Physics 760: Electricity and Magnetism

Assignment 2

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Contents

Problem 1 (Jackson ed. 3 Problem 2.23)

a

A hollow cube has conducting walls defined by six planes $x = 0$, $y = 0$, $z = 0$, and $x = a$, $y = a$, $z = a$. The walls $z = 0$ and $z = a$ are held at a constant potential V. The other four sides are at zero potential.
Find the potential $\Phi(x, y, z)$ at any point inside the cube.
The solution to a problem of this type is found by solving Laplace's equation, $\nabla^2 = 0$ everywhere in space where charges are absent. Given the rectangular symmetry of the problem I will propose a solution in Cartesian coordinates. That is, I will propose a solution of the form: $V(x, y, z) = V_x(x)V_y(y)V_z(z)$. Inserting this into Laplace's equation above yields:

$$\begin{aligned}\nabla^2 V = V_y(y)V_z(z)\frac{\partial^2 V_x(x)}{\partial x^2} + V_x(x)V_z(z)\frac{\partial^2 V_y(y)}{\partial y^2} + V_x(x)V_y(y)\frac{\partial^2 V_z(z)}{\partial z^2} = 0\\ 0 = \frac{1}{V_x(x)}\frac{\partial^2 V_x(x)}{\partial x^2} + \frac{1}{V(y)}\frac{\partial^2 V_y(y)}{\partial y^2} + \frac{1}{V_z(z)}V_x(x)V_y(y)\frac{\partial^2 V_z(z)}{\partial z^2}\end{aligned}$$

This implies that each term in the sum is a constant such that the sum of all of the terms is zero for any particular x, y or z.

$$\frac{\partial^2 V_x(x)}{\partial x^2} = -\alpha^2 V_x(x) \quad , \quad \frac{\partial^2 V_y(y)}{\partial y^2} = -\beta^2 V_y(y) \quad , \quad \frac{\partial^2 V_z(z)}{\partial z^2} = -\gamma^2 V_z(z)$$

$$\begin{aligned}V_x(x) &= A\exp(-i\alpha x) + B\exp(i\alpha x)\\ V_y(y) &= C\exp(-i\beta y) + D\exp(i\beta y)\\ V_z(z) &= E\exp(\sqrt{\alpha^2+\beta^2}\,z) + F\exp(-\sqrt{\alpha^2+\beta^2}\,z)\end{aligned}$$

Solving for A and B using $V_x(x)$'s two boundary conditions:

$$V_x(a) = 0 = A + B \quad , \quad 0 = A(\exp(-i\alpha a) - \exp(i\alpha a)) = -2iA\sin(\alpha a) \quad , \quad \alpha = \frac{n\pi}{a} \quad , \quad n = 0, 1, 2, ...$$

Solving for C and D using $V_y(y)$'s two boundary conditions:

$$V_y(a) = 0 = C + D \quad , \quad 0 = C(\exp(-i\alpha a) - \exp(i\alpha a)) = -2iC\sin(\alpha a) \quad , \quad \alpha = \frac{m\pi}{a} \quad , \quad n = 0, 1, 2, ...$$

Solving for E and F using $V_z(z)$'s two boundary conditions:

$$\begin{aligned}\Phi(x, y, z) &= \sum_{oddn>0}\sum_{oddm>0}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi x}{a})(A_{nm}\exp(-\theta z) + B_{nm}\exp(\theta z))\\ \Phi(x, y, 0) = V &= \sum_{oddn>0}\sum_{oddm>0}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi x}{a})(A_{nm} + B_{nm})\\ \Phi(x, y, a) = V &= \sum_{oddn>0}\sum_{oddm>0}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi x}{a})(A_{nm}\exp(-\theta a) + B_{nm}\exp(\theta a))\end{aligned}$$

