Condensed Matter Field Theory notes

Taper

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Abstract

Notes of book [AS10], and another book [GR96] for information about path integral.

Contents

1 Todo

1. Understand in what case can the Gaussian integral formula can be applied. In another word, understand the analytical continuation of the Gaussian integral. See for example, buttom of pp.343 of [GR96].

`todo:analytical-c`

2. Understand the Wick Rotation, cf. pp.356(ch11.5) of [GR96]. This is related to todo 1
3. Understand how the constant term in the path integral of a Feynman Kernal will(or will not) affect the physics. Understand the mathematical rationale to support this. (cf. buttom of pp.344 of [GR96].
4. I have doubt about the correctness of pp.110 eq 3.28 till pp.111 (Construction recipe of the path integral), especially about his argument, the size of the *Planck cell*.

Bonus objectives:

1. Find abou the similarity between Path Integral of a free particle and the solution to a classical diffusion equation (cf. pp.112, footnote 1 of [AS10]).

2. Those marked *todo* in [AS10].

2 pp.33 eq.1.43

sec:Table

In page 33 of [AS10], the author derives a difference of action, when we have a symmetry transformation paraterized by ω_a:

$$x_\mu \to x'_\mu = x_\mu + \frac{\partial x_\mu}{\omega_a}|_{\omega=0}\omega_a(x) \tag{2.0.1}$$

$$\phi^i(x) \to \phi'(x') = \phi^i(x) + \omega_a(x)F^i_a[\phi] \tag{2.0.2}$$

We have:

$$\mathcal{L} = \mathcal{L}(\phi^i(x), \partial_{x_\mu}\phi^i(x)) \tag{2.0.3}$$

$$\mathcal{L}' = \mathcal{L}'(\phi'^i(x'), \partial_{x'_\mu}\phi'^i(x')) \tag{2.0.4}$$

$$= \mathcal{L}\left(\phi^i + F^i_a\omega_a, \left(\delta_{\mu\nu} - \partial_{x_\mu}(\omega_a\partial_{\omega_a}x_\mu)\right)\partial_{x_\nu}(\phi^i + F^i_a\omega_a)\right) \tag{2.0.5}$$

And

$$\Delta S = \int \mathrm{d}^m x'\, \mathcal{L}' - \int \mathrm{d}^m x\, \mathcal{L} \tag{2.0.6}$$

eq:dS-integrand

$$\begin{aligned} &= \int \mathrm{d}^m x \left(1 + \partial_{x_\mu}\left(\omega_a\partial_{\omega_a}x_\mu\right)\right) \\ &\times \mathcal{L}\left(\phi^i + F^i_a\omega_a, \left(\delta_{\mu\nu} - \partial_{x_\mu}\left(\omega_a\partial_{\omega_a}x_\mu\right)\right)\partial_{x_\nu}(\phi^i + F^i_a\omega_a)\right) \\ &- \int \mathrm{d}^m x\, \mathcal{L}(\phi^i(x), \partial_{x_\mu}\phi^i(x)) \end{aligned} \tag{2.0.7}$$

Then he argues that, "for constant parameters ω_a the action difference Δa vanishes". Therefore "the leading contribution to the action difference of a symmetry transformation must be linear in the derivative $\partial_{x_\mu}\omega_a$".

Then he writes that "A straightforward expansion of the formula above for ΔS shows that these terms are given by"

$$\Delta S = -\int \mathrm{d}^m x\, j^a_\mu(x)\partial_{x_\mu}\omega_a \tag{2.0.8}$$

where j^a_μ is:

$$j^a_\mu = \left(\frac{\partial\mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)}\partial_{x_\nu}\phi^i - \mathcal{L}\delta_{\mu\nu}\right)\frac{\partial x_\nu}{\partial\omega_a} - \frac{\partial\mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)}F^i_a \tag{2.0.9}$$

I am partically confused about how to do the "straightforward expansion". I guess I should do $\frac{\partial}{\partial(\partial_{x_\mu}\omega_a)}$ to the integrand inside expression for ΔS, though I don't really understand the reason. Even so, the integrand contains terms like $\partial_{x_\mu}\partial_{\omega_a}x_\mu$, which I don't know how to deal with.

