Introduction to Noise Processes
ECE730/QIC890-T33
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Problem Set 4

Due: June 28, 2016, 8:30 am

1. **Carson's Theorem**

   Carson's theorem describes the power spectral density of a pulse train whose pulses arrive randomly, namely, according to a Poisson arrival process. We derive an analogous result for a pulse train arising from a regulated arrival process.

   Consider a random process consisting of the pulse train

   $$x(t) = \sum_{k=-\infty}^{\infty} a_k f(t - t_k).$$

   In other words, $x(t)$ consists of a sum of pulses of shape $f(t)$; pulse k is located at $t = t_k$ with an amplitude a_k. The amplitudes a_k are random variables with ensemble average $\langle a_k \rangle$ and variance $\sigma_{a_k}^2$.

   Here, we assume that pulses only arrive at evenly-spaced moments in time. To be precise,

   $$t_k = k\Delta t.$$

   The amplitude of the pulse at t_k is given by a_k; it is certainly possible for $a_k = 0$, in which case no pulse occurs around t_k.

   Show that the power spectral density for $x(t)$ is given by

   $$S_x(\omega) = 2\nu\sigma_a^2 |F(i\omega)|^2 + 4\pi \overline{x(t)}^2 \delta(\omega),$$

   assuming that all pulse amplitudes are independent and have equal mean and variance; *i.e.*, $\sigma_a^2 = \sigma_{a_k}^2$ and $\langle a \rangle = \langle a_k \rangle$. In addition, assume that $\Delta t \to 0$. The pulse arrival rate is defined as $\nu = 1/\Delta t$, and $F(i\omega)$ is the Fourier transform of $f(t)$.

2. **$1/f$ Noise**

   In class, we discussed the capture and release of charged particles at trapping centers as a possible physical model to explain the $1/f$ noise phenomenon. Problem 2 walks through the details of the model.

   Suppose a conducting channel with many free charged carriers. If a free carrier falling into a trap, it is no longer mobilized and modules the current. Such modulation of carrier numbers forms random telegraph signal with a Poisson point process shown in Fig. 1. The probability of observing m telegraphic signals in the time interval T is given by

   $$p(m, T) = \frac{(\nu T)^m}{m!} \exp(-\nu T),$$

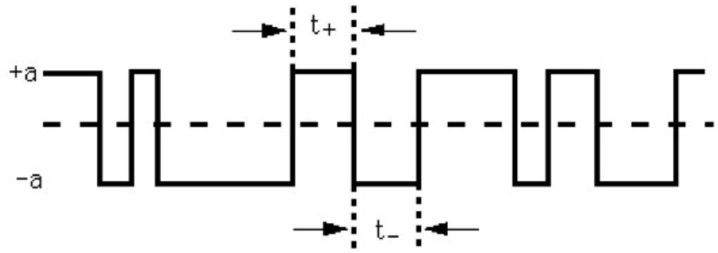

Figure 1: A random telegraph signal produced by a carrier trap.

where nu is the mean rate of transitions per second. If τ_+ and τ_- are the average times spent in the upper and lower states clearly denoted as in Fig. 1, the probability distributions of the upper state time t_+ and lower state time t_- are

$$p(t_\pm) = \tau_\pm^{-1} \exp(-\frac{t_\pm}{\tau_\pm}).$$

The product $x(t)x(t+\tau)$ is equal to $+a^2$ if an even number of transitions occur in the interval $(t, t+\tau)$ and $-a^2$ if an odd number of transitions occurs in the same interval.

(1) Autocorrelation function $\phi_\tau(x)$
Please show that the autocorrelation function $\phi_\tau(x)$ is given by $a^2 \exp(-2\nu\tau)$.

(2) The Power spectrum $S_X(\omega)$
Using the Wiener-Khintchine theorem, show that the power spectral density $S_X(\omega)$ is given by

$$S_X(\omega) = a^2 \frac{4\tau_z}{1+\omega^2\tau_z^2},$$

where $\tau_z = 1/2\nu$ is the time constant of the trap.
Suppose the τ_z is distributed according to the probability density function $p(\tau_z)$ ($\int_0^\infty p(\tau_z)dz = 1$), the power spectral density of the total carrier number fluctuation, $S_n(\omega)$ is given by

$$S_n(\omega) = 4\phi_n(\tau = 0) \int_0^\infty \frac{\tau_z p(\tau_z)}{(1+\omega^2\tau_z^2} d\tau_z . (Eq.1)$$

(3) Probability density function $p(\tau_z)$
Suppose the carrier trap occurs by the tunneling of charged carriers form a conducting layer to traps inside the oxide layer at depth w, the time constant obeys

$$\tau_z = \tau_0 \exp(\gamma w),$$

where τ_0 and γ are constants.
If the traps are uniformly distributed between the depth w_1 and w_2, corresponding to the time constants τ_1 and τ_2, the probability function $p(\tau_z)$ is proportional to $1/\tau_z$ in this region $\tau_1 \le \tau_z \le \tau_2$. What is the probability function calculated?

