

# Miscellaneous notes for D. Huybrechts's Complex Geometry

Taper

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## Abstract

Miscellaneous notes for D. Huybrechts's book *Introduction to Complex Geometry*, include some homeworks done.

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## 1 The structure of almost complex structures on $\mathbb{R}^4$ (exercise 1.2.1)

In exercise 1.2.1, it says that the set of all compatible almost complex structures on a euclidean space of dimension  $2n$ , is two copies of  $S^2$ .

To show it, I tried first a straight calculation. Assuming the almost complex structure  $I = (a_{ij})$ . Then we have:

Section 1.2 of Intro to Complex Geometry.

Exercises:

1.2-1.

Q: Let  $(V, \langle, \rangle)$  : euclidian space of  $\dim = 4$ .

Show:  $\{ \text{all compatible almost complex structures} \}$   
 $\cong$   
 $\text{two copies of } S^2 \cong \text{two balls.}$

Recap: compatible:  $I: I^2 = -1, \langle I(v), I(w) \rangle = \langle v, w \rangle$ .

Choose an orthogonal basis:  $e_1, \dots, e_4$

Let  $I = (a^i_j)$   $I^2 = a^i_j a^j_k = -\delta^i_k$

also  $\langle, \rangle \cong \delta^i_j$   $\otimes$

$$\langle I(v), I(w) \rangle = (a^i_j v^j) \cdot \delta^i_k (a^k_l w^l) = v^j \delta^i_k w^k$$

( $\forall \vec{v}, \vec{w}$ )

Hence  $a^i_j \delta^i_k a^k_l = \delta_{jl}$  or  $a^i_j a^i_k = \delta_{jk}$

For example:

$$\sum_j a^1_j a^j_2 = 0 = \sum_j a^j_1 a^j_2 \Rightarrow \sum_{j=2,4} a^1_j a^j_2 = \sum_{j=2} a^j_1 a^j_2$$

This can be generalized:

$$\sum_{\substack{j=1 \\ j \neq k}}^4 a^k_j a^j_l = \sum_{\substack{j=1 \\ j \neq k}}^4 a^j_k a^j_l \quad (k \neq l)$$

} 16 sets of eq.

also:  $a^i_j \sum_{j=1}^4 a^j_i a^j_i = - \sum_{j=1}^4 a^j_i a^j_i$

Figure 1: Draft

Then I discover this too hard to work, because too many equations are involved, and none of them could be eliminated by other. Meanwhile, I found a post in Math.SE about this [2]. Here are several important concepts for understanding that post.

## 1.1 Understand $\frac{GL(2n, \mathbb{R})}{GL(n, \mathbb{C})}$

### 1.1.1 Why $M_n = \frac{GL(2n, \mathbb{R})}{GL(n, \mathbb{C})}$

This is a note of my question on Math.SE [5], which explains that we can identify the set of almost complex structures with  $\frac{GL(2n, \mathbb{R})}{GL(n, \mathbb{C})}$ .

First, I try to do it when  $n = 1$ . I inject a complex number  $a + bi$  by identify it with  $a\mathbb{I} + b\mathbb{J}$ , where  $\mathbb{I}$  is the identify matrix and  $\mathbb{J}$  is  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . I take one set of basis of  $GL(2, \mathbb{R})$  as:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(I think this is a basis because the following matrix is non-singular:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

) Then the  $\frac{GL(2n, \mathbb{R})}{GL(n, \mathbb{C})}$  becomes equivalent classes represented by

$$\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$$

However, I don't know how to link this with an almost complex structure.

I have a feeling that I might have been in the wrong direction. It was pointed out that  $GL(2n, \mathbb{R})$  is not even a vector space. So what I did is in fact nonsense.

Below is one correct answer I got:

An almost-complex structure is a matrix  $J$  such that  $J^2 = -I$  is the negative identity. As you said, one example of such a matrix  $J$  is

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Interpreting  $GL(n, \mathbb{C})$  as a subgroup of  $GL(2n; \mathbb{R})$  depends on having fixed such an almost-complex structure. Once we have a matrix  $J$ , we can call a matrix  $A \in GL(2n; \mathbb{R})$  complex-linear if it commutes with  $J$ , i.e.  $AJA^{-1} = J$ .

