Physics 760: Electricity and Magnetism

Assignment 1

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Contents

Note: I will be using $k \equiv \frac{1}{4\pi\epsilon_0}$ throughout this homework set. I will clear up any ambiguity that may arise.

Problem 1

Two point charges (each of charge Q) are positioned at z=R and z=-R on the z-axis. A circular ring of radius R is: centered at the origin, sits on the x-y plane and has a total charge of -2Q uniformly distributed on its circumference.

(a)

Determine the potential along the z-axis (for $r > R$).
In order to determine the potential everywhere in space I will first determine the electric field everywhere in space. This allows me to answer part b) immediately. Of course, I could take the gradient of the potential and get the electric field for part b). Sue me.

First, let's consider the two point charges (because they are super easy). The electric field generated by a point charge (in SI units, used for the remainder of this assignment) looks like $\vec{E} = k\frac{Q}{|\vec{r}-\vec{r'}|^3}(\vec{r} - \vec{r'})$. Considering a point on the positive Z axis, where $z > R$. Both charges are of value +Q. Thus, the electric field points upward along the z-axis. The electric field from the point charges, then, above this z value is: $\vec{E}_{z_{pc}}(z > R) = k\frac{Q}{(z-R)^2}\hat{z} + k\frac{Q}{(z+R)^2}\hat{z}$. When $z < -R$ this expression must indicate that the field points downward along the negative z axis. Thus, $\vec{E}_{z_{pc}}(z < -R) = -k\frac{Q}{(z+R)^2}\hat{z} - k\frac{Q}{(z-R)^2}\hat{z}$. Note that the magnitude of this expression is the same as the first. Only the sign changes. This makes sense.

Now, let's consider the ring of charge. Given that this is a distribution of charge along a line segment (of length $2\pi R$) the charge density on the ring can be given by $\lambda = \frac{-2Q}{2\pi R}$. Let's consider the effect of an infinitesimal amount of charge $dq = \lambda R d\theta$ located on the ring of charge. We will determine the field generated by this infinitesimal amount of charge at a position along the z axis where, still, $z > R$.

$$d\vec{E} = k\frac{\lambda R d\theta}{|z\hat{z} - R\hat{\rho}|^3}(z\hat{z} - R\hat{\rho})$$

Here, $\hat{\rho}$ is the unit vector that points radially towards the center of the ring in the XY plane

$$dE_z = d\vec{E}_z \cdot \hat{z} = k\frac{\lambda R d\theta}{(z^2 + R^2)^{1.5}} z$$
$$d\vec{E}_{z_{ring}}(z > R) = k\frac{\lambda R d\theta}{(z^2 + R^2)^{1.5}} z\hat{z}$$

Now, we need to add up all of the charge. I'll use a polar integral around the circumference of the ring.

$$E_{z_{ring}}(z > R) = k\frac{\lambda R}{(z^2 + R^2)^{1.5}} z \int_0^{2\pi} d\theta$$
$$E_{z_{ring}}(z > R) = 2\pi k\frac{\lambda R}{(z^2 + R^2)^{1.5}} z$$

Of course, $2\pi R\lambda = -2Q$.

$$\vec{E}_{z_{ring}}(z > R, z < -R) = -k\frac{2Q}{(z^2+R^2)^{1.5}}z\hat{z}$$

$k \equiv \frac{1}{4\pi\epsilon_0}$ so I could reduce this expression. I won't. The z in the numerator takes care of the direction. When $z > R$ the field is directed downward; when $z < -R$ the field is directed upward. when The field for $z < -R$ is the same, but it is directed in the $\hat{+z}$ direction.

Thus, the total electric field in the z-direction is $\vec{E}_z = \vec{E}_{z_{ring}} + \vec{E}_{z_{pc}}$ which can be written as $\vec{E}_z(z > R, z < -R) = (-k\frac{2Q}{(z^2+R^2)^{1.5}}z + k\frac{Q}{(z-R)^2} + k\frac{Q}{(z+R)^2})\hat{z}(\theta(z+R) - 1)$. Here, $\theta(z)$ is a Heaviside step function in z. That is, the function returns 1 when the argument is non-negative and 0 when the argument is negative. This toggles the unit vector symbol correctly. Note that this field expression is only valid (as stated) when z lies above R or below -R. The field in the region along the z axis where $z > -R$ and $z < R$ has not been determined. It has not been requested.

