QIC 885 - QEP HW4 Solutions

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Abstract

What follows are my (John Rinehart's) solutions for the fourth problem set assigned as a part of Quantum Electronics and Photonics (course number: QIC 885) taught by Dr. Hamed Majedi in the Winter term of 2014.

PROBLEM 1

Consider an object in a one-dimensional simple harmonic oscillator (SHO) that is subjected to a constant force, F for $t > 0$. The object has been initiated in the ground state at $t = 0$. Using the the Heisenberg picture:

a) Find the expectation value of the position of the object. b) Find the expectation value of the momentum.

Problem 1 Solution

There are a few ways to approach this problem. I have avoided some techniques because I doubt their validity. I will highlight a couple techniques I believe are valid. The first technique will use definitions motivated by classical system responses to constant forces; it will explicitly avoid the use of a Hamiltonian. The second technique will attempt to attain the same results by use of a time-dependent Hamiltonian and the Schrodinger equation.

Technique One : Consider the force to act on the particle in such a way that the following holds $\frac{d\hat{p}}{dt} = F\hat{\mathbb{I}}$. This implies that $\hat{p}(t) = Ft\mathbb{I} + \hat{C}$, where $\hat{C}$ is the initial momentum operator $\hat{C} = \hat{p}(0) = \hat{p}_0$. Thus, the Heisenberg representation of $\hat{p}(t) = Ft\mathbb{I} + \hat{p}_0$. Assuming the mass of the object is time-independent (only valid for short times in order that the constant force does not bring the particle into a relativistic regime) $\frac{d\hat{x}(t)}{dt} = \frac{\hat{p}(t)}{m} = \frac{Ft\mathbb{I}+\hat{p}_0}{m}$. Now, this implies that $\hat{x}(t) = \frac{Ft^2\mathbb{I}}{2m} + \frac{\hat{p}_0}{m} + \hat{x}_0$. The commutator $[\hat{x}(t), \hat{p}(t)]$ should be $i\hbar$, as usual. Confirming this:

$$
\begin{aligned}
&[\frac{Ft^2\mathbb{I}}{m} + \frac{\hat{p}_0}{m} + \hat{x}_0, Ft\mathbb{I} + \hat{p}_0] && (1)\\
=&(\frac{Ft^2\mathbb{I}}{m} + \frac{\hat{p}_0}{m} + \hat{x}_0)(Ft\mathbb{I} + \hat{p}_0) - (Ft\mathbb{I} + \hat{p}_0)(\frac{Ft^2\mathbb{I}}{m} + \frac{\hat{p}_0}{m} + \hat{x}_0) && (2)\\
=&(\frac{F^2t^3\mathbb{I} + Ft^2\hat{p}_0}{m}) + (\frac{Ft\hat{p}_0 + \hat{p}_0^2}{m}) + (Ft\hat{x}_0 + \hat{x}_0\hat{p}_0) && (3)\\
&-(\frac{F^2t^3\mathbb{I} + Ft\hat{p}_0}{m} + Ft\hat{x}_0) - (\frac{Ft^2\hat{p}_0 + \hat{p}_0^2}{m} + \hat{p}_0\hat{x}_0) && (4)\\
=&\hat{x}_0\hat{p}_0 - \hat{p}_0\hat{x}_0 && (5)\\
=&[\hat{x}_0, \hat{p}_0] && (6)\\
=&i\hbar && (7)
\end{aligned}
$$

Now, to determine the expectation value I just stick the (time-dependent) operators between the (initial) state bra and ket:

$\langle\hat{x}(t)\rangle = \langle\psi_0|\,\hat{x}(t)\,|\psi_0\rangle = \langle\psi_0|\,\frac{Ft^2\mathbb{I}}{2m} + \frac{\hat{p}_0}{m} + \hat{x}_0\,|\psi_0\rangle = \frac{Ft^2}{2m} + \langle\psi_0|\,\frac{\hat{p}_0}{m}\,|\psi_0\rangle + \langle\psi_0|\,\hat{x}_0\,|\psi_0\rangle$. Now, for all energy eigenstates of the harmonic oscillator, the expected value of the position is zero. A similar case is true for the expectation value of the momentum operator for an energy eigenstate. This can be verified by representing the position and momentum operators in terms of the ladder operators. The ladder operators will generate kets that are orthogonal to the bras. Thus, the inner product will be zero for an energy eigenstate. All this being said,

$\langle\psi_0|\,\hat{x}(t)\,|\psi_0\rangle = \frac{Ft^2}{2m}$, in accordance with classical expectation (Ehrenfest's theorem).

