

Noetherian Ring

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Abstract

A note about Noetherian Ring, majorly from the book [1].

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1 Module

(pp.117 to 118 of [1])

Definition 1.1 (Module). *Let A : ring. M is a nomen left A -module if and only if*

- 1. M is an abelian group, usually written additively.*
- 2. there exists an operation of A on M , written as a multiplicative monoid, such that, for any $a, b \in A$, any $x, y \in M$, we have:*

$$(a + b)x = ax + bx \quad (1.0.1)$$

$$a(x + y) = ax + ay \quad (1.0.2)$$

By definition of an operation, we have $1x = x$. Also, it can be easily derived that $a(-x) = -ax$, and $0x = 0$.

Example 1.1. Examples of modules

1. A is a module over itself.
2. Any commutative group is a \mathbb{Z} -module.
3. Any left ideal of A is a module over A , i.e. a left A -module.
4. A vector space V over K , is basically a K -module, with the additional structure of K being a field.

5. Let V be a vector space. Let R be the ring of all linear maps of V into itself. Then V is also a module over R .

Definition 1.2 (Submodule). *A submodule M is an additive subgroup such that $AN \subset N$*

Definition 1.3 (factor module). *Let M be an A -module, and N a submodule. A factor module M/N is the factor group M/N (for the additive group structure) equipped with a module structure. The action of A on M/N is defined by $a(x + N) = ax + N$. This is well defined, since if y is in the same coset as x , then ay is in the same coset as ax .*

(pp.119 of [1])

2 Noetherian

Definition 2.1 (Noetherian Module). *Let A : a ring. M : a left A -module. M is called Noetherian if M satisfies any of the following conditions:*

1. *Every submodule of M is finitely generated.*
2. *Every ascending sequences of submodules of M*

$$M_1 \subset M_2 \subset \dots$$

such that $M_i \neq M_{i+1}$, is finite.

3. *Every non-empty set S of submodules of M has a maximal element.*

The equivalence of the above conditions are proved in page 413 to 414 of [1].

Definition 2.2 (Noetherian Ring). *A ring A is Noetherian if and only if it is Noetherian when viewed as a left module over itself.*

(pp.415 of [1])

Some theorems Here are some theorems mentioned in Chapter X, section 2 of [1].

The propositions 2.1, 2.2, 2.1 expresses the "Noetherian relation" between M and its submodules. The proposition 2.3 relates a Noetherian ring A and the A -modules. The proposition 2.4 relates two rings. The proposition 2.5 relates a commutative Noetherian ring and its multiplicative subset. The following diagram summarized these relations.

$$\begin{array}{ccccc}
 S^{-1}A & \xleftarrow[\text{prop.2.5}]{} & A & \xrightarrow{\text{prop.2.3}} & M \\
 & & \downarrow \text{prop.2.4} & & \uparrow \text{prop.:2.1,2.2,coro.2.1} \\
 & & B & & \text{submodules}
 \end{array}$$

(A, B : ring, S : A 's multiplicative subset. M : a A -module.)

The structure of being Noetherian is consistent between a module and its submodules, factor modules, in the sense of the following two propositions.

Proposition 2.1. *Let M be a Noetherian A -module, then every submodule and every factor module of M is Noetherian.*

(pp.414 of [1])

Proposition 2.2. *Let M be a module, N be a submodule. If N and M/N are Noetherian, then M is Noetherian.*

(pp.414 of [1])

The above statements could be summarized by saying that, given an exact sequence:

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

M is Noetherian if and only if M' and M'' are Noetherian. This could be seen by two immediate fact of an exact sequence:

$$M' \cong \text{Im} f, \quad M/\text{Ker} g \cong M''$$

Corollary 2.1. *A finite direct sum of Noetherian modules is Noetherian. Specifically, let M be a module, let N, N' be two submodules. If $M = N + N'$ and if both N, N' are Noetherian, then M is Noetherian.*

(pp.415 of [1])

Proposition 2.3. *Let A be a Noetherian ring, and let M be a finitely generated A -module. Then M is Noetherian.*

(pp.415 of [1])

Proposition 2.4. *Let A be a ring which is Noetherian, and let $\phi : A \rightarrow B$ be a surjective ring-homomorphism. Then B is Noetherian.*

(pp.415 of [1]) In colloquial term, a surjective homomorphism induces a Noetherian ring.

Proposition 2.5. *Let A be a commutative Noetherian ring, and let S be a multiplicative subset of A . Then $S^{-1}A$ is Noetherian.*

(pp.415 of [1])

3 Examples

- The polynomial Ring
- The ring of power series
- The ring of formal power series is **NOT** Noetherian. See this post

The essential point is that the polynomial ring in infinitely many variables is the ascending union of subrings $K[x_1, \dots, x_n]$, since any polynomial can involve only finitely-many indeterminates. Each of these rings is a UFD, and it is easy to see that a polynomial in which x_N does not appear has only factorizations in which x_N does not appear, again because everything takes place inside some polynomial ring in finitely-many variables. But the ring is not Noetherian, because the ideal generated by all the indeterminates is certainly not finitely-generated.

4 Anchor

Nomenclature

M Module A left module., page 1

factor module ., page 2

Noetherian Module ., page 2

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Submodule ., page 2

References

[1] S. Lang. Algebra. 3rd. Springer.

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