Quantum Electronics & Photonics

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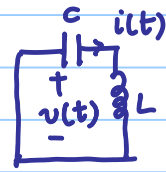
Solution to Problem Set 3

Problem 1)



(20)

a) $$H = \frac{1}{2} L\, i^2(t) + \frac{1}{2} C\, v^2(t) = \frac{1}{2m} p^2(t) + \frac{1}{2} m \omega^2 x^2(t)$$

b) $$\begin{cases} \dfrac{d}{dt} v(t) = \dfrac{1}{C}\, i(t) \\[2mm] \dfrac{d}{dt} i(t) = \dfrac{-1}{L}\, v(t) \end{cases} \qquad \begin{cases} \dfrac{d}{dt} x(t) = \dfrac{1}{m}\, p(t) \\[2mm] \dfrac{d}{dt} p(t) = -m \omega_0^2 x(t) \end{cases}$$

$$\begin{cases} i(t) \longleftrightarrow p(t) \\ v(t) \longleftrightarrow x(t) \end{cases}$$

c) The Lagrangian that is consistent whith these classical equations is:

$$\mathcal{L} = \frac{1}{2} L \left(C \frac{dv}{dt} \right)^2 - \frac{1}{2} C v^2$$

We take $\underline{x(t) = v(t)} \longrightarrow$

$$p(t) = \frac{\partial}{\partial \dot{x}} \mathcal{L} = L C^2 \frac{\partial v}{\partial t} = L C\, i(t) \longrightarrow \boxed{i(t) = \frac{1}{LC} p(t)}$$

d) $$[\hat{x}, \hat{p}] = i\hbar$$

$$\hat{x} = \hat{v} \longrightarrow \hat{p} = -i\hbar \frac{\partial}{\partial v} \longrightarrow \hat{i} = \frac{1}{LC}\hat{p}$$

$$[\hat{v}, \hat{i}] = iLC\hbar \Rightarrow \boxed{[\hat{v}, \hat{i}] = \frac{i}{\omega_0^2}\hbar}$$

or $\omega_0^2 = \frac{1}{LC}$

e) $$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p})$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} - i\hat{p})$$

if $\begin{cases} \hat{x} = \hat{v} \\ \hat{i} = \frac{1}{LC}\hat{p} \end{cases} \Rightarrow m = LC^2$ from part (a)

$$\rightarrow \begin{cases} \hat{a} = \sqrt{\frac{C}{2\hbar\omega}}\,\hat{v} + i\sqrt{\frac{L}{2\hbar\omega}}\,\hat{i} \\ \hat{a}^\dagger = \sqrt{\frac{C}{2\hbar\omega}}\,\hat{v} - i\sqrt{\frac{L}{2\hbar\omega}}\,\hat{i} \end{cases}$$

f) $$\begin{cases} \hat{v} = \sqrt{\frac{2\hbar\omega}{C}}(\hat{a} + \hat{a}^\dagger) \\ \hat{i} = -i\sqrt{\frac{\hbar\omega}{2L}}(\hat{a} - \hat{a}^\dagger) \end{cases}$$

g) $$E_n = \hbar\omega(n + 1/2) = \frac{\hbar}{\sqrt{LC}}(n + 1/2) \quad n = 0, 1, 2, \cdots$$

h) $(\Delta x)(\Delta p) \geqslant \frac{\hbar}{2} \Rightarrow (\Delta\hat{i})(\Delta\hat{v}) \geqslant \frac{\hbar}{2LC} \Rightarrow$

$$\boxed{(\Delta\hat{i})(\Delta\hat{v}) \geqslant \frac{\hbar\omega_0^2}{2}}$$

The uncertainties in $\hat{i}$ & $\hat{v}$ or quantum fluctuation are more pronounced in high frequency.

i) Suppoce that $C = a$ pF & $L = b$ nH $\Rightarrow$

$$\omega_0 = \frac{3.16\times10^{10}}{\sqrt{ab}} \, (\text{rad}/s) \Rightarrow \hbar\omega_0 = \frac{3.33\times10^{-24}}{\sqrt{ab}} \, (J)$$

$$\Delta v = \frac{1.3\times10^{-6}}{a^{3/4}b^{1/4}} \, (V), \quad \Delta i = \frac{4.1\times10^{-8}}{a^{1/4}b^{3/4}} \, (A)$$

where I used $\frac{1}{4}\hbar\omega_0 = \frac{1}{2}L(\Delta i)^2$ and $\frac{1}{4}\hbar\omega_0 = \frac{1}{2}C(\Delta v)^2$.

