_n(\mathbb{C})$ to denote a Clifford Algebra over $\mathbb{C}$. We use $\mathcal{M}(n, K)$ to denote the algebra of $n \times n$ matrices over $K = \mathbb{C}$ or $K = \mathbb{R}$. We use $\mathbb{1}_n$ to denote the $n \times n$ identify matrix.

3.3.1 Symmetry Constraint

sec:Symmetry Constraint

The Hamiltonian with extra mass term is

$$H = M\tilde{\gamma}_0 + \sum_{j=1}^{D} m_j\tilde{\gamma}_j + \sum_{i=1}^{d} k_i\gamma_i \tag{3.3.1}$$

eq:sym-spemt

The symmetry properties 2.2.3a,2.2.3b give the following conditions on the gamma matrices:

$$\{T, \gamma_i\} = 0,\ [C, \tilde{\gamma}_j] = 0 \tag{3.3.2a}$$
$$\{C, \tilde{\gamma}_j\} = 0,\ [C, \gamma_i] = 0 \tag{3.3.2b}$$
$$\{S, \tilde{\gamma}_j\} = \{S, \gamma_i\} = 0 \tag{3.3.2c}$$

With this, we can establish an isomorphism between "Hamiltonian and Symmetry Operators" and "Clifford Algebra over $\mathbb{R}$ or $\mathbb{C}$". But let's first see some examples in action.

3.3.2 Examples of Constructing Isomorphism

sec:iso-Examples

For class AI ($T^2 = 1, C = 0, S = 0$) in $d = 1, D = 0$, we define:

$$G_{\text{AI}} = \{i, T, \tilde{\gamma}_0, \gamma_1\} \tag{3.3.3}$$

We have its (anti)commutation relations listed in Table 1.

Now with $J_4 = T\tilde{\gamma}_0\gamma_1$, $J_3 = iT\tilde{\gamma}_0$, we could verify that all $\{i, T, J_3, J_4\}$ anticommute with each other, and $J_3^2 = J_4^2 = 1$. So together they generate the algebra $\mathrm{C}\ell_{3,1}(\mathbb{R}) \cong \mathrm{C}\ell_{2,2}(\mathbb{R})$. This is consistent with the isomorphism $G_{\text{AI}} \cong \mathrm{C}\ell_{2+D,1+d}(\mathbb{R})$ shown later in Table 4.

Table 1: Generators in G_{AI} with $d = 1, D = 0$

tab:generator-AI

	i	T	$\tilde{\gamma}_0$	γ_1
i	$i^2 = -1$	$\{i, T\} = 0$	$[i, \tilde{\gamma}_0] = 0$	$[i, \gamma_1] = 0$
T		$T^2 = 1$	$[T, \tilde{\gamma}_0] = 0$	$\{T, \gamma_1\} = 0$
$\tilde{\gamma}_0$			$\tilde{\gamma}_0^2 = 1$	$\{\tilde{\gamma}_0, \gamma_1\} = 0$
γ_1				$\gamma_1^2 = 1$

Another example. For class AII ($T^2 = -1, C = 0, S = 0$) in $d = 1, D = 0$, we have:

$$G_{\text{AI}} = \{i, T, \tilde{\gamma}_0, \gamma_1\} \tag{3.3.4}$$

We have its (anti)commutation relations listed in Table 2. Now with $J_4 = T\tilde{\gamma}_0\gamma_1$,

Table 2: Generators in G_{AII} with $d = 1, D = 0$

tab:generator-AII

	i	T	$\tilde{\gamma}_0$	γ_1
i	$i^2 = -1$	$\{i, T\} = 0$	$[i, \tilde{\gamma}_0] = 0$	$[i, \gamma_1] = 0$
T		$T^2 = -1$	$[T, \tilde{\gamma}_0] = 0$	$\{T, \gamma_1\} = 0$
$\tilde{\gamma}_0$			$\tilde{\gamma}_0^2 = 1$	$\{\tilde{\gamma}_0, \gamma_1\} = 0$
γ_1				$\gamma_1^2 = 1$

$J_3 = iT\tilde{\gamma}_0$, we could verify that all $\{i, T, J_3, J_4\}$ anticommute with each other, and $J_3^2 = J_4^2 = -1$. So together they generate the algebra $C\ell_{0,4}(\mathbb{R})$. Notice that, assume we have $C\ell_{1,3}(\mathbb{R}) = \{K_1, K_2, K_3, K_1'\}$, and $K_i^2 = -1$, $(K_1')^2 = 1$. The following map

$$J_1 = K_1,\ J_2 = K_2,\ J_3 = K_3,\ J_4 = K_1K_2K_3K_1' \tag{3.3.5}$$

has the property shown in the table 3. Therefore, $C\ell_{0,4}(\mathbb{R}) \cong C\ell_{1,3}(\mathbb{R})$, which is

Table 3: Map from $C\ell_{1,3}(\mathbb{R})$ to $C\ell_{0,4}(\mathbb{R})$.

tab:map-cl13-2-cl04

	$J_1 = K_1$	$J_2 = K_2$	$J_3 = K_3$	$J_4 = K_1K_2K_3K_1'$
J_1	-1	$\{\}$	$\{\}$	$\{\}$
J_2		-1	$\{\}$	$\{\}$
J_3			-1	$\{\}$
J_4				-1

where $\{\}$ means anticommute.

consistent with $G_{\text{AII}} \cong C\ell_{d,3+D}(\mathbb{R})$ in Table 4.

3.3.3 Constructing the Isomorphism (Complex Symmetry Classes)

sec:Complex Classes-iso

The general steps for constructing the isomorphism is presented here. First, we need to treat the situation of complex classes and real classes differently. The complex classes are isomorphic to complex Clifford Algebras, while the real classes are isomorphic to real Clifford Algebras. The reason for this distinction is that, in real classes, there are complex conjugate operator K. First, there is no way to represent complex conjugation simply as multiplication of complex matrices. Second, complex matrices come with a nature definition of complex conjugate †, whereas $K^\dagger$ is ill-defined [13]. Therefore, we use real Clifford Algebras when dealing with real classes (which is the reason why they are named real classes).

For complex classes, let $G_\#$ be the group generated by elements $\{\gamma_i, \tilde{\gamma}_j, S\}$ ($i = 1, \cdots, d$, $j = 0, 1$) in each symmetry class # (so S exists only in class AIII). We may choose a phase such that $S^2 = 1$, as mentioned earlier. Then obviously $G_\#$ will be a Clifford Algebra:

$$G_A = \text{generated by } \{\gamma_i, \tilde{\gamma}_j\} \cong \mathrm{C}\ell_{d+D+1}(\mathbb{C}) \tag{3.3.6}$$

$$G_{\text{AIII}} = \text{generated by } \{\gamma_i, \tilde{\gamma}_j, S\} \cong \mathrm{C}\ell_{d+D+2}(\mathbb{C}) \tag{3.3.7}$$

`sec:Real Classes-iso`

3.3.4 Constructing the Isomorphism (Real Symmetry Classes)

For real classes, we need to include i, and symmetry operators T, C in our group $G_\#$. It turns out that we need to pick a phase such that $\{T, C\} = 0$, which is possible as mentioned earlier. Now we demonstrate the proof of isomorphism in class AII.[14]

We first note that, given a real vector space V, we can complexify a real space in two ways. The first is trivially taking the tensor product $V \otimes_{\mathbb{R}} \mathbb{C}$, which does not suit our purpose. The second is to find an almost complex structure J, which is a $\mathbb{R}$-linear map that squares to $-\mathbb{1}$, i.e. $J^2 = -\mathbb{1}$. With this almost complex structure J, then V admits in a natural way the structure of a complex vector space $V_{\mathbb{C}}$ [Dan05]. Also, a $\mathbb{R}$-linear map A on V is $\mathbb{C}$-linear($\mathbb{C}$-antilinear) if and only if A commutes (anticommutes) with J. Therefore, we can model the antiunitary operators on a real vector space naturally.

