Lecture 21: Qubits and resonators

QIC880, Adrian Lupascu

(Dated: 2011/11/20)

I. INTRODUCTION

The interaction between two-level systems and cavities is an important area of research in quantum optics. This is important paradigm which addresses the most basic process of interaction between matter of light, namely the coherent interaction between a single atom and a single photon.The name of this area of research is cavity quantum electrodynamics (cavity QED).

The main goal of the research in this field was for a long time attaining the strong coupling regime, in which the strength of the interaction between the atom and the resonator (which is the rate of the coherent exchange of one quanta of energy when the two systems are resonant) is larger than the decoherence rates of the atom and the resonator. Two main routes were explored towards this goal:

- neutral atoms for which the transition of interest was a visible (or near visible) transition between the ground state and an electronic excited state [1]. One particular point of interest here is that techniques of atomic cooling of trapping enable the cooling of the motional degree of freedom for the atom. This platform also provides an interface between photons and atoms, relevant for quantum computing and communication.

- Rydberg atoms coupled to a superconducting resonator [2]. In this case the atoms are prepared in special states called circular Rydberg states for which the only radiative transitions are in the microwave range and therefore have very long decay times. The strength of the coupling is also very strongly enhanced by the strong dipole moments.

Interest in cavity-QED physics using superconducting circuits started in the early 2000s with proposals for implementations in *e.g.* [3] and earlier papers (see introduction in [3]). The proposal in [3] has been the most common implementation of cavity QED with superconducting devices (a field called circuit QED or cQED for short). The resonator is a coplanar waveguide resonator, which has the advantage that it has a very long decay time and allows for strong coupling to a qubit. The first demonstration of coherent coupling was done in [4]. There are two main directions of research related to cavity qed. The first is the use of this platform for quantum computing (see [5] for the implementation of an algorithm using cqed). The second is exploring the physics of strong coupling between two level systems and resonators, beyond what can be achieved in regular quantum optics (one example is the strong dispersive regime, see [6]).

II. RESONATOR MODES

A. Distributed resonators

In general termination of a transmission line results in a discrete set of allowed electromagnetic modes. We can consider the situation in Fig. 1, where a transmission line (TL) is terminated at $z = -L/2$ by impedance Z_1 and at $z = L/2$ by impedance Z_2. The general solution for the voltage and current is given by

$$V(z,t) = \sum_{\omega} \left(V_{\omega}^{+} e^{i\omega(z/\bar{c}-t)} + V_{\omega}^{-} e^{i\omega(-z/\bar{c}-t)} \right) \tag{1}$$

and

$$I(z,t) = \sum_{\omega} \left(\frac{V_{\omega}^{+}}{Z_0} e^{i\omega(z/\bar{c}-t)} - \frac{V_{\omega}^{-}}{Z_0} e^{i\omega(-z/\bar{c}-t)} \right). [1] \tag{2}$$

$\bar{c}$ is the speed of light in the TL.

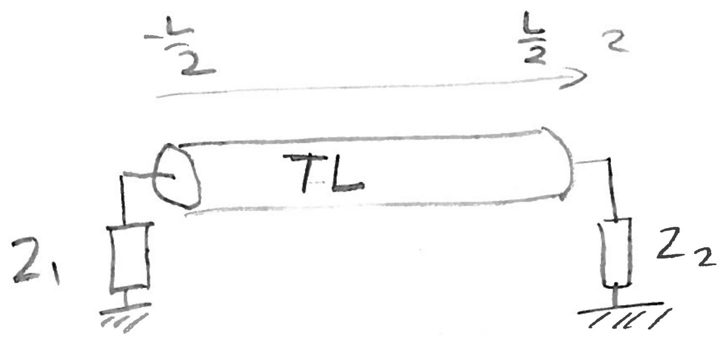

FIG. 1. Transmission line terminated by impedance Z_1 at left end ($z = -L/2$) and by impedance Z_2 at left end ($z = L/2$).

The following boundary conditions have to hold for *all* the frequencies ω:

[1] These are complex quantities, one has to keep in mind that only the real part matters.