Solution. The reality is a bit more complicated. We first do a first order expasion to get the infinitesimal difference:

$$
\begin{aligned}
&\mathcal{L}' - \mathcal{L} && (2.0.10)\\
\approx&\frac{\partial \mathcal{L}}{\partial \phi^i} F_a^i \omega_a + \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left[\partial_\mu \left(F_a^i \omega_a\right) - \partial_\mu \left(\omega_a \frac{\partial x_\nu}{\partial \omega_a}\right) \partial_\nu \left(\phi^i + F_a^i \omega_a\right)\right] \\
= \quad & \omega_a \left[\frac{\partial \mathcal{L}}{\partial \phi^i} F_a^i + \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left(\partial_\mu F_a^i - \partial_\mu (\frac{\partial x_\nu}{\partial \omega_a}) \partial_\nu (\phi^i + F_a^i \omega_a)\right)\right] && (2.0.11)\\
+&\partial_\mu \omega_a \left[\frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left(F_a^i - \frac{\partial x_\nu}{\partial \omega_a} \partial_\nu (\phi^i + F_a^i \omega_a)\right)\right] && (2.0.12)
\end{aligned}
$$

eq:l-l-omega

eq:l-l-pmu-omega

We also discover the integrand in Eq.2.0.6 to be

$$
\begin{aligned}
&\left(1 + \partial_\mu (\omega_a \frac{\partial x_\mu}{\partial \omega_a})\right) \mathcal{L}' - \mathcal{L} && (2.0.13)\\
=&\left(1 + \partial_\mu (\omega_a \frac{\partial x_\mu}{\partial \omega_a})\right) (\mathcal{L}' - \mathcal{L}) + \left(\partial_\mu (\omega_a \frac{\partial x_\mu}{\partial \omega_a})\right) \mathcal{L} && (2.0.14)
\end{aligned}
$$

eq:integrand-l-density

For the first term $\left(1 + \partial_\mu (\omega_a \frac{\partial x_\mu}{\partial \omega_a})\right) (\mathcal{L}' - \mathcal{L})$, the $(\mathcal{L}' - \mathcal{L})$ already has terms of first order of ω_a and of first order of $\partial_\nu \omega_a$. For our purpose, the second order terms $(\partial_\nu (F_a^i \omega_a))$ from item 2.0.11 and item 2.0.12 can be ignored. Also, the item $(\partial_\mu (\omega_a \frac{\partial x_\mu}{\partial \omega_a}))(\mathcal{L}' - \mathcal{L})$ in eq.2.0.14 can also be ignored.

Therefore the integrand in Eq.2.0.6 becomes

$$
\begin{aligned}
&(\mathcal{L}' - \mathcal{L}) + \left(\partial_\mu (\omega_a \frac{\partial x_\mu}{\partial \omega_a})\right) \mathcal{L} && (2.0.15)\\
=&\omega_a \left[\frac{\partial \mathcal{L}}{\partial \phi^i} F_a^i + \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left(\partial_\mu F_a^i - (\partial_\mu \frac{\partial x_\nu}{\partial \omega_a}) \partial_\nu (\phi^i + F_a^i \omega_a)\right) + (\partial_\nu \frac{\partial x_\mu}{\partial \omega_a}) \mathcal{L}\right] && (2.0.16)\\
+&\partial_\mu \omega_a \left[\frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left(F_a^i - \frac{\partial x_\nu}{\partial \omega_a} \partial_\nu (\phi^i + F_a^i \omega_a)\right) + \frac{\partial x_\mu}{\partial \omega_a} \mathcal{L}\right] && (2.0.17)
\end{aligned}
$$

