Notes about the Minimal Dimensional Method to Classify Topological Phases (preliminary)

Taper

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Abstract

In this paper, I will review the use of minimal Dirac Hamiltonian method (developed by Dr. Chiu in [Chi13]), which accounts for the mechanism behind the homotopic classification of topological insulators. I start with a introduction to the general framework of "Tenfold Way", and briefly outline the two approaches to classification of topological insulators. Then, I will argue step by step how Minimal Dirac Hamiltonian approach expose the mechanism of homotopic classification of topological insulators.

Contents

sec:Outline

1 Outline

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The classification of topological insulators in non-interacting system is in effect a classification of $\cancel{N \times N}$ Hamiltonian matrix. ~~The~~ simple case considered here is the classification done by considering how the Hamiltonian matrix responses to those anti-unitary symmetries, when we effectively ignored all unitary symmetries. Specifically, when we module out any unitary symmetries of the Hamiltonian, the number of meaningful anti-unitary symmetries are limited, and we only classify topological insulators based on how our Hamiltonian matrix responses to these anti-unitary symmetries.

There are two classical approaches towards this classification. The first is to examine the existence of Anderson delocalization at the boundary of the insulator, because the existence of surface conducting state is the signature of topological insulators and the Anderson localization is the obstruction that surface conducting states must overcome. This approach uses the Nonlinear Sigma Models (NLσMs) to describe the surface Hamiltonian and consider the when can a localization-breaking term can be added to NLσMs.

Be more specific about this and check some facts.

The second approach is to use homotopy theory to classify classes Hamiltonians under continuous deformation, more specifically, under adiabatical change of Hamiltonians. This method is most elegant but most mathematical demanding approach compared with others, because calculating homotopy groups, and homotopy theories in general are notorious hard. Also, one has to use K-theory, because of something physical is enlarging those distinct classes of Hamiltonian. To be more specific.

But the key to understand topological insulators, are not necessarily the Hamiltonian of the whole bulk, but the physics happening when the bulk gap closes. For example, the Chern number is the integration on a compact manifold without boundary. Therefore, it should always be zero because of the Stokes theorem, unless inside this manifold there is somewhere where the formula for Chern number got "blown up", and it is just this locally "blown up" point that is responsible for the nonzero Chern number, and for the non-triviality of topological insulators.

Therefore, we can detect the different types of topological insulators when we are focused in only the crossing point of the band, or, when we shifted the energy zero level to the crossing point, in only the low energy physics. We expect a locally linear crossing energy spectrum around this point, so we expect $E = \pm\sqrt{\mathbf{p}^2 + m^2}$, with m being the parameter controlling the closing or opening of this spectrum. One simple model with only first-order derivatives to describe such a low energy physics is the Dirac Hamiltonian.

Argument based on Klein-Gorden equation to get the Dirac Hamiltonian

An amazing fact happens, when one consider the gamma matrices in Dirac Hamiltonian, the anti-unitary symmetry operators, and their anti-commutation relations. Together they are proved to be isomorphic to some Clifford Algebra over $\mathbb{R}$ or $\mathbb{C}$, depending on the situation. And consequently our Hamiltonian and symmetry operators, all become some specific representations of their corresponding Clifford Algebra. Also, it will be illustrated later that the minimal matrix dimension of such representation, i.e. the minimal dimension of the Dirac Hamiltonian, can tell the specific type of this topological insulator. Therefore, as a whole all the classification work is reduced to the consideration of minimal matrix dimension of the representation of Clifford Algebra, in different spatial dimension.

Cautious, we have two dimensions here: matrix and spatial.

Structure of this work This work are divided into several parts.

1. **Ten-fold Way** introduces the result of Ten-fold way.
2. **Original Approach** gives brief idea of how the Nonlinear Sigma Models Approach and the Homotopy/K-theory approach work.
3. **Minimal Dirac Hamiltonian** gives comprehensive account of how the Minimal Dirac Hamiltonian approach works, and how the ten-fold way is derived in this approach.
4. **Outlook** outlines the possible direction in classification under unitary symmetries, and the problems we faced when interaction is turned on.

2 Ten-fold Way

The classification of topological phases with different non-unitary symmetries in non-interacting picture, are in essence, the classification of $N \times N$ matrices by its response to three discrete symmetries: time reversal symmetry (T), charge conjugation/particle hole symmetry (C), and chiral/sub-lattice symmetry (S). As for those unitary symmetries, one can in principle, block diagonalize the Hamiltonian $H = \text{diag}(H^{\lambda_1}, H^{\lambda_2}, \cdots)$, such that each block H^{λ_i} is labeled by irreducible representations λ_i, and it has no memory of the unitary symmetries. Therefore, our classification is applied to those blocks and still holds.

It will be find that, those T, C are the only two meaningful non-unitary symmetries of the Single-particle system, and there are in total 10 different ways in which the Single-particle Hamiltonian could respect the three symmetries T, C, S. Now we explain in detail, about the reason why we care about, and only care about these three symmetries T, C, and S.

2.1 The symmetries of Single-particle Hamiltonian

ngle-particle Hamiltonian

According to Wigner's theorem, symmetries of physical system can either be unitary represented, or anti-unitarily represented. Here we show that in the case of anti-unitary symmetries, there are only 2 different kinds of them that a Single-particle Hamiltonian in first-quantized space can have. Actually, there can be more anti-unitary symmetries than this. But under reasonable assumptions, we can limit our discussion in only this two types.

The Single-particle Hamiltonian acting on Fock space is

$$\hat{H} = \sum_{A,B} \hat{\psi}_A^\dagger H_{AB} \hat{\psi}_B \tag{2.1.1}$$

eq:H-2nd

where H_{AB} are just complex numbers, $\hat{\psi}_A^\dagger$ and $\hat{\psi}_B$ are creation and annihilation operators acting on Fock space.

Here and henceforth, we will add a hat ˆ to all operators in Fock space to stress that it acts on Second-quantized Fock space. A symmetry of the Hamiltonian, represented as an operator $\hat{U}$, must have:

$$\hat{U}\hat{H}\hat{U}^{-1} = \hat{H} \tag{2.1.2}$$

eq:sym-in-2nd-1

eq:sym-in-2nd-permute

We expect $\hat{U}$ to change the creation/annihilation operators in two different ways. First, it may only permute the creation/annihilation operators:

$$\hat{\psi}'_A = \hat{U}\hat{\psi}_A\hat{U}^{-1} = \sum_B (u^\dagger)_{AB}\hat{\psi}_B \tag{2.1.3a}$$ eq:sym-in-2nd-permute-1

$$\hat{\psi}'^\dagger_A = \hat{U}\hat{\psi}^\dagger_A\hat{U}^{-1} = \sum_B \hat{\psi}^\dagger_B u_{AB} \tag{2.1.3b}$$ eq:sym-in-2nd-permute-2

eq:sym-cc

Where u is some matrix implementing the permutation. Or, it may interchange the role of creation/annihilation operators:

$$\hat{\psi}'_A = \hat{U}\hat{\psi}_A\hat{U}^{-1} = \sum_B (u^*)^\dagger_{AB}\hat{\psi}^\dagger_B \tag{2.1.4a}$$ eq:sym-cc-1

$$\hat{\psi}'^\dagger_A = \hat{U}\hat{\psi}^\dagger_A\hat{U}^{-1} = \sum_B \hat{\psi}_B u^*_{BA} \tag{2.1.4b}$$ eq:sym-cc-2

Where u^* is some other matrix implementing the interchange. We write complex conjugation u^* instead of u for convenience. In both cases (permute or interchange), to conserve the anticommutation relation between $\hat{\psi}^\dagger_A$ and $\hat{\psi}_B$ operators, one can easily show that u should be a unitary matrix.

Also, $\hat{U}$ may be linear or anti-linear. The different combinations of these conditions give us 1 unitary symmetry, 2 anti-unitary symmetries, and 1 special symmetry, to be explained below.

