Group Theory in Physics (Course Note)

Taper

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Abstract

This is our course note for the course about group theoy, with its application in physics.

Contents

Information about the class

1. Final project: Combine group theory with you research.
2. Mid-term and final exam.
3. No homework, cause it is already graduate level.
4. Professor Fei Ye. (Phone: 88018229, 228 in Research building 2), T.A. Zhe Zhang. (110 Research building 2).

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He first introduces several common examples of symmetries in our life and physics. Omitted, with one exception:

He mentions that there is one more symmetry in the Hydrogen Hamiltonian: the Laplace-Runge-Lenz symmetry. (So its symmetry group is not just $SO(3)$, but two copies of $SO(3)$ that forms a $SO(4)$. And using the representation of $SO(4)$, the complete spectrum of Hydrogen Hamiltonian is solved. Hence this $SO(4)$ is the largest symmetry of Hydrogen Hamiltonian.

1.1 Digression about Lenz vector

Since the class is too boring, I checked about the Lenz vector via Google and found this Math.SE question [2]

The first answer to that post is:

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1) **Problem**. The Kepler Problem has Hamiltonian

$$H \ := \ \frac{p^2}{2m} - \frac{k}{q},$$

where m is the 2-body reduced mass. The [Laplace–Runge–Lenz vector](http://en.wikipedia.org/wiki/Laplace

$$A^j \ := \ a^j + km\frac{q^j}{q}, \qquad a^j \ := \ (\mathbf{L}\times\mathbf{p})^j \ = \ \mathbf{q}\cdot\mathbf{p}\, p^j - p^2\, q^j, \qquad \mathbf{L} \ := \ \mathbf{q}\times\mathbf{p}.$$

2) **Action**. The Hamiltonian Lagrangian is

$$L_H \ := \ \dot{\mathbf{q}}\cdot\mathbf{p} - H,$$

and the action is

$$S[\mathbf{q},\mathbf{p}] \;=\; \int \mathrm{d}t\; L_H.$$

The non-zero fundamental canonical Poisson brackets are

$$\{q^i,p^j\} \;=\; \delta^{ij}.$$

3) **Inverse Noether's Theorem**. Quite generally in the Hamiltonian formulation, given a constant of motion Q, then the infinitesimal variation

$$\delta \;=\; \varepsilon\{Q,\cdot\}$$

is a global off-shell symmetry of the action S (modulo boundary terms). Here ε is an infinitesimal global parameter, and $X_Q = \{Q,\cdot\}$ is a Hamiltonian vector field with Hamiltonian generator Q. The full Noether current is (minus) Q, see e.g. my answer to [this question](http://physics.stackexchange.com/q/8626/2451). (The words **on-shell**. and **off-shell**. refer to whether the equations of motion are satisfied or not.)
4) **Variation**. Let us check that the three Laplace–Runge–Lenz components A^j are Hamiltonian generators of three continuous global off-shell symmetries of the action S. In detail, the infinitesimal variations $\delta = \varepsilon_j\{A^j,\cdot\}$ read

$$\delta q^i \;=\; \varepsilon_j\{A^j,q^i\}, \qquad \{A^j,q^i\} \;=\; 2p^iq^j - q^ip^j - \mathbf{q}\cdot\mathbf{p}\;\delta^{ij},$$

$$\delta p^i \;=\; \varepsilon_j\{A^j,p^i\}, \qquad \{A^j,p^i\} \;=\; p^ip^j - p^2\,\delta^{ij} + km\left(\frac{\delta^{ij}}{q} - \frac{q^iq^j}{q^3}\right),$$

$$\delta t \;=\; 0,$$

where ε_j are three infinitesimal parameters.
5) Notice for later that

$$\mathbf{q}\cdot\delta\mathbf{q} \;=\; \varepsilon_j(\mathbf{q}\cdot\mathbf{p}\;q^j - q^2\;p^j),$$

$$\mathbf{p}\cdot\delta\mathbf{p} \;=\; \varepsilon_j km(\frac{p^j}{q} - \frac{\mathbf{q}\cdot\mathbf{p}\;q^j}{q^3}) \;=\; -\frac{km}{q^3}\mathbf{q}\cdot\delta\mathbf{q},$$

$$\mathbf{q}\cdot\delta\mathbf{p} \;=\; \varepsilon_j(\mathbf{q}\cdot\mathbf{p}\;p^j - p^2\;q^j) \;=\; \varepsilon_j a^j,$$

$$\mathbf{p}\cdot\delta\mathbf{q} \;=\; 2\varepsilon_j(p^2\;q^j - \mathbf{q}\cdot\mathbf{p}\;p^j) \;=\; -2\varepsilon_j a^j\;.$$

6) The Hamiltonian is invariant

$$\delta H \;=\; \frac{1}{m}\mathbf{p}\cdot\delta\mathbf{p} + \frac{k}{q^3}\mathbf{q}\cdot\delta\mathbf{q} \;=\; 0,$$

showing that the Laplace–Runge–Lenz vector A^j is classically a constant of motion

$$\frac{dA^j}{dt} \approx \{A^j, H\} + \frac{\partial A^j}{\partial t} = 0.$$

(We will use the $\approx$ sign to stress that an equation is an on-shell equation.)

7) The variation of the Hamiltonian Lagrangian L_H is a total time derivative

$$\begin{aligned}
\delta L_H &= \delta(\dot{\mathbf{q}} \cdot \mathbf{p}) = \dot{\mathbf{q}} \cdot \delta\mathbf{p} - \dot{\mathbf{p}} \cdot \delta\mathbf{q} + \frac{d(\mathbf{p} \cdot \delta\mathbf{q})}{dt} \\
&= \varepsilon_j \left(\dot{\mathbf{q}} \cdot \mathbf{p}\, p^j - p^2\, \dot{q}^j + km \left(\frac{\dot{q}^j}{q} - \frac{\mathbf{q} \cdot \dot{\mathbf{q}}\, q^j}{q^3} \right) \right) \\
&\quad -\varepsilon_j \left(2\dot{\mathbf{p}} \cdot \mathbf{p}\, q^j - \dot{\mathbf{p}} \cdot \mathbf{q}\, p^j - \mathbf{p} \cdot \mathbf{q}\, \dot{p}^j \right) - 2\varepsilon_j \frac{da^j}{dt} \\
&= \varepsilon_j \frac{df^j}{dt}, \qquad f^j := A^j - 2a^j,
\end{aligned}$$

and hence the action S is invariant off-shell up to boundary terms.

8) **Noether current**. The bare Noether current j^k is

$$j^k := \frac{\partial L_H}{\partial \dot{q}^i}\{A^k, q^i\} + \frac{\partial L_H}{\partial \dot{p}^i}\{A^k, p^i\} = p^i\{A^k, q^i\} = -2a^k.$$

The full Noether current J^k (which takes the total time-derivative into account) becomes (minus) the Laplace–Runge–Lenz vector

$$J^k := j^k - f^k = -2a^k - (A^k - 2a^k) = -A^k.$$

J^k is conserved on-shell

$$\frac{dJ^k}{dt} \approx 0,$$

due to Noether's first Theorem. Here k is an index that labels the three symmetries.

However, I don't really understand the content inside. I asked professor Ye whether we can find some physics about this conserved quantity, and he answered with no.

The next answer is also interesting:

While Kepler second law is simply a statement of the conservation of angular momentum (and as such it holds for all systems described by central forces), the first and the third laws are special and are linked with the unique form of the newtonian potential $-k/r$. In particular, Bertrand theorem assures that *only* the newtonian potential and the harmonic potential kr^2 give rise to closed orbits (no precession). It is natural to think that this must be due to some kind of symmetry of the problem. In fact, the particular symmetry of the newtonian potential is described exactly by the conservation of the RL vector (it can be shown that the RL vector is conserved iff the potential is central and newtonian). This, in turn, is due to a more general symmetry: if conservation of angular momentum is linked to the group of special orthogonal transformations in 3-dimensional space $SO(3)$, conservation of the RL vector must be linked to a 6-dimensional group of symmetries, since in this case there are apparently six conserved quantities (3 components of L and 3 components of $\mathcal{A}$). In the case of bound orbits, this group is $SO(4)$, the group of rotations in 4-dimensional space.

