Group Theory in Physics (Course Note)

Taper

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Abstract

This is our course note for the course about group theoy, with its application in physics.

Contents

1. Combine group theory with you research.
2. Mid-term and final.
3. No homework, cause it is already graduate level.
4. Professor Fei Ye. (Phone: 88018229, 228 in Research building 2), T.A. Zhe Zhang. (110 Research building 2).

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He first introduces several common examples of symmetries in our life and physics. Omitted, with one exception:

He mentions that there is one more symmetry in the Hydrogen Hamiltonian: the Laplace-Runge-Lenz symmetry. (So its symmetry group is not just $SO(3)$, but two copies of $SO(3)$ that forms a $SO(4)$. And using the representation of $SO(4)$, the complete spectrum of Hydrogen Hamiltonian is solved. Hence this $SO(4)$ is the largest symmetry of Hydrogen Hamiltonian.

1.1 Digression about Lenz vector

Since the class is too boring, I checked about the Lenz vector via Google and found this Math.SE question [2]

The first answer to that post is:

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1) **Problem**. The Kepler Problem has Hamiltonian

$$H \; := \; \frac{p^2}{2m} - \frac{k}{q},$$

where m is the 2-body reduced mass. The [Laplace–Runge–Lenz vector](http://en.wikipedia.org/wiki/Laplace

$$A^j \; := \; a^j + km\frac{q^j}{q}, \qquad a^j \; := \; (\mathbf{L}\times\mathbf{p})^j \; = \; \mathbf{q}\cdot\mathbf{p}\, p^j - p^2\, q^j, \qquad \mathbf{L} \; := \; \mathbf{q}\times\mathbf{p}.$$

2) **Action**. The Hamiltonian Lagrangian is

$$L_H \; := \; \dot{\mathbf{q}}\cdot\mathbf{p} - H,$$

and the action is

$$S[\mathbf{q},\mathbf{p}] \; = \; \int \mathrm{d}t \; L_H.$$

The non-zero fundamental canonical Poisson brackets are

$$\{q^i, p^j\} \; = \; \delta^{ij}.$$

3) **Inverse Noether's Theorem**. Quite generally in the Hamiltonian formulation, given a constant of motion Q, then the infinitesimal variation

$$\delta \; = \; \varepsilon\{Q, \cdot\}$$

is a global off-shell symmetry of the action S (modulo boundary terms). Here ε is an infinitesimal global parameter, and $X_Q = \{Q, \cdot\}$ is a Hamiltonian vector field with Hamiltonian generator Q. The full Noether current is (minus) Q, see e.g. my answer to [this question](http://physics.stackexchange.com/q/8626/2451). (The words **on-shell**. and **off-shell**. refer to whether the equations of motion are satisfied or not.)

4) **Variation**. Let us check that the three Laplace–Runge–Lenz components A^j are Hamiltonian generators of three continuous global off-shell symmetries of the action S. In detail, the infinitesimal variations $\delta = \varepsilon_j\{A^j, \cdot\}$ read

$$\delta q^i \; = \; \varepsilon_j\{A^j, q^i\}, \qquad \{A^j, q^i\} \; = \; 2p^i q^j - q^i p^j - \mathbf{q}\cdot\mathbf{p}\; \delta^{ij},$$

$$\delta p^i \;=\; \varepsilon_j\{A^j,p^i\}, \qquad \{A^j,p^i\} \;=\; p^ip^j{-}p^2\,\delta^{ij}{+}km\left(\frac{\delta^{ij}}{q}-\frac{q^iq^j}{q^3}\right),$$

$$\delta t \;=\; 0,$$

where ε_j are three infinitesimal parameters.

