Feynman Diagram
Part I - Interaction Picture

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Abstract

In this article, we introduces the basic Interaction Picture in quantum mechanics, then introduce the idea of "adiabatic turn on", which is supported by Gell-Mann and Low theorem (proved in appendix).

The style of the article might be fastidious. Yet I have try to maintain some physical viewpoints.

1 Note

This note is intended for learning Feynman Diagram techniques in many-body physics. However, due to the limited time and the growing size of this document, I have to split it into small parts. This first part provides one foundational material for the beginning of Feynman Diagrams, i.e. Interaction Picture of quantum mechanics. Further updates will be found on my blog [1] within this summer vocation.

2 Interaction Picture

The interaction Picture is a very important way to view physical systems as interacting systems growing from the underlying ground state systems. In Interaction Picture, both the wave functions and the operators are time dependent. The Hamiltonian is separated into the ground state part and the interaction part:

$$H = H_0 + V$$

where

- H_0 : is taken to be an exactly solvable Hamiltonian. Note that H_0 should be bilinear in $C^\dagger$ and C for Wick's theoren to be applicable. [1]

The operators and states are related to Schrödinger's Picture by

$$|\Psi_I\rangle = e^{iH_0t}\,|\Psi_s\rangle = e^{iH_0t}e^{-iHt}\,|\Psi_H\rangle \tag{2.0.1}$$

$$\hat{O}_1 = e^{iH_0t}\hat{O}_S e^{-iH_0t} \tag{2.0.2}$$

where subscripts denoting:

- I:Interaction Picture
- S: Schrödinger Picture
- H: Heisenberg Picture

Remark: we see that the "inner product" value is unaltered by this:

$$\langle\Psi_I|\hat{O}_S|\Psi_S\rangle = \langle\Psi_S|e^{-iH_0t}\cdot(e^{iH_0t}\hat{O}e^{-iH_0t})\cdot e^{iH_0t}|\Psi_S\rangle = \langle\Psi_S|\hat{O}_S|\Psi_S\rangle$$

Physically, this leaves the probabilities (the only important value) unchanged.

The evolution of states is governed by equation of motion. In Schrödinger Picture, it is the Schrödinger's equation. In Interaction Picture, we will mainly deal the evolution operator.

2.1 Equation of Motion for Observables

Before we talk about evolution operator, we mention two useful results. The first is a equation of motion for observables in Interaction Picture.

[1] See page 79, Sec 2.4 of [3]

Since: $\hat{O}_I(t) = e^{i\hat{H}_0 t}\hat{O}_S e^{-i\hat{H}_0 t}$, we have:

$$
\begin{aligned}
& i \cdot \frac{\partial \hat{O}_I(t)}{\partial t} \\
&= i\left[i\hat{H}_0 e^{i\hat{H}_0 t}\hat{O}_S e^{-i\hat{H}_0 t}] + e^{i\hat{H}_0 t}\hat{O}_S(-i\hat{H}_0)e^{-i\hat{H}_0 t}\right] \\
&= -\hat{H}_0(e^{i\hat{H}_0 t}\hat{O}_S e^{-i\hat{H}_0 t}) + e^{i\hat{H}_0 t}\hat{O}_S e^{-i\hat{H}_0 t}\hat{H}_0 \\
&= [\hat{O}_I(t), \hat{H}_0] \qquad (2.1.1)
\end{aligned}
$$

This looks very similar to the Heisenberg's equation, which is reasonable since in both pictures the operators is time-dependent.

2.2 Equation of Motion for wave function

Another important result is the following equation:

$$
\begin{aligned}
i\frac{\partial}{\partial t}|\Psi_I(t)\rangle =& i\frac{\partial}{\partial t}(e^{i\hat{H}_0 t}|\Psi_S(t)\rangle) && (2.2.1)\\
=& i(i\hat{H}_0)\cdot|\Psi_I(t)\rangle + e^{i\hat{H}_0 t}\cdot\frac{\partial}{\partial t}|\Psi_S(t)\rangle && (2.2.2)\\
=& -\hat{H}_0|\Psi_I(t)\rangle + e^{i\hat{H}_0 t}(H_0 + V)|\Psi_S(t)\rangle && (2.2.3)\\
=& [-\hat{H}_0 + e^{iH_0 t}(\hat{H}_0 + V)e^{-i\hat{H}_0 t}]|\Psi_I(t)\rangle && (2.2.4)\\
=& \hat{V}_I|\Psi_I(t)\rangle && (2.2.5)
\end{aligned}
$$

However, in Interaction Picture, there is one operator which is even more important than the above equations. It is the evolution operator to be defined below.

