Tue.

1. Classical Measurement theory : Bayesian statistical inference

↳ "how our <u>knowledge</u> about the value of some quantity $\underline{x}$ changes when we obtain a piece of data relating to x"

our knowledge about x = probability distribution $P(x)$

↳ the likelihood that x will have various values $\underline{x}$

overall how certain or uncertain we are about $\underline{\underline{x}}$

↳ "state-of-~~the~~ knowledge of x"

<u>How</u> $P(x)$ changes when we obtain the value y?

<u>Bayes rule</u>

$$\underbrace{P(x,y)}_{\text{joint prob. for 2 random variables}} = P(x|y)\underbrace{P(y)}_{\text{marginal probability irrespective of x-value}} = P(y|x)\underbrace{P(x)}_{\text{state of knowledge prior to x.}}$$

joint prob. for 2 random variables

marginal probability irrespective of x-value

$$P(y) = \int_{-\infty}^{\infty} P(x,y)\,dx$$

~~prob~~ State of knowledge prior to x.

$$P(x) = \int_{-\infty}^{\infty} P(x,y)\,dy$$

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)}$$

$P(x)$ → prior knowledge

$P(x|y)$: posteria probability

$P(y)$: normalization factor

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Example Suppose a measurement of a discrete variable x, either 0 or 1

if we measure, the outcomes are either $y=0$ or $y=1$

Suppose $y=1$,

Bayes' rule says

$$p(x|1) = \frac{P(1|x)P(x)}{\sum_{x'} P(1|x')P(x')} = \frac{P(1|x)P(x)}{P(1|0)P(0)+P(1|1)P(1)}$$

posterior

How can we obtain a better understanding from this expression.

Assume the likelihood function

$P(y\|x)$	$y=0$	$y=1$
$x=0$	α	$1-\alpha$
$x=1$	$1-\alpha$	α

if $\alpha \approx 1$, the 2-values of x give very different distributions for the measurement result y

$x=0$ & $y=0$: $P(0|0) = 1$

& $y=1$: $P(1|0) = 0$ ∴ we get a lot about x

$x=1$ $y=0$: $P(0|1) = 0$

$y=1$: $p(1|1) = 1$

if $\frac{1}{2} < \alpha < 1$, $x=0$ $y=0$: $P(0|0) > \frac{1}{2}$ ⇐ higher.

$y=1$: $P(1|0) < \frac{1}{2}$

$x=1$ $y=0$: $P(0|1) < \frac{1}{2}$

$y=1$: $P(1|1) > \frac{1}{2}$ ⇒ higher

Suppose $P(x) = \frac{1}{2}$ ∴ no knowledge of x

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What happens when we get $y=1$?

$$\underbrace{p(x|1)}_{\text{posterior}} = \frac{p(1|x)\,p(1)}{p(1|0)p(0)+p(1|1)p(1)} \Rightarrow \cancel{p(0|1) = 1-}\alpha < \frac{1}{2}$$

$$p(1|1) = \alpha > \frac{1}{2}$$

→ most likely than $x=1$ our state of the knowledge improves.

~~P(X|1)~~

Bayes' thm tells that

the values of X that are more likely are those for which the result we have obtained is the more likely outcome.

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2. Quantum Measurement Theory

$|\psi\rangle$ = the state of a quantum system.

= a vector in a complex vector space.

$= \sum_n C_n |n\rangle$ → basis. orthonormal.

C_n → complex coeff. $\sum_n |C_n|^2 = 1$

~~basis~~ Measurement postulate = projection postulate

When we measure, we will find the system to be in one of these basis ~~states~~.

If the system is initially in the state $|\psi\rangle$,

the probability that we will find state $|n\rangle$ is given by $|C_n|^2$

"von Neumann measurement."

in QM., states of knowledge is described by "density matrices"

$$\langle X \rangle = \langle \psi | X | \psi \rangle = \mathrm{Tr}\left[X |\psi\rangle\langle\psi| \right]$$

if our probability ftn in the state m $|\phi_m\rangle$ is p_m,

$\langle X \rangle$ → expectation value of a physical observable

$$= \sum_m p_m \langle \phi_m | X | \phi_m \rangle$$

$$= \sum_m p_m \,\mathrm{Tr}\left[X |\phi_m\rangle\langle\phi_m| \right]$$

$$= \mathrm{Tr}\Big[X \underbrace{\Big[\sum_m p_m |\phi_m\rangle\langle\phi_m| \Big]}_{\equiv\ \rho\ =\ \text{density matrices}} \Big]$$

$$= \mathrm{Tr}[X\rho]$$

ρ : sufficient to fully characterize the future behavior of a q. system

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In the absence of measurement,

how to describe the evolution of ρ?