Case 1: Unitary Symmetry Assume the symmetry just permutes the creation/annihilation operators, as in equation 2.1.3, and assume it is linear in Second-quantized Hamiltonian, i.e. $\hat{U}i\hat{U}^{-1} = i$. The permutation relation 2.1.3 plugged into equation 2.1.2, gives

$$uHu^{-1} = H \tag{2.1.5}$$

Therefore, in this case the symmetry is unitarily realized in First-quantized Hamiltonian H_{AB}.

Case 2: Anti-unitary T Symmetry Assume the symmetry just permutes the creation/annihilation operators, as in equation 2.1.3, but assume it is anti-linear in Second-quantized Hamiltonian, i.e. $\hat{U}i\hat{U}^{-1} = -i$. The permutation relation 2.1.3 plugged into equation 2.1.2, gives a different result, since $\hat{U}H_{AB}\hat{U}^{-1} = H^*_{AB}$ now.

$$uH^*u^\dagger = H \tag{2.1.6}$$

Or:

$$uKHu^tK = uKH(uK)^{-1} = H \tag{2.1.7}$$

where K is complex conjugation. This symmetry is called Time-reversal symmetry, and is realized in First-quantized Hamiltonian as an anti-unitary operator $T = uK$.

Case 3: Anti-unitary C Symmetry Assume the symmetry just interchange the creation/annihilation operators, as in equation 2.1.4, but assume it is linear in

Second-quantized Hamiltonian, i.e. $\hat{U} i \hat{U}^{-1} = i$. The interchange relation 2.1.4 plugged into equation 2.1.2, gives: [1]

$$u(H - \frac{1}{2}\operatorname{tr}(H))^t u^\dagger = -(H - \frac{1}{2}\operatorname{tr}(H)) \tag{2.1.8}$$

eq:sym-C-cond

Taking the trace of above equality will give $2\operatorname{tr}(H) = N\operatorname{tr}(H)$, since in solids $N >> 2$, we must have $\operatorname{tr}(H) = 0$. Then the above equality simplifies into [2]:

$$uH^* u^\dagger = uKH(uK)^{-1} = -H \tag{2.1.9}$$

This type of symmetry is called charge-conjugation symmetry. It is also called particle-hole symmetry in condensed matter physics. It is realized as $C = uK$ with $CHC^{-1} = -H$ for 1st-quantized single-particle Hamiltonian.

Case 4: Unitary S Symmetry Assume the symmetry now interchange the creation/annihilation operators, as in equation 2.1.4, and assume it is anti-linear in Second-quantized Hamiltonian, i.e. $\hat{U} i \hat{U}^{-1} = -i$. The interchange relation 2.1.4 plugged into equation 2.1.2, gives:

$$u(H - \frac{1}{2}\operatorname{tr}(H))u^\dagger = -(H - \frac{1}{2}\operatorname{tr}(H)) \tag{2.1.10}$$

eq:sym-S-cond

and since $N >> 2$, we have $\operatorname{tr}(H) = 0$. Then

$$uHu^\dagger = -H \tag{2.1.11}$$

This symmetry will be called the chiral symmetry, denoted $\hat{S} = u$. It is unitarily realized in First-quantized Hamiltonian, but since $\{S, H\} = 0$ instead of $[S, H] = 0$, it is not a traditional symmetry that we are used to. Also, it is easy to see that S is a combination of T and C,[3] and the symmetry property of Hamiltonian under T or C uniquely defined the symmetry property of Hamiltonian under S. But there is one exception. When the Hamiltonian does not obey T and C, it may or may not obey $S = TC$ as a whole.

Squaring of T, C, S Squaring of T/C symmetry operators should be proportional to identity $\mathbb{1}$, hence it is only a phase as they are unitary. This can be viewed from two perspectives. First, we expect the system, after applying twice of symmetry operation T/C, should come back to the same state, except a possible phase difference. Second, T^2/C^2 commutes with all T/C symmetric Hamiltonians (easy to derive) in all irreducible representations of unitary symmetries (to be explained later), therefore by Schur's lemma, they must be proportional to a constant, which is a phase.

It is easy to find that this phase $e^{i\delta}$should be ± 1. For example, let $T = uK$, then $T^2 = uu^*$, and $(uu^*)u = u(u^*u)$ gives us $e^{2i\delta} = 1$, hence $e^{i\delta} = \pm 1$.

But the square of S is tricky. Since $S = TC = u_T u_C^*$, and each unitary matrix u_T and u_C has a phase freedom (as they acts on creation/annihilation operators), we can always pick a phase such that S^2 is some phase we want. Sometimes we pick $S^2 = 1$. Sometime we want $\{T, C\} = 0$, and pick some specific phase for S^2.[4]

[1] Note that in calculating this, $\sum_i \hat{\psi}_i^\dagger \hat{\psi}_i = \mathbb{1}$ on 1st-quantized single-particle Hilbert space.
[2] note that $H^t = H^*$ for Hermitian H
[3] In fact, we could have defined $S = TC$ or $S = CT$, and obtained the same result.
[4] This is a bit troublesome. First we denote $T^2 = \varepsilon_T$, $C^2 = \varepsilon_C$, then it can be found that $(u_T)^t =$

Why Ten Classes For the three unitary symmetries under consideration, as will be mentioned, S symmetry is a combination of T and C, $S = TC$. It will be obtained that $T^2 = \pm 1$, $C^2 = \pm 1$, and S^2 is undetermined (depending on the phase choice[5] we give for T and C). Therefore, there are in total 10 different possible ways that this 3 symmetries could be combined together: We denote $T^2 = 0$ ($C^2 = 0$) to symbolize that the system does not follow $T(C)$ symmetry. Then $T^2 = 0, \pm 1$ and $C^2 = 0, \pm 1$ gives us $3 \times 3 = 9$ different ways. But if $T^2 = C^2 = 0$, S may or may not conserve. Therefore, we have $9 - 1 + 2 = 10$ different possible ways. In each way the topological invariants of the Hamiltonian respecting these symmetries are tabulated in figure . Each class is given a name (A, AII, etc.), and their properties in different spatial dimensions respect a periodic structure. For the upper two classes, they are named complex classes and respect a 2-fold period. The Lower eight classes are named real classes and respect a 8-fold period, which is why we only listed the first $0 \to 7$ spatial dimensions.

add the master table

2.2 Dealing with Unitary Symmetries

g with Unitary Symmetries

As mentioned, we could module out those unitary symmetries. This statement is made precise by the following theorem about unitary symmetries of the Hamiltonian.

Theorem 2.1 (Diagonalization of Hamiltonian in unitary representation)**.** *This space $\mathcal{V}$ decomposes into a direct sum of vector spaces $\mathcal{V}_\lambda$ associated with the irrep (irreducible representations, labeled by λ) of G_0.*

$$\mathcal{V} = \oplus_\lambda m_\lambda \mathcal{V}_\lambda \tag{2.2.1}$$

where m_λ denotes the multiplicity of λth irrep. Denote the dimension of each irrep as d_λ.

In each vector space $\mathcal{V}_\lambda$, one can choose a (orthogonal) basis of the form:

$$|v_\alpha^{(\lambda)}\rangle \otimes |w_k^{(\lambda)}\rangle \tag{2.2.2}$$

where

- *G_0 acts only only $|w^{(\lambda)}\rangle_k$, $k = 1, \cdots, d_\lambda$,*
- *H acts only on $|v^{(\lambda)}\rangle_\alpha$, $\alpha = 1, \cdots, m_\lambda$.*

Therefore, with all unitary symmetries ignored, we are classifying how Hamiltonian will be like when it respects, in 10 different ways, the combinations of T, C, S symmetries.

