Quantum Electronics and Photonics

Assignment 4

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Problem 1

In this problem we frequently use two basic properties of the Pauli matrices:

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij} \tag{1}$$

$$[\sigma_k, \sigma_l] = \sigma_k\sigma_l - \sigma_l\sigma_k = 2i\sum_m \varepsilon_{klm}\sigma_m \tag{2}$$

where ε_{klm} is Levi-Civita symbol.

(a)

This problem can be simply solved using the nice properties of the Pauli matrices. We first develop a simple instruction to a more general problem. We claim that a 2×2 matrix X (not necessarily Hermitian, nor unitary) can be written as:

$$X = \sum_i a_i\sigma_i + a_0 = \mathbf{a}.\boldsymbol{\sigma} + a_0 \tag{3}$$

Actually we are going to show that the Pauli matrices and unity matrix form a complete set of independent basis to expand any arbitrary 2×2 matrices defined in complex field ($\mathbb{C}$). First we prove that they are independent. Assume that they are expanding a null matrix, we will show all expansion coefficents are zero:

$$\sum_i \sigma_i\alpha_i + \alpha_0 = 0 \tag{4}$$

If we calculate the anticommutation of above equation and one of the Pauli matrices and by employing (1) we arrive at:

$$\left\{\sum_i \sigma_i\alpha_i + \alpha_0, \sigma_k\right\} = 2\alpha_k + 2\alpha_0\sigma_k = 0 \tag{5}$$

Please note that all Pauli matrices are traceless, that is:

$$\mathrm{Tr}(\sigma_i) = \sum_k \sigma_{kk} = 0 \tag{6}$$

So we have:

$$\mathrm{Tr}\left(2\alpha_k + 2\alpha_0\sigma_k\right) = 4\alpha_k = 0 \tag{7}$$

So α_ks are zero and from (5) α_0 is zero as well. So this set of matrices are independent and consequently all a_is in (3) can be uniqely determined. To evaluate the expansion coefficients we use traceless property of the pauli matrices. First we have to prove the following identity:

$$\mathrm{Tr}\left(\sigma_i\sigma_j\right) = 2\delta_{ij} \tag{8}$$

Proof: Using (1) we have:

$$\mathrm{Tr}\left\{\sigma_i, \sigma_j\right\} = \mathrm{Tr}\left(\sigma_i\sigma_j\right) + \mathrm{Tr}\left(\sigma_j\sigma_i\right) = 4\delta_{ij} \tag{9}$$

The trace of multiplication of two matrices in independent of the order of multiplication:

$$\mathrm{Tr}\left(AB\right) = \sum_k (AB)_{kk} = \sum_k\sum_j A_{kj}B_{jk} = \sum_j\sum_k B_{jk}A_{kj} = \sum_j (BA)_{jj} = \mathrm{Tr}\left(BA\right) \tag{10}$$

therefore:

$$\mathrm{Tr}\left\{\sigma_i, \sigma_j\right\} = 2\,\mathrm{Tr}\left(\sigma_i\sigma_j\right) = 4\delta_{ij} \quad \Longrightarrow \quad \mathrm{Tr}\left(\sigma_i\sigma_j\right) = 2\delta_{ij} \tag{11}$$

Problem 1 continued on next page. . .

Using this important identity we can evaluate a_is in (3). By mutiplying the both sides of that equation by σ_i we get:

$$\mathrm{Tr}\,(\sigma_i X) = 2a_i \tag{12}$$
$$\mathrm{Tr}\,(X) = 2a_0 \tag{13}$$

Now we can return back to our main problem. The density matrix as a 2 matrix (like X) can be writen as:

$$\rho = |\Psi(t)\rangle\langle\Psi(t)| = \frac{1}{2}\left(\xi + \sum_i \Psi_i\sigma_i\right) \tag{14}$$

Using (12) and (13) we have:

$$\xi = \mathrm{Tr}(\rho) = \mathrm{Tr}\,(|\Psi(t)\rangle\langle\Psi(t)|) = \sum_b \langle b|\Psi(t)\rangle\langle\Psi(t)|b\rangle = \langle\Psi(t)|\Psi(t)\rangle = 1 \tag{15}$$

