Sheaf Cohomology

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June 24, 2016

Abstract

Sheaf theory is a powerful tool developed in the 40's of last century and widely used in the area of algebraic geometry and complex geometry . By sheaves and cohomology of sheaves, we can express clearly the obstructions to the problems of going from local solutions to global solutions. This small essay is intended to give a brief outline to the basic sheaf theory presented in the book [1].

Contents

1 Sheafification

For every presheaf $\mathcal{F}$ one can always form a sheaf associated to it. This sheaf is unique up to an sheaf isomorphism. The procedure is presented below. Throughout this section, X is our topological space.

Definition 1.1. *The sheaf $\prod \mathcal{F}$ of a presheaf $\mathcal{F}$ is defined as the sheaf that gives, for any open set $U \subset X$*

$$\Pi(U) = \prod\nolimits_{x \in U} \mathcal{F}_x.$$

The restriction map is obvious and the abelian group structure is pointwise. This obvious constitutes a sheaf.

Definition 1.2. *The sheaf* ***associated*** *with a presheaf $\mathcal{F}$, denoted by $\mathcal{F}^+$, called the* ***sheafification*** *of $\mathcal{F}$, is the sheaf such that,*

$$\mathcal{F}^+(U) = \{(s_u) \in \prod\nolimits_{u \in U} \mathcal{F}_u \text{ such that } (*)\}$$

where $(*)$ is the property:

$(*)$ *For every $x \in U$, there exists an open neighbourhood $x \in V \subset U$, and a section $t \in \mathcal{F}(V)$ such that for all $y \in V$ we have $s(y) = t_y$.*

$\mathcal{F}^+$ is clearly a subset of $\prod \mathcal{F}$. It becomes a sheaf because it naturally inherts the sheaf structure of $\prod \mathcal{F}$, and its definition allows us to look at a section only locally. These meanings are best illustrate in the following example.

Example 1.1. Let $\mathbb{R}$ denotes the pre-sheaf of constant real-valued functions. It is not a sheaf because the any sections f, g of $\mathbb{R}$ on disjoint sets U, V agrees on their intersectin, the empty set. Yet they can not be glued. But the sheafification of it is the sheaf of locally constant functions. In the new sheaf f, g can be glued freely because we only need to look at them locally.

The sheaf $\mathcal{F}^+$ is better than the crude idea $\prod \mathcal{F}$ because of the following lemmata:

Lemma 1.1. $\mathcal{F}_x \cong \mathcal{F}^+_x$

Proof. Observe that an element of $(\prod \mathcal{F})_x$ is characterized by being the same in some local region. An element of $\mathcal{F}^+_x$ is the same as some $t \in \mathcal{F}(V)$ or possibly another $t' \in \mathcal{F}(V')$. Such an ambiguity is removed by $t \sim t'$ in $\mathcal{F}_x$. □

Lemma 1.2. *Let $\mathcal{F}$ be a presheaf and $\mathcal{G}$ be a sheaf. Any map $\phi : \mathcal{F} \to \mathcal{G}$ factors uniquely as $\mathcal{F} \to \mathcal{F}^+ \to \mathcal{G}$.*

Proof. We first construct a diagram:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "f_1"] \arrow[d, "\phi"] & \mathcal{F}^+ \arrow[r, "f_2"] \arrow[d, "j"] & \prod(\mathcal{F}) \arrow[d, "f_3"] \\
\mathcal{G} \arrow[r, "f_4"] & \mathcal{G}^+ \arrow[r, "f_5"] & \prod(\mathcal{G})
\end{tikzcd}$$

The maps f_2 and f_3 are inclusions. f_3 is the map induced by ϕ_x. j is the map defined by f_2 and f_3 and f_5, with the inverse of f_5 defined by projection. f_1 and f_4 are naturally defined. Notice that by lemma 2.1 and lemma 1.1, we have $\mathcal{G} = \mathcal{G}^+$. Hence f_4 is actually an isomorphism. To prove that j is the required map, we need to show that the left part of the above diagram commutes. This is obvious if we careful tract the image of a section $s \in \mathcal{F}(U)$, and notice the fact $(\phi_U(s))_x = \phi_x(s_x)$. □

Note: There are other means of sheafification, which can be found in lemma 4.4 of [1]. The above lemma ensures that all such sheafifications are equivalent.

