Quantum Electronics and Photonics

Assignment 3

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Contents

Problem 1

(a)

To construct the Hamiltonian describing time evolution of the dynamical varibales we can simply consider the total energy of the system as a constant of motion however we firmly follow a more general approach. we first try to redrive the second order differential equation which determines equation of motion based on Lagrange equation. Primary KVL and KCL equations manifest themself in defining conjugate varibales. Quite generally we should first pick up minimum number of varibles which can completely describe the working point in configuration space. We choose $\mathcal{Q} = v$ as the sole dynamical quantity. i is related to v by:

$$i = C\frac{dv}{dt} = C\dot{\mathcal{Q}} \tag{1}$$

Please note that we could also choose the cahrge of the capacitor (Q) and total magnetic flux inisde the

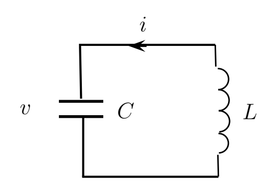

Figure 1: parallel LC network

inductor (Φ) as the dynamical variables. We define the action as the difference of magnetic and electrical energies as:

$$\mathcal{L} = \frac{1}{2}Li^2 - \frac{1}{2}Cv^2 = \frac{1}{2}C\left[\frac{1}{\omega_0^2}\dot{\mathcal{Q}}^2 - \mathcal{Q}^2\right] \tag{2}$$

where ω_0 is defined as

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Lagrange equation for this action is:

$$\frac{d}{dt}\frac{\partial\mathcal{L}(\mathcal{Q},\dot{\mathcal{Q}})}{\partial\dot{\mathcal{Q}}} - \frac{\partial\mathcal{L}(\mathcal{Q},\dot{\mathcal{Q}})}{\partial\mathcal{Q}} = 0 \quad\Longrightarrow\quad \ddot{\mathcal{Q}} = -\omega_0^2\mathcal{Q} \quad\Longrightarrow\quad \frac{d^2v}{dt^2} = -\omega_0^2 v \tag{3}$$

This equation is exactly same as the equation which can be derived based on KVL and KCL equations. Thus our Lagrangian works well and we can go ahead to cunstruct the Hamiltonian. First note that the conjugate varible corresponde to $\mathcal{Q}$ is:

$$\mathcal{P} = \frac{\partial\mathcal{L}}{\partial\dot{\mathcal{Q}}} = \frac{C\dot{\mathcal{Q}}}{\omega_0^2} = LCi \tag{4}$$

Now the Hamiltonian is:

$$\mathcal{H} = \mathcal{P}\dot{\mathcal{Q}} - \mathcal{L}(\mathcal{Q},\dot{\mathcal{Q}})\Big|_{\dot{\mathcal{Q}}=\omega_0^2\mathcal{P}/C} = \frac{\omega_0^2\mathcal{P}^2}{2C} + \frac{1}{2}C\mathcal{Q}^2 \tag{5}$$

Interestingly canonical equations of motion in phase space are:

$$\dot{\mathcal{Q}} = \frac{\partial\mathcal{H}(\mathcal{Q},\mathcal{P})}{\partial\mathcal{P}} = \frac{\omega_0^2\mathcal{P}}{C} \quad\Longrightarrow\quad C\frac{dv}{dt} = i \quad \text{(KCL)} \tag{6}$$

$$\dot{\mathcal{P}} = -\frac{\partial\mathcal{H}(\mathcal{Q},\mathcal{P})}{\partial\mathcal{Q}} = -C\mathcal{Q} \quad\Longrightarrow\quad L\frac{di}{dt} = -v \quad \text{(KVL)} \tag{7}$$

Problem 1 continued on next page. . .