By convention, the electric potential is zero at infinity. Also, by convention, the electric potential is defined in terms of the electric field as $V(x_1, x_2, x_3) = -\int_\infty^{x_1}\int_\infty^{x_2}\int_\infty^{x_3}\vec{E}(x_1', x_2', x_3')\cdot dl\hat{dl}$. Here, $\vec{dl}$ is an infinitesimal displacement in the direction of the path that takes me from (∞, ∞, ∞) to (x_1, x_2, x_3). Such an integral is called a "path integral". Let's consider a path integral along the z-axis to some position where $z > R$. I will integrate the electric field along that line and determine the electric potential along the z axis for all points where $z > R$.

$$\begin{aligned} V(z_0 > R) & \int_\infty^{z_0>R}(k\frac{-2Q}{(z^2+R^2)^{1.5}}z + k\frac{Q}{(z-R)^2} + k\frac{Q}{(z+R)^2})\hat{z}\cdot dz(-\hat{z}) \\ & \int_\infty^{z_0>R}(k\frac{2Q}{(z^2+R^2)^{1.5}}z)dz - \int_\infty^{z_0>R}k\frac{Q}{(z-R)^2} - k\frac{Q}{(z+R)^2}dz \end{aligned}$$

At this time it is useful to cite the following integral identity: $\int_\infty^a \frac{x}{(x^2+b^2)^{1.5}}dx = -(b^2+a^2)^{-.5}$

$$V(z_0 > R) = -\frac{k2Q}{(z_0^2+R^2)^{1.5}} + k\frac{Q}{z_0-R} + k\frac{Q}{z_0+R}$$

However, given the symmetry of the problem, traversing the path from $-\infty \to z < -R$ would result in the same potential. I would be climbing electric field vectors from the positive point charges, still. I would be traveling in the direction of the field lines from the charged ring. So, I would have the same potential.

Thus, $V(z > R) = V(z < -R) = -\frac{2kQ}{(z^2+R^2)^{1.5}} + k\frac{Q}{z-R} + k\frac{Q}{z+R}$.

(b)

Determine the E-field along the z-axis (for $r > R$). This solution can be taken immediately from part a).

$\vec{E_z} = \vec{E}_{z_{ring}} + \vec{E}_{z_{pc}}$ which can be written as $\vec{E_z}(z > R, z < -R) = (-k\frac{2Q}{(z^2+R^2)^{1.5}}z + k\frac{Q}{(z-R)^2} + k\frac{Q}{(z+R)^2})\hat{z}(\theta(z+R)-1)$

(c)

In the case that $z >> R$ (but not infinity), what is a good approximation for the Electric Potential along the x-axis? (it's not zero, write the potential as an expansion and keep the first non-zero term).

This problem requires that I expand the potential. I will consider a fixed z. I will then consider expanding about small R/z. I will expand about $R/z = 0$. I will then keep the lowest order terms. To make my life a little easier, though, I will utilize the following property of Maclaurin series expansions. Consider $h(\alpha) = f(\alpha) + g(\alpha)$. The expansion of $h(\alpha)$ about $\alpha = 0$ is $\sum_n \frac{d^n h}{d^n \alpha}|_{\alpha=0}\alpha^n = \sum_n \frac{d^n(f+g)}{d^n\alpha}|_{\alpha=0}\alpha^n = \sum_n \frac{d^n f}{d^n \alpha}\alpha^n + \sum_n \frac{d^n g}{d^n \alpha}|_{\alpha=0}\alpha^n$. The last expression is simply the sum of the two expansions, though.

Thus, $V(z) = -\frac{2kQ}{z(1+(\frac{R}{z})^2)} + k\frac{Q}{z(1-\frac{R}{z})} + k\frac{Q}{z(1+\frac{R}{z})}$. It can easily be shown that the expansion of a function of the form $(1+x^2)^{-.5}$ for sufficiently small x is $1 - \frac{x^2}{2} + \frac{3}{8}x^4$. Similarly, a function of the form $(1 \pm x)^{-1}$ for sufficiently small x resembles $1 \mp x + x^2$.

Expanding the potential about $\frac{R}{z} = 0$ yields, then, to second order in $\frac{R}{z}$: $V_{O(2)}(z) = -\frac{2kQ}{z}\left(1 - (\frac{1}{2}\frac{R}{z})^2\right) + \frac{kQ}{z}(1 - \frac{R}{z} + (\frac{R}{z})^2) + \frac{kQ}{z}(1 + \frac{R}{z} + (\frac{R}{z})^2) = 3k\frac{Q}{z}(\frac{R}{z})^2$.

