# Dirac Hamiltonian

Taper

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**Abstract**

This aims for recapitulate important information about Dirac Hamiltonian as is written in Sakurai's text [SN11], chapter 8.[1] This is intended to learn to help me deal with a Lattice Dirac Hamiltonian.

## Contents



The starting point is still the Schrodinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle \tag{0.0.1}$$

We will be using **Natural Units** from now on.

## 1 Klein-Gorden Equations

sec:Klien-Gorden Equations

The Klein-Gorden equation starts with the the relativistic energy (of a particle with momentum $\mathbf{p}$ and mass $m$):

$$E_p = +\sqrt{p^2 + m^2} \tag{1.0.2}$$

The plus sign here hints that we may have a negative energy solution.

We do not quantize this equation directly, because the square root is hard to represent in operators. Instead, Klein-Gorden equation starts with $H^2$:

$$H^2 = p^2 + m^2 \overset{\text{quantized as}}{\rightarrow} -\boldsymbol{\nabla}^2 + m^2 \tag{1.0.3}$$

On the other hand, the Schrodinger equation applying twice gives:

$$-\frac{\partial^2}{\partial t^2} = H^2 \tag{1.0.4}$$

[1] Be careful that some version of this book is missing chapter 8... I don't understand why...

Hence we have the Klein-Gorden equation:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\psi(x,t) = 0 \tag{1.0.5}$$ eq:kg-eq

It can be simplified in 4-vector notation as:

$$\left(\partial_\mu \partial^\mu + m^2\right)\psi(x,t) = 0 \tag{1.0.6}$$ eq:kg-eq-4vec

or most succinctly as,

$$\left(\partial^2 + m^2\right)\psi(x,t) = 0 \tag{1.0.7}$$

The author looked at this equation from three different aspects.

**Producing the free particle solution** It is first checked that the free-particle state:

$$\psi \propto \exp(-i(Et - \mathbf{p}\cdot\mathbf{x})) = e^{-ip^\mu x_\mu} \tag{1.0.8}$$

is a solution to the Klein-Gorden equation, provided that:

$$E^2 = \mathbf{p}^2 + m^2 \tag{1.0.9}$$

**Probability density** By analogy with non-relativistic case, the four-vector current is defined as:

$$j^\mu = \frac{i}{2m}[\psi^* \partial^\mu \psi - (\partial^\mu \psi)^* \psi] \tag{1.0.10}$$

It is easy to show $\partial_\mu j^\mu = 0$[2]. Then, this is a conserved current. The time-component of this is the density:

$$\rho \equiv j^0 = \frac{i}{2m}\left[\psi^* \partial_t \psi - (\partial_t \psi)^* \psi\right] \tag{1.0.11}$$

If one check this equation, one find that $\rho$ is proportionally to the imaginary of some number which, one has no clear reason to say whether it is positive or negative. Therefore, the Klein-Gorden equation leaves one in a bad position to interprate the probability nature of quantum mechanics (what is a "negative probability" anyway?).

**Coupling to electromagnetic field** The author discusses a little bit about why, when there is electromagnetic field, we have the replacement

$$p \rightarrow p + eA \tag{1.0.12}$$

in page 490, footnote (be careful that the author uses the notation $e = |e|$, whereas I used the notation $e = -|e|$, i.e. $e$ is just $q$, the charge).

This coupling effectively gives the replacement:

$$\partial_\mu \quad \rightarrow \quad D_\mu \tag{1.0.13}$$

where $D_\mu \equiv \partial_\mu - ieA_\mu$.

The Klein-Gorden equation now becomes:

$$[D_\mu D^\mu + m^2]\psi(x,t) = 0 \tag{1.0.14}$$ eq:kg-emfield

This equation revels a connection between $e \rightarrow -e$ and $\psi \rightarrow \psi^*$[3].

[2] Just note that normally we have $A^\mu B_\mu = A_\mu B^\mu$

[3] You will find this if you complex conjugate the Klein-Gorden equation.

**Two component interpretation** The author also mentions a way to split the equation into two equations. If we define $\phi(x,t)$ and $\chi(x,t)$ as in eq.(8.1.15) in p.491, then it is straightforward to check that, the Klein-Gorden equation is equivalent to:

$$iD_t\Upsilon = \left[-\frac{1}{2m}\mathbf{D}^2(\tau_3 + i\tau_2) + m\tau_3\right]\Upsilon \tag{1.0.15}$$

where $(D_t, \mathbf{D}) = D_\mu$, $\tau_i$ are pauli matrices, and $\Upsilon$ is the two component wave function:

$$\Upsilon \equiv \begin{pmatrix}\phi(x,t)\\ \chi(x,t)\end{pmatrix} \tag{1.0.16}$$

This formulation enables a mysterious interpretation of negative energy, negative probability, and particles and antiparticles. For more information, please look at pp.491 494 of [SN11].