5) Notice for later that

$$\mathbf{q}\cdot\delta\mathbf{q} \;=\; \varepsilon_j(\mathbf{q}\cdot\mathbf{p}\;q^j - q^2\;p^j),$$

$$\mathbf{p}\cdot\delta\mathbf{p} \;=\; \varepsilon_j km(\frac{p^j}{q}-\frac{\mathbf{q}\cdot\mathbf{p}\;q^j}{q^3}) \;=\; -\frac{km}{q^3}\mathbf{q}\cdot\delta\mathbf{q},$$

$$\mathbf{q}\cdot\delta\mathbf{p} \;=\; \varepsilon_j(\mathbf{q}\cdot\mathbf{p}\;p^j - p^2\;q^j) \;=\; \varepsilon_j a^j,$$

$$\mathbf{p}\cdot\delta\mathbf{q} \;=\; 2\varepsilon_j(p^2\;q^j - \mathbf{q}\cdot\mathbf{p}\;p^j) \;=\; -\,2\varepsilon_j a^j\;.$$

6) The Hamiltonian is invariant

$$\delta H \;=\; \frac{1}{m}\mathbf{p}\cdot\delta\mathbf{p}+\frac{k}{q^3}\mathbf{q}\cdot\delta\mathbf{q} \;=\; 0,$$

showing that the Laplace–Runge–Lenz vector A^j is classically a constant of motion

$$\frac{dA^j}{dt} \;\approx\; \{A^j,H\}+\frac{\partial A^j}{\partial t} \;=\; 0.$$

(We will use the $\approx$ sign to stress that an equation is an on-shell equation.)

7) The variation of the Hamiltonian Lagrangian L_H is a total time derivative

$$\begin{aligned}\delta L_H \;&=\; \delta(\dot{\mathbf{q}}\cdot\mathbf{p}) \;=\; \dot{\mathbf{q}}\cdot\delta\mathbf{p}-\dot{\mathbf{p}}\cdot\delta\mathbf{q}+\frac{d(\mathbf{p}\cdot\delta\mathbf{q})}{dt}\\
&=\; \varepsilon_j\left(\dot{\mathbf{q}}\cdot\mathbf{p}\;p^j - p^2\;\dot{q}^j + km\left(\frac{\dot{q}^j}{q}-\frac{\mathbf{q}\cdot\dot{\mathbf{q}}\;q^j}{q^3}\right)\right)\\
&\quad -\varepsilon_j\left(2\dot{\mathbf{p}}\cdot\mathbf{p}\;q^j - \dot{\mathbf{p}}\cdot\mathbf{q}\;p^j - \mathbf{p}\cdot\mathbf{q}\;\dot{p}^j\right) - 2\varepsilon_j\frac{da^j}{dt}\\
&=\; \varepsilon_j\frac{df^j}{dt}, \qquad f^j \;:=\; A^j-2a^j,\end{aligned}$$

and hence the action S is invariant off-shell up to boundary terms.

8) **Noether current**. The bare Noether current j^k is

$$j^k \;:=\; \frac{\partial L_H}{\partial\dot{q}^i}\{A^k,q^i\}+\frac{\partial L_H}{\partial\dot{p}^i}\{A^k,p^i\} \;=\; p^i\{A^k,q^i\} \;=\; -\,2a^k.$$

The full Noether current J^k (which takes the total time-derivative into account) becomes (minus) the Laplace–Runge–Lenz vector

$$J^k \;:=\; j^k - f^k \;=\; -\,2a^k - (A^k - 2a^k) \;=\; -\,A^k.$$

J^k is conserved on-shell

$$\frac{dJ^k}{dt} \;\approx\; 0,$$

due to Noether's first Theorem. Here k is an index that labels the three symmetries.

However, I don't really understand the content inside. I asked professor Ye whether we can find some physics about this conserved quantity, and he answered with no.

