Miscellaneous notes for D. Huybrechts's Complex Geometry

Taper

September 20, 2016

Abstract

Miscellaneous notes for D. Huybrechts's book *Introduction to Complex Geometry*, include some homeworks done.

Contents

1 The structure of almost complex structures on $\mathbb{R}^4$ (exercise 1.2.1)

In exercise 1.2.1, it says that the set of all compatible almost complex structures on a euclidean space of dimension $2n$, is two copies of S^2.

To show it, I tried first a straight calculation. Assuming the almost complex structure $I = (a_{ij})$. Then we have:

Section 1.2 of Intro → Complex Geometry.

Exercises:

1.2-1.

Q: Let $(V, \langle , \rangle)$: euclidian space of $\dim = \underset{\sim}{4}$.

Show: { all compatible almost complex structures }
$$\simeq$$
two copies of S^2 $\cancel{*}$ = ~~a~~ two balls.

Recap: compatible : I : $I^2 = -1$, $\langle I(v), I(w) \rangle = \langle v, w \rangle$

Choose an orthogonal basis: $e_1 \cdots e_4$

Let $I = (\cancel{a_j}\, a^i{}_j)$ $\boxed{I^2 = a^i{}_j a^j{}_k = -\delta^i{}_k}$

also $\langle , \rangle \approx \delta^i{}_j$ $\cancel{a}$

$$\langle I(v), I(w) \rangle = (a^i{}_j v^j) \cdot \delta^i{}_k (a^k{}_l w^l) = v^i \delta^i{}_k w^k$$

$(\forall \vec{v}, \vec{w})$

Hence $a^i{}_j \delta^i{}_k a^k{}_l = \delta_{jl}$ or $\boxed{a^i{}_j a^i{}_k = \delta_{jk}}$

For example:

$$\sum_j a^1_j a^j_2 = 0 = \sum_j a^j_1 a^j_2 \Rightarrow \sum_{j=2}^{4} a^1_j a^j_2 = \sum_{j=2}^{4} a^j_1 a^j_2$$

This can be generalized: $\displaystyle \sum_{\substack{j=1 \\ j \neq k}}^{4} a^k_j a^j_l = \sum_{\substack{j=1 \\ j \neq k}}^{4} a^j_k a^j_l \quad (k \neq l)$ } 16 sets of eq.

Also: $\displaystyle \cancel{a^i} \sum_{j=1}^{4} a^i_j a^j_{\cancel{k}\, i} = -\sum_{j=1}^{4} \cancel{a^i_j}\, a^j_i a^j_i$

Figure 1: Draft

Then I discover this too hard to work, because too many equations are involed, and none of them could be eliminiated by other. Meanwhile, I found a post in Math.SE about this [2]. Here are several important concepts for understanding that post.

1.1 Understand $\frac{GL(2n,\mathbb{R})}{GL(n,\mathbb{C})}$

1.1.1 Why $M_n = \frac{GL(2n,\mathbb{R})}{GL(n,\mathbb{C})}$

This is a note of my question on Math.SE [5], which explains that we can identify the set of almost complex structures with $\frac{GL(2n,\mathbb{R})}{GL(n,\mathbb{C})}$.

First, I try to do it when $n = 1$. I inject a complex number $a + bi$ by identify it with $a\mathbb{I}+b\mathbb{J}$, where $\mathbb{I}$ is the identify matrix and $\mathbb{J}$ is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. I take one set of basis of $\mathrm{GL}(2,\mathbb{R})$ as:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(I think this is a basis because the following matrix is non-singular:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

) Then the $\frac{\mathrm{GL}(2n,\mathbb{R})}{\mathrm{GL}(n,\mathbb{C})}$ becomes equivalent classes represented by

$$\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$$

However, I don't know how to link this with an almost complex structure.

I have a feeling that I might have been in the wrong direction. It was pointed out that $\mathrm{GL}(2n,\mathbb{R})$ is not even a vector space. So what I did is in fact nonsense.

Below is one correct answer I got:

An almost-complex structure is a matrix J such that $J^2 = -I$ is the negative identity. As you said, one example of such a matrix J is

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Interpreting $GL(n,\mathbb{C})$ as a subgroup of $GL(2n;\mathbb{R})$ depends on having fixed such an almost-complex structure. Once we have a matrix J, we can call a matrix $A \in GL(2n;\mathbb{R})$ complex-linear if it commutes with J, i.e. $AJA^{-1} = J$.