j) If $\hbar\omega \sim k_B T \Rightarrow T \sim \frac{0.24}{\sqrt{ab}}$ (K)

if we have (μV) level voltages & (nA) level currents around 5 GHz.

k) $\hat{H}\,\Psi(v,t) = \frac{1}{2} L \hat{i}^2 \,\Psi(v,t) + \frac{1}{2} C \hat{v}^2 \Psi(v,t)$

$$i\hbar \frac{\partial}{\partial t} \Psi(v,t) = \frac{-\hbar^2}{2LC^2} \frac{\partial^2}{\partial v^2} \Psi(v,t) + \frac{1}{2} C v^2 \Psi(v,t)$$

l) If we consider TI-SE that is found above

$$\Psi_0(v) = \sqrt[4]{\frac{L^{1/2} C^{3/4}}{\pi \hbar}} \exp\left[-\frac{C^{3/2} L^{1/2}}{2\hbar} v^2\right] \Rightarrow$$

$E_0 = \frac{1}{2}\hbar\omega_0$

$$\Psi_0(v) = \sqrt[4]{\frac{C}{\pi\hbar\omega_0}} \exp\left[\frac{-Cv^2}{2\hbar\omega_0}\right]$$ ✓.

Problem 2)

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 - x F(t) - p\, G(t)$$

a)

Since (x, p) are dynamical variables, then:

$$\begin{cases} \overset{\circ}{x} = \dfrac{\partial H}{\partial p} = \dfrac{p}{m} - G(t) \\ \overset{\circ}{p} = -\dfrac{\partial H}{\partial x} = -m\omega^2 x + F(t) \end{cases}$$

Combining the above two equations leads to:

$$\overset{\circ\circ}{x} = \frac{1}{m} \overset{\circ}{p} - \overset{\circ}{G} \quad \text{or}$$

$$\boxed{\frac{d^2}{dt^2} x(t) = -\omega^2 x(t) + \frac{F(t)}{m} - \frac{d}{dt} G(t)}$$

b)

Using the ladder operators as

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + i \frac{\hat{p}}{m\omega}\right) \quad \& \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - i \frac{\hat{p}}{m\omega}\right)$$

we can find:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger) \quad \& \quad \hat{p} = -im\omega \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} - \hat{a}^\dagger)$$

Therefore the Hamiltonian reads:

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) - \sqrt{\frac{\hbar}{2m\omega}} \left(F(t) + im\omega G(t)\right) \hat{a}$$

$$- \sqrt{\frac{\hbar}{2m\omega}} \left(F(t) - im\omega G(t)\right) \hat{a}^\dagger$$

Defining:

$$f(t) = \frac{1}{\sqrt{2m\hbar\omega}} F(t) - i\sqrt{\frac{m\omega}{2\hbar}}\, G(t) \Rightarrow$$

$$\boxed{\hat{H} = \hbar\omega\left(\hat{a}^\dagger \hat{a} + \tfrac{1}{2}\right) - \hbar f^*(t)\,\hat{a} - \hbar f(t)\,\hat{a}^\dagger}$$

c) The Heisenberg equation of motions for $\hat{a}$ & $\hat{a}^\dagger$ is:

$$\begin{cases} \frac{d}{dt}\hat{a}(t) + i\omega\hat{a}(t) = i f^*(t) \\ \frac{d}{dt}\hat{a}^\dagger(t) - i\omega\hat{a}^\dagger(t) = -i f(t) \end{cases}$$

$$\begin{cases} \hat{a}(t) = a(t=0)\, e^{-i\omega t} + i e^{-i\omega t}\int_0^t e^{i\omega\tau} f^*(\tau)\, d\tau \\ \hat{a}^\dagger(t) = \hat{a}^\dagger(t=0)\, e^{i\omega t} + i e^{i\omega t}\int_0^t e^{-i\omega\tau} f(\tau)\, d\tau \end{cases}$$

d) Lets define $\hat{b}(t)$ & $\hat{b}^\dagger(t)$ for $t > T$

as $$\begin{cases} b(t) = \hat{a}(t) = e^{-i\omega t}(\hat{a} + C) \\ \hat{b}^\dagger(t) = \hat{a}^\dagger(t) = e^{i\omega t}(\hat{a}^\dagger + C^*) \end{cases} \quad t > T$$