2.3 The evolution operator in Interaction picture

Note: From now on, we work mostly in Interaction Picture. Hence the subscript I will be neglected, when it is convenient, for operators or states in Interaction Picture.

The evolution operator $\hat{U}(t, t_0)$ is defined s.t.

$$|\Psi(t)\rangle = \hat{U}(t, t_0)|\Psi(t_0)\rangle \qquad (2.3.1)$$

We observe that $\hat{U}$ behaves just like the time-evolution operator in Schrödinger Picture: it brings a state at time t_0 to a later time t. For finite time, it can be given a formal expression as follows:

By equation (2.0.1) on the preceding page we have

$$|\Psi_I(t)\rangle = e^{iH_0t}\,|\Psi_S(t)\rangle \tag{2.3.2}$$

$$= e^{iH_0t}\cdot e^{iH(t-t_0)}\,|\Psi_S(t_0)\rangle \qquad \text{(by Schrödinger equation)} \tag{2.3.3}$$

$$= e^{iH_0t}\cdot e^{-iH(t-t_0)}\cdot e^{iH_0t_0}\,|\Psi_I(t_0)\rangle \qquad \text{(by 2.0.1)} \tag{2.3.4}$$

so:

$$\hat{U}(t,t_0) = e^{iH_0t}e^{-iH(t-t_0)}e^{iH_0t_0} \tag{2.3.5}$$

The evolution operator has following properties:

1. $\hat{U}(t_0,t_0) = 1$

   Physically, this means that a state is not evolving.

2. $\hat{U}(t_2,t_1)\hat{U}(t_1,t_0) = \hat{U}(t_2,t_0)$:

   Physically, an evolving state should obviously be continuous in time, hence this property. It can also be demonstrated mathematically:

$$\begin{aligned}\hat{U}(t_0,t_1)\hat{U}(t_1,t_0) =& e^{iH_0t_2}e^{-iH(t_2-t_1)}e^{-iH_0t_1}\cdot e^{iH_0t_1}e^{-iH(t_1-t_0)}e^{-iH_0t_0}\\ =& e^{iH_0t_2}e^{-iH(t_2-t_0)}e^{-iH_0t_0}\end{aligned}$$

3. $\hat{U}(t,t_0)\cdot\hat{U}(t_0,t) = 1$

   This is a result of 2.

4. $\hat{U}(t,t_0)$ is unitary:

   Because $U(t,t_0)$ is a transformation of basis that brings all basis at time t_0 to time t.

   Also, one can easily see that mathematically: $\hat{U}(t,t_0)\cdot\hat{U}^\dagger(t,t_0) = 1$

5. $U^{-1}(t,t_0) = U^\dagger(t,t_0) = U(t_0,t)$

   This means that "inverse" = "dagger" for $\hat{U}$. This is result of 4.

3 Perturbative Solution for $\hat{U}$

The operator $\hat{U}$ has another more useful form which is the basis for perturbative analysis of complex interacting systems. To obtain such form, we first derive an integral equation for $\hat{U}$ By equation (2.2.5) on the previous page:

$$i\frac{\partial}{\partial t} \cdot \hat{U}(t,t_0) \left|\Psi_I(t_0)\right\rangle = (\hat{V})_I \hat{U}(t,t_0) \left|\Psi_I(t_0)\right\rangle \tag{3.0.6}$$

$$\rightarrow \qquad i\frac{\partial}{\partial t}\hat{U}(t,t_0) = -i\hat{V}_I U(t,t_0) \tag{3.0.7}$$

$$\rightarrow \qquad \hat{U}(t,t_0) - \hat{U}(t_0,t_0) = -i \cdot \int_{t_0}^{t} \hat{V}_I U(t,t_0) dt \tag{3.0.8}$$

$$or \qquad \hat{U}(t,t_0) = 1 - i\int_{t_0}^{t} \hat{V}_I U(t_1,t_0) dt_1 \tag{3.0.9}$$

The integral equation is solved by iterative methods. We first start with a primitive guess:

