# Miscellaneous notes for D. Huybrechts's Complex Geometry

Taper

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**Abstract**

Miscellaneous notes for D. Huybrechts's book *Introduction to Complex Geometry*, include some homeworks done.

## Contents



## 1 The structure of almost complex structures on $\mathbb{R}^{2n}$

In exercise 1.2.1, it says that the set of all compatible almost complex structures on a euclidean space of dimension $2n$, is two copies of $S^2$.

To show it, I tried first a straight calculation. Assuming the almost complex structure $I = (a_{ij})$. Then we have:

Section 1.2 of Intro → Complex Geometry.

Exercises:

1.2-1.

Q: Let $(V, \langle , \rangle)$ : euclidian space of $\dim = \underset{\sim}{4}$.

Show: $\{$ all compatible almost complex structures $\}$

$\cong$

two copies of $S^2$ $\cancel{\ }$ = $\cancel{a}$ two balls.

Recap: compatible : $I$ : $I^2 = -1$, $\langle I(v), I(w) \rangle = \langle v, w \rangle$

Choose an orthogonal basis: $e_1 \cdots e_4$

Let $I = (\overset{a^i_j}{\cancel{a_j}})$ $\boxed{I^2 = a^i_j a^j_k = -\delta^i_k}$

also $\langle , \rangle \approx \delta^i_j$ $\cancel{a}$

$\langle I(v), I(w) \rangle = (a^i_j v^j) \cdot \delta^i_k (a^k_l w^l) = v^i \delta^i_k w^k$

$(\forall \vec{v}, \vec{w})$

Hence $a^i_j \delta^i_k a^k_l = \delta_{jl}$ or $\boxed{a^i_j a^i_k = \delta_{jk}}$

For example:

$\sum_j a^1_j a^j_2 = 0 = \sum_j a^j_1 a^j_2 \Rightarrow \sum_{j=2}^{4} a^1_j a^j_2 = \sum_{j=2}^{4} a^j_1 a^j_2$

This can be generalized: $\sum_{\substack{j=1 \\ j \neq k}}^{4} a^k_j a^j_l = \sum_{\substack{j=1 \\ j \neq k}}^{4} a^j_k a^j_l \quad (k \neq l)$ $\Big\}$ $\underline{\text{16 sets of eq.}}$

Also: $\cancel{a^i}$ $\sum_{j=1}^{4} a^i_j a^j_i = -\sum_{j=1}^{4} \cancel{a^i_j} a^j_i a^j_i$

Figure 1: Draft

Then I discover this too hard to work, because too many equations are involed, and none of them could be eliminiated by other. Meanwhile, I found a post in Math.SE about this [2].

## 1.1 Understand $\frac{GL(2n,\mathbb{R})}{GL(n,\mathbb{C})}$

Here I digressed to another problem. I will come back later.

To understand that post, I read this [3]. However, the answer in the second post is not perfect:

> The claim is: If $V$ is an $n$-dimensional complex vector space with underlying $2n$-dimensional real vector space $W$, then the canonical group monomorphism $\mathrm{GL}(V) \to \mathrm{GL}(W)$ lands inside $\mathrm{GL}^+(W) = \{f \in \mathrm{GL}(W) : \det(f) > 0\}$. The purpose of this abstract reformulation is that we may use operations on vector spaces in order to simplify the problem: If $V'$ is another finite-dimensional complex vector space with underlying real vector space $W'$, the diagram
>
> $$\begin{array}{ccc} \mathrm{GL}(V) \times \mathrm{GL}(V') & \to & \mathrm{GL}(W) \times \mathrm{GL}(W') \\ \downarrow & & \downarrow \\ \mathrm{GL}(V \oplus V') & \to & \mathrm{GL}(W \oplus W') \end{array} \tag{1.1.1}$$
>
> commutes, and the image of $\mathrm{GL}^+(W) \times \mathrm{GL}^+(W')$ is contained in $\mathrm{GL}^+(W \oplus W')$. Therefore, if some element in $\mathrm{GL}(V \oplus V')$ lies in the image of $\mathrm{GL}(V) \times \mathrm{GL}(V')$, it suffices to consider the components. Combining this with the fact that $\mathrm{GL}(V)$ is generated by elementary matrices (after chosing a basis of $V$), we may reduce the whole problem to the following three types of matrices:
>
> - the $1 \times 1$-matrices $(\lambda)$,
>
> - the $2 \times 2$-matrices $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$,
>
> - and the $2 \times 2$-matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
>
> Write $\lambda = a + ib$ with $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Then, the complex $1\times1$-matrix $(\lambda)$ becomes the real $2\times2$-matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, which has determinant $a^2+b^2 > 0$. The complex $2\times2$-matrix $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$ becomes the real $4 \times 4$-matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & -b & 1 & 0 \\ b & a & 0 & 1 \end{pmatrix}$, which has determinant 1. Finally, the complex $2 \times 2$-matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ becomes the real $4 \times 4$-matrix $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, which has deter-

minant 1.

