Lecture 10: The phase qubit[a]

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[a] This is part of the material in the lecture; the complement is the USEQIP lecture notes.

I. NOTIONS OF QUANTUM COMPUTING AND QUANTUM CONTROL

In this section we introduce basic notions of quantum computing and quantum control, in particular those relevant to the discussion of single qubits.

A. Quantum computing

Quantum computing employs quantum mechanics for processing information. In this course we will be concerned primarily with the circuit model of quantum computing. In this model, a physical system is considered formed by a set of quantum bits. A quantum bit (qubit) is a two level system, which can be initialized, controlled, and measured. At the beginning of a computation, the bits are initialized in a well defined state. Information processing is achieved through a controlled unitary evolution of the state of the n-qubit system. Finally, relevant information is extracted through measurement.

The controlled quantum evolution of the qubits can be implemented in different ways. The best approach depends on the physical system considered, the interaction between qubits, and the specific operations to be implemented. A fundamental theorem in quantum computing states that any unitary operation can be implemented to any required accuracy by the successive application of single and two-qubit gates, with each gate chosen from a *universal set of quantum gates* [1].

Before introducing universal sets of gates, we discuss conventions for representations of operations. We choose a particular *computational basis* of the qubit, with the two states labeled as $|0\rangle$ and $|1\rangle$. The order (relevant in the matrix representation of operations) is by convention usually taken to be $0, 1$. For a n qubit system, the computational basis is formed by states of the form $|i_1 i_2 .. i_n\rangle$, with i_j =0 or 1. The convention for ordering the basis for the n qubit system is to take states in increasing order of the binary string $i_1 i_2 .. i_n$. With this chosen order, any operator A for a n qubit system is represented by a matrix A with elements A_{ij}, $i, j = \overline{0, 2^n - 1}$, with

$$\mathrm{A}_{ij} = \langle b(i)|A|b(j)\rangle \tag{1}$$

where $b(i)$ is the n-bit binary string equal to i. For example, a single qubit operator O^1 is represented as

$$\mathrm{O}^1 = \begin{bmatrix} \langle 0|A|0\rangle & \langle 0|A|1\rangle \\ \langle 1|A|0\rangle & \langle 1|A|1\rangle \end{bmatrix}. \tag{2}$$

One particular universal set of quantum gates is formed by the following combination of single and two qubit gates:

1. single qubit gates: a sufficient set is formed by the gates Hadamard (H), Phase (S), and $\pi./8$ (T) gates. They are given by

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{3}$$

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \tag{4}$$

   and

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}. \tag{5}$$

2. one two-qubit gate: the CNOT gate

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{6}$$

For further reading on quantum gates, see [1].

B. Bloch sphere representation of qubit states

Single qubit states can be represented using the Bloch sphere. Any single qubit state can be written, up to a irrelevant global phase, as

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle. \tag{7}$$

Based on this, any qubit state can be represented by the point, on the unit sphere, with coordinates

$$p_x = \sin\theta\cos\phi, \tag{8}$$

$$p_y = \sin\theta\sin\phi, \tag{9}$$

$$p_z = \cos\theta. \tag{10}$$

This is the Bloch sphere representation of a quantum state (see Fig. 1). The $|0\rangle/|1\rangle$ state is at the North/South pole of the sphere.

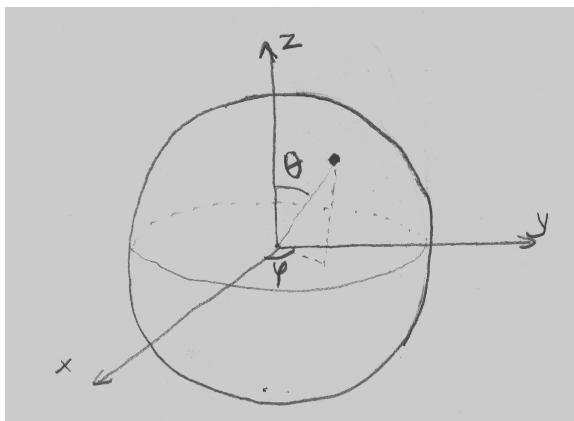

FIG. 1. Representation of the state of a qubit on the Bloch sphere.

C. State evolution

We introduce the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{11}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{12}$$

and

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

The Hamiltonian of a qubit can be written as a linear combination with real coefficients of these Pauli matrices:

$$H = \frac{\hbar}{2} \boldsymbol{\omega}(t) \boldsymbol{\sigma}, \tag{14}$$

with $\boldsymbol{\omega}(t)$ the vector of real coefficients and $\boldsymbol{\sigma}$ the vector with components the Pauli matrices. The reason for this is the fact that the Pauli matrices together with the identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{15}$$

form a complete set, in terms of which any 2x2 matrix can be expressed as a linear combination. The Hamiltonian is a Hermitian matrix, so the coefficients must be real (as the identity and the Pauli matrices are themselves Hermitian). Finally, we can drop the identity matrix, which is just a shift of the energy levels, without any consequence for the dynamics.

We can express the position on the Bloch sphere in terms of expectation values of the Pauli operators. We have

$$p_x = \langle \sigma_x \rangle, \tag{16}$$

$$p_y = \langle \sigma_y \rangle, \tag{17}$$

$$p_z = \langle \sigma_z \rangle \tag{18}$$

or in more compact notation

$$\boldsymbol{p} = \langle \boldsymbol{\sigma} \rangle. \tag{19}$$

We can represent the time evolution of the qubit by the evolution of the polarization vector $\boldsymbol{p}(t)$.

It is convenient for the following to use the Heisenberg representation. In this representation the wavefunction is fixed to its value at some time t_0 and the time-dependence is transferred to the operators. Thus the evolution of an operator A is given by

$$A_H(t) = U(t, t_0)^\dagger A(t) U(t, t_0) \tag{20}$$

where we included the possibility of some intrinsic time dependence in the operator by writing $A(t)$. $U(t, t_0)$ is the evolution operator, defined by

$$U(t_0, t_0) = I \tag{21}$$

and

$$i\hbar \frac{d}{dt} U(t, t_0) = H(t) U(t, t_0). \tag{22}$$

We find

$$\frac{dA_H}{dt} = \frac{i}{\hbar}[H, A_H]. \tag{23}$$

We can now determine the time dependence of the Pauli matrix operators. We use the commutation relations

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k. \tag{24}$$

with ϵ_{ijk} and a summation assumed over dummy indices. Using 23 we find

$$\frac{d\langle \boldsymbol{\sigma}_H \rangle}{dt} = \boldsymbol{\omega}(t) \times \langle \boldsymbol{\sigma}_H \rangle. \tag{25}$$

This can also be written as

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{\omega}(t) \times \boldsymbol{p}. \tag{26}$$

The polarization thus precesses around the instantaneous direction of the vector $\boldsymbol{\omega}(t)$ with an angular velocity given by $|\boldsymbol{\omega}(t)|$.

II. THE PHASE QUBIT

For circuit quantization, as well as elements on operation: following the summer school notes (USEQIP 2011).

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2000).