(5.7.6) of Sakurai P356 = (2.18) of Makan P68

$$\hat{U}^{(1)}(t,t_0) = 1 - \int_{t_0}^{t} \hat{V}(t')dt'$$

Here, $\hat{U}^{(1)}$ is the first order solution.
Then one improves on this with

$$\begin{aligned} \hat{U}^{(2)}(t,t_0) =& 1 - \int_{t_0}^{t} \hat{V}(t')\hat{U}^{(1)}(t',t_0)dt' && (3.0.10)\\ =& 1 - \int_{t_0}^{t} \hat{V}(t_2)(1 - \int_{t_0}^{t_2} \hat{V}(t_1)dt_1)dt_2 && (3.0.11) \end{aligned}$$

This is called it the second order solution.
And one improves again:

$$\begin{aligned} U^{(3)} =& 1 - \int_{t_0}^{t} dt_3 \cdot \hat{V}(t_3) \cdot \hat{U}^{(2)}(t_3,t_0) && (3.0.12)\\ =& 1 - \int_{t_0}^{t} dt_3 \hat{V}(t_3) \cdot [1 - \int_{t_0}^{t_3} dt_2 \hat{V}(t_2) \cdot (1 - \int_{t_0}^{t_2} dt_1 \hat{V}(t_1))dt_1] && (3.0.13) \end{aligned}$$

Inductively we get the expression:

$$\hat{U}(t,t_0) = \sum_{n=0}^{\infty} (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \hat{V}(t_1) \cdots \hat{V}(t_n) \tag{3.0.14}$$

We remark that we have already obtain U in the form of

$$U = \Sigma_1 (-i)^n a_n$$

If we can find another value $b_n = n!a_n$, then

$$U = \sum_i \frac{(-i)^n}{n!} b_n = \text{ (Formally writing as) } e^{-ib}$$

This is related to other forms of Wick's theorem, which we will not discuss until the subsequent parts of this series.

Now we change the expression for $\hat{U}$ a little bit. Let us take the case of $n = 2$ as an example. The main part is the integral:

$$\int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \hat{V}_I(t_2)\hat{V}_I(t_1)$$

The integration is taken diagrammatically:

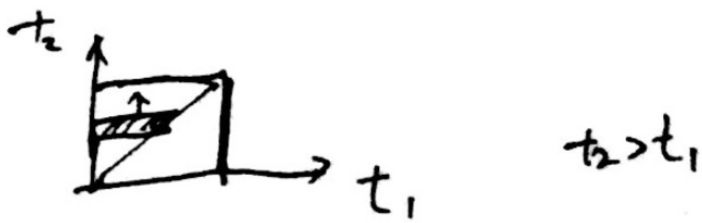

Figure 1: $t_2 > t_1$

If we relabel t_1 to t_2, t_2 to t_1, we get:

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdot \hat{V}_I(t_1)\hat{V}_I(t_2)$$

Then the integration is changed into:

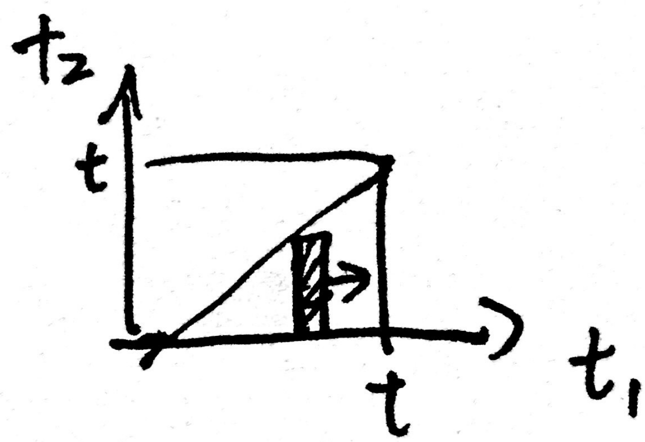

Figure 2: $t_1 > t_2$

We observe that doing the two integration is equivalent to doing a full integral over the region marked by ($t_0 \leqslant t_1 \leqslant t$ and $t_0 \leqslant t_2 \leqslant t$),

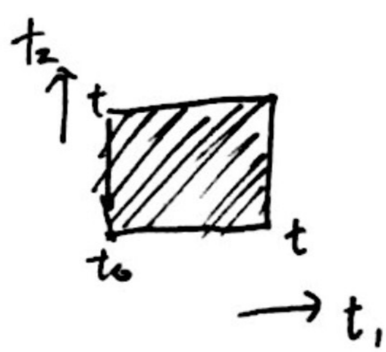

Figure 3: Whole region

while restricting the value of integration in the proper region and putting the operators in correct order.