Differently Realized Anti-unitary Symmetries It should be noted that there is still a freedom of unitary matrix that implements the T or C symmetry. Therefore, we could have, for example, two different time-reversal symmetries $T_1 = u_1 K$ and $T_2 = u_2 K$. They are obviously related by a unitary matrix, say $T_1 = u_{12} T_2$. And we could easily get that u_{12} is also a unitary symmetry of the Hamiltonian (if the Hamiltonian respect both T_1 and T_2). Therefore, upon enlarging the

$\varepsilon_T u_T$, $(u_C)^t = \varepsilon_C u_C$. Then, with $S = u_T u_C^*$, we have $S^\dagger = \varepsilon_C \varepsilon_T u_C u_T^* = \varepsilon_C \varepsilon_T CT$, or $CT = \varepsilon_C \varepsilon_T S^\dagger$. Since $TC + CT = 0$, we have $S + \varepsilon_C \varepsilon_T S^\dagger = 0$, or $S^2 = -\varepsilon_C \varepsilon_T$.

[5] To be explained later.

symmetry group G_0 to include the element u_{12} and repeat the process described in the theorem above, we only have one time-reversal symmetry. Similar analysis could be done for the C symmetry as well.

Translational Symmetry Since we are classifying in the solids, it is natural to expect translational symmetry to present. Also, the addition of translational symmetry do not alter our classification, which is based on non-unitary symmetries, but also make this classification more convenient to be done.

With translational symmetry added, To be added

The Hamiltonian in k-space that preserve these three discrete symmetries should follow[6]:

$$TH(k)T^{-1} = H(-k) \tag{2.2.3a}$$ eq:T-sym-Hk

$$CH(k)C^{-1} = -H(-k) \tag{2.2.3b}$$ eq:C-sym-Hk

$$SH(k)S^{-1} = -S(k) \tag{2.2.3c}$$ eq:S-sym-Hk

2.3 The Tenfold Way

sec:The Tenfold Way

Add the table, and the meaning of each Hamiltonian, and the class of each evolutionary operators

2.4 The Two Original Approaches

e Two Original Approaches

Add this

3 Classification by Dirac Mass Terms

imal Hamiltonian Approach

Dirac Hamiltonian and Topological Property of the Bulk We first note that different topological phases are distinguished by the closing and then opening of the gap, i.e. a quantum phase transition. If we adiabatically flatten the energy spectrum to two bands of energy $\pm E$, the transition (closing of gap) will happen at some point where locally the energy is of a Dirac cone type. Therefore, the Dirac Hamiltonian captures this transition behavior. Then, a class of topological phases, or more specifically, a class of Hamiltonian matrix that shares the same topological invariant, can be *represented* by the Dirac Hamiltonian in this class.

3.1 Crossing Spectrum and Edge Mode

ing Spectrum and Edge Mode

Here we show that the crossing spectrum characteristic of topological insulators can be characterized by a Dirac type Hamiltonian, and with this Dirac Hamiltonian we could find an Edge mode solution built into it.

Crossing Spectrum and Dirac Hamiltonian In some local region around the crossing point, the energy spectrum should look like:

$$E = \pm\sqrt{\mathbf{k}^2 + M^2} \tag{3.1.1}$$

[6]It is easy to anticipate the change $k \to -k$ for anti-unitary symmetries, since they change the phase factor when one Fourier transform the Hamiltonian.

where **k** is the crystal momentum and M is some constant responsible for the opening and closing of the gap, or in some cases, a parameter in which we can control to open of close the gap.

As has been analysed in most Quantum Mechanics textbook[7], this energy-momentum relationship requires a Dirac type Hamiltonian. Then, the Dirac Hamiltonian we use is:

$$H_{\text{Dirac}} = M\gamma_0 + \sum_{i=1}^{d} k_i \gamma_i \tag{3.1.2}$$

eq:dirac-H

where d is the spatial dimension of the physical system, γ_i obeys the Clifford Algebra anti-commutation relation:

$$\{\gamma_i, \gamma_j\} = 2\delta_{i,j}\mathbb{1} \tag{3.1.3}$$

Edge Mode and M Now we explain why the Dirac Hamiltonian3.1.2 is useful for capturing the topological phase. First, it has a gapless state. Due to the anti-commutation relation of γ_i, we can show[8] that the energies of Dirac Hamiltonian are:

$$E_{\pm} = \pm\sqrt{M^2 + \sum_{i=1}^{d} k_i^2} \tag{3.1.4}$$

Therefore, as M varies from negative to 0 to positive, the system goes from gaped to gapless and again to gaped state. This shows that the M is parameter controlling the quantum phase transition, and the system with $M < 0$ or $M > 0$ possesses two different quantum phase.

In addition, we can analyse its edge mode. Assuming the two material of different quantum phases touches each other in the dth direction. Then we should replace k_d by $-i\partial_{x_d}$, since the translational invariance is broken in that direction. Now the eigenvalue equation is:

$$(M\gamma_0 + \sum_{i=1}^{d-1} k_i\gamma_i - i\partial_{x_d}\gamma_d)\Phi = E\Phi \tag{3.1.5}$$

Past experience teaches us the ansatz (assuming $M(x_d)$ goes to $\pm\infty$ as $x_d \to \pm\infty$, so that the leading term below does not blow up the wavefunction):

$$\Phi = e^{-\int_0^{x_d} M(x'_d)\mathrm{d}x'_d}\phi(x_1, \cdots, x_{d-1}) \tag{3.1.6}$$

which after plugging inside the eigenvalue equation, leads to

$$\left(M(x_d)(\gamma_0 + i\gamma_d) + \sum_{i=1}^{d-1} k_i\gamma_i\right)\phi = E\phi \tag{3.1.7}$$

After left multiplication of γ_0, we have:

$$\left(M(\mathbb{1} + i\gamma_0\gamma_d) + \sum_{i=1}^{d-1} k_i\gamma_i\right)\phi = E\gamma_0\phi \tag{3.1.8}$$

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[7] See for example, p.99 of [Gre97]. Although their discussion is in dimension $d = 3$, it is easy to generalize it to any dimension.

[8] Note that $H_{\text{Dirac}}^2 = E^2\mathbb{1}$, this tells us it can only have eigenvalues $E_{\pm}$. Note that this also explains why we choose not $\gamma_i^2 = -1$, which would give non-sensible result $H_{\text{Dirac}}^2 = -E^2\mathbb{1}$.

Since $i\gamma_0\gamma_d$ squares to 1, it has eigenvalues ± 1. Also, $i\gamma_0\gamma_d$ and γ_0 anticommute, so they share the same eigenspaces with γ_0 mapping all the $+1$ eigenvectors of $i\gamma_0\gamma_d$ to the -1 eigenvectors and vice versa. Now we want this state to be a surface state, so we choose the -1 eigenvectors (denoted ϕ_-) of $i\gamma_0\gamma_d$ to kill the bulk term. This leads to:

$$\gamma_0 \sum_{i=1}^{d-1} k_i \gamma_i \phi_- = \gamma_0 E \phi_- \tag{3.1.9}$$

As mentioned, γ_0 switches the two eigenspaces, so we must necessarily have a surface Dirac Hamiltonian, by projecting all those matrices to the -1 eigenspaces:

$$H_{\text{surf}} \phi_- = \sum_{i=1}^{d-1} \Gamma_{i-} \phi_- = E \phi_- \tag{3.1.10}$$

eq:H-Dirac-surf

3.2 From Homotopy Classification to Minimal Dirac Hamiltonian Method

Homotopy to Minimal Dirac

Spectral Flattening The classification of topological insulators concerns different classes of Hamiltonian that cannot be adiabatically changed to each without closing or opening of a gap. Hence, we could in general adiabatically change our parameters such that the spectrum of Hamiltonian is simplified into two bands of energy ± 1. Under this condition, we have $H^2 = 1$, which already hints that the Hamiltonian itself is a viable candidate for generating Clifford Algebras.

