Dirac Hamiltonian

Taper

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Abstract

This aims for recapitulate important information about Dirac Hamiltonian as is written in Sakurai's text [SN11], chapter 8.[1] This is intended to learn to help me deal with a Lattice Dirac Hamiltonian.

Contents

The starting point is still the Schrodinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle \tag{0.0.1}$$

We will be using **Natural Units** from now on.

1 Klein-Gorden Equations

sec:Klien-Gorden Equations

The Klein-Gorden equation starts with the the relativistic energy (of a particle with momentum $\mathbf{p}$ and mass m):

$$E_p = +\sqrt{p^2 + m^2} \tag{1.0.2}$$

The plus sign here hints that we may have a negative energy solution.

We do not quantize this equation directly, because the square root is hard to represent in operators. Instead, Klein-Gorden equation starts with H^2:

$$H^2 = p^2 + m^2 \overset{\text{quantized as}}{\rightarrow} -\boldsymbol{\nabla}^2 + m^2 \tag{1.0.3}$$

On the other hand, the Schrodinger equation applying twice gives:

$$-\frac{\partial^2}{\partial t^2} = H^2 \tag{1.0.4}$$

[1] Be careful that some version of this book is missing chapter 8... I don't understand why...

Hence we have the Klein-Gorden equation:

$$\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 + m^2\right)\psi(x,t) = 0 \tag{1.0.5}$$

eq:kg-eq

It can be simplified in 4-vector notation as:

$$\left(\partial_\mu \partial^\mu + m^2\right)\psi(x,t) = 0 \tag{1.0.6}$$

eq:kg-eq-4vec

or most succinctly as,

$$\left(\partial^2 + m^2\right)\psi(x,t) = 0 \tag{1.0.7}$$

The author looked at this equation from three different aspects.

Producing the free particle solution It is first checked that the free-particle state:

$$\psi \propto \exp(-i(Et - \mathbf{p}\cdot\mathbf{x})) = e^{-ip^\mu x_\mu} \tag{1.0.8}$$

is a solution to the Klein-Gorden equation, provided that:

$$E^2 = \mathbf{p}^2 + m^2 \tag{1.0.9}$$

Probability density By analogy with non-relativistic case, the four-vector current is defined as:

$$j^\mu = \frac{i}{2m}[\psi^* \partial^\mu \psi - (\partial^\mu \psi)^* \psi] \tag{1.0.10}$$

It is easy to show $\partial_\mu j^\mu = 0$[2]. Then, this is a conserved current. The time-component of this is the density:

$$\rho \equiv j^0 = \frac{i}{2m}\left[\psi^* \partial_t \psi - (\partial_t \psi)^* \psi\right] \tag{1.0.11}$$

If one check this equation, one find that ρ is proportionally to the imaginary of some number which, one has no clear reason to say whether it is positive or negative. Therefore, the Klein-Gorden equation leaves one in a bad position to interprate the probability nature of quantum mechanics (what is a "negative probability" anyway?).

Coupling to electromagnetic field The author discusses a little bit about why, when there is electromagnetic field, we have the replacement

$$p \to p + eA \tag{1.0.12}$$

in page 490, footnote (be careful that the author uses the notation $e = |e|$, whereas I used the notation $e = -|e|$, i.e. e is just q, the charge).

This coupling effectively gives the replacement:

$$\partial_\mu \quad \to \quad D_\mu \tag{1.0.13}$$

where $D_\mu \equiv \partial_\mu - ieA_\mu$.

The Klein-Gorden equation now becomes:

$$[D_\mu D^\mu + m^2]\psi(x,t) = 0 \tag{1.0.14}$$

eq:kg-emfield

This equation revels a connection between $e \to -e$ and $\psi \to \psi^*$[3].

[2] Just note that normally we have $A^\mu B_\mu = A_\mu B^\mu$

[3] You will find this if you complex conjugate the Klein-Gorden equation.

