Introduction to Noise Processes

ECE730/QIC890-T33

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Problem Set 1 Suggested Solution

1. **Statistically Stationary vs. Statistically Non-Stationary Processes**

   In class, we discussed stationarity. If $\langle x(t_1)\rangle$ and $\langle x(t_1)^2\rangle$ are independent of the time t_1 and if $\langle x(t_1)x(t_2)\rangle$ is independent of absolute times t_1 and t_2 but dependent only on the time difference $\tau = t_2 - t_1$, such a noise process is called as "statistically stationary process". The "statistically-nonstationary" process does not hold the aforementioned conditions. Hence, the concept of ensemble averaging is valid, whereas the concept of time averaging fails.

   (1) Please show the example of statistically stationary and statistically non-stationary processes.

   (Answers) An example of a stationary process is aircraft engine noise in level flight, which keeps the engine at a certain operation condition. A random walk process would be a statistically non-stationary process (1 point).

   (2) Can you make a statistically non-stationary process to a statistically stationary process? If yes, how? If no, why?

   (Answers) In statistics, there are a few techniques to transform a statistically non-stationary process to a statistically stationary process: taking differences, defining a time-gated function, or using square roots. It is to perform such techniques until the statistically non-stationary process becomes a stationary process (1 point).

   (3) Here is a member function $x(t) = \sin(\omega t + \theta)$ from a stochastic process specified by a transformation of variables. θ is a random variable with uniform distribution over the internal $0 < \theta < 2\pi$, with the probability $P(\theta) = \frac{1}{2\pi}$. Is $x(t)$ ergodic in the mean? Please justify your answers.

   (Answers) Let's compute the time-average and ensemble-average:

   1. Time-Average (1 point)

$$\begin{aligned}
\overline{x(t)} &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \sin(\omega t + \theta)dt, \\
&= \lim_{T\to\infty} \frac{1}{T}\left(-\frac{\cos(\omega t + \theta)}{\omega}\right)\Big|_{-T/2}^{T/2}, \\
&= \lim_{T\to\infty} \frac{1}{\omega T}(-\cos(\omega T/2 + \theta) + \cos(-\omega T/2 + \theta)), \\
&= 0.
\end{aligned}$$

2. Ensemble-Average (1 point)

$$\begin{aligned}
\langle x(t)\rangle &= \int_{-\infty}^{\infty} x(t)P(\theta)d\theta, \\
&= \int_0^{2\pi} \sin(\omega t+\theta)\frac{1}{2\pi}d\theta, \\
&= \frac{1}{2\pi}(-\cos(\omega t+2\pi)+\cos(-\omega t)), \\
&= 0.
\end{aligned}$$

Since the time-average $\overline{x(t)}$ is equal to the ensemble-average $\langle x(t)\rangle$, $x(t)$ is ergodic in the mean.

Is $x(t)$ ergodic in the auto-correlation?Please justify your answers.

(Answers) Let's compute the time-average and ensemble-average:

1. Time-Average (1 point)

$$\begin{aligned}
\phi_x(\tau) &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)dt, \\
&= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} \sin(\omega t+\theta)\sin(\omega(t+\tau)+\theta)dt, \\
&= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} \frac{1}{2}(\cos(\omega t+\theta-(\omega(t+\tau)+\theta))-\cos(\omega t+\theta+\omega(t+\tau)+\theta))dt, \\
&= \lim_{T\to\infty}\frac{1}{2T}(\cos(\omega\tau)T-\frac{1}{2\omega}\sin(2\omega t+\omega\tau+2\theta)|_{-T/2}^{T/2}, \\
&= \frac{\cos(\omega\tau)}{2}-\frac{\cos(\omega\tau+2\theta)}{2}\lim_{T\to\infty}\frac{\sin(\omega T)}{\omega T}), \\
&= \frac{\cos(\omega\tau)}{2}.
\end{aligned}$$

2. Ensemble-Average (1 point)

$$\begin{aligned}
\langle x(t)x(t+\tau)\rangle &= \int_{-\infty}^{\infty} \sin(\omega t+\theta)\sin(\omega t+\omega\tau+\theta)P(\theta)d\theta, \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos(\omega\tau)-\cos(2\theta+2\omega(t+\tau))}{2}d\theta, \\
&= \frac{1}{4\pi}(\cos(\omega\tau)\cdot 2\pi-\sin(2\theta+2\omega(t+\tau))|_0^{2\pi}), \\
&= \frac{\cos(\omega\tau)}{2}, \\
&= \phi_x(\tau).
\end{aligned}$$

Thus, $x(t)$ is ergodic in the autocorrelation.