Problem 2

Using Dirac delta functions in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities p(x).

(a)

In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R.
Without loss of generality I will assume that the sphere is centered at the origin of my spherical coordinate system. To solve this problem I just need to meet the following requirements: $Q = \int_{r>R} \rho(x')d\tau'$ and that $0 = \int_{r<R} \rho(x')d\tau'$. I will propose the following charge distribution $\rho(x') = \frac{Q}{4\pi R^2}\delta(|\vec{r}| - R)$. Sticking this into a spherical integral: $\int_0^\pi \int_0^{2\pi} \int_0^\infty \frac{Q}{4\pi R^2}\delta(|\vec{r}| - R)r^2 sin(\theta)drd\phi d\theta = 4\pi\frac{Q}{4\pi R^2}R^2 = Q$.. Here, I have chosen ϕ to be my azimuthal angle and θ to be my colatitude or polar angle. This result is good. My delta function is positioned at the right place. If I was to integrate over a sphere within the charged sphere, I would pick up no charge (delta function is zero, there). If I was to integrate over a sphere larger than the charged sphere then I would pick up Q charge (delta function would force $\int r^2\delta(|\vec{r}| - R)dr \to R^2$).

(b)

In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b.
Let's propose the following solution and just see if it works. I don't have a great way to explain my mental process for generating these charge densities. I'm not stealing them. I promise. My roommate helped me think through these problems but I used no solutions manuals or internet resources. Consider $\rho(\vec{r'}) = \frac{\lambda}{2\pi b}\delta(|\vec{r'}| - b)$. Sticking this into an integral over a cylinder of Δz height and infinite radius: $\int_{allxyspace,andsome\Delta zheight} \frac{\lambda}{2\pi b}\delta(\vec{r'}| - R)r'dr'd\theta' dz = \frac{\lambda}{2\pi b}b2\pi\Delta z = \lambda\Delta z$. This is the amount of charge I expect

to get when I do this integral. When my radius is too small I grab no charge. When my radius is just past b then I pick up $\lambda \Delta z$ charge.

(c)

In cylindrical coordinates, a charge Q spread uniformly over a flat circular disc of negligible thickness and radius R.

Again, proposing a volume charge density: $\rho(\vec{r'}) = \frac{Q}{\pi R^2}\delta(z)\int_{-\infty}^{R-|\vec{r}|}\delta(t)dt$. The integral of the delta function $(\int_{-\infty}^{R-|\vec{r}|}\delta(t)dt)$ is just a representation of the Heaviside step function. That is: $\int_{-\infty}^{R-|\vec{r}|}\delta(t)dt = H(R-|\vec{r}|)$. I have used a new notation $H(x)$ for a Heaviside step function of x instead of the more traditional $\theta(x)$ to avoid confusion with θ being used as an angular coordinate.

Sticking this volume charge density into an integral over a cylinder of infinite height and radius a ¡ R: $\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^a \frac{Q}{\pi R^2}\delta(z)(\int_{-\infty}^{R-|\vec{r}|}\delta(t))rdrd\theta dzdt = \frac{Q}{\pi R^2}\frac{a^2}{2}2\pi = Q\frac{a^2}{R^2}$. This is exactly what I would expect. Once, $a = R$ I collect all of the charge.

(d)

The same as part (c), but using spherical coordinates.

A snarky solution would just have me replace $r \to rsin(\theta)$ and $z \to rcos(\theta)$ and $\theta \to \phi$. This would result in the expression : $\rho(\vec{r'}) = \frac{Q}{\pi R^2}\delta(Rcos(\theta))\int_{-\infty}^{R-|r\vec{sin}(\theta)|}\delta(t)dt$. Technically, this is correct. However, evaluating functions of direct delta distributions is non-trivial. Therefore, I will try to recast this expression in a form which utilizes products of dirac delta distributions over the various coordinates.

