Notes of Basic Topolgy

Taper

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Abstract

A note of Basic Topology, based on *Basic Topology* by M.A. Armstrong.

Contents

There are several parts that I will skipped for convenience. Those include chapter 1 - Introduction, chapter 2 - Continuity, chapter 3 - Compactness and Connectedness, and chapter 4 - Identification Spaces. Below is some especially confusing part that I would like to note:

1 Special Notes

sec:Special-Notes

About map In book [1], a map is defined as a continuous function (page 32), which is confusing. In this note, I will not use this convention and will always states continuity clearly.

Basic facts about maps Assuming domain $f = X$, codomain $f = Y$.

$$f(U \cup V) = f(U) \cup f(V) \tag{1.0.1}$$

$$f(U \cap V) \subseteq f(U) \cap f(V) \tag{1.0.2}$$

$$f(U^c) \supseteq f(U)^c, \text{ i.e. } f(U)^c \subseteq f(U^c) \tag{1.0.3}$$

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V) \tag{1.0.4}$$

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V) \tag{1.0.5}$$

$$f^{-1}(U^c) = [f^{-1}(U)]^c \tag{1.0.6}$$

Smallest the Largest Topolgy The set of all possible topolgies on X is partially ordered by inclusion. For a certain characteristics $\mathcal{C}$, it is possible to have the smallest or the largest one.

The **smallest topolgy** $\mathcal{T}_{\min}$ is the one such that, for any $\mathcal{T}'$ satisfying $\mathcal{C}$, $\mathcal{T}_{\min} \subseteq \mathcal{T}'$. The **largest topolgy** $\mathcal{T}_{\max}$ is the one such that, for any $\mathcal{T}'$ satisfying $\mathcal{C}$, $\mathcal{T}' \subseteq \mathcal{T}_{\max}$. Synonyms of these two words are:

- Larger: stronger, finer.
- Smaller: weaker, coarser.

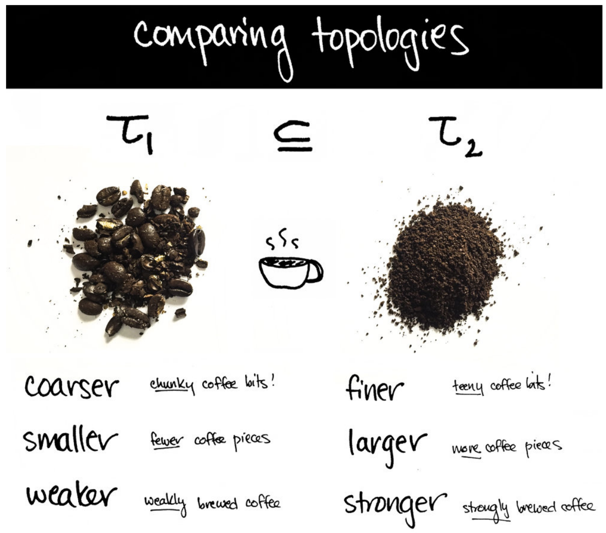


Figure 1: Comparing topologies and coffee (Credit: math3ma)

For example, assuming we have

$$f : X \to Y \tag{1.0.7}$$

where f is any function.

If X has topolgy $\mathcal{T}_X$, we ask then what kind of topolgy on Y will make f a continuous function. First, all $f^{-1}(V)$, with $V \in \mathcal{T}_Y$ should be open in X. So, the easiest choice is to make $\mathcal{T}_{Y,\min} = \{\varnothing, Y\}$, this is the smallest topolgy. Also, any set $V \in Y$ such that $f^{-1}(V) \notin \mathcal{T}_X$ should not be in $\mathcal{T}_Y$. Then the largest topolgy is $\mathcal{T}_{Y,\max} = \{V \subset Y | f^{-1}(V) \in \mathcal{T}_X\}$.

If Y has topolgy $\mathcal{T}_Y$, we also ask what kind of topolgy on X will make f a continuous function. First, all $V \in \mathcal{T}_Y$, their preimage $f^{-1}(V)$ must be in $\mathcal{T}_X$. So the smallest topolgy is $\mathcal{T}_{X,\min} = \{f^{-1}(V) | V \in \mathcal{T}_Y\}$. Than what about the largest topolgy? We consider, what kind of sets cannot be inside $\mathcal{T}_X$. First, can $(f^{-1}(V))^c = f^{-1}(V^c)$ be in $\mathcal{T}_X$? Yes. Since unless the space is connected, there can be sets being both open and closed (other than X and $\varnothing$). Any other restrictions? No that I can think of. So, the largest topolgy $\mathcal{T}_{X,\max} = 2^X$, the set of all subsets of X. (The notation 2^X is taken from the page 4 of book [2].

A summary:

Table 1: Largest and Smallest Topolgies

$X \xrightarrow{f} Y$	Smallest	Largest
Given $\mathcal{T}_X$	$\mathcal{T}_{Y,\min} = \{\varnothing, Y\}$	$\mathcal{T}_{Y,\max} = \{V \subset Y \vert f^{-1}(V) \in \mathcal{T}_X\}$
Given $\mathcal{T}_Y$	$\mathcal{T}_{X,\min} = \{f^{-1}(V) \vert V \in \mathcal{T}_Y\}$	$\mathcal{T}_{X,\max} = 2^X$
No constraint	$\{\varnothing, X\}$	2^X

Facts about subspace/induced topolgy Let Y be a subspace of a topological space X wit induced topolgy.

Fact 1.1. A set $H \subseteq Y$ is open in Y if and only if $H = F \cap Y$ for some open set F in X.

Fact 1.2. A set $H \subseteq Y$ is closed in Y if and only if $H = F \cap Y$ for some closed set F in X.

Fact 1.3. A set H is open/closed in $X \Rightarrow H$ is open/closed in Y. But the converse may not be true. The converse statement depends on whether Y is open or closed in X.

1.1 Lebesgue lamme

`sec:Lebesgue lamme`

This is a very important lemma, which is why I gave it a seperate section. It is labeled (3.11) in book [1].

`lemma:lebesgue-lemma`

Theorem 1.1 (Lebesgue Lemma)**.** *Let X be a compact metric space and let $\mathscr{F}$ be an open cover of X. Then there exists a real number $\delta > 0$ (called the* ***Lebesgue number*** *of $\mathscr{F}$) such that any subset of X of diameter less than δ is contained in some member of $\mathscr{F}$.*

2 A Brief Note of Chapter 4 - Identification Spaces

`sec:Brief-Note-Chapter-4`

2.1 Identification topology

`ec:Identification topology`

Definition 2.1 (Identification Topology)**.** Let X be a topological space and let $\mathscr{P}$ be a family of disjoint nonempty subsets of X such that $\cup \mathscr{P} = X$. Such a family is usually called a partition of X. Let Y be a new space whose points are the members of $\mathscr{P}$. Let $\pi : X \to Y$ sends each point of X to the subset of $\mathscr{P}$. Define a topolgy $\mathcal{T}_Y$ on Y to be the largest topolgy such that the π is continuous. This $\mathcal{T}_Y$ is called the idetification topolgy. And Y is called the **identification space** .

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d, equal] \\ & \mathscr{P} \end{tikzcd}$$

Theorem 2.1. *Let Y be an idetification space defined as above and let Z be an arbitrary topological space. A function $f : Y \to Z$ is continuous if and ony if the composition $f \circ \pi : X \to Z$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[rr, bend left, "f\circ\pi"] \arrow[dr] & Y \arrow[r, "f"] \arrow[d, equal] & Z \\ & \mathscr{P} & \end{tikzcd}$$

Definition 2.2 (Identification Map)**.** Let $f : X \to Y$ be an onto continuous map and suppose that the topolgy on Y is the largest for which f is continuous. Then we call f an identification map.

The naming ”identification map” is because:

Theorem 2.2. *Any function $f : X \to Y$ gives rise to a partition of X whose members are the subsets $\{f^{-1}(y)\}$, where $y \in Y$. Let Y_* denote the identification space associated with this partition, and $\pi : X \to Y_*$ the usual continuous map.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] & Y \\ \{f^{-1}(y)\} \arrow[r, equal] & Y_* \end{tikzcd}$$

If f is an identification map, then:

1. *the spaces Y and Y_* are homeomorphic;*
2. *a function $g : Y \to Z$ is continuous if and only if the composition $g \circ f : X \to Z$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] \arrow[rr, bend left, "g \circ f"] & Y \arrow[r, "g"] \arrow[d, equal] & Z \\ \{f^{-1}(y)\} \arrow[r, equal] & Y_* & \end{tikzcd}$$

Theorem 2.3. *Let $f : X \to Y$ be an onto continuous map. If f maps open sets of X to open sets of Y, or closed sets to closed sets, then f is an identification map, i.e. $\mathcal{T}_y$ is the largest topolgy such that f is continuous.*

coro:idmap-coro

Corollary 2.1. *Let $f : X \to Y$ be an onto continuous map. If X is compact and Y is Hausdorff, then f is an identification map.*

Definition 2.3 (Torus)**.** Torus is the unit square $[0, 1] \times [0, 1]$, with 1. opposite edge identified; 2. four edge points identified.

Remark 2.1. The identification map and corollary 2.1 can be used to show that torus is homeomorphic to two copies of circles:$S^1 \times S^1$. This is mentioned in page 68 of [1].

Definition 2.4 (Cone CX)**.** The cone of any space CX is formed from $X \times I$, where I is the unit interval $[0, 1]$, with certain identification. The identification shrinks all points in one surface into one point. This is discussed in page 68 of [1].

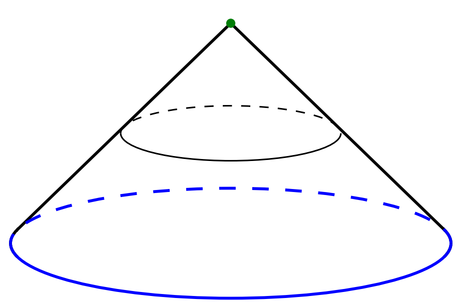


Figure 2: Cone of a Circle (Wikipedia)

Remark 2.2. There is another definition of cone CX when X in imbeded into $\mathbb{E}^n$, may be found on page 68 of [1]. Cone constructed in this way is called a geometric cone. It is made up of all straight line segments that join $v = (0, 0, \cdots, 1) \in \mathbb{E}^{n+1}$ to some point of X.

Lemma 2.1. *The geometric cone on X is homeomorphic to CX.*

Definition 2.5 (Quotient Space)**.** Let X be a topological space, A be its subspace. Then X/A menas the X with subspace A identified to a point.

1. the set A.
2. the individual points of $X \setminus A$.

Remark 2.3. In this notation, CX becomes $(X \times I)/(X \times \{1\})$.

Fact 2.1.

$$B^n/S^{n-1} \cong S^n \tag{2.1.1}$$

where $\cong$ menas homeomorphic. This is proved on page 69. Intuitively, this is like wrap a lower dimension ball surround the higher dimension ball.

Definition 2.6 ($f \cup g$)**.** Let X, Y $f \cup g$ subsets of a topological space and give each of X, Y, and$X \cup Y$ the induced topolgy. If $f : X \to Z$ and $g : Y \to Z$ are functions which agree on the intersection of X and Y, we can define

$$\begin{aligned} f \cup g : X \cup Y \to Z \qquad (2.1.2) \\ (f \cup g)(x) = f(x),\ x \in X \\ (f \cup g)(x) = g(x),\ x \in Y \end{aligned}$$

We say that $f \cup g$ are formed by 'glueing together' the functions f and g.

Lemma 2.2 (Glueing lemma (closed))**.** *If X and Y are closed in $X \cup Y$, and if both f and g are continuous, then $f \cup g$ are continuous.*

Similarly,

Lemma 2.3 (Glueing lemma (open))**.** *If X and Y are open in $X \cup Y$, and if both f and g are continuous, then $f \cup g$ are continuous.*

These two lemmas are seen as a special case of the following theorem, explained in page 70.

Define $X + Y$ to be the disjoint union of spaces X, Y. Define $j : X + Y \to X \cup Y$ which restrict to either X or Y is just the inclusion in $X \cup Y$.

Theorem 2.4. *If j is an identification map, and if both $f : X \to Z$ and $g : X \to Z$ are continuous, then $f \cup g : X \cup Y \to Z$ is continuous.*

$$\begin{tikzcd}
X + Y \arrow[r, "j"] & X \cup Y \arrow[r, "f \cup g"] & Z \\
X \arrow[urr, "f"] & Y \arrow[ur, bend right, "g"'] &
\end{tikzcd}$$

This can be generalized as follows. Let $X_\alpha, \alpha \in A$ be a family of subsets of a topological space and give each X_α and the union $\cup X_\alpha$, the induced topolgy. Let Z be a space and suppose we are given maps $f_\alpha : X_\alpha \to Z$, one for each α in A, such that if $\alpha, \beta \in A$,

$$f_\alpha\Big|_{X_\alpha \cap X_\beta} = f_\beta\Big|_{X_\alpha \cap X_\beta}$$

Define function $F : \cup X_\alpha \to Z$ by glueing together f_α. Let $\oplus X_\alpha$ be the disjoint unin of spaces X_α. Let $j : \oplus X_\alpha \to \cup X_\alpha$ be similarly defined.

Theorem 2.5. *If j is an identification map, and if each f_α is continuous, then F is continuous.*

Note: When j is the identification map, then $\cup X_\alpha$ has the identification topolgy instead of the subspace topology. The two will be quite different, as discussed on page 70 to 71 of [1].

Definition 2.7 (Projective space P^n)**.** A discussion of real P^n may be found on page 71.

Attaching maps and $X \cup_f Y$ Let:

$$Y \supseteq A \xrightarrow{f} X \tag{2.1.3}$$

where X, Y are topological spaces, f is continuous. We identify the disjoint union $X + Y$ using f, partitioning them into:

1. pairs of points $\{a, f(a)\}$ where $a \in A$;
2. individual points of $Y \setminus A$;
3. individual points of $X \setminus \mathrm{Im}(f)$.

The result identification space is denoted $X \cup_f Y$, and f is called the attaching map. This process can also be viewed as:

$$X \cup_f Y = (X \amalg Y)/\{f(A) \sim A\} \tag{2.1.4}$$

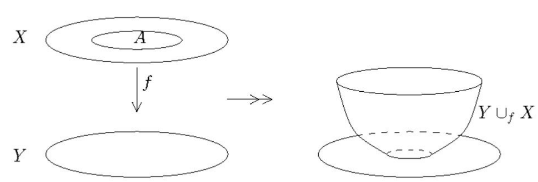


Figure 3: Attaching Space (credit: nLab

Example 2.1. P^2 can be seen as attaching a closed disc D to the boundary of M, a Mobius strip, as discussed in page 72 of [1]. Geometrically, this simply shrinks the boundary of M into a point. And an ant travelling around this point can point out the direction just as in P^2.

Remark 2.4. It is remarked that properties such as compactness, connectedness, and path-connectedness is inherited in identification. However, Hausdorff-ness is not. An counter example can be found in page 72 of [1].