it is simply given by evolving each of its component states

$\therefore |\psi(t)\rangle = U|\psi(0)\rangle$ by U = a unitary operator

$$\therefore \rho(t) = \sum_m p_m |\phi_m(t)\rangle\langle\phi_m(t)| = \sum_m p_m U|\phi_m(0)\rangle\langle\phi_m(0)|U^\dagger$$

$$= \underline{U\rho(0)U^\dagger}$$

~~off diagonal~~ all element $\rho_{jk} = \langle j|\rho|k\rangle = \sum_m p_m c_{jm} c_{km}^*$

diagonal $\rho_{jj} = \langle j|\rho|j\rangle = \sum_m p_m |c_{jm}|^2$

know that $|c_{jm}|^2$ = the conditional probability of finding the system in $|j\rangle$

given that its initial state is $|\phi_m\rangle$

$\therefore$ ρ_{jj} = the total probability of finding the system in the state $\underline{|j\rangle}$

States

① pure state : the ρ consists of only a single state

$$\rho = |\psi\rangle\langle\psi| = \sum_{j,k} c_{jm} c_{km}^* |j\rangle\langle k|$$

② statistical mixture $\neq$ superposition.

$$\rho = \sum_j p_j |j\rangle\langle j|$$

completely mixed : there is no information about the system

since each of eigenstates are equally likely

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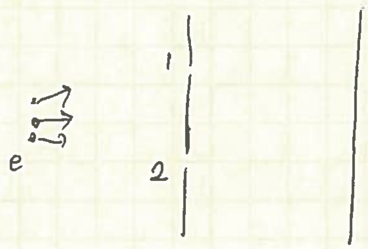
·mixture vs superposition

⇓ mixture: a system really in one of the states, but we do not know which state it is

⇓ superposition: it is definitely <u>not</u> in ~~the~~ any of the states

<u>example</u> Double slit experiment



Suppose one slit is blocked-up.

electron passes through the open slit ⟶ then reaching the screen.

this state is set $|\psi_1\rangle$

& probability density at x is given by $P_1(x) = |\langle\psi_1|x\rangle|^2$

electron passes through the other slit by blocking the opposite

$$P_2(x) = |\langle\psi_2|x\rangle|^2$$

Now open both slots.

if half electrons goe through each slit,

$$P(x) = \frac{1}{2}P_1(x) + \frac{1}{2}P_2(x) \quad \Leftarrow \text{mixture of two states}$$

$$\therefore \rho = \frac{1}{2}\left(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|\right)$$

$$P_{mix}(x) = Tr\left[|x\rangle\langle x|\rho\right] = \frac{1}{2}P_1(x) + \frac{1}{2}P_2(x) = P(x)$$

<u>If</u> after electron passes through the slits, it is in the superposition states $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$

$$P_{sup}(x) = |\langle\psi|x\rangle|^2 = \frac{1}{2}P_1(x) + \frac{1}{2}P_2(x) + \underbrace{Re\left[\langle x|\psi_1\rangle\langle\psi_2|x\rangle\right]}_{\text{interference coherence}}$$

$$\neq P(x)$$

both states at <u>once</u>!

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Given a state-of-knowledge ρ,

von Neumann measurement of the system is expressed as follows.

for a basis $\{|n\rangle\}$, the Projection operators are written as.

$$P_n \equiv |n\rangle\langle n|$$

If the system is measured to be in the state $|m\rangle$ with probability p_m,

$$p_m = \langle m|\rho|m\rangle = \mathrm{Tr}[P_m \rho P_m] = \mathrm{Tr}[(P_m)^2 \rho]$$

cyclic property of the trace.

state-of-knowledge after a measurement

$$\therefore \tilde{\rho}_m = |m\rangle\langle m| = \frac{P_m \rho P_m}{\mathrm{Tr}[P_m \rho P_m]} = \frac{P_m \rho P_m}{p_m}$$

Von Neumann measurement = a special case of more general measurement.

general kind of measurement

Consider a quantum system we like to measure = the target system.

a second system = probe ← prepared in some state indep of the target

(target) — (probe) ⇒ perform von Neumann measurement on the "probe"

interaction ⇝ makes the probe state correlated with the target

Suppose the probe state is $|0\rangle$, given by $\{|n\rangle\}$

$$\rho_{comb,\,initial} = \underbrace{|0\rangle\langle 0|}_{\text{probe}} \otimes \underbrace{\rho}_{\text{target}} \leftarrow \text{initial, target}$$