2. **Ergodicity**

We learned ergodicity in the mean and in the autocorrelation in class.

(1) Please write down an example of the processes which are ergodic in the mean and/or the autocorrelation.

(Answers) A Gaussian white noise is a statistically stationary and erdogic process with zero mean and zero autocorrelation. In this case, the power spectrum of the Gaussian white noise is constant over all frequencies, namely 'white noise' (1 point).

(2) Suppose the autocorrelation of a noise process is ergodic. Is the process ergodic in the mean? Please justify your answers.

(Answers) Not necessarily. One example of the process is

$$x(t) = \begin{cases} a & \text{case 1 with probability} \quad p = 1/2; \\ -a & \text{case 2 with probability} \quad p = 1/2. \end{cases}$$

The autocorrelation of $x(t)$ is a^2, namely both the time-average and the ensemble-average are the same to be a^2. However, the time-averaged mean is a or $-a$ depending on the case, whereas the ensemble-averaged mean is 0. Hence, the process is not ergodic in the mean (1 point).

3. **Parseval Theorem**

(1) If $x_1(t)$ and $x_2(t)$ have Fourier transform $X_1(i\omega)$ and $X_2(i\omega)$, prove the equality of the equation:

$$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(i\omega)X_2^*(i\omega)d\omega.$$

(Answers) Using the definition of the Fourier transform,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(i\omega)\exp(i\omega t)d\omega,$$

the complex conjugate $x^*(t)$ satisfies the following,

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(-i\omega)\exp(-i\omega t)d\omega.$$

When $x(t)$ is a real function of time, $X(i\omega) = X^*(i\omega)$.

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\begin{aligned}
\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt &= \int_{-\infty}^{\infty}(\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(i\omega)\exp(i\omega t)d\omega)(\frac{1}{2\pi}\int_{-\infty}^{\infty} X_2^*(-i\omega')\exp(-i\omega' t)d\omega')dt, \\
&= (\frac{1}{2\pi})^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_1(i\omega)X_2^*(i\omega')d\omega d\omega'\int_{-\infty}^{\infty} dt\exp(i(\omega-\omega')dt), \\
&= (\frac{1}{2\pi})^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_1(i\omega)X_2^*(i\omega')d\omega d\omega' 2\pi\delta(\omega-\omega'), \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(i\omega)X_2^*(i\omega)d\omega \qquad \text{(Q.E.D. 1 point)}.
\end{aligned}
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(2) Suppose $x_1(t) = x_T(t+\tau)$ and $x_2(t) = x_T(t)$. The notation $x_T(t)$ implies $x(t)$ if $|t| \le T/2$ and 0 if $|t| > T/2$, where T is a finite-valued measurement time interval.

Show that

$$\int_{-\infty}^{\infty} x_T(t+\tau)x_T(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(i\omega)|^2\exp(i\omega\tau)d\omega.$$

(Answers) Note that

$$x(t+\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(i\omega)\exp(i\omega(t+\tau))d\omega,$$

which mentions that the Fourier transform of $x(t+\tau)$ has $\exp(i\omega\tau)$ factor to the Fourier transform of $x(t)$.

Therefore,

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\begin{aligned}
\int_{-\infty}^{\infty} x_T(t+\tau)x_T(t)dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_T(i\omega)X_T^*(i\omega)\exp(i\omega\tau)d\omega, \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(i\omega)|^2\exp(i\omega\tau)d\omega \qquad \text{(Q.E.D. 1 point)}.
\end{aligned}
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