Solution of Problem Set 4

Problem 1)

a) In the basis of $\{|0\rangle, |1\rangle\} \rightarrow$

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \Rightarrow \rho = |\psi\rangle\langle\psi| \Longrightarrow$$

$$\rho = \begin{pmatrix} |\psi_1|^2 & \psi_1 \psi_2^* \\ \psi_1^* \psi_2 & |\psi_2|^2 \end{pmatrix} \qquad (4\text{-}1)$$

Note that $Tr(\rho) = 1 = |\psi_1|^2 + |\psi_2|^2$

a)

Now writing down $\rho = \frac{1}{2}(\mathbb{1} + \vec{\psi}.\hat{\sigma})$

where ρ is given in (4-1), we have:

$$\vec{\psi}.\hat{\sigma} = \psi_x . \sigma_x + \psi_y . \sigma_y + \psi_z . \hat{\sigma}_z$$

$$\begin{cases} \psi_x = \psi_1 \psi_2^* + \psi_1^* \psi_2 \\ \psi_y = i(\psi_1 \psi_2^* - \psi_1^* \psi_2) \\ \psi_z = |\psi_1|^2 - |\psi_2|^2 \end{cases} \qquad (4\text{-}2)$$

b)

$$\dot{\rho} = \frac{d}{dt}|\psi(t)\rangle\langle\psi(t)| + |\psi(t)\rangle \frac{d}{dt}\langle\psi(t)|$$

$$\dot{\rho} = -\frac{i}{\hbar} \left(H|\Psi(t)\rangle\langle\Psi(t)| - |\Psi(t)\rangle\langle\Psi(t)|H \right)$$

$$\dot{\rho} = -\frac{i}{\hbar} [\hat{H}, \rho]$$

Since $\hat{H} = \frac{1}{2} (H_0 + \vec{H} \cdot \vec{\sigma})$

$$\begin{cases} \vec{H} = \hat{x} (H_{12} + H_{12}^*) + \hat{y}\, i (H_{12} + H_{12}^*) + \hat{z} (H_{11} - H_{12}) \\ \vec{\Psi} = \hat{x} \Psi_x + \hat{y} \Psi_y + \hat{z} \Psi_z \end{cases}$$

we can show that $\dot{\rho} = -\frac{i}{\hbar} [\hat{H}, \rho]$ is equivalent to:

$$\frac{d}{dt} \vec{\Psi} = \frac{1}{\hbar} \vec{H} \times \vec{\Psi} \qquad (4\text{-}3)$$

c)

Note that by taking dot of this equation with H, the component along H vector is a constant of motion, i.e. $\Psi_{\parallel}(t) = \left(\hat{H} \cdot \Psi(t)\right) \hat{H} = \Psi_{\parallel}(0)$

then we can write down:

$$\frac{d}{dt} \Psi_{\perp} = \frac{1}{\hbar} \vec{H} \times \Psi_{\perp} \quad \text{or} \quad \frac{d^2 \Psi_{\perp}}{dt^2} = -\left(\frac{H^2}{\hbar^2}\right) \Psi_{\perp} \qquad (4\text{-}4)$$

wher $\Psi_{\perp}$ is the normal component of Ψ.

Note $\hat{H}^2 = H_x^2 + H_y^2 + H_z^2 = (H_{11} - H_{22})^2 + 4|H_{12}|^2$

$$= (E_1 - E_2)^2 = \hbar^2 \omega^2$$

The solution of eq.(3-4-4) is:

$$\Psi_\perp(t) = \Psi_\perp(0)\cos\omega t + \hat{H} \times \Psi(0)\sin\omega t$$

or

$$\boxed{\Psi(t) = \hat{H}\cdot\Psi(0)\,\hat{H} + \left[\Psi(0) - \hat{H}\cdot\Psi(0)\,\hat{H}\right]\cos\omega t + \hat{H}\times\Psi(0)\sin\omega(t)}$$

The period of motion is $\dfrac{2\pi\hbar}{|E_2 - E_1|}$.

