# Complex Geometry - Index of Notations and ideas

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# Contents

# Part I

# Indices of Notation

# Chapter 1

# Book

## 1.1 1. Local Theory

### 1.1.1 1.1 Holomorphic Functions of Several Variables

**Note**: the content covered by this seciton is geared for accompanying my personal notes of lecture 1.

holomorphic: pp.1. pp4. Def 1.1.1. pp.10(Def.1.1.8).
Cauchy-Riemann equations: pp.2
$\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$: $\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ , $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$
Maximum principle: pp.3.
Identity theorem: pp.3.
Riemann extension theorem: pp.3. pp.9 (Prop. 1.1.7).
Riemann mapping theorem: pp.3.
Liouville theorem: pp.4.
Residue theorem: pp.4.
polydiscs $B_\epsilon(\omega)$: $\{z||z_i - \omega_i| < \epsilon\}$. pp.4.
Hartogs' theorem: Prop. 1.1.4. pp.6.
Weierstrass preparation theorem (WPT): Prop. 1.1.6. pp.8.
Weierstrass polynomial: Def. 1.1.5. pp.7.
$Z(f)$: zero set of $f$. pp.9.
biholomorphic: pp.10.
(complex) Jacobian, regular, regular value: Def. 1.1.9. pp.10.
IFT. Inverse function theorem: Prop 1.1.10 pp.11.
IFT. Implicit function theorem: Prop 1.1.11. pp.10.
$\mathcal{O}_{\mathbb{C}^n}$: sheaf of holomorphic functions on $\mathbb{C}^n$. Def. 1.1.14. pp.14.
$\mathcal{O}_{\mathbb{C}^n,z}$: Def. 1.1.14. pp.14.
$\mathcal{O}^*_{\mathbb{C}^n,0}$: units of $\mathcal{O}_{\mathbb{C}^n,0}$. pp.14.
UFD, unique factorization domain, irreducible: Def. 1.1.16. pp.14.
Gauss Lemma: pp.14.
Weierstrass division theorem: Prop. 1.1.17. pp.15.
germ of set: (**pp.18**)

$Z(f)$: germ of zero set of $f$. (**pp.18**)
analytic germ: $Z(f_1, \cdots, f_k)$. (**pp.18**)
analytic subset: locally are zero sets. (**pp.18**)
$I(X)$: the set of all $f \in \mathcal{O}_{\mathbb{C}^n,0}$ with $X \subset Z(f)$. (**pp.18**)

### 1.1.2 1.2 Complex and Hermitian Structures

almost complex structure, $I$: $I^2 = -\mathrm{id}$. (**pp.25**)
$V^{1,0}$ and $V^{0,1}$: the $\pm i$ eigenspaces of $I$. (**pp.25**)
$\bigwedge^{p,q} V := \bigwedge^p V^{1,0} \oplus_{\mathbb{C}} \bigwedge^q V^{0,1}$. (**pp.27**)
$\prod^k, \prod^{p,q}$: natural projections. (**pp.28**)
$\mathbf{I}$:$=\sum_{p,q} i^{p-q} \cdot \prod^{q,p}$. (**pp.28**)
compatible:an almost complex structure $I$ is compatible with the scalar product $<,>$, if $< I(v), I(w) >=< v, w >$. (**pp.28**)
Conformal equivalence(between scalar product): (**pp.29**)
fundamental form, $\omega$:$=< I(\ ), () >$ . (**pp.29**)

