Quantize electrical circuits

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Göran Johansson

In this lecture I will discuss how to derive the quantum mechanical hamiltonian governing the dynamics of an electrical circuit, starting from the circuit diagram. A classical electrical circuit is fully described by its *nodes* and the *voltages* across and *currents* through all *branches* of the circuit. In a analog to classical mechanics, these variables plays the roles of velocities and forces, respectively. All branches should be given an orientation, to keep track of the sign of the voltage and current (see Fig. 1). The branch voltages and currents are not independent variables, but are constrained by the circuit topology and Kirchoff's rules, i.e. the sum of voltages across any loop is zero and sum of currents flowing into a node is zero. One may proceed by introducing node variables, which are then dependent on the *spanning tree* used (see Devoret 1995 for details).

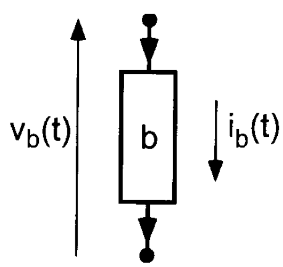

Figure 1: *From Devoret 1995 Les Houches lectures.*

1 Some circuit elements

The different circuit elements are defined by different relations between the currents and voltages. We will mainly deal with three different elements.

1.1 Capacitance

A *capacitance* is an element which stores charge and is characterized by its capacitance C. The voltage $v(t)$ across the capacitance is proportional to the charge $q(t)$ stored,

$$v(t) = \frac{q(t)}{C}. \tag{1}$$

The energy of the capacitance is stored in the electric field

$$E_C = \frac{Cv^2}{2} = \frac{q^2}{2C}. \tag{2}$$

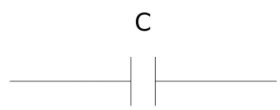

Figure 2: *Circuit symbol for capacitance.*

1.2 Inductance

An *inductance* is an element which develops a voltage when you change the current through it according to,

$$v(t) = L\partial_t i(t).,\tag{3}$$

where the proportionality constant L is the inductance. Equivalently, one can also talk about the magnetic flux

$$\Phi(t) = \int_0^t v(t')dt', \tag{4}$$

and that the inductance stores a flux $\Phi(t)$ proportional to the current $i(t)$ through the inductance,

$$i(t) = \frac{\Phi(t)}{L}. \tag{5}$$

The energy of the inductance is stored in the magnetic field

$$E_L = \frac{Li^2}{2} = \frac{\Phi^2}{2L}. \tag{6}$$

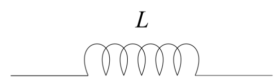

Figure 3: *Circuit symbol for inductance.*

1.3 Josephson junction

The *Josephson junction* (JJ) is a circuit element made up of a tunnel junction between two superconductors. It is characterized by its capacitance C_J and its so-called Josephson energy E_J. According to the DC Josephson relation, the current through the JJ is determined by the phase difference ϕ between the superconducting condensates on the two sides of the junction,

$$\phi(t) = \frac{2e}{\hbar}\int_0^t V(t')dt' + \phi(0), \tag{7}$$

which is given by the time-integral of the voltage across the junction. The time-integral of the voltage is also a magnetic flux Φ, and the normalization is sometimes written in therms of the magnetic flux quantum $\Phi_0 = h/2e$,

$$\phi = \frac{2e}{\hbar}\Phi = 2\pi\frac{\Phi}{\Phi_0}. \tag{8}$$

Now, the current through the JJ junction is

$$i_J(t) = I_c \sin\phi(t), \tag{9}$$

where the critical current I_c is proportional to the Josephson energy

$$I_c = \frac{2e}{\hbar}E_J = \frac{2\pi}{\Phi_0}E_J. \tag{10}$$

We see that for small phase differences $|\phi(t)| \ll 1$,

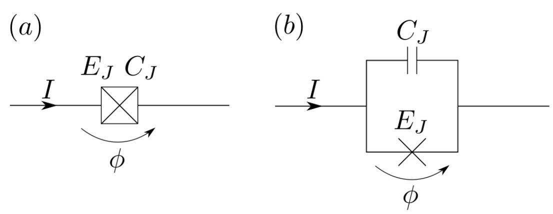

Figure 4: *Circuit symbol for a) a Josephson junction and b) an ideal Josephson junction in parallel with its capacitance.*

