

Note on the Irreducible Triangulations of the Klein Bottle

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Abstract

We give the complete list of the 29 irreducible triangulations of the Klein bottle. We show how the construction of Lawrencenko and Negami, which listed only 25 such irreducible triangulations, can be modified at two points to produce the 4 additional irreducible triangulations of the Klein bottle.

Key words: Irreducible triangulation, Klein bottle
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1 Introduction

A *triangulation* of a closed surface is a simple graph embedded in the surface so that each face is a triangle and so that any two faces share at most one edge. Two triangulations G and G' of a surface are *equivalent* if there is a homeomorphism h with $h(G) = G'$.

Let $e = ac$ be an edge in a triangulation G and abc and acd be the two faces which have e as a common edge. The *contraction* of e is obtained by deleting ac , identifying vertices a and c , removing one of the multiple edges ab or cb , and removing one of the multiple edges ad or cd . An edge e of a triangulation G is *contractible* if the contraction of e yields another triangulation of the surface in which G is embedded. If an edge e is contained in a three cycle other than the two which bound the faces which share e , then the contraction of e produces multiple edges. So, for a triangulation G , not K_4 embedded in the sphere, an edge e of G is not contractible if and only if e is contained in at least three cycles of length 3. A triangulation is said to be *irreducible* if it has no contractible edge.

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This note assumes the reader's familiarity with Lawrencenko and Negami's paper [2] where their Theorem 1 claims the existence of exactly 25 irreducible triangulations of the Klein bottle. Theorem 1 corrects this claim; it requires Lemma 2, a modification of Lemma 6 of [2], which claimed the existence of exactly 21 irreducible triangulations of handle type. All other results of [2] remain valid.

Theorem 1 *There are exactly 29 nonequivalent irreducible triangulations of the Klein bottle.*

Lemma 2 *There are exactly 25 nonequivalent irreducible triangulations of handle type.*

In Section 2 we list all 29 irreducible triangulations of the Klein bottle and show that each of the new ones is not equivalent to any of the others. A complete proof that these 29 irreducible triangulations are all of the irreducible triangulations of the Klein bottle requires repeating the construction in [2] along with two modifications. We give only the modifications required. In Section 3 we examine a subcase which was overlooked in [2] and which leads to the additional triangulation Kh25. In Section 4 we examine in more detail than was given in [2] a subcase which leads to the additional triangulations Kh22, Kh23, and Kh24.

2 List of Irreducible Triangulations

To show that there are at least 29 irreducible triangulations of the Klein bottle we exhibit them. Figures 1, 2, 3, and 4 show the complete list. Figures 1, 2, and 4 are the same as Figs. 13, 14, and 15 of [2]. In Figs. 1, 2, and 3 the pair of horizontal sides of each rectangle are identified in parallel and the pair of vertical sides are identified in antiparallel to obtain an actual triangulation of the Klein bottle. In [2] such triangulations are classified as *handle type*. In Fig. 4 each triangulation can be obtained from two copies of irreducible triangulations of the projective plane by removing one triangular face from each copy and pasting them together along boundaries of the removed faces. For each of the two hexagons in each graph identify each antipodal pair of vertices on the boundary to obtain the triangulation of the projective plane less one face. When these identifications have been made for both parts of a graph a triangulation of the Klein bottle is produced. This type of triangulation is classified as *crosscap type* in [2].

The irreducible triangulations shown in Fig. 3, which are denoted Kh22, Kh23, Kh24, and Kh25, were not listed in [2]. It can be seen that they are indeed irreducible by checking that each edge e is contained in a cycle of length 3