Problem 2) (10)

a) For a single SHO, $H = \hbar\omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) = \hbar\omega\left(n+\frac{1}{2}\right)$

$$Z = Tr\left(e^{-\beta H}\right)$$

$$= \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)} = \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}} = \frac{1}{2\sinh\left(\frac{\beta\hbar\omega}{2}\right)}$$

$$Z = \frac{1}{2\sinh\left(\frac{\beta\hbar\omega}{2}\right)} = \frac{1}{2\sinh\left(\frac{\hbar\omega}{2k_BT}\right)}$$

$$Z_N = Z^N = \frac{1}{2^N \sinh^N\left(\frac{\hbar\omega}{2k_BT}\right)}$$

Average energy = $\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z$

$$\langle E\rangle = N\frac{\hbar\omega}{2}\coth\left(\frac{\hbar\omega}{2k_BT}\right)$$

b) if $\hbar\omega \gg k_BT \Rightarrow \frac{\langle E\rangle}{N} \simeq \frac{\hbar\omega}{2}$ Quantum Noise

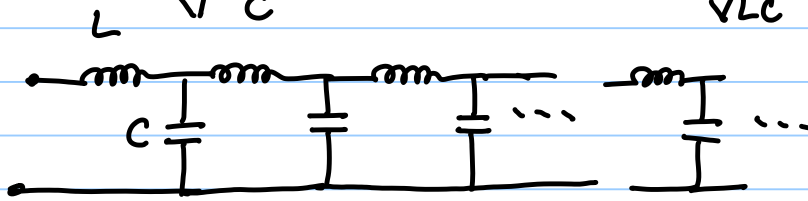
if $\hbar\omega \ll k_BT \Rightarrow \frac{\langle E\rangle}{N} \simeq \hbar\omega \frac{e^{-\hbar\omega/k_BT}}{1-e^{-\hbar\omega/k_BT}}$ Thermal Noise c)

$$\rho = \sum_{n=0}^{\infty} \rho_n |\psi_n\rangle\langle\psi_n| = \sum_{n=0}^{\infty} c\, e^{-\beta E_n} |\psi_n\rangle\langle\psi_n|$$

but $\mathrm{Tr}(\rho) = 1 \Rightarrow c = \dfrac{1}{\sum_n e^{-\beta E_n}} \Rightarrow \rho = \dfrac{1}{\sum e^{-\beta E_n}} \sum e^{-\beta E_n} |\psi_n\rangle\langle\psi_n|$

$$\Rightarrow \rho = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$$

d) $R = \sqrt{\dfrac{L}{C}}$ where $\omega = \dfrac{1}{\sqrt{LC}}$



e) $\dfrac{\langle E\rangle}{N} \simeq \dfrac{\hbar\omega}{2} \coth\left(\dfrac{\hbar\omega}{2k_B T}\right)$ $\qquad \omega = \dfrac{1}{\sqrt{LC}}$

f) $\dfrac{\langle E\rangle}{N} = \dfrac{\hbar\omega}{2} \coth\left(\dfrac{\hbar\omega}{2k_B T}\right) = \hbar\omega\left(\dfrac{1}{2} + \dfrac{e^{-\hbar\omega/k_B T}}{1 - e^{-\hbar\omega/k_B T}}\right)$

$\dfrac{\hbar\omega}{2}$: Quantum Noise

$\dfrac{e^{-\hbar\omega/k_B T}}{1 - e^{-\hbar\omega/k_B T}}$: Thermal Noise

g) In general the correlation function for voltage can be written as:

$$\langle V_n(t+\tau)\, V_n(t)\rangle = \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{i\omega\tau}\, S(\omega)\, d\omega$$

where $S(\omega)$ is spectral density of noise, in which for this problem is simply

$$S(\omega) = R\,\frac{\hbar\omega}{2}\coth\left(\frac{\hbar\omega}{2k_BT}\right)$$

Since $P = \frac{v^2}{R} \propto \int E(\omega)\,d\omega$.

