Notes of Basic Topolgy

Taper

November 19, 2016

Abstract

A note of Basic Topology, based on *Basic Topology* by M.A. Armstrong.

Contents

There are several parts that I will skipped for convenience. Those include chapter 1 - Introduction, chapter 2 - Continuity, chapter 3 - Compactness and Connectedness, and chapter 4 - Identification Spaces. Below is some especially confusing part that I would like to note:

1 Special Notes

sec:Special-Notes

About map In book [1], a map is defined as a continuous function (page 32), which is confusing. In this note, I will not use this convention and will always states continuity clearly.

Basic facts about maps Assuming domain $f = X$, codomain $f = Y$.

$$f(U \cup V) = f(U) \cup f(V) \tag{1.0.1}$$
$$f(U \cap V) \subseteq f(U) \cap f(V) \tag{1.0.2}$$
$$f(U^c) \supseteq f(U)^c, \text{ i.e. } f(U)^c \subseteq f(U^c) \tag{1.0.3}$$
$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V) \tag{1.0.4}$$
$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V) \tag{1.0.5}$$
$$f^{-1}(U^c) = [f^{-1}(U)]^c \tag{1.0.6}$$

Smallest the Largest Topolgy The set of all possible topolgies on X is partially ordered by inclusion. For a certain characteristics $\mathcal{C}$, it is possible to have the smallest or the largest one.

The **smallest topolgy** $\mathcal{T}_{\min}$ is the one such that, for any $\mathcal{T}'$ satisfying $\mathcal{C}$, $\mathcal{T}_{\min} \subseteq \mathcal{T}'$. The **largest topolgy** $\mathcal{T}_{\max}$ is the one such that, for any $\mathcal{T}'$ satisfying $\mathcal{C}$, $\mathcal{T}' \subseteq \mathcal{T}_{\max}$. Synonyms of these two words are:

- Larger: stronger, finer.
- Smaller: weaker, coarser.

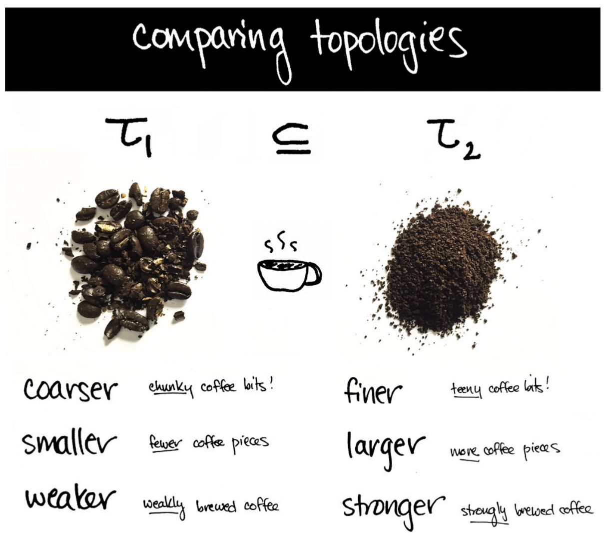

Figure 1: Comparing topologies and coffee (Credit: math3ma)

For example, assuming we have

$$f : X \to Y \tag{1.0.7}$$

where f is any function.

If X has topolgy $\mathcal{T}_X$, we ask then what kind of topolgy on Y will make f a continuous function. First, all $f^{-1}(V)$, with $V \in \mathcal{T}_Y$ should be open in X. So, the easiest choice is to make $\mathcal{T}_{Y,\min} = \{\varnothing, Y\}$, this is the smallest topolgy. Also, any set $V \in Y$ such that $f^{-1}(V) \notin \mathcal{T}_X$ should not be in $\mathcal{T}_Y$. Then the largest topolgy is $\mathcal{T}_{Y,\max} = \{V \subset Y | f^{-1}(V) \in \mathcal{T}_X\}$.