Now the state of n quanta is $|n, b\rangle$

and this can be generated as:

$$|n, b\rangle = \frac{(\hat{b}^\dagger)^n}{\sqrt{n!}}|0, b\rangle$$

as $|0,b\rangle$ is a coherent state, since

$\hat{b}\,|0,b\rangle = e^{-i\omega t}(\hat{a}+c)|0,b\rangle = 0$ being eigenstate of an annihilation operator.

Therefore:

$$|0,b\rangle = e^{-i\omega t}\, e^{-|c|^2/2}\, e^{-c\hat{a}^\dagger}\,|0,a\rangle$$

Note state $|0,a\rangle$ is the state annihilated by $\hat{a}$ and gives the ground state of Hamiltonian at $t<0$.

Now we need

$$\langle n,b|0,a\rangle =$$

$$e^{-in\omega t}\, e^{-|c|^2/2}\,\langle 0,a|\, e^{-c^*\hat{a}}\,\frac{(\hat{a}+c)^n}{\sqrt{n!}}\,|0,a\rangle$$

$$= e^{-in\omega t}\, e^{-|c|^2/2}\,\frac{c^n}{\sqrt{n!}}$$

$$\Rightarrow \boxed{P(n) = \langle n,b|0,a\rangle^2 = e^{-|c|^2}\,\frac{|c|^{2n}}{n!}}$$

This is a Poisson distribution.

Problem 3)

a) $H = \frac{p^2}{2m} + \frac{1}{2}k(x-x_0)^2 = \frac{p^2}{2m} + \frac{1}{2}kx^2 + \frac{1}{2}kx_0^2 - kxx_0$

Since $\frac{1}{2}kx_0^2$ is constant, the total Hamiltonian

can be written now as

$$\boxed{H = \frac{p^2}{2m} + \frac{1}{2}kx^2 - kx_0 x = H_0 + \alpha x}$$

b) $\begin{cases} H_0 = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2}) \\ \alpha x = \alpha\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}) \end{cases} \Rightarrow$

In Schrödinger picture

$$\hat{H} = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2}) + \alpha\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a})$$

$$= \hbar\omega\left[\hat{a}^\dagger\hat{a} + c(\hat{a}^\dagger + \hat{a}) + \frac{1}{2}\right]$$

where $c = \alpha\sqrt{\frac{\hbar}{2m\omega}}$

if $\hat{b} = \hat{a} + c$ & $\hat{b} = \hat{a}^\dagger + c \Rightarrow$

$$\hat{H} = \hbar\omega(\hat{b}^\dagger\hat{b} + \frac{1}{2} - c^2)$$

The ground state has an energy

$E_0 = \hbar\omega(\frac{1}{2} - c^2)$ & $b|0\rangle = 0$

Schrödinger ground state is

$$|\Psi_S(t)\rangle = e^{-i\omega t(\frac{1}{2} - c^2)}|0\rangle$$

The equation for the state in the Dirac picture is:

$$i\hbar \frac{\partial}{\partial t} \Psi_D(t) = \alpha\, x_D(t)\, \Psi_D(t)$$

where $\Psi_D(0) = \Psi_S(0)$ $\Rightarrow$

$$\boxed{\Psi_D(t) = \Psi_S(t) - \frac{i\alpha}{\hbar}\int_0^t \Psi_D(\tau)\, x_D(\tau)\, d\tau}$$

ground state wave function

This equation is a Volterra-type equation that can be solved by iterating, (Dyson Series).