`sec:Topological-Groups`

2.2 Topological Groups

In simple words, **topological groups** are objects that has both a topolgy on it and a group structure in it. And the two structures must be compatible. Specifically, the multiplication map $a \cdot b$ and the inverse map $a \to a^{-1}$ are continuous. Homomorphisms between are both group-homomorphisms and topological-homomorphisms (continuous maps). Isomorphisms are both group-isomorphisms and topolgy-isomorphisms (homeomorphisms). A sub-(topological group) is both a subgroup and has subspace topolgy. For convenience of language, use $\mathcal{TPG}$ denotes the category of topological groups. [1]

Example 2.2. The $\mathbb{R}$ is a topological group. The $\mathbb{Z}$ with discrete topology form the sub-(topological group) of $\mathbb{R}$. The quotient $\mathbb{R}/\mathbb{Z}$ forms a topological group. The map $f : \mathbb{R} \to S^1$ induces a homeomorphism $\mathbb{R}/\mathbb{Z} \cong S^1$, which is also a group isomorphisms, i.e. it is a $\mathcal{TPG}$-isomorphism.

Example 2.3. Similarly, R^n.

Example 2.4. The circle is also one. The group structure is combination of degrees.

Example 2.5. Any group with discrete topology.

Example 2.6. The torus considered as the product of two circles. (Take the producttopology and the product group structure.

Example 2.7. Three sphere S^3 considered as the unit sphere in the space of quaterions $\mathbb{H}$.

Remember this? :

$$\begin{tikzcd}
 & i \arrow[dr] \arrow[d, phantom, "\#"] & \\
k \arrow[ur] & & j \arrow[ll]
\end{tikzcd}$$

The unit sphere are unit quaterions, see more Versor.

Example 2.8. The **orthogonal group** $\mathrm{O}(n)$, of $n \times n$ orthogonal real matrices. It is easy to check that $\mathrm{O}(n-1)$ is a sub-$\mathcal{TPG}$ of $\mathrm{O}(n)$.

Definition 2.8 (Left translation L_x)**.** For $x \in G$, the function

$$L_x : G \to G \tag{2.2.1}$$
$$g \mapsto xg \tag{2.2.2}$$

is called a left translation by x. Similarly we have **right translation** R_x .

Fact 2.2. L_x and R_x are homeomorphisms (But not group-isomorphisms).

Remark 2.5. This shows that a topological group has a certain homogeneity as a topological space. For if $x, y \in G$, then $L_{yx^{-1}}$ maps x to y and is a homeomorphism. Therefore G *exhibits the same topological structure locally near each point.*

Theorem 2.6. *Let G is a topological group, let K be a connected component of G which contains the identity element. Then K is a closed normal subgroup of G.*

[1] This notation is nowhere popular or accepted. I use it to only to save space and time.

Fact 2.3. If $G = \mathrm{O}(n)$, then $K = \mathrm{SO}(n)$.

Theorem 2.7. *In a connected topological group, any neighbourhood of the identity element is a set generates the whole group.*

The two theorems above is summarised as

Table 2: caption

topology		group/topology
e + connected	$\Rightarrow$	closed & normal subgroup
e + neighbourhood	$\Rightarrow$	generator

A bit more examples about matrices:

Example 2.9. $\mathbb{M}(n)$ the $n \times n$ matrices, is not a topological group. But its subspace GL(n) , specifically, $\mathrm{GL}(\mathrm{n}, \mathbb{R})$ or $\mathrm{GL}(\mathrm{n}, \mathbb{C})$, is a topological group. This is demonstrated in page 76, theorem 4.12.

Fact 2.4. GL(n) is not compact. It has two idsjoint nonempty open sets: those with positive and those with negative determinants.

Theorem 2.8. $\mathrm{O}(n)$ *and* $\mathrm{SO}(n)$ *are closed and compact.* $\mathrm{SO}(n)$ *is a sub-*$\mathcal{TPG}$ *of* $\mathrm{O}(n)$.

Fact 2.5. $\mathrm{SO}(2) \cong S^1$ and $\mathrm{SO}(3) \cong P^3$. Here $\cong$ means isomorphisms of topological groups.

Remark 2.6. These two facts established on page 77. The first one can be eaily guess. Since a rotation is obviously determined by a rotation degree on S^1. Mathematically we have

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cong e^{i\theta} \tag{2.2.3}$$

The second one is proved mathematical in book [1]. But it has a physical argument. Remember we have the the homogeneous coordinates for P^3, such as $[1, \theta_x, \theta_y, \theta_z]$. As indicated in my labels, the three free coordinates θ_i can be regarded as rotation in 3-dimensional space. This rotation preserves the orientation, so it is in SO, not in O.

2.3 Orbit Space

sec:Orbit-Space

Definition 2.9 (Group Action on Topology Space)**.** A topological group G is said to act as a group of homeomorphisms on a space X if each group element (let $g, h \in G$) induces a homeomorphism of the space in such a way that:

1. $(hg)(x) = h(g(x))$, $\forall x \in X$;
2. $e(x) = x$, $\forall x \in X$, where $e = gg^{-1}$;
3. the function $G \times X \to X$,$(g, x) \mapsto g(x)$ is continuous.

The subset of X, consisting of $g(x)$ for all $g \in G$, is called an **orbit** of $x \in X$, written $O(x)$. Thought, it more convenient to write it just as Gx , as in textbooks of abstract algebra.

Fact 2.6. A common fact in abstract algebra here is: each orbit Gx is disjoint. If two $Gx \cap Gy \neq \varnothing$, then $Gx = Gy$.

By above fact, orbits partitions X, hence we can form the Identification space, with every elements in X identified with their brothers in the same orbit. The result is **orbit space** X/G .

ex:R-over-Z-T

Example 2.10. $\mathbb{Z}$ acts on $\mathbb{R}$ by addition $x \mapsto x+n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$. It partitioned $\mathbb{R}$ into intervals, for each $x \in X$, $x \sim x+n, \forall n \in \mathbb{Z}$. The orbit space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1.

An action G on X is called **transitive** , if and only if the orbit space X/G is the trivial point $\{1\}$. Or equivalently, the only orbit is the whole space, i.e. $Gx = G$, $\forall x \in G$.

Example 2.11. The orthogonal action $O(n)$ on S^{n-1} is transitive. Physically, this is saying that $\forall x \in S^{n-1}$, it can be rotated into $\forall y \in S^{n-1}$. A mathematical proof is on page 79 of [1]

A lot of examples from book [1]

Example 2.12. Extending example 2.10:

$$\mathbb{E}^2/(\mathbb{Z} \times \mathbb{Z}) = T \text{ (torus)} \tag{2.3.1}$$

Here $=$ means homeomorphism.

Example 2.13.

$$S^n/\mathbb{Z}_2 = P^n \tag{2.3.2}$$

Here $=$ means homeomorphism.

Example 2.14 (Three ways of $\mathbb{Z}_2$ acting on T)**.** The detailed procedure is to be found on page 91 of [1]. Here's a picture to visualize the action:

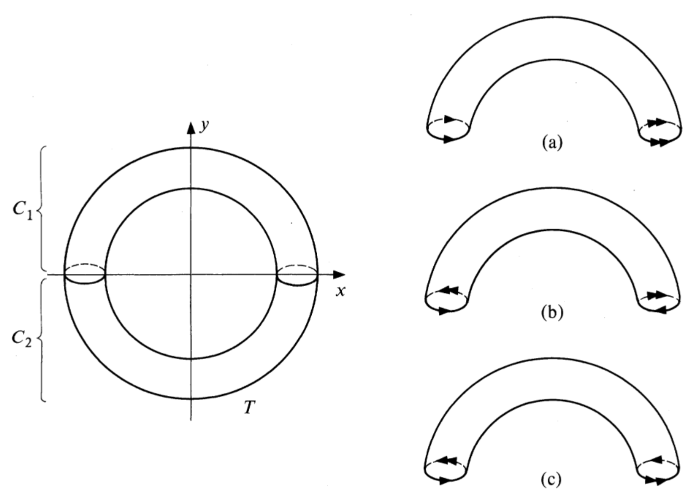


Figure 4:

The results are (a) a sphere; (b) a torus; (c) a Klein bottle.

Example 2.15. If G is a topological group, and H is $\mathcal{TPG}$-subgroup. Then, the left cosets of right cosets can be canonically seen as orbits. See more on page 81, example 4.

Example 2.16.

$$\mathrm{O}(n)/\mathrm{O}(n-1) = S^{n-1} \tag{2.3.3}$$

$$\mathrm{SO}(n)/\mathrm{SO}(n-1) = S^{n-1} \tag{2.3.4}$$

Here $=$ means homeomorphism. The first is established mathematically in page 82 of [1]. The second is mentioned there, indicating a similar proof.

Here I give an argument. Consider a unit vector y in S^{n-1}, if we want to rotate another unit vector e_1 to y, since the action is transitive, we can easily find a $A \in \mathrm{O}(n)$ to do this. But in addition, we can also find that $A \cdot B$, where $B \in \mathrm{O}(n-1)$ rotates the space around e_1 (thus leaving e_1 un-affected) also do our job. So there is an $\mathrm{O}(n-1)$ redundancy in $\mathrm{O}(n) \to S^{n-1}$. Similar for the second relation.

Theorem 2.9. *Let G acts on X and suppose that both G and X/G are connected, then X is connected.*

Fact 2.7. Using the theorem above, one can deduce that: $\mathrm{SO}(1)$ is connected, S^{n-1} is connected, so $\mathrm{SO}(n)$ is connected.

Next, the book [1] (page 82 to 85) introduces several three spaces (**Lens space** , **irrational flow** on T torus, **fundamental region** or in my word *space filling shapes*) and two group **Euclidean group** (page 84) and **plane-crystallographic group** (page 85). To save time, I leave here only some pictures:

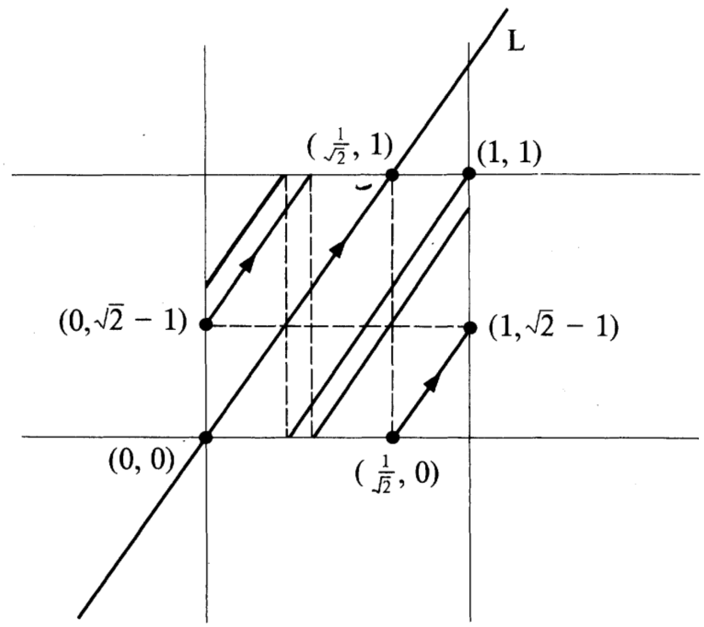


Figure 5: Irrational Flow on T

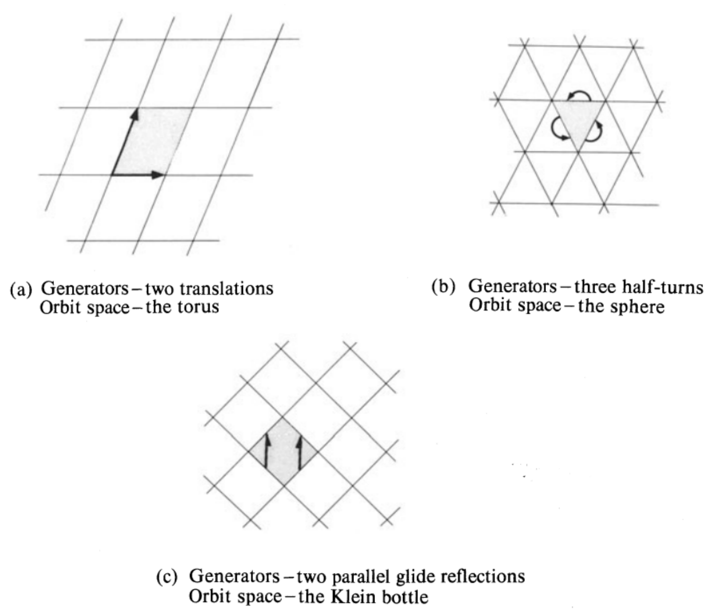


Figure 6: Space-filling Shapes

apter-5-Fundamental-Groups

sec:Homotopic-maps

3 Chapter 5 - Fundamental Groups

3.1 Homotopic maps

By a **loop** we mena a continuous map $\alpha : I \to X$ such that $\alpha(0) = \alpha(1)$. We can view is also as a continuous map $\alpha : S^1 \to X$. It is said to be based at the point $\alpha(0)$. Two loops α and β with the same **base point** can be multiplied, and their product is defined on page 87. A visualization of $\alpha \cdot \beta$ is here:

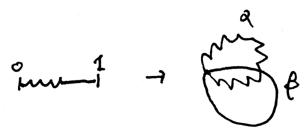


Figure 7: Multiply loops

But this product is not sufficient to become a group. At least, the multiplication is not associative:

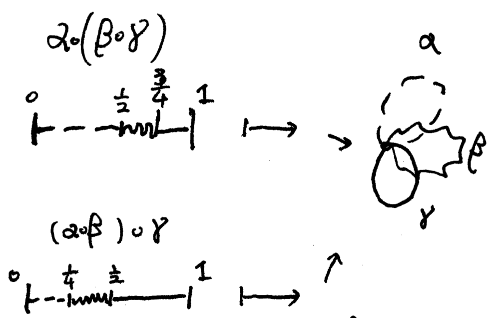


Figure 8: Multiplication of loop is not associative

But clearly the two result are exactly if we do care how long they occupy on the interval I, if the interval I is considered as a time parameter. So we define the following homotopy relation between loops. If we can find a family $\{f_r\}$ of maps, one for each $r \in [0, 1]$, such that $f_0 = \alpha$, $f_1 = \beta$, then we say that the loops α and β are homotopic. Schematically,

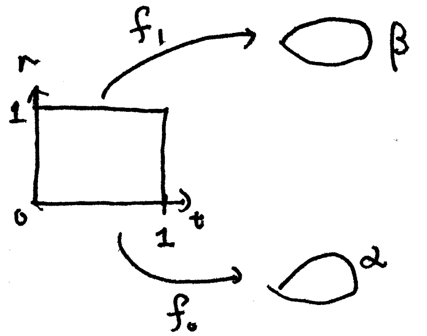


Figure 9: Homotopy for loops

This relation can be generalized to any continuous maps:

Definition 3.1 (Homotopic)**.** Let $f, g : X \to Y$ be continuous maps. Then f is homotopic to g if there exists a map $F : X \times I \to Y$ such that $F(x, O) = f(x)$ and $F(x, 1) = g(x)$ for alt points $x \in X$.