(The idea is that  $\mathbb{C}$ -linear maps  $T$  are just real linear maps with the additional property that  $T(iv) = iT(v)$  for all vectors  $v$ )

Given any matrix  $A \in GL(2n; \mathbb{R})$ , we get another almost-complex structure  $AJA^{-1}$ . This is the same almost-complex structure  $J$  if and

only if  $A \in GL(n; \mathbb{C})$ . On the other hand, all almost-complex structures are similar (although it may take some work to be convincing that they are similar over  $\mathbb{R}$  and not only  $\mathbb{C}$ ) since they are diagonalizable with the same eigenvalues  $\pm i$ . That gives you a bijection

$$GL(2n; \mathbb{R})/GL(n; \mathbb{C}) \longrightarrow \{\text{almost - complex structures}\}$$

under which a class  $A \cdot GL(n; \mathbb{C})$  corresponds to the almost-complex structure  $AJA^{-1}$ .

I questioned him:

1. Why  $AJA^{-1}$  is the same almost-complex structure  $J$  if and only if  $A \in GL(n; \mathbb{C})$ .
2. Why all almost-complex structures are similar over  $\mathbb{R}$ .

He responded that:

1. is the definition of  $GL(n; \mathbb{C})$  as matrices  $A$  with  $AJA^{-1} = J$ .
2. comes from the fact that any real matrices that are similar over  $\mathbb{C}$  are already similar over  $\mathbb{R}$ . This isn't trivial but it has been asked and answered many times on this site: here is one reference [6].

Inside that reference, the following theorem is proved:

**Theorem 1.1.** *Let  $E$  be a field, let  $F$  be a subfield, and let  $A$  and  $B$  be  $n \times n$  matrices with coefficients in  $F$ . If  $A$  and  $B$  are similar over  $E$ , they are similar over  $F$ .*

However, I still have doubts about the following question: For  $A \in GL(2n, \mathbb{R})$ , if  $AJA^{-1} = J$ , can we conclude that  $A$  is inside  $GL(n, \mathbb{C})$ ?

The following is my solution:

**Lemma 1.1.** *There exists a injection  $\phi$  of  $GL(n, \mathbb{C}) \hookrightarrow GL(2n, \mathbb{R})$  such that:*

$$\phi(iB) = \phi(i)\phi(B) \tag{1.1.1}$$

for any  $B \in GL(n, \mathbb{C})$ . Also, for any  $A \in GL(2n, \mathbb{R})$  we have  $AJA^{-1} = J$  if and only if  $A \in \text{Im}(\phi)$ , where  $J \equiv \phi(i)$ .

*Proof.* The  $\phi$  is construct as follows. Let  $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , define  $H(x + iy)$  for  $x, y \in \mathbb{R}$  as

$$H(x + iy) = xI + yJ \tag{1.1.2}$$

Then:

$$\phi(A)_{ij} \equiv H(a_{ij}) \tag{1.1.3}$$

Then:

$$\phi(i) = \begin{pmatrix} J & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & J \end{pmatrix} \tag{1.1.4}$$

By direct simple calculation (remember to use the technique of block multiplication), we have:  $\phi(iB) = \phi(i)\phi(B)$ . for any  $B \in GL(n, \mathbb{C})$ . This shows also  $BJB^{-1} = J$ , since  $iB = Bi$ .

To prove the converse, we see that the following matrices forms a basis of  $2n \times 2n$  real matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

They are denoted, from left to right as  $I, J_0, K, L$ . Let any  $A \in GL(2n, \mathbb{R})$ , we can partition  $A$  into a matrix of  $2 \times 2$  matrices  $(a_{ij})$ . Each matrix can be expressed as  $a_{ij} = x_{ij}I + y_{ij}J_0 + z_{ij}K + t_{ij}L$ . Then if  $AJA^{-1} = J$ , by direct calculation we find:

$$(z_{ij}K + t_{ij}L)J_0 = J_0(z_{ij}K + t_{ij}L)$$

then also by direct calculation, it can be easily found that  $z_{ij} = t_{ij} = 0$ . Hence  $A \in \text{Im}(\phi)$ .  $\square$

