Lecture 8: Josephson effect - gauge invariance, classical phenomena

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I. SIGNIFICANCE OF THE PHASE PARAMETER

Take γ_1 and γ_2 to be the superconducting phases on the two sides of a Josephson junction. As this is the phase of a particle in a wavefunction, the time derivative has to be the negative of the energy - which in this case is the chemical potential (up to a constant offset). We have

$$\hbar\dot{\phi}_{1,2} = -\mu_{1,2}. \tag{1}$$

. If we use next

$$\mu_1 - \mu_2 = 2e(V_1 - V_2) = 2eV, \tag{2}$$

where we introduced the voltage difference V, we obtain

$$V = -\frac{\hbar}{2e}(\dot{\phi}_1 - \dot{\phi}_2). \tag{3}$$

This result suggests that the phase parameter which appears in the Josephson relation is equal to the difference between the phases in superconductors 1 and 2.

A different approach to understanding the significance of the phase is provided by the derivation of Josephson effect due to Bloch [1]. We consider a superconducting ring with a tunnel barrier (see Fig. 1). The energy levels are invariant under change of the externally applied flux Φ by multiples of $\frac{h}{|e|} = 2\Phi_0$ (this is a consequence of gauge invariance and single valuedness of the phase, see [1]). The free energy is therefore periodic:

$$F(\Phi, T) = \sum_n \left[A_n \cos(2\pi n \frac{\Phi}{2\Phi_0}) \right] \tag{4}$$

The odd terms vanish (the ring "looks the same" upside down).

Consider the contour indicated in Fig. 1, taken well enough inside the superconductor

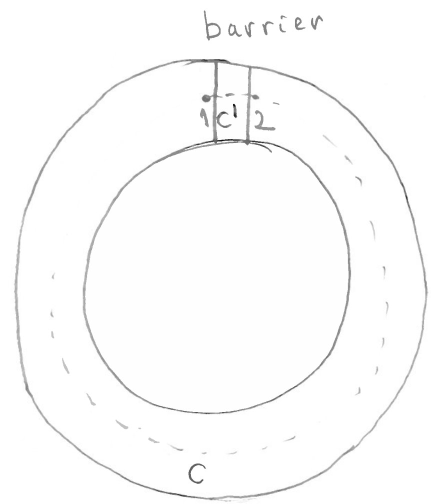

FIG. 1. Superconducting ring interrupted by a barrier.

that the current is negligible:

$$\mathbf{0} = \boldsymbol{j} = \frac{e}{m}|\Psi|^2(\hbar\nabla\varphi - 2e\boldsymbol{A}) \tag{5}$$

$$\varphi_2 - \varphi_1 = -\frac{2|e|}{\hbar}\int_{\text{ring, } 1\to 2} \vec{A}d\vec{l} \approx -2\pi\frac{1}{\Phi_0}\oint \vec{A}d\vec{l} = -2\pi\frac{\Phi}{\Phi_0} \tag{6}$$

$$\Delta\varphi \equiv -2\pi\frac{\Phi}{\Phi_0} \tag{7}$$

$$F(\Delta\varphi, T) = \sum_n A_n \cos\left(\frac{n}{2}\Delta\varphi\right) \tag{8}$$

Consider the variation of F induced by a change in flux. The change in flux induces an electric field. We take I_+ and V_+ to be respectively the current and the induced voltage in trigonometric direction:

$$\delta F = I_+V_+\delta t = -I_+\delta\Phi = I_+\frac{\Phi_0}{2\pi}\delta(\Delta\varphi). \tag{9}$$

Then we have (dropping the subscript which indicates direction)

$$I = \frac{2\pi}{\Phi_0}\frac{\partial F}{\partial(\Delta\varphi)} = \frac{2\pi}{\Phi_0}\frac{\partial}{\partial(\Delta\varphi)}\left[\sum_n A_n \cos(\frac{n}{2}\Delta\varphi)\right] \tag{10}$$

The term with $n = 2$ gives the usual Josephson effect.