If Y has topolgy $\mathcal{T}_Y$, we also ask what kind of topolgy on X will make f a continuous function. First, all $V \in \mathcal{T}_Y$, their preimage $f^{-1}(V)$ must be in $\mathcal{T}_X$. So the smallest topolgy is $\mathcal{T}_{X,\min} = \{f^{-1}(V) | V \in \mathcal{T}_Y\}$. Than what about the largest topolgy? We consider, what kind of sets cannot be inside $\mathcal{T}_X$. First, can $(f^{-1}(V))^c = f^{-1}(V^c)$ be in $\mathcal{T}_X$? Yes. Since unless the space is connected, there can be sets being both open and closed (other than X and $\varnothing$). Any other restrictions? No that I can think of. So, the largest topolgy $\mathcal{T}_{X,\max} = 2^X$, the set of all subsets of X. (The notation 2^X is taken from the page 4 of book [2].

A summary:

Table 1: Largest and Smallest Topolgies

$X \xrightarrow{f} Y$	Smallest	Largest
Given $\mathcal{T}_X$	$\mathcal{T}_{Y,\min} = \{\varnothing, Y\}$	$\mathcal{T}_{Y,\max} = \{V \subset Y \| f^{-1}(V) \in \mathcal{T}_X\}$
Given $\mathcal{T}_Y$	$\mathcal{T}_{X,\min} = \{f^{-1}(V) \| V \in \mathcal{T}_Y\}$	$\mathcal{T}_{X,\max} = 2^X$
No constraint	$\{\varnothing, X\}$	2^X

Facts about subspace/induced topolgy Let Y be a subspace of a topological space X wit induced topolgy.

Fact 1.1. A set $H \subseteq Y$ is open in Y if and only if $H = F \cap Y$ for some open set F in X.

Fact 1.2. A set $H \subseteq Y$ is closed in Y if and only if $H = F \cap Y$ for some closed set F in X.

Fact 1.3. A set H is open/closed in $X \Rightarrow H$ is open/closed in Y. But the converse may not be true. The converse statement depends on whether Y is open or closed in X.

sec:Brief-Note-Chapter-4

2 A Brief Note of Chapter 4 - Identification Spaces

ec:Identification topology

2.1 Identification topology

Definition 2.1 (Identification Topology)**.** Let X be a topological space and let $\mathscr{P}$ be a family of disjoint nonempty subsets of X such that $\cup\mathscr{P} = X$. Such a family is usually called a partition of X. Let Y be a new space whose points are the members of $\mathscr{P}$. Let $\pi : X \to Y$ sends each point of X to the subset of $\mathscr{P}$. Define a topolgy $\mathcal{T}_Y$ on Y to be the largest topolgy such that the π is continuous. This $\mathcal{T}_Y$ is called the idetification topolgy. And Y is called the **identification space** .

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d, equal] \\ & \mathscr{P} \end{tikzcd}$$

Theorem 2.1. *Let Y be an idetification space defined as above and let Z be an arbitrary topological space. A function $f : Y \to Z$ is continuous if and ony if the composition $f \circ \pi : X \to Z$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[rr, bend left, "f\circ\pi"] \arrow[dr] & Y \arrow[r, "f"] \arrow[d, equal] & Z \\ & \mathscr{P} & \end{tikzcd}$$

Definition 2.2 (Identification Map)**.** Let $f : X \to Y$ be an onto continuous map and suppose that the topolgy on Y is the largest for which f is continuous. Then we call f an identification map.

The naming "identification map" is because:

Theorem 2.2. *Any function $f : X \to Y$ gives rise to a partition of X whose members are the subsets $\{f^{-1}(y)\}$, where $y \in Y$. Let Y_* denote the identification space associated with this partition, and $\pi : X \to Y_*$ the usual continuous map.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] & Y \\ \{f^{-1}(y)\} \arrow[r, equal] & Y_* \end{tikzcd}$$

If f is an identification map, then:

1. *the spaces Y and Y_* are homeomorphic;*
2. *a function $g : Y \to Z$ is continuous if and only if the composition $g \circ f : X \to Z$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] \arrow[rr, bend left, "g \circ f"] & Y \arrow[r, "g"] \arrow[d, equal] & Z \\ \{f^{-1}(y)\} \arrow[r, equal] & Y_* & \end{tikzcd}$$