Let us denote the generators in Clifford Algebra as J_j and $\tilde{J}_i$, where $J_j^2 = -\tilde{J}_i^2 = -\mathbb{1}$. Since this almost complex structure J squares to $-\mathbb{1}$ and anticommutes with antiunitary symmetry operators, we naturally take it to be the first generator in our Clifford Algebra, $J_1 = J$.

Then we discuss some tips that will be useful for later calculation.

Tips

1. Since all generator anticommute, all matrices either commute or anticommute. So the Algebra is pretty simple.
2. Since $[A, BC] = \pm[A, CB]$ and $\{A, BC\} = \pm\{A, CB\}$, the commutation or anticommutation does not depends on the order of the matrices BC or CB. So one might rearrange them in these orders whichever is convenient.
3. For $A = \{J_1, \cdots, J_n\}, B = \{K_1, \cdots, K_m\}$, we have $AB = (-1)^{mn}BA$. However, if A and B has something in common, then the above condition breaks. This is like adding an "impurity" in it to change the commutation/anticommutation relations.
4. The restriction given by $\sum_i k_i\gamma_i$, and $\sum_j m_j\tilde{\gamma}_j$ (will be shown later) are much restrictive that the candidates for symmetry operator are only a small finite set.

`enum:tips-4`

[13] Think about this. $\langle c\phi | K\psi\rangle$ could have different values depending on whether we move K to the left as $K^\dagger$ first, or we move c out of the inner-product first.

[14] For reader unfamiliar with the complexfication of real vector space, and the realification of complex vector space, I recommend a quick reading of section 12 of [SKM89].

Class AII Now we prove the isomorphism. First, we try to construct each gamma matrices in Hamiltonian. All matrices commute with i, so in real vector space, they commute with $J_1 = J$. Then, each gamma matrices should have an even number of Clifford Algebra generators other than J_1 itself. Similarly, antiunitary symmetry operators should have an odd number of Clifford Algebra generators other than J_1. We "dope" the gamma matrices with some J_1 to make it commute/anticommute with symmetry operators. More explicitly, bearing in mind that class AII has only $T^2 = -1$, we take a quick look into Hamiltonian 3.2.2 and symmetry conditions 3.3.2, and they give us the inspiration to set:

$$H_{\text{AII}} = mJ_1J_2J_3 + \sum_{j=1}^{D} m_j J_1J_2J_{3+j} + \sum_{i=1}^{d} k_i J_2\tilde{J}_i \tag{3.3.8}$$

Here $\tilde{\gamma}_j$ are $J_1J_2J_{3+j}$, which square to 1. γ_i are $J_2\tilde{J}_i$. The Clifford Algebra is at least $\mathrm{C}\ell_{d,3+D}(\mathbb{R})$. And within this algebra, only J_2 is a possible candidate for symmetry operators, which satisfy equations 3.3.2 (use tips 4 for calculation). J_2 is found to be a time reversal operator, and $J_2^2 = -1$ confirms that this Hamiltonian belongs to class AII. Therefore, the map:

$$\begin{aligned} f : \mathrm{C}\ell_{d,3+D}(\mathbb{R}) &\to G_{\text{AII}} \\ J_1J_2J_{3+j} &\to \tilde{\gamma}_j,\ (j = 0, 1, \cdots, D) \\ J_2\tilde{J}_i &\to \gamma_i,\ (i = 0, 1, \cdots, d) \end{aligned} \tag{3.3.9}$$

is a map from Clifford Algebra $\mathrm{C}\ell_{d,3+D}(\mathbb{R})$ to symmetry class AII. The inverse map can be found easily. We first solve some formality problems. A complex number can be written as a real number by identifying i with $J = -i\sigma_y$ and 1 with $\mathbb{1}$, the identity matrix:

$$a + bi \to a\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + b\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = a\mathbb{1} + bJ \tag{3.3.10}$$

Therefore, all complex matrices $\gamma_i, \tilde{\gamma}_j$ are identified with real matrices $\Gamma_i, \tilde{\Gamma}_j$, of twice the size of $\gamma_i, \tilde{\gamma}_j$. More explicitly, $(a_{mn} + ib_{mn})$ is identified as $A + iB = (a_{mn}) + i(b_{mn}) \to \mathbb{1}_2 \otimes (a_{mn}) + J \otimes (b_{mn})$. On important matrix is the matrix J_1 which is identified as i, and is equal to:

$$J_1 \equiv -i\sigma_y \otimes \mathbb{1}_{n\times n} \tag{3.3.11}$$

The complex conjugate K is then a matrix anticommuting with J_1. Then $T \to \Gamma_T$, $C \to \Gamma_C$ for some real matrices anticommute with $i \to J_1$. We also note that, as mentioned earlier, we make a phase choice of T and C such that:

$$\{\Gamma_T, \Gamma_C\} = 0 \tag{3.3.12}$$

Now, we write symbolically $J_1J_2J_{3+j} = J_1\Gamma_TJ_{3+j} = \tilde{\Gamma}_j$, then clearly $J_{3+j} = \Gamma_TJ_1\tilde{\Gamma}_j$. Similarly, $\tilde{J}_i = \Gamma_T\Gamma_i$. So the map:

$$\begin{aligned} f^{-1} : G_{\text{AII}} &\to \mathrm{C}\ell_{d,3+D}(\mathbb{R}) \\ i\gamma_y \otimes \mathbb{1}_{n\times n} &\to J_1 \\ \Gamma_T &\to J_2 \\ \Gamma_TJ_1\Gamma_j &\to J_{3+j} \\ \Gamma_T\tilde{\Gamma}_i &\to J_i \end{aligned} \tag{3.3.13}$$

is the desired inverse map.

Class CII The class CII has only one $C^2 = -1$ more than class AII, therefore, we only need to enlarge the Clifford Algebra to include one more J_{4+D}, and set it as the C symmetry operator. The rest is exactly the same as in class AII.

All other classes can be treated similarly, so we do not repeat the calculation and only list the result in Table 4.

