QIC880 F2013

Oscillator with applied force

$$\underset{\uparrow \text{ mass}}{m}\ \underset{\uparrow\text{ position}}{\ddot{x}} = -\underset{\uparrow\text{ spring constant}}{k}x + \underset{\uparrow\text{ force (time dependent)}}{f(t)}$$

Lagrangean leading to eq above

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + f(t)x$$

Hamiltonian:

$$\left.\begin{aligned}\mathcal{H} &= \dot{x}p - \mathcal{L}\\ \underset{\uparrow\text{ momentum}}{p} &= m\dot{x}\end{aligned}\right\} \mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2 - f(t)x$$

In terms of creation/annihilation operators

$$x = \sqrt{\frac{m\omega_0}{2\hbar}}(a+a^+)$$

$\omega_0 = \sqrt{\frac{k}{m}}$ is the oscillation frequency

The quantum Hamiltonian

$$H = \hbar\omega_0\left(a^+a+\frac{1}{2}\right) - \underbrace{f(t)\sqrt{\frac{m\omega_0}{2\hbar}}(a+a^+)}_{\equiv \hbar F(t)(ia^+ - ia)}$$

$$\hbar F(t) \equiv \sqrt{\frac{m\omega_0}{2\hbar}}\, f(t)$$

$$\left.\begin{aligned}-a^+ &\to ia^+\\ -a &\to -ia\end{aligned}\right\} \text{change of basis}$$

Excitation of a cavity/qubit by driving a coupled line

Consider a semi-infinite line, coupled to the cavity. Driving of the line is usually localized in space

$$H_d = F(t) V(x)$$

$$V(x) = \sum e^{ikx} a_k + hc$$

$\Rightarrow$ the kind of state that is created is a multi mode coherent state.

$$D(\{\alpha(\omega)\}) \mid vac\rangle$$

$$D(\{\alpha(\omega)\}) = \exp \int d\omega [\alpha(\omega) b^+(\omega) - \alpha^*(\omega) b(\omega)]$$

$$= \prod_{\omega} e^{\alpha(\omega) b^+(\omega) - \alpha^*(\omega) b(\omega)}$$

$\uparrow$ all the modes with $\alpha(\omega) \neq 0$ are displaced.

Assume the force is applied over $(-\tau, 0)$ time interval. The state at time $t > 0$ is

$$U_0(t,0)\, D(\{\alpha(\omega)\}) \mid Vac\rangle$$

$$= U_0(t,0)\, D\, U_0^+ \, U_0 \mid vac\rangle$$

$$U_0 = e^{-i H_0 t/\hbar}$$

$$H_0 = \sum_\omega \hbar\omega\left(b^+(\omega)\, b(\omega) + \frac{1}{2}\right)$$

$$U_0 |vac\rangle = |vac\rangle$$

up to a phase factor

$$U_0(t,0)\, D(\{\alpha(\omega)\})\, U_0^+ = D(\{\alpha(\omega)\, e^{-i\omega t}\})$$

We want to go to a frame that undoes this time dep. displacement

$$U_f(t,0) = D(\{-\bar{\alpha}(\omega,t)\})$$

$$\bar{\alpha}(\omega,t) = \alpha(\omega)\, e^{-i\omega t}$$

In this frame the state of the field is the ground state.

$$U_f^+\, b(\omega)\, U_f^+ = D^+(\{\alpha(\omega,t)\})\, b(\omega)\, D(\{\alpha(\omega,t)\})$$

$$= b(\omega) + \bar{\alpha}(\omega,t)$$

$$\tilde{H}_f = U_f H U_f^+ + i\hbar\, \dot{U}_f U_f^+$$

$$U_f H U_f^+ = \sum_\omega \hbar\omega\left[\left(b^+(\omega) + \bar{\alpha}^*(\omega,t)\right)\left(b(\omega) + \bar{\alpha}(\omega,t)\right) + \frac{1}{2}\right]$$

$$\dot{D}(\{-\bar{\alpha}(\omega,t)\}) = \sum_\omega \left(-\dot{\bar{\alpha}}(\omega,t)\, b^+(\omega) + \dot{\bar{\alpha}}^*(\omega,t)\, b(\omega)\right) D \ldots$$

$$= \sum_\omega \left[+ i\omega\, \bar{\alpha}(\omega,t)\, b^+(\omega) - i\omega\, \bar{\alpha}^*(\omega,t)\, b(\omega)\right] D \ldots$$

$$\Rightarrow \tilde{H}_f = \sum_\omega \hbar\omega \left[b^+(\omega) * b(\omega) + \tfrac{1}{2} + |\alpha(\omega,t)|^2 \right]$$

