ECE 770-T14: Quantum Electronics & Photonics

E&CE Dept., University of Waterloo

Instructor: A. Hamed Majedi, Apr. 14, 2008, Final Exam, Duration: 2.5 hours

Problem 1: **Schrodinger Equation in The Presence of Electrostatic Field** (20 points)

Consider an electron, with mass, m, and charge, e, in the presence of an eletrostatic field that is described by its scalar potential, $\phi_(x, y, z) = V_o(x^2 + y^2 + z^2)$.

a) Write down the time-independent Schrodinger equation for the electron in the Cartesian component system.

b) Using the method of separation of variables, construct the solution to the Schrodinger equation.

c) Find the allowed energy levels in terms of m, e, and V_o.

Problem 2: **An Electron in Time-Harmonic Magnetic Field** (25 points)

An electron, with gyromagnetic ratio, γ is at rest in a time-harmonic magnetic field, $\mathbf{B} = B_o \sin(\omega t)\mathbf{z}$, where B_o and ω are constants.

a) Construct the time-dependent spin Hamiltonian matrix for this system.

b) Consider the electron's eigenstate in terms of the eigenspinors, $|\chi(t)\rangle = \alpha(t)\,|\uparrow\rangle + \beta(t)\,|\downarrow\rangle = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$, solve the time-dependent Schrodinger equation to find $|\chi(t)\rangle$, if the eletron starts out in the spin-up state with respect to the x axis, that is $|\chi(0)\rangle = \alpha(0)\,|\uparrow\rangle + \beta(0)\,|\downarrow\rangle$.

c) Find the expectation value of S_y.

Problem 3: **Cavity Resonator Filled with Nonlinear Material** (30 points)

Consider a 1D lossless cavity resonator that is uniformly filled with a material that exhibit third order optical nonlinearity. The cavity is assumed to consist of a single-mode optical field. In the presence of the nonlinearity the Hamiltonian becomes:

$$\hat{H} = \hbar\omega_o \hat{a}^\dagger \hat{a} - \frac{\hbar\kappa}{2}\hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a}$$

where κ is a real number that represents the optical nonlinearity. The Hamiltonian describes the photons interacting with each other via the optical nonlinearity by means of ladder operators, $\hat{a}^\dagger$ and $\hat{a}$.

a) Having the commutation relationship $[\hat{a}, \hat{a}^\dagger] = 1$ and the definition of the number operator $\hat{N} = \hat{a}^\dagger \hat{a}$, prove two commutation relations: $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$ and $[\hat{a}, \hat{N}] = \hat{a}$.

b) Show that $[\hat{N}, \hat{H}] = 0$.

c) Using the result of part (b), Find all the eigenstates and the corresponding eigenvalues of the Hamiltonian.

d) Write down the Heisenberg time evolution equation (Heisenberg equation of motion) for number operator and solve it to find the $\hat{N}(t)$ for $t \geq 0$ in terms of the initial number operator at $t = 0$, namely $\hat{N}$. Interpret your result.

e) Solve the Heisenberg time evolution equation for the annihilation operator, $\hat{a}(t)$, in terms of the initial annihilation operator at $t = 0$, namely $\hat{a}$.

$\Big($Note that $[\hat{a}, \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a}] = 2\hat{a}^\dagger \hat{a}\hat{a}\Big)$

f) Using the results obtained in part (d) and (e), write down the time-dependent creation operator, $\hat{a}^\dagger$, in terms of the initial creation operator at $t = 0$, namely $\hat{a}^\dagger$.

g) Suppose at time $t = 0$, the quantum state of the field is given by $|n\rangle$. At time $T > 0$, a single photon is somehow lost from the cavity and its frequency is measured by a spectrometer. Assuming a perfect spectrometer, what should be the result of this frequency measurement? (i.e. what frequencies could be measured by the spectrometer and with what probabilities.)

Problem 4: **Optical Detection in a Single Deep Quantum Well** (25 points)

Consider a single quantum well of width d as depicted in Fig. 1. We assume that the well is sufficiently deep and narrow to be considered as an infinite quantum well. This well also admits unbound (delocalized) states, where the electrons may take on any value of positive energy without being affected by the presence of the well. To avoid problems involving the normalization of these wavefunctions in the continuum, we introduce a fictitious square quantum well with infinite potential of width L within which the continuum electrons are trapped. Of course, this well is both deep and wide.

a) Write down the ground state wavefunction of the narrow and deep quantum well of width d, namely $|\psi_i\rangle$.

b) Write down the even and odd wavefunctions of the wide and deep quantum well of width L, namely $|\psi_f\rangle$.

c) Consider an electron in the ground state of the quantum well of width d, in the $|\psi_i\rangle$ state. We would like to calculate the steady state transtion rate between the ground state, and the continuum, in the $|\psi_f\rangle$ under the effect of a sinusoidally varying dipole transition: $H_p(z,t) = eE_o z\cos(\omega t)$. Calculate the nonzero transition matrix element $\langle\psi_i|H_p|\psi_f\rangle$.

d) Considering the proper density of states in the quantum well of width L, calculate the steady-state transition rate from the ground state of quantum well of width, d, to the continuum of the quantum well of width L.

e) Does your answer in part (d) depend on L? Interpret your result.

Note that: $2\int x\cos ax \sin bx dx = \frac{x}{b-a}\cos(b-a)x - \frac{1}{(b-a)^2}\sin(b-a)x + \frac{x}{b+a}\cos(b+a)x$
$- \frac{1}{(b+a)^2}\sin(b-a)x$ for $b > a$.

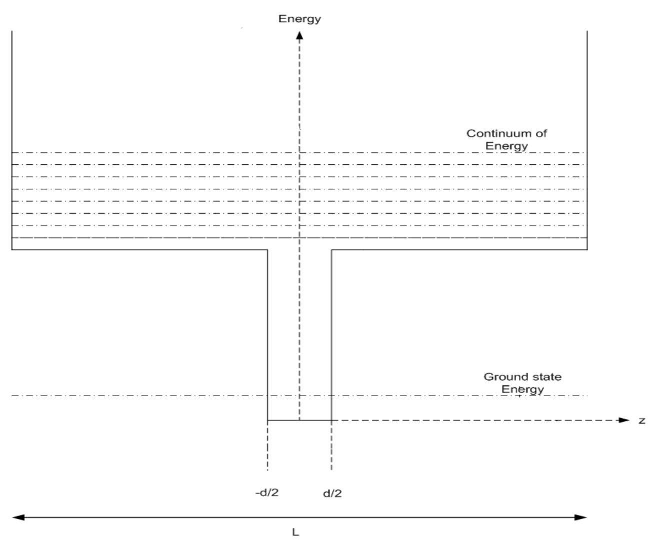

Figure 1: A Simple Model for Optical Detection in a Single Deep Quantum Well