ECE 671 Fall 2015 Assignment 1

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Foreword

This problem set had me using a few tools at different times. I used Keysight's Advanced Design Suite (ADS) at a few points during my homework, for simulations. I have also used Mathematica, extensively, to assist me with algebra and plugging in values. Each problem is more or less self-contained in the written text. However, if you want to see calculations or simulations of anything that is referenced in the problem please consult the appendix for the associated problem. I have almost as many appendices as problems to account for this extra information. Please don't hesitate to ask me any questions you may have concerning my work. I have done my best to make it clear, but sometimes clear is only "clear" to one's self.

Power and Waves

Available Power

The maximum power available from the source is determined by the condition where the source is conjugately matched to the load (this is when $Z_s = Z_{in}^*$). Thus, a naive analysis of the power available from the source can be calculated as follows.

Available Power: Naive Approach

The source impedance is specified as 100 Ω. Thus, a conjugately matched input impedance would be given by $(100\ \Omega)^*$. This load will divide the voltage of the source. Thus, the power available from the source is given by:

$$\begin{aligned}
P_{avl} &= \Re(V_{in} I_{in}^*) \\
&= \Re\left(\frac{|V_{in}|^2}{Z_{in}^*}\right) \\
&= |V_{in}|^2 \, \Re\left(\frac{1}{Z_s}\right) \\
&= \frac{|V_{in}|^2}{|Z_s|^2} \Re\left(Z_s^*\right) \\
&= \frac{|V_{in}|^2}{|Z_s|^2} \Re\left(Z_s\right)
\end{aligned}$$

But, $V_{in} = V_s \frac{Z_s}{Z_s + Z_{in}}$. Using this:

$$
\begin{aligned}
P_{avl} &= \frac{|V_s|^2}{|Z_s|^2}\frac{|Z_s|^2}{|Z_s+Z_{in}|}\Re\left(Z_s\right)\\
&= \frac{|V_s|^2}{|Z_s+Z_{in}|^2}\Re\left(Z_s\right)\\
&= \frac{|V_s|^2}{4\left(\Re\left(Z_s\right)\right)^2}\Re\left(Z_s\right)\\
&= \frac{\left|V_s^2\right|}{4\Re\left(Z_s\right)}
\end{aligned}
$$

Now, this makes sense and we could use this to obtain an expression for the power available from the source. Let's consider that and obtain an answer, for this problem, for the power available from the source. The derivation above assumed that the voltages and currents were specified as RMS quantities. If they are provided as peak values (like in this problem) we need to scale both the current and voltage by $2^{-1/2}$ placing a $\frac{1}{2}$ in front of the previous expression. Given that the source impedance is specified as a real $100\,\Omega$ we can solve for the power available from the source:

$$
P_{avl} = \frac{|20\,\mathrm{V}|^2}{2\cdot 4\cdot 100\,\Omega} = 500\,\mathrm{mW}
$$

This is the naive solution.

Available Power: Alternative Solution

It makes sense. Another way to solve this problem is to consider the voltage that would be incident on the network V_1^+ and realize that the power transmitted from the source down a transmission line of characteristic impedance Z_{c_1} is:

$$
P_{trans} = \Re\left(V_1 I_1^*\right) = \Re\left(\left(V_1^+ + V_1^-\right)\frac{\left(V_1^+ - V_1^-\right)^*}{Z_{c_1}^*}\right)
$$

Then, realizing that the wave reflected off of the interface between the source impedance, Z_s, and the input impedance Z_{in} is

$$
V_1^- = \Gamma_{in}V_1^+ = \frac{Z_{in}-Z_s}{Z_{in}+Z_s}V_1^+
$$

we can rewrite the previous expression:

$$
P_{trans} = \left|V_1^+\right|^2\Re\left(\left(1+\Gamma_{in}\right)\frac{\left(1-\Gamma_{in}^*\right)}{Z_{c_1}^*}\right) = \left|V_1^+\right|^2\left(1-|\Gamma_{in}|^2\right)\frac{\Re\left(Z_{c_1}\right)}{\left|Z_{c_1}\right|^2} \tag{1}
$$

If we can find an expression for $V_1^+(V_s)$ then we have something with which to compare our naive solution. This is not too hard. The voltage across the input of the network can be expressed through a voltage divider as:

$$
V_{in} = V_1^+ + V_1^- = V_1^+(1+\Gamma_{in}) = V_s\frac{Z_{in}}{Z_s+Z_{in}}
$$

Thus,

$$
V_1^+ = V_s\frac{1}{1+\Gamma_{in}}\frac{Z_{in}}{Z_s+Z_{in}}
$$

Finally, we can write that the power delivered, in general, from a source with complex impedance Z_c to a network with impedance Z_{in} is:

$$P_{del} = |V_s|^2 \frac{|Z_{in}|^2}{|Z_s + Z_{in}|^2} \frac{1 - |\Gamma_{in}|^2}{|1 + \Gamma_{in}|^2} \frac{\Re(Z_{c_1})}{|Z_{c_1}|^2}$$

The case in which the maximum amount of power is delivered to the network is when the network is conjugately matched to the source $Z_{in} = Z_s^*$. This constrains $\Gamma_{in} = 0$ (for real Z_{in} and Z_c). Since, in this problem, $Z_{in} \in \mathbb{R}$. Then we can write the following for the P_{avl}, the maximum amount of deliverable power to the network:

$$\begin{aligned} P_{avl} = P_{del}|_{Z_s = Z_{in}^*} = P_{del} &= |V_s|^2 \frac{Z_s^2}{(2\Re(Z_s))^2} \frac{1 - |0|^2}{|1 + 0|^2} \frac{\Re(Z_s)}{|Z_s|^2} \\ &= |V_s|^2 \frac{1}{4\Re(Z_s)} \end{aligned}$$

Again, if the voltage is given in peak values and not in RMS quantities then a factor of $\frac{1}{2}$ will have to be inserted to account for this. Note that this solution is the same as the naive solution presented earlier. It has the distinct advantage of presenting us with the forward and reverse travelling waves which we will need later.

Calculating Port Waves and Powers (Try #1: Z_c-less)

Port 1 Waves

I will now proceed to calculate the port 1 power waves. Please note that the assumptions made in this problem set are different than the assumptions made in most reference texts regarding this material. The most significant departure in this solution set is the assumption that the input reflection coefficient is not referenced to a characteristic impedance but to the source impedance. The reason for this is because the problem does not explicitly state that a transmission line connects the source to the two port network and there is no reason to assume that there is one. The two port network scattering parameters could, if desired, be referred back to the source impedance such as to allow for the following analysis to be perfectly justified. So, to calculate the port 1 power waves I will use the following relationships:

$$\sqrt{Z_c}\,(a_1 + b_1) = V_s \frac{Z_{in}}{Z_{in} + Z_s} \tag{2}$$

$$b_1 = \Gamma_{in} a_1 \tag{3}$$

$$\Gamma_{in} = \frac{Z_{in} - Z_s}{Z_{in} + Z_s} \tag{4}$$

Note that 4 differs from the typical definition of Γ_{in} because I have assumed that the input impedance has been referred back to the reference plane at the source impedance. That is, there is no explicit transmission line that connects the source impedance to the two port network. Thus, the two port network is "directly connected" to the source and this is the first reflection plane in the signal propagation from the source. Z_c in 2 refers to some choice of a "reference impedance" for the system under consideration. It does not change the voltage values under consideration. This arbitrary choice of a reference impedance just affects the normalization of these voltages. Combining (2) and (3) yields:

$$a_1 = \frac{V_s}{\sqrt{Z_c}} \frac{1}{(1 + \Gamma_{in})} \frac{Z_{in}}{Z_{in} + Z_s} \tag{5}$$

Rearranging (4) for Z_{in} produces $Z_{in} = Z_s \frac{1+\Gamma_{in}}{1-\Gamma_{in}}$. Substituting this into (5) yields:

$$
\begin{aligned}
a_1 &= \frac{V_s}{\sqrt{Z_c}(1+\Gamma_{in})} \frac{Z_s \frac{1+\Gamma_{in}}{1-\Gamma_{in}}}{Z_s \frac{1+\Gamma_{in}}{1-\Gamma_{in}} + Z_s} \\
&= \frac{V_s}{\sqrt{Z_c}(1+\Gamma_{in})} \frac{Z_s\,(1+\Gamma_{in})}{Z_s\,(1+\Gamma_{in}) + Z_s\,(1-\Gamma_{in})} \\
&= \frac{V_s}{2\sqrt{Z_c}} \qquad (6)
\end{aligned}
$$

This result is surprising to anyone who studies microwave circuits. The result for the incident wave usually relies on the characteristic impedance of the transmission line and the source impedance. However, because the input impedance has been referred back to the plane of the source impedance the dependence on the transmission line drops out. It is surprising (even to the author) that there exists no dependence on the source impedance. But, this is what the math dictates. The reflected wave (and, thus, the total voltage across the input network) definitely does depend on the relationship between the source and the input impedance. The reflected wave b_1 is easily obtained from this using (3).

$$
b_1 = \frac{V_s \Gamma_{in}}{2\sqrt{Z_c}} \qquad (7)
$$

Notice that if $\Gamma_{in} = 1$ (an open) that $a_1 = \frac{V_s}{2\sqrt{Z_c}}$ and $b_1 = \frac{V_s}{2\sqrt{Z_c}}$ such that $V_{load} = \sqrt{Z_c}(a_1 + b_1) = V_s$ (reflects completely in phase). This makes sense; $\Gamma_{in} = 1$ corresponds to Z_{in} being an open. This would mean no current flows through the circuit and $V_{in} = V_s$.

If $\Gamma_{in} = -1$ (a short)

$$
\begin{aligned}
a_1 &= \frac{V_s}{2\sqrt{Z_c}} \\
b_1 &= -\frac{V_s}{2\sqrt{Z_c}} \\
V_{load} &= 0
\end{aligned}
$$

If $\Gamma_{in} = 0$ (a matched load)

$$
\begin{aligned}
b_1 &= 0 \\
a_1 &= \frac{V_s}{2\sqrt{Z_c}} \\
V_{load} &= \frac{V_s}{2}
\end{aligned}
$$

Thus, this recovers the well-known results in the limiting cases.

Port 2 Waves

Calculating the port 2 power waves I will use the following relationships:

$$
\begin{aligned}
b_2 &= S_{21} a_1 + S_{22} a_2 \qquad (8) \\
a_2 &= \Gamma_l b_2 \qquad (9) \\
a_1 &= \frac{V_s}{2\sqrt{Z_c}} \qquad (10)
\end{aligned}
$$

Combining (8) and (9) yield:

$$b_2 = \frac{S_{21} a_1}{1 - S_{22}\Gamma_l} \tag{11}$$

Combining (11) and (10) yields:

$$b_2 = \frac{V_s}{2\sqrt{Z_c}} \frac{S_{21}}{1 - S_{22}\Gamma_l} \tag{12}$$

Obtaining a_2 from (9) and (12) is trivial:

$$a_2 = \frac{V_s}{2\sqrt{Z_c}} \frac{S_{21}\Gamma_l}{1 - S_{22}\Gamma_l} \tag{13}$$

Note that Γ_l is usually defined as $\Gamma_l = \frac{Z_l - Z_c}{Z_l + Z_c}$ and requires some notion of a characteristic impedance. Why can we not do the same thing we did with the input impedance and refer the impedance of the load to port 2 of the two-port network. We could! In fact, this is usually what is done. Based on the output impedance of the two-port network and the impedance of the load we could determine this reflection coefficient. Thus, instead of defining the reflection coefficient of the load in this way, we will consider looking back from the load to determine the output impedance of the network, Z_{out}. Then, we will calculate the reflection coefficient at the interface between the two-port network and the load as $\Gamma_l = \frac{Z_l - Z_{out}}{Z_l + Z_{out}}$. Note that $\Gamma_{out} = -\Gamma_l = \frac{Z_{out} - Z_l}{Z_{out} + Z_l}$ (because we're looking the other way). To determine what Γ_{out} is in terms of the parameters of the problem consider that:

$$\frac{b_2}{a_2} = \Gamma_{out} = S_{21}\frac{a_1}{a_2} + S_{22}$$

by 8. We get a relationship between a_1 and a_2 by considering the following three things:

$$\begin{aligned} b_1 &= a_1 S_{11} + a_2 S_{12} \\ a_1 &= \Gamma_s b_1 \\ \Gamma_s &= \frac{Z_s - Z_{in}}{Z_s + Z_{in}} \end{aligned}$$