The derivation by Bloch makes the connection with the SC phase more transparent. One consequence is that we have to use the gauge invariant phase γ (this will become more clear in the discussion below on the 1 junction interferometer):

$$\gamma = \varphi_2 - \varphi_1 + \frac{2e}{\hbar}\int_{\text{barrier},2\to1} \vec{A}d\vec{l} \tag{11}$$

$$= \Delta\varphi - 2\pi\frac{1}{\Phi_0}\int_{\text{barrier},2\to1} \vec{A}d\vec{l}. \tag{12}$$

With this definition of γ:

$$\gamma = -2\pi\frac{\Phi}{\Phi_0} \tag{13}$$

for the ring system discussed above. This relation and equivalent relations for multi-junction loops are essential for quantum interferometers (RF-SQUID and DC-SQUID).

II. JOSEPHSON JUNCTIONS IN THE CLASSICAL REGIME

$$H_J = \frac{E_C}{2}\hat{n}^2 - E_J \cos\widehat{\gamma} \tag{14}$$

In general, both n and γ are uncertain, due to quantum fluctuations. In the following, we are concerned with large junctions, for which

$$\frac{E_J}{E_C} \sim S^2 \gg 1 \tag{15}$$

where S is the surface. In this limit we can write

$$H_J \cong \frac{E_C}{2}\hat{n}^2 + \frac{E_J}{2}\hat{\gamma}^2 \tag{16}$$

The fluctuations in the ground state can be estimated if we write:

$$\frac{E_J}{2}\langle\gamma^2\rangle = \frac{\hbar\omega_{\text{res}}}{4} \quad \text{half of the “vacuum energy”} \tag{17}$$

$$\omega_{\text{res}} = \sqrt{E_C E_J} \tag{18}$$

$$\sqrt{\langle\gamma^2\rangle} = \frac{1}{\sqrt{2}}\left(\frac{E_C}{E_J}\right)^{1/4} \tag{19}$$

Take some typical numbers for Al/AlO_x/Al junctions:

$$\tilde{C} = 100\,\text{fF}/\mu\text{m}^2 \tag{20}$$

$$\tilde{I}_c = 10\,\mu\text{A}/\mu\text{m}^2 \tag{21}$$

$$\frac{E_J}{E_C} = \frac{2.067\times10^{-15}/2\pi\times10\times10^{-6}}{\frac{(2\times1.6\times10^{-19})^2}{100\times10^{-15}}}(S[\mu A])^2 = 3200\times(S[\mu A])^2 \tag{22}$$

$\Rightarrow$ for micron or larger junctions, phase fluctuations are small, this is the *classical regime.*

III. ONE JUNCTION INTERFEROMETER (RF-SQUID)

Consider again Fig. 1. We have

$$\Phi = (\int_{C,1\to 2} + \int_{C',2\to 1})\vec{A}d\vec{l} \tag{23}$$

Assume C is well within the superconductor where the current vanishes:

$$\mathbf{0} = \boldsymbol{j} = \frac{e}{m}|\Psi|^2(\hbar\nabla\varphi - 2e\boldsymbol{A}) \tag{24}$$

$$\int_{C,1\to 2}\vec{A}d\vec{l} = \frac{\hbar}{2e}\int_{C,1\to 2}\nabla\varphi d\vec{l} = -\frac{\Phi_0}{2\pi}(\varphi_2 - \varphi_1) \tag{25}$$

$$-\frac{2\pi}{\Phi_0}\Phi = \underbrace{\varphi_2 - \varphi_1 - \frac{2\pi}{\Phi_0}\int_{C',2\to 1}\vec{A}d\vec{l}}_{\gamma-\text{gauge invariant phase}} \tag{26}$$