Just to fix the notation, the RL vector is:

$$\mathcal{A} = \mathbf{p} \times \mathbf{L} - \frac{km}{r}\mathbf{x} \tag{1.1.1}$$

Calculate its total derivative:

$$\frac{d\mathcal{A}}{dt} = -\nabla U \times (\mathbf{x} \times \mathbf{p}) + \mathbf{p} \times \frac{d\mathbf{L}}{dt} - \frac{k\mathbf{p}}{r} + \frac{km(\mathbf{p}\cdot\mathbf{x})}{r^3}\mathbf{x} \tag{1.1.2}$$

Make use of Levi-Civita symbol to develop the cross terms:

$$\epsilon_{sjk}\epsilon_{sil} = \delta_{ji}\delta_{kl} - \delta_{jl}\delta_{ki} \tag{1.1.3}$$

Finally:

$$\frac{d\mathcal{A}}{dt} = \left(\mathbf{x}\cdot\nabla U - \frac{k}{r}\right)\mathbf{p} + \left[(\mathbf{p}\cdot\mathbf{x})\frac{k}{r^3} - 2\mathbf{p}\cdot\nabla U\right]\mathbf{x} + (\mathbf{p}\cdot\mathbf{x})\nabla U \tag{1.1.4}$$

Now, if the potential $U = U(r)$ is central:

$$(\nabla U)_j = \frac{\partial U}{\partial x_j} = \frac{dU}{dr}\frac{\partial r}{\partial x_j} = \frac{dU}{dr}\frac{x_j}{r} \tag{1.1.5}$$

so

$$\nabla U = \frac{dU}{dr}\frac{\mathbf{x}}{r} \tag{1.1.6}$$

Substituting back:

$$\frac{d\mathcal{A}}{dt} = \frac{1}{r}\left(\frac{dU}{dr} - \frac{k}{r^2}\right)\left[r^2\mathbf{p} - (\mathbf{x}\cdot\mathbf{p})\mathbf{x}\right] \tag{1.1.7}$$

Now, you see that if U has *exactly* the newtonian form then the first parenthesis is zero and so the RL vector is conserved. Maybe there's some slicker way to see it (Poisson brackets?), but this works anyway.

1.2 Coming back to the course

After mentioning the Poincáre group, he produces to review some concepts about linear algebra:

1. The axioms of linear space, using quantum mechanics as basic example (Omitted).
2. Some common concepts of linear space: linear-independence, subspace, direct sum, linear operators, its matrix representation. (Omitted)
3. Introducing the complete antisymmetric tensor $\epsilon^{a_1,\cdots,a_n}$. Some properties:

$$\frac{1}{(m-n)!}\sum_{a_{n+1},\cdots,a_m}\epsilon_{a_1,\cdots,a_n,a_{n+1},a_m}\epsilon_{b_1,\cdots,b_n,a_{n+1},a_m} = \sum_{p_1,\cdots,p_n}\epsilon_{p_1,\cdots,p_n}\delta_{a_1,b_{p_1}}\cdots\delta_{a_n,b_{p_n}} \tag{1.2.1}$$

$$\epsilon_{ab}\epsilon_{rs} = \delta_{ar}\delta_{bs} - \delta_{as}\delta_{br} \tag{1.2.2}$$

$$\sum_d \epsilon_{abd}\epsilon_{rsd} = \delta_{ar}\delta_{bs} + \delta_{as}\delta_{br} \tag{1.2.3}$$

4. Some special matrices.
5. Fact: If $R\Gamma = \Gamma R$, and Γ is diagonal. (let $\mu \neq \nu$) Then if $\Gamma_{\mu\mu} \neq \Gamma_{\nu\nu}$, we have: $R_{\mu\nu} = R_{\nu\mu} = 0$. On the other hand, if $R_{\mu\nu} \neq 0$, then $\Gamma_{\mu\mu} = \Gamma_{\nu\nu}$. This is obviously from:

$$\sum_j R^i_j\Gamma^j_k = \sum_j \Gamma^i_j R^j_k \Longrightarrow R^i_k\Gamma^k_k = \Gamma^i_i R^i_k$$

   where the first is automatically summed, and the second is not.
6. A linear functional is closed w.r.t. a vector space. (Omitted)
7. ... then this linear functional can be expressed as a matrix w.r.t to a basis of this vector space. (Omitted)
8. Invariant subspace. (Omitted)
9. Transformation of basis. (Omitted)
10. Direct sum of operators:
    Let vector spaces $L = L_1 \oplus L_2$, with $L = \langle e_i\rangle$, $L_1 = \langle e'_1, \cdots e'_n\rangle$,$L_2 = \langle e'_{n+1}, \cdots, e'_m\rangle$, $e'_\nu = \sum_\mu e_\mu S_{\mu\nu}$. Assume that L_1, L_2 are invariant w.r.t A, an linear operator. If:

$$Ae'_\mu = \sum_{\nu=1}^{m} e'_\nu R'_{\nu\mu} \tag{1.2.4}$$

we have obviously:

$$Ae'_\mu = \sum_{\nu=1}^{n} e'_\nu R'_{\nu\mu} \text{for } \mu \in \{1 \cdots n\} \tag{1.2.5}$$

$$Ae'_\mu = \sum_{\nu=n}^{m} e'_\nu R'_{\nu\mu} \text{for } \mu \in \{n \cdots m\} \tag{1.2.6}$$

i.e., A's matrix representation has two diagonal blocks. Using this fact, A after a linear transformation (by S), could be written as $R_1 \oplus R_2$, where the meaning of R_1/R_2 is obvious.

11. Eigenvalues and the characteristic equation. (Omitted) Some properties:
    (a) Trace $= \sum_i \lambda_i$
    (b) Determinant $= \prod_i \lambda_i$
    (c) Geometric multiplicity $\leq$ Algebraic multiplicity, or

    $$\dim V_{\lambda_1} \leq n_1$$

    .

12. Inner product and orthonormal basis. (Omitted) Here we define matrix Ω to be, when a basis $\{e_i\}$is given:

    Definition 1.1.

    $$\Omega_{ij} \equiv \langle e_i, e_j \rangle \tag{1.2.7}$$

13. Adjoint operator:
    Let A be a linear operator represented by matrix A^i_j. Let its adjoint $A^\dagger$ be represented by R^i_J. Then using $\langle A^\dagger e_j, e_i \rangle = \langle e_j, Ae_i \rangle$, we will get $(R^k_j)^* \Omega_{ki} = \Omega_{jk} A^k_i$, i.e. $(R^T)^* \Omega = \Omega A$, so:

    $$R = \Omega^{-1} A^\dagger \Omega \tag{1.2.8}$$

    where we have used the fact that $\Omega^\dagger = \Omega$.
    Note that $(R^k_j)^* \Omega_{ki}$ is not $\Omega^T R^*$. (Be careful and you will find out why.)
    This is very different from my previous naive concept when Ω is not identity matrix, i.e. when the basis is not orthonormal.

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He first introduces some important matrices:

2.1 Some Important Matrices

Unitary matrix Eigenvalues of Unitary matrices has modulus 1, i.e. $|\lambda| = 1$. This can be proved directly. Also, Unitary matrices are unitarily diagonalizable. This is a result of the following Spectral Theorem:

Theorem 2.1 (Spectral Theorem)**.** *One matrix A is normal (i.e. $A^\dagger A = AA^\dagger$), if and only if it is unitarily diagonalizable.*

Proof. If A is normal, then by Schur decomposition, we can write $A = UTU^\dagger$, here U is unitary and T is upper-triangular. Using the condition of being normal, one can show directly that T is in fact also normal. Now we show that any triangular matrix that is normal must be diagonal. Observe that we have $\langle e_i, T^\dagger T e_i\rangle = \langle e_i, TT^\dagger e_i\rangle$, i.e. $\langle T^\dagger e_i, T^\dagger e_i\rangle = \langle Te_i, Te_i\rangle$. This is saying that the norm of the first column of $A^\dagger$ is equal to the norm of the first column of A. Obviously A has to be diagonal.

The converse is obvious. □

Also, unitary matrix's eigenvector corresponding to different eigenvalues are orthogonal. This is a direct result of fact mentioned above.

Hermitian matrices They have real eigenvalues and orthogonal eigenvectors (proof omitted). Also, if $\det(R^\dagger R) \neq 0$, then $R^\dagger R > 0$, i.e. it is positive-definite.

This is wong: An example is that the matrix Ω introduced in the previous lecture has $\det(\Omega^\dagger \Omega) = \det(\Omega)$, hence $\det(\Omega) = 1$ (it cannot be 0), hence it is positive definite.

Actually $\det(\Omega^\dagger\Omega) \neq \det(\Omega)$, because

$$\sum_\rho |e_\rho\rangle\langle e_\rho| \neq 1 (\text{unless the basis is orthonormal}) \tag{2.1.1}$$

Therefore we need anthoer argument for Ω being positive-definite. It is provided in page 11 of [1].