U (unitary operator) for the interaction btw the target and probe

$$U = \sum_{nn'kk'} u_{nk,n'k'} |n\rangle|s_k\rangle\langle n'|\langle s_{k'}|$$

basis for the target

$$= \sum_{nn'} A_{nn'} \otimes |n\rangle\langle n'| \quad \text{where} \quad A_{nn'} = \sum_{kk'} u_{nk,n'k'} |s_k\rangle\langle s_{k'}|$$

sub-block

$$1 = \langle 0|UU^{\dagger}|0\rangle = \sum A_{n}^{\dagger} A_{n}$$

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Now apply U to the initial state & project onto the probe state $|n\rangle$

then the final state of the combined system

$$\tilde{\sigma} = \underline{(|n\rangle\langle n| \otimes I)}\, U(|0\rangle\langle 0| \otimes \rho)\, U^{\dagger} \underline{(|n\rangle\langle n| \otimes I)}$$

$$= |n\rangle\langle n| \otimes \underline{A_n \rho A_n^{\dagger}}$$

$\therefore$ $\tilde{\rho_n}$ (the final state of the target) $= \dfrac{A_n \rho A_n^{\dagger}}{Tr[A_n^{\dagger} A_n \rho]}$

after interaction, the density matrix

$$\rho_U = U(|0\rangle\langle 0| \otimes \rho) U^{\dagger}$$

the sum of the diagonal elements in the probe state $|n\rangle$,

$$p_n = Tr\left[|n\rangle\langle n| \otimes I\right] \rho_U (|n\rangle\langle n| \otimes I)\Big]$$

$$= Tr\left[(|n\rangle\langle n| \otimes I) U (|0\rangle\langle 0| \otimes \rho) U^{\dagger} (|n\rangle\langle n| \otimes I)\right]$$

$$= Tr[\tilde{\sigma}] = Tr[|n\rangle\langle n| \otimes A_n \rho A_n^{\dagger}] = \underbrace{Tr[|n\rangle\langle n|]}_{=1} Tr[A_n \rho A_n^{\dagger}]$$

$$= Tr[A_n^{\dagger} A_n \rho]$$

a complete description of what happens to a q. sys. under a general measurement process

Fundamental thm of Q. measurement

every set of operators $\{A_n\}$, $n = 1, \cdots, N$ satisfy $\sum A_n^{\dagger} A_n = I$ describes a possible measurement on a q. system via

$$\tilde{\rho_n} = \frac{A_n \rho A_n^{\dagger}}{p_n}, \quad p_n = Tr[A_n^{\dagger} A_n \rho]$$

$\tilde{\rho_n}$ → after measurement

p_n → prob. of obtaining the result <u>n</u>

ρ → before measurement

$\{A_n\}$ = measurement operators

· Understanding Q. Measurement.

Consider $\rho = \sum_n p_n |n\rangle\langle n|$ a statistical mixture

this case, the ~~state-of-knowledge~~ is same as a classical state of knowledge

$\because |n\rangle$ is distinguishable

Now consider A_j, the measurement operators, diagonal in $\{|n\rangle\}$

then $\rho = \sum_n \underbrace{P(n)}_{\text{we want to know}} |n\rangle\langle n|$, $A_j = \sum_n A(j,n)|n\rangle\langle n|$

the final (posterior) density matrix.

$$\tilde{\rho}_j = \frac{A_j \rho A_j^{\dagger}}{\mathrm{Tr}[A_j^{\dagger} A_j \rho]} = \frac{\sum_n A^2(j,n) P(n) |n\rangle\langle n|}{\mathcal{N}} = \sum_n P(n|j)|n\rangle\langle n|$$

$\boxed{A^2(j;n) = P(j|n)}$

likelihood for

$P(n|j) = \frac{P(j|n)P(n)}{\mathcal{N}}$

$\therefore$ just like "classical measurements"

A_j diagonal component =

$[A_j, A_k] = [A_j, \rho] = 0 \quad \forall j,k$ when two operators commute

Since measurement operators commute with the density matrix,

the action on the diagonal elements of the density matrix "semiclassical measurements"

= the action of Bayesian inference.

