1 Compression of Quantum Channels

The channel fidelity of a quantum channel is given by $F_{channel}(\Xi, \sigma) = \sqrt{\sum\limits_{k=1}^{N} |\langle A_k, \sigma\rangle|^2}$, where $\Xi(X) = \sum\limits_{k=1}^{N} A_k X A_k^*$.

(Φ, Ψ) is an (n, α, δ)-quantum compression scheme if $F_{channel}(\Psi\Phi, \rho^{\otimes n}) > 1 - \delta$.

[Schumacher's Theorem] Let

$$\rho \in D((C)^{\Sigma}), \quad \alpha \geq 0, \delta > 0$$

1. If $\alpha > H(\rho)$ then there exists an (n, α, δ)-compression scheme for ρ for all but finitely many $n \in \mathbb{N}$.
2. If $\alpha < H(\rho)$ then there exists an (n, α, δ)-compression scheme for ρ for at most finitely many $n \in \mathbb{N}$.

Proof sketch: Start with a spectral decomposition

$$\rho = \sum_{a \in \Sigma} p(a) x_a x_a^*$$

for some $p \in P(\Sigma)$ and $\{x_a : a \in \Sigma\}$ an orthonormal basis of X.

Remember that the typical sequences of length n are : $T_{n,\epsilon}(p) = \{a_1 \dots a_n \in \Sigma^n : 2^{-n(H(\rho)+\epsilon)} < p(a_1) \dots p(a_n) < 2^{-n(H(p)-\epsilon)}\}$

In a similar way we define a typical projector $\Pi_{n,\epsilon} = \sum_{a_1 \dots a_n \in T_{n,\epsilon}} x_{a1} x_{a1}^* \otimes \dots \otimes x_{an} x_{an}^*$.

Note that $\langle \Pi_{n,\epsilon}, \rho^{\otimes n}\rangle = Pr[a_1 \dots a_n \in T_{n,\epsilon} \xrightarrow{\text{(as n}\to\infty)} 1$, where each $a \in \Sigma$ independently chosen according to p.

Now, we define an isometry $A = \sum\limits_{a_1 \cdots a_n \in T_{n,\epsilon}} e_{f(a_1 \cdots a_n)} x^*_{a_1 \cdots a_n}$. $f : \Sigma^n \to \Sigma^m$ is an encoding function from a classical compression scheme.

$A \in L(\mathcal{X}_1 \otimes \dots \otimes \mathcal{X}_n, \mathcal{Y}_1 \otimes \dots \otimes \mathcal{Y}_n)$.

If we consider $\Phi(X) = AXA^*$ we don't have a channel because it doesn't necessarily preserve trace. But, we can make it a channel by modifying it a bit $\Phi(X) = AXA^* + \langle \mathbb{1} - \Pi_{n,\epsilon}, X\rangle\sigma$ for any $\sigma \in D(\mathcal{Y}_1 \otimes \dots \otimes \mathcal{Y}_n)$. If this is a channel then it should be the case that $Tr(\Phi(X)) = Tr(X) = Tr(A^*AX) + \langle \mathbb{1} - \Pi_{n,\epsilon}, X\rangle$. But, $A^*A = \Pi_{n,\epsilon}$ so $Tr(\Phi(X)) = Tr(X)$.

We could propose some $\Psi'(Y) = A^*YA$ to undo what Φ does. However, this may not be a channel so we should add another correction term: $\Psi = A^*YA + \langle \mathbb{1} - AA^*, Y\rangle\xi$ for any state $\xi \in D(\mathcal{X}_1 \otimes \dots \otimes \mathcal{X}_n)$. We should check to make sure that this actually undoes Ψ. We want to show that $F_{channel}(\Psi\Phi, \rho^{\otimes n})$. So let's consider what $(\Psi\Phi)(X)$ is.

If I knew that $\Phi(X) = \sum_k A_k X A_k^*$ (Kraus rep) and $\Phi(Y) = \sum_l B_l Y B_l^*$ then $(\Psi\Phi(X) = \sum_{k,l} (B_l A_k) X (B_l A_k)^*$.

So, $(\Phi\Psi)(X) = (A^*A)X(AA^*)^* + \cdots$ more things like $B_k X B_k^*$.

Now,

$$\begin{aligned} F_{channel}(\Psi\Phi, \rho^{\otimes n}) &= \sqrt{|\langle A^*A, \rho^{\otimes n}\rangle|^2 + \text{non-negative terms}} \\ &\geq \langle A^*A, \rho^{\otimes n}\rangle \\ &= \langle \Pi_{n,\epsilon}, \rho^{\otimes n}\rangle \to 1 \end{aligned}$$

So, the compression is good and this works!

2 Quantum Entropy

2.1 Properties of the Von-Neumann entropy

1. The Shannon entropy is continuous : $H(p) = -\sum_{a\in\Sigma} p(a)\log(p(a))$. The von Neumann entropy, $H(\rho) = H(\lambda(\rho))$, equals the entropy of a vector of eigenvalues. Since eigenvalues are continuous so the von Neumann entropy is continuous (you can understand this by considering constructing an arbitrary characteristic polynomial to obtain the eigenvalues. Since the von Neumann entropy is the composition of two continuous functions then it is continuous. It's easy to understand this when applied to hermitian operators whose eigenvalues can be ordered ($||\lambda(A)-\lambda(B)||_1 \leq ||A-B||_1$).

2.

[Fannes-Audenaert] Consider two density operators ρ_0, ρ_1 and $\delta = .5||\rho_0 - \rho_1||_1$. You can conclude that $|H(\rho_0) - H(\rho_1)| \leq \delta\log(n-1) + H(\delta, 1-\delta)$. This puts an upper bound on the difference in entropy when you perturb a density operator a little bit.

2.2 Quantum relative entropy

$D(P||Q) = Tr\left(P\log(P) - P\log(Q)\right)$ for $P, Q \in Pos(\mathcal{X})$. If $ker(Q) \subseteq ker(P)$ then this is well defined and I can find $D(P||Q)$. However, when this is not the case we allow $D(P||Q) = \infty$.

One way to think about this function is as one that tells you how much compression "loss" you have if you design a compression scheme for some ρ and you used that on some σ.

[Klein's inequality] Imagine you have $\rho, \sigma \in D(\mathcal{X})$ then $D(\rho||\sigma) \geq 0$ with equality if and only if $\rho = \sigma$.

Proof: Let's assume (without loss of generality) that $\rho, \sigma > 0$ (positive definite). Let's assume that we have spectral decompositions of ρ and σ.

$$\rho = \sum_{a\in\Sigma} p(a) u_a u_a^*$$

$$\sigma = \sum_{a\in\Sigma} q(a) v_a v_a^*$$

. Now,

$$D(\rho||\sigma) = \frac{1}{\ln(2)} \sum_{a,b} |\langle u_a, v_b \rangle|^2 (p(a)ln(a) - p(a)ln(b)$$

It can be shown that $ln(x) < x - 1$ (draw a 2d plot

$$= -p\ln(q/p) \geq -p(\frac{q}{p} - 1) = p - q$$
$$D(\rho||\sigma) \geq \frac{1}{\ln(2)} \sum_{a,b} |\langle u_a, v_b \rangle|^2 (p(a) - q(b)) = 0$$