Therefore, the term we seek, i.e. the coefficient of $\partial_\mu \omega_a$ is

$$
\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \left(F_a^i - \frac{\partial x_\nu}{\partial \omega_a} \partial_\nu (\phi^i + F_a^i \omega_a)\right) + \frac{\partial x_\mu}{\partial \omega_a} \mathcal{L} && (2.0.18)\\
=&\left(\mathcal{L} \delta_{\mu\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} \partial_\nu \phi^i\right) \frac{\partial x_\nu}{\partial \omega_a} + \frac{\partial \mathcal{L}}{\partial(\partial_{x_\mu}\phi^i)} F_a^i && (2.0.19)
\end{aligned}
$$

which is what we expect in equation 1.43 of [AS10].

Question: as for why we should ignore the term with ω_a, there are two posts ([1], [2]) might be useful for a thought.

confusion

I had great doubt about this problem. Though I have posted an answer on [1], I don't think that answer is satisfactory.

3 About $\dot{q}$ and imaginary time

The $\dot{q}$ in all the path integrals, especially eq.3.6 and eq.3.8, is in fact a shorthand for the divided difference $\frac{q_{n+1}-q_n}{\Delta t}$ as in pp.99 (the buttom). It is not exactly the same as $\dot{q}$. pp.343 of [GR96] also mentioned that in this sense, the Lagrangian in all path integrals is not identical with the ordinary Lagrange function. Though, I still do not know if this matters at all.

In the most common imaginary time transformation, such as those mentioned in pp.106 of [AS10], and pp.358 of [GR96], we have the transformation $t \to -i\tau$. This in effect change all the Δt in, e.g. eq 3.5 (pp.99) of [AS10], to $-i\Delta\tau$. Therefore, the divided difference

$$\frac{q_{n+1}-q_n}{\Delta t} \to \frac{q_{n+1}-q_n}{-i\Delta\tau}$$

Therefore,

$$\begin{aligned} \dot{q} &\to i\partial_\tau q \\ \dot{q}^2 &\to -\partial_\tau^2 q \end{aligned}$$

4 Eq. 3.5

It is not so obvious to get Eq.3.5 in pp.99 of [AS10]. Here is my notes.

According to the book, Eq.3.3 is turned into (I set $\hbar = 1$ occasionally, though sometimes I forgot that I have set $\hbar = 1$, orz):

$$\begin{aligned} \langle q_f| \int \mathrm{d}q_N\, \mathrm{d}p_N\, |q_N\rangle \langle q_N|p_N\rangle \langle p_N| e^{-i\hat{T}\Delta t} e^{-i\hat{V}\Delta t} \times \\ \int \mathrm{d}q_{N-1}\, \mathrm{d}p_{N-1}\, |q_{N-1}\rangle \langle q_{N-1}|p_{N-1}\rangle \langle p_{N-1}| e^{-i\hat{T}\Delta t} e^{-i\hat{V}\Delta t} \times \dots \\ \int \mathrm{d}q_1\, \mathrm{d}p_1\, |q_1\rangle \langle q_1|p_1\rangle \langle p_1| e^{-i\hat{T}\Delta t} e^{-i\hat{V}\Delta t} |q_i\rangle \end{aligned} \tag{4.0.20}$$

Notice that

T has only p, V has only q

$$\langle q|p\rangle = \frac{\exp(iqp/\hbar)}{\sqrt{2\pi\hbar}} \tag{4.0.21}$$

$$\langle p_N| e^{-i\hat{T}\Delta t} = \langle p_N| e^{-iT(p_N)\Delta t} \tag{4.0.22}$$

$$e^{-i\hat{V}\Delta t} |q_{N-1}\rangle = e^{-iV(q_{N-1})\Delta t} |q_{N-1}\rangle \tag{4.0.23}$$