$$\omega = \frac{i}{2} \sum_i z^i \wedge \bar{z}^i = \sum_i x^i \wedge y^i.$$

(Local calculation could be found on **pp.31**)
hermitian form $(,)$:$=<,> -i \cdot \omega$. (**pp.30**)
Lefschetz operator $L$: $\bigwedge^* V_{\mathbb{C}}^* \to \wedge^* V_{\mathbb{C}}^*$, given by $\alpha \to \omega \wedge \alpha$. (**pp.31**)
Hodge star $*$-operator: $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \cdot \mathrm{vol}$. (**pp.33**)
dual Lefschetz operator $\Lambda$: $\langle \Lambda\alpha, \beta \rangle = \langle \alpha, L\beta \rangle$, degree $-2$, bidegree $(-1, -1)$, $\Lambda = *^{-1} \circ L \circ *$. (**pp.33 to 34**)
Counting operator $H$: $H = \sum_{k=0}^{2n} (k-n) \cdot \prod^k$, where $\dim_{\mathbb{R}} = 2n$. (**pp.34**)
commutator [A,B]:$= A \circ B - B \circ A$. (**pp.34**)

**Commutators**

$$[H, L] = 2L,\ [H, \Lambda] = -2\Lambda,\ [L, \Lambda] = H.$$

(**pp.34**)

$$[L^i, \Lambda](\alpha) = i(k - n + i - 1)L^{i-1}(\alpha), \text{for all} \alpha \in \bigwedge^k V^*$$

(**pp.35**)
primitive element in $\bigwedge^k V^*$: $\alpha$ is primitive if and only if $\Lambda\alpha = 0$. (**pp.36**)
$P^k \subset \bigwedge^k V^*$: is the subspace of all primitive elements. (**pp.36**)
Hodge-Riemann pairing $Q$:

$$\bigwedge^k V^* \times \bigwedge^k V^* \to \mathbb{R},\ (\alpha, \beta) \mapsto (-1)^{k(k-1)} 2\alpha \wedge \beta \wedge w^{n-k}$$

Note: here we identify $\bigwedge^{2n} V^*$ with $\mathbb{R}$ by the volumn form vol. Also the $\mathbb{C}$-linear extension of this is still denoted $Q$.

## 1.2 2.Complex Manifolds

### 1.2.1 2.1 Complex and Hermitian Structures

almost complex structure: (**pp.25**)

### 1.2.2 2.2 Holomorphic Vector Bundles

$\tau_X$: holomorphic tangent bundle of a complex manifold $X$ (Def 2.2.14 at pp. 71).

$\Omega_X$, $\Omega_X^p$: holomorphic cotangent bundle and holomorphic $p$-forms. (Def 2.2.14 at pp. 71)

$K_X$:=det$(\Omega_X) = \Omega_X^n$, the canonical bundle of $X$. (Def 2.2.14 at pp. 71)

### 1.2.3 2.6 Differential Calculus on Complex Manifolds

$\wedge_{\mathbb{C}}^k X := \wedge^k (T_{\mathbb{C}} X)^*$. (Def 2.6.7 at pp. 105)

$\wedge^{p,q} X := \wedge^p (T^{1,0}X)^* \bigoplus_{\mathbb{C}} \wedge^q (T^{0,1}X)^*$. (Def 2.6.7 at pp. 105)

$\mathcal{A}_{X,\mathbb{C}}^k$,$\mathcal{A}_X^{p,q}$: sheaves of section of the above correspond items. (Def 2.6.7 at pp. 105)

$\mathcal{A}^{p,q}(E)$: the sheaf of $p,q$-forms with values in $E$, a complex vector bundle. (Def 2.6.22 at pp.109). Note that in particular, $\mathcal{A}^0(E)$ is the sheaf of sections of $E$.

### 1.2.4 Appendix B: Sheaf Cohomology

- pre-sheaf: Def B.0.19, pp. 287.
- $\mathcal{C}'_{\mathcal{M}}$: the pre-sheaf of continuous functions on $M$. Example B.0.20, pp. 287.
- sheaf: Def B.0.21, at pp.288.
- $\underline{\mathbb{R}}$,$\underline{\mathbb{Z}}$: constant sheaves, Sometimes written simply as $\mathbb{R}$, $\mathbb{Z}$ respectively. pp. 288.
- $\mathcal{E}$: actually a $\mathcal{C}_M^0$-modules. Sometimes identified as $E$. pp.288.
- (pre)-sheaf homomorphism: Def B.0.23. pp.288.
- Ker$(\phi)$,Im$(\phi)$,Coker$(\phi)$: as pre-sheaves in pp.288. sheaves in pp.289, Def B.0.26.
- injective, surjective of sheaf-homomorphism:pp.289.
- complex, exact complex: Def B.0.27. pp.289
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# Chapter 2

# My lecture Notes

## 2.1 Lecture 2016 Lecture 1

The first few lectures are not well noted, hence I delegate the task of recording the theorems and notations to the book's correspoding section:section 1.1.1 on page 7.

