# Notes for General Relativity

Taper

September 22, 2016

**Abstract**

(None)

## Contents



# 1 The Principle of Relativity

Landau's book [1] describes the speed of light, as a speed of the maximum velocity of propagation of interaction. He perceives velocity as describing the propagation of interaction.

In another aspect, I think that time does not exist. I perceive time as a agent of the environmental effect, communicating information and coordinating movement between the system under consideration and its envrionment. Therefore, a single existance is eternal by nature. Perhaps before the universe originates, time is a meaningless concept. In my view, the light is the only reliable tool to serve function of communication and coordination.

Fram either perspective, the speed of light is inherently constent. However, from my view, it is not clear why this speed should be a maximum.

## 1.1 Invariance of Interval

To do dynamics, we necessarily need a measure of distance. Here introduces the distance in spacetime - interval. It is:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \tag{1.1.1}$$

**Note** that from now on, I will try to use natural units as often as possible.

Such a measure should be invariant in different Lorentz frames. Landau proves it by postulating a priori that:

$$\mathrm{d}s^2 = a\,\mathrm{d}s'^2 \tag{1.1.2}$$

This is suggested by $\mathrm{d}s = 0$ is invariant (invariance of the speed of light), and $\mathrm{d}s$ and $\mathrm{d}s'$ should be infinitesimals of the same order. From this postulation, it is straightforward to argue that $a$ should be a constant and is equal to 1.

Using this property, one can classify interval between events as being *timelike, spacelike,* and *lightlike*, by whether $ds > 0$, $ds < 0$ or $ds = 0$. A mnemonic tip is that for timelike intervals, the "time difference" is dominant, and for spacelike intervals, the spatial difference is dominant.

The following figure gives an indication of how the timelike, spacelike, lightlike classification is related to causality:

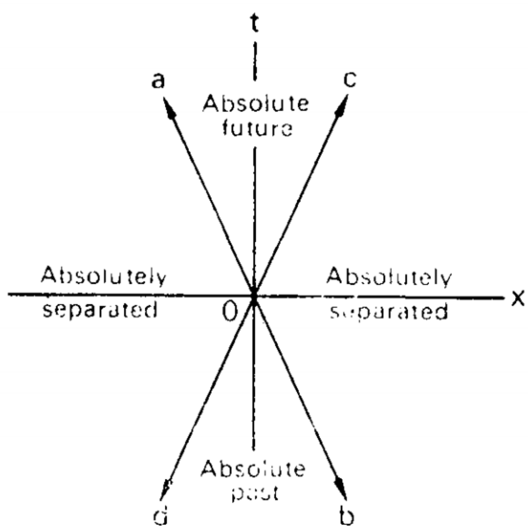


Figure 1: Spacetime regions

Only those events in the absolute future and in the absolute past regions can have a causal link with the people at $O$, due to the limit of speed of propagation. And the future and past combined together as the timelike region.

Another difference of timelike and spacelike regions lies in the prefix "absoluate". All events in timelike region cannot be simultaneous with $O$ in any reference frame, since the interval must have a nonzero time component. Similarly all events in spacelike region must happen in different place with the $O$. An additional requirement, which comes naturally from law of causality, is that all events in the future must remain absolutely in the future in any reference frame. Similarly we have an absolute past.

Interestingly, the concept of being simultaneous with $O$, being before or after $O$ in time, are relative for events in spacelike region. However, since there can be no causal link between $O$ and those events, this relativity does not pose a challenge to causality.

Lastly, the cone formed by all events with $ds = 0$ is called the *light cone*. Events in it is very special that it deserve to devote a separate section to discuss it, which will be done later in this note.

## 1.2 Proper Time

By

$$dt'^2 = dt^2 - dx^2 = dt^2 - (v\,\mathrm{d}t)^2$$

we have

$$\Delta t' = \int_{t_1}^{t_2} \mathrm{d}t\sqrt{1-v^2} \tag{1.2.1}$$

This shows the time dilation effect of a moving clock. Also, by this we can always calculate the time experienced by a clock by $t = \frac{1}{c}\int \mathrm{d}s$ (SI unit).

The Landau's book [1] explains a classical paradox about time dilation, which is omitted here.