$$\tag{4.0.24}$$

Also,

$$\begin{aligned} &\langle q_N|p_N\rangle \langle p_N| e^{-i\hat{T}\Delta t} e^{-i\hat{V}\Delta t} |q_{N-1}\rangle = \frac{e^{iq_N p_N/\hbar}}{\sqrt{2\pi\hbar}} \langle p_N| e^{-iT(p_N)\Delta t} e^{-iV(q_{N-1})\Delta t} |q_{N-1}\rangle \\ =&\frac{e^{iq_N p_N/\hbar}}{\sqrt{2\pi\hbar}} \langle p_N|q_{N-1}\rangle e^{-iT(p_N)\Delta t} e^{-iV(q_{N-1})\Delta t} = \frac{e^{ip_N(q_N-q_{N-1})/\hbar}}{2\pi\hbar} e^{-i[T(p_N)+V(q_{N-1})]\Delta t} \end{aligned} \tag{4.0.25}$$

etc. Now we have to pay special attentiont to the start and end. For the start, we have a

$$\int \mathrm{d}q_N \, \langle q_f | q_N \rangle = \int \mathrm{d}q_N \, \delta(q_N - q_f)$$

So every q_N is replaced by q_f. For the end, we have

$$\langle q_1 | p_1 \rangle \langle p_1 | \, e^{-i\hat{T}\Delta t} e^{-i\hat{V}\Delta t} \, | q_i \rangle = e^{-i[T(p_1)+V(q_i)]} \frac{e^{ip_1(q_1-q_i)}}{2\pi\hbar}$$

Together we have the whole thing into:

$$\begin{aligned}
&\int \mathrm{d}q_1 \cdots \mathrm{d}q_{N-1} \, \mathrm{d}p_1 \, \mathrm{d}p_N \, \frac{1}{(2\pi\hbar)^N} \times \\
&\quad e^{i\left[p_1(q_1-q_i)+\cdots p_N(q_N-q_{N-1})\right]} \times \\
&\quad e^{-i\left[T(p_1)+\cdots+T(p_N)+V(q_i)+V(q_1)+\cdots+V(q_{N-1})\right]}
\end{aligned} \tag{4.0.26}$$

which is exactly eq.(3.5) in book.

5 Eq 9.4

The Hamiltonian for particle on a ring is claimed to be (Eq. 9.1 of [AS10], pp. 498):

$$H = \frac{1}{2}(-i\partial_\phi - A)^2 = \frac{1}{2}(p - A)^2 \tag{5.0.27}$$

The book [AS10] claims that

$$L = \frac{1}{2}\dot{\phi}^2 - iA\dot{\phi} \tag{5.0.28}$$

I am quite confused, especially about the appearance of $\dot{\phi}$. Can any explain a bit?

How I tried: Since the inverse of a Legendre transformation is Legendre transformation itself,

$$\text{Denote } x \equiv \frac{\partial H}{\partial p} = p - A, \text{ so,} \tag{5.0.29}$$

$$p = x + A, \quad H = \frac{1}{2}x^2, \text{ so,} \tag{5.0.30}$$

$$L = xp - H = x(x + A) - \frac{1}{2}x^2 = \frac{1}{2}x^2 + xA \tag{5.0.31}$$

So my calculation found that the Lagrangian of above Hamiltonian is:

$$L = \frac{1}{2}x^2 + xA \tag{5.0.32}$$

where

$$x = \frac{\partial H}{\partial p} \tag{5.0.33}$$

References

[AS10] Alexander. Altland and Ben BD Ben Simons. *Condensed Matter Field Theory (Second Edition)*. Cambridge University Press, 2010. URL: http://www.cambridge.org/us/academic/subjects/physics/condensed-matter-physics-nanoscience-and-mesoscopic-physics/condensed-matter-field-theory-2nd-edition?format=HB{&}isbn=9780521769754.

[GR96] Walter Greiner and Joachim Reinhardt. *Field Quantization*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1996. URL: http://link.springer.com/10.1007/978-3-642-61485-9, doi:10.1007/978-3-642-61485-9.

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