For example:

equation (3) on page 5 is: $\int_{t_0}^{t} dt_2 \cdot \int_{t_0}^{t_2} dt_1 \theta(t_1 - t_2)\hat{V}_I(t_2)\hat{V}_I(t_1)$

equation (3) on the previous page is: $\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdot \theta(t_1 - t_2)\hat{V}_I(t_2)\hat{V}_I(t_1)$

where $\theta(t_1 - t_2)$ is restricting the to region $t_2 > t_1$. Also, I put the latest time in the front.

3.1 Time Ordering Operator T

We introduce a time ordering operator T to signify the process we described above. When T acts on a set of operators, it automatically sorts them in descending time order.

For example, if $t_3 > t_2 > t_1$, then

$$T(A(t_1)B(t_3)C(t_2)) = B(t_3)C(t_2)A(t_1)$$

Then

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 = T(\hat{V}(t_1)\hat{V}(t_2)) \tag{3.1.1}$$
$$= (\text{integration of } \hat{V}_I(t_1)\hat{V}_I(t_2) \text{ when } t_1 > t_2)$$
$$+(\text{integration of } \hat{V}(t_2)\hat{V}(t_1) \text{ when } t_2 > t_1)$$
$$= (\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \hat{V}_I(t_1)\hat{V}_I(t_2)$$
$$+(\int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \hat{V}_I(t_2)\hat{V}_I(t_1)) \tag{3.1.2}$$

But equation (3) on the preceding page is equal to equation (3) on page 5, because $\int_{t_0}^{t} dt_2 \cdot \int_{t_0}^{t_2} dt_1 \hat{V}_I(t_2)\hat{V}_I(t_1) = \frac{1}{2} \cdot \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdot T[\hat{V}_2(t_1)\hat{V}_2(t_2)]$

Similarly, for arbitrary n, we have

$$\int_{t_0}^{t} dt_n \int_{t_0}^{t_n} dt_(n-1) \cdots \int_{t_0}^{t} dt_1 \cdot \hat{V}_I(t_n) \cdots \hat{V}_I(t_1) = \frac{1}{n!} \cdot \int_{t_0}^{t} dt_n \cdots \int_{t_0}^{t} dt_1 \cdot T(\hat{V}_I(t_n) \cdots \hat{V}_I(t_1))$$

The factor $n!$ is that we have in total $n!$ different ways to rearrange $t_1, \cdots, t_n$. And each arrangement $t_{i_1}, \cdots, t_{i_n}$ corresponds to a region in $[t_0, t_0]^n$ where $t_0 \leqslant t_{i_1} < \cdots < t_{i_n} \leqslant t$.

Thus we the most useful expression for operator $\hat{U}$:

$$\hat{U}(t, t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n T[\hat{V}(t_1) \cdots \hat{V}(t_n)] \tag{3.1.3}$$

4 Adiabatic Switch on

The next idea for interacting many-body system is to use the notion of adiabatic switch on. In the Interaction Picture, we assume our Hamiltonian is:

$$H = H_0 + V$$

We wish to extract the eigenstate of H from that of H_0. Hence we imagine that we have

$$H' = H_0 + e^{-\epsilon|t|}V$$

So we have $\lim_{t\to\infty} H' = \lim_{t\to\infty} H' = H_0$.

In this scenario, we have

$$|\Psi(t = \pm\infty)\rangle = |\Phi_0\rangle$$

where $|\Phi_0\rangle$ can be exactly solved.

Then using the evolution operator U, we can bring up the states from $-\infty$ to now $t = 0$.

$$|\Psi(t = 0)\rangle = U_\epsilon(0, -\infty)\,|\Psi(t = -\infty\rangle \tag{4.0.4}$$

$$= U_\epsilon(0, -\infty)\,|\Psi(\infty)\rangle \tag{4.0.5}$$

This is the solution to $H = H_0 + V$, since $e^{-\epsilon\cdot 0} = 0$.

However, there is a problem with this. Because the real physical system is in the Hamiltonian $H = H_0 + V$ all the time. And our system dives in the Hamiltonian H only at one instant of time. If the result of over calculation is to be meaningful, it must be independent on ϵ.