Consider, $\rho(\vec{r'}) = \frac{Q}{\pi R^2 r'}\delta(\theta - \frac{\pi}{2})(\int_{-\infty}^{R-|\vec{r'}|}\delta(t)dt)$. Sticking this into a volume integral similar to the one in (c) yields: $\int_0^{\pi}\int_0^{2\pi}\int_0^a \frac{Q}{\pi R^2 r'}\delta(\theta-\frac{\pi}{2})(\int_{-\infty}^{R-|\vec{r'}|}\delta(t)dt)r'^2 sin(\theta)drd\phi d\theta = \int_0^a \frac{Q}{\pi R^2}r'2\pi dr' = 2\pi\frac{Q}{\pi R^2}\frac{a^2}{2} = \frac{Qa^2}{R^2}$. This is just as expected. When $a \to R$ I obtain the total charge Q built up on the surface.

Problem 3

Each of three charged spheres of radius a, one conducting, one having a uniform charge density within its volume, and one having a spherically symmetric charge density that varies radially as r^n ($n > -3$), has a total charge Q. Use Gauss's theorem to obtain the electric fields both inside and outside each sphere. Sketch the behavior of the fields as a function of radius for the first two spheres, and for the third with n = -2, +2.

(a)

Let's break the cases into different problems. The first problem is to find the electric fields over all space for a charge distribution formed by a conducting sphere of radius a. A conductor naturally spreads its static charge uniformly over the surface of the sphere. Thus, no charge resides within the sphere. Gauss' law tells us that $\oint \vec{E}\cdot\vec{da} = \frac{1}{\epsilon_0}\int_{corresponding\ volume}\rho(\vec{r'})d\tau'$. Where $\rho(r')$ is the volume charge density over space. Thus, $\oint_{sphere\ contained\ within\ the\ charged\ sphere}\vec{E}\cdot\vec{da} = 0$. This does not mean, necessarily that the electric field is zero. It only means that the amount of electric field (both negative - entering the surface - and positive - exiting the surface). However, given the symmetry of the problem at hand, all of the electric field lines will point in the radial direction. Thus, $\vec{E}\cdot\vec{da} = EA$, where A is the surface area of the sphere. However, if I were to choose a non-zero volume over which to integrate this dot product (the electric flux) then I would find that A is non-zero. However, in the interior of the sphere, there is no charge. Thus, the electric field must be zero within the interior. Outside of the shell, I have acquired all of the charge. The area of the surface over which I have performed this flux integral is $4\pi r^2$, where r is an arbitrary radius strictly larger

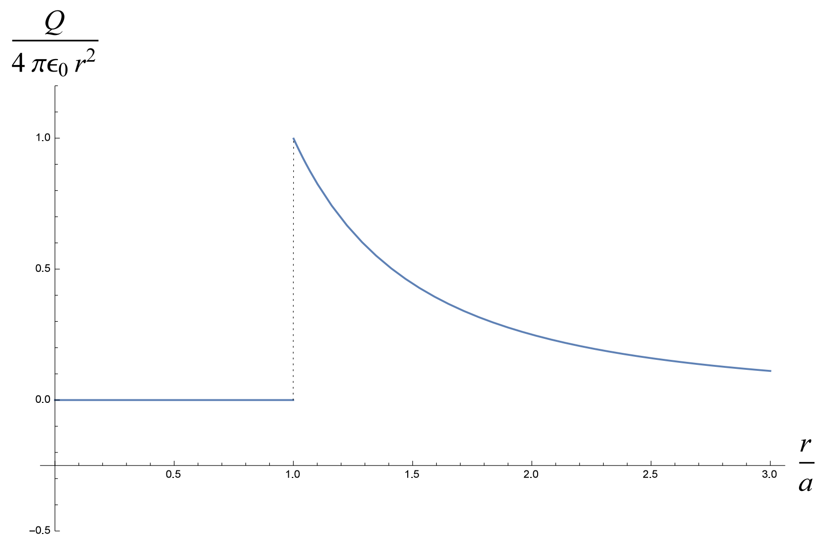

Figure 1: Magnitude of the electric field for a charged conducting sphere.

than a, the radius of the charged sphere.
Now, $\vec{E}(\vec{r}) = \frac{Q}{4\pi a^2\epsilon_0}\hat{r}H(|\vec{r}| - a)$. The Heaviside step function "turns on" my electric field only once my distance away from the origin, r, becomes greater than a.
Sketching this as a function of $|\vec{r}|$, I obtain the following result shown in figure 1. In this figure I have allowed $\frac{Q}{4\pi\epsilon_0} = 1$. The horizontal axis is given in terms of r/a.