Interestingly, we have the property that, for all lines between two events, the longest one (i.e. the path that has the longest interval), is the straight line, contrary to the classical case. To see this, we note that any two events (assumed to be causally linked) can be connected by a flying clock with uniform speed.

## 1.3 Lorentz Transformation

Next we have to consider the coordinate transformations in four-dimensional spacetime. The formula can be obtained solely by the restricting transformation to be an isometry.

These transformations are categorize into two groups: parallel displacement and rotation. Only formula for rotation is worth consideration. Rotation in spacetime, simiilar to their counterparts in $\mathbb{R}^3$, can be decomposed according to six axes of rotation, each representing rotation in plane $xy, xz, yz, xt, yt, zt$. The first of this is only ordinary spatial rotation. The rest are called *Lorentz Transformations*. And there is really only one important formula, one for rotation around $xt$ plane:

$$x = x'\cosh\psi + t\sinh\psi,\; t = x'\sinh\psi + t\cosh\psi \tag{1.3.1}$$

where $\psi$ is the "angle of rotation". However, this "angle" do not have much physical interpretation. Instead, the following formula is more useful, which relates the coordinate in frame $K$, to anther fram $K'$ moving in uniform velocity $v$ w.r.t $K$:

$$
\begin{aligned}
x &= \frac{x' + vt'}{\sqrt{1-v^2}} \\
y &= y',\, z = z' \\
t &= \frac{t' + vx'}{\sqrt{1-v^2}} \qquad (1.3.2)
\end{aligned}
$$

Note that in matrix form, (using the popular notations $\Lambda^{\nu}{}_{\mu}, \gamma,$ and $\beta$):

$$
\Lambda^{\nu}{}_{\mu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (1.3.3)
$$

Aided with this formula, Landau discusses the *Lorentz contraction* phenomenon in special relativity and defines the proper length $l_0$, defined as the length in its rest frame. It is related to its length measured in another frame $K'$ by:

$$
l' = l_0\sqrt{1-v^2} \qquad (1.3.4)
$$

Finally he mentions that, since Lorentz transformations are in fact rotations, they naturally do not commute, unless the rotation is fixed in only one plane.

## 1.4 Transformation of velocities

Note that for convenience, I will adopt the popular Lorentz factor $\gamma$, defined as:

$$
\gamma \equiv \frac{1}{\sqrt{1-V^2}} \qquad (1.4.1)
$$

To get the similar transofmrations for velocities is quite easy, and the result is:

$$
v_x = \frac{v'_x + V}{1 + v'_x V} \qquad (1.4.2)
$$

$$
v_y = \frac{v'_y}{\gamma\,(1 + v' x V)} \qquad (1.4.3)
$$

$$
v_z = \frac{v'_z}{\gamma\,(1 + v' x V)} \qquad (1.4.4)
$$

$$
\qquad (1.4.5)
$$

In the special case when the motion is parallel to the $x$-axis, we have (let $v_x$ just be $v$):

$$
v = \frac{v' + V}{1 + v'V} \qquad (1.4.6)
$$

Landau mentions that when $V$ is significantly smaller than the velocity of light, we have the following approximate formula:

$$
\vec{v} = \vec{v'} + \vec{V} - (\vec{V} \cdot \vec{v'})\vec{v'} \qquad (1.4.7)
$$

## 1.5 Four-vectors

This section is basically an recapitulation of important definitions and results in tensor analysis in spacetime. Since I am quite familiar with this topic, I will only cover something I felt new.

**Complete antisymmetric tensor $e^{iklm}$** It is defined such that any interchange of indices gives a minus sign. We requires that:

$$e^{0123} = +1 \tag{1.5.1}$$

Therefore:

$$e_{0123} = -1 \,, e^{iklm} e_{iklm} = -24$$

Also, $e^{iklm}$ is only a pseudotensor:

*Proof.* Under transformation (let $\Lambda$ be the transformation matrix) we have: $\tilde{e}^{0123} = \Lambda^0_i \Lambda^1_k \Lambda^2_l \Lambda^3_m e^{iklm}$ Notice that $e^{iklm} = \mathrm{sgn}(iklm)$ by our definition. Therefore, the right hand side above is exactly the definition of determinant. Hence $\tilde{e}^{0123} = \det(\Lambda)$.

Also, one can easily show that $\tilde{e}^{nqpr}$ is also totally antisymmetric, since $e^{iklm}$ is totally antisymmetric.

