$$e^{i<\phi(t)>} = e^{-\frac{1}{2}\int_0^\tau dt_1 \int_0^\tau dt_2 \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} S(\omega) d\omega}$$

Now, for this problem, considering the pulse sequence, I can break the integrals over time into the sum of two integrals. The first integral is over the first free-precession period and the second integral is over the second free-precession period.

Allow the time origin to be placed at the beginning of the first free-precession period so that the integral from 0 to τ becomes a sum that resembles the following:

$$e^{-\frac{1}{2}(\int_0^{\tau_1} dt_1 + \int_{\tau_1+t_\pi}^{\tau_1+t_\pi+\tau_2} dt_1)(\int_0^{\tau_1} dt_2 + \int_{\tau_1+t_\pi}^{\tau_1+t_\pi+\tau_2} dt_2)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} S(\omega) d\omega}$$

Pulling my time integrals inside the frequency integrals yields the following:

$$e^{-\frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty}(\int_0^{\tau_1} e^{-i\omega t} dt + \int_{\tau_1+t_\pi}^{\tau_1+t_\pi+\tau_2} e^{-i\omega t} dt)^2 S(\omega) d\omega}$$

The "square" comes from the understanding that the two time integrals will yield the same quantity when integrated from 0 to τ. Thus, the result of performing the integral twice will just be to square the integral evaluated once. Performing the time integral results in the following :

$$e^{-\frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty}(\int_0^{\tau_1} e^{-i\omega t} dt + \int_{\tau_1+t_\pi}^{\tau_1+t_\pi+\tau_2} e^{-i\omega t} dt)^2 S(\omega) d\omega}$$

$$e^{-\frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty}(e^{\frac{-i\omega\tau_1}{2}}\tau_1 sinc(\frac{\omega\tau_1}{2}) + e^{-i\omega(\tau_1+t_\pi+\frac{\tau_2}{2})}\tau_2 sinc(\frac{\omega\tau_2}{2}))^2 S(\omega) d\omega}$$

$$e^{-\frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty}(e^{-i\omega\tau_1}\tau_1^2 sinc^2(\frac{\omega\tau_1}{2}) + e^{-i\omega(2\tau_1+2t_\pi+\tau_2)}\tau_2^2 sinc^2(\frac{\omega\tau_2}{2}) + e^{-i\omega(\frac{3\tau_1}{2}+t_\pi+\frac{\tau_2}{2})}\tau_1\tau_2 sinc(\frac{\omega\tau_1}{2}) sinc(\frac{\omega\tau_2}{2})) S(\omega) d\omega}$$

My questions are these: Is this logic sound (even if the math isn't expressed as formally as possible)? Also, am I allowed to shift my time coordinates for the two integrals in such a way as to avoid having all these complex exponentials float around? I want to make my filter function real so that it's easier to analyze. Am I allowed to change my first integral bounds to $-\tau_1/2$ to $\tau_1/2$ and, likewise, place the second integral's bounds in the middle of the second free precession (eliminating the imaginary exponentials)?