Table 4: Mapping Relations between symmetry classes and Clifford Algebras. Here $J_j^2 = -\mathbb{1}, \tilde{J}_i^2 = \mathbb{1}$, and they generates the Clifford Algebra. Also, only one direction of mapping is shown. The inverse map can be easily constructed accordingly. The complex class are also added for convenience. tab:map-sym-cl

Class(#)	T	C	S	Mappings	$G_\# \cong$
D	0	+	0	$\tilde{J}_1 \to \Gamma_C$, $\tilde{J}_1 J_{2+j} \to \tilde{\Gamma}_j$, $J_1 \tilde{J}_1 \tilde{J}_{1+i} \to \Gamma_i$	$\mathrm{C}\ell_{1+d,2+D}(\mathbb{R})$
DIII	−	+	1	$\tilde{J}_1 \to \Gamma_C$, $\tilde{J}_{3+D} \to \Gamma_T$, $\tilde{J}_1 J_{2+j} \to \tilde{\Gamma}_j$, $J_1 \tilde{J}_1 \tilde{J}_{1+i} \to \Gamma_i$	$\mathrm{C}\ell_{1+d,3+D}(\mathbb{R})$
AII	−	0	0	$J_2 \to \Gamma_T$, $J_1 J_2 J_{3+j} \to \tilde{\Gamma}_j$, $J_2 \tilde{J}_i \to \Gamma_i$	$\mathrm{C}\ell_{d,3+D}(\mathbb{R})$
CII	−	−	1	$J_2 \to \Gamma_T$, $J_{4+D} \to \Gamma_C$, $J_1 J_2 J_{3+j} \to \tilde{\Gamma}_j$, $J_2 \tilde{J}_i \to \Gamma_i$	$\mathrm{C}\ell_{d,4+D}(\mathbb{R})$
C	0	−	0	$J_2 \to \Gamma_C$, $J_2 \tilde{J}_{1+j} \to \tilde{\Gamma}_j$, $J_1 J_2 J_{i+2} \to \Gamma_i$	$\mathrm{C}\ell_{1+D,2+d}(\mathbb{R})$
CI	+	−	1	$J_2 \to \Gamma_C$, $\tilde{J}_{2+D} \to \Gamma_T$, $J_2 \tilde{J}_{1+j} \to \tilde{\Gamma}_j$, $J_1 J_2 J_{i+2} \to \Gamma_i$	$\mathrm{C}\ell_{2+D,2+d}(\mathbb{R})$
AI	+	0	0	$\tilde{J}_1 \to \Gamma_T$, $J_1 \tilde{J}_1 \tilde{J}_{2+j} \to \tilde{\Gamma}_j$, $\tilde{J}_1 J_{i+1} \to \Gamma_i$	$\mathrm{C}\ell_{2+D,1+d}(\mathbb{R})$
BDI	+	+	1	$\tilde{J}_1 \to \Gamma_T$, $\tilde{J}_{3+D} \to \Gamma_C$, $J_1 \tilde{J}_1 \tilde{J}_{2+j} \to \tilde{\Gamma}_j$, $\tilde{J}_1 J_{i+1} \to \Gamma_i$	$\mathrm{C}\ell_{3+D,1+d}(\mathbb{R})$
A	0	0	0		$\mathrm{C}\ell_{d+D+1}(\mathbb{C})$
AIII	0	0	1		$\mathrm{C}\ell_{d+D+2}(\mathbb{C})$

ssification in 1-dimension

3.4 Classification in 1-dimension

sec:Model Systems

3.4.1 Model Systems

Classification of 1 dimension is the simplest, since the minimal matrix dimension will mostly be 2, which means we can use the familiar Pauli matrices directly.

Class A We have the extension from $\mathrm{C}\ell_2(\mathbb{C})$ to $\mathrm{C}\ell_3(\mathbb{C})$. We have for example:

$$H_A = M\sigma_x + k_x\sigma_y \tag{3.4.1}$$

Obviously, there is an SPEMT $m\sigma_x$. So this phase is trivial.

Class AIII We have the extension from $\mathrm{C}\ell_3(\mathbb{C})$ to $\mathrm{C}\ell_4(\mathbb{C})$. We have for example:

$$H_A = M\sigma_x + k_x\sigma_y \tag{3.4.2}$$

However, the symmetry operator S takes the rest of Pauli matrices σ_z, and there is no other matrix possible for the extra mass term. So the state is

topologically non-trivial. Now we consider the state with arbitrary copies of it.

$$H_A = M\sigma_x \otimes \mathbb{1}_n + k_x\sigma_y \otimes \mathbb{1}_n, S = \sigma_z \otimes \mathbb{1}_n \tag{3.4.3}$$

where n is some positive integer. Since gamma matrices are basically tensor products of Pauli matrices, there is no SPEMT term. Hence this is a $\mathbb{Z}$ topological insulator.

The above are for complex classes. For real classes, we have to be careful, since one generator J_1 is taken up by i.

Class AII We have the extension from $\mathrm{C}\ell_{1,3}(\mathbb{R})$ to $\mathrm{C}\ell_{1,4}(\mathbb{R})$. Explicitly, we could let

$$H_{\mathrm{AII}} = M\sigma_x \otimes \mathbb{1}_2 + k_x\sigma_y \otimes \mathbb{1}_2 \tag{3.4.4}$$

with $T = \mathbb{1}_2 \otimes \sigma_x K$. The SPEMT is $m\sigma_z \otimes \sigma_x$. So this is a trivial insulator.

Class C We have the extension from $\mathrm{C}\ell_{1,3}(\mathbb{R})$ to $\mathrm{C}\ell_{2,3}(\mathbb{R})$. Explicitly, we could let

$$H_{\mathrm{C}} = M\sigma_z \otimes \mathbb{1}_2 + k_x\sigma_y \otimes \mathbb{1}_2 \tag{3.4.5}$$

with $C = \sigma_x \otimes \sigma_y K$. The SPEMT is $m\sigma_x \otimes \sigma_x$. So this is a trivial insulator.

Class CI We have the extension from $\mathrm{C}\ell_{2,3}(\mathbb{R})$, to $\mathrm{C}\ell_{3,3}(\mathbb{R})$. Explicitly, we could let

$$H_{\mathrm{CI}} = M\sigma_z \otimes \mathbb{1}_2 + k_x\sigma_y \otimes \mathbb{1}_2 \tag{3.4.6}$$

with $C = \sigma_x \otimes \sigma_y K, T = K$. The SPEMT is again $m\sigma_x \otimes \sigma_x$. So this is a trivial insulator.

Class AI We have the extension from $\mathrm{C}\ell_{2,2}(\mathbb{R})$ to $\mathrm{C}\ell_{3,2}(\mathbb{R})$. Explicitly, we could let

$$H_{\mathrm{AI}} = M\sigma_x + k_x\sigma_y \tag{3.4.7}$$

with $T = K$. The SPEMT is $m\sigma_z$. So this is a trivial insulator.

