Introduction to Noise Processes

ECE730/QIC890-T33

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Problem Set 2

Due: May 17, 2016, 8:30 am

1. **Wiener-Khintchine Theorem**

   The average power of a noisy function $x_T(t)$ is defined by

$$\lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}[x_T(t)]^2dt = \lim_{T\to\infty}\frac{1}{2\pi}\int_{0}^{\infty}\frac{2|X_T(i\omega)|^2}{T}d\omega,$$

   where $x_T(t)$ a gated function is defined by

$$x_T(t) = \begin{cases} x(t) & \text{if } |t| < T/2; \\ 0 & \text{if otherwise} \end{cases},$$

   T is a measurement time interval and $X_T(i\omega)$ is the Fourier Transform of $x_T(t)$.

   (1) If $x_T(t)$ is a statistically stationary process, show that the average power of a noisy function is independent of T and a constance universal quantity.

   (2) If $x_T(t)$ is a statistically nonstationary process, show that the average power is dependent on T.

   For the statistically nonstationary process, we are not allowed to take the limit of $T \to \infty$. In this case, we introduce ensemble averaging which is taken first for many identical gated function $x_T(t)$. Then, the order of $\lim_{T\to\infty}$ and $\int_0^\infty d\omega$ can be interchanged. Now, we can define the unilateral power spectral density $S_x(\omega)$ is defined as

$$S_x(\omega) = \lim_{T\to\infty}\frac{2\langle|X_T(i\omega)|^2\rangle}{T}.$$

   (3) Recall the formula in Problem Set 1, 3(2)

$$\int_{-\infty}^{\infty}x_T(t+\tau)x_T(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X_T(i\omega)|^2\exp(i\omega\tau)d\omega \qquad \text{(Eq. 1)}.$$

   Suppose $\tau \neq 0$. One can also divide both sides of Eq. (1) by T, take an ensemble average, and take a limit of $T \to \infty$. Show your steps to reach the following relation,

$$\lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\langle x_T(t+\tau)x_T(t)\rangle dt = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}|X_T(i\omega)|^2\cos(\omega\tau)d\omega \qquad \text{(Eq. 2)}.$$

(4) We know that the left-hand side of Eq. (2) is the ensemble averaged autocorrelation function $\langle \phi_x(\tau) \rangle$. Now we obtain the relation of the ensemble averaged autocorrelation $\langle \phi_x(\tau) \rangle$ and the unilateral power spectral density $S_x(\omega)$,

$$\langle \phi_x(\tau) \rangle = \frac{1}{2\pi} \int_0^\infty S_x(\omega) \cos(\omega\tau) d\omega \qquad \text{(Eq. 3)}.$$

Show that the inverse relation of Eq. (3) is written as,

$$S_x(\omega) = 4 \int_0^\infty \langle \phi_x(\tau) \rangle \cos(\omega\tau) d\tau \qquad \text{(Eq. 4)}.$$

Equations (3) and (4) are known as the Wiener-Khintchine theorem.

2. **Unilateral power spectral density** $S_x(\omega)$

In class, we examined one example with a noisy waveform $x(t)$, which is a wide-sense statistically stationary. The autocorrelation function $\phi_x(\tau)$ has a form of

$$\phi_x(\tau) = \phi_x(0) \exp(-\frac{|\tau|}{\tau_1}),$$

where τ_1 is a relaxation time constant.

Compute the unilateral power spectral density $S_x(\omega)$ using the Wiener-Khintchine theorem.

3. **Mathematical Identity**

Show that

$$\lim_{a \to 0} \int_0^\infty \frac{1 - \cos(\omega t)}{\omega^2 + a^2} d\omega = \frac{\pi}{2} t.$$