Solution to Problem Set 4

P1) In order to map out the eigenvectors of the spatial part of spin operators to spherical coordinate system, we can use Bloch sphere. Lets do this in general

$$\hat{H} = -\gamma \vec{B}\cdot\hat{S} = E_0 \, \hat{\sigma}\cdot\hat{n}$$

where $\hat{S} = \frac{\hbar}{2}\hat{\sigma}$ & $\vec{B} = B_0 \hat{n}$

$$n = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$\hat{\sigma} = \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}$$

$$\hat{\sigma}\cdot\hat{n} = \sigma_x \sin\theta\cos\varphi + \sigma_y \sin\theta\sin\phi + \sigma_z \cos\theta$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad . \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{\sigma}\cdot\hat{n} = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}$$

Lets call $\begin{pmatrix} a \\ b \end{pmatrix}$ eigenfunctions of H with eigenvalues of λ, then

$$H\begin{pmatrix} a' \\ b' \end{pmatrix} = \lambda \begin{pmatrix} a' \\ b' \end{pmatrix} \quad \text{or} \quad \hat{\sigma}\cdot\hat{n}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix} \Longrightarrow$$

$$\Rightarrow \begin{cases} a(\cos\theta - \lambda) + b e^{-i\varphi}\sin\theta = 0 \\ a e^{i\varphi}\sin\theta - b(\lambda + \cos\theta) = 0 \end{cases} \Longrightarrow$$

$$\begin{vmatrix} \cos\theta-\lambda & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -(\lambda+\cos\theta) \end{vmatrix} = 0 \rightarrow \lambda^2=1 \rightarrow \boxed{\lambda=\pm 1}$$

For $\lambda=1 \longrightarrow |n\uparrow\rangle = \begin{pmatrix} \cos\theta/2 \\ e^{i\varphi}\sin\theta/2 \end{pmatrix}$

For $\lambda=-1 \longrightarrow |n\downarrow\rangle = \begin{pmatrix} -e^{-i\varphi}\sin\theta/2 \\ \cos\theta \end{pmatrix}$

For the problem since $\hat{n}=\hat{z} \longrightarrow \begin{cases} \varphi=0 \\ \theta=\theta \end{cases}$

If the object going to second S-G apparatus have spin up $\longrightarrow$

$|z\uparrow\rangle = \alpha|n\uparrow\rangle + \beta|n\downarrow\rangle$

$\alpha = \langle n\uparrow|z\uparrow\rangle = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)\begin{pmatrix}1\\0\end{pmatrix} = \cos\frac{\theta}{2}$

$\beta = \langle n\downarrow|z\uparrow\rangle = \left(-\sin\theta/2, \cos\frac{\theta}{2}\right)\begin{pmatrix}1\\0\end{pmatrix} = -\sin\frac{\theta}{2}$

therefore the ratio of the numbers of the two beams leaving the S-G will be $\boxed{\frac{|\alpha|^2}{|\beta|^2} = \cot^2\frac{\theta}{2}}$

If the object going to second S-G have spin down:

$|z\downarrow\rangle = \alpha|n\uparrow\rangle + \beta|n\downarrow\rangle \longrightarrow$

$\alpha = \sin\frac{\theta}{2}$
$\beta = \cos\frac{\theta}{2}$ $\longrightarrow$ $\boxed{\frac{|\alpha|^2}{|\beta|^2} = \tan^2\frac{\theta}{2}}$

Problem 2)

$$V(r) = \begin{cases} U & r<R \\ 0 & r>R \end{cases}$$

a)
The interaction term in classical Schrödinger equation is : $V(r)\Psi(r)$

By generalization to field operators we have $\hat{\Psi}_\sigma$, then interaction will be between pairs of object

$$\hat{V}(r) = U \sum_{\sigma,\sigma'} \int d^3x \int_{|x-x'|<R} d^3x' \frac{1}{2}\left[\Psi^\dagger_\sigma(x)\, \Psi^\dagger_{\sigma'}(x')\, \Psi_{\sigma'}(x')\, \Psi_\sigma(x)\right]$$

b) $$\hat{C}_{\vec{k}\sigma} = \int d^3x\, \Psi_\sigma(x)\, e^{-i\vec{k}\cdot\vec{x}}$$

$$[\hat{C}_{\vec{k}\sigma}, \hat{C}^\dagger_{\vec{k}'\sigma'}] = \int d^3x\, d^3x' \left[\Psi_\sigma(x), \Psi^\dagger_{\sigma'}(x')\right]_{\mp} e^{-i(\vec{k}\cdot\vec{x}-\vec{k}'\vec{x}')}$$

$$= \delta_{\sigma\sigma'} \int d^3x\, e^{-i(\vec{k}-\vec{k}')\vec{x}}$$

$$= \delta_{\sigma\sigma'} (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

$$V = \frac{1}{2}\int \frac{d^3k\, d^3k'}{(2\pi)^9} d^3q\, V(q) \left[\hat{C}^\dagger_{\vec{k}+q,\sigma}\, \hat{C}^\dagger_{k'-q,\sigma'}\, , \hat{C}_{k'\sigma'}\, \hat{C}_{\vec{k}\sigma}\right]$$

where $$V(\vec{q}) = \int d^3x\, V(x)\, e^{i\vec{q}\cdot\vec{x}}$$

P3)

a) $J_p = Re\left\{ \Psi^* \left(-i\frac{\hbar}{m}\nabla - \frac{q}{m}\vec{A}\right)\Psi \right\}$