Class CII We have extension from $\mathrm{C}\ell_{1,4}(\mathbb{R})$ to $\mathrm{C}\ell_{1,5}(\mathbb{R})$. We let:

$$H_{\mathrm{CII}} = M\sigma_x \otimes \mathbb{1}_2 + k_x\sigma_y \otimes \mathbb{1}_2 \tag{3.4.8}$$

with $T = \mathbb{1}_2 \otimes \sigma_y K, C = \sigma_z \otimes \sigma_y K$. There is no SPEMT[15]. Consider arbitrary copies of it:

$$H_{\mathrm{CII}} = (M\sigma_x \otimes \mathbb{1}_2 + k_x\sigma_y \otimes \mathbb{1}_2) \otimes \mathbb{1}_n \tag{3.4.9}$$

with $T = \mathbb{1}_2 \otimes \sigma_y \otimes \mathbb{1}_n K, C = \sigma_z \otimes \sigma_y \otimes \mathbb{1}_n K$. There is still no SPEMT[16]. Therefore, this is a $\mathbb{Z}$ topological insulator.

Class BDI We have extension from $\mathrm{C}\ell_{3,2}(\mathbb{R})$ to $\mathrm{C}\ell_{4,2}(\mathbb{R})$. We let:

$$H_{\mathrm{BDI}} = M\sigma_z + k_x\sigma_y \tag{3.4.10}$$

with $T = K, C = \sigma_x K$. Obviously, there is no SPEMT. For arbitrary copies of it:

$$H_{\mathrm{BDI}} = M\sigma_z \otimes \mathbb{1}_n + k_x\sigma_y \otimes \mathbb{1} \tag{3.4.11}$$

with $T = \mathbb{1}_{n+2}K, C = \sigma_x \otimes \mathbb{1}_n K$. The SPEMT must be of the form $\sigma_x \otimes \Delta$ to anticommute with H_{BDI}. But to commute with T leads to Δ being real, and to anticommute with C leads to Δ being complex. Hence there is no SPEMT. This is a $\mathbb{Z}$ topological insulator.

[15] To anticommute with $\sigma_x \otimes \mathbb{1}_2$ and $\sigma_y \otimes \mathbb{1}_2$, it must be of the form $\sigma_z \otimes ?$. But none of the remaining choice are acceptable, considering T and C.

[16] The SPEMT has to be of the form $\sigma_z \otimes \sigma_\alpha \otimes A_{n\times n}$. Commuting with T makes A completely imaginary. But it cannot anticommute with C.

Class D We let:

$$H_{\mathrm{D}} = M\sigma_z + k_x\sigma_y \tag{3.4.12}$$

with $C = \sigma_x K$. Obviously, there is no SPEMT. Now consider two copies of it:

$$H_{\mathrm{D}} = (M\sigma_z + k_x\sigma_y) \otimes \mathbb{1}_2 \tag{3.4.13}$$

with $C = \sigma_x \otimes \mathbb{1}_2 K$. There is one SPEMT $m\sigma_z \otimes \sigma_y$. Hence, this is a $\mathbb{Z}_2$ topological insulator.

To lighten the notation a bit, we introduce $\tau_i = s_i = \sigma_i$ ($i = 0, 1, 2, 3$). But τ_i, s_i, and σ_i all act on the different spaces.

Class DIII We let:

$$H_{\mathrm{DIII}} = M\sigma_z s_0 + k_x\sigma_y s_0 \tag{3.4.14}$$

with $C = \sigma_x K, T = s_y K$. Any SPEMT anticommute with Hamiltonian has the form $\sigma_x s_i$. For it to anticommute with C, it is $\sigma_x s_y$. But this does not commute with T. Hence, there is no SPEMT. Now consider two copies of it:

$$H_{\mathrm{DIII}} = (M\sigma_z s_0 + k_x\sigma_y s_0)\tau_0 \tag{3.4.15}$$

with $C = \sigma_x K$, $T = s_y K$. There is one SPEMT $m\sigma_x s_x \tau_y$. So this is a $\mathbb{Z}_2$ topological insulator.

In summary, we shown examples of different classes of topological insulators in $d = 1$, summarized in Table 5.

Table 5: Topological Insulators in $d = 1$

tab:ti-d=1

Class(#)	T	C	S	$d = 1$
A	0	0	0	0
AIII	0	0	1	$\mathbb{Z}$
D	0	+	0	$\mathbb{Z}_2$
DIII	−	+	1	$\mathbb{Z}_2$
AII	−	0	0	0
CII	−	−	1	$\mathbb{Z}$
C	0	−	0	0
CI	+	−	1	0
AI	+	0	0	0
BDI	+	+	1	$\mathbb{Z}$

3.4.2 Irreducible Representation Arguments for Classification

guments for Classification

The above examples are only valid in special points of the Brillouin Zone. However, in real materials, one will be faced with Hamiltonians of various form:

$$H = m\gamma_0 \pm k_1\gamma_1 \pm k_2\gamma_2 \cdots \tag{3.4.16}$$

Here, we look at this problem from a representation theory point of view. It is well known that representation of Clifford Algebras are completely reducible, and there are only 1 or 2 nontrivial (dimension > 1) inequivalent irreducible representations of Clifford Algebras (see Appendix A). The dimension of the representation of relevant Clifford Algebras are listed in Table 6. There are only three types of change occurs in that table:

1. **n** $\to$ **n** or **n** $\to$ **n₂** Basically, this means that in the same dimension we could always have a SPEMT present. So there will never be stable edge states.
2. **n** $\to$ **2n** or **n₂** $\to$ **2n** In this case, a single copy of Hamiltonian is stable against perturbation. Therefore, we consider two copies of the Hamiltonian and see if the double-size Hamiltonian can be gapped by perturbation. If one denote the one irreducible representation in dimension n by Γ, then we are most importantly faced with two types of double-size Hamiltonian:

$$H_{\text{doubled},1} = (M\tilde{\Gamma}_0 + \sum_i k_i\Gamma_i) \otimes \mathbb{1}_{2\times 2} \tag{3.4.17}$$

or

$$H_{\text{doubled},2} = \begin{pmatrix} M\tilde{\Gamma}_0 + \sum_i k_i\Gamma_i & 0 \\ 0 & \pm M\tilde{\Gamma}_0 + \sum_i \pm k_i\Gamma_i \end{pmatrix} \tag{3.4.18}$$

Where the sign $\pm$ for $H_{\text{doubled},2}$ is such that $H_{\text{doubled},2} \neq H_{\text{doubled},1}$. The first Hamiltonian is called two equivalent copies of the original Hamiltonian, whereas the second one is called two inequivalent copies of the original Hamiltonian.[17] For the second copy, we could always find a SPEMT to gap the system. For example, for

$$H_{\text{doubled},2} = \begin{pmatrix} M\tilde{\Gamma}_0 + \sum_i k_i\Gamma_i & 0 \\ 0 & -M\tilde{\Gamma}_0 + \sum_i k_i\Gamma_i \end{pmatrix} = M\tilde{\Gamma}_0\otimes\sigma_z + (\sum_i k_i\Gamma_i)\otimes\mathbb{1}_{2\times 2} \tag{3.4.19}$$

It can be gapped by a SPEMT such as $\Gamma_1 \otimes \sigma_x$.

For the first type, the situation is different and it distinguishes the $\mathbb{Z}$ and $\mathbb{Z}_2$ topological insulator. We first not that, a SPEMT in this case must be of the form $\Gamma \otimes A_{2\times 2}$, where Γ anticommutes with Γ_i and $\tilde{\Gamma}_0$, A is some 2×2 matrix.

