L1 On-chip resonators Microwave photons

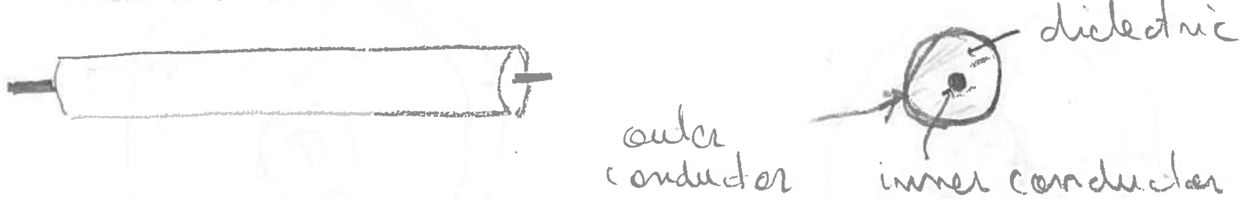
Telegrapher equations for transmission lines

Transmission lines (TL):

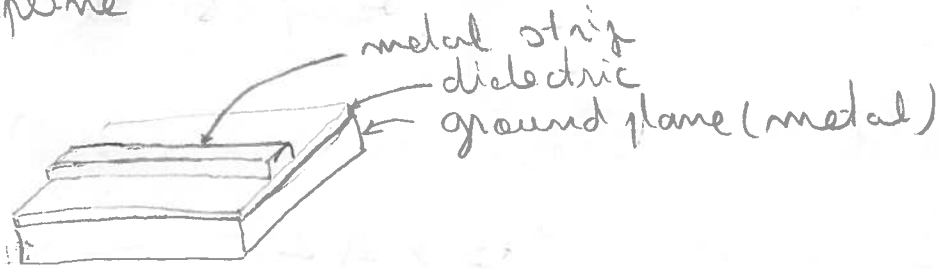
- metallic structures guiding the propagation of EM waves
- spatial extent L can be (much) larger than the wavelength corresponding to propagation in free space, $\lambda = c/\nu$, at frequency ν (i.e. not lumped).
- uniform transmission lines: locally translationally invariant

Examples of TL:

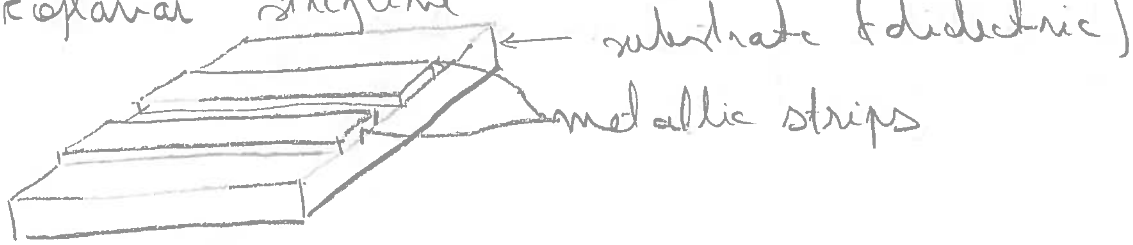
- coaxial cable



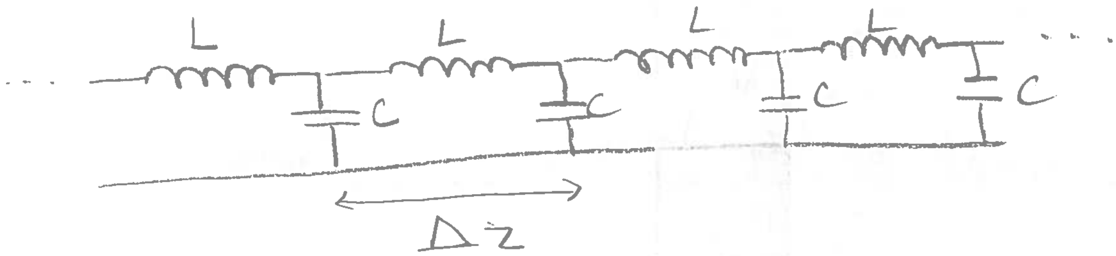
- stripline



- coplanar stripline



Discrete element model for a TL



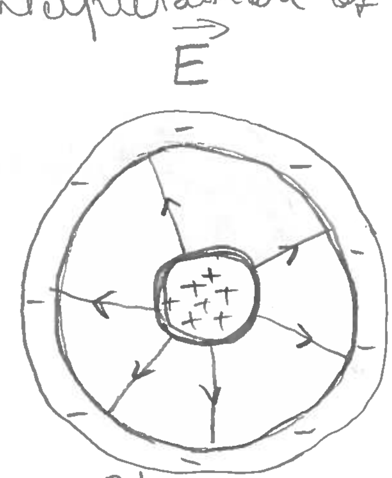
(1) $L = \tilde{L}\,\Delta z$

(2) $C = \tilde{C}\,\Delta z$

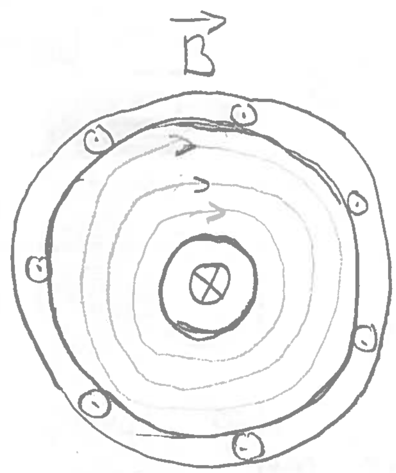
$\tilde{L}\,/\,\tilde{C}$: specific inductance/capacitance of the TL

For a typical coaxial cable: $\tilde{L} = 10\,nH/cm$,
$\tilde{C} = 1\,pF/cm$

Interpretation of the model



Charge



Current

To determine $\tilde{L}$ & $\tilde{C}$ one can use energetic arguments:

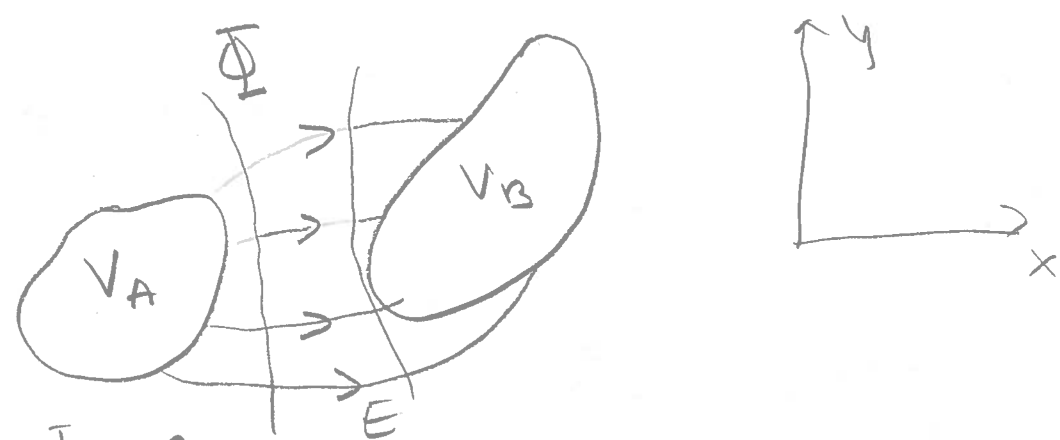
(3) $\frac{1}{2}\tilde{L} I^2 \Delta z = \int \frac{1}{2}\mu H^2 dS \times \Delta z$

(4) $\frac{1}{2}\tilde{C} V^2 \Delta z = \int \frac{1}{2}\varepsilon E^2 dS \times \Delta z$

In general it can be shown that TEM (transverse electromagnetic) waves on a transmission line can be described using the LC model. See Pozar, Sect. 3.1 for a detailed account of the equivalence of Maxwell equations / LC model.

Outline of the procedure:

- Solve Laplace equation with BC defined by the electrodes



(5) $$\Delta \Phi = 0$$

(6) $$\Phi |_{\Gamma_A} = V_A$$

(7) $$\Phi |_{\Gamma_B} = V_B$$

- Calculate the electric field

(8) $$\vec{E}_t (x,y) = - \nabla_t \Phi (x,y)$$ (depends on $\Delta V = V_A - V_B$ only!)