2 Morphisms of sheaves

Lemma 2.1. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a sheaf homomorphism. Then ϕ is a isomorphism if and only if for any $x \in X$, we have $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism.*

Proof. The necessity is clear.

Conversely, if ϕ_x is an isomorphism for all $x \in X$, $U \subset X$ is open. We proof,

$\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective: For $s \in \mathcal{F}(U)$, if $\phi_U(s) = 0$. Then $(\phi_U(s))_x = \phi_x(s_x) = 0$, i.e. $s_x = 0$. Hence $s = 0$ by the uniqueness of gluing. Thus the kernal is 0 and the map is injective.

$\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective: For $t \in \mathcal{G}(U)$, we can use the representative elements of t_x to find an open covering $\{U_i\}$ of U and $s_i \in F(U_i)$ such that $\phi(s_i) = t|_{U_i}$. Then $\phi(s_i)_{U_i \cap U_j} - \phi(s_j)_{U_i \cap U_j} = 0$. By injective we have $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, hence s_i can be glued into $s \in F(U)$, whose image is clearly t. So ϕ_U is surjective. □

Definition 2.1. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then ϕ is injective (resp. surjective) if for every $x \in X$, the morphism ϕ_x is injective (resp. surjective).*

Remark 2.1. *This definition of injectivity is compatible with lemma 2.1. However, a sheaf morphism ϕ being surjective, does not guarantee that the induced morphism ϕ_U is surjective, as can be seen in the proof of lemma 2.1.*

Lemma 2.2. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a sheaf morphism. Then the presheaf*

$$U \mapsto Ker(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U))$$

is a sheaf, written $Ker(\phi)$. $Ker(\phi) = 0$ if and only if ϕ is injective.

Proof. That $Ker(\phi)$ is a sheaf has already been proved in class.

The sufficiency for the second assumption is already proved in the proof of lemma 2.1. The necessity for the second assumption follows from the observation that $\phi_x(s_x) = (\phi_U(s))_x$. □

On the other hand, the image presheaf $Im(\phi)$ is not necessarily a sheaf. Usualy one has to deal with its sheafification, as in the following lemma.

Lemma 2.3. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves, then the sheaf associated to the presheaf*

$$U \mapsto Im(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U))$$

also written as $Im(\phi)$, is equal to $\mathcal{G}$ if and only if ϕ is surjective.

Proof. The necessity is obvious by lemma 1.1 and lemma 2.1.

Conversely, if ϕ is surjective, By lemma 1.2, there is a unique map $j : Im(\phi) \to \mathcal{G}$. Since j is already injective on the presheaf, j is injective. Now since ϕ_x is surjective, j_x is surjective. If s is a section of $\mathcal{G}$ on U, there thus exists a covering of U by open sets V , and sections t_V of $Im(\phi)$, such that $j(t_V) = s|_V$. As j is injective, the t_V coincide on the intersections, so there exists a section t of $Im(\phi)$ such that $t_V = t_U$. Then $s = j(t)$ and

$$j : Im(\phi)(U) \to \mathcal{G}(U)$$

is surjective. Therefore j is an isomorphism. $\square$

3 Resolution

Definition 3.1. *A resolution of a sheaf $\mathcal{F}$ is a complex $0 \to \mathcal{F}^0 \to \mathcal{F}^1 \to \dots$ together with a homomorphism $\mathcal{F} \to \mathcal{F}^0$ such that*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \mathcal{F}^2 \longrightarrow \cdots$$

is an exact complex of sheaves.

The Čech resolution Let $\mathcal{F}$ be a sheaf over X, and let U_i, $i \in \mathbb{N}$ be a countable covering by open sets of X. For each finite set $I \subset N$, set

$$U_I = \bigcap_{i \in I} U_i.$$

If $V \xrightarrow{j} X$is the inclusion of an open set, then whenever G is a sheaf over V, we define the sheaf $j_*\mathcal{G}$ by the formula

$$j_*\mathcal{G}(U) = \mathcal{G}(V \bigcap U).$$

We also introduce the sheaf $j_*\mathcal{F}$, sometimes written $\mathcal{F}_V$; it is called the restriction of $\mathcal{F}$ to V. To an open set $U \subset V$, this sheaf associates $\mathcal{F}(U)$. For every open set U_I of X, let j_I be the inclusion of U_I in X, and let