In above equation $|b\rangle$ are the eigenstates a given Hermitian operator. For the other coefficients we obtain:

$$\Psi_i = \mathrm{Tr}\,(\sigma_i\rho) \tag{16}$$

Now we can calculate every coefficient explicitly:

$$\Psi_1 = \mathrm{Tr}\left[(|0\rangle\langle 1| + |1\rangle\langle 0|)\,|\Psi(t)\rangle\langle\Psi(t)|\right] = 2\Re\left\{\langle 1|\Psi(t)\rangle\langle 0|\Psi(t)\rangle^*\right\} \tag{17}$$
$$\Psi_2 = i\,\mathrm{Tr}\left[(|0\rangle\langle 1| - |1\rangle\langle 0|)\,|\Psi(t)\rangle\langle\Psi(t)|\right] = 2\Im\left\{\langle 1|\Psi(t)\rangle\langle 0|\Psi(t)\rangle^*\right\} \tag{18}$$
$$\Psi_3 = \mathrm{Tr}\left[(|1\rangle\langle 1| - |0\rangle\langle 0|)\,|\Psi(t)\rangle\langle\Psi(t)|\right] = \langle 1|\Psi(t)\rangle^2 - \langle 0|\Psi(t)\rangle^2 \tag{19}$$

So we arrive at:

$$\rho = \frac{1}{2}\left(1 + \mathbf{\Psi}.\boldsymbol{\sigma}\right) \tag{20}$$

(b)

The equation of motion for the density matrix is:

$$i\hbar\frac{\partial\rho}{\partial t} = [\mathcal{H}, \rho] \tag{21}$$

The Hamiltonian is given by:

$$\mathcal{H} = \frac{1}{2}\left(1 + \mathbf{H}.\boldsymbol{\sigma}\right) \tag{22}$$

By inserting (22) and (20) into the equation of motion we obtain:

$$i\hbar\frac{d}{dt}\left\{\frac{1}{2}\left(1 + \mathbf{\Psi}(t).\boldsymbol{\sigma}\right)\right\} = \frac{i\hbar}{2}\frac{d\mathbf{\Psi}}{dt}.\boldsymbol{\sigma} = \frac{1}{4}\left[1 + \mathbf{H}.\boldsymbol{\sigma}, 1 + \mathbf{\Psi}.\boldsymbol{\sigma}\right] \tag{23}$$

therefore:

$$\frac{d\mathbf{\Psi}}{dt}.\boldsymbol{\sigma} = \frac{1}{2}\left[\sum_k H_i\sigma_k, \sum_l \Psi_l\sigma_l\right] = \frac{1}{2}\sum_{k,l} H_k\Psi_l\left[\sigma_k, \sigma_l\right] = \frac{i}{i\hbar}\sum_{k,l,m}\varepsilon_{klm}H_k\Psi_l\sigma_m \tag{24}$$

From elementary vector analysis the Triple product of three vectors is:

$$(\mathbf{A}\times\mathbf{B}).\mathbf{C} = \sum_{l,m} A_k B_l C_m \varepsilon_{klm}$$

hence:

$$\boldsymbol{\sigma}.\frac{d\mathbf{\Psi}}{dt} = \frac{1}{\hbar}\boldsymbol{\sigma}.(\mathbf{H}\times\mathbf{\Psi}) \tag{25}$$

Since the Pauli matrices are linearly independent we immediately conclude:

$$\frac{d\mathbf{\Psi}}{dt} = \frac{1}{\hbar}(\mathbf{H}\times\mathbf{\Psi}) \tag{26}$$

(c)

To solve (26) we simply assume that the coordinate system has been properly rotated so that the**real** vector $\mathbf{H}$ has been aligned in the $\hat{z}$ direction. This assumption dosn't degrade the solution because the final solution is independent of the coordinate system describing the problem. A linear transformation $\mathcal{T}$ exists that produces the desired rotation:

$$\mathbf{H'} = \mathbf{T}^t \mathbf{H} = |\mathbf{H}|\,\hat{z} \qquad \mathbf{\Psi'} = \Psi_1'\hat{x} + \Psi_2'\hat{y} + \Psi_3'\hat{z} = \mathbf{T}^t\mathbf{\Psi} \tag{27}$$

From (26) we obtain:

$$\frac{d}{dt}\mathbf{\Psi'} = \frac{1}{\hbar}\,|\mathbf{H}|\,\hat{z}\times\mathbf{\Psi'} \tag{28}$$

Three coupled equations are:

$$\frac{d\Psi_1'}{dt} = -\frac{1}{\hbar}\,|\mathbf{H}|\,\Psi_2' \tag{29}$$
$$\frac{d\Psi_2'}{dt} = \frac{1}{\hbar}\,|\mathbf{H}|\,\Psi_1' \tag{30}$$
$$\frac{d\Psi_3'}{dt} = 0 \tag{31}$$