The map F is called a **homotopy** from f to g, and we write $f \simeq_F g$. In addition, if f and g agree on some $A \subset X$, we may wish to deform f to g without altering the values of f on A. In this case we ask for a homotopy F from f to g with the additional property that

$$F(a, t) = f(a) \text{ for all } a \in A, \text{ for all } t \in I \tag{3.1.1}$$

when such a homotopy exists, we say the f **is homotopic to** g **relative to** A and write $f \simeq_F g$ **rel** A .

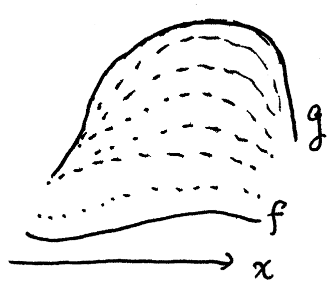


Figure 10: Homotopy Scheme

When f and g are loops, then the homotopic relation for loops are just saying that $f \simeq g$ rel $\{0, 1\}$.

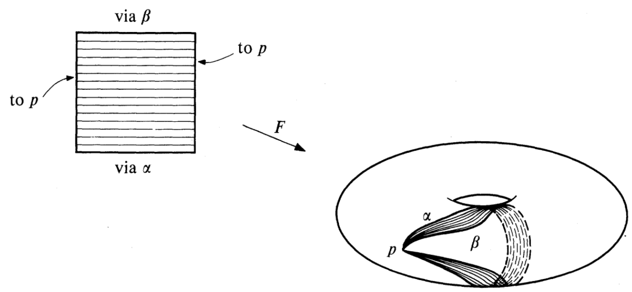


Figure 11: Homotopy for loops 2 (pp. 88 of [1])

Example 3.1. The author shows on page 88 of [1] that: when C is a convex subset of a euclidean space, let $f, g : X \to C$ be continuous maps, then $f \simeq_F g$, where F is $F(x,t) = (1-t)f(x) + tg(x)$. Note that if f and g agree on a subset A of X, then this homotopy is a homotopy relative to A. This F is called a **straight-line homotopy** .

Example 3.2. Let $f, g : X \to S^n$ be continuous maps. We can take S^n to be the unit sphere in $\mathbb{E}^{n+1}$, and think of f, g as continuous maps into $\mathbb{E}^{n+1}$, then we may form a similar "straight-line homotopy" from f to g by:

$$F(x,t) = \frac{(1-t)f(x) + tg(x)}{||(1-t)f(x) + tg(x)||} \tag{3.1.2}$$

Notice that the ball B^{n+1} is a convex set, so the numerator lies inside the ball. When normalized (as in $F(x,t)$), the numerator is a point on the sphere.

Example 3.3. This is example is best illustrated by pictures:

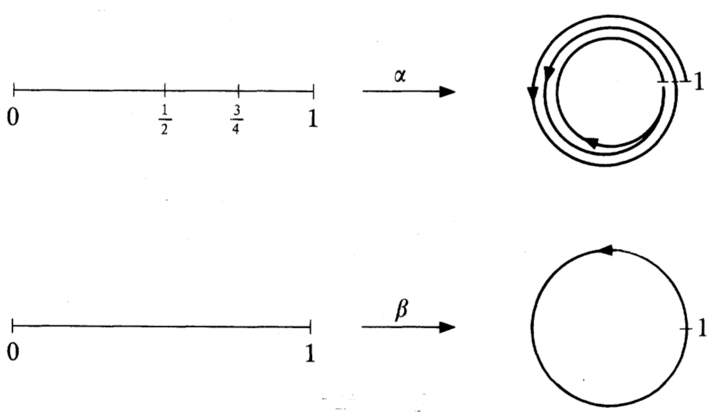


Figure 12: Loops to loops (pp. 89 of [1])

Geometrieally, α winds eaeh of the segments $[O, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{4}]$, $[\frac{3}{4}, 1]$ once round the eirc1e, the first two being wound in an anticlockwise direetion, and the third clockwise. The loop β simply winds the whole interval $[0, 1]$ once round the circle anticlockwise.

The book [1] gives a homotopy F between α and β on page 89. But it is best to imagine α and β being metal coils, and this F just describes the process when one magically strach and unfold the coil from α to β.

Notice that this coil is connected head to tail, so it is essential that there is not pole inside the coil in order that one can unfold the coil from α to β.

I think we already feel this, but the book proves it on page 90, that

Lemma 3.1. *The relation of 'homotopy' is an equivalence relation on the set of all maps from X to Y.*

Also

Lemma 3.2. *The relation of 'homotopy relative to a subset A of X' is an equivalence relation on the set of all maps from X to Y which agree with some give map on A.*

The book also mentions that

Lemma 3.3. *Homotopy behaves well with respect to composition of maps*

which means precisely that:

- If $f \simeq_F g$ rel A, then $hf \simeq_{hF} hg$ rel A.

$$\begin{tikzcd} A \subset X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \arrow[r, "h"] & Z \end{tikzcd}$$

- If $g \simeq_G h$ rel B, then $gf \simeq_F hf$ rel $f^{-1}B$ via the homotopy $F(x, t) = G(f(x), t)$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & Z \\ \cup & \cup & \\ f^{-1}B & B & \end{tikzcd}$$

3.2 Construction of the fundamental group

n-of-the-fundamental-group

Theorem 3.1. *The set of homotopy classes of loops in X based at $p \in X$ forms a group under the multiplication $\langle\alpha\rangle \cdot \langle\beta\rangle = \langle\alpha \cdot \beta\rangle$. The identity and inverse elements are defined on page 92 and 93 of [1].*

Remark 3.1. Notice that the fundamental group dependes heavily on the base point. Especially when the space X is disconnected. A natural intuition is that when the space is path-connected, then any two paths can be connected. If two points can be connected, then two loops based on different points can be connected. That why we have the following.

Theorem 3.2. *If X is a path-connected then $\pi_1(X, p)$ $\pi_1(X, q)$ are isomorphic for any two points $p, q \in X$.*

Proof. The proof is on page 94 of [1]. □

Definition 3.2 (f_*)**.** Suppose we have a continuous map $f : X \to Y$, f can induce a map ($p \in X, q \in Y$ and $q = f(p)$).

$$f_* : \pi_1(X,p) \to \pi_1(Y,q) \tag{3.2.1}$$
$$\langle \alpha \rangle \mapsto \langle f \circ \alpha \rangle \tag{3.2.2}$$

This map is actually a homomorphism.

Fact 3.1. By construction, we have for

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$(g \circ f)_* = g_* \circ f_* \tag{3.2.3}$$

Fact 3.2. With a homeomorphism $h : X \to Y$ and the above fact, we see that homeomorphic spaces have isomorphic fundamental groups.

sec:Sectin-5.3

3.3 Sectin 5.3

This section calculates the following facts:

#	Space	Fundamental group
1	Conves subset of $\mathbb{E}^n$	{e}
2	Circle	$\mathbb{Z}$
3	S^1	$\mathbb{Z}$
4	S^n, $n \geq 2$	{e}
5	Torus $S^1 \times S^1$	$\mathbb{Z} \times \mathbb{Z}$
6	$P^n, , n \geq 2$	$\mathbb{Z}_2$
7	Klein bottle	$\{a, b \mid a^2 = b^2\}$
8	Len space $L(p,q)$	$\mathbb{Z}_p$

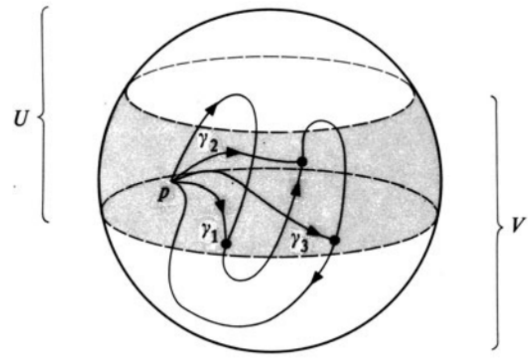
Convex subset of $\mathbb{E}^n$ is on page 96 of [1].

Definition 3.3 (simply connected)**.** A path-connected space whose fundamental group is trivial is said to be simply connected.

The n-sphere S^n is on page 99. To prove it, we require a theorem:

Theorem 3.3. *Let X be a space which can be written as the union of two simply connected open sets U, V in such a way that $U \cap V$ is path-connected. Then X is simply connected.*

Proof. Proof is on page 99 of [1]. It requires the Lebesgue lemma 1.1. A picture illustrating the proof is provided here:



□

Then, the two cover of S^n are simply $U = S^n - \{x\}$ and $V = S^n - \{y\}$, where x, y are two distinct points on S^n. Since U and V are homeomorphic to $\mathbb{E}^n$, S^n is simply-connected.

sec:Continued-Section-5.3

3.4 Continued Section 5.3

In this section, I will follow the lecture by professor Li Qin. We show that

Theorem 3.4.

$$\pi_1(S^1) = \mathbb{Z}$$

Proof. Given any integer $n \in \mathbb{Z}$, we give a loop by associatng each n

$$\begin{aligned} \pi : \mathbb{R} &\to S^1 \\ t &\mapsto e^{2\pi i t} \end{aligned}$$

Define

$$\gamma_n : [0, 1] \to \mathbb{R} \tag{3.4.1}$$

$$s \mapsto ns \tag{3.4.2}$$

Then $\phi_n := \pi \circ \gamma_n$ has the property that $\phi_n(0) = 1$, $\phi_n(1) = 1$. So we obtain

$$\phi : \mathbb{Z} \to \pi_1(S^1) \tag{3.4.3}$$

Geometrically, we ϕ_n loops around S^1 in n turns.

We need to prove that ϕ is a isomorphism. This is done by:

1. Prove that ϕ is a homomorphism;
2. Prove that ϕ is bijective.

First,

$$\begin{aligned} \gamma_n &: s \mapsto ns \\ \gamma_m &: s \mapsto ms \end{aligned}$$

We need

$$\langle \pi \circ \gamma_{m+n} \rangle = \langle \pi \circ \gamma_m \rangle \langle \pi \circ \gamma_n \rangle$$

Define $\sigma : [0,1] \to \mathbb{R}$, $s \mapsto \gamma_n(s) + m$, this is a translation of real line. Then $\pi \circ \sigma = \pi \circ \gamma_n$. Then

$$\langle \pi \circ \gamma_m \rangle \langle \pi \circ \gamma_n \rangle = \langle \pi \circ \gamma_m \rangle \langle \pi \circ \sigma \rangle = \langle \pi \circ (\gamma_m \circ \sigma) \rangle$$

γ_{m+n} has the same domain and codomain of $\gamma_m \circ \sigma$, and they obviously share the same start and the same end point. Therefore these two path are homotopic relative to $\{0,1\}$. Therefore

$$\langle \pi \circ \gamma_{m+n} \rangle = \langle \pi \circ (\gamma_m \circ \sigma) \rangle \tag{3.4.4}$$

Or

$$\phi_{m+n} = \phi_m \phi_n \tag{3.4.5}$$

Second, we need to show that this map is surjective. Notice that $\pi : \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$, is like a projection of a circulatory path onto a circle S^1. This map is locally homeomorphic. We can find a cover of S^1 as the combination of

$$U = S^1 \setminus \{-1\}$$
$$V = S^1 \setminus \{1\}$$

Then $\pi^{-1}(V)$ are the intervals on $\mathbb{R}$ excluding the whole integer points. Similarly, $\pi^{-1}(U)$ are those intervals on $\mathbb{R}$ excluding those half-integer points. In each of those intervals the map π is bijective. Now we need a lemma:

thm:path-lifting-lemma-1

Lemma 3.4 (Path-lifting lemma)**.**

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\pi"] \\ {[0,1]} \arrow[ur, "f"] \arrow[r, "\sigma"] & S^1 \end{tikzcd} \tag{3.4.6}$$

Assuming we have π and σ, both are continuous maps. More specifically, σ is a path in S^1 which begins at the point $1 \in S^1$. Then there is a unique path $\tilde{\sigma}$ in $\mathbb{R}$ which begins at $0 \in \mathbb{R}$ and satisfies $\pi \circ \tilde{\sigma} = \sigma$.

Proof. The proof is on page 97 to 98 of [1]. The class gives me enough intuition to understand the proof.

The intuition is that, by Lebesgue lemma, we can divide the interval $[0,1]$ fine enough such that each divided part is maped to only one of the cover U or V. We thus break a path σ into small paths σ_i. Each σ_i can be lifted into a path $\tilde{\sigma}_i$ in $\mathbb{R}$. But such lifting can be arbitrary because the inverse of π is not a good function. To resolve this ambiguity, one requires the first path should starts with $0 \in \mathbb{R}$, and the second should be continuously connected to the first, and so is the third, fourth, etc. This fixes the ambiguity and the paths $\tilde{\sigma}_i$ when connected give the required path $\tilde{\sigma}$. □

Note that $\tilde{\sigma}(0) = 0$, $\tilde{\sigma}(1)$ is an integer. Now for any loop $\gamma : [0,1] \to S^1$ based at 1, we can find a lifting $\tilde{\gamma} : [0,1] \to \mathbb{R}$ such that $\tilde{\gamma}(0) = 0$,$\tilde{\gamma}(1) = n$, and $\gamma = \pi \circ \tilde{\gamma}$. Then $\tilde{\gamma} \cong \gamma_n$ rel $\{0,1\}$, also $\langle \pi \circ \tilde{\gamma} \rangle = \langle \pi \circ \gamma_n \rangle$. Hence for any path γ we find a n such that $\gamma = \phi(n)$. So *the map is surjective.*

We need another lemma to prove that it is injective.

Lemma 3.5 (Homotopy-lifting lemma)**.** *If $F : [0,1] \times [0,1] \to S^1$ is a map such that $F(0,t) = F(1,t) = 1$ for $0 \le t \le 1$, then there exists a unique $\widetilde{F} : [0,1] \times [0,1] \to \mathbb{R}$ such that*

$$\pi \circ \widetilde{F} = F \tag{3.4.7}$$

$$\widetilde{F}(0,t) = 0,\ 0 \le t \le 1 \tag{3.4.8}$$

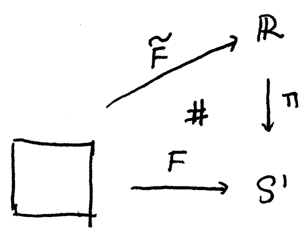


Figure 13: Homotopy lifting lemma-1

Proof. The proof is on page 98 of [1]. The idea is exactly the same as in lemma 3.4. □

Now we proof that the map ϕ is injective. Suffice to prove that $\mathrm{Ker}(\phi)$ is trivial. Suppose $\phi(n) = \pi \circ \gamma_n$ is homotopic to the constant loop. Then choose a homotopy F from $\pi \circ \gamma_n$ to the constant loop. By the homotopy-lifting lemma we can find $\widetilde{F} : [0,1] \times [0,1] \to \mathbb{R}$ such that $\pi \circ \widetilde{F} = F$. Also $\widetilde{F}(0,t) = 0$. We can find the vertical bottom is 0 and verticl top is γ_n. Right line is integers and can only be 0.