$$\mathcal{F}_I := j_{I_*}\mathcal{F}_{U_I}.$$

We then define

$$\mathcal{F}^k = \bigoplus_{|I|=k+1} \mathcal{F}_I$$

and $d : \mathcal{F}^k \to \mathcal{F}^{k+1}$ by the formula

$$(d\sigma)_{j_0,\dots,j_{k+1}} = \sum_i (-1)^i \sigma_{j_0,\dots,\check{j}_i,\dots,j_{k+1}|U \bigcap U_{j_0,\dots,j_{k+1}}}, j_0, < \cdots < j_{k+1}$$

which is valid for $\sigma = (\sigma_I)$,$\sigma_I \in \mathcal{F}_I(U) = \mathcal{F}(U \bigcap U_I)$.We easily check that $d \circ d = 0$. Let us also define $j : \mathcal{F} \to \mathcal{F}^0$ by $j(\sigma)_i = \sigma_{|U \bigcap U_i}$ for $\sigma \in \mathcal{F}(U)$.

Proposition 3.1. *The complex*

$$0 \to \mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{F}^n \xrightarrow{d} \mathcal{F}^{n+1} \cdots \tag{3.0.1}$$

is a resolution of $\mathcal{F}$.

We call this resolution the Čech resolution of $\mathcal{F}$ associated to the covering U_i of X. The functorial nature of this resolution renders it very useful.

Proof. The injectivity of j is due to the property of uniqueness of the sections of F having given restriction to the U_i. The fact that Im j can be identified with the kernel of d on $\mathcal{F}^0$ is exactly equivalent to the fact that sections of F on $U \bigcap U_i$ which coincide on the intersections glue together to form a section of $\mathcal{F}$ on U. The exactness in general can be checked stalk by stalk, as follows. Let $x \in X$, and let i be such that $x \in U_i$. We then define

$$\delta : \mathcal{F}_x^k \to \mathcal{F}_x^{k-1}$$

for $k \geq 1$ by the following formula. An element $\sigma \in \mathcal{F}_x^k$ is represented by a series of germs $\sigma_I \in \mathcal{F}(V_I \bigcap U_I)$ for $|I| = k+1$, where V_I is an open set containing x which we can assume is contained in U_i. We then define $\delta\sigma$ by

$$(\delta\sigma)_{i_0,\ldots,i_{k-1}} = \epsilon\sigma_{i,i_0,\ldots,i_{k-1}}, i_0 < \cdots < i_{k-1} \tag{3.0.2}$$

where ϵ is the signature of the permutation reordering the set $\{i, i_0, \ldots, i_{k-1}\}$. We use the convention that $\sigma_{i,i_0,\ldots,i_{k-1}} = 0$ if $i \in \{i, i_0, \ldots, i_{k-1}\}$. To see that equation 3.0.2 makes sense, we need to see that the right-hand term defines a germ of a section of $j_{i_0,\ldots,i_{k-1}*}\mathcal{F}$ on the neighbourhood of x. But as each V_I is contained in U_i, we have $V_{i,i_0,\ldots,i_{k-1}} \bigcap U_{i_0,\ldots,i_{k-1}} = V_{i,i_0,\ldots,i_{k-1}} \bigcap U_{i,i_0,\ldots,i_{k-1}}$, so that $\sigma_{i,i_0,\ldots,i_{k-1}}$ can be seen as a section of $j_{i_0,\ldots,i_{k-1}*}\mathcal{F}$ on $V_{i,i_0,\ldots,i_{k-1}}$. We immediately check that $d \circ \delta + \delta \circ d = Id$ on $\mathcal{F}_x^k$ for $k \geq 1$. This implies the exactness of the complex 3.0.1 at the point x. □

de Rham resolution Let X be a $\mathcal{C}^\infty$ differentiable manifold. $\mathcal{A}^k$ be the sheaf of $\mathcal{C}^\infty$ differential forms, i.e. the sheaf of sections of the bundle $\Omega^k_{X,\mathbb{R}}$. The exterior differential d is a morphism of sheaves $\mathcal{A}^k$. Notice that the constant sheaf $\mathbb{R}$ is natually included in $\mathcal{A}^0$, and the kernal of $d : \mathcal{A}^0 \to \mathcal{A}^1$ consist precisely of the locally constant functions. Thus we have the following theorem:

Proposition 3.2. *The complex*

$$0 \to \mathcal{A}^0 \to \mathcal{A}^1 \to \ldots \to \mathcal{A}^n \to 0$$

where $n = dimX$ is a resolution of the constant sheaf $\mathbb{R}$.