(b)

The Hamiltonian describing time evolution in a one dimensional SHO is:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{8}$$

where x and p are conjugate varibles. In both (5) and (8) Hamiltonians are elipitical functions of two conjugate varibles. In fact both Hamiltonians are equal if we do the following replacements:

$$H \leftrightarrow \mathcal{H} \qquad x \leftrightarrow \mathcal{Q} \qquad p \leftrightarrow \mathcal{P} \qquad m \leftrightarrow LC^2 \qquad \omega \leftrightarrow \omega_0 \tag{9}$$

(c)

The conh=jugate varible to $\mathcal{Q} = v$ has been calculated in part (a) as we showed:

$$\mathcal{P} = \frac{\partial \mathcal{L}}{\partial \dot{\mathcal{Q}}} = \frac{C\dot{\mathcal{Q}}}{\omega_0^2} = LCi \tag{10}$$

(d)

To quantize the LC circuit we start form the Driac's quantization rules. We first take a look at fundamental Poison backets in the classical problem:

$$\{\mathcal{Q}, \mathcal{P}\} = \frac{\partial \mathcal{Q}}{\partial \mathcal{Q}}\frac{\partial \mathcal{P}}{\partial \mathcal{P}} - \frac{\partial \mathcal{P}}{\partial \mathcal{Q}}\frac{\partial \mathcal{Q}}{\partial \mathcal{P}} = 1 \tag{11}$$

According to the Dirac's quantization rule we should just replace:

$$\frac{1}{j\hbar}[\quad,\quad] \leftrightarrow \{\quad,\quad\} \tag{12}$$

Please note that we have used j as the unit imaginary number. This quantization implies that $\mathcal{Q}$ and $\mathcal{P}$ should be treated as operators. So we have:

$$\left[\hat{\mathcal{Q}}, \hat{\mathcal{P}}\right] = \left[\hat{v}, LC\hat{i}\right] = j\hbar \tag{13}$$

This commutation relation is the starting point to develop the whole *Lie algebra*. So we should have the same quantization and generally algebra in a SHO and quantum LC circuit.

(e)

As it's explained in the previous part we have the same algebra in SHO and quantum LC network. So we can simply use the result of our discussions in SHO. we should just use the substitutions given in (9). So we have:

$$a_{SHO} = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{jp}{m\omega}\right) \quad \Longrightarrow \quad \mathcal{A} = \sqrt{\frac{LC^2\omega_0}{2\hbar}}\left(\mathcal{Q} + \frac{j\mathcal{P}}{LC^2\omega_0}\right) \tag{14}$$

$$a^\dagger_{SHO} = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{jp}{m\omega}\right) \quad \Longrightarrow \quad \mathcal{A}^\dagger = \sqrt{\frac{LC^2\omega_0}{2\hbar}}\left(\mathcal{Q} - \frac{j\mathcal{P}}{LC^2\omega_0}\right) \tag{15}$$

Problem 1 continued on next page...

or

$$\mathcal{A} = \sqrt{\frac{C}{2\hbar\omega_0}} \left(\hat{v} + \frac{j\hat{i}}{C\omega_0} \right) \qquad \text{annihilation operator} \tag{16}$$

$$\mathcal{A}^\dagger = \sqrt{\frac{C}{2\hbar\omega_0}} \left(\hat{v} - \frac{j\hat{i}}{C\omega_0} \right) \qquad \text{creation operator} \tag{17}$$

The Hamiltonian can be rewriten in terms of creation and annihilation operators as:

$$\mathcal{H} = \hbar\omega_0 \left(\mathcal{A}^\dagger \mathcal{A} + \frac{1}{2} \right) \tag{18}$$

Numbering operator is defined as:

$$\mathcal{N} = \mathcal{A}^\dagger \mathcal{A} \tag{19}$$

(f)

from (16) and (17)

$$\hat{v} = \sqrt{\frac{\hbar\omega_0}{2C}} \left(\mathcal{A} + \mathcal{A}^\dagger \right) \tag{20}$$

$$\hat{i} = -j\sqrt{\frac{\hbar\omega_0}{2L}} \left(\mathcal{A} - \mathcal{A}^\dagger \right) \tag{21}$$

(g)

Using the results of the quantization of SHO we can write:

$$E_n = \left(n + \frac{1}{2} \right) \hbar\omega_0 = \left(n + \frac{1}{2} \right) \hbar \frac{1}{\sqrt{LC}} \tag{22}$$

(h)

Current and voltage as conjugate dynamical varibles are incompatible. So the following uncertainty relations prohibite the precise measurment of both v and i simultaniously:

$$\langle (\Delta v)^2 \rangle \langle (\Delta i)^2 \rangle \geq \frac{1}{4} \left| \langle \left[\hat{v}, \hat{i} \right] \rangle \right|^2 = \frac{\hbar^2}{4L^2C^2} \tag{23}$$