What about the interaction with a cavity? Consider

$$b(\omega)\, a^+$$

$$\rightarrow \left(b(\omega) + \alpha(\omega) e^{-i\omega t} \right) a^+$$

This is a driving field acting on the cavity mode a

1 Qubit in cQED, 1 drive

$$H = H_{sc} + H_D$$

$$H_{sc} = \omega_r a^+ a + \frac{\omega_a}{2}\sigma_z - g(a^+\sigma_- + \sigma_+ a)$$

$$H_D(t) = \varepsilon(t) e^{-i\omega_d t} a^+ + \varepsilon^*(t) e^{i\omega_d t} a$$

Transformation

$$D(\alpha) = e^{\alpha a^+ - \alpha^* a}$$

with α time-dependent. Note: transf. of state is $D(-\alpha)$

$$\tilde{H} = D^+(\alpha) H D(\alpha) - i D^+(\alpha)\dot{D}(\alpha)$$

- $\dot{D}(\alpha) = D(\alpha) a^+\dot{\alpha} - D(\alpha) a \dot{\alpha}^*$

$*$ $-iD^+(\alpha)\dot{D}(\alpha) = -i\dot{\alpha} a^+ + i\dot{\alpha}^* a$

$*$ $D^+(\alpha) H_D D(\alpha) = \varepsilon(t) e^{-i\omega_d t}(a^+ + \alpha^*) + \varepsilon^*(t) e^{i\omega_d t}(a + \alpha)$

$*$ $D^+(\alpha) H_{sc} D(\alpha) = \omega_r (a^+ + \alpha^*)(a + \alpha) + \frac{\omega_a}{2}\sigma_z - g\left[(a^+ + \alpha^*)\sigma_- + \sigma_+(a + \alpha)\right]$

Gathering all terms:

- $\omega_r a^+ a + \frac{\omega_a}{2}\sigma_z - g(a^+\sigma_- + a\sigma_+)$ constant
- $-g(\alpha^*\sigma_- + \alpha\sigma_+)$ qubit term controlled by driving the field
- $a^+ \times (-i\dot{\alpha} + \varepsilon(t)e^{-i\omega_d t} + \omega_r \alpha)$ $+ h.c.$
- Scalar terms - leading to global phase

We choose α to cancel terms proportional to a and a^+ (so no drive on resonator; in lab frame, there is a change in resonator state though).

We have

$$\dot{\alpha} = -i\omega_r \alpha - i\,\varepsilon(t) e^{-i\omega_d t}$$

For constant drive ($\varepsilon(t) = \varepsilon$) we can solve the equation above by taking

$$\bar{\alpha} = \alpha\, e^{i\omega_r t}$$

This gives

$$\dot{\bar{\alpha}} = -i\,\varepsilon(t) e^{i(\omega_r - \omega_d)t}$$

$$\Rightarrow \quad \bar{\alpha} = -\frac{i\varepsilon}{i\Delta_r} e^{i\Delta_r t} + \text{const}$$

$$\Delta_r \equiv \omega_r - \omega_d$$

$$\Rightarrow \quad \alpha(t) = -\frac{\varepsilon}{\Delta_r} e^{i\Delta_r t} e^{-i\omega_r t} + \text{const}\ e^{-i\omega_r t}$$

We next do one more transformation: both the qubit and the resonators move at the drive frequency:

$$\tilde{H} \rightarrow \quad \bar{H} = U H U^{\dagger} + i \dot{U} U^{\dagger}$$

with

$$U = e^{i\omega_d a^{\dagger} a}\, e^{i\omega_d \frac{\sigma_z}{2}}$$

$$a \rightarrow a\, e^{-i\omega_d t}$$

$$a^{\dagger} \rightarrow a^{\dagger} e^{i\omega_d t}$$

$$\sigma \rightarrow \sigma\, e^{-i\omega_d t}$$

$$\sigma^{+} \rightarrow \sigma^{+} e^{i\omega_d t}$$

The driving term becomes

$\alpha\sigma_+ + hc =$

$\alpha e^{i\omega_d t}\sigma_+ + hc =$

$-\frac{\varepsilon}{\Delta_r}\sigma_+ + hc$

The total Hamiltonian is

$$\bar{H} = \Delta_r a^+ a + \frac{\Delta_a}{2}\sigma_z - g(a^+\sigma_- + \sigma_+ a) + \frac{\Omega_R}{2}\sigma_x$$

The Rabi frequency is

$$\Omega_R = 2\frac{\varepsilon g}{\Delta_r}.$$

This can be written as

$$\Omega_R = 2g\sqrt{\bar{n}}$$

with $\bar{n} = \left(\frac{\varepsilon}{\Delta_r}\right)^2$ (this simply follows from the equation for α, $n = |\alpha|^2$ for a coherent state).

