Notes of Quantum Field Theory in a Nutshell

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August 14, 2016

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Chapter 1

Part I: Motivation and Foundations

1.1 I.2 Path Integral Formulation of Quantum Physics

Here the path integral formulation of quantum mechanics is introduced. The intuition comes from a limiting case of the traditional double slit electron interference experiment. (**pp.7 to 10**) Then it calculates the transition probability $\langle q_F|e^{-iHT}|q_I\rangle$ by divide it into a infinite of steps:

$$\langle q_F|e^{-iHT}|q_I\rangle = \lim_{N\to\infty} \langle q_F|e^{iH\delta t}e^{iH\delta t}\cdots e^{iH\delta t}|q_I\rangle \,(\text{with } N\delta t = T) \tag{1.1.0.1}$$

For illustration, it calculates this value when $H = \frac{\hat{p}^2}{2m}$. The result is that:

$$\langle q_F|e^{-iHT}|q_I\rangle = \int Dq(t)e^{i\int_0^T dt\frac{1}{2}m\dot{q}^2}$$

where:

$$\int Dq(t) \equiv \lim_{N\to\infty}\left(\frac{-im}{2\pi\delta t}\right)^{N/2}\left(\prod_{k=1}^{N-1}\int dq_k\right) \tag{1.1.0.2}$$

It notes that when $H = \hat{p}^2/2m + V(\hat{q})$, the final result would have been:

$$\begin{aligned}\langle q_F|e^{-iHT}|q_I\rangle &= \int Dq(t)e^{i\int_0^T dt\frac{1}{2}m\dot{q}^2 - V(q)}\\ &= \int Dq(t)e^{i\int_0^T dtL(\dot{q},q)}\end{aligned} \tag{1.1.0.3}$$

where L is the Lagrangian of the system.

Often, the value we need to calculate is $\langle F|e^{-iHT}|I\rangle$. Using $\int |q\rangle \langle q|\, dq = 1$, we have:

$$\langle F|e^{-iHT}|I\rangle = \int dq_F \int dq_I \, \langle F|q_F\rangle \langle q_F|e^{-iHT}|q_I\rangle \langle q_I|I\rangle \tag{1.1.0.4}$$

The value $\langle 0|e^{-iHT}|0\rangle$ is denoted Z. This part mentions that one often effect a change of coordinate $t \to -it$, called *Wick rotation*, to obtain:

$$Z = \int Dq(t) e^{-\int_0^T dt H(\dot{q},q)} \tag{1.1.0.5}$$

where H is the Hamiltonian of the system. The mathematical rigorous aspect is often ignored.

It also discuss how this formulation could explain the classical limit of quantum mechanics, i.e. classical mechanics, in a very direct manner. This is related to the saddle point approximation to the integral 1.1.0.3.

Unclear point Why is $\int dq\, |q\rangle \langle q| = 1$ while $\int \frac{p}{2\pi} |p\rangle \langle p| = 1$. What does it mean by saying "to see that the normalization is correct" (pp. 10 and 11). Why is effecting the Wick rotation is "somewhat rigorous"?

Corresponding pages in draft pp. 1 to 3.

1.1.1 Appendix 1 - Dirac Delta function and ε as infinitesimal small value

Here the Dirac Delta function is defined as the limit of another function $d_K(x)$. Since:

$$d_K(x) \equiv \int_{-K/2}^{K/2} \frac{dk}{2\pi} e^{ikx} = \frac{1}{\pi x} \sin \frac{Kx}{2} \tag{1.1.1.1}$$

$$\int_{-\infty}^{\infty} dx \; d_K(x) = 1 \tag{1.1.1.2}$$

Hence we de fine $\delta(x) = \lim_{K\to\infty} d_K(x)$. Other important formula include:

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tag{1.1.1.3}$$

$$\frac{1}{x + i\varepsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x) \tag{1.1.1.4}$$

$$\delta(x) = \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2} \tag{1.1.1.5}$$

Here ε is a infinitesimal value. $\mathcal{P}$ denotes the principal value integral, defined by:

$$\int dx \mathcal{P}\frac{1}{x} f(x) = \lim_{\varepsilon\to 0} \int dx \frac{x}{x^2 + \varepsilon^2} f(x) \tag{1.1.1.6}$$

1.1.2 Appendix 2 - Wick theorem in Gaussian Integral

This part introduces some very important formulae, listed below:

(It is very important that A is a real symmetric matrix.)

