1 Majorization

Next week we'll talk about entanglement. This week we'll talk about Nielsen's theorem.

- Doubly-stochastic operators

Consider some alphabet Σ and some linear mapping $L(\mathcal{R})^{\Sigma}$. This linear mapping being identified with rows and columns indexed by Σ.

An operator $A \in L(\mathcal{R}^{\Sigma})$ is called "doubly stochastic" if the following hold:

- $A(a,b) \in [0,1]$ for all $a, b \in \Sigma$
- $\sum\limits_{a \in \Sigma} A(a,b) = 1$ for every $b \in \Sigma$
- $\sum\limits_{b \in \Sigma} A(a,b) = 1$ for every $a \in \Sigma$

Ex: Suppose $\Pi \in Sym(\Sigma)$. Now, define $V_\Pi(a,b) = 1$ if $a = \Pi(b)$ and equals 0 otherwise. This describes the class of permutation operators, e.g. :

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

. Also, for $p \in P(Sym(\Sigma))$, $A = \sum\limits_{\Pi \in Sym(\Sigma)} p(\pi) V_\Pi$ is doubly stochastic (by convexity).

There exist a theorem (Birkhoff-von Neumann theorem): An operator $A \in L(\mathbb{R}^{\Sigma})$ is doubly stochastic if and only if $A = \sum\limits_{\pi \in Sym(\Sigma)} p(\pi) V_\pi$ for some choice of $p \in P(Sym(\Sigma))$.

2 Majorization for real vectors

Definition: For vectors $u, v\ in \mathbb{R}^{\Sigma}$ we say that u majorizes v written as $u \succ v$ ($v \prec u$) if and only if there exists a doubly sthocastic $A \in L(\mathbb{R}^{\Sigma}$ such that $v = Au$.

Notation: $u \in \mathbb{R}^{\Sigma}$ and $n = |\Sigma|$. Define $r(u) = (r_1(u), \ldots, r_2(u))$ to be the unique vector such that

- $r_1(u) \geq \cdots \geq r_n(u)$
- $r_1(u), \ldots, r_n(u) = u(a) : a \in \Sigma$ (as multisets)

Consider the following theorem where $u, v \in \mathbb{R}^{\Sigma}$. The following are equivalent:

1. $u \succ v$ (i.e., $v = Au$ for A doubly stochastic)
2. For every $k \in 1, \ldots, n$ it holds that

$$\sum_{j=1}^{k} r_j(u) \geq \sum_{j=1}^{k} r_j(v)$$

   and also that $\sum_{j=1}^{n} r_j(u) = \sum_{j=1}^{n} r_j(v)$.

3. It holds that $v = Bu$ for $B \in L(\mathbb{R})^{\Sigma}$ given by $B(a,b) = |U(a,b)|^2$ for some unitary $U \in U(\mathbb{C})^{\Sigma}$.

We can easily show 1 from 2 by using the Birkhoff-von Neumann theorem by writing each doubly stochastic operator as a convex combination of permutation operators.

Going from 2 to 3 is difficult. You can show it by performing induction on n.

Going from 3 to 1 is trivial because if 3 holds then you can just let A = B.

This is majorization for real vectors. We can also discuss majorization in the context of Hermitian operators.

3 Majorization of Hermitian operators

Definition: A channel $\Phi \in C(\mathcal{X})$ is a mixed-unitary channel if and only if there exists an alphabet Σ, a probability vector $p \in P(\Sigma)$ and a collection of unitary operators $U_a : A \in \Sigma$ such that $\Phi(X) = \sum_{a \in \Sigma} p(a) U_a X U_a^*$ (for all $X \in L(\mathcal{X})$.

Definition: Let's suppose that $A, B \in Herm(\mathcal{X})$. We say that A majorizes B, $A \succ B$ or $B \prec A$, if only if there exists a mixed unitary channel $\Phi \in C(\mathcal{X})$ such that $B = \Phi(A)$.

Theorem: Let's suppose that we have $A, B \in Herm(\mathcal{X})$. It holds that $A \succ B$ if and only if their vectors of eigenvalues have the following relationship: $\lambda(A) \succ \lambda(B)$. Let's prove this theorem.

Proof of this theorem: Let $n = dim(\mathcal{X})$. Consider the spectral decompositions of two operators A and B as $A = \sum_{k=1}^{n} \lambda_k(A) v_k v_k^*$ and $B = \sum_{k=1}^{n} \lambda_k(B) u_k u_k^*$.

Right to Left

Assume, first, that $\lambda(A) \succ \lambda(B)$. It follows that

$$\lambda_j(B) = \sum_{\pi \in S_n} p(\pi) \lambda_{\pi(j)}(A)$$

for some $p \in P(S_n)$. Now, let's define $U_\pi = \sum_k u_k v_{\pi(k)}^*$ for each $\pi \in S_n$. Now, we'll average these: $\sum_\pi p(\pi) U_\pi A U_\pi^*$. We'll bustitute our spectral decomposition of A so that the previous sum becomes:

$$\begin{aligned}
&\sum_{\pi} p(\pi)U_{\pi}(\sum_{k=1}^{n} \lambda_k(A)v_k v_k^*)U_{\pi}^* \\
&= \sum_{\pi} p(\pi) \sum_{k=1}^{n} \lambda_{\pi(k)}(A)u_k u_k^* \\
&= \sum_{k=1}^{n} \lambda_k(B)u_k u_k^* \\
&= B
\end{aligned}$$

Going from left to right

Suppose, on the other hand, that $A \succ B$:

$$B = \sum_{i=1}^{m} p_i U_i A U_i^*$$

We have that $\lambda_j(B) = u_j^* B u_j = \sum_{i=1}^{m} p_i u_j^* U_i$. This can be written as:

$$\begin{aligned}
&\sum_{k=1}^{n}\sum_{i=1}^{m} p_i(u_j^* U_i v_k)(v_k^* U_i^* u_j)\lambda_k(A) \\
=&\sum_{k=1}^{n}\sum_{i=1}^{m} p_i |u_j^* U_i v_k|^2 \lambda_k(A) \\
=&\sum_{k=1}^{n} D(j,k)\lambda_k(A)
\end{aligned}$$

Where $D(j,k) = \sum_{i=1}^{m} p_i |u_j^* U_i v_k|^2$ is doubly stochastic. This is equivalent to saying that $D\lambda(A) = \lambda(B)$.

The following are equivalent conditions:

1. $A \succ B$ (i.e. $B = \Phi(A)$ for Φ being a mixed unitary)
2. $B = \Phi(A)$ for Φ being a unital channel
3. $B = \Phi(A)$ for Φ being a positive, trace-preserving and unital channel (note that Φ does not need to be completely positive)

Consider the following proposition: Suppose $\rho, \sigma \in D(\mathcal{X})$ satisfy $\rho \succ \sigma$. It holds that $H(\sigma) \geq H(\rho)$.

Proof: We know that $\rho \succ \sigma$ implies that $\sigma = \sum_{k=1}^{n} p_k U_k \rho U_k^*$. Then we use the concavity of the entropy to write:

$$\begin{aligned} H(\sigma) &= H(\sum_{k=1}^{n} p_k U_k \rho U_k^*) \\ &\geq \sum_{k=1}^{n} p_k H(U_k \rho U_k^*) \\ &= \sum_{k=1}^{n} p_k H(\rho) \\ &= H(\rho) \end{aligned}$$