With these we have:

$$\tilde{e}^{nqpr} = \det(\Lambda) e^{nqpr}$$

But $e^{iklm}$ is defined without reference to basis, therefore we have actually:

$$\tilde{e}^{nqpr} \equiv e^{npqr} \tag{1.5.2}$$

Note that since $\Lambda$ must be orthonormal, we have $\det(\Lambda) = \pm 1$. Therefore when $\det(\Lambda) = 1$, the two requirement coincide and $e^{iklm}$ behaves like a tensor, and when $\det(\Lambda) = -1$, it behaves like a pseudotensor. $\square$

By similar arguments, we see that $e^{iklm} e^{npqr}$ forms a true tensor, so does $e^{iklm} e_{prst}$, etc.

Also, Landau says that the only tensor that is invariant in all corrdinate system is the unit tensor $\sigma^i_k$. So all the above tensor should be able to be expressed in terms of this. The author then provides the following formulae (they are too complicated to type that I just paste the screenshot here):

$$e^{iklm} e_{prst} = -\begin{vmatrix} \delta^i_p & \delta^i_r & \delta^i_s & \delta^i_t \\ \delta^k_p & \delta^k_r & \delta^k_s & \delta^k_t \\ \delta^l_p & \delta^l_r & \delta^l_s & \delta^l_t \\ \delta^m_p & \delta^m_r & \delta^m_s & \delta^m_t \end{vmatrix}, \qquad e^{iklm} e_{prsm} = -\begin{vmatrix} \delta^i_p & \delta^i_r & \delta^i_s \\ \delta^k_p & \delta^k_r & \delta^k_s \\ \delta^l_p & \delta^l_r & \delta^l_s \end{vmatrix},$$

$$e^{iklm} e_{prlm} = -2(\delta^i_p \delta^k_r - \delta^i_r \delta^k_p), \qquad e^{iklm} e_{pklm} = -6\delta^i_p.$$

Figure 2: Formulae for ee

And:

$$e^{prst}A_{ip}A_{kr}A_{ls}A_{mt} = -Ae_{iklm} \tag{1.5.3}$$

$$e^{iklm}e^{prst}A_{ip}A_{kr}A_{ls}A_{mt} = 24\det(A) \tag{1.5.4}$$

Next Landau introduces the dual tensors. Since this is important and the case of Lorentzian spacetime is somewhat different from the usual case, this topic deserves a separate treatment in the following section.

## 1.6 Digression: inverse of $\Lambda$

Here's a simple formula to get the inverse of Lorentz transformation.

$$(\Lambda^{-1})^{\nu}{}_{\mu} = \Lambda_{\mu}{}^{\nu} \tag{1.6.1}$$

i.e., we drag down the first index and raise the second index, using of course the metric $g$. To put it more precise, we have:

$$\begin{aligned} \Lambda^{i}{}_{\sigma}\Lambda_{j}{}^{\sigma} &= \delta^{i}_{j} \\ \Lambda_{\sigma}{}^{i}\Lambda^{\sigma}{}_{j} &= \delta^{i}_{j} \end{aligned} \tag{1.6.2}$$

*Proof.* This can be proved by noting $(\Lambda V, \Lambda W) = (V, W)$, i.e.

$$\Lambda^{\mu}{}_{\nu}V^{\nu}g_{\mu\mu'}\Lambda^{\mu'}{}_{\nu'}W^{\nu'} = V^{\mu}g_{\mu\mu'}W^{\mu'}$$

By choosing $V$ and $W$ to have appropriate components, one can prove that $\Lambda^{\mu}{}_{i}g_{\mu\mu'}\Lambda^{\mu'}{}_{j} = g_{ij}$. Also notice that $g^2 = I$. Then by multiply $g^{ik}$ to the left, we can get:

$$g^{ik}\Lambda^{\mu}{}_{k}g_{\mu\mu'}\Lambda^{\mu'}{}_{j} = \delta^{i}_{j} \tag{1.6.3}$$

□

Actually, the above proof shows that this relation is much general, can be applied in all space transformations that preserve some metric!

## 1.7 Dual Tensors

## 1.8 Light's life

The life of a proton must be miserable.

Possible sources: 1, 2. 3.

# References

[1] The Classical Theory of Fields

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