Orthogonal matrix For an orthogonal matrix over $\mathbb{C}$, it is quite troublesome. For example, if $Ra = \lambda a$ and $\lambda \neq \pm 1$, then we have $a^T a = 0$, which is quite bad because this force a to have complex components.

Orthogonal matrix over $\mathbb{R}$ In this case, we have similar result. But it is easy to show that for an orthogonal matrix R having only real elements, then its eigenvalues $\lambda = \pm 1$.

Then he proceeds to direct product.

Direct product and also the Kronecker Product of two matrices. Properties (let $T = R \otimes S$):

1. $\dim T = \dim R \times \dim S$
2. $\mathrm{tr}(T) = \mathrm{tr}(R)\mathrm{tr}(S)$
3. $\otimes$ commutes with the operation of inverse, transpose, and transpose conjugation.
4. $$\frac{d}{d\alpha}(R(\alpha) \otimes S(\alpha)) = R'(\alpha) \otimes S(\alpha) + R(\alpha) \otimes S'(\alpha) \tag{2.1.2}$$
5. when the dimentions are the same:
   (a) $(R_1 \otimes S_1)(R_2 \otimes S_2) = (R_1 R_2) \otimes (S_1 S_2)$

2.2 Symmetry and Group

Finally we arrived in the group theory.

Symmetry examples Dipole transition. $\langle\phi_f|\hat{P}|\phi_i\rangle$, must happen when the parity of ϕ_i and ϕ_f is of opposite parity. (pp.18 of [1])

Group

Definition 2.1 (Group)**.** Omitted.

Some basic properties (Omitted).

Definition 2.2 (Abel Group)**.** Omitted.

Definition 2.3 (Cardinality of group $\#A$)**.** Omitted.

Multiplication table Facts: group of order $1, 2$, and 3 are unique up to an isomorphism.

Definition 2.4 (Cyclic group, generators)**.** Omitted.

固有转动是指的那些 $det(M) > 0$ 的转动. 用C_n来表示他们.
Also, 周期 of R is just $\langle R\rangle$.
Let σ for spatial reflection.

Definition 2.5 $(C_N, \bar{C}_N)$**.** $\bar{C}_N = C_N * \sigma$

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3.1 Common Concepts in Group

Introducing to various groups:S_4, V_4, D_3, all omited. (**pp.22-23 of [1]**)

D_n group. See pp. 25-26 of the book [1]. Note that here the n refers to the n-polygon, not that the group is of order n. For the mathematicians, they might be comfortable with D_n means the dihedral group of order n, but is actually the group of symmetries of $n/2$-polygon.

Definition 3.1 (Subgroup)**.** Omitted.

Fact 3.1. One only has to check the closeness for determining a subgroup, if it is of finite order.

However, for group of infinite order, one has to check the existance of unit and inverse elements.

Examples of subgroup (**pp.26 of [1]**)
Noteworth:C_6 has three copies of D_2, this can be intuitively guessed by the fact that a hexago has three rectanle in it.

Definition 3.2 (Coset)**.** Omitted.

Properties of coset (omitted).

Definition 3.3 (Index of subgroup)**.** Omitted.

Proposition 3.1. *Two elements R and T belongs to the same coset kH, if and only if $R^{-1}T \in H$.*

Proof. Omitted. □

Definition 3.4 (Normal/Invariant subgroup)**.** A subgroup is normal or nnvariant subgrouproup(also invariant), if and only if for any $x \in G$, we have $xH = Hx$.

Fact 3.2. If H has index 2, then it must be normal/invariant. This is obvious.

Definition 3.5 (Quotient)**.** Omitted.

Note that quotient group (a.k.a. factor group) is only defined for a normal subgroup.

Example 3.1. D_3 (Using the multiplication table). Omitted because it is too complex to be typed down here.

3.2 Conjugacy classes

Definition 3.6 (Conjugate)**.** If exists $S \in G$, s.t. $R' = S^{-1}RS$, then we say R' is conjugate to R.

see (**pp.28-30 of [1]**) This is clearly an equivalence relationship. By this we can define conjugate class, denoted by $C = \{R_1, \cdots\}$, then we have the characterization $C = \{s^{-1}R_i s | \forall s \in G\}$, for any R_i. We then have the following facts (all are obvious):

Fact 3.3. The unit class formed just by the unit element.

Fact 3.4. The inverse class formed by just all the inverse element. $C^{-1} = \{R_i^{-1}\}$. If $C \cap C^{-1} \neq \varnothing$, then $C = C^{-1}$, C is called self-inverse.

Fact 3.5. The order of elements in a class is just the same.

Fact 3.6. For $\forall T, S \in G$, TS and ST are conjugate to each other. This means that all elements symmetric on the multiplication table is conjugate to each other.

Fact 3.7. For two elements R, R' conjugate to each other, both can be expressed by the products of two elements in two different way. (Isn't this too obvious to mention.)

Fact 3.8. Let G be a rotation group. Suppose it has an axis of the order of n, with its operation denoted as R,, we can get a new axis by the following steps. (Supose we have another rotation S),

1. S^{-1}, rotate m back to n.
2. R, rotate about n around $2\pi/n$,
3. S rotate n to m

Result: $S^{-1}RS$ rotate around a new axis m about $2\pi/n$. So $R' = S^{-1}RS$ and R is calle the equivalent axis.

Also, if $m = -n$, then they are called polar axis to each other.

Fact 3.9. C_n, which is an abel group, every element form a conjugate class by itself. Specifically, e and $R^{n/2}$ are self inverse, if n is even.

Proposition 3.2. *For an normal subgroup, every conjugate element is also inside the same normal subgroup. This indicates that an normal subgroup can be decomposed into a series of sum of conjugate classes.*

Proof. If $R \in H$, then we show that $S^{-1}RS \in H$, this is obviously since it belongs to $S^{-1}HS$. □

Example 3.2. For D_3 : E, D, F, A, B, C, their orders are respetively $1, 3, 3, 2, 2, 2$. We have the following conjugate classes:

1. $\{E\}$. Is self inverse.
2. $\{D, F\}$. D is conjugate to F. We can see this physically by looking at rotation from the front or the below. This class is also self-inverse.
3. $\{A, B, C\}$, is clearly a self-inverse conjugate class.

Example 3.3 (D_6)**.** Ramiliarize one with the formulae for D_n. Hint: use the order of elements to find classes of conjugate. Then use the proposition 3.2 to find the subgroups.

4 Skipped Two lectures

Due to GRE physics preparation. Concepts that may have been covered:

- Conjugacy classes
- Representation of Groups.
  - Character of representation. (Notes for this part is delayed to be contained in the next lecture's note).
  - Equivalence between representation.
- Transformation of Fields
- Group Algebra and Regular Representation.

4.1 Conjugacy classes (continued)

One important group is the symmetric group. It is important because that every finite group of order n can be embedded inside the symmetric group S_n (Cayley's theorem) [1].

The conjugacy classes in symemtric is pretty easy to find. What we need to know is the following observation (from [3])

Key Point 4.1. Consider two permutation π and σ:

$$\pi = \begin{pmatrix} 1 & \cdots & n \\ \pi(1) & \cdots & \pi(n) \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{pmatrix} \tag{4.1.1}$$

Then we have by direct calculation:

$$\sigma\pi\sigma^{-1} = \begin{pmatrix} \sigma(1) & \cdots & \sigma(n) \\ \sigma(\pi(1)) & \cdots & \sigma(\pi(n)) \end{pmatrix} \tag{4.1.2}$$

Therefore, the cycle structure of π is unchanged under any "conjugacy transformation".

[1] However, it is hard sometimes to find the smallest possible symmetric group to embed into. For example, S_3 has 6 elements and can thus be embedded into S_6. But obviously it can be embedded better just into itself S_3.

