Lecture 22: Applications of circuit QED - quantum control

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(Dated: 2011/11/21)

I. RESONANT AND DISPERSIVE REGIME IN CAVITY QED

In experiments with cQED (and cavity QED in general) ports coupled to the resonator are needed for various purposes: controlling the state of the resonator and/or the qubit, probing spectroscopically the coupled system, etc. Before considering coupled ports, we first analyze the spectrum of the coupled qubit (atom) - resonator system in two regimes: resonant and dispersive.

A. Resonant regime

The JC Hamiltonian (with the RWA taken into account) only couples levels in the subspaces $\mathcal{E}_n = \{|g\rangle|n\rangle, |e\rangle|n-1\rangle\}$. The Hamiltonian in this subspace is

$$\mathcal{H}_n^{JC} = \hbar \begin{pmatrix} n\omega_r & g\sqrt{n} \\ g\sqrt{n} & (n-1)\omega_r + \omega_a \end{pmatrix} \tag{1}$$

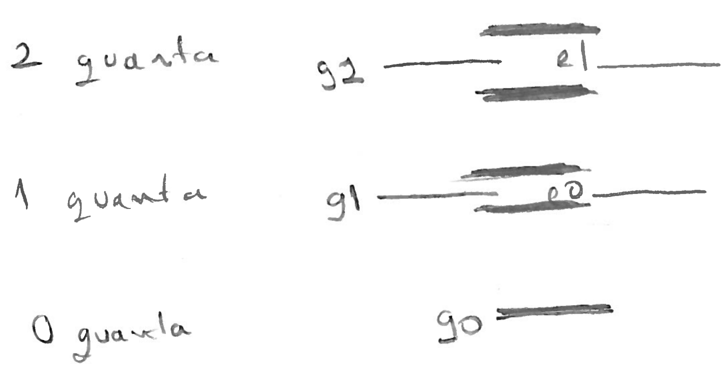

FIG. 1. The JC ladder of levels in the resonant case; the thick lines denote the energy levels with coupling taken into account.

At resonance ($\omega_r = \omega_a$), the structure of levels is shown in Fig. 1. The eigenvalues of $\mathcal{H}_n^{JC}$ are

$$E_n^{\pm} = \hbar \left(n\omega_a \pm g\sqrt{n} \right) \tag{2}$$

and the eigenstates are

$$|\pm\rangle_n = \frac{1}{\sqrt{2}}\left(|g\rangle|n\rangle \pm |e\rangle|n-1\rangle\right). \tag{3}$$

In the manifold $n = 1$ the eigenstates are

$$|-\rangle_1 = \frac{1}{\sqrt{2}}\left(|g\rangle|1\rangle - |e\rangle|0\rangle\right) \tag{4}$$

$$|+\rangle_1 = \frac{1}{\sqrt{2}}\left(|g\rangle|1\rangle + |e\rangle|0\rangle\right). \tag{5}$$

$$\tag{6}$$

These states contain both atom and photon excitations. The energy level splitting

$$E_1^+ - E_1^- = \hbar g \tag{7}$$

is called the vacuum Rabi splitting. Its spectroscopic observation requires the strong coupling regime. For superconducting circuits, this was achieved by Wallraff *et al* in 2004 [1]..

B. Dispersive regime

In general (off resonance) the JC Hamiltonian in manifold $\mathcal{E}_n$ is

$$\mathcal{H}_n^{JC} = \hbar \begin{pmatrix} n\omega_r & g\sqrt{n} \\ g\sqrt{n} & n\omega_r + \Delta \end{pmatrix}. \tag{8}$$

with the detuning $\Delta = \omega_a - \omega_r$. The eigenenergies are

$$\frac{E_n^\pm}{\hbar} = n\omega_r + \frac{\Delta}{2} \pm \sqrt{\left(\frac{\Delta}{2}\right)^2 + \left(g\sqrt{n}\right)^2} = n\omega_r + \frac{\Delta}{2} \pm \frac{\Delta}{2}\sqrt{1 + \left(\frac{2g\sqrt{n}}{\Delta}\right)^2}. \tag{9}$$

In the strong resonant regime one has the two following expressions

$$\frac{E_n^+}{\hbar} = n\omega_r + \Delta + n\frac{g^2}{\Delta} = n\left(\omega_r + \frac{g^2}{\Delta}\right) + \Delta \tag{10}$$

and

$$\frac{E_n^-}{\hbar} = n\omega_r - n\frac{g^2}{\Delta} = n\left(\omega_r - \frac{g^2}{\Delta}\right). \tag{11}$$

This is the same as the spectrum for two different harmonic oscillators. The spectrum is shown in Figure. 2.

