Condensed Matter Field Theory notes

Taper

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Abstract

Notes of book [AS10].

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1 pp.33 eq.1.43

sec:Table

In page 33 of [AS10], the author derives a difference of action, when we have a symmetry transformation paraterized by ω_a:

$$x_\mu \to x'_\mu = x_\mu + \frac{\partial x_\mu}{\omega_a}|_{\omega=0}\omega_a(x) \tag{1.0.1}$$

$$\phi^i(x) \to \phi'(x') = \phi^i(x) + \omega_a(x)F^i_a[\phi] \tag{1.0.2}$$

We have:

$$\mathcal{L} = \mathcal{L}(\phi^i(x), \partial_{x_\mu}\phi^i(x)) \tag{1.0.3}$$

$$\mathcal{L}' = \mathcal{L}'(\phi'^i(x'), \partial_{x'_\mu}\phi'^i(x')) \tag{1.0.4}$$

$$= \mathcal{L}\left(\phi^i + F^i_a\omega_a, \left(\delta_{\mu\nu} - \partial_{x_\mu}(\omega_a\partial_{\omega_a}x_\mu)\right)\partial_{x_\nu}(\phi^i + F^i_a\omega_a)\right) \tag{1.0.5}$$

And

$$\Delta S = \int \mathrm{d}^m x' \, \mathcal{L}' - \int \mathrm{d}^m x \, \mathcal{L} \tag{1.0.6}$$

eq:dS-integrand

$$\begin{aligned} &= \int \mathrm{d}^m x \left(1 + \partial_{x_\mu} \left(\omega_a \partial_{\omega_a} x_\mu\right)\right) \\ &\times \mathcal{L}\left(\phi^i + F^i_a \omega_a, \left(\delta_{\mu\nu} - \partial_{x_\mu}\left(\omega_a \partial_{\omega_a} x_\mu\right)\right) \partial_{x_\nu}(\phi^i + F^i_a\omega_a)\right) \\ &- \int \mathrm{d}^m x \, \mathcal{L}(\phi^i(x), \partial_{x_\mu}\phi^i(x)) \end{aligned} \tag{1.0.7}$$

Then he argues that, "for constant parameters ω_a the action difference Δa vanishes". Therefore "the leading contribution to the action difference of a symmetry transformation must be linear in the derivative $\partial_{x_\mu}\omega_a$".

Then he writes that "A straightforward expansion of the formula above for ΔS shows that these terms are given by"

$$\Delta S = -\int \mathrm{d}^m x \, j^a_\mu(x) \partial_{x_\mu} \omega_a \tag{1.0.8}$$

where j^a_μ is:

$$j^a_\mu = \left(\frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \partial_{x_\nu}\phi^i - \mathcal{L}\delta_{\mu\nu}\right) \frac{\partial x_\nu}{\partial \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} F^i_a \tag{1.0.9}$$

I am partically confused about how to do the "straightforward expansion". I guess I should do $\frac{\partial}{\partial(\partial_{x_\mu}\omega_a)}$ to the integrand inside expression for ΔS, though I don't really understand the reason. Even so, the integrand contains terms like $\partial_{x_\mu}\partial_{\omega_a} x_\mu$, which I don't know how to deal with.

Solution. The reality is a bit more complicated. We first do a first order expasion to get the infinitesimal difference:

$$\mathcal{L}' - \mathcal{L} \tag{1.0.10}$$

$$\approx \frac{\partial \mathcal{L}}{\partial \phi^i} F^i_a \omega_a + \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left[\partial_\mu \left(F^i_a \omega_a\right) - \partial_\mu \left(\omega_a \frac{\partial x_\nu}{\partial \omega_a}\right) \partial_\nu \left(\phi^i + F^i_a \omega_a\right)\right]$$

$$= \quad \omega_a \left[\frac{\partial \mathcal{L}}{\partial \phi^i} F^i_a + \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left(\partial_\mu F^i_a - \partial_\mu (\frac{\partial x_\nu}{\partial \omega_a}) \partial_\nu(\phi^i + F^i_a \omega_a)\right)\right] \tag{1.0.11}$$

eq:l-l-omega

$$+\partial_\mu \omega_a \left[\frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left(F^i_a - \frac{\partial x_\nu}{\partial \omega_a} \partial_\nu(\phi^i + F^i_a\omega_a)\right)\right] \tag{1.0.12}$$

