Quantum Field Theory in Condensed Matter Physics

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Abstract

This is my study notes of various books, listed in the reference.

Contents

1 Second Quantization

sec:Second-Quantization

The [Nag99] introduces second quantization in a heuristic, non-rigorous way. It starts from conjecturing that

- $N_n \approx \hat{N}_n$,

  N_n represents N_n times repreatance of the experiments done in on a single particle. $\hat{N}_n$ represents the results from one experiments done on a system of $\hat{N}_n$ non-interacting particles.

  This equality means that we expect these two kinds of experiments to be approximately equivalent.

With such spirit, we first examine the single-particle state and tries to promote something to many-body picture. In single particle state, we have

$$\psi(\mathbf{r}, t) = \sum_n a_n(t)\phi_n(\mathbf{r}) \tag{1.0.1}$$

as decomposing any wave function into orthonormal basis and concentrate the dynamics property on the coefficients $a_n(t)$. And we could found [1]

$$\frac{\mathrm{d}a_n(t)}{\mathrm{d}t} = \frac{\partial \langle \hat{H} \rangle}{\partial(i\hbar a_n^*)} \tag{1.0.2}$$

$$\frac{\mathrm{d}(i\hbar a_n^*(t))}{\mathrm{d}t} = -\frac{\partial \langle \hat{H} \rangle}{\partial a_n} \tag{1.0.3}$$

[1] Eq (1.2.5) to eq(1.2.10) of [Nag99]

where $\langle H\rangle = \langle\psi|H|\psi\rangle$. These equations are analogous to Hamilton's canonical equations with $a_n \Leftrightarrow x$, and $i\hbar a_n^* \Leftrightarrow p$. Then propose that we can promote a_n and a_n^* as operators. The promotion is not analysed but we can give a quick analogy as:

- $N_n \equiv N|a_n|^2$ is promoted to $\hat{N}_n$, which is an observable that counts the total number of particles in state n.
- $\sqrt{N}a_n \to \hat{A}_n$, $\sqrt{N}a_n^* \to \hat{A}_n^\dagger$. So that $\hat{N}_n = \hat{A}_n^\dagger \hat{A}_n$.

And

$$\begin{cases} [\hat{A}_n, \hat{A}_m^\dagger] = \delta_{n,m}, [\hat{A}_n, \hat{A}_m] = [\hat{A}_n^\dagger, \hat{A}_n^\dagger] = 0, & \text{for bosons} \\ \{\hat{A}_n, \hat{A}_n^\dagger\} = \delta_{n,m}, \{\hat{A}_n, \hat{A}_m\} = \{\hat{A}_n^\dagger, \hat{A}_n^\dagger\} = 0, & \text{for fermions} \end{cases} \tag{1.0.4}$$

and many other usual commutative/anti-commutative relations.

The *wave picture* starts from promoting ψ and ψ^* into **field operators** :

$$\hat{\psi}(\mathbf{r}) \equiv \sum_n \hat{A}_n \phi_n(\mathbf{r}) \tag{1.0.5}$$

$$\hat{\psi}^\dagger(\mathbf{r}) \equiv \sum_n \hat{A}_n^\dagger \phi_n^*(\mathbf{r}) \tag{1.0.6}$$

This definition is again analogous to the single particle picture. Calculation[2] shows for bosons

$$[\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}') \tag{1.0.7}$$

$$[\hat{\psi}(\mathbf{r}), \hat{\psi}(\mathbf{r}')] = [\hat{\psi}^\dagger(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')] = 0 \tag{1.0.8}$$

with commutators [] replaced with anti-commutators {} for fermions. The **particle density operator** is defined as $n(\mathbf{r}) = \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})$. and the rest are old things.

The operator new to me is the **phase operator** $\hat{\theta}_n$ for bosons [3]. It has the property that:

$$\hat{A}_n^\dagger = \sqrt{\hat{N}_n} e^{-i\hat{\theta}_n/\hbar} \tag{1.0.9}$$

$$\hat{A}_n = \sqrt{\hat{N}_n} e^{i\hat{\theta}_n/\hbar} \tag{1.0.10}$$

$$[\hat{N}_n, \hat{\theta}_n] = i\hbar \tag{1.0.11}$$ eq:Ntheta

It is shonw that equation 1.0.11 leads to $[\hat{A}_n, \hat{A}_n^\dagger] = 1$. It is shown also that

$$\exp\left(\frac{i}{\hbar}\hat{\theta}_n\right)(\hat{N}_n)^m \exp\left(-\frac{i}{\hbar}\hat{\theta}_n\right) = (\hat{N}_n + 1)^m \tag{1.0.12}$$

$$\exp\left(\frac{i}{\hbar}\hat{\theta}_n\right)|N_n\rangle \propto |N-1\rangle \tag{1.0.13}$$ eq:eithetaN

It is concluded, because of the above equation 1.0.13 and that there is not state $|-1\rangle$, the $\hat{\theta}_n$ *is not Hermitian.* [4]

[2] Page 15 of [Nag99]
[3] The phase for fermions is complicated and is delegated to the task of chapter 5 of [Nag99].
[4] See page 17 of [Nag99]

References

[Nag99] Naoto Nagaosa. *Review of Quantum Mechanics and Basic Principles of Field Theory.* Springer Berlin Heidelberg, Berlin, Heidelberg, 1999.

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