Extension Problem with General Examples The general theme of homotopy classification of classifying topological insulators are done by the considering the *extension problem*, i.e. different ways to extend from a Clifford containing only symmetry operators, to a Clifford Algebra containing Hamiltonians. For example, for class A in 0-dimension, the empty space is nothing, hence $C\ell_0(\mathbb{C})$[9]. And $H^2 = 1$ as mentioned earlier, hence the extension problem is from $C\ell_0(\mathbb{C}) \to C\ell_1(\mathbb{C})$, whose possible representations form some mathematical structure called *classifying space* C_0 [MF13]. And since we are in 0-dimension, all operators are maps starting from $T^0 = S^0$. So our classified objects are different classes of maps from $S^0 \to C_0$, which is mathematically captured by the 0-th homotopy group $\pi_0(C_0) = \mathbb{Z}$. For class AIII which has only chiral symmetry operator S, we choose a phase such that $S^2 = 1$, i.e. making it a candidate for generators of Clifford Algebra. The symmetry condition $\{H, S\} = 0$ tells us now we have a extension problem of $C\ell_1(\mathbb{C}) \to C\ell_2(\mathbb{C})$, whose classifying space is C_1, and has $\pi_0(C_1) = 0$. Therefore, in 0-dimension, the class A has $\mathbb{Z}$ topological insulators and class AIII has trivial topological insulators. The case for real symmetry classes are not too complicated and is concisely mentioned in [MF13].

However, the method to obtain such classifying spaces are mathematical daunting and requires K-theory. We try to simplify it by looking at Dirac Hamiltonians, since they should capture the essence of topological states.

Extension Problem with Dirac Hamiltonian The Dirac Hamiltonian

$$H = \vec{k} \cdot \vec{\gamma} + m\tilde{\gamma}_0 \tag{3.2.1}$$

[9]The reason for using different Clifford Algebra ($C\ell(\mathbb{R})$ or $C\ell(\mathbb{C})$ for different symmetry classes will be explained in section 3.3.

has a gap closing and opening mass term $m\tilde{\gamma}_0(r)$, depending on some parameter r. Suppose that we have a domain wall squeezed by two bulk regions A and B. Now as the parameter r changes freely from region A to region B, the symmetry condition will force the matrix γ_0 to explorer some space having the same homotopy type of some classifying space (again!). For example, for class A in 2-dimension, the Hamiltonian without mass term $(k_1\gamma_1 + k_2\gamma_2)$ consists of two gamma matrices, generating a $\mathrm{C}\ell_2(\mathbb{C})$, whereas adding the mass term, we have $\mathrm{C}\ell_3(\mathbb{C})$. Since there is no symmetry in class A, the extension problem concerns different ways to extend the algebra $\mathrm{C}\ell_2(\mathbb{C}) \to \mathrm{C}\ell_3(\mathbb{C})$, which tells us the number of unitarily non-equivalent mass terms. Detailed examples can be found in section III.C.1 of [CTSR16].

Stable Classification An unmentioned hypothesis above are that we are classifying strong topological insulators. They are topological insulators that robust against disorder [FKM07]. There is a consensus that in classification of strong topological insulators, we replace the Brillouin Zone T^d by S^d [KG15], and we do it in a stable way, i.e., we choose our classification to be independent of and insensitive to the addition of irrelevant trivial bands. The first replacement allow us to use homotopy groups, which classify maps with domain in spheres. The "stable way" allows the use of K-theory (sec.III.C.1 of [CTSR16]).

The View from SPEMT Let us consider a modified Dirac Hamiltonian with extra mass term:

$$H = M\tilde{\gamma}_0 + \sum_{i=1}^{d} k_i\gamma_i + \sum_{j=1}^{D} m_j\tilde{\gamma}_j \tag{3.2.2}$$

eq:H-spemt

Here, the parameter M characterized the domain wall between different phases of topological insulators. The extra mass term m_j represents a perturbation caused by disorder. I argue that we can do our classification in the following way. First, we consider the case when $D = 0$, i.e. without perturbation, and we consider how many different mass terms we could have. Second, we consider adding an extra mass term which respect the symmetry of the Hamiltonian ($D = 1$). Whether this symmetry preserving extra mass term (SPEMT) can be added or not. If a SPEMT can be added, then the system is not robust against perturbation and gap may be opened by disorder. So it is trivial topological insulator. If a SPEMT cannot be added, then this system is topological non-trivial.

One best thing about this classification is that it can be done by considering only the minimal matrix dimension of those gamma matrices in Hamiltonian. The argument is that, complex gamma matrices are of even dimension (except the trivial cases of $\mathrm{C}\ell_{0/1}(\mathbb{C})$)[10]. Therefore, gamma matrices of different matrix dimension are built by tensor products of Pauli matrices.Now, to increase the dimension of the matrix, is equivalent to tensor them. One can add more bands of the same type, or add more bands of a different type in the smaller dimension, or add trivial bands. In all cases, the triviality of topological insulators can be detected in minimal dimension, since adding trivial bands can be ignored (insensitive to addition of trivial bands), and the possible different types of matrices are limited.

make it more rigorous!

However, to distinguish between a $\mathbb{Z}$ topological insulator and a $\mathbb{Z}_2$ topological insulator, we need to increase the matrix dimension to consider multiple copies of

[10]See this post [Phy], or the p.12 and Theorem in p.5 of this [Wes98].

the Dirac Hamiltonian. If the topological state is stable for an arbitrary copies, then this is a $\mathbb{Z}$ topological insulator. If the topological state is stable only of an odd number of copies, this is a $\mathbb{Z}_2$ topological insulator.

Another way to view this way of classification, is to look at the surface Dirac Hamiltonian 3.1.10. Then the above mentioned approach is to see if the surface mode can be gapped, so as to detect the topological properties of the bulk.

3.3 Isomorphism between Symmetry Classes and Clifford Algebras

sec:Isomorphism

As with other classifications, we construct isomorphisms between Dirac matrices and Clifford Algebras, after which we are faced with an extension problem. The isomorphism is constructed now.

3.3.1 Symmetry Constraint

sec:Symmetry Constraint

The Hamiltonian with extra mass term is

$$H = M\tilde{\gamma}_0 + \sum_{j=1}^{D} m_j \tilde{\gamma}_j + \sum_{i=1}^{d} k_i \gamma_i \tag{3.3.1}$$

eq:sym-spemt

The symmetry properties 2.2.3a,2.2.3b give the following conditions on the gamma matrices:

$$\{T, \gamma_i\} = 0,\ [C, \tilde{\gamma}_j] = 0 \tag{3.3.2a}$$
$$\{C, \tilde{\gamma}_j\} = 0,\ [C, \gamma_i] = 0 \tag{3.3.2b}$$
$$\{S, \tilde{\gamma}_j\} = \{S, \gamma_i\} = 0 \tag{3.3.2c}$$

With this, we can establish a isomorphism between "Hamiltonian and Symmetry Operators" and "Clifford Algebra over $\mathbb{R}$ or $\mathbb{C}$". But let's first see some examples in action.

3.3.2 Examples of Constructing Isomorphism

sec:iso-Examples

For class AI ($T^2 = 1, C = 0, S = 0$) in $d = 1, D = 0$, we define:

$$G_{\text{AI}} = \{i, T, \tilde{\gamma}_0, \gamma_1\}$$

We have its (anti)commutation relations listed in Table 1.

Table 1: Generators in G_{AI} with $d = 1, D = 0$

tab:generator-AI

	i	T	$\tilde{\gamma}_0$	γ_1
i	$i^2 = -1$	$\{i, T\} = 0$	$[i, \tilde{\gamma}_0] = 0$	$[i, \gamma_1] = 0$
T		$T^2 = 1$	$[T, \tilde{\gamma}_0] = 0$	$\{T, \gamma_1\} = 0$
$\tilde{\gamma}_0$			$\tilde{\gamma}_0^2 = 1$	$\{\tilde{\gamma}_0, \gamma_1\} = 0$
γ_1				$\gamma_1^2 = 1$

Now with $J_4 = T\tilde{\gamma}_0\gamma_1$, $J_3 = iT\tilde{\gamma}_0$, we could verify that all $\{i, T, J_3, J_4\}$ anticommute with each other, and $J_3^2 = J_4^2 = 1$. So together they generate the algebra $C\ell_{3,1}(\mathbb{R}) \cong C\ell^g((2,2),\mathbb{R})$. This will be consistent with the isomorphism $G_{\text{AI}} \cong C\ell_{2+D,1+d}(\mathbb{R})$ in Table 4.

