

# Notes about the Minimal Dimensional Method to Classify Topological Phases (preliminary)

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## Abstract

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# 1 Outline

The classification of topological insulators in non-interacting system is in effect a classification of  $N \times N$  Hamiltonian matrix. The simple case considered here is the classification done by considering how the Hamiltonian matrix responds to those anti-unitary symmetries, when we effectively ignored all unitary symmetries. Specifically, when we module out any unitary symmetries of the Hamiltonian, the number of meaningful anti-unitary symmetries are limited, and we only classify topological insulators based on how our Hamiltonian matrix responds to these anti-unitary symmetries.

There are two classical approaches towards this classification. The first [SRFL08] is to examine the existence of Anderson delocalization at the boundary of the insulator. The existence of robust surface conducting state is the signature of topological insulators. When some random impurity potentials could be present in these surfaces, the Anderson localization is the obstruction that surface conducting states must overcome. This approach uses the Nonlinear Sigma Models (NL $\sigma$ Ms) to describe the surface Hamiltonian and consider the when can a localization-breaking term can be added to NL $\sigma$ Ms.

The second approach [Kit09] is to use see the classification as an extension problem of Clifford Algebras. The anti-unitary symmetries already possess some behavior similar to generators of Clifford Algebras. And sometimes Hamiltonians could be regarded as yet another generator of Clifford Algebra [MF13]. The  $K$ -theory approach is to put the symmetries in the Clifford Algebra first, and consider adding the Hamiltonian that preserve such symmetries to the Clifford Algebra. So the problem of classification of Hamiltonians becomes a problem of extending the Clifford Algebra.

But the key to understand topological insulators, are not necessarily the Hamiltonian of the whole bulk, but the physics happening when the bulk gap closes. For example, the Chern number is the integration on a compact manifold without boundary. Therefore, it should always be zero because of the Stokes theorem, unless inside this manifold there is somewhere where the formula for Chern number got "blown up", and it is just this locally "blown up" point (caused by the crossing) that is responsible for the nonzero Chern number, and for the non-triviality of topological insulators. More generally, we believe that there is a bulk-boundary correspondence that the properties of the bulk is reflected by the properties in the boundary.

Therefore, we can detect the different types of topological insulators when we are focused in only the crossing point of the band, or, when we shifted the energy zero level to the crossing point, in only the low energy physics. We expect a locally linear crossing energy spectrum around this point, so we expect  $E = \pm\sqrt{\mathbf{p}^2 + m^2}$ , with  $m$  being the parameter controlling the closing or opening of this spectrum. One simple model with only first-order derivatives to describe such a low energy physics is the Dirac Hamiltonian. Also, the traditional discussion of Klein-Gordon equation all tells us to look at Dirac Hamiltonian.

Using the Dirac Hamiltonian, it is discovered by Dr. Chiu [Chi13] that the classification of topological insulators could be brought done into very simple applications of classification and representation theory of Clifford Algebras, which have been well studied with excellent resources provided online. In his classification, he established a isomorphism from  $G_{\text{symmetries class}}$ , the algebra generated by symmetry operators and Dirac Hamiltonians, to some Clifford Algebras. Consequently, the

Hamiltonian and symmetry operators, all become some specific representations of their corresponding Clifford Algebras. Also, it will be argued later that the minimal matrix dimension of such representation, i.e. the minimal dimension of the Dirac Hamiltonian, can tell the specific type of this topological insulator. Therefore, as a whole all the classification work is reduced to the consideration of minimal matrix dimension of the representation of Clifford Algebra, in different spatial dimension.

This work may be considered as a note to Chiu's dissertation [Chi13]. However, I do not guarantee that my interpretation is the same as his as I have modified and amended some parts to make the work more, in my opinion, congruous and systematic.

**Structure of this work** This work are divided into several parts.

1. **Ten-fold Way** Introduces the result of Ten-fold way: why we care only about anti-unitary symmetries, why there are ten different classes.
2. **Classification by Dirac Mass Hamiltonians** gives comprehensive account of how the Minimal Dirac Hamiltonian approach works, and how the ten-fold way is derived in this approach.
3. **Looking Further** outlines the possible direction in classification under unitary symmetries, and some other results.

## 2 Ten-fold Way

We begin our discussion with a background before this classification. The classification of topological phases with different non-unitary symmetries in non-interacting picture, are in essence, the classification of  $N \times N$  matrices by its response to three discrete symmetries: time reversal symmetry ( $T$ ), charge conjugation/particle hole symmetry ( $C$ ), and chiral/sub-lattice symmetry ( $S$ ). As for those unitary symmetries, one can in principle, block diagonalize the Hamiltonian  $H = \text{diag}(H^{\lambda_1}, H^{\lambda_2}, \dots)$ , such that each block  $H^{\lambda_i}$  is labeled by irreducible representations  $\lambda_i$ , and it has no memory of the unitary symmetries. Therefore, our classification is applied to those blocks and still holds.

It will be find that, those  $T, C$  are the only two meaningful non-unitary symmetries of the Single-particle system, and there are in total 10 different ways in which the Single-particle Hamiltonian could respect the three symmetries  $T, C, S$ . Now we explain in detail, about the reason why we care about, and only care about these three symmetries  $T, C$ , and  $S$ .

### 2.1 The symmetries of Single-particle Hamiltonian

According to Wigner's theorem, symmetries of physical system can either be unitary represented, or anti-unitarily represented. Here we show that in the case of anti-unitary symmetries, there are only 2 different kinds of them that a Single-particle Hamiltonian in first-quantized space can have. Actually, there can be more anti-unitary symmetries than this. But under reasonable assumptions, we can limit our discussion in only this two types.

The Single-particle Hamiltonian acting on Fock space is

$$\hat{H} = \sum_{A,B} \hat{\psi}_A^\dagger H_{AB} \hat{\psi}_B \quad (2.1.1)$$

eq:H-2nd

where  $H_{AB}$  are just complex numbers,  $\hat{\psi}_A^\dagger$  and  $\hat{\psi}_B$  are creation and annihilation operators acting on Fock space.

Here and henceforth, we will add a hat to all operators in Fock space to stress that it acts on Second-quantized Fock space. A symmetry of the Hamiltonian, represented as an operator  $\hat{U}$ , must have:

$$\hat{U}\hat{H}\hat{U}^{-1} = \hat{H} \quad (2.1.2)$$

eq:sym-in-2nd-1

We expect  $\hat{U}$  to change the creation/annihilation operators in two different ways. First, it may only permute the creation/annihilation operators:

$$\hat{\psi}'_A = \hat{U}\hat{\psi}_A\hat{U}^{-1} = \sum_B (u^\dagger)_{AB} \hat{\psi}_B \quad (2.1.3a)$$

eq:sym-in-2nd-permute-1

$$\hat{\psi}'^\dagger_A = \hat{U}\hat{\psi}^\dagger_A\hat{U}^{-1} = \sum_B \hat{\psi}^\dagger_B u_{AB} \quad (2.1.3b)$$

eq:sym-in-2nd-permute-2

Where  $u$  is some matrix implementing the permutation. Or, it may interchange the role of creation/annihilation operators:

$$\hat{\psi}'_A = \hat{U}\hat{\psi}_A\hat{U}^{-1} = \sum_B (u^*)^\dagger_{AB} \hat{\psi}^\dagger_B \quad (2.1.4a)$$

eq:sym-cc-1

$$\hat{\psi}'^\dagger_A = \hat{U}\hat{\psi}^\dagger_A\hat{U}^{-1} = \sum_B \hat{\psi}_B u^*_{BA} \quad (2.1.4b)$$

eq:sym-cc-2

Where  $u^*$  is some other matrix implementing the interchange. We write complex conjugation  $u^*$  instead of  $u$  for convenience. In both cases (permute or interchange), to conserve the anticommutation relation between  $\hat{\psi}_A^\dagger$  and  $\hat{\psi}_B$  operators, one can easily show that  $u$  should be a unitary matrix.

Also,  $\hat{U}$  may be linear or anti-linear. The different combinations of these conditions give us 1 unitary symmetry, 2 anti-unitary symmetries, and 1 special symmetry, to be explained below.

**Case 1: Unitary Symmetry** Assume the symmetry just permutes the creation/annihilation operators, as in equation 2.1.3, and assume it is linear in Second-quantized Hamiltonian, i.e.  $\hat{U}i\hat{U}^{-1} = i$ . The permutation relation 2.1.3 plugged into equation 2.1.2, gives

$$uHu^{-1} = H \quad (2.1.5)$$

Therefore, in this case the symmetry is unitarily realized in First-quantized Hamiltonian  $H_{AB}$ .

**Case 2: Anti-unitary T Symmetry** Assume the symmetry just permutes the creation/annihilation operators, as in equation 2.1.3, but assume it is anti-linear in Second-quantized Hamiltonian, i.e.  $\hat{U}i\hat{U}^{-1} = -i$ . The permutation relation 2.1.3 plugged into equation 2.1.2, gives a different result, since  $\hat{U}H_{AB}\hat{U}^{-1} = H_{AB}^*$  now.

$$uH^*u^\dagger = H \quad (2.1.6)$$

Or:

$$uKHu^tK = uKH(uK)^{-1} = H \quad (2.1.7)$$

where  $K$  is complex conjugation. This symmetry is called Time-reversal symmetry, and is realized in First-quantized Hamiltonian as an anti-unitary operator  $T = uK$ .