Now

$$\boxed{\langle v_n^2 \rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R\,\hbar\omega\,\coth\left(\frac{\hbar\omega}{2k_BT}\right)d\omega}$$

If $\hbar\omega \ll k_BT$, using the approximated $S(\omega)$ we have

$$\langle v_n^2 \rangle = \frac{R}{2\pi}\int_{-\Delta\omega}^{\Delta\omega} \hbar\omega\,\frac{e^{-\hbar\omega/k_BT}}{1-e^{-\hbar\omega/k_BT}}\,d\omega = \frac{2}{\pi}Rk_BT\,\Delta\omega$$

or $\boxed{\langle v_n^2 \rangle = 4k_BTR\,\Delta f}$

Problem 3)

We set $|\Psi(t)\rangle = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$ & $H = -\vec{\mu}\cdot\vec{B}$

The Schrodinger equation is

$$i\hbar \frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi\rangle \Rightarrow$$

$$i\hbar \frac{\partial}{\partial t}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = -\mu B \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\omega t} \\ \sin\theta\, e^{i\omega t} & -\cos\theta \end{pmatrix}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \quad (3-1)$$

where

$$\vec{\mu}\cdot\vec{B} = \mu_B(\sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}) \cdot B(\sin\theta\cos\omega t\, \hat{x} + \sin\theta \sin\omega t\, \hat{} + \cos\theta\, \hat{z})$$

has been used.

If we write $\begin{cases} \hbar q = \mu B \cos\theta \\ \hbar p = \mu B \sin\theta \end{cases} \Longrightarrow$ (3-1) becomes:

$$\begin{cases} \frac{d}{dt} a(t) = iq\, a(t) + ip\, e^{-i\omega t}\, b(t) & \text{(a)} \\ \frac{d}{dt} b(t) = ip\, e^{i\omega t}\, a(t) - iq\, b(t) & \text{(b)} \end{cases} \quad (3-2)$$

by differentiating from (4-2a) and use (3-2b),

$$\frac{d^2}{dt^2} a + i\omega \frac{d}{dt} a + (p^2 + q^2 + \omega q)\, a = 0 \quad (3-3)$$

The solution of eq. (3-3) is :

$$a(t) = a_1 e^{i\Omega_1 t} + a_2 e^{i\Omega_2 t} \qquad (3\text{-}4)$$

where $\Omega_{1,2} = -\frac{\omega}{2} \pm \sqrt{\frac{\omega^2}{4} + p^2 + q^2 + \omega q}$ and

$$b(t) = \frac{\Omega_1 - q}{p} a_1 e^{i(\Omega_1 + \omega) t} + \frac{\Omega_2 - q}{p} a_2 e^{i(\Omega_2 + \omega) t} \qquad (3\text{-}5)$$

Having the initial condition implies that we have:

$$\Psi(0) = \begin{pmatrix} a(0) \\ b(0) \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ \frac{\Omega_1 - q}{p} a_1 + \frac{\Omega_2 - q}{p} a_2 \end{pmatrix}$$

therefore a_1 & a_2 is known.

$$|\Psi(t)\rangle = \begin{pmatrix} a_1 e^{i\Omega_1 t} + a_2 e^{i\Omega_2 t} \\ \frac{\Omega_1 - q}{p} a_1 e^{i(\Omega_1 + \omega) t} + \frac{\Omega_2 - q}{p} a_2 e^{i(\omega + \Omega_2) t} \end{pmatrix}$$

Problem 4) (65 marks)

a) $J_p = \mathrm{Re}\left\{ \Psi^* \left(-i\frac{\hbar}{m}\nabla - \frac{q}{m}\vec{A}\right)\Psi \right\}$

If $\Psi = \sqrt{n}\, e^{i\theta} \Rightarrow \nabla\vec{\Psi} = \left(\frac{1}{2n}\nabla n + i\nabla\theta\right)\Psi$

$\Rightarrow \Psi^*\left(\frac{-i\hbar}{m}\nabla - \frac{q}{m}\vec{A}\right)\Psi = \Psi^*\left[\frac{-i\hbar}{2mn}\nabla n + \hbar\nabla\theta\right]\Psi$

$$= \sqrt{n}\, e^{-i\theta}\left[\frac{-i\hbar}{2mn}\nabla n + \frac{\hbar\nabla\theta}{m} - \frac{q}{m}A\right]\sqrt{n}\, e^{i\theta}$$

$$J_p = n\left(\frac{\hbar}{m}\nabla\theta - \frac{q}{m}\vec{A}\right) \Longrightarrow$$

$$\boxed{J = q\, J_p = q\, n(r,t)\left[\frac{\hbar}{m}\nabla\theta(r,t) - \frac{q}{m}\vec{A}(r,t)\right]}$$

b) The Schrodinger eq. in the presence of EM field is:

$$i\hbar\frac{\partial}{\partial t}\Psi = \frac{1}{2m}\left(-i\hbar\nabla - qA\right)^2\Psi + q\Phi\Psi \quad (b\text{-}1)$$