$$i_J(t) \approx I_c\phi(t) = E_J\left(\frac{2\pi}{\Phi_0}\right)^2\Phi, \tag{11}$$

the JJ acts like an linear inductance with

$$L_J = \frac{1}{E_J}\left(\frac{\Phi_0}{2\pi}\right)^2, \tag{12}$$

while in general for larger phase-differences it is a *non-linear* inductance, giving rise to anharmonic spectra of the system where it is included.

2 An LC oscillator in a Lagrangian formulation

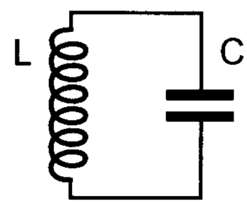

Figure 5: *An LC oscillator circuit.*

As in standard circuit theory, we can analyze the LC oscillator circuit simply by introducing the node voltages. However, in the case we have Josephson junctions in the circuit it is more convenient to use the node fluxes as coordinates. We consider a simple LC oscillator as in Fig. 5, where the lower node is grounded. In this case, we have only one active node, the upper one, with flux Φ. We get the equations of motion by equating the two currents: i) the sum of currents arriving from the inductive elements connected to that node and ii) the sum of currents going into the capacitive elements connected to that node:

$$\frac{\Phi}{L} = -C\dot{v} = -C\ddot{\Phi}.$$

Now, taking the mechanical analogue, where Φ is the coordinate of a particle, we see that this equation for current conservation is similar to Newton's equation of motion for a particle with a quadratic potential energy

$$E_L = \frac{\Phi^2}{2L}, \tag{13}$$

given by the inductance, and a kinetic energy

$$E_C = \frac{Cv^2}{2} = \frac{C\dot{\Phi}^2}{2}, \tag{14}$$

given by the capacitance. The Lagrangian of that system is given by the kinetic energy minus the potential energy

$$\mathcal{L} = \frac{C\dot{\Phi}^2}{2} - \frac{\Phi^2}{2L},$$

written in terms of coordinate Φ and velocity $\dot{\Phi}$. To connect to quantum mechanics we then replace velocity by conjugate momentum

$$q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi},$$

which has the unit of charge. We get the hamiltonian through the Legendre transformation,

$$H = \sum_n \dot{\Phi}_n q_n - \mathcal{L},$$

where the sum goes over all active nodes, which for our LC-oscillator gives

$$H = \dot{\Phi} q - \mathcal{L} = \frac{q^2}{C} - \left(\frac{C\dot{\Phi}^2}{2} - \frac{\Phi^2}{2L}\right) = \frac{q^2}{C} - \left(\frac{q^2}{2C} - \frac{\Phi^2}{2L}\right) = \frac{q^2}{2C} + \frac{\Phi^2}{2L},$$

which we recognize as the harmonic oscillator hamiltonian. So far we haven't done anything more than rewritten the classical equations of motion for the circuit in terms of a hamiltonian.

3 Quantization

The act of quantizing the the circuit is now very simple. We simply 'promote' the variables Φ and q to quantum mechanical operators, satisfying the canonical commutation relation

$$[\Phi, q] = \Phi q - q\Phi = i\hbar.$$

We then have a quantum mechanical description of the LC circuit, which indeed is a quantum mechanical harmonic oscillator. For the simple LC-oscillator it was straightforward to guess what is the conjugate momentum to Φ, but in more complicated circuits the above scheme can be followed to systematically derive the hamiltonian of the circuit.

In the exercise class you will see how to derive the hamiltonian of a current biased Josephson junction and in the hand-in you will derive the hamiltonian of the single Cooper-pair box.