Polar decomposition: bare measurements and feedback

any operator A may be written as $P\,U$ ← unitary operator → does ~~not~~ change eigen values

↑ positive operator

↓ a complete set of eigenvalues ≥ 0

e.g. a density matrix, $P = P^{\dagger}$

$A_n = U_n P_n$, $p_n = \mathrm{Tr}[(P_n)^2 \rho]$ $\therefore \tilde{\rho}_n = U_n \left(\frac{P_n \rho P_n}{p_n}\right) U_n^{\dagger}$

posterior

$\sum_n A_n^{\dagger} A_n = \sum_n (P_n)^2 = 1$

$\therefore U_n$ do not change the eigenvalues of ρ,

no action on the information extraction

P_n: information capturing

measurement by $\{P_n\}$

forced the system to

& generate the U_n upon n^{th} result

→ "feedback"

Example : a simple semi-classical measurement. for a 2-state system.

$$\rho = p_0 |0\rangle\langle 0| + p_1 |1\rangle\langle 1|$$

then. a measurement ~~is~~ described by the operators

$$A_0 = \sqrt{K}\,|0\rangle\langle 0| + \sqrt{1-K}\,|1\rangle\langle 1|$$

$$A_1 = \sqrt{1-K}\,|0\rangle\langle 0| + \sqrt{K}\,|1\rangle\langle 1|$$

if $K=1$ (or $K=0$), both A_0 and A_1 = projectors.

∴ after the measurement, we know the state of the system w/ 100%

if $K = \frac{1}{2}$, A_0 and $A_1 = I$ identity operator.

∴ measurement has no action, ∴ no knowledge

For $\frac{1}{2} < K < 1$, the measurement is an "incomplete" measurement, which changes p_0 and p_1, but does not reduce to 0.

Polar decomposition thm.

any operator A may be written as the product of a positive operator P & a unitary operator U

P has a complete set of eigen values, all eigenvalues ≥ 0.

$P = P^\dagger$

U : gives "time evolution of the system"

↳ does not change the eigen values but eigenvectors.

↳ does not change the amount of our uncertainty of a system but where in the state space it is most likely to be

$UU^\dagger = U^\dagger U = I$

Suppose a q. measument by a set of operators $\{A_n\}$, i.e. $A_n = U_n P_n$.

the probability of finding the n^{th} result $\Rightarrow p_n = \mathrm{Tr}\,[(P_n)^2 \rho]$

$\therefore$ the final state $\tilde{\rho}_n = U_n \left(\frac{P_n \rho P_n}{p_n} \right) U_n^\dagger$

only requirement $= \sum_n A_n^\dagger A_n = \sum_n (P_n)^2 = I$

Note ① U_n do not change the eigenvalues of ρ.

$\therefore$ no role in the information extraction.

The information gathering properties are from P_n positive operators

interpretation $\tilde{\rho}_n = U_n \underbrace{\left(\frac{P_n \rho P_n}{p_n} \right)} U_n^\dagger$

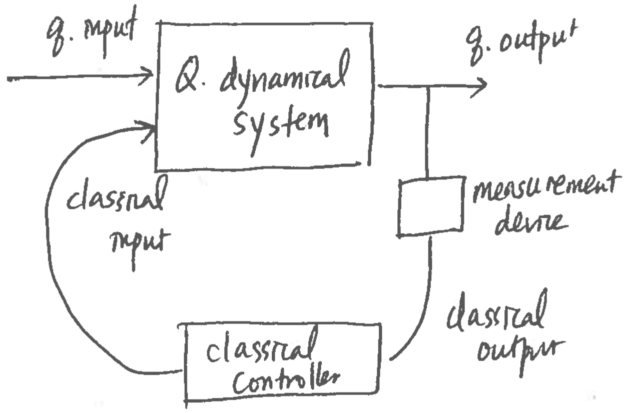
$\Rightarrow$ we made ↑ measurement described by $\{P_n\}$ and applied forces to the system to generate the evolution U_n upon getting the nth result

Applying forces to a system conditional upon the result of a measurem

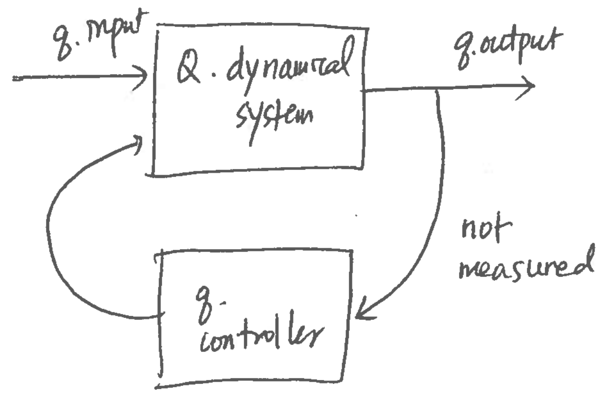
= "feedback"

