Solution of Problem Set 5

Problem 1)

Referring to section 7-7 in lecture 9, for a single-mode cavity along the z-axis, we can write the electric field operator as:

a) $\hat{E}_x(z,t) = E_0 (\hat{a} + \hat{a}^\dagger) \sin kz$

$$\rightarrow \langle \hat{E}_x \rangle = \langle n | \hat{E}_x | n \rangle$$

$$= E_0 \sin kz \left\{ \langle n | \hat{a} | n \rangle + \langle n | \hat{a}^\dagger | n \rangle \right\}$$

$$= E_0 \sin kz \left\{ \sqrt{n} \langle n | n-1 \rangle + \sqrt{n+1} \langle n | n+1 \rangle \right\}$$

$$\Rightarrow \langle \hat{E}_x \rangle = 0.$$

$$\langle \hat{E}_x^2 \rangle = \langle n | \hat{E}_x^2 | n \rangle$$

$$= E_0^2 \sin^2 kz \langle n | \hat{a}^{\dagger 2} + \hat{a}^2 + \hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger | n \rangle$$

$$= E_0^2 \sin^2 kz \langle n | \hat{a}^{\dagger 2} + \hat{a}^2 + 2\hat{a}^\dagger \hat{a} + 1 | n \rangle$$

$$= E_0^2 \sin^2 kz \,(2n+1) \Rightarrow$$

$$\langle \hat{E}_x^2 \rangle = 2E_0^2 \sin^2 kz \,(n + 1/2)$$

$$\sigma_{E_x} = \sqrt{\langle \hat{E}_x^2 \rangle - \langle \hat{E}_x \rangle^2} = \sqrt{2}\, E_0 \sin kz \sqrt{(n+1/2)}$$

$$\sigma_{E_x} = E_0 \sin kz \sqrt{2n+1}$$

b)

$$[\hat{N},\hat{E}_x] = \hat{N}\hat{E}_x - \hat{E}_x\hat{N} = E_0 \sin kz\left[\hat{N}(\hat{a}^\dagger+\hat{a}) - (\hat{a}^\dagger+\hat{a})\hat{N}\right]$$

$$= E_0 \sin kz\left[\hat{a}^\dagger\hat{a}(\hat{a}^\dagger+\hat{a}) - (\hat{a}^\dagger+\hat{a})\hat{a}^\dagger\hat{a}\right]$$

$$= E_0 \sin kz\left[\hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{a}^\dagger(\hat{a})^2 - (\hat{a}^\dagger)^2\hat{a} - \hat{a}\hat{a}^\dagger\hat{a}\right]$$

$$= E_0 \sin kz\left\{\hat{a}^\dagger\underbrace{(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a})}_{1} + \underbrace{(\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger)}_{-1}\hat{a}\right\}$$

$$\boxed{[\hat{N},\hat{E}_x] = E_0 \sin kz\,(\hat{a}^\dagger - \hat{a})}$$

c) Back to lecture 5:

$$\boxed{\sigma_N\,\sigma_{E_x} \geq \frac{1}{2}E_0\,|\sin kz|\,|\langle\hat{a}^\dagger - \hat{a}\rangle|}$$

Remember that if $[\hat{A},\hat{B}]=\hat{C} \rightarrow \sigma_A\sigma_B \geq \frac{1}{2}|\langle\hat{C}\rangle|$

Problem 2)

a) $[\hat{a}, \hat{a}^\dagger] = 1 \Rightarrow \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1 \Longrightarrow$

$\hat{a}^\dagger \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^\dagger\hat{a} = \hat{a}^\dagger \longrightarrow \hat{N}\hat{a}^\dagger - \hat{a}^\dagger\hat{N} = \hat{a}^\dagger$

$\Rightarrow [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$ ✓.