From the fisrt two equations we get:

$$\frac{d^2\Psi_1'}{dt^2} = -\omega^2\Psi_1' \quad\Longrightarrow\quad \Psi_1'(t) = \Psi_1'(0)\cos\omega t + \frac{1}{\omega}\dot{\Psi}_1'(0)\sin\omega t \tag{32}$$
$$\frac{d^2\Psi_2'}{dt^2} = -\omega^2\Psi_2' \quad\Longrightarrow\quad \Psi_2'(t) = \Psi_2'(0)\cos\omega t + \frac{1}{\omega}\dot{\Psi}_2'(0)\sin\omega t \tag{33}$$

where:

$$\omega = \frac{|\mathbf{H}|}{\hbar}$$

using the equation of motions we can write:

$$\Psi_1'(t) = \Psi_1'(0)\cos\omega t - \Psi_2'(0)\sin\omega t \tag{34}$$
$$\Psi_2'(t) = \Psi_2'(0)\cos\omega t + \Psi_1'(0)\sin\omega t \tag{35}$$

$\Psi_3'(t)$ is simply a constant function.

Problem 2

(a)

To calculate the average energy of the system composed on N identical SHOs, we can simply use the partition function defined as

$$Z = \mathrm{Tr}\left(e^{-\beta\hat{H}}\right)$$

So for a system of identical harmonic oscillators we obtain:

$$Z = \sum_{n=0}^{\infty} e^{-\beta(n+1/2)\hbar\omega} = \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}} = \frac{2}{\sinh\frac{\beta\hbar\omega}{2}} \tag{1}$$

So the ensemble average of the energy of the system is:

$$\langle E\rangle = -\frac{\partial \ln Z}{\partial\beta} = \frac{\partial}{\partial\beta}\ln\left(\sinh\frac{\beta\hbar\omega}{2}\right) = \frac{\hbar\omega}{2}\coth\frac{\beta\hbar\omega}{2} \tag{2}$$

(b)

Assume that $\hbar\omega \gg k_B T$ or equivalently $\hbar\omega\beta \gg 1$ using large argument approximation of cotangent hyperbolic function we obtain:

$$\hbar\omega\beta \gg 1 \quad \Longrightarrow \quad \coth\left(\frac{\hbar\omega\beta}{2}\right) \approx 1 \quad \Longrightarrow \quad \langle E \rangle \approx \frac{\hbar\omega}{2} \tag{3}$$

There is an interesting interpretation for the approximate ensemble average of energy for large values of $\hbar\omega$. Every harmonic oscillator has *zero point energy* which is $\hbar\omega/2$, that is in the equilubrium state the energy of every oscillator should be greater than or equal to $\hbar\omega/2$. Maximization of entropy prohibites arbitary distribution of energy and and most probable energy levels are smaller than $k_B T$. So when $E_0 \gg k_B T$ all energy states have to be in their ground state. So the average of this uniform distribution is approximately $\hbar\omega/2$. This zero point energy is responsible for one the of sources of noise in an electrical resistor.

Now lets to think about the second asymptotic case. Assume that $\hbar\omega \ll k_B T$ or $\beta\hbar\omega \ll 1$. Using small argument approximation of cotangent hyperbolic function we get:

$$\beta\hbar\omega \ll 1 \quad \Longrightarrow \quad \coth\frac{\beta\hbar\omega}{2} \approx \frac{1}{\sinh\frac{\beta\hbar\omega}{2}} \approx \frac{2}{\beta\hbar\omega} \tag{4}$$

So we have:

$$\hbar\omega\beta \ll 1 \quad \Longrightarrow \quad \coth\left(\frac{\hbar\omega\beta}{2}\right) \approx \frac{2}{\beta\hbar\omega} \quad \Longrightarrow \quad \langle E \rangle \approx \frac{1}{\beta} = k_B T \tag{5}$$

Unlike the fisrt case, here a huge group of microstates are possible and the average energy is so close to $k_B T$.

(c)

The density matrix in the equilibrium can be diagonalized by eigen states of the Hamiltonian:

$$\frac{\partial\rho}{\partial t} = 0 \quad \Longrightarrow \quad i\hbar\frac{\partial\rho}{\partial t} = [\rho, H] = 0 \tag{6}$$

So the density matrix is:

$$\rho = \frac{\sum_n \exp(-\beta E_n)|n\rangle\langle n|}{\sum_n \exp(-\beta E_n)} = 2\sinh(\frac{\beta\hbar\omega}{2})\sum_n \exp(-\beta E_n)|n\rangle\langle n| \tag{7}$$

where $|n\rangle$ are the eigenstates of the Hamiltonian.

(d)

From the transmission line theory a resistor can be modeled by a ladder network composed of an infinite number of L and C as shown in the figure.

