Solution for HW9

Taper

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Abstract

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Problem 1

For any smooth function f of $\vec{x}$, we have

$$\langle\vec{x}|\, f(\hat{\vec{x}})\, |\vec{x'}\rangle = \langle\vec{x}| \sum_{n=0}^{\infty} c_n \hat{\vec{x}}^n \, |\vec{x'}\rangle = \sum_{n=0}^{\infty} c_n \vec{x}^n \delta(x - x') = f(\vec{x})\delta(\vec{x} - \vec{x}')$$

Also use the equation (190) in the lecture notes:

$$\langle\vec{x}| \frac{P^2}{2m} |\psi\rangle = -\frac{\hbar^2}{2m}\nabla^2 \langle\vec{x}|\psi\rangle \tag{0.0.1}$$

The Schrödinger's equation gives

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t} \langle\vec{x}|\psi\rangle &= \langle\vec{x}|H|\psi\rangle \\
&= \langle\vec{x}| \frac{P^2}{2m} + V(\vec{x}) \,|\psi\rangle \\
&= -\frac{\hbar^2}{2m}\nabla^2 \langle\vec{x}|\psi\rangle + \int \mathrm{d}\vec{x}' \, \langle\vec{x}|\, V(\vec{x}) \,|\vec{x}'\rangle \langle\vec{x}'|\psi\rangle \\
&= -\frac{\hbar^2}{2m}\nabla^2 \langle\vec{x}|\psi\rangle + \int \mathrm{d}\vec{x}' \, V(\vec{x})\delta(\vec{x} - \vec{x}') \langle\vec{x}'|\psi\rangle \\
&= -\frac{\hbar^2}{2m}\nabla^2 \langle\vec{x}|\psi\rangle + V(\vec{x}) \langle\vec{x}|\psi\rangle
\end{aligned}$$

Identifying $\psi(\vec{x}, t)$ and $|\vec{x}|\psi\rangle$, then the above equation is just the normal wave equation (263).

Problem 2

For simplicity, take $H_{11} = H_{22} = 0$. Then

$$H = \begin{pmatrix} 0 & H_{12} \\ 0 & 0 \end{pmatrix} \tag{0.0.2}$$

This matrix has rank 1, so it does not have a full eigenvector-spectrum. In addition, its eigenvalues are find by solving characteristic equation to be 0, which are not valid eigenvalues. So it does not have energy eigenstates. This means that the Hamiltonian is not likely to be physical.

For example, let's solve the Schrödinger's equation:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi\\ \phi\end{pmatrix} &= H\begin{pmatrix}\psi\\ \phi\end{pmatrix} \\ &= \begin{pmatrix}0 & H_{12}\\ 0 & 0\end{pmatrix}\begin{pmatrix}\psi\\ \phi\end{pmatrix} \\ &= \begin{pmatrix}H_{12}\phi\\ 0\end{pmatrix} \end{aligned}$$

We see: $\phi(t) \equiv \phi(0)$, $\psi(t) = -i\int_0^t \mathrm{d}t'\, H_{12}(t')\phi(0)/\hbar + c$. But $\psi(t)$ is not normalizable, becuase c is a constant while $-i\phi(0)t/\hbar$ diverges as $t \to \infty$. This means every measurable quantity (i.e when takeing the inner product $\langle|A|\rangle$) will diverge, clearly not physically possible if the system is allowed to evolve infinitely.

However, non-Hermitian Hamiltonian maybe used to describe a system which is not closed, such as a system which always dissipates energy to the environment, or which always absorbs energy from the environment. Such Hamiltonian will have complex eigenvalues, such as $E + i\varepsilon$ with $\{E, \varepsilon\} \subset \mathbb{R}^*$ (they must be non-zero). Then as seen in the time-evolution operator $e^{-i(E+i\varepsilon)t/\hbar} = e^{-iEt/\hbar}e^{\varepsilon t/\hbar}$, the system will decay (dissipates energy) or grow (absorbs energy) as time goes on, depending on the sign of ε.

As a final remark, non-Hermitian system in general does not preserve the probability, as would expected from the above argument. But this is also obvious in the following equality:

$$i\hbar\partial_t \langle\phi|\phi\rangle = \langle\phi|\, H - H^\dagger\, |\phi\rangle \tag{0.0.3}$$

This is equality can be obtained from following three equations:

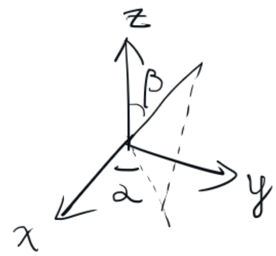
$$\begin{aligned} i\hbar\partial_t |\phi\rangle &= H|\phi\rangle \\ -i\hbar\langle\partial_t\phi| &= \langle\phi|\, H^\dagger \\ \partial_t\langle\phi|\phi\rangle &= \langle\phi|\,\partial_t\,|\phi\rangle + \langle\partial_t\phi|\phi\rangle \end{aligned}$$

Problem 3

In general, the eigenstate of $S \cdot \hat{n}$ with eigen value $\frac{\hbar}{2}$ is

$$e^{-iS_z\alpha/\hbar}e^{-iS_y\beta/\hbar}\,|+\rangle \tag{0.0.4}$$

where $|+\rangle$ in the canonical basis is $\begin{pmatrix}1\\ 0\end{pmatrix}$, α and β are shown here:



So in this case, we have

$$|\mathbf{n}\rangle = e^{-iS_y\gamma/\hbar}\,|+\rangle = e^{-i\sigma_y\gamma/2}\begin{pmatrix}1\\0\end{pmatrix} = (\cos(\gamma/2) - i\sigma_y\sin(\gamma/2))\begin{pmatrix}1\\0\end{pmatrix}$$
$$= \begin{pmatrix}\cos(\gamma/2)\\ \sin(\gamma/2)\end{pmatrix}$$

The eigenstate of S_x with eigenvalue $\hbar/2$ is $\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$ (setting $\gamma = \pi/2$), so the probability getting $\hbar/2$ when measuring S_x is:

$$|\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\cdot\begin{pmatrix}\cos(\gamma/2)\\ \sin(\gamma/2)\end{pmatrix}|^2 = \frac{1}{2}(\cos(\gamma/2)+\sin(\gamma/2))^2 \tag{0.0.5}$$

Also, the dispersion is

$$\begin{aligned}\langle(S_x - \langle S_x\rangle)^2\rangle &= \langle S_x^2\rangle - \langle S_x\rangle^2\\ &= \begin{pmatrix}\cos(\gamma/2) & \sin(\gamma/2)\end{pmatrix}\cdot\hbar^2/4\sigma_x^2\begin{pmatrix}\cos(\gamma/2)\\ \sin(\gamma/2)\end{pmatrix} - \left[\begin{pmatrix}\cos(\gamma/2) & \sin(\gamma/2)\end{pmatrix}\cdot\hbar/2\sigma_x\begin{pmatrix}\cos(\gamma/2)\\ \sin(\gamma/2)\end{pmatrix}\right]^2\\ &= \sin^2\left(\frac{\gamma}{2}\right) + \cos^2\left(\frac{\gamma}{2}\right) - 4\sin^2\left(\frac{\gamma}{2}\right)\cos^2\left(\frac{\gamma}{2}\right)\\ &= 1 - \sin^2(\gamma) = \cos^2(\gamma)\end{aligned}$$