$\hat{a}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{a} = \hat{a} \Rightarrow \hat{a}\hat{N} - \hat{N}\hat{a} = \hat{a}$

$[\hat{a}, \hat{N}] = -\hat{a}$ ✓.

b) $[\hat{N}, \hat{H}] = [\hat{N}, \hbar\omega_0 \hat{N} - \frac{k}{2}\hat{a}^\dagger\hat{N}\hat{a}]$

$[\hat{N}, \hbar\omega\hat{N}] - \frac{k}{2}[\hat{N}, \hat{a}^\dagger\hat{N}\hat{a}] = 0 - \frac{k}{2}[\hat{N}, \hat{a}^\dagger\hat{N}\hat{a}]$

We need to calculate $[\hat{N}, \hat{a}^\dagger\hat{N}\hat{a}] = 0$

$[\hat{N}, \hat{a}^\dagger\hat{N}\hat{a}] = \hat{N}\hat{a}^\dagger\hat{N}\hat{a} - \hat{a}^\dagger\hat{N}\hat{a}\hat{N}$

From $[\hat{a}^\dagger, \hat{N}] = \hat{a}^\dagger \rightarrow \hat{a}^\dagger\hat{N} - \hat{N}\hat{a}^\dagger = \hat{a}^\dagger \Longrightarrow$

$\hat{a}^\dagger N \hat{N}\hat{a} - \underline{\hat{N}\hat{a}^\dagger\hat{N}\hat{a}} = \hat{a}^\dagger\hat{N}\hat{a}$ (a)

From $[\hat{a}, \hat{N}] = \hat{a}$

$\underline{\hat{a}^\dagger\hat{N}\hat{a}\hat{N}} - \hat{a}^\dagger\hat{N}\hat{N}a = \hat{a}^\dagger\hat{N}\hat{a}$ (b)

$(-a) - b \Rightarrow \hat{N}\hat{a}^\dagger\hat{N}\hat{a} - \hat{a}^\dagger\hat{N}\hat{a}\hat{N} = 0 \Longrightarrow$

$$[\hat{N}, \hat{a}^\dagger \hat{N} \hat{a}] = 0 \Rightarrow [\hat{N}, \hat{H}] = 0. \checkmark.$$

c) Since $\hat{H}$ & $\hat{N}$ commute, then they share the same eigenstates & eigenvalues, thus:

$$\hat{H}|n\rangle = \left[\hbar\omega_0 \hat{N} - \frac{\hbar k}{2} \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a}\right]|n\rangle$$

$$= \hbar\omega_0 \hat{N}|n\rangle - \frac{\hbar k}{2} \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a}|n\rangle$$

$$= \hbar\omega_0 n|n\rangle - \frac{\hbar k}{2} \hat{a}^\dagger \hat{a}^\dagger \hat{a}\sqrt{n}\,|n-1\rangle$$

$$= \hbar\omega_0 n|n\rangle - \hbar\frac{k}{2}\sqrt{n}\,\hat{a}^\dagger \hat{a}^\dagger \hat{a}|n-1\rangle \rightarrow \sqrt{n-1}\,|n-2\rangle$$

$$= n\hbar\omega_0|n\rangle - \frac{\hbar k}{2}\sqrt{n(n-1)}\,\hat{a}^\dagger \hat{a}^\dagger |n-2\rangle$$

$$= n\hbar\omega_0|n\rangle - \frac{\hbar k}{2}\sqrt{n}\,(n-1)\,\hat{a}^\dagger|n-1\rangle$$

$$= n\hbar\omega_0|n\rangle - \frac{\hbar k}{2}\, n(n-1)\,|n\rangle$$

$$\boxed{\hat{H}|n\rangle = \left\{n\hbar\omega_0 - \frac{\hbar k}{2}\, n(n-1)\right\}|n\rangle}$$

Eigenstates are $|n\rangle$ with eigenvalues

$n\hbar\omega_0 - \hbar\frac{k}{2}n(n-1)$.

d) $i\hbar \frac{d}{dt} \hat{N}(t) = [\hat{N}(t), \hat{H}(t)] = 0 \Rightarrow$

$\hat{N}(t) = \hat{N}$. Conservation of photon number!

e) $i\hbar \frac{d}{dt} \hat{a}(t) = [\hat{a}(t), \hat{H}(t)]$

$i\hbar \frac{d}{dt} \hat{a}(t) = \hbar\omega_0 \hat{a}(t) - \hbar k \hat{a}^\dagger \hat{a} \hat{a}$

$\frac{d}{dt} \hat{a}(t) = -i\omega_0 \hat{a} + ik \hat{N}(t) \hat{a}(t) \Rightarrow$

$\frac{d\hat{a}}{\hat{a}} = (-i\omega + ik\hat{N}) dt \Rightarrow$

$$\hat{a}(t) = \hat{a} \exp[-i\omega_0 t + ikt\hat{N}]$$

f) $\hat{N} = \hat{a}^\dagger(t) \hat{a}(t) \Rightarrow$

$$\hat{a}^\dagger(t) = \hat{a}^\dagger \exp[i\omega_0 t - ikt\hat{N}]$$

g) Energy lost from the cavity is

$n\hbar\omega_0 - \frac{\hbar k}{2} n(n-1) - \left[(n-1)\hbar\omega_0 - \frac{\hbar k}{2}(n-1)(n-2)\right]$

$= \hbar\omega_0 - \hbar k(n-1) = \hbar(\omega_0 - k(n-1))$

The frequency of the photon is $\omega_0 - k(n-1)$

& measurement probability is 1.