**Case 3: Anti-unitary C Symmetry** Assume the symmetry just interchange the creation/annihilation operators, as in equation 2.1.4, but assume it is linear in Second-quantized Hamiltonian, i.e.  $\hat{U}i\hat{U}^{-1} = i$ . The interchange relation 2.1.4 plugged into equation 2.1.2, gives: <sup>1</sup>

$$u(H - \frac{1}{2} \text{tr}(H))^t u^\dagger = -(H - \frac{1}{2} \text{tr}(H)) \quad (2.1.8) \quad \boxed{\text{eq:sym-C-cond}}$$

Taking the trace of above equality will give  $2 \text{tr}(H) = N \text{tr}(H)$ , since in solids  $N \gg 2$ , we must have  $\text{tr}(H) = 0$ . Then the above equality simplifies into <sup>2</sup>:

$$uH^*u^\dagger = uKH(uK)^{-1} = -H \quad (2.1.9)$$

This type of symmetry is called charge-conjugation symmetry. It is also called particle-hole symmetry in condensed matter physics. It is realized as  $C = uK$  with  $CHC^{-1} = -H$  for 1st-quantized single-particle Hamiltonian.

**Case 4: Unitary S Symmetry** Assume the symmetry now interchange the creation/annihilation operators, as in equation 2.1.4, and assume it is anti-linear in Second-quantized Hamiltonian, i.e.  $\hat{U}i\hat{U}^{-1} = -i$ . The interchange relation 2.1.4 plugged into equation 2.1.2, gives:

$$u(H - \frac{1}{2} \text{tr}(H))u^\dagger = -(H - \frac{1}{2} \text{tr}(H)) \quad (2.1.10) \quad \boxed{\text{eq:sym-S-cond}}$$

and since  $N \gg 2$ , we have  $\text{tr}(H) = 0$ . Then

$$uHu^\dagger = -H \quad (2.1.11)$$

This symmetry will be called the chiral symmetry, denoted  $\hat{S} = u$ . It is unitarily realized in First-quantized Hamiltonian, but since  $\{S, H\} = 0$  instead of  $[S, H] = 0$ , it is not a traditional symmetry that we are used to. Also, it is easy to see that  $S$  is a combination of  $T$  and  $C$ ,<sup>3</sup> and the symmetry property of Hamiltonian under  $T$  or  $C$  uniquely defined the symmetry property of Hamiltonian under  $S$ . But there is one exception. When the Hamiltonian does not obey  $T$  and  $C$ , it may or may not obey  $S = TC$  as a whole.

**Squaring of  $T, C, S$**  Squaring of  $T/C$  symmetry operators should be proportional to identity  $\mathbb{1}$ , hence it is only a phase as they are unitary. This can be viewed from two perspectives. First, we expect the system, after applying twice of symmetry operation  $T/C$ , should come back to the same state, except a possible phase difference. Second,  $T^2/C^2$  commutes with all  $T/C$  symmetric Hamiltonians (easy to derive) in all irreducible representations of unitary symmetries (to be explained later), therefore by Schur's lemma, they must be proportional to a constant, which is a phase.

It is easy to find that this phase  $e^{i\delta}$  should be  $\pm 1$ . For example, let  $T = uK$ , then  $T^2 = uu^*$ , and  $(uu^*)u = u(u^*u)$  gives us  $e^{2i\delta} = 1$ , hence  $e^{i\delta} = \pm 1$ .

But the square of  $S$  is tricky. Since  $S = TC = u_T u_C^*$ , and each unitary matrix  $u_T$  and  $u_C$  has a phase freedom (as they acts on creation/annihilation operators),

<sup>1</sup> Note that in calculating this,  $\sum_i \hat{\psi}_i^\dagger \hat{\psi}_i = \mathbb{1}$  on 1st-quantized single-particle Hilbert space.

<sup>2</sup> note that  $H^t = H^*$  for Hermitian  $H$

<sup>3</sup> In fact, we could have defined  $S = TC$  or  $S = CT$ , and obtained the same result.

we can always pick a phase such that  $S^2$  is some phase we want. Sometimes we pick  $S^2 = 1$ . Sometime we want  $\{T, C\} = 0$ , and pick some specific phase for  $S^2$ .<sup>4</sup>

**Why Ten Classes** For the three unitary symmetries under consideration, as will be mentioned,  $S$  symmetry is a combination of  $T$  and  $C$ ,  $S = TC$ . It will be obtained that  $T^2 = \pm 1$ ,  $C^2 = \pm 1$ , and  $S^2$  is undetermined (depending on the phase choice<sup>5</sup> we give for  $T$  and  $C$ ). Therefore, there are in total 10 different possible ways that this 3 symmetries could be combined together: We denote  $T^2 = 0$  ( $C^2 = 0$ ) to symbolize that the system does not follow  $T(C)$  symmetry. Then  $T^2 = 0, \pm 1$  and  $C^2 = 0, \pm 1$  gives us  $3 \times 3 = 9$  different ways. But if  $T^2 = C^2 = 0$ ,  $S$  may or may not conserve. Therefore, we have  $9 - 1 + 2 = 10$  different possible ways. In each way the topological invariants of the Hamiltonian respecting these symmetries are tabulated in Figure.1 in [Lud16]. Each class is given a name (A, AII, etc.), and their properties in different spatial dimensions respect a periodic structure. For the class A and class AII, they are named complex classes and respect a 2-fold period. For the other eight classes are named real classes and respect a 8-fold period, which is why we only listed the first  $0 \rightarrow 7$  spatial dimensions in Table 7.

## 2.2 Dealing with Unitary Symmetries

As mentioned, we could module out those unitary symmetries. This statement is made precise by the following theorem about unitary symmetries of the Hamiltonian.

**Theorem 2.1** (Diagonalization of Hamiltonian in unitary representation). *This space  $\mathcal{V}$  decomposes into a direct sum of vector spaces  $\mathcal{V}_\lambda$  associated with the irrep (irreducible representations, labeled by  $\lambda$ ) of  $G_0$ .*

$$\mathcal{V} = \oplus_\lambda m_\lambda \mathcal{V}_\lambda \quad (2.2.1)$$

where  $m_\lambda$  denotes the multiplicity of  $\lambda$ th irrep. Denote the dimension of each irrep as  $d_\lambda$ .

In each vector space  $\mathcal{V}_\lambda$ , one can choose a (orthogonal) basis of the form:

$$|v_\alpha^{(\lambda)}\rangle \otimes |w_k^{(\lambda)}\rangle \quad (2.2.2)$$

where

- $G_0$  acts only on  $|w_k^{(\lambda)}\rangle$ ,  $k = 1, \dots, d_\lambda$ ,
- $H$  acts only on  $|v_\alpha^{(\lambda)}\rangle$ ,  $\alpha = 1, \dots, m_\lambda$ .

Therefore, with all unitary symmetries ignored, we are classifying how Hamiltonian will be like when it respects, in 10 different ways, the combinations of  $T, C, S$  symmetries.

<sup>4</sup>This is a bit troublesome. First we denote  $T^2 = \varepsilon_T$ ,  $C^2 = \varepsilon_C$ , then it can be found that  $(u_T)^t = \varepsilon_T u_T$ ,  $(u_C)^t = \varepsilon_C u_C$ . Then, with  $S = u_T u_C^*$ , we have  $S^\dagger = \varepsilon_C \varepsilon_T u_C u_T^* = \varepsilon_C \varepsilon_T C T$ , or  $C T = \varepsilon_C \varepsilon_T S^\dagger$ . Since  $T C + C T = 0$ , we have  $S + \varepsilon_C \varepsilon_T S^\dagger = 0$ , or  $S^2 = -\varepsilon_C \varepsilon_T$ .

<sup>5</sup>To be explained later.

**Differently Realized Anti-unitary Symmetries** It should be noted that there is still a freedom of unitary matrix that implements the  $T$  or  $C$  symmetry. Therefore, we could have, for example, two different time-reversal symmetries  $T_1 = u_1 K$  and  $T_2 = u_2 K$ . They are obviously related by a unitary matrix, say  $T_1 = u_{12} T_2$ . And we could easily get that  $u_{12}$  is also a unitary symmetry of the Hamiltonian (if the Hamiltonian respect both  $T_1$  and  $T_2$ ). Therefore, upon enlarging the symmetry group  $G_0$  to include the element  $u_{12}$  and repeat the process described in the theorem above, we only have one time-reversal symmetry. Similar analysis could be done for the  $C$  symmetry as well.

**Translational Symmetry** Since we are classifying in the solids, it is natural to expect translational symmetry to present. Also, the addition of translational symmetry do not alter our classification, which is based on non-unitary symmetries, but also make this classification more convenient to be done.

With translational symmetry added, the classification can be made the will yield the same result.<sup>6</sup>

The Hamiltonian in  $k$ -space that preserve these three discrete symmetries should follow<sup>7</sup>:

$$TH(k)T^{-1} = H(-k) \quad (2.2.3a) \quad \text{eq:T-sym-Hk}$$

$$CH(k)C^{-1} = -H(-k) \quad (2.2.3b) \quad \text{eq:C-sym-Hk}$$

$$SH(k)S^{-1} = -S(k) \quad (2.2.3c) \quad \text{eq:S-sym-Hk}$$

## 2.3 The Tenfold Way

The Tenfold Way, is originally the ten different ways to write down a Hamiltonian when there are ten different combinations of symmetries that the system respect.