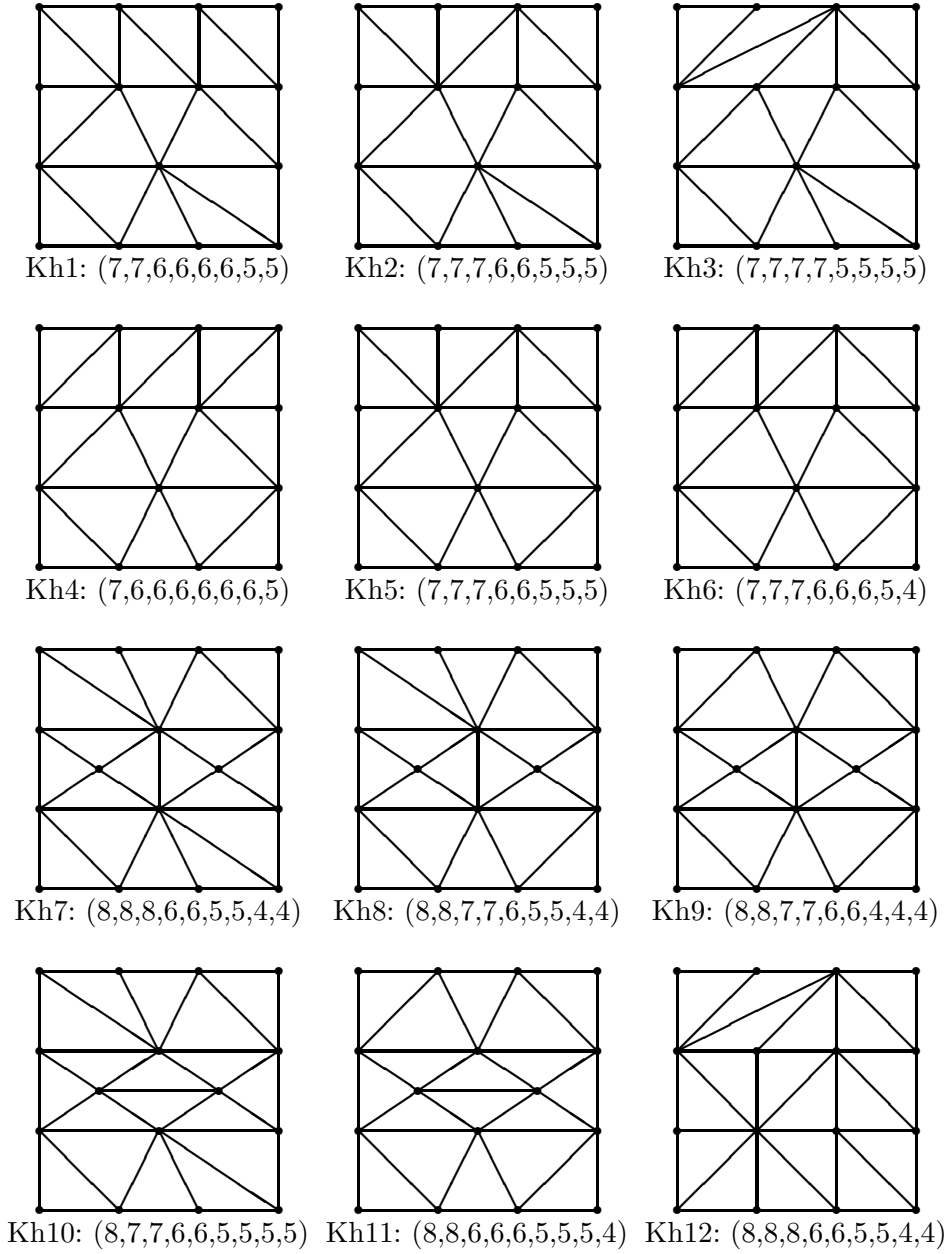


Fig. 1. Irreducible triangulations of the Klein bottle, Kh1 - Kh12

other than the two which bound the faces incident to e .