Paricularly uncertainty in nth energy eigenstate is:

$$\langle (\Delta v)^2 \rangle_n = \langle n|v^2|n\rangle - \langle n|v|n\rangle^2 = \frac{\hbar\omega_0}{2C} \langle n|\mathcal{A}^2 + \mathcal{A}^{\dagger 2} + \mathcal{A}\mathcal{A}^\dagger + \mathcal{A}^\dagger\mathcal{A}|n\rangle - \frac{\hbar\omega_0}{2C} \langle n|\mathcal{A} + \mathcal{A}^\dagger|n\rangle^2 \tag{24}$$

$$\langle (\Delta i)^2 \rangle_n = \langle n|i^2|n\rangle - \langle n|i|n\rangle^2 = \frac{\hbar\omega_0}{2L} \langle n|\mathcal{A}^2 + \mathcal{A}^{\dagger 2} + \mathcal{A}\mathcal{A}^\dagger + \mathcal{A}^\dagger\mathcal{A}|n\rangle - \frac{\hbar\omega_0}{2L} \langle n|\mathcal{A} + \mathcal{A}^\dagger|n\rangle^2 \tag{25}$$

Hence:

$$\left. \begin{array}{l} \langle (\Delta v)^2 \rangle_n = \frac{\hbar\omega_0}{2C}(2n+1) \\ \langle (\Delta i)^2 \rangle_n = \frac{\hbar\omega_0}{2L}(2n+1) \end{array} \right\} \quad \Longrightarrow \quad \langle (\Delta v)^2 \rangle_n \langle (\Delta i)^2 \rangle_n = \frac{\hbar^2}{4L^2C^2}(2n+1)^2 \tag{26}$$

Clearly qunatum uncertainty which reprsents quantum mechanical effects is proportional to ω_0^4 this menas that quantum mechanical effects are more important at higher frequencies comparable to ω_0. Dimensional analysis is a direct method to show this limit.

(i)

To evaluate quantum flactuation level in an eigenstate of energy we use equation (26). We have:

$$\sqrt{\langle(\Delta v)^2\rangle_n} = \sqrt{\frac{\hbar\omega_0}{2C}(2n+1)} \approx 1.28\mu\text{V}\sqrt{2n+1} \tag{27}$$

$$\sqrt{\langle(\Delta i)^2\rangle_n} = \sqrt{\frac{\hbar\omega_0}{2L}(2n+1)} \approx 40.7\text{nA}\sqrt{2n+1} \tag{28}$$

(j)

We assume that quanta of thermal radiation is $k_B T$. k_B is Boltzman's constant. To measure the the quantized energy levels we should conduct our experiment in an enviornment in which termal energy quanta is lower than the half of the first energy level:

$$k_B T < \frac{\hbar\omega_0}{2} \qquad \Longrightarrow \qquad T < 0.12\text{K} \tag{29}$$

(k)

Assume that the voltage wave function is reptresented by $\Psi(v,t)$. since $\mathcal{Q}$ and $\mathcal{P}$ commutation relation is the same as $x-p$ commutation relation we can use the same math. So we can just replace $\mathcal{P} \leftrightarrow -j\hbar\frac{\partial}{\partial\mathcal{Q}}$. Hamiltonian is also generator of time translation so we can write:

$$j\hbar\frac{\partial\Psi(v,t)}{\partial t} = -\frac{\omega_0^2\hbar^2}{2C}\frac{\partial^2\Psi(v,t)}{\partial v^2} + \frac{1}{2}Cv^2\Psi(v,t) \tag{30}$$

(l)

The voltage wavefunction associated with the ground state of the LC network is similar to x-space wave function of SHO. we should just use (9). Actually we have:

$$\mathcal{A}|0\rangle = 0 \qquad \Longrightarrow \qquad v\Psi_0(v) + \frac{\hbar}{LC^2\omega_0}\frac{\partial\Psi_0(v)}{\partial v} = 0 \tag{31}$$

And explicitly:

$$\Psi_0(v) = \left(\frac{LC^2\omega_0}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{1}{2}\frac{LC^2\omega_0 v^2}{\hbar}\right] \tag{32}$$

Problem 2

(a)

The Hamiltonian for a *driven simple harmonic oscilator* is:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 - xF(t) - pG(t) \tag{1}$$