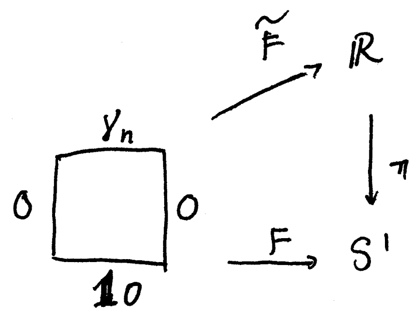


Figure 14: Schematic Draft

Hence γ_n starts at 0 and ends at 0. One can find that $\gamma_n \cong 0$. This completes the proof of injectivity. Hence completes the whole proof. □

We have an application,

thm:brouwer-fixpoint-B2

Theorem 3.5 (Brouwer Fixed Point theorem). *A contiuous map $f : B^2 \to B^2$ (B^2 : 2D-closed dicks) must have a fixed point. That is, $\exists x \in B^2$ such that $f(x) = x$.*

Proof. Assuming that this theorem is false, that is $\forall x \in B^2$, $x \neq f(x)$, then we have a straight path from $f(x)$ to x. We can extends this path to cuts the boundary of B^2 at $h(x)$. This is for all $x \in B^2$, hence we have a map $h : B^2 \to S^1$. Also, $h|_{S^1}$ is obvious an identity map. But S^1 can be included inside B^2, so we have:

$$S^1 \to B^2 \to S^1 \tag{3.4.9}$$

Hence we have a series of homomorphism of fundamental groups:

$$\pi_1(S^1) \to \pi_1(B^2) \to \pi_1(S^1) \tag{3.4.10}$$

and the composite is identity map. But observe that B^2 is a convex set and hence its fundamental group is trivial. But S^1 has non-trivial fundamental group. It is then impossible to form such a chain of homomorphism whose product is identity map. Contradiction! □

Remark 3.2. This theorem can be extended to higher dimensional case. But the proof cannot be the same because for higher dimension $\pi_1(S^n)$ is no longer non-trivial.

Another application, which we need a theorem to help:

Theorem 3.6.

$$\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \otimes \pi_1(X, y_0) \tag{3.4.11}$$

Proof. We use the projection maps: P_1 and P_2. Then, the map

$$(P_1)_* : \pi_1(X \times Y) \to \pi_1(X)$$
$$(P_2)_* : \pi_1(X \times Y) \to \pi_1(Y)$$

and their composition formed into

$$\langle \alpha \rangle \mapsto (\langle P_1 \circ \alpha \rangle, \langle P_2 \circ \alpha \rangle)$$

this map is surjective, injective, and is homomorphism. The detail can be found on page page 101 of [1]. $\square$

Fact 3.3. By this theorem, the two objects S^2 and $S^1 \times S^1$ is not homeomorphic, since their fundamental groups are not the same (the former is trivial and the later is $\mathbb{Z} \times \mathbb{Z}$).

3.5 Homotopy Type

sec:Homotopy-Type

Notice that a homeomorphic map will make two space have the same fundamental group, but two space having the same fundamental group may not be homeomorphic. For example, the plane and the S^2 both have the same $\pi_1 = \{e\}$, but the plane is not compact while the 2-sphere is, i.e. they are not homeomorphic.

Notice also that homeomorphic requires $fg^{-1} = \mathbb{1}$, whereas under homotopic relation, we may require $fg^{-1} \cong \mathbb{1}$. Therefore we may also define a new type of relationship between topological spaces:

Definition 3.4 (Homotopy type)**.** Two spaces X and Y have the same homotopy type, or they are homotopy equivalent, if there exist maps:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

such that $g \circ f \cong \mathbb{1}_X$, and $f \circ g \cong \mathbb{1}_Y$.

Fact 3.4. This relationship must be an equivalence relation on topological spaces, as confirmed by lemma (5.16) on page 103 of [1].

Remark 3.3. The topological spaces X and Y are not necessarily path-connected! But we have:

Fact 3.5. If X and Y have the same homotopy type then X is path-connected if and only if Y is path-connected. (This is noted in a footnote on page 107 of [1].)

Definition 3.5 (Deformation retraction)**.** Let A be a subspace of X. Let a homotopy $G : X \times I \to X$ which is relative to A and for which

$$\begin{cases} G(x, 0) = x \\ G(x, 1) \in A \end{cases}$$

for all $x \in X$. Then G will be called a deformation retraction of X onto A.

Remark 3.4. If there is a deformation retraction of X onto A, then of course X and A have the same homotopy type.

Following is an example of a deformation retraction:

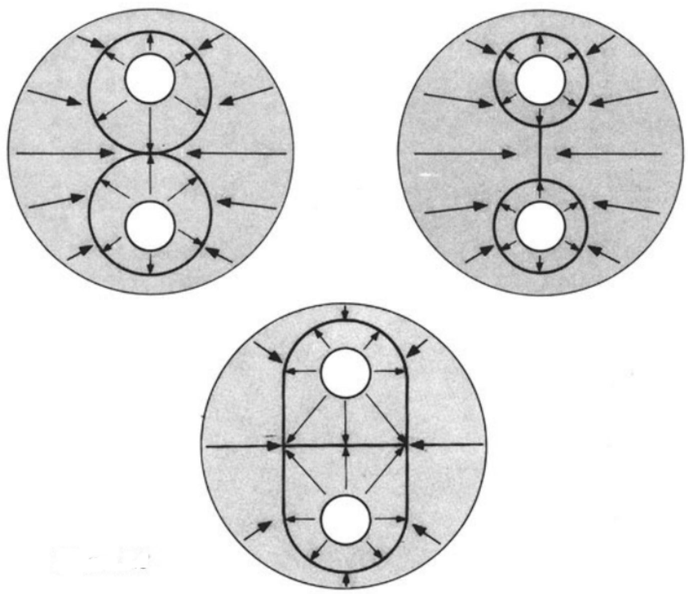


Figure 15: Three Deformation Retractions (from [1])

Exmaples from the book:

Example 3.4. Homeomorphic spaces have the same homotopy type.

Example 3.5. Any convex subset of a euclidean space is homotopy equivalent to a point.

-space-and-sphere-homotopy **Example 3.6.** $\mathbb{E}^n \setminus \{0\}$ has the homotopy type of S^{n-1}. This is shown on page 104 of [1], and is illustrated there by:

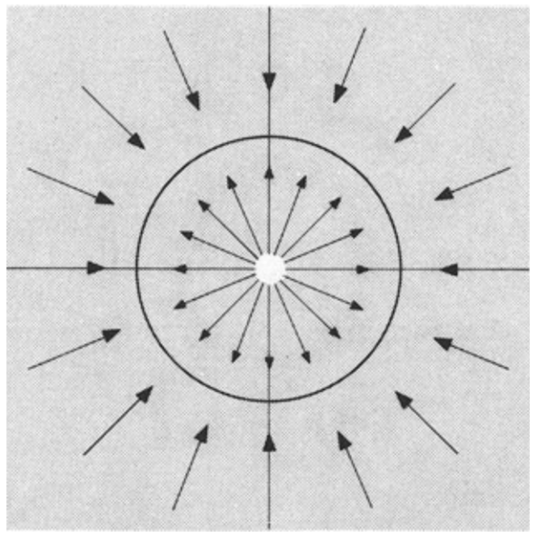


Figure 16: $E^n \setminus \{0\}$ and S^{n-1}

Theorem 3.7. *If $f \cong_F g : X \to Y$ then $g_* : \pi_1(X,p) \to \pi_1(Y, g(p))$ is equal to the composition*

$$\pi_1(X,p) \xrightarrow{f_*} \pi_1(Y, f(p)) \xrightarrow{\gamma_*} \pi_1(Y, g(p)) \tag{3.5.1}$$

where γ is the path joining $f(p)$ to $g(p)$ in Y defined by $\gamma(s) = F(p,s)$.

Proof. Proof can be found on page 105 of [1]. □

Theorem 3.8. *If two path-connected spaces are of the same homotopy type, then they have isomorphic fundamental group.*

Proof. Proof can be found on page 106 of [1]. Here's a note using the notation in that book:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

It should be noticed that the continuity allows one to identify the diagram:

$$\begin{tikzcd} \pi_1(X,p) \arrow[r, "(gf)_*"] \arrow[rr, bend right, "\mathbb{1}"'] & \pi_1(X, gf(p)) \arrow[r, "\gamma_*^{-1}"] & \pi_1(X,p) \end{tikzcd}$$

with the diagram:

$$\begin{tikzcd} \pi_1(X) \arrow[r, "(gf)_*"] \arrow[rr, bend right, "\mathbb{1}"'] & \pi_1(X) \arrow[r, "\gamma_*^{-1}"] & \pi_1(X) \end{tikzcd}$$

□

Fact 3.6. Using the above fact, we can find that the Möbius strip, the cylinder, the punctured plane $\mathbb{E}^2 \setminus \{0\}$, and the solid torus, all have the homotopy type of a circle S^1, and consequiently have $\mathbb{Z}$ as fundamental group.

Fact 3.7. Also, $\mathbb{E}^n \setminus \{0\}$ deformation-retracts onto S^{n-1} and is therefore a simply connected space when $n \geq 3$.

Definition 3.6 (Contractible)**.** A space X is called contractible if the identity map $\mathbb{1}_X$ is homotopic to the constant map at some point of X.

Theorem 3.9.

1. *A space is contractible if and only if it has the homotopy type of a point.*
2. *A contractible space is simply connected.*
3. *Any two maps into a contractible space are homotopic.*
4. *If X is contractible, then $\mathbb{1}_X$ is homotopic to the constant map at x for any $x \in X$.*

Example 3.7. Here is a contractible space that is really hard to imagine. It is called the topologist's dunce hat. It is formed by identifying the sides of a triangle in the manner indicated in the following figure:

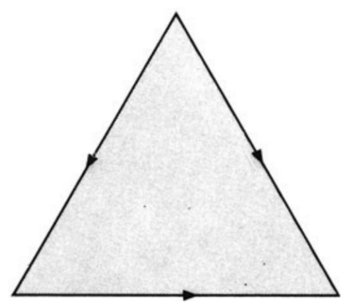


Figure 17: Topologist's dunce hat

fig:topo-d-hat

You may imagine identifying a pair of sides, then identify the identitied edge to the resting side. The direction for identifying is really important. This space is contractible. Despite some hard words said by the book, it is really simple to see on figure 17. Draw any loop on it and shrink it. Noticing that all edges are identified, though in some odd direction.

Remark 3.5. An identity function $\mathbb{1}_X$ on a contractible space X is homotopic to constant function $\mathbb{1}_p$ on a point $p \in X$. But this homotopy may not keep the point p fixed. That is, we do not necessarily have $\mathbb{1}_X \cong \mathbb{1}_p$ rel p.

An example on page l08 of [1]. In a space called the topologist's comb:

$$K = \{\frac{1}{n} | n \in \mathbb{N} \setminus \{0\}\}$$
$$\text{comb space} = (\{0\} \times [0,1]) \cup (K \times [0,1]) \cup ([0,1] \times \{0\})$$

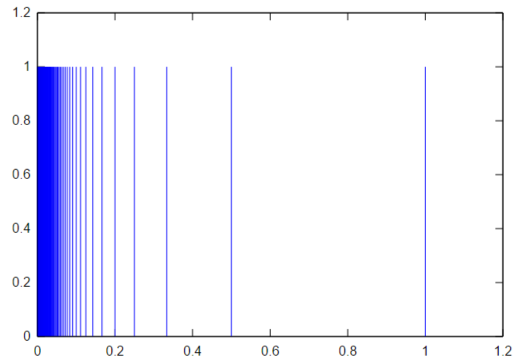


Figure 18: Topologist's Comb (from Wikipedia)

We cannot continuously defrom the identity function $\mathbb{1}_X$ to the constant function on $p = (0, 1)$. Intuitively, there are infinite points around p that are infinitesimally close to it. But the identity function must shrink from all these infinite points to the base line $[0, 1]$ without afftecting the value on p. This sounds unlikely.

Analytically, the argument is provided by Pedro in his Math.SE post:

> " **Idea:** Take the sequence $x_n = (\frac{1}{n}, 1)$, it converges to x_0. If existed such homotopy $H(x, t)$ then the sequences $H(x_n, t)$ would still converge to x_0.
> You have for each neighborhood U of x_0 a number $N_{(U,t)}$ and $\epsilon_{(U,t)}$ such that $H(x_n, s) \in U$ for $n > N_{(U,t)}$ and $|t - s| < \epsilon_{(U,t)}$.
> Covering the interval I with $(t - \epsilon_{(U,t)}, t + \epsilon_{(U,t)})$ and taking a finite subcover you obtain a number N such that $H(x_n, t) \in U$ for any $n > N$ and $t \in [0, 1]$.
> Now you can use disconnectedness of a small neighborhood to show that the homotopy can't take the elements of the sequence to x_0. "

Note: The teacher decided to temporarily switched to the next chapter, which is about a technique to patch the whole space. He will come back to the remaining sections of chapter 5 later.

4 Chapter 6 - Triangulation

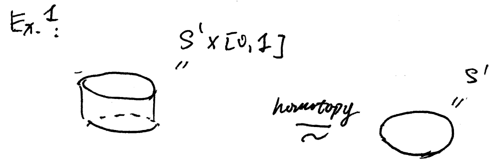
sec:Chapter6-Triangulation

4.1 Section 6.1 Triangulating Spaces

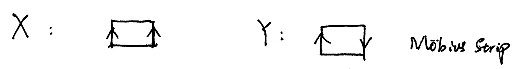
sec:Triangulating-Spaces

He gave to examples first,

Example 4.1. A trivial example:



Example 4.2. A non trivial one. That is, the Möbius Strip also has the homotopy type of a ciclr S^1.



X, Y can be both shrink to S^1., though shrink Y, one must be careful to check that the shrinking function is compatable on the identified edges.

But for complicated spaces, its homotopy type will not be so easy to calculate in with simple imagination. Here we will introduce a new technique to actually calculate the homotopy type of topological spaces.

The technique is called Triangulation. The most visual example comes from the computer vision technology (though I cannot find a picture by directly Googling triangulation). The idea we wants to stress is that the Triangulation is like use small patches of triangles to patch and cover the surface of 3D smooth objects. Increase the overall number of patches and make each patch get smaller. In the limit of this process one might get to recover the original image.

Definition 4.1 (Simplex of $\dim k$)**.** The stanford simplex of $\dim k$ in R^{k+1} is defined as follows. Let $v_i = (0, \cdots, 0, 1, 0, \cdots 0)$, where 1 is in $i+1$ coordinates. That is:

$$v_0 = (1, 0, \cdots)$$

$$v_1 = (0, 1, 0 \cdots)$$

$$\cdots$$

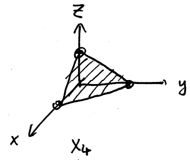
$$v_0 = (0, \cdots, 0, 1)$$

Then the simplex is the smallest convex set containing $\{v_0, \cdots, v_k\}$ in $\mathbb{R}^{k+1}$. It is also called a **k-simplex** .