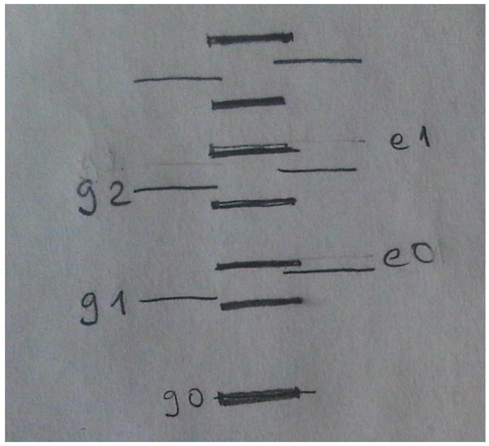

FIG. 2. The JC ladder of levels in the dispersive case; the thick lines denote the energy levels with coupling taken into account.

It can be shown that the following transformation for the Hamiltonian

$$H \to UHU^{\dagger} \equiv H' \tag{12}$$

with

$$U = \exp\left[\frac{g}{\Delta}\left(a\sigma^{+} - a^{\dagger}\sigma^{-}\right)\right] \tag{13}$$

brings the Hamiltonian to the following form:

$$H' = \hbar\left(\omega_r + \frac{g^2}{\Delta}\sigma_z\right)a^{\dagger}a + \frac{\hbar}{2}\left(\omega_a + \frac{g^2}{\Delta}\right)\sigma_z. \tag{14}$$

with terms left aside being second order in the ratio g/Δ.

II. HAMILTONIAN OF A QUBIT IN THE CAVITY WITH DRIVING THROUGH THE CAVITY

A. Coherent states

Coherent states are extremely important in optics. The notion of coherent state arises naturally when considering the Hamiltonian of a resonator driven by a classical force term, that is

$$H_r = \hbar\omega_r a^{\dagger}a + \hbar i\left(F(t)a^{\dagger} - F^{*}(t)a\right). \tag{15}$$

(it can be shown that a classical force leads to a term of this type).

We go to the rotating frame in which the wavefunction is the Schrodinger wavefunction on which we act with the unitary operator

$$U = \exp\left[i\omega_r t a^\dagger a\right]. \tag{16}$$

In this frame the Hamiltonian is

$$\widetilde{H} = UHU^\dagger + i\hbar\dot{U}U^\dagger. \tag{17}$$

We have

$$U\omega_r a^\dagger a U^\dagger = \omega_r a^\dagger a \tag{18}$$

and

$$i\dot{U}U^\dagger = -\omega_r a^\dagger a. \tag{19}$$

For the transformation of the a and $a^\dagger$ terms we use the operator expansion theorem [2]. For any operators A and B one has

$$e^{xA}Be^{-xA} = B + \frac{x^1}{1!}\left[A, B\right] + \frac{x^2}{2!}\left[A, \left[A, B\right]\right] + .. \tag{20}$$

Based on this one has

$$UaU^\dagger = e^{-i\omega_r t}a \tag{21}$$

and

$$Ua^\dagger U^\dagger = e^{i\omega_r t}a^\dagger. \tag{22}$$

The transformed Hamiltonian is therefore

$$\widetilde{H} = i\hbar\left(F(t)e^{i\omega_r t}a^\dagger - F(t)e^{-i\omega_r t}a\right). \tag{23}$$

The evolution operator over an infinitesimal time interval $[t, t+\delta t]$ in the rotating frame is of the form

$$\widetilde{U}(t, t+\delta t) = e^{\alpha(t,t+\delta t)a^\dagger - \alpha^*(t,t+\delta t)a}. \tag{24}$$

An operator $D(\alpha)$ of the form

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} \tag{25}$$

is called a displacement operator. The total evolution operator in the rotating frame is the limit of a product of displacement operators. Such a product of displacement operators is a displacement operator as well. So the unitary evolution in the frame is a displacement operator.

To show that the product of two displacement operators is a displacement operator we use the Campbell-Baker-Haussdorf theorem [2]: for two operators A and B which are such that they both commute with their commutator, one has

$$e^{A+B} = e^A e^B e^{-[A,B]/2} \tag{26}$$

$$= e^B e^A e^{[A,B]/2}. \tag{27}$$

One can apply this theorem to show the product the displacement operator $D(\alpha+\alpha')$ is equal to $D(\alpha)D(\alpha')$ up to a phase factor.