The triangulations Kh22 through Kh25 are not equivalent to any of the other triangulations. In fact, they are not isomorphic as graphs to any of the other triangulations. The degree sequence of each triangulation is shown in Figs. 1 through 4. The degree sequences of Kh23 and Kh25 are unique. The degree sequence of Kh22 and Kh10 are the same. However, in Kh10 the two vertices of degree 6 are adjacent, while in Kh22 the two vertices of degree 6 are not adjacent. Similarly, the degree sequences of Kh24 and Kh15 are the same. In

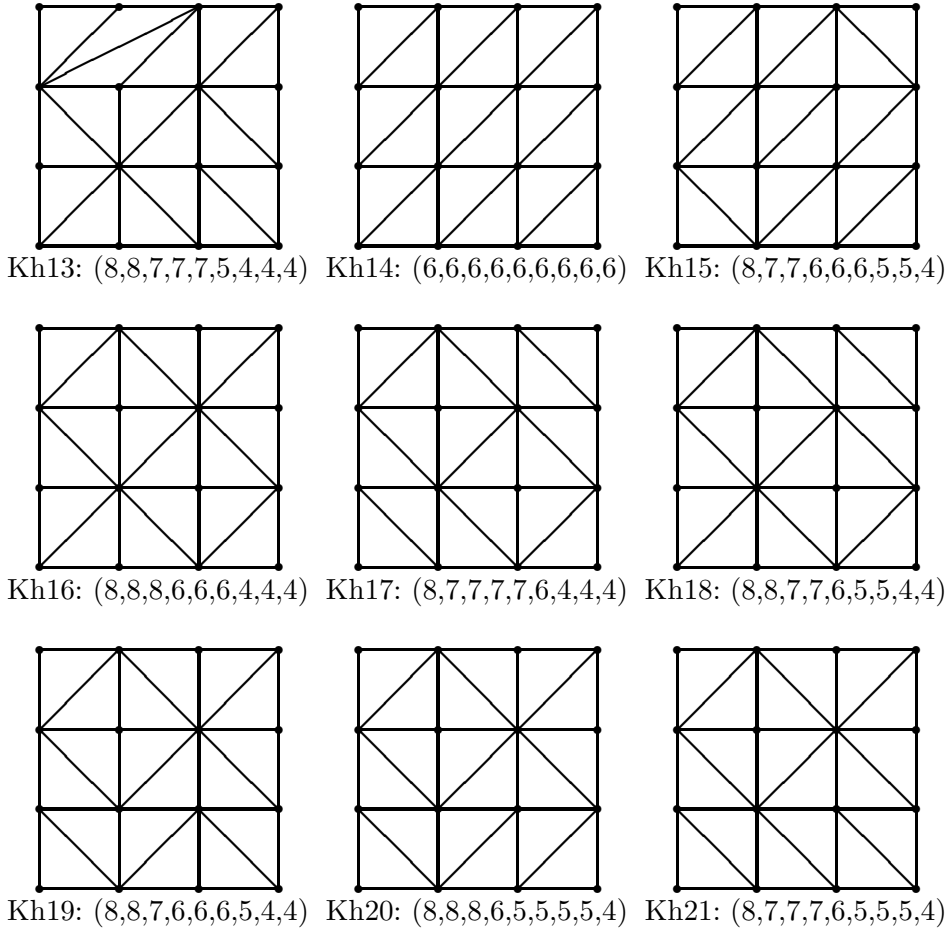


Fig. 2. Irreducible triangulations of the Klein bottle, Kh13 - Kh21

Kh15 the two vertices of degree 5 are adjacent, while in Kh24 the two vertices of degree 5 are not adjacent.

3 Construction of Kh25

The construction of Lawrencenko and Negami can be modified to produce all 29 irreducible triangulations of the Klein bottle and to show that there are no others. We do not need to modify the basic proof of Lemma 6 of [2]. Instead we correct some of the details of their proof. Assuming that the reader is familiar with the proof in [2] we do not give its details here.