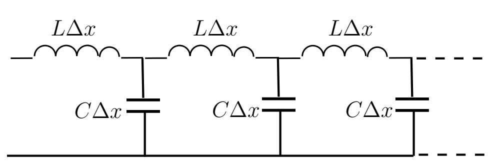

Figure 1: an infinite LC ladder network which models a resistor

It can be readily shown that when $\Delta x \to 0$, the input impedance behaves like a pure resistor. The equivalent input resistance is:

$$R = Z_c = \sqrt{\frac{L}{C}} \tag{8}$$

(e)

As discussed in the previous part a resistor can be modeled by a series of coupled LC resonators. Each resonator can be treated quantum mechanicly. However quantization of the problem is complicated: first, because of coupling and secondly because of undetermined value of Δx. Please note that L and C are the inductance and capacitance per unite length and we have an ambiguous resonance frequency for each single resonator. To overcome these difficulties in the quantization process we can take a look at the problem in the frequency domain. Assume that the structure is distributed in space ,that is, in each Δx length interval we have put one inductor and one capacitor. From the transmission line theory a monochromatic wave propagates in this structure without distortion. This means that for every Δx step we have a definite and fixed value for the phase change in frequency domain. This definite value of phase change allows us to write the Hamiltonian as an infinite sum of independent simple harmonic oscillators. Actually in frequency domain, the resistor can be treated as a sea of identical simple harmonic oscillators and we can simply apply the statistics developed the previous parts. As in standard circuit theory, we can analyze the LC oscillator

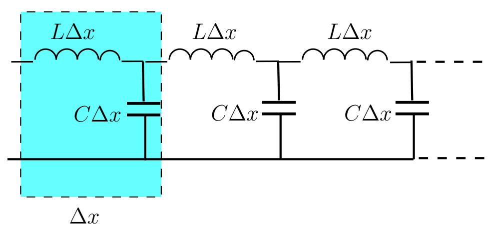

Figure 2: The circuit of a discretized transmission line

circuit simply by introducing the node voltages. However here it is more convenient to use the node fuxes as coordinates. Since just the final results are important for us we avoid presenting all mathematical details in our derivations. Assume that the flux in the node n is shown by Φ_n so Hamiltonian is:

$$H = \sum_n \frac{P_n^2}{2C\Delta x} + \frac{(\Phi_n - \Phi_{n-1})^2}{2L\Delta x} \approx \int dx \left\{ \frac{\mathcal{P}^2}{2C} + \frac{1}{2L} \left| \frac{\partial \phi}{\partial x} \right|^2 \right\} \tag{9}$$

where

$$P_n = C\Delta x C\dot{\Phi}_n$$

$\phi(x)$ and $\mathcal{P}(x)$ are continuous versions of Φ_n and $P_n/\Delta x$.

As we can see the current continuous version of the Hamiltonian contains $\dfrac{\partial \phi}{\partial x}$. Now we can reformulate the problem in the Fourier space. From the basic transmission line theory we have a one to one correspondence between the frequency and the wave number. Assume that the wave number is reprsented by k which is related to ω by a simple disperssion equation:

$$\omega = |k|\, v = |k| \frac{1}{\sqrt{LC}} \tag{10}$$

Using *Parseval's Theorem* we can express H as:

$$H = \frac{1}{2\pi} \int_{k_1}^{k_2} dk \left\{ \frac{\left|\widetilde{\mathcal{P}}\right|^2}{2C} + \frac{k^2}{2L} \left|\widetilde{\phi}\right|^2 \right\} \tag{11}$$

where $\widetilde{\mathcal{P}}$ and $\widetilde{\phi}$ deisgnate the Fourier space representation of $\mathcal{P}$ and ϕ respectively. k_1 and k_2 represent lower and upper frequencies:

$$k_1 = \omega_1 \sqrt{LC} \qquad k_2 = \omega_2 \sqrt{LC} \tag{12}$$

As we can see interestingly (11) describes an the Hamiltonian in a set of uncoupled simple harmonic oscillators. So we can apply the statistics developed in the previous parts. The average energy of this sea of SHO's is simply:

$$\langle E\rangle = \int_{\omega_1}^{\omega_2} \frac{\hbar\omega}{2}\coth\frac{\beta\hbar\omega}{2} \tag{13}$$

(f)

Interestingly we have two mian sources of noise in the resistor:

1. Zero point energy in the simple harmonic oscillators: every SHO introduces a minimum energy of $\hbar\omega/2$. This minimum energy can be considered as the main source of noise.
2. Thermal distribution: simple harmonic oscillators can be excited in higher energy states as a result of thermal distribution

(g)

When $\hbar\omega \ll k_B T$ we can use (5) approximation. Combining every thing together we arrive at:

$$\langle E\rangle = \frac{1}{2}R\langle v_n^2\rangle \approx \int_B k_B T d\omega = k_B T B \tag{14}$$