In addition, we have a Gell-Mann and Low theorem, which requires a length proof provided in appendix A. It states that:

If the following quantity exists to all orders in perturbation theory, (denoting $|\Psi_0\rangle :\equiv U_\epsilon(0, -\infty)\,|\Phi_0\rangle$.)

$$\frac{\hat{U}_\epsilon(0, -\infty)\,|\Phi_0\rangle}{\langle\Phi_0|\,\hat{U}_\epsilon(0, -\infty)\,|\Phi_0\rangle} \equiv \frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle}$$

then $|\Psi_0\rangle$ is an eigenstate of $\hat{H}$ in the following sense:

$$\hat{H}\frac{|\Phi_0\rangle}{\langle\Phi_0|\Psi_0\rangle} = E\frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle}$$

The equation (6.44) could be recast into a new form.

$$\langle\Phi_0|\,\hat{H}\frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle} = E\cdot\frac{\langle\Phi_0|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle}$$

Since $H = H_0 + V$ and $\langle\Phi_0|H_0|\Psi_0\rangle = E_0\cdot\langle\Phi_0|\Psi_0\rangle$, we have

$$\frac{\langle\Phi_0|V|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle} = E - E_0 \tag{4.0.6}$$

Thus, in the limit $\epsilon \to 0$ we have a way to calculate the energy of our physical system's E by 4.0.6. This is also intuitively satisfying, since when ϵ is small, $\hat{H}'$ approximates $\hat{H}$ in a long enough time. If we imagine that we turn on H' from $t = -\infty$ very slowly, then we are confident that our calculation is giving us the real wave function and its energy.

5 Conclusion

An abrupt stop has to be put here, but we leaves a few points on the next part now. We will define Green functions, which are related to a variety of physical phenomenons. Then the above idea of adiabatic switch on gives us a formula to calculate Green functions. This formula, combined with perturbative solution to $\hat{U}$ and the Wick's thoerem, gives us a systematical way of calculating Green functions. Feynman Diagrams will then be introduced to give a graphical representation of each Green functions.

6 Credits

Since almost all of the material are taken from chapter 3 of Fetter's book [2], citation has been omitted throughout the whole article.

A Proof of Gell-Mann and Low Theorem

The theorem states:

Theorem 1. *If the following quantity exists to all orders in perturbation theory:*

$$\frac{\hat{U}_\epsilon(0, -\infty) |\Phi_0\rangle}{\langle\Phi_0| \hat{U}_\epsilon(0, -\infty) |\Phi_0\rangle} \equiv \frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle}$$

then $|\Psi_0\rangle$ is an eigenstate of $\hat{H}$ in the following sense:

$$\hat{H}\frac{|\Phi_0\rangle}{\langle\Phi_0|\Psi_0\rangle} = E\frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle}$$

Proof:
Here and afterwards, we will set $|\Psi_0(\epsilon)\rangle \equiv \hat{U}_\epsilon(0, -\infty) |\Phi_0\rangle$.
Consider the following:

$$\begin{aligned}
(\hat{H}_0 - E_0) |\Psi_0(\epsilon)\rangle =& (\hat{H}_0 - E_0)\hat{U}_\epsilon(0, -\infty) |\Psi_0\rangle && \text{(A.0.7)}\\
=& \Big(\hat{H}_0\hat{U}_\epsilon(0, -\infty) - \hat{U}_\epsilon(0, -\infty)E_0\Big) |\Psi_0\rangle && \text{(A.0.8)}\\
=& \Big(\hat{H}_0\hat{U}_\epsilon(0, -\infty) - \hat{U}_\epsilon(0, -\infty)\hat{H}_0\Big) |\Psi_0\rangle && \text{(A.0.9)}\\
=& [\hat{H}_0, \hat{U}_\epsilon(0, -\infty)] |\Psi_0\rangle && \text{(A.0.10)}
\end{aligned}$$

The expression for $\hat{U}_\epsilon(0,-\infty)$ is explicitly:

$$\hat{U}_\epsilon(0,-\infty) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{0} dt_1 \cdots \int_{-\infty}^{0} dt_n e^{\epsilon(t_1+\cdots+t_n)} T[\hat{V}(t_1)\cdots\hat{V}(t_n)] \tag{A.0.11}$$

So we need $[\hat{H}_0, \hat{V}(t_1)\cdots V(\hat{t}_n)]$ to get $[\hat{H}_0, \hat{U}_\epsilon(0,-\infty)]$.

Lemma 1. $[H, ABC\cdots] = ([H,A]BC\cdots) + (A[H,B]C\cdots) + \cdots$

Proof:
Obviously we have:

$$[H, AB] = [H,A]B + A[H,B] \tag{A.0.12}$$

Then for more terms we can, for example:

$$\begin{aligned}[H, ABC] =& [H,AB]C + AB[H,C] \\ =& [H,A]BC + A[H,B]C + AB[H,C]\end{aligned}$$

Thus one may obviously obtain the lemma by using equation (A.0.12) recursively.