Application of a displacement operator to the vacuum leads to the so called coherent states:

$$|\alpha\rangle \equiv D(\alpha)|0\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-|\alpha|^2/2}|0\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger}|0\rangle, \tag{28}$$

where we used the Campbell-Baker-Haussdorf theorem and then the property $e^{-\alpha^* a}|0\rangle = |0\rangle$. We have

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{n!} a^{\dagger n}|0\rangle. \tag{29}$$

This definition shows also how a coherent state $|\alpha\rangle$ can be defined as an eigenstate of the annihilation operator a with eigenvalue α:

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{30}$$

B. Coupling of a cavity to the outside world

Consider a superconducting resonator coupled to a transmission line (this is a semi-infinite transmission line terminated at the position of the cavity but one may have as well an infinite transmission line). The coupling between the resonator and the transmission line can be written as follows:

$$H_{coupling} = \sum_\omega C(\omega)(a b_\omega^\dagger + a^\dagger b_\omega) \tag{31}$$

where $a/a^\dagger$ are the annihilation/creation operator for the cavity and the $b_\omega/b_\omega^\dagger$ are annihilation/creation operators for the transmission line mode at frequency ω (note: for an infinite line the possibility for the two propagation directions has to be taken into account). $C(\omega)$ is a frequency dependent coupling constant.

The fact that the coupling assumes the linear form above can be understood with respect to the situation shown in Fig. 3. The coupling between the cavity and the transmission line is a term of the form $1/2C_c(V_{TL} - V_{cav})^2$ where V_{TL} and V_{cav} are the voltage in the transmission line at the coupling point and the voltage of the cavity respectively. Since both these voltage operators are linear in creation/annihilation operators, the expression 31 for the coupling follows. The last ingredient is assuming the rotating wave approximation.

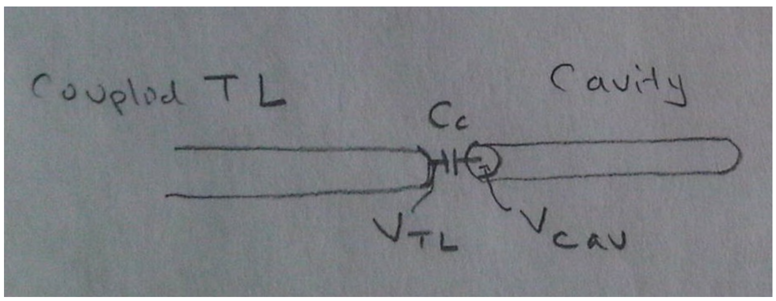

FIG. 3. Cavity coupled to a transmission line (TL).

In most experiments in cQED (and cavity QED for that matter) the transmission lines coupled to the cavity (or equivalently coupling input modes in optics) are used to send to the cavity wave packets which can be used to probe the state of the cavity and/or change the state of the qubit. When using "classical" sources of electromagnetic radiation, modeling the state of the field using coherent states is a good approximation. For the multimode transmission line, a generalized coherent state is defined as the action of a multimode displacement operator [3]

$$D(\{\alpha(\omega)\}) = \exp\left[\int d\omega(\alpha(\omega)b^\dagger(\omega) - \alpha^*(\omega)b(\omega))\right] \tag{32}$$

on the vacuum state.

Solving the problem of the cavity coupled to the transmission line has to take into account the initial state of the electromagnetic field in the resonator. We can assume these states to be separable, with the TL field in a coherent state. We can next do a transformation of the state, which is applying the negative displacement operator which brings the state of the TL back to the ground state. The effect on the Hamiltonian in this frame will be a shift of the creation and annihilation operators. Lets consider this for a single mode. We have

$$D^\dagger(\alpha) a D(\alpha) = a + \alpha \tag{33}$$

and similarly

$$D^\dagger(\alpha) a^\dagger D(\alpha) = a^\dagger + \alpha^*. \tag{34}$$

Application of the shift operators involved in the transformation to the new frame will result in the appearance of terms of the type $\alpha(\omega)a^\dagger$ and $\alpha^*(\omega)a$. The amplitudes α are time dependent (their time dependence is given as $e^{-i\omega t}$. This means that the effective result of the coupling will be a driving term for the qubit. This is the driving term in Eq. 3.1 of reference [4].

III. SINGLE QUBIT CONTROL IN CQED

Following Sec III (Intro part and sections A and B) of Blais et al [4].

IV. COUPLING QUBITS IN CIRCUIT QED

Following parts of sections IV and V of Blais et al [4].

[1] A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, R. S. Huang, J. Majer, S. Kumar, S. M. Girvin, and R. J. Schoelkopf, Nature **431**, 162 (2004).

[2] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, 1995).

[3] S. M. Bernett and P. M. Radmore, *Methods in Theoretical Quantum Optics* (OXFORD SCIENCE PUBLICATIONS, 1997).

[4] A. Blais, J. Gambetta, A. Wallraff, D. I. Schuster, S. M. Girvin, M. H. Devoret, and R. J. Schoelkopf, Phys. Rev. A **75**, 032329 (2007).