In this appendix, we will review important properties of spatial & temporal Fourier transforms and provide some useful mathematical relationships.

A1) Spatial Fourier Transform:

$$F(\vec{r}) \underset{FT^+}{\overset{FT^-}{\rightleftarrows}} F_k(\vec{k})$$

$$F_k(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} F(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, d^3r = FT^-\{F(r)\}$$

$$F(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} F_k(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}\, d^3k = FT^+\{F_k(k)\}$$

a) $FT^{\pm}\{\alpha F(r) + \beta G(r)\} = \alpha\, FT^{\pm}\{F(r)\} + \beta\, FT^{\pm}\{G(r)\}$

b) $F(\frac{\vec{r}}{a}) \longleftrightarrow |a|^3 F_k(a\vec{k})$

$F(a\vec{r}) \longleftrightarrow \frac{1}{|a|^3} F_k(\frac{\vec{k}}{a})$ $\quad a \in \mathbb{R}$

c) If $F(\vec{r}) = \ell(x)\, m(y)\, n(z)$

and $\ell(x) \longleftrightarrow L(k_x)$
$m(y) \longleftrightarrow M(k_y)$
$n(z) \longleftrightarrow N(k_z)$

then

$$F(r) \longleftrightarrow L(k_x)\, M(k_y)\, N(k_z)$$

e) $F(\vec{r} \pm r_0) \longleftrightarrow e^{\pm i \vec{k}\cdot\vec{r}_0} F_k(\vec{k})$

$e^{\pm i \vec{k}\cdot r_0} F(r) \longleftrightarrow F_k(k \mp k_0)$

f) $\int_{-\infty}^{+\infty} F(r)\, d^3r = (2\pi)^{3/2} F_k$

$\int_{-\infty}^{+\infty} F_k(k)\, d^3k = (2\pi)^{3/2} F(0)$

g) $\int_{-\infty}^{+\infty} F^*(r)\, G(r)\, d^3r = \int_{-\infty}^{+\infty} F_k^*(k)\, G_k(k)\, d^3k$

h) $FT^+\{ f(r) \} = \frac{1}{(2\pi)^{3/2}} F_k(-\vec{k})$

$FT^-\{ F_k(k) \} = (2\pi)^{3/2} F(-r)$

i) If $F(r)$ is a function of $|\vec{r}|$

$\int_{-\infty}^{+\infty} e^{\pm i \vec{k}\cdot\vec{r}} f(|\vec{r}|)\, d^3r = \int_0^{\infty} f(|r|) \frac{\sin \vec{k}\cdot\vec{r}}{|k||r|} 4\pi |r|^2\, dr$

j) If $f(\vec{r}) * G(\vec{r}) = \int F(\vec{r}_1)\, G(\vec{r}-\vec{r}_1)\, d^3r_1$

$F(r)\, G(r) \longleftrightarrow F_k(k) * G_k(k)$

$F(r) * G(r) \longleftrightarrow F_k(k)\, G_k(k)$

$\vec{k}$) $F(\vec{r}) \longleftrightarrow F_k(\vec{k})$

$\vec{\nabla}\cdot F(r) \longleftrightarrow i\vec{k}\cdot F_k(\vec{k})$

$\vec{\nabla} \times F(r) \longleftrightarrow i\vec{k} \times F_k(k)$

Some useful functions & their transforms:

$F(\vec{r})$	$F_k(\vec{k})$
$\delta(\vec{r}-\vec{r}_0)$	$\frac{1}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{r}_0}$
$e^{-ar^2} \quad a \in R$	$\frac{1}{(\sqrt{2}a)^3} e^{-k^2/4a^2}$
$e^{\pm iar^2}$	$-\frac{(1\mp i)}{4a^3} e^{\mp i \frac{k^2}{4a^2}}$
$\frac{1}{4\pi r} e^{-ar}$	$\frac{1}{(2\pi)^{3/2}} \frac{1}{k^2+a^2}$
$F(r) = \begin{cases} A & \lvert r\rvert \leq R \\ 0 & \text{elsewhere} \end{cases}$	$\frac{\sqrt{\pi}R^3}{6} A \left\{ \frac{3(\sin kR - kR\cos kR)}{(kR)^3} \right\}$
$F(r) = \begin{cases} \frac{A}{4\pi a R^2} & R \leq r \leq R+a, \ a \ll R \\ 0 & \text{elsewhere} \end{cases}$	$\frac{A}{(2\pi)^{3/2}} \frac{\sin \vec{k}\cdot\vec{R}}{\vec{k}\cdot\vec{R}}$
$\sum \delta(\vec{r}-n\vec{a})$ $n = 0, \pm1, \pm2, \ldots$	$\frac{1}{\sqrt{2\pi}\,a} \sum \delta(\vec{k}-\vec{K})$ $\vec{K}\cdot\vec{a} = 2\pi m \ ; \ m = 0, \pm1, \pm2, \ldots$