Theorem 2.3. *Let $f : X \to Y$ be an onto continuous map. If f maps open sets of X to open sets of Y, or closed sets to closed sets, then f is an identification map, i.e. $\mathcal{T}_y$ is the largest topolgy such that f is continuous.*

`coro:idmap-coro`

Corollary 2.1. *Let $f : X \to Y$ be an onto continuous map. If X is compact and Y is Hausdorff, then f is an identification map.*

Definition 2.3 (Torus)**.** Torus is the unit square $[0, 1] \times [0, 1]$, with 1. opposite edge identified; 2. four edge points identified.

Remark 2.1. The identification map and corollary 2.1 can be used to show that torus is homeomorphic to two copies of circles:$S^1 \times S^1$. This is mentioned in page 68 of [1].

Definition 2.4 (Cone CX)**.** The cone of any space CX is formed from $X \times I$, where I is the unit interval $[0, 1]$, with certain identification. The identification shrinks all points in one surface into one point. This is discussed in page 68 of [1].

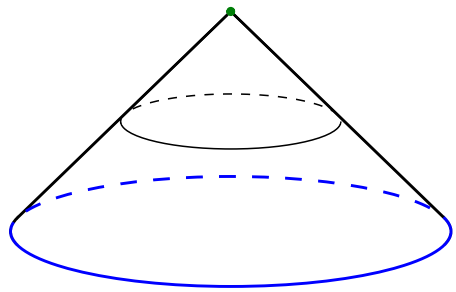

Figure 2: Cone of a Circle (Wikipedia)

Remark 2.2. There is another definition of cone CX when X in imbeded into $\mathbb{E}^n$, may be found on page 68 of [1]. Cone constructed in this way is called a geometric cone. It is made up of all straight line segments that join $v = (0, 0, \cdots, 1) \in \mathbb{E}^{n+1}$ to some point of X.

Lemma 2.1. *The geometric cone on X is homeomorphic to CX.*

Definition 2.5 (Quotient Space)**.** Let X be a topological space, A be its subspace. Then X/A menas the X with subspace A identified to a point.

1. the set A.
2. the individual points of $X \setminus A$.

Remark 2.3. In this notation, CX becomes $(X \times I)/(X \times \{1\})$.

Fact 2.1.

$$B^n/S^{n-1} \cong S^n \tag{2.1.1}$$

where $\cong$ menas homeomorphic. This is proved on page 69. Intuitively, this is like wrap a lower dimension ball surround the higher dimension ball.

Definition 2.6 ($f \cup g$)**.** Let X, Y $f \cup g$ subsets of a topological space and give each of X, Y, and$X \cup Y$ the induced topolgy. If $f : X \to Z$ and $g : Y \to Z$ are functions which agree on the intersection of X and Y, we can define

$$\begin{aligned} f \cup g : X \cup Y \to Z \\ (f \cup g)(x) = f(x),\, x \in X \\ (f \cup g)(x) = g(x),\, x \in Y \end{aligned} \tag{2.1.2}$$

We say that $f \cup g$ are formed by 'glueing together' the functions f and g.

Lemma 2.2 (Glueing lemma (closed))**.** *If X and Y are closed in $X \cup Y$, and if both f and g are continuous, then $f \cup g$ are continuous.*

Similarly,

Lemma 2.3 (Glueing lemma (open))**.** *If X and Y are open in $X \cup Y$, and if both f and g are continuous, then $f \cup g$ are continuous.*

These two lemmas are seen as a special case of the following theorem, explained in page 70.

Define $X+Y$ to be the disjoint union of spaces X, Y. Define $j: X+Y \to X \cup Y$ which restrict to either X or Y is just the inclusion in $X \cup Y$.