We introduce next the kinetic inductance (important in qubit circuits). It is useful to go to the opposite limit, of constant current density in the ring. The LHS of Eq. 25 has a term

$$\frac{m}{2e^2|\Psi|^2}\int_{C,1\to 2}\vec{j}d\vec{l} = \frac{m}{2e^2|\Psi|^2 S}\int_{C,1\to 2}S\vec{j}d\vec{l} = \frac{m}{2e^2|\Psi|^2 S}lI \tag{27}$$

We introduced S — cross-section area (assumed constant). We define

$$L_K \equiv \frac{me}{2e^2|\Psi|^2 S} \tag{28}$$

L_K has dimensions of inductance.

One can show that:

$$\frac{1}{2}L_K I^2 = \int_{\text{volume supercond.}} d\vec{r}\, n_{\text{CP}}\frac{1}{2}m_{\text{CP}}v_{\text{CP}}^2 \tag{29}$$

with n_{CP}, m_{CP}, and v_{CP} the Cooper pair density, mass, and velocity. The RHS of this equation is the total kinetic energy of the Cooper pairs in the wire. This equality shows that the kinetic energy can be expressed in a way which is analogous to a magnetic energy.

We have

$$-\frac{2\pi}{\Phi_0}(\Phi + L_K I) = \gamma \tag{30}$$

$$\Phi = \Phi_x + LI \tag{31}$$

Φ_x: external flux.

L: ring geometric self inductance.

$$\Rightarrow -\frac{2\pi}{\Phi_0}\left[\Phi_x + \underbrace{(L + L_K)}_{L'}I\right] = \gamma. \tag{32}$$

This shows that it is possible to incorporate the energy of motion of Cooper pairs, or the kinetic inductance, into a total effective inductance L'.

There are two ways to represent equation 32 (we use L to denote the total inductance):
— current

$$i \equiv \frac{I}{I_c} = \sin\gamma = -\sin\frac{2\pi}{\Phi_0}(\Phi_x + LI_c\frac{I}{I_c}) = -\sin(2\pi f_x + \beta i) \tag{33}$$

$$\beta = 2\pi\frac{LI_c}{\Phi_0} \quad \text{— screening parameter for interferometers}$$

— phase

$$\gamma = -2\pi f_x - \beta\sin\gamma \tag{34}$$

The solution can be understood using a graphical method. We have

$$\frac{1}{\beta}(-\gamma - 2\pi f_x) = \sin\gamma. \tag{35}$$

The graphical solution is shown in Fig. 2.

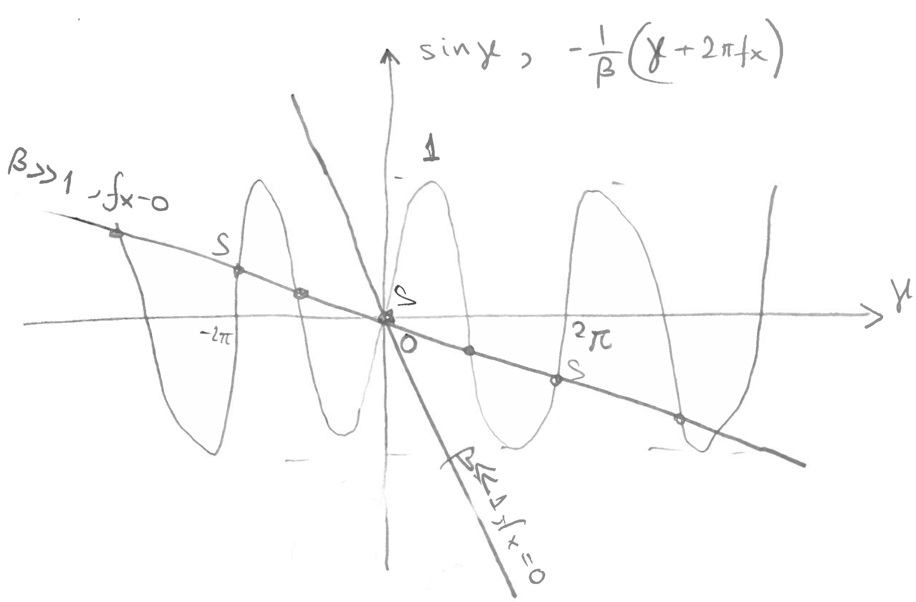

FIG. 2. Graphical solution of the equation for a one-junction interferometer.