Choosing the Coulomb gauge, i.e. $\vec{\nabla}\cdot\vec{A} = 0$ & $\Phi = 0$

(b1) reduces to:

$$i\hbar\frac{\partial}{\partial t}\Psi = \frac{-\hbar^2}{2m}\nabla^2\Psi + i\frac{\hbar q}{m}\vec{A}\cdot\vec{\nabla}\Psi + \frac{q^2}{2m}A^2\Psi \quad (b\text{-}2)$$

Given $\Psi = \sqrt{n}\, e^{i\theta}$, we calculate each term:

$$i\hbar \frac{\partial}{\partial t}\Psi = i\hbar \left(\frac{1}{2n} \frac{\partial n}{\partial t} + i \frac{\partial \theta}{\partial t} \right) \Psi \triangleq W_0 \Psi \quad (b\text{-}3)$$

$$\nabla^2 \Psi = \nabla \left[\left(\frac{\nabla n}{2n} + i \nabla \theta \right) \Psi \right] =$$

$$\left(\frac{n.\nabla^2 n - (\nabla n)^2}{2n^2} + i \nabla^2 \theta + \left(\frac{\nabla n}{2n} \right)^2 - (\nabla \theta)^2 + \frac{\nabla n . \nabla \theta}{n} \right) \Psi$$

$$\triangleq W_1 \Psi \quad (b\text{-}4)$$

$$i\hbar \frac{q}{m} \vec{A}.\nabla \Psi = \frac{i\hbar q}{m} \vec{A} \left(\frac{\nabla n}{2n} + i \nabla \theta \right) \Psi \triangleq W_2 \Psi \quad (b\text{-}5)$$

Plugging (b-3), (b-4) & (b-5) into (b-2), yields:

$$W_0 = -\frac{\hbar^2}{2m} W_1 + W_2 + \frac{q^2 A^2}{2m} \quad (b\text{-}6)$$

Taking imaginary part from (b-6), we arrive at

$$\frac{\hbar}{2n} \frac{\partial n}{\partial t} = \frac{-\hbar^2}{2m} \left(\nabla^2 \theta + \frac{\nabla n . \nabla \theta}{n} \right) + \frac{\hbar q}{m} \vec{A} \frac{\nabla n}{2n} \quad (b\text{-}7)$$

Multiplying (b-7) by $\frac{2nq}{\hbar}$, we have:

$$\frac{\partial}{\partial t}(nq) = \frac{-\hbar q}{m} \underbrace{\left[n \nabla^2 \theta + \nabla n \nabla \theta \right]}_{\nabla(n.\nabla\theta)} + \frac{q^2}{m} \vec{A}.\vec{\nabla} n \quad (b\text{-}8)$$

$\Rightarrow$ if $\rho = q \Psi^* \Psi = q n \longrightarrow$

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\left[nq\left(\frac{\hbar}{m}\vec{\nabla}\theta - \frac{q}{m}\vec{A}\right)\right] = -\vec{\nabla}\cdot\vec{J} \Rightarrow$$

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

c) $|\Psi(r,t)|^2 = n(r,t)$ is the number density or the number of charge carriers per unit volume.

d, e)

Using the result in part (a), we get:

$$J = n^* q^* \left(\frac{\hbar}{m^*}\nabla\theta - \frac{q^*}{m^*}\vec{A}\right) \Rightarrow$$

$$\frac{m^*}{n^*(q^*)^2} J = \Lambda J = \frac{\hbar}{q^*}\vec{\nabla}\theta - \vec{A} \qquad (e\text{-}1) \Rightarrow$$

$$\vec{\nabla}\times(\Lambda J) = \vec{\nabla}\times\left(\frac{\hbar}{q^*}\vec{\nabla}\theta - \vec{A}\right) = -\vec{\nabla}\times\vec{A}$$