## 2 Dirac Equation

sec:Dirac Equation

The Dirac equation aims to change the Klein-Gorden equation into a first order PDE. There are two approaches to derive the equation. The first can be found in section 8.2 of [SN11], and section II.1 of [Zee10]. The second approach is found in page 99 (chapter 2) of [Gre97], and is preferred in my opinion. After all, they lead to the same equation:

$$(i\gamma^\mu\partial_\mu - m)\psi(x,t) = 0 \tag{2.0.17}$$

eq:dirac-eq

Where $\gamma^\mu$ are matrices satisfying:

$$\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu} \tag{2.0.18}$$

In other words, $\gamma^\mu$ is a representation of the Clifford algebra $\mathrm{Cl}_{1,3}(\mathbb{R})$. We can cast this equation into another form:

$$i\frac{\partial}{\partial t}ket\psi = H\left|\psi\right\rangle = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m \tag{2.0.19}$$

where $\mathbf{p}$ is the momentum, $\boldsymbol{\alpha}$ and $\beta$ satisfy a similar relation:

$$\frac{1}{2}\{\alpha_i, \alpha_j\} = \delta_{ij} \tag{2.0.20}$$

$$\{\alpha_i, \beta\} = 0 \tag{2.0.21}$$

$$\alpha_i^2 = \beta^2 = 0 \tag{2.0.22}$$

They are related to $\gamma^\mu$ matrices by:

$$\alpha_i = \gamma^0\gamma^i \quad \text{and} \quad \beta = \gamma^0 \tag{2.0.23}$$

A detailed formula for $\gamma$ matrices can be found in p.94 of [Zee10] or in the section . A detailed formula for $\alpha_i, \beta$ matrices can be found in p.496 of [SN11].

**The density problem** The dirac formulation has a appearent advantage. The current defined by

$$j^\mu \equiv \bar{\psi}\gamma^\mu\psi \tag{2.0.24}$$

where

$$\bar{\psi} \equiv \psi^\dagger\beta \tag{2.0.25}$$

is conserved. And the probability density $\rho$ is

$$\rho \equiv j^0 = \psi^\dagger\beta\gamma^0\psi = \psi^\dagger\psi \tag{2.0.26}$$

It is positive-definite.

**Coupled with a charge** When coupled with electromagnetic field, and with a the vector potential $\mathbf{A} = 0$, it becomes

$$H = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m - e\Phi \tag{2.0.27}$$

where $\Phi$ is the scalar potential.

sec:Gamma-matrices

# 3 Gamma matrices

It could be quite easy to find representations of Clifford Algebra in any dimension. First I note that the size of the Gamma matrices of $4n \times 4n$, i.e., the dimension could/must [4] be $4n$. So we could have a very easy construction by repeatedly using the Pauli matrices. The Pauli matrices follow:

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \tag{3.0.28}$$

So the following $4 \times 4$ matrices are good Gamma matrices:

$$\gamma_{1/2/3} = \begin{pmatrix} & \sigma_{1/2/3} \\ -\sigma_{1/2/3} & \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} \tag{3.0.29}$$

Note the minus sign in the $-\mathbb{1}$ of $\gamma_0$ is important to produce the correct anticommutation relations.

Next, for higher dimension 8, we can set:

$$\gamma'_{1/2/3} = \begin{pmatrix} & \gamma_{1/2/3} \\ -\gamma_{1/2/3} & \end{pmatrix}, \quad \gamma'_0 = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} \tag{3.0.30}$$

And repeating the above process, we can get the Gamma matrices in any $4n$ dimension.

[4] I am not sure about this. In this [1] says it must be $4n$, the [2] says there could be Gamma matrix in odd dimension

# References


[Gre97] Professor Dr. Walter Greiner. *Relativistic Quantum Mechanics*. Springer Berlin Heidelberg, 1997. URL: `https://link.springer.com/book/10.1007{%}2F978-3-662-03425-5`, `doi:10.1007/978-3-662-03425-5`.

[SN11] J. J. (Jun John) Sakurai and Jim. Napolitano. *Modern quantum mechanics*. Addison-Wesley, 2011.

[Zee10] A. Zee. *Quantum Field Theory in a Nutshell: (Second Edition)*. In a Nutshell. Princeton University Press, 2010. URL: `https://books.google.co.jp/books?id=n8Mmbjtco78C`.


# References


[1] `https://physics.stackexchange.com/questions/53318/dimension-of-dirac-gamma-matrices`

[2] `http://demonstrations.wolfram.com/DiracMatricesInHigherDimensions/`


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