1b) Using the results of previous: $\langle\hat{p}(t)\rangle = \langle\psi_0|\,\hat{p}(t)\,|\psi_0\rangle = \langle\psi_0|\,Ft\mathbb{I} + \hat{p}_0\,|\psi_0\rangle = Ft$

NOW DO FOR HAMILTONIAN CASE.

PROBLEM 2

Consider a simple harmonic oscillator and a new operator defined as $\hat{G}(t) = m\hat{x}(t)cos(\omega t) - \hat{p}(t)sin(\omega t)$.

a) Can this operator be simultaneously diagonalized with the Hamiltonian? Justify your answer.

b) Find the equation of motion for $\hat{G}(t)$. Can this operator be treated as a constant of the particle's motion?

c) Solve the equation of motion, if the initial position and momentum are both known.

Problem 2 Solution

If two matrices are simultaneously diagonalizable then they share a set of eigenvectors. If they share a set of eigenvectors then it is trivially shown that these matrices commute. Thus, if $\hat{G}(t)$ is simultaneously diagonizable with the Hamiltonian then it will commute with the Hamiltonian.

$$\begin{aligned}
[\hat{G}, \hat{H}] =& [f(t)\hat{x}(t) + g(t)\hat{p}(t), \kappa\hat{x}^2 + \gamma\hat{p}^2] \\
=& \kappa f(t)[\hat{x}(t), \hat{x}^2] + \gamma f(t)[\hat{x}(t), \hat{p}^2] + \kappa g(t)[\hat{p}(t), \hat{x}^2] + \gamma g(t)[\hat{p}(t), \hat{p}^2]
\end{aligned}$$

Now, though not explicitly stated, $\hat{G}(t)$ has been given in the Heisenberg representation (as have $\hat{x}(t)$ and $\hat{p}(t)$). Thus, take the Hamiltonian operators $\hat{x}^2$ and $\hat{p}^2$ to be given by their Heisenberg representation, also : $\hat{x}^2 \to \hat{x}(t)^2$ and $\hat{p}^2 \to \hat{p}(t)^2$.

$$
\begin{aligned}
&[\hat{G}, \hat{H}] = \gamma f(t)[\hat{x}(t), \hat{p}(t)^2] + \kappa g(t)[\hat{p}(t), \hat{x}(t)^2] \\
&= \gamma f(t)(\hat{x}(t)\hat{p}(t)\hat{p}(t) - \hat{p}(t)\hat{p}(t)\hat{x}(t)) \\
&\qquad\qquad + \kappa g(t)(\hat{p}(t)\hat{x}(t)\hat{x}(t) - \hat{x}(t)\hat{x}(t)\hat{p}(t))
\end{aligned}
$$

Using the fact that $\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$

$$
\begin{aligned}
&= \gamma f(t)((i\hbar + \hat{p}(t)\hat{x}(t))\hat{p}(t) - \hat{p}(t)(\hat{x}(t)\hat{p}(t) - i\hbar)) \\
&\qquad\qquad + \kappa g(t)((\hat{x}(t)\hat{p}(t) - i\hbar)\hat{x}(t) - \hat{x}(t)(i\hbar + \hat{p}(t)\hat{x}(t))) \\
&= \gamma f(t)(2i\hbar\hat{p}(t)) + \kappa g(t)(-2i\hbar\hat{x}(t)) \\
&= 2i\hbar(\gamma f(t)\hat{p}(t) - \kappa g(t)\hat{x}(t)) \\
&\neq 0
\end{aligned}
$$

Thus, since $\hat{G}(t)$ does not commute with the Hamiltonian, it is not simultaneously diagonalizable with the Hamiltonian.

2b) The equation of motion of an operator is given by the Schrodinger equation. SHOW HERE. It can be shown that a valid time transformation (one that preserves inner products/expectation values) is given by a unitary transformation of the following form: $O_H(t) = e^{\frac{-i\hat{H}t}{\hbar}} O_S(t) e^{\frac{i\hat{H}t}{\hbar}}$, where $O_H(t)$ and $O_S(t)$, respectively, represent the Heisenberg-picture and Schrodinger picture of the operator. Thus, it must be the case that $\frac{dO_H(t)}{dt} = e^{\frac{i\hat{H}t}{\hbar}}(\frac{i\hat{H}O_S(t)}{\hbar} + \frac{dO_S(t)}{dt} + \frac{-i\hat{H}O_S(t)}{\hbar})e^{\frac{-i\hat{H}t}{\hbar}} = e^{\frac{i\hat{H}t}{\hbar}}(\frac{i}{\hbar}[\hat{H}, O_S(t)] + \frac{dO_S(t)}{dt})e^{\frac{-i\hat{H}t}{\hbar}} = \frac{i}{\hbar}[\hat{H}, O_H(t)] + e^{\frac{i\hat{H}t}{\hbar}}\frac{dO_S(t)}{dt}e^{\frac{-i\hat{H}t}{\hbar}} = \frac{i}{\hbar}[\hat{H}, O_H(t)] + \frac{\partial O}{\partial t}$. Assuming the operator I have been given is one in the Schrodinger picture (where the state carries the time dependence) and using the results of the previous problem (2a) I have:

$$
\frac{dG(t)}{dt} = 2(\gamma f(t)\hat{p}(t) - \kappa g(t)\hat{x}(t)) + (\hat{x}(t)\frac{\partial f(t)}{\partial t} + \hat{p}(t)\frac{\partial g(t)}{\partial t})
$$

Now, evaluating the left-hand side of that expression is possible given I know the form of $\hat{G}(t)$

$$\frac{d\hat{G}(t)}{dt} = \frac{df(t)}{dt}\hat{x}(t) + f(t)\frac{d\hat{x}(t)}{dt} + \frac{dg(t)}{dt}\hat{p}(t) + g(t)\frac{\hat{p}(t)}{dt}$$

Realizing that $\frac{df(t)}{dt} = \frac{\partial f(t)}{dt}$ since f is a sole function of t (similarly with g(t))

$$f(t)\dot{\hat{x}}(t) + g(t)\dot{\hat{p}}(t) = 2\gamma f(t)\hat{p}(t) - 2\kappa g(t)\hat{x}(t)$$

PROBLEM 5

A spinless object is described by the wavefunction $\psi = A(x + y + 2z)e^{-\alpha r}$ where A and α are real constant numbers and $r = \sqrt{x^2 + y^2 + z^2}$.

1. What is the total angular momentum of the object?

   Note that the position representation of the angular momentum operators L_x, L_y and L_z are $L_x = (-i\hbar)(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})$, $L_y = (-i\hbar)(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z})$ and $L_z = (-i\hbar)(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$. Let's consider the partial derivatives of ψ with respect to each of its independent variables.

   $\frac{\partial \psi}{\partial x} = e^{-a\sqrt{x^2+y^2+z^2}}(1 - ax\frac{(x+y+2z)}{\sqrt{x^2+y^2+z^2}})$
   $\frac{\partial \psi}{\partial y} = e^{-a\sqrt{x^2+y^2+z^2}}(1 - ay\frac{(x+y+2z)}{\sqrt{x^2+y^2+z^2}})$
   $\frac{\partial \psi}{\partial z} = e^{-a\sqrt{x^2+y^2+z^2}}(2 - az\frac{(x+y+2z)}{\sqrt{x^2+y^2+z^2}})$

So, now, the component of x momentum is:

$$\hat{L}_x\psi = -i\hbar(y * e^{-a\sqrt{x^2+y^2+z^2}}(2 - az\frac{(x+y+2z)}{\sqrt{x^2+y^2+z^2}}) - z * e^{-a\sqrt{x^2+y^2+z^2}}(1 - ay\frac{(x+y+2z)}{\sqrt{x^2+y^2+z^2}})) \tag{8}$$

$$= -i\hbar e^{-a\sqrt{x^2+y^2+z^2}}(2y - z) \tag{9}$$

$$\hat{L}_y\psi = -i\hbar(z * e^{-a\sqrt{x^2+y^2+z^2}}(1 - ax\frac{(x+y+2z)}{\sqrt{x^2+y^2+z^2}}) - x * e^{-a\sqrt{x^2+y^2+z^2}}(2 - az\frac{(x+y+2z)}{\sqrt{x^2+y^2+z^2}})) \tag{10}$$

$$= -i\hbar e^{-a\sqrt{x^2+y^2+z^2}}(z - 2x) \tag{11}$$

$$\hat{L}_z\psi = -i\hbar(x * e^{-a\sqrt{x^2+y^2+z^2}}(1 - ay\frac{(x+y+2z)}{\sqrt{x^2+y^2+z^2}}) - y * e^{-a\sqrt{x^2+y^2+z^2}}(1 - ax\frac{(x+y+2z)}{\sqrt{x^2+y^2+z^2}})) \tag{12}$$

$$= -i\hbar e^{-a\sqrt{x^2+y^2+z^2}}(x - y) \tag{13}$$

The total angular momentum can be expressed as $L = \sqrt{L_x^2 + L_y^2 + L_Z^2}$.

$$L_x^2 = -i\hbar(-i\hbar)(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}) \tag{14}$$

2. What is the expectation value of the z-component of the angular momentum?

3. If the z-component of the angular momentum was measured, what is the probability of obtaining $+\ \hbar$?