Using above lemma, one has:

$$\begin{aligned}[\hat{H}_0, \hat{V}(t_1)\cdots\hat{V}(t_n)] = & [\hat{H}_0, \hat{V}(t_1)]\hat{V}(t_2)\cdots \\ & + \cdots \\ & + \hat{V}(t_1)\cdots\hat{V}(t_{n-1})[\hat{H}_0, \hat{V}(t_n)]\end{aligned} \tag{A.0.13}$$

Also, in Interaction Picture, one has:

$$i\frac{\partial \hat{V}(t)}{\partial t} = [\hat{V}(t), \hat{H}_0]$$

or:

$$\frac{1}{i}\frac{\partial \hat{V}(t)}{\partial t} = [\hat{H}_0, \hat{V}(t)]$$

Hence the above commutators are turned into partial derivatives:

$$[\hat{H}_0, \hat{V}(t_1)\cdots\hat{V}(t_n)] = \frac{1}{i}\left(\frac{\partial}{\partial t_1} + \cdots + \frac{\partial}{\partial t_n}\right)\hat{V}(t_1)\cdots\hat{V}(t_n) \tag{A.0.14}$$

We need an additional fact:

$$T[\left(\frac{\partial}{\partial t_1} + \cdots + \frac{\partial}{\partial t_n}\right)\hat{V}(t_1)\cdots\hat{V}(t_n)] = \left(\frac{\partial}{\partial t_1} + \cdots + \frac{\partial}{\partial t_n}\right)T[\hat{V}(t_1)\cdots\hat{V}(t_n)] \tag{A.0.15}$$

This can be seen on an example. Notice that $T[\hat{V}(t_1)\hat{V}(t_2)] = \theta(t_1-t_2)\hat{V}(t_1)\hat{V}(t_2) + \theta(t_2-t_1)\hat{V}(t_2)\hat{V}(t_1)$, where $\theta(t)$ is Heaviside step function. We also have: $\frac{\partial}{\partial t}\theta(t) = \delta(t)$. So when $t_1 \neq t_2$, $T\frac{\partial}{\partial t_1} = \frac{\partial}{\partial t_1}T$, i.e. it commutes with T. When $t_1 = t_2$, this is obviously also correct as well. Hence:

$$T[\frac{\partial}{\partial t_1}\hat{V}(t_1)\hat{V}(t_2)] = \frac{\partial}{\partial t_1}T[\hat{V}(t_1)\hat{V}(t_2)]$$

Similar arguments will give us equation (A.0.15), since T is essentially a series of $\theta(t)$ functions.

Using the above fact and A.0.14, one turns A.0.10 into (Denote $\int_{-\infty}^{0} dt_1 \cdots \int_{-\infty}^{0} dt_n$ as *(Integrate)*):

$$\begin{aligned}
&(\hat{H}_0 - E_0)\,|\Psi_0(\epsilon)\rangle = \\
&\left[\hat{H}_0, 1 + \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\ \textit{(Integrate)}\ e^{\epsilon(t_1+\cdots+t_n)}T[\hat{V}(t_1)\cdots\hat{V}(t_n)]\right] \\
&\qquad = \sum_{n=1}^{\infty}\frac{(-i)^n}{i\cdot n!}\ \textit{(Integrate)}\ e^{\epsilon(t_1+\cdots+t_n)}\left(\frac{\partial}{\partial t_1}+\cdots+\frac{\partial}{\partial t_n}\right)T[\hat{V}(t_1)\cdots\hat{V}(t_n)] \\
&\qquad = \sum_{n=1}^{\infty}-\frac{(-i)^{n-1}}{n!}\ \textit{(Integrate)}\ e^{\epsilon(t_1+\cdots+t_n)}\left(\frac{\partial}{\partial t_1}+\cdots+\frac{\partial}{\partial t_n}\right)T[\hat{V}(t_1)\cdots\hat{V}(t_n)]
\end{aligned} \tag{A.0.16}$$

Integration by parts gives us, for example:

$$\int_{-\infty}^{0} dt_1 e^{\epsilon t_1}\frac{\partial}{\partial t_1}T[\hat{V}(t_1)] = e^{\epsilon t_1}T[\hat{V}(t_1)]\Big|_{\infty}^{0} - \epsilon\int_{\infty}^{0} dt_1 e^{\epsilon t_1}T[\hat{V}(t_1)]$$