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dx_1 dx_2\cdots dx_N e^{-\frac{1}{2}x^T Ax+J^T x} = \left(\frac{(2\pi)^N}{\det A}\right)^{\frac{1}{2}} e^{\frac{1}{2}J^T A^{-1} J} \tag{1.1.2.1}$$

$$\int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}ax^2+Jx} = \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} e^{\frac{J^2}{2a}} \tag{1.1.2.2}$$

$$\langle x_i x_j\cdots x_k x_l\rangle \equiv \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dx_1 dx_2\cdots dx_N e^{-\frac{1}{2}x^T Ax} x_i x_j\cdots x_k x_l}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dx_1 dx_2\cdots dx_N e^{-\frac{1}{2}x^T Ax}}$$

$$= \sum_{\text{Wick}} (A^{-1})_{ab}\cdots(A^{-1})_{cd} \tag{1.1.2.3}$$

For example:

$$\langle x^{2n}\rangle \equiv \frac{\int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}ax^2} x^{2n}}{\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2}} = \frac{1}{a^n}(2n-1)!!$$

$$\langle x_i x_j x_k x_l\rangle = (A^{-1})_{ij}(A^{-1})_{kl} + (A^{-1})_{ik}(A^{-1})_{jl} + (A^{-1})_{il}(A^{-1})_{jk}$$

Corresponding pages in draft pp. 4 to 6.

1.2 I.3 From Mattress to Field

Note After reading this section and several comments online, I realize that this book put more emphasize on physical intuition than on mathematical rigor. To illustrate this point, I quote a sentence from A. Zee's homepage at UCSB, which comes from a review of this book (QFT in a Nutshell):

> "It is often deeper to know why something is true rather than to have a proof that it is true."

By above reason, I gave up tracking the calculations done in this book.

Mattress in continuum limit It start with taking the limit of lattice spacing $l \to 0$, aiming to write down the field equation from the mattress perspective. The process is transparant and neatly summarized in the "promotion table" on page 19:

function	$q \to \phi$
atom position	$a \to x$
dynamic function	$q_a(t) \to \phi(t,\mathbf{x}) = \phi(x)$
summation	$\sum_a \to \int d^D x$

In detail, $\sum_a \frac{1}{2} m_a \dot{q}_a^2$ becomes $\int d^2x \frac{1}{2}\sigma(\partial\phi/\partial t)^2$., $\sum_{ab} k_{ab}\frac{1}{2}(q_a - q_b)^2$ becomes $\int d^2x\rho(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2})$, and

$$\begin{aligned} S(q) \to S(\phi) &\equiv \int_0^T dt \int d^2x \mathcal{L}(\phi) \\ &= \int_0^T dt \int d^2x \frac{1}{2}\left\{\sigma(\frac{\partial\phi}{\partial t})^2 - \rho\left[(\frac{\partial\phi}{\partial x})^2 + \frac{\partial\phi}{\partial x})^2\right] - \tau\phi^2 - \xi\phi^4 + \cdots\right\} \end{aligned} \tag{1.2.0.4}$$

Then he notes about the Landau-Ginzburg view:

> we start with the desired symmetry, say Lorentz invariance if we want to do particle physics, decide on the fields we want by specifying how they transform under the symmetry (in this case we decided on a scalar field ϕ), and then write down the action involving no more than two time derivatives (because we don't know how to quantize actions with more than two time derivatives)

Also it is customary to write $d = D + 1$ and speak of a $(D + 1)$-dimensional spacetime. When $D = 0$, quantum field theory reduces to normal quantum mechanics, which deal with only a point (in the mattress sense, we have only one particle).

Then he also notes that the Lagrangian can only have the form $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$.[1].

The classical limit of this formulation can be obtained by method similar to previous section. A saddle point approximation to the integral:

$$Z = \int D\phi e^{i/\hbar \int d^4x \mathcal{L}(phi)} \tag{1.2.0.5}$$

will find the classical field equation:

$$\partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} - \frac{\delta\mathcal{L}}{\delta\phi} = 0 \tag{1.2.0.6}$$

Corresponding pages in draft pp. 7 to 8.

Vacuum and Response It is said that the vacuum in quantum field theory is a stormy sea of quantum fluctuations.

Then, it discuss a process, very similar to the idea in linear response theory in condensed matter physics. The idea is to set up a source and a sink in which particles are created and annihilated. It correspondes to adding a term $sum_a J_a(t) q_a$ to the potential, or in the continuum limit, a term $\int d^D x J(x)\phi(x)$.