Multiple solutions exist for $\beta \gg 1$. Not all the solutions are stable. When we will discuss the determination of energy for these circuits we will see that the energy is given by

$$U(\gamma) = -\varphi_0 I_c \cos\gamma + \frac{\varphi_0^2}{2L}(\gamma + 2\pi f_x)^2 \tag{36}$$

— condition of extremum

$$\frac{\partial U}{\partial \gamma} = 0 \Leftrightarrow \varphi_0 I_c \sin\gamma + \frac{\varphi_0^2}{L}(\gamma + 2\pi f_x) = 0 \tag{37}$$

$$\Leftrightarrow \gamma + 2\pi f_x = -\beta \sin\gamma \tag{38}$$

— condition of minimum

$$\frac{\partial^2 U}{\partial \gamma^2} > 0 \Leftrightarrow \varphi_0 I_c \cos\gamma + \frac{\varphi_0^2}{L} > 0 \tag{39}$$

$$\Leftrightarrow \beta \cos\gamma > -1 \tag{40}$$

Points indicated by S on the diagram are stable.

IV. THE TWO-JUNCTION INTERFEROMETER (DC-SQUID)

The two-junction interferometer is shown in Fig. 3.

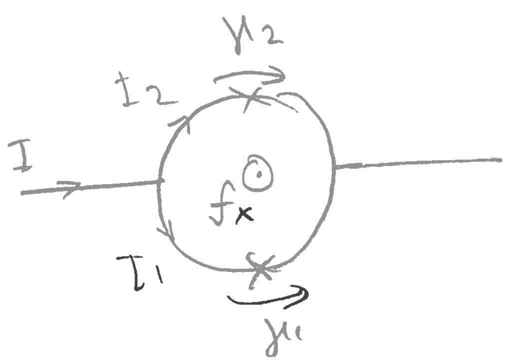

FIG. 3. Two junction interferometer.

We use arguments similar to the case of a single-junction interferometer to write down the relation between the phases γ_1 and γ_2 across the two junctions, the applied magnetic flux $\Phi_x = f_x \Phi_0$, and the self-inductance of the loop. We have

$$\gamma_1 - \gamma_2 = -2\pi f. \tag{41}$$

with

$$f = f_x + f_{\text{induced}} \tag{42}$$

The induced magnetic flux is given by

$$f_{\text{induced}} = \frac{LI_{\text{circ}}}{\Phi_0} \tag{43}$$

with the circulating current

$$I_{\text{circ}} = \frac{I_1 - I_2}{2}, \tag{44}$$

with I_1 and I_2 the currents in the two arms. Note that expression 43 applies when the SQUID is symmetric: the two arms couple to the loop equally and there is no direct coupling from the bias current I_B. Combining all these relations, we write

$$\gamma_1 - \gamma_2 = -2\pi f_x - \frac{\pi L}{\Phi_0}\left(I_{c1}\cos\gamma_1 - I_{c2}\cos\gamma_2\right). \tag{45}$$

Current conservation implies

$$I_B = I_{c1}\cos\gamma_1 + I_{c2}\cos\gamma_2 \tag{46}$$

Equations 45 and 46 can be solved to obtain the values of the phase on the two junctions. This is difficult in general for the following reasons:

- non-algebraic equations
- similar to the case of the one junction interferometer, some solutions may not be stable