The triangulation Kh25 was missed in determining the partial structures inside the rectangle R_{01} as defined in [2]. In *Step 2. Recognizing the inside of R_{01}* of [2], in the last full paragraph on Page 279, Line -11 the authors incorrectly stated that “there is no chord incident to y_1 outside the polygon”

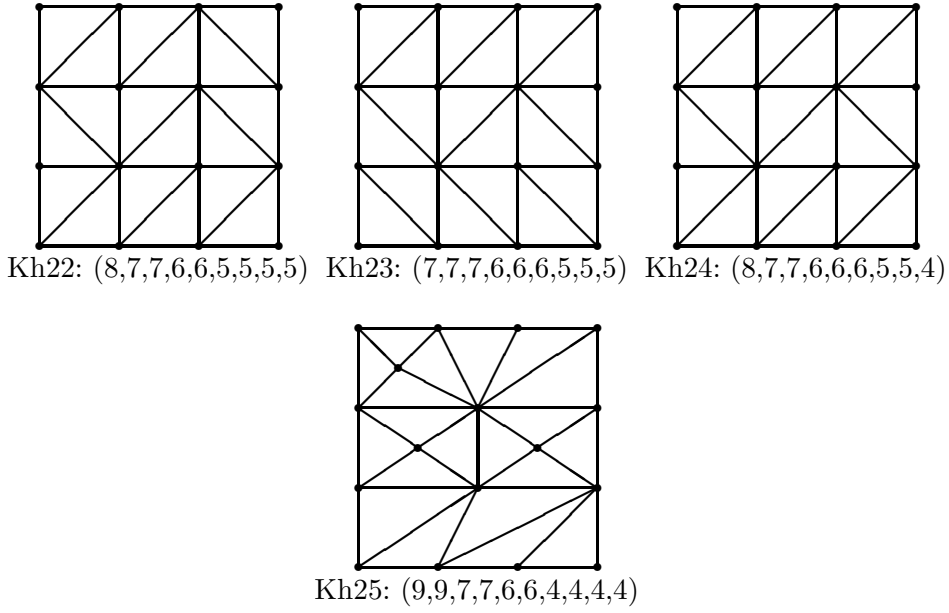


Fig. 3. Irreducible triangulations of the Klein bottle, Kh22 - Kh25

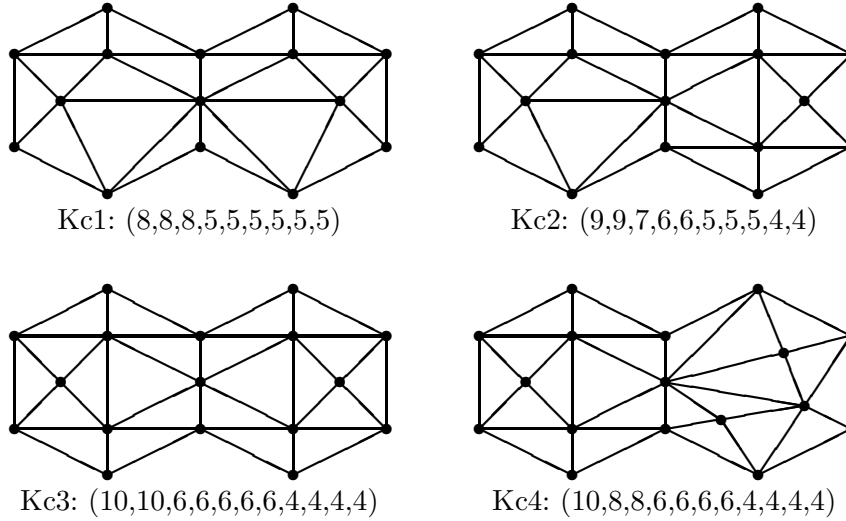


Fig. 4. Irreducible triangulations of the Klein bottle, Kc1 - Kc4

$W_2 \cup y_1 b' a' x_1 x_2$ and used this conclusion to the end of that paragraph. However, one needs to consider both the presence and the absence of such a chord.

First assume that vertex y_1 is not adjacent to a' or to a'' . Then there is no chord incident to y_1 outside the polygon $W_2 \cup y_1 b' a' x_1 x_2$. The path W_1 has length 1, that is, $W_1 = x_1 y_1$ by Lemma 3 of [2]. Similarly, $W_2 = x_2 y_1$ and each of the two quadrilateral regions bounded by $a' b' y_1 x_1$ and $a'' c'' y_1 x_2$ contains only one diagonal by Lemma 3 of [2]. In this case, by assumption $y_1 a'$ and $y_1 a''$ are not diagonals, thus we have the partial structure $R_{01} - 6$.