- At $z = -L/2$

$$V_\omega^+ e^{-i\omega L/2\bar{c}} + V_\omega^- e^{i\omega L/2\bar{c}} = Z_1(\omega)\left(\frac{V_\omega^+}{Z_0} e^{-i\omega L/2\bar{c}} - \frac{V_\omega^-}{Z_0} e^{i\omega L/2\bar{c}}\right) \tag{3}$$

- At $z = L/2$

$$V_\omega^+ e^{i\omega L/2\bar{c}} + V_\omega^- e^{-i\omega L/2\bar{c}} = Z_2(\omega)\left(\frac{V_\omega^+}{Z_0} e^{i\omega L/2\bar{c}} - \frac{V_\omega^-}{Z_0} e^{-i\omega L/2\bar{c}}\right) \tag{4}$$

Requiring that equations (3) and (4) have solutions with non-vanishing values of V_ω^+ and V_ω^- , a discrete set of allowed frequencies ω is obtained. To each of these frequencies, there corresponds one value of $\frac{V_\omega^+}{V_\omega^-}$.

The most typical resonator in superconducting circuits is the one in which the TL is open ended at both ends. In this case the boundary conditions are

$$V_\omega^+ e^{-i\omega L/2\bar{c}} - V_\omega^- e^{i\omega L/2\bar{c}} = 0 \tag{5}$$

$$V_\omega^+ e^{i\omega L/2\bar{c}} - V_\omega^- e^{-i\omega L/2\bar{c}} = 0 \tag{6}$$

The allowed set of discrete frequencies is

$$\omega_n = \frac{\bar{c}}{2L} 2\pi n \tag{7}$$

with n a positive integer. One also has

$$\frac{V_\omega^+}{V_\omega^-} = (-1)^n. \tag{8}$$

B. Lumped resonators

Lumped resonators have been used in the context of superconducting circuits to various ends, cQED being only one of them. One major advantage of lumped resonator circuits is the fact that given a certain resonance frequency ω_0 the product LC of the inductance and capacitance forming the circuit is given, however one still has, in principle, the freedom to very their ratio L/C or equivalently the impedance $Z_{LC} = \sqrt{L/C}$ over a wide range. This has the advantage that one can favor situations in which either coupling of the current or voltage can be favored through design. A second advantage of lumped resonators is that the space occupied on a chip is smaller which is advantageous from a practical point of view.

Reaching high quality factors with lumped resonators is in general problematic, more so than with distributed resonators. Overlap capacitors are in general quite lossy due to the fact that amorphous dielectrics have in general high dissipation; the high loss is in many cases connected to two level systems (see [7] and reference 13 in that paper; this is an area developing quite rapidly). More recently, increasing the quality factors relied on using interdigitated capacitors or crystalline dielectric slabs.

III. CHARGE QUBIT COUPLED TO A DISTRIBUTED RESONATOR

A. Quantization of a resonator

We consider an open-ended resonator for which the allowed values of k are (7)

$$k_n = \frac{\pi}{L} n. \tag{9}$$

For the voltage in the resonator we have

$$V(z,t) = \sum_{n>0} \left[V_n^+ e^{i(k_n z - \omega t)} + V_n^- e^{i(-k_n z - \omega t)} + cc \right]. \tag{10}$$

We next use the relation

$$\frac{V_\omega^+}{V_\omega^-} = r_n \tag{11}$$

between the two voltage components with $r_n = (-1)^n$ (see 8). We also define the mode function

$$f_n^V(z) = \left(e^{ik_n z} + r_n e^{-ik_n z} \right) \times \frac{1}{2} \times \begin{cases} -i & n \,\text{is odd} \\ 1 & n \,\text{is even} \end{cases}. \tag{12}$$

The $\frac{1}{2}$ factor was inserted to make the maximum of this function 1 whereas the $-i/1$ factor was used to make the function real. This results in

$$V(z) = \sum_n \left(V_n + V_n^* \right) f_n^V(z) \tag{13}$$

where the time dependence is in V_n. The voltage mode function is given by

$$f_n^V(z) = \begin{cases} \sin \frac{\pi}{L} nz & n \,\text{is odd} \\ \cos \frac{\pi}{L} nz & n \,\text{is even} \end{cases}. \tag{14}$$

