$\cancel{P_m = \langle \gamma}$

$$P_m = \langle \gamma \rangle_m = \int_0^\infty \gamma \left|\psi_m^R(t)\right|^2 dt$$

$$\langle m \rangle := \sum_m m P_m$$ → the object of this paper.

$\cancel{i\hbar \partial_t \psi = H\psi \qquad i\hbar \partial_t \psi^* = H^* \psi^*}$ $\qquad -i\hbar \langle \partial_t \psi | = \langle \psi | H^\dagger$

$$i\hbar\, \partial_t |\psi\rangle = H|\psi\rangle \qquad -i\hbar \left(\partial_t |\psi\rangle\right)^\dagger = \langle \psi | H^\dagger$$

$$i\hbar \frac{d}{dt} \langle \psi | \psi \rangle = i\hbar \left(\langle \partial_t \psi | \psi \rangle + \langle \psi | \partial_t \psi \rangle \right)$$

$$= \langle \psi |\ -H^\dagger + H\ | \psi \rangle$$

$$i\hbar\, \partial_t \langle \psi_m^R | \psi_m^R \rangle = i\hbar \left(\langle \partial_t \psi_m^R | \psi_m^R \rangle + \langle \psi_m^R | \partial_t \psi_m^R \rangle \right)$$

$\cancel{= i\hbar}$

$$i\hbar\, \partial_t |\psi_m^R\rangle = \underline{\tilde{\varepsilon}_R |\psi_m^R\rangle} + v|\psi_m^L\rangle + v'|\psi_{m-1}^L\rangle$$

$$\tilde{\varepsilon}_R = \varepsilon_R - i\hbar\gamma/2$$

so $$i\hbar\, \partial_t \langle \psi_m^R | \psi_m^R \rangle = -\langle \psi_m^R |\ \varepsilon_R + i\hbar\gamma/2\ | \psi_m^R \rangle + \langle \psi_m^R | \varepsilon_R - i\hbar\gamma/2 | \psi_m^R \rangle$$

$$= -i\hbar\gamma \langle \psi_m^R | \psi_m^R \rangle$$

$\Rightarrow$

$$\frac{\partial_t \langle \psi_m^R | \psi_m^R \rangle}{\langle \psi_m^R | \psi_m^R \rangle} = -\gamma$$

$\Rightarrow$ If $|\psi^R\rangle = \sum_m |\psi_m^R\rangle$

Then $\partial_t \langle\psi^R|\psi^R\rangle = \sum_m -\gamma \langle\psi_m^R|\psi_m^R\rangle$

Also

$$i\hbar\,\partial_t |\psi_m^L\rangle = \underbrace{\varepsilon_L |\psi_m^L\rangle + v|\psi_m^R\rangle + v'|\psi_{m+1}^R\rangle}_{\text{No Complex Term}}$$

$\Downarrow$

$$\partial_t \langle\psi_m^L|\psi_m^L\rangle = 0$$

$\Downarrow$

If $|\psi\rangle = |\psi^L\rangle + |\psi^R\rangle$, with

$$|\psi^L\rangle = \sum_m |\psi_m^L\rangle$$

Then

$$\partial_t \langle\psi|\psi\rangle = -\sum_m \gamma \langle\psi_m^R|\psi_m^R\rangle$$

Physically, this means all leakage comes from "R".

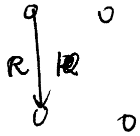
$$\sum_m P_m = \sum_m \int_0^\infty \gamma \langle\psi_m^R|\psi_m^R\rangle\, dt = 1\ ?$$

I don't think it is certain that this system will leak!

