Solution for HW5 20161102

Taper

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Abstract

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1 Uncertainty of electron velocity

2 Proof

Proof. Assuming an orthonormal basis labeled by n: $|n\rangle$. Using Einstein Summation convention, one finds:

$$\begin{aligned}\mathrm{Tr}(AB) &= \langle n|AB|n\rangle = \langle n|A|m\rangle \langle m|B|n\rangle = \langle m|B|n\rangle \langle n|A|m\rangle \\ &= \langle m|BA|m\rangle = \mathrm{Tr}(BA)\end{aligned}$$

Hence

$$\mathrm{Tr}(XYZ) = \mathrm{Tr}((XY)Z) = \mathrm{Tr}(Z(XY)) = \mathrm{Tr}((ZX)Y) = \mathrm{Tr}(YZX)$$

□

3 Proof

Proof.

$$\langle [A,B]\rangle = \langle AB - BA\rangle = \langle AB\rangle - \langle BA\rangle = \langle AB\rangle - \langle AB\rangle^* = 2i\,\mathrm{Im}(\langle AB\rangle)$$

Hence it is imaginary or zero. Similar, by replacing the $-$ sign above with $+$ sign, one easily finds:

$$\langle \{A,B\}\rangle = 2\,\mathrm{Re}(\langle AB\rangle)$$

So it is real. □

4 Diagonalization

A is real and symmetric, hence it is diagonalizable:

$$\det\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix} = \det\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 0 & -14+3\lambda & -\lambda-2 \end{pmatrix}$$

$$= (1-\lambda)\left[(5-\lambda)(-\lambda-2)+14-3\lambda\right] - (-10\lambda+4) = -\lambda^3+7\lambda^2-36$$

The roots are $\lambda_1 = -2$, $\lambda_2 = 3$, $\lambda_3 = 6$. For $\lambda = -2$, we have

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Or

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Or

$$\begin{pmatrix} 0 & -20 & 0 \\ 1 & 7 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Obviously the corresponding eigenvector is $\alpha\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, where α is any nonzero complex number. The case for $\lambda = 3$ and $\lambda = 6$ can be similar solved by examing the following two equations:

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0, \quad \begin{pmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

and the result is summarized as

Eigenvalue	*Eigenvector*
-2	$\alpha\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$
3	$\beta\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$
6	$\gamma\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

where α, β, γ are arbitrary nonzero complex numbers. Therefore, take the three eigenvector as basis (to get an orthonormal basis, we can let $\alpha = \frac{1}{\sqrt{2}}$, $\beta = \frac{1}{\sqrt{3}}$, $\gamma = \frac{1}{\sqrt{6}}$.), we have:

$$X^{-1}AX = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

where

$$X = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & -\beta & 2\gamma \\ \alpha & \beta & \gamma \end{pmatrix}$$