The next answer is also interesting:

While Kepler second law is simply a statement of the conservation of angular momentum (and as such it holds for all systems described by central forces), the first and the third laws are special and are linked with the unique form of the newtonian potential $-k/r$. In particular, Bertrand theorem assures that *only* the newtonian potential and the harmonic potential kr^2 give rise to closed orbits (no precession). It is natural to think that this must be due to some kind of symmetry of the problem. In fact, the particular symmetry of the newtonian potential is described exactly by the conservation of the RL vector (it can be shown that the RL vector is conserved iff the potential is central and newtonian). This, in turn, is due to a more general symmetry: if conservation of angular momentum is linked to the group of special orthogonal transformations in 3-dimensional space $SO(3)$, conservation of the RL vector must be linked to a 6-dimensional group of symmetries, since in this case there are apparently six conserved quantities (3 components of L and 3 components of $\mathcal{A}$). In the case of bound orbits, this group is $SO(4)$, the group of rotations in 4-dimensional space.

Just to fix the notation, the RL vector is:

$$\mathcal{A} = \mathbf{p} \times \mathbf{L} - \frac{km}{r}\mathbf{x} \tag{1.1.1}$$

Calculate its total derivative:

$$\frac{d\mathcal{A}}{dt} = -\nabla U \times (\mathbf{x} \times \mathbf{p}) + \mathbf{p} \times \frac{d\mathbf{L}}{dt} - \frac{k\mathbf{p}}{r} + \frac{km(\mathbf{p}\cdot\mathbf{x})}{r^3}\mathbf{x} \tag{1.1.2}$$

Make use of Levi-Civita symbol to develop the cross terms:

$$\epsilon_{sjk}\epsilon_{sil} = \delta_{ji}\delta_{kl} - \delta_{jl}\delta_{ki} \tag{1.1.3}$$

Finally:

$$\frac{d\mathcal{A}}{dt} = \left(\mathbf{x}\cdot\nabla U - \frac{k}{r}\right)\mathbf{p} + \left[(\mathbf{p}\cdot\mathbf{x})\frac{k}{r^3} - 2\mathbf{p}\cdot\nabla U\right]\mathbf{x} + (\mathbf{p}\cdot\mathbf{x})\nabla U \tag{1.1.4}$$

Now, if the potential $U = U(r)$ is central:

$$(\nabla U)_j = \frac{\partial U}{\partial x_j} = \frac{dU}{dr}\frac{\partial r}{\partial x_j} = \frac{dU}{dr}\frac{x_j}{r} \tag{1.1.5}$$

so

$$\nabla U = \frac{dU}{dr}\frac{\mathbf{x}}{r} \tag{1.1.6}$$

Substituting back:

$$\frac{d\mathcal{A}}{dt} = \frac{1}{r}\left(\frac{dU}{dr} - \frac{k}{r^2}\right)\left[r^2\mathbf{p} - (\mathbf{x}\cdot\mathbf{p})\mathbf{x}\right] \tag{1.1.7}$$

Now, you see that if U has *exactly* the newtonian form then the first parenthesis is zero and so the RL vector is conserved. Maybe there's some slicker way to see it (Poisson brackets?), but this works anyway.

1.2 Coming back to the course

After mentioning the Poincáre group, he produces to review some concepts about linear algebra:

1. The axioms of linear space, using quantum mechanics as basic example (Omitted).
2. Some common concepts of linear space: linear-independence, subspace, direct sum, linear operators, its matrix representation. (Omitted)
3. Introducing the complete antisymmetric tensor $\epsilon^{a_1,\cdots,a_n}$. Some properties:

$$\frac{1}{(m-n)!}\sum_{a_{n+1},\cdots,a_m}\epsilon_{a_1,\cdots,a_n,a_{n+1},a_m}\epsilon_{b_1,\cdots,b_n,a_{n+1},a_m} = \sum_{p_1,\cdots,p_n}\epsilon_{p_1,\cdots,p_n}\delta_{a_1,b_{p_1}}\cdots\delta_{a_n,b_{p_n}} \tag{1.2.1}$$

$$\epsilon_{ab}\epsilon_{rs} = \delta_{ar}\delta_{bs} - \delta_{as}\delta_{br} \tag{1.2.2}$$

$$\sum_d \epsilon_{abd}\epsilon_{rsd} = \delta_{ar}\delta_{bs} + \delta_{as}\delta_{br} \tag{1.2.3}$$