## 3 Classification by Dirac Mass Terms

**Dirac Hamiltonian and Topological Property of the Bulk** We first note that different topological phases are distinguished by the closing and then opening of the gap, i.e. a quantum phase transition. If we adiabatically flatten the energy spectrum to two bands of energy  $\pm E$ , the transition (closing of gap) will happen at some point where locally the energy is of a Dirac cone type. Therefore, the Dirac Hamiltonian captures this transition behavior. Then, a class of topological phases, or more specifically, a class of Hamiltonian matrix that shares the same topological invariant, can be *represented* by the Dirac Hamiltonian in this class.

### 3.1 Crossing Spectrum and Edge Mode

Here we show that the crossing spectrum characteristic of topological insulators can be characterized by a Dirac type Hamiltonian, and with this Dirac Hamiltonian we could find an Edge mode solution built into it.

<sup>6</sup>This is still a mystery to me which this is true. The exact reason is related to some isomorphism between certain homotopy groups. Interested reader could see Ludwig's lecture here for a clue.

<sup>7</sup>It is easy to anticipate the change  $k \rightarrow -k$  for anti-unitary symmetries, since they change the phase factor when one Fourier transform the Hamiltonian.

sec:The Tenfold Way

imal Hamiltonian Approach

ing Spectrum and Edge Mode

Add the table, and the meaning of each Hamiltonian, and the class of each evolutionary operators, if I have time

**Crossing Spectrum and Dirac Hamiltonian** In some local region around the crossing point, the energy spectrum should look like:

$$E = \pm \sqrt{\mathbf{k}^2 + M^2} \quad (3.1.1)$$

where  $\mathbf{k}$  is the crystal momentum and  $M$  is some constant responsible for the opening and closing of the gap, or in some cases, a parameter in which we can control to open or close the gap.

As has been analysed in most Quantum Mechanics textbook<sup>8</sup>, this energy-momentum relationship requires a Dirac type Hamiltonian. Then, the Dirac Hamiltonian we use is:

$$H_{\text{Dirac}} = M\gamma_0 + \sum_{i=1}^d k_i \gamma_i \quad (3.1.2)$$

eq:dirac-H

where  $d$  is the spatial dimension of the physical system,  $\gamma_i$  obeys the Clifford Algebra anti-commutation relation:

$$\{\gamma_i, \gamma_j\} = 2\delta_{i,j} \mathbb{1} \quad (3.1.3)$$

**Edge Mode and  $M$**  Now we explain why the Dirac Hamiltonian 3.1.2 is useful for capturing the topological phase. First, it has a gapless state. Due to the anti-commutation relation of  $\gamma_i$ , we can show<sup>9</sup> that the energies of Dirac Hamiltonian are:

$$E_{\pm} = \pm \sqrt{M^2 + \sum_{i=1}^d k_i^2} \quad (3.1.4)$$

Therefore, as  $M$  varies from negative to 0 to positive, the system goes from gaped to gapless and again to gaped state. This shows that the  $M$  is parameter controlling the quantum phase transition, and the system with  $M < 0$  or  $M > 0$  possesses two different quantum phase.

In addition, we can analyse its edge mode. Assuming the two material of different quantum phases touches each other in the  $d$ th direction. Then we should replace  $k_d$  by  $-i\partial_{x_d}$ , since the translational invariance is broken in that direction. Now the eigenvalue equation is:

$$(M\gamma_0 + \sum_{i=1}^{d-1} k_i \gamma_i - i\partial_{x_d} \gamma_d) \Phi = E \Phi \quad (3.1.5)$$

Past experience teaches us the ansatz (assuming  $M(x_d)$  goes to  $\pm\infty$  as  $x_d \rightarrow \pm\infty$ , so that the leading term below does not blow up the wavefunction):

$$\Phi = e^{-\int_0^{x_d} M(x'_d) dx'_d} \phi(x_1, \dots, x_{d-1}) \quad (3.1.6)$$

which after plugging inside the eigenvalue equation, leads to

$$\left( M(x_d)(\gamma_0 + i\gamma_d) + \sum_{i=1}^{d-1} k_i \gamma_i \right) \phi = E \phi \quad (3.1.7)$$

<sup>8</sup>See for example, p.99 of [Gre97]. Although their discussion is in dimension  $d = 3$ , it is easy to generalize it to any dimension.

<sup>9</sup>Note that  $H_{\text{Dirac}}^2 = E^2 \mathbb{1}$ , this tells us it can only have eigenvalues  $E_{\pm}$ . Note that this also explains why we choose not  $\gamma_i^2 = -1$ , which would give non-sensible result  $H_{\text{Dirac}}^2 = -E^2 \mathbb{1}$ .



After left multiplication of  $\gamma_0$ , we have:

$$\left( M(\mathbb{1} + i\gamma_0\gamma_d) + \sum_{i=1}^{d-1} k_i\gamma_i \right) \phi = E\gamma_0\phi \quad (3.1.8)$$

Since  $i\gamma_0\gamma_d$  squares to 1, it has eigenvalues  $\pm 1$ . Also,  $i\gamma_0\gamma_d$  and  $\gamma_0$  anticommute, so they share the same eigenspaces with  $\gamma_0$  mapping all the  $+1$  eigenvectors of  $i\gamma_0\gamma_d$  to the  $-1$  eigenvectors and vice versa. Now we want this state to be a surface state, so we choose the  $-1$  eigenvectors (denoted  $\phi_-$ ) of  $i\gamma_0\gamma_d$  to kill the bulk term. This leads to:

$$\gamma_0 \sum_{i=1}^{d-1} k_i\gamma_i\phi_- = \gamma_0 E\phi_- \quad (3.1.9)$$

As mentioned,  $\gamma_0$  switches the two eigenspaces, so we must necessarily have a surface Dirac Hamiltonian, by projecting all those matrices to the  $-1$  eigenspaces:

$$H_{\text{surf}}\phi_- = \sum_{i=1}^{d-1} \Gamma_i\phi_- = E\phi_- \quad (3.1.10)$$

eq:H-Dirac-surf

## 3.2 From Homotopy Classification to Minimal Dirac Hamiltonian Method

Homotopy to Minimal Dirac

**Spectral Flattening** The classification of topological insulators concerns different classes of Hamiltonian that cannot be adiabatically changed to each without closing or opening of a gap. Hence, we could in general adiabatically change our parameters such that the spectrum of Hamiltonian is simplified into two bands of energy  $\pm 1$ . Under this condition, we have  $H^2 = 1$ , which already hints that the Hamiltonian itself is a viable candidate for generating Clifford Algebras.

**Extension Problem with General Examples** The general theme of homotopy classification of classifying topological insulators are done by the considering the *extension problem*, i.e. different ways to extend from a Clifford containing only symmetry operators, to a Clifford Algebra containing Hamiltonians. For example, for class A in 0-dimension, the empty space is nothing, hence  $C\ell_0(\mathbb{C})$ <sup>10</sup>. And  $H^2 = 1$  as mentioned earlier, hence the extension problem is from  $C\ell_0(\mathbb{C}) \rightarrow C\ell_1(\mathbb{C})$ , whose possible representations form some mathematical structure called *classifying space*  $C_0$  [MF13]. And since we are in 0-dimension, all operators are maps starting from  $T^0 = S^0$ . So our classified objects are different classes of maps from  $S^0 \rightarrow C_0$ , which is mathematically captured by the 0-th homotopy group  $\pi_0(C_0) = \mathbb{Z}$ . For class AIII which has only chiral symmetry operator  $S$ , we choose a phase such that  $S^2 = 1$ , i.e. making it a candidate for generators of Clifford Algebra. The symmetry condition  $\{H, S\} = 0$  tells us now we have a extension problem of  $C\ell_1(\mathbb{C}) \rightarrow C\ell_2(\mathbb{C})$ , whose classifying space is  $C_1$ , and has  $\pi_0(C_1) = 0$ . Therefore, in 0-dimension, the class A has  $\mathbb{Z}$  topological insulators and class AIII has trivial topological insulators. The case for real symmetry classes are not too complicated and is concisely mentioned in [MF13].

<sup>10</sup>The reason for using different Clifford Algebra ( $C\ell(\mathbb{R})$  or  $C\ell(\mathbb{C})$ ) for different symmetry classes will be explained in section 3.3.

However, the method to obtain such classifying spaces are mathematical daunting and requires  $K$ -theory. We try to simplify it by looking at Dirac Hamiltonians, since they should capture the essence of topological states.

### Extension Problem with Dirac Hamiltonian The Dirac Hamiltonian

$$H = \vec{k} \cdot \vec{\gamma} + m\tilde{\gamma}_0 \quad (3.2.1)$$

has a gap closing and opening mass term  $m\tilde{\gamma}_0(r)$ , depending on some parameter  $r$ . Suppose that we have a domain wall squeezed by two bulk regions  $A$  and  $B$ . Now as the parameter  $r$  changes freely from region  $A$  to region  $B$ , the symmetry condition will force the matrix  $\gamma_0$  to explore some space having the same homotopy type of some classifying space (again!). For example, for class A in 2-dimension, the Hamiltonian without mass term ( $k_1\gamma_1 + k_2\gamma_2$ ) consists of two gamma matrices, generating a  $C\ell_2(\mathbb{C})$ , whereas adding the mass term, we have  $C\ell_3(\mathbb{C})$ . Since there is no symmetry in class A, the extension problem concerns different ways to extend the algebra  $C\ell_2(\mathbb{C}) \rightarrow C\ell_3(\mathbb{C})$ , which tells us the number of unitarily non-equivalent mass terms. Detailed examples can be found in section III.C.1 of [CTSR16].