## 2.2 Lecture 4 (20160307) Complex Manifold

**Note**: we use abbrevation *mnfd* for *manifold*.

pp. A:

- Holomorphic Atlas
- Holmorphic chart
- Complex mnfd

pp. B:

- Holomorphic function
- $\mathcal{O}_X$: sheaf of holomorhic functions on a complex mnfd $X$.

pp. C:

- Hartdogs' theorem: on complex mnfd.
- Holomorphic functions on complex mnfd:

pp. D:

- Complex Lie group
- Complex Projective Space, $\mathbb{CP}^n$, or just $\mathbb{P}^n$.

pp. E:

- Topology in $\mathbb{P}^n$
- Mnfd structure on $\mathbb{P}^n$, atlas, and the **canonical covering**

pp. F

- Grassmannian mnfd.

## 2.3 Lecture 5 Submanifolds (20160308)

pp. A:

- Affine Hypersurface (actually this is not quite different from the usual $\mathbb{C}^n$.)

  Part 2. **Sheaf Theory**
  pp. A:

- pre-sheaf
- $\mathcal{O}_X(U)$
- $\mathcal{O}_X^*(U)$

pp. B:

- $C^\infty$
- $\underline{\mathbb{Z}}$, sometimes simply denoted as $\mathbb{Z}$: sheaf of localy constant $\mathbb{Z}$-valued functions.
- Sheaf

pp. D:

- sheaf-morphisms

pp. E:

- Section
- Ker$(\phi)$ - sheaf of kernals.

pp. F:

- Im$(\phi)$ is a presheaf, but not a sheaf.
- Im$(\phi)$: the sheafification of Im$(\phi)$ above. Note that we use the same notation to denote both.

## 2.4 Lecture 6 Sheaf & Cohomology (20160315)

pp. A:

- Stalk $\mathcal{F}_x$.
- germ
- Directed partial order set
- Directed System

pp. B:

- Directed limit

pp. C:

- Exact Complex/ Exact Sequence.
- Exponential sequence *(mentioned under the definition of exact sequence).*
- 

pp. D:

- Čech cohomology

pp. E:

- q-cochain
- coboundary operator. $\delta$.
- $Z^p(U, \mathcal{F}) =$ Ker.
- $B^p(U, \mathcal{F}) =$ Im.
- $\check{H}^p(U, \mathcal{F}) = \frac{Ker}{Im}$.

### 2.4.1 Notes of Čech Cohomology with Coeficients in a Sheaf

pp.1:

- q-simplex $\sigma$.
- support $|\sigma|$.
- q-cocain
- $C^q(U, \mathcal{F})$
- Coboundary Operator $\delta$.

pp.2,3,4:

- Cochain Complex
- Čech cohomology
- cocycle
- cochain
- $\check{H}^p(U, \mathcal{F}), Z^p(U, \mathcal{F}), B^p(U, \mathbb{F})$.
- $\check{H}^0(\{u_i\}, \mathcal{F}) = \mathcal{F}(X)$.

## 2.5 Lecture 7 Vector Bundle (20160321)

pp.1,2:

- Vector Bundle
- Trivializing covering, $\{(U_i, \tau_i)\}$.
- trivializing maps, trivializes.
- VB-equivalent of trivializing maps.
- E: total space, X: base space.

pp. 3,5:

- transition maps.
- fibre.
- $\mathcal{O}(-1)$
- cocycle condition.
- $\mathcal{T}_X$, Holomorphic tangent bundle.

pp. 8:

- s: section of a holomorphic vector bundle.
- $\mathcal{E}$: sheaf of sections of holomorphic vector bundle. $\mathcal{E}(U)$.