In the following we consider the simplest case, in which the junctions are symmetrical ($I_{c1} = I_{c2} \equiv I_c$) and the inductance L is very small, more precisely

$$\frac{2\pi L I_c}{\Phi_0} \ll 1. \tag{47}$$

We have

$$\gamma_1 - \gamma_2 \approx -2\pi f_x. \tag{48}$$

Using 48 and 46 we obtain

$$I_B = 2I_c \cos\pi f_x \sin\frac{\gamma_1 + \gamma_2}{2}, \tag{49}$$

which shows that the maximum supercurrent, called also the *critical current* of the DC-SQUID, is given by

$$I_{\text{c,SQUID}} = 2I_c|\cos\pi f_x|. \tag{50}$$

This is plotted in Fig. 4.

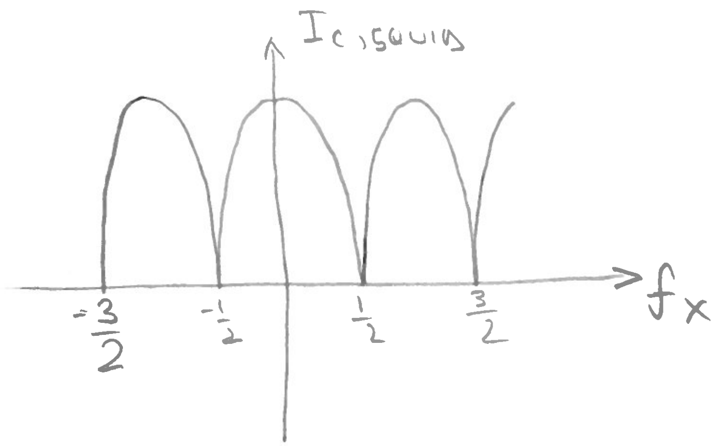

FIG. 4. Modulation of the critical current of a DC SQUID.

The dependence of the critical current on the applied magnetic flux provides the basis for magnetic field measurements. DC-SQUIDS are extremely sensitive magnetometers. Example: take ΔB to be the increment of magnetic field which induces a change in the magnetic flux in the SQUID of one flux quantum. We have $\Delta B \times S = \Phi_0$ with S the area of the SQUID. If we take $S = 1000\,\mu\text{m}$ we have $\Delta B = 2\,\mu\text{T}$. A change in magnetic field which is a very small fraction of ΔB is in fact detectable, if the measurement time is long enough.

V. THE RCSJ MODEL

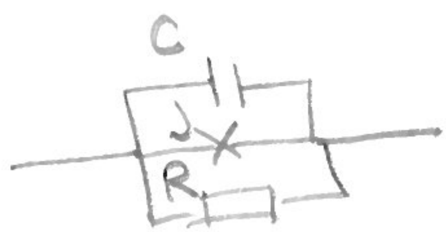

FIG. 5. RCSJ model for a Josephson junction.

A Josephson junction can be modeled by the resistively and capacitively shunted junction (RCSJ) model, represented in Fig. 5. There are three contributions to the current through the junction:

- the tunneling current

$$I_J = I_c \cos\gamma \tag{51}$$

- the displacement current

$$I_C = \frac{d}{dt}(CV) \tag{52}$$

- the resistive channel

$$I_R = \frac{V}{R} \tag{53}$$

  For now we assume the resistance R to be a constant. Note that when the voltage V across the junction exceed $2\Delta/e$ a finite resistance component to the current arises due to transfer of charge through creation of excited states in one of the superconductor.. This component of the resistance is nonlinear. A separate component to the resistance arises at finite temperature, when excitations in the superconductor lead to single-particle tunneling. A constant resistance can be assumed when a shunt resistance is added to each junction, with this resistance taken so small that it always dominates the larger nonlinear component due to tunneling

We next consider the behaviour of the junction when biased by a current source I_B. Using 51, 52 and 53 we obtain

$$I_B = I_c \cos\gamma + \frac{d}{dt}(CV) + \frac{V}{R}. \tag{54}$$

We use the second Josephson relation to express V as a function of the time derivative of the phase. We find

$$C\phi_0^2\ddot{\gamma} = -\frac{\partial U}{\partial \gamma} - \frac{\phi_0^2}{R}\dot{\gamma} \tag{55}$$

where we introduced the *tilted washboard potential*

$$U(\gamma) = -\phi_0 I_c \cos\gamma - \phi_0 I_B \gamma. \tag{56}$$

We also introduced the reduced flux quantum ϕ_0. The last term on the RHS of 55 is a friction term.