(The idea is that $\mathbb{C}$-linear maps T are just real linear maps with the additional property that $T(iv) = iT(v)$ for all vectors v)

Given any matrix $A \in GL(2n;\mathbb{R})$, we get another almost-complex structure AJA^{-1}. This is the same almost-complex structure J if and only if $A \in GL(n;\mathbb{C})$. On the other hand, all almost-complex structures are similar (although it may take some work to be convincing that they are similar over $\mathbb{R}$ and not only $\mathbb{C}$) since they are diagonalizable with the same eigenvalues $\pm i$. That gives you a bijection

$$GL(2n;\mathbb{R})/GL(n;\mathbb{C}) \longrightarrow \{\text{almost} - \text{complex structures}\}$$

under which a class $A \cdot GL(n;\mathbb{C})$ corresponds to the almost-complex structure AJA^{-1}.

I questioned him:

1. Why AJA^{-1} is the same almost-complex structure J if and only if $A \in \mathrm{GL}(n;\mathbb{C})$.
2. Why all almost-complex structures are similar over $\mathbb{R}$.

He responded that:

1. is the definition of $\mathrm{GL}(n;\mathbb{C})$ as matrices A with $AJA^{-1} = J$.
2. comes from the fact that any real matrices that are similar over $\mathbb{C}$ are already similar over $\mathbb{R}$. This isn't trivial but it has been asked and answered many times on this site: here is one reference [6].

Inside that reference, the following theorem is proved:

Theorem 1.1. *Let E be a field, let F be a subfield, and let A and B be nxn matrices with coefficients in F. If A and B are similar over E, they are similar over F.*

However, I still have doubts about the following question: For $A \in \mathrm{GL}(2n,\mathbb{R})$, if $AJA^{-1} = J$, can we conlude that A is inside $\mathrm{GL}(n,\mathbb{C})$?

The following is my solution:

Lemma 1.1. *There exists a injection ϕ of $\mathrm{GL}(n,\mathbb{C}) \hookrightarrow \mathrm{GL}(2n,\mathbb{R})$ such that:*

$$\phi(iB) = \phi(i)\phi(B) \tag{1.1.1}$$

for any $B \in \mathrm{GL}(n,\mathbb{C})$. Also, for any $A \in \mathrm{GL}(2n,\mathbb{R})$ we have $AJA^{-1} = J$ if and only if $A \in \mathrm{Im}(\phi)$, where $J \equiv \phi(i)$.

Proof. The ϕ is construct as follows. Let $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, define $H(x+iy)$ for $x, y \in \mathbb{R}$ as

$$H(x+iy) = xI + yJ \tag{1.1.2}$$

Then:

$$\phi(A)_{ij} \equiv H(a_{ij}) \tag{1.1.3}$$

Then:

$$\phi(i) = \begin{pmatrix} J & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & J \end{pmatrix} \tag{1.1.4}$$

By direct simple calculation (remember to use the technique of block multiplication), we have: $\phi(iB) = \phi(i)\phi(B)$. for any $B \in \mathrm{GL}(n, \mathbb{C}$. This shows also $BJB^{-1} = J$, since $iB = Bi$.

To prove the converse, we see that the following matrices forms a basis of $2x2$ real matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

They are denoted, from left to right as I, J_0, K, L. Let any $A \in \mathrm{GL}(2n, \mathbb{R})$, we can partition A into a matrix of $2x2$ matrices (a_{ij}). Each matrix can the be expressed as $a_{ij} = x_{ij}I + y_{ij}J_0 + z_{ij}K + t_{ij}L$. Then if $AJA^{-1} = J$, by direct calculation we find:

Fun fact: $[K, J_0] = \sigma_z$, $[L, J_0] = \sigma_x$, the pauli matrices!

$$(z_{ij}K + t_{ij}L)J_0 = J_0(z_{ij}K + t_{ij}L)$$

then also by direct calculation, it can be easily found that $z_{ij} = t_{ij} = 0$. Hence $A \in \mathrm{Im}(\phi)$. □

1.1.2 Why $GL(n, \mathbb{C}) \hookrightarrow GL^+(2n, \mathbb{R})$

To understand that post [2], I also read this [3]. In it, it asks how to prove that

$$GL(n, \mathbb{C}) \hookrightarrow GL^+(2n, \mathbb{R}) \tag{1.1.5}$$

for any n. The questioner gives the intuition for this fact:

> how about since as Lie groups, $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$ and $GL(n, \mathbb{C})$ is connected but $GL(2n, \mathbb{R})$ has two connected components, one for positive determinant and one for negative determinant? And the identity has positive determinant, so it must lie in that component.