A coherent state is a superposition of number states with probability $\frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$.
Spectrometer measures $\omega_0 - k(n-1)$ with probability $\frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$ for $n = 1, 2, \ldots$. For a single photon will be $|\alpha|^2 e^{-|\alpha|^2}$. ✓.

Problem 3)

$$|\psi\rangle = A\left[|\alpha\rangle + |-\alpha\rangle\right]$$

a)

$$\langle\psi|\psi\rangle = 1 \Rightarrow 1 = |A|^2\left[\langle\alpha|\alpha\rangle + \langle-\alpha|-\alpha\rangle + \langle-\alpha|\alpha\rangle + \langle\alpha|-\alpha\rangle\right]$$

$$|A|^2\left[2 + 2e^{-2|\alpha|^2}\right] = 1 \rightarrow \boxed{|A| = \frac{1}{\sqrt{2+2e^{-2|\alpha|^2}}}}$$

Remember $\langle\alpha|\beta\rangle = \exp\left(\frac{-1}{2}|\alpha-\beta|^2\right)$ for coherent states.

b) if $\alpha \to \infty \Rightarrow A = \frac{1}{\sqrt{2}} \Rightarrow \boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\alpha\rangle + |-\alpha\rangle\right]}$

c) Since $\langle n|\alpha\rangle = \frac{|\alpha|^n}{\sqrt{n!}} e^{-|\alpha|^2/2} \Rightarrow$

$$\langle n|\psi\rangle = \frac{1}{\sqrt{2}} \frac{|\alpha|^n}{\sqrt{n!}} \left(1 + (-1)^n\right) e^{-|\alpha|^2/2} \Rightarrow$$

$$P_n = \begin{cases} e^{-|\alpha|^2/2} \dfrac{|\alpha|^{2n}}{n!} & n \text{ is even.} \\ 0 & n \text{ is odd.} \end{cases}$$

d)

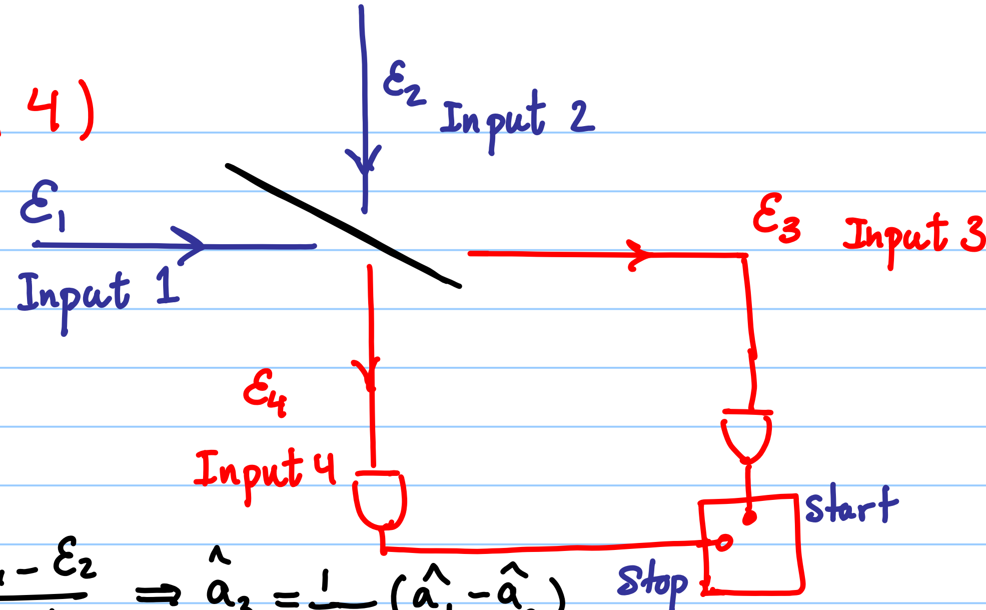
$$\hat{\rho} = |\psi\rangle\langle\psi| = \frac{1}{2}\Big(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle\alpha| + |\alpha\rangle\langle-\alpha| + |-\alpha\rangle\langle-\alpha| \Big)$$