eq:l-l-pmu-omega

We also discover the integrand in Eq.1.0.6 to be

$$\left(1 + \partial_\mu(\omega_a \frac{\partial x_\mu}{\partial \omega_a})\right) \mathcal{L}' - \mathcal{L} \tag{1.0.13}$$

$$= \left(1 + \partial_\mu(\omega_a \frac{\partial x_\mu}{\partial \omega_a})\right) (\mathcal{L}' - \mathcal{L}) + \left(\partial_\mu(\omega_a \frac{\partial x_\mu}{\partial \omega_a})\right) \mathcal{L} \tag{1.0.14}$$

eq:integrand-l-density

For the first term $\left(1 + \partial_\mu(\omega_a \frac{\partial x_\mu}{\partial \omega_a})\right)(\mathcal{L}' - \mathcal{L})$, the $(\mathcal{L}' - \mathcal{L})$ already has terms of first order of ω_a and of first order of $\partial_\nu \omega_a$. For our purpose, the second order terms $(\partial_\nu(F^i_a\omega_a))$ from item 1.0.11 and item 1.0.12 can be ignored. Also, the item $(\partial_\mu(\omega_a \frac{\partial x_\mu}{\partial \omega_a}))(\mathcal{L}' - \mathcal{L})$ in eq.1.0.14 can also be ignored.

Therefore the integrand in Eq.1.0.6 becomes

$$(\mathcal{L}' - \mathcal{L}) + \left(\partial_\mu(\omega_a \frac{\partial x_\mu}{\partial \omega_a})\right)\mathcal{L} \tag{1.0.15}$$

$$= \omega_a \left[\frac{\partial \mathcal{L}}{\partial \phi^i} F_a^i + \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left(\partial_\mu F_a^i - (\partial_\mu \frac{\partial x_\nu}{\partial \omega_a}) \partial_\nu (\phi^i + F_a^i \omega_a) \right) + (\partial_\nu \frac{\partial x_\mu}{\partial \omega_a}) \mathcal{L} \right] \tag{1.0.16}$$

$$+ \partial_\mu \omega_a \left[\frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left(F_a^i - \frac{\partial x_\nu}{\partial \omega_a} \partial_\nu (\phi^i + F_a^i \omega_a) \right) + \frac{\partial x_\mu}{\partial \omega_a} \mathcal{L} \right] \tag{1.0.17}$$

Therefore, the term we seek, i.e. the coefficient of $\partial_\mu \omega_a$ is

$$\frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left(F_a^i - \frac{\partial x_\nu}{\partial \omega_a} \partial_\nu (\phi^i + F_a^i \omega_a) \right) + \frac{\partial x_\mu}{\partial \omega_a} \mathcal{L} \tag{1.0.18}$$

$$= \left(\mathcal{L}\delta_{\mu\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \partial_\nu \phi^i \right) \frac{\partial x_\nu}{\partial \omega_a} + \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} F_a^i \tag{1.0.19}$$

which is what we expect in equation 1.43 of [AS10].

Question: as for why we should ignore the term with ω_a, there are two posts ([1], [2]) might be useful for a thought.

confusion

I had great doubt about this problem. Though I have posted an answer on [1], I don't think that answer is satisfactory.

2 Eq. 3.5

It is not so obvious to get Eq.3.5 in pp.99 of [AS10]. Here is my notes.

According to the book, Eq.3.3 is turned into (I set $\hbar = 1$ occasionally, though sometimes I forgot that I have set $\hbar = 1$, orz):

$$\begin{aligned} &\langle q_f | \int \mathrm{d}q_N \, \mathrm{d}p_N \, |q_N\rangle \langle q_N | p_N \rangle \langle p_N | \, e^{-i\hat{T}\Delta t} e^{-i\hat{V}\Delta t} \times \\ &\quad \int \mathrm{d}q_{N-1} \, \mathrm{d}p_{N-1} \, |q_{N-1}\rangle \langle q_{N-1} | p_{N-1} \rangle \langle p_{N-1} | \, e^{-i\hat{T}\Delta t} e^{-i\hat{V}\Delta t} \times \dots \\ &\quad \int \mathrm{d}q_1 \, \mathrm{d}p_1 \, |q_1\rangle \langle q_1 | p_1 \rangle \langle p_1 | \, e^{-i\hat{T}\Delta t} e^{-i\hat{V}\Delta t} \, |q_i\rangle \end{aligned} \tag{2.0.20}$$