Three dimensional lattice of points

$$F(\vec{r}) = A \sum \delta(r - n_1\vec{a} - n_2\vec{b} - n_3\vec{c}) \qquad F_k(k) = B \sum \delta(\vec{k} - m_1\vec{a}^* - m_2\vec{b}^* - m_3^*\vec{c})$$

$n_1, n_2, n_3 = 0, \pm1, \pm2, \ldots \qquad m_1, m_2, m_3 = 0, \pm1, \pm2, \ldots$

$$\frac{B}{A} = \frac{(2\pi)^{3/2}}{\lvert a \cdot b \times c \rvert} = \frac{\lvert a^* \cdot b^* \times c^* \rvert}{(2\pi)^{3/2}}$$

$$a^* = 2\pi \frac{b \times c}{a \cdot b \times c} \quad , \qquad a = 2\pi \frac{b^* \times c^*}{a^* \cdot b^* \times c^*}$$

$$a \cdot a^* = b \cdot b^* = c \cdot c^* = 2\pi$$

$$a \cdot b^* = a \cdot c^* = b \cdot a^* = b \cdot c^* = c \cdot a^* = c \cdot b^* = 0$$

$|a \cdot b \times c|$: volume of unit cell in $\vec{r}$ space

$|a^* \cdot b^* \times c^*|$: volume of unit cell in $\vec{k}$ space

A2) Temporal Fourier Transform:

$$f(t) \underset{FT^{+}}{\overset{FT^{-}}{\rightleftarrows}} f_\omega(\omega)$$

$$f_\omega(\omega) = \frac{1}{\sqrt{2\pi}} \int f(t)\, e^{i\omega t}\, dt = \mathcal{F}T^{-}\{f(t)\}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int f_\omega(\omega)\, e^{-i\omega t}\, d\omega = \mathcal{F}T^{+}\{f_\omega(\omega)\}$$

a) $f(t-a) \longleftrightarrow e^{ia\omega} f_\omega(\omega)$

$e^{\pm 2\pi i a} f(t) \longleftrightarrow f_\omega(\omega \pm 2\pi a)$

b) $f(at) \longleftrightarrow \frac{1}{|a|} f_\omega\left(\frac{\omega}{a}\right)$

c) $f_\omega(t) \longleftrightarrow f(-\omega)$

d) $\frac{d^n}{dt^n} f(t) \longleftrightarrow (-i\omega)^n f_\omega(\omega)$

e) $t^n f(t) \longleftrightarrow (-i)^n \frac{d^n}{d\omega^n} f_\omega(\omega)$

f) $f(t) * g(t) \longleftrightarrow \sqrt{2\pi}\, f_\omega(\omega)\, g_\omega(\omega)$

$f(t)$	$f_\omega(\omega)$
1	$\sqrt{2\pi}\,\delta(\omega)$
$\delta(t)$	$\dfrac{1}{\sqrt{2\pi}}$
e^{-iat}	$\sqrt{2\pi}\,\delta(\omega-a)$
$\cos(at)$	$\sqrt{2\pi}\,\dfrac{\delta(\omega-a)+\delta(\omega+a)}{2}$
$\sin(at)$	$-i\sqrt{2\pi}\,\dfrac{\delta(\omega+a)-\delta(\omega-a)}{2}$
t^n	$(-i)^n\sqrt{2\pi}\,\delta^{(n)}(\omega)$
$\dfrac{1}{t}$	$i\sqrt{\dfrac{\pi}{2}}\,\mathrm{sgn}(\omega)$
$u(t)$	$\sqrt{\dfrac{\pi}{2}}\left(\dfrac{-1}{i\pi\omega}+\delta(\omega)\right)$
$\sum_{n=-\infty}^{+\infty}\delta(t-nT)$	$\dfrac{\sqrt{2\pi}}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\dfrac{2\pi}{T}k\right)$
$\dfrac{1}{t^n}$	$i\sqrt{\dfrac{\pi}{2}}\,\dfrac{(i\omega)^{n-1}}{(n-1)!}\,\mathrm{sgn}(\omega)$
$\mathrm{sgn}(t)$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{i}{\omega}$
$\cos(at^2)$	$\dfrac{1}{\sqrt{2a}}\cos\left(\dfrac{\omega^2}{4a}-\dfrac{\pi}{4}\right)$
$\sin(at^2)$	$\dfrac{-1}{\sqrt{2a}}\sin\left(\dfrac{\omega^2}{4a}-\dfrac{\pi}{4}\right)$