Noetherian Ring

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Abstract

About Noetherian Ring.

Contents

1 Module

(**pp.117 to 118 of [1]**)

Definition 1.1 (Module)**.** *Let A: ring. M is a nomen left A-module if and only if*

1. *M is an abelian group, usually written additively.*
2. *there exists an operation of A on M, written as a multiplicative monoid, such that, for any $a, b \in A$, any $x, y \in M$, we have:*

$$(a+b)x = ax + bx \tag{1.0.1}$$
$$a(x+y) = ax + ay \tag{1.0.2}$$

By definition of an operation, we have $1x = x$. Also, it can be easily derived that $a(-x) = -ax$, and $0x = 0$.

Example 1.1. Examples of modules

1. A is a module over itself.
2. Any commutative group is a $\mathbb{Z}$-module.
3. Any left ideal of A is a module over A, i.e. a left A-module.
4. A vector space V over K, is basically a K-module, with the additional structure of K being a field.

5. Let V be a vector space. Let R be the ring of all linear maps of V into itself. Then V is also a module over R.

Definition 1.2 (Submodule)**.** *A submodule M is an additive subgroup such that $AN \subset N$*

Definition 1.3 (factor module)**.** *Let M be an A-module, and N a submodule. A factor module M/N is the factor group M/N (for the additive group structure) equipped with a module structure. The action of A on M/N is defined by $a(x+N) = ax+N$. This is well defined, since if y is in the same coset as x, then ay is in the same coset as ax.*

(**pp.119 of [1]**)

2 Noetherian

Definition 2.1 (Noetherian Module)**.** *Let A: a ring. M: a left A-module. M is called Noetherian if M satisfies any of the following conditions:*

1. *Every submodule of M is finitely generated.*
2. *Every ascending sequences of submodules of M*

$$M_1 \subset M_2 \subset \cdots$$

   such that $M_i \neq M_{i+1}$, is fininte.

3. *Every non-empty set S of submodules of M has a maximal element.*

The equivalence of the above conditions are proved in page 413 to 414 of [1].

Definition 2.2 (Noetherian Ring)**.** *A ring A is Noetherian if and only if it is Noetherian when viewed as a left module over itself.*

(**pp.415 of [1]**)
The structure of being Noetherian is consistent between a module and its submodules, factor modules, in the sense of the following two propositions.

Proposition 2.1. *Let M be a Noetherian A-module, then every submodule and every factor module of M is Noetherian.*

(**pp.414 of [1]**)

Proposition 2.2. *Let M be a module, N be a submodule. If N and M/N are Noetherian, then M is Noetherian.*

(**pp.414 of [1]**)
The above statements could be summarized by saying that, given an exact sequence:

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

M is Noetherian if and only if M' and M'' are Noetherian. This could be seen by two immediate fact of an exact sequence:

$$M' \cong \mathrm{Im} f,\ \ M/\mathrm{Ker} g \cong M''$$

3 Anchor

References

[1] S. Lang. Algebra. 3rd. Springer.

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