Final Exam

ECE 770-T14/QIC 885: Quantum Electronics & Photonics

University of Waterloo

Instructor: A. Hamed Majedi, Winter 2013, Duration: 2.5 hours

Problem 1 (50 points):

Consider a spinless object of mass m and charge q that is constrained to move in a circle of radius a in the $x-y$ plane.

a) Determine the quantized energy levels for the following cases:

a1) The motion of the object is nonrelativistic with a constant linear momentum.

Either Schrodinger equation or the Sommerfeld-Bohr quantization rule can be used. Since Schrodinger equation is used in the following parts, I use the Bohr-Sommerfeld quantization. As $\oint \mathbf{p} \cdot d\mathbf{l} = nh$, then, $|\mathbf{p}|(2\pi a) = nh$, and $E = \dfrac{|\mathbf{p}|^2}{2m}$, the quantized energy levels are:

$$\boxed{E_n = \frac{n^2\hbar^2}{2ma^2}} \tag{1}$$

a2) The motion of the object is strongly relativistic, namely $E \approx pc$, c is the speed of light.

For a relativistic object the energy-momentum relationship is $E^2 = |\mathbf{p}|^2c^2 + m_o^2c^4$, then the quantized energy levels based on Bohr-Sommerfeld quantization are $E = \sqrt{\dfrac{n^2\hbar^2c^2}{a^2} + m_o^2c^4}$. For extremely relativistic object as $E \approx |\mathbf{p}|c$, then:

$$\boxed{E_n \approx \frac{n\hbar c}{a}} \tag{2}$$

a3) The motion of the object is influenced by a constant magnetic field, B_o, perpendicular to the plane of the circle, e.g. in $+z$ direction.

Hint: Use the circular gauge for the vector magnetic potential.

In the presence of the magnetic field, the momentum of the object is changed to $\mathbf{p} + q\mathbf{A}$, where $\mathbf{A}$ is the vector magnetic potential. Using the circular gauge, $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$, since $r = a$, then the momentum becomes $|\mathbf{p}|^2 + \frac{1}{2}qB_o a = \sqrt{2mE} + \frac{1}{2}qB_o a$ and the quantized energy levels become:

$$\boxed{E_n = \frac{1}{2m}\Big(\frac{n\hbar}{a} - \frac{qB_o a}{2}\Big)^2} \tag{3}$$

b) Now consider the motion of the object that is influenced by a strong electric field, $\mathbf{E} = E_o\hat{x}$, in $+x$ direction so that $qE_o >> \dfrac{\hbar^2}{ma^2}$.

b1) Write down the Hamiltonian of the system in a cylindrical coordinate system. Note that the electric potential energy can be written as $V = -\int q\mathbf{E}\cdot d\mathbf{r}$.

Reminder: $(x = r\cos\phi, y = r\sin\phi)$ and (r, ϕ) are Cartesian and cylindrical coordinates, respectively.

The electric potential energy is $V = -\int q\mathbf{E}\cdot d\mathbf{r} = -qE_o a\cos\phi$, then

$$\boxed{\hat{H} = -\frac{\hbar^2}{2ma^2}\frac{d^2}{d\phi^2} - qE_o a\cos\phi} \tag{4}$$

b2) Given the electric field is strong, the probability that the object is moving around $\phi = 0$ is large and $\cos\phi \approx 1 - \frac{\phi^2}{2}$, rewrite the Hamiltonian in part b1).

$$\boxed{\hat{H} = -\frac{\hbar^2}{2ma^2}\frac{d^2}{d\phi^2} - qE_o a\big(1 - \frac{\phi^2}{2}\big)} \tag{5}$$

b3) Write down the time-independent Schrodinger equation for the wavefunction of the system, namely $\Psi(\phi)$.

$$\boxed{-\frac{\hbar^2}{2ma^2}\frac{d^2}{d\phi^2}\psi_(\phi) + \frac{1}{2}qE_o a\phi^2\psi(\phi) = (E + qE_o a)\psi(\phi)} \tag{6}$$

The equation (6) is the same as the TI-SE in the presence of quadratic potential and the same as simple harmonic oscillator.

b4) Determine the eigenstates of the system, i.e. $\psi(\phi)$.

The eigenfunctions of simple harmonic oscillator can be used directly with new parameters $M = ma^2$ and $\omega = \sqrt{\frac{qE_o a}{M}} = \sqrt{\frac{qE_o}{ma}}$.

$$\boxed{\psi(\phi) = \sqrt[4]{\frac{M\omega}{\pi\hbar}}\frac{1}{\sqrt{(2^n n!)}}H_n\Big(\sqrt{\frac{M\omega}{\hbar}}\phi\Big)\exp\Big(-\frac{M\omega\phi^2}{2\hbar}\Big)} \tag{7}$$

b5) Determine the quantized energy levels.