Two component interpretation The author also mentions a way to split the equation into two equations. If we define $\phi(x,t)$ and $\chi(x,t)$ as in eq.(8.1.15) in p.491, then it is straightforward to check that, the Klein-Gorden equation is equivalent to:

$$iD_t\Upsilon = \left[-\frac{1}{2m}\mathbf{D}^2(\tau_3 + i\tau_2) + m\tau_3\right]\Upsilon \tag{1.0.15}$$

where $(D_t, \mathbf{D}) = D_\mu$, τ_i are pauli matrices, and Υ is the two component wave function:

$$\Upsilon \equiv \begin{pmatrix}\phi(x,t)\\ \chi(x,t)\end{pmatrix} \tag{1.0.16}$$

This formulation enables a mysterious interpretation of negative energy, negative probability, and particles and antiparticles. For more information, please look at pp.491 494 of [SN11].

2 Dirac Equation

sec:Dirac Equation

The Dirac equation aims to change the Klein-Gorden equation into a first order PDE. There are two approaches to derive the equation. The first can be found in section 8.2 of [SN11], and section II.1 of [Zee10]. The second approach is found in page 99 (chapter 2) of [Gre97], and is prefered in my opinion. After all, they lead to the same equation:

$$(i\gamma^\mu\partial_\mu - m)\psi(x,t) = 0 \tag{2.0.17}$$

eq:dirac-eq

Where γ^μ are matrices satisfying:

$$\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu} \tag{2.0.18}$$

In other words, γ^μ is a representation of the Clifford algebra $\mathrm{Cl}_{1,3}(\mathbb{R})$. We can cast this equation into another form:

$$i\frac{\partial}{\partial t} ket\psi = H\left|\psi\right\rangle = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m \tag{2.0.19}$$

where $\mathbf{p}$ is the momentum, $\boldsymbol{\alpha}$ and β satisfy a similar relation:

$$\frac{1}{2}\{\alpha_i, \alpha_j\} = \delta_{ij} \tag{2.0.20}$$

$$\{\alpha_i, \beta\} = 0 \tag{2.0.21}$$

$$\alpha_i^2 = \beta^2 = 0 \tag{2.0.22}$$

They are related to γ^μ matrices by:

$$\alpha_i = \gamma^0\gamma^i \quad \text{and} \quad \beta = \gamma^0 \tag{2.0.23}$$

A detailed formula for γ matrices can be found in p.94 of [Zee10]. A detailed formula for α_i, β matrices can be found in p.496 of [SN11].

The density problem The dirac formulation has a appearent advantage. The current defined by

$$j^\mu \equiv \bar{\psi}\gamma^\mu\psi \tag{2.0.24}$$

where

$$\bar{\psi} \equiv \psi^\dagger\beta \tag{2.0.25}$$

is conserved. And the probability density ρ is

$$\rho \equiv j^0 = \psi^\dagger\beta\gamma^0\psi = \psi^\dagger\psi \tag{2.0.26}$$

It is positive-definite.

Coupled with a charge When coupled with electromagnetic field, and with a the vector potential $\mathbf{A} = 0$, it becomes

$$H = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m - e\Phi \tag{2.0.27}$$

where Φ is the scalar potential.

References

[Gre97] Professor Dr. Walter Greiner. *Relativistic Quantum Mechanics*. Springer Berlin Heidelberg, 1997. URL: https://link.springer.com/book/10.1007{%}2F978-3-662-03425-5, doi:10.1007/978-3-662-03425-5.

[SN11] J. J. (Jun John) Sakurai and Jim. Napolitano. *Modern quantum mechanics*. Addison-Wesley, 2011.

[Zee10] A. Zee. *Quantum Field Theory in a Nutshell: (Second Edition)*. In a Nutshell. Princeton University Press, 2010. URL: https://books.google.co.jp/books?id=n8Mmbjtco78C.

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