Quantum feedback schemes

① Measurement-based feedback
(classical control)



② coherent feedback
(Q. control)



full Q. loop

Given a q. control system:

$$\dot{x} = f(x) + \sum_i g_i(x) u_i(x)$$

1. classical control $\quad \dot{x} = f(x) + \sum_i g_i(x) u_i(x)\ ;\ y = h(x)$

design $\quad u_i(x) = K(h(x))$

2. Q. control $\quad \dot{x} = f(x) + \sum_i g_i(x) u_i(x) + \underbrace{\sum_j t_j(x) v_j(x)}_{\text{adding}}$

design $\quad u_i(x),\ v_j(x)$

- coherent states $|\alpha\rangle$ $\qquad$ $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ ← QM definition
  - e.g. laser emission
  - Poissonian number statistics

$$P(n) = \frac{e^{-\langle n\rangle}\langle n\rangle^n}{n!}, \quad \langle n\rangle = |\alpha|^2$$

$\therefore g^{(2)}(\tau) = 1$ for all τ.

- Thermal light.
  - number statistics

$$P(n) = \left(\frac{\langle n\rangle}{\langle n\rangle+1}\right)^n \frac{1}{\langle n\rangle+1}$$

<u>bunched</u>! $g^{(2)}(0) = 2$.

- Non-classical light
  - single photons
  - entangled pairs
  - 2-mode squeezing

e.g. Fock (#) states

squeezed vacuum states

NOON states $\quad |\psi_{NOON}\rangle = \dfrac{|N\rangle_a|0\rangle_b + e^{iN\theta}|0\rangle_a|N\rangle_b}{\sqrt{2}}$

superposition of N-ptl in mode <u>a</u> w/ 0-ptl in mode <u>b</u>

ptl = bosons (or photons)

↓

useful for q. sensing & q. metrology

squeezed coherent state

⟨ amplitude-squeezed state.
phase-squeezed

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- Photon number states or Fock states

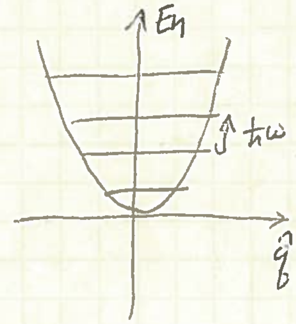
$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}\,[\hat{a}^{+} + \hat{a}] \qquad \hat{p} = i\sqrt{\frac{\hbar\omega}{2}}\,[\hat{a}^{+} - \hat{a}]$$

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \omega^2\hat{q}^2) = \hbar\omega\left(\hat{a}^{+}\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)$$

✓ $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$: eigen~~state~~ energy of energy operator $\hat{H}$

✓ Fock states = eigen states of energy operator

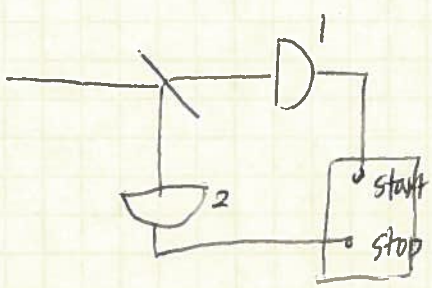
∴ a <u>photon</u> is the single excitation of oscillator



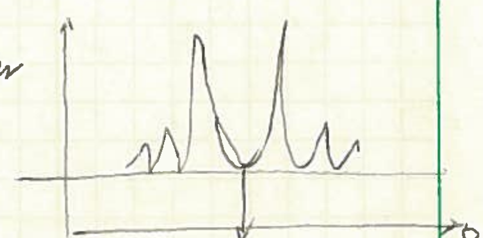
- Quadrature of light : Photon Counting

← correlation function of single photons

$$g^{(2)}(\tau) = \frac{\langle n_1(t)\, n_2(t+\tau)\rangle}{\langle n_1(t)\rangle\langle n_2(t+\tau)\rangle}$$



Time-interval counter
or
correlator



$g^{(2)}(\tau=0) < 0.5$
single-photon generation

$g^{(2)}(0) = 1$: random no correlation

$g^{(2)}(0) > 1$ bunching : photons arrive together

$g^{(2)}(0) < 1$ anti-bunching : <u>photons repel</u>

$g^{(2)}(\tau) \longrightarrow 1$ at long times

- Classical light
  - Coherent light e.g. lasers
  - Thermal light

- Non-classical light
  - single photon emitter
  - entangled photons in pairs