Solution for HW8

Taper

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Abstract

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Problem 1

Observe that the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Has eigenvalues $\{-\sqrt{2}, \sqrt{2}\}$, and eigenvectors

$$\begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}$$

So the eigenvalues and eigenvectors of H are

$$E_+ = -\sqrt{2}a, \qquad |E_+\rangle = \frac{1}{\sqrt{4-2\sqrt{2}}}\left[(1-\sqrt{2})\,|1\rangle + |2\rangle\right] \tag{0.0.1}$$

$$E_- = \sqrt{2}a, \qquad |E_-\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}\left[(1+\sqrt{2})\,|1\rangle + |2\rangle\right] \tag{0.0.2}$$

Problem 2

Proof. Write them using Pauli matrices:

$$S_x = \frac{\hbar}{2}\sigma_x,\ S_y = \frac{\hbar}{2}\sigma_y,\ S_y = \frac{\hbar}{2}\sigma_z$$

So we have

$$[S_i, S_j] = \frac{\hbar^2}{4}[\sigma_i, \sigma_j]$$

$$\{S_i, S_j\} = \frac{\hbar^2}{4}\{\sigma_i, \sigma_j\}$$

Now let's calculate:

i	j	$[\sigma_i, \sigma_j]$	$\{\sigma_i, \sigma_j\}$
x	x	0	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\mathbb{1}$
y	y	0	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\mathbb{1}$
z	z	0	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\mathbb{1}$
x	y	$\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2i\sigma_z$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
z	x	$\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 2i\sigma_y$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
y	x	$\begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = 2i\sigma_z$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

The rest can be calculated using simple rules like $[A, B] = -[B, A]$ and $\{A, B\} = \{B, A\}$. The above table tells us the relation $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$, $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$. Hence

$$[S_i, S_j] = \frac{\hbar^2}{2} i\varepsilon_{ijk}\sigma_k = i\hbar\varepsilon_{ijk}S_k$$

$$\{S_i, S_j\} = \frac{\hbar^2}{2}\delta_{ij} = i\hbar\delta_{ij}$$

□

Problem 3

The Heisenberg equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}A(t) = [A(t), H(t)]$$

tells us that

$$\begin{aligned}
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}S_x(t) &= [S_x(t), \omega S_z] = -\omega i\hbar S_y \\
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}S_y(t) &= [S_y(t), \omega S_z] = \omega i\hbar S_x \\
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}S_z(t) &= [S_z(t), \omega S_z] = 0
\end{aligned}$$

Next

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}S_x(t) &= -\omega(\omega S_x) \\
\frac{\mathrm{d}^2}{\mathrm{d}t^2}S_y(t) &= \omega(-\omega S_y)
\end{aligned}$$

One see that the solution of S_x and S_y is inextricably related. Then we fix S_x first.

$$\begin{aligned}
&S_x = Ae^{i\omega t} + Be^{-i\omega t} \qquad (0.0.3)\\
&S_x(0) = A + B\\
&\frac{\mathrm{d}}{\mathrm{d}t}S_x(0) = -\omega S_y(0) = Ai\omega - Bi\omega\\
\Rightarrow &A = \frac{1}{2}(S_x(0) + iS_y(0)), \quad B = \frac{1}{2}(S_x(0) - iS_y(0)) \qquad (0.0.4)
\end{aligned}$$

Similary, we have

$$\begin{aligned}
&S_y = Ce^{i\omega t} + De^{-i\omega t}\\
&C = \frac{1}{2}(S_y(0) - iS_x(0)), \quad D = \frac{1}{2}(S_y(0) + iS_x(0)) \qquad (0.0.5)
\end{aligned}$$

The rest is obvious:

$$S_z(t) \equiv S_z(0) \qquad (0.0.6)$$