What do I mean by cycle structure? Let's see an example. Suppose we have a cycle (123):

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 2 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd}$$

Then the conjugated map $\sigma(123)\sigma^{-1}$ will have:

$$\begin{tikzcd} & \sigma(1) \arrow[dr] & \\ \sigma(2) \arrow[ur] & & \sigma(3) \arrow[ll] \end{tikzcd}$$

So it has a cycle of $(\sigma(1)\sigma(2)\sigma(3))$. We see that in general, the cycle type is unchanged under conjugacy transformation. [2]

On the other hand, all elements of the same cycle type belong to the same conjugacy class. For example, consider two permutations: $(i_1 i_2 i_3)(i_4 i_5)$ and $(\pi_{i_1}\pi_{i_2}\pi_{i_3})(\pi_{i_4}\pi_{i_5})$, where π permutes the five numbers $i_1 \cdots i_5$. Consider the map:

$$\sigma \equiv \begin{pmatrix} i_1 & \cdots & i_5 \\ \pi_{i_1} & \cdots & \pi_{i_5} \end{pmatrix} \tag{4.1.3}$$

Then

$$\begin{aligned} &\sigma(i_1 i_2 i_3)(i_4 i_5)\sigma^{-1} = \\ &\begin{pmatrix} i_1 & \cdots & i_5 \\ \pi_{i_1} & \cdots & \pi_{i_5} \end{pmatrix} (i_1 i_2 i_3)(i_4 i_5) \begin{pmatrix} \pi_{i_1} & \cdots & \pi_{i_5} \\ i_1 & \cdots & i_5 \end{pmatrix} \\ &= (\pi_{i_1}\pi_{i_2}\pi_{i_3})(\pi_{i_4}\pi_{i_5}) \end{aligned}$$

The proof for the general case is similar. So we have the theorem below:

Theorem 4.1 (Cycle type determines conjugacy class)**.** *Two permutations are conjugate in the symmetric group if and only if they have the same cycle type.*

Remark 4.1. Since a cycle type is just a set of unordered integer partition of number n. The above theorem means that the set of conjugacy classes in the symmetric group on a finite set is in bijection with the set of unordered integer partitions of the size of the set.

4.2 Group Representation

Definition 4.1 (Representation)**.** A representation of a group G is a continuous homomorphism D from G to the group of automorphisms of a vector space V:

$$D : G \mapsto \mathrm{Aut}V \tag{4.2.1}$$

V is called the *represetation space*, and the *dimension of the representation* is the dimension of V.

[2] For a permutation, its cycle type is determined by decomposing it into independent cycles. For independent cycles, the cycle type is unique. For example, (135)(24) has a cycle type of $(3, 2)$, or $(2, 3)$ since the order is irrelevant. But if we decompose it into non-independent cycles, we cannot determine its cycle type. For example, (135) = (15)(13), which is ambiguous if we were to tell the cycle type.

Related concepts. (The following is copied from [3].)

- There is always the representation $D(g) = 1$ for all g. If $\dim V = 1$, this is called the *trivial representation* .
- The matrix groups, i.e. $\mathrm{GL}(n, K)$ and subgroups, naturally have the representation "by themselves", i.e. by$n \times n$ matrices acting on K_n and satisfying the defining constraints (e.g. nonzero determinant). This is loosely called the *fundamental or defining representation.*
- Two representations D and D' are called *equivalent* if they are related by a similarity transformation, i.e. if there is an operator S such that
$$SD(g)S^{-1} = D'(g) \tag{4.2.2}$$
for all g. Note that S does not depend on g! Two equivalent representation can be thought of as the same representation in different bases. We will normally regard equivalent representations as being equal.
Note also here S represents a transformation of vector. If we know the transformation of basis X, then
$$X^{-1}D(g)X = D'(g) \tag{4.2.3}$$
- A representation is called *faithful* if it is injective, i.e. $\ker D = \{e\}$, or in other words, if $D(g_1) \neq D(g_2)$ whenever $g_1 \neq g_2$.
- If V is equipped with a (positive definite) scalar product, D is unitary if it preserves that scalar product, i.e if
$$\langle u, v\rangle = \langle D(g)u, D(g)v\rangle \tag{4.2.4}$$
for all $g \in G$. (Here we assume that V is a complex vector space, as that is the most relevant case. Otherwise one could define orthogonal representations etc.)

4.2.1 Regular Representation

Now we construce a simple representation for all groups. It utilizes a vector space constructed from group itself called Group Algebra.

Definition 4.2 (Group algebra/Monoid algebra)**.** Let A be a commutative ring, G be a monoid, written multiplicatively. Then the monoid ring $A[G]$ consists of those finite formal linear combinations v of the form:
$$v = \sum_{g\in G} v_g g \tag{4.2.5}$$
where $v_g \in A$. The v_g are seen as coefficients and g are seen as the basis vectors that can be multiplied. Soe the addition is defined as:
$$v + w = \sum_{g\in G}(v_g + w_g)g \tag{4.2.6}$$
And the multiplication is
$$vw = \sum_{g,g'\in G}(v_g w_{g'})gg' \tag{4.2.7}$$
If G is a group then the corresponding $A[G]$ is called a *group algebra.*

A rigorious definition would use the concept of function to define the "formal linear combination". Please refer to page 105 (section II.3) of [4].

Remark 4.2. Note that both A and G can be naturally embeded into $A[G]$. This will helps us to define a representation of group G later.

Example 4.1. (From page 106 of [4].) Polynomial rings are special cases. In n variables, consider a multiplicative free abelian group of rank n. Let $X_l, \cdots X_n$ be generators. Let G be the multiplicative subset consisting of elements $X_1^{v_1}, \cdots, X_n^{v_n}$, where $v_i \leq 0$ for all i. Then G is a monoid, and it is eas to verify that $A[G]$ is just $A[X_1, \cdots, X_n]$.

Here we take A to be $\mathbb{C}$, then we have clearly $\mathbb{C}[G]$ a vector space equipped with a bilinear map (the product), i.e. an algebra over a field. Its dimension is clear equal to the order of G. We can also give it an inner product by:

$$\langle v, w \rangle = \sum_{g \in G} v_g^* w_g \tag{4.2.8}$$

Now we can define the regular representation:

Definition 4.3 (Regular Representation)**.** A regular representation of a group G is the following endomorphism of $\mathbb{C}[G]$:

$$D_{\text{reg}} : v \mapsto g \cdot v \tag{4.2.9}$$

where $v \in \mathbb{C}[G]$, $g \in G \hookrightarrow \mathbb{C}[G]$.

Remark 4.3. This representation can be just seen as a permutation of basis vectors, because:

$$g \cdot v = \sum_{h \in G} v_h (gh) = \sum_{h' \in G} v_{g^{-1}h'}\, h' \tag{4.2.10}$$

Therefore, it is unitary.

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5.1 Unitarity of Representation

Theorem 5.1 (Unitary Representation)**.** *For finite groups and for compact Lie groups, all representations are equivalent to a unitary representation.*

Remark 5.1. Before the formal proof, I remarked that since any finite group can be embedded inside the symmetric group (Cayley's theorem), and the symmetric has clearly a unitary representation (perhaps the easiest representation one could ever construct besides the trivial representation), one could naturally guess whether it is possible that any representation could be turned into a unitary one.

As for the compact Lie group case, it is just mentioned in [3] and will not be proved here. It is mentioned in page 25 of [3] that:

> For compact Lie groups, however, there exists a (unique) translationally invariant measure, so we can replace $\sum_g \to \int \mathrm{d}g$ and the integral is convergent because of compactness. Then the proof directly carries over.

Proof. For $D(g)$, we need to find a X such that $\bar{D}(g) \equiv X^{-1}D(g)X$ is unitary.

Since

$$1 = \bar{D}^\dagger \bar{D}$$

One will find

$$(XX^\dagger)^{-1} = D^\dagger (XX^\dagger)^{-1} D$$

Then let

$$H \equiv \sum_{s \in G} D^\dagger(s) D(s) \tag{5.1.1}$$

One can verify that

$$D^\dagger(g) H D(g) = H \tag{5.1.2}$$

Now we construct X from H. We have

$$H = (XX^\dagger)^{-1} \tag{5.1.3}$$

Notice we have H is Hermitian by above equation. Also, H is positive definite (easily seen from the definition of H and $a^\dagger Ha \geq 0$. Also H is of full rank.).

Then we have $UHU^{-1} = \mathrm{diag}\{\gamma_1, \gamma_2, \cdots\}$ and $\gamma_i > 0$. The rest for constructing X should be obvious. □

Remark 5.2 (Examples of Non-unitary representation)**.** It is remard in [3] that for infinite and non-compact groups, the representation are not unitary in general. He gives two examples:

- The group $\mathbb{Z}^*$ acting on $\mathbb{C}$ by mulplication is certain not equivalent to any unitary representation.
- Arugment for a representation of non-compact Lie group cannot be made unitary by a similarity transformation:

  > The group of unitary operators on a finite dimensional vector space is isomorphic to $U(n)$ (for complex vector spaces, $O(n)$ otherwise), and is hence compact. But there cannot be a bijective continuous map between a compact and a non-compact space, so faithful finite-dimensional representations of non-compact groups will be non-unitary.

  But then he mentions something about a perculiar case, the Lorentz group, which I do not understand.