Similarly:

$$\begin{aligned}
&\textit{(Integrate)}\ e^{\epsilon(t_1+\cdots+t_n)}\frac{\partial}{\partial t_1}T[\hat{V}(t_1)\cdots\hat{V}(t_n)] \\
&= \hat{V}(0)\ \textit{(Integrate without } t_1\textit{)}\ e^{\epsilon(t_2+\cdots+t_n)}T[\hat{V}(t_2)\cdots\hat{V}(t_n)] \\
&- \epsilon\ \textit{(Integrate)}\ e^{\epsilon(t_1+\cdots+t_n)}T[\hat{V}(t_1)\cdots\hat{V}(t_n)]
\end{aligned} \tag{A.0.17}$$

For other "$\frac{\partial}{\partial t_i}$", the procedure is essentially the same, and they all contribute to the same value, as can be seen by relabeling the dummy indices "$t_1 \cdots t_n$". Hence A.0.16 is:

$$
\begin{aligned}
(\hat{H}_0 - E_0)\,|\Psi_0\rangle = \sum_{n=1}^{\infty} -\frac{(-i)^{n-1}}{n!} \cdot n \times \\
&\Big\{ \hat{V}(0) \textit{ (Integrate without } t_1\textit{)}\ e^{\epsilon(t_2+\cdots+t_n)} T[\hat{V}(t_2)\cdots\hat{V}(t_n)] \\
&- \epsilon \textit{ (Integrate) } e^{\epsilon(t_1+\cdots+t_n)} T[\hat{V}(t_1)\cdots\hat{V}(t_n)] \Big\}\,|\Phi_0\rangle \\
&(n \text{ for } n \text{ same contributions from } \int_{-\infty}^{0} dt_1 \cdots \int_{-\infty}^{0} dt_n.) \\
&= -\hat{V}(0)\hat{U}_\epsilon(0,-\infty)\,|\Phi_0\rangle \text{ (the first term)} \\
&+ \epsilon \sum_{n=1}^{\infty} \frac{(-i)^{n-}}{(n-1)!} \textit{ (Integrate) } e^{\epsilon(t_1+\cdots+t_n)} T[\hat{V}(t_1)\cdots\hat{V}(t_n)]\,|\Phi_0\rangle
\end{aligned}
$$

The second term is dealt with as follows:

$$
\begin{aligned}
\frac{(-i)^{n-1}}{(n-1)!} \textit{ (Integrate) } T[\hat{V}(t_1)\cdots\hat{V}(t_n)] (\text{assuming } \hat{V} = g\cdot\hat{V}') \\
&= \frac{(-i)^{n-1}}{(n-1)!} \textit{ (Integrate) } g^n T[\hat{V}'(t_1)\cdots\hat{V}'(t_n)] \\
&= ig\frac{\partial}{\partial g}\frac{(-i)^n}{n!} g^n \textit{ (Integrate) } T[\hat{V}'(t_1)\cdots\hat{V}'(t_n)] \\
&= ig\frac{\partial}{\partial g}\frac{(-i)^n}{n!} g^n \textit{ (Integrate) } T[\hat{V}(t_1)\cdots\hat{V}(t_n)]
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\text{second term} &= \epsilon \sum_{n=1}^{\infty} ig\frac{\partial}{\partial g}\frac{(-i)^n}{n!} \textit{ (Integrate) } T[\hat{V}(t_1)\cdots\hat{V}(t_n)]\,|\Phi_0\rangle \\
&= i\epsilon g\frac{\partial}{\partial g}\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \textit{ (Integrate) } T[\hat{V}(t_1)\cdots\hat{V}(t_n)]\,|\Phi_0\rangle \\
&\quad (\text{since } \frac{\partial}{\partial g}(\text{zeroth term}) = 0) \\
&= i\epsilon g\frac{\partial}{\partial g}\hat{U}_\epsilon(0,-\infty)\,|\Phi_0\rangle \\
&= i\epsilon g\frac{\partial}{\partial g}\,|\Psi_0(\epsilon)\rangle
\end{aligned}
$$

Together we have (noting that $\hat{V}(0)$ is actually the interacting part of real Hamiltonian $\hat{V}$):

$$
(\hat{H}_0 - E_0)\,|\Psi_0(\epsilon)\rangle = -\hat{V}\,|\Psi_0(\epsilon)\rangle + i\epsilon\frac{\partial}{\partial g}\,|\Psi_0(\epsilon)\rangle \tag{A.0.18}
$$