Now the quantized energy levels are as the same as simple harmonic oscillator, i.e. $E_n + qE_o a = (n + \frac{1}{2})\hbar\omega$, therefore:

$$\boxed{E_n = -qE_o a + \left(n + \frac{1}{2}\right)\hbar\sqrt{\frac{qE_o}{ma}}} \tag{8}$$

c) If there is an experiment that determines few first quantized energy levels, such as spectroscopy, how can you identify that the system under the experiment is falling in to each categories, i.e. a1), a2), a3) or b5) ?

The case of a1) has n^2 dependence and a3) has both n^2 and n dependence that can be distinguished from others and each other. Although the a2) and b5) exhibit linear dependence on n but the only way through energy measurement is to get the ground state, as for a2) the ground state is zero but nonzero value for the case of b5).

Problem 2 (20 points):

Consider a system of free electron gas. A fraction p is known to have their z-component of spin in the up direction, i.e. $|\uparrow\rangle$ state, and remainder are randomly in both up and down direction with equal probability.

a) Write down the density matrix describing the system.

By having the spin up and down states as:

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then the density matrix is:

$$\boxed{\rho = \sum p_i|\Psi_i\rangle\langle\Psi_i| = p|\uparrow\rangle\langle\uparrow| + (1-p)\left[\frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow|\right]}$$

by having the spinor basis, then the matrix density can be calculated as:

$$\rho = \frac{1}{2}\begin{pmatrix} 1+p & 0 \\ 0 & 1-p \end{pmatrix} = \frac{1}{2}\left[\mathbf{I} + p\sigma_z\right]$$

where **I** is the identity matrix.

b) Find the expectation value of spin in x, y and z directions.

$$\boxed{\langle \hat{S}_x \rangle = \mathrm{Tr}(\rho \hat{S}_x) = \frac{\hbar}{4}\big[\sigma_x + p\sigma_x\sigma_z\big] = 0}$$

$$\boxed{\langle \hat{S}_y \rangle = \mathrm{Tr}(\rho \hat{S}_y) = \frac{\hbar}{4}\big[\sigma_y + p\sigma_y\sigma_z\big] = 0}$$

$$\boxed{\langle \hat{S}_z \rangle = \mathrm{Tr}(\rho \hat{S}_z) = \frac{\hbar}{4}\big[\sigma_z + p\sigma_z\sigma_z\big] = \frac{\hbar}{2}p}$$

Problem 3 (30 points):

Consider an electromagnetic field intensity operator $\hat{I}(z) = I_o \exp(gz\hat{N})$ representing an intensity of a traveling wave along $z > 0$ direction with constant I_o and g, where $\hat{N}$ is the number operator. Calculate the uncertainty in the measurement of the field intensity operator in a coherent state basis.

Using the expansion of the coherent states in the number state basis, i.e. $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$, the expectation value of field intensity operator can be written as:

$$\langle\alpha|\hat{I}|\alpha\rangle = e^{-|\alpha|^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}}\langle m|\hat{I}|n\rangle \tag{9}$$

We need to calculate the $\langle m|\hat{I}|n\rangle$. Using Taylor Expansion the $e^{gz\hat{N}} = \sum_{j=0}^{\infty}\frac{(gz\hat{N})^j}{j!}$, having the orthonormality of number state basis, i.e. $\langle m|n\rangle = \delta_{mn}$, and the fact that $\hat{N}^n|n\rangle = n^n|n\rangle$, we have then:

$$\langle\alpha|\hat{I}(z)|\alpha\rangle = I_o e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{(\alpha^2 e^{gz})^n}{n!} = I_o \exp\left[|\alpha|^2(e^{gz}-1)\right]$$

We need to calculate the $\langle\alpha|\hat{I}^2(z)|\alpha\rangle$, which can be done simply by using previous result:

$$\langle\alpha|\hat{I}^2(z)|\alpha\rangle = I_o^2 \exp\left[|\alpha|^2(e^{2gz}-1)\right]$$

Therefore the uncertainty is:

$$\boxed{\sigma_I = \sqrt{\langle\alpha|\hat{I}^2(z)|\alpha\rangle - \big(\langle\alpha|\hat{I}(z)|\alpha\rangle\big)^2} = I_o\Big(\exp\Big[|\alpha|^2(e^{2gz}-1)\Big] - \exp\Big[2|\alpha|^2(e^{gz}-1)\Big]\Big)}$$