- Calculate the magnetic field

(9) $$\vec{B}_t (x,y) = \frac{1}{Z_{TEM}} \vec{z} \times \vec{E}_t (x,y)$$

$$Z_{TEM} = \sqrt{\frac{\mu}{\varepsilon}}$$

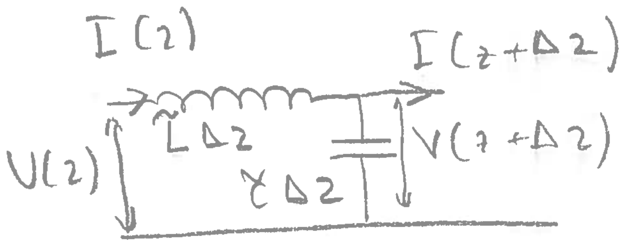
Relation between voltage/current and field quantities:

(10) $$V_A - V_B = \int_A^B \vec{E}\, d\vec{\ell}$$

$$I_{A,B} = \int_{\tilde{\Gamma}_{A,B}} \vec{H}\, d\vec{\ell}$$ ($\tilde{\Gamma}_{A(B)}$ is a contour encircling conductor A (B)).

Along the direction (z axis) of the waveguide: $e^{i(\omega t - kz)}$ factors account for propagation.

Propagation on a transmission line: telegrapher equations



(11) KVL: $$V(z) = \tilde{L}\,\Delta z\,\frac{dI(z)}{dt} + V(z+\Delta z)$$

(12) KCL: $$I(z) = \tilde{C}\,\Delta z\,\frac{dV(z+\Delta z)}{dt} + I(z+\Delta z)$$

Taylor expansion and the limit $\Delta z \to 0$ gives

(13) $$\frac{\partial V}{\partial z} + \tilde{L}\frac{\partial I}{\partial t} = 0$$

(14) $$\frac{\partial I}{\partial z} + \tilde{C}\frac{\partial V}{\partial t} = 0$$

Consider harmonic functions, eg $V(z,t) = \tilde{V}_\omega(z) e^{-i\omega t}$

(15) $$\frac{\partial \tilde{V}_\omega}{\partial z} - i\omega \tilde{L}\,\tilde{I}_\omega = 0$$

(16) $$\frac{\partial \tilde{I}_\omega}{\partial z} - i\omega \tilde{C}\,\tilde{V}_\omega = 0$$

Combining (15) & (16) gives:

(17) $$\frac{\partial^2 \tilde{V}_\omega}{\partial z^2} + \omega^2 \tilde{L}\tilde{C}\,\tilde{V}_\omega = 0$$

(18) $$\frac{\partial^2 \tilde{I}_\omega}{\partial z^2} + \omega^2 \tilde{L}\tilde{C}\,\tilde{I}_\omega = 0$$

There are two possible solutions for the eqs. above

(19) $$\begin{cases} \tilde{V}_\omega(z) = \bar{V}^+_\omega\, e^{ik_+(\omega)z} \\ \tilde{V}_\omega(z) = \bar{V}^-_\omega\, e^{ik_-(\omega)z} \end{cases}$$

(20) $$\begin{cases} \tilde{I}_\omega(z) = \bar{I}^+_\omega\, e^{ik_+(\omega)z} \\ \tilde{I}_\omega(z) = \bar{I}^-_\omega\, e^{ik_-(\omega)z} \end{cases}$$

Here the wave vector

(21) $$k_+(\omega) = \omega\sqrt{\tilde{L}\tilde{C}}$$

(22) $$k_-(\omega) = -\omega\sqrt{\tilde{L}\tilde{C}}$$

Note that (15) (or (16)) define relations between $V^{+/-}_\omega$ and $I^{+/-}_\omega$:

(23) $$\frac{\bar{V}^+_\omega}{\bar{I}^+_\omega} = Z_0$$

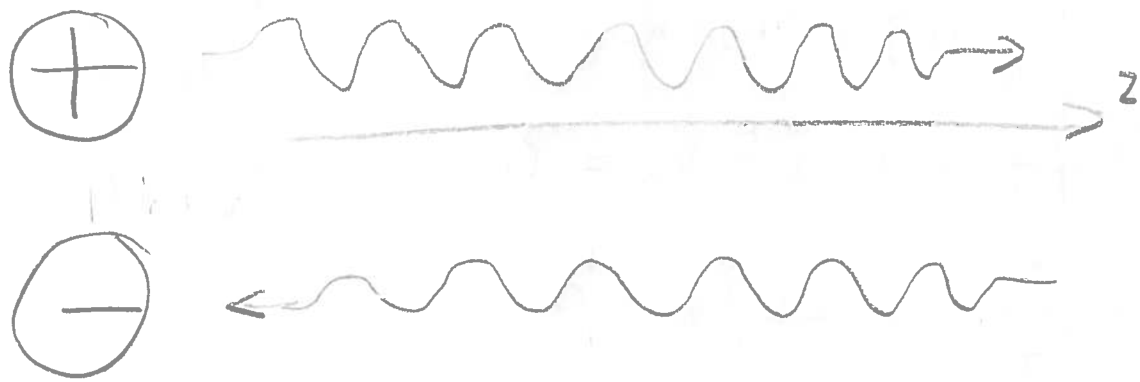
(24) $$\frac{\bar{V}^-_\omega}{\bar{I}^-_\omega} = -Z_0$$

with

(25) $$Z_0 = \sqrt{\frac{\tilde{L}}{\tilde{C}}}$$

the characteristic wave impedance.

The (+) solutions propagate along the positive/negative z axis



(26) $$v^{+/-}_{phase}(\omega) = \frac{\omega}{k^{+/-}(\omega)} = \pm \frac{1}{\sqrt{\tilde{L}\tilde{C}}}$$

(27) $$v^{+/-}_{group}(\omega) = \frac{d\omega}{dk^{+/-}(\omega)} = \pm \frac{1}{\sqrt{\tilde{L}\tilde{C}}}$$

$v_{group} = v_{phase} \Rightarrow$ no dispersion (in this model)

Quantization of the field in transmission lines

(28) $$V(z,t)=\sum_{\omega,s}\left[\bar{V}_\omega^s\, e^{i(k_s(\omega)z-\omega t)}+\bar{V}_\omega^{s*}\, e^{-i(k_s z-\omega t)}\right],\ s=+/-$$

This form is a result of - linearity of (13) & (14)
- imposing V to be real

One can include the time variation in the dynamical quantities

(29) $$V(z,t)=\sum_{\omega,s}\left(V_\omega^s(t)\, e^{ik_s(\omega)z}+V_\omega^{s*}(t)\, e^{-ik_s(\omega)z}\right)$$

Dropping the time dependence gives

(30) $$V(z)=\sum_{\omega,s}\left[V_\omega^s\, e^{ik_s(\omega)z}+V_\omega^{s*}\, e^{-ik_s(\omega)z}\right]$$

Assume periodic boundary conditions (PBC):
$z=-\frac{L}{2}$ and $z=\frac{L}{2}$ are equivalent points.

(31) $$e^{ik_s\left(-\frac{L}{2}\right)}=e^{ik_s\frac{L}{2}} \Rightarrow k_s L=2\pi n \quad (32)$$

Next: count the modes by the n index

(33) $$V(z)=\sum_n\left(V_n e^{ik_n z}+V_n^* e^{-ik_n z}\right),\ k_n=\frac{2\pi n}{L}$$

$n\in\mathbb{Z}$ (both positive & negative)

Next step: calculate the total field energy in the portion $[-\frac{L}{2},\frac{L}{2}]$ of the line

(34) $$E_e=\int_{-\frac{L}{2}}^{L/2}\frac{1}{2}\tilde{C}V^2 dz=\frac{1}{2}\tilde{C}L\sum_n\left(V_nV_{-n}+V_nV_n^*+V_n^*V_n+V_n^*V_{-n}^*\right.$$

For the magnetic energy:

(35) $$E_m=\int_{-\frac{L}{2}}^{L/2}\frac{1}{2}\mathcal{L}I^2 dz$$

(36) $$I(z) = \sum_n \left(\frac{V_n}{Z_n} e^{ik_n z} + \frac{V_n^*}{Z_n} e^{-ik_n z} \right)$$