Substituting the second of the above two equations into the first and solving for $\frac{a_1}{a_2}$ yields:

$$\frac{a_1}{a_2} = \frac{S_{12}\Gamma_l}{1 - \Gamma_s S_{11}}$$

so that:

$$\Gamma_{out} = S_{22} + \frac{S_{12}S_{21}\Gamma_s}{1 - S_{11}\Gamma_s}$$

This implies (and it can be shown that)

$$\Gamma_{in} = S_{11} + \frac{S_{12}S_{21}\Gamma_l}{1 - S_{22}\Gamma_l}$$

Note that we don't have an expression that allows us to calculate Γ_{out} or Γ_{in}, yet. $\Gamma_{out} = -\Gamma_l$ depends on $\Gamma_s = -\Gamma_{in}$ but Γ_{in} depends on Γ_l. Thus, substituting our expression for $\Gamma_s = -\Gamma_l$ in to Γ_{in} will allow us to solve for Γ_{in} in terms of everything else. This turns out to be very difficult to solve by hand (as I've experienced). Using Mathematica to help with generating the algebraic solution yields not one but two solutions for both Γ_{in} and Γ_{out} which are as follows:

Quantity	Expression
Γ_{in_1}	$-\frac{-S_{12}^2S_{21}^2+2S_{11}S_{12}S_{22}S_{21}-\sqrt{4\left(S_{12}S_{21}S_{22}-S_{11}\left(S_{22}^2+1\right)\right)^2+\left(-\left(S_{22}^2+1\right)S_{11}^2+2S_{12}S_{21}S_{22}S_{11}-S_{12}^2S_{21}^2+S_{22}^2+1\right)^2}-\left(S_{11}^2-1\right)\left(S_{22}^2+1\right)}{2\left(S_{11}\left(S_{22}^2+1\right)-S_{12}S_{21}S_{22}\right)}$
Γ_{in_2}	$-\frac{-S_{12}^2S_{21}^2+2S_{11}S_{12}S_{22}S_{21}+\sqrt{4\left(S_{12}S_{21}S_{22}-S_{11}\left(S_{22}^2+1\right)\right)^2+\left(-\left(S_{22}^2+1\right)S_{11}^2+2S_{12}S_{21}S_{22}S_{11}-S_{12}^2S_{21}^2+S_{22}^2+1\right)^2}-\left(S_{11}^2-1\right)\left(S_{22}^2+1\right)}{2\left(S_{11}\left(S_{22}^2+1\right)-S_{12}S_{21}S_{22}\right)}$
Γ_{out_1}	$\frac{S_{12}^2S_{21}^2-2S_{11}S_{12}S_{22}S_{21}+\sqrt{4\left(S_{12}S_{21}S_{22}-S_{11}\left(S_{22}^2+1\right)\right)^2+\left(-\left(S_{22}^2+1\right)S_{11}^2+2S_{12}S_{21}S_{22}S_{11}-S_{12}^2S_{21}^2+S_{22}^2+1\right)^2}+\left(S_{11}^2+1\right)\left(S_{22}^2-1\right)}{2\left(S_{11}^2+1\right)S_{22}-2S_{11}S_{12}S_{21}}$
Γ_{out_2}	$\frac{S_{12}^2S_{21}^2-2S_{11}S_{12}S_{22}S_{21}-\sqrt{4\left(S_{12}S_{21}S_{22}-S_{11}\left(S_{22}^2+1\right)\right)^2+\left(-\left(S_{22}^2+1\right)S_{11}^2+2S_{12}S_{21}S_{22}S_{11}-S_{12}^2S_{21}^2+S_{22}^2+1\right)^2}+\left(S_{11}^2+1\right)\left(S_{22}^2-1\right)}{2\left(S_{11}^2+1\right)S_{22}-2S_{11}S_{12}S_{21}}$

Although these expressions are extremely complicated you can see that the main difference between the two solutions for Γ_{in} and Γ_{out} is the sign in front of the radical. These are the two roots of a quadratic solution. In the event that the two solutions are different, numerically, the proper Γ_{in} and Γ_{out} must be chosen in such a way that problem still makes sense. For passive circuits, it must be the case that $|\Gamma_{in}| \leq 1$ and $|\Gamma_{out}| \leq 1$.

Power to the Load

To determine the power delivered to the load we begin with the definition of real power (note that I assume, immediately, that the voltages and currents are provided as peak quantities):

$$P_{load} = \frac{1}{2}\Re\left(V_{load}I_{load}^*\right)$$

In this case, $V_{load} = \sqrt{Z_c}\left(a_2 + b_2\right)$ and $I_{load} = \sqrt{Z_c}\frac{b_2+a_2}{\sqrt{Z_l}}$. So P_{load} can be rewritten as follows:

$$P_{load} = \frac{1}{2}\Re\Big(Z_c\left(a_2 + b_2\right)\frac{\left(b_2^* + a_2^*\right)}{Z_l}\Big)$$

We have expressions for both b_2 and a_2 so we're done.

To calculate the power reflected to the source I need to take the incident power and subtract the amount that is delivered to the input network.

$$P_{source} = \frac{1}{2}\Re\left(V_{source}I_{source}^*\right)$$

$$V_{source} = V_s\frac{Z_s}{Z_s + Z_{in}}$$

and

$$I_{source} = \frac{V_s}{Z_s + Z_{in}}$$

.

Substituting these two expressions into P_{source} yields:

$$\begin{aligned} P_{source} &= \frac{1}{2}\Re\left(\frac{V_s Z_s}{Z_s + Z_{in}} \frac{V_s^*}{Z_S^* + Z_{in}^*}\right) \\ &= \frac{|V_S|^2}{2}\Re\left(\frac{Z_s}{|Z_s + Z_{in}|^2}\right) \\ &= \frac{|V_s|^2}{2\,|Z_s + Z_{in}|^2}\Re(Z_s) \end{aligned}$$

Note that in a similar way I could determine the amount of power delivered to the input network and, also, to the circuit as a whole.

$$P_{network} = \frac{1}{2}\Re\left(\left(V_s \frac{Z_{in}}{Z_{in} + Z_s}\right)\left(\frac{V_s}{Z_{in} + Z_s}\right)^*\right)$$

$$P_{del} = P_{network} + P_{source} = \frac{1}{2}\Re\left(V_s\left(\frac{(a_1 + b_1)\sqrt{Z_c}}{Z_{in}}\right)^*\right)$$

Plugging in Numbers

We will now plug the following numbers into equations 6, 7, 12, 13 to obtain a_1, b_1, b_2 and a_2, respectively.

Variable	Value
$Z_l = Z_c$(by choice)	$50\,\Omega$
Z_s	$100\,\Omega$
$\Gamma_{in} = -\Gamma_s = \frac{Z_s - Z_{in}}{Z_s + Z_{in}}$	$\Gamma_{in_1} \approx -8.79 + j458 \cdot 10^{-3}$ (invalid) or $\Gamma_{in_2} \approx 114 \cdot 10^{-3} + j5.92 \cdot 10^{-3}$
$\Gamma_{out} = -\Gamma_l = \frac{Z_l - Z_{out}}{Z_l + Z_{out}}$	$\Gamma_{out_1} \approx -5.96 + j433 \cdot 10^{-3}$ (invalid) or $\Gamma_{out_2} \approx 167 \cdot 10^{-3} + j12.1 \cdot 10^{-3}$
S_{11}	$.1\angle -30^\circ$
S_{12}	$.4\angle -75^\circ$
S_{21}	$.95\angle -45^\circ$
S_{22}	$.15\angle -10^\circ$
V_s	$20\,\mathrm{V}\angle 0^\circ$
$Z_{in} = Z_s \frac{1+\Gamma_{in}}{1-\Gamma_{in}}$	$\approx (126 + 1.51\mathrm{i})\,\Omega$

Plugging these numbers in yields:

Quantity	Value
a_1	$\sqrt{2}\,\mathrm{V}/\sqrt{\Omega}$ (exactly)
b_1	$\approx 160 + \mathrm{j}8.37\,\mathrm{mV}/\sqrt{\Omega}$
a_2	$\approx -166 + \mathrm{j}143\,\mathrm{mV}/\sqrt{\Omega}$ (exactly)
b_2	$\approx 929 - \mathrm{j}924\,\mathrm{mV}/\sqrt{\Omega}$
P_{load}	$\approx 596\,\mathrm{mW}$
P_{source}	$\approx 393\,\mathrm{mW}$
$P_{network}$	$\approx 493\,\mathrm{mW}$
P_{del}	$\approx 886\,\mathrm{mW}$

Note that more the source produced $\approx 946\,\mathrm{mW}$ of power. Note that for $Z_{in} = 126\,\Omega$ and $Z_s = 100\,\Omega$ that $P_{src} = 885\,\mathrm{mW}$. Looking at the table above this is also the sum of the amount of reflected power, P_{ref}, and the power delivered to the network, $P_{network}$ (as it should be). However, the amount of power delivered to the load is greater than that delivered to the input network. This implies that the two-port network was supplying power to the load. This can not be. I don't know why this is happening. I have checked my formulae multiple times. Any assistance that could be shed on this matter would be greatly appreciated.

Calculating Port Waves and Powers "Properly" (Try #2: With Z_c)

The approach we will take, now, will follow the more traditional paths that are taken to solve these problems. We will begin by calculating each of the incident and reflected waves.

Port 1 Waves

$V_1 = (a_1 + b_1)\sqrt{Z_c} = V_s \frac{Z_{in}}{Z_s + Z_{in}}$. $b_1 = \Gamma_{in} a_1$.

$$a_1 = \frac{V_s}{\sqrt{Z_c}} \frac{Z_{in}}{Z_s + Z_{in}} \frac{1}{1 + \Gamma_{in}}$$

Now,

$$b_1 = \Gamma_{in} a_1 = \frac{V_s}{\sqrt{Z_c}} \frac{Z_{in}}{Z_s + Z_{in}} \frac{\Gamma_{in}}{1 + \Gamma_{in}}$$

Note that the expressions given for Γ_{in} provided earlier in terms of the scattering parameters and Γ_l are still valid. But, we must change Γ_{in} to refer to an characteristic impedance Z_{c_1} at the input and we must change Γ_l to refer to a characteristic impedance Z_{c_2} at the output. These may be the same, but in general they do not have to be.

$$\Gamma_{in} = \frac{Z_{in} - Z_{c_1}}{Z_{in} + Z_{c_1}} \quad \text{and} \quad \Gamma_l = \frac{Z_l - Z_{c_2}}{Z_l + Z_{c_2}}$$

Port 2 Waves

Now, a_1 and b_1 are completely defined and are able to be calculated. The reference impedance Z_c defined in both of them should be taken with respect to the characteristic impedance at the input side, Z_{c_1}. b_2 can be calculated in a similar manner as before:

$$b_2 = a_1 S_{21} + a_2 S_{22}$$

But, $a_2 = \Gamma_l b_2$.

$$b_2 = a_1 \frac{S_{21}}{1 - \Gamma_l S_{22}} = \frac{V_s}{\sqrt{Z_{c_2}}} \frac{Z_{in}}{Z_s + Z_{in}} \frac{1}{1 + \Gamma_{in}} \frac{S_{21}}{1 - \Gamma_l S_{22}}$$

$$a_2 = \Gamma_l b_2 = a_1 \frac{S_{21}\Gamma_l}{1 - \Gamma_l S22} = \frac{V_s}{\sqrt{Z_{c_2}}} \frac{Z_{in}}{Z_s + Z_{in}} \frac{\Gamma_l}{1 + \Gamma_{in}} \frac{S_{21}}{1 - \Gamma_l S_{22}}$$

Plugging in Numbers (These Actually Work)

Performing a similar thing as before, I will fill the table below with known quantities. Below that I will supply the calculated quantities.