Theorem 2.4. *If j is an identification map, and if both $f : X \to Z$ and $g : X \to Z$ are continuous, then $f \cup g : X \cup Y \to Z$ is continuous.*

$$\begin{tikzcd} X+Y \arrow[r, "j"] & X \cup Y \arrow[r, "f \cup g"] & Z \\ X \arrow[urr, "f"] & Y \arrow[ur, bend right, "g"'] & \end{tikzcd}$$

This can be generalized as follows. Let $X_\alpha, \alpha \in A$ be a family of subsets of a topological space and give each X_α and the union $\cup X_\alpha$, the induced topolgy. Let Z be a space and suppose we are given maps $f_\alpha : X_\alpha \to Z$, one for each α in A, such that if $\alpha, \beta \in A$,

$$f_\alpha\Big|_{X_\alpha \cap X_\beta} = f_\beta\Big|_{X_\alpha \cap X_\beta}$$

Define function $F : \cup X_\alpha \to Z$ by glueing together f_α. Let $\oplus X_\alpha$ be the disjoint unin of spaces X_α. Let $j : \oplus X_\alpha \to \cup X_\alpha$ be similarly defined.

Theorem 2.5. *If j is an identification map, and if each f_α is continuous, then F is continuous.*

Note: When j is the identification map, then $\cup X_\alpha$ has the identification topolgy instead of the subspace topology. The two will be quite different, as discussed on page 70 to 71 of [1].

Definition 2.7 (Projective space P^n)**.** A discussion of real P^n may be found on page 71.

Attaching maps and $X \cup_f Y$ Let:

$$Y \supseteq A \xrightarrow{f} X \tag{2.1.3}$$

where X, Y are topological spaces, f is continuous. We identify the disjoint union $X+Y$ using f, partitioning them into:

1. pairs of points $\{a, f(a)\}$ where $a \in A$;
2. individual points of $Y \setminus A$;
3. individual points of $X \setminus \mathrm{Im}(f)$.

The result identification space is denoted $X \cup_f Y$, and f is called the attaching map. This process can also be viewed as:

$$X \cup_f Y = (X \amalg Y)/\{f(A) \sim A\} \tag{2.1.4}$$

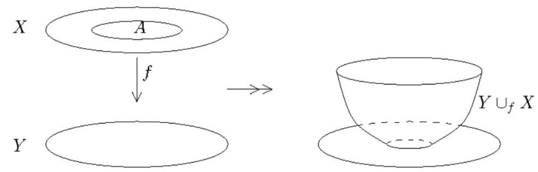

Figure 3: Attaching Space (credit: nLab

Example 2.1. P^2 can be seen as attaching a closed disc D to the boundary of M, a Mobius strip, as discussed in page 72 of [1]. Geometrically, this simply shrinks the boundary of M into a point. And an ant travelling around this point can point out the direction just as in P^2.

Remark 2.4. It is remarked that properties such as compactness, connectedness, and path-connectedness is inherited in identification. However, Hausdorff-ness is not. An counter example can be found in page 72 of [1].

sec:Topological-Groups

2.2 Topological Groups

In simple words, **topological groups** are objects that has both a topolgy on it and a group structure in it. And the two structures must be compatible. Specifically, the multiplication map $a \cdot b$ and the inverse map $a \to a^{-1}$ are continuous. Homomorphisms between are both group-homomorphisms and topological-homomorphisms (continuous maps). Isomorphisms are both group-isomorphisms and topolgy-isomorphisms (homeomorphisms). A sub-(topological group) is both a subgroup and has subspace topolgy. For convenience of language, use $\mathcal{TPG}$ denotes the category of topological groups. [1]

Example 2.2. The $\mathbb{R}$ is a topological group. The $\mathbb{Z}$ with discrete topology form the sub-(topological group) of $\mathbb{R}$. The quotient $\mathbb{R}/\mathbb{Z}$ forms a topological group. The map $f : \mathbb{R} \to S^1$ induces a homeomorphism $\mathbb{R}/\mathbb{Z} \cong S^1$, which is also a group isomorphisms, i.e. it is a $\mathcal{TPG}$-isomorphism.