Problem 3

The spinor part of the Hamiltonian describing time evolution of an half spin system such as electron inside a uniform but time-varying magnetic filed is:

$$H = -\boldsymbol{\mu}.\mathbf{B} = -\frac{gq}{2m}\mathbf{S}.\mathbf{B} = -\mu_0\boldsymbol{\sigma}.\mathbf{B} \tag{1}$$

where g is gyromagnetic ration and μ_0 is defined as:

$$\mu_0 = \frac{qg}{2m} \tag{2}$$

Explicitly the Hamiltonian is:

$$H = -\mu_0 B_0 \begin{pmatrix} \cos\theta & \sin\theta\,[\cos\omega t - i\sin\omega t] \\ \sin\theta\,[\cos\omega t + i\sin\omega t] & -\cos\theta \end{pmatrix} \tag{3}$$

For notational convenience we write the Hamiltonian in terms of two new parameters:

$$H = -\hbar\begin{pmatrix} \omega_1 & \gamma e^{-i\omega t} \\ \gamma e^{i\omega t} & -\omega_1 \end{pmatrix} \tag{4}$$

where:

$$\hbar\omega_1 = \mu_0 B_0\cos\theta$$
$$\hbar\gamma = \mu_0 B_0\sin\theta$$

The Schrödinger equation now reads:

$$i\hbar\frac{d}{dt}\begin{pmatrix}c_1(t)\\c_2(t)\end{pmatrix}=-\hbar\begin{pmatrix}\omega_1 & \gamma e^{-i\omega t}\\ \gamma e^{i\omega t} & -\omega_1\end{pmatrix}\begin{pmatrix}c_1(t)\\c_2(t)\end{pmatrix} \tag{5}$$

In the absense of the time dependent part of the Hamiltonian the solution is simply $c_1(t)=A\exp(i\omega_1 t)$ and $c_2(t)=B\exp(-i\omega_1 t)$. It's reasonable to keep the exponential part for the new Hamiltonian. Two coupled first order differential equations given in (5) can be extremely simplified by the following change of varibles:

$$c_1(t)=A(t)\exp(i\omega_1 t) \qquad c_2(t)=B(t)\exp(-i\omega_1 t) \tag{6}$$

Two new coupled equations are:

$$\begin{cases} i\frac{dA(t)}{dt} = -\gamma\exp\left[-i(\omega+2\omega_1)t\right]B(t)\\ i\frac{dB(t)}{dt} = -\gamma\exp\left[+i(\omega+2\omega_1)t\right]A(t)\end{cases} \tag{7}$$

If we solve $B(t)$ from the first equation and put it in the second one, we obtain:

$$\frac{d^2A}{dt^2}+i\alpha\frac{dA}{dt}+\gamma^2A=0 \tag{8}$$

where α has been defined for notational convenience:

$$\alpha=\omega+2\omega_1 \tag{9}$$

because of symmetry the same equation holds for $B(t)$ we should change the sign of α:

$$\frac{d^2B}{dt^2}-i\alpha\frac{dB}{dt}+\gamma^2B=0 \tag{10}$$

The general solution for two differential equations are simply:

$$A(t)=A_+\exp(i\Omega_+t)+A_-\exp(i\Omega_-t) \tag{11}$$

$$B(t)=B_+\exp(-i\Omega_+t)+B_-\exp(-i\Omega_-t) \tag{12}$$

where Ω_+ and Ω_- are defined as:

$$\Omega_\pm=-\frac{\alpha}{2}\pm\sqrt{\frac{\alpha^2}{4}+\gamma^2} \tag{13}$$

The constant coefficients can be writen in terms of $A(0)$ and $B(0)$:

$$\begin{cases} A(0) = A_++A_-\\ B(0) = B_++B_-\\ \left.\frac{dA}{dt}\right|_{t=0}=i\gamma B(0) = +i\Omega_+A_++i\Omega_-A_-\\ \left.\frac{dB}{dt}\right|_{t=0}=i\gamma A(0) = -i\Omega_+B_+-i\Omega_-B_-\end{cases} \tag{14}$$

These eqquation lead to:

$$A_+=\frac{\Omega_+A(0)-\gamma B(0)}{\Omega_+-\Omega_-} \tag{15}$$

$$A_-=-\frac{\Omega_+A(0)-\gamma B(0)}{\Omega_+-\Omega_-} \tag{16}$$

$$B_+=-\frac{\Omega_-B(0)+\gamma A(0)}{\Omega_+-\Omega_-} \tag{17}$$

$$B_-=\frac{\Omega_+B(0)+\gamma a(0)}{\Omega_+-\Omega_-} \tag{18}$$

This problem is a general spin procession problem which shows that a time varying magnetic filed which rotates along a prescribed axis changes the spin state by two diferent frequency components.