Variable	Value
$Z_l = Z_{c_1}$(by choice)	$50\,\Omega$
Z_s	$100\,\Omega$
Γ_{in}	$86.6 \cdot 10^{-3} - j50 \cdot 10^{-3}$
S_{11}	$.1\underline{/-30°}$
S_{12}	$.4\underline{/-75°}$
S_{21}	$.95\underline{/-45°}$
S_{22}	$.15\underline{/-10°}$
V_s	$20\,\mathrm{V}\underline{/0°}$
$Z_{in} = Z_{c_1}\frac{1+\Gamma_{in}}{1-\Gamma_{in}}$	$\approx (118 - 12.0\mathrm{i})\,\Omega$

Plugging these numbers (see Mathematica in appendix) in yields:

Quantity	Value
a_1	$1\,\mathrm{V}/\sqrt{\Omega}$
b_1	$\approx 86.6 - \mathrm{j}50.0\,\mathrm{mV}/\sqrt{\Omega}$
a_2	$\approx 0 + \mathrm{j}0\,\mathrm{mV}/\sqrt{\Omega}$ (exactly)
b_2	$\approx 672 - \mathrm{j}672\,\mathrm{mV}/\sqrt{\Omega}$
P_{load}	$\frac{1}{2}\Re\left(\sqrt{Z_{c_2}}\left(a_2+b_2\right)\left(\sqrt{Z_{c_2}}\frac{a_2+b_2}{Z_l}\right)^*\right) \approx 451\,\mathrm{mW}$
P_{source}	$\frac{1}{2}\Re\left(\left(\frac{V_s Z_s}{Z_{in}+Z_s}\right)\left(\frac{V_s}{Z_{in}+Z_s}\right)^*\right) \approx 418\,\mathrm{mW}$
$P_{network}$	$\frac{1}{2}\Re\left(\left(V_s\frac{Z_{in}}{Z_{in}+Z_s}\right)\left(\frac{V_s}{Z_{in}+Z_s}\right)^*\right) \approx 495\,\mathrm{mW}$
P_{del}	$\approx \frac{1}{2}\Re\left(V_s\left(\frac{V_s}{Z_{in}+Z_s}\right)^*\right) \approx 913\,\mathrm{mW}$

Note that, now, $P_{source} + P_{network}$ (the power delivered to the source plus that delivered to the load and the two port network) is equal to the delivered power, as it should be, **and** $P_{load} < P_{network}$, as it should be.

Different Tuning Circuits

Type 1: Single Open Stub Tuner

The goal is to match a load impedance to a transmission line by placing a stub of a certain length l_s a certain distance l_l away from a load. To be matched means that the input impedance looks like the characteristic impedance Z_c. To the source, the two paths (the stub and the load) will seem to be connected in parallel. Thus, quite generally:

$$Z_{in} = Z_{stub}||Z_l'$$

where Z_l' is the impedance of the load transformed by a certain length l_l down the line. A load of impedance Z_l looks the following impedance when we're a length "l" down the line:

$$Z_l'(l) = Z_c\frac{Z_l + jZ_c\tan\beta l}{Z_c + jZ_l\tan\beta l}$$

However, because we are combining parallel impedances it will be easier to deal with admittances:

$$Y_{in} = Y_{stub} + Y_l'$$

where

$$Y_l'(l_l) = Y_c \frac{Y_l + jY_c \tan \beta l_l}{Y_c + jY_l \tan \beta l_l}$$

Our goal is to make $\Re(Y_{in}) = Z_c$ and $\Im(Y_{in}) = 0$. We have an expression for Y_l' already. We'd like an expression for Y_{stub}. However, Y_{stub} is just a infinite impedance (zero admittance) load. Thus, we can use the same equation as before before:

$$\begin{aligned} Y_{stub}(l_s) &= Y_c \frac{Y_s + jY_c \tan \beta l_s}{Y_c + jY_s \tan \beta l_s} \\ &= Y_c \frac{0 + jY_c \tan \beta l_s}{Y_c + j0 \tan \beta l_s} \\ &= jY_c \tan \beta l_s \end{aligned}$$

Unfortunately, the load can not, in general, be simplified any further. Thus, the input admittance is:

$$Y_{in} = Y_c \left(\frac{Y_l + jY_c \tan \beta l_l}{Y_c + jY_l \tan \beta l_l} + j \tan \beta l_s \right)$$

From this, we can consider the following two equations:

$$\begin{aligned} 1 &= \Re(Y_{in}) \\ 0 &= \Im(Y_{in}) \end{aligned}$$

These two equations can be solved simultaneously for pairs of l_l and l_s that satisfy them. Given the transcendental nature of these two solutions it is not, in general, possible to find an analytic solution. However, computers are more than capable, these days, of determining the roots of such equations with much ease. Mathematica was employed to find the following roots:

$$\beta_{l_l} \approx .955 \text{ radians} \qquad \beta_{l_s} \approx 2.53 \text{ radians}$$

These values were used in ADS to simulate the frequency response of the single stub tuner.

Type 2: LC Matching Network

The goal in constructing a matching network for the load using an LC network is going to be similar as that that was performed in the previous section. Namely, our goal is going to be to make $Z_{in} = Z_c$ such that $\Re(Z_{in}) = Z_c$ and $\Im(Z_{in}) = 0$. It is easy, in this case to construct the input impedance as

$$Z_{in} = Z_A + Z_B || Z_l$$

The only knowledge we have, currently, regarding Z_A and Z_B is that both impedances are purely imaginary. We know that Z_l is purely real ($100\,\Omega$). Thus, we start by considering $\Re(Z_{in})$.

$$\begin{aligned} \Re(Z_{in}) = Z_c &= \Re(\frac{Z_B Z_l}{Z_B + Z_l}) \\ &= \Re(\frac{jX_B R_l}{jX_B + R_l}) \\ &= \frac{X_B^2 R_l}{X_B^2 + R_l^2} \end{aligned}$$

Considering the imaginary part of Z_{in}

$$\Im\left(Z_{in}\right) = 0 = \Im\left(jX_A + \frac{jX_B R_l}{jX_B + R_l}\right) \tag{1}$$
$$= \Im\left(jX_A + \frac{jX_B R_l\left(R_l - jX_B\right)}{X_B^2 + R_l^2}\right) \tag{2}$$
$$= X_A + \frac{X_B R_l^2}{X_B^2 + R_l^2} \tag{3}$$

The real part of Z_{in} only involves Z_B. So, we can easily solve for Z_B that way. Doing so yields:

$$X_B^2 = \frac{R_l^2 Z_c}{R_l - Z_c}$$

Since we have constrained X_B to be real (so that the impedance Z_B is purely reactive) we know that we must take the positive root. We also know that this only works if $R > Z_c$. If the load is smaller than the characteristic impedance than an LC matching network will not work. You must add some resistance.

Now, X_A is easily determined:

$$\begin{aligned} X_A &= -\frac{X_B R_l^2}{X_B^2 + R_l^2} \\ &= -\frac{X_B}{\frac{Z_c}{R_l - Z_c} + 1} \\ &= -\frac{X_B\left(R_l - Z_c\right)}{Z_c + R_l - Z_c} \\ &= -\frac{X_B\left(R_l - Z_c\right)}{R_l} \\ &= -\sqrt{\frac{Z_c}{R_l - Z_c}}\left(R_l - Z_c\right) \\ &= -\sqrt{Z_c\left(R_l - Z_c\right)} \end{aligned}$$

If one allows, again, $R_l = \alpha Z_c$ then

$$X_B^2 = \frac{\alpha^2 Z_c^2}{\alpha - 1}$$

and

$$X_A = -Z_c\sqrt{\alpha - 1}$$

Plugging in the relevant values $\alpha = \frac{R_l}{Z_c} = 2$ and $Z_c = 2$ yields the following $50\,\Omega$

$$X_A = -Z_c = -50\,\Omega \quad \text{and} \quad X_B = 2Z_c = 100\,\Omega$$

This implies that Z_A is a capacitor of value $Z_A = jX_A = \frac{1}{j\omega C_A}$ and Z_B is an inductor of value $Z_B = jX_B = j\omega L_B$. Thus:

$$C_A = -\frac{1}{\omega X_A} = \frac{1}{50\omega}\,\text{F} \qquad\qquad L_B = \frac{X_B}{\omega} = \frac{100}{\omega}\,\text{H}$$

The reflection coefficient for the load and the matching network can be found by considering $\Gamma_{in} = \frac{Z_{in} - Z_c}{Z_{in} + Z_c}$. In this case, Γ_{in} is a function of frequency.

$$
\begin{aligned}
Z_{in} &= Z_A + Z_B || Z_l \\
&= jX_A + \frac{jX_B R_l}{jX_B + R_l} \\
&= \frac{1}{j\omega \frac{1}{50\omega_m}}\ \Omega + \frac{j\omega \frac{100}{\omega_m} R_l}{j\omega \frac{100}{\omega_m} + R_l}\ \Omega \\
&= \frac{50\omega_m}{j\omega}\ \Omega + \frac{j\omega 100 R_l}{j\omega 100 + \omega_m R_l}\ \Omega
\end{aligned}
$$

In the above, ω_m is the frequency at which we have chosen to match our network. Let's consider a parameter $\gamma = \frac{\omega}{\omega_m}$ which we will adjust from $.5\omega_m \to 1.5\omega_m$. The previous expression for Z_{in} can be written, now, as:

$$Z_{in} = \frac{50}{j\gamma}\ \Omega + \frac{j100\gamma R_l}{j100\gamma + R_l}\ \Omega$$

Since 100 bears a nice relationship with $R_l = 100\Omega$ we can simplify Z_{in} even further in this particular case:

$$Z_{in} = \frac{50}{j\gamma}\ \Omega + \frac{j\gamma}{j\gamma + 1}\ \Omega$$

Let's assume an operation frequency of 1 GHz as an example. This determines a capacitance of $\frac{1}{50 \cdot 2 \cdot \pi \cdot 1 \cdot 10^9}$ F ≈ 3.18 pF and an inductance of $\frac{100}{2\pi \cdot 10^9}\ \Omega \approx 15.9$ nH.

These values were used in ADS to simulate the reflection coefficient as a function of frequency.

Type 3: Double Open Stub Tuner

This problem is very similar to Problem 2a with the single stub tuner. The only difference, now is that we have two stubs instead of one. We will follow a very similar path as we did for problem 2a to solve for the matching network.

$$Y_{in} = Y_{stub_2} + Y'_{stub_1} + Y'_l \tag{4}$$

Where, here, Y'_{stub_1} is the impedance of the stub 1 transformed from the end of stub 1 to the start of stub 2. Like before:

$$Y'_l(l_{l_1}) = Y_c \frac{Y_l + jY_c \tan \beta l_{l_1}}{Y_c + jY_l \tan \beta l_{l_1}}$$

Where, unlike before, we have to transform the impedance of stub 1 (which is l_{s_1} long) and the load a length of l_{l_1} along the line. $Y_{stub_1}(l_{s_1}) = jY_c \tan \beta l_{s_1}$. Right at the first stub, though, the admittance of both the first stub and the load is:

$$Y_1 = jY_c \tan \beta l_{s_1} + Y_c \frac{Y_l + jY_c \tan \beta l_{l_1}}{Y_c + jY_l \tan \beta l_{l_1}}$$

Thus, the total admittance at the 2nd stub (closer to the source, farther from the load) is:

$$Y_2 = Y_c \frac{Y_1(l_{s_1}, l_{l_1}) + jY_c \tan \beta l_{l_2}}{Y_c + jY_1(l_{s_1}, l_{l_1}) \tan \beta l_{l_2}}$$

This admittance must relate to the admittance of stub 2 in the following way:

$$\Re(Y_2) = Y_c$$
$$\Im(Y_2) = -\Im(Y_{stub_2}) = -\Im(jY_c \tan \beta l_{s_2})$$

There are four unknowns in the above equation for the admittance: $l_{s_1}, l_{s_2}, l_{l_1}$ and l_{l_2}. We have been given, in addition to the problem statement, the information that $l_{l_2} = l_{l_1} = \frac{\lambda}{8}$. I'm going to assume that both stubs are the same length and allow the distance from stub 1 to the load to vary. Using Mathematica, again, to solve for these roots yields:

$$\beta l_{s_2} \approx 65.8^\circ \quad \text{and } \beta l_{s_1} \approx 54.1^\circ$$

Comparison of Different Matching Networks

It seems that based on the bandwidths of the reflection coefficients alone, the LC matching network is the best option.