   (a) **n** $\to$ **2n**: In this case, there will be a SPEMT to gap the doubled Hamiltonian. So this is a $\mathbb{Z}_2$ topological insulator. To be more specific, assume that we have the extension problem of $\mathrm{C}\ell_{p,q}(\mathbb{R}) \to \mathrm{C}\ell_{p,q+1}(\mathbb{R})$, where $\dim(\mathrm{C}\ell_{p,q}(\mathbb{R})) = n$, $\dim(\mathrm{C}\ell_{p,q+1}(\mathbb{R})) = 2n$. It can be checked in Table 6 that $\dim(\mathrm{C}\ell_{p+1,q}(\mathbb{R})) = n_2$, i.e. an extra Γ_{p+1} is present in the same dimension. This extra term form a SPEMT in the doubled Hamiltonian as $\Gamma_{p+1} \otimes i\sigma_y$.
   Similarly, if we have $\mathrm{C}\ell_{p,q}(\mathbb{R}) \to \mathrm{C}\ell_{p+1,q}(\mathbb{R})$, an extra term Γ_{q+1} from $\mathrm{C}\ell_{p,q+1}(\mathbb{R})$ is present in the same dimension, and it forms a SPEMT in the doubled Hamiltonian as $\Gamma_{q+1} \otimes i\sigma_y$.

   (b) **n₂** $\to$ **2n**: In this case, unfortunately no SPEMT can be found from either $\mathrm{C}\ell_{p+1,q}(\mathbb{R})$ or $\mathrm{C}\ell_{p,q+1}(\mathbb{R})$, as can be checked in Table 6 that they all double the dimension.

In this way we confirmed the classification in Table 5. And we note that, as long as $p - q \bmod 8$ is fixed, all the above discussion will be the same. This come as a property of the representation of Clifford Algebras (see Appendix A for details). Also, if one is interested in how those model system mentioned before can be reflected in the representation of Clifford Algebras, one could see Appendix B for one example.

[17] The exact reason for "equivalent" and "inequivalent" would be found in representation theory or Clifford Algebras, but it is not important for our discussion here.

Table 6: Matrix dimensions related to the extension problem for different classes in $d = 1$. Note that a subscript 2 as in 2_2, means that there are two inequivalent representations in the same matrix dimension. Note also that since the complexification of a real matrix reduce the dimension by $\frac{1}{2}$, we write only half of the dimension as shown in Appendix A.

tab:mat-dim-allClass-1d

Name of Class	Extension of $C\ell$	Dimension Change	Topological Invariant
A	$C\ell_2(\mathbb{C}) \to C\ell_3(\mathbb{C})$	$2 \to 2_2$	0
AIII	$C\ell_3(\mathbb{C}) \to C\ell_4(\mathbb{C})$	$2_2 \to 4$	$\mathbb{Z}$
AII	$C\ell_{1,3}(\mathbb{R}) \to C\ell_{1,4}(\mathbb{R})$	$4 \to 4_2$	0
C	$C\ell_{1,3}(\mathbb{R}) \to C\ell_{2,3}(\mathbb{R})$	$4 \to 4$	0
CI	$C\ell_{2,3}(\mathbb{R}) \to C\ell_{3,3}(\mathbb{R})$	$4 \to 4$	0
AI	$C\ell_{2,2}(\mathbb{R}) \to C\ell_{3,2}(\mathbb{R})$	$2 \to 2_2$	0
CII	$C\ell_{1,4}(\mathbb{R}) \to C\ell_{1,5}(\mathbb{R})$	$4_2 \to 8$ with $C\ell_{2,4}(\mathbb{R}) \sim 8$.	$\mathbb{Z}$
BDI	$C\ell_{3,2}(\mathbb{R}) \to C\ell_{4,2}(\mathbb{R})$	$2_2 \to 4$ with $C\ell_{3,3}(\mathbb{R}) \sim 4$.	$\mathbb{Z}$
D	$C\ell_{2,2}(\mathbb{R}) \to C\ell_{2,3}(\mathbb{R})$	$2 \to 4$ with $C\ell_{3,2}(\mathbb{R}) \sim 2_2$.	$\mathbb{Z}_2$
DIII	$C\ell_{2,3}(\mathbb{R}) \to C\ell_{2,4}(\mathbb{R})$	$4 \to 8$ with $C\ell_{3,3}(\mathbb{R}) \sim 4$.	$\mathbb{Z}_2$

3.5 Classification in Arbitrary Dimension

on in Arbitrary Dimension

We now show that, when combined with some properties, the classification in $d = 1$ can be generalized to classification in arbitrary dimensions. In K-theoretic classification, the extension problem is not affected when tensored with some other algebras. Specifically, for complex classes, the extension problem of $C\ell_n(\mathbb{C}) \to C\ell_m(\mathbb{C})$ is the same as the extension problem of $C\ell_n(\mathbb{C}) \otimes C\ell_2(\mathbb{C}) \to C\ell_m(\mathbb{C}) \otimes C\ell_2(\mathbb{C})$ (See for example, sec III.C.1 of [CTSR16]). For real classes, similar property holds. This insensitivity to tensoring a new algebra, can be understood in SPEMT background. Because $C\ell_2(\mathbb{C}) \cong \mathcal{M}(2, \mathbb{C})$, which only multiply the matrix dimension of both sides by a constant 2. So the pattern of $n \to n, n \to n_2$, $n \to 2n$, and $n_2 \to 2n$, mentioned in previous section, remains unaltered. For real classes, instead of $C\ell_2(\mathbb{C})$, we have $C\ell_{1,1}(\mathbb{R}) \cong \mathcal{M}(2, \mathbb{R})$, or $C\ell_{2,2}(\mathbb{R}) \cong C\ell_{1,1}(\mathbb{R}) \otimes C\ell_{1,1}(\mathbb{R}) \cong \mathcal{M}(4, \mathbb{R})$. So the reasoning is the same.

Now, let us write $\sim$ to represent the equivalence between two extension problems. The Complex Clifford Algebra has a periodicity of 2:

$$C\ell_{n+2}(\mathbb{C}) \cong C\ell_n(\mathbb{C}) \otimes C\ell_2(\mathbb{C}) \tag{3.5.1}$$

The Real Clifford Algebra has a periodicity of 8:

$$C\ell_{p+8,q}(\mathbb{R}) = C\ell_{p,q+8}(\mathbb{R}) = C\ell_{p,q}(\mathbb{R}) \otimes \mathcal{M}(16, \mathbb{R}) \tag{3.5.2}$$

This two relation means that, the extension problem will be the same with respect to a periodicity of 2 for complex symmetry classes, and 8 for real symmetry classes. Hence the classification of topological insulators need only be done with $d = 1, 2$ for complex symmetry classes, and $d = 0, 1, \cdots, 7$ for real symmetry classes, i.e.

eq:cli-periodic

$$G_{\#}(d = d_0) \sim G_{\#}(d = (d_0 \mathrm{mod} 2)) \quad \text{(complex class)} \tag{3.5.3a}$$

$$G_{\#}(d = d_0) \sim G_{\#}(d = (d_0 \mathrm{mod} 8)) \quad \text{(real class)} \tag{3.5.3b}$$

