Tue

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$\boxed{\text{Lectures}}$: EIT 3141, Tuesdays 8:30-11:20 am

Part 1: 8:30-9:45 am

Part 2: 10:00-11:20 am

Office Hours: Th 3-4 pm, QNC 4404.

Problem Sets 4-5, 50% . $S(t) = S(0) \times (10-t) \times 0.1$ if $t < t_s$, 0 if $t > t_s$

Final Exam. 50%

$\boxed{\text{key words}}$ syllabus.

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- Noise ?

What is Noise ?

Is Noise Bad or Good ?

What are the synonyms of "Noise" ?

- randomness deviations
- error
- fluctuations ...

$\boxed{\text{Noise}}$ ≡ Spontaneous fluctuations in currents, voltages and other physical quantities of a system under test.

→ "limits" an ultimate sensitivity in any measurement

In 1998, R. Landauer "The noise is the signal." Nature 392, 658 (1998)

- Why do we have "noise" in a system data?

- Noise : "statistical" quantity

⇒ does not make sense to argue a single event at a certain time

- Precision vs Accuracy

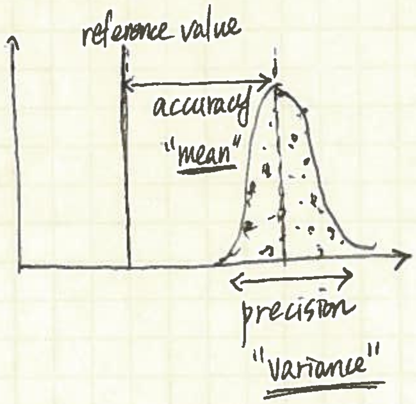
precision = a description of "random errors"
a measure of statistical variability

the degree to which repeated measurements under unchanged condition to show the same results
"reproducibility", "repeatability"

vs

accuracy = a description of "systematic errors"
a measure of statistical bias

the degree of closeness of measurements of a quantity



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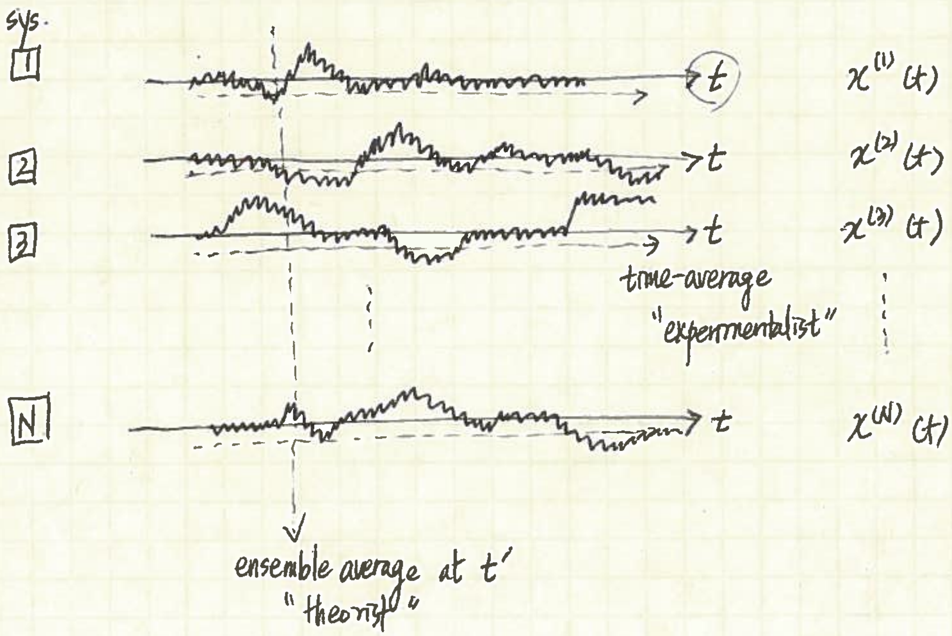
Two kinds of Statistics