Ex. $k=0$: a point

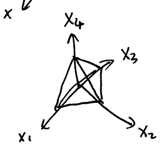
$k=1$: a closed line segment connecting v_0, v_1.

$k=2$:



is a triangle

$k=3$:



pyramid

is a ~~octa~~hedral

tetra

四面体.

Example 4.3.

Fact 4.1. Any point in a k-simplex is of the form:

$$\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_k v_k \tag{4.1.1}$$

where each $\lambda_i \geq 0$, and $\sum_{i=0}^{k} \lambda_k = 1$.

Example 4.4. There is a special point in the k-simplex:

$$\frac{1}{k+1} \sum_{i=0}^{k} v_i \tag{4.1.2}$$

When $k = 2$, this is just the usual center of gravity of triangle.

We want to study those space which is the union of a finite collection of simplexes which fit together nicely in some Euclidean space. These are called "triangulable spaces". More specifically:

Definition 4.2 (faces)**.** If A and B are simplexes, and if the vertices of B form a subset of vertices of A. Then B is called a face of A, written as $B < A$.

Definition 4.3 (Simplicial Complex)**.** A **finite** collection of simplexes in some Euclidean space in $\mathbb{R}^n$ is called a simplicial complex, if

1. whenever a simplex lies in this collection, then so does its faces
2. whenever two simplexes in this collection intersect, their intersection is a common face of these two simplexes.

Example 4.5.

Definition 4.4 (Topology on simplicial complex)**.** The topology on a simplicial complex is given by the subspace topology of Euclidean space. Let K be a simplicial complex, we denote $|K|$ as the topology of K. This $|K|$ is called a **polyhedron** .

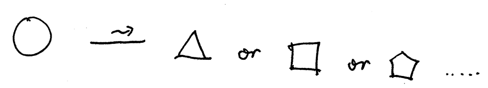
Definition 4.5 (Triangulation and Triangulable)**.** A triangulation on a topological space X consists of a simplicial complex K, and a homeomorphism h:

$$h : |K| \to X$$

A space is called triangulable if such simplicial complex $|K|$ and homeomorphism h exists.

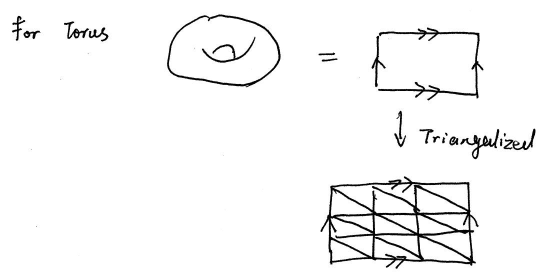
Fact 4.2. If X is triangulable, then X is compact and can be made into a metric space. (Notice that the simplicial complex is a finite collection.)

Remark 4.1. Trigulation is in general not unique. For example:

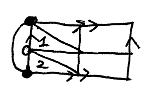


`ex:triangu-torus`

Example 4.6. Here is a triangulation on the torus:



Notice that the following diagram is not a triangulation of torus:



because when $\triangle_1$ intersects $\triangle_2$, ~~it is not~~ the intersection (2 points) is not a face!

Lemma 4.1. *Let K be a simplicial complex in $\mathbb{R}^n$, then*

1. *$|K|$ is compact.*
2. *Each point of $|K|$ lies in the interior of exactly one simplex of K.*
3. *If we take the simplexes of K seperately, and give their union the identification topology, we obtain $|K|$. Thus we have two equivalent ways to view the topology on K.*

4. *If $|K|$ is connected, then $|K|$ is also path-connected.*

Proof. 1. Obviously.

2. Since K is connected by simplexes, a point p must obviously be contained in some simplex. Also, since on each simplex p must be on the interior of it, or interior of its components. We only need to prove that the containing simplex is unique. Suppose p is the interior of simplexes A and B. Notice that their intersection must be their faces. The only face of A or B containing interior points is A or B itself. Hence $A = B$.
3. By definition of identification topology, we can see that in the identified space, a subset $C \subset |K|$, C is closed $\Leftrightarrow C \cap |A|$ is closed in $|A|$ for all simplexes $A \in K$. The rest of the proof is obvious.
4. We need only to show that $|K|$ is locally path-connected. But any point $p \in K$, $p \in$ interior of some A where $A \in K$. That is, we can find a neighbourhood U of p, and U is contained in A ($U \subset A$). (More specifically, we can find ε such that $B_\varepsilon(p) \cup |K| = B_\varepsilon(p) \cup |A|$) But a simplex A is obviously locally path-connected. Hence U (or $B_\varepsilon(p)$) is path connected, hence $|K|$ is path-connected.

□

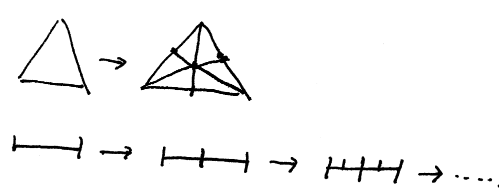
4.2 Section 6.2 Barycentric Division

sec:Barycentric-Division

Now we begin to make the triangulation more and more detailed. We start from a simplicial complex K and constructing a new simplicial complex K_1, such that

1. $|K| = |K_1|$, homeomorphically.
2. diameter $|K_1| <$ diameter $|K|$.

The general construction method is called Barycentric division, visualized as:



We first define a concept

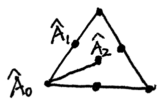
Definition 4.6 ($\hat{A}$ Bary center)**.** Let A be a simplex, then the bery center of A, denoted $\hat{A}$, is the point:

$$\hat{A} := \frac{1}{k+1}(v_0 + \cdots + v_k) \tag{4.2.1}$$

Then, the general process is: The new space K_1 is such that

1. The vertices of K_1 are bary centers of **ALL** simplexes of K. Note the the bary center of a 0-simplex is just itself. So in general, K_1 has more vertices of K.
2. A collection $\hat{A}_0, \cdots, \hat{A}_k$ of such bary centers will form a vertex of a k-simplex in K_1 if and only if there is a permutation σ of $\{0, 1, \cdots, k\}$ such that
$$A_{\sigma 0} < A_{\sigma 1} < \cdots < A_{\sigma k} \tag{4.2.2}$$

Remark 4.2. The second point above tries to do the following



we have: $A_0 < A_1$

a point

an edge of original triangle

$A_0 < A_2$ the original triangle.

Also $A_0 < A_1 < A_2$

Now we give a precise description of how this Bary center division will work. We first describe precisely two characteristics of a simplicial complex.

Definition 4.7 (dimension of a simplicial complex)**.** The dimension of a simplicial complex K is the maximum of the dimensions of all its simplexes.

Definition 4.8 (mesh $\mu(K)$)**.** The mesh of a simplicial complex K is the maximum of the diameters of all its simplexes.

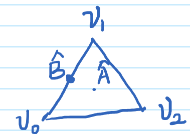
As we expected, the mesh (which carries the intuition of the smallness of one simplicial complex) of a simplicial complex will decrease. The following lemma confirmed this.

Lemma 4.2. *The sollection of simplexes described above forms a simplicial complex. If it is denoted by K^1, and is called the first barycentric subdivision of K, then K^1 has the following properties:*

1. *each simplex of K^1 is contained in a simplex of K;*
2. *Their topology are the same: $|K^1| = |K|$*
3. *If the dimension of K is n, then*
$$\mu(K^1) \leq \frac{n}{n+1}\mu(K) \tag{4.2.3}$$

Proof. The proof can be found in page 126 to 127 of [1]. But the proof of the last point is not so clear and is illustrated here:

$\hat{A}$, $\hat{B}$ forms an edge



$B < A$, $\Rightarrow$ B is a face of A

$$\Rightarrow \operatorname{dist}(\hat{B}, \hat{A}) < \max_i \left(\operatorname{dist}(\hat{A}, v_i) \right)$$

also $\quad \operatorname{dist}(\hat{A}, v_i) = \left| \frac{1}{k+1}(v_0 + \cdots + v_k) - v_i \right|$

$$= \left| \underbrace{\frac{1}{k+1}(v_0 - v_i) + \cdots + \frac{1}{k+1}(v_k - v_i)}_{\text{One of them is zero}} \right|$$

$$\leq \frac{k}{k+1} \max_j (v_j - v_i)$$

$$\leq \frac{k}{k+1} \mu(A) \leq \frac{n}{n+1} \mu(K)$$

□

Remark 4.3. Notice that the number $\frac{n}{n+1} < 1$, hence after an infinite number of Barycentric subdivision, the mesh (averaged diameter) of a simplicial complex K^m will goes to zero!

:Simplicial-approximation

4.3 Section 6.3 Simplicial approximation

The purpose here is to approximate any continuous map $f : A \to B$ with linear map, in a way that is compatible with the triangulations on both A and B.

Definition 4.9 (Simplicial map)**.** Let K and L be simplicial complexes. A continuous map $s : K \to L$ is called simplicial, if it takes simplexes of K *linearly* to simplexes of L. Specifically, being *linear* means that:

For $\forall A \in K$, we have $B = s(A) \in L$. Also, if $A = \{v_0, \cdots, v_k\}$, then $B = \{s(v_0), \cdots, s(v_k)\}$. Lastly, if $x = \sum_{i=0}^{k} \lambda_i v_i$ is a a point in A, with

$\lambda_i \geq 0$ and $\sum_{i=0}^{k} \lambda_i = 1$, then

$$s(x) = \sum_{i=0}^{k} \lambda_i s(v_i)$$

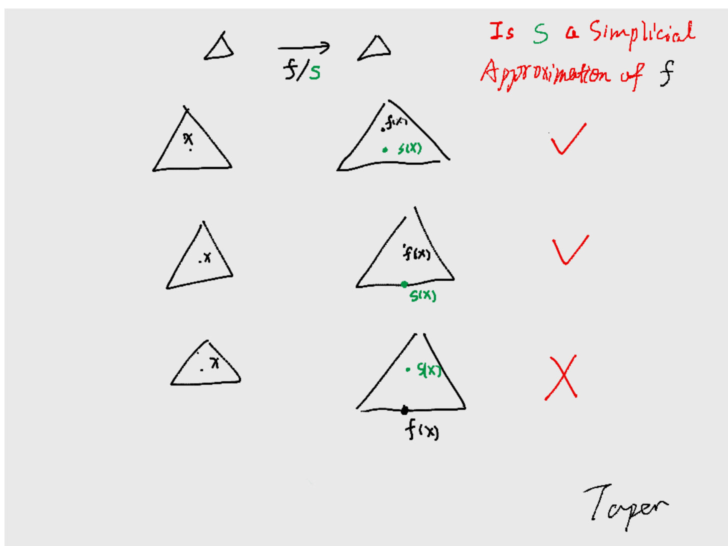
In a word, s maps simplexes to simplexes, vertices, to vertices, and points linearly to points.

Remark 4.4. It is easy to see that the a simplicial map is uniquely determined by its value on all the vertices. By this, there can only be a finite number of such kind of maps, since there are only a finite number of vertices.

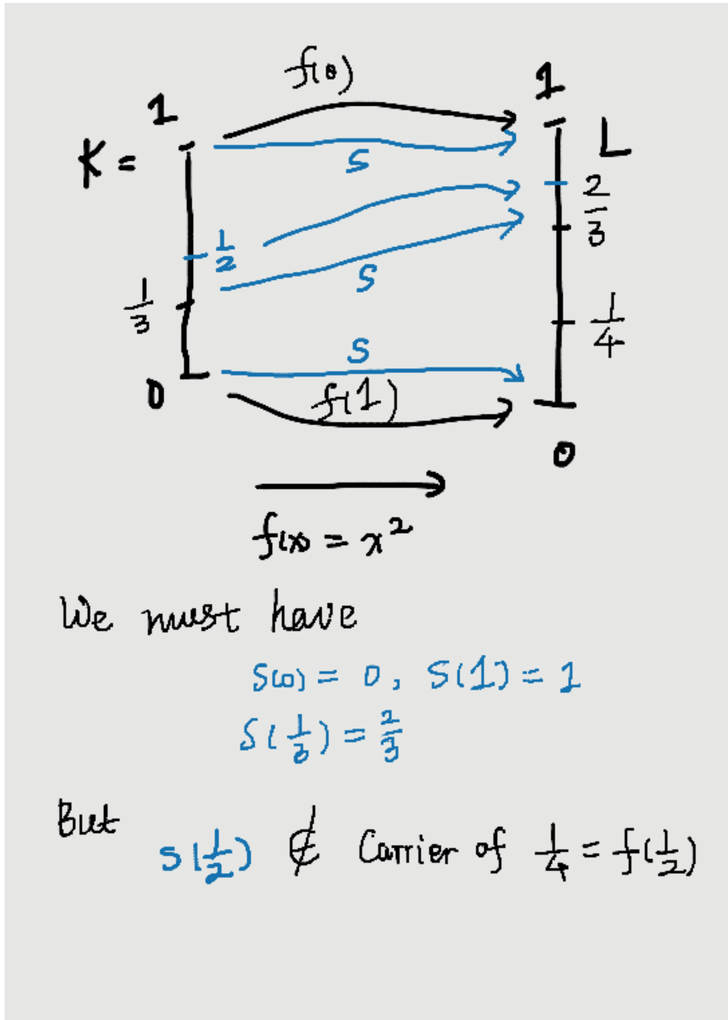
Now we want to find such linear maps to approximate a give function $f : |K| \to |L|$. Given a point $x \in |K|$, we define the **carrier of** $f(x)$ to be the unique simplex of L that contain $f(x)$ as its interior point.

Definition 4.10 (Simplicial Approximation)**.** A simplicial map $s : |K| \to |L|$ is called a simplicial approximation of $f : |K| \to |L|$ if and only if $s(x) \in$ carrier of $f(x)$, $\forall x \in |K|$.

