Solution for HW7

Taper

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Abstract

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1 Prove

$$[x, p^n] = in\hbar p^{n-1}$$

Proof.

$$\begin{aligned}
[x, p^n] &= p[x, p^{n-1}] + [x, p]p^{n-1} = p[x, p^{n-1}] + i\hbar p^{n-1} \\
&= p[x, p^{n-2}] + i\hbar p^{n-1} + i\hbar p^{n-1} \\
&\cdots \\
&= ni\hbar p^{n-1}
\end{aligned}$$

□

$$[p, x^n] = -in\hbar x^{n-1}$$

Proof.

$$\begin{aligned}
[p, x^n] &= x[p, x^{n-1}] + [p, x]x^{n-1} = x[p, x^{n-1}] - i\hbar x^{n-1} \\
&= x[p, x^{n-1}] - i\hbar x^{n-1} - i\hbar x^{n-1} \\
&\cdots \\
&= -in\hbar x^{n-1}
\end{aligned}$$

□

$$[x, f(p)] = i\hbar \frac{\partial f(p)}{\partial p}$$

Proof. Any smooth function of p can be taylor-expanded into a series of the form $\sum_n c_n p^n$, since the commutator is linear and $[x, p^n] = i\hbar n p^{n-1}$, one see the result will be of the form $i\hbar \sum_n c_n n p^{n-1}$, which is exactly the expansion of the taylor expansion of $\frac{\partial f(p)}{\partial p}$. This observation concludes the proof. □

$$[p, g(x)] = -i\hbar \frac{\partial f(x)}{\partial x}$$

Proof. The proof is exactly the same of the one above, if one do a transformation $x \to p$, $p \to x$, and $i \to -i$. $\square$

Problem 2

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & \text{ib} \\ 0 & -\text{ib} & 0 \end{pmatrix} \tag{1.0.1}$$

The eigenvalues of A are obviously $a, -a, -a$. The eigenvalues of B is found by characteristic equations $\mathrm{Det}(B - \lambda \mathbb{1}) = 0$ to be $\{-b, b, b\}$. So B's spectrum is degenerate too.

It can be found that

$$AB = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & -iab \\ 0 & iab & 0 \end{pmatrix}, BA = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & -iab \\ 0 & iab & 0 \end{pmatrix},$$

The eigenvectors of A are easy to guess, they are just the three natural basis $\{e_1, e_2, e_3\}$.

It is easy to find (by solving $BX = \lambda_i X$) B's eigenvectors:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$$

Since both are degenerate, we can linearly combine the degenerate eigenvectors of A to form that of B, but since A's eigenvectors are natural basis, and in the only non-degenerate case, the two matrix share the same eigenvector (notice also that those degenerate eigenvectors of B does not "mess up" with the non-degenerate eigenvector). So we do not have to do anything and those B's eigenvectors will also diagolize A. They are the required eigenvectors.