Lossy Matching Networks

The first thing to do is to find the amount of loss in the matching network. This loss will be due to two non-ideal conditions for the circuit elements

1. The inductor is lossy with series lossy element R_s
2. The capacitor is lossy with parallel lossy element R_p

Thus, the loss in R_s can be found by driving the now-matched input network with a voltage V_{in} and finding the following:

$$P_s = \frac{1}{2}\Re(V_s I_s^*)$$

$V_s = V_{in}\frac{R_s}{Z_{in}}$ and $I_s = \frac{V_{in}}{Z_{in}}$
The power delivered to the series resistor is now

$$P_s = \frac{1}{2}\frac{|V_{in}|^2 R_s}{|Z_{in}|^2}$$

The power delivered to the parallel resistor is a bit more tricky. We can consider the voltage across the parallel branch (with R_p, R_l and Z_C). The voltage across that branch is

$$V_p = V_{in}\frac{Z_c||R_l||R_p}{Z_{in}}$$

The current through R_p, I_p, is:

$$I_p = \frac{V_p}{R_p}$$

Thus, we can write the power delivered to this resistor as:

$$P_p = \frac{1}{2}\frac{|V_p|^2}{R_p}$$

And the total power delivered to these two circuits elements is just

$$P_s + P_p = \frac{1}{2}\frac{|V_{in}|^2 R_s}{|Z_{in}|^2} + \frac{1}{2}\frac{|V_{in}|^2 \left|Z_c||R_l||R_p\right|^2}{R_p |Z_{in}|^2} \tag{1}$$

This is exact. However, if we consider the case where we are close to the matching frequency then $Z_A \approx -Z_B$ and most of the energy is dissipated in the resistors (not reflected or stored in the inductor and capacitor). Also, if the inductors and capacitors are any good, it should be the case that $R_p >> R_l >> R_s$ (a large shunt resistor on the capacitor implies that it's very good). This means that if we write an expression for the power lost near the matching frequency it would look like:

$$P_{lost} = \left(V_s \frac{R_s}{R_s + R_p||R_l}\right)^2 \frac{1}{R_s} + \left(V_s \frac{R_p||R_l}{R_s + R_p||R_l}\right)^2 \frac{1}{R_p}$$

But, because $R_p >> R_l$ then $R_p||R_l \approx R_l$.

$$P_{lost} \approx V_s^2 \left(\frac{R_s}{R_l + R_s}\right)^2 \frac{1}{R_s} + V_s^2 \left(\frac{R_l}{R_s + R_l}\right)^2 \frac{1}{R_p} = V_s^2 / (R_s + R_l)$$

Expressing this in terms of the transformation ratio $\eta = R_l/R_{in}$ (where $R_{in} \approx R_s + R_l$) yields:

$$P_{lost} \approx V_s^2 \frac{1}{R_{in}} = V_s^2 \frac{\eta}{R_l}$$

Notice that as the transformation ratio increases, the loss increases.

We know the value of R_l is $50\,\Omega$ since this type of matching network is designed to lower the impedance of the load. Now, we get to pick the sign of the susceptance. We will select it to be positive such as to maintain the geometry of the capacitor shunting the load. Realizing that $Z_c = 10\,\Omega$ and $R_l = 50\,\Omega$ we have

$$B = \frac{1}{50}\sqrt{5-1} = \frac{2}{50}\ \mathrm{S}$$

At 3 GHz this corresponds to a capacitor of value: $\frac{B}{\omega} = C$:

$$C = \frac{\frac{2\ \mathrm{S}}{50}}{2\pi \cdot 3\ \mathrm{GHz}} \approx 2.12\,\mathrm{pF}$$

Using Pozar's 5.3b to find X, the series impedance of the matching network yields:

$$\begin{aligned}
X &= \frac{1}{B} + \frac{X_l Z_c}{R_l} - \frac{Z_0}{BR_l} \\
&= \frac{1}{B} - \frac{Z_c}{BR_l} \\
&= 25\,\Omega - \frac{10\ \Omega}{\frac{2S}{50} 50\ \Omega} \\
&= 20\,\Omega
\end{aligned}$$

This is an inductor of value $L = \frac{X}{\omega}$:

$$L = \frac{20\ \Omega}{2\pi \cdot 3\ \mathrm{GHz}} \approx 1.06\,\mathrm{nH}$$

If the Q factors of the inductor and capacitor are 15 and 30, respectively, this implies a value of $R_s = \frac{X_l}{Q} = \frac{20\ \Omega}{15} = \frac{4\ \Omega}{3}$ and $R_p = Q_c X_c = 20 \cdot 25\ \Omega = 500\ \Omega$.

Plugging in the values I have, the power delivered to the load is:

$$P_{load} = \frac{1}{2}\left(|V_s|^2 \frac{\left|Z_c||R_l||R_p\right|^2}{\left|Z_{in}\right|^2}\right)\frac{1}{R_l}$$

Dividing this by the power delivered to the whole network:

$$P_{del} = \frac{V_s^2}{2}\Re\left(\frac{1}{Z_{in}^*}\right)$$

Calculating the following on Mathematica I obtain:

$$\text{Insertion Loss } = -10\log_{10}\frac{P_{load}}{P_{del}} \approx .93 \text{ dB}$$

This aligns well with the simulated value $-20\log_{10}|S_{21}|$ which (apart from ignoring reflection) demonstrates the same idea. At the matching frequency, the amount of reflection is very small (< 20 dB). The amount of forward gain reported by ADS is $-.968$ dB, i.e. a loss of .968 db. Apart from the reflection, these values are consistent with one another.

Problem 4: 4-port Scattering Problem

The scattering matrix provided, along with the knowledge of how ports 3 and 4 are associated yields the following relationships (note that we assume a matched load for simplicity):

$$V_1^- = S_{11}V_1^+ + S_{14}V_4^+ \tag{1}$$
$$V_2^- = S_{22}V_2^+ + S_{23}V_3^+ \tag{2}$$
$$V_3^- = S_{32}V_2^+ + S_{33}V_3^+ \tag{3}$$
$$V_4^- = S_{41}V_1^+ + S_{44}V_4^+ \tag{4}$$
$$V_4^- = e^{-j\beta l}V_3^+ \tag{5}$$
$$V_3^- = e^{-j\beta l}V_4^+ \tag{6}$$
$$V_2^+ = 0 \tag{7}$$

Equations 1 to 4 come from the scattering matrix provided and the definition of scattering parameters. Equations 5 to 6 are given by the fact that the output of port 3 is connected to the input of port 4 and that the output of port 4 is connected to the input of port 4. Note that after a travelling a length l down the line, the forward voltage wave picks up a phase of $e^{-j\beta l}$.

$$V^+(l) = e^{-j\beta l}V^+(0)$$

Equation 7 comes from the fact that we're using a matched load at the output so that there is no reflection back to the 2-port network.

What we want to know is $\frac{V_2}{V_1} = \frac{V_2^-}{V_1^+(1+\Gamma_{in})}$. To determine Γ_{in} we will first divide 1 by V_1^+.

$$\Gamma_{in} = \frac{V_1^-}{V_1^+} = S_{11} + S_{14}\frac{V_4^+}{V_1^+}$$

Now, all that is needed is a relationship between $V_4^+(V_1^+, V_1^-)$ and we've solved Γ_{in}. We will find this by first applying 7 to the 2 and 3 yielding the following reduced equations:

$$V_2^- = S_{23}V_3^+ \tag{8}$$
$$V_3^- = S_{33}V_3^+ \tag{9}$$

Using 9 with 5 and 6 yields the fellowing:

$$V_4^+ = S_{33}e^{2j\beta l}V_4^- \tag{10}$$

Substituting this back into 4:

$$V_4^- = S_{41}V_1^+ + S_{44}(S_{33}e^{2j\beta l}V_4^-)$$

Now, we have an expression for $V_4^-(V_1^+)$ which is close to what we want.

$$V_4^- = \frac{S_{41}V_1^+}{1 - S_{33}S_{44}e^{2j\beta l}}$$

However, we know from 10 that we can finally write:

$$V_4^+ = \frac{S_{33}S_{41}e^{2j\beta l}V_1^+}{1 - S_{33}S_{44}e^{2j\beta l}}$$

Finally, we have:

$$\Gamma_{in} = S_{11} + \frac{S_{33}S_{41}e^{2j\beta l}}{1 - S_{33}S_{44}e^{2j\beta l}} \tag{11}$$

We're almost done! We just need $V_2 = V_2^-$. But, $V_2 = S_{23}V_3^+$ by 2 and $V_3^+ = e^{j\beta l}V_4^-$ by 5. This allows us to write:

$$V_2 = V_2^- = S_{23}V_3^+ = S_{23}e^{j\beta l}V_4^- = \frac{S_{23}S_{41}e^{j\beta l}V_1^+}{1 - S_{33}S_{44}e^{2j\beta l}}$$

And since $V_1 = V_1^+ + V_1^- = V_1^+(1 + \Gamma_{in})$ we can write $\frac{V_2}{V_1} = \frac{f(V_1^+)}{V_1^+(1+\Gamma_{in})}$.

$$\frac{V_2}{V_1} = \frac{S_{23}S_{41}e^{j\beta l}}{(1 + \Gamma_{in})\,(1 - S_{33}S_{44}e^{2j\beta l})}$$

βl was given as $45° = \frac{\pi}{4}$ radians. Plugging in the following values for the scattering parameters and βl yields Γ_{in} and $\frac{V_2}{V_1} = \text{Gain}$.

$$\frac{V_2}{V_1} \approx 42.4 \cdot 10^{-3}\underline{/152.2°}$$

The phase delay can be determined immediately. The phase difference between the input and the output is 167.7°.

The insertion loss is a little bit harder to determine. Let's consider insertion loss as a logarithmic quantity that relates the power delivered to the device after the insertion of the 2-port network as compared to before it was inserted.

Before the device is inserted it received $\Re(V_l I_l^*)$ amount of power (assuming the voltages and currents are given as RMS quantities). $V_l = V_1^+$ since the load is matched to the transmission line. $I_l = \frac{V_1^+}{Z_c}$ since $V_l^- = 0$. Thus, the power that used to be delivered to the load is:

$$P_{before} = \frac{|V_1^+|^2}{Z_c}$$

After the two-port device is inserted it receives $V_l = \text{Gain} \cdot V_1^+$ incident voltage (still there is no reflected voltage). Thus, the amount of power delivered to the device after the two-port is inserted is:

$$P_{after} = \frac{\left|V_1^+\right|^2 |\text{Gain}|^2}{Z_c}$$

The insertion loss will be defined as follows:

$$IL_{dB} = 10\log_{10}(\frac{P_{before}}{P_{after}})$$

Note that this definition will be dictate that IL_{dB} is a positive quantity since P_{before} (for passive devices) will be greater than P_{after}. Thus, there is no "wonkiness" with signs in determining the insertion loss. Plugging in our expressions for P_{before} and P_{after} yields:

$$IL_{dB} = -20 \cdot \log_{10}(|\text{Gain}|)$$

Plugging in the magnitude of the gain ($\sim 42.4 \cdot 10^{-3}$) we obtain an insertion loss of

$$IL_{dB} \approx 27.5 \text{ dB}$$

Series-Connected Two-Port Networks

What's important to realize in this problem is that the same current that flows out of the transistor's emitter flows into the inductor. Thus, the inductor (a 2-port device) is connected in series with the transistor (a 2-port device). A nice way to solve for the 'black box" that represents series-connected two port devices is to use the impedance matrix. The relationship between the impedance matrix and the voltages and currents of the two port is:

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

So, if we consider two two-port networks given by two impedance matrices Z_a and Z_b respectively and we know that $I_{1_a} = I_{1_b}$ and that $I_{2_a} = I_{2_b}$ then we can say that

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_{1_a} \\ V_{2_a} \end{pmatrix} + \begin{pmatrix} V_{1_b} \\ V_{2_b} \end{pmatrix} = Z_a \begin{pmatrix} I_{1_a} \\ I_{2_a} \end{pmatrix} + Z_b \begin{pmatrix} I_{1_b} \\ I_{2_b} \end{pmatrix} = (Z_a + Z_b) \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

We are given scattering parameters for the transistor and we don't know the scattering parameters or the impedance parameters for the inductor so we might seem stuck. However, as shown in the appendix it is possible to transform between scattering parameters and impedance parameters. So, if we transform the transistor parameters to impedance parameters and then add the to-be-determined inductor impedance parameters we can add these two impedance matrices together and go back to scattering parameters.