Problem 4

(a)

The Schrödinger equation in the presence of electromagnetic field is:

$$\frac{1}{2m}\left[-i\hbar\nabla - q\mathbf{A}\right].\left[-i\hbar\nabla - q\mathbf{A}\right]\Psi(\mathbf{r},t) + q\Phi\Psi(\mathbf{r},t) = i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} \tag{1}$$

The electrical current desnsity function should describe the follow of the electrical charge so it should be consistent with clssical counterpart. It's reasonable to assume that density of electrical chrage should be $\rho = q\left|\Psi\right|^2$. Moreover current and charge density should satisfy continuity equation:

$$\nabla.\mathbf{J} = -\frac{\partial\rho}{\partial t} = -q\frac{\partial\left|\Psi(\mathbf{r},t)\right|}{\partial t} \tag{2}$$

Mutiplying (1) by $\Psi^*(\mathbf{r},t)$ leads to:

$$\Re\left\{\Psi^*\frac{\partial\Psi}{\partial t}\right\} = \frac{1}{2m}\Re\left\{i\hbar\Psi^*\nabla^2\Psi - \frac{i}{\hbar}q^2\mathbf{A}^2\left|\Psi\right|^2 + q\Psi^*\nabla.(\mathbf{A}\Psi) + q\Psi^*\mathbf{A}.\nabla\Psi\right\} \tag{3}$$

Making use of divercence identity we have:

$$\nabla.\left(\mathbf{A}\left|\Psi\right|^2\right) = \Psi^*\nabla.(\mathbf{A}\Psi) + \Psi^*\mathbf{A}.\nabla\Psi + \Psi\nabla.(\mathbf{A}\Psi^*) + \Psi^*\mathbf{A}.\nabla\Psi^* \tag{4}$$

therefore:

$$\frac{\partial\left|\Psi\right|^2}{\partial t} = \Psi^*\frac{\partial\Psi}{\partial t} + \Psi\frac{\partial\Psi^*}{\partial t} = \frac{\hbar}{2mi}\nabla.\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) - \frac{q}{m}\nabla.\left(\mathbf{A}\left|\Psi\right|^2\right) \tag{5}$$

So from the continuty equation the best choice for $\mathbf{J}$ is:

$$\mathbf{J} = \frac{q\hbar}{m}\Im\left\{\Psi^*\nabla\Psi\right\} - \frac{q^2}{m}\mathbf{A}\left|\Psi\right|^2 \tag{6}$$

If the wave function of the system is given by:

$$\Psi(r,t) = \sqrt{n(\mathbf{r},t)}\exp\left[i\theta(\mathbf{r},t)\right] \tag{7}$$

So the gradient of Ψ is:

$$\nabla\Psi(\mathbf{r},t) = \frac{\nabla n(\mathbf{r},t)}{2\sqrt{n(\mathbf{r},t)}}\exp\left[i\theta(\mathbf{r},t)\right] + i\nabla\theta(\mathbf{r},t)\Psi(\mathbf{r},t) \tag{8}$$

Inserting (8) into (6) leads to the following equality:

$$\mathbf{J} = \frac{q\hbar}{m}\Im\left\{\frac{\nabla n(\mathbf{r},t)}{2\sqrt{n(\mathbf{r},t)}} + in(\mathbf{r},t)\nabla\theta(\mathbf{r},t)\right\} - \frac{q^2}{m}n(\mathbf{r},t)\mathbf{A} \tag{9}$$

So we arrive at:

$$\mathbf{J} = qn(\mathbf{r},t)\left[\frac{\hbar}{m}\nabla\theta(\mathbf{r},t) - \frac{q}{m}\mathbf{A}(\mathbf{r},t)\right] \tag{10}$$

(b)

In fact we derived (10) based on continuty equation , however we can check it here. If the divergence operator actos on $\mathbf{J}$ then:

$$\nabla.\mathbf{J} = q\nabla n(\mathbf{r},t).\left[\frac{\hbar}{m}\nabla\theta(\mathbf{r},t) - \frac{q}{m}\mathbf{A}(\mathbf{r},t)\right] + qn(\mathbf{r},t)\left[\frac{\hbar}{m}\nabla^2\theta(\mathbf{r},t) - \frac{q}{m}\nabla.\mathbf{A}(\mathbf{r},t)\right] \tag{11}$$

and equivalent chrage density is:

$$\Im\left\{i\hbar\Psi^*\frac{\partial\Psi}{\partial t}\right\} = \frac{1}{2m}\Im\left\{-\hbar^2\Psi^*\nabla^2\Psi + q^2\mathbf{A}^2\left|\Psi\right|^2 + i\hbar q\left|\Psi\right|^2\nabla.\mathbf{A} + 2i\hbar q\Psi^*\mathbf{A}.\nabla\Psi\right\} \tag{12}$$