Notice that

T has only p, V has only q

$$\langle q | p \rangle = \frac{\exp(iqp/\hbar)}{\sqrt{2\pi\hbar}} \tag{2.0.21}$$

$$\langle p_N | \, e^{-i\hat{T}\Delta t} = \langle p_N | \, e^{-iT(p_N)\Delta t} \tag{2.0.22}$$

$$e^{-i\hat{V}\Delta t} \, |q_{N-1}\rangle = e^{-iV(q_{N-1})\Delta t} \, |q_{N-1}\rangle \tag{2.0.23}$$

(2.0.24)

Also,

$$\langle q_N|p_N\rangle \langle p_N| e^{-i\hat{T}\Delta t} e^{-i\hat{V}\Delta t} |q_{N-1}\rangle = \frac{e^{iq_N p_N/\hbar}}{\sqrt{2\pi\hbar}} \langle p_N| e^{-iT(p_N)\Delta t} e^{-iV(q_{N-1})\Delta t} |q_{N-1}\rangle$$
$$= \frac{e^{iq_N p_N/\hbar}}{\sqrt{2\pi\hbar}} \langle p_N|q_{N-1}\rangle e^{-iT(p_N)\Delta t} e^{-iV(q_{N-1})\Delta t} = \frac{e^{ip_N(q_N-q_{N-1})/\hbar}}{2\pi\hbar} e^{-i[T(p_N)+V(q_{N-1})]\Delta t} \tag{2.0.25}$$

etc. Now we have to pay special attentiont to the start and end. For the start, we have a

$$\int \mathrm{d}q_N \, \langle q_f|q_N\rangle = \int \mathrm{d}q_N \, \delta(q_N - q_f)$$

So every q_N is replaced by q_f. For the end, we have

$$\langle q_1|p_1\rangle \langle p_1| e^{-i\hat{T}\Delta t} e^{-i\hat{V}\Delta t} |q_i\rangle = e^{-i[T(p_1)+V(q_i)]} \frac{e^{ip_1(q_1-q_i)}}{2\pi\hbar}$$

Together we have the whole thing into:

$$\begin{aligned} &\int \mathrm{d}q_1 \cdots \mathrm{d}q_{N-1} \, \mathrm{d}p_1 \, \mathrm{d}p_N \, \frac{1}{(2\pi\hbar)^N} \times \\ &e^{i\left[p_1(q_1-q_i)+\cdots p_N(q_N-q_{N-1})\right]} \times \\ &e^{-i\left[T(p_1)+\cdots+T(p_N)+V(q_i)+V(q_1)+\cdots+V(q_{N-1})\right]} \end{aligned} \tag{2.0.26}$$

which is exactly eq.(3.5) in book.

3 Eq 9.4

The Hamiltonian for particle on a ring is claimed to be (Eq. 9.1 of [AS10], pp. 498):

$$H = \frac{1}{2}(-i\partial_\phi - A)^2 = \frac{1}{2}(p - A)^2 \tag{3.0.27}$$

The book [AS10] claims that

$$L = \frac{1}{2}\dot{\phi}^2 - iA\dot{\phi} \tag{3.0.28}$$

I am quite confused, especially about the appearance of $\dot{\phi}$. Can any explain a bit?

How I tried: Since the inverse of a Legendre transformation is Legendre transformation itself,

$$\text{Denote } x \equiv \frac{\partial H}{\partial p} = p - A, \text{ so,} \tag{3.0.29}$$

$$p = x + A, \quad H = \frac{1}{2}x^2, \text{ so,} \tag{3.0.30}$$

$$L = xp - H = x(x + A) - \frac{1}{2}x^2 = \frac{1}{2}x^2 + xA \tag{3.0.31}$$

So my calculation found that the Lagrangian of above Hamiltonian is:

$$L = \frac{1}{2}x^2 + xA \tag{3.0.32}$$

where

$$x = \frac{\partial H}{\partial p} \tag{3.0.33}$$

References

[AS10] Alexander. Altland and Ben BD Ben Simons. *Condensed matter field theory*. Cambridge University Press, 2010.

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