**Stable Classification** An unmentioned hypothesis above are that we are classifying strong topological insulators. They are topological insulators that robust against disorder [FKM07]. There is a consensus that in classification of strong topological insulators, we replace the Brillouin Zone  $T^d$  by  $S^{d+1}$ , and we do it in a stable way, i.e., we choose our classification to be independent of and insensitive to the addition of irrelevant trivial bands. The first replacement allow us to use homotopy groups, which classify maps with domain in spheres. The "stable way" allows the use of  $K$ -theory (sec.III.C.1 of [CTSR16]).

**The View from SPEMT** Let us consider a modified Dirac Hamiltonian with extra mass term:

$$H = M\tilde{\gamma}_0 + \sum_{i=1}^d k_i\gamma_i + \sum_{j=1}^D m_j\tilde{\gamma}_j \quad (3.2.2)$$

eq:H-spemt

Here, the parameter  $M$  characterized the domain wall between different phases of topological insulators. The extra mass term  $m_j$  represents a perturbation caused by disorder. I argue that we can do our classification in the following way. First, we consider the case when  $D = 0$ , i.e. without perturbation, and we consider how many different mass terms we could have. Second, we consider adding an extra mass term which respect the symmetry of the Hamiltonian ( $D = 1$ ). Whether this symmetry preserving extra mass term (SPEMT) can be added or not. If a SPEMT can be added, then the system is not robust against perturbation and gap may be opened by disorder. So it is trivial topological insulator. If a SPEMT cannot be added, then this system is topological non-trivial.

One best thing about this classification is that it can be done by considering only the minimal matrix dimension of those gamma matrices in Hamiltonian. The argument is that, complex gamma matrices are of even dimension (except the trivial cases of  $C\ell_{0/1}(\mathbb{C})$ )<sup>12</sup>. Therefore, gamma matrices of different matrix dimension

<sup>11</sup>The exact reason can be found in Kitaev's classification [Kit09] or another article [KG15] which discuss it from a different view.

<sup>12</sup>See this post [Phy], or the p.12 and Theorem in p.5 of this [Wes98].

are built by tensor products of Pauli matrices.<sup>13</sup> Now, to increase the dimension of the matrix, is equivalent to tensor them. One can add more bands of the same type, or add more bands of a different type in the smaller dimension, or add trivial bands. In all cases, the triviality of topological insulators can be detected in minimal dimension, since adding trivial bands can be ignored (insensitive to addition of trivial bands), and the possible different types of matrices are limited.

However, to distinguish between a  $\mathbb{Z}$  topological insulator and a  $\mathbb{Z}_2$  topological insulator, we need to increase the matrix dimension to consider multiple copies of the Dirac Hamiltonian. If the topological state is stable for an arbitrary copies, then this is a  $\mathbb{Z}$  topological insulator. If the topological state is stable only of an odd number of copies, this is a  $\mathbb{Z}_2$  topological insulator.

Another way to view this way of classification, is to look at the surface Dirac Hamiltonian 3.1.10. Then the above mentioned approach is to see if the surface mode can be gapped, so as to detect the topological properties of the bulk.

### 3.3 Isomorphism between Symmetry Classes and Clifford Algebras

sec:Isomorphism

As with other classifications, we construct isomorphisms between Dirac matrices and Clifford Algebras, after which we are faced with an extension problem. The isomorphism is constructed now.

#### 3.3.1 Symmetry Constraint

The Hamiltonian with extra mass term is

$$H = M\tilde{\gamma}_0 + \sum_{j=1}^D m_j \tilde{\gamma}_j + \sum_{i=1}^d k_i \gamma_i \quad (3.3.1)$$

The symmetry properties 2.2.3a, 2.2.3b give the following conditions on the gamma matrices:

$$\{T, \gamma_i\} = 0, [C, \tilde{\gamma}_j] = 0 \quad (3.3.2a)$$

$$\{C, \tilde{\gamma}_j\} = 0, [C, \gamma_i] = 0 \quad (3.3.2b)$$

$$\{S, \tilde{\gamma}_j\} = \{S, \gamma_i\} = 0 \quad (3.3.2c)$$

With this, we can establish a isomorphism between "Hamiltonian and Symmetry Operators" and "Clifford Algebra over  $\mathbb{R}$  or  $\mathbb{C}$ ". But let's first see some examples in action.

#### 3.3.2 Examples of Constructing Isomorphism

For class AI ( $T^2 = 1, C = 0, S = 0$ ) in  $d = 1, D = 0$ , we define:

$$G_{\text{AI}} = \{i, T, \tilde{\gamma}_0, \gamma_1\}$$

We have its (anti)commutation relations listed in Table 1.

<sup>13</sup>Or, the non-trivial irreducible representations (turns out there are at most 2) are built by tensoring Pauli matrices.

sec:Symmetry Constraint

eq:sym-spemt

sec:iso-Examples

Table 1: Generators in  $G_{AI}$  with  $d = 1, D = 0$ 

	$i$	$T$	$\tilde{\gamma}_0$	$\gamma_1$
$i$	$i^2 = -1$	$\{i, T\} = 0$	$[i, \tilde{\gamma}_0] = 0$	$[i, \gamma_1] = 0$
$T$		$T^2 = 1$	$[T, \tilde{\gamma}_0] = 0$	$\{T, \gamma_1\} = 0$
$\tilde{\gamma}_0$			$\tilde{\gamma}_0^2 = 1$	$\{\tilde{\gamma}_0, \gamma_1\} = 0$
$\gamma_1$				$\gamma_1^2 = 1$

tab:generator-AI

Now with  $J_4 = T\tilde{\gamma}_0\gamma_1$ ,  $J_3 = iT\tilde{\gamma}_0$ , we could verify that all  $\{i, T, J_3, J_4\}$  anticommute with each other, and  $J_3^2 = J_4^2 = 1$ . So together they generate the algebra  $\mathcal{Cl}_{3,1}(\mathbb{R}) \cong \mathcal{Cl}^g((2, 2), \mathbb{R})$ . This will be consistent with the isomorphism  $G_{AI} \cong \mathcal{Cl}_{2+D, 1+d}(\mathbb{R})$  in Table 4.

Another example. For class AII ( $T^2 = -1, C = 0, S = 0$ ) in  $d = 1, D = 0$ , we have:

$$G_{AI} = \{i, T, \tilde{\gamma}_0, \gamma_1\} \quad (3.3.3)$$

We have its (anti)commutation relations listed in Table 2. Now with  $J_4 = T\tilde{\gamma}_0\gamma_1$ ,

Table 2: Generators in  $G_{AII}$  with  $d = 1, D = 0$ 

	$i$	$T$	$\tilde{\gamma}_0$	$\gamma_1$
$i$	$i^2 = -1$	$\{i, T\} = 0$	$[i, \tilde{\gamma}_0] = 0$	$[i, \gamma_1] = 0$
$T$		$T^2 = -1$	$[T, \tilde{\gamma}_0] = 0$	$\{T, \gamma_1\} = 0$
$\tilde{\gamma}_0$			$\tilde{\gamma}_0^2 = 1$	$\{\tilde{\gamma}_0, \gamma_1\} = 0$
$\gamma_1$				$\gamma_1^2 = 1$

tab:generator-AII

$J_3 = iT\tilde{\gamma}_0$ , we could verify that all  $\{i, T, J_3, J_4\}$  anticommute with each other, and  $J_3^2 = J_4^2 = -1$ . So together they generate the algebra  $\mathcal{Cl}_{0,4}(\mathbb{R})$ . Notice that, assume we have  $\mathcal{Cl}_{1,3}(\mathbb{R}) = \{K_1, K_2, K_3, K'_1\}$ , and  $K_i^2 = -1, (K'_1)^2 = 1$ . The following map

$$J_1 = K_1, J_2 = K_2, J_3 = K_3, J_4 = K_1K_2K_3K'_1 \quad (3.3.4)$$

has the property shown in the table 3. Therefore,  $\mathcal{Cl}_{0,4}(\mathbb{R}) \cong \mathcal{Cl}_{1,3}(\mathbb{R})$ , which is

Table 3: Map from  $\mathcal{Cl}_{1,3}(\mathbb{R})$  to  $\mathcal{Cl}_{0,4}(\mathbb{R})$ .

	$J_1 = K_1$	$J_2 = K_2$	$J_3 = K_3$	$J_4 = K_1K_2K_3K'_1$
$J_1$	$-1$	$\{\}$	$\{\}$	$\{\}$
$J_2$		$-1$	$\{\}$	$\{\}$
$J_3$			$-1$	$\{\}$
$J_4$				$-1$

tab:map-cl13-2-cl04

where  $\{\}$  means anticommute.

consistent with  $G_{AII} \cong \mathcal{Cl}_{d, 3+D}(\mathbb{R})$  in Table 4.

### 3.3.3 Constructing the Isomorphism (Complex Symmetry Classes)

The general steps for constructing the isomorphism is presented here. First, we need to treat the situation of complex classes and real classes differently. The com-

sec:Complex Classes-iso

plex classes are isomorphic to complex Clifford Algebras, while the real classes are isomorphic to real Clifford Algebras. The reason for this distinction is that, in real classes, there are complex conjugate operator  $K$ . First, there is not way to represent complex conjugation simply as multiplication of complex matrices. Second, complex matrices come with a nature definition of complex conjugate  $\dagger$ , whereas  $K^\dagger$  is ill-defined<sup>14</sup>. Therefore, we use real Clifford Algebras when dealing with real classes (which is the reason why they are named real classes).