Fig. 5. Additional structures inside R_{01} .

Now by symmetry, without loss of generality, assume that y_1 is adjacent to a'' . The edge x_2y_1 is not a chord outside the polygon $W_1 \cup x_1x_2a''y_1$, thus W_2 has length 1, that is, $W_2 = x_2y_1$. If W_1 has length 1, then the polygon $y_1b'a'x_1$ either has a diagonal $b'x_1$ and we have the partial structure $R_{01}-5$, or there is one internal vertex in $y_1b'a'x_1$ and we have an additional partial structure $R_{01}-9$ which is shown in Fig. 5. The polygon $y_1b'a'x_1$ in $R_{01}-9$ has an internal vertex and x_1b'' must be an external chord in R_{02} . If, on the other hand, W_1 has length greater than 1, then there is exactly one vertex z on W_1 and inside the polygon $y_1b'a'x_1x_2$ by Lemma 3 of [2]. In this case we show that z is adjacent to all the vertices of the polygon $y_1b'a'x_1x_2$. Since $W_1 = x_1zy_1$ is a minimum length path joining x_1 and y_1 , the edge x_1y_1 is not a diagonal of the polygon $x_1zy_1x_2$. There is no interior vertex in the polygon $x_1zy_1x_2$, hence zx_2 is a diagonal. The polygon $y_1b'a'x_1z$ has no interior vertex and neither y_1a' nor y_1x_1 are edges, hence $b'z$ is an edge in $y_1b'a'x_1z$. If $b'x_1$ is an edge in R_{01} , then the polygon $b'x_1x_2y_1$ would contain vertex z ; hence by Lemma 3 of [2], there would be a chord x_1y_1 outside the polygon $b'x_1x_2y_1$, which is impossible. Therefore, $b'x_1$ is not an edge in R_{01} and za' is the diagonal of the polygon $b'a'x_1z$. In this case we have another additional partial structure $R_{01}-10$ which is shown in Fig. 5. The polygon $y_1b'a'x_1x_2$ in $R_{01}-10$ has an internal vertex, and x_1b'' must be an external chord in R_{02} .

In *Step 4. Composing partial structures in triangulations* of [2], the additional configurations using the partial structures $R_{01}-9$ and $R_{01}-10$ must be classified. Both $R_{01}-9$ and $R_{01}-10$ require the edge x_1b'' in R_{02} , thus the additional configurations are $\{[i, 1, -k] : i = 9, 10; k = 1, 2, 3\}$ and $\{[i, j, -k] : i = 9, 10; j = 2, 3; k = 7, 8\}$. The configuration $[9, 3, -8]$ is added to group (i) and is the additional irreducible triangulation Kh25. The configuration $[9, 3, -7]$ is equivalent to $[9, 3, -8]$. The configuration $[10, 1, -1]$, which is equivalent to $[5, 2, -5]$, is contrary to assumption (III). The configurations $[9, 1, -1]$, $[9, 1, -2]$, and $[9, 1, -3]$ are added to group (v). The configurations $[9, 2, -7]$, $[9, 2, -8]$, $[10, 1, -2]$, $[10, 1, -3]$, $[10, 2, -7]$, $[10, 2, -8]$, $[10, 3, -7]$, and $[10, 3, -8]$ are added to group (vii).