We can then use one of telegrapher's equations, that is

$$\frac{\partial I}{\partial z} = -\widetilde{C}\frac{\partial V}{\partial t} \tag{15}$$

to find

$$I(z) = -\widetilde{C}\sum_n (-i)\frac{\omega_n}{k_n}\left(V_n - V_n^*\right) f_n^I(z) \tag{16}$$

where the current mode functions are

$$f_n^I(z) = \begin{cases} -\cos\frac{\pi}{L}nz & n \,\text{is odd} \\ \sin\frac{\pi}{L}nz & n \,\text{is even} \end{cases}. \tag{17}$$

Both the current and voltage mode functions satisfy the following orthogonality conditions

$$\int_{-L/2}^{L/2} dz f_n^{V,I}(z) dz f_m^{V,I}(z) = \frac{L}{2}\delta_{n,m} \tag{18}$$

Based on this, we can calculate the total energy for the electric and magnetic field, as the integral in the capacitors and the inductors respectively over the length of the line:

$$E_e = \frac{1}{2}\widetilde{C}\frac{L}{2}\sum_n (V_n + V_n^*)^2 \tag{19}$$

$$E_m = \frac{1}{2}\widetilde{C}\frac{L}{2}\sum_n (V_n - V_n^*)^2. \tag{20}$$

As a final step in the quantization line the following substitutions are made

$$V_n \rightarrow \sqrt{\frac{\hbar\omega_n}{\widetilde{C}L}}a_n \tag{21}$$

$$V_n^* \rightarrow \sqrt{\frac{\hbar\omega_n}{\widetilde{C}L}}a_n^\dagger. \tag{22}$$

With this substitution the Hamiltonian becomes

$$H_{\mathrm{CPW}} = \sum_n \hbar\omega_n\left(a_n^\dagger a_n + \frac{1}{2}\right). \tag{23}$$

In terms of creation and annihilation operators, the voltage operator for the transmission line becomes

$$V(z) = \sum_n \sqrt{\frac{\hbar\omega_n}{\widetilde{C}L}}\left(a_n + a_n^\dagger\right) f_n^V(z). \tag{24}$$

B. Hamiltonian of the qubit coupled to the resonator

Figure 2 shows a Cooper pair box (CPB) coupled to a coplanar waveguide (CPW) resonator. The CPB Hamiltonian was derived in previous lectures:

$$H_{CPB} = \frac{E_C}{2}\left(n - n_g\right)^2 - E_J \cos\gamma. \tag{25}$$

Here n_g is the gate charge parameter related to the gate voltage through

$$n_g = \frac{C_g V_g}{2e}, \tag{26}$$

with C_g the gate capacitance and V_g the gate voltage.

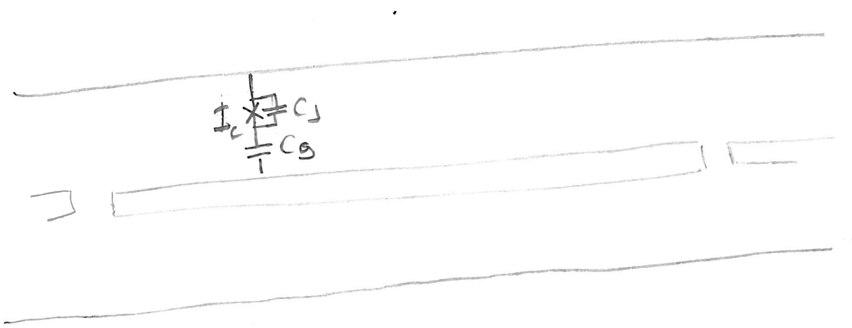

FIG. 2. CPB coupled to CPW.

