Physics 760

Assignment 3

Dr. Stefan Kycia

John Rinehart

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Problem 1

a

Charge multipole expansions are given in terms of an integral over the charge density as follows:

$$q_{lm} = \int\limits_{\text{all space}} Y^*_{lm}(\theta', \phi') r'^l \rho(\vec{x'}) d\tau'$$

Now, this homework problem has a charge density which can be expressed as follows:

$$\rho(\vec{x'}) = \frac{q\delta(|\vec{x'}|)\delta(\theta' - \pi/2)}{r'^2 sin^2\theta'} \big(\delta(\phi') + \delta(\phi' - \pi/2) - \delta(\phi - \pi) - \delta(\phi - 3\pi/2)\big)$$

Additionally, the spherical harmonics, Y_{lm}s, can be written in terms of associated Legendre polynomials as:

$$Y^*_{lm}(\cos\theta) = \gamma_{lm} P_{lm}(\cos\theta) \exp(-im\phi)$$

Given all of this.

$$q_{lm} = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} Y_{lm}(\theta', \phi') r'^l \left(\frac{q\delta(|\vec{x'}|)\delta(\theta' - \pi/2)}{r'^2 sin^2\theta'} \big(\delta(\phi') + \delta(\phi' - \pi/2) - \delta(\phi - \pi) - \delta(\phi - 3\pi/2)\big)\right) r'^2 sin^2\theta' \,\mathrm{d}r' \,\mathrm{d}\theta' \,\mathrm{d}\phi'$$

$$q_{lm} = qa^l Y^*_{lm}(\pi/2, 0) + Y^*_{lm}(\pi/2, \pi/2) - Y^*_{lm}(\pi/2, \pi) - Y^*_{lm}(\pi/2, 3\pi/2)$$
$$Y_{lm}(\theta, \phi) = \gamma_{lm} P_{lm}(\cos\theta) \exp(-im\phi), \quad So, \quad Y_{lm}(\pi/2, a) = \gamma_{lm} P_{lm}(0) \exp(-ima)$$

For a $= 0, \pi/2, \pi, 3\pi/2$ the expression for $\exp(-ima)$ can be reduced as $1, (-i)^m, (-1)^m, i^m$, respectively. Given this, the expression for the multipoles can be recast as follows:

$$q_{lm} = qa^l \gamma_{lm} P_{lm}(0) \big((1 - (-1)^m) - (i^m - (-i)^m)\big)$$

This can be easily seen to be zero for even m. Consider $f(m) = (1 - (-1)^m) - (i^m - (-i)^m$. $f(1) = 2 - 2i$, $f(3) = 2 + 2i$, $f(5) = 2 - 2i$, $f(7) = 2 + 2i$, $\cdots$. Additionally, though, $P_{lm}(x)$ is odd for any combination of l and m that is also odd. Since m is constrained to be odd for nonzero q_{lm} then l must be constrained to be odd as well for nonvanishing q_{lm} Thus, the final expression for q_{lm} can be reduced:

$$q_{2j+1,2k+1} = 2qa^{2j+1}\big\{1 + (-i)^{2k+1}\big\}\gamma_{1,1} P_{2j+1,2k+1}(0)$$

To meet the problem demands

$$q_{1,1} = 2qa^3(1 - i)\gamma_{1,1} P_{1,1}(0) = -\sqrt{\frac{3}{2\pi}} qa^3(1 - i)$$
$$q_{1,-1} = 2qa^3(1 - i)\gamma_{1,-1} P_{1,-1}(0) = \sqrt{\frac{3}{2\pi}} qa^3(1 + i)$$

b

In a similar fashion as before, we can construct $\rho(r,\theta,\phi) = \frac{q\delta(\phi')}{r'^2 \sin^2\theta'}\left(\delta(\theta')\delta(|\vec{r'} - a\hat{z}|) + \delta(\theta'-\pi)\delta(|\vec{r'} + a\hat{z}|) - 2\delta(\theta')\delta(|\vec{r'}|\right)$. Thus, the $q_{lm}s$ are given by:

$$\int\limits_{\text{all space}} \frac{qr'^l\delta(\phi')}{r'^2\sin^2\theta'} Y^*_{l,m}(\theta',\phi')\Big(\delta(|\vec{r'}| - a)\delta(\theta') + \delta(|\vec{r'}| - a|)\delta(\theta'-\pi) - 2\delta(r')\delta(\theta')\Big)$$

Reducing this expression yields:

$$q_{lm} = -2q\delta_{l,0} + qa^l(Y^*_{l,m}(0,0) + Y^*_{l,m}(\pi,0))$$

Since, $Y_{l,m}(\theta,\phi) = \gamma_{l,m}P_{l,m}(\cos\theta)\exp(-im\phi)$ then $Y_{l,m}(0,0)$ can be written as $\gamma_{l,m}P_{l,m}(1)$ and $Y_{l,m}(\pi,0)$ can be written as $\gamma_{l,m}P_{l,m}(-1)$.

$$q_{l,m} = -2q\delta_{l,0} + qa^l\gamma_{l,m}(P_{l,m}(1) + P_{l,m}(-1))$$

Now, $P_{l,m}$ is odd if $l+m$ is odd. Thus, in order to have nonvanishing $q_{l,m}$ it must be the case that the sum of l and m must be even. Additionally, it must be the case the $P_{l,m}(1)$ and $P_{l,m}(-1) \neq 0$. Note that for all associated Legendre polynomials for which m is nonzero, $P_{l,m}(\pm 1) = 0$. This can be seen in that $P_{l,l} = (-1)^l(2l-1)!!(1-x^2)^{l/2}$. This is a restatement of the fact that this charge distribution exhibits azimuthal symmetry. Thus, $q_{0,0}$ dies and ⇔q$_{1,-1}$ and $q_{1,1}$ die (as a consequence of $P_{1,\pm1}(\pm 1) = 0$). Note that $q_{1,0}$ dies, too. Thus, the first set of nonvanishing $q_{l,m}$ are:

$$q_{2,0} = qa^2\gamma_{2,0}(P_{2,0}(1) + P_{2,0}(-1)) = qa^2\sqrt{\frac{5}{4\pi}}(1+1) = qa^2\sqrt{5/\pi}$$