4 Construction of Kh22, Kh23, and Kh24

In *Step 5. Classifying triangulations up to equivalence* (the first paragraph on page 283) of the proof of Lemma 6 of [2], diagonals were added to the partial

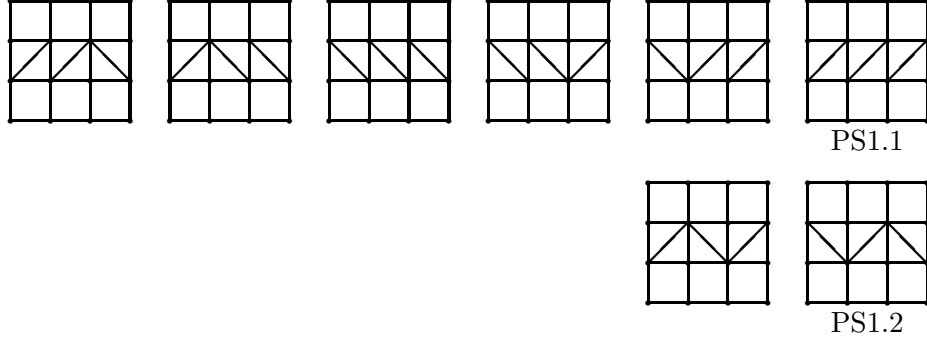


Fig. 6. PS1 with diagonals in R_{12} .

structures and the results were classified up to equivalence. No details were given regarding how this classification was accomplished. Only 8 irreducible triangulations (Kh14 through Kh21) which can be obtained by adding diagonals to the partial structure PS1 (Fig. 8 of [2]) were listed in [2]. There are 3 additional irreducible triangulations (Kh22 through Kh24) which can be obtained from PS1. We examine in detail how to add diagonals to PS1 and classify all the resulting triangulations. This procedure requires that we check only 16, not all 2^9 , configurations obtained by adding diagonals to the nine quadrilateral regions of PS1.

Consider the possible configurations for diagonals in R_{12} of PS1. Figure 6 shows all eight partial structures obtained by adding diagonals to R_{12} of PS1. Going from left to right, each partial structure is transformed into the next by removing the left column, reflecting it vertically, and pasting it onto the right side. Thus we can assume that PS1 has one of the two rightmost partial structures, PS1.1 or PS1.2, of Fig. 6.

Consider the partial structure PS1.1 (Fig. 7a). The edge by_1 is a diagonal of the polygon $bax_1y_1z_1c$. By Lemma 3 of [2] there must be a path of length 2 outside this polygon connecting b and y_1 . Thus y_1x_2 and x_2b must be edges of PS1.1. Likewise, bx_1 and x_1z_2 must be edges of PS1.1 because of the diagonal z_2b in the polygon $z_2y_2cbax_2$. We then have the partial structure in Fig. 7b. If y_2a is a diagonal of the polygon y_2x_2ac in Fig. 7b, then az_1 must also be an edge of PS1.1. Also, if az_1 is a diagonal of the polygon acz_1x_1 in Fig. 7b, then y_2a must also be an edge of PS1.1. So the two possible completed structures from PS1.1 are Kh14 and Kh15.

We now determine the number of nonequivalent ways to complete the partial structure PS1.2, thus avoiding checking all 2^6 configurations. We examine two operations on PS1.2 which maintain its basic structure but which permute the quadrilateral regions labeled with the letters shown in Fig. 8. Reading from left to right in Fig. 8 and recalling that the left end of R_{02} is identified with the right end of R_{01} , the order of the letter is ABCDEF. If PS1.2 is reflected both vertically and horizontally, then it is still a type PS1.2 partial structure. The

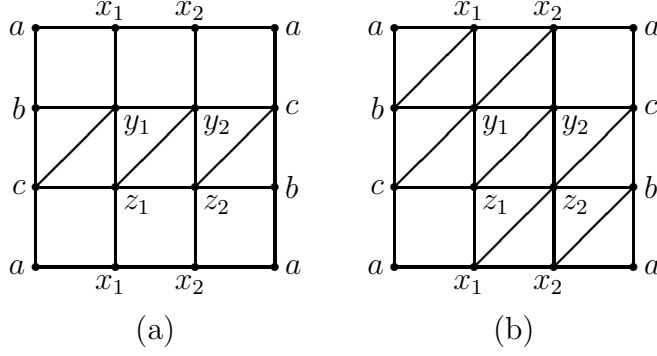


Fig. 7. Partial structure PS1.1 and required edges.