Anyway, he summarised in the following:

> To summarise, for finite groups and for compact Lie groups, all representations are equivalent to a unitary one (and we will usually take them to be unitary from now on). For infinite groups and non-compact Lie groups, on the other hand, finite-dimensional faithful representations are never unitary. Finally, some non-compact groups may have representations which are unitary with respect to a non-definite scalar product, such as the Lorentz group

From class, Ye Fei proved that:

Theorem 5.2. *For any two equivalent representation, there is always a unitary matrix to relate the two, i.e. exits Y unitary, s.t.*

$$\bar{D}(g) = Y^{-1} D(g) Y$$

Proof. Suppose we have two unitary representation $D(g)$ and $\bar{D}(g)$, related by

$$\bar{D}(g) = X^{-1} D(g) X$$

where X is not necessarily unitary.

Let $H \equiv X^\dagger X$, it is easy to prove that H is Hermitian and positive definite. Direct calculation shows that

$$\bar{D}^{-1}(g) H \bar{D}(g) = H$$

That is $\bar{D}(g)$ and H commute. Then we construct Y from H. Let V be such that

$$V^{-1} H V = \Gamma \tag{5.1.4}$$

where Γ is a diagonal matrix of H's eigenvalues. Obviously V has to be unitary. Define $\bar{Y}$ to be

$$\bar{Y} \equiv V \sqrt{\Gamma} V^{-1} \tag{5.1.5}$$

By direct calculation, we have

$$(X\bar{Y})^\dagger (X\bar{Y}) = \mathbb{1}$$

One can show that

$$\bar{Y}^{-1} \bar{D} \bar{Y} = \bar{D}$$

with laborious calculation. Then one can easily verify that $Y \equiv \bar{Y} X$ is the required unitary transformation. □

5.2 Reduciblility of Representations

Definition 5.1 (Reducible/Irreducible Representation)**.** A representation D is called *reducible* if V contains an invariant subspace. Otherwise D is called irreducible.

A representation is called *fully reducible* if V can be written as the direct sum of irreducible invariant subspaces, i.e. $V = V_1 \oplus \cdots \oplus V_p$, all the V_i are invariant and the restriction of D to each V_i is irreducible.

Example 5.1. The regular representation mentioned before is reducible. For example, the vector

$$V = \sum_{g \in G} g \tag{5.2.1}$$

spans an invariant subspace of the operator D_{reg}.

Example 5.2.

- The representation of finite group is obviously fully reducible or irreducible (since a unitary matrix cannot have off diagonal blocks).

- The representation of translation group is not fully reducible. For example, for translation we have: $T_a T_b = T_{a+b}$, One can confirm that the following representation obeys the above relationship:

$$T_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

but this is obviously not fully reducible.

Definition 5.2 (Interwiner)**.** Given two representations D_1 and D_2 acting on V_1 and V_2, an intertwiner between D_1 and D_2 is a linear operator

$$F : V_1 \mapsto V_2 \tag{5.2.2}$$

which "commutes" with G in the sense that

$$FD_1(g) = D_2(g)F \tag{5.2.3}$$

for all $g \in G$.

(From pp.30 of [3])

The existence of an intertwiner has a number of consequences. First, D_1 and D_2 are equivalent exactly if there exists an invertible intertwiner. Second, the kernel and the image of F are invariant subspaces: Assume $v \in \operatorname{Ker} F$, i.e. $Fv = 0$. Then

$$FD_1 v = D_2 F v = D_2 0 = 0 \tag{5.2.4}$$

so $D_1 v \in \operatorname{Ker} F$. On the other hand, let $w_2 = F w_1$ be an arbitrary element of the image of F. Then from the definition we have

$$D_2 w_2 = D_2 F w_1 = F D_1 w_1 \tag{5.2.5}$$

which is again in the image of F. Now if D_1 is irreducible, the only invariant subspaces, hence the only possible kernels, are $\{0\}$ and V_1 itself, so F is either injective or zero. Similarly, if D_2 is irreducible, F is either surjective or zero. Taking these statements together, we arrive at Schur's Lemma:

Lemma 5.1 (Schur's lemma I)**.** *An intertwiner between two irreducible representations is either an isomorphism, in which case the representations are equivalent, or zero*

An important special case is the one where $D_1 = D_2$. In that case, we see that F is essentially unique. More precisely, we have the following theorem, also often called Schur's Lemma:

Lemma 5.2 (Schur's lemma II)**.** *If D is an irreducible finite-dimensional representation on a complex vector space and there is an endomorphism F of V which satisfies*

$$FD(g) = D(g)F \tag{5.2.6}$$

for all $g \in G$, then F is a multiple of the identity, $F = \lambda \mathbb{1}$

Proof. Note that F has at least one eigenvector v with eigenvalue λ. (This is where we need V to be a complex vector space: A real matrix might have complex eigenvalues, and hence no real eigenvectors.) Clearly, $F - \lambda\mathbb{1}$ is also an intertwiner, and it is not an isomorphism since it annihilates v. Hence, by Schur's Lemma, it vanishes, thus $F = \lambda\mathbb{1}$. □

(From the book [1])

5.3 Orthogonality Relations and Counting Irreducible Representations

Theorem 5.3 (Orthogonality Theorem for Representations)**.** *For finite group G, let $D^i(G)$ and $D^j(G)$ be its two irreducible representation. Then, as a vector in group algebra, they have the following orthogonal relationship:*

$$\sum_{h\in G} D^i_{\mu\rho}(h^{-1})D^j_{\nu\lambda}(h) = \frac{N}{d_j}\delta_{ij}\delta_{\mu\nu}\delta_{\rho\lambda} \tag{5.3.1}$$

N is the order of the group, and d_j is the dimension of representation $D^j(G)$. If in addition, the two representations are unitary, we have

$$\sum_{h\in G} D^{i*}_{\mu\rho}(h)D^j_{\nu\lambda}(h) = \frac{N}{d_j}\delta_{ij}\delta_{\mu\nu}\delta_{\rho\lambda} \tag{5.3.2}$$

Proof. **Note**: The following proof is clumsy and only applies for the unitary case. For a good proof, please refer to page 44 of [3].

Let

$$Y^{\mu\nu}_{\rho\lambda} \equiv \delta_{\rho\lambda}\delta_{\mu\nu} \tag{5.3.3}$$

Then let

$$X^{\mu\nu} = \sum_{h\in G} D^i(h^{-1})Y^{\mu\nu}D^j(h)$$

One can find by direct calculation

$$X^{\mu\nu}_{\rho\lambda} = \sum_{h\in G} D^{i*}_{\mu\rho}(h)D^j_{\nu\lambda}(h) \tag{5.3.4}$$

And also through direct calculation, one finds

$$D^i(s)X^{\mu\nu} = X^{\mu\nu}D^j(s)$$

for any $s \in G$. Then $X^{\mu\nu}$ is a interwiner. So the case for $i \neq j$ is obvious. When $i = j$, we have:

$$X = \lambda\mathbb{1}$$

Now we find the λ, i.e. the eigenvalue of $X^{\mu\nu}$.

Now since

$$X^{\mu\nu}_{\rho\lambda} = \lambda^{\mu\nu}\delta_{\rho\lambda}$$

One can find two fact by direct calculation:

$$\sum_\rho X^{\mu\nu}_{\rho\lambda} = d_j\lambda^{\mu\nu}$$

$$\sum_\rho X^{\mu\nu}_{\rho\lambda} = N\delta^{\mu\nu}$$

Hence $\lambda^{\mu\nu} = \frac{N}{d_j}\delta^{\mu\nu}$. □

The above relation gives us the first clue to the total number of irreducible representations by the following corollary.

Corollary 5.1.

$$\sum_j d_j^2 = N = |G| \tag{5.3.5}$$

where the index j runs over all possible irreducible representations.

Proof. Define the following vector in group algebra

$$v^i_{\mu\nu} = \sqrt{\frac{d_j}{N}} \sum_{h \in G} \left(D^i(h) \right)_{\mu\nu} h \tag{5.3.6}$$

Then by the orthogonality theorem, this vector is orthonomal in vector space/group algebra $\mathbb{C}[G]$. But the dimension of this vector space is N, and for each representation D^i we have d_i^2 vectors like the one above, which are all orthogonal to each other. Note that orthogonal implies linear independence, hence we have:

$$\sum_j d_j^2 \leq N$$

To make the inequality an equality, it is suffice to prove that such $v^i_{\mu\nu}$ forms a basis of the group algebra $\mathbb{C}[G]$. In page 45 to 46 of [3], he shows that for any $g \in G \hookrightarrow \mathbb{C}[G]$, one has

$$g = \sum_{h \in G} \left(D_{\text{reg}}(h) \right)_{ge} h = \sum_{h \in G} \left[\sum_{j,\mu\nu} c^j_{\mu\nu} \left(D^j(h) \right)_{\mu\nu} \right] h \tag{5.3.7}$$

where e is the unit in the group. c are some hard to tell constants. To show this, [3] uses that fact that D_{reg} is unitary, so it is completely reducible into irreducible components, which have to be (some of) the D^i, i.e. $D_{\text{reg}} = U(D^{i_1} \oplus D^{i_2} \oplus \cdots)U^\dagger$. The detailed proof is not reproduced here.