① Time-average (="experimentalist's statistics")

$$\overline{x^{(i)}(t)} = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^{(i)}(t)\,dt \qquad \text{mean} = \text{1st-order time-average}$$

$$\overline{x^{(i)}(t)^2} = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} [x^{(i)}(t)]^2\,dt \qquad \text{mean square} = \text{2nd-order time-average}$$

$$\phi_x^{(i)}(\tau) \equiv \overline{\underset{\substack{\uparrow \\ \text{original} \\ \text{noise wavefn}}}{x^{(i)}(t)}\ \underset{\substack{\uparrow \\ \text{delayed} \\ \text{noise wavefn}}}{x^{(i)}(t+\tau)}} = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^{(i)}(t)\,x^{(i)}(t+\tau)\,dt$$

auto-correlation function

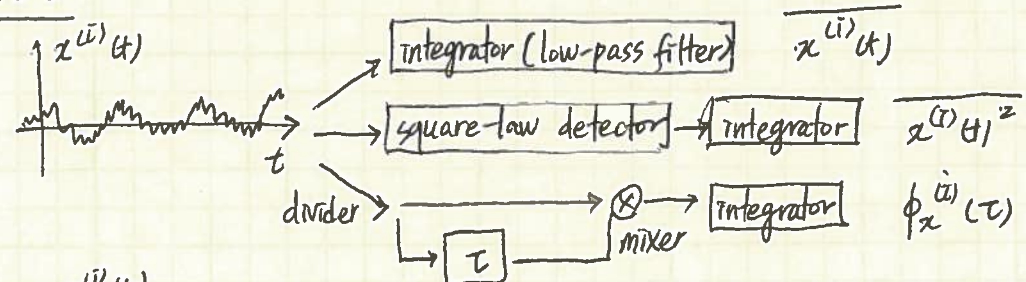
auto-correlation fn = time average of the original and delayed noise wavefn.

$\boxed{\text{when } \tau=0 \Rightarrow \text{"mean square"}}$

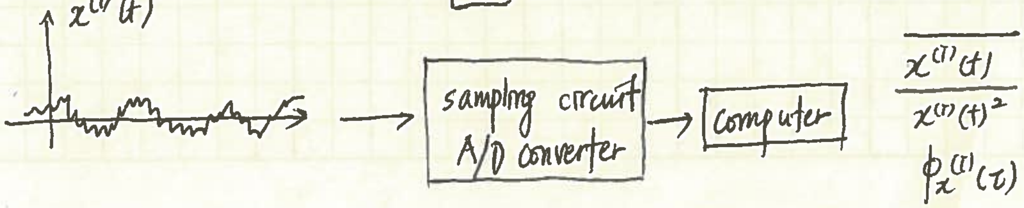
= the correlation btw values of the process at different times of "τ"

How to measure?

① Analog



② Digital



② Ensemble-average (= "theorist's statistics) "given t"

$$\langle x(t_1)\rangle = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} x^{(i)}(t_1) = \int_{-\infty}^{\infty} x_1\, p_1(x_1,t_1)\,dx_1$$ mean = 1st-order ensemble average

↑ probability density fn.

$$\langle x(t_1)^2\rangle = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\left[x^{(i)}(t_1)\right]^2 = \int_{-\infty}^{\infty} x_1^2\, p_1(x_1,t_1)\,dx_1$$ mean square = 2nd-order ensemble average

$$\langle x(t_1)x(t_2)\rangle = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} x^{(i)}(t_1)\,x^{(i)}(t_2) = \int_{-\infty}^{\infty} x_1 x_2\, p_2(x_1,x_2;t_1,t_2)\,dx_1\,dx_2$$

covariance = a measure of how much 2 random variables change together

cf) correlation coeff. = normalized version of the correlation

- $p_1(x_1,t_1)$ = first-order probability density function

  $p_1(x_1,t_1)\,dx_1$: probability that x is found in the range between x_1 and x_1+dx_1 at t_1

- $p_2(x_1,x_2;t_1,t_2)$ = second-order joint probability density function

  $p_2(x_1,x_2;t_1,t_2)\,dx_1 dx_2$: probability that x is found in the range btw x_1 and x_1+dx_1 at t_1 and x is found in the range btw x_2 and x_2+dx_2 at t_2

(Theory) $\boxed{\text{Fokker-Planck equation}}$: calculation of probability distributions

⇒ obtain the probability density functions p_1 and p_2.

Q: How can we close the gap between "Time-Average" and "Ensemble-Average"?

When is "theoretical predictions from ensemble averaging" equivalent to "experimental results" from time averaging?

Need (Introduce) "Ergodicity" & "Statistical stationarity"

→ equivalent when and only when the system is a so-called "ergodic ensemble"

ensemble averaging and time averaging are identical for a statistically-stationary system

different for a statistically-nonstationary system

Introduce "Ergodicity" & "statistical stationarity"

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$\boxed{\text{Ergodicity}}$

in mathematics, a dynamical system has the same behavior averaged over time as averaged over the space of all system's state (phase space)

in statistics, a random process for which the time average of one sequence of events is the same as the ensemble average.