Determining the impedance parameters of the inductor is not too hard. To determine the diagonal entries we need to drive one port of the inductor while leaving the other port open. The voltage with which we drive the inductor will determine a current through the inductor. The voltage we choose at port one divided by the current we measure through port one determines Z_{11} and likewise for Z_{22}. By inspection it is easy to determine that Z_{11} and Z_{22} are both $j\omega L$. For the off diagonal terms we need to drive one port with current and measure the voltage built up across the other port when it is open. However, again, any current, I_1 with which we drive port 1 will determine a voltage across port 2 that is $j\omega L I_1$. So, $Z_{21} = Z_{12} = Z_{11} = Z_{22} = j\omega L$.

$$Z_I = \begin{pmatrix} j\omega L & j\omega L \\ j\omega L & j\omega L \end{pmatrix}$$

Z_I is the impedance matrix of the inductor. To obtain the impedance parameters for the transistor we consider the following equations:

$$Z_{11} = Z_c \frac{(1 + S_{11})(1 - S_{22}) + S_{12}S_{21}}{(1 - S_{11})(1 - S_{22}) - S_{12}S_{21}} \tag{1}$$

$$Z_{12} = Z_c \frac{2S_{12}}{(1 - S_{11})(1 - S_{22}) - S_{12}S_{21}} \tag{2}$$

$$Z_{21} = Z_c \frac{2S_{21}}{(1 - S_{11})(1 - S_{22}) - S_{12}S_{21}} \tag{3}$$

$$Z_{22} = Z_c \frac{(1 - S_{11})(1 + S_{22}) + S_{12}S_{21}}{(1 - S_{11})(1 - S_{22}) - S_{12}S_{21}} \tag{4}$$

Plugging in the numbers that were given for the transistor yields:

$$Z_T \approx \begin{pmatrix} 26.6\underline{/-60.4^\circ} & 0 \\ 0 & 98.6\underline{/-79.7^\circ} \end{pmatrix}$$

If the inductor is a 100 pH inductor then it has an impedance of $j(2\pi \cdot 10 * 10^9).1 \cdot 10^{-9} = 2\pi\ \Omega$ at 10 GHz. Adding the impedance matrix of the inductor, Z_I to that of the transistor, Z_T, yields:

$$Z_T + Z_I \approx \begin{pmatrix} 20.8\underline{/-62.1^\circ} & j2\pi \\ j2\pi & 92.8\underline{/-68.3^\circ} \end{pmatrix}$$

Using the following relationships to convert these impedance parameters to scattering parameters yields:

$$S_{I+T} \approx \begin{pmatrix} .717\underline{/-138^\circ} & .083\underline{/153^\circ} \\ .083\underline{/153^\circ} & .723\underline{/-55.0^\circ} \end{pmatrix}$$

Scattering Parameters of an Ideal Transformer

The thing to understand about a transformer with an 1:n turns ratio is that the relationship between port 2's voltage and port 1's voltage is:

$$\frac{V_2}{V_1} = n$$

Similarly, the currents through both ports are related in the following way:

$$\frac{I_1}{I_2} = n$$

Now, in order to find the scattering parameters I will need to use these relationships in addition to the following idea: When I drive port 1, there will be no incident voltage on port 2, because it's matched. Likewise, when I drive port 2, there will be no incident voltage wave on port 1, because it's matched. To start, we will find S_{11} and S_{21}. When we do this we will drive port 1. This implies that $V_2^+ = 0$ so that we can write the following for the transformer relationships:

$$n(V_1^+ + V_1^-) = V_2^- \quad \text{Voltage Relationship} \tag{1}$$

$$V_1^- - V_1^+ = nV_2^- \quad \text{Current Relationship} \tag{2}$$

$$S_{11} = \frac{V_1^-}{V_1^+} = \frac{1 + n^2}{1 - n^2}$$

$$S_{21} = \frac{V_2^-}{V_1^+} = \frac{n\left(V_1^+ + V_1^-\right)}{V_1^+} = n\left(1 + S_{11}\right) = \frac{2n}{1 - n^2}$$

To find S_{22} and S_{12} I will drive port 2 which implies that $V_1^+ = 0$. This gives me the following transformer relationships:

$$nV_1^- = V_2^- + V_2^+ \quad \text{Voltage Relationship} \tag{3}$$
$$V_1^- = n\left(V_2^- - V_2^+\right) \quad \text{Current Relationship} \tag{4}$$

$$S_{22} = \frac{V_2^-}{V_2^+} = \frac{n^2+1}{n^2-1}$$

$$S_{12} = \frac{V_1^-}{V_2^+} = \frac{n\left(V_2^- - V_2^+\right)}{V_2^+} = n\left(S_{22}-1\right) = \frac{2n}{n^2-1}$$

Thus, finally,

$$S_{\text{Transformer}} = \begin{pmatrix} \frac{1+n^2}{1-n^2} & \frac{2n}{1-n^2} \\ \frac{2n}{n^2-1} & \frac{n^2+1}{n^2-1} \end{pmatrix}$$

Problem 1 Appendix

Mathematica

Values used everywhere

In[60]:=
$$S_{11} = .1\,\mathrm{Exp}[-I\,30\,°];$$
$$S_{12} = .4\,\mathrm{Exp}[-I\,75\,°];$$
$$S_{21} = .95\,\mathrm{Exp}[-I\,45\,°];$$
$$S_{22} = .15\,\mathrm{Exp}[-I\,10\,°];$$
$$V_s = 20\,\mathrm{V};$$
$$Z_s = 100\,\Omega;$$
$$Z_c = 50\,\Omega;$$
$$Z_l = 50\,\Omega;$$

Attempt at a reference-impedance-free solution:

In[63]:=
$$\mathrm{Gammas} = \mathrm{FullSimplify}\left[\mathrm{Solve}\left[\left\{\Gamma_{ins} == S_{11} - \frac{S_{12}\,S_{21}\,\Gamma_{outs}}{1 + S_{22}\,\Gamma_{outs}},\ \Gamma_{outs} == S_{22} - \frac{S_{12}\,S_{21}\,\Gamma_{ins}}{1 + S_{11}\,\Gamma_{ins}}\right\},\ \{\Gamma_{ins},\ \Gamma_{outs}\}\right]\right];$$
$$\{\Gamma_{in1},\ \Gamma_{out1}\} = (\{\Gamma_{ins},\ \Gamma_{outs}\}\ /.\ \mathrm{Gammas})[[1]]$$
$$\{\Gamma_{in2},\ \Gamma_{out2}\} = (\{\Gamma_{ins},\ \Gamma_{outs}\}\ /.\ \mathrm{Gammas})[[2]]$$
$$Z_{in} = Z_s \frac{1 + \Gamma_{in2}}{1 - \Gamma_{in2}}$$
$$\Gamma_{in} = \Gamma_{in2}$$
$$\Gamma_l = -\Gamma_{out2}$$

Solve::ratnz : Solve was unable to solve the system with inexact coefficients.
The answer was obtained by solving a corresponding exact system and numericizing the result. >>

Out[64]= $\{-8.78584 + 0.458246\,i,\ -5.95836 + 0.433468\,i\}$

Out[65]= $\{0.113511 + 0.00592043\,i,\ 0.166948 + 0.0121454\,i\}$

Out[66]= $(125.599 + 1.50666\,i)\ \Omega$

Out[67]= $0.113511 + 0.00592043\,i$

Out[68]= $-0.166948 - 0.0121454\,i$

In[69]:=
$$a_1 = \frac{V_s}{2\sqrt{Z_c}}$$

Out[69]= $\sqrt{2}\ \mathrm{V}/\sqrt{\Omega}$

In[70]:= $\mathbf{b_1 = \frac{V_s}{2\sqrt{Z_c}} \Gamma_{in}}$

Out[70]= $(0.160528 + 0.00837275\, i)\ V/\sqrt{\Omega}$

In[71]:= $\mathbf{a_2 = \frac{V_s}{2\sqrt{Z_c}} \frac{S_{21}\,\Gamma_L}{1 - S_{22}\,\Gamma_L}}$

Out[71]= $$\frac{\left((0.95 - 0.95\, i)\ V/\sqrt{\Omega}\right)\Gamma_L}{1 - (0.147721 - 0.0260472\, i)\ \Gamma_L}$$

In[72]:= $\mathbf{b_2 = \frac{V_s}{2\sqrt{Z_c}} \frac{S_{21}}{1 - S_{22}\,\Gamma_L}}$

Out[72]= $$\frac{(0.95 - 0.95\, i)\ V/\sqrt{\Omega}}{1 - (0.147721 - 0.0260472\, i)\ \Gamma_L}$$

In[73]:= $\mathbf{P_{load} = UnitConvert\left[\frac{1}{2} Re\left[\sqrt{Z_c}\ (a_2 + b_2)\ Conjugate\left[\frac{\sqrt{Z_c}\ (a_2 + b_2)}{Z_L}\right]\right], \text{"milliWatts"}\right]}$

Thread::tdlen : Objects of unequal length in {{0.852279 − 0.0260472 *i*, 1 + Conjugate[Γ_L]}, Quantity[0.95 + 0.95 *i*, {Ohms, Volts}]} + {{0.852279 − 0.0260472 *i*, 1 + Conjugate[Γ_L]}, Conjugate[Γ_L], Quantity[0.95 + 0.95 *i*, {Ohms, Volts}]} cannot be combined. ≫

Thread::tdlen : Objects of unequal length in {Quantity[0.95 − 0.95 *i*, {Ohms, Volts}], {0.852279 + 0.0260472 *i*, 1 + Γ_L}} + {Quantity[0.95 − 0.95 *i*, {Ohms, Volts}], {0.852279 + 0.0260472 *i*, 1 + Γ_L}, Γ_L} cannot be combined. ≫

Thread::tdlen : Objects of unequal length in {{0.852279, 1 + Re[Γ_L]}, Re[Quantity[0.95 + 0.95 *i*, {Ohms, Volts}]]} + {{0.852279, 1 + Re[Γ_L]}, Re[Γ_L], Re[Quantity[0.95 + 0.95 *i*, {Ohms, Volts}]]} cannot be combined. ≫

General::stop : Further output of Thread::tdlen will be suppressed during this calculation. ≫

Out[73]= $$\text{UnitConvert}\left[\frac{1}{2} \text{Re}\left[\frac{1}{\text{Conjugate}[Z_L]} \left(\frac{(0.95 + 0.95\, i)\ V/\sqrt{\Omega}}{1 - (0.147721 + 0.0260472\, i)\ \text{Conjugate}[\Gamma_L]} + \frac{\text{Conjugate}[\Gamma_L]\left((0.95 + 0.95\, i)\ V/\sqrt{\Omega}\right)}{1 - (0.147721 + 0.0260472\, i)\ \text{Conjugate}[\Gamma_L]}\right) (50\ \Omega) \left(\frac{(0.95 - 0.95\, i)\ V/\sqrt{\Omega}}{1 - (0.147721 - 0.0260472\, i)\ \Gamma_L} + \frac{\left((0.95 - 0.95\, i)\ V/\sqrt{\Omega}\right)\Gamma_L}{1 - (0.147721 - 0.0260472\, i)\ \Gamma_L}\right)\right], \text{milliWatts}\right]$$

In[74]:= $P_{ref} = \text{UnitConvert}\left[\frac{1}{2}\frac{\text{Abs}[V_s]^2}{\text{Abs}[Z_s + Z_{in}]^2}\text{Re}[Z_s], \text{"milliWatts"}\right]$

Out[74]= 392.949 mW

In[75]:= $P_{used} = \text{UnitConvert}\left[\frac{1}{2}\text{Re}\left[V_s\ \text{Conjugate}\left[\frac{(a_1 + b_1)}{Z_{in}}\ \text{Sqrt}[Z_c]\right]\right], \text{"milliWatts"}\right]$

Out[75]= 886.489 mW

In[76]:= $P_{network} = \text{UnitConvert}\left[\frac{1}{2}\text{Re}\left[\left(V_s \frac{Z_{in}}{Z_{in} + Z_s}\right)\text{Conjugate}\left[\frac{V_s}{Z_{in} + Z_s}\right]\right], \text{"milliWatts"}\right]$

Out[76]= 493.54 mW

In[77]:= $P_{network} + P_{ref}$

Out[77]= 886.489 mW

In[78]:=

Reference Impedance Solution

Out[78]= Reference Impedance Solution

In[79]:= $Z_{c_1} = 100\ \Omega;\ Z_{c_2} = 50\ \Omega;\ \Gamma_1 = \frac{Z_1 - Z_{c_2}}{Z_1 + Z_{c_2}};\ \Gamma_{in} = S_{11} + \frac{S_{12} S_{21} \Gamma_1}{1 - S_{22} \Gamma_1};\ \Gamma_s = \frac{Z_s - Z_{c_1}}{Z_s + Z_{c_1}};$

$Z_{in} = Z_{c_1}\frac{(1 + \Gamma_{in})}{1 - \Gamma_{in}};$

$a_1 = \text{UnitConvert}\left[N\left[\frac{V_s}{\sqrt{Z_{c_1}}}\frac{Z_{in}}{Z_s + Z_{in}}\frac{1}{1 + \Gamma_{in}}\right], \text{"SI"}\right]$

Out[81]= $(1. + 1.38778 \times 10^{-17}\ i)\ V/\sqrt{\Omega}$

In[82]:= $b_1 = \Gamma_{in}\ a_1$

Out[82]= $(0.0866025 - 0.05\ i)\ V/\sqrt{\Omega}$

In[83]:= $b_2 = a_1 \frac{S_{21}}{1 - \Gamma_1 S_{22}}$

Out[83]= $(0.671751 - 0.671751\ i)\ V/\sqrt{\Omega}$

In[84]:= $a_2 = b_2\ \Gamma_1$

In[85]:= $P_{load} = \text{UnitConvert}\left[\frac{1}{2}\,\text{Re}\left[\sqrt{Z_{c_2}}\,(a_2 + b_2)\,\text{Conjugate}\left[\sqrt{Z_{c_2}}\,\frac{a_2 + b_2}{Z_1}\right]\right], \text{"milliWatts"}\right]$