$\nabla^2\Psi$ in terms of n and θ is:

$$\nabla^2\Psi = e^{i\theta}\left[\nabla^2\sqrt{n} + \frac{i}{\sqrt{n}}\nabla n.\nabla\theta + i\sqrt{n}\nabla^2\theta - \sqrt{n}(\nabla\theta)^2\right] \tag{13}$$

So the left hand side of (12) can be simplified as:

$$\text{LHS} = \frac{\hbar}{2m}\left\{-\hbar(\nabla n.\nabla\theta) + \hbar\nabla^2\theta + qn\nabla.\mathbf{A} + q\mathbf{A}.\nabla n\right\} \tag{14}$$

From (11) and (14) we conclude:

$$\nabla.\mathbf{J} = -2q\Re\left\{\Psi^*\frac{\partial\Psi}{\partial t}\right\} = -\frac{\partial\rho}{\partial t} \tag{15}$$

(c)

The continuity equation implies that $|\Psi|^2$ reprsents the spatial distribution of the charge. In fact amplitude of the wave function in bohm interpretation of qunatum mechanics represents the relative probabilty to find the particle in spacific location. So this quantity can be consider as a measure of the cloud of the charge distribution.

(e)

In a smaple of superconductor the wave function is

$$\Psi(\mathbf{r},t) = \sqrt{n^*}e^{i\theta(\mathbf{r},t)} \tag{16}$$

where n^* is constant over space and time. If we apply the curl operator on (10) we arrive at:

$$\nabla\times\mathbf{J} = q^*n^*\left[\frac{\hbar}{m}\nabla\times\nabla\theta(\mathbf{r},t) - \frac{q^*}{m}\nabla\times\mathbf{A}\right] \tag{17}$$

The first term on the right hand side of (17) vanishes and just the second term remains. since $\mathbf{B} = \nabla\times\mathbf{A}$ we obtain:

$$\nabla\times\mathbf{J} = \frac{{q^*}^2n^*}{m}\mathbf{B} \tag{18}$$

or:

$$\nabla\times(\Lambda\mathbf{J}) = -\mathbf{B} \qquad \Lambda = \frac{m}{{q^*}^2n^*} \tag{19}$$

(f)

If we apply time derivaative operator on both sides of (19) we get:

$$\nabla\times\left(\frac{\partial(\Lambda\mathbf{J})}{\partial t}\right) = -\frac{\partial\mathbf{B}}{\partial t} = \nabla\times\mathbf{E} \tag{20}$$

We have employed Faraday's law in the last step. This equation suggets that:

$$\nabla\times\left[\frac{\partial(\Lambda\mathbf{J})}{\partial t} - \mathbf{E}\right] = 0 \tag{21}$$

In equilibrium state total density of free charges is zero and consequently $\nabla.\mathbf{E} = 0$. This allows us to write:

$$\nabla.\left[\frac{\partial(\Lambda\mathbf{J})}{\partial t} - \mathbf{E}\right] = \Lambda\frac{\partial}{\partial t}\nabla.\mathbf{J} - \nabla.\mathbf{E} = 0 \tag{22}$$

Note that $\nabla.\mathbf{J} = -q\dfrac{\partial n^*}{\partial t} = 0$. According to the Helmholtz's theorem in vector analysis, a vector in uniquely specified by giving its divergence and its curl within a simply connected region and its normal component over the boundary. In this simple case we write:

$$\frac{\partial}{\partial t}(\Lambda\mathbf{J}) = \mathbf{E} \tag{23}$$

In the absense of electrical field we have:

$$\mathbf{E} = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial t}(\Lambda\mathbf{J}) = 0 \quad \Longrightarrow \quad \mathbf{J} = \mathbf{C} \tag{24}$$

So we may have follow of chargem in the absense of electrical field.

(g)

By integrating (10) over an arbitrary closed contour we get:

$$\Lambda\oint_C \mathbf{J}.d\mathbf{l} = \frac{\hbar}{q^*}\oint_C \nabla\theta.d\mathbf{l} - \oint_C \mathbf{A}.d\mathbf{l} = \frac{\hbar}{q^*}(\theta_2 - \theta_1) - \int_S \nabla\times\mathbf{A}.d\mathbf{s} \tag{25}$$