(37) $Z_n = \text{sign}(n) Z_0$ (positive/negative for +/- propagation)

(38) $$E_m = \frac{1}{2} \tilde{C} L \sum_n \left(-V_n V_{-n} + V_n V_n^* + V_n^* V_n - V_n^* V_{-n}^* \right)$$

Finally, the total energy

(39) $$E = E_e + E_m$$

(40) $$E = 2 \tilde{C} L \sum_n V_n V_n^*$$

Now define

(41) $$q_n \equiv e_n (V_n + V_n^*)$$

(42) $$p_n \equiv \dot{q}_n = e_n(-i\omega_n)(V_n - V_n^*)$$

when the constant e_n is to be conveniently adjusted later.

From (41) & (42) $V_n = \frac{1}{e_n} \left(q_n + i \frac{p_n}{\omega_n} \right)$ (43)

This leads to

(44) $$E = 2 \tilde{C} L \sum_n \frac{1}{e_n^2} \left(\frac{p_n^2}{\omega_n^2} + q_n^2 \right)$$

By choosing $e_n^2 = 4 \tilde{C} L / \omega_n^2$

(45) $$E = \sum_n \left(\frac{1}{2} p_n^2 + \frac{1}{2} \omega_n^2 q_n^2 \right)$$

It can be shown that

(46) $$H(\{p_n\}, \{q_n\}) = \sum_n \left(\frac{1}{2} p_n^2 + \frac{1}{2} \omega_n^2 q_n^2 \right).$$

is a proper Hamilton function for the transmission line.

Indeed

$$\left.\begin{array}{l}\dfrac{\partial H}{\partial p_n} = p_n \\ p_n = \dot{q}_n\end{array}\right\} \Rightarrow \frac{\partial H}{\partial p_n} = \dot{q}_n$$

$$\left.\begin{array}{l}\dfrac{\partial H}{\partial q_n} = \omega_n^2 q_n \\ \dot{p}_n = -\omega_n^2 q_n\end{array}\right\} \Rightarrow \frac{\partial H}{\partial q_n} = -\dot{p}_n$$

The different modes are harmonic-oscillator-like, or bosonic.

(47) $$H = \sum_n H_n$$

(48) $$H_n = \frac{p_n^2}{2} + \frac{1}{2}\omega_n^2 q_n^2$$

For an overview of the properties of the harmonic oscillator, see Cohen & Tannoudji Quantum Mechanics, Chap V.

For each mode n:

- quantised energy levels

(49) $$E_n^m = \hbar\omega_n\left(m + \frac{1}{2}\right), \quad m = 0, 1, 2, \ldots$$

- energy eigenstates are non-degenerate
- it is convenient to work with creation & annihilation operators

(50) $$a_n = \frac{1}{\sqrt{2}}\left(\omega_n q_n + i p_n\right)$$

(51) $$a_n^+ = \frac{1}{\sqrt{2}}\left(\omega_n q_n - i p_n\right)$$

Properties of the creation / annihilation operators

A) $a_n |0\rangle_n = 0$ ($|0\rangle_n$ is the ground state for mode n)
(52)

B) The Hamiltonian for mode n can be written

(53) $H_n = \hbar\omega_n \left(N_n + \frac{1}{2}\right)$, with the number operator

(54) $N_n = a_n^+ a_n$

C) (55) $[N_n, a_n] = -a_n$

(56) $[N_n, a_n^+] = a_n^+$

As a consequence

(57) $a_n |m\rangle_n = \sqrt{m}\, |m-1\rangle_n$

(58) $a_n^+ |m\rangle_n = \sqrt{m+1}\, |m+1\rangle_n$

States of the field; single photons

The eigenstates of the Hamiltonian (47) are the Fock states:

$$(59) \quad \prod_n |m_n\rangle_n$$

In this representation, m_n can be called the number of photons in mode n.