For $x_D(t)$, we need $U_0(t) = \exp\left[-i\frac{H_0}{\hbar}t\right]$

$$U_0(t) = \exp\left[-i\omega t\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)\right]$$

$$x_D(t) = U_0^\dagger(t)\, x\, U_0(t)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\, e^{i\omega t(\hat{a}^\dagger \hat{a} + \frac{1}{2})} (\hat{a}^\dagger + \hat{a})\, e^{-i\omega t(\hat{a}^\dagger \hat{a} + \frac{1}{2})}$$

Since $[\hat{a}, \hat{H}] = \hbar\omega \hat{a}$ & $[\hat{a}^\dagger, \hat{H}] = \hbar\omega \hat{a}^\dagger$

$$\begin{cases} \hat{a}\, e^{-i\omega t \hat{a}^\dagger \hat{a}} = e^{-i\omega t}\, e^{-i\omega t \hat{a}^\dagger \hat{a}}\, \hat{a} \\ \hat{a}^\dagger\, e^{-i\omega t \hat{a}^\dagger \hat{a}} = e^{i\omega t}\, e^{-i\omega t \hat{a}^\dagger \hat{a}}\, \hat{a}^\dagger \end{cases} \Rightarrow$$

$$x_D(t) = \sqrt{\frac{\hbar}{2m\omega}}\left[e^{-i\omega t}\,\hat{a} + e^{i\omega t}\,\hat{a}^\dagger\right]$$

$$\boxed{x_D(t) = x_S \cos\omega t + \frac{P_S}{m\omega}\sin\omega t}$$

c) $$\begin{cases} i\hbar \dfrac{d}{dt} x_D(t) = [\hat{x}_D, \hat{H}_0] = \dfrac{i\hbar}{m} P_D \\ i\hbar \dfrac{dP_D(t)}{dt} = [\hat{P}_D, \hat{H}_0] = -i\hbar k x_D \end{cases} \Longrightarrow$$

d) $$\frac{d^2 x_D}{dt^2} = \frac{1}{m}\frac{dP_D}{dt} = -\frac{k}{m} x_D(t) = -\omega^2 x_D(t) \Longrightarrow$$

$$\boxed{x_D(t) = x_S \cos\omega t + \frac{P_D}{m\omega}\sin\omega t}$$

Problem 4)

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2(t)\, \hat{x}^2 = \hbar\omega(t) \left[\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2}\right] \quad (3\text{-}2\text{-}1)$$

$$\hat{a}(t) = \sqrt{\frac{m\omega(t)}{2\hbar}}\, \hat{x} + i \frac{1}{\sqrt{2m\hbar\omega(t)}}\, \hat{p} \quad (3\text{-}2\text{-}2)$$

$$\hat{a}^\dagger(t) = \sqrt{\frac{m\omega(t)}{2\hbar}}\, \hat{x} - i \frac{1}{\sqrt{2m\hbar\omega(t)}}\, \hat{p} \quad (3\text{-}2\text{-}3)$$

a) The Heisenberg equations of motion can be obtained as:

$$\frac{d}{dt}\hat{A} = \frac{1}{i\hbar}\left[\hat{A}, \hat{H}\right] + \frac{\partial}{\partial t}\hat{A}(t)$$

Therefore:

$$\frac{d}{dt}\hat{a}(t) = \frac{1}{i\hbar}\left[\hat{a}(t), \hat{H}(t)\right] + \frac{\partial}{\partial t}\hat{a}(t) \quad (3\text{-}2\text{-}4)$$

We need to calcute $\left[\hat{a}(t), \hat{H}(t)\right]$ as

$$\left[\hat{a}(t), \hbar\omega(t)\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\right] = \hbar\omega\left[\hat{a}, \hat{a}^\dagger\hat{a} + \frac{1}{2}\right] = \hbar\omega\, \hat{a}(t) \quad (3\text{-}2\text{-}5)$$

since $\left[\hat{a}, \hat{a}^\dagger\right] = 1$

Therefore Plugging (3-2-5) into (3-2-4), we have:

$$\frac{d}{dt}\hat{a} = \frac{\omega}{i}\hat{a} + \frac{\partial}{\partial t}\hat{a} \qquad (3\text{-}2\text{-}6)$$