4. Some special matrices.

5. Fact: If $R\Gamma = \Gamma R$, and Γ is diagonal. (let $\mu \neq \nu$) Then if $\Gamma_{\mu\mu} \neq \Gamma_{\nu\nu}$, we have: $R_{\mu\nu} = R_{\nu\mu} = 0$. On the other hand, if $R_{\mu\nu} \neq 0$, then $\Gamma_{\mu\mu} = \Gamma_{\nu\nu}$. This is obviously from:

$$\sum_j R^i_j \Gamma^j_k = \sum_j \Gamma^i_j R^j_k \Longrightarrow R^i_k \Gamma^k_k = \Gamma^i_i R^i_k$$

where the first is automatically summed, and the second is not.

6. A linear functional is closed w.r.t. a vector space. (Omitted)

7. ... then this linear functional can be expressed as a matrix w.r.t to a basis of this vector space. (Omitted)

8. Invariant subspace. (Omitted)

9. Transformation of basis. (Omitted)

10. Direct sum of operators:
Let vector spaces $L = L_1 \oplus L_2$, with $L = \langle e_i \rangle$, $L_1 = \langle e'_1, \cdots e'_n \rangle$,$L_2 = \langle e'_{n+1}, \cdots, e'_m \rangle$, $e'_\nu = \sum_\mu e_\mu S_{\mu\nu}$. Assume that L_1, L_2 are invariant w.r.t A, an linear operator. If:

$$Ae'_\mu = \sum_{\nu=1}^{m} e'_\nu R'_{\nu\mu} \tag{1.2.4}$$

we have obviously:

$$Ae'_\mu = \sum_{\nu=1}^{n} e'_\nu R'_{\nu\mu} \text{for } \mu \in \{1 \cdots n\} \tag{1.2.5}$$

$$Ae'_\mu = \sum_{\nu=n}^{m} e'_\nu R'_{\nu\mu} \text{for } \mu \in \{n \cdots m\} \tag{1.2.6}$$

i.e., A's matrix representation has two diagonal blocks. Using this fact, A after a linear transformation (by S), could be written as $R_1 \oplus R_2$, where the meaning of R_1/R_2 is obvious.

11. Eigenvalues and the characteristic equation. (Omitted) Some properties:
    (a) Trace $= \sum_i \lambda_i$
    (b) Determinant $= \prod_i \lambda_i$
    (c) Geometric multiplicity $\leq$ Algebraic multiplicity, or

    $$\text{dim}V_{\lambda_1} \leq n_1$$

    .

12. Inner product and orthonormal basis. (Omitted) Here we define matrix Ω to be, when a basis $\{e_i\}$is given:

Definition 1.1.

$$\Omega_{ij} \equiv \langle e_i, e_j \rangle \tag{1.2.7}$$

13. Adjoint operator:

    Let A be a linear operator represented by matrix A^i_j. Let its adjoint $A^\dagger$ be represented by R^i_J. Then using $\langle A^\dagger e_j, e_i\rangle = \langle e_j, Ae_i\rangle$, we will get $(R^k_j)^*\Omega_{ki} = \Omega_{jk}A^k_i$, i.e. $(R^T)^*\Omega = \Omega A$, so:

    $$R = \Omega^{-1}A^\dagger\Omega \tag{1.2.8}$$

    where we have used the fact that $\Omega^\dagger = \Omega$.

    Note that $(R^k_j)^*\Omega_{ki}$ is not $\Omega^T R^*$. (Be careful and you will find out why.)

    This is very different from my previous naive concept when Ω is not identity matrix, i.e. when the basis is not orthonormal.