(b)

I must attack the same problem where, now, the charge density is uniform within the volume. Now, as I integrate over the volume of the sphere I acquire charge. But, by symmetry considerations I still have a radial electric field distribution. Thus, $\oint_{area}\vec{E}\cdot\vec{da} = \frac{1}{\epsilon_0}\int_{vol}\rho(r')d\tau'$. The left hand side evaluates to $E(r)4\pi r^2 = \rho * \frac{4}{3\epsilon_0}\pi r^3$. This simplifies to $E(r) = \frac{\rho r}{3\epsilon_0}$. This holds until I get outside of the charged sphere. Then, I have contained all of the charge $Q = \rho * \frac{4\pi a^3}{3}$. Thus, $E(r > a) = \frac{4\pi a^3}{3\epsilon_0}\rho\frac{1}{4\pi r^2} = \frac{\rho a^3}{3\epsilon_0 r^2}$. As a function of $|\vec{r}|$, then, the electric field grows linearly until r = a. Then, the field drops off inverse quadratically. This is plotted in figure 2. The axes and constants have been similarly scaled.

(c)

I must now obtain the charge distribution for a nonuniform charge distribution. I have been told that the charge density varies as r^n. This implies that $\rho(r)\ \alpha\ r^nH(a - |\vec{r}|)$ or that $\rho(r) = Cr^nH(a - |\vec{r}|)$, where C is a normalization constant. C is constrained by the fact that $\int_{allspace}\rho(r')d\tau' = Q = 4\pi C\frac{a^{n+3}}{n+3}$. Thus, $C = \frac{Q(n+3)}{4\pi a^3 a^n}$. Identifying $\frac{Q}{4\pi a^3}$ as a sort of volume density, I will assign it the label ρ_0. Thus, $\rho(r) = \rho_0\frac{n+3}{a^n}r^n$; the units work. Now, skipping some algebra $E_n(r)4\pi r^2 = \rho_0\frac{n+3}{a^n}\frac{r^{n+3}}{n+3}4\pi$. So, $E(r) = \rho_0\frac{n+3}{a^n}\frac{r^{n+1}}{n+3}$. This holds until $r \to a$ at the radius of the sphere. After that point, the sphere looks like a point charge. So, the field will drop off inverse quadratically.

When n=2, $E_2(r < a) = \rho_0\frac{5}{a^5}\frac{r^3}{8}$. For n = -2, $E_{-2}(r < a) = \rho_0\frac{1}{a^{-3}}r^{-1}$. Thus, for n=2, the field rises cubically until the boundary, then it decays quadratically. For n=-2, the field drops as inverse r until the boundary. Then, it drops off quadratically. The plots for $E_2(r)$ and $E_{-2}(r)$ are given in figures 3 and 4, respetively. Similar scaling has been applied.

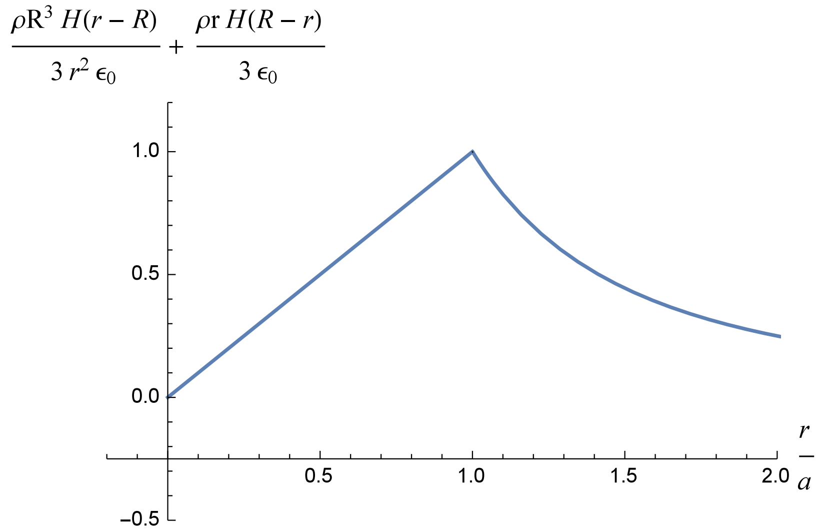

Figure 2: Magnitude of the electric field for a uniformly charged sphere.

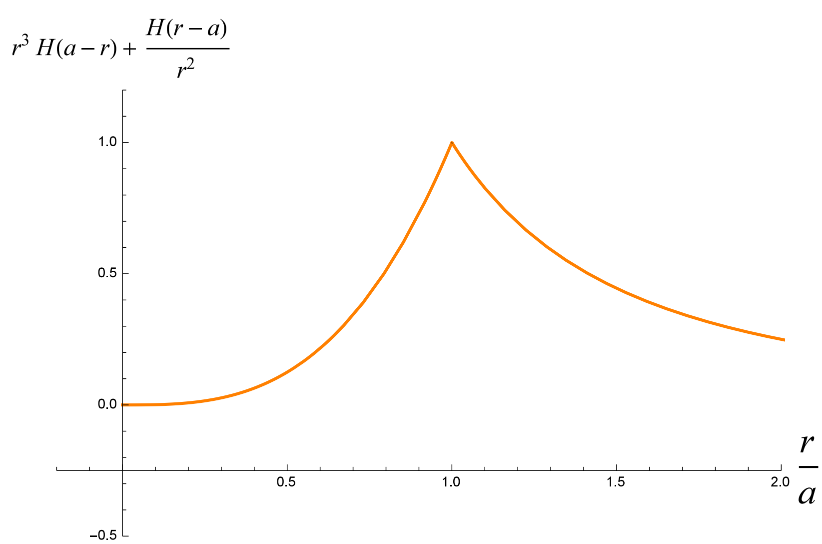

Figure 3: Magnitude of the electric field for a charged sphere whose volume charge density varies as r^2.

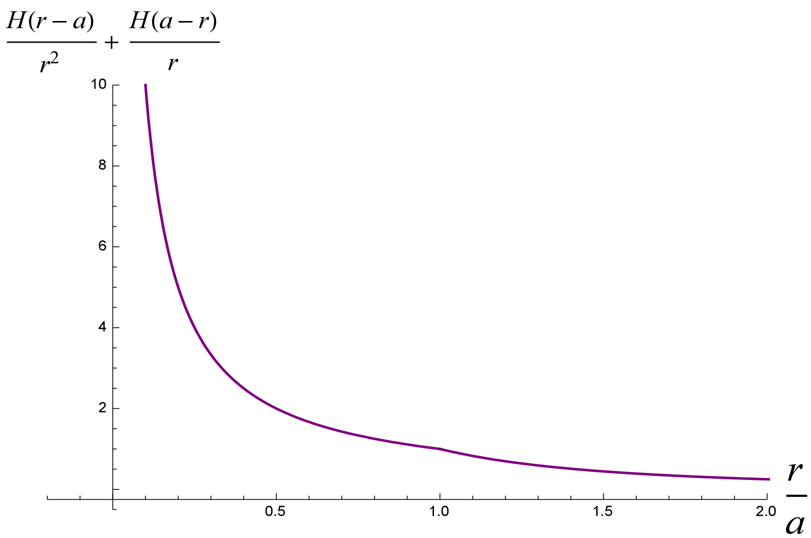

Figure 4: Magnitude of the electric field for a charged sphere whose volume charge density varies as r^-2.

Problem 4

The time-averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}(1+\frac{\alpha r}{2}) \tag{1}$$

where q is the magnitude of the electronic charge, and $a^{-1} = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.

What I will do to solve this will be to take the Laplacian of both sides of this equation. I will use the following identities: $\nabla^2(fg) = g\nabla^2 f + f\nabla^2 g + 2\nabla f \cdot \nabla g$ and $\nabla^2\frac{1}{r} = -4\pi\delta^3(\vec{r})$ and $\nabla\frac{1}{r} = -\frac{\hat{r}}{r^2}$.

$$\begin{aligned}\rho &= -\epsilon_0\nabla^2\Phi \\ &= A\big(\frac{1}{r}\nabla^2 e^{-\alpha r} + e^{-\alpha r}\nabla^2\frac{1}{r} + 2\nabla 1r \cdot \nabla e^{-\alpha r} + \frac{\alpha}{2}\nabla^2 e^{-\alpha r}\big)\end{aligned}$$

A here has absorbed the quantities $\frac{-\epsilon_0 q}{4\pi\epsilon_0}$.Using the identities above and the following results, $\nabla^2 e^{-\alpha r} = \frac{e^{-\alpha r}\alpha}{x}(x\alpha - 2)$ and $\nabla e^{-\alpha r} = -\alpha e^{-\alpha r}\hat{r}$:

$$\begin{aligned}&= A\big(\frac{1}{r}(\frac{e^{-\alpha r}\alpha}{r}(r\alpha-2)) - e^{-\alpha r}4\pi\delta^3(\vec{r}) - 2\frac{\hat{r}}{r^2}\cdot(-\alpha e^{-\alpha r}\hat{r}) + \frac{\alpha}{2}\frac{e^{-\alpha r}\alpha}{r}(r\alpha-2)\big) \\ &= A\big(\frac{e^{-\alpha r}\alpha}{r^2}(r\alpha-2) - e^{-\alpha r}4\pi\delta^3(\vec{r}) + 2\alpha\frac{e^{-\alpha r}}{r^2} + \frac{\alpha^3}{2}e^{-\alpha r} - \frac{e^{-\alpha r}\alpha^2}{r}\big) \\ &= A(-e^{-\alpha r}4\pi\delta^3(\vec{r}) + \frac{\alpha^3}{2}e^{-\alpha r}) \\ &= -\epsilon_0\frac{q}{4\pi\epsilon_0}(-e^{-\alpha r}4\pi\delta^3(\vec{r}) + \frac{\alpha^3}{2}e^{-\alpha r}) \\ &= q\delta^3(\vec{r}) - qe^{-\alpha r}\frac{1}{a_0^3\pi}\end{aligned}$$

I have managed to eliminate the exponential tacked on the δ^3 term because it has no effect on the integral over which the δ^3 would have significance. There, the exponential just reduces to the constant 1.

The first term looks like a positive point charge located at the origin. The second term looks like a negative charge distribution that is smeared over all of space. If I integrate that second term over all space I get -q: $4\pi \frac{q}{\pi a_0^3} \int_0^\infty e^{-\alpha r} r^2 dr = -q$. So, this looks like the electron term.

Actually, as an aside, the ground state probability distribution of the electron's position in space is $\Psi(r) = \frac{1}{\sqrt{\pi} a_0^{1.5}} e^{\frac{r}{a_0}}$. The probability density of the electron is given by $\rho(x) = |\Psi(x)|^2(-q) = -qe^{-\alpha r} \frac{1}{a_0^3 \pi}$. This is the same answer as I obtained above, through analyzing classical electrodynamics equations. This is an example of the oft-cited Ehrenfest theorem in which averages over the quantum regime approach the results obtained in the classical regime.

Problem 5

A simple capacitor is a device formed by two insulated conductors adjacent to each other. If equal and opposite charges are placed on the conductors, there will be a certain difference of potential between them. The ratio of the magnitude of the charge on one conductor to the magnitude of the potential difference is called the capacitance (in SI units it is measured in farads). Using Gauss's law, calculate the capacitance of:

(a)

Two large, flat, conducting sheets of area A, separated by a small distance d.

Using a small Gaussian surface on either side of the sheet I find that the electric field looks like $E * 2A = \frac{\sigma a}{\epsilon_0}$, where the factor of 2A on the left side came from the fact that there is flux through both sides of my pill box. Thus, the electric field from a sheet of charge can be approximated in the large A, small d limit as $E = \frac{Q}{2A\epsilon_0}$ pointing perpendicular to the plates. Thus, the electric field inside, due to both the positive and negative charge is $E_{inside} = \frac{Q}{A\epsilon_0}$. The potential gained by traveling from one plate to another is just $\int \vec{E} \cdot \vec{dl} = \frac{2Qd}{a\epsilon_0}$. This was obtained by taking a path perpendicular to the plates (the direction in which the field faces). The ratio of the charge on one plate to the change in potential is : $\frac{Q}{V} = \frac{A\epsilon_0}{d}$.

(b)

Two concentric conducting spheres with radii a, b ($b > a$);

Using the results from Problem 1, I know that the potential outside of a sphere of charge is $\frac{kQ}{r}$, where r is my distance from the center of the sphere of charge and Q is the charge on the sphere. Thus, the change in potential between distances a and b with $b > a$ is $|\Delta V| = \frac{kQ}{\frac{1}{a} - \frac{1}{b}}$, where the difference has been taken in order to make the result a positive quantity. So, the capacitance, which is given by $C = \frac{Q}{V} = \frac{1}{4\pi\epsilon_0} \frac{1}{\frac{1}{a} - \frac{1}{b}}$.

(c)

Two concentric conducting cylinders of length L, large compared to their radii a,b ($b > a$).

Problem 5 continued on next page...

Using a Gaussian cylinder as my test volume and assuming that the field is radially-distributed (reasonable given the conditions of the problem statement). Thus $E(r)2\pi r\Delta z = \frac{\sigma 2\pi a\Delta z}{\epsilon_0}$. Here, σ is the charge per unit area on the surface of the cylinder. Thus, $E(r) = \frac{\sigma a}{\epsilon_0 r}$. But, if we consider the charge per unit length λ that runs along the length of the cylinder : $\sigma = \frac{\lambda}{2\pi a}$. Thus $E(r) = \frac{\lambda}{2\pi\epsilon_0 r}$. $\lambda = \frac{Q}{L}$. The potential gained from moving out along the radius will go as the integral of $E(r)$ over r. Thus, the expression obtained will be a natural log. $\Delta V(r) = \frac{Q}{2\pi L\epsilon_0} ln(b/a)$ where b is a distance chosen to be larger than a so as to make the change in potential a positive quantity. Now, $C = \frac{Q}{V} = \frac{Q}{\frac{Q}{2\pi L\epsilon_0} ln(b/a)} = \frac{2\pi L\epsilon_0}{ln(b/a)}$.

(d)

What is the inner diameter of the outer conductor in an air-filled coaxial cable whose center conductor is a cylindrical wire of diameter 1 mm and whose capacitance is:

- $3 * 10^{-11}\frac{F}{m}$?
- $3 * 10^{-12}\frac{F}{m}$?

To be general I will solve for the inner diameter as a function of an arbitrary capacitance. From problem 5 (c) I know that $C/L = \frac{2\pi\epsilon_0}{ln(b/a)}$. Thus, $b = ae^{\frac{2\pi\epsilon_0}{C/L}}$. I am given b. $2\pi\epsilon_0 \approx 5.563 * 10^-11 F/m$. Thus, substituting the proper values: $b_{3*10^{-11}F/m} \approx 1mm * e^{1.85} \approx 6.36mm$ and $b_{3*10^{-12}F/m} \approx 1mm * e^{18.5} \approx 108km$.

Problem 6

Prove that $\nabla(\frac{1}{|\vec{x}-\vec{y}|}) = \frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|^3}$. Use this to prove that $\nabla x\vec{E} = 0$ and why we can define scalar potential as: $\vec{E} = \nabla V$.

Considering $\nabla(\frac{1}{|\vec{x}-\vec{y}|})$.

$$\begin{aligned}
\nabla(\frac{1}{|\vec{x}-\vec{y}|}) &= \nabla(\frac{1}{\sqrt{(x_1^2-y_1^2)^2+(x_2^2-y_2^2)+(x_3^3-y_3^2)}}) \\
&= \frac{\partial}{\partial x_1}\frac{1}{\sqrt{(x_1^2-y_1^2)^2+(x_2^2-y_2^2)+(x_3^3-y_3^2)}}\hat{x_1} \\
&+ \frac{\partial}{\partial x_2}\frac{1}{\sqrt{(x_1^2-y_1^2)^2+(x_2^2-y_2^2)+(x_3^3-y_3^2)}}\hat{x_2} \\
&+ \frac{\partial}{\partial x_3}\frac{1}{\sqrt{(x_1^2-y_1^2)^2+(x_2^2-y_2^2)+(x_3^3-y_3^2)}}\hat{x_3} \\
&= \frac{(x_1-y_1)\hat{x_1}+(x_2-y_2)\hat{x_2}+(x_3-y_3)\hat{x_3}}{\big((x_1^2-y_1^2)^2+(x_2^2-y_2^2)+(x_3^3-y_3^2)\big)^{1.5}} \\
&= \frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|^3}
\end{aligned}$$

Considering the integral formulation of $\vec{E} = k\int\frac{\rho(x')}{|\vec{x}-\vec{x'}|^3}(\vec{x}-\vec{x'})d\tau'$. This can be rewritten using the identity derived earlier: $\vec{E} = k\nabla\int\rho(\vec{x'})\frac{1}{|\vec{x}-\vec{x'}|}d\tau'$. Thus, the electric field has been expressed as the gradient of a scalar function (the result of the integral). However, a well-known (and easy to prove) vector identity claims that $\nabla\times\nabla f = 0$ for any scalar function f. Thus, the $\nabla\times\vec{E} = 0$.