Another useful property is:

$$\mathrm{C}\ell_{p+1,q+1}(\mathbb{R}) \cong \mathrm{C}\ell_{p,q}(\mathbb{R}) \otimes \mathcal{M}(2,\mathbb{R}) \tag{3.5.4}$$

This property tells us that Clifford Algebra basically depends only on the difference $p-q$, or combined with previous periodicity, $p-q$ (mod 8). For example, we have (let n be an arbitrary integer): $\mathrm{C}\ell_{p+1,q+1}(\mathbb{R}) \sim \mathrm{C}\ell_{p,q}(\mathbb{R})$. With this, one can derive two chains of relations:

$$\begin{aligned} &\mathrm{C}\ell_{1+(D+n),2+D}(\mathbb{R}) \sim \mathrm{C}\ell_{1+(D+1+n),3+D}(\mathbb{R}) \\ \sim &\mathrm{C}\ell_{(D+2+n),3+D}(\mathbb{R}) \sim \mathrm{C}\ell_{(D+3+n),4+D}(\mathbb{R}) \end{aligned} \tag{3.5.5}$$

and

$$\begin{aligned} &\mathrm{C}\ell_{1+D,2+(D+4+n)}(\mathbb{R}) \sim \mathrm{C}\ell_{2+D,2+(D+5+n)}(\mathbb{R}) \\ \sim &\mathrm{C}\ell_{2+D,1+(D+6+n)}(\mathbb{R}) \sim \mathrm{C}\ell_{3+D,1+(D+7+n)}(\mathbb{R}) \end{aligned} \tag{3.5.6}$$

Now we try to connect the two chains together. We note further that the Clifford Algebra has the following properties:

$$\mathrm{C}\ell_{p+1,q}(\mathbb{R}) \cong \mathrm{C}\ell_{q+1,p}(\mathbb{R}) \tag{3.5.7}$$
$$\mathrm{C}\ell_{q,p+2}(\mathbb{R}) \cong \mathrm{C}\ell_{p,q}(\mathbb{R}) \otimes \mathcal{M}(2,\mathbb{R}) \tag{3.5.8}$$

Then:

$$\begin{aligned} &\mathrm{C}\ell_{1+D,2+(D+4+n)}(\mathbb{R}) \cong \mathrm{C}\ell_{D+4+n,1+D}(\mathbb{R}) \otimes \mathcal{M}(2,\mathbb{R}) \\ \cong &\mathrm{C}\ell_{2+D,D+3+n}(\mathbb{R}) \otimes \mathcal{M}(2,\mathbb{R}) \cong \mathrm{C}\ell_{D+1+n,2+D}(\mathbb{R}) \otimes \mathcal{M}(4,\mathbb{R}) \end{aligned} \tag{3.5.9}$$

Therefore, $\mathrm{C}\ell_{1+D,2+(D+4+n)}(\mathbb{R}) \sim \mathrm{C}\ell_{1+(D+n),2+D}(\mathbb{R})$, connecting the two chains. If we compare this carefully with Table 4, then we would realize that we actually managed to prove the dimension-shift feature of classification:

$$\begin{aligned} &G_{\mathrm{D}}(d=d_0) \sim G_{\mathrm{DIII}}(d=d_0+1) \\ \sim\ &G_{\mathrm{AII}}(d=d_0+2) \sim G_{\mathrm{CII}}(d=d_0+3) \\ \sim\ &G_{\mathrm{C}}(d=d_0+4) \sim G_{\mathrm{CI}}(d=d_0+5) \\ \sim\ &G_{\mathrm{AI}}(d=d_0+6) \sim G_{\mathrm{BDI}}(d=d_0+7) \end{aligned} \tag{3.5.10}$$

eq:cli-Chain1

We need one more property:

$$\mathrm{C}\ell_{1+d,2+D}(\mathbb{R}) \cong \mathrm{C}\ell_{3+D,d}(\mathbb{R}) = \mathrm{C}\ell_{3+D,1+(d-1)}(\mathbb{R}) \tag{3.5.11}$$

Then

$$G_{\mathrm{D}}(d=d_0) \sim G_{\mathrm{BDI}}(d=d_0-1) \tag{3.5.12}$$

eq:cli-Chain2

With equivalences 3.5.3,3.5.10,3.5.12, we see that we need only the result in a dimension, to obtain the whole table of classification of all real classes. Similar fact holds for complex classes:

$$G_{\mathrm{AIII}}(d=d_0) \sim G_{\mathrm{A}}(d=d_0-1) \tag{3.5.13}$$

which is obvious.

In a word, our classification for topological insulators in 1 spatial dimension is sufficient to generate the whole Table 7. This is a remarkable consequence of K-theory.

The original classification table											
AZ class$\backslash d$	0	1	2	3	4	5	6	7	T	C	S
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	0
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	1
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$+$	0	0
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$+$	$+$	1
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	0	$+$	0
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$-$	$+$	1
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$-$	0	0
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$-$	$-$	1
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$-$	0
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$+$	$-$	1

Table 7: The original classification table of topological insulators and superconductors. Note that in this document we could not yet differentiate between $\mathbb{Z}$ and $2\mathbb{Z}$ topological invariants.

tab:master-table2

4 Looking Further

sec:Looking Further

There has be a great deal of progress made in classification of topological insulator and superconductors. We first comment that our discussion could be easily generalized to some topological superconductors because they naturally admits a matrix like Hamiltonian using Nambu spinors.

An unmentioned hypothesis above is that we are classifying strong topological insulators. They are topological insulators that are robust against disorder [FKM07]. There is a consensus that in classification of strong topological insulators, we replace the Brillouin Zone T^d by S^d[18], and we do it in a stable way, i.e., we choose our classification to be independent of and insensitive to the addition of irrelevant trivial bands. The first replacement allow us to use homotopy groups, which classify maps with domain in spheres. The "stable way" allows the use of K-theory (sec.III.C.1 of [CTSR16]).

However, our discussion falls short in two respects. First, it cannot distinguish between $\mathbb{Z}$ and $2\mathbb{Z}$ cases. Secondly, we do not give explicit formulae for calculating topological invariants in different cases. The former problem can be addressed in Ludwig's [SRFL08], whereas the latter requires considerable work, and requires different topological invariant in different cases (see sec. III.B of [CTSR16]).

There has been progress made both in incorporating defects into classification, as well as incorporating unitary symmetries into the classification. The defect's classification has been incorporated beautifully into the same table of Tenfold Way. On the other hand, introduction of unitary symmetries gives a more fruitful result. For example, introduction of crystal symmetries gives a new class of topological insulators, dubbed topological crystalline insulators. [AF15] In the case of reflection symmetry, some new topological invariants denoted as $M\mathbb{Z}$, called mirror topological invariants, are needed to classify topological matters [CYR13].

Lastly, the discussion of topological states in interaction picture extends to the

[18]The exact reason can be found in Kitaev's classification [Kit09] or another article [KG15] which discuss it from a different view.

case of symmetry-protected phases (SPTs), with which I am not familiar. For all these new concepts, the review [CTSR16] is an excellent source of information.

