QIC 710 Talk Notes

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Contents

Chapter 1

Classical Information Theory

1.1 Definitions

$$C = \lim_{T \to \infty} \frac{\log(N(T))}{T} \tag{1.1}$$

$$H = \sum_{i=1}^{n} p_i \log p_i \tag{1.2}$$

$$H(x, y) = -\sum_i \sum_j p(i, j) \log p(i, j) \tag{1.3}$$

$$H_x(y) = -\sum_i \sum_j p(i, j) \log p(j|i) \tag{1.4}$$

$$H_y(x) = -\sum_i \sum_j p(i, j) \log p(i|j) \tag{1.5}$$

1.2 Basic Results

Proof.

$$p = p_1^{p_1 N} p_2^{p_2 N} p_3^{p_3 N} ... p_n^{p_n N} \tag{1.6}$$

$$\log(p) = N(p_1 \log(p_1) + p_2 \log(p_2) + p_3 \log(p_3) + ... + p_n \log(p_n)) \tag{1.7}$$

$$\log(p) = -NH \tag{1.8}$$

$$p = 2^{-NH} \tag{1.9}$$

$\square$

Proof.

$$\# = \frac{N!}{n_1!n_2!n_3!...n_n!} \tag{1.10}$$

$$\log(\#) = N\log(N) - n_1\log(n_1) - n_2\log(n_2) - ... - n_n\log(n_n) \tag{1.11}$$

$$\log(\#) = N\log(N) - N(\log(N(p_1 + p_2 + ... + p_n)) + \sum_i p_i \log(p_i)) \tag{1.12}$$

$$\log(\#) = HN \tag{1.13}$$

$$\# = 2^{HN} \tag{1.14}$$

□

Proof.

Talk about the rate of the channel. Talk about how it reduces to the noiseless case.

$$\tag{1.15}$$

$$R = H(x) - H_y(x) \tag{1.16}$$

Discuss the phenemonon when $p_{fail} = 1\%$. Then move on the the maximum channel capacity.

$$\tag{1.17}$$

$$C = \max_x (H(x) - H_y(x)) \tag{1.18}$$

$$C_{erasure} = H(x) - H(X|Y) \tag{1.19}$$

optimal H(x) = 1

$$\tag{1.20}$$

$$= 1 - \sum_{i,j} p(i,j)\log(p(i|j)) \tag{1.21}$$

$$= 1 - \sum_{i,j} p(i|j)p(j)\log(p(i|j)) \tag{1.22}$$

$$\begin{aligned} = 1 - \Big(p(0)\Big(p(0|0)\log p(0|0) + p(1|0)\log p(1|0)\Big) \\ + p(1)\Big(p(0|1)\log p(0|1) + p(1|1)\log p(1|1)\Big) \\ + p(e)\Big(p(0|e)\log p(0|e) + p(1|e)\log p(1|e)\Big)\Big) \end{aligned} \tag{1.23}$$

$$\begin{aligned} = 1 - \Big(\frac{1-p}{2} 1\log 1 + 0\log 0\Big) + \frac{1-p}{2}\Big(0\log 0 + 1\log 1\Big) \\ + p\Big(.5\log .5 + .5\log .5\Big)\Big) \end{aligned} \tag{1.24}$$

$$= 1 - p \tag{1.25}$$

□

Chapter 2

Quantum Information Theory

2.1 Definitions and Properties

$$S(\rho) = -Tr(\rho \log(\rho)) = -\sum_i \lambda_i \log \lambda_i \tag{2.1}$$

$$S(|\Psi\rangle \langle\Psi|) = 0 \tag{2.2}$$

$$\mathcal{E}(\rho) = \sum_k |e_k\rangle U[\rho \otimes |e_0\rangle \langle e_0|]U^\dagger |e_k\rangle = \sum_k E_k \rho E_k^\dagger, E_k \equiv \langle e_k| U |e_0\rangle \tag{2.3}$$

2.2 Quantum Channels

2.2.1 $C_{1,1}$ Single Use

$$C_{1,1} = \max_{\rho_i} \big(H(X:Y) = H(Y) - H(Y|X) = H(X) - H(X|Y) \big) \tag{2.4}$$

2.2.2 $C_{1,\infty}$ - Infinite Use

$$C_{1,\infty} = \max_{\rho_i} \Big(S(\sum_i p_i \sigma_i) - \sum_i p_i S(\sigma_i) \Big) = \chi \tag{2.5}$$

the Holevo information

$$\tag{2.6}$$

This is the well-known Helov-Schumacher-Westmoreland (HSW) theorem

$$\tag{2.7}$$

2.2.3 C_E : Entangelement-Assisted Channels

Holds for noiseless channels. Defined over classical information that is sent.

$$C_E(\mathcal{N}) = \max_{\rho \in \mathcal{H}_{in}} \Big(S(\rho) + S(\mathcal{N}(\rho)) - S((\mathcal{N} \otimes \mathcal{I})(\Phi_\rho)) \Big) \tag{2.8}$$

Φ_ρ is an element of $\mathcal{H}_i n \otimes \mathcal{H}_R$ such that $Tr_R \Phi_\rho = \rho$.

2.3 Proofs

2.3.1 Holevo's Bound

$$H(X:Y) \leq S(\rho) - \sum_x p_x S(\rho_x) \tag{2.9}$$

$$\rho^{PQM} = \sum_x p_x |x\rangle \langle x| \otimes \rho_x \otimes |0\rangle \langle 0| \tag{2.10}$$

$$\mathcal{E}(\sigma \otimes |0\rangle \langle 0|) \equiv \sum_y \sqrt{E_y} \sigma \sqrt{E_y} \otimes |y\rangle \langle y| \tag{2.11}$$

$$S(P:Q) = S(P:Q,M) \tag{2.12}$$

since M is initially uncorrelated with P,Q

$$\tag{2.13}$$

$$S(P:Q,M) \geq S(P':Q',M') \tag{2.14}$$

applying a quantum operation can't increase mutual information between P and Q,M

$$\tag{2.15}$$

$$S(P':Q',M') \geq S(P':M') \tag{2.16}$$

tracing out Q can't increase mutual information

$$\tag{2.17}$$

$$S(P':M') \leq S(P:Q) \tag{2.18}$$

$$S(P:Q) = S(P) + S(Q) - S(P,Q) = H(p_x) + S(\rho) - (H(p_x) + \sum_x p_x S(\rho_x)) \tag{2.19}$$

$$\rho^{P'Q'M'} = \sum_{xy} p_x |x\rangle \langle x| \otimes \sqrt{E_y} \rho_x \sqrt{E_y} \otimes |y\rangle \langle y| \tag{2.20}$$

Note that

$$p(x,y) = p_x p(y|x) = p_x tr(\sqrt{E_y} \rho_x \sqrt{E_y}) \tag{2.21}$$

$$\rho^{P'M'} = \sum_{xy} p(x,y) |x\rangle \langle x| \otimes |y\rangle \langle y| \tag{2.22}$$

$$S(\rho^{P'M'}) = -Tr \rho^{P'M'} \log \rho^{P'M'} = -\sum_{xy} p(x,y) \log p(x,y) = H(X:Y) \tag{2.23}$$

Thus,

$$H(X:Y) \leq S(\rho) - \sum_x p_x S(\rho_x) \tag{2.24}$$