Bloch Theorem

$$\psi^L_m = e^{ik\cdot r} u(r)$$

$$\psi^R_m = \frac{1}{2\pi}\oint dk\, e^{ikm} \cdot \psi^R_k$$



$$\psi^R_k(r) = \psi^R_R(\vec{r} + \vec{R})$$

$$\psi^R_m(r+R) = \frac{1}{2\pi}\oint dk\, e^{ikm}\, \psi^R_k$$

$$i\hbar\, \partial_t \left(\frac{1}{2\pi}\oint dk\, e^{ikm}\, \psi^R_k\right) =$$

$$\psi^L_m = e^{ikma}\, u(r)$$

$$\psi^L_{m+1} = e^{ika} \cdot \psi^L_m$$

$$i\hbar\, \partial_t \frac{1}{2\pi}\oint dk\, e^{ikm}\, \psi^R_k = \varepsilon_L \frac{1}{2\pi}\oint dk\, e^{ikm}\, \psi^L_k + v\, \frac{1}{2\pi}\oint dk\, e^{ikm}\, \psi^R_k + v' \frac{1}{2\pi}\oint dk\, e^{ik(m+1)}\, \cancel{\psi^R_{m+1}}\, \psi^R_k$$

$$i\hbar\, \partial_t\, \psi^L_k = \varepsilon_L\, \psi^L_k + v\, \psi^R_k + v' e^{ik\cdot 1}\, \psi^R_k$$

$$i\hbar\, \partial_t\, \psi^R_k = \tilde{\varepsilon}_R\, \psi^R_k + v\, \psi^L_k + v' e^{-ik\cdot 1}\, \psi^L_k$$

$$i\hbar\, \partial_t \begin{pmatrix} \psi^L_k \\ \psi^R_k \end{pmatrix} = \begin{pmatrix} \varepsilon_L & v + v' e^{ik\cdot 1} \\ v + v' e^{-ik 1} & \tilde{\varepsilon}_R \end{pmatrix} \begin{pmatrix} \psi^L_k \\ \psi^R_k \end{pmatrix}$$

$$i\hbar\,\partial_t \langle \psi^L_k | \psi^L_k \rangle = \cancel{\quad} -\langle \psi^L_k | \mathcal{E}_L | \psi^L_k \rangle + \langle \psi^L_k | \mathcal{E}_L | \psi^L_k \rangle$$

$$= 0$$

$$i\hbar\,\partial_t \langle \psi^R_k | \psi^R_k \rangle = -\langle \psi^R_k | \cancel{\mathcal{E}} \tilde{\mathcal{E}}^*_R | \psi^R_k \rangle + \langle \psi^R_k | \tilde{\mathcal{E}}_R | \psi^R_k \rangle$$

$$= 2\,\mathrm{Im}\,\tilde{\mathcal{E}}_R \langle \psi^R_k | \psi^R_k \rangle = -i\hbar\gamma \langle \psi^R_k | \psi^R_k \rangle$$

so

$$\partial_t P_k(t) := \cancel{\tfrac{1}{2}}\, \partial_t \left(\langle \psi^L_k | \psi^L_k \rangle + \langle \psi^R_k | \psi^R_k \rangle \right)$$

$$= -\gamma \langle \psi^R_k | \psi^R_k \rangle$$

PR(4) ~~Eq (4)~~

$$-\frac{i}{2\pi}\oint dk\, \partial_k (e^{ikm}) |\psi^R_k\rangle = -i(i\cdot m)\, \frac{1}{2\pi}\oint dk\, e^{ikm} |\psi^R_k\rangle$$

$$= m |\psi^R_m\rangle$$

~~Assuming $e^{ikm} |\psi^R_k\rangle \to 0$ as~~ k

$$-\frac{i}{2\pi}\oint dk\, \partial_k(e^{ikm})\, |\psi^R_k\rangle = -i$$

$$= \left(e^{ikm}|\psi^R_k\rangle\right)\Big|_{k=\partial BZ} - \oint dk\, e^{ikm} \left(\partial_k |\psi^R_k\rangle\right)$$

e^{ikm} at $\partial BZ = 0.$ $\cancel{\text{ie}}$

$$= \frac{i}{2\pi}\oint dk\, e^{ikm}\, \partial_k |\psi^R_k\rangle$$

$$\langle \Delta m \rangle = \sum_m m \cdot \int_0^\infty \gamma \langle \psi_m^R | \psi_m^R \rangle \, dt$$