Example 2.3. Similarly, R^n.

Example 2.4. The circle is also one. The group structure is combination of degrees.

Example 2.5. Any group with discrete topology.

Example 2.6. The torus considered as the product of two circles. (Take the producttopology and the product group structure.

Example 2.7. Three sphere S^3 considered as the unit sphere in the space of quaterions $\mathbb{H}$.

Remember this? :

$$\begin{tikzcd} & i \arrow[dr] & \\ k \arrow[ur] & \# & j \arrow[ll] \end{tikzcd}$$

[1] This notation is nowhere popular or accepted. I use it to only to save space and time.

The unit sphere are unit quaterions, see more Versor.

Example 2.8. The **orthogonal group** $\mathrm{O}(n)$, of $n \times n$ orthogonal real matrices. It is easy to check that $\mathrm{O}(n-1)$ is a sub-$\mathcal{TPG}$ of $\mathrm{O}(n)$.

Definition 2.8 (Left translation L_x)**.** For $x \in G$, the function

$$L_x : G \to G \tag{2.2.1}$$

$$g \mapsto xg \tag{2.2.2}$$

is called a left translation by x. Similarly we have **right translation** R_x .

Fact 2.2. L_x and R_x are homeomorphisms (But not group-isomorphisms).

Remark 2.5. This shows that a topological group has a certain homogeneity as a topological space. For if $x, y \in G$, then $L_{yx^{-1}}$ maps x to y and is a homeomorphism. Therefore *G exhibits the same topological structure locally near each point.*

Theorem 2.6. *Let G is a topological group, let K be a connected component of G which contains the identity element. Then K is a closed normal subgroup of G.*

Fact 2.3. If $G = \mathrm{O}(n)$, then $K = \mathrm{SO}(n)$.

Theorem 2.7. *In a connected topological group, any neighbourhood of the identity element is a set generates the whole group.*

The two theorems above is summarised as

Table 2: caption

topology		group/topology
e + connected	$\Rightarrow$	closed & normal subgroup
e + neighbourhood	$\Rightarrow$	generator

A bit more examples about matrices:

Example 2.9. $\mathbb{M}(n)$ the $n \times n$ matrices, is not a topological group. But its subspace GL(n) , specifically, $\mathrm{GL}(\mathrm{n}, \mathbb{R})$ or $\mathrm{GL}(\mathrm{n}, \mathbb{C})$, is a topological group. This is demonstrated in page 76, theorem 4.12.

Fact 2.4. GL(n) is not compact. It has two idsjoint nonempty open sets: those with positive and those with negative determinants.

Theorem 2.8. $\mathrm{O}(n)$ *and* $\mathrm{SO}(n)$ *are closed and compact.* $\mathrm{SO}(n)$ *is a sub-$\mathcal{TPG}$ of* $\mathrm{O}(n)$.

Fact 2.5. $\mathrm{SO}(2) \cong S^1$ and $\mathrm{SO}(3) \cong P^3$. Here $\cong$ means isomorphisms of topological groups.

Remark 2.6. These two facts established on page 77. The first one can be eaily guess. Since a rotation is obviously determined by a rotation degree on S^1. Mathematically we have

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cong e^{i\theta} \tag{2.2.3}$$

The second one is proved mathematical in book [1]. But it has a physical argument. Remember we have the the homogeneous coordinates for P^3, such as $[1, \theta_x, \theta_y, \theta_z]$. As indicated in my labels, the three free coordinates θ_i can be regarded as rotation in 3-dimensional space. This rotation preserves the orientation, so it is in SO, not in O.

2.3 Orbit Space

sec:Orbit-Space

Definition 2.9 (Group Action on Topology Space)**.** A topological group G is said to act as a group of homeomorphisms on a space X if each group element (let $g, h \in G$) induces a homeomorphism of the space in such a way that:

1. $(hg)(x) = h(g(x))$, $\forall x \in X$;
2. $e(x) = x$, $\forall x \in X$, where $e = gg^{-1}$;
3. the function $G \times X \to X$,$(g, x) \mapsto g(x)$ is continusous.