## 2.6 Lecture 8 Almost Complex Structures (20160322)

pp. 1,2:

- $I$: Almost Complex Structure. $I^2 = -1$. Sometime $J$ is used in place of $I$.
- $V_{\mathbb{C}} := V \otimes \mathbb{C}$.
- $I_{\mathbb{C}}$: $I$ extending to $V_{\mathbb{C}}$. Usually abbreviated simply as $I$.
- $V^{1,0}$:= $\ker(I + i)$.
- $V^{0,1}$:= $\ker(I - i)$.

## 2.7 Lecture 9 Exterior Algebra on Complex Manifold (20160329)

pp.1,2:

- $V^*$: dual of $V$.
- $\{dx^i, dy^i\}$.
- $J^*$: $J$ extending to dual space.
- $dz^i, d\bar{z}^i$.

pp. 3:

- $S^k(V)$, $\Lambda^k(V)$.
- $s$ and $a$, symmetrization and anti-symmetrization of a tensor.
- $\Lambda^* V$.

pp. 4:

- $\Lambda^n T^*_{\mathcal{C}} X$.
- $\Lambda^* T^*_{\mathcal{C}} X$.
- $\Lambda^{p,q} T^*_{\mathcal{C}} X$.

pp. 5,6:

- $\mathcal{A}$: sheaf of section of cotangent bundle.
- $\mathcal{A}^n(U)$, $\mathcal{A}^{p,q}(U)$.
- $\Lambda$ on $\mathcal{A}$.
- $d$: de Rham differential.
- $\partial, \bar{\partial}$.

## 2.8 Lecture 10 Debeault Cohomology (20160406)

pp. 1:

- $\mathcal{H}^{p,q}(X)$.
- $f^*$: pull-back. Various defintion from pp.1 to pp.4.

pp. 5,6,7:

- $\mathcal{A}^{p,q}(U, E) := \Gamma(U, \Lambda^{p,q} T^*_{\mathbb{C}} X \otimes E)$.
- $\bar{\partial}_E$
- $\mathcal{H}^{p,q}(X, E)$.
- $\bar{\partial}$-Poincaré lemma in one variable.

## 2.9 Lecture 11 (20160412)

pp.1,2,3:

- $\bar{\partial}$-Poincaré lemma in n-dimension
- $\Omega^p_X$: holomorphic p-forms. On pp.2.
- $\check{H}^q(X, \Omega^p)$(Čech) $\cong \mathcal{H}^{p,q}_{\bar{\partial}}(X)$(Dolbeault). On pp.3.

pp. 6,7:

- Analytic Subvarity.
- Analytic Hybersurface.
- Cousin's Problem.

## 2.10 Lecture 12 Hermitian Structure on Manifold Manifold (20160418)

pp. 1,2,3:

- $I$ compatible with $< -, - >$.
- $\omega$: Fundamental form associated with $<,>$ and $I$. $\omega(v, w) :=< I(v), w >$.
- Conformal Equivalence.
- $<,>$: Hermitian Inner Product.

pp. 4:

- $(,)$: s.t. $(v,w) :=< v,w > -i\omega(v,w) =< v,w > -i < I(v),w >$

pp. 5:

- $<,>_{\mathbb{C}}$ be s.t.$< v \otimes \alpha, w \otimes \beta >:= \alpha\bar{\beta} < v,w >$.

pp. 6:

- $\frac{1}{2}(,) =<,>_{\mathbb{C}} |_{V^{1,0}}$

pp. 7,8:

- Local computations: $z_i$,$h_{ij}$,
- $\omega = (...dx^i...dy^i)$
- $\omega$, Fundamental form on Riemannian Mnfd.
- Kähler mnfd: $d\omega \equiv 0$.

