ECE 770-T14/QIC 885: Quantum Electronics & Photonics

Winter 2013, Problem Set 4, Instructor: A. H. Majedi

1- Density Matrix and Time Evolution of Qubit

Consider a two-state system, such as a qubit with states $|0\rangle$ and $|1\rangle$. The Hamiltonian matrix in the orthonormal basis $\{|0\rangle, |1\rangle\}$ is a Hermitian 2×2 matrix that can be written in terms of the three Pauli Matrices, σ_x, σ_y and σ_z and the unit matrix as $\mathcal{H} = \frac{1}{2}(H_0 + \mathbf{H}.\sigma)$. Note that $\mathbf{H} = \mathbf{x}(H_{12} + H_{12}^*) + \mathbf{y}i(H_{12} - H^*12) + \mathbf{z}(H_{11} - H22)$ where H_{ij} are the matrix elements of the original Hamiltonian matrix.

a) Show that the density matrix can be written as $\rho = |\psi(t)\rangle\langle\psi(t)| = \frac{1}{2}(1 + \mathbf{\Psi}.\sigma)$ and identify the components of $\mathbf{\Psi}$.

b) Show that Schrodinger equation, or corresponding time-evolution equation of motion for the density matrix is:

$$\frac{d}{dt}\mathbf{\Psi} = \frac{1}{\hbar}\mathbf{H} \times \Psi$$

c) Show that the equation of motion is periodic and solve for $\mathbf{\Psi}$.

2-Quantum Noise in an Electrical Resistor

Canonical Partition Function is an important quantity in statistical mechanics that encodes the statistical properties of a canonical ensemble in thermodynamical equilibrium. Canonical ensemble is a thermodynamically large system that is in a constant contact to a thermal reservoir with a temperature T in which both the number of objects in the system and the volume of the system are constant. If the system can occupy n microstates with total energy E_n the canonical partition function is $Z = \sum_n e^{-\beta E_n}$, where $\beta = \dfrac{1}{k_B T}$ and k_B is Boltzmann's constant. The ensemble average of the energy of such system is then $\langle E \rangle = -\dfrac{\partial \ln Z}{\partial \beta}$.

Now consider a canonical quantum system that is an ensemble of N identical simple harmonic oscillators. The partition function of a single harmonic oscillator with Hamiltonian $H = \hbar\omega(n + 1/2)$ as a quantum microstate can be found as $Z =$Tr$(e^{-\beta\hat{H}})$.

a) Find the average energy of this system.

b) Discuss the result in part a) where $\hbar\omega >> k_B T$ and $\hbar\omega << k_B T$.

c) Assuming that the probability in which the system is in each microstate is proportional to $e^{-\beta E_n}$ write down the density matrix of any mixed state for this system.

d) Consider a resistor, R. Find the electrical model of the resistor using capacitors and inductors, e.g. series of LC circuits.

e) Use the canonical partition function to find the average energy noise in a resistor.

f) Interpret your results by specifying two main sources of quantum noise you have found in section e).

g) Find the expression for the mean square voltage noise, namely $\langle v_n^2 \rangle$, on the resistor, where $\hbar\omega << k_B T$ over the frequency bandwidth B.

3- A spin half-integer with magnetic moment μ is placed in a uniform and time-varying magnetic field, $\mathbf{B} = B_o\big(\mathbf{x}\sin\theta\cos\omega t + \mathbf{y}\sin\theta\sin\omega t + \mathbf{z}\cos\theta\big)$ rotating with frequency ω about an axis making an angle θ with the z-axis. If the initial wave function at $t = 0$ is known, calculate the wave function at a later time t.

4- Macroscopic Quantum Model of Superconductivity

If the wave function of a system of particles with electric charge q is given by $\psi(\mathbf{r}, t) = \sqrt{n(\mathbf{r}, t)}e^{i\theta(\mathbf{r},t)}$,

a) By using the Schrodinger equation in the presence of EM field, show that the electric current density vector is

$$\mathbf{J} = qn(\mathbf{r}, t)\left(\frac{\hbar}{m}\nabla\theta(\mathbf{r}, t) - \frac{q}{m}\mathbf{A}(\mathbf{r}, t)\right) \tag{1}$$

b) Using the imaginary portion of Schrodinger equation and the result of part (a), show that the following continuity equation must hold.

$$\nabla.\mathbf{J} + \frac{\partial\rho}{\partial t} = 0 \tag{2}$$

where $\rho(\mathbf{r}, t) = qn(\mathbf{r}, t)$.

c) Based on part (b), what is the physical interpretation of $|\psi(\mathbf{r}, t)|^2$?

