ECE770-T14/QIC 885: Quantum Electronics & Photonics

Problem Set 3, Winter 2014, Instructor: A. Hamed Majedi

Problem 1- Consider an object in a one -dimensional Simple Harmonic Oscillator (SHO) that is also acted by a constant force, F at $t > 0$. If the object has been initiated in the ground state of the SHO without the force, i.e. at $t = 0$, using the Heisenberg picture

a) Find the expectation value of the position.

b) Find the expectation value of the momentum.

Problem 2- Consider a simple harmonic oscillator and a new operator defined as $\hat{G}(t) = m\omega\hat{x}(t)\cos\omega t - \hat{p}(t)\sin\omega t$.

a) Can this operator be simultaneously diagonalized with the Hamiltonian? Justify your answer.

b) Find the equation of motion for $\hat{G}(\mathrm{t})$. Can this operator be treated as the constant of motion?

c) Solve the equation of motion, if the initial position and momentum is known.

Problem 3- Consider a simple harmonic oscillator that is suddenly displaced from its equilibrium point, namely $x = 0$ to $x = x_o$ e.g. much faster than the oscillation period. An example of such system can be an electron in a simple harmonic oscillator subject to a constant electric field.

a) Write down the Hamiltonian of the displaced harmonic oscillator.

b) Use the Dirac picture to find $|\Psi_D(t)\rangle$ and position operator $\hat{x}_D(t)$.

Problem 4- Consider the Lagrangian, $\mathcal{L} = \int \mathscr{L} d^3\mathbf{r}$ with

$$\mathscr{L} = \frac{i\hbar}{2}(\Psi^*\dot{\Psi} - \dot{\Psi}^*\Psi) - \frac{\hbar^2}{2m}\nabla\Psi^*.\nabla\Psi - V(\mathbf{r})\mathbf{\Psi}^*\mathbf{\Psi} \tag{1}$$

In this Lagrangian $\Psi(\mathbf{r})$ is a complex classical field called the **Schrodinger matter field**.

a) Choose proper dynamical variables (canonical conjugate variables) that the Lagrangian equation associated with equation(1) coincide with the Schrodinger equation. Justify your choice of dynamical variables.

b) Let $\Psi(\mathbf{r}) = \Psi_r(\mathbf{r}) + \Psi_i(\mathbf{r})$, express $\mathscr{L}$ as a function of Ψ_r and Ψ_i and their temporal derivative.

c) Consider the new Lagrangian $\mathcal{L}'$ as $\mathcal{L}' = \mathcal{L} + \frac{d}{dt}\int \hbar\Psi_r(\mathbf{r})\Psi_i(\mathbf{r})d^3\mathbf{r}$ and show that this choice does not depend on $\dot{\Psi}_i(\mathbf{r})$.

d) Show that it is possible to use the Lagrange equation relative to Ψ_i to eliminate Ψ_i from $\mathcal{L}'$. The new lagrangian which is obtained is only a function of Ψ_r and its temporal derivative and is denoted $\hat{\mathcal{L}}'$. Note that all these steps are necessary as the original Lagrangian has an excess of dynamical variables, since the Schrodinger equation is a first-order equation in time.

e) Show that, for a real motion

$$\frac{\partial \mathcal{L}'}{\partial \Psi_r} = \frac{\partial \hat{\mathcal{L}}'}{\partial \dot{\Psi}_r} \tag{2}$$

and derive the conjugate momentum of Ψ_r.

f) Express Ψ as a function of Ψ_r and its conjugate momentum.

g) Find the Hamiltonian $\mathcal{H}$ associate with $\hat{\mathcal{L}}'$. Is your result the same as the expectation value of total energy in quantum mechanics?

h) Proceed with the canonical quantization of the theory by replacing Ψ and Ψ^* by $\hat{\Psi}$ and $\hat{\Psi}^\dagger$.

i) Using canonical commutation relationships between the Ψ_r and its conjugate momentum, compute the commutators of $\hat{\Psi}(\mathbf{r})$ and $\hat{\Psi}^\dagger(\mathbf{r}')$ and $\hat{\Psi}(\mathbf{r})$ and $\hat{\Psi}(\mathbf{r}')$.

Problem 5- A spinless object is described by the wavefunction $\psi = A(x + y + 2z)e^{-\alpha r}$ where A and α are real constant numbers and $r = \sqrt{x^2 + y^2 + z^2}$.

a) What is the total angular momentum of the object?

b) What is the expectation value of the z-component of the angular momentum?

c) If the z-component of the angular momentum were measured, what is the probability of obtaining $+\hbar$?

Due: Wednesday March 12, 2014. (before starting the class)