When do we need "ergodicity"?

we have only 1 sample ftn of a stochastic process instead of the entire ensemble

if the process is ergodic,
even w/ 1 sample ftn, all statistical information can be derived from time averaging

① Ergodic in the mean

$$\overline{x(t)} = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt = \langle x(t)\rangle$$

ensemble-average using the "1st-order" PDF $P_1(x,t)$

calculation w/ a particular member $x(t)$ & time averaging

$\cancel{\text{time}}$ t-independent

$\therefore$ constant

$\therefore$ ergodicity of mean implies the stationarity of the mean

(vice versa doesn't hold the truth.)

② Ergodic in the autocorrelation

$$\phi_x(\tau) = \overline{x(t)x(t+\tau)} = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,x(t+\tau)\,dt = \langle x(t)\,x(t+\tau)\rangle$$

calculation w/ $x(t)$

using the 2nd-order PDF.

HW Q

Consider the example of the processes which are ergodic in the mean,
the autocorrelation
or/and
both.

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statistically-stationary process vs. statistically-nonstationary process.

$\langle \overline{x(t_1)} \rangle$, $\langle x(t_1)^2 \rangle$: independent of the time t_1

left. not true

$\langle x(t_1)\, x(t_2) \rangle$: independent of t_1 & t_2
but dependent on $\tau = |t_2 - t_1|$

"concept of ensemble average" is valid,
but not time average.

no preferred origin in time

$\boxed{\text{Wiener-Khintchine theorem}}$

A stochastic process is wide-sense stationary
if its mean value is constant
and its autocorrelation function depends only on $\tau = t_2 - t_1$
time difference.

← "revisit-later"

Example statistically-stationary process

suppose select a battery at random & measure Voltage

$v(t)$ = a member ftn selected from a certain sub-group of constant battery voltage

$\langle v_{AAA}(t) \rangle \approx 1.5\,V$ indep. of time t_1

$\langle v_{AAA}^2(t) \rangle \approx$ constant

$\langle v_{AAA}(t_1)\, v_{AAA}(t_2) \rangle \approx$ constant.

∴ statistically stationary

Is this ergodic? No, why?

Example statistically-stationary & ergodic?

$x(t) = \sin(\omega t + \theta)$, $0 \le \theta \le 2\pi$, $\boxed{P(\theta) = \frac{1}{2\pi}}$ ergodic in the mean?
↑ random variable,

Proof $\overline{x(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \sin(\omega t + \theta)\, dt \approx 0$,
time-average

$\langle x(t) \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x^{(i)}(t)$
$= \int_{-\infty}^{\infty} x(t) p(\theta)\, d\theta$ $\sin(\omega t + \theta_1) + \cdots$
$= \int \sin(\omega t + \theta) \frac{1}{2\pi} \approx 0.$

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$\boxed{\text{Fourier Analysis}}$. time-domain vs frequency-domain

When $x(t)$ is absolutely integrable, i.e., $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$,

the Fourier transform of $x(t)$ is defined by, $X(i\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$,

the inverse Fourier transform is $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(i\omega)\, e^{i\omega t} d\omega$

(pf) $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(i\omega) e^{i\omega t} d\omega$
$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t} \int_{-\infty}^{\infty} x(t') e^{-i\omega t'} dt'$
$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, x(t') e^{i\omega(t-t')} dt'$
$= \int_{-\infty}^{\infty} x(t')\, \delta(t'-t)\, dt'$
$= x(t)$

if $x(t)$ is real,
$\mathrm{Re}(X(i\omega))$ = even fn of ω
$\mathrm{Im}(X(i\omega))$ = odd fn of ω.
$X(i\omega) = X^*(-i\omega)$

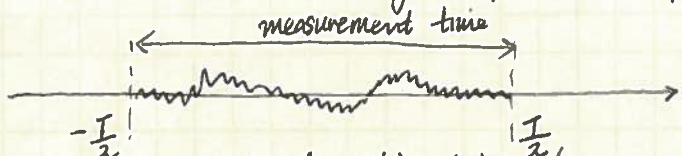
- Does the Fourier transform of a statistically-stationary process exist?

No. Why? $\int_{-\infty}^{\infty} |x(t)|\, dt$ is not absolutely integrable

$\therefore$ No Fourier Transform exists

How to overcome this problem?