Then, g is a linear combination with coefficients in $\left(D^j(h) \right)_{\mu\nu}$, hence a linear combination of $v^i_{\mu\nu}$. Since g can be any basis in the group algebra, this shows that the $\{v^i_{\mu\nu}\}$ is complete. □

Remark 5.3. The above proof shows in some sense that the regular representation contains all irreducible representation as components. Since all $v^i_{\mu\nu}$ appears in the regular representation.

5.4 Characters

Definition 5.3 (Character of Representation)**.** Given a representation D of G over vector space on field K, the character $\chi : G \to K$ is defined as the trace

$$\chi(g) = \operatorname{Tr} D(g) \tag{5.4.1}$$

Remark 5.4. Since trace is invariant under similarity transformation (remember that $\operatorname{Tr} AB = \operatorname{Tr} BA$), trace can be used to distinguishes between non-equivalent representations. Hence it is called the character of that representation.

Using the orthogonality relation in Section 5.3 it is esay to see we have

Corollary 5.2.

$$\sum_{h \in G} \chi^i(h^{-1})\chi^j(h) = N\delta_{ij} \tag{5.4.2}$$

Or when the representations are unitary:

$$\sum_{h \in G} \chi^{i*}(h)\chi^j(h) = N\delta_{ij} \tag{5.4.3}$$

Here again $N = |G|$.

Let the conjugacy classes be labeled by K_a, $a = 1, \cdots, k$. k is the number of conjugacy classes. Let n_a be the number of group elements in each conjugacy class K_a. Consider one thing:

- Characters is invariant within each conjugacy classes

One can derive (pp. 47 of [3])

$$\sum_a n_a \chi^{i,a}(\chi^{j,a})^* = N\delta^{ij} \tag{5.4.4}$$

here $\chi^{i,a}$ menas the character in the i's irreducible representation of the group element in a's conjugacy class. Then the vectors (labeled by each irreducible representation i)

$$\frac{1}{\sqrt{N}}(\sqrt{n_1}\chi^{i,1}, \cdots, \sqrt{n_k}\chi^{i,k}) \tag{5.4.5}$$

are orthonomal. Notice that given a k-dimensional space, we in effect show cases r linear independence vectors. Therefore we must have

$$r \leq k \tag{5.4.6}$$

5.4.1 Finding Components of Representations

Remark 5.5. It's easy to see that if $D = aD^1 \oplus bD^2$, one has $\chi = a\chi^1 + b\chi^2$.

Now we consider the coefficients in the general case. Suppose a representation D has:

$$D = \oplus a^i D^i \tag{5.4.7}$$

where the sum is taken over all irreducible representations D^i. We have:

$$X = \sum_i a^i \chi^i \tag{5.4.8}$$

It is calculated (page 47 to 48 of [3]) using equation 5.4.4, that

$$\sum_a n_a \chi^{i,a*} X^a = N a^i \tag{5.4.9}$$

so

$$a^i = \frac{1}{N}\sum_a n_a (\chi^{i,a})^* X^a \tag{5.4.10}$$

From now on, we always denote the class of identity by K_1, and the corresponding character in that representation by $\chi^{i,1}$. Clearly $\chi^{i,1} \equiv d_i$.

Example 5.3. If D is the regular representation. It is easy to see that

$$X_{\text{reg}}(g) = N\delta_{g,e} \tag{5.4.11}$$

where e is the unit element in the group. Then

$$a^i = \frac{1}{N}\sum_a n_a(\chi^{i,a})^* X^a_{\text{reg}} = \frac{1}{N}(\chi^{i,1})^* X^1_{\text{reg}} = \frac{1}{N} d_i N = d_i \tag{5.4.12}$$

Remark 5.6 (Criterion for irreduciblility)**.** The above result gives us one way to assess the level of irreduciblility of a representation. Direct calculation shows (with the help of equation 5.4.10)

$$\sum_a n_a(X^a)^* X^a = N\sum_i (a^i)^2 \tag{5.4.13}$$

So for a irreducible representation X, we have $\sum_a n_a(X^a)^* X^a = N$. For other representations, we have $\sum_a n_a(X^a)^* X^a = N/2N/3N/\cdots$. In this way one somehow measures qualitatively how complex this representation is.

It is also very easy to use the idea of equation 5.4.8 to the following result:

Theorem 5.4. *Two representations are equivalent if and only if they have the same character function.*

5.4.2 Conjugacy Classes and Representations

Now we proves an refined version of inequality 5.4.6:

Theorem 5.5. *For a group G, we have*

$$r = k \tag{5.4.14}$$

Here r is the number of different irreducible representations of G, k is the number of different conjugacy classes of G.

Example 5.4. It is interesting for symmetric group. Because we have known from remark 4.1 that the number of symmetric group's conjugacy class is an combinatorical problem. So its number of irreducible representation is also a combinatorical number.

Proof. To prove the theorem, we somehow reverse the logic. We want to show that there are k vectors, labled by conjugacy classes, in a r-dimensional vector space. These k vectors forms someting resembles $\sum_i \chi^{i,a}\chi^{i,b} \approx \delta^{ab}$.

We first look at the regular representation. The previous section shows (by equation 5.4.8, 5.4.11, and 5.4.12):

$$\sum_{i\in \text{ I.R.}} \chi^{i,1}\chi^{i,a} = N\delta_{a,e} \tag{5.4.15}$$

Here i denotes different irreducible representations, a for conjugacy classes. The delta function is 1 only when a is the unit conjugacy class.

This shows that character of the identity, considered as a r-component vector in the space of representation, is orthogonal to the other characters. We have to extend this to all classes.

Now consider the class vectors:

Definition 5.4 (Class vector $\mathcal{K}_a$)**.**

$$\mathcal{K}_a = \sum_{g \in K_a} g \tag{5.4.16}$$

where a represents the conjugacy class labeled by K_a.

It has the following property: it is invariant under conjugation

$$h\mathcal{K}_a h^{-1} = \mathcal{K}_a \tag{5.4.17}$$

since a conjugacy class are those elements that can be related by a conjugation. From this, we have obvioiusly

$$g\mathcal{K}_a\mathcal{K}_b g^{-1} = \mathcal{K}_a\mathcal{K}_b \tag{5.4.18}$$

Thus $\mathcal{K}_a\mathcal{K}_b$ is also a vector whose components are related to each other with conjating like relations. It is then a linear combination of class vectors:

$$\mathcal{K}_a\mathcal{K}_b = \sum_c c_{abc}\mathcal{K}_c \tag{5.4.19}$$

where c_{abc} are coefficients. (it is mentioned that such a form will make those classes an algebra, named class algebra, but now I will not talk on that.

Now, consider the class vectors $\mathcal{K}_a$ in certain irreducible representation μ:

$$D^a_{(\mu)} := \sum_{g \in \mathcal{K}_a} D_{(\mu)}(g) \tag{5.4.20}$$

It commutes with all $D_{(\mu)}(h)$, $h \in G$, since

$$D_{(\mu)}(h) \sum_{g \in \mathcal{K}_a} D_{(\mu)}(g) D_{(\mu)}(h^{-1}) = \sum_{g \in \mathcal{K}_a} D_{(\mu)}(g)$$

Then by Schur's lemma, $D^a_{(\mu)} = \lambda^a_\mu \mathbb{1}$. Take the trace of $D^a_{(\mu)}$, one will easily find:

$$\lambda^a_\mu = \frac{n_a \chi^a_\mu}{\chi^1_\mu} \tag{5.4.21}$$

Meanwhile, by the algebra like relation 5.4.19, one finds:

$$\lambda^a_\mu \lambda^b_\mu = \sum_c c_{abc}\lambda^c_\mu \tag{5.4.22}$$

combine this with the expression for λ: 5.4.21, one has:

$$n_a n_b \chi^a_\mu \chi^b_\mu = \chi^1_\mu \sum_c c_{abc} n_c \chi^c_\mu \tag{5.4.23}$$

Now, one can see this pattern and I would not expatiate on it. The only remaining elements are equation 5.4.15 and the fact that

$$c_{abe} = \delta_{b,a^{-1}} n_a \tag{5.4.24}$$

where e is the unit element in group G, and a^{-1} is the inverse class of a. This is like saying that only the product of a with its inverse class can have unit elements. With these one can easily calculate and get

$$\sum_{\mu \in \text{I.R.}} \chi_\mu^{a*} \chi_\mu^b = \frac{|G|}{n_a} \delta^{ab} \tag{5.4.25}$$

So we have:

$$\sqrt{\frac{n_a}{|G|}} (\chi_1^a, \chi_2^a, \cdots, \chi_r^a) \tag{5.4.26}$$

a series of k orthonomal r-dimensional vectors. Hence

$$k \leq r \tag{5.4.27}$$

Then, together with previous result, we have:

$$r = k \tag{5.4.28}$$

□

Another proof provided by the book [1].