Another example. For class AII ($T^2 = -1, C = 0, S = 0$) in $d = 1, D = 0$, we have:

$$G_{\text{AI}} = \{i, T, \tilde{\gamma}_0, \gamma_1\} \tag{3.3.3}$$

We have its (anti)commutation relations listed in Table 2. Now with $J_4 = T\tilde{\gamma}_0\gamma_1$,

Table 2: Generators in G_{AII} with $d = 1, D = 0$

tab:generator-AII

	i	T	$\tilde{\gamma}_0$	γ_1
i	$i^2 = -1$	$\{i, T\} = 0$	$[i, \tilde{\gamma}_0] = 0$	$[i, \gamma_1] = 0$
T		$T^2 = -1$	$[T, \tilde{\gamma}_0] = 0$	$\{T, \gamma_1\} = 0$
$\tilde{\gamma}_0$			$\tilde{\gamma}_0^2 = 1$	$\{\tilde{\gamma}_0, \gamma_1\} = 0$
γ_1				$\gamma_1^2 = 1$

$J_3 = iT\tilde{\gamma}_0$, we could verify that all $\{i, T, J_3, J_4\}$ anticommute with each other, and $J_3^2 = J_4^2 = -1$. So together they generate the algebra $C\ell_{0,4}(\mathbb{R})$. Notice that, assume we have $C\ell_{1,3}(\mathbb{R}) = \{K_1, K_2, K_3, K_1'\}$, and $K_i^2 = -1$, $(K_1')^2 = 1$. The following map

$$J_1 = K_1,\ J_2 = K_2,\ J_3 = K_3,\ J_4 = K_1K_2K_3K_1' \tag{3.3.4}$$

has the property shown in the table 3. Therefore, $C\ell_{0,4}(\mathbb{R}) \cong C\ell_{1,3}(\mathbb{R})$, which is

Table 3: Map from $C\ell_{1,3}(\mathbb{R})$ to $C\ell_{0,4}(\mathbb{R})$.

tab:map-cl13-2-cl04

	$J_1 = K_1$	$J_2 = K_2$	$J_3 = K_3$	$J_4 = K_1K_2K_3K_1'$
J_1	-1	$\{\}$	$\{\}$	$\{\}$
J_2		-1	$\{\}$	$\{\}$
J_3			-1	$\{\}$
J_4				-1

where $\{\}$ means anticommute.

consistent with $G_{\text{AII}} \cong C\ell_{d,3+D}(\mathbb{R})$ in Table 4.

3.3.3 Constructing the Isomorphism (Complex Symmetry Classes)

sec:Complex Classes-iso

The general steps for constructing the isomorphism is presented here. First, we need to treat the situation of complex classes and real classes differently. The complex classes are isomorphic to complex Clifford Algebras, while the real classes are isomorphic to real Clifford Algebras. The reason for this distinction is that, in real classes, there are complex conjugate operator K. First, there is not way to represent complex conjugation simply as multiplication of complex matrices. Second, complex matrices come with a nature definition of complex conjugate †, whereas $K^\dagger$ is ill-defined [11]. Therefore, we use real Clifford Algebras when dealing with real classes (which is the reason why they are named real classes).

[11] Think about this. $\langle c\phi|K\psi\rangle$ could have different values depending on whether we move K to the left as $K^\dagger$ first, or we move c out of the inner-product first.

For complex classes, let $G_{\#}$ be the group generated by elements $\{\gamma_i, \tilde{\gamma}_j, S\}$ ($i = 1, \cdots, d$, $j = 0, 1$) in each symmetry class # (so S exists only in class AIII). We may choose a phase such that $S^2 = 1$, as mentioned earlier. Then obviously $G_{\#}$ will be a Clifford Algebra:

$$G_A = \text{generated by } \{\gamma_i, \tilde{\gamma}_j\} \cong \mathrm{C}\ell_{d+D+1}(\mathbb{C}) \tag{3.3.5}$$

$$G_{\text{AIII}} = \text{generated by } \{\gamma_i, \tilde{\gamma}_j, S\} \cong \mathrm{C}\ell_{d+D+2}(\mathbb{C}) \tag{3.3.6}$$

3.3.4 Constructing the Isomorphism (Real Symmetry Classes)

sec:Real Classes-iso

For real classes, we need to include i, and symmetry operators T, C in our group $G_{\#}$. It turns out that we need to pick a phase such that $\{T, C\} = 0$, which is possible as is mentioned earlier. Now we demonstrate the proof of isomorphism in class AII.

We first note that, given a real vector space V, we can complexify a real space in two ways. The first is trivially taking the tensor product $V \otimes_{\mathbb{R}} \mathbb{C}$, which does not suit our purpose. The second is to find an almost complex structure J, which is a $\mathbb{R}$-linear map that squares to $-\mathbb{1}$, i.e. $J^2 = -\mathbb{1}$. With this almost complex structure J, then V admits in a natural way the structure of a complex vector space $V_{\mathbb{C}}$ [Dan05]. Also, a $\mathbb{R}$-linear map A on V is $\mathbb{C}$-linear($\mathbb{C}$-antilinear) if and only if A commutes (anticommutes) with J. Therefore, we map model the antiunitary operators on a real vector space naturally.

Let us denote the generators in Clifford Algebra as J_j and $\tilde{J}_i$, where $J_j^2 = -\tilde{J}_i^2 = -\mathbb{1}$. Since this almost complex structure J squares to $-\mathbb{1}$ and anticommutes with antiunitary symmetry operators, we naturally take it to be the first generator in our Clifford Algebra, $J_1 = J$.

Then we discuss some tips that will useful for later calculation.

Tips

1. Since all generator anticommute, all matrices either commute or anticommute. So the Algebra is pretty simple.
2. Since $[A, BC] = \pm[A, CB]$ and $\{A, BC\} = \pm\{A, CB\}$, the commutation or anticommutation does not depends on the order of the matrices BC or CB. So one might rearrange them in the order whichever is convenient.
3. For $A = \{J_1, \cdots, J_n\}, B = \{K_1, \cdots, K_m\}$, we have $AB = (-1)^{mn} BA$. However, if A and B has something in common, then the above condition breaks. This is like adding an "impurity" in it to change the commutation/anticommutation relations.
4. The restriction given by $\sum_i k_i \gamma_i$, and $\sum_j m_j \tilde{\gamma}_j$ (will be shown later) are much restrictive that the candidates for symmetry operator are only a small finite set.

enum:tips-4

Class AII Now we prove the isomorphism. First, we try to construct each gamma matrices in Hamiltonian. All matrices commute with i, so in real vector space, they commute with $J_1 = J$. Then, each gamma matrices should have an even number of Clifford Algebra generators other than J_1 itself. Similarly, antiunitary symmetry operators should have an odd number of Clifford Algebra generators other than

J_1. We "dope" the gamma matrices with some J_1 to make it commute/anticommute with symmetry operators. More explicitly, bearing in mind that in class AII has only $T^2 = -1$, we take a quick look into Hamiltonian 3.2.2 and symmetry conditions 3.3.2, and they give us the inspiration to set:

$$H_{\text{AII}} = mJ_1J_2J_3 + \sum_{j=1}^{D} m_j J_1J_2J_{3+j} + \sum_{i=1}^{d} k_i J_2\tilde{J}_i \tag{3.3.7}$$