Out[85]= 451.25 mW

In[86]:= $P_{source} = \text{UnitConvert}\left[\frac{1}{2}\,\text{Re}\left[\left(V_s\,\frac{Z_s}{Z_{in} + Z_s}\right)\,\text{Conjugate}\left[\frac{V_s}{Z_{in} + Z_s}\right]\right], \text{"milliWatts"}\right]$

Out[86]= 418.397 mW

In[87]:= $P_{network} = \text{UnitConvert}\left[\frac{1}{2}\,\text{Re}[(a_1 + b_1)\,\text{Conjugate}[a_1 - b_1]], \text{"milliWatts"}\right]$

Out[87]= 495. mW

In[89]:= $P_{del} = \text{UnitConvert}\left[\frac{1}{2}\,\text{Re}\left[\frac{V_s{}^2}{\text{Conjugate}[Z_{in} + Z_s]}\right], \text{"milliWatts"}\right]$

Out[89]= 913.397 mW

In[90]:= $P_{network} + P_{source}$

Out[90]= 913.397 mW

Problem 2 Appendix

Mathematica

LC Matching Network

$Z_c = 50;\ Z_l = 100;\ \alpha = \frac{Z_l}{Z_c};\ Y_l = \frac{1}{Z_l};\ Y_c = \frac{1}{Z_c};$

```
(*x will represent βl_l and y will represent βl_s*)
```

$X_b = \text{Sqrt}\left[\frac{\alpha^2 Z_c{}^2}{\alpha - 1}\right]$

100

$X_a = -Z_c\,\text{Sqrt}[\alpha - 1]$

-50

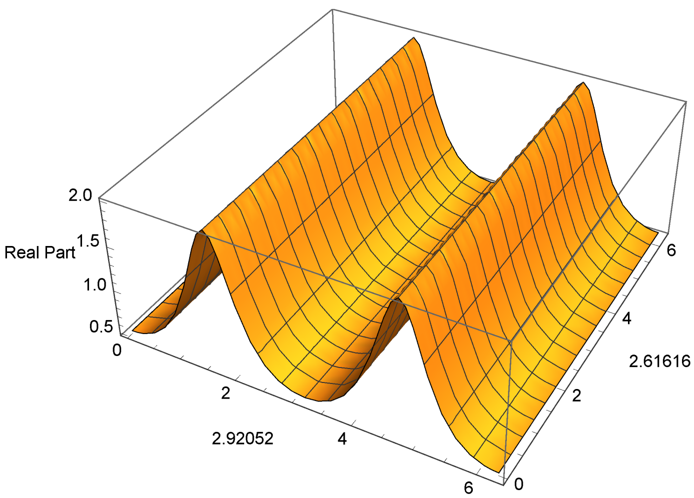
Single Open Stub

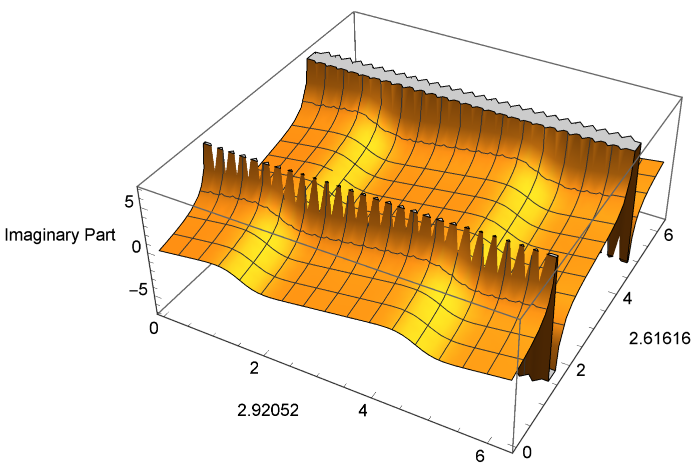
$\text{RealEq}[x\_, y\_] := \text{Re}\left[\frac{(Y_l + I\, Y_c\, \text{Tan}[x])}{Y_c + I\, Y_l\, \text{Tan}[x]} + I\, \text{Tan}[y]\right]$

$\text{ImagEq}[x\_, y\_] := \text{Im}\left[\frac{(Y_l + I\, Y_c\, \text{Tan}[x])}{Y_c + I\, Y_l\, \text{Tan}[x]} + I\, \text{Tan}[y]\right]$

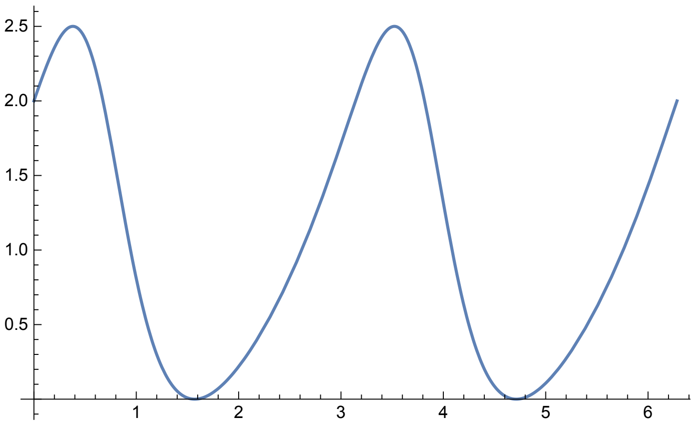
```
FindRoot[{RealEq[x, y] == 1, ImagEq[x, y] == 0}, {x, 1}, {y, 2.5}]
Plot3D[RealEq[x, y], {x, 0, 2 π}, {y, 0, 2 π}, AxesLabel → {β_{l_l}, β_{l_s}, "Real Part"}]
Plot3D[ImagEq[x, y], {x, 0, 2 π}, {y, 0, 2 π}, AxesLabel → {β_{l_l}, β_{l_s}, "Imaginary Part"}]
```

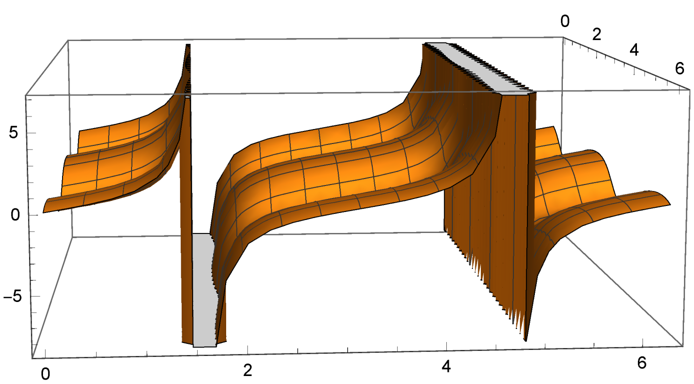
```
{x → 0.955317, y → 2.52611}
```






```
d = Pi / 4;
StubAdmittance[l_] := I Y_c Tan[l];
LoadAdmittance[length_] := Y_c (Y_l + I Y_c Tan[length]) / (Y_c + I Y_l Tan[length]);
Y1[t_, l_] := StubAdmittance[t] + LoadAdmittance[l];
Y2[t_, l_] := Y_c (Y1[t, l] + I Y_c Tan[l]) / (Y_c + I Y1 [t, l] Tan[l]);
Plot[Re[Y2[x, d] / Y_c], {x, 0, 2 Pi}]
```





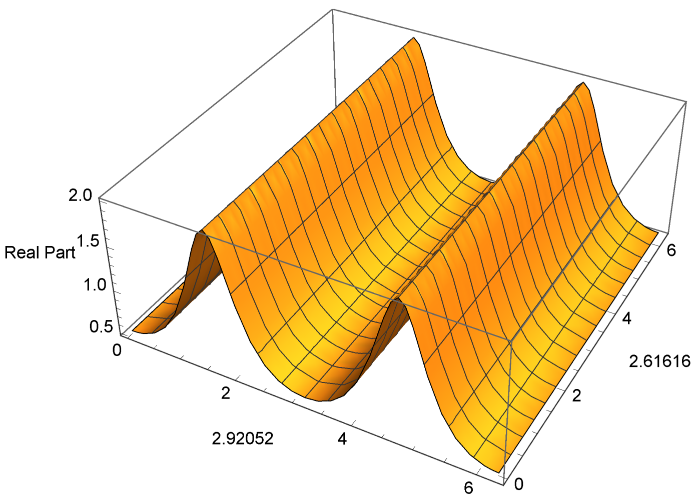
{0.943655, 1.14837}

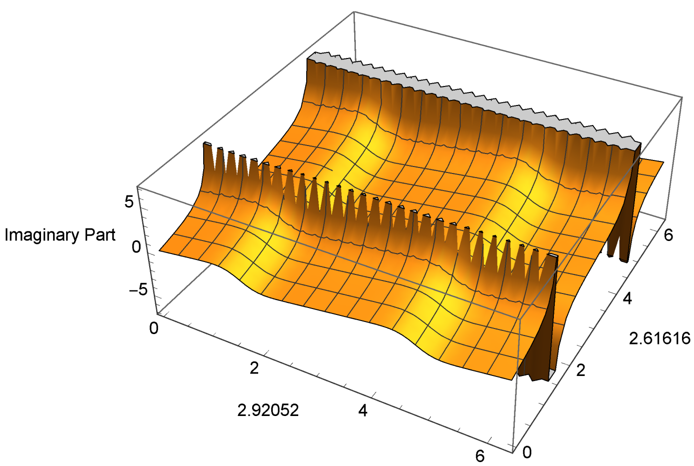
{β_{1_s}, β_{1_l}} * 180 / Pi

{54.0675, 65.7966}

```
FindRoot[{RealEq[x, y] == 1, ImagEq[x, y] == 0}, {x, 1}, {y, 2.5}]
Plot3D[RealEq[x, y], {x, 0, 2 π}, {y, 0, 2 π}, AxesLabel → {β1_l, β1_s, "Real Part"}]
Plot3D[ImagEq[x, y], {x, 0, 2 π}, {y, 0, 2 π}, AxesLabel → {β1_l, β1_s, "Imaginary Part"}]
```

```
{x → 0.955317, y → 2.52611}
```





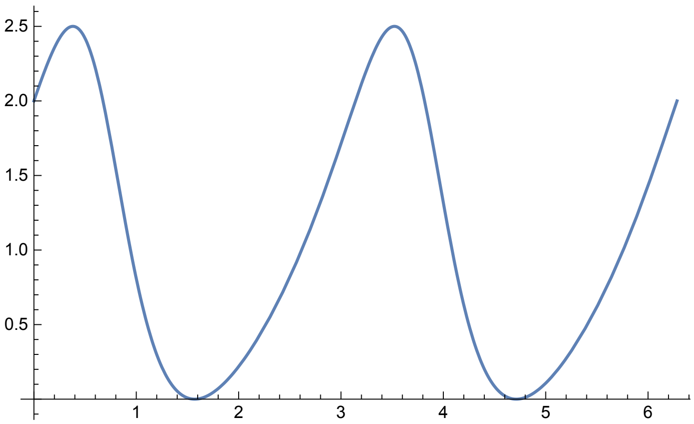

```
d = Pi / 4;
StubAdmittance[l_] := I Yc Tan[l];
```

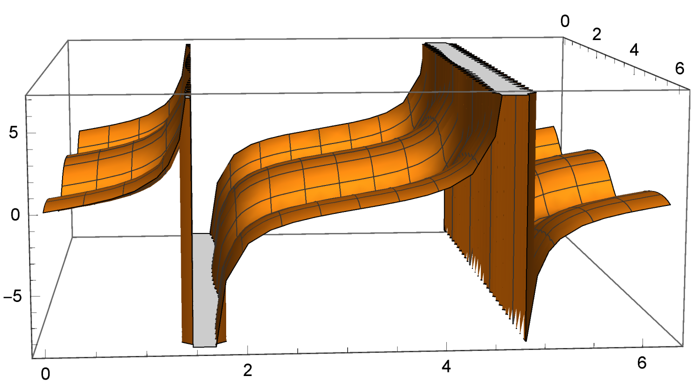
$$\texttt{LoadAdmittance[length\_] := } Y_c \, \frac{(Y_l + I\, Y_c\, \texttt{Tan[length]})}{Y_c + I\, Y_l\, \texttt{Tan[length]}};$$

```
Y1[t_, l_] := StubAdmittance[t] + LoadAdmittance[l];
```

$$\texttt{Y2[t\_, l\_] := } Y_c \, \frac{(\texttt{Y1[t, l]} + I\, Y_c\, \texttt{Tan[l]})}{Y_c + I\, \texttt{Y1 [t, l] Tan[l]}};$$

```
Plot[Re[Y2[x, d] / Yc], {x, 0, 2 Pi}]
```