For complex classes, let  $G_\#$  be the group generated by elements  $\{\gamma_i, \tilde{\gamma}_j, S\}$  ( $i = 1, \dots, d, j = 0, 1$ ) in each symmetry class  $\#$  (so  $S$  exists only in class AIII). We may choose a phase such that  $S^2 = 1$ , as mentioned earlier. Then obviously  $G_\#$  will be a Clifford Algebra:

$$G_A = \text{generated by } \{\gamma_i, \tilde{\gamma}_j\} \cong \text{Cl}_{d+D+1}(\mathbb{C}) \quad (3.3.5)$$

$$G_{\text{AIII}} = \text{generated by } \{\gamma_i, \tilde{\gamma}_j, S\} \cong \text{Cl}_{d+D+2}(\mathbb{C}) \quad (3.3.6)$$

sec:Real Classes-iso

### 3.3.4 Constructing the Isomorphism (Real Symmetry Classes)

For real classes, we need to include  $i$ , and symmetry operators  $T, C$  in our group  $G_\#$ . It turns out that we need to pick a phase such that  $\{T, C\} = 0$ , which is possible as is mentioned earlier. Now we demonstrate the proof of isomorphism in class AII.

We first note that, given a real vector space  $V$ , we can complexify a real space in two ways. The first is trivially taking the tensor product  $V \otimes_{\mathbb{R}} \mathbb{C}$ , which does not suit our purpose. The second is to find an almost complex structure  $J$ , which is a  $\mathbb{R}$ -linear map that squares to  $-\mathbb{1}$ , i.e.  $J^2 = -\mathbb{1}$ . With this almost complex structure  $J$ , then  $V$  admits in a natural way the structure of a complex vector space  $V_{\mathbb{C}}$  [Dan05]. Also, a  $\mathbb{R}$ -linear map  $A$  on  $V$  is  $\mathbb{C}$ -linear( $\mathbb{C}$ -antilinear) if and only if  $A$  commutes (anticommutes) with  $J$ . Therefore, we map model the antiunitary operators on a real vector space naturally.

Let us denote the generators in Clifford Algebra as  $J_j$  and  $\tilde{J}_i$ , where  $J_j^2 = -\tilde{J}_i^2 = -\mathbb{1}$ . Since this almost complex structure  $J$  squares to  $-\mathbb{1}$  and anticommutes with antiunitary symmetry operators, we naturally take it to be the first generator in our Clifford Algebra,  $J_1 = J$ .

Then we discuss some tips that will useful for later calculation.

#### Tips

1. Since all generator anticommute, all matrices either commute or anticommute. So the Algebra is pretty simple.
2. Since  $[A, BC] = \pm[A, CB]$  and  $\{A, BC\} = \pm\{A, CB\}$ , the commutation or anticommutation does not depends on the order of the matrices  $BC$  or  $CB$ . So one might rearrange them in the order whichever is convenient.
3. For  $A = \{J_1, \dots, J_n\}, B = \{K_1, \dots, K_m\}$ , we have  $AB = (-1)^{mn} BA$ . However, if  $A$  and  $B$  has something in common, then the above condition breaks. This is like adding an "impurity" in it to change the commutation/anticommutation relations.

<sup>14</sup>Think about this.  $\langle c\phi | K\psi \rangle$  could have different values depending on whether we move  $K$  to the left as  $K^\dagger$  first, or we move  $c$  out of the inner-product first.

enum:tips-4

4. The restriction given by  $\sum_i k_i \gamma_i$ , and  $\sum_j m_j \tilde{\gamma}_j$  (will be shown later) are much restrictive that the candidates for symmetry operator are only a small finite set.

**Class AII** Now we prove the isomorphism. First, we try to construct each gamma matrices in Hamiltonian. All matrices commute with  $i$ , so in real vector space, they commute with  $J_1 = J$ . Then, each gamma matrices should have an even number of Clifford Algebra generators other than  $J_1$  itself. Similarly, antiunitary symmetry operators should have an odd number of Clifford Algebra generators other than  $J_1$ . We "dope" the gamma matrices with some  $J_1$  to make it commute/anticommute with symmetry operators. More explicitly, bearing in mind that in class AII has only  $T^2 = -1$ , we take a quick look into Hamiltonian 3.2.2 and symmetry conditions 3.3.2, and they give us the inspiration to set:

$$H_{\text{AII}} = mJ_1J_2J_3 + \sum_{j=1}^D m_j J_1J_2J_{3+j} + \sum_{i=1}^d k_i J_2\tilde{J}_i \quad (3.3.7)$$

Here  $\tilde{\gamma}_j$  are  $J_1J_2J_{3+j}$ , which square to 1.  $\gamma_i$  are  $J_2\tilde{J}_i$ . The Clifford Algebra is at least  $\text{Cl}_{d,3+D}(\mathbb{R})$ . And within this algebra, only  $J_2$  is a possible candidate for symmetry operators, which satisfy equations 3.3.2 (use tips 4 for calculation).  $J_2$  is found to be a time reversal operator, and  $J_2^2 = -1$  confirms that this Hamiltonian belongs to class AII. Therefore, the map:

$$\begin{aligned} f : \text{Cl}_{d,3+D}(\mathbb{R}) &\rightarrow G_{\text{AII}} \\ J_1J_2J_{3+j} &\rightarrow \tilde{\gamma}_j, (j = 0, 1, \dots, D) \\ J_2\tilde{J}_i &\rightarrow \gamma_i, (i = 0, 1, \dots, d) \end{aligned} \quad (3.3.8)$$

is a map from Clifford Algebra  $\text{Cl}_{d,3+D}(\mathbb{R})$  to symmetry class AII. The inverse map can be found easily. We first solve some formality problems. A complex number can be written as a real number by identifying  $i$  with  $J = i\sigma_y$  and 1 with 1, the identity matrix:

$$a + bi \rightarrow a \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + b \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = a\mathbb{1} + bJ \quad (3.3.9)$$

Therefore, all complex matrices  $\gamma_i, \tilde{\gamma}_j$  are identified with real matrices  $\Gamma_i, \tilde{\Gamma}_j$ , of twice the size of  $\gamma_i, \tilde{\gamma}_j$ . More explicitly,  $(a_{mn} + ib_{mn})$  is identified as  $A + iB = (a_{mn}) + i(b_{mn}) \rightarrow (a_{mn}) + J_1(b_{mn})$ , where

$$J_1 \equiv i\sigma_y \otimes \mathbb{1}_{n \times n} \quad (3.3.10)$$

The complex conjugate  $K$  is then a matrix anticommute with  $J_1$ , we define:

$$K \equiv \sigma_z \otimes \mathbb{1}_{n \times n} \quad (3.3.11)$$

Then  $T \rightarrow \Gamma_T, C \rightarrow \Gamma_C$  for some real matrices. We also note that, as mentioned earlier, we make a phase choice of  $T$  and  $C$  such that

$$\{\Gamma_T, \Gamma_C\} = 0 \quad (3.3.12)$$

Now, we write symbolically  $J_1 J_2 J_{3+j} = J_1 \Gamma_T J_{3+j} = \tilde{\Gamma}_j$ , then clearly  $J_{3+j} = \Gamma_T J_1 \tilde{\Gamma}_j$ . Similarly,  $\tilde{J}_i = \Gamma_T \Gamma_i$ . So the map:

$$\begin{aligned} f^{-1} : G_{\text{AII}} &\rightarrow \mathcal{Cl}_{d,3+D}(\mathbb{R}) \\ i\gamma_y \otimes \mathbb{1}_{n \times n} &\rightarrow J_1 \\ \Gamma_T &\rightarrow J_2 \\ \Gamma_T J_1 \Gamma_j &\rightarrow J_{3+j} \\ \Gamma_T \tilde{\Gamma}_i &\rightarrow J_i \end{aligned} \quad (3.3.13)$$

is the desired inverse map.

**Class CII** The class CII has only one  $C^2 = -1$  more than class AII, therefore, we only need to enlarge the Clifford Algebra to include one more  $J_{4+D}$ , and set it as the  $C$  symmetry operator. The rest is exactly the same as in class AII.

All other classes can be treated similarly, so we do not repeat the calculation and only list the result in Table 4.

Table 4: Mapping Relations between symmetry classes and Clifford Algebras. Here  $J_j^2 = -\mathbb{1}$ ,  $\tilde{J}_i^2 = \mathbb{1}$ , and they generates the Clifford Algebra. Also, only one direction of mapping is shown. The inverse map can be easily constructed accordingly. The complex class are also added for convenience.