Here $\tilde{\gamma}_j$ are $J_1J_2J_{3+j}$, which square to 1. γ_i are $J_2\tilde{J}_i$. The Clifford Algebra is at least $\mathrm{C}\ell_{d,3+D}(\mathbb{R})$. And within this algebra, only J_2 is a possible candidate for symmetry operators, which satisfy equations 3.3.2 (use tips 4 for calculation). J_2 is found to be a time reversal operator, and $J_2^2 = -1$ confirms that this Hamiltonian belongs to class AII. Therefore, the map:

$$\begin{aligned} f : \mathrm{C}\ell_{d,3+D}(\mathbb{R}) &\to G_{\text{AII}} \\ J_1J_2J_{3+j} &\to \tilde{\gamma}_j,\ (j = 0, 1, \cdots, D) \\ J_2\tilde{J}_i &\to \gamma_i,\ (i = 0, 1, \cdots, d) \end{aligned} \tag{3.3.8}$$

is a map from Clifford Algebra $\mathrm{C}\ell_{d,3+D}(\mathbb{R})$ to symmetry class AII. The inverse map can be found easily. We first solve some formality problems. A complex number can be written as a real number by identifying i with $J = i\sigma_y$ and 1 with $\mathbb{1}$, the identity matrix:

$$a + bi \to a\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + b\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = a\mathbb{1} + bJ \tag{3.3.9}$$

Therefore, all complex matrices $\gamma_i, \tilde{\gamma}_j$ are identified with real matrices $\Gamma_i, \tilde{\Gamma}_j$, of twice the size of $\gamma_i, \tilde{\gamma}_j$. More explicitly, $(a_{mn} + ib_{mn})$ is identified as $A + iB = (a_{mn}) + i(b_{mn}) \to (a_{mn}) + J_1(b_{mn})$, where

$$J_1 \equiv i\sigma_y \otimes \mathbb{1}_{n\times n} \tag{3.3.10}$$

The complex conjugate K is then a matrix anticommute with J_1, we define:

$$K \equiv \sigma_z \otimes \mathbb{1}_{n\times n} \tag{3.3.11}$$

Then $T \to \Gamma_T$, $C \to \Gamma_C$ for some real matrices. We also note that, as mentioned earlier, we make a phase choice of T and C such that

$$\{\Gamma_T, \Gamma_C\} = 0 \tag{3.3.12}$$

Now, we write symbolically $J_1J_2J_{3+j} = J_1\Gamma_T J_{3+j} = \tilde{\Gamma}_j$, then clearly $J_{3+j} = \Gamma_T J_1\tilde{\Gamma}_j$. Similarly, $\tilde{J}_i = \Gamma_T\Gamma_i$. So the map:

$$\begin{aligned} f^{-1} : G_{\text{AII}} &\to \mathrm{C}\ell_{d,3+D}(\mathbb{R}) \\ i\gamma_y \otimes \mathbb{1}_{n\times n} &\to J_1 \\ \Gamma_T &\to J_2 \\ \Gamma_T J_1\Gamma_j &\to J_{3+j} \\ \Gamma_T\tilde{\Gamma}_i &\to J_i \end{aligned} \tag{3.3.13}$$

is the desired inverse map.

Class CII The class CII has only one $C^2 = -1$ more than class AII, therefore, we only need to enlarge the Clifford Algebra to include one more J_{4+D}, and set it as the C symmetry operator. The rest is exactly the same as in class AII.

All other classes can be treated similarly, so we do not repeat the calculation and only list the result in Table 4.

Table 4: Mapping Relations between symmetry classes and Clifford Algebras. Here $J_j^2 = -\mathbb{1}, \tilde{J}_i^2 = \mathbb{1}$, and they generates the Clifford Algebra. Also, only one direction of mapping is shown. The inverse map can be easily constructed accordingly. The complex class are also added for convenience. tab:map-sym-cl

Class(#)	T	C	S	Mappings	$G_\# \cong$
D	0	+	0	$\tilde{J}_1 \to \Gamma_C$, $\tilde{J}_1 J_{2+j} \to \tilde{\Gamma}_j$, $J_1 \tilde{J}_1 \tilde{J}_{1+i} \to \Gamma_i$	$C\ell_{1+d,2+D}(\mathbb{R})$
DIII	−	+	1	$\tilde{J}_1 \to \Gamma_C$, $\tilde{J}_{3+D} \to \Gamma_T$, $\tilde{J}_1 J_{2+j} \to \tilde{\Gamma}_j$, $J_1 \tilde{J}_1 \tilde{J}_{1+i} \to \Gamma_i$	$C\ell_{1+d,3+D}(\mathbb{R})$
AII	−	0	0	$J_2 \to \Gamma_T$, $J_1 J_2 J_{3+j} \to \tilde{\Gamma}_j$, $J_2 \tilde{J}_i \to \Gamma_i$	$C\ell_{d,3+D}(\mathbb{R})$
CII	−	−	1	$J_2 \to \Gamma_T$, $J_{4+D} \to \Gamma_C$, $J_1 J_2 J_{3+j} \to \tilde{\Gamma}_j$, $J_2 \tilde{J}_i \to \Gamma_i$	$C\ell_{d,4+D}(\mathbb{R})$
C	0	−	0	$J_2 \to \Gamma_C$, $J_2 \tilde{J}_{1+j} \to \tilde{\Gamma}_j$, $J_1 J_2 J_{i+2} \to \Gamma_i$	$C\ell_{1+D,2+d}(\mathbb{R})$
CI	+	−	1	$J_2 \to \Gamma_C$, $\tilde{J}_{2+D} \to \Gamma_T$, $J_2 \tilde{J}_{1+j} \to \tilde{\Gamma}_j$, $J_1 J_2 J_{i+2} \to \Gamma_i$	$C\ell_{2+D,2+d}(\mathbb{R})$
AI	+	0	0	$\tilde{J}_1 \to \Gamma_T$, $J_1 \tilde{J}_1 \tilde{J}_{2+j} \to \tilde{\Gamma}_j$, $\tilde{J}_1 J_{i+1} \to \Gamma_i$	$C\ell_{2+D,1+d}(\mathbb{R})$
BDI	+	+	1	$\tilde{J}_1 \to \Gamma_T$, $\tilde{J}_{3+D} \to \Gamma_C$, $J_1 \tilde{J}_1 \tilde{J}_{2+j} \to \tilde{\Gamma}_j$, $\tilde{J}_1 J_{i+1} \to \Gamma_i$	$C\ell_{3+D,1+d}(\mathbb{R})$
A	0	0	0		$C\ell_{d+D+1}(\mathbb{C})$
AIII	0	0	1		$C\ell_{d+D+2}(\mathbb{C})$

3.4 Classification in 1-dimension

ssification in 1-dimension

Classification of 1 dimension is the simplest, since the minimal matrix dimension will mostly be 2, which means we can use the familiar Pauli matrices directly.

Class A We have the extension from $C\ell_2(\mathbb{C})$ to $C\ell_3(\mathbb{C})$. Since $C\ell_2(\mathbb{C}) \cong \mathcal{M}(2, \mathbb{C})$, we have for example:

$$H_A = M\sigma_x + k_x\sigma_y \tag{3.4.1}$$

Obviously, there is an SPEMT $m\sigma_x$. So this phase is trivial.