{0.943655, 1.14837}

$\{\beta_{1_s}, \beta_{1_l}\} * 180 / \mathrm{Pi}$

{54.0675, 65.7966}

ADS

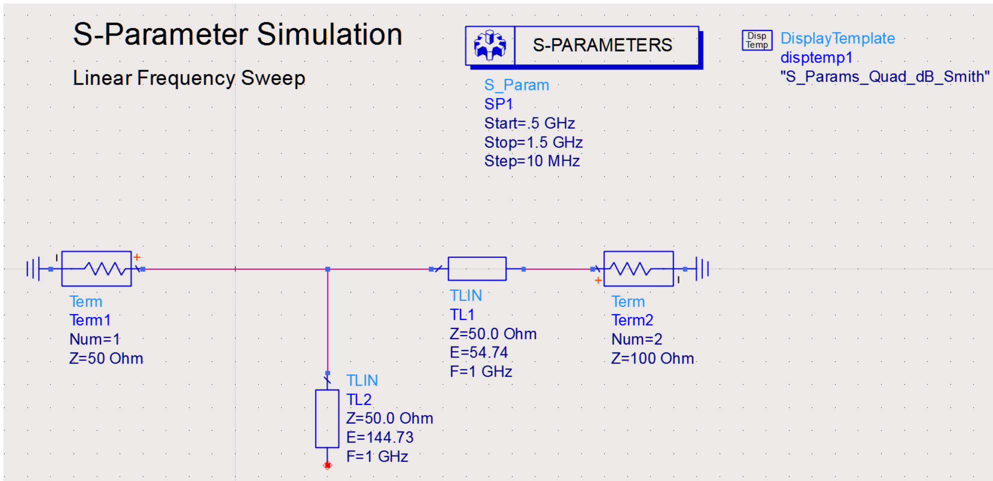


Figure 1: Schematic for the single open stub tuner.

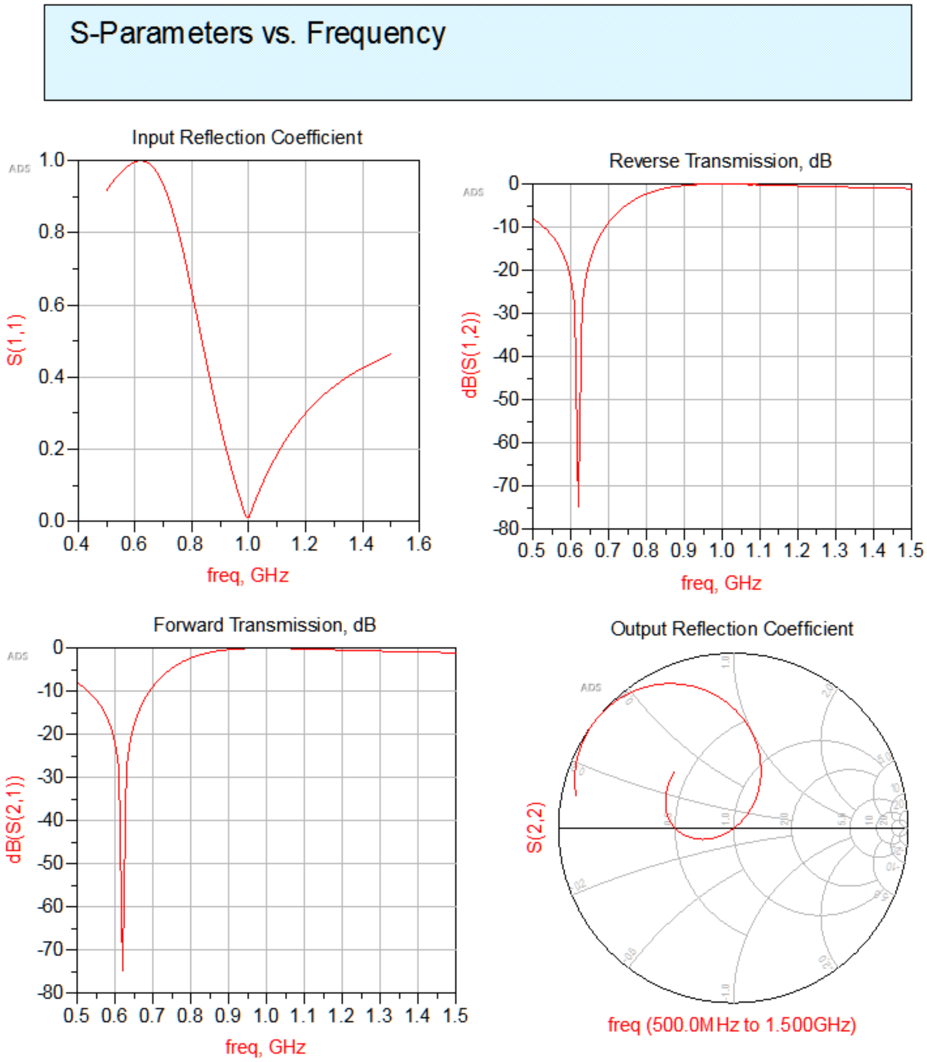


Figure 2: Results for the single open stub tuner.

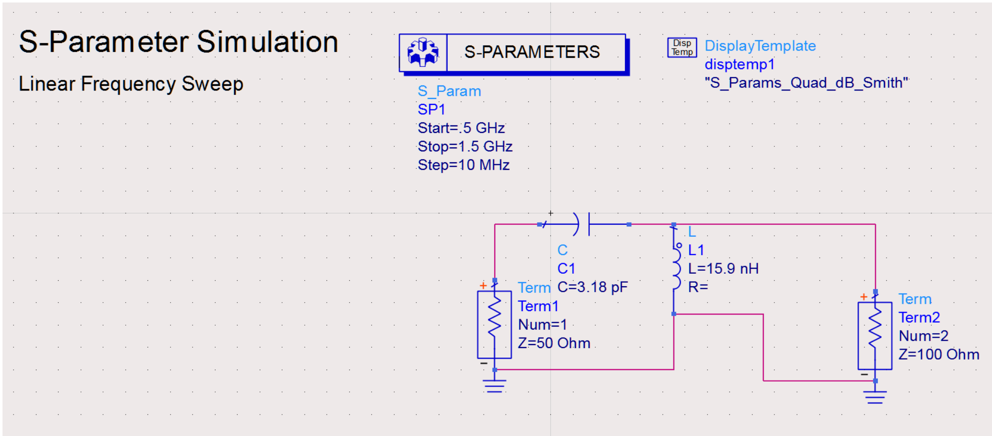


Figure 3: Schematic for the LC Matching Network.

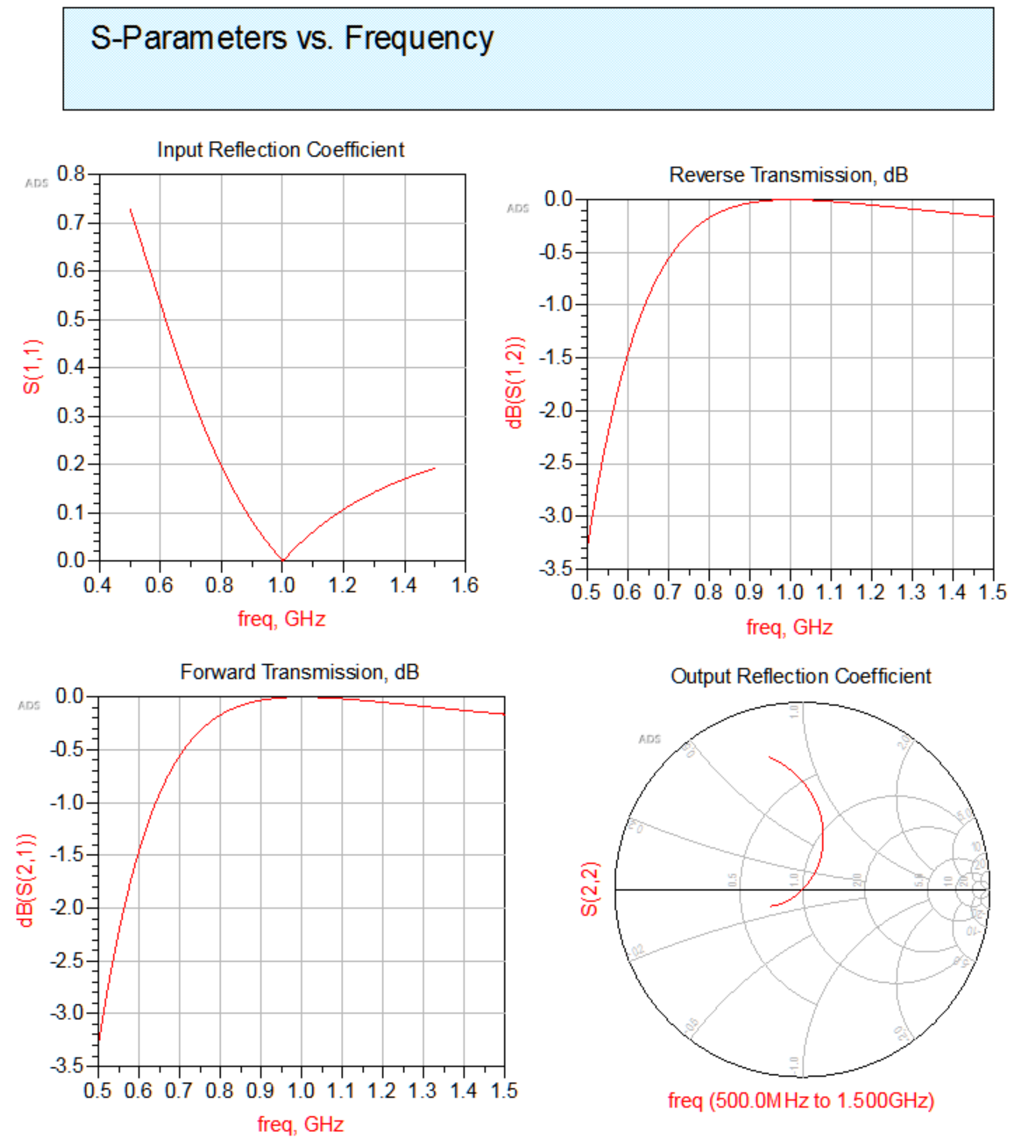


Figure 4: Results for the LC Matching Network.

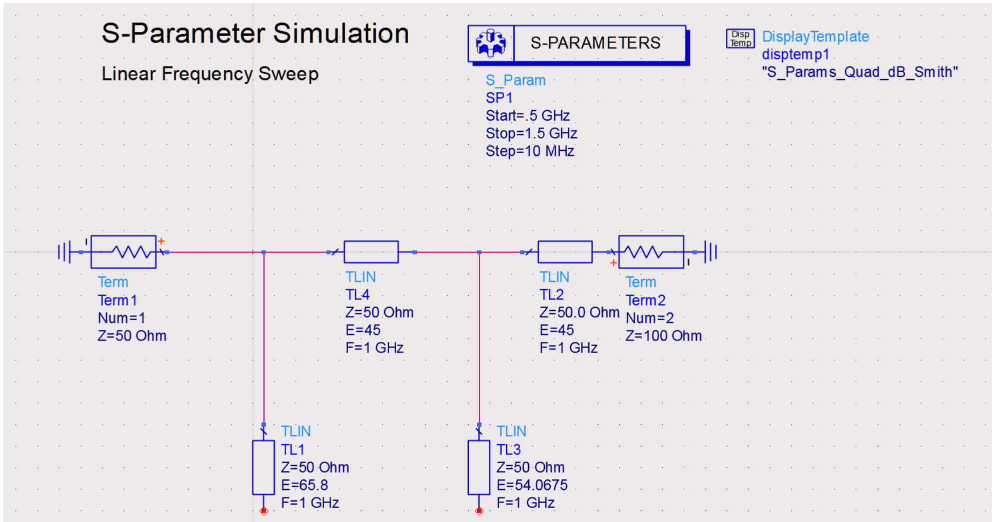


Figure 5: Schematic for the double open stub tuner.