## 2.11 Lecture 13 Kähler Manifold (20160419)

pp.1:

- Local computation: $\omega = (...dz^i...d\bar{z}^i)$

pp.4:

- Fubini-Study Metric on $\mathbb{CP}^n$.

## 2.12 Lecture 14 Hodge Theory (20160425)

pp.1:

- $<,>$ on $\Lambda^k V$
- vol: volumn element.
- $*$: Hodge Star Operator.

pp.4:

- $L$: Lefschetz Operator
- $\Lambda$: adjoint of $L$. $\Lambda = *^{-1} \circ L \circ *$.

pp.5 :

- $*, L, \Lambda$ on Kähler mnfd.
- $d^* := (-1)^{m*(k+1)+1} * \circ d \circ *$, adjoint of $d$. On a Kähler mnfd, $d^* = - * \circ d \circ *$
- $\Delta := d^* \circ d + d \circ d^*$.

pp. 6:

- $\bar{\partial}^*, \partial^*$: Similar to the above for $d$.
- $\Delta_\partial, \Delta_{\bar{\partial}}$: Similar to the above for $d$.

## 2.13 Lecture 15 Hodge Theory on Manifold (20160426)

pp.1:

- $(,)$ on $\mathcal{A}^*(X)$. $(\alpha, \beta) := \int_X g_{\mathbb{C}}(\alpha, \beta) vol$

pp.3:

- $\mathcal{H}^k(X, g)$: d-harmonic forms. Sometimes we replace $\mathcal{H}$ with $\mathscr{H}$ for harmonic forms, so is for symbols below.
- $\mathcal{H}^k_{\bar{\partial}}(X, g)$: $\bar{\partial}$-harmonic forms. (Be careful to distinguish this with Dolbeault Cohomology groups).
- $\mathcal{H}^k_{\partial}(X, g)$: $\partial$-harmonic forms.

pp. 5:

- $\mathcal{H}^k_d(X, g) \cong \mathcal{H}^{2n-k}_d(X, g)$, Poincaré duality
- $\mathcal{H}^{p,q}_{\bar{\partial}}(X, g) \cong \left(\mathcal{H}^{n-p,n-q}_{\bar{\partial}}(X, g)\right)^*$, both are harmonic forms, called Serre Duality.

pp. 6,7:

- $\mathcal{A}^{p,q} = \bar{\partial}\mathcal{A}^{p,q-1}(X) \oplus \bar{\partial}^*\mathcal{A}^{p,q+1}(X) \oplus \mathcal{H}^{p,q}_{\bar{\partial}}(X, g)$:
  Hodge decomposition
- $\mathcal{H}^{p,q}_{\bar{\partial}}$(harmonic forms) $\cong \mathcal{H}^{p,q}_{\bar{\partial}}(X)$(Dolbeault Cohomology group)
- $\mathcal{H}^{p,q}_{d}$(harmonic forms) $\cong \mathcal{H}^{p,q}_{dR}(X)$(de Rham Cohomology group)

pp. 8:

- A lot of isomorphisms between *de Rham*, *Dolbeault* and *harmonic forms*.

## 2.14 Lecture 16 Harmonic forms on Kähler Manifold (20160503)

pp.1:

- $\Delta_{\partial} = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta_d$, for Kähler mnfd.

## 2.15 Lecture 17 Hermitian Vector Bundle (20160510)

pp.1,3:

- Hermitian Vector Bundle. pp.1
- Antilinear map. pp.3.

- Hermitian Inner Product on $\mathcal{A}^{p,q}(X, E)$. pp.4
- $\bar{*}_E$ Hodge Operator on Hermitian vector bundle. pp.5.
- $\bar{\partial}_E^*$

pp. 8:

- Kadaira-Serre Duality.