Class AIII We have the extension from $C\ell_3(\mathbb{C})$ to $C\ell_4(\mathbb{C})$. Since $C\ell_3(\mathbb{C}) \cong \mathcal{M}(2, \mathbb{C}) \oplus \mathcal{M}(2, \mathbb{C})$, we use Pauli matrices. For example:

$$H_A = M\sigma_x + k_x\sigma_y \tag{3.4.2}$$

However, the symmetry operator S takes rest Pauli matrices σ_z, and there is not other matrix possible for the extra mass term (this can be viewed alternatively, from $C\ell_4(\mathbb{C}) \cong \mathcal{M}(4, \mathbb{C})$, which only has irreducible representation in dimension 1 or 4). So the state is topologically non-trivial. Now we consider the state with arbitrary copies of it.

$$H_A = M\sigma_x \otimes \mathbb{1}_n + k_x\sigma_y \otimes \mathbb{1}_n, S = \sigma_z \otimes \mathbb{1}_n \tag{3.4.3}$$

where n is some positive integer. Since any matrix A, A could be expressed as a linear combination of terms like $B \otimes C$, where B is 2×2 dimensional[12]. Due to the linearity of anti-commutator and some basic properties of Kronecker product, there is no SPEMT term. Hence this is a $\mathbb{Z}$ topological insulator.

The above are for complex classes. For real classes, we have to be careful, since one generator J_1 is taken up by i.

Class AII We have the extension from $C\ell_{1,3}(\mathbb{R}) \cong \mathcal{M}(2,\mathbb{R}) \otimes \mathbb{H}$, to $C\ell_{1,4}(\mathbb{R}) \cong \mathcal{M}(2,\mathbb{R}) \otimes (\mathbb{H} \oplus \mathbb{H})$. This already hints that we will have SPEMT. Explicitly, we could let

$$H_{\text{AII}} = M\sigma_x \otimes \mathbb{1}_2 + k_x \sigma_y \otimes \mathbb{1}_2 \tag{3.4.4}$$

with $T = \mathbb{1}_2 \otimes \sigma_x K$. The SPEMT is $m\sigma_z \otimes \sigma_x$. So this is a trivial insulator.

Class C We have the extension from $C\ell_{1,3}(\mathbb{R}) \cong \mathcal{M}(2,\mathbb{R}) \otimes \mathrm{H}$, to $C\ell_{2,3}(\mathbb{R}) \cong \mathcal{M}(4,\mathbb{R}) \otimes \mathbb{C}$. Explicitly, we could let

$$H_{\text{C}} = M\sigma_z \otimes \mathbb{1}_2 + k_x \sigma_y \otimes \mathbb{1}_2 \tag{3.4.5}$$

with $C = \sigma_x \otimes \sigma_y K$. The SPEMT is $m\sigma_x \otimes \sigma_x$. So this is a trivial insulator.

Class CI We have the extension from $C\ell_{2,3}(\mathbb{R}) \cong \mathcal{M}(4,\mathbb{R}) \otimes \mathbb{C}$, to $C\ell_{3,3}(\mathbb{R}) \cong \mathcal{M}(6,\mathbb{R})$. If one realize that $\mathbb{C}$ is realized by 2×2 matrices, then this should have SPEMT. Explicitly, we could let

$$H_{\text{CI}} = M\sigma_z \otimes \mathbb{1}_2 + k_x \sigma_y \otimes \mathbb{1}_2 \tag{3.4.6}$$

with $C = \sigma_x \otimes \sigma_y K, T = K$. The SPEMT is again $m\sigma_x \otimes \sigma_x$. So this is a trivial insulator.

Class AI We have the extension from $C\ell_{2,2}(\mathbb{R}) \cong \mathcal{M}(4,\mathbb{R})$, to $C\ell_{3,2}(\mathbb{R}) \cong \mathcal{M}(4,\mathbb{R}) \otimes (\mathbb{R} \oplus \mathbb{R})$. This hints that we have SPEMT. Explicitly, we could let

$$H_{\text{AI}} = M\sigma_x + k_x \sigma_y \tag{3.4.7}$$

with $T = K$. The SPEMT is $m\sigma_z$. So this is a trivial insulator.

Class CII We have extension from $C\ell_{1,4}(\mathbb{R}) \cong \mathcal{M}(2,\mathbb{R}) \otimes (\mathbb{H} \oplus \mathbb{H})$, to $C\ell_{1,5}(\mathbb{R}) \cong \mathcal{M}(2,\mathbb{R}) \otimes \mathcal{M}(2,\mathbb{H})$. We let:

$$H_{\text{CII}} = M\sigma_x \otimes \mathbb{1}_2 + k_x \sigma_y \otimes \mathbb{1}_2 \tag{3.4.8}$$

with $T = \mathbb{1}_2 \otimes \sigma_y K, C = \sigma_z \otimes \sigma_y K$. There is no SPEMT[13]. Consider arbitrary copies of it:

$$H_{\text{CII}} = (M\sigma_x \otimes \mathbb{1}_2 + k_x \sigma_y \otimes \mathbb{1}_2) \otimes \mathbb{1}_n \tag{3.4.9}$$

with $T = \mathbb{1}_2 \otimes \sigma_y \otimes \mathbb{1}_n K, C = \sigma_z \otimes \sigma_y \otimes \mathbb{1}_n K$. There is still no SPEMT[14]. Therefore, this is a $\mathbb{Z}$ topological insulator.

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[12] For example, just take $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

[13] To anticommute with $\sigma_x \otimes \mathbb{1}_2$ and $\sigma_y \otimes \mathbb{1}_2$, it must be of the form $\sigma_z \otimes ?$. But none of the remaining choice are acceptable, considering T and C.

[14] The SPEMT has to be of the form $\sigma_z \otimes \sigma_\alpha \otimes A_{n\times n}$. Commuting with T makes A completely imaginary. But it cannot anticommute with C.

Class BDI We have extension from $C\ell_{3,2}(\mathbb{R}) \cong \mathcal{M}(4,\mathbb{R}) \otimes (\mathbb{R} \oplus \mathbb{R})$, to $C\ell_{4,2}(\mathbb{R}) \cong \mathcal{M}(8,\mathbb{R})$. We let:

$$H_{\text{BDI}} = M\sigma_z + k_x\sigma_y \tag{3.4.10}$$

with $T = K$, $C = \sigma_x K$. Obviously, there is no SPEMT. For arbitrary copies of it:

$$H_{\text{BDI}} = M\sigma_z \otimes \mathbb{1}_n + k_x\sigma_y \otimes \mathbb{1} \tag{3.4.11}$$

with $T = \mathbb{1}_{n+2}K$, $C = \sigma_x \otimes \mathbb{1}_n K$. The SPEMT must be of the form $\sigma_x \otimes \Delta$ to anticommute with H_{BDI}. But to commute with T leads to Δ being real, and to anticommute with C leads to Δ being complex. Hence there is no SPEMT. This is a $\mathbb{Z}$ topological insulator.

Class D We let:

$$H_{\text{D}} = M\sigma_z + k_x\sigma_y \tag{3.4.12}$$

with $C = \sigma_x K$. Obviously, there is no SPEMT. Now consider two copies of it:

$$H_{\text{D}} = (M\sigma_z + k_x\sigma_y) \otimes \mathbb{1}_2 \tag{3.4.13}$$

with $C = \sigma_x \otimes \mathbb{1}_2 K$. There is one SPEMT $m\sigma_z \otimes \sigma_y$. Hence, this is a $\mathbb{Z}_2$ topological insulator.

To lighten the notation a bit, we introduce $\tau_i = s_i = \sigma_i$ $(i = 0,1,2,3)$. But τ_i, s_i, and σ_i all act on the different spaces.

Class DIII We let:

$$H_{\text{DIII}} = M\sigma_z s_0 + k_x\sigma_y s_0 \tag{3.4.14}$$

with $C = \sigma_x K$, $T = s_y K$. Any SPEMT anticommute with Hamiltonian has the form $\sigma_x s_i$. For it to anticommute with C, it is $\sigma_x s_y$. But this does not commute with T. Hence, there is no SPEMT. Now consider two copies of it:

$$H_{\text{DIII}} = (M\sigma_z s_0 + k_x\sigma_y s_0)\tau_0 \tag{3.4.15}$$

with $C = \sigma_x K$, $T = s_y K$. There is one SPEMT $m\sigma_x s_x \tau_y$. So this is a $\mathbb{Z}_2$ topological insulator.