### 1.1.2 Why $GL(n, \mathbb{C}) \hookrightarrow GL^+(2n, \mathbb{R})$

To understand that post [2], I also read this [3]. In it, it asks how to prove that

$$GL(n, \mathbb{C}) \hookrightarrow GL^+(2n, \mathbb{R}) \quad (1.1.5)$$

for any  $n$ . The questioner gives the intuition for this fact:

how about since as Lie groups,  $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$  and  $GL(n, \mathbb{C})$  is connected but  $GL(2n, \mathbb{R})$  has two connected components, one for positive determinant and one for negative determinant? And the identity has positive determinant, so it must lie in that component.

Someone answered that question:

The claim is: If  $V$  is an  $n$ -dimensional complex vector space with underlying  $2n$ -dimensional real vector space  $W$ , then the canonical group monomorphism  $GL(V) \rightarrow GL(W)$  lands inside  $GL^+(W) = \{f \in GL(W) : \det(f) > 0\}$ . The purpose of this abstract reformulation is that we may use operations on vector spaces in order to simplify the problem: If  $V'$  is another finite-dimensional complex vector space with underlying real vector space  $W'$ , the diagram

$$\begin{array}{ccc} GL(V) \times GL(V') & \rightarrow & GL(W) \times GL(W') \\ \downarrow & & \downarrow \\ GL(V \oplus V') & \rightarrow & GL(W \oplus W') \end{array} \quad (1.1.6)$$

commutes, and the image of  $GL^+(W) \times GL^+(W')$  is contained in  $GL^+(W \oplus W')$ . Therefore, if some element in  $GL(V \oplus V')$  lies in the image of  $GL(V) \times GL(V')$ , it suffices to consider the components. Combining this with the fact that  $GL(V)$  is

Fun fact:  
 $[K, J_0] = \sigma_z$ ,  
 $[L, J_0] = \sigma_x$ ,  
the pauli matrices!

generated by elementary matrices (after choosing a basis of  $V$ ), we may reduce the whole problem to the following three types of matrices:

- the  $1 \times 1$ -matrices  $(\lambda)$ ,
- the  $2 \times 2$ -matrices  $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$ ,
- and the  $2 \times 2$ -matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

Write  $\lambda = a + ib$  with  $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ . Then, the complex  $1 \times 1$ -matrix  $(\lambda)$  becomes the real  $2 \times 2$ -matrix  $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ , which has determinant  $a^2 + b^2 > 0$ . The complex  $2 \times 2$ -matrix  $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$

becomes the real  $4 \times 4$ -matrix  $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & -b & 1 & 0 \\ b & a & 0 & 1 \end{pmatrix}$ , which has

determinant 1. Finally, the complex  $2 \times 2$ -matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  be-

comes the real  $4 \times 4$ -matrix  $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ , which has determinant 1.

However, this proof is not complete because, to build the proof from  $\mathbb{R}^2$  to  $\mathbb{R}^{2n}$ , it requires, in his argument, that any element in  $\text{GL}(V \oplus V)$  is in the image of  $\text{GL}(V) \times \text{GL}(V')$ , which is not the case.

On the other hand, it seems that this property can be proved directly by calculation. The following will be a notes of a paper [4], which one comment mentions in the Math.SE post [3].

### 1.1.3 Determinants of Block Matrices

This paper tries to prove the theorem:

**Theorem 1.2.** *Let  $R$  be a commutative subring of  ${}^nF^n$ , where  $F$  is a field (or a commutative ring), and let  $M \in {}^mR^m$ . Then*

$$\det_F \mathbf{M} = \det_F(\det_R \mathbf{M}) \quad (1.1.7)$$

In particular, we have:

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \det_F(AD - BC) \quad (1.1.8)$$

Note that, that the ring being is commutative excludes some ambiguity. For example, when the ring  $4$  is not commutative, then the quantity:

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \quad (1.1.9)$$

is not well-defined. It can be  $AD - BC$ , or  $DA - CB$ , etc.