Bit-flip gate

Using this tranformation, corresponding to the dispersive regime

$$U = e^{\frac{g}{\Delta}(a^+\sigma_- - a\sigma_+)}$$

Hausdorf expansion up to 2^{nd} order:

$$e^{-\lambda X} H e^{\lambda X} = H + \lambda[H, X] + \frac{\lambda^2}{2!}[[H, X], X]$$

$$\lambda = \frac{g}{\Delta}$$

$$X = (a^+\sigma_- - a\sigma_+)$$

$$H = \Delta_r a^+a + \frac{1}{2}\left(\Delta a + 2\chi\left(a^+a + \frac{1}{2}\right)\right)\sigma_z + \frac{\Omega_r}{2}\sigma_x$$

a^+a is negligible: this is not the real numbers of photons, but the number in the frame.

This is the common method to do qubit operations in cQED.

Phase gate using AC Stark shift

$\Omega_R \ll \Delta_a$

Starting with

$$H = \Delta_r a^\dagger a + \frac{\Delta_a}{2}\sigma_z - g(a^\dagger\sigma_- + a\sigma_+) + \frac{\Omega_R}{2}\sigma_x$$

Using transformation

$$U = e^{\beta^*\sigma_+ - \beta\sigma_-} \qquad \beta = \frac{\Omega_R}{2\Delta_A}$$

$$U^\dagger H U = e^{-X^*} H e^{X} \cong H + [H,X] + \frac{1}{2}[[H,X],X] + \dots$$

$$X = \beta^*\sigma_+ - \beta\sigma_-$$

$$[\sigma_+,\sigma_z] = \left[\frac{\sigma_x - i\sigma_y}{2},\sigma_z\right] = \frac{1}{2}[-2i\sigma_y - i(2i\sigma_x)] = -i\sigma_y + \sigma_x = \frac{1}{2}\sigma_+$$

$$[\sigma_-,\sigma_z] = \left[\frac{\sigma_x + i\sigma_y}{2},\sigma_z\right] = \frac{1}{2}\left[-2i\sigma_y + i(2i\sigma_x)\right] = -i\sigma_y - \sigma_x = -\frac{1}{2}\sigma_-$$

$$[\sigma_z, X] = \beta^*\left(-\frac{1}{2}\sigma_+\right) - \beta\,\frac{1}{2}\sigma_- = -\frac{1}{2}(\beta^*\sigma_+ + \beta\sigma_-)$$

$$\equiv X_+$$

$$[X_+, X] = \beta^*\beta[\sigma_+,\sigma_-] - \beta\beta^*[\sigma_-,\sigma_+]$$

$$[\sigma_+,\sigma_-] = \left[\frac{\sigma_x - i\sigma_y}{2}, \frac{\sigma_x + i\sigma_y}{2}\right] = \frac{1}{4}[i \times 2i\sigma_z - i(-2i\sigma_z)] = -\sigma_z$$

$$[X_+, X] = -2|\beta|^2\sigma_z$$

$$[[\sigma_z, X], X] = |\beta|^2\sigma_z$$

$$[\sigma_-, X] = \beta^*\sigma_z$$

$$[[\sigma_-, X], X] = \beta^*\left(\beta^*\left(-\frac{1}{2}\sigma_+\right) - \beta\,\frac{1}{2}\sigma_-\right) = -\frac{\beta^*}{2}X_+$$

$$[\sigma_+, X] = +\beta\sigma_z$$

$$[[\sigma_+, X], X] = -\frac{\beta}{2}X_+$$

$$[\sigma_x, X] = [\sigma_+ + \sigma_-, X] = -\beta(-\sigma_z) + \beta^*\sigma_z = (\beta+\beta^*)\sigma_z$$

$$[[\sigma_x, X], X] = -\frac{\beta+\beta^*}{2}X_+$$

$$\tilde{H} \cong \Delta_r a^\dagger a + \frac{\Delta_a}{2}\sigma_z - g(a^\dagger\sigma_- + a\sigma_+) + \frac{\Omega_R}{2}\sigma_x$$

$$+ \frac{\Delta_a}{2}\left(-\frac{1}{2}X_+\right) - g\left[a^\dagger \beta^* + a\beta\right]\sigma_z + \frac{\Omega_R}{2}(\beta+\beta^*)\sigma_z$$

$$+ \frac{\Delta_a}{2}|\beta|^2\sigma_z - g\left[a^\dagger\left(-\frac{\beta^*}{2}\right) + a\left(-\frac{\beta}{2}\right)\right]X_+ + \frac{\Omega_R}{2}\left(-\frac{\beta+\beta^*}{2}\right)X_+$$

$$\sigma_z : \quad \frac{\Delta_a}{2} + \frac{\Omega_R}{2}(\beta+\beta^*) + \frac{\Delta_a}{2}|\beta|^2$$

This shows how the second correction arises.