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He first introduces some important matrices:

Unitary matrix Eigenvalues of Unitary matrices has modulus 1, i.e. $|\lambda| = 1$. This can be proved directly. Also, Unitary matrices are unitarily diagonalizable. This is a result of the following Spectral Theorem:

Theorem 2.1 (Spectral Theorem)**.** *A matrix A, which is normal (i.e. $A^\dagger A = AA^\dagger$), if and only if it is unitarily diagonalizable.*

Proof. If A is normal, then by Schur decomposition, we can write $A = UTU^\dagger$, here U is unitary and T is upper-triangular. Using the condition of being normal, one can show directly that T is in fact also normal. Now we show that any triangular matrix that is normal must be diagonal. Observe that we have $\langle e_i, T^\dagger Te_i\rangle = \langle e_i, TT^\dagger e_i\rangle$, i.e. $\langle T^\dagger e_i, T^\dagger e_i\rangle = \langle Te_i, Te_i\rangle$. This is sayin g that the norm of the first column of $A^\dagger$ is equal to the norm of the first column of A. Obviously A has to be diagonal.

The converse is obvious. □

Also, unitary matrix's eigenvector corresponding to different eigenvalues are orthogonal. This is a direct result of fact mentioned above.

Hermitian matrices They have real eigenvalues and orthogonal eigenvectors (proof omitted). Also, if $\det(R^\dagger R) \neq 0$, then $R^\dagger R > 0$, i.e. it is positive-definite.

This is wong: An example is that the matrix Ω introduced in the previous lecture has $\det(\Omega^\dagger\Omega) = \det(\Omega)$, hence $\det(\Omega) = 1$ (it cannot be 0), hence it is positive definite.

Actually $\det(\Omega^\dagger\Omega) \neq \det(\Omega)$, because

$$\sum_\rho |e_\rho\rangle\langle e_\rho| \neq 1 (\text{unless the basis is orthonormal}) \tag{2.0.9}$$

Therefore we need anthoer argument for Ω being positive-definite. It is provided in page 11 of [1].

Orthogonal matrix For an orthogonal matrix over $\mathbb{C}$, it is quite troublesome. For example, if $Ra = \lambda a$ and $\lambda \neq \pm 1$, then we have $a^T a = 0$, which is quite bad because this force a to have complex components.

Orthogonal matrix over $\mathbb{R}$ In this case, we have similar result. But it is easy to show that for an orthogonal matrix R having only real elements, then its eigenvalues $\lambda = \pm 1$.

Then he proceeds to direct product.

Direct product and also the Kronecker Product of two matrices. Properties (let $T = R \otimes S$):

1. $\mathrm{dim}T = \mathrm{dim}R \times \mathrm{dim}S$
2. $\mathrm{tr}(T) = \mathrm{tr}(R)\mathrm{tr}(S)$
3. $\otimes$ commutes with the operation of inverse, transpose, and transpose conjugation.
4. $$\frac{d}{d\alpha}(R(\alpha) \otimes S(\alpha)) = R'(\alpha) \otimes S(\alpha) + R(\alpha) \otimes S'(\alpha) \tag{2.0.10}$$
5. when the dimentions are the same:
   (a) $(R_1 \otimes S_1)(R_2 \otimes S_2) = (R_1 R_2) \otimes (S_1 S_2)$

Finally we arrived in the group theory.

Symmetry examples Dipole transition. $\langle \phi_f | \hat{P} | \phi_i \rangle$, must happen when the parity of ϕ_i and ϕ_f is of opposite parity. (pp.18 of [1])

Group

Definition 2.1 (Group)**.** *Omitted.*

Some basic properties (Omitted).

Definition 2.2 (Abel Group)**.** *Omitted.*

Definition 2.3 (Cardinality of group $\#A$)**.** *Omitted.*

Multiplication table Facts: group of order $1, 2$, and 3 are unique up to an isomorphism.

Definition 2.4 (Cyclic group, generators)**.** *Omitted.*

Definition 2.5 ($C_N, \bar{C}_N$)**.** $\bar{C}_N = C_N * \sigma$

3 Anchor

References

[1] Zhongqi Ma, Group Theory in Physics

[2] What symmetry causes the Runge-Lenz vector to be conserved?

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