Using the orthogonality of sine functions I can multiply both sides by two functions $f(x) = sin(\frac{k\pi x}{a})$ and $g(y) = sin(\frac{l\pi y}{a})$ and integrate over the domain of the box to obtain the following two results. Note that I have used the following two integral identities: $\int_0^a sin(\frac{k\pi x}{a})dx = \frac{2a}{k\pi}$ for odd k (the integral is zero for even k) and $\int_0^a sin^2(\frac{k\pi x}{a})dx = \frac{a}{2}$.

$$\int_0^a \int_0^a V sin(\frac{k\pi x}{a}) sin(\frac{l\pi y}{a}) dxdy = \int_0^a sin^2(\frac{k\pi x}{a}) sin^2(\frac{l\pi y}{a})(A_{kl} + B_{kl})dxdy$$

$$\frac{16V^2}{kl\pi} = A_{kl} + B_{kl}$$

By similar analysis I can write the following below:

$$\frac{16V^2}{kl\pi} = A_{kl}\exp(-\theta a) + B_{kl}\exp(\theta a)$$

Note above the change in variables $\sqrt{\alpha^2 + \beta^2} = \sqrt{(\frac{n\pi x}{a})^2 + (\frac{m\pi y}{a})^2} = \theta$. Combining these two equations into a matrix equation and inverting:

$$\begin{pmatrix} P \\ P \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \exp(-\theta a) & \exp(\theta a) \end{pmatrix} \begin{pmatrix} A_{kl} \\ B_{kl} \end{pmatrix}$$

Above I have used the value P to absorb all the relevant constants: $P = \frac{16V}{kl\pi}$

$$\tag{1}$$

By inverting this 2x2 matrix we obtain:

$$\begin{pmatrix} A_{kl} \\ B_{kl} \end{pmatrix} = \frac{1}{exp(\theta a) - exp(-\theta a)} \begin{pmatrix} exp(\theta a) & -1 \\ -\exp(-\theta a) & 1 \end{pmatrix} \begin{pmatrix} P \\ P \end{pmatrix}$$

Now it is clear that $A_{kl} = P\frac{(\exp(\theta a)-1)}{\exp(\theta a)-\exp(-\theta a)}$ and that $B_{kl} = P\frac{(1-\exp(-\theta a))}{\exp(\theta a)-\exp(-\theta a)}$. Exponential identities can be used to reduce these expressions: $A_{kl} = P\frac{\exp(\theta a)}{1+\exp(\theta a)}$ and $B_{kl} = P\frac{1}{1+\exp(\theta a)}$.
Thus, the potential can be written as :

$$\boxed{\Phi(x,y,z) = \sum_{oddn}\sum_{oddm} \frac{16V}{nm\pi}\frac{1}{1+\exp(\theta a)} \sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{a})(\exp(\theta a)exp(-\theta z) + \exp(\theta z))}$$
.

b

Evaluate the potential at the center of the cube numerically, accurate to three significant figures. How many terms in the series is it necessary to keep in order to attain this accuracy? Compare your numerical result with the average value of the potential on the walls. See Problem 2.28.

Having evaluated this, I obtained $\approx \frac{V}{3}$. The value I obtained to three significant figures was $.333V$. This compares well with the average wall-potential: $V + V/6 = \frac{V}{3}$.Below, I have included a table of the contribution of each term in the double sum to the value of the potential at the center of the cube. These were obtained using Matlab code included in the appendix. The m and n values associated with the contribution of the mn^{th} term to the sum determine the ordering. I.E. The larger the "weight" of the term in the sum, the higher that value of that m and n value in the sum (they are ordered higher).