Example 4.7. Trivial examples:



Example 4.8. Now we show that when simplicial approximation does not exits. The set up is as follows:



Hence, there is no simplicial approximation of f. But, it will be shown below that, if we divide K small enough $(K^1, K^2, \cdots)$, we will eventually be able to find a simplicial approximation of f in some K^n.

thm:SAT

Theorem 4.1 (Simplicial approximation theorem)**.** *Let $f : |K| \to |L|$ be a continuous map between polyhedras formed by simplicial complexes $|K|$ and $|L|$. If m is chosen large enough, then there is a simplicial approximation $s : |K^m| \to |L|$ to $f : |K^m| \to |L|$.*

The proof will use a lemma. Define the **open star** v of a simplicial complex K, as the union of the interiors of those simplexes of K having v as a vertex. This union is denoted by $\mathrm{star}(v, K)$. An example is:

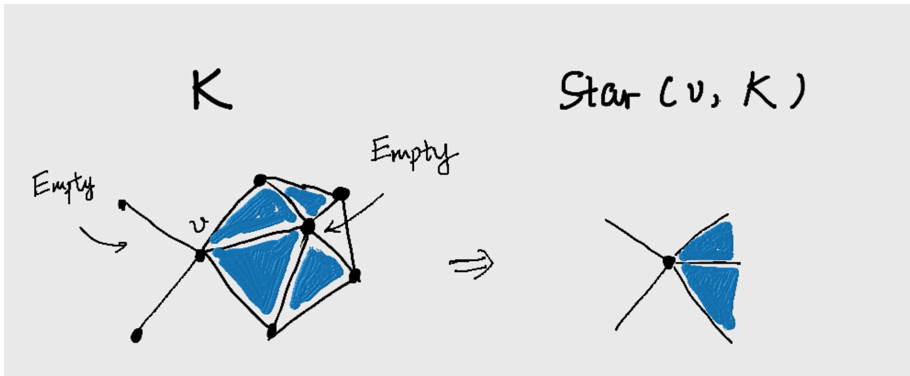


Figure 19: Example of star

So it really feels like a star spreading out from v. We have

Lemma 4.3. *Vertices $v_0, v_1, \cdots, v_k$ of a simplicial complex K are the vertices of a simplex of K if and only if the intersection of their open stars is nonempty, i.e.*

$$\cup_0^k \operatorname{star}(v_i, K) \neq \varnothing$$

Proof. The proof can be found in page 130 of [1]. It is quite trivial. $\square$

Then we have the **proof of theorem 4.1**:

Proof. The proof is found in page 130 to 131 of [1]. The idea is that if for each vertex u of K, we can find a vertex v of L such that

$$f(\operatorname{star}(u, K)) \subseteq \operatorname{star}(v, L) \qquad (*)$$ eq:proof-temp

Then the simplicial map $s(u) = v$ will be easily verified to be a simplicial approximation of f.

To make such (*) happen, we naturally have to divide K^m small enough. We can achieve such small division is guaranteed by the Lebesgue lemma 1.1, since each subdivision makes the averaged diameter smaller. $\square$

4.4 Section 6.4 The edge group of a complex

sec:edge-group

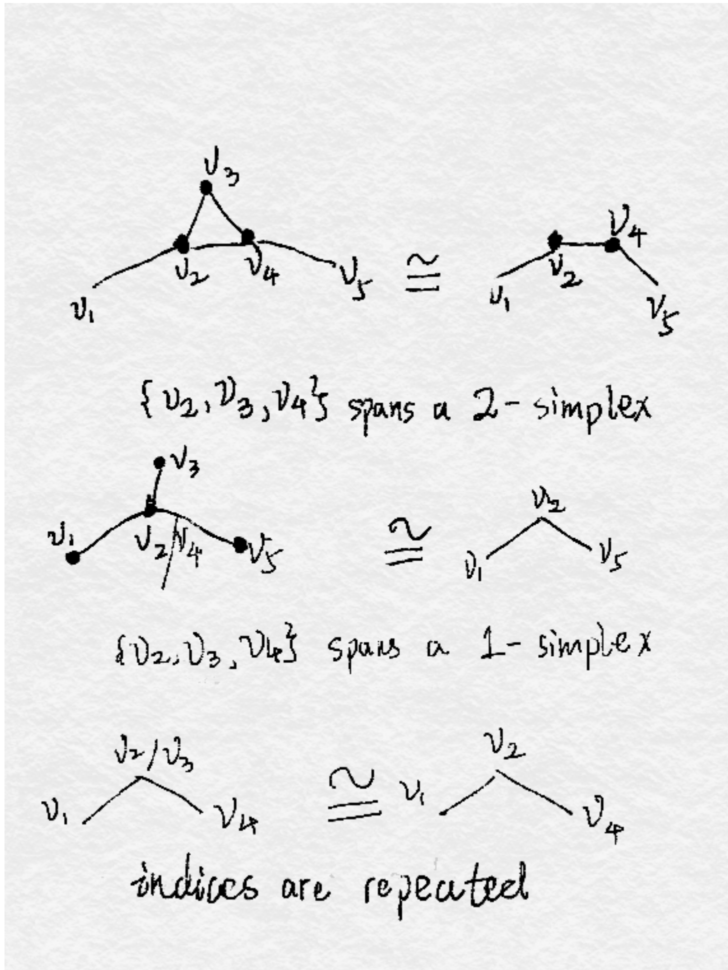
The definition of edge group is so similar to that of fundamental group (actually, edge group is made to simplify computation of fundamental group). Therefore, I will not waste too much words here. One with reasonable intuition could just skim through this section.

The following excerpt from book [1] best captures the idea:

> " We calculated the fundamental groups of one or two spaces in Chapter 5, but our calculations, though efficient for the examples given there, were rather *ad*

hoc. If we agree to work with triangulable spaces, we can be much more systematic. We shall show how to read off generators and relationst for the fundamental group from a triangulation of the space.
Let X be a path-connected triangulable space, take a specific triangulation $h : |K| \to X$, and replace X by $|K|$ (we are at liberty to do this since the fundamental group is a topological invariant). Now the advantage of a polyhedron $|K|$ is that the elements of its fundamental group can be represented by loops which are made up of edges of K. Using such 'edge loops' we shall construct a group, called the edge group of the complex $|K|$, which can be computed and wh ich is isomorphic to the fundamental group of $|K|$. "

The edge pathes $v_0, v_1, \cdots, v_k$, edge loops $v, v_1, \cdots, v_{k-1}, v$ are defined similar to pathes and loops. The equivalence relation between edge loops are defined as represented by the following drawing:



The equivalence class of edge pathes and edge loops are denoted $\{v_0, v_1, \cdots, v_k\}$, and $\{v, v_1, \cdots, v_{k-1}, v\}$, respectively. The multiplication of edge loops is defined simply as

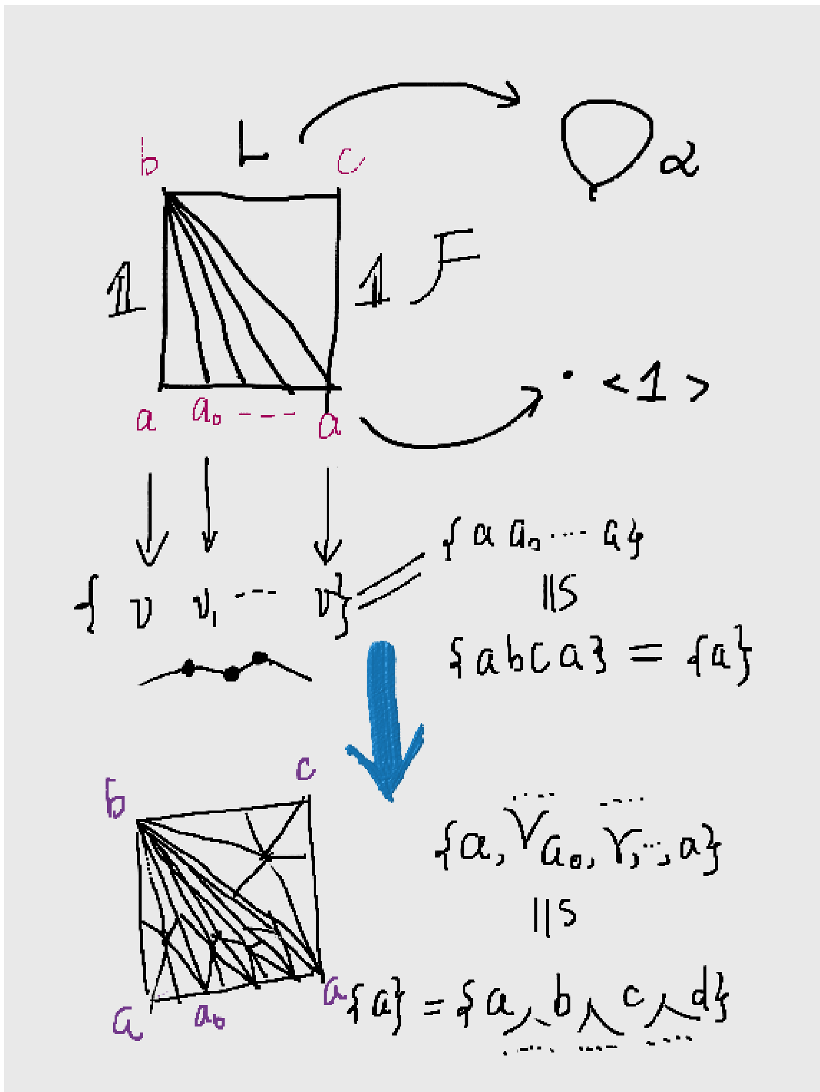
$$\{v, v_1, \cdots, v\}\{v, w_1, \cdots, v\} = \{v, v_1, \cdots, v, w_1, \cdots, v\} \tag{4.4.1}$$

It is trivially verified that such edge loops form a group, called the **edge group based at** v, $E(K, v)$. It is also trivially verified that such group does not depend on the choice of base point v, since $|K|$ is path-connected.

Theorem 4.2. *$E(K, v)$ is isomorphic to $\pi_1(|K|, v)$*

Proof. The proof can be found on page 133 of [1]. The essential idea is that an edge loop is just a loop. And a loop is just a continuous map $f : I \to |K|$, we can easily use the simplicial approximation $s : I^m \to |K|$ of f (with appropriately m), to find a corresponding edge loop.

The uniqueness of the loop corresponding to an edge loop is found again by treating $L := I^2$ as a simplicial complex and apprximate again the homotopy $F : L \to |K|$. The resulted S will again gives us a equivalence relation to the constant edge loop. (Note, we have to use the fact the simplicial maps preserves the equivalence relation between edge pathes). The following pictorially produces the proof:



∎

Remark 4.5. This theorem already tells us that the Edge group $E(K, v)$ is indepedent on the choice of base point v, since our polyhedron $|K|$ is always assumed to be path-connected.

Based on this edge group, we will try to calculate more explicitly the structure of this group. Here we provide a way to get the generators and relations directly from the edge group.

Key Point 4.1. The intuition is that the structure of this group is determined only by those 1-simplexes, 2-simplexes, and the equivalence relations between edge pathes. All higher dimensional simplexes are not relevent to this group structure.

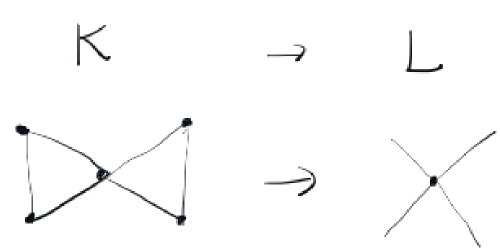
We let L the subcomplex of a simplicial complex K. This subcomplex L is required to contain all the vertices of K and such that its polyhedron $|L|$ is path-connected and simply-connected. Can we find such a subcomplex? The intuitive answer is yes due to the path-connectedness nature of $|K|$. The rigorous answer needs the help of a concept **tree**. A tree T is a one-dimensional subcomplex of K whose polyhedron $|T|$ is both path-connected and simply-connected. The following lemma affirms that we can find our L to be exactly the maximal tree in K:

Lemma 4.4. *A maximal tree contains all the vertices of K.*

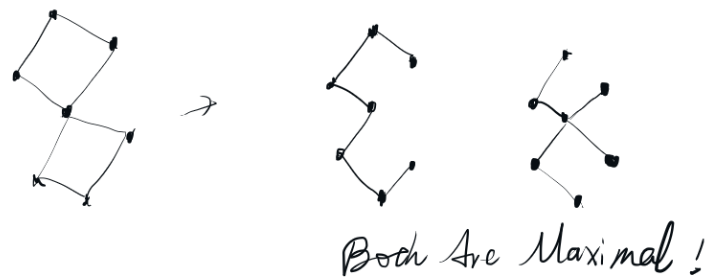
Proof. We first observe that a tree must exists, unless the dimension of K is 0 (in which case, there is entirely no need for the following discussions, and the lemma is still true because there is not even a maximal tree!). Let T be a maximal tree in K, which means that if T' is a tree and contains T, than $T' = T$. The next step is simply and is expatiated on page 134 of [1]. The idea is to grow a tree by adding more vertices, which is possible because $|K|$ is path-connected and a path can be easily turned into an edge path. Note that add a simply-connected space in this way will keep the space simply-connected, just do not add a vertices twice too creat a loop.

Obviously one will get a tree which is maximal and contains all the vertices. Such a process also affirms that the maximal tree will exist (but may not be unique). □

Example 4.9. Here is the maximal tree of a triangulation:



Example 4.10. The maximal tree is usually not unique!



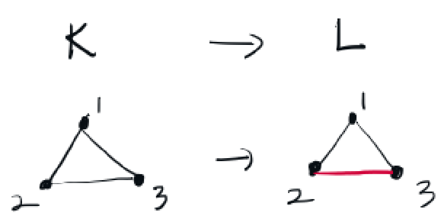
Now since $|L|$ is simply-connected, it will not contribute to the edge group. What contributes to the edge group is those edges that are not in this $|L|$. This motivates us to define a new group:

Definition 4.11 ($G(K, L)$)**.** Let L be the maximal tree in K (assume $|K|$ is path-connected). Then $G(K, L)$ is defined by generators g_{ij} for each **ordered** pair of vertices v_i, v_j such that $\{v_i, v_j\}$ spans a simplex (0-simplex or 1-simplex) in K. Those generators are subject to the following conditions:

1. $g_{ij} = 1$, the identity element, if and only if $\{v_i, v_j\}$ spans a simplex in L.
2. $g_{ij}g_{jk} = g_{ik}$ if and only if $\{v_i, v_j, v_k\}$ spans a simplex (0 or 1 or 2 simplex) in K.

ex:torus-GLKgroup

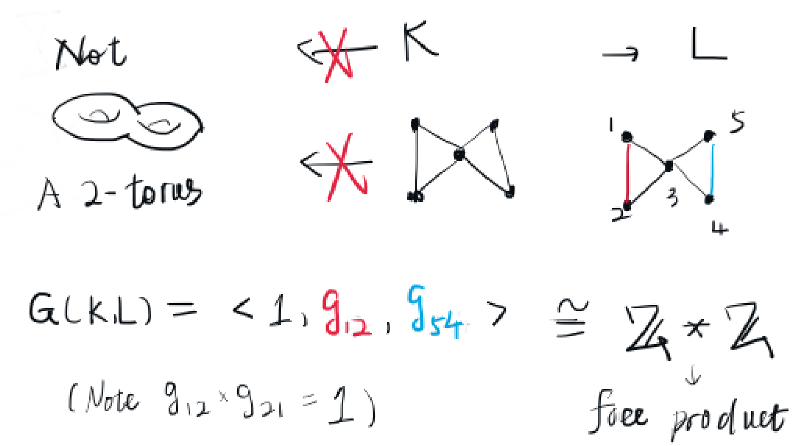
Example 4.11. Again, for circle:



$$G(K,L) = \langle 1, g_{23} \rangle_{\times} \cong \mathbb{Z} = \langle 0, 1 \rangle_{+}$$

ex:ex4-L-in-two-torus

Example 4.12. And for another:



Note that this does not gives a triangulation of two-torus! The reason will be explained later in example 4.13.