The subset of X, consisting of $g(x)$ for all $g \in G$, is called an **orbit** of $x \in X$, written $O(x)$. Thought, it more convenient to write it just as Gx , as in textbooks of abstract algebra.

Fact 2.6. A common fact in abstract algebra here is: each orbit Gx is disjoint. If two $Gx \cap Gy \neq \varnothing$, then $Gx = Gy$.

By above fact, orbits partitions X, hence we can form the Identification space, with every elements in X identified with their brothers in the same orbit. The result is **orbit space** X/G .

ex:R-over-Z-T

Example 2.10. $\mathbb{Z}$ acts on $\mathbb{R}$ by addition $x \mapsto x + n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$. It partitioned $\mathbb{R}$ into intervals, for each $x \in X$, $x \sim x + n, \forall n \in \mathbb{Z}$. The orbit space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1.

An action G on X is called **transitive** , if and only if the orbit space X/G is the trivial point $\{1\}$. Or equivalently, the only orbit is the whole space, i.e. $Gx = G$, $\forall x \in G$.

Example 2.11. The orthogonal action $O(n)$ on S^{n-1} is transitive. Physically, this is saying that $\forall x \in S^{n-1}$, it can be rotated into $\forall y \in S^{n-1}$. A mathematical proof is on page 79 of [1]

A lot of examples from book [1]

Example 2.12. Extending example 2.10:

$$\mathbb{E}^2/(\mathbb{Z} \times \mathbb{Z}) = T \text{ (torus)} \tag{2.3.1}$$

Here = means homeomorphism.

Example 2.13.

$$S^n/\mathbb{Z}_2 = P^n \tag{2.3.2}$$

Here = means homeomorphism.

Example 2.14 (Three ways of $\mathbb{Z}_2$ acting on T)**.** The detailed procedure is to be found on page 91 of [1]. Here's a picture to visualize the action:

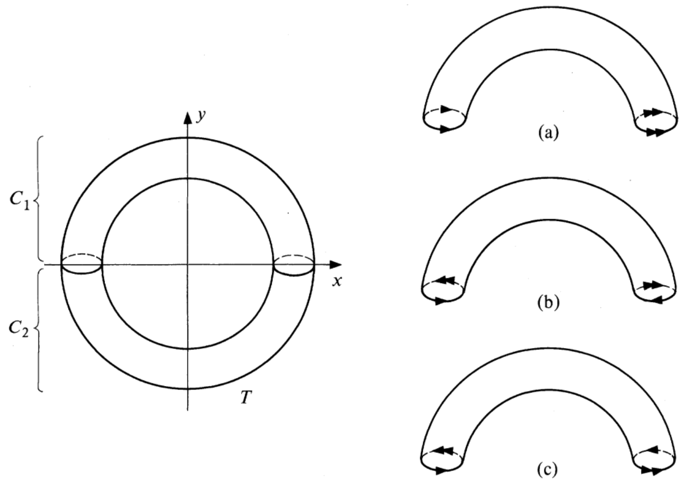

Figure 4:

The results are (a) a sphere; (b) a torus; (c) a Klein bottle.

Example 2.15. If G is a topological group, and H is $\mathcal{TPG}$-subgroup. Then, the left cosets of right cosets can be canonically seen as orbits. See more on page 81, example 4.

Example 2.16.

$$\mathrm{O}(n)/\mathrm{O}(n-1) = S^{n-1} \tag{2.3.3}$$

$$\mathrm{SO}(n)/\mathrm{SO}(n-1) = S^{n-1} \tag{2.3.4}$$

Here = means homeomorphism. The first is established mathematically in page 82 of [1]. The second is mentioned there, indicating a similar proof.