Term #	m	n	Term's Weight	Cumulative Sum
1	1	1	0.34754584866469	0.347545848664695
2	3	1	-0.00752383905851	0.340022009606181
3	1	3	-0.00752383905851	0.332498170547667
4	3	3	0.00045954463592	0.332957715183595
5	5	1	0.00021547126164	0.333173186445239
6	1	5	0.00021547126164	0.333388657706883
7	5	3	-2.27484874789757	0.333365909219404
8	3	5	-2.27484874789757	0.333343160731925
9	7	1	-6.94949638519198	0.333336211235540
10	1	7	-6.94949638519198	0.333329261739155

Thus, it seems as if I need the first 4 terms in my sub in order to reach 3 significant digits. The 5th term has a weight of $2 \cdot 10^{-4}$ which is not strong enough to alter the third significant digit.

c

Find the surface-charge density on the surface z = a.
To find the surface charge density $\sigma(x, y)$ I will take advantage of the fact that due to the discontinuity in the electric field, the change in potential with respect to the normal direction at a surface is proportional to the surface charge density. That is $\frac{\partial V}{\partial n} = \frac{\sigma}{\epsilon_0}$. In this case, the normal vector is in the z direction. So, I must take a partial derivative of my potential with respect to z and multiply it by ϵ_0 and evaluate it at $z = a$.

$$\sigma(x,y) = \epsilon_0 \sum_{odd\ n} \sum_{odd\ m} \frac{16V}{nm\pi} \frac{1}{1+\exp(\theta a)} \sin(\frac{n\pi x}{a}) \sin(\frac{m\pi y}{a}) \frac{\partial}{\partial z}\big(\exp(\theta a) exp(-\theta z) + \exp(\theta z)\big)\Big|_{z=a}$$
$$\sigma(x,y) = \epsilon_0 \sum_{odd\ n} \sum_{odd\ m} \frac{16V}{nm\pi} \frac{1}{1+\exp(\theta a)} \sin(\frac{n\pi x}{a}) \sin(\frac{m\pi y}{a})\big(-\theta \exp(\theta a) exp(-\theta a) + \theta \exp(\theta a)\big)$$
$$\sigma(x,y) = \epsilon_0 \sum_{odd\ n} \sum_{odd\ m} \frac{16V}{nm\pi} \frac{1}{1+\exp(\theta a)} \sin(\frac{n\pi x}{a}) \sin(\frac{m\pi y}{a})\theta\big(-1 + \exp(\theta a)\big)$$

Realizing that $\frac{e^x - 1}{e^x + 1} = \tanh(x/2)$ this last expression can be rewritten

$$\sigma(x,y) = \epsilon_0 \sum_{odd\ n} \sum_{odd\ m} \frac{16V}{nm\pi} \sin(\frac{n\pi x}{a}) \sin(\frac{m\pi y}{a}) \tanh(a/2)\theta$$

Problem 2

Problem 2 (Jackson ed. 3 Problem 3.1) **Two concentric spheres have radii a, b ($b > a$) and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V. The other hemispheres are at zero potential Determine the potential in the region $a < r < b$ as a**

Problem 2 continued on next page. . .

series in Legendre polynomials. Include terms at least up to $l = 4$. Check your solution against known results in the limiting cases $b \to \infty$, and $a \to 0$.

The boundary conditions are expressed in terms of potentials. Thus, I should use Laplace's equation and the Legendre polynomials to determine the potential everywhere between the two spheres. Due to the azimuthal symmetry of the problem I can assume that "$m = 0$" for my associated Legendre polynomials.

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)} P_l(\cos\theta) \right)$$

Using orthogonality of the Legendre polynomials we can solve for A_l and B_l. We have two boundary conditions. At $r = a$ $\Phi = V$ and at $r = b$ $\Phi =$. This sets up the two equations (note that $\int\limits_{-1}^{1} P_l(\cos\theta)P_m(\cos\theta)d\cos\theta = \frac{2}{2l+1}\delta_{lm}$).

$$\frac{2k+1}{2}\int_{\theta=0}^{\theta=\pi} V(a,\theta)P_k(\cos\theta)d\cos\theta = A_k a^k + B_k a^{-(k+1)}$$
$$\frac{2k+1}{2}\int_{\theta=0}^{\theta=\pi} V(b,\theta)P_k(\cos\theta)d\cos\theta = A_k b^k + B_k b^{-(k+1)}$$