Definition 5.5 (Group function)**.** A $\mathbb{C}$-linear function $F : \mathbb{C}[G] \to \mathbb{C}$ is called a group function.

He first shows that:

Theorem 5.6. *The matrix elements of unitary irreducible representations of finite groups may form an complete orthogonal basis of group algebra. Any group function can be expanded in terms of them.*

Proof. Let $D^j_{\mu\nu}(g)$, where $g \in G$, j labels irreducible representations, $\mu\nu$ labels matrix element, be one such element. Then we have, for any function $F : G \to \mathbb{C}$,

$$F(h) = \sum_{j,\mu\nu} C^j_{\mu\nu} D^j_{\mu\nu}(h) \tag{5.4.29}$$

$$C^j_{\mu\nu} = \frac{d_j}{|G|} \sum_{h \in G} D^j_{\mu\nu}(h)^* F(h) \tag{5.4.30}$$

This can be easily get using the orthonomal relationship. Also notes that we have $\sum_j d_j^2 = |G| = \text{Dim}(\mathbb{C}[G])$, confirming that the dimension of basis agree with the dimension of group algebra. □

Definition 5.6 (Class function)**.** A class function is a group function f, whose satisfies:

$$f(sgs^{-1}) = f(g) \tag{5.4.31}$$

for any $s, g \in G$.

Similarly, we have:

Theorem 5.7. *The character $\chi^j(g)$,where j labels irreducible representation, $g \in G$, of unitary irreducible representations of finite groups may form an complete orthogonal basis of class functions.*

Proof. This is easy if one realize that, since $f(sgs^{-1} = f(g)$ for any $s, g \in G$, one has

$$f(g) = \frac{1}{|G|} \sum_{s \in G} f(sgs^{-1}) \tag{5.4.32}$$

The rest is to use result mentioned in proving the group function basis above. One finally get:

$$f(g) = \sum_j \left(\frac{1}{d_j} \sum_\mu C^j_{\mu\mu} \right) \chi^j(g) \tag{5.4.33}$$

where $C^j_{\mu\mu}$ is defined in equation 5.4.30. Or we can write it in another way:

$$F(g) = \sum_j C_j \chi^j(g) \tag{5.4.34}$$

$$C_j = \frac{1}{|G|} \sum_{h \in G} \chi^j(h)^* F(h) \tag{5.4.35}$$

the formula for C_j is also easily found by simply calculating $\sum_{h \in G} \chi^j(h)^* F(h)$. □

Note that the above result is enough to show that the number of irreducible representations is the same as the dimension of class functions (consider it as a vector space).

The book [1] also rephrase it in page 63, as:

> Define two matrices $U_{m,n}$ and $V_{m,n}$, whose elements are defined by:
>
> $$U_{g,j\mu\nu} := \sqrt{\frac{d_j}{|G|}} D^j_{\mu\nu}(g) \tag{5.4.36}$$
>
> $$V_{a,j} := \sqrt{\frac{n_a}{|G|}} \chi^j_a \tag{5.4.37}$$
>
> The two matrices are unitary. This is true by equation 5.3.2 and equation 5.4.25. Note here $g \in G$, j labels different unitary irreducible representations, μ, ν labels the matrix element in that representation, a labels conjugacy classes. The labels $j\mu\nu$ together form the second index of matrix $U_{m,n}$. For convenience, I type all the orthonomal relationship together:
>
> Equation 5.3.2:
>
> $$\sum_{h \in G} D^{i*}_{\mu\rho}(h) D^j_{\nu\lambda}(h) = \frac{N}{d_j} \delta_{ij} \delta_{\mu\nu} \delta_{\rho\lambda} \tag{5.4.38}$$
>
> Equation 5.4.25:
>
> $$\sum_{\mu \in \mathrm{I.R.}} \chi^{a*}_\mu \chi^b_\mu = \frac{|G|}{n_a} \delta^{ab} \tag{5.4.39}$$

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6.1 Finding Representation

One important work in group theory is to find all its irreducible representations. The book [1] suggests we first find the characters of irreducible representations, and list them in a character table, and then find the matrix of those irreducible representations.

He also notes that, a unfaithful representation of a group is equivalent to a faithful representation of a certain quotient group. Since quotient group is less complex than the original group, finding irreducible representation of normal subgroups is very useful for analysing a group.

Fact 6.1. Also, a the direct product of two irreducible representations is also a representation. If the dimension of one of the representation is 1, then their product is still a irreducible representation.

Fact 6.2. Another helpful point is that, for all irreducible representations, the identity element is always mapped to the identity matrix. So the character for this unit class is just the dimension of the representation.

6.2 Self-conjugate representation and Real Representation

It's easy to see that for any representation D, conjugating ever matrix $D(g)^*$ will give another representation. We define

Definition 6.1 (Conjugate Representation)**.** The conjugate representation of a given representation D is obtained by complex conjugate all its matrices $D(g)^*$, $g \in G$.

Definition 6.2 (Self-conjugate Representation)**.** A representation D is called self-conjugate if and only if it is equivalent (but not necessarily equal) to its conjugate representation D^*.

Fact 6.3. The character of a self-conjugate representation is a real function. Since (assume $XD(g)X^{-1} = D^*(g)$):

$$\mathrm{Tr}\big(XD(g)X^{-1}\big) = \mathrm{Tr}\big(X^{-1}XD(g)\big) = \mathrm{Tr}(D(g)) = \mathrm{Tr}(D^*(g)) = \mathrm{Tr}(D(g))^*$$

We have a similar concept of real representation:

Definition 6.3 (Real representation)**.** A representation D is called real if and only if it is equivalent to a representation D^{real}, such that for any $g \in G$, the matrix $D^{\mathrm{real}}(g)$ is real.

Observe that a real representation is obviously self-conjugate, but a self-conjugate representation is not necessarily real. We have two indicator for its exact classifications.

Firstly,

Theorem 6.1. *Let D be a unitary irreducible representation of a finite group. Let D^* be its conjugate representation. Assume that D is self-conjugate. Let X be the matrix that relates the two. Then X is either symmetric or anti-symmetric. X is symmetric if and only if D is real. X is anti-symmetric if and only if D is self-conjugate but is not real.*

Secondly,

Theorem 6.2. *for any irreducible representation D of finite group:*

$$\frac{1}{|G|}\sum_{g\in G}\chi(g^2)=\begin{cases}1, & D \text{ is real}\\ -1, & D \text{ is self-conjugate but is not real}\\ 0, & D \text{ is not self-conjugate}\end{cases} \tag{6.2.1}$$

The proof of these two facts are presented in page 68 of [1]. Note that there is more general version of the second fact, called *Frobenius–Schur indicator*, see Wikipedia page.

6.3 Subduced Representation

To be finished.

Let G be a group. Let $\mathcal{C}_a$ denotes it conjugacy classes. Let D^j be its irreducible representations, with dimension m_j, character χ_a^j. Assume it has a subgroup $H=\{T_1=E,T_2,\cdots,T_h\}$, where $h=|H|$. Let $n=|G|/h$ be its index. Denotes its coset by R_rH. With $R_1:=E$, any element in G can be denotes as R_rT_t (assume R_r is chosen). Let $\bar{\mathcal{C}}_\beta$ denotes conjugacy classes of the subgroup H, with character $\bar{\chi}_\beta^k$, dimension $\bar{m}_k$.

7 Point Group

Here I collect the result related to point group in physical space.

Definition 7.1 (Point group)**.** A group is called a point group if it contains of rotations that leaves a common point unchanged.

Definition 7.2 (Proper point group)**.** A point group is called proper if its elements keep the orientation of the volumn element.