Since the measurement time is finite,
we can introduce "the gated function" $x_T(t)$



$$x_T(t) = \begin{cases} x(t) & |t| \le T/2 \\ 0 & |t| > T/2 \end{cases} \xrightarrow[\text{Transform}]{\text{Fourier}} X_T(i\omega) = \int_{-\infty}^{\infty} x_T(t) e^{-i\omega t} dt$$

$$x_T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_T(i\omega) e^{i\omega t} d\omega$$

Now $\int_{-\infty}^{\infty} |x_T(t)| < \infty$, absolutely integrable

$\therefore$ the Fourier transform of the $x_T(t)$ exists.

Lists of Theorems

① Parseval Theorem

② ~~Wiener~~ Energy Theorem

③ Wiener-Khintchine Theorem

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1. Parseval Theorem

$$x_1(t) \longleftrightarrow X_1(i\omega)$$
$$x_2(t) \longleftrightarrow X_2(i\omega)$$

~~consider the "auto correlation" ftn~~

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \int_{-\infty}^{\infty} dt\, x_1(t)\, \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, X_2(i\omega)^* e^{-i\omega t}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, X_2(i\omega)^* \boxed{\int_{-\infty}^{\infty} dt\, x_1(t)\, e^{-i\omega t}}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(i\omega)\, X_2(i\omega)^*\, d\omega$$

consider the "auto correlation" ftn

$$x_1(t) = x_T(t+\tau) \qquad x_2(t) = x_T(t)$$

$$\int_{-\infty}^{\infty} x_T(t+\tau)\, x_T(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(i\omega)|^2 e^{i\omega\tau}\, d\omega$$

$$\int_{-\infty}^{\infty} x_T(t+\tau)\, e^{-i\omega t}\, dt = X_T(i\omega)\, e^{i\omega\tau}.$$

$$\int_{-\infty}^{\infty} x_T(t+\tau)\, \frac{1}{2\pi}\int_{-\infty}^{\infty} X(i\omega) e^{i\omega \ldots}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(i\omega) \int_{-\infty}^{\infty} x_T(t+\tau) e^{i\omega \ldots}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_T(\tau') e^{i\omega(t' \ldots}$$

2. Energy theorem

If $\tau = 0$, $\boxed{\int_{-\infty}^{\infty} |x_T(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(i\omega)|^2\, d\omega}$ = the total energy

$X_T(i\omega)$ = the (complex) amplitude of the harmonic component $(e^{i\omega t})$ in a gated ftn $x_T(t)$

$|X_T(i\omega)|^2$ = the energy density of this harmonic component with units of energy per Hz

$\int_{-\infty}^{\infty} |x_T(t)|^2\, dt$ = the total energy.

= increases linearly with T for a statistically-stationary process

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3. Wiener-Khintchine theorem

Consider the ensemble-averaged autocorrelation ftn.

① statistically-stationary process

$$\lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\langle x_T(t+\tau)\,x_T(t)\rangle\,dt = \lim_{T\to\infty}\frac{1}{2\pi}\int_0^{\infty}\frac{2\langle|X_T(i\omega)|^2\rangle}{T}\cos(\omega\tau)\,d\omega$$

: statis

② statistically-nonstationary process

$$\frac{1}{T}\int_{-\infty}^{\infty}\langle x_T(t+\tau)\,x_T(t)\rangle\,dt = \frac{1}{2\pi}\int_0^{\infty}\frac{2\langle|X_T(i\omega)|^2\rangle}{T}\cos(\omega\tau)\,d\omega$$

Wiener-Khintchine theorem : the autocorrelation function of a wide-sense stationary random process has a spectral decomposition given by the power spectrum of that process.

stationary

$$\langle\phi_x(\tau)\rangle = \frac{1}{2\pi}\int_0^{\infty}S_x(\omega)\cos(\omega\tau)\,d\omega$$

$$S_x(\omega) = 4\int_0^{\infty}\langle\phi_x(\tau)\rangle\cos(\omega\tau)\,d\tau$$

HW proof

$$S_x(\omega) = \lim_{T\to\infty}\frac{2\langle|X_T(i\omega)|^2\rangle}{T}$$

non-stationary

$$\langle\phi_x(\tau,T)\rangle = \frac{1}{2\pi}\int_0^{\infty}S_x(\omega,T)\cos(\omega\tau)\,d\omega$$

$$S_x(\omega,T) = 4\int_0^{T}\langle\phi_x(\tau,T)\rangle\cos(\omega\tau)\,d\tau$$

Fourier Transform pairs

stationary

$$\langle 2\phi_x(\tau)\rangle \longleftrightarrow S_x(\omega)$$

non-stationary

$$\langle 2\phi_x(\tau,T)\rangle \longleftrightarrow S_x(\omega,T)$$