These two integrals can be substantially reduced by realizing that the potential (hence, the integrand) is zero for half of the integration.

$$V\int_{\theta=0}^{\theta=\pi/2} P_k(\cos\theta)d\cos\theta = A_k a^k + B_k a^{-(k+1)}$$
$$V\int_{\theta=\pi/2}^{\theta=\pi} P_k(\cos\theta)d\cos\theta = A_k a^k + B_k a^{-(k+1)}$$

In general, these integrals have no closed form analytic solution (truth be told, these integrals can be expressed in terms of gamma functions, but this is unnecessarily complicated and not particularly enlightening. I will, at this time, change integration variables from $\cos\theta \to x$.

$$V\int_{1}^{0} P_k(x)dx = A_k a^k + B_k a^{-(k+1)}$$
$$V\int_{0}^{-1} P_k(x)dx = A_k a^k + B_k a^{-(k+1)}$$

Now, a useful property for evaluating Legendre polynomials is the following: $P_l(x) = \dfrac{1}{2l+1}\frac{d}{dx}\left(P_{l+1}(x) + P_{l-1}(x)\right)$. One might be concerned regarding the $P_0(x)$ case. However, $P_0(x) = 1$ from $x = -1 \to 1$. Thus, the expression for the $P_0(x)$ case is given below. After, the $P_l(x)$ cases will be evaluated (where $l > 1$).

$$\frac{V}{2}\int_{1}^{0} P_0(x)dx = A_k a^k + B_k a^{-(k+1)} = -V$$
$$2V\int_{0}^{-1} P_0 x dx = A_k b^k + B_k b^{-(k+1)} = -V$$

For $k > 1$ we consider the first integral (the one whose bounds are $1 \to 0$), $\frac{V(2k+1)}{2} \int_1^0 P_k(x)dx$

$$\frac{V(2k+1)}{2} \int_1^0 P_k(x)dx = \gamma_k \frac{d}{dx}\left(P_{k+1}(x) + P_{k-1}(x)\right) \quad \gamma_k \equiv \frac{1}{2k+1}$$

Note that $\dfrac{2k+1}{2} = \frac{\gamma_k}{2}$.

$$\frac{V}{2} \int_1^0 P_k(x)dx = \frac{d}{dx}\left(P_{k+1}(x) + P_{k-1}(x)\right)$$

Throwing in a couple more Legendre polynomial properties (thank you, Wikipedia!) $P_l(1) = 1 \forall l$ and $(-1)^l P_l(-x) = P_l(x) \forall l$. In plain English, the second identity states that if l is odd that P_l is odd. So, $P_l(-1) = -1$ for odd l. For even l, $P_l(-1) = 1$. Considering even $l = 2p$ $p = 0, 1, 2, ...$

$$\begin{aligned}\frac{V(2(2p)+1)}{2} \int_1^0 P_{2p}(x)dx &= \gamma_{2p} \int_1^0 \frac{d}{dx}\left(P_{2p+1}(x) + P_{2p-1}(x)\right) dx \\ &= \gamma_{2p}\big(P_{2p+1}(1) - P_{2p-1}(1) + P_{2p-1}(0) - P_{2p+1}(0)\big)\end{aligned}$$

According to the properties above this can be easily seen to be zero for all p. Now, considering odd $l = 2p+1$ $p = 0, 1, 2, ...$

$$\begin{aligned}\frac{V(2(2p+1)+1)}{2} \int_1^0 P_{2p+1}(x)dx &= \gamma_k \int_1^0 \frac{d}{dx}\left(P_{2p+2}(x) + P_{2p}(x)\right) dx \\ &= \gamma_{2p+1}\big(P_{2p+2}(0) - P2p(0) + P_{2p}(1) - P_{2p+2}(1)\big)\end{aligned}$$

By the properties listed above we can reduce this to the following expression:

$$= \gamma_{2p+1}\big(P_{2p+2}(0) - P_{2p}(0)\big)$$

As there is no closed form solution for this expression (outside of the use of gamma functions) this integral is for all intents and purposes completed. For the case where $l = 0$ the integral simplifies to -1 since $P_0(x) = 1$ from $x = -1 \to 1$.