[1] in a footnote, he also mentions a possible additional term $U(\phi)(\partial\phi)^2$. It is related to a particle whose mass depends on position. This will be considered later

Free field theory Here he centers at the the Lagrangian

$$\mathcal{L}(\phi) = \frac{1}{2}\left((\partial\phi)^2 - m^2\phi^2\right) \tag{1.2.0.7}$$

The corresponding theory is called the **free** or **Guassian theory**. Using 1.2.0.6, one can easily obtain the Klein-Gorden equation:

$$(\partial^2 + m^2)\phi = 0 \tag{1.2.0.8}$$

The plain wave solution is of the form:

$$\omega^2 = \vec{k}^2 + m^2 \tag{1.2.0.9}$$

He notes that by transfer back to the SI unit, one will recognize above equations the same as the $E = mc^2$ identity.

Next, we evaluate the disturbed case of free field theory:

$$Z = \int D\phi e^{i\int d^4x\left(\frac{1}{2}[(\partial\phi)^2 - m^2\phi^2] + J\phi\right)} \tag{1.2.0.10}$$

The integration is rather lengthy. It involves a trick called "to imagine discretizing spacetime". In my opinion, such a trick is to the best, a fancy intuition. There maybe mathematical proof, but is neglected by the author. However, the result is surprisingly rich in intuition and analogy. It start with the discretized analogy with (1.1.2.1). One is gradually lead to believe that the result of:

$$Z = \int D\phi e^{i\int d^4x\left(-\frac{1}{2}(\partial^2 + m^2)\phi + J\phi\right)} \tag{1.2.0.11}$$

is:

$$Z(J) = \mathcal{C}e^{-(i/2)\int\int d^4x d^4y J(x)D(x-y)J(y)} \tag{1.2.0.12}$$

where $\mathcal{C}$ is just some factor which will proven to be unimportant. $D(x-y)$ is the solution of:

$$-(\partial^2 + m^2)D(x-y) = \delta^{(4)}(x-y) \tag{1.2.0.13}$$

The following is devoted to the function $D(x)$.

One checks by direct substitution that:

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2 - m^2 + i\varepsilon} \tag{1.2.0.14}$$

is the solution to 1.2.0.13.

We also denote the exponent in 1.2.0.12 as $W(J)$. So:

$$W(J) = -\frac{1}{2}\int\int d^4x d^4y J(x)D(x-y)J(y) \tag{1.2.0.15}$$

$$Z = \mathcal{C}e^{iW(J)} \tag{1.2.0.16}$$

Free propagator The function $D(x)$ is also called a propagator. It "plays an essential role in quantum field theory. As the inverse of a differential operator it is clearly closely related to the Green's function you encountered in a course on electromagnetism".

Then the author tries to explain how to evaluate $D(x)$, which I fail to reproduce. The final result is equation (23) on page 24:

$$D(x) = -i \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(e^{-i(w_k t - \vec{k}\cdot\vec{x})}\theta(x^0) + e^{i(w_k t - \vec{k}\cdot\vec{x})}\theta(-x^0) \right) \quad (1.2.0.17)$$

He notes that

> Physically, $D(x)$ describes the amplitude for a disturbance in the field to propagate from the origin to x.

In different region of spacetime, the value of $D(x)$ varies greatly. When x is in timelink region, $D(x)$ is the superposition of plane waves, while the phase is opposite between $x^0 = t < 0$ and $x^0 = t > 0$. When x is in the spacelike region, the $D(x)$ exhibits a strange behavior that the particle can leak outside lightcone, over a distance of order m^{-1}. This is consistent with the Heisenberg's uncertainty principle.

Corresponding pages in draft pp. 9 to 11.

Unclear points : I cannot follow all the calculation for $D(x)$, because my result is different from the book's. Also, the physics picture about $D(x)$ on page 24, is badly reasoned. The argument is tenuous, in my opinion.

1.3 I.4 From Field to Particle to Force

Note: discussion in this section is based on the *free field theory.*

From field to particle With the Fourier transformation, several physical properties could be extrated from $W(J)$. The Fourier transformed $W(J)$ is:

$$W(J) = -\frac{1}{2} \int \frac{d^4}{(2\pi)^4} J(k)^* \frac{1}{k^2 - m^2 + i\epsilon} J(k) \quad (1.3.0.18)$$

If separating J into $J(x) = J_1(x) + J_2(x)$, then naturally, the Fourier transformed calculation will include terms of the form: $J_1^* J_1, J_1^* J_2, J_2^* J_1, J_2^* J_2$. We see that the denominator in $W(J)$ contains $k^2 - m^2$. Somehow the author wants to argue that the term $J_2^* J_1$ describes the influence of a particle of mass m propagating from the source 1 to the sink 2. But his language is loose.

This part also introduces the jargon *on mass shell* and the language of "virtual particles" moving in k-space.