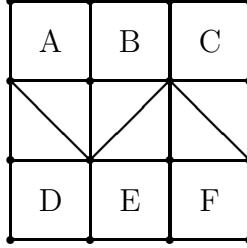


Fig. 8. Partial structure PS1.2.

order of the letters is now FEDCBA and the permutation is $(AF)(BE)(CD)$. If the left two columns of PS1.2 are removed, reflected vertically, and pasted on the right, then the result is still a type PS1.2 partial structure. The order of the letters is now CDEFAB and the permutation is $(ACE)(BDF)$. These two operations define a permutation group

$$\{(A)(B)(C)(D)(E)(F), (ACE)(BDF), (AEC)(BFD), \\ (AF)(BE)(CD), (AB)(CF)(DE), (AD)(BC)(EF)\}$$

acting on $\{A, B, C, D, E, F\}$. We color each element of $\{A, B, C, D, E, F\}$ with one of two colors 0 and 1. Select a letter from $\{A, B, C, D, E, F\}$. This letter is the label of one quadrilateral region in R_{01} or R_{02} which is adjacent to one quadrilateral region in R_{12} which has a fixed diagonal in PS1.2. If the diagonals in these two quadrilateral regions share a vertex (they are “perpendicular”), then the letter is given the color 1. If the diagonals in these two quadrilateral regions do not share a vertex (they are “parallel”), then the letter is given the color 0. The coloring has been defined so that the colors do not change under the group operations described above. It can be shown that the number of equivalence classes of colorings is 16.

Table 1 lists one coloring from each equivalence class along with the resulting complete structure obtained from PS1.2. From this table we see that the partial structure PS1.2 and thus PS1 produces the irreducible triangulations Kh14 through Kh24 and no others.

Table 1
Complete structures obtained from PS1.2

Coloring	Complete structure
000000	equivalent to Kh14
000001	has a contractible edge
000011	equivalent to Kh15
000101	has a contractible edge
000110	equivalent to Kh24
001001	has two contractible edges
111000	Kh19
001101	has a contractible edge
101010	Kh23
100101	Kh22
101101	Kh24
111001	Kh20
111010	Kh21
110011	Kh17
110111	Kh18
111111	Kh16

5 Remarks

In this section we reconsider three results appearing in [1,2,3] whose proofs are based on the list of irreducible triangulations of the Klein bottle. Their proofs require that the irreducible triangulations of the Klein bottle have certain properties. Since, as we will observe, the additional triangulations, Kh22 through Kh25, also have these properties, these results remain valid.

Firstly, Theorem 10 of [2] states that *if an irreducible triangulation of the Klein bottle can be embedded in the torus, then it is equivalent to Kh1*. In its proof the set of all the irreducible triangulations of the Klein bottle are partitioned into four subsets. The structure of the triangulations in each subset is considered in turn. Since each of the additional triangulations, Kh22 through Kh25, can be placed in one of these four subsets, the proof can be easily modified. This result is then used by Lawrencenko and Negami in [3] to construct all graphs which are triangulations of both the torus and the Klein bottle.

Secondly, by checking Kh22 through Kh25 we see, as was observed in [2], that every irreducible triangulation of the Klein bottle still includes:

- a disjoint pair of longitudes and a meridian which crosses each of the longitudes only once,
- a meridian and an equator which cross each other at precisely two vertices,
- a Hamilton cycle which is trivial on the Klein bottle,
- a Hamilton cycle which is a meridian,
- a Hamilton cycle which is a longitude, and
- a Hamilton cycle which is an equator.

Thirdly, Theorem 12 of [2] states that *a triangulation of the Klein bottle includes two disjoint meridians if and only if it does not include an equator of length 3*. Its proof uses the fact that every irreducible triangulation of handle type includes two disjoint meridians. This fact is true for Kh22 through Kh25; hence the proof needs no change. Theorem 12 of [2] is used by Brunet, Nakamoto, and Negami in [1] to prove that every 5-connected triangulation of the Klein bottle has a Hamilton cycle which is contractible.

Acknowledgements

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