Before the proof of the main theorem, it establishes several facts:

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \det_F \mathbf{A} \det_F \mathbf{D} \quad (1.1.10)$$

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} = \det_F \mathbf{A} \det_F \mathbf{D} \quad (1.1.11)$$

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{pmatrix} = \det_F -\mathbf{C} \det_F \mathbf{B} \quad (1.1.12)$$

$$\det_F \mathbf{A} \det_F \mathbf{D} = \det_F \mathbf{I}_n \det_F (\mathbf{A}\mathbf{D}) \quad (1.1.13)$$

He first builds up a seemingly simplified, but is actually different version of the main theorem:

**Theorem 1.3.** *Let  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in {}^n F^n$ . Let  $\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$ .*

*If  $\mathbf{CD} = \mathbf{DC}$ , then,*

$$\det_F \mathbf{M} = \det_F (\mathbf{AD} - \mathbf{BC}) \quad (1.1.14)$$

and similar results:

$$\text{if } \mathbf{AC} = \mathbf{CA} \text{ then, } \det_F \mathbf{M} = \det_F (\mathbf{AD} - \mathbf{CB}) \quad (1.1.15)$$

$$\text{if } \mathbf{BD} = \mathbf{DB} \text{ then, } \det_F \mathbf{M} = \det_F (\mathbf{DA} - \mathbf{BC}) \quad (1.1.16)$$

$$\text{if } \mathbf{AB} = \mathbf{BA} \text{ then, } \det_F \mathbf{M} = \det_F (\mathbf{DA} - \mathbf{CB}) \quad (1.1.17)$$

These equalities can be proved easily by the following:

$$\begin{pmatrix} D & 0 \\ -C & i \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} AD - BC & B \\ CD - DC & D \end{pmatrix} = \begin{pmatrix} AD - BC & B \\ 0 & D \end{pmatrix} \text{ when } C, D \text{ commutes}$$

$$\begin{pmatrix} D & -B \\ 0 & i \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} DA - BC & DB - BD \\ C & D \end{pmatrix} = \begin{pmatrix} DA - BC & 0 \\ C & D \end{pmatrix} \text{ when } D, B \text{ commutes.}$$

The author also gives an illuminative explanation for why

$$(\det_F \mathbf{M} - \det_F (\mathbf{AD} - \mathbf{BC})) \det_F \mathbf{D} = 0$$

necessarily implies:

$$\det_F \mathbf{M} = \det_F (\mathbf{AD} - \mathbf{BC})$$

However, I am dubious about this conclusion, since I think it needs in addition that the polynomial ring  $F[x]$  has not nonzero zero divisor.

Having demonstrated the above simple case, the author continues to prove the main theorem. He proves by induction. He first uses:

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} \begin{pmatrix} d\mathbf{I} & 0 \\ -c & 1 \end{pmatrix} = \begin{pmatrix} A_0 & b \\ 0 & d \end{pmatrix} \quad (1.1.18)$$

where  $A, A_0 \in {}^{m-1}R^{m-1}, b \in {}^{m-1}R, c \in R^{m-1}, d \in R$ . Therefore, (let  $M = \begin{pmatrix} A & b \\ c & d \end{pmatrix}$ ) with similar reason mentioned before, he shows if:

$$\det_F \mathbf{A}_0 = \det_F (\det_R \mathbf{A}_0) \quad (1.1.19)$$

(which is true by induction) then:

$$\det_F \mathbf{M} = \det_F(\det_{\mathbf{R}} \mathbf{M}) \quad (1.1.20)$$

Proof completes.

He also mentions a corollary:

**Corollary 1.1.** *Let  $\mathbf{P} \in {}^n F^n$  and  $\mathbf{Q} \in {}^m F^m$ , then*

$$\det_F(\mathbf{P} \otimes \mathbf{Q}) = (\det_F \mathbf{P})^m (\det_F \mathbf{Q})^n \quad (1.1.21)$$

The proof is quite straightforward and is omitted.