Generally, it is extremely difficult, if not impossible, to obtain the overall wave function of a system of many particles. For example, in metals, there are in the order of 10^{22} electrons per cubic centimeter. For sufficiently pure metals, the wave function of a single electron can be taken as a plane wave $e^{i\mathbf{k}.\mathbf{r}}$. So the overall wave function is a superposition of many plane waves with different wave vectors $\mathbf{k}$. Nevertheless, if all of the electrons had the same plane wave eigenfunction, the overall wavefunction of the system would be $\psi = Ae^{i\mathbf{k}.\mathbf{r}}$, with A some constant proportional to the number of the electrons in the sample. On its face, it sounds contrary to the Pauli's exclusion principle, that prohibits the electrons to share the same quantum state. Therefore, they must possess different wave functions under normal circumstances; and the wave function ψ introduced above seems invalid. The simplest way to overcome this restriction, is to assume that every two electrons bond together to form electron pairs, namely superelectrons. Superelectrons are exempt from the Pauli's exclusion principle, and thus the wave function ψ is quantum mechanically valid to describe a system of superelectrons with charge $q^* = 2e$ and $m^* = 2m$.

In fact, the situation explained above, is the simplified description of superconductors. However, it looks somehow strange that how two electrons with repulsive Coulomb force are able to form a bond. The qualitative answer is easy; there is an attractive force between the electrons and the positively charged lattice, which at low temperatures overcompensate the Coulomb repulsive force of the two electrons. Rigorous treatment of superconductivity is of course beyond the scope of this course; however, superconductors are one of the best manifestations of laws of quantum mechanics over macroscopic scales, i.e. we can measure them directly in the lab. In the next part, we explore some of the basic properties of superconductors based on their macroscopic wave function and the Schrodinger equation.

d) A sample of superconductor is represented by its wavefunction $\psi = \sqrt{n^*}e^{i\theta(\mathbf{r},t)}$, where n^* is the number density of the superelectrons and is assumed to be a constant over the space and time.

e) Using the result of the previous problem show that

$$\nabla \times (\Lambda \mathbf{J}) = -\mathbf{B} \tag{3}$$

where $\Lambda \equiv \frac{m^*}{(q^*)^2 n^*}$. This is called the second London's equation and is one of the two constitutive relations governing the electrodynamics of superconductors, without which Maxwell's equations are inadequate to account for the electrodynamics of superconductors.

f) Using Maxwell's equations and the second London's equation prove the first London's equation as

$$\frac{\partial}{\partial t}(\Lambda \mathbf{J}) = \mathbf{E} \tag{4}$$

It is easy to see that (4) implies that it is possible to have constant current in superconductors in the absence of electric field. This is in contrast to Ohm's law which characterizes conductors with their conductivity σ, whereas superconductors are characterized by Λ which is called London constant.

g) By integrating (1) over a closed contour and using the requirement that the wavefunction ψ is single-valued in the space, show that

$$\oint_C (\Lambda \mathbf{J}).d\mathbf{l} + \int_S \mathbf{B}.d\mathbf{S} = n\Phi_0 \tag{5}$$

where $\Phi_0 \equiv \frac{h}{2e}$ and is called flux quantum and n is an integer. Equation (5) implies that magnetic flux is quantized in superconductors.

We have already seen, in the lectures, that electrons can tunnel through a potential barrier. One of the physical realizations of this phenomenon is when two conductors are separated by a thin oxide layer. For a junction made of similar normal metals on both sides under no external bias, the probability of tunneling of electrons from left to right equals to that of right to left and hence no net current flows across the junction. Now consider that rather than normal metals we have superconductors on both sides. These types of junctions with very thin oxide layers are called Josephson junctions and the tunneling associated with the superelectrons is called Josephson tunneling.

h) Consider two superconducting samples represented by $\psi_L = \sqrt{n^*}e^{i\theta_L(\mathbf{r},t)}$ and $\psi_R = \sqrt{n^*}e^{i\theta_R(\mathbf{r},t)}$, respectively. If we bring the two samples into close proximity in order to make a Josephson junction the wavefunctions of the superconductors couple together due to tunneling of superelectrons. One can assume that the two Schrodinger equations governing the two superconductors are

$$i\hbar\frac{\partial \psi_L}{\partial t} = E_L\psi_L + K\psi_R \tag{6}$$

$$i\hbar\frac{\partial\psi_R}{\partial t} = E_R\psi_R + K\psi_L \tag{7}$$

where $E_{L,R}$ are the energy eigenvalues of the schrodinger for corresponding isolated superconductor and K is the coupling factor.

i) Show that the current and voltage across the junction obey the basic Josephson junction equations

$$J = J_c\sin(\theta) \tag{8}$$

$$\frac{\partial\theta}{\partial t} = \frac{eV}{\hbar} \tag{9}$$

where $\theta \equiv \theta_R - \theta_L$, $E_L - E_R \equiv eV$ and J_c is a constant proportional to K.

j) What is the current flowing through the junction if there is no external voltage? (d.c. Josephson effect)

k) What is the current flowing through the junction if the external voltage is constant $V = V_0$? (a.c. Josephson effect)

Due: Wednesday March 27, 2013. (in class)