## 2.16 Lecture 18 Connection (20160516)

- $\bigtriangledown$: connection. pp.1.
- Trivial connections. pp.2
- $\mathcal{A}^1(M, End(E)) := \Gamma(M, \Lambda^1 M \otimes End(E))$. pp.3. Also, one may find how elements in this sheaf act on $\mathcal{A}^0(M)$ on pp.173, inside proof of proposition 4.2.3.
- $s \in \mathcal{A}^0(E)$ is Parrallel/flat/constant $\Leftrightarrow \Delta(s) = 0$. pp.4.
- $\Delta = d + A$. pp.4.
- $\Delta$ be compatible with hermitian structure on $E$. pp.5.
- $\Delta$ be compatible with holomorphic vector bundle. pp.6.
- $A = \bar{H}^{-1}\partial H$. Chern connection. pp.6.

## 2.17 Lecture 19 Holomorphic Connection & Curvature(20160517)

- Holomorphic Connecction. pp.1.
- $At(E)$: Atiyah class of $E$. pp.2.
- $\Delta^k$. pp.4.
- $F_\Delta$: curvature associated with $\Delta$. pp.5.
- $F_\Delta = dA + A \wedge A$: Cartan structure equation. pp.6.
- First Chern class of complex line bundle.

## 2.18 Lecture 20 Divisors & (Holomorphic) Line Bundles (20160524)

- Analytic Subvariety. pp.1.
- Regular/Smooth Point. pp.1.
- Singular Point. pp.2.
- Irreducible analytic subvariety. pp.2.
- $dim(Y)$: dimension of analytic subvariety. pp.2. Also pp.4.
- Affine algebraic varieties. pp.3.
- Projective algebraic varieties. pp.3.
- Hypersurface. pp.4.
- Divisor, $Div(X)$:=group of all divisors. pp.5.
- Effective divisor. pp.6.
- $Ord_Y(f)$: order of function. pp.6. Also pp.8.
- Meromorphic function on complex mnfd.
- $(f)$: divisor given by a global meromorphic function.
- Principal divisor. pp.8.

## 2.19 Lecture 21 Divisors & (Holomorphic) Line Bundles (20160530)

- $H^0(X, K_X^*/\mathcal{O}_X^*) \cong Div(X)$. pp.1.
- $Pic(X)$: Picard group, all holomorphic line bundles. pp.3.
- $Pic(X) \cong \check{H}^1(X, \mathcal{O}_X^*)$. pp.3.
- $\mathcal{O}(D)$: line bundle given by divisor $D$. pp.5.
- Linear equivalent of divisors.
- $*$: used only in this section to denoted the map:

  $$(Div(X)/Pic(X)) \hookrightarrow Pic(X)$$

  pp.6.
- $Z(s)$: divisor constructed from nonzero section $s \in H^0(X, L)$ for a line bundle $L$.

## 2.20 Lecture 22 Divisors & (Holomorphic) Line Bundles (20160606)

- Base point of a line bundle. pp.4.
- $Bs(L)$:= set of all base points of line bundle $L$. pp.4.
- $\mathcal{O}(1), \mathcal{O}(k)$. pp.6.

# Part II

# Indices of Results

Theorems, Remarks, etc.

# Chapter 3

# Local Theory

## 3.1 1.1 Holomorphic Functions of Several Variables

**Proposition 3.1.1.** *The local ring $\mathcal{O}_{\mathbb{C}^n,0}$ is a UFD.*

(**pp.14 of [1]**)

**Proposition 3.1.2.** *Weierstrass division theorem Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ and $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ be a Weierstrass polynomial of degree $d$. Then there exist $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ of degree $< d$ and $h \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $f = g \cdot h + r$. The functions $h$ and $r$ are uniquely determined.*

(**pp.15 of [1]**)

**Proposition 3.1.3.** *The local UFT $\mathcal{O}_{\mathbb{C}^n,0}$ is Noetherian.*

(**pp.16 of [1]**)

**Corollary 3.1.1.** *Let $g \in \mathcal{O}_{\mathbb{C}^n,0}$ be an irreducible function. If $f \in \mathcal{O}_{\mathbb{C}^n,0}$ vanishes on $Z(g)$, then $g$ divides $f$.*