More generally, we have

Theorem 4.3. *$G(K, L) \cong E(K, v)$, as a group isomorphism.*

Proof. The proof in page 135 to 136 of [1] constructs the following homomorphisms:

$$G(K, L) \underset{\theta}{\overset{\phi}{\rightleftarrows}} E(K, v)$$

It is pretty simple to get these two maps and confirms that their composites are identity functions on each space. □

Remark 4.6. We have now the following chain of homomorphisms:

$$\pi_1(|K|, v) \cong E(K, v) \cong G(K, L)$$

Corollary 4.1. *For any path-connected, triangulable space, its fundamental group is finitely presented, i.e., the fundamental group has only a finite number of generators and a finite number of relations.*

For example, we have demonstrated in example 4.11 that its $G(K, L) \cong \mathbb{Z}_2$, which is just its fundamental group.

In addition, armed with this tool, we can answer an additional question. We want to know what is the fundamental group of two torus connected together:

$\pi_1($  $) = \mathbb{Z} \times \mathbb{Z}$

What about

$\pi_1($  $) = ?$ (is it $\mathbb{Z}^4$?)

This can be done using the following theorem:

Theorem 4.4 (Van Kampen's theorem)**.** *Let J, K be two simplicial complexes in some Euclidean space which intersect in a common subcomplex, and suppose all $|J|$, $|K|$, $|J \cap K|$ are path-connected. Let j, k be the inclusion maps:*

$$|J \cap K| \xhookrightarrow{j} |J|$$

$$|J \cap K| \xhookrightarrow{k} |K|$$

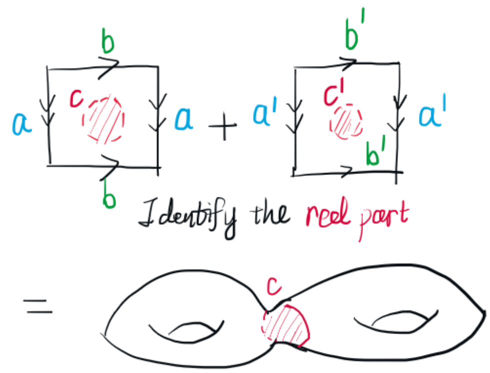
Take a vertice $v \in J \cap K$ as the base point. The fundamental group $\pi_1(|J \cup K|, v)$ is obtained from the free product $\pi_1(|J|, v) * \pi_1(|K|, v)$ by adding the relations $j_*(z) = k_*(z)$ for all $z \in \pi_1(|J \cap K|, v)$.

Proof. The proof can be found on page 137 to 138 of [1]. Please do read the paragraph just before this theorem to get a picture of the proof.

Note also that a 2-simplex cannot possiblly cross the intersection part $J \cap K$. Please check why I mention this when reading the proof on the book [1]. □

ex:pi1-two-torus

Example 4.13. Here's how to find the fundamental group for two-torus connected:



$$c = aba^{-1}b^{-1} \xlongequal{\text{identify}} c' = a'b'a'^{-1}b'^{-1}$$

$$\pi_1 = \{1, a, a', b, b' \mid aba^{-1}b^{-1} = a'b'a'^{-1}b'^{-1}\}$$

The result confirms that the triangulation of a two-torus is as simple as in example 4.12.

Example 4.14. Next we consider the projective space $\mathbb{R}P^2$, the space is homeomorphic to half the sphere S^2, with the antipodal points on the boundary identified:

$\mathbb{R}P^2$

Then it can be regarded as a 2-disc with it boundary attached to the boundary of a Möbius strip. This construction will tell us that the fundamental group of $\mathbb{R}P^2$ is $\mathbb{Z}_2$.

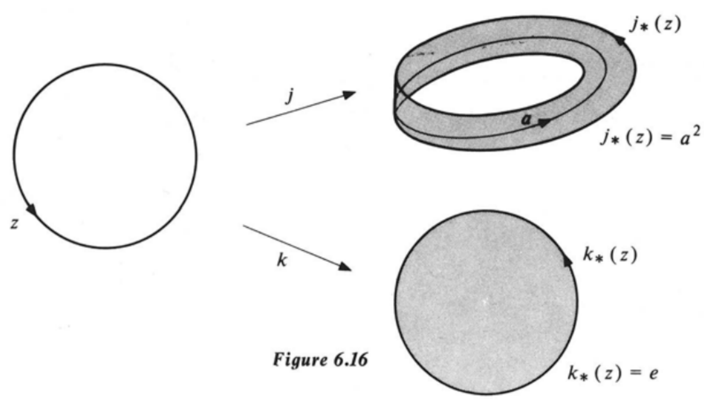


Figure 20: Page 139, Fig 6.16 of [1]

The teacher jumps right to chapter 8, which introduces an important tool, the homology groups.

5 Chapter 8 - Simplicial Homology

sec:Simplicial-Homology

We first notice that

$$\pi_1(S^n) = \pi_1(\{*\}) = \{1\} \tag{5.0.2}$$

for $n \geq 2$. But clearly S^n is not homotopic to the one point space $\{*\}$, i.e. S^n is not contractible! Therefore, the fundamental homotopy group does not solve the problem of determine the homotopy of a space. The homology groups will be an new direction to go.

The details is covered in page 173 of [1]. Sadly, even if one can computes all the homology groups (or all the homotopy groups), one cannot determine uniquely the homotopy type of the space in general, unless one has some additional conditions. For example, we have the Whitehead's Theorem: Given a continuous map from $X \to Y$, two "good" topological spaces, if f induces isomorphisms on all homotopy group $\pi_n(X) \to \pi_n(Y)$, $(n \geq 1)$, and induces bijection on $\pi_0(X) \to \pi_0(Y)$, then f is a homotopic equivalence. Is there similar good theorem for homology groups? Perhaps there is, but it is still not good enough for we still need to find such a f.

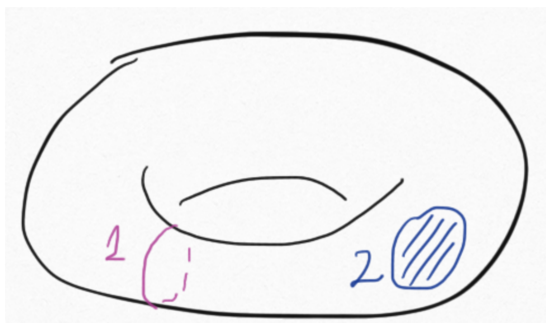
Anyway, let's proceed to the discussion of simplicial homology. **Note**: all the following discussions are based on a given triangulation K of the space.

5.1 Section 8.1 - Cycles and boundaries

sec:Cycles-and-boundaries

Geometrically, the homology groups aims to encode the information about k-dimensional holes in a space, as holes are important in determining the homotopy type of a space.

Then we notice that the boundary and bulk relation is also important. What I mean is in the following picture. The first loop encircles a part that does not belong to the torus, whereas the second loop encloses a solid part in the torus. One sees that the first loop is not nullhomotopic, i.e. it cannot be homotopic contracted to the trivial loop. On the other hand, loop 2 can. Look more generally, if a loop is the boundary of simple part (e.g. a simplex), then it is homotopic to the trivial loop. In this case we physicists can regard it as the boundary of a good bulk, g simplex. However, if a loop has empty bulk, i.e. it encloses no simplex, it is not a boundary of something, then it will be a nontrivial element in the fundamental group. This is why boundaries are important in our consideration.



Before discussion, I should make clear that if we only care about information of holes of certain dimension q, then we only have to care about composition of simplexes of the same dimension q that encloses certain area (be it empty or not). For example, in fundamental groups, we only care about holes looks like S^1. So in most cases we only care about series of 1-simplexes that is connected to form a loop, as in the case of edge groups. We call these things that we care about a closed q-cycles, to be defined precisely later. The set of all closed q-cycles are denoted Z_q.

Next is how to consider boundaries algebraically. For example, give a set of vertices, how to seperate the boundary out of this? This is especially troublesome given that we will have boundaries of different dimensions. There are the 0-dimension boundary of endpoints, the 1-dimension boundary of lines enclosing area, the 2-dimension boundaries called faces.

The boundary operator ∂ aims to solve this problem, it gives a systematic way to obtaining $(d-1)$-dimension boundary from a d-dimension simplex. The simple examples are provided in page 174 to 175 of [1].

Lastly, as mentioned, closed q-cycles that is the boundary of some solid part may become trivial, they are called q-boundaries. Closed q-cycles that are not the boundary a solid part will be nontrivial. Therefore, we might want to consider a quotient space:

$$H_q = Z_q/B_q$$

I believe I have done a good job in summarizing the key point, now is time to read the first section to get some practical ideas.

sec:Homology-groups

5.2 Section 8.2 Homology groups

This section formalize the intuitions from the previous section. It defines **oriented simplex** , i.e. simplex denoted by a specific $\{v_0, v_1, \vdots\}$ in this order. An change of order will change its orientation. Such change of orientation is denoted by a minus sign, and we have:

$$(v_0, v_1, \cdots) = \operatorname{sign}\theta(v_{\theta(0)}, v_{\theta(1)}, \cdots) \tag{5.2.1}$$

where θ is any permutation.

Then $C_q(K)$ is the free abelian group generated by the oriented q-simplexes of K, subject to the relations $\sigma + \tau = 0$, whenever σ and τ are just the same simplex with opposite orientations. An element of this group is called a **q-chain** . Note that the rank of $C_q(K)$ is equal to the number of q-simplexes in K.

The boundary homomorphism ∂ is defined such that

$$\partial(v_0, \cdots, v_q) = \sum_{i=0}^{q}(-1)^i(v_{(0)}, \cdots, \hat{v}_{(i)}, \cdots, v_{(q)}) \tag{5.2.2}$$

where $(v_{(0)}, \cdots, \hat{v}_{(i)}, \cdots, v_{(q)})$ is shorthand for the oriented $(q-1)$-simplex obtained by deleting the vertex v_i. It is easy to verify that $\partial^2 = 0$

In the special case when $q = 0$, we define that boundary of a single vertex is 0 and that $C_{-1}(K) = 0$.

We define the closed chains $Z_q(K)$ as

$$Z_q(K) = \operatorname{Ker}(\partial : C_q(K) \to C_{q-1}(K)). \tag{5.2.3}$$

And the boundary $B_q(K)$ is then

$$B_q(K) = \operatorname{Im}(\partial : C_{q+1}(K) \to C_q(K)). \tag{5.2.4}$$

B_q will be a subgroup of Z_q because $\partial^2 = 0$, this allows us to define the **q-th homology group of K** as:

$$H_q(K) = Z_q(K)/B_q(K) \tag{5.2.5}$$

The elements of $H_q(K)$ are written $[z]$ for $z \in Z_q$, called the **homology class** of z. Two q-cycles differ by a boundary will have the same homology class and is called **homologous cycles** . We note that $H_q(K)$ is by definition abelian and finitely generated.

Of course, we need to verify various things in the above statements, they can be found in the textbook [1].

sec:Section 8.3 Examples

5.3 Section 8.3 Examples

I will mention two theorems first:

Theorem 5.1. *$H_0(K)$ is a free abelian group whose rank is the number of path-connected components of $|K|$ (or just components, as $|K|$ is path-connected).*

Proof. This is quite obvious. □

Theorem 5.2. *If $|K|$ is connected, then $H_1(K)$ is isomorphic to the abelianization of its fundamental group $\pi_1(K)$.*

Proof. Found on page 182 of [1]. □

The two theorems indicates again that those homology groups are independent of the choice of triangulation, and they should be topological invariants.

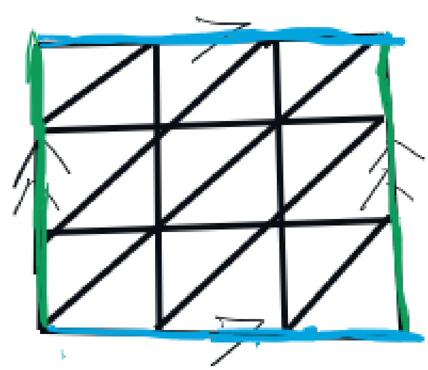
Then I will turn attention to specific examples.

key:2cycle-orientation

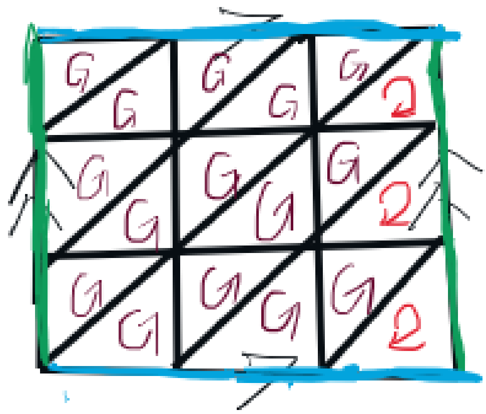
Key Point 5.1. Observe that every cycle is actually a closed object. It encircles or encloses an aresa. A closed 1-cycle is actually a loop. A closed 2-cycle will always enclose an area, so it is actually a sphere. A closed 2-cycle will give an orientation on its surface. For example, suppose that we have a closed 2-cycle $\lambda_i \sigma_i$, and assume that there is no repeated 2-simple in this collection. For a given 2-simplex σ_1 in it, there will have to be 3 simplexes surrounding it to cancal out σ_1's edges. These 4 2-simplexes are said to have compatible orientation. If one draw them on a paper, one will see that they are of the same orientation. Continue this argument to include all simplexes and one will see this 2-cycle actually gives an orientation on its surface.

Example 5.1. Torus

Consider the triangulation of torus as in exmaple 4.6.



The zeroth homology group $H_0(K)$ marks the connected components, hence it is $H_0(K) = \mathbb{Z}$. The first homology group H_1 is about those closed loops. Almost all loops are boundaries. But there are two (the green one and the blue one), each of which cannot be a boundary. For example, I managed to make the orientation such that the green one is not cancelled, but this leaves a part of the blue one also un-cancelled.

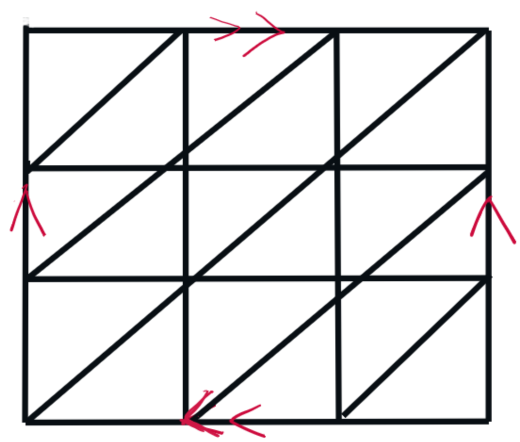


This two loops generates the $H_1(K) = \mathbb{Z} \times \mathbb{Z}$, and are also the non-trivial objects in its fundamental group (a coindence, $\pi_1(T^2) = H_1(T^2)$).