When coupled to the CPW, the CPB "senses" the transmission line voltage. If the CPB is placed at position z_0 in the waveguide, the gate charge parameter can be replaced as

$$n_g \rightarrow n_{g0} + \frac{C_g}{2e} V_{CPW}(z_0), \tag{27}$$

with n_{g0} a constant gate charge which can be imposed by a voltage source. With this taken into account the total Hamiltonian of the coupled system can be written as

$$H = H_{CPB} + H_{CPW} + H_{int} \tag{28}$$

with

$$H_{CPB} = \frac{E_C}{2}\left(n - n_{g0}\right)^2 - E_J \cos\gamma, \tag{29}$$

$$H_{\text{CPW}} = \sum_n \hbar\omega_n \left(a_n^\dagger a_n + \frac{1}{2}\right), \tag{30}$$

$$H_{int} = E_c(n - n_{g0})\frac{C_g}{2e} V_{CPW}(z_0). \tag{31}$$

We consider biasing of the CPB at the symmetry point $n_{g0} = 1/2$ and the large charging energy regime ($E_c/E_J \gg 1$). The energy eigenstates are

$$|g\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle - |0\rangle\right), \tag{32}$$

$$|e\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |0\rangle\right). \tag{33}$$

In this basis

$$n - \frac{1}{2} = \frac{1}{2}\sigma_x. \tag{34}$$

Finally, the interaction Hamiltonian can be written as

$$H_{int} = \beta e V_0 \sigma_x (a + a^\dagger) \tag{35}$$

with

$$\beta = \frac{C_g}{C_g + C_J} \tag{36}$$

and

$$V_0 = \sqrt{\frac{\hbar\omega_r}{\widetilde{C}L}} f^V(z_0). \tag{37}$$

These are some typical experimental parameters:

- $\beta = 0.1$
- $V_0 = 2\,\mu$V, for coupling to the antinode for mode $n = 2$
- $\omega_r = 2\pi \times 6$ GHz

Reference [3] provides more details on this circuit, in particular the comparison on different figures of merit for cavity QED systems for circuit QED.

IV. THE JAYNES CUMMINGS MODEL

The Jaynes -Cummings (JC) model describes the interaction between a two-level system and a bosonic mode [8]. The JC Hamiltonian is

$$H = H_a + H_c + H_{int} \tag{38}$$

with

$$H_a = -\frac{\hbar\omega_a}{2}\sigma_z, \tag{39}$$

$$H_c = \hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right), \tag{40}$$

$$H_{int} = \hbar g\left(a^\dagger\sigma^- + a\sigma^+\right). \tag{41}$$

Here ω_a and ω_r are respectively the atom and bosonic mode frequencies, σ_z is an operator which has the Pauli Z matrix representation in the energy eigenbasis representation, $\sigma^+ = |e\rangle\langle g|$ and $\sigma^- = |g\rangle\langle e|$ are also qubit operators with $|g\rangle$ and $|e\rangle$ the ground and excited energy eigenstates respectively, a and $a^\dagger$ the annihilation and creation operators for the bosonic mode, and g the strength of the coupling between the two level systems and the bosonic mode.

In addition to the coherent Hamiltonian, one also has to consider the decay rates of the atom and the bosonic mode, denoted by κ_a and κ_c respectively[2]. The strong coupling regime is attained when the strength of the coupling is larger than the two decay rates.

The JC model is of wide interest in the field of quantum control, since it provides a very accurate description of very different types of physical systems.

- cavity QED. The implementation are diverse, including the discussed atomic physics and superconducting circuit based, but also other systems such as quantum dots in optical cavities

- ion traps: in this case the two level systems are the reduction to two levels connected by an electric dipole transition and the bosonic mode is the vibration mode of ions in a Paul trap

- nanomechanical resonators coupled to superconducting qubits

The Hamiltonian 35 has the form $\sigma_x(a + a^\dagger)$ which is different of the form $a^\dagger\sigma^- + a\sigma^+$ in (41). The latter form is valid when the coupling $g \ll \Delta$ where $\Delta = \omega_a - \omega_r$ is the detuning.

In the following we discuss the solution of the JC model in two different regimes: the resonant and the dispersive regime.