### 1.1.4 Why $GL(n, \mathbb{C}) \hookrightarrow GL^+(2n, \mathbb{R})$ (continued)

With above theorem, the proof of equation 1.1.5 is straight forward. Since for  $(a_{ij}) = A \in GL(n, \mathbb{C})$ , it injects into  $GL(2n, \mathbb{R})$  as matrices of the form:

$$\begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & H a_{ij} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

where:

$$H(z \equiv x + iy) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

Since  $H(a_{ij})$  commutes with each other (proved by calculation), we can use the theorem in previous part to show that:

$$\det_{\mathbb{R}}(A) = \det_{\mathbb{R}}(\det_{\mathbb{C}}(A)) = |\det_{\mathbb{C}}(A)|^2 \geq 0$$

Notice that I have been sloopy in language, but the meaning should be clear.

## 1.2 Math.SE answer in $M_2$ is two copies of $S^2$

Following is an answer [7] in Math.SE about this question:

---

As you noted,  $M$  is \*not\* diffeomorphic to  $S^2 \amalg S^2$  for dimension reasons.

On the other hand, what is true is  $M$  is homotopy equivalent to  $S^2 \amalg S^2$ .

(The following argument is partly adapted from a [paper][1] of Montgomery)

To see this, it's enough to show that  $Gl^+(4, \mathbb{R})/Gl(2, \mathbb{C})$  is homotopy equivalent to  $S^2$ , where  $Gl^+$  denotes those matrices of positive determinant.

Now, consider the subgroups  $U(2) \subseteq Gl(2, \mathbb{C})$  and  $SO(4) \subseteq Gl^+(4, \mathbb{R})$ .

It's relatively well known that  $Gl(2, \mathbb{C})$  is diffeomorphic to  $U(2) \times \mathbb{R}^4$  and that  $Gl^+(4, \mathbb{R})$  is diffeomorphic to  $SO(4) \times \mathbb{R}^{10}$ .



Further, in the usual inclusion  $Gl(2, \mathbb{C}) \rightarrow Gl^+(4, \mathbb{R})$ ,  $U(2)$  becomes a subgroup of  $SO(4)$ .

Now, the chain of subgroups  $U(2) \subseteq SO(4) \subseteq Gl^+(4, \mathbb{R})$  gives rise to a homogeneous fibration

$$SO(4)/U(2) \rightarrow Gl^+(4, \mathbb{R})/U(2) \rightarrow Gl^+(4, \mathbb{R})/SO(4).$$

In light of the above diffeomorphisms,  $Gl^+(4, \mathbb{R})/SO(4)$  is diffeomorphic to  $\mathbb{R}^{10}$ . Since Euclidean spaces are contractible, it follows that the fibration is trivial, so  $Gl^+(4, \mathbb{R})/U(2)$  is diffeomorphic to  $SO(4)/U(2) \times \mathbb{R}^{10}$ . In particular,  $SO(4)/U(2)$  is homotopy equivalent to  $Gl^+(4, \mathbb{R})/U(2)$ .

Now, consider the chain of subgroups  $U(2) \subseteq Gl(2, \mathbb{C}) \subseteq Gl^+(4, \mathbb{R})$ . This gives rise to a homogeneous fibration

$$Gl(2, \mathbb{C})/U(2) \rightarrow Gl^+(4, \mathbb{R})/U(2) \rightarrow Gl^+(4, \mathbb{R})/Gl(2, \mathbb{C}).$$

In this case, the fiber is diffeomorphic to  $\mathbb{R}^4$ , which immediately implies that  $Gl^+(4, \mathbb{R})/U(2)$  is homotopy equivalent to  $Gl^+(4, \mathbb{R})/Gl(2, \mathbb{C})$ .

---

(Paused reading here)

Putting the last two paragraphs together, we now know that  $SO(4)/U(2)$  is homotopy equivalent to  $Gl^+(4, \mathbb{R})/Gl(2, \mathbb{C})$ .

To finish off the argument, we need to show that  $SO(4)/U(2)$  is diffeomorphic to  $S^2$ . To see this, first note that  $U(2)$  intersects the center  $Z(SO(4)) = \{\pm I\}$  of  $SO(4)$ . It follows that

$$SO(4)/U(2) \cong [SO(4)/Z(SO(4))]/[U(2)/(Z(SO(4)) \cap U(2))].$$

But  $SO(4)/Z(SO(4)) \cong SO(3) \times SO(3)$  and  $U(2)/(Z(SO(4)) \cap U(2)) \cong SO(3) \times S^1$ . So,  $SO(4)/U(2) \cong (SO(3) \times SO(3))/(SO(3) \times S^1) \cong SO(3)/S^1$ .