4 The transmission line

Using the same method, we can derive the quantum mechanical hamiltonian of an one-dimensional transmission line with characteristic inductance/capacitance per unit length being L_0/C_0. We consider a TL of length L and discretize it with $M = 2N + 1$ nodes numbered with integers from $-N$ to N with equal spacing $\Delta x = L/2N$. From inspection of the discretized circuit in Fig. 6 we can write down the expression for the capacitive (kinetic) energy

$$E_C = \Delta x C_0 \sum_{n=-N}^{N} \frac{\dot{\Phi}_n^2}{2}, \tag{15}$$

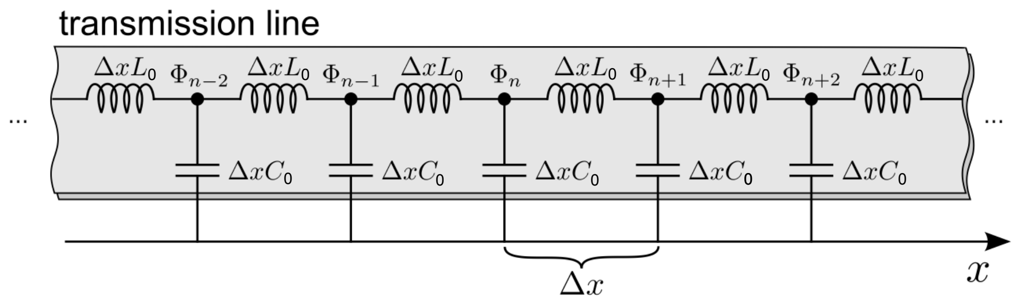

Figure 6: *The circuit of a discretized transmission line with characteristic inductance/capacitance per unit length being L_0/C_0. (Adapted from Robert Johansson's thesis.)*

as well as the potential (inductive) energy

$$E_L = \sum_{n=-N}^{N} \frac{(\Phi_n - \Phi_{n-1})^2}{2\Delta x L_0}, \tag{16}$$

giving the Lagrangian

$$\mathcal{L} = \sum_{n=-N}^{N} \Delta x C_0 \frac{\dot{\Phi}_n^2}{2} - \frac{(\Phi_n - \Phi_{n-1})^2}{2\Delta x L_0}, \tag{17}$$

where we used periodic boundary conditions $\Phi_{-N-1} \equiv \Phi_N$.

The conjugate momentum to the node flux Φ_n is

$$q_n = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_n} = \Delta x C_0 \dot{\Phi}_n. \tag{18}$$

We get the hamiltonian from the Legendre transformation

$$H = \sum_{n=-N}^{N} \dot{\Phi}_n q_n - \mathcal{L} = \sum_{n=-N}^{N} \frac{q_n^2}{2\Delta x C_0} + \frac{(\Phi_n - \Phi_{n-1})^2}{2\Delta x L_0}, \tag{19}$$

which is the hamiltonian of an infinite sum of coupled harmonic oscillators.

4.1 Diagonalizing the discrete hamiltonian

We can diagonalize the discrete hamiltonian Eq. (19) by changing the coordinates from Φ_n to the Fourier transformed ones

$$\Phi_\kappa = \frac{1}{\sqrt{M}} \sum_{n=-N}^{N} e^{-i2\pi(\kappa/M)n} \Phi_n \Leftrightarrow \Phi_n = \frac{1}{\sqrt{M}} \sum_{\kappa=-N}^{N} e^{i2\pi(\kappa/M)n} \Phi_\kappa,$$

where for real Φ_n we have $\Phi_{-\kappa} = \Phi_\kappa^*$. This change of basis is unitary and we will use the orthogonality in the form of

$$\sum_{n=-N}^{N} e^{i2\pi(\kappa/M)n} = M\delta_{\kappa 0},$$

where δ_{mn} is Kronecker's delta, which e.g. leads to

$$\sum_{n=-N}^{N} \dot{\Phi}_n^2 = \frac{1}{M} \sum_{n=-N}^{N} \sum_{\kappa=-N}^{N} \sum_{\kappa'=-N}^{N} e^{i2\pi((\kappa+\kappa')n/M)} \dot{\Phi}_\kappa \dot{\Phi}_{\kappa'} = \sum_{\kappa=-N}^{N} \dot{\Phi}_\kappa \dot{\Phi}_{-\kappa} = \sum_{\kappa=-N}^{N} |\dot{\Phi}_\kappa|^2.$$