(**pp.16 of [1]**)

**Lemma 3.1.1.** *For any germ $X \subset \mathbb{C}^n$ the set $I(X) \subset \mathcal{O}_{\mathbb{C}^n,0}$ is an ideal. If $(A) \subset \mathcal{O}_{\mathbb{C}^n,0}$ denotes the ideal generated by the subset $A \subset \mathcal{O}_{\mathbb{C}^n,0}$, then $Z(A) = Z((A))$ and $Z(A)$ is analytic.*

(**pp.18 of [1]**)

**Lemma 3.1.2.** *If $X_1 \subset X_2$, then $I(X_2) \subset I(X_1)$. If $I_1 \subset I_2$, then $Z(I_2) \subset Z(I_1)4$. For any analytic germ $X$ one has $Z(I(X)) = X$. For any ideal $I \subset \mathcal{O}_{\mathbb{C}^n,0}$, one has $I \subset I(Z(I))$.*

(**pp.18 of [1]**)

## 3.2 1.2 Complex and Hermitian Structures

**Lemma 3.2.1.** *If $I$ is an almost complex structure on a real vector space $V$, then $V$ admits in a natural way the structure of a complex vector space*

(**pp.25 of [1]**)

**Remark 3.2.1.** *An almost complex structure can only exist on an even dimensional real vector space.*

**Corollary 3.2.1.** *Any almost complex structure on $V$ induces a natural orientation on $V$.*

(**pp.25 of [1]**)

**Lemma 3.2.2.** *Let $V$ be a real vector space endowed with an almost complex structure $I$. Then*

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

*Complex conjugation on $V_{\mathbb{C}}$ induces an $\mathbb{R}$-linear isomorphism $V^{1,0} \cong V^{0,1}$.*

(**pp.26 of [1]**)

**Remark 3.2.2.** *Two almost complex structures on $V_{\mathbb{C}}$: $I$ and $i$, coincide on the subspace $V^{1,0}$ but differ by a sign on $V^{0,1}$.*

**Lemma 3.2.3.** *Let $V$ be a real vector space endowed with an almost complex structure $I$. Then the dual space $V^* = Hom_{\mathbb{R}}(V, \mathbb{R})$ has a natural almost complex structure given by $I(f)(v) = f(I(v))$. The induced decomposition on $(V^*)_{\mathbb{C}} = Hom_{\mathbb{R}}(V, \mathbb{C}) = (V_{\mathbb{C}})^*$ is given by*

$$(V^*)^{1,0} = \{f \in Hom_{\mathbb{R}}(V, \mathbb{C}) | f(I(v)) = if(v)\} = (V^{1,0})^*$$

$$(V^*)^{0,1} = \{f \in Hom_{\mathbb{R}}(V, \mathbb{C}) | f(I(v)) = -if(v)\} = (V^{0,1})^*$$

*Also note that $(V^*)^{1,0} = Hom_{\mathbb{C}}((V, I), \mathbb{C})$.*

**Proposition 3.2.1.** *For a real vector space $V$ endowed with an almost complex structure $I$, one has:*

1. *$\bigwedge^{p,q} V$ is in a canonical way a subsapce of $\bigwedge^{p+q} V_{\mathbb{C}}$.*
2. *$\bigwedge^{k} V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V$.*
3. *Complex conjugation on $\bigwedge^{*} V_{\mathbb{C}}$ defines a ($\mathbb{C}$-linear) isomorphism $\bigwedge^{p,q} V \cong \bigwedge^{q,p} V$, i.e. $\overline{\bigwedge^{p,q} V} = \bigwedge^{q,p} V$.*
4. *The exterior prodoct is of bidegree $(0,0)$.*

(**pp.27 of [1]**)

**Remark 3.2.3.** *Local calculation of* $V^{1,0}$,$(V^*)^{1,0}$

$$z_i = \frac{1}{2}(x_i - y_i),\ \bar{z}_i = \frac{1}{2}(x_i + iy_i)$$

$$z^i = x^i + iy^i,\ \bar{z}^i = x^i - iy^i$$

$$I(z_i) = iz_i,\ I(z^i) = iz^i$$

(**pp.27 to 28 of [1]**)