If $\Psi = \sqrt{n}\, e^{i\theta} \Rightarrow \vec{\nabla}\Psi = \left(\frac{1}{2n}\nabla n + i\nabla\theta\right)\Psi$

$$\Rightarrow \Psi^*\left(\frac{-i\hbar}{m}\nabla - \frac{q}{m}\vec{A}\right)\Psi = \Psi^*\left[\frac{-i\hbar}{2mn}\nabla n + \hbar\nabla\theta\right]\Psi$$

$$= \sqrt{n}\, e^{-i\theta}\left[\frac{-i\hbar}{2mn}\nabla n + \frac{\hbar\nabla\theta}{m} - \frac{q}{m}A\right]\sqrt{n}\, e^{i\theta}$$

$$J_p = n\left(\frac{\hbar}{m}\nabla\theta - \frac{q}{m}\vec{A}\right) \Rightarrow$$

$$\boxed{J = q\, J_p = q\, n(r,t)\left[\frac{\hbar}{m}\nabla\theta(r,t) - \frac{q}{m}\vec{A}(r,t)\right]}$$

b) The Schrodinger eq. in the presence of EM field is:

$$i\hbar\frac{\partial}{\partial t}\Psi = \frac{1}{2m}\left(-i\hbar\nabla - qA\right)^2\Psi + q\Phi\Psi \quad (b\text{-}1)$$

Choosing the Coulomb gauge, i.e. $\vec{\nabla}.\vec{A} = 0$ & $\Phi = 0$

(b1) reduces to:

$$i\hbar\frac{\partial}{\partial t}\Psi = \frac{-\hbar^2}{2m}\nabla^2\Psi + i\frac{\hbar q}{m}\vec{A}.\vec{\nabla}\Psi + \frac{q^2}{2m}A^2\Psi \quad (b\text{-}2)$$

Given $\Psi = \sqrt{n}\, e^{i\theta}$, we calculate each term:

$$i\hbar \frac{\partial}{\partial t}\Psi = i\hbar\left(\frac{1}{2n}\frac{\partial n}{\partial t} + i\frac{\partial \theta}{\partial t}\right)\Psi \triangleq W_0 \Psi \quad (b\text{-}3)$$

$$\nabla^2 \Psi = \nabla\left[\left(\frac{\nabla n}{2n} + i\nabla\theta\right)\Psi\right] =$$

$$\left(\frac{n.\nabla^2 n - (\nabla n)^2}{2n^2} + i\nabla^2\theta + \left(\frac{\nabla n}{2n}\right)^2 - (\nabla\theta)^2 + \frac{\nabla n.\nabla\theta}{n}\right)\Psi$$

$$\triangleq W_1 \Psi \quad (b\text{-}4)$$

$$i\hbar \frac{q}{m}\vec{A}.\nabla\Psi = \frac{i\hbar q}{m}\vec{A}\left(\frac{\nabla n}{2n} + i\nabla\theta\right)\Psi \triangleq W_2\Psi \quad (b\text{-}5)$$

Plugging (b-3), (b-4) & (b-5) into (b-2), yields:

$$W_0 = -\frac{\hbar^2}{2m}W_1 + W_2 + \frac{q^2A^2}{2m} \quad (b\text{-}6)$$

Taking imaginary part from (b-6), we arrive to

$$\frac{\hbar}{2n}\frac{\partial n}{\partial t} = \frac{-\hbar^2}{2m}\left(\nabla^2\theta + \frac{\nabla n.\nabla\theta}{n}\right) + \frac{\hbar q}{m}\vec{A}\,\frac{\nabla n}{2n} \quad (b\text{-}7)$$

Multiplying (b-7) by $\frac{2nq}{\hbar}$, we have:

$$\frac{\partial}{\partial t}(nq) = \frac{-\hbar q}{m}\underbrace{\left[n\nabla^2\theta + \nabla n\nabla\theta\right]}_{\nabla(n.\nabla\theta)} + \frac{q^2}{m}\vec{A}.\vec{\nabla} n \quad (b\text{-}8)$$

$\Rightarrow$ if $\rho = q\,\Psi^*\Psi = q\,n \longrightarrow$

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\left[nq\left(\frac{\hbar}{m}\vec{\nabla}\theta - \frac{q}{m}\vec{A}\right)\right] = -\vec{\nabla}\cdot\vec{J} \Rightarrow$$

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

c) $|\Psi(r,t)|^2 = n(r,t)$ is the number density or the number of charge carriers per unit volume.

d, e)

