Noetherian Ring

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Abstract

A note about Noetherian Ring, majorly from the book [1].

Contents

1 Module

sec:Module

(**pp.117 to 118 of [1]**)

Definition 1.1 (Module). Let A: ring. M is a left A-module if and only if

1. M is an abelian group, usually written additively.

2. there exists an operation of A on M, written as a multiplicative monoid, such that, for any $a, b \in A$, any $x, y \in M$, we have:

$$(a + b)x = ax + bx \tag{1.0.1}$$
$$a(x + y) = ax + ay \tag{1.0.2}$$

This captures the notion of being *linear*.

By definition of an operation, we have $1x = x$. Also, it can be easily derived that $a(-x) = -ax$, and $0x = 0$.

Example 1.1. Examples of modules

1. A is a module over itself.

2. Any commutative group is a $\mathbb{Z}$-module.

3. Any left ideal of A is a module over A, i.e. a left A-module.

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4. A vector space V over K, is basically a K-module, with the additional structure of K being a field.

5. Let V be a vector space. Let R be the ring of all linear maps of V into itself. Then V is also a module over R, since R are just linear maps.

Example 1.2. One can also pass the module structure M from function whose codomain is M. For example, when M is an A-module and S is an nonempty set. The set of all maps $\text{Map}(S, M)$ can be made into an A-module. Let $a \in A$, a acts on $f \in \text{Map}(S, M)$ by $(af)(s) := a(f(s))$.

Definition 1.2 (Submodule)**.** A submodule M is an additive subgroup such that $AN \subset N$

Definition 1.3 (factor module)**.** Let M be an A-module, and N a submodule. A factor module M/N is the factor group M/N (for the additive group structure) equipped with a module structure. The action of A on M/N is defined by $a(x + N) = ax + N$. This is well defined, since if y is in the same coset as x, then ay is in the same coset as ax.

(**pp.119 of [1]**)

2 Noetherian

Definition 2.1 (Noetherian Module)**.** Let A: a ring. M: a left A-module. M is called Noetherian if M satisfies any of the following conditions:

1. Every submodule of M is finitely generated.

2. Every ascending sequences of submodules of M

$$M_1 \subset M_2 \subset \cdots$$

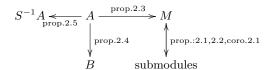such that $M_i \neq M_{i+1}$, is fininte.

3. Every non-empty set S of submodules of M has a maximal element.

The equivalence of the above conditions are proved in page 413 to 414 of [1].

Definition 2.2 (Noetherian Ring)**.** A ring A is Noetherian if and only if it is Noetherian when viewed as a left module over itself.

(**pp.415 of [1]**)

Some theorems Here are some theorems mentioned in Chapter X, section 2 of [1].

The propositions 2.1, 2.2, 2.1 expresses the "Noetherian relation" between M and its submodules. The proposition 2.3 relates a Noetherian ring A and the A-modules. The proposition 2.4 relates two rings. The proposition 2.5 relates a commutative Noetherian ring and its multiplicative subset. The following diagram summarized these relations.

$$S^{-1}A \xleftarrow[\text{prop.2.5}]{} A \xrightarrow{\text{prop.2.3}} M$$

$$\downarrow{\text{prop.2.4}} \qquad \updownarrow{\text{prop.:2.1,2.2,coro.2.1}}$$

$$B \qquad \text{submodules}$$

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$(A, B$: ring, S: A's multiplicative subset. M: a A-module.)

The structure of being Noetherian is consistent between a module and its submodules, factor modules, in the sense of the following two propositions.

Proposition 2.1. *Let M be a Noetherian A-module, then every submodule and every factor module of M is Noetherian.*

(**pp.414 of** [**1**])

Proposition 2.2. *Let M be a module, N be a submodule. If N and M/N are Noetherian, then M is Noetherian.*

(**pp.414 of** [**1**])

The above statements could be summarized by saying that, given an exact sequence:

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

M is Noetherian if and only if M' and M'' are Noetherian. This could be seen by two immediate fact of an exact sequence:

$$M' \cong \mathrm{Im} f, \ M/\mathrm{Ker} g \cong M''$$

Corollary 2.1. *A finite direct sum of Noetherian modules is Noetherian. Specifically, let M be a module, let N, N' be two submodules. If $M = N + N'$ and if both N, N' are Noetherian, then M is Noetherian.*

(**pp.415 of** [**1**])

Proposition 2.3. *Let A be a Noetherian ring, and let M be a finitely generated A-module. Then M is Noetherian.*

(**pp.415 of** [**1**])

Proposition 2.4. *Let A be a ring which is Noetherian, and let $\phi : A \to B$ be a surjective ring-homomorphism. Then B is Noetherian.*

(**pp.415 of** [**1**]) In colloquial term, a surjective homomorphism induces a Noetherian ring.

Proposition 2.5. *Let A be a commutative Noetherian ring, and let S be a multiplicative subset of A. Then $S^{-1}A$ is Noetherian.*

(**pp.415 of** [**1**])

3 Examples

- The polynomial Ring
- The ring of power series
- The ring of formal power series is **NOT** Noetherian. See this post

    The essential point is that the polynomial ring in infinitely many variables is the ascending union of subrings $K[x_1, \ldots, x_n]$, since any polynomial can involve only finitely-many indeterminates. Each of these rings is a UFD, and it is easy to see that a polynomial in which x_N does not appear has

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only factorizations in which x_N does not appear, again because everything takes place inside some polynomial ring in finitely-many variables. But the ring is not Noetherian, because the ideal generated by all the indeterminates is certainly not finitely-generated.

4   Anchor

Nomenclature

References

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[1] S. Lang. Algebra. 3rd. Springer.

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