Since $\vec{\nabla}\times\vec{\nabla}\theta = 0 \Rightarrow$ $\vec{\nabla}\times(\Lambda J) = -\vec{B}$

f) Now if we take $\frac{\partial}{\partial t}$ from $\vec{\nabla}\times(\Lambda J) = -\vec{B}$ we have:

$$\vec{\nabla}\times\left(\frac{\partial}{\partial t}\Lambda J\right) = -\frac{\partial}{\partial t}\vec{B} = \vec{\nabla}\times\vec{E} \Rightarrow$$

$$\boxed{E = \frac{\partial}{\partial t}\Lambda J}$$

g) Consider a superconducting sample. Using eq. (e-1), we integrate it over a closed contour C within the superconducting sample, as:

$$\oint_C \Lambda\vec{J}\cdot d\vec{l} + \oint \vec{A}\cdot d\vec{l} = \frac{\hbar}{q^*}\oint \nabla\theta\cdot d\vec{l} \quad (g\text{-}1)$$

Applying the Green's identity to the second integral in the left:

$$\oint_C \Lambda\vec{J}\cdot d\vec{l} + \iint_S \vec{\nabla}\times\vec{A}\cdot d\vec{S} = \frac{\hbar}{q^*}\,\theta\Big|_{a^-}^{a^+} \quad (g\text{-}2)$$

It is required that the Ψ function be single-valued in the space, hence $\theta\Big|_{a^-}^{a^+} = 2\pi n \quad n\in\pm\mathbb{N} \Rightarrow$

$$\oint \Lambda\vec{J}\cdot d\vec{l} + \int_S \vec{B}\cdot d\vec{S} = \frac{2\pi n\hbar}{q^*} = n\frac{h}{q^*} = n\frac{h}{2e} = n\underline{\Phi_0}$$

h)

We start from the coupled Schrodinger equations and the given form for Ψ_L & Ψ_R :

$$i\hbar\left(\frac{\partial n_L^*}{\partial t}\cdot\frac{1}{2n_L^*}+i\frac{\partial\theta_L}{\partial t}\right)\Psi_L = E_L\Psi_L + K\Psi_R \quad (h\text{-}1)$$

$$i\hbar\left(\frac{\partial n_R^*}{\partial t}\cdot\frac{1}{2n_R^*}+i\frac{\partial\theta_R}{\partial t}\right)\Psi_R = E_R\Psi_R + K\Psi_L \quad (h\text{-}2)$$

i)

Dividing (h-1) & (h-2), by Ψ_L & Ψ_R, respectively:

$$\frac{i\hbar}{2n_L^*}\frac{\partial n_L^*}{\partial t} - \hbar\frac{\partial\theta_L}{\partial t} = E_L + Ke^{i\theta} \quad (h\text{-}3)$$

$$\frac{i\hbar}{2n_R^*}\frac{\partial n_R^*}{\partial t} - \hbar\frac{\partial\theta_R}{\partial t} = E_R + Ke^{i\theta} \quad (h\text{-}4)$$

Taking real part from (h-3) & (h-4) and subtracting from each other, we get:

$$\hbar\frac{\partial\theta}{\partial t} = E_L - E_R = eV \Longrightarrow \boxed{\frac{\partial\theta}{\partial t} = \frac{eV}{\hbar}} \quad (h\text{-}5)$$

We do the same with the imaginary parts. Note that while $n_L^* = n_R^*$, their time-derivative are not equal, but rather $\frac{\partial n_L^*}{\partial t} = -\frac{\partial n_R^*}{\partial t}$, since the extraction of one electron from one side means adding one electron to the other side.

Therefore:

$$\frac{\hbar}{2n^*}\frac{\partial n^*}{\partial t} = K\sin\theta \Rightarrow \boxed{J = J_c \sin\theta}$$

where $J_c \triangleq \frac{2n^* q^*}{\hbar} K$

In fact the source compensates for excess carriers produced by the term $\frac{\partial n^*}{\partial t}$ in order to make n^* fixed.

j) If $V_0 = 0 \Rightarrow \frac{\partial \theta}{\partial t} = 0 \longrightarrow \theta = \theta_0 \Rightarrow$

$J = J_c \sin \theta_0$

A constant current flows even in the absence of voltage!

k) If $V = V_0 \rightarrow \frac{\partial \theta}{\partial t} = \frac{eV_0}{\hbar} \Rightarrow$

$\theta = \frac{eV_0}{\hbar} t + \theta_0 \longrightarrow \boxed{J(t) = J_c \sin\left(\frac{eV_0}{\hbar} t + \theta_0\right)}$

A pure sinusoidal wave is generated by a constant bias!