An arbitrary pure (or not mixed) state of the field can be written as a superposition of states of the form (59):

$$(60) \quad |\psi\rangle = \sum_{\{m_n\}} c(\{m_n\}) \prod |m_n\rangle_n$$

A single-photon (state) is a state $|\psi\rangle$ of the field for which

$$(61) \quad a) \ \langle\psi| N_{total} |\psi\rangle = 1 \quad \text{(one photon on average)}$$

$$(62) \quad b) \ \langle\psi| (N_{total} - \bar{N}_{total})^2 |\psi\rangle = 0$$

$$\text{with } \bar{N}_{total} = \langle\psi| N_{total} |\psi\rangle$$

Here the operator $N_{total} = \sum_n N_n$ (63) corresponds to the total number of photons.

Examples of single-photon states

(A) Fock states for a single mode

$|1\rangle_n$.

– fixed frequency $\omega_n = \frac{1}{\sqrt{LC}} \frac{2\pi n}{L}$ (see (32) & (21)) (63)

– not localised in space

(B) Spatially localized photons

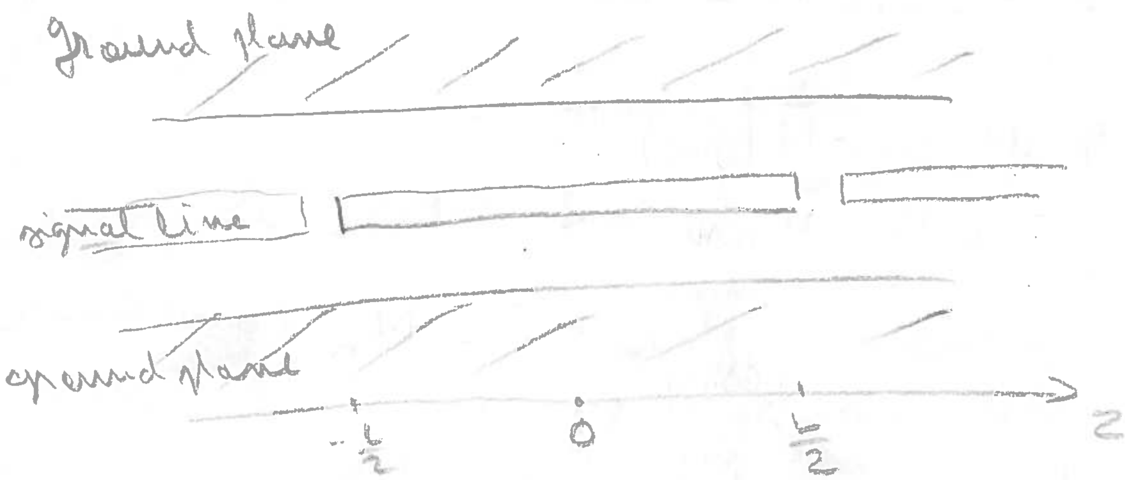
(64) $$|\psi\rangle = C \sum_n e^{-(k_n - k_{n_0})^2/2\sigma^2} \, |1_n, \{0\}_{n' \neq n}\rangle$$

This is a wavepacket:

a) frequency spread $\Delta\omega = \frac{1}{\sqrt{\tilde{L}\tilde{C}}}\,\sigma$

b) spatial extent $\Delta x \cong \frac{1}{\Delta\omega} = \sqrt{\tilde{L}\tilde{C}}\,\frac{1}{\sigma}$

Trapped photons in microwave resonators



This is a coplanar waveguide structure (CPW) – used in many experiments with qubits.

What is the mode structure?

Note on TEM vs TLs - lecture 16

TEM: $\vec{E} = \vec{E}_t(x,y)\, f_E(z)\, e^{-i\omega t}$

$\vec{H} = \vec{H}_t(x,y)\, f_H(z)\, e^{-i\omega t}$

$\nabla \times \vec{E} = -\mu \dot{\vec{H}}$

* On z axis: $\nabla_t \times E_t = 0$

$\Rightarrow \quad \frac{\partial E_{ty}}{\partial x} - \frac{\partial E_{tx}}{\partial y} = 0 \Rightarrow E_t = -\nabla_t \Phi_t$

* On x axis

$\frac{\partial}{\partial y} \cancel{E_z} - \frac{\partial}{\partial z} E_y = -\mu(-i\omega) H_x$

$f_E(z) = f_H(z) = e^{i\beta z}$

$H_x = \frac{i\beta}{\mu i\omega} E_y$