The conjugate momentum to the node flux Φ_κ is

$$q_\kappa = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_\kappa} = \Delta x C_0 \dot{\Phi}_{-\kappa}. \tag{20}$$

For the potential energy we get

$$\begin{aligned}
&\sum_{n=-N}^{N} (\Phi_n - \Phi_{n-1})^2 = \\
= \; & \frac{1}{M} \sum_{n=-N}^{N} \sum_{\kappa=-N}^{N} \sum_{\kappa'=-N}^{N} \left(1 - e^{-i2\pi(\kappa/M)}\right) \left(1 - e^{-i2\pi(\kappa'/M)}\right) e^{i2\pi((\kappa+\kappa')n/M)} \Phi_\kappa \Phi_{\kappa'} = \\
= \; & \sum_{\kappa=-N}^{N} \left|1 - e^{-i2\pi(\kappa/M)}\right|^2 |\Phi_\kappa|^2 \approx \sum_{|\kappa| \ll M} \left(\frac{2\pi\kappa}{M}\right)^2 |\Phi_\kappa|^2,
\end{aligned} \tag{21}$$

where the last approximation is OK if we include only modes with wavelengths ($\lambda \propto L/|\kappa|$) much larger than the discretization length Δx, i.e. $|\kappa| \ll M$. We now have a diagonalized Hamiltonian

$$H = \sum_{\kappa=-N}^{N} \frac{|q_\kappa|^2}{2\Delta x C_0} + \left|1 - e^{-i2\pi(\kappa/M)}\right|^2 \frac{|\Phi_\kappa|^2}{2\Delta x L_0}. \tag{22}$$

Now, in taking the continuum limit $N \to \infty$ we need to use not the charge q_κ which goes to zero, but instead the charge density $\tilde{q}_\kappa = q_\kappa/\Delta x$,

$$H \approx \Delta x \sum_{|\kappa| \ll M} \frac{|\tilde{q}_\kappa|^2}{2C_0} + \left(\frac{2\pi\kappa}{M\Delta x}\right)^2 \frac{|\Phi_\kappa|^2}{2L_0}.$$

Thus, the hamiltonian consists of a sum of independent harmonic oscillators characterized with their quantum number κ. In real space, the eigenstates are plane waves $e^{\pm ikx \pm i\omega_k t}$ with wave vector

$$k = \frac{2\pi\kappa}{M\Delta x} = \kappa\frac{2\pi}{L},$$

and frequency $\omega_k = v|k|$, where $v = 1/\sqrt{L_0 C_0}$ is the wave velocity of the waveguide. Introducing creation and annihilation operators for these modes the hamiltonian can be written

$$H = \int_{-\infty}^{\infty} \hbar\omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right) dk. \tag{23}$$

4.2 The massless Klein-Gordon field

From the discrete hamiltonian Eq. (19) one may also derive equations of motion for the field,

$$\dot{q}_n = \left[q_n, \frac{\Delta x}{2L_0} \frac{(\Phi_{n+1} - \Phi_n)^2 - (\Phi_n - \Phi_{n-1})^2}{(\Delta x)^2} \right] \Rightarrow \dot{q}(x,t) = \frac{\mathrm{d}x}{L_0} \frac{\partial^2 \Phi(x,t)}{\partial x^2} \tag{24}$$

where

$$q(x,t) = C_0 \mathrm{d}x \dot{\Phi}(x,t). \tag{25}$$

The phase in the transmission lines thus obeys the massless scalar Klein-Gordon equation

$$\frac{\partial^2 \Phi(x,t)}{\partial t^2} - \frac{1}{L_0 C_0} \frac{\partial^2 \Phi(x,t)}{\partial x^2} = 0, \tag{26}$$

which has solutions in the form of right and leftgoing travelling waves which formally can be written as

$$\Phi = \Phi_{\leftarrow} \left(\frac{x}{v} + t \right) + \Phi_{\rightarrow} \left(-\frac{x}{v} + t \right), \tag{27}$$

where $v = 1/\sqrt{L_0 C_0}$ is the velocity of the waves in the transmission line.