From particle to force With two delta source $J_a(x) = \delta^{(3)}(\vec{x} - \vec{x}_a)$, one can describe two massive lumps setting on $\vec{x}_1$ and $\vec{x}_2$ and not moving at all. These two lumps, coupled with the free field ϕ, will attract each other. To show the attraction, the author calculates explicitly the $W(J)^2$, and identifies $iW(J) \equiv -iET$ (the identification is substantiate by $\langle 0|e^{-iET}|0\rangle = Z = e^{iW(J)}$). Finally, it shows:

$$E = -\frac{1}{4\pi r}e^{-mr} \tag{1.3.0.19}$$

This is actually the Yukawa model for the attraction between nucleons in the atomic nucleus. A plot is in due:

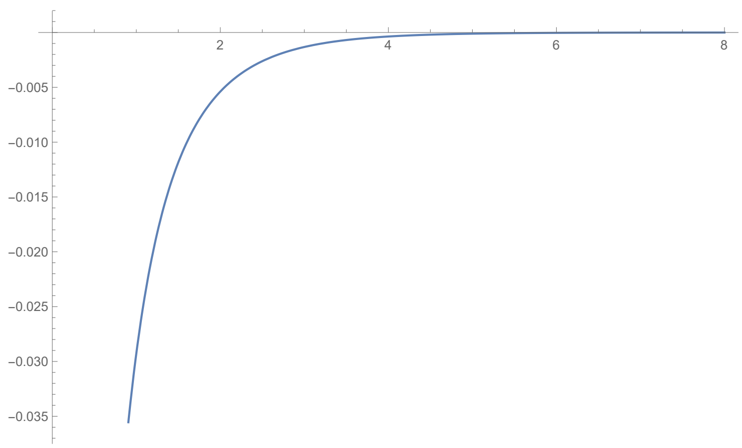

Figure 1.1: Plot of Yukawa potential $-\frac{e^{-r}}{4\pi r}$

It shows us several information:

- In the long distance, the potential drops off over the distance scale of $1/m$.
- In the short range, the force is $1/r^2$.
- The energy gets smaller when the two sources come closer. It tends to $-\infty$ as $r \to 0$.

Origin of force The energy calculated above, and the previous *sink and source* interpretation, lead us to believe that force is the result of an exchange of particles.

[2] Specifically, he calculates the $J_1^* J_2$ and $J_2^* J_1$ terms, ignoring the $J_1^* J_1$ and $J_2^* J_2$ terms, because the latter appears in $W(J)$ regardless of the existance of the other particle.

Connected versus disconnected Here the author notes the Feynman's heuristic language to discribe the integral related to $J(x)D(x-y)J(y)$ as a graph:

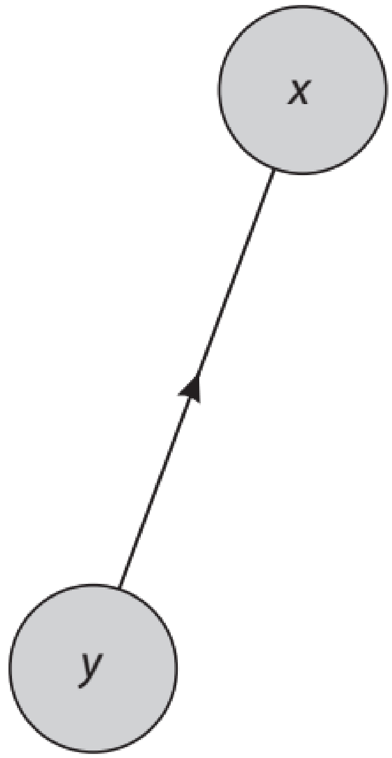

Figure 1.2: $J(x)D(x-y)J(y)$

It also introduces the concept of connectedness or disconnectedness in the graph (omitted here).

Unclear Points Marked in the text.

1.4 I.5 Coulomb an Newton: Repulsion and Attraction

This section tries to replicate the previous section's procedure to produce a repulsive force. It starts with the Maxwell's Lagrangian for electromagnetism $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. However, the authors adds an additional mass term and let the mass $m \to 0$ in the end. This is declared to be avoiding the complication of gauge invariance. The result is surprisingly simple. We have:

$$
\begin{aligned}
D_{\nu\lambda}(k) &= \frac{-g_{\nu\lambda} + k_\nu k_\lambda / m^2}{k^2 - m^2} \\
W(J) &= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} J^\mu(k)^* \frac{1}{k^2 - m^2 + i\epsilon} J_\mu(k) \\
E &= \frac{1}{4\pi r} e^{-mr}
\end{aligned}
\tag{1.4.0.20}
$$

This clearly represent the idea of repulsive Coulomb force.

Chapter 2

Bibliography

Bibliography

[1] A. Zee. Quantum Field Theory in a Nutshell 2ed. PUP.

Chapter 3

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