Someone answered that question:

> The claim is: If V is an n-dimensional complex vector space with underlying $2n$-dimensional real vector space W, then the canonical group monomorphism $\mathrm{GL}(V) \to \mathrm{GL}(W)$ lands inside $\mathrm{GL}^+(W) = \{f \in \mathrm{GL}(W) : \det(f) > 0\}$. The purpose of this abstract reformulation is that we may use operations on vector spaces in order to simplify the problem: If V' is another finite-dimensional complex vector space with underlying real vector space W', the diagram
>
> $$\begin{array}{ccc} \mathrm{GL}(V) \times \mathrm{GL}(V') & \to & \mathrm{GL}(W) \times \mathrm{GL}(W') \\ \downarrow & & \downarrow \\ \mathrm{GL}(V \oplus V') & \to & \mathrm{GL}(W \oplus W') \end{array} \tag{1.1.6}$$
>
> commutes, and the image of $\mathrm{GL}^+(W) \times \mathrm{GL}^+(W')$ is contained in $\mathrm{GL}^+(W \oplus W')$. Therefore, if some element in $\mathrm{GL}(V \oplus V')$ lies in the image of $\mathrm{GL}(V) \times \mathrm{GL}(V')$, it suffices to consider the components. Combining this with the fact that $\mathrm{GL}(V)$ is

generated by elementary matrices (after chosing a basis of V), we may reduce the whole problem to the following three types of matrices:

- the 1×1-matrices (λ),
- the 2×2-matrices $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$,
- and the 2×2-matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Write $\lambda = a + ib$ with $(a, b) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then, the complex 1×1-matrix (λ) becomes the real 2×2-matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, which has determinant $a^2+b^2 > 0$. The complex 2×2-matrix $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$ becomes the real 4×4-matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & -b & 1 & 0 \\ b & a & 0 & 1 \end{pmatrix}$, which has determinant 1. Finally, the complex 2×2-matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ becomes the real 4×4-matrix $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, which has determinant 1.

However, this proof is not complete because, to build the proof from $\mathbb{R}^2$ to $\mathbb{R}^{2n}$, it requries, in his argument, that any element in $\mathrm{GL}(V \oplus V)$ is in the image of $\mathrm{GL}(V) \times \mathrm{GL}(V')$, which is not the case.

On the other hand, it seems that this property can be proved directly by calculation. The following will be a notes of a paper [4], which one comment mentions in the Math.SE post [3].

1.1.3 Determinants of Block Matrices

This paper tries to prove the theorem:

Theorem 1.2. *Let R be a commutative subring of ${}^nF^n$, where F is a field (or a commutative ring), and let $M \in {}^mR^m$. Then*

$$det_F \mathbf{M} = det_F(det_R \mathbf{M}) \tag{1.1.7}$$

In particular, we have:

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \det_F(AD - BC) \tag{1.1.8}$$

Note that, that the ring being is commutative excludes some ambiguity. For example, when the ring 4 is not commutative, then the quantity:

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \tag{1.1.9}$$

is not well-defined. It can be $AD-BC$, or $DA-CB$, etc.

Before the proof of the main theorem, it establishes several facts:

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \det_F \mathbf{A} \det_F \mathbf{D} \tag{1.1.10}$$

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} = \det_F \mathbf{A} \det_F \mathbf{D} \tag{1.1.11}$$

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{pmatrix} = \det_F - \mathbf{C} \det_F \mathbf{B} \tag{1.1.12}$$

$$\det_F \mathbf{A} \det_F \mathbf{D} = \det_F \mathbf{I}_n \det_F (\mathbf{AD}) \tag{1.1.13}$$

He first builds up a seemingly simplified, but is actually different version of the main theorem:

Theorem 1.3. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in {}^nF^n$. *Let* $\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$.