Canonical equations of motion reads:

$$\dot{x} = \frac{\partial H(x,p,t)}{\partial p} = \frac{p}{m} - G(t) \tag{2}$$

$$\dot{p} = -\frac{\partial H(x,p,t)}{\partial x} = -m\omega^2 x + F(t) \tag{3}$$

Combining (2) and (3) leads to the following differential equation for $x(t)$:

$$\ddot{x} = \frac{1}{m}\dot{p} - \frac{dG(t)}{dt} = -\omega^2 x + \frac{F(t)}{m} - \frac{dG(t)}{dt} \tag{4}$$

(b)

We define ladder operators like an unperturbed SHO:

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right) \tag{5}$$

x and p can be calculated in terms of a and $a^\dagger$:

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right) \qquad p = -im\omega\sqrt{\frac{\hbar}{2m\omega}}\left(a - a^\dagger\right) \tag{6}$$

The commutation relation between a and $a^\dagger$ is:

$$[a, a^\dagger] = \frac{m\omega}{2\hbar}\left[x + \frac{ip}{m\omega}, x - \frac{ip}{m\omega}\right] = \frac{1}{i\hbar}[x,p] = 1 \tag{7}$$

First two terms in the right hand side of (1) can be simply replaced by $\hbar\omega(a^\dagger a + 1/2)$ and for the other terms we use (6) so we get:

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) - \sqrt{\frac{\hbar}{2m\omega}}\left\{F(t) + im\omega G(t)\right\}a - \sqrt{\frac{\hbar}{2m\omega}}\left\{F(t) - im\omega G(t)\right\}a^\dagger \tag{8}$$

(c)

In the Heisenberg picture operators are time dependent and equations of motion are simply:

$$\frac{da}{dt} = \frac{1}{i\hbar}[a, H] + \frac{\partial a}{\partial t} \tag{9}$$

$$\frac{da^\dagger}{dt} = \frac{1}{i\hbar}[a^\dagger, H] + \frac{\partial a^\dagger}{\partial t} \tag{10}$$

By inserting the Hamiltonian from (8) and using a and $a^\dagger$ commutation relation we obtain:

$$\frac{da}{dt} = -i\omega[a, a^\dagger a + \frac{1}{2}] - \sqrt{\frac{\hbar}{2m\omega}}\left\{F(t) - im\omega G(t)\right\} = -i\omega a - \sqrt{\frac{\hbar}{2m\omega}}\left\{F(t) - im\omega G(t)\right\} \tag{11}$$

$$\frac{da^\dagger}{dt} = +i\omega[a^\dagger, a^\dagger a + \frac{1}{2}] - \sqrt{\frac{\hbar}{2m\omega}}\left\{F(t) + im\omega G(t)\right\} = i\omega a - \sqrt{\frac{\hbar}{2m\omega}}\left\{F(t) + im\omega G(t)\right\} \tag{12}$$

To solve these differential equations we use a new operator which is defined by:

$$Q = ae^{i\omega t} \tag{13}$$

Using this definition (11) and (12) are translated as

$$\frac{da}{dt} = -i\omega Q e^{-i\omega t} + \frac{dQ}{dt}e^{-i\omega t} = -i\omega Q e^{-i\omega t} - \sqrt{\frac{\hbar}{2m\omega}}\left\{F(t) - im\omega G(t)\right\} \tag{14}$$

$$\frac{da^\dagger}{dt} = +i\omega Q^\dagger e^{i\omega t} + \frac{dQ^\dagger}{dt}e^{+i\omega t} = +i\omega Q^\dagger e^{i\omega t} - \sqrt{\frac{\hbar}{2m\omega}}\left\{F(t) + im\omega G(t)\right\} \tag{15}$$

So simply we get:

$$Q(t) = Q(0) - \sqrt{\frac{\hbar}{2m\omega}}\int_{t'=0}^{t}\left\{F(t') - im\omega G(t')\right\}e^{i\omega t'}dt' \tag{16}$$

therefore

$$a(t) = a(0)e^{-i\omega t} - \sqrt{\frac{\hbar}{2m\omega}}e^{-i\omega t}\int_{t'=0}^{t}\left\{F(t') - im\omega G(t')\right\}e^{i\omega t'}dt' \tag{17}$$

$$a^\dagger(t) = a^\dagger(0)e^{i\omega t} - \sqrt{\frac{\hbar}{2m\omega}}e^{i\omega t}\int_{t'=0}^{t}\left\{F(t') + im\omega G(t')\right\}e^{-i\omega t'}dt' \tag{18}$$