A Various Properties for Clifford Algebras

ies for Clifford Algebras

We have used various properties of Clifford Algebras from various sources. Here I summarized those properties for the convenience of readers.

Isomorphism Relations The best source of isomorphism between Clifford Algebras is actually the Wikipedia page about *Classifications of Clifford Algebras*. For a more authoritative account, I am using the book [Vd16]. The results used in this document are to be found in chapter 4 of that book.

Representation Theory on Complex Clifford Algebras This paper [Wes98] has a good discussion on the irreducible representations of complex Clifford Algebras, in p.5, p.12. They can be summarized as:

Theorem A.1. *Consider $\mathrm{C}\ell_N(\mathbb{C})$. For both $N = 2n$ and $N = 2n + 1$, the dimension d of the irreducible representation is uniquely given by*

$$d = 2^n (N = 2n \text{ or } N = 2n + 1) \tag{A.0.1}$$

Moreover, for $N = 2n$, the irreducible space is unique, while for $N = 2n + 1$, we have two inequivalent irreducible representations with the same dimension d.

Representation Theory on Real Clifford Algebra The two papers [Oku91b] [Oku91a] are great in this respect. Their result used in this document could be summarized as:

Theorem A.2. *Let $N = p + q$. Consider $\mathrm{C}\ell_{p,q}(\mathbb{R})$. For $N = 2n$, we have:*

$$d = \begin{cases} 2^n_1, & \text{for } p - q = 0 \text{ or } 2, \text{ mod } 8 \\ 2^{n+1}_1, & \text{for } p - q = 4 \text{ or } 6, \text{ mod } 8 \end{cases} \tag{A.0.2}$$

For $N = 2n + 1$, we have:

$$d = \begin{cases} 2^n_2, & \text{for } p - q = 1, \text{ mod } 8 \\ 2^{n+1}_1, & \text{for } p - q = 3 \text{ or } 7, \text{ mod } 8 \\ 2^{n+1}_2, & \text{for } p - q = 5, \text{ mod } 8 \end{cases} \tag{A.0.3}$$

Where we have used a subscript 1 to denote that there is only one unique irreducible representation, and a subscript 2 to denote that there are two inequivalent irreducible representations.

Below collects a list of dimensions that I have used in the discussion of Table 6:

$$\begin{aligned} &\mathrm{C}\ell_{1,2}(\mathbb{R}) \sim 2^2_1,\ \mathrm{C}\ell_{1,3}(\mathbb{R}) \sim 4^2_1,\ \mathrm{C}\ell_{1,4}(\mathbb{R}) \sim 4^2_2,\ \mathrm{C}\ell_{1,5}(\mathbb{R}) \sim 8^2_1, \\ &\mathrm{C}\ell_{2,2}(\mathbb{R}) \sim 2^2_1,\ \mathrm{C}\ell_{2,3}(\mathbb{R}) \sim 4^2_1,\ \mathrm{C}\ell_{3,2}(\mathbb{R}) \sim 2^2_2,\ \mathrm{C}\ell_{2,4}(\mathbb{R}) \sim 8^2_1, \\ &\mathrm{C}\ell_{3,3}(\mathbb{R}) \sim 4^2_1,\ \mathrm{C}\ell_{4,2}(\mathbb{R}) \sim 4^2_1. \end{aligned} \tag{A.0.4}$$

eq:Cl-dimenList

B A Case Study of Class D at $d = 1$

sec:a case study

Throughout the document we never gave a explicit example of constructing the actual matrices and symmetry operators from representations of Clifford Algebras. However, a real example is crucial to understand the mechanism. Here I provide a calculated example, from which I gained the intuition to distinguish between a $\mathbb{Z}$ and a $\mathbb{Z}_2$ topological insulator.

Class D has the extension problem: $C\ell_{2,2}(\mathbb{R}) \to C\ell_{2,3}(\mathbb{R})$, dimension changes as $2 \to 4$. Now we write a simple irreducible representation of $C\ell_{2,2}(\mathbb{R})$ as:

$$J_1 = -i\sigma_y \otimes \mathbb{1}_2, J_2 = \sigma_z \otimes i\sigma_y, \tilde{J}_1 = \sigma_x \otimes \mathbb{1}_2, \tilde{J}_2 = \sigma_z \otimes \sigma_x \tag{B.0.1}$$

eq:old-realified-J

It can be calculated from Table 4 (or directly use table A.2 from [Chi13]) that:

$$\begin{aligned} \Gamma_C &= \tilde{J}_1, \\ \tilde{\Gamma}_0 &= \tilde{J}_1 J_2, \\ \Gamma_1 &= J_1 \tilde{J}_1 \tilde{J}_2 \end{aligned} \tag{B.0.2}$$

since our complex structure $J_1 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, the real vector space is spanned by basis $\{e_i, J_1 e_i\}$. So the complexfication can be done by looking at how each matrix acts on e_i. The process is done manually and the result shows that:

$$\begin{aligned} J_1 &\to i \\ \Gamma_C &\to C = iK \\ \tilde{\Gamma}_0 &\to \tilde{\gamma}_0 = -\sigma_x \\ \Gamma_1 &\to \gamma_1 = -\sigma_y \end{aligned} \tag{B.0.3}$$

So our Hamiltonian and symmetry operator in minimal dimension is

$$H = -(M\sigma_y + k_1\sigma_x), C = iK \tag{B.0.4}$$

there is certainly no SPEMT. Consider two copies of it:

$$H = -(M\sigma_y + k_1\sigma_x) \otimes \mathbb{1}_2, C = iK \tag{B.0.5}$$

There is clearly one SPEMT, $m\sigma_z \otimes \sigma_y$. Let me show how this SPEMT is reflected in representation of Clifford Algebras. We now have in the doubled dimension:

$$\begin{aligned} \tilde{\gamma}_0 &= -\sigma_y \otimes \mathbb{1}_2 = i(i\sigma_y \otimes \mathbb{1}_2) \\ \tilde{\gamma}_1 &= \sigma_z \otimes \sigma_y = i\sigma_z \otimes (-i\sigma_y) \\ \gamma_1 &= -\sigma_x \otimes \mathbb{1}_2 \end{aligned} \tag{B.0.6}$$

These new gamma matrices, together with i and $C = iK$, are realified as

$$\begin{aligned} i &\to J_1 = -i\sigma_y \otimes \mathbb{1}_4 \\ C &\to \Gamma_C = \sigma_x \otimes \mathbb{1}_4 \\ \tilde{\gamma}_0 &\to \tilde{\Gamma}_0 = -i\sigma_y \otimes i\sigma_y \otimes \mathbb{1}_2 \\ \tilde{\gamma}_1 &\to \tilde{\Gamma}_1 = -i\sigma_y \otimes \sigma_z \otimes (-i\sigma_y) \\ \gamma_1 &\to \Gamma_1 = -\mathbb{1}_1 \otimes \sigma_x \otimes \mathbb{1}_1 \end{aligned} \tag{B.0.7}$$

Then, we reverse the process and use Table 4 to get

$$\begin{aligned}
J_1 &= -i\sigma_y \otimes \mathbb{1}_4 \\
J_2 &= \sigma_z \otimes i\sigma_y \otimes \mathbb{1}_2 \\
J_3 &= -\sigma_z \otimes \sigma_z \otimes i\sigma_y \\
\tilde{J}_1 &= \sigma_x \otimes \mathbb{1}_4 \\
\tilde{J}_2 &= \sigma_z \otimes \sigma_x \otimes \mathbb{1}_2
\end{aligned} \tag{B.0.8}$$

eq:new-realified-J

Comparing the old Js(B.0.1) and new Js(B.0.8), clearly there is only one more player J_3. And J_3 comes from $\sigma_z \otimes \sigma_z$, which is an element extending $\mathrm{C}\ell_{2,2}(\mathbb{R}) \to \mathrm{C}\ell_{3,2}(\mathbb{R})$. This the reason why Class D in $d = 1$ is $\mathbb{Z}_2$, not $\mathbb{Z}$, as mentioned in Section 3.4.2.