[2] In general one has to consider different decay rates even for the atom or bosonic mode alone; for example one has the relaxation and dephasing rate both showing up in general

A. Resonant regime

The JC Hamiltonian only couples levels in the subspaces $\mathcal{E}_n = \{|g\rangle|n\rangle, |e\rangle|n-1\rangle\}$. The structure of levels is shown in Fig. 3. The Hamiltonian in this subspace is

$$\mathcal{H}_n^{JC} = \hbar \begin{pmatrix} n\omega_r & g\sqrt{n} \\ g\sqrt{n} & (n-1)\omega_r + \omega_a \end{pmatrix} \tag{42}$$

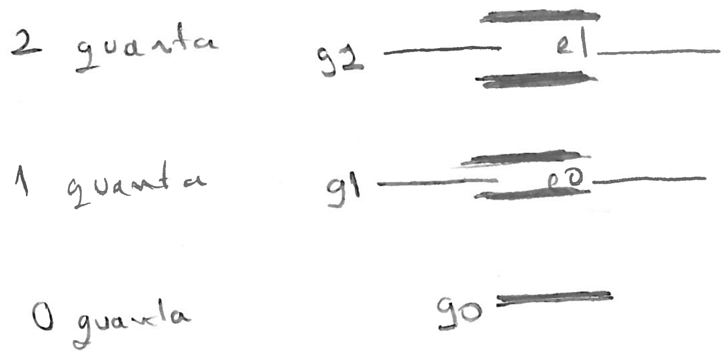

FIG. 3. The JC ladder of levels in the resonant case; the thick lines denote the energy levels with coupling taken into account.

At resonance, the eigenvalues of $\mathcal{H}_n^{JC}$ are

$$E_n^{\pm} = \hbar \left(n\omega_a \pm g\sqrt{n} \right) \tag{43}$$

and the eigenstates are

$$|\pm\rangle_n = \frac{1}{\sqrt{2}} \left(|g\rangle|n\rangle \pm |e\rangle|n-1\rangle \right). \tag{44}$$

In the manifold $n = 1$ the eigenstates are

$$|-\rangle_1 = \frac{1}{\sqrt{2}} \left(|g\rangle|1\rangle - |e\rangle|0\rangle \right) \tag{45}$$

$$|+\rangle_1 = \frac{1}{\sqrt{2}} \left(|g\rangle|1\rangle + |e\rangle|0\rangle \right). \tag{46}$$

$$\tag{47}$$

These states have both atom and photon components. The energy level splitting

$$E_1^+ - E_1^- = \hbar g \tag{48}$$

is called the vacuum Rabi splitting. Its spectroscopic observation requires the strong coupling regime. For superconducting qubits, this was achieved by Wallraff *et al* in 2004 [4]..

B. Dispersive regime

In general the JC Hamiltonian in manifold $\mathcal{E}_n$ is

$$\mathcal{H}_n^{JC} = \hbar \begin{pmatrix} n\omega_r & g\sqrt{n} \\ g\sqrt{n} & n\omega_r + \Delta \end{pmatrix} . \tag{49}$$

The eigenenergies are

$$\frac{E_n^{\pm}}{\hbar} = n\omega_r + \frac{\Delta}{2} \pm \sqrt{\left(\frac{\Delta}{2}\right)^2 + \left(g\sqrt{n}\right)^2} = n\omega_r + \frac{\Delta}{2} \pm \frac{\Delta}{2}\sqrt{1 + \left(\frac{2g\sqrt{n}}{\Delta}\right)^2} . \tag{50}$$

In the strong dispersive regime one has:

$$\frac{E_n^{+}}{\hbar} = n\omega_r + \Delta + n\frac{g^2}{\Delta} = n\left(\omega_r + \frac{g^2}{\Delta}\right) + \Delta \tag{51}$$

and

$$\frac{E_n^{-}}{\hbar} = n\omega_r - n\frac{g^2}{\Delta} = n\left(\omega_r - \frac{g^2}{\Delta}\right) . \tag{52}$$

This is the same as the spectrum for two different harmonic oscillators.

It can be shown that the following transformation for the Hamiltonian

$$H \rightarrow UHU^{\dagger} \equiv H' \tag{53}$$

with

$$U = \exp\left[\frac{g}{\Delta}\left(a\sigma^{+} + a^{\dagger}\sigma^{-}\right)\right] \tag{54}$$

brings the Hamiltonian under the following form:

$$H' = \hbar\left(\omega_r + \frac{g^2}{\Delta}\sigma_z\right) + \frac{\hbar}{2}\left(\omega_a + \frac{g^2}{\Delta}\right)\sigma_z \tag{55}$$

when terms of higher order in g/Δ are neglected.

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