Problem Set 1

QIC750 Winter 2014

Due: February 6, 2014

1 Commutator Identities

Prove the following commutator identities:

a) $[A, B] = -[B, A]$
b) $[A + B, C] = [A, C] + [B, C]$
c) $[A, BC] = [A, B]C + B[A, C]$
d) $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ (The Jacobi Identity)

2 Coherent States (Sakurai 2.18)

A coherent state of a 1D simple harmonic oscillator is defined to be an eigenstate of the annihilation operator a:

$$a|\lambda\rangle = \lambda|\lambda\rangle \tag{1}$$

where λ is complex.

a) What is the average number of excitations in the state $\overline{n} = \langle\lambda|N|\lambda\rangle$?
b) Using the expression for the Heisenberg operator $x(t)$ from class, calculate $\langle\lambda|x(t)|\lambda\rangle$.
c) Show that

$$|\lambda\rangle = \exp(-|\lambda|^2/2)\exp(\lambda a^\dagger)|0\rangle \tag{2}$$

is a normalized coherent state.

d) Write $|\lambda\rangle$ as

$$|\lambda\rangle = \sum_{n=0}^{\infty} f(n)|n\rangle. \tag{3}$$

Show that the probability distribution $|f(n)|^2$ with respect to n is a Poisson distribution.

3 Pauli Operators (Sakurai 1.8)

In the study of two-level systems, the Pauli operators play a central role. They can be defined in the orthonormal basis $\{|+\rangle, |-\rangle\}$ as:

$$\sigma_x = |+\rangle\langle-| + |-\rangle\langle+| \quad , \quad \sigma_y = -i|+\rangle\langle-| + i|-\rangle\langle+| \quad , \quad \sigma_z = |+\rangle\langle+| - |-\rangle\langle-|. \tag{4}$$

Using these definitions, prove that:

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \quad , \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij} \tag{5}$$

where ϵ_{ijk} is the Levi-Civita tensor and $\{A, B\} = AB + BA$ is known as the anticommutator.

4 An example of degeneracy (Sakurai 1.23)

Consider the two observables A and B represented in a certain basis by the matrices

$$A \doteq \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \quad , \quad B \doteq \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} \tag{6}$$

with a and b both real.

a) Obviously A exhibits a degenerate spectrum. Does B also exhibit a degenerate spectrum?

b) Show that A and B commute.

c) Find a new set of basis states (vectors) which are simultaneous eigenstates of A and B. Specify the eigenvalues of A and B for each of the three eigenstates. Does your specification of the eigenvalues uniquely characterize each eigenstate?