Performing similar analysis on the other integral yields the following:

$$\int_{-1}^0 \frac{V(2l+1)}{2} P_l(x)dx = \begin{cases} 0 & \text{l is even and } l > 0 \\ -\gamma_l\big(P_{l+1}(0) - P_{l-1}(0)\big) & \text{l is odd} \\ -1 & \text{l is 0} \end{cases}$$

Succinctly put, the first result is:

$$\frac{V(2l+1)}{2} \int_1^0 P_l(x)dx = \begin{cases} 0 & \text{l is even and } l > 0 \\ \gamma_l\big(P_{l+1}(0) - P_{l-1}(0)\big) & \text{l is odd} \\ -1 & \text{l is 0} \end{cases}$$

Given the similarity of the two expressions I will allow the expression $\gamma_l\big(P_{l+1}(0) - P_{l-1}(0)\big) = \kappa_l$. So, expressing the two equations for A_k and B_k (for odd k) in a matrix form yields:

$$V\gamma_k\kappa_k\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}a^k & a^{-(k+1)}\\ b^k & b^{-(k+1)}\end{pmatrix}\begin{pmatrix}A_k\\B_k\end{pmatrix}$$

Inverting this, we obtain:

$$\begin{pmatrix}A_k\\B_k\end{pmatrix}=\frac{V\gamma_k\kappa_k}{a^kb^{-(k+1)}-b^ka^{-(k+1)}}\begin{pmatrix}b^{-(k+1)} & -a^{-(k+1)}\\ -b^k & a^k\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}$$

For the case where $k=0$, this expression reduces to $A_k=-1$ and $B_k=1$ as seen by allowing $k=0$ in the above expression and realizing that $\gamma_0=\frac{1}{2(0)+1}=1$ and Now, writing an expression for A_k and B_k and substituting this into the original expression for $\Phi(r,\theta)$ would be extremely cumbersome at this point. Ra

Problem 3

Problem 3 (Jackson ed. 3 Problem 3.2) **A spherical surface of radius R has charge uniformly distributed over its surface with a density $\frac{Q}{4\pi R^2}$, except for a spherical cap at the north pole, defined by the cone $\theta=\alpha$.**

a

Show that the potential inside the spherical surface can be expressed as

$$\frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left(P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)\right)\frac{r^l}{R^{l+1}}P_l(\cos\theta)$$

where, for $l=0$, $P_l(\cos\alpha)=-1$. What is the potential outside?
I will integrate over the charge density in the usual way to obtain the potential.

$$V(\vec{r})=\frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\sigma\delta(|\vec{r}|-R)H(\alpha-\theta)|\vec{r}-\vec{r'}|^{-1}r'^2d\Omega'$$

Here, we have allowed $\Omega'\equiv\sin\theta' d\theta' d\phi'$ and H(x) is the Heaviside step function in x. Now, we can use the following identity:

$$|\vec{x}-\vec{x'}|^{-1}=4\pi\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}P_l\cos\gamma\quad\text{where r < R and }\gamma\text{ is the angle between }\vec{r}\text{ and }\vec{r'}$$

$$V(|\vec{r}|<R)=\frac{\sigma R^2}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}\int_0^{2\pi}\int_{\alpha}^{\pi}P_l(\cos\gamma)d\Omega'$$

In general, γ could be a function of both θ' and ϕ' so this could be a very hard integral. However, the following result, known as the "spherical harmonic addition theorem" allows us to express

$$P_l(\cos\gamma)=P_l(\cos\theta')P_l(\cos\theta)+2\sum_{m=-l}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta')P_l^m(\cos\theta)\cos(m(\phi'-\phi))$$

