Solution of Problem Set 1

P1)

$$\Psi_1(x,t) = A_1 \frac{e^{-i(kx-\omega t)}}{1+x^2} \quad \& \quad \Psi_2(x,t) = A_2 \frac{e^{-i(kx-\omega t+\pi x)}}{1+x^2}$$

Since $\int_{-\infty}^{+\infty} \frac{1}{(1+x^2)^2}\,dx = \pi/2$, then normalization condition,

i.e. $\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 dx = 1$, yields: $\boxed{A_1^2 = A_2^2 = \frac{2}{\pi}}$

a) $$I_a(x) = \frac{1}{2}|\Psi_1(x,t)|^2 + \frac{1}{2}|\Psi_2(x,t)|^2 = \frac{2}{\pi(1+x^2)^2}$$

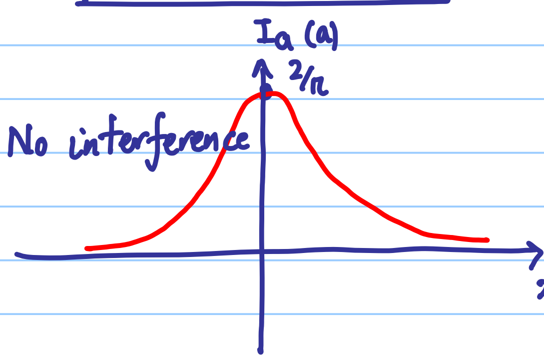
b) $$I_b(x) = |\Psi_1(x,t) + \Psi_2(x,t)|^2$$

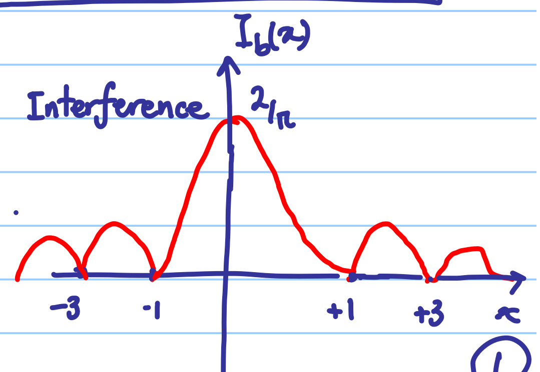
$$= \frac{2}{\pi(1+x^2)^2}\left| e^{-i(kx-\omega t)} + e^{-i(kx-\omega t+\pi x)} \right|^2$$

$$= \frac{2}{\pi(1+x^2)^2}(1+e^{i\pi x})(1+e^{-i\pi x})$$

$$= \frac{2}{\pi(1+x^2)^2}\cos^2\left(\frac{\pi}{2}x\right)$$

$$\Rightarrow \boxed{I_a(x) = \frac{2}{\pi(1+x^2)}} \quad , \quad \boxed{I_b(x) = \frac{2}{\pi(1+x^2)^2}\cos^2\left(\frac{\pi}{2}x\right)}$$





P2)

$$\Psi(z,t) = A e^{-\alpha|z|} e^{-i\omega t}$$

a) $\int_{-\infty}^{+\infty} |\Psi(z,t)|^2 dz = 1 \Rightarrow$

$$2\int_0^{\infty} |A|^2 e^{-2\alpha z} dz = \frac{|A|^2}{\alpha} = 1 \Rightarrow \boxed{A = \sqrt{\alpha}}$$

As α & A are real positive constants.

b) $\sigma_z = \sqrt{\langle z^2\rangle - \langle z\rangle^2}$, thus.

$$\langle z\rangle = \int_{-\infty}^{+\infty} z\,|\Psi(z,t)|^2 dz = \int_{-\infty}^{+\infty} z\,|A|^2 e^{-2\alpha z} dz = 0$$

odd integrand

$$\langle z^2\rangle = \int_{-\infty}^{+\infty} z^2 |\Psi(z,t)|^2 dz = 2|A|^2 \int_0^{+\infty} z^2 e^{-2\alpha z} dz = \frac{1}{2\alpha^2}.$$

$$\sigma_z = \sqrt{\frac{1}{2\alpha^2}} = \frac{\sqrt{2}}{2}\,\frac{1}{\alpha}. \qquad \boxed{\sigma_z = \frac{\sqrt{2}}{2\alpha}}$$

c) $$\Phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A e^{-\alpha|z|} \cdot e^{-ikz} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A e^{-\alpha|z|} \left[\cos kz - i \sin kz\right] dz$$

The $\cos kz$ integrand is even and the $\sin kz$ integrand is odd, so the former survives and the later one vanishes, so :

$$\Phi(k) = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\alpha|z|} \left(\frac{e^{-ikz} + e^{ikz}}{2} \right) dz$$

$$= \frac{A}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\alpha|z|} \left(e^{-ikz} + e^{ikz} \right) dz$$

$$= \frac{A}{\sqrt{2\pi}} \left[\frac{e^{(ik-\alpha)z}}{ik-\alpha} + \frac{e^{-(ik+\alpha)z}}{-(ik+\alpha)} \right]_0^{\infty} = \sqrt{\frac{\alpha}{2\pi}} \frac{2\alpha}{k^2+\alpha^2}$$

$$\boxed{\Phi(k) = \sqrt{\frac{\alpha}{2\pi}} \frac{2\alpha}{k^2+\alpha^2}}$$

d) $$\Psi(z,t) = \frac{2}{\sqrt{2\pi}} \sqrt{\frac{\alpha^3}{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{k^2+\alpha^2} e^{i(kz - \frac{\hbar k^2}{2m})t} dk$$

$$\Psi(z,t) = \frac{\alpha^{3/2}}{\pi} \int_{-\infty}^{+\infty} \frac{1}{k^2+\alpha^2} e^{i(kz - \frac{\hbar k^2}{2m})t} dk$$

e) For large α, $\Psi(z,0)$ is sharp spike & thus $\Phi(k) \simeq \sqrt{\frac{2}{\pi\alpha}}$ is flat and broad, thus position is well-defined but not the momentum.