(d)

It's assumed that $F(t)$ and $G(t)$ just act in $0 < t < T$ and the initial state is the ground state of the original Hamiltonian. To find the final state at $t = T$ we use Schödinger picture. We try to find time evolution operator which can predict state vector in an arbitary time. From schrödinger equation we have:

$$i\hbar\frac{\partial U(t,0)}{\partial t} = HU(t,0) \tag{19}$$

In above equation $U(t,0)$ is the time evolution operator statevector at a given time is:

$$|\Psi(t)\rangle = U(t,0)|\Psi(t=0)\rangle \tag{20}$$

First note that:

$$[H(t_1), H(t_2)] = [H_0 - xF(t_1) - pG(t_1), H_0 - xF(t_2) - pG(t_2)] \neq 0 \tag{21}$$

Since Hamiltonian in two different time are not commuative we can not solve (19) by direct integartion similar to the analytic function in a simple commutative complex space. Instead we have to use *Dyson series* [1]. In this case time evolution operator is:

$$U(t,0) = 1 + \sum_{n=1}^{\infty}\left(\frac{-i}{\hbar}\right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n H(t_1)H(t_2)\cdots H(t_n) \tag{22}$$

The desired quantity is the correlation of the state $|n\rangle$ and the evolved wavefunction at $t = T$.

$$p(n) = |\langle n|\Psi(T)\rangle|^2 = |\langle n|\Psi(t)\rangle|^2 \qquad \text{for } t > T \tag{23}$$

We may calculate U_{n0} to get $p(n)$:

$$U_{n0}(T,0) = \delta_{n0} + \sum_{m=1}^{\infty}\left(\frac{-i}{\hbar}\right)^m \int_0^T dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m \langle n|H(t_1)H(t_2)\cdots H(t_m)|0\rangle \tag{24}$$

If we insert unity operator between H's we arrive at:

$$\begin{aligned} &U_{n0}(T,0) = \delta_{n0} + \\ &\quad \sum_{m=1}^{\infty}\sum_{k_1,k_2,\cdots}\left(\frac{-i}{\hbar}\right)^m \int_0^T dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m \langle n|H(t_1)|k_1\rangle\langle k_1|H(t_2)|k_2\rangle\cdots\langle k_{m-1}|H(t_m)|0\rangle \end{aligned} \tag{25}$$

Please note that H connects contiguous states , i.e. the Hamiltonian at any instanse of time can just connect $|i\rangle$ to $|i\rangle$, $|i-1\rangle$ and $|i+1\rangle$. This statement means that we need at least n H's to get a nonzero corrlation in (26). So the first n terms in the summation are zero so:

$$U_{n0}(T,0) = \delta_{n0} + \sum_{m=n}^{\infty} \sum_{k_1,k_2,\cdots} \left(\frac{-i}{\hbar}\right)^m \int_0^T dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m \langle n|H(t_1)|k_1\rangle\langle k_1|H(t_2)|k_2\rangle \cdots \langle k_{m-1}|H(t_m)|0\rangle \quad (26)$$

The matrix elements of $H(t)$ can be explicitly calculated through the use of (8) :

$$\langle k|H(t)|l\rangle = \hbar\omega\delta kl - \sqrt{\frac{l\hbar}{2m\omega}}\{F(t) + im\omega G(t)\}\delta_{k,l-1} - \sqrt{\frac{(l+1)\hbar}{2m\omega}}\{F(t) - im\omega G(t)\}\delta_{k,l+1} \quad (27)$$

Problem 3

(a)

It's assumed that a simple harmonic oscilator suddenly displaced from its equilibrium point. We can appoximately model this sudden change as a perturbing potential whose time dependence is simply a step function:

$$H = \begin{cases} \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 & t<0 \\ \frac{p^2}{2m} + \frac{1}{2}m\omega^2(x-x_0)^2 & t>0 \end{cases} \quad (1)$$

Or:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + u(t)\left\{\frac{1}{2}m\omega^2x_0^2 - m\omega x_0 x\right\} = H_0 + V(t) \quad (2)$$

In (2) $u(t)$ is the unite step function.