Definition 7.3 (Improper point group)**.** Improper point group is a point group that contains both improper and proper rotations.

Definition 7.4 (Spatial inversion σ)**.**

Remark 7.1. Let G be a improper point group. An improper rotation in it can been seen as the direct product of a proper rotation and a spatial inversion σ. (Note that σ commutes with any rotation, and $\sigma^2=\mathbb{1}$.) Therefore, a improper point group will always contain some proper rotation by $\sigma^2=\mathbb{1}$, and of course, the e is always proper. Also, the subset of proper rotation will form a subgroup H. And such division will make $G/H\cong\mathbb{Z}_2$.

However, a improper point group does not necessarily contains σ. For example, assuming a proper point group has a subgroup H of index 2, then multiply every elements of the coset by σ will gives us a group denoted σH. Then $H\otimes\sigma H$ gives us a improper point group that does not contain σ.

Definition 7.5 (P-type and I-type improper point group)**.** A P-type improper point group is the improper point group that does not contain σ, spatial inversion. On the other hand, if it does contain σ, it is called an I-type improper point group.

7.1 Rotation C_n

C_n is isomorphic to $\mathbb{Z}_n$, here the C emphasize that it represent the group generated by the rotation of $2\pi/n$. This group is abelian.

Conjugacy Classes Obvious each element form a conjugacy class on its only. Moreover, when n is even, the conjugacy class of rotation π is self-conjugate.

Invariant Subgroup When n is prime, there is no nontrivial invariant subgroup. When n can be factorized as $n_1 n_2$, both not equals 1, then C_n is the product of C_m and C_n, both are invariant subgroup.

Irreducible representation (pp.64-65 of [1])
Since it has n elements and n conjugacy classes, with $\sum_{j=1}^{n} d_j^2 = n$, each irreducible representation is 1-dimensional. Let R denotes the generator of rotation by $2\pi/n$, then by calculating $D^j(R)^n = D^j(1)$, one can easily find that for each irreducible representation j:

$$D^j(R) = \exp(-i2\pi j/N) \tag{7.1.1}$$

where $j = 0, 1, \cdots, n$. Since R is its generator, the rest of the representation is determined. Hence we have all the irreducible representations of C_n. The table of characters may be found on page 65 of [1].

Note that since all representations are 1-dimensional, their products will keep being irreducible. So having two set of irreducible representations of C_n and C_m, will give us all the irreducible representations of C_{nm}. For example:

7.2 Dihedral Group D_n

Definition 7.6 (Dihedral Group D_n)**.** Dihedral group D_n is the symmetric group of regular n-gons for $n \geq 3$. For $n = 1/2$, it is usually not defined. I found that in book [1], D_2 is isomorphic to Klein Group, D_1 is of course the trivial group.

But we can have another more mathematical definition of Dihedral group: To be added

Conjugacy Classes There is some important difference between the case when n is odd and when n is even. When n is odd, the 2-fold axis are all linked with the vortices of the n-polygon. On the contrary, when n is even, there is two different 2-fold axis. The first $n/2$ axises are those linked with the vortices, the second $n/2$ axises are those cut through the opposite edges.

Note that by fact 3.8 one can see that such difference will note make the two classes in even n the same conjugacy class. On the contrary, by the same fact 3.8 one sees that those 2-fold axises are in the same conjugacy class when n is odd.

Also denotes one $2\pi/n$ rotation around the N-fold axis by T, then by this very same fact 3.8, T and T^{-1} lies in the same conjugacy class, related by a reflection around a 2-fold axis. This is obvious a self-conjugate class. (Note that, when only $2\pi/n$ rotation is considered, they form an abelian group and each rotation forms a conjugacy class.) Similarly for the conjugacy class $\{T^m, T^{-m}\}$, where m is an integer.

In summary, this discussion shows that the self-conjugate classes in D_n are:

Fact 7.1. When n is odd (Let $n = 2n' + 1$): The identity class, the class of all 2-fold axis. The class of $\{T^m, T^{-m}\}$, where $m = 1, 2, \cdots, n'$. (Why $m \leq n'$? Think about the case when $n = 5, n' = 2$.). In total $n' + 2 = \frac{n+3}{2}$ conjugacy classes/irreducible representations. All conjugacy classes are self-conjugate.

Fact 7.2. When n is even (Let $n = 2n'$): The identity class, the two classes of all 2-fold axis, as discussed above. The class of $\{T^m, T^{-m}\}$, where $m = 1, 2, \cdots, n'$. In total $n' + 3 = \frac{n+6}{2}$. All conjugacy classes are self-conjugate.

Invariant Subgroups (**pp.30 of [1]**) By the distinction between edge and vortices type of 2-fold axises discussed before, one can see that there is difference in invariant subgroup when n is odd or even. For example, when $n = 5$ is odd, D_5 has only C_5 as a notrivial subgroup. But when $n = 6$ is even, it has C_6 (then C_2 and C_3), two copies of D_3 (each formed by edge and vortex type of 2-fold axis).

More generally, there importnat subgroup of index for D_n. When n is odd (let $n'2n' + 1$), there are one subgroup of index 2, i.e. C'_n. When n is even (let $n' = n/2$), there are three invariant subgroup of index 2: C'_n, D_n, and D'_n. Denote n 2-fold axis as S_j, rotation around n-fold axis by $2\pi/n$ as T, then:

$$D_n = \{E, T^2, T^4, \cdots, T^{2n-2}, S_0, S_2, \cdots, S_{2n-2}\} \tag{7.2.1}$$

$$D'_n = \{E, T^2, T^4, \cdots, T^{2n-2}, S_1, S_3, \cdots, S_{2n-1}\} \tag{7.2.2}$$

Note that this list is not complete. For example, D_6 contains other non-trivial invariant subgroups, listed in page 30 of [1].

Character Table (**pp.66 of [1]**)

For D_2, it is isomorphic to the group $\{1, \sigma, \tau, \sigma\tau\}$, where σ is spatial inversion, τ is time-reversal. Notice that this group is abelian, so it obvious has 4 irreducible representations. It is easy to get its character table, which may be found on page 66, table 3.6 of [1].

For D_3, it is non-abelian. As discussed previously, it has 3 conjugacy classes, so it has 3 irreducible representations. Notice that $D_3/C_3 \cong C_2$, so it inherets the two simple irreducible representations of C_2. This gives two lines in the character table. Also, $6 - 1^2 - 1^2 = 2^2$, so we have one 2-dimensional irreducible representation left. Next, by fact 6.2, we can determine one element in the character for 2-dimensional irreducible representation. The rest two empty blanks can be filled using the orthogonal relations for the character table.

For general D_n, see the part for irreducible representations.

Irreducible representation (**pp.71-74 of [1]**)

The general procedure is to use the invariant subgroup of index 2 mentioned above to get the 1-dimensional representations. And use the relation $\sum_j m_j^2 = |D_n|$ and $\sum_j 1 =$ number of conjugacy class to "guess" the dimensional of remaining irreducible representations. Finally use subgroup C_n to construct those representations (induced representation) and reduce these into irreducible ones. But how to reduce them into irreducible one is actually the most difficult part, in my opinion.

For odd n, let $n = 2n' + 1$. We have:

$$\sum_{j=1} n' + 2m_j^2 = 4n' + 2 \tag{7.2.3}$$

There is two 1-dimensional irreducible representations, formed using the invariant subgroup $C_{2n'+1}$. Since $4n' + 2 = 2^2 n' + 2$, one guesses that the remaining ones are n 2-dimensional irreducible representation. They are constructed (construction process could be found in [1]) to be (notice $D_{2n'+1}$ is generated by $T_{2n'+1}$, the rotation $2\pi/(2n'+1)$, and C_2', any one of the 2-fold reflection):

$$\bar{D}^{E_j}(C_{2n'+1}) = \begin{pmatrix} \cos\left(\frac{2j\pi}{2n'+1}\right) & -\sin\left(\frac{2j\pi}{2n'+1}\right) \\ \sin\left(\frac{2j\pi}{2n'+1}\right) & \cos\left(\frac{2j\pi}{2n'+1}\right) \end{pmatrix} \tag{7.2.4}$$

$$\bar{D}^{E_j}(C_2') = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.2.5}$$

where $j = 1, 2, \cdots, n$.

The case for even n is simialr (let $n' = n/2$). Except that here one as three invariant subgroup of index 2 (see dicusson about invariant subgroup above). So it has three 1-dimensional irreducible representations, and $n-1$ 2-dimensional irreducible representations.

When n is odd, let $n = 2n'$. (TBD)

8 Anchor

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