Write it in another way:

$$(\hat{H}_0 + \hat{V} - E_0)\,|\Psi_0(\epsilon)\rangle = (\hat{H} - E_0)\,|\Psi_0(\epsilon)\rangle$$
$$= i\epsilon g\frac{\partial}{\partial g}\,|\Psi_0(\epsilon)\rangle \tag{A.0.19}$$

Multiplying left with $\frac{\langle\Phi_0|}{\langle\Phi_0|\Psi_0(\epsilon)\rangle}$, we have:

$$RHS = i\epsilon g\frac{\langle\Phi_0|\frac{\partial}{\partial g}|\Psi_0(\epsilon)\rangle}{\langle\Phi_0|\Psi_0(\epsilon)\rangle}$$
$$(\text{since } \frac{\partial}{\partial g}\langle\Phi_0| = 0)$$
$$= i\epsilon g\frac{\frac{\partial}{\partial g}\langle\Phi_0|\Psi_0(\epsilon)\rangle}{\langle\Phi_0|\Psi_0(\epsilon)\rangle}$$
$$= i\epsilon g\frac{\partial}{\partial g}\ln\langle\Phi_0|\Psi_0(\epsilon)\rangle \tag{A.0.20}$$

while:

$$LHS = \frac{\langle\Phi_0|\hat{H}|\Psi_0(\epsilon)\rangle}{\langle\Phi_0|\Psi_0(\epsilon)\rangle} = E - E_0 \tag{A.0.21}$$

So we have:

$$i\epsilon g\frac{\partial}{\partial g}\ln\langle\Phi_0|\Psi_0(\epsilon)\rangle = E - E_0 \tag{A.0.22}$$

From above expression, we notice that $\lim_{\epsilon\to 0^+}\epsilon\ln\langle\Phi_0|\Psi_0(\epsilon)\rangle$ is finite, otherwise we would have $E = E_0$ when $\epsilon \to 0^+$, which is absurd. Hence, $ln\,\langle\Phi_0|\Psi_0(\epsilon)\rangle \propto \frac{1}{\epsilon}$, or $\langle\Phi_0|\Psi_0(\epsilon)\rangle \propto e^{1/\epsilon}$.

Now we calculate

$$\begin{aligned}
&(H - E_0 - i\epsilon g\frac{\partial}{\partial g})\frac{|\Psi_0(\epsilon)\rangle}{\langle\Phi_0|\Psi_0(t)\rangle} \\
&= \frac{(\hat{H} - E_0 - i\epsilon g\frac{\partial}{\partial g})\,|\Psi_0(\epsilon)\rangle}{\langle\Phi_0|\Psi_0(\epsilon)\rangle} + i\epsilon g\frac{|\Psi_0(\epsilon)\rangle\,\frac{\partial}{\partial g}\langle\Phi_0|\Psi_0(\epsilon)\rangle}{\langle\Phi_0|\Psi_0(\epsilon)\rangle^2} \\
&= (\text{by A.0.19 the first term vanishes})\quad 0 + \frac{|\Psi_0(\epsilon)\rangle}{\langle\Phi_0|\Psi_0(\epsilon)\rangle}\cdot i\epsilon g\frac{\partial}{\partial g}\ln\langle\Phi_0|\Psi_0(\epsilon)\rangle \\
&= (\text{by A.0.22})\quad \frac{|\Psi_0(\epsilon)\rangle}{\langle\Phi_0|\Psi_0(\epsilon)\rangle}\cdot(E - E_0)
\end{aligned} \tag{A.0.23}$$

That is:

$$\left(\hat{H} - E - i\epsilon g\frac{\partial}{\partial g}\right)\frac{|\Psi_0(\epsilon)\rangle}{\langle\Phi_0|\Psi_0(\epsilon)\rangle} = 0 \tag{A.0.24}$$

Now by assumption $\frac{|\Psi_0(\epsilon)\rangle}{\langle\Phi_0|\Psi_0(\epsilon)\rangle}$ is finite when $\epsilon \to 0^+$, we have:

$$\hat{H}\frac{|\Phi_0\rangle}{\langle\Phi_0|\Psi_0\rangle} = E\frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle} \tag{A.0.25}$$

Q.E.D.

References

[1] `http://we-taper.github.io/`

[2] Alexander L. Fetter, John Dirk Walecka, Quantum Theory of Many-Particle Systems

[3] Gerald D. Mahan, Many-Particle Physics (3rd)