By noting that the charge distribution has no azimuthal asymmetry we can assume that there must be no dependence on ϕ in the final answer. This implies that $m=0$. Substituting the reduced expression into the integral we find:

$$\tag{1}$$

$$V(|\vec{r}| < R) = \frac{\sigma R^2}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}\int\limits_0^{2\pi}\int\limits_\alpha^{\pi} P_l(\cos\theta')P_l(\cos\theta)d\Omega'$$

Now, I will seize the extremely powerful property of Legendre polynomials that I have used in the last problem:

$$P_n(x) = \frac{1}{2n+1}\frac{d}{dx}\left(P_{n+1}(x) - P_{n-1}(x)\right)$$

.

$$\begin{aligned}
V(|\vec{r}| < R) &= \frac{\sigma R^2}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}P_l(\cos\theta)\int\limits_0^{2\pi}\int\limits_{\theta=\alpha}^{\theta=\pi}\frac{1}{2l+1}\frac{d}{d\cos\theta'}\left(P_{l+1}(\cos\theta') - P_{l-1}(\cos\theta')\right)d\sin\theta' d\theta' d\phi' \\
V(|\vec{r}| < R) &= \frac{2\pi\sigma R^2}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}P_l(\cos\theta)\frac{1}{2l+1}\left(P_{l+1}(\cos\theta')|_{\theta=\alpha}^{\theta=\pi} - P_{l-1}(\cos\theta')|_{\theta=\alpha}^{\theta=\pi}\right)
\end{aligned} \tag{2}$$

Realizing that $P_{l+1}(-1) - P_{l-1}(-1) = 0$ for all l the final expression is obtained.

$$\begin{aligned}
V(|\vec{r}| < R) &= \frac{2\pi\sigma R^2}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\frac{r^l}{R^{l+1}}P_l(\cos\theta)\big(P_{l+1}(\alpha) - P_{l-1}(\alpha)\big) \\
V(|\vec{r}| < R) &= \frac{2\pi Q R^2}{4\pi\epsilon_0 4\pi R^2}\sum_{l=0}^{\infty}\frac{1}{2l+1}\frac{r^l}{R^{l+1}}P_l(\cos\theta)\left(P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right) \\
V(|\vec{r}| < R) &= \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\frac{r^l}{R^{l+1}}P_l(\cos\theta)\left(P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right)
\end{aligned}$$

The field outside of the sphere would follow the same procedure except for that, now, the roles of R and r would interchange with respect to equation 1 listed above. Thus, the potential can easily be re-expressed for the case outside the sphere as : $V(|\vec{r}| > R) = \frac{Q}{8\pi\epsilon_0}\sum\limits_{l=0}^{\infty}\frac{1}{2l+1}\frac{R^l}{r^{l+1}}P_l(\cos\theta)\left(P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right)$

b

Find the magnitude and the direction of the electric field at the origin. In order to find $\vec{E} = -\nabla V$ I need to identify ∇ in spherical coordinates. According to Wikipedia, this is : $\frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi}$. The electric potential for this problem only depends on r and θ. This allows the derivative over ϕ to be neglected. Thus, the electric field is:

$$\begin{aligned}
V &= \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\frac{\big(P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\big)}{R^{l+1}}r^l P_l(\cos\theta) \\
-\vec{E} &= A\sum_{l=0}^{\infty}\gamma_l P_l(\cos\theta)\frac{\partial r^l}{\partial r}\hat{r} + A\sum_{l=0}^{\infty}\gamma_l r^l\frac{\partial p_l(\cos\theta)}{\partial\theta}\hat{\theta}
\end{aligned}$$

Above, γ_l has absorbed the constants within the sum.