Class(#)	T	C	S	Mappings	$G_{\#} \cong$
D	0	+	0	$\tilde{J}_1 \rightarrow \Gamma_C, \tilde{J}_1 J_{2+j} \rightarrow \tilde{\Gamma}_j, J_1 \tilde{J}_1 \tilde{J}_{1+i} \rightarrow \Gamma_i$	$\mathcal{Cl}_{1+d,2+D}(\mathbb{R})$
DIII	-	+	1	$\tilde{J}_1 \rightarrow \Gamma_C, \tilde{J}_{3+D} \rightarrow \Gamma_T, \tilde{J}_1 J_{2+j} \rightarrow \tilde{\Gamma}_j, J_1 \tilde{J}_1 \tilde{J}_{1+i} \rightarrow \Gamma_i$	$\mathcal{Cl}_{1+d,3+D}(\mathbb{R})$
AII	-	0	0	$J_2 \rightarrow \Gamma_T, J_1 J_2 J_{3+j} \rightarrow \tilde{\Gamma}_j, J_2 \tilde{J}_i \rightarrow \Gamma_i$	$\mathcal{Cl}_{d,3+D}(\mathbb{R})$
CII	-	-	1	$J_2 \rightarrow \Gamma_T, J_{4+D} \rightarrow \Gamma_C, J_1 J_2 J_{3+j} \rightarrow \tilde{\Gamma}_j, J_2 \tilde{J}_i \rightarrow \Gamma_i$	$\mathcal{Cl}_{d,4+D}(\mathbb{R})$
C	0	-	0	$J_2 \rightarrow \Gamma_C, J_2 \tilde{J}_{1+j} \rightarrow \tilde{\Gamma}_j, J_1 J_2 J_{i+2} \rightarrow \Gamma_i$	$\mathcal{Cl}_{1+D,2+d}(\mathbb{R})$
CI	+	-	1	$J_2 \rightarrow \Gamma_C, \tilde{J}_{2+D} \rightarrow \Gamma_T, J_2 \tilde{J}_{1+j} \rightarrow \tilde{\Gamma}_j, J_1 J_2 J_{i+2} \rightarrow \Gamma_i$	$\mathcal{Cl}_{2+D,2+d}(\mathbb{R})$
AI	+	0	0	$\tilde{J}_1 \rightarrow \Gamma_T, J_1 \tilde{J}_1 \tilde{J}_{2+j} \rightarrow \tilde{\Gamma}_j, \tilde{J}_1 J_{i+1} \rightarrow \Gamma_i$	$\mathcal{Cl}_{2+D,1+d}(\mathbb{R})$
BDI	+	+	1	$\tilde{J}_1 \rightarrow \Gamma_T, \tilde{J}_{3+D} \rightarrow \Gamma_C, J_1 \tilde{J}_1 \tilde{J}_{2+j} \rightarrow \tilde{\Gamma}_j, \tilde{J}_1 J_{i+1} \rightarrow \Gamma_i$	$\mathcal{Cl}_{3+D,1+d}(\mathbb{R})$
A	0	0	0		$\mathcal{Cl}_{d+D+1}(\mathbb{C})$
AIII	0	0	1		$\mathcal{Cl}_{d+D+2}(\mathbb{C})$

### 3.4 Classification in 1-dimension

#### 3.4.1 Model Systems

Classification of 1 dimension is the simplest, since the minimal matrix dimension will mostly be 2, which means we can use the familiar Pauli matrices directly.

**Class A** We have the extension from  $\mathcal{Cl}_2(\mathbb{C})$  to  $\mathcal{Cl}_3(\mathbb{C})$ . Since  $\mathcal{Cl}_2(\mathbb{C}) \cong \mathcal{M}(2, \mathbb{C})$ , we have for example:

$$H_A = M\sigma_x + k_x\sigma_y \quad (3.4.1)$$

Obviously, there is an SPENT  $m\sigma_x$ . So this phase is trivial.

**Class AIII** We have the extension from  $C\ell_3(\mathbb{C})$  to  $C\ell_4(\mathbb{C})$ . Since  $C\ell_3(\mathbb{C}) \cong \mathcal{M}(2, \mathbb{C}) \oplus \mathcal{M}(2, \mathbb{C})$ , we use Pauli matrices. For example:

$$H_A = M\sigma_x + k_x\sigma_y \quad (3.4.2)$$

However, the symmetry operator  $S$  takes rest Pauli matrices  $\sigma_z$ , and there is not other matrix possible for the extra mass term (this can be viewed alternatively, from  $C\ell_4(\mathbb{C}) \cong \mathcal{M}(4, \mathbb{C})$ , which only has irreducible representation in dimension 1 or 4). So the state is topologically non-trivial. Now we consider the state with arbitrary copies of it.

$$H_A = M\sigma_x \otimes \mathbb{1}_n + k_x\sigma_y \otimes \mathbb{1}_n, S = \sigma_z \otimes \mathbb{1}_n \quad (3.4.3)$$

where  $n$  is some positive integer. Since gamma matrices are basically tensor products of Pauli matrices, there is no SPEMT term. Hence this is a  $\mathbb{Z}$  topological insulator.

The above are for complex classes. For real classes, we have to be careful, since one generator  $J_1$  is taken up by  $i$ .

**Class AII** We have the extension from  $C\ell_{1,3}(\mathbb{R}) \cong \mathcal{M}(2, \mathbb{R}) \otimes \mathbb{H}$ , to  $C\ell_{1,4}(\mathbb{R}) \cong \mathcal{M}(2, \mathbb{R}) \otimes (\mathbb{H} \oplus \mathbb{H})$ . This already hints that we will have SPEMT. Explicitly, we could let

$$H_{AII} = M\sigma_x \otimes \mathbb{1}_2 + k_x\sigma_y \otimes \mathbb{1}_2 \quad (3.4.4)$$

with  $T = \mathbb{1}_2 \otimes \sigma_x K$ . The SPEMT is  $m\sigma_z \otimes \sigma_x$ . So this is a trivial insulator.

**Class C** We have the extension from  $C\ell_{1,3}(\mathbb{R}) \cong \mathcal{M}(2, \mathbb{R}) \otimes \mathbb{H}$ , to  $C\ell_{2,3}(\mathbb{R}) \cong \mathcal{M}(4, \mathbb{R}) \otimes \mathbb{C}$ . Explicitly, we could let

$$H_C = M\sigma_z \otimes \mathbb{1}_2 + k_x\sigma_y \otimes \mathbb{1}_2 \quad (3.4.5)$$

with  $C = \sigma_x \otimes \sigma_y K$ . The SPEMT is  $m\sigma_x \otimes \sigma_x$ . So this is a trivial insulator.

**Class CI** We have the extension from  $C\ell_{2,3}(\mathbb{R}) \cong \mathcal{M}(4, \mathbb{R}) \otimes \mathbb{C}$ , to  $C\ell_{3,3}(\mathbb{R}) \cong \mathcal{M}(6, \mathbb{R})$ . If one realize that  $\mathbb{C}$  is realized by  $2 \times 2$  matrices, then this should have SPEMT. Explicitly, we could let

$$H_{CI} = M\sigma_z \otimes \mathbb{1}_2 + k_x\sigma_y \otimes \mathbb{1}_2 \quad (3.4.6)$$

with  $C = \sigma_x \otimes \sigma_y K, T = K$ . The SPEMT is again  $m\sigma_x \otimes \sigma_x$ . So this is a trivial insulator.

**Class AI** We have the extension from  $C\ell_{2,2}(\mathbb{R}) \cong \mathcal{M}(4, \mathbb{R})$ , to  $C\ell_{3,2}(\mathbb{R}) \cong \mathcal{M}(4, \mathbb{R}) \otimes (\mathbb{R} \oplus \mathbb{R})$ . This hints that we have SPEMT. Explicitly, we could let

$$H_{AI} = M\sigma_x + k_x\sigma_y \quad (3.4.7)$$

with  $T = K$ . The SPEMT is  $m\sigma_z$ . So this is a trivial insulator.



**Class CII** We have extension from  $C\ell_{1,4}(\mathbb{R}) \cong \mathcal{M}(2, \mathbb{R}) \otimes (\mathbb{H} \oplus \mathbb{H})$ , to  $C\ell_{1,5}(\mathbb{R}) \cong \mathcal{M}(2, \mathbb{R}) \otimes \mathcal{M}(2, \mathbb{H})$ . We let:

$$H_{\text{CII}} = M\sigma_x \otimes \mathbb{1}_2 + k_x\sigma_y \otimes \mathbb{1}_2 \quad (3.4.8)$$

with  $T = \mathbb{1}_2 \otimes \sigma_y K$ ,  $C = \sigma_z \otimes \sigma_y K$ . There is no SPEMT<sup>15</sup>. Consider arbitrary copies of it:

$$H_{\text{CII}} = (M\sigma_x \otimes \mathbb{1}_2 + k_x\sigma_y \otimes \mathbb{1}_2) \otimes \mathbb{1}_n \quad (3.4.9)$$

with  $T = \mathbb{1}_2 \otimes \sigma_y \otimes \mathbb{1}_n K$ ,  $C = \sigma_z \otimes \sigma_y \otimes \mathbb{1}_n K$ . There is still no SPEMT<sup>16</sup>. Therefore, this is a  $\mathbb{Z}$  topological insulator.

**Class BDI** We have extension from  $C\ell_{3,2}(\mathbb{R}) \cong \mathcal{M}(4, \mathbb{R}) \otimes (\mathbb{R} \oplus \mathbb{R})$ , to  $C\ell_{4,2}(\mathbb{R}) \cong \mathcal{M}(8, \mathbb{R})$ . We let:

$$H_{\text{BDI}} = M\sigma_z + k_x\sigma_y \quad (3.4.10)$$

with  $T = K$ ,  $C = \sigma_x K$ . Obviously, there is no SPEMT. For arbitrary copies of it:

$$H_{\text{BDI}} = M\sigma_z \otimes \mathbb{1}_n + k_x\sigma_y \otimes \mathbb{1} \quad (3.4.11)$$

with  $T = \mathbb{1}_{n+2} K$ ,  $C = \sigma_x \otimes \mathbb{1}_n K$ . The SPEMT must be of the form  $\sigma_x \otimes \Delta$  to anticommute with  $H_{\text{BDI}}$ . But to commute with  $T$  leads to  $\Delta$  being real, and to anticommute with  $C$  leads to  $\Delta$  being complex. Hence there is no SPEMT. This is a  $\mathbb{Z}$  topological insulator.