The Dolbeault resolution Let X be a complex manifold and $E \to X$ be a holomorphic vector bundle. Let $\mathcal{E}$ be the associated sheaf of free $\mathcal{O}_X$-module. Let $\mathcal{A}^{0,q}(E)$ be the sheaf of $\mathcal{C}^\infty$ sections of $\Omega^{p,q} \otimes E$. Then the $\bar{\partial}$ operator defines morphisms $\mathcal{A}^{0,q} \to \mathcal{A}^{0,q}$. Note that the kernal $\bar{\partial} : \mathcal{A}^{0,0} \to \mathcal{A}^{0,1}$ is $\mathcal{E}$. Note also that by $\bar{\partial}$-Poincàre lemma, sections in above is $\bar{\partial}$-closed if and only if it is $\bar{\partial}$-exact. Hence we have

Proposition 3.3. *The complex*

$$0 \to \mathcal{A}^{0,0}(E) \to \mathcal{A}^{0,1}(E) \to \ldots \to \mathcal{A}^{0,n}(E) \to 0$$

where $n = dim_{\mathbb{C}}X$ is a resolution of the sheaf $\mathcal{E}$.

4 Abelian Categories

Definition 4.1. *An abelian category $\mathcal{C}$ is a category satisfying the following conditions:*

- For every pair of objects A, B of $\mathcal{C}$, $\mathrm{Hom}(A, B)$ is an abelian group, and the composition of morphisms

  $$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$$

  is bilinear for these abelian groupstructures.

- Every morphism $\phi : A \to B$ admits a kernel and a cokernel; the kernel of ϕ is an object $\mathcal{C}$ written $\mathrm{Ker}\phi$, equipped with a morphism $\chi : C \to A$, such that for every object M of $\mathcal{C}$, left composition with χ induces an isomorphism

  $$\mathrm{Hom}(M, C) \cong \{\psi \in \mathrm{Hom}(M, A) | \phi \circ \psi = 0\}.$$

  Diagramatically:
  $$\begin{tikzcd}
  A \arrow[r, "\phi"] & B \\
  \mathrm{Ker}\phi = C \arrow[u, "\chi"] & M \arrow[ul, "\psi"'] \arrow[l]
  \end{tikzcd}$$

  Similarly, the cokernel of ϕ is an object D, written $\mathrm{Coker}\phi$, equipped with a morphism $\chi : B \to D$, such that for every object M of C, right composition with χ induces an isomorphism

  $$\mathrm{Hom}(D, M) \cong \{\psi \in \mathrm{Hom}(B, M) | \psi \circ \phi = 0\}$$

  Diagramatically:
  $$\begin{tikzcd}
  A \arrow[d, "\phi"] & M \\
  B \arrow[ur, "\psi"] & D = \mathrm{Coker}\phi \arrow[l, "\chi"'] \arrow[u]
  \end{tikzcd}$$

- A morphism $\phi : A \to B$ is said to be **injective** if $\mathrm{Hom}(\mathrm{Ker}\phi, A) = \{0\}$.

- The image of a morphism ϕ can be defined as the cokernel of its kernel or as the kernel of its cokernel. (*image* in the sense of isomorphism.)

- Direct sums exist; the direct sum $A \otimes B$ is such that for every object M of $\mathcal{C}$, we have

  $$\mathrm{Hom}(M, A \otimes B) = \mathrm{Hom}(M, A) \otimes \mathrm{Hom}(M, B)$$

  $$\mathrm{Hom}(A \otimes B, M) = \mathrm{Hom}(A, M) \otimes \mathrm{Hom}(B, M)$$

The standard examples of abelian categories are the category of abelian groups and their morphisms, and the category of modules over a given ring. If X is a topological space, the category of sheaves of abelian groups or sheaves of $\mathcal{A}$-modules, where $\mathcal{A}$ is a sheaf of rings over X, is also abelian

Example 4.1. Let X be a topological space. The functor Γ from the category of sheaves of abelian groups over X to the category of abelian groups, which to a sheaf F of abelian groups over X associates the group of its global sections $\mathcal{F}(X)$, is left-exact.