In summary, we have classified topological insulators in $d = 1$, summarized in Table 5.

add reflection about the disadvantage of this method (no $2\mathbb{Z}$). And representation aspect

3.5 Classification in Arbitrary Dimension

on in Arbitrary Dimension

We now show combined with some properties, the classification in $d = 1$ can be generalized to classification in arbitrary dimensions. In K-theoretic classification, the extension problem is not affected when tensored with some other algebras. Specifically, for complex classes, the extension problem of $C\ell_n(\mathbb{C}) \to C\ell_m(\mathbb{C})$ is the same as the extension problem of $C\ell_n(\mathbb{C}) \otimes C\ell_l(\mathbb{C}) \to C\ell_m(\mathbb{C}) \otimes C\ell_l(\mathbb{C})$ (See for example, sec III.C.1 of [CTSR16]). For real classes, similar property holds. This insensitive to tensoring a new algebra, can be understand in SPEMT background.

understand this

Now, let us write $\sim$ to represent the equivalence between two extension problems. The Complex Clifford Algebra has a periodicity of 2:

$$C\ell_{n+2}(\mathbb{C}) \cong C\ell_n(\mathbb{C}) \otimes C\ell_2(\mathbb{C}) \tag{3.5.1}$$

Table 5: Topological Insulators in $d = 1$ tab:ti-d=1

Class(#)	T	C	S	$d = 1$
A	0	0	0	0
AIII	0	0	1	$\mathbb{Z}$
D	0	$+$	0	$\mathbb{Z}_2$
DIII	$-$	$+$	1	$\mathbb{Z}_2$
AII	$-$	0	0	0
CII	$-$	$-$	1	$\mathbb{Z}$
C	0	$-$	0	0
CI	$+$	$-$	1	0
AI	$+$	0	0	0
BDI	$+$	$+$	1	$\mathbb{Z}$

The Real Clifford Algebra has a periodicity of 8:

$$\mathrm{C}\ell_{p+8,q}(\mathbb{R}) = \mathrm{C}\ell_{p,q+8}(\mathbb{R}) = \mathrm{C}\ell_{p,q}(\mathbb{R}) \otimes \mathcal{M}(16, \mathbb{R}) \tag{3.5.2}$$

This two relation means that, the extension problem will be the same with respect to a periodicity of 2 for complex symmetry classes, and 8 for real symmetry classes. Hence the classification of topological insulators need only be done with $d = 1, 2$ for complex symmetry classes, and $d = 0, 1, \cdots, 7$ for real symmetry classes, i.e.

eq:cli-periodic

$$G_{\#}(d = d_0) \sim G_{\#}(d = (d_0 \mathrm{mod} 2)) \quad \text{(complex)} \tag{3.5.3a}$$
$$G_{\#}(d = d_0) \sim G_{\#}(d = (d_0 \mathrm{mod} 8)) \quad \text{(real)} \tag{3.5.3b}$$

Another useful property is:

$$\mathrm{C}\ell_{p+1,q+1}(\mathbb{R}) \cong \mathrm{C}\ell_{p,q}(\mathbb{R}) \otimes \mathcal{M}(2, \mathbb{R}) \tag{3.5.4}$$

This property tells us that Clifford Algebra basically depends only on the difference $p - q$, or combined with previous periodicity, $p - q$ (mod 8). For example, we have (let n be an arbitrary integer): $\mathrm{C}\ell_{p+1,q+1}(\mathbb{R}) \sim \mathrm{C}\ell_{p,q}(\mathbb{R})$. With this, one can derive easily two chains of relation:

$$\begin{aligned} &\mathrm{C}\ell_{1+(D+n),2+D}(\mathbb{R}) \sim \mathrm{C}\ell_{1+(D+1+n),3+D}(\mathbb{R}) \\ \sim &\mathrm{C}\ell_{(D+2+n),3+D}(\mathbb{R}) \sim \mathrm{C}\ell_{(D+3+n),4+D}(\mathbb{R}) \end{aligned} \tag{3.5.5}$$

and

$$\begin{aligned} &\mathrm{C}\ell_{1+D,2+(D+4+n)}(\mathbb{R}) \sim \mathrm{C}\ell_{2+D,2+(D+5+n)}(\mathbb{R}) \\ \sim &\mathrm{C}\ell_{2+D,1+(D+6+n)}(\mathbb{R}) \sim \mathrm{C}\ell_{3+D,1+(D+7+n)}(\mathbb{R}) \end{aligned} \tag{3.5.6}$$

Now we tries to connect the two chains together. We note further that the Clifford Algebra has the following properties:

$$\mathrm{C}\ell_{p+1,q}(\mathbb{R}) \cong \mathrm{C}\ell_{q+1,p}(\mathbb{R}) \tag{3.5.7}$$
$$\mathrm{C}\ell_{q,p+2}(\mathbb{R}) \cong \mathrm{C}\ell_{p,q}(\mathbb{R}) \otimes \mathcal{M}(2, \mathbb{R}) \tag{3.5.8}$$

The original classification table											
AZ class$\backslash d$	0	1	2	3	4	5	6	7	T	C	S
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	0
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	1
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$+$	0	0
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$+$	$+$	1
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	0	$+$	0
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$-$	$+$	1
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$-$	0	0
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$-$	$-$	1
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$-$	0
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$+$	$-$	1

Table 6: The original classification table of topological insulators and superconductors `tab:master-table2`

Then:

$$\begin{aligned}&\mathrm{C}\ell_{1+D,2+(D+4+n)}(\mathbb{R}) \cong \mathrm{C}\ell_{D+4+n,1+D}(\mathbb{R}) \otimes \mathcal{M}(2,\mathbb{R})\\ \cong&\mathrm{C}\ell_{2+D,D+3+n}(\mathbb{R}) \otimes \mathcal{M}(2,\mathbb{R}) \cong \mathrm{C}\ell_{D+1+n,2+D}(\mathbb{R}) \otimes \mathcal{M}(4,\mathbb{R})\end{aligned} \tag{3.5.9}$$

Therefore, $\mathrm{C}\ell_{1+D,2+(D+4+n)}(\mathbb{R}) \sim \mathrm{C}\ell_{1+(D+n),2+D}(\mathbb{R})$, connecting the two chains. If we compare this carefully with Table 4, then we would realize that we actually managed to prove the dimension-shift feature of classification:

$$\begin{aligned}&G_{\mathrm{D}}(d=d_0) \sim G_{\mathrm{DIII}}(d=d_0+1)\\ &\sim G_{\mathrm{AII}}(d=d_0+2) \sim G_{\mathrm{CII}}(d=d_0+3)\\ &\sim G_{\mathrm{C}}(d=d_0+4) \sim G_{\mathrm{CI}}(d=d_0+5)\\ &\sim G_{\mathrm{AI}}(d=d_0+6) \sim G_{\mathrm{BDI}}(d=d_0+7)\end{aligned} \tag{3.5.10}$$ `eq:cli-Chain1`

We need one more property:

$$\mathrm{C}\ell_{1+d,2+D}(\mathbb{R}) \cong \mathrm{C}\ell_{3+D,d}(\mathbb{R}) = \mathrm{C}\ell_{3+D,1+(d-1)}(\mathbb{R}) \tag{3.5.11}$$

Then

$$G_{\mathrm{D}}(d=d_0) \sim G_{\mathrm{BDI}}(d=d_0-1) \tag{3.5.12}$$ `eq:cli-Chain2`

With equivalences 3.5.3,3.5.10,3.5.12, we see that we need only the result in a dimension, to obtain the whole of classification of all real classes. Similar fact holds for complex classes:

$$G_{\mathrm{AIII}}(d=d_0) \sim G_{\mathrm{A}}(d=d_0-1) \tag{3.5.13}$$

which is too trivial to be mentioned here.

In a word, our classification for topological insulators in 1 spatial dimension is sufficient to generate the whole Table 6. This is a remarkable consequence of K-theory.

4 License

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