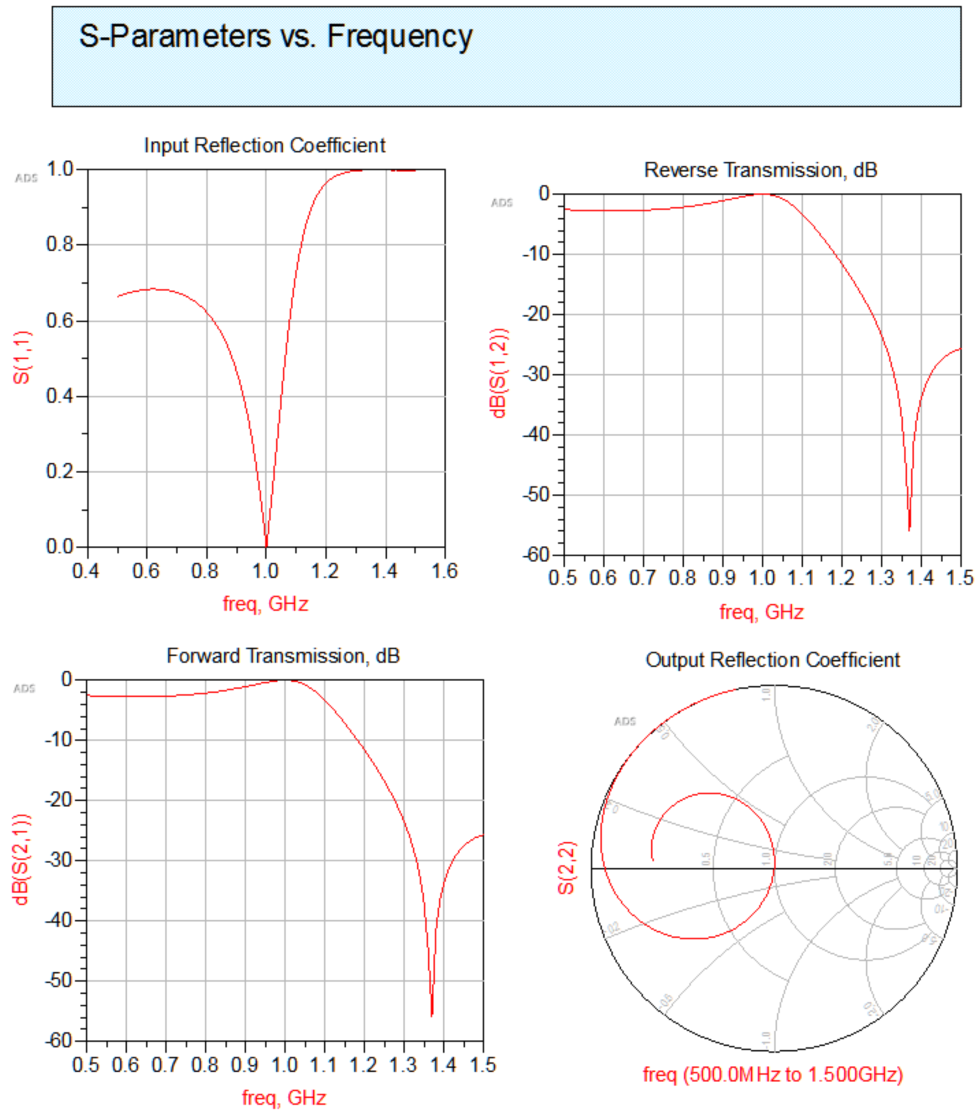


Figure 6: Results for the double open stub tuner.

Problem 3 Appendix

Mathematica

In[11]:= $R_l = 50\ \Omega;\ Z_c = 10\ \Omega;\ \omega = 2\,\text{Pi}\ 3\ \text{GHz};\ \text{Cap} = \frac{\left(\frac{2\ \text{S}}{50}\right)}{\omega};\ L = \frac{20\ \Omega}{\omega};\ R_s = 4\ \Omega/3;\ R_p = 500\ \Omega;$

In[12]:= `ParallelImpedance = UnitConvert[1 / (I ω Cap + 1 / R_l + 1 / R_p), "Ohms"]`

Out[12]= $\left(\frac{5500}{521} - \frac{10\,000\,i}{521}\right)\Omega$

In[13]:= `InputImpedance = N[ R_s + I ω L + ParallelImpedance]`

Out[13]= $(11.89 + 0.806142\,i)\ \Omega$

In[14]:= $P_{load} = \frac{V_s{}^2}{2}\ \text{Re}\left[\frac{\text{ParallelImpedance}}{\text{InputImpedance}}\ \text{Conjugate}\left[\frac{\text{ParallelImpedance}}{\text{InputImpedance}}\ \frac{1}{R_l}\right]\right]$

Out[14]= $(0.033787/\Omega)\ V_s^2$

In[15]:= $P_{in} = \frac{V_s{}^2}{2}\ \text{Re}\left[1/\text{Conjugate}[\text{InputImpedance}]\right]$

Out[15]= $(0.0418599/\Omega)\ V_s^2$

In[16]:= `10 Log10[(P_load / P_in)]`

Out[16]= -0.930479

In[17]:= `UnitConvert[N[Cap], "Farads"]`

Out[17]= 2.12207×10^{-12} F

In[18]:= `UnitConvert[N[L], "Henries"]`

Out[18]= 1.06103×10^{-9} H

ADS

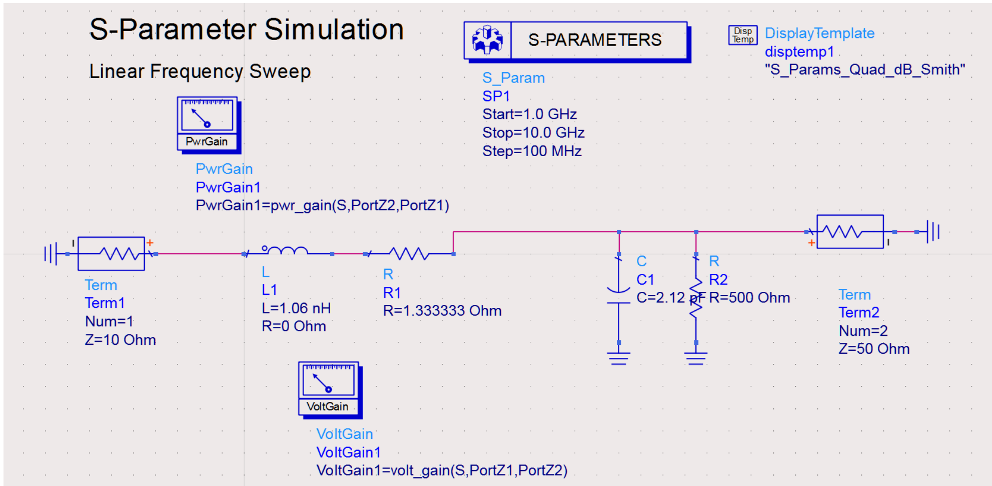


Figure 7: Schematic for the lossy matching network problem.

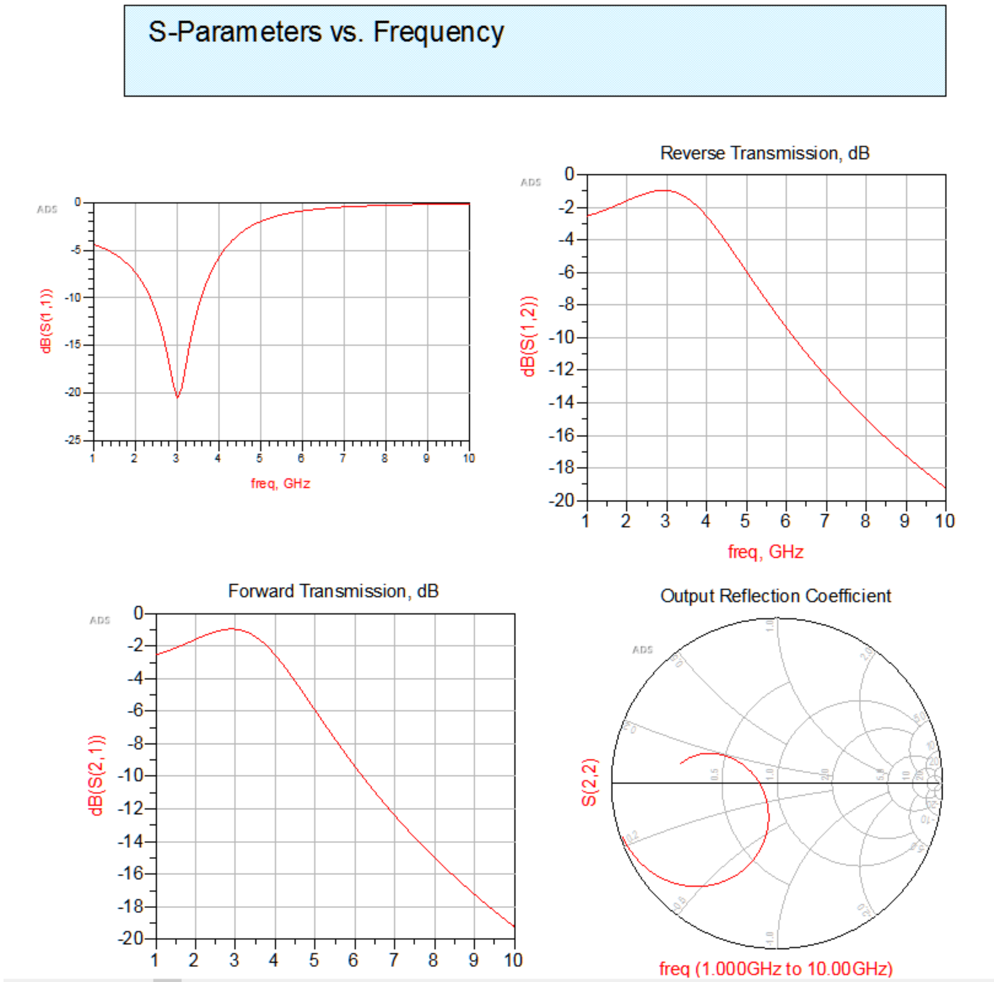


Figure 8: Results for the lossy matching network problemc.

Problem 4 Appendix

Mathematica

```
S11 = .2 Exp[I 50 °];
S14 = .4 Exp[I - 45 °];
S22 = .6 Exp[I 45 °];
S23 = .7 Exp[I - 45 °];
S32 = .7 Exp[I - 45 °];
S33 = .6 Exp[I 45 °]; S41 = .4 Exp[I - 45 °];
S44 = .5 Exp[I 45 °];
β = 45 °
```

45 °

$$\Gamma_{in} = S_{11} + \frac{(S_{33}\ \mathrm{Exp}[I\ 2\ \beta]\ S_{41})}{1 - S_{33}\ S_{44}\ \mathrm{Exp}[I\ 2\ \beta]}$$

0.0463152 + 0.135283 i

$$\mathrm{Gain} = \frac{S_{23}\ \mathrm{Exp}[I\ \beta]\ S_{41}}{(1 + \Gamma_{in})\ (1 - S_{33}\ S_{44}\ \mathrm{Exp}[I\ 2\ \beta])}$$

-0.037549 + 0.019777 i

```
Abs[Gain]
Arg[Gain] * 180 / π
```

0.0424389

152.224

```
-20 Log10[Abs[Gain]]
```

27.4447

Problem 5 Appendix

Mathematica

```
S11 = .73 Exp[-I 126 °];
S12 = 0;
S21 = 0;
S22 = .75 Exp[-I 52 °];
Zc = 50;
denom = (1 - S11) (1 - S22) - S12 S21;
toPolar[Z_] := With[{n = Abs[Z], a = 180 * Arg[Z] / π}, Defer[n ⅇ^(ⅈ a)]];
Z11 = Zc ((1 + S11) (1 - S22) + S12 S21)/denom
Z12 = 2 Zc S12/denom
Z21 = 2 Zc S12/denom
Z22 = Zc ((1 + S22) (1 - S11) + S12 S21)/denom
```

```
9.76761 - 24.6995 ⅈ

0. + 0. ⅈ

0. + 0. ⅈ

34.2328 - 92.4884 ⅈ
```

```
f = 10 * 10^9; L = .1 * 10^-9;
InductorImpedance = I 2 π f L;

Z11out = Z11 + InductorImpedance
Z12out = Z12 + InductorImpedance
Z21out = Z21 + InductorImpedance
Z22out = Z22 + InductorImpedance
```

```
9.76761 - 18.4164 ⅈ

0. + 6.28319 ⅈ

0. + 6.28319 ⅈ

34.2328 - 86.2052 ⅈ
```

```
Δ = (Z11out + Zc) (Z22out + Zc) - Z12out Z21out;
S11out = toPolar[((Z11out - Zc) (Z22out + Zc) - Z12out Z11out)/Δ]
S12out = toPolar[(2 Z12out Zc)/Δ]
[2 Z21out Zc]
```

$0.08315602149436857` e^{i\,152.5225906137076`}$

$0.08315602149436857` e^{i\,152.5225906137076`}$

$0.7229654599579297` e^{i\,(-54.60089626858122`)}$