Let A, B, C be three objects of $\mathcal{C}$, and let $\phi : A \to B, \psi : B \to C$ be morphisms.

Definition 4.2. *We say that the sequence*

$$0 \to A \overset{\phi}{\to} B \overset{\psi}{\to} C \to 0$$

is a short exact sequence if $A \overset{\phi}{\to} B$ *is isomorphic to the kernel of* ψ *and* $B \overset{\psi}{\to} C$ *is isomorphic to the cokernel of* ϕ.

Note:The kernel Ker ψ of ψ is an object of $\mathcal{C}$ equipped with a morphism $\chi : Ker\psi \to B$. The isomorphism above is an isomorphism $i : A \cong Ker\psi$ such that $\chi \circ i = \phi$. The analogous notion holds for the cokernel, i.e.

$$A \overset{i}{\cong} \mathrm{Ker}\psi \overset{\chi}{\to} B.$$

Definition 4.3. *An object* I *of an abelian category is called injective if for every injective morphism* $A \overset{j}{\to} B$ *and for every morphism* $\phi : A \to I$, *there exists a morphism* $\psi : B \to I$ *such that* $\psi \circ j = \phi$.

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "\phi"] & B \arrow[dl, "\psi"] \\ I & \end{tikzcd}$$

Example 4.2. The injective objects in the category of abelian groups are the divisible groups G, i.e. those such that for every $g \in G$ and every $n \in \mathbb{N}^*$, there exists $g' \in G$ such that $ng' = g$.

Definition 4.4. *The degree* i *cohomology of a complex* $(M^\cdot, d_M)$ *is the object*

$$H^i(M^\cdot) := Coker\left(d_M^{i-1} : M^{i-1} \to Kerd_M^i\right)$$

Definition 4.5. *A homotopy* H *between two morphisms of complexes* $\phi^\cdot : (M^\cdot, d_M) \to (N^\cdot, d_N)$ *and* $\psi^\cdot : (M^\cdot, d_M) \to (N^\cdot, d_N)$ *is a collection of morphisms*

$$H^\cdot : M^\cdot \to N^{\cdot - 1}$$

satisfying

$$H^{i+1} \circ d_M^i + d_N^{i-1} \circ H^i = \phi^i - \psi^i, \forall i \geq 0. \qquad (4.0.3)$$

If there exists a homotopy between two morphisms of complexes $\phi^\cdot : (M^\cdot, d_M) \to (N^\cdot, d_N)$ and $\psi^\cdot : (M^\cdot, d_M) \to (N^\cdot, d_N)$, then the induced morphisms $H^i(\phi^\cdot)$ and $H^i(\psi^\cdot)$ are equal. Indeed, relation 4.0.3 shows that $\phi^i \to \psi^i : Kerd_M^i \to N^i$ factors through d_N^{i-1}, and thus induces 0 in Hom(Ker d_M^i,Coker d_N^{i-1}).

Definition 4.6. *A complex $M^i, i \geq 0$ is called a resolution of an object A of $\mathcal{C}$ if $Im\ d^i = Ker\ d^{i+1}$, and there exist an injective morphism $j : A \to M^0$ such that $j : A \to M^0$ is isomorphic to $Ker\ d^0$.*

Definition 4.7. *An abelian category $\mathcal{C}$ has sufficently many injective objects if every object A of $\mathcal{C}$ admits an injective morphism $j : A \to I$, where I is injective.*

Lemma 4.1. *If $\mathcal{C}$ has sufficently many injective objects, then every object of $\mathcal{C}$ admits an injective resolution, i.e. a resolution $I^{\cdot}$ by a complex all of whose objects are injective objects.*

Proof. See Lemma 4.26 of book [1]. □

Proposition 4.1. *Let $I^{\cdot}$, $A \xrightarrow{i} I^0$, and $J^{\cdot}$, $B \xrightarrow{j} J^0$ be resolutions of A, B respectively, and let $\phi : A \to B$ be a morphism. Then if the second resolution is injective, there exists a morphism of complexes $\phi^{\cdot} : I^{\cdot} \to J^{\cdot}$ satisfying $\phi^0 \circ i = j \circ \phi$. Moreover, if we have two such morphisms $\phi^{\cdot}$ and $\psi^{\cdot}$, there exists a homotopy $H^{\cdot}$ between $\phi^{\cdot}$ and $\psi^{\cdot}$.*