Here I give an argument. Consider a unit vector y in S^{n-1}, if we want to rotate another unit vector e_1 to y, since the action is transitive, we can easily find a $A \in \mathrm{O}(n)$ to do this. But in addition, we can also find that $A \cdot B$, where $B \in \mathrm{O}(n-1)$ rotates the space around e_1 (thus leaving e_1 un-affected) also do our job. So there is an $\mathrm{O}(n-1)$ redundancy in $\mathrm{O}(n) \to S^{n-1}$. Similar for the second relation.

Theorem 2.9. *Let G acts on X and suppose that both G and X/G are connected, then X is connected.*

Fact 2.7. Using the theorem above, one can deduce that: SO(1) is connected, S^{n-1} is connected, so $\mathrm{SO}(n)$ is connected.

Next, the book [1] (page 82 to 85) introduces several three spaces (**Lens space** , **irrational flow** on T torus, **fundamental region** or in my word *space filling shapes*) and two group **Euclidean group** (page 84) and **plane-crystallographic group** (page 85). To save time, I leave here only some pictures:

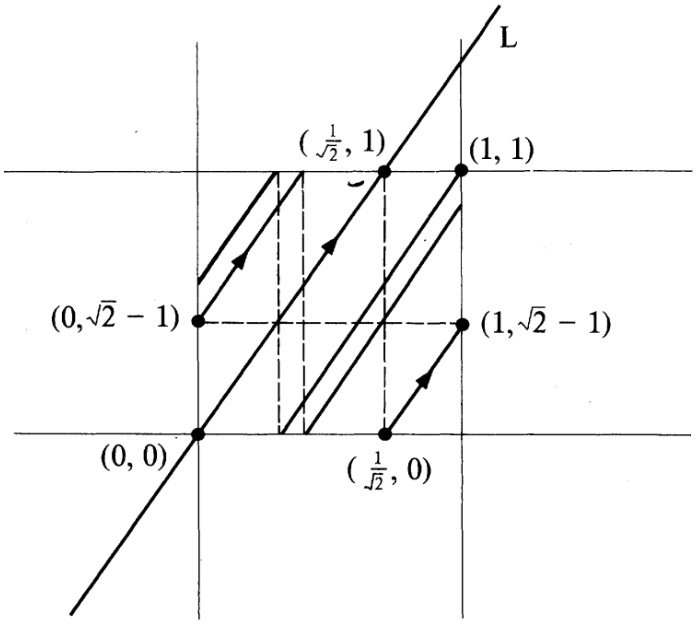

Figure 5: Irrational Flow on T

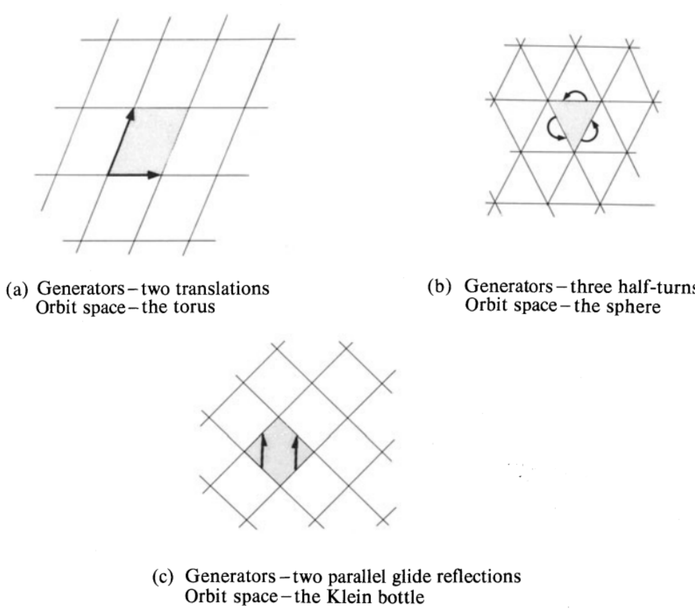

Figure 6: Space-filling Shapes

sec:Anchor

3 Anchor

References

book [1] M.A. Armstrong. Basic Topology. 2ed.

Singer.Thorpe [2] I.M. Singer, J.A. Thorpe. Lecture Notes on Elementary Topology and Geometry. UTM.

4 License