No we need to calculate $\frac{\partial}{\partial t}\hat{a}$. We use (3-2-2)

$$\frac{\partial}{\partial t}\hat{a} = \frac{1}{2\omega}\frac{\partial\omega}{\partial t}\left[\sqrt{\frac{m\omega(t)}{2\hbar}}\,\hat{x} - i\frac{1}{\sqrt{2m\hbar\omega(t)}}\hat{p}\right]$$

$$= \frac{1}{2\omega}\frac{\partial\omega}{\partial t}\hat{a}^{\dagger} \qquad (3\text{-}2\text{-}7)$$

Inserting (3-2-7) in (3-2-6) yields:

$$\frac{d}{dt}\hat{a} = -i\omega\hat{a} + \frac{\omega^{\circ}}{2\omega}\hat{a}^{\dagger} \qquad (3\text{-}2\text{-}8) \qquad \text{where } \omega^{\circ} = \frac{\partial}{\partial t}\omega$$

by taking dagger of (3-2-8), we have

$$\frac{d}{dt}\hat{a}^{\dagger} = i\omega\hat{a}^{\dagger} + \frac{\omega^{\circ}}{2\omega}\hat{a} \qquad (3\text{-}2\text{-}9)$$

Thus equations of motion are:

$$\boxed{\begin{aligned} \frac{d}{dt}\hat{a} &= -i\omega\hat{a} + \frac{\omega^{\circ}}{2\omega}\hat{a}^{\dagger} \\ \frac{d}{dt}\hat{a}^{\dagger} &= i\omega\hat{a}^{\dagger} + \frac{\omega^{\circ}}{2\omega}\hat{a} \end{aligned}}$$

Note that you can check that these equations are correct by writting out the Heisenberg equations for $\hat{x}$ & $\hat{p}$ as:

$$\frac{d}{dt}\hat{x} = \frac{\hat{p}}{m} \qquad \& \qquad \frac{d\hat{p}}{dt} = -m\omega^2(t)\,\hat{x}.$$

b) Bogoliubov transformation:

$$\hat{a}(t) = e^{-i\alpha(t)}\,\hat{a}(0)\cosh\beta(t) + e^{i\gamma(t)}\,\hat{a}^{\dagger}(0)\sinh\beta(t)$$

$$\hat{a}^{\dagger}(t) = e^{-i\gamma(t)}\,\hat{a}(0)\sinh\beta(t) + e^{i\alpha(t)}\,\hat{a}^{\dagger}(0)\cosh\beta(t)$$

We need to compute $\langle n|\hat{H}(t)|n\rangle$ as:

$$\langle n|\hat{H}(t)|n\rangle = \langle n|\hbar\omega(t)\left(\hat{a}^{\dagger}\hat{a}+\tfrac{1}{2}\right)|n\rangle =$$

$$\langle n|\hbar\omega(t)\left(\sinh^2\beta(t)\,\hat{a}(0)\hat{a}^{\dagger}(0) + \cosh^2\beta(t)\,\hat{a}^{\dagger}(0)\hat{a}(0) + \tfrac{1}{2}\right)|n\rangle \quad (3\text{-}2\text{-}10)$$

Now using the fact that $H(0)|n\rangle = (n+\tfrac{1}{2})\hbar\omega(0)|n\rangle$

$$\Rightarrow \begin{cases} \hat{a}(0)|n\rangle = \sqrt{n}\,|n-1\rangle \\ \hat{a}^{\dagger}(0)|n\rangle = \sqrt{n+1}\,|n+1\rangle \end{cases} \quad (3\text{-}2\text{-}11) \Longrightarrow$$

Inserting (3-2-11) in (3-2-10), we have:

$$\langle n|\hat{H}(t)|n\rangle = \hbar\omega(t)\left((n+1)\sinh^2\beta(t) + n\cosh^2\beta(t) + \tfrac{1}{2}\right)$$

$$\boxed{\langle n|\hat{H}(t)|n\rangle = (n+\tfrac{1}{2})\hbar\omega(t)\cosh 2\beta(t)} \quad (3\text{-}2\text{-}12)$$

$$\Rightarrow \boxed{f(t) = \cosh 2\beta(t)}$$

c) Using bogoliubov transformation & inserting them in (3-2-8) & (3-2-9), we have:

$$\frac{d}{dt}\hat{a} = (-i\alpha^{\circ}\cosh\beta + \beta^{\circ}\sinh\beta)e^{-i\alpha}a(0) +$$
$$(i\gamma^{\circ}\sinh\beta + \beta^{\circ}\cosh\beta)e^{i\gamma}\hat{a}(0)$$
$$= -i\omega\left(e^{-i\alpha}\cosh\beta\,\hat{a}(0) + e^{i\gamma}\sinh\beta\,\hat{a}^{\dagger}(0)\right) +$$
$$\frac{\omega^{\circ}}{\omega}\left(e^{-i\gamma}\sinh\beta\,\hat{a}(0) + e^{i\alpha}\cosh\beta\,\hat{a}^{\dagger}(0)\right) \quad (3\text{-}2\text{-}13)$$

Now using $[\hat{a}(0), \hat{a}^{\dagger}(0)] = 1$ & constructing factors of $\hat{a}(0)$ & $\hat{a}^{\dagger}(0)$ in (3-2-13), we get:

$$(-i\alpha^{\circ}\cosh\beta + \beta^{\circ}\sinh\beta)e^{-i\alpha} = -i\omega e^{-i\alpha}\cosh\beta + \frac{\omega^{\circ}}{2\omega}e^{-i\gamma}\sinh\beta$$
$$(i\gamma^{\circ}\sinh\beta + \beta^{\circ}\sinh\beta)e^{i\gamma} = -i\omega e^{i\gamma}\sinh\beta + \frac{\omega^{\circ}}{2\omega}e^{i\alpha}\cosh\beta$$

or

$$\begin{cases}(-i\alpha^{\circ} + i\omega)\cosh\beta + \left(\beta^{\circ} - \frac{\omega^{\circ}}{2\omega}e^{i(\alpha-\gamma)}\right)\sinh\beta = 0 \\ (i\gamma^{\circ} + i\omega)\sinh\beta + \left(\beta^{\circ} - \frac{\omega^{\circ}}{2\omega}e^{i(\alpha-\gamma)}\right)\cosh\beta = 0\end{cases} \Rightarrow$$

Separating real & imaginary we get three coupled differential equations as:

$$\begin{cases} (\alpha^\circ - \omega)\cosh\beta + \dfrac{\omega^\circ}{2\omega}\sin(\alpha-\gamma)\sinh\beta = 0 \\ (\gamma^\circ + \omega)\sinh\beta - \dfrac{\omega^\circ}{2\omega}\sin(\alpha-\gamma)\cosh\beta = 0 \\ \beta^\circ - \dfrac{\omega^\circ}{2\omega}\cos(\alpha-\gamma) = 0 \end{cases} \qquad (3\text{-}2\text{-}14)$$

Upon having $\alpha(0)$, $\beta(0)$ & $\gamma(0)$, we can solve above equations.

d)

Note that $\langle n|\hat{x}(t)|n\rangle = \langle n|\hat{p}(t)|n\rangle = 0$

as both $\hat{x}$ & $\hat{p}$ are linear operators in $\hat{a}(0)$ & $\hat{a}^\dagger(0)$.

$$\langle n|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega(t)}\langle n|(\hat{a}^\dagger(0)+\hat{a}(0))^2|n\rangle$$

$$= \frac{\hbar}{2m\omega(t)}\langle n|\cosh\beta\sinh\beta\,(e^{-i(\alpha-\gamma)}+e^{i(\alpha-\gamma)})$$

$$[\hat{a}^\dagger(0)a(0)+\hat{a}(0)\hat{a}^\dagger(0)] + (\cosh^2\beta+\sinh^2\beta)[a(0)a^\dagger(0)+$$

$$\hat{a}^\dagger(0)\hat{a}(0)]|n\rangle$$

$$\langle n|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega(t)}(2n+1)\left[\cosh(2\beta)+\cos(\alpha-\gamma)\sinh(2\beta)\right]$$

$$\langle n|\hat{p}^2|n\rangle = \frac{-m\hbar\omega(t)}{2}\langle n|[\hat{a}^\dagger(0)-\hat{a}(0)]^2|n\rangle$$

$$= \frac{m\hbar\omega(t)}{2}(2n+1)\left[\cosh(2\beta)-\cos(\alpha-\gamma)\sinh(2\beta)\right]$$

$$(\Delta x)^2 = \frac{\hbar}{2m\omega(t)}(2n+1)\left[\cosh(2\beta) + \cos(\alpha-\gamma)\sinh(2\beta)\right]$$

$$(\Delta p)^2 = \frac{m\hbar\omega(t)}{2}(2n+1)\left[\cosh(2\beta) - \cos(\alpha-\gamma)\sinh(2\beta)\right]$$

$$\Rightarrow \boxed{(\Delta x)(\Delta p) = (2n+1)\frac{\hbar}{2}\sqrt{\cosh^2(2\beta) - \cos^2(\alpha-\gamma)\sinh^2(2\beta)}}$$