But the standard action of  $SO(3)$  on  $S^2$  is transitive with stabilizer  $S^1$ , so  $SO(3)/S^1 \cong S^2$ .

[1]: <http://www.ams.org/journals/proc/1950-001-04/S0002-9939-1950-0037311-6/S0002-9939-1950-0037311-6.pdf>

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I honestly know almost nothing about the concepts this response mentioned. Therefore, I try to dismantle the response into several parts:

Facts he mentioned that I am not familiar

1.  $Gl(2, \mathbb{C})$  is diffeomorphic to  $U(2) \times \mathbb{R}^4$ .
2.  $Gl^+(4, \mathbb{R})$  is diffeomorphic to  $SO(4) \times \mathbb{R}^{10}$
3. In the usual inclusion  $Gl(2, \mathbb{C}) \rightarrow Gl^+(4, \mathbb{R})$ ,  $U(2)$  becomes a subgroup of  $SO(4)$ .

Concepts to be learnt:

1. fibration of above Lie groups

2. Highlight area 1: can fibration kill a subgroup?
3. Highlight area 2: contractible and fibration?
4. And the following sentence.
5. then the next sentence: diffeomorphism and homotopy?
6. How does a "chain of subgroups" gives rise to a fibration.

The following notes are aim at understanding the above sentences.

### 1.3 Lifting Property

#### Lift of morphisms

**Definition 1.1** (Lift of morphisms). *The **lift** of a morphism  $f : Y \rightarrow B$  along an epimorphism<sup>1</sup>. (or more general map)  $p : X \rightarrow B$  is a morphism  $\tilde{f} : Y \rightarrow X$  such that  $f = p \circ \tilde{f}$ .*

$$\begin{array}{ccc} X & \xrightarrow{p} & B \\ \tilde{f} \uparrow & \nearrow f & \\ Y & & \end{array}$$

**Definition 1.2** (Lift property). *We say that  $f$  has a **left lifting property** w.r.t  $g$ , or equivalently that  $g$  has a **right lifting property** w.r.t  $f$ , if and only if for every commutative diagram below:*

$$\begin{array}{ccc} a & \xrightarrow{u} & c \\ \downarrow f & & \downarrow g \\ b & \xrightarrow{v} & d \end{array}$$

*there is an arrow  $\gamma$ , s.t. both triangles in the following diagram commutes.*

$$\begin{array}{ccc} a & \xrightarrow{u} & c \\ \downarrow f & \nearrow \gamma & \downarrow g \\ b & \xrightarrow{v} & d \end{array}$$

*Such an arrow  $\gamma$  is called a **lift** or a **solution** to the lifting problem  $(u, v)$ . If such  $\gamma$  is unique, i.e. we have:*

$$\begin{array}{ccc} a & \xrightarrow{u} & c \\ \downarrow f & \nearrow \gamma & \downarrow g \\ b & \xrightarrow{v} & d \end{array}$$

*Then we say  $f$  is **orthogonal** to  $g$ , denoted  $f \perp g$ .*

ref [8].

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<sup>1</sup>Epimorphism is the category analogy of surjective functions in set theory

## 2 Anchor

### References

- [1] D Huybrechts's Introduction to Complex Geometry.
- [2] set of almost complex structures on  $\mathbb{R}^4$  as two disjoint spheres.
- [3] Does  $GL(n, \mathbb{C})$  inject into  $GL^+(2n, \mathbb{R})$  for all  $n$ ?
- [4] John R. Silvester, Determinants of Block Matrices. Available in WebArchive link: <https://web.archive.org/web/20140505161153/http://www.mth.kcl.ac.uk/~jrs/gazette/blocks.pdf>
- [5] Making sense of  $\frac{GL(2n, \mathbb{R})}{GL(n, \mathbb{C})}$
- [6] Similar matrices and field extensions
- [7] set of almost complex structures on  $\mathbb{R}^4$  as two disjoint spheres
- [8] nLab Lift

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