For small α, $\Psi(z,0)$ is broad & flat while the momentum $\Phi(k) \simeq \sqrt{\frac{2\alpha^3}{\pi}} \frac{1}{k^2}$ is a sharp narrow spike, hence the momentum is well-defined not the position.

P3)

$$\Psi(z, t=0) = \delta(z-a)$$

$$\Phi(k, \omega=0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \delta(z-a)\, e^{-ikz}\, dz = \frac{1}{\sqrt{2\pi}}\, e^{-ika}$$

Note that the dispersion relation for free object is:

$$E = \hbar\omega = \frac{\hbar^2 k^2}{2m} \Rightarrow \omega = \frac{\hbar k^2}{2m}$$

$$\Rightarrow \Phi(k, \omega) = \frac{1}{\sqrt{2\pi}}\, e^{-ika} \cdot e^{-\frac{i\hbar k^2 t}{2m}} = \Phi(k)$$

$$\Psi(z,t) = \frac{1}{\sqrt{2\pi}} \int \Phi(k)\, e^{ikz}\, dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(z-a)}\, e^{-i\frac{\hbar k^2}{2m}t}\, dk$$

$$= \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(z-a)}\, e^{-i\left(\frac{\hbar t}{2m} - i\epsilon\right)k^2}\, dk$$

$$= \lim_{\epsilon \to 0} \frac{1}{2\pi} \sqrt{\frac{\pi}{\epsilon + i\hbar t/2m}}\, e^{i\frac{m(z-a)^2}{2\hbar t}}$$

$$= \sqrt{\frac{m}{2\pi i \hbar t}}\, \exp\left(\frac{i m (z-a)^2}{2\hbar t}\right)$$

we use $\epsilon \to 0^+$ to get the correct square root.

P4)

a)

$V(z) = V_0\,\delta(z)$

In TI-SE, $\frac{d^2\psi}{dz^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0$

For bound state $\Rightarrow E<0 \Rightarrow k^2 = \frac{2m}{\hbar^2}|E| \Rightarrow$

$$\boxed{\frac{d^2\psi}{dz^2} - k^2\psi - \frac{2mV_0}{\hbar^2}\delta(z) = 0} \quad (P4-1)$$

By integrating equation (P4-1) over z from $-\epsilon$ to $+\epsilon$

where $\epsilon \ll 1$, we get:

$$\psi'(\epsilon) - \psi'(-\epsilon) - k^2\int_{-\epsilon}^{+\epsilon}\psi\,dz = \frac{2mV_0}{\hbar^2}\psi(z=0)$$

If $\epsilon \to 0^+ \therefore \psi'(\epsilon) - \psi'(-\epsilon) = \frac{2mV_0}{\hbar^2}\psi(z=0)$

If $z \neq 0 \Rightarrow \psi(z) \propto e^{-k|z|}$ with $k>0 \Rightarrow$

$$\psi'(z) = \begin{cases} -ke^{-kz} & z>0 \\ ke^{-kz} & z<0 \end{cases} \Rightarrow$$

$$\psi'(\epsilon) - \psi'(-\epsilon) = -2k\,\psi(z=0) = \frac{2mV_0}{\hbar^2}\psi(z=0) \Rightarrow$$

$\boxed{k = -\frac{mV_0}{\hbar^2}}$ This requires $\boxed{V_0 < 0}$ ✓

b) The energy of bound state is $E = \frac{-\hbar^2k^2}{2m} = \frac{-mV_0^2}{2\hbar^2}$

the binding energy is $\boxed{\frac{mV_0^2}{2\hbar^2} = E_b}$

c)

$$\psi(z) = \sqrt{\frac{-mV_0}{\hbar^2}}\ \exp\left(\frac{mV_0}{\hbar^2}|z|\right) \quad , \; k = \frac{-mV_0}{\hbar^2}$$

$$P(z<z_0) = 2\left(\frac{mV_0}{\hbar^2}\right)\int^{z_0} e^{\mp 2kz}\,dz = \frac{2mV_0}{k\hbar^2}\left(1-e^{\mp 2kz_0}\right).$$

$$P(-\infty<z<\infty) = \frac{2mV_0}{\hbar^2}\int_0^{\infty} e^{-2kz}\,dz = \frac{2mV_0}{\hbar^2 k}$$

$$1 - e^{\mp 2kz_0} = 0.9 \longrightarrow 2kz_0 = \ln 10 \Rightarrow$$

$$\boxed{z_0 = -\frac{\hbar^2 \ln 10}{2mV_0}}$$

note $V_0 < 0$. ✓

P5)

a) $\oint \vec{p} \cdot d\vec{\ell} = nh \Rightarrow \begin{cases} |P|(2\pi a) = nh \\ E = \dfrac{|P|^2}{2m} \end{cases} \Rightarrow \boxed{E_n = \dfrac{n^2 \hbar^2}{2ma^2}}$

b) If $E = |P|^2 c^2 + m_0^2 c^4 \Rightarrow \boxed{E_n = \sqrt{\dfrac{n^2 \hbar^2 c^2}{a^2} + m_0^2 c^4}}$

For extremely relativistic object $\Rightarrow E \simeq |P|c \Rightarrow$

$\boxed{E_n \simeq \dfrac{n \hbar c}{a}}$

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