Noetherian Ring

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Abstract

A note about Noetherian Ring, majorly from the book [1].

Contents

1 Module

(pp.117 to 118 of [1])

Definition 1.1 (Module)**.** *Let A: ring. M is a left A-module if and only if*

1. *M is an abelian group, usually written additively.*
2. *there exists an operation of A on M, written as a multiplicative monoid, such that, for any $a, b \in A$, any $x, y \in M$, we have:*

$$(a+b)x = ax + bx \tag{1.0.1}$$
$$a(x+y) = ax + ay \tag{1.0.2}$$

By definition of an operation, we have $1x = x$. Also, it can be easily derived that $a(-x) = -ax$, and $0x = 0$.

Example 1.1. Examples of modules

1. A is a module over itself.
2. Any commutative group is a $\mathbb{Z}$-module.
3. Any left ideal of A is a module over A, i.e. a left A-module.
4. A vector space V over K, is basically a K-module, with the additional structure of K being a field.

5. Let V be a vector space. Let R be the ring of all linear maps of V into itself. Then V is also a module over R.

Definition 1.2 (Submodule)**.** *A submodule M is an additive subgroup such that $AN \subset N$*

Definition 1.3 (factor module)**.** *Let M be an A-module, and N a submodule. A factor module M/N is the factor group M/N (for the additive group structure) equipped with a module structure. The action of A on M/N is defined by $a(x+N) = ax + N$. This is well defined, since if y is in the same coset as x, then ay is in the same coset as ax.*

(**pp.119 of [1]**)

2 Noetherian

Definition 2.1 (Noetherian Module)**.** *Let A: a ring. M: a left A-module. M is called Noetherian if M satisfies any of the following conditions:*

1. *Every submodule of M is finitely generated.*
2. *Every ascending sequences of submodules of M*

$$M_1 \subset M_2 \subset \cdots$$

such that $M_i \neq M_{i+1}$, is fininte.

3. *Every non-empty set S of submodules of M has a maximal element.*

The equivalence of the above conditions are proved in page 413 to 414 of [1].

Definition 2.2 (Noetherian Ring)**.** *A ring A is Noetherian if and only if it is Noetherian when viewed as a left module over itself.*

(**pp.415 of [1]**)

Some theorems Here are some theorems mentioned in Chapter X, section 2 of [1].

The propositions 2.1, 2.2, 2.1 expresses the "Noetherian relation" between M and its submodules. The proposition 2.3 relates a Noetherian ring A and the A-modules. The proposition 2.4 relates two rings. The proposition 2.5 relates a commutative Noetherian ring and its multiplicative subset. The following diagram summarized these relations.

$$\begin{tikzcd}
S^{-1}A & A \arrow[l, "\text{prop.2.5}"'] \arrow[r, "\text{prop.2.3}"] \arrow[d, "\text{prop.2.4}"] & M \arrow[d, leftrightarrow, "\text{prop.:2.1,2.2,coro.2.1}"] \\
 & B & \text{submodules}
\end{tikzcd}$$

(A, B: ring, S: A's multiplicative subset. M: a A-module.)

The structure of being Noetherian is consistent between a module and its submodules, factor modules, in the sense of the following two propositions.

Proposition 2.1. *Let M be a Noetherian A-module, then every submodule and every factor module of M is Noetherian.*

(**pp.414 of [1]**)

Proposition 2.2. *Let M be a module, N be a submodule. If N and M/N are Noetherian, then M is Noetherian.*

(**pp.414 of [1]**)

The above statements could be summarized by saying that, given an exact sequence:

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

M is Noetherian if and only if M' and M'' are Noetherian. This could be seen by two immediate fact of an exact sequence:

$$M' \cong \mathrm{Im} f,\ M/\mathrm{Ker} g \cong M''$$

Corollary 2.1. *A finite direct sum of Noetherian modules is Noetherian. Specifically, let M be a module, let N, N' be two submodules. If $M = N + N'$ and if both N, N' are Noetherian, then M is Noetherian.*

(**pp.415 of [1]**)

Proposition 2.3. *Let A be a Noetherian ring, and let M be a finitely generated A-module. Then M is Noetherian.*

(**pp.415 of [1]**)

Proposition 2.4. *Let A be a ring which is Noetherian, and let $\phi : A \to B$ be a surjective ring-homomorphism. Then B is Noetherian.*

(**pp.415 of [1]**) In colloquial term, a surjective homomorphism induces a Noetherian ring.

Proposition 2.5. *Let A be a commutative Noetherian ring, and let S be a multiplicative subset of A. Then $S^{-1}A$ is Noetherian.*

(**pp.415 of [1]**)

3 Examples

- The polynomial Ring
- The ring of power series
- The ring of formal power series is **NOT** Noetherian. See this post

> The essential point is that the polynomial ring in infinitely many variables is the ascending union of subrings $K[x_1, \ldots, x_n]$, since any polynomial can involve only finitely-many indeterminates. Each of these rings is a UFD, and it is easy to see that a polynomial in which x_N does not appear has only factorizations in which x_N does not appear, again because everything takes place inside some polynomial ring in finitely-many variables. But the ring is not Noetherian, because the ideal generated by all the indeterminates is certainly not finitely-generated.

4 Anchor

Nomenclature

M Module A left module., page 1

factor module ., page 2

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Submodule ., page 2

References

[1] S. Lang. Algebra. 3rd. Springer.

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