If $\mathbf{CD} = \mathbf{DC}$, *then,*

$$det_F \mathbf{M} = det_F(\mathbf{AD} - \mathbf{BC}) \tag{1.1.14}$$

and similar results:

$$\text{if } \mathbf{AC} = \mathbf{CA}\text{then, } \det_F \mathbf{M} = \det_F(\mathbf{AD} - \mathbf{CB}) \tag{1.1.15}$$

$$\text{if } \mathbf{BD} = \mathbf{DB}\text{then, } \det_F \mathbf{M} = \det_F(\mathbf{DA} - \mathbf{BC}) \tag{1.1.16}$$

$$\text{if } \mathbf{AB} = \mathbf{BA}\text{then, } \det_F \mathbf{M} = \det_F(\mathbf{DA} - \mathbf{CB}) \tag{1.1.17}$$

These equalities can be proved easily by the following:

$$\begin{pmatrix} D & 0 \\ -C & i \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathrm{AD-BC} & B \\ \mathrm{CD-DC} & D \end{pmatrix} = \begin{pmatrix} \mathrm{AD-BC} & B \\ 0 & D \end{pmatrix} \text{ when } C, D \text{ commutes}$$

$$\begin{pmatrix} D & -B \\ 0 & i \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathrm{DA-BC} & \mathrm{DB-BD} \\ C & D \end{pmatrix} = \begin{pmatrix} \mathrm{DA-BC} & 0 \\ C & D \end{pmatrix} \text{ when } D, B \text{ commutes.}$$

The author also gives an illuminative explanation for why

$$(\det_F \mathbf{M} - \det_F(\mathbf{AD} - \mathbf{BC})) \det_F \mathbf{D} = 0$$

necessarily implies:

$$\det_F \mathbf{M} = \det_F(\mathbf{AD} - \mathbf{BC})$$

However, I am dubious about this conclusion, since I think it needs in addition that the polynomial ring $F[x]$ has not nonzero zero divisor.

Having demonstrated the above simple case, the author continues to prove the main theorem. He proves by induction. He first uses:

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathrm{dI} & 0 \\ -c & 1 \end{pmatrix} = \begin{pmatrix} A_0 & b \\ 0 & d \end{pmatrix} \tag{1.1.18}$$

where $A, A_0 \in {}^{m-1}R^{m-1}, b \in {}^{m-1}R, c \in R^{m-1}, d \in R$. Therefore, (let $M = \begin{pmatrix} A & b \\ c & d \end{pmatrix}$) with similar reason mentioned before, he shows if:

$$\det_F \mathbf{A_0} = \det_F(\det_{\mathbf{R}} \mathbf{A_0}) \tag{1.1.19}$$

(which is true by induction) then:

$$\det_F \mathbf{M} = \det_F(\det_{\mathbf{R}} \mathbf{M}) \tag{1.1.20}$$

Proof completes.

He also mentions a corollary:

Corollary 1.1. *Let* $\mathbf{P} \in {}^nF^n$ *and* $\mathbf{Q} \in {}^mF^m$*, then*

$$det_F(\mathbf{P} \otimes \mathbf{Q}) = (det_F \mathbf{P})^m (det_F \mathbf{Q})^n \tag{1.1.21}$$

The proof is quite straightforward and is omitted.

1.1.4 Why $GL(n, \mathbb{C}) \hookrightarrow GL^+(2n, \mathbb{R})$ (continued)

With above theorem, the proof of equation 1.1.5 is straight forward. Since for $(a_{ij}) = A \in \mathrm{GL}(n, \mathbb{C})$, it injects into $\mathrm{GL}(2n, \mathbb{R})$ as matrices of the form:

$$\begin{pmatrix} ... & ... & ... \\ ... & Ha_{\mathrm{ij}} & ... \\ ... & ... & ... \end{pmatrix}$$

where:

$$H(z \equiv x + iy) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

Since $H(a_{ij})$ commutes with each other (proved by calculation), we can use the theorem in previous part to show that:

$$\det_{\mathbb{R}}(A) = \det_{\mathbb{R}}(\det_{\mathbb{C}}(A)) = |\det_{\mathbb{C}}(A)|^2 \geq 0$$

Notice that I have been sloopy in language, but the meaning should be clear.

2 Anchor

References

[1] D Huybrechts's Introduction to Complex Geometry.

[2] set of almost complex structures on $\mathbb{R}^4$ as two disjoint spheres.

[3] Does $GL(n, \mathbb{C})$ inject into $GL^+(2n, \mathbb{R})$ for all n?

[4] John R. Silvester, Determinants of Block Matrices. Available in WebArchive link: `https://web.archive.org/web/20140505161153/http://www.mth.kcl.ac.uk/~jrs/gazette/blocks.pdf`

[5] Making sense of $\frac{\mathrm{GL}(2n,\mathbb{R})}{\mathrm{GL}(n,\mathbb{C})}$

[6] Similar matrices and field extensions

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