$$= \sum_m \int_0^\infty \gamma \underline{\langle \psi_m^R |} \left(\underline{m |\psi_m^R \rangle} \right) dt$$

$$\underline{m |\psi_m^R \rangle} = \frac{i}{2\pi} \oint dk \, e^{ikm} \partial_k |\psi_k^R \rangle$$

$$\underline{\langle \psi_m^R |} = \frac{1}{2\pi} \oint dk \, e^{-ikm} \langle \psi_k^R |$$

$$\frac{i}{(2\pi)^2} \oiint dk_1 dk_2 \, \cancel{e^{ik_1 - km}} \, e^{i(k_1-k_2)m} \langle \psi_{k_2}^R | \partial_{k_1} \psi_{k_1}^R \rangle$$

Assuming γ independent of m!

$$\left(\cancel{\int\delta} \sum_m e^{i(k_1-k_2)m} = \delta(k_1-k_2) \cdot \square \right.$$

$\square$ is some number I forget what it should be $\Big)$.

$$= \int_0^\infty \gamma \cdot \frac{i}{(2\pi)^2} \cdot \oiint dk_1 dk_2 \, \square \delta(k_1-k_2) \cdot \langle \psi_{k_2}^R | \partial_{k_1} \psi_{k_1}^R \rangle$$

$$= i\gamma \int_0^\infty \frac{\square}{2\pi} \oint \frac{dk}{2\pi} \langle \psi_k^R | \partial_k | \psi_k^R \rangle.$$

~~Therefore~~,

From expression (5) on p P_2 of the paper, $\square$ should be 2π, which is quite STRANGE!

Polar.

$$\text{设}\quad \psi_k^R(t) = \underset{\substack{\|\\ \text{取}\ \|\psi_k^R(t)\|}}{u_k(t)}\, e^{i\theta_k(t)}$$

Assuming $\quad u_k(t) = \underbrace{\|\psi_k^R(t)\|}_{\downarrow} > 0 \quad \forall t>0.$

$\psi_k^R(t)$ is non-vanishing → always filled.

Follows from Eq. (4) ?:

$$\underline{i\hbar\,\partial_t \begin{pmatrix} |\psi_k^L\rangle \\ |\psi_k^R\rangle \end{pmatrix} = \begin{pmatrix} \varepsilon_L & A_k \\ A_k^* & \tilde{\varepsilon}_R \end{pmatrix} \begin{pmatrix} |\psi_k^L\rangle \\ |\psi_k^R\rangle \end{pmatrix}}$$

$$i\hbar\partial_t \langle\psi_k^R|\psi_k^R\rangle = -\langle\psi_k^R|\tilde{\varepsilon}_R^*|\psi_k^R\rangle + \langle\psi_k^R|\tilde{\varepsilon}_R|\psi_k^R\rangle$$

$$= -i\hbar\gamma\langle\psi_k^R|\psi_k^R\rangle$$

or $\quad \partial_t(u_k^2) = -\gamma(u_k^2) \qquad u_k^2 = e^{-\gamma t}u_k^2(0) > 0$, if $u_k^2(t=0)>0$

$$\partial_t \vec{X} = A\vec{X} \Rightarrow \vec{X} = e^{tA}X_0$$

i.e. $$\begin{pmatrix} |\psi_k^L\rangle \\ |\psi_k^R\rangle \end{pmatrix} = e^{\frac{-it}{\hbar}\begin{pmatrix} \varepsilon_L & A_k \\ A_k^* & \tilde{\varepsilon}_R \end{pmatrix}} \cdot \begin{pmatrix} |\psi_k^L(t=0)\rangle \\ |\psi_k^R(t=0)\rangle \end{pmatrix}.$$

$$\begin{pmatrix} \varepsilon_L & A_k \\ A_k^* & \tilde{\varepsilon}_R \end{pmatrix} = \begin{pmatrix} \varepsilon_L & v+v'e^{ik1} \\ v+v'e^{-ik1} & \varepsilon_R - i\hbar\gamma/2 \end{pmatrix}$$