**Class D** We let:

$$H_{\text{D}} = M\sigma_z + k_x\sigma_y \quad (3.4.12)$$

with  $C = \sigma_x K$ . Obviously, there is no SPEMT. Now consider two copies of it:

$$H_{\text{D}} = (M\sigma_z + k_x\sigma_y) \otimes \mathbb{1}_2 \quad (3.4.13)$$

with  $C = \sigma_x \otimes \mathbb{1}_2 K$ . There is one SPEMT  $m\sigma_z \otimes \sigma_y$ . Hence, this is a  $\mathbb{Z}_2$  topological insulator.

To lighten the notation a bit, we introduce  $\tau_i = s_i = \sigma_i$  ( $i = 0, 1, 2, 3$ ). But  $\tau_i$ ,  $s_i$ , and  $\sigma_i$  all act on the different spaces.

**Class DIII** We let:

$$H_{\text{DIII}} = M\sigma_z s_0 + k_x\sigma_y s_0 \quad (3.4.14)$$

with  $C = \sigma_x K$ ,  $T = s_y K$ . Any SPEMT anticommute with Hamiltonian has the form  $\sigma_x s_i$ . For it to anticommute with  $C$ , it is  $\sigma_x s_y$ . But this does not commute with  $T$ . Hence, there is no SPEMT. Now consider two copies of it:

$$H_{\text{DIII}} = (M\sigma_z s_0 + k_x\sigma_y s_0)\tau_0 \quad (3.4.15)$$

with  $C = \sigma_x K$ ,  $T = s_y K$ . There is one SPEMT  $m\sigma_x s_x \tau_y$ . So this is a  $\mathbb{Z}_2$  topological insulator.

In summary, we shown examples of different classes of topological insulators in  $d = 1$ , summarized in Table 5.

<sup>15</sup>To anticommute with  $\sigma_x \otimes \mathbb{1}_2$  and  $\sigma_y \otimes \mathbb{1}_2$ , it must be of the form  $\sigma_z \otimes ?$ . But none of the remaining choice are acceptable, considering  $T$  and  $C$ .

<sup>16</sup>The SPEMT has to be of the form  $\sigma_z \otimes \sigma_\alpha \otimes A_{n \times n}$ . Commuting with  $T$  makes  $A$  completely imaginary. But it cannot anticommute with  $C$ .

Table 5: Topological Insulators in  $d = 1$ 

Class(#)	T	C	S	$d = 1$
A	0	0	0	0
AIII	0	0	1	$\mathbb{Z}$
D	0	+	0	$\mathbb{Z}_2$
DIII	-	+	1	$\mathbb{Z}_2$
AII	-	0	0	0
CII	-	-	1	$\mathbb{Z}$
C	0	-	0	0
CI	+	-	1	0
AI	+	0	0	0
BDI	+	+	1	$\mathbb{Z}$

### 3.4.2 Irreducible Representation Arguments for Classification

The above examples are only valid in special points of the Brillouin Zone. However, in real materials, one will be faced with Hamiltonians of various form:

$$H = m\gamma_0 \pm k_1\gamma_1 \pm k_2\gamma_2 \cdots \quad (3.4.16)$$

Here, we look at this problem from a representation theory point of view. It is well known that representation of Clifford Algebras are completely reducible, and there are only 1 or 2 nontrivial (dimension  $> 1$ ) inequivalent irreducible representations of Clifford Algebras (see 5). The dimension of the representation of relevant Clifford Algebras are listed in Table 6. There are only three types of change occurs in that table:

$\mathbf{n} \rightarrow \mathbf{n}$  or  $\mathbf{n} \rightarrow \mathbf{n}_2$  Basically, this means that in the same dimension we could always have a SPEMT present. So there will never be stable edge states.

$\mathbf{n} \rightarrow 2\mathbf{n}$  This means that, although in the same dimension we have always stable edge mode. When we take two copies of the Hamiltonian, i.e., when there are two edge modes. There will a new gamma matrix in the doubled-dimension acting as a SPEMT to gap out the system. Hence in general, an odd copies of the edge mode is stable whereas an even copies of the edge mode is not. Hence this is a  $\mathbb{Z}_2$  topological insulator.

$\mathbf{n}_2 \rightarrow 2\mathbf{n}$  This is similar to the above  $\mathbb{Z}_2$  case. However, since we have two inequivalent representations in a single edge mode, we could the doubled system could have came from different representations, which could not be gapped out by the extra SPEMT in doubled-dimension (otherwise we would have two inequivalent representations in the doubled-dimension). Hence, this is a  $\mathbb{Z}$  topological insulator.

In this way we confirmed the classification in Table 5.

Table 6: Matrix dimensions related to the extension problem for different classes in  $d = 1$ . Note that a subscript 2 as in  $2_2$ , means that there are two inequivalent representations in the same matrix dimension.

Name of Class	Extension of $Cl$	Dimension Change	Topological Invariant
A	$Cl_2(\mathbb{C}) \rightarrow Cl_3(\mathbb{C})$	$2 \rightarrow 2_2$	0
AIII	$Cl_3(\mathbb{C}) \rightarrow Cl_4(\mathbb{C})$	$2_2 \rightarrow 4$	$\mathbb{Z}$
AII	$Cl_{1,3}(\mathbb{R}) \rightarrow Cl_{1,4}(\mathbb{R})$	$4 \rightarrow 4_2$	0
C	$Cl_{1,3}(\mathbb{R}) \rightarrow Cl_{2,3}(\mathbb{R})$	$4 \rightarrow 4$	0
CI	$Cl_{2,3}(\mathbb{R}) \rightarrow Cl_{3,3}(\mathbb{R})$	$4 \rightarrow 4$	0
AI	$Cl_{2,2}(\mathbb{R}) \rightarrow Cl_{3,2}(\mathbb{R})$	$2 \rightarrow 2_2$	0
CII	$Cl_{1,4}(\mathbb{R}) \rightarrow Cl_{1,5}(\mathbb{R})$	$4_2 \rightarrow 8$	$\mathbb{Z}$
BDI	$Cl_{3,2}(\mathbb{R}) \rightarrow Cl_{4,2}(\mathbb{R})$	$2_2 \rightarrow 4$	$\mathbb{Z}$
D	$Cl_{2,2}(\mathbb{R}) \rightarrow Cl_{2,3}(\mathbb{R})$	$2 \rightarrow 4$	$\mathbb{Z}_2$
DIII	$Cl_{2,3}(\mathbb{R}) \rightarrow Cl_{2,4}(\mathbb{R})$	$4 \rightarrow 8$	$\mathbb{Z}_2$

tab:mat-dim-allClass-1d

### 3.5 Classification in Arbitrary Dimension

We now show combined with some properties, the classification in  $d = 1$  can be generalized to classification in arbitrary dimensions. In  $K$ -theoretic classification, the extension problem is not affected when tensored with some other algebras. Specifically, for complex classes, the extension problem of  $Cl_n(\mathbb{C}) \rightarrow Cl_m(\mathbb{C})$  is the same as the extension problem of  $Cl_n(\mathbb{C}) \otimes Cl_2(\mathbb{C}) \rightarrow Cl_m(\mathbb{C}) \otimes Cl_2(\mathbb{C})$  (See for example, sec III.C.1 of [CTSR16]). For real classes, similar property holds. This insensitive to tensoring a new algebra, can be understood in SPEMT background. Because  $Cl_2(\mathbb{C}) \cong \mathcal{M}(2, \mathbb{C})$ , which only multiply the matrix dimension of both sides by a constant 2. So the pattern of  $n \rightarrow n, n \rightarrow n_2, n \rightarrow 2n$ , and  $n_2 \rightarrow 2n$ , mentioned in previous section, remains unaltered. For real classes, instead of  $Cl_2(\mathbb{C})$ , we have  $Cl_{1,1}(\mathbb{R}) \cong \mathcal{M}(2, \mathbb{R})$ , or  $Cl_{2,2}(\mathbb{R}) \cong Cl_{1,1}(\mathbb{R}) \otimes Cl_{1,1}(\mathbb{R}) \cong \mathcal{M}(4, \mathbb{R})$ . So the reasoning is the same.