$$= A\sum_{l=0}^{\infty}\gamma_l l r^{l-1}P_l(\cos\theta)\hat{r} + A\sum_{l=0}^{\infty}\gamma_l \frac{1}{r}r^l\frac{\partial p_l(\cos\theta)}{\partial\theta}\hat{\theta}$$
$$= A\sum_{l=0}^{\infty}\gamma_l l r^{l-1}P_l(\cos\theta)\hat{r} + A\sum_{l=0}^{\infty}\gamma_l \frac{1}{r}r^l\frac{\partial p_l(\cos\theta)}{\partial\cos\theta}\frac{\partial\cos\theta}{\partial\theta}\hat{\theta}$$
$$= A\sum_{l=0}^{\infty}\gamma_l l r^{l-1}P_l(\cos\theta)\hat{r} - A\sum_{l=0}^{\infty}\gamma_l r^{l-1}\frac{\partial p_l(\cos\theta)}{\partial\cos\theta}\sin\theta\hat{\theta}$$

Now, I want the solution at this expression at $r = 0$. There, it really doesn't matter what value θ takes on, physically. Thus, I will select a θ that makes my life easy to work with. I will allow $\theta = 0$ such that $\cos\theta = 1$ since for all l $P_l(1) = 1$.

$$= A\sum_{l=0}^{\infty}\gamma_l l r^{l-1}P_l(\cos\theta)\hat{r} - A\sum_{l=0}^{\infty}\gamma_l r^{l}\frac{\partial p_l(\cos\theta)}{\partial\cos\theta}\sin\theta\hat{\theta}$$

Now, it seems that the potential is zero at $r = 0$ since r^l occurs in both sums. And the second sum is zero by the angular constraint mentioned earlier. The first sum would be zero except for the $l = 1$ term. This raises the undefined condition whereby 0^0 results in the sum. What value this should have is hotly debated in mathematical literature. However, it can be reasoned that $0^0 = 1$ since $\lim_{r\to 0} r^0 = \lim_{r\to 0} 1 = 1$. Thus, one term in the sum survives and, thus, the electric field at the origin is:

$$\vec{E}(0,0,0) = A\gamma_1\hat{r}$$

Now, $A = \frac{Q}{8\pi\epsilon_0}$ and $\gamma_1 = \frac{P_2(\cos\alpha)-P_0(\cos\alpha)}{R^{l+1}2l+1}$.
$P_2(\cos\alpha) = .5(3\cos^2\alpha - 1)$; $P_0(\cos\alpha) = 1$.
So, now, $\vec{E}(0,0,0) = \frac{Q}{8\pi\epsilon_0}(1.5\cos^2\alpha - 1.5)\frac{1}{3R^2}\hat{r} = \frac{Q}{16\pi\epsilon_0}\frac{\cos^2\alpha - 1}{R^2}\hat{r} = \frac{Q}{16\pi\epsilon_0}\frac{\sin^2\alpha}{R^2}$.

c

Discuss the limiting forms of the potential (part a) and electric field (part b) as the spherical cap becomes A) very small, and B) so large that the area with charge on it becomes a very small cap at the south pole.

Problem 4

Problem 4 (Jackson ed. 3 Problem 3.4a)
The first thing to realize is that no longer is there azimuthal symmetry in this problem. Thus, $\Phi(r,\theta,\phi) = \sum_{0}^{\infty}(A_l r^l + B_l r^{-(l+1)})Y_{lm}(\theta,\phi)$. I have been given $V(R,\phi)$. Now, I need to use orthogonality and slick mathematical tricks to get my answer into a tractable form. $V(R,\phi)$ alternates sign based on how many "slices" the sphere has in it (there are 2n wedges for n slices). Thus, the potential can be expressed in the following form if the sphere is aligned such that at $\phi = 0$ the dividing line between V and -V is along the x-axis and for $\phi \in (0, \pi/n)$ the potential is positive.

$$\Phi(\Phi, R) = \begin{cases} V & \phi\in\left(\frac{\pi 2j}{n}, \frac{\pi(2j+1)}{n}\right) \\ V & \phi\in\left(\frac{\pi(2j+1)}{n}, \frac{\pi(2j+2)}{n}\right) \end{cases} \quad \text{for } j = 0,1,2,...,n-1$$