Diagramatically:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\phi"] & I^0 \arrow[r, "i"] \arrow[d, "\phi^0"] & I^1 \arrow[r] \arrow[d] & ... \\ B \arrow[r, "j"] & J^0 \arrow[r, "j"] & J^1 \arrow[r] & ... \end{tikzcd}$$

Proof. See Proposition 4.27 of book [1]. □

In particular, appying this proposition to the case where $I^{\cdot}$ and $J^{\cdot}$ are two injective resolution of A, we obtain morphism $\phi^{\cdot} : I^{\cdot} \to J^{\cdot}$ and $\psi^{\cdot} : J^{\cdot} \to I^{\cdot}$ such that $\psi^{\cdot} \circ \phi^{\cdot}$ and $\phi^{\cdot} \circ \psi^{\cdot}$ are morphisms of complexes (from $I^{\cdot}$ to itself and from $J^{\cdot}$ to itself respectively), which are both homotopic to the identity. We then say that $\phi^{\cdot}$ is a homotopy equivalence. Thus, we see that an injective resolution is unique up to homotopy equivalence.

5 Derived functors

Let $\mathcal{C}$ and $\mathcal{C}'$ be two abelian categories, and let F be a left-exact functor from $\mathcal{C}$ to $\mathcal{C}'$. Assume that $\mathcal{C}$ has sufficiently many injective objects.

Theorem 5.1. *For every object M of $\mathcal{C}$, there exist objects $R^iF(M)$, $i \geq 0$ in $\mathcal{C}'$, determined up to isomorphism , satisfying the following conditions*

- *We have $R^0F(M) = F(M)$.*
- *For every short exact sequence*

$$0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$$

in $\mathcal{C}$, we can construct a long exact sequence (i.e. an exact complex) in $\mathcal{C}'$:

$$0 \to F(A) \xrightarrow{\phi} F(B) \xrightarrow{\psi} F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to \cdots .$$

- *For every injective object I of $\mathcal{C}$, we have $R^iF(I) = 0$, $i > 0$.*

Proof. See Theorem 4.28 of book [1]. $\square$

Lemma 5.1. *If*

$$0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$$

is a short exact sequence in $\mathcal{C}$, then there exist injective resolutions $I^\cdot$, $J^\cdot$, $K^\cdot$ of A, B, C respectively, and an exact sequence of complexes

$$0 \to I^\cdot \xrightarrow{\phi^\cdot} J^\cdot \xrightarrow{\psi^\cdot} K^\cdot \to 0$$

with $\phi^0 \circ i = j \circ \phi$, $\psi^0 \circ j = k \circ \psi$.

Proof. See Lemma 4.29 of book [1]. $\square$

Proposition 5.1. *If $\phi : A \to B$ is a morphism in $\mathcal{C}$, and $I^\cdot$, $J^\cdot$ are injective resolutions of A and B respectively, then there exists a canonical morphism induced by ϕ,*

$$R^iF(\phi) : R^iF(A) \to R^iF(B)$$

where the derived objects are computed using the chosen resolutions.

Proof. See Proposition 4.30 of book [1]. $\square$

In practice, injective resolutions are difficult to manipulate. The following result shows how to replace injective resolutions by resolutions satisfying a weaker condition.

Definition 5.1 (F-acyclic object)**.** *We say that an object M of $\mathcal{C}$ is acyclic for the functor F (or F-acyclic) if we have $R^iF(M) = 0$ for all $i > 0$.*

Proposition 5.2. *Let $M^\cdot$, $i : A \to M^0$ be a resolution of A, where the M^i are acyclic for the functor F. Then $R^iF(A)$ is equal to the cohomology $H^i(F(M^\cdot))$ of the complex $F(M^\cdot)$.*

Proof. See Proposition 4.32 of book [1]. $\square$

6 Sheaf cohomology

From now on, we consider the category of sheaves of abelian groups over a topological space X, and the functor Γ of "global sections" which to $\mathcal{F}$ associates$\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$, with values in the category of abelian groups.