References

[AF15] Yoichi Ando and Liang Fu. Topological Crystalline Insulators and Topological Superconductors: From Concepts to Materials. *Annual Review of Condensed Matter Physics*, 6(1):361–381, mar 2015. URL: `http://arxiv.org/abs/1501.00531http://dx.doi.org/10.1146/annurev-conmatphys-031214-014501http://www.annualreviews.org/doi/10.1146/annurev-conmatphys-031214-014501`, `arXiv:1501.00531`, `doi:10.1146/annurev-conmatphys-031214-014501`.

[Chi13] Ching-Kai Chiu. *The classification of topological insulators and superconductors for non-spatial and spatial symmetries*. PhD thesis, University of Illinois at Urbana-Champaign, 2013. URL: `https://www.ideals.illinois.edu/handle/2142/44335`.

[CTSR16] Ching Kai Chiu, Jeffrey C Y Teo, Andreas P. Schnyder, and Shinsei Ryu. Classification of topological quantum matter with symmetries. *Reviews of Modern Physics*, 88(3):1–63, 2016. URL: `http://hdl.handle.net/2142/44335`, `arXiv:1505.03535`, `doi:10.1103/RevModPhys.88.035005`.

[CYR13] Ching Kai Chiu, Hong Yao, and Shinsei Ryu. Classification of topological insulators and superconductors in the presence of reflection symmetry. *Physical Review B - Condensed Matter and Materials Physics*, 88(7):075142, aug 2013. URL: `http://link.aps.org/doi/10.1103/PhysRevB.88.075142`, `arXiv:1303.1843`, `doi:10.1103/PhysRevB.88.075142`.

[Dan05] Daniel Huybrechts. *Complex Geometry - An Introduction*. Universitext. Springer-Verlag, Berlin/Heidelberg, 2005. URL: `https://link.springer.com/book/10.1007{%}2Fb137952`, `doi:10.1007/b137952`.

[FKM07] Liang Fu, C. L. Kane, and E. J. Mele. Topological insulators in three dimensions. *Physical Review Letters*, 98(10):1–4, 2007. `arXiv:0607699`, `doi:10.1103/PhysRevLett.98.106803`.

[Gre97] Professor Dr. Walter Greiner. *Relativistic Quantum Mechanics*. Springer Berlin Heidelberg, 1997. URL: `https://link.springer.com/book/10.1007{%}2F978-3-662-03425-5`, `doi:10.1007/978-3-662-03425-5`.

[KG15] Ricardo Kennedy and Charles Guggenheim. Homotopy theory of strong and weak topological insulators. *Phys. Rev. B*, 91(24):245148, 2015. URL: `https://link.aps.org/doi/10.1103/PhysRevB.91.245148`, `doi:10.1103/PhysRevB.91.245148`.

[Kit09] Alexei Kitaev. Periodic table for topological insulators and superconductors. *AIP Conf. Proc.*, 1134:22–30, 2009. URL: `https://arxiv.org/abs/0901.2686http://aip.scitation.org/doi/abs/10.1063/1.3149495`, `arXiv:0901.2686`, `doi:10.1063/1.3149495`.

[Lud16] Andreas W. W. Ludwig. Topological phases: classification of topological insulators and superconductors of non-interacting fermions, and beyond. *Physica Scripta*, T168(1):014001, dec 2016. URL: `http://arxiv.org/abs/1512.08882http://stacks.iop.org/1402-4896/2016/i=T168/a=014001?key=crossref.861694c20b0f47efa9e2b7f7c29e6033`, `arXiv:1512.08882`, `doi:10.1088/0031-8949/2015/T168/014001`.

[MF13] Takahiro Morimoto and Akira Furusaki. Topological classification with additional symmetries from Clifford algebras. *Physical Review B - Condensed Matter and Materials Physics*, 88(12):1–16, 2013. `arXiv:1306.2505`, `doi:10.1103/PhysRevB.88.125129`.

[Oku91a] Susumu Okubo. Real representations of finite Clifford algebras. I. Classification. *Journal of Mathematical Physics*, 32(7):1657–1668, jul 1991. URL: `http://dx.doi.org/10.1063/1.529277http://aip.scitation.org/doi/10.1063/1.529277`, `doi:10.1063/1.529277`.

[Oku91b] Susumu Okubo. Real representations of finite Clifford algebras. II. Explicit construction and pseudo-octonion. *Journal of Mathematical Physics*, 32(7):1669–1673, jul 1991. URL: `http://dx.doi.org/10.1063/1.529278`, `doi:10.1063/1.529278`.

[Phy] Physics Stack Exchange. Dimension of Dirac gamma matrices. URL: `https://physics.stackexchange.com/questions/53318/dimension-of-dirac-gamma-matrices`.

[SKM89] P K Suetin, A I Kostrikin, and Y I Manin. *Linear Algebra and Geometry*. Algebra, logic and applications. Taylor & Francis, 1989. URL: `https://books.google.co.jp/books?id=3TZETTPyOQ0C`.

[SRFL08] Andreas P Schnyder, Shinsei Ryu, Akira Furusaki, and Andreas W W Ludwig. Classification of topological insulators and superconductors in three spatial dimensions. *Phys. Rev. B*, 78(19):195125, mar 2008. URL: `https://link.aps.org/doi/10.1103/PhysRevB.78.195125http://arxiv.org/abs/0803.2786http://dx.doi.org/10.1103/PhysRevB.78.195125`, `arXiv:0803.2786`, `doi:10.1103/PhysRevB.78.195125`.

[Vd16] Jr. Vaz Jayme and Jr. da Rocha Roldão. *An Introduction to Clifford Algebras and Spinors*. Oxford University Press, Oxford, 2016. URL: `http://www.oxfordscholarship.com/view/10.1093/acprof:oso/9780198782926.001.0001/acprof-9780198782926`, `doi:10.1093/acprof:oso/9780198782926.001.0001`.

[Wes98] P West. Supergravity, Brane Dynamics and String Duality. *Arxiv*, page 109, 1998. URL: `http://arxiv.org/abs/hep-th/9811101`, `arXiv:9811101`.