The requirement that the wave function should be single-valued in the space forces us to write:

$$\theta_2 - \theta_1 = 2N\pi \qquad N \text{ is an integer} \tag{26}$$

Since $\mathbf{B} = \nabla\times\mathbf{A}$ we obtain:

$$\oint_C (\Lambda\mathbf{J}).d\mathbf{l} = -\int_S \mathbf{B}.d\mathbf{s} + \frac{2N\pi\hbar}{2q^*} \tag{27}$$

or:

$$\oint_C (\Lambda\mathbf{J}).d\mathbf{l} + \Phi_S = N\frac{h}{2q^*} = N\Phi_0 \tag{28}$$

(i)

Two superconducting smaples are repsented by:

$$\Psi_L = \sqrt{n_L}e^{i\theta_L(\mathbf{r},t)}$$
$$\Psi_R = \sqrt{n_R}e^{i\theta_R(\mathbf{r},t)}$$

Two schrödinger equation governing the two superconductors are:

$$i\hbar\frac{\partial\Psi_L}{\partial t} = E_L\Psi_L + K\Psi_R \tag{29}$$
$$i\hbar\frac{\partial\Psi_R}{\partial t} = E_R\Psi_R + K\Psi_L \tag{30}$$

It's expected that current across the junction represents the rate of change in the equivalent charge density inside each region. Because of charge conservation we intuitively conclude that:

$$J_{R\to L} = -J_{L\to R} = qL_x\frac{\partial\left|\Psi_L\right|^2}{\partial t} = -qL_x\frac{\partial\left|\Psi_R\right|^2}{\partial t} \tag{31}$$

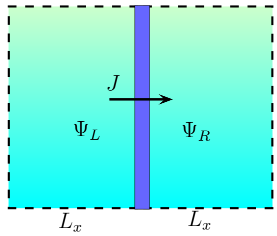

Figure 3: A simple Josephson junction

Please note that we have two main assumptions: it's assumed that n_L and n_R remain uniform inside each region during charge transmission between two sides . Moreover the cross section of the junction is large enough. This assumption allows use the treat the current across the junction perpendicular and relatively uniform. Using (29) we can write:

$$\frac{\partial |\Psi_L|^2}{\partial t} = \Psi_L^* \frac{\partial \Psi_L}{\partial t} + \Psi_L \frac{\partial \Psi_L^*}{\partial t} = \frac{1}{i\hbar}\left[E_L |\Psi_L|^2 + K\Psi_L^*\Psi_R\right] - \frac{1}{i\hbar}\left[E_L |\Psi_L|^2 + K\Psi_L\Psi_R^*\right] = \frac{2K}{\hbar}\Im\{\Psi_L^*\Psi_R\} \tag{32}$$

So we arrive at:

$$J_{R\to L} = \frac{2KqL_x}{\hbar}\Im\{\Psi_L^*\Psi_R\} = \frac{2KqL_x}{\hbar}\sqrt{n_R n_L}\sin(\theta_R - \theta_L) \tag{33}$$

In the case of two identical conductors we get:

$$J = J_c \sin\theta \qquad J_c = \frac{2KqL_x n^*}{\hbar} \qquad \theta = \theta_R - \theta_L \tag{34}$$

Now we can calculate time evolution of θ using two coupled dynamic equations:

$$i\hbar\frac{\partial}{\partial t}(\Psi_R^*\Psi_L) = i\hbar\frac{\partial \Psi_L}{\partial t}\Psi_R^* + i\hbar\frac{\partial \Psi_R^*}{\partial t}\Psi_L = E_L\Psi_R^*\Psi_L + K|\Psi_R|^2 - E_R\Psi_R^*\Psi_L - K|\Psi_L|^2 \tag{35}$$

In the case of identical concuctors we have:

$$i\hbar\left[i\frac{\partial}{\partial t}(\theta_L - \theta_R) + \frac{\partial n^*}{\partial t}\right]e^{i(\theta_L-\theta_R)} = (E_L - E_R)e^{i(\theta_L-\theta_R)} \tag{36}$$

therefore:

$$\frac{\partial \theta}{\partial t} = \frac{E_L - E_R}{\hbar} = \frac{eV}{\hbar} \tag{37}$$

(j)

If there is no external voltage from (37) we conclude that is constant so

$$\frac{\partial \theta}{\partial t} = 0 \quad \Longrightarrow \quad \theta = c \tag{38}$$

So from (34) we arrive at:

$$J = J_c \sin c \tag{39}$$

Interestingly J is not zero!!!.

(k)

If the external voltage is constant then:

$$\frac{\partial\theta}{\partial t} = \frac{eV_0}{\hbar} \quad \Longrightarrow \quad \theta = \frac{eV_0}{\hbar}t + \alpha \tag{40}$$

where α is an arbitary constant. So the current density is

$$J = J_c \sin\left(\frac{eV_0}{\hbar}t + \alpha\right) \tag{41}$$

This means that we have a sinusoidal current when the applied voltage is constant!!!!.