Lemma 6.1. *The category of sheaves of abelian groups has sufficiently many injective objects.*

Proof. See section 4.3 of book [1]. $\square$

Definition 6.1 (flasque sheaf)**.** *A sheaf $\mathcal{F}$ is said to be flasque if for every pair of open sets $V \subset U$, the restriction map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.*

Proposition 6.1. *Flasque sheaves are acyclic for the functor Γ .*

Proof. See Proposition 4.34 of book [1]. $\square$

Lemma 6.2 (Godement resolution)**.** *The Godement resolution of $\mathcal{F}$ is constructed by considering the inclusion of $\mathcal{F}$ into the sheaf $\mathcal{F}_{God}$*

$$U \mapsto \mathcal{F}_{God}(U) = \bigoplus_{x \in U} \mathcal{F}_x$$

where the sum is actually the infinite direct product. This sheaf is obviously flasque. We then inject the quotient $\mathcal{F}_{God}/F$ into the flasque sheaf $(\mathcal{F}_{God}/\mathcal{F})_{God}$, and so on.

Proof. See page 103 of book [1]. $\square$

Definition 6.2 (fine sheaf)**.** *A fine sheaf $\mathcal{F}$ over X is a sheaf of $\mathcal{A}$-modules, where $\mathcal{A}$ is a sheaf of rings over X satisfying the property: For every open cover U_i ,$i \in I$ of X , there exists a partition of unity f_i ,$i \in I$, $\Sigma f_i = 1$ (where the sum is locally finite), subordinate to this covering.*

Proposition 6.2. *If $\mathcal{F}$ is a fine sheaf , we have $H^i(X, \mathcal{F}) = 0$, $\forall i > 0$.*

Proof. See Proposition 4.36 of book [1]. $\square$

Corollary 6.1. *Let X be a $\mathcal{C}^\infty$ manifold. Then*

$$H^k(X, \mathbb{R}) = Ker(d : A^k(X) \to A^{k+1}(X))/Im(d : A^{k-1}(X) \to A^k(X))$$

where $A^i(X)$ is the real vector space of differential forms of degree i. A similar statement holds for the complex cohomology.

Proof. See Corollary 4.37 of book [1]. $\square$

Corollary 6.2. *Let E be a holomorphic vector bundle over a complex manifold X, and let $\mathcal{E}$ be the sheaf of holomorphic sections of E. Then*

$$H^q(X, \mathcal{E}) = Ker(\bar{\partial} : A^{0,q}(E) \to A^{0,q+1}(E))/Im(\bar{\partial} : A^{0,q-1}(E) \to A^{0,q}(E))$$

Proof. See Corollary 4.38 of book [1]. $\square$

Corollary 6.3. *If E is as above , we have $H^q(X, E) = 0$ for $q > n = dimX$.*

Proof. See Corollary 4.39 of book [1]. $\square$

We will now introduce the Čech cohomology, which is extremely useful in practice, since it gives a uniform way of computing cohomology groups, unlike the de Rham type resolutions, which specifically concern constant sheaves over manifolds. Let $\mathcal{F}$ be a sheaf of abelian groups over a topological space X. Let $U = (U_i)_{i \in \mathbb{N}}$ be a countable ordered open covering of X.

Definition 6.3. *Define $\check{H}^q(\mathcal{U}, \mathcal{F})$ to be the qth cohomology group of the complex of global sections*

$$C^q(\mathcal{U}, \mathcal{F}) = \bigoplus_{|I|=q+1} \mathcal{F}(U_I)$$

of the Čech complex associated to the covering $\mathcal{U}$.

Theorem 6.1. *If the open sets* $U_I = \bigcap_{i \in I} U_i$ *satisfy* $H^q(U_I, \mathcal{F}) = 0$ *for all* $q > 0$*, then*

$$H^q(X, \mathcal{F}) = \check{H}^q(\mathcal{U}, \mathcal{F}), \forall q \geq 0$$

Proof. See Theorem 4.41 of book [1]. □

Theorem 6.2. *If* X *is separable, then by passage to the direct limit, the morphisms*

$$\check{H}^q(\mathcal{U}, \mathcal{F}) \to H^q(X, \mathcal{F})$$

induce an isomorphism

$$\lim_{\vec{\mathcal{U}}} \check{H}^q(\mathcal{U}, \mathcal{F}) \cong H^q(X, \mathcal{F})$$

Proof. See Theorem 4.44 of book [1]. □

References

[1] C. Voisin. Hodge Theory and Complex Algebraic Geometry I, Cambridge University Press.

[2] Daniel Huybrechts. Complex Geometry - An Introduction, Springer.