This proof is not complete because, to build the proof from $\mathbb{R}^2$ to $\mathbb{R}^{2n}$, it requries, in his argument, that any element in $\mathrm{GL}(V \oplus V)$ is in the image of $\mathrm{GL}(V) \times \mathrm{GL}(V')$, which is not the case.

On the other hand, it seems that this property can be proved directly by calculation. The following will be a notes of a paper [4], which one comment mentions in the Math.SE post [3].

### 1.1.1 Determinants of Block Matrices

This paper tries to prove the theorem:

**Theorem 1.1.** *Let $R$ be a commutative subring of ${}^nF^n$, where $F$ is a field (or a commutative ring), and let $M \in {}^mR^m$. Then*

$$det_F\mathbf{M} = det_F(det_R\mathbf{M}) \tag{1.1.2}$$

In particular, we have:

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \det_F(AD - BC) \tag{1.1.3}$$

Note that, that the ring being is commutative excludes some ambiguity. For example, when the ring 4 is not commutative, then the quantity:

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \tag{1.1.4}$$

is not well-defined. It can be $AD - BC$, or $DA - CB$, etc.

Before the proof of the main theorem, it establishes several facts:

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \det_F\mathbf{A}\det_F\mathbf{D} \tag{1.1.5}$$

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} = \det_F\mathbf{A}\det_F\mathbf{D} \tag{1.1.6}$$

$$\det_F \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{pmatrix} = \det_F-\mathbf{C}\det_F\mathbf{B} \tag{1.1.7}$$

$$\det_F\mathbf{A}\det_F\mathbf{D} = \det_F\mathbf{I}_n\det_F(\mathbf{AD}) \tag{1.1.8}$$

He first builds up a seemingly simplified, but is actually different version of the main theorem:

**Theorem 1.2.** *Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in {}^nF^n$. Let $\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$.*

*If $\mathbf{CD} = \mathbf{DC}$, then,*

$$det_F\mathbf{M} = det_F(\mathbf{AD} - \mathbf{BC}) \tag{1.1.9}$$

and similar results:

$$\text{if } \mathbf{AC} = \mathbf{CA}\text{then, } \det_F\mathbf{M} = \det_F(\mathbf{AD} - \mathbf{CB}) \tag{1.1.10}$$

$$\text{if } \mathbf{BD} = \mathbf{DB}\text{then, } \det_F\mathbf{M} = \det_F(\mathbf{DA} - \mathbf{BC}) \tag{1.1.11}$$

$$\text{if } \mathbf{AB} = \mathbf{BA}\text{then, } \det_F\mathbf{M} = \det_F(\mathbf{DA} - \mathbf{CB}) \tag{1.1.12}$$

These equalities can be proved easily by the following:

$$\begin{pmatrix} D & 0 \\ -C & i \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathrm{AD}-\mathrm{BC} & B \\ \mathrm{CD}-\mathrm{DC} & D \end{pmatrix} = \begin{pmatrix} \mathrm{AD}-\mathrm{BC} & B \\ 0 & D \end{pmatrix} \text{ when } C, D \text{ commutes}$$

$$\begin{pmatrix} D & -B \\ 0 & i \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathrm{DA}-\mathrm{BC} & \mathrm{DB}-\mathrm{BD} \\ C & D \end{pmatrix} = \begin{pmatrix} \mathrm{DA}-\mathrm{BC} & 0 \\ C & D \end{pmatrix} \text{ when } D, B \text{ commutes.}$$

The author also gives an illuminative explanation for why

$$(\det_F \mathbf{M} - \det_F(\mathbf{AD}-\mathbf{BC}))\det_F \mathbf{D} = 0$$

necessarily implies:

$$\det_F \mathbf{M} = \det_F(\mathbf{AD}-\mathbf{BC})$$

## 2 Anchor

## References


[1] D Huybrechts's Introduction to Complex Geometry.

[2] set of almost complex structures on $\mathbb{R}^4$ as two disjoint spheres.

[3] Does $GL(n, \mathbb{C})$ inject into $GL^+(2n, \mathbb{R})$ for all $n$?

[4] John R. Silvester, Determinants of Block Matrices. Available in WebArchive link: `https://web.archive.org/web/20140505161153/http://www.mth.kcl.ac.uk/~jrs/gazette/blocks.pdf`


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