Using Mathematica.

$$= [v+v'\cos(k\cdot 1)]\sigma_x - v'\sin(k\cdot 1)\sigma_y + \frac{1}{4}(2E_L - 2E_R + i\hbar\gamma)\sigma_z$$

$$+\left(\frac{E_L+E_R}{2} - \frac{i\hbar}{4}\gamma\right)\mathbb{I}.$$

Since $\mathbb{I}$ commutes with other σ_i. we have $e^{\frac{-it}{\hbar}\left(\frac{E_L+E_R}{2} - i\frac{\hbar}{4}\gamma\right)}$ factor $\quad e^{X+Y} = e^X e^Y$ if $[X,Y]=0$.

Also by: $$\exp\left(\frac{-i\vec{\sigma}\cdot\hat{n}\phi}{2}\right) = \cos\left(\frac{\phi}{2}\right)\mathbb{I} - i\vec{\sigma}\cdot\hat{n}\sin\left(\frac{\phi}{2}\right).$$

We do not care about the details of ϕ, $\hat{n}$ first.

The ~~ϕ~~ 1^{st} term ~~gives~~ $\cos(\frac{\phi}{2})\,\mathbb{I}$ scales $\begin{pmatrix} |\psi_k^L\rangle \\ |\psi_k^R\rangle \end{pmatrix}$.

The second term will ~~mix~~ mixes up the two wave-functions.

But notice that the dependence of time t is in $\sin(\frac{\phi}{2})$, or in $\frac{\phi}{2}$.

$\frac{\phi}{2}$ is the length of the vector:

$$-\frac{it}{\hbar}\begin{pmatrix} v+v'\cos k\cdot 1 \\ -v'\sin(k\cdot 1) \\ \frac{1}{4}(2E_L-2E_R+i\hbar\gamma) \end{pmatrix}$$

~~Som~~ Somehow I can't get the result in Note [18] at the end.

$$\partial_k(u_k^2) = 2u_k\partial_k u_k$$

so $\oint dk\, u_k\partial_k u_k = \frac{1}{2}\oint dk\, \partial_k(u_k^2) = u_k^2\Big|$ at two ~~same~~ points of the same value $=0$

$$\langle \Delta m\rangle = i\gamma\int_0^\infty dt \oint \frac{dk}{2\pi}\cdot \underbrace{\psi_k^{R*}}_{u_k(t)e^{-i\theta_k(t)}}\,\partial_k\left(U_k(t)e^{i\theta_k(t)}\right)$$

$$= i\gamma\int_0^\infty dt\oint\frac{dk}{2\pi}\, u_k(t)e^{-i\theta_k(t)}\cdot\left[(\partial_k U_k(t))\,e^{i\theta_k(t)} + u_k(t)\,e^{i\theta_k(t)}\cdot i(\partial_k\theta_k(t))\right]$$

$$\oint dk\,(U_k\partial_k U_k)=0$$

$$= i\gamma\int_0^\infty dt\oint\frac{dk}{2\pi}\, U_k^2(t)\cdot i\,\partial_k\theta_k(t) = -\gamma\int_0^\infty dt\oint\frac{dk}{2\pi}\, U_k^2(t)\,\partial_k\theta_k(t).$$

$$= \oint\frac{dk}{2\pi}\int_0^\infty(\partial_t P_k)(\partial_k\theta_k(t)).$$

Eq. (7)

It's easy to get Eq(8) from Eq(7).

To get Eq(9)., we need to ~~assume~~ <u>ASSUME</u> $\theta_k(t)$ is smooth in k and t. so that $\partial_t \partial_k \theta_k(t) = \partial_k \partial_t \theta_k(t)$.

Also, the boundary term in (9) is

a : lattice constant

$$\oint dk\, \partial_k \left(P_k\, \partial_t \theta_k(t) \right) = P_k\, \partial_t \theta_k(t) \left(\Big|_{k=\frac{\pi}{a}} - \Big|_{k=-\frac{\pi}{a}} \right) = 0$$

due to periodicity.