(4) Power Spectral Density of the total number fluctuation $S_n(\omega)$
Please compute $S_n(\omega)$ using (Eq. 1).

(5) Interpretation of $S_n(\omega)$
Please find the expressions of $S_n(\omega)$ for the following three cases: (i) $\omega\tau_2 \gg 1$ (ii) $1/\tau_2 < \omega < 1/\tau_1$ (iii) $0 \le \omega\tau_1 \ll 1$

3. **Noise Figure of a 2-port Linear Network**

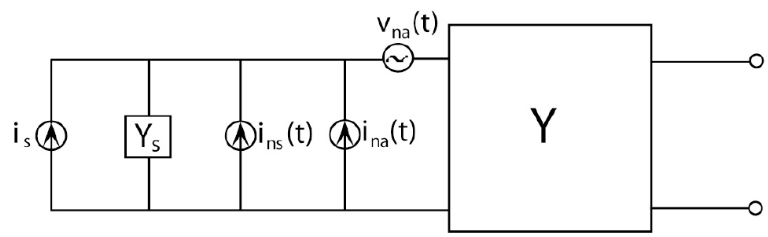

Figure 2: A two-port network to compute the noise figure.

Figure 2 is a diagram of a two-port circuit, which transfers a signal input $i_s(t)$ to the output using an input admittance Y_s and a noisy network Y. $i_{ns}(t)$ is the noise generated in the input admittance Y_s and the external noise sources are denoted as $i_{na}(t)$ and $v_{na}(t)$, which are added to the input port. We assume that the input noise $i_{ns}(t)$ is independent of $i_{na}(t)$ and $v_{na}(t)$ in the two port.

(1) Noise Figure
By definition, the noise figure of a linear two-port is defined as

$$F = \frac{\text{total output noise power per unit bandwidth}}{\text{output noise power per unit bandwidth due to input noise}}.$$

Please show the steps of each line that the noise figure of the whole system is expressed as

$$\begin{aligned} F &= \frac{\langle |I_{ns} + I_{na} + Y_s V_{na}|^2 \rangle}{\langle |I_{ns}|^2 \rangle}, \\ &= 1 + \frac{S_{ia}(\omega)}{S_{is}(\omega)} + |Y_s|^2 \frac{S_{va}(\omega)}{S_{is}(\omega)} + 2\Re(\Gamma_{iv} Y_s^*) \frac{\sqrt{S_{ia}(\omega) S_{va}(\omega)}}{S_{is}(\omega)}, \end{aligned}$$

where $S_{ia}(\omega), S_{va}(\omega), S_{is}(\omega)$ are the power spectral density of $i_{na}(t), v_{na}(t), i_{ns}(t)$ respectively, and Γ_{iv} is the normalized cross-correlation spectral density (or coherence function) between $i_{na}(t)$ and $v_{na}(t)$,

$$\Gamma_{iv}^*(\omega) = \frac{\langle I_{na}^* V_{na} \rangle}{\sqrt{|I_{na}|^2 |V_{na}|^2}} = \frac{S_{iva}(\omega)}{\sqrt{S_{ia}(\omega) S_{va}(\omega)}}.$$

(2) Cross-correlation function Γ_{iv}

Suppose the current noise generator $i_{na}(t)$ has two contributions, one of which, $i_{nb}(t)$ is uncorrelated with $v_{na}(t)$ and the other is fully correlated with $v_{na}(t)$. In terms of the correlation admittance Y_c between $i_{na}(t)$ and $v_{na}(t)$, we can write in the frequency domain,

$$I_{na} = I_{nb} + Y_c V_{na}.$$

Please show that

$$\Gamma_{iv} = Y_c \sqrt{\frac{\langle |V_{na}|^2 \rangle}{\langle |I_{na}|^2 \rangle}} = \frac{Y_c}{\sqrt{G_{ni} G_{nv}}},$$

where G_{ni} and G_{nv} are the equivalent noise conductance to express power spectral densities $S_{ia}(\omega)$ and $S_{va}(\omega)$ at a given temperature Θ.

(3) Expression of F in terms of equivalent noise conductance G

Please express the noise figure in part(1) in terms of equivalent noise conductances. You can introduce G_c, B_c are the real and imaginary parts of Y_c and G_s and B_s are the real and imaginary parts of Y_s.

(4) Minimum F at the optimum source admittance Y_s

Please find the expression of minimum F by optimizing the source admittance and the conductance matching conditions of G_s and B_s.

(5) Minimum F at the optimum correlation admittance Y_c

Please find the expression of F by optimizing the correlation admittance and the conductance matching condition of G_c and B_c.