(b)

In Dirac picture (iteraction picture) the state of the wave function is manipulated by the original Hamiltonian as:

$$|\Psi_D(t)\rangle = \exp\left(\frac{iH_0t}{\hbar}\right)|\Psi_s(t)\rangle \quad (3)$$

where $|\Psi_s(t)\rangle$ is the state ket in the Schrödinger picture. Evolution of the wavefunctiuon in the interaction picture can be determined by:

$$i\hbar\frac{\partial}{\partial t}|\Psi_D(t)\rangle = V_I|\Psi_D(t)\rangle \quad (4)$$

In this equation V_I designates the purterbing potential in the interaction picture:

$$V_I = \exp\left(\frac{iH_0t}{\hbar}\right)V(t)\exp\left(\frac{-iH_0t}{\hbar}\right)$$

$V_I(t)$ can be explicitly calculated through the use of this identity:

$$[A,B] = \lambda A \quad \Longrightarrow \quad Ae^B = e^\lambda e^B A \quad (5)$$

since $[\hat{a}_S, H] = \hbar\omega\hat{a}_S$ and $[\hat{a}_S^\dagger, H] = -\hbar\omega\hat{a}_S^\dagger$ we get:

$$\left.\begin{array}{l} \hat{a}_S \exp\left(\frac{-iH_0t}{\hbar}\right) = e^{-i\omega t}\exp\left(\frac{-iH_0t}{\hbar}\right)\hat{a}_S \\ \hat{a}_S^\dagger \exp\left(\frac{-iH_0t}{\hbar}\right) = e^{+i\omega t}\exp\left(\frac{-iH_0t}{\hbar}\right)\hat{a}_S^\dagger \end{array}\right\} \Longrightarrow V_I = \frac{1}{2}m\omega^2x_0^2 - x_0\sqrt{\frac{1}{2}m\omega\hbar}\left(e^{-i\omega t}\hat{a}_S + e^{+i\omega t}\hat{a}_S^\dagger\right) \quad (6)$$

Problem 3 continued on next page. . .

Now we consider the commutation relation between $V_I(t_1)$ and $V_I(t_2)$:

$$[V_I(t_1), V_I(t_2)] = -\frac{1}{2}m\omega\hbar[e^{-i\omega t_1}\hat{a}_S + e^{+i\omega t_1}\hat{a}_S^\dagger, e^{-i\omega t_2}\hat{a}_S + e^{+i\omega t_2}\hat{a}_S^\dagger] = -im\omega\hbar\sin\omega(t_2 - t_1) \tag{7}$$

This commutation relation shows that we can not find evolution operator in close form and we can just use *Dyson series* or iterative techniques. Evolution of the wave function in the interaction picture can be fomulated as follows. Assume that $|\Psi_D(t)\rangle$ is expanded in terms of the initial Hamiltonian eigenstates as:

$$|\Psi_D(t)\rangle = \sum_n c_n(t)|n\rangle$$

If we apply the unity operator in (4) we get:

$$i\hbar\frac{\partial}{\partial t}\langle n|\Psi_D(t)\rangle = \sum_m \langle n|V_I|m\rangle\langle m|\Psi_D(t)\rangle \tag{8}$$

This equation can be simplified as:

$$i\hbar\frac{dc(t)}{dt} = \sum_n V_{Inm}c_m(t) \tag{9}$$

where V_{Inm} is:

$$V_{Inm} = V_{nm}\exp\left(it\frac{E_m - E_n}{\hbar}\right) = \langle n|V(t)|m\rangle\exp(i\omega_{nm}t) \tag{10}$$

Hence:

$$i\hbar\begin{pmatrix}\dot{c}_1\\ \dot{c}_2\\ \vdots\end{pmatrix} = \begin{pmatrix}V_{11} & V_{12}e^{i\omega_{12}t} & \cdots\\ V_{21}e^{i\omega_{21}t} & V_{22} & \cdots\\ \vdots & \vdots & \vdots\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ \vdots\end{pmatrix} \tag{11}$$

$\hat{x}_D$ which is the Dirac picture representation of $\hat{x}$ operator is:

$$\hat{x}_D(t) = \exp\left(\frac{iH_0t}{\hbar}\right)\hat{x}_S\exp\left(\frac{-iH_0t}{\hbar}\right) \tag{12}$$

In above equation index S stands for schrödinger picture. Now we can use the ladder operator :

$$\hat{a}_S = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_S + \frac{i\hat{p}_S}{m\omega}\right)$$

Then:

$$H_0 = \left(\hat{a}_S^\dagger\hat{a}_S + \frac{1}{2}\right)\hbar\omega \tag{13}$$

$$V = \frac{1}{2}m\omega^2x_0^2 - m\omega^2x_0(\hat{a}_S + \hat{a}_S^\dagger)\sqrt{\frac{\hbar}{2m\omega}} \tag{14}$$

So $\hat{x}_D$ can be written as:

$$\hat{x}_D(t) = \sqrt{\frac{\hbar}{2m\omega}}\exp\left(\frac{iH_0t}{\hbar}\right)(\hat{a}_S + \hat{a}_S^\dagger)\exp\left(\frac{-iH_0t}{\hbar}\right) \tag{15}$$

Using the same identity employed in the previous part we have:

$$\left.\begin{array}{l}\hat{a}_S\exp\left(\frac{-iH_0t}{\hbar}\right) = e^{-i\omega t}\exp\left(\frac{-iH_0t}{\hbar}\right)\hat{a}_S\\ \hat{a}_S^\dagger\exp\left(\frac{-iH_0t}{\hbar}\right) = e^{+i\omega t}\exp\left(\frac{-iH_0t}{\hbar}\right)\hat{a}_S^\dagger\end{array}\right\} \Longrightarrow \hat{x}_D(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(e^{-i\omega t}\hat{a}_S + e^{+i\omega t}\hat{a}_S^\dagger\right) \tag{16}$$

So $\hat{x}_D(t)$ can be explicitly calculated in terms of $\hat{x}_S$ and $\hat{p}_S$:

$$\hat{x}_D(t) = \frac{1}{2}\left\{e^{-i\omega t}\left(\hat{x}_s + \frac{i\hat{p}_S}{m\omega}\right) + e^{+i\omega t}\left(\hat{x}_s - \frac{i\hat{p}_S}{m\omega}\right)\right\} = \hat{x}_S\cos\omega t + \frac{\hat{p}_S}{m\omega}\sin\omega t \tag{17}$$

(c)

The equation of motion for position operator in the Heisenberg picture is:

$$\frac{d\hat{x}_H}{dt} = \frac{1}{i\hbar}[\hat{x}_H, H] + \frac{\partial \hat{x}_H}{\partial t} = \frac{1}{i\hbar}\left[\hat{x}_H, \frac{\hat{p}_H^2}{2m} + \frac{1}{2}m\omega^2(\hat{x}_H - x_0)^2\right] = \frac{1}{i2m\hbar}[\hat{x}_H, \hat{p}_H^2] = \frac{\hat{p}_H}{m} \tag{18}$$

similarly for momentum operator we have:

$$\frac{d\hat{p}_H}{dt} = \frac{1}{i\hbar}[p_H, H] + \frac{\partial \hat{p}_H}{\partial t} = \frac{1}{i\hbar}\left[\hat{p}_H, \frac{\hat{p}_H^2}{2m} + \frac{1}{2}m\omega^2(\hat{x}_H - x_0)^2\right] = \frac{m\omega^2}{2i\hbar}\left[\hat{p}_H, \hat{x}_H^2 - 2x_0\hat{x}_H\right] = -m\omega^2(\hat{x}_H - x_0) \tag{19}$$

(d)

Two coupled differntial equations (18) and (19) determines the evolution of the operators in the Heisenberg picture. we can decouple the equations using time differentiation:

$$\frac{d^2\hat{x}_H}{dt^2} = -\omega^2(\hat{x}_H - x_0) \qquad \dot{\hat{x}}_H(0) = \frac{\hat{p}(0)}{m} \tag{20}$$

$$\frac{d^2\hat{p}_H}{dt^2} = -\omega^2\hat{p}_H \qquad \dot{\hat{p}}_H(0) = -m\omega^2(\hat{x}_H(0) - x_0) \tag{21}$$

Ultimately we arrive at:

$$\hat{x}_H(t) = x_0 + (\hat{x}_H(0) - x_0)\cos\omega_0 t + \frac{\hat{p