Hence Eq(9) is correct

Eq(9).

Start from considering a general H

$$H = E_0\sigma_0 + W_1\sigma_1 + W_2\sigma_2 + \Delta\sigma_3 = \begin{pmatrix} E_0+\Delta & W_1 - iW_2 \\ W_1+iW_2 & E_0-\Delta \end{pmatrix}$$

Eigenvalue: $\lambda_\pm = \sqrt{\Delta^2+|W|^2}$

Take $W = |W|e^{i\theta}$

$\sin\theta = \dfrac{|W|}{\sqrt{\Delta^2+|W|^2}}$, $\cos\theta = \dfrac{\Delta}{\sqrt{\Delta^2+|W|^2}}$,

$$|E_+\rangle = e^{i\phi/2}\begin{pmatrix} \cos\theta/2 \\ e^{-i\phi}\sin\theta/2 \end{pmatrix}, \quad |E_-\rangle = \begin{pmatrix} -e^{i\phi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$$

Now the system is in a

For ~~the~~ moment, suppose the author's claim is true.

Evaluate I_0 When $t=0$, $(\psi^L_m = \delta_{m0}) \rightarrow \frac{1}{2\pi}\oint dk\, e^{ikm}\psi^L_k = \delta_{m,0}$

$\Downarrow$

$\psi^L_k = \cancel{x}\cdot 1.$

$P_k = |\psi^L_k(t)|^2 + |\psi^R_k(t)|^2$. $P_k(t=0) = 1$

1 at $t=0$. 0 at $t=0$.

This is enough to confirm that Eq(11) is correct. Eq 11

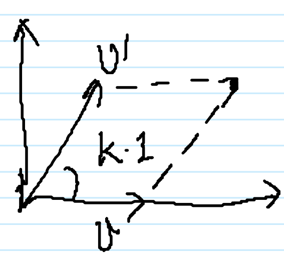
From (4),

$$i\hbar\, d\psi^R_k = (A^*_k \psi^L_k + \varepsilon_R \psi^R_k)\, dt$$

At $t=0$, $\psi^L_k = 1$ $\psi^R_k = 0$, so $i\hbar\,\psi^R_k(dt) = A^*\cdot 1\cdot dt$

$\psi^R_k(dt) = -\frac{i}{\hbar}A^* dt$, its angle: $\theta_k(0^+) = \theta_k(dt)$

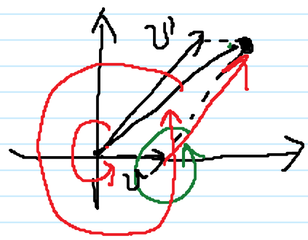
Now: $A_k = v + v' e^{ik\cdot 1}$,



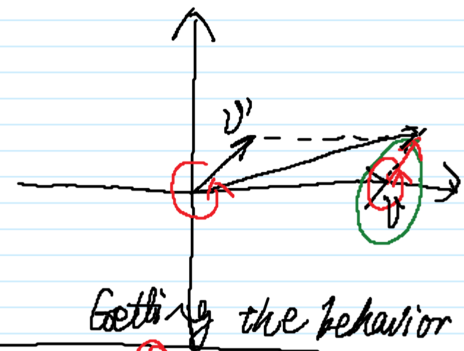
If $v' > v$

If $v > v'$

If $\upsilon' > \upsilon$



If $\upsilon > \upsilon'$



Getting the behavior of
Quantization of <m>

$$\bar{\tau} = -\int_0^\infty t \frac{d}{dt}\langle\psi|\psi\rangle\, dt = \int_0^\infty \langle\psi|\psi\rangle\, dt =$$

$$= -\int_0^\infty t \oint \frac{dk}{2\pi}\, \partial_t P_k\, dt = \int_0^\infty \frac{dk}{2\pi} P_k\, dt$$

Eq (12)