Now, let us write  $\sim$  to represent the equivalence between two extension problems. The Complex Clifford Algebra has a periodicity of 2:

$$Cl_{n+2}(\mathbb{C}) \cong Cl_n(\mathbb{C}) \otimes Cl_2(\mathbb{C}) \quad (3.5.1)$$

The Real Clifford Algebra has a periodicity of 8:

$$Cl_{p+8,q}(\mathbb{R}) = Cl_{p,q+8}(\mathbb{R}) = Cl_{p,q}(\mathbb{R}) \otimes \mathcal{M}(16, \mathbb{R}) \quad (3.5.2)$$

This two relation means that, the extension problem will be the same with respect to a periodicity of 2 for complex symmetry classes, and 8 for real symmetry classes. Hence the classification of topological insulators need only be done with  $d = 1, 2$  for complex symmetry classes, and  $d = 0, 1, \dots, 7$  for real symmetry classes, i.e.

eq:cli-periodic

$$G_\#(d = d_0) \sim G_\#(d = (d_0 \bmod 2)) \quad (\text{complex class}) \quad (3.5.3a)$$

$$G_\#(d = d_0) \sim G_\#(d = (d_0 \bmod 8)) \quad (\text{real class}) \quad (3.5.3b)$$

Another useful property is:

$$Cl_{p+1,q+1}(\mathbb{R}) \cong Cl_{p,q}(\mathbb{R}) \otimes \mathcal{M}(2, \mathbb{R}) \quad (3.5.4)$$

This property tells us that Clifford Algebra basically depends only on the difference  $p - q$ , or combined with previous periodicity,  $p - q \pmod{8}$ . For example, we have (let  $n$  be an arbitrary integer):  $C\ell_{p+1,q+1}(\mathbb{R}) \sim C\ell_{p,q}(\mathbb{R})$ . With this, one can derive easily two chains of relation:

$$\begin{aligned} C\ell_{1+(D+n),2+D}(\mathbb{R}) &\sim C\ell_{1+(D+1+n),3+D}(\mathbb{R}) \\ \sim C\ell_{(D+2+n),3+D}(\mathbb{R}) &\sim C\ell_{(D+3+n),4+D}(\mathbb{R}) \end{aligned} \quad (3.5.5)$$

and

$$\begin{aligned} C\ell_{1+D,2+(D+4+n)}(\mathbb{R}) &\sim C\ell_{2+D,2+(D+5+n)}(\mathbb{R}) \\ \sim C\ell_{2+D,1+(D+6+n)}(\mathbb{R}) &\sim C\ell_{3+D,1+(D+7+n)}(\mathbb{R}) \end{aligned} \quad (3.5.6)$$

Now we tries to connect the two chains together. We note further that the Clifford Algebra has the following properties:

$$C\ell_{p+1,q}(\mathbb{R}) \cong C\ell_{q+1,p}(\mathbb{R}) \quad (3.5.7)$$

$$C\ell_{q,p+2}(\mathbb{R}) \cong C\ell_{p,q}(\mathbb{R}) \otimes \mathcal{M}(2, \mathbb{R}) \quad (3.5.8)$$

Then:

$$\begin{aligned} C\ell_{1+D,2+(D+4+n)}(\mathbb{R}) &\cong C\ell_{D+4+n,1+D}(\mathbb{R}) \otimes \mathcal{M}(2, \mathbb{R}) \\ \cong C\ell_{2+D,D+3+n}(\mathbb{R}) \otimes \mathcal{M}(2, \mathbb{R}) &\cong C\ell_{D+1+n,2+D}(\mathbb{R}) \otimes \mathcal{M}(4, \mathbb{R}) \end{aligned} \quad (3.5.9)$$

Therefore,  $C\ell_{1+D,2+(D+4+n)}(\mathbb{R}) \sim C\ell_{1+(D+n),2+D}(\mathbb{R})$ , connecting the two chains. If we compare this carefully with Table 4, then we would realize that we actually managed to prove the dimension-shift feature of classification:

$$\begin{aligned} G_D(d = d_0) &\sim G_{DIII}(d = d_0 + 1) \\ &\sim G_{AII}(d = d_0 + 2) \sim G_{CII}(d = d_0 + 3) \\ &\sim G_C(d = d_0 + 4) \sim G_{CI}(d = d_0 + 5) \\ &\sim G_{AI}(d = d_0 + 6) \sim G_{BDI}(d = d_0 + 7) \end{aligned} \quad (3.5.10)$$

eq:cli-Chain1

We need one more property:

$$C\ell_{1+d,2+D}(\mathbb{R}) \cong C\ell_{3+D,d}(\mathbb{R}) = C\ell_{3+D,1+(d-1)}(\mathbb{R}) \quad (3.5.11)$$

Then

$$G_D(d = d_0) \sim G_{BDI}(d = d_0 - 1) \quad (3.5.12)$$

eq:cli-Chain2

With equivalences 3.5.3,3.5.10,3.5.12, we see that we need only the result in a dimension, to obtain the whole of classification of all real classes. Similar fact holds for complex classes:

$$G_{AIII}(d = d_0) \sim G_A(d = d_0 - 1) \quad (3.5.13)$$

which is too trivial to be mentioned here.

In a word, our classification for topological insulators in 1 spatial dimension is sufficient to generate the whole Table 7. This is a remarkable consequence of  $K$ -theory.

The original classification table											
AZ class\ $d$	0	1	2	3	4	5	6	7	T	C	S
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	0
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	1
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	+	0	0
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	+	+	1
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	0	+	0
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	−	+	1
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	−	0	0
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	−	−	1
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	−	0
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	+	−	1

Table 7: The original classification table of topological insulators and superconductors. Note that in this document we could not yet differentiate between  $\mathbb{Z}$  and  $2\mathbb{Z}$  topological invariants.

tab:master-table2

## 4 Looking Further

sec:Looking Further

There has been a great deal of progress made in classification of topological insulator and superconductors. We first comment that our discussion could be easily generalized to some topological superconductors because they naturally admit a matrix like Hamiltonian using Nambu spinors.

Also, our discussion falls short in two respects. First, it cannot distinguish between  $\mathbb{Z}$  and  $2\mathbb{Z}$  cases. Secondly, we do not give explicit formulae for calculating topological invariants in different cases. The former problem can be addressed in Ludwig's [SRFL08], while the latter requires considerable work, and provides different topological invariants in different cases (see sec. III.B of [CTSR16]).

There has been progress made both in incorporating defects into classification, as well as incorporating unitary symmetries into the classification. The defect's classification has been incorporated beautifully into the same table of Tenfold Way. On the other hand, introduction of unitary symmetries gives a more fruitful result. For example, introduction of crystal symmetries gives a new class of topological insulators, dubbed topological crystalline insulators. [AF15] the case of reflection symmetry, some new topological invariants denoted as  $M\mathbb{Z}$ , called mirror topological invariants, are needed to classify topological matters [CYR13].

Lastly, the discussion of topological states in interaction picture extends to the case of symmetry-protected phases (SPTs), of which I am not familiar with. For all these new concepts, the review [CTSR16] is an excellent source of information.

## 5 Various Properties for Clifford Algebras

We have used various properties of Clifford Algebras from various sources. Here I summarized those properties for the convenience of readers.

Properties for Clifford Algebras

**Isomorphism Relations** The best source of isomorphism between Clifford Algebras is actually the Wikipedia page about *Classifications of Clifford Algebras*. For a more authoritative account, I am using the book [Vd16]. The results used in this document are to be found in chapter 4 of that book.

**Representation Theory on Complex Clifford Algebras** This paper [Wes98] has a good discussion on the irreducible representations of complex Clifford Algebras, in p.5, p.12. They can be summarized as:

**Theorem 5.1.** *Consider  $\text{Cl}_N(\mathbb{C})$ . For both  $N = 2n$  and  $N = 2n + 1$ , the dimension  $d$  of the irreducible representation is uniquely given by*

$$d = 2^n (N = 2n \text{ or } N = 2n + 1) \quad (5.0.14)$$

Moreover, for  $N = 2n$ , the irreducible space is unique, while for  $N = 2n + 1$ , we have two inequivalent irreducible representations with the same dimension  $d$ .

**Representation Theory on Real Clifford Algebra** The two papers [Oku91b] [Oku91a] are great in this respect. Their result used in this document could be summarized as:

**Theorem 5.2.** *Consider  $\text{Cl}_{p,q}(\mathbb{R})$ . For  $N = 2n$ , we have:*

$$d = \begin{cases} 2_1^n, & \text{for } p - q = 0 \text{ or } 2, \text{ mod } 8 \\ 2_1^{n+1}, & \text{for } p - q = 4 \text{ or } 6, \text{ mod } 8 \end{cases} \quad (5.0.15)$$

For  $N = 2n + 1$ , we have:

$$d = \begin{cases} 2_2^n, & \text{for } p - q = 1, \text{ mod } 8 \\ 2_1^{n+1}, & \text{for } p - q = 3 \text{ or } 7, \text{ mod } 8 \\ 2_2^{n+1}, & \text{for } p - q = 5, \text{ mod } 8 \end{cases} \quad (5.0.16)$$

Where we have used a subscript  $_1$  to denote that there is only one unique irreducible representation, and a subscript  $_2$  to denote that there are two inequivalent irreducible representations.

Below collects a list of dimensions that I have used in the discussion of Table 6:

$$\begin{aligned} \text{Cl}_{1,2}(\mathbb{R}) &\sim 2_1^2, \text{Cl}_{1,3}(\mathbb{R}) \sim 4_1^2, \text{Cl}_{1,4}(\mathbb{R}) \sim 4_2^2, \text{Cl}_{1,5}(\mathbb{R}) \sim 8_1^2, \\ \text{Cl}_{2,2}(\mathbb{R}) &\sim 2_1^2, \text{Cl}_{2,3}(\mathbb{R}) \sim 4_1^2, \text{Cl}_{3,2}(\mathbb{R}) \sim 2_2^2, \text{Cl}_{2,4}(\mathbb{R}) \sim 8_1^2, \\ \text{Cl}_{3,3}(\mathbb{R}) &\sim 4_1^2, \text{Cl}_{4,2}(\mathbb{R}) \sim 4_1^2. \end{aligned}$$

## 6 Online Version

This document may be updated online at [here](#).

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