When the resistance R is very large, the IV characteristic of the JJ is as represented schematically in Fig. 6. Hysteresis appears, as a consequence of the fact that with weak dissipation, even a small tilt in the potential is sufficient to sustain the rolling down of the phase degree of freedom.

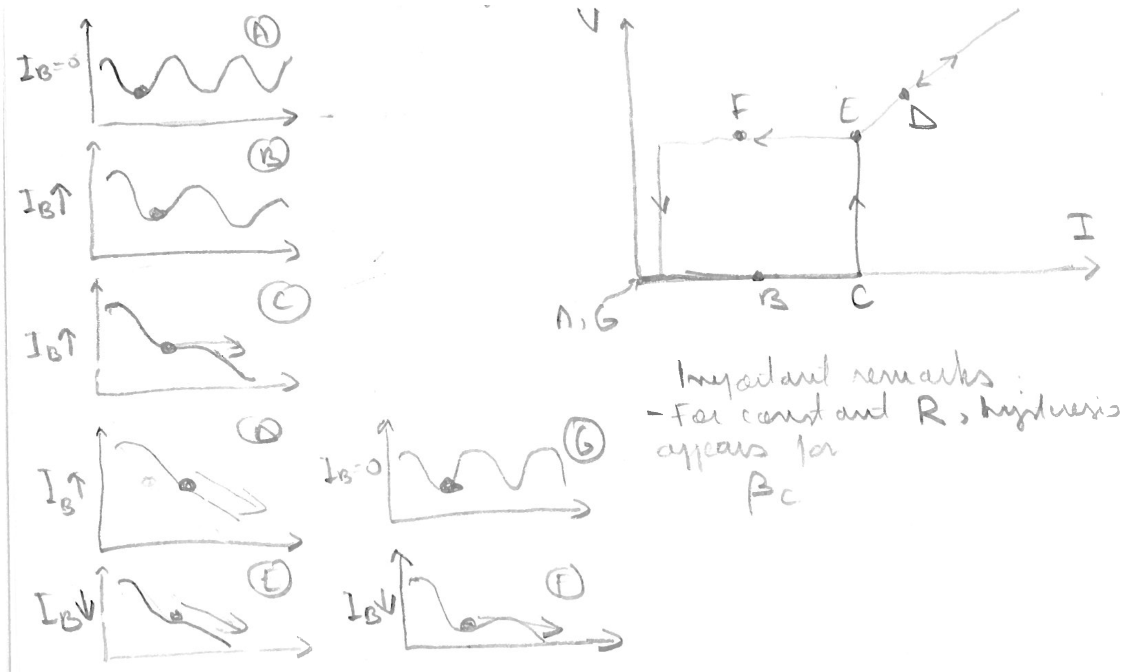

FIG. 6. Representation of the IV characteristic of a hysteretic junction, together with a representation of the situation in the different positions on the curve.

When the resistance is low enough the IV characteristic become non-hysteretic. The crossover between the two regimes takes place at a value of the Stewart McCumber parameter, defined as $\beta_c = \frac{2eI_cR^2C}{\hbar}$, of 1 [2].

[1] F. Bloch, Phys. Rev. B **2**, 109 (1970).

[2] A. Barone and G. Paterno, *Physics and Applications of the Josephson effect* (John Wiley and Sons, 1982).