The 2-cycles are easy to find. Because our space torus has no boundary, we have to add more and more 2-simplexes until the whole space is covered with a 2-chain. Then in this case, there is only one generator for $Z_2(K)$. Also, there is not 3-simplex, so the $B_2 = 0$. Hence $H_2(K) = Z_2(K) = \mathbb{Z}$.

Example 5.2. Klein bottle T^2

As for the Klein bottle, I have had a hard time getting and confirming that the following is a triangulation: [2]



The $H_0(K)$ is clearly $\mathbb{Z}$.

The 1-cycles on this triangulation are hard to analyse. So I will use the fact that its $\pi_1 = \langle a, b | ba = a^{-1}b \rangle$. The commutator is generated by

[2] I found two articles talking about minimal triangulations of Klein bottle. They are: link1 and link2. The pdf files is also included.

$aba^{-1}b^{-1} = a^2$ and $bab^{-1}a^{-1} = a^{-2}$. So in the abelianization, $[a] = [a^{2n+1}]$, $[a^{2n}] = [1]$. So $[a]$ generates $\mathbb{Z}_2$. The $[b]$ is not affected, so $[b]$ generates $\mathbb{Z}$. It's easy to confirm that $[ab] = [ba]$, i.e. the abelianization group is abelian. All in all, $H_1(K) = \mathbb{Z} \times \mathbb{Z}_2$.

The $H_2(K) = 0$, since there is no way to orient the Klein in a reasonable way (see key point 5.1).

Example 5.3. Cone Space

Now the book considers a complex K which is a cone, i.e. K is isomorphic to a complex of the form CL where the dimentions of L is $(\dim K-1)$.

It is easy to see that $H_0(K) = \mathbb{Z}$, since cone is always connected. It is shown in page 181 of book [1] that $H_n(K) = 0$, for $n \geq 1$. This is reasonable because cones are actually contractible, so there is not much interesting information about higher dimensional spheres in cones to be recorded. The book shows this by constructing a map $d : C_q(K) \to C_{q+1}(K)$, such that

$$\partial d(\sigma) = \sigma - d\partial(\sigma) \tag{5.3.1}$$

It's easy to see that this means every closed cycle is a boundary.

Example 5.4. $(n + 1)$-simplexes and $\sum^n \cong S^n$

Let Δ^{n+1} denotes a $(n + 1)$-simplex. In this case, we consider it as a simplicial complex with all of its faces. Let $\sum^n$ denotes the simplexes which lies in the boundary of this $(n + 1)$-simplex Δ^{n+1}. Notice this is equivalent to say that $\sum^n$ is the all remaing simplex obtained by removing the interior points of Δ^{n+1}. For example, $\sum^1$ is the 3 edges and 3 vertices of the triangle. We see that $\sum^n$ is isomorphic to S^n.

Every Δ^{n+1} can be thought of as a cone (except when $n = -1$, which is too trivial to be discussed here). So the homology classes of Δ^{n+1} are $H_0(\Delta^{n+1}) = \mathbb{Z}$, $H_q(\Delta^{n+1}) = 0$, $q \geq 1$.

The homology classes of $\sum^n$ can be calculated using Δ^{n+1}, because they differ by only a single $n + 1$-simplex, then most of their chains are the same.

When $n = 0$, the case is trivial and $H_0(\sum^0) = \mathbb{Z} \oplus \mathbb{Z}$, $H_q(\sum^0) = 0$ for $q \geq 0$.

When $n \geq 1$, then since H_q does not involve simplexes of dimension higher than $q+1$, we have $H_q(\sum^n) = H_q(\Delta^{n+1})$, for $0 \leq q \leq n-1$. Hence $H_0(\sum^n) = \mathbb{Z}$. $H_q(\sum^n) = 0$, for $1 \leq q \leq n - 1$. By using $H_n(\Delta^{n+1}) = 0$, it is easy to get $H_n(\sum^n) = \mathbb{Z}$. (book [1] page 182). The rest of the Homology groups will of course be 0.

It is interesting to check that $H_n(\sum^n) = \mathbb{Z}$, not 0 for a one point space. If we are able to show that Homology group is a topological invariant, then we know now Homology groups can be used to distinguish spheres with a point.

5.4 Section 8.4 Simplicial maps

sec:Simplicial-maps

Now we extend a simplicial map $K \to L$ to a homomorphism between homology groups. This is easily done by first constructing $s : C_q(K) \to C_q(L)$, and then confirming $s\partial = \partial s$, or just $[s, \partial] = 0$. This is a very general procedure and the condition will be frequently seen in Homological algebra or Cohomological algebra. In face, let me first introduces Homological concepts first.

The collection of groups and homomorphisms:

$$\cdots \xrightarrow{\partial} C_q(K) \xrightarrow{\partial} C_{q-1}(K) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(K) \xrightarrow{\partial} 0$$

will be called a **chain complex** of K, written $C(K)$. A **chain map** $\phi : C(K) \to C(L)$ is a set of group homomorphisms $\phi : C_q(K) \to C_q(L)$. Such a map will induce a homomorphism of Homology groups when $\phi\partial = \partial\phi$ (easily proved). This induced homomorphism between Homology groups is dentoed ϕ_*.

In our specific case where s are simplicial maps $K \to L$, it is easy to extend it to $s : C_q(K) \to C_q(L)$, since s has already been required to map simplexes to simplexes. We only need to pay attention when σ a generator (i.e. a q-simplex) of $C_q(K)$, is mapped to a simplex of inferior quaility, i.e. of lower dimension $< q$. Then it is nature to define $s(\sigma) = 0$ in this case. It is verifed (page 185 of [1]) that this map is well defined, is a homomorphism, and has the property $s\partial = \partial s$. So s induces a homomorphism $s_* : H_q(K) \to H_q(L)$.

5.5 Section 8.5 Stellar Subdivision and Section 8.6 Invariance

ar-Subdivision-Invariance

Let me draw a **general blueprint**. We aims to show that simplicial homology groups are actually a topological property. The ultimate aim is to prove these theorems

Theorem 5.3. *Any continuous function $f : |K| \to |L|$ induces a homomorphism $f_* : H_q(K) \to H_q(L)$ in for each $q \geq 0$.*

Theorem 5.4. *If f is the identity map of $|K|$, then each $f_* : H_q(K) \to H_q(K)$ is the identity homomorphism, and if we have two maps*

$$|K| \xrightarrow{f} |L| \xrightarrow{g} |M|$$

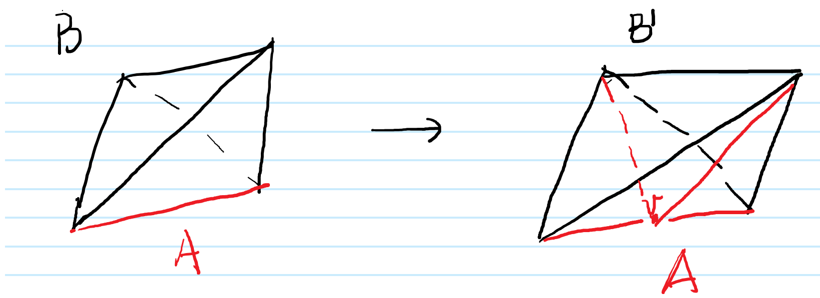
then $(g \circ f)_* = g_* \circ f_* : H_q(K) \to H_q(M)$ for all $q \geq 0$.

Theorem 5.5. *If $f, g : |K| \to |L|$ are homotopic maps, then $f_* = g_* : H_q(K) \to H_q(L)$ for all q.*

Corollary 5.1. *If $|K|$ and $|L|$ have the same homotopy type, then K and L have isomorphic homology groups.*

These theorems can be found on page 189 of the book [1]. But unfortunately, this course is too limited to talk about their proof in detail. So here we provide the general picture. The starting point will be of course, approximating a continuous function f with a simplicial map s. But in doing so, we have to do barycentric subdivision on the codomain. So if we can ensure that the homology groups are unaltered in barycentric subdivision, then of course simplicial maps will preserve homology groups, as we have seen in section 5.4. And it is nature, though it requires proof, that the function f approximated by s will also have the same property of preserving simplicial maps.

This section (section 8.5) deals with proving that the barycentric subdivision does not alter Homology groups. But it uses a simpler step. In general, a lot more vertices will be introduced in one barycentric subdivision. But we can look at a simper notion first, i.e. a **stellar subdivision** . Stellar meanings relating to stars, and a stellar subdivision quite resembles a star's shining lines. A first example is given on top of page 186 of [1]. Here I reproduced the figure 8.6 in a more "star" like depiction:



The benefit of stellar subdivision is that one can do a complete barycentric subdivision through performing a series of steller subdivision in a specific order. Amazingly the book [1] do not provide a proof of this, except that it gives an example using figure 8.7 on page 186. But the proof should be easy.

I have omitted a precise description of stellar subdivision here, because it is best understood along with the process to obtain a barycentric subdivision through stellar subdivision. They are found in page 186 of [1]. Just keep in mind that $A_0 < A_1 < \cdots < A_k$, for a simplex $\{\hat{A}_0, \hat{A}_1, \cdots, \hat{A}_k\}$ in a barycentric subdivision.

Then the book proves that

Theorem 5.6. *If K' is obtained from K by a single stellar subdivision, then K' and K have isomorphic Homology groups*

and hence

Corollary 5.2. *Barycentric subdivision does not change the Homology groups of a complex*

The proof can be in page 186 to 188 of [1]. I also omit the proof here, but I warn anyone who wish to study the prooves in the next section to study the proofs here carefully, because a important map (subdivision chain map) will be constructed here.

As an effort to prevent this tacit talk from being a transient memory, I will mention several applications.

Remark 5.1. First, Homology group is a homotopic property. This means we can calculate the Homology group of a big complex space by shrinking it into a simple space.

Key Point 5.2. The Homology groups of S^n are:

$$H_0(S^n) = H_n(S^n) = \mathbb{Z}$$

$$H_q(S^n) = 0, \text{for } q \neq 0, n$$

Theorem 5.7. *S^m is not homotopic to S^n if $m \neq n$.*

Proof. Obviously. □

Theorem 5.8. *Two Euclidean spaces are homeomorphic if and only if they have the same dimension*

Proof. As mentioned in example 3.6, the homotype type of S^{m-1} and $\mathbb{E}^m \setminus \{0\}$ are the same. If there is a homotopic equivalence from $\mathbb{E}^m \to \mathbb{E}^n$, it would induces homotopic equivalence on S^m and S^n. Hence $\mathbb{E}^m$ is homotopic to $\mathbb{E}^n$ if and only if $m = n$. □

Theorem 5.9 (Brouwer fixed-point theorem)**.** *A map from B^n to itself must leave at least one point fixed.*

Proof. The proof is exactly as it is in theorem 3.5, except that we have to use Homology groups instead of the fundamental group to distinguish different spheres. □

6 Chapter 9 - Degree and Lefschetz Number

egree-and-Lefschetz-Number

Professor Li Qin skipped many parts before. In this chapter, he also mentioned only important theorems in the first two sections.

The language in this note will be quite loose. Proofs will be omitted, so will some theorems, as this was how my teacher taught me. I am not sure if there are more valuable techniques in these sections. Hence this can be only regarded as a guideline/introduction to the book.

6.1 Section 9.1 Maps of spheres

sec:Maps-of-spheres

In this section, we restrict our attention only on spheres S^n. The goal is investigate the homotopy classes of continuous functions between spheres. We will see, in the end, that we can obtain some information by examining the how these continuous functions acting on a simple triangulation of sphere.

We first define the degree of a continuous function $f : S^n \to S^n$. Since $H_n(S^n) = \mathbb{Z}$, we define $\deg f$ to be the number $f_*(1)$, in $f_* : H_n(S^n) \to H_n(S^n)$. Or, equivalently, the number λ such that $f([z]) = \lambda[z]$, where $[z]$ is the generator of $H_n(S^n) = \mathbb{Z}$.

Then we discover several facts:

Fact 6.1. The degree does not depend on the choice of triangulation of sphere.

Fact 6.2. Homotopic continuous functions have the same degree.

Fact 6.3. $\deg(f \circ g) = \deg f \times \deg g$

Fact 6.4. By above fact, if f is a homeomorphism, $\deg f = \pm 1$.

Fact 6.5. Obvious, the degree of identity map of S^n is 1, that of constant map is 0. Hence the identity map is not homotopic to the constant map.

The above facts can be found in page 195 of [1].

Since degree is triangulation independent, we will now stick to a very convenient triangulation constructed in page 196 of [1]. It is the most straightforward one, and is denoted by Σ (remember, we have constructed similar things in chapter 8, denoted Σ^n). The vertices are $v_i = (0, \cdots, 0, 1, 0, \cdots, 0)$, for $i = 1, 2, \cdots, n+1$, and the only 1 is on the i-th coordinate. Similary we have $v_{-i} = 0, \cdots, 0, -1, 0, \cdots, 0)$. Whenever $|i_1| < |i_2| < \cdots < |i_s|$, with $1 \leq s \leq n+1$, the vertices $v_{i_1}, v_{i_2}, \cdots, v_{i_s}$ spans a simplex in Σ. The collection of all such simplexes constitutes the simplicial complex Σ. A graphical illustration can be found on page 196 of [1].

The degree of a continuous function f can than be easily determined on this triangulation. But I will only mention that this is demonstrated in theorem (9.1) on page 196 of [1].

Then we discover a sequel of facts, each depending on the previous one:

Fact 6.6. The antipodal map of S^n has degree $(-1)^{n+1}$

Fact 6.7. A continuous function $f : S^n \to S^n$ which has no fixed points must have degree $(-1)^{n+1}$.

Fact 6.8. If n is even, and if $f : S^n \to S^n$ is homotopic to the identity, then f has a fixed point.

Using this we can derive an interesting theorem:

Theorem 6.1 (Hairy ball theorem)**.** *S^n admits a continuous nonvanishing vector field if and only if n is odd.*

Proof. Let v be a vector field on S^n, for each $v(\mathbf{x})$, it can be regarded just as a normal vector in $\mathbb{E}^{n+1}$ since each tangent space is locally a $\mathbb{R}^{n+1}$.

The continuous function:

$$f(\mathbf{x}) = \frac{\mathbf{x} + v(\mathbf{x})}{||\mathbf{x} + v(\mathbf{x})||}$$

is homotopic (hint, add a parameter t in front of $v(\mathbf{x})$), to identity map. Hence $f(\mathbf{x})$ admits a fix point, i.e. there is a point $\mathbf{x}$ such that $v(\mathbf{x}) = 0$. □

Notice that in proof above, the function $f(\mathbf{x})$ actually help us regard a vector filed as a speed field, a field that moves each point on the sphere infinitesimally.

sec:Anchor

7 Anchor

References

book [1] M.A. Armstrong. Basic Topology. 2ed.

Singer.Thorpe [2] I.M. Singer, J.A. Thorpe. Lecture Notes on Elementary Topology and Geometry. UTM.

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