The Klein-Gordon equation of motion is easily solved by introducing the Fourier transformed operators in the transmission line

$$\Phi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(k,t) e^{ikx} dk, \tag{28}$$

and we get an ordinary differential equation for each value of the wave-number k. Solving Eq. (26) is reduced to the problem of solving

$$\ddot{\Phi}(k,t) + v^2 |k|^2 \Phi(k,t) = 0, \tag{29}$$

which is the classical equation of motion for a harmonic oscillator with the dispersion relation $\omega_k = v|k|$. In analogy with the ordinary harmonic oscillator it is convenient to work with the creation and annihilation operators a and $a^\dagger$ instead of charges and phases. However, since we are dealing with a field equation each Fourier component is treated independently and we write

$$\Phi(k,t) \quad = \quad \sqrt{\frac{\hbar}{2C_0\omega_k}} \Big(a_k(t) + a_{-k}^\dagger(t) \Big), \tag{30}$$

where a and $a^\dagger$ obey the canonical commutation relations $[a_k, a_{k'}^\dagger] = \delta(k-k')$ and $[a_k, a_{k'}] = 0$, which give the statistics of the transmission line excitations (photons). The harmonic oscillator nature of the transmission line makes the phase of the creation and annihilation operators rotate with angular frequency ω_k, $a_k(t) = a_k e^{-i\omega_k t}$, $a_k^\dagger(t) = a_k^\dagger e^{i\omega_k t}$ and we arrive at the final expression for the left and right going fields $\Phi_\leftarrow$ and $\Phi_\rightarrow$ in terms of a_k and $a_k^\dagger$

$$\Phi_\rightarrow(x,t) = \sqrt{\frac{\hbar}{4\pi C_0}} \int_0^\infty \frac{dk}{\sqrt{\omega_k}} \left(a_k^\rightarrow e^{-i(\omega_k t - kx)} + (a_k^\rightarrow)^\dagger e^{i(\omega_k t - kx)} \right), \quad (31)$$

$$\Phi_\leftarrow(x,t) = \sqrt{\frac{\hbar}{4\pi C_0}} \int_0^\infty \frac{dk}{\sqrt{\omega_k}} \left(a_k^\leftarrow e^{-i(\omega_k t + kx)} + (a_k^\leftarrow)^\dagger e^{i(\omega_k t + kx)} \right), \quad (32)$$

where we used only positive $k > 0$ and indicate the direction with arrow indices. One may also use frequencies instead of wavenumbers

$$\Phi_\rightarrow(x,t) = \sqrt{\frac{\hbar Z_0}{4\pi}} \int_0^\infty \frac{d\omega}{\sqrt{\omega}} \left(a_\omega^\rightarrow e^{-i(\omega t - k_\omega x)} + H.C. \right), \quad (33)$$

$$\Phi_\leftarrow(x,t) = \sqrt{\frac{\hbar Z_0}{4\pi}} \int_0^\infty \frac{d\omega}{\sqrt{\omega}} \left(a_\omega^\leftarrow e^{-i(\omega t + k_\omega x)} + H.C. \right), \quad (34)$$

where $Z_0 = \sqrt{L_0/C_0}$ is the characteristic impedance of the transmission line.

5 Transmission Lines Modeling Dissipation

In a em closed quantum system the time-evolution is unitary. Different initial states thus maps to different final states. In a system with dissipation we may have that all initial states will evolve into the same final state, e.g. the ground state in the case of relaxation. To model dissipation we need an *open* quantum system, e.g. by including a semi-infinite transmission line. A qubit can then relax by emitting photons into the transmission line, which then propagate away from the qubit and never comes back. Different initial states will give different photon states in the transmission line, but they will a finally leave the qubit in the ground state. By tracing out the transmission line, ignoring the different photon states, one may then describe the dissipative dynamics of the qubit. One may note that in order for the photons never to come back, the transmission line needs to be semi-infinite, having an dense set of photon modes.

6 References and Reading suggestions

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- "An Introduction to Quantum Field Theory" by Peskin and Schröder,
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