**Lemma 3.2.4.** *For any* $m \leq dim_{\mathbb{C}} V^{1,0}$*, one has*

$$(-2i)^m (z_1 \wedge \bar{z}_1) \wedge \cdots \wedge (z_m \wedge \bar{z}_m) = (x_1 \wedge y_1) \wedge \cdots \wedge (x_m \wedge y_m).$$

*For* $m = dim_{\mathbb{C}} V^{1,0}$*, this defines a positive oriented volume form for the natural orientation of* $V$*.*

*Also*

$$\left(\frac{i}{2}\right)^m (z^1 \wedge \bar{z}^1) \wedge \cdots \wedge (z^m \wedge \bar{z}^m) = (x^1 \wedge y^1) \wedge \cdots \wedge (x^m \wedge y^m).$$

**Proposition 3.2.2** (Lefschetz decomposition)**.** *There exists a direct sum decomposition of the form:*

$$\bigwedge^k V^* = \bigoplus_{i \geq 0} L^i(P^{k-2i}) \tag{3.2.0.1}$$

*Also,* $P^k = \alpha \in \bigwedge^k V^* | L^{n-k+1}\alpha = 0$*, for* $k \leq n$*. Naturally* $P^k = 0$ *for* $k > 0$*.*

*We also have several morphisms induced by* $L$*, which is illustrated in the following graph adapted from the book:*

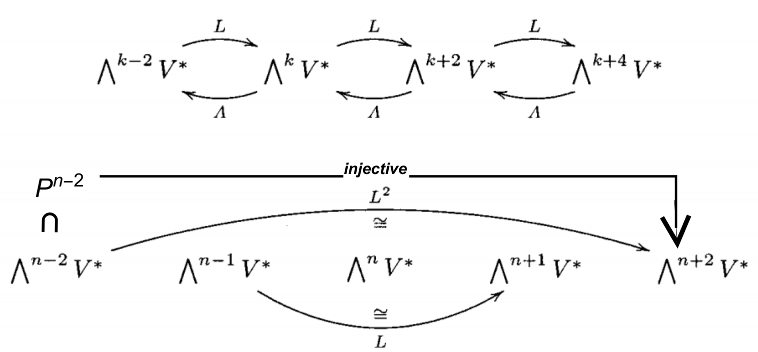


Figure 3.1: Morphisms

(**pp.36 of [1]**)

As shown in the theorem, the map $\Lambda^{n-k}$ is produce a mirror effect in $\bigwedge^* V^*$, very similar to the Hodge $*$. The next proposition relates the two:

**Proposition 3.2.3.** *For all $\alpha \in P^k$, we have:*

$$*L^j\alpha = (-1)^{\frac{k(k+2)}{2}} \frac{j!}{(n-k-j)!} \cdot L^{n-k-j} I(\alpha). \tag{3.2.0.2}$$

Particularly, when $j = k = 0$, we have $*1 = \mathrm{vol} = \frac{\omega^n}{n!}$, or,

$$n!\mathrm{vol} = \omega^n \tag{3.2.0.3}$$

(**pp.37 of [1]**)

**Corollary 3.2.2** (Hodge—Riemann bilinear relation)**.**

$$Q(\bigwedge^{p,q} V^*, \bigwedge^{p',q'} V^*) = 0 \tag{3.2.0.4}$$

*for $(p,q) \neq (p',q')$, and*

$$i^{p-q} Q(\alpha, \bar{\alpha}) = (n-(p+q))! \cdot \langle \alpha, \alpha \rangle_{\mathbb{C}} > 0 \tag{3.2.0.5}$$

*for $0 \neq \alpha \in P^{p,q}$, with $p + q \leq n$.*

(**pp.39 of [1]**)

# Bibliography

[1] Complex Geometry

# Part III

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