Using the result in part (a), we get:

$$J = n^* q^* \left(\frac{\hbar}{m^*}\nabla\theta - \frac{q^*}{m^*}\vec{A}\right) \Rightarrow$$

$$\frac{m^*}{n^*(q^*)^2} J = \Lambda J = \frac{\hbar}{q^*}\vec{\nabla}\theta - \vec{A} \qquad (e\text{-}1) \Rightarrow$$

$$\vec{\nabla}\times(\Lambda J) = \vec{\nabla}\times\left(\frac{\hbar}{q^*}\vec{\nabla}\theta - \vec{A}\right) = -\vec{\nabla}\times\vec{A}$$

since $\vec{\nabla}\times\vec{\nabla}\theta = 0 \Rightarrow \vec{\nabla}\times(\Lambda J) = -\vec{B}$

f) Now if we take $\frac{\partial}{\partial t}$ from $\vec{\nabla}\times(\Lambda J) = -\vec{B}$

we have:

$$\vec{\nabla}\times\left(\frac{\partial}{\partial t}\Lambda J\right) = -\frac{\partial}{\partial t}\vec{B} = \vec{\nabla}\times\vec{E} \Rightarrow$$

$$\boxed{E = \frac{\partial}{\partial t}\Lambda J}$$

g) Consider a superconducting sample. Using eq. (e-1), we integrate it over a closed contour C within the superconducting sample, as:

$$\oint_C \Lambda\vec{J}\cdot\vec{dl} + \oint \vec{A}\cdot\vec{dl} = \frac{\hbar}{q^*}\oint \nabla\theta\cdot dl \qquad (g\text{-}1)$$

Applying the Green's identity to the second integral in the left:

$$\oint_C \Lambda\vec{J}\cdot\vec{dl} + \iint_S \vec{\nabla}\times\vec{A}\cdot\vec{ds} = \frac{\hbar}{q^*}\theta\Big|_{a^-}^{a^+} \qquad (g\text{-}2)$$

It is required that the Ψ function be single-valued in the space, hence $\theta\Big|_{a^-}^{a^+} = 2\pi n \qquad n\in\pm\mathbb{N} \Rightarrow$

$$\oint \Lambda\vec{J}\cdot\vec{dl} + \int_S \vec{B}\cdot\vec{ds} = \frac{2\pi n\hbar}{q^*} = n\frac{h}{q^*} = n\frac{h}{2e} = n\Phi_0$$

h)

We start from the coupled Schrodinger equations and the given form for Ψ_L & Ψ_R :

$$i\hbar\left(\frac{\partial n_L^*}{\partial t}\cdot\frac{1}{2n_L^*}+i\frac{\partial\theta_L}{\partial t}\right)\Psi_L = E_L\Psi_L + K\Psi_R \quad (h\text{-}1)$$

$$i\hbar\left(\frac{\partial n_R^*}{\partial t}\cdot\frac{1}{2n_R^*}+i\frac{\partial\theta_R}{\partial t}\right)\Psi_R = E_R\Psi_R + K\Psi_L \quad (h\text{-}2)$$

i)

Dividing (h-1) & (h-2), by Ψ_L & Ψ_R, respectively:

$$\frac{i\hbar}{2n_L^*}\frac{\partial n_L^*}{\partial t} - \hbar\frac{\partial\theta_L}{\partial t} = E_L + Ke^{i\theta} \quad (h\text{-}3)$$

$$\frac{i\hbar}{2n_R^*}\frac{\partial n_R^*}{\partial t} - \hbar\frac{\partial\theta_R}{\partial t} = E_R + Ke^{i\theta} \quad (h\text{-}4)$$

Taking real part from (h-3) & (h-4) and subtracting from each other, we get:

$$\hbar\frac{\partial\theta}{\partial t} = E_L - E_R = eV \Rightarrow \boxed{\frac{\partial\theta}{\partial t} = \frac{eV}{\hbar}} \quad (h\text{-}5)$$

We do the same with the imaginary parts. Note that while $n_L^* = n_R^*$, their time-derivative are not equal, but rather $\frac{\partial n_L^*}{\partial t} = -\frac{\partial n_R^*}{\partial t}$, since the extraction of one electron from one side means adding one electron to the other side.

Therefore:

$$\frac{\hbar}{2n^*}\frac{\partial n^*}{\partial t} = K \sin\theta \implies \boxed{J = J_c \sin\theta}$$

where $J_c \triangleq \frac{2n^* q^*}{\hbar} K$

In fact the source compensates for excess carriers produced by the term $\frac{\partial n^*}{\partial t}$ in order to make n^* fixed.

j) If $V_0 = 0 \implies \frac{\partial \theta}{\partial t} = 0 \longrightarrow \theta = \theta_0 \implies$

$J = J_c \sin\theta_0$

A constant current flows even in the absence of voltage!

k) If $V = V_0 \longrightarrow \frac{\partial \theta}{\partial t} = \frac{eV_0}{\hbar} \implies$

$\theta = \frac{eV_0}{\hbar} t + \theta_0 \longrightarrow \boxed{J(t) = J_c \sin\left(\frac{eV_0}{\hbar} t + \theta_0\right)}$

A pure sinusoidal wave is generated by a constant bias!