e) $$Q(\mu) = \frac{1}{2\pi}\langle\mu|\rho|\mu\rangle$$

$$= \frac{1}{4\pi}\left|\langle\mu|\alpha\rangle + \langle\mu|-\alpha\rangle\right|^2$$

$$\boxed{Q(\mu) = \frac{1}{4\pi} e^{-|\alpha|^2 - |\mu|^2}\left|e^{\mu^*\alpha} + e^{-\mu^*\alpha}\right|^2}$$

Problem 4) (20)



a)

$$\mathcal{E}_3 = \frac{\mathcal{E}_1 - \mathcal{E}_2}{\sqrt{2}} \Rightarrow \hat{a}_3 = \frac{1}{\sqrt{2}}(\hat{a}_1 - \hat{a}_2)$$

$$\mathcal{E}_4 = \frac{\mathcal{E}_1 + \mathcal{E}_2}{\sqrt{2}} \Rightarrow \hat{a}_4 = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2)$$

Input State to HBT is $|\Psi\rangle = |\psi_1, 0_2\rangle = |\psi_1\rangle|0\rangle_2$

where $|\psi_1\rangle$ is an arbitrary input state to port 1

and $|0\rangle_2$ is vacuum state input to port 2.

Therefore $$g^{(2)}(0) = \frac{\langle \hat{N}_3(t)\hat{N}_4(t)\rangle}{\langle \hat{N}_3(t)\rangle\langle \hat{N}_4(t)\rangle}$$

$$g^{(2)}(0) = \frac{\langle \hat{a}_3^\dagger \hat{a}_3 \hat{a}_4^\dagger \hat{a}\rangle}{\langle \hat{a}_3^\dagger \hat{a}_3\rangle\langle \hat{a}_4^\dagger \hat{a}_4\rangle}$$

We need to calculate three terms, so

$$\langle \hat{a}_3^\dagger \hat{a}_3\rangle = \langle \Psi | \hat{a}_3^\dagger \hat{a}_3 | \Psi\rangle =$$

$$= \langle \psi_1 | \langle 0_2 | \hat{a}_3^\dagger \hat{a}_3 | \psi_1\rangle | 0_2\rangle$$

$$= \langle \psi_1 | \langle 0_2 | \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 | \psi_1 \rangle | 0 \rangle_2$$

$$= \langle \psi_1 | \hat{a}_1^\dagger \hat{a}_1 | \psi_1 \rangle$$

$$\langle \hat{a}_3^\dagger \hat{a}_3 \rangle = \langle \psi_1 | \hat{N}_1 | \psi_1 \rangle$$

Note $\hat{a}_2 | 0_2 \rangle = 0$

like wise

$$\langle \hat{a}_4^\dagger \hat{a}_4 \rangle = \langle \psi_1 | \hat{N}_1 | \psi_1 \rangle$$

The numerator:

$$\langle \hat{a}_3^\dagger \hat{a}_3 \hat{a}_4^\dagger \hat{a}_4 \rangle = \langle \Psi | \hat{a}_3^\dagger \hat{a}_3 \hat{a}_4^\dagger \hat{a}_4 | \Psi \rangle$$

$$= \langle \Psi | (\hat{a}_1^\dagger - \hat{a}_2^\dagger)(\hat{a}_1 - \hat{a}_2)(\hat{a}_1^\dagger + \hat{a}_2^\dagger)(\hat{a}_1 + \hat{a}_2) | \Psi \rangle$$

This has 16 terms but most of them are zero.

$$\langle \hat{a}_3^\dagger \hat{a}_3 \hat{a}_4^\dagger \hat{a}_4 \rangle = \langle \psi_1 | \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 | \psi_1 \rangle$$

$$= \frac{1}{4} \langle \psi_1 | \hat{N}_1 (\hat{N}_1 - 1) | \psi_1 \rangle$$

$$g^{(2)}(0) = \frac{\langle \hat{N} (\hat{N} - 1) \rangle}{\langle \hat{N} \rangle^2}$$

b) if $\begin{cases} \hat{N} | N \rangle = n | N \rangle \\ | \psi_1 \rangle = | N \rangle \end{cases} \Rightarrow g^{(2)}(0) = 1 - \frac{1}{n}$