Note that if $\cos^2(\alpha-\gamma) = 1 \Rightarrow (\Delta x)(\Delta p) = (2n+1)\hbar/2$

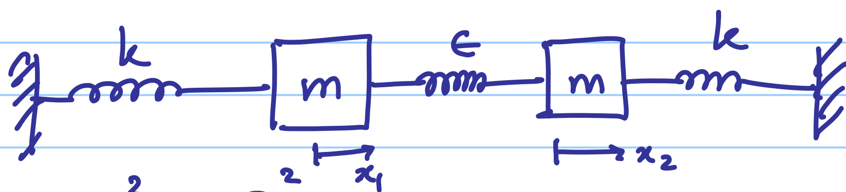
So the Heisenberg uncertainty equation returns original SHO. In this case, where $\cos(\alpha-\gamma) = 1 \longrightarrow$

$$\begin{cases} \langle n|\hat{x}^2|n\rangle = \dfrac{\hbar}{m\omega(t)}(n+1/2)\,e^{2\beta} \\ \langle n|\hat{p}^2|n\rangle = m\hbar\omega\,(n+1/2)\,e^{-2\beta} \end{cases}$$

Thus if $\beta > 0$, variable p is squeezed & if $\cos(\alpha-\gamma) = -1$ & $\beta > 0$, variable x is squeezed.

Problem 5)

a) Consider two identical mass-spring SHO as connected through another spring with constant ϵ, then



$$H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} \epsilon (x_1 - x_2)^2$$

$$\Longrightarrow H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + \frac{1}{2}(k_0+\epsilon) x_1^2 + \frac{1}{2}(k_0+\epsilon) x_2^2 \underbrace{- \epsilon x_1 x_2}_{\text{interaction term}}.$$

b) $$H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \eta x_1 x_2$$

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + \frac{1}{2} m\omega^2 x_1^2 + \frac{1}{2} m\omega^2 x_2^2 + \eta x_1 x_2$$

c) In order to decouple these two SHOs, we use the following transformation:

$$\begin{cases} x_1 = \frac{1}{\sqrt{2}}(y_1 + y_2) \\ x_2 = \frac{1}{\sqrt{2}}(y_1 - y_2) \end{cases} \Rightarrow$$

$y_1 =$

where

This choices can be found via demanding

$$\begin{cases} [\hat{y}_1, \hat{P}_1] = i\hbar \\ [\hat{y}_2, \hat{P}_2] = i\hbar \\ [\hat{y}_1, \hat{P}_2] = [\hat{y}_2, \hat{P}_1] = 0 \end{cases}$$

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial y_2^2} + \frac{1}{2}(m\omega^2+\eta)y_1^2 \Longrightarrow$$

$$+\frac{1}{2}(m\omega^2-\eta)\,y_2^2$$

$$E_{n,n'} = \left(n+\frac{1}{2}\right)\hbar\sqrt{\omega^2+\frac{\eta}{m}} + \left(n'+\frac{1}{2}\right)\hbar\sqrt{\omega^2-\frac{\eta}{m}}$$

$n, n' = 0, 1, 2, \cdots$

d)

$$E_{n,n'} = \left(n+\frac{1}{2}\right)\hbar\omega\left(1+\frac{\eta}{k}\right)^{1/2} + \left(n'+\frac{1}{2}\right)\hbar\omega\left(1-\frac{\eta}{k}\right)^{1/2}$$

$$E_{n,n'} \simeq \left(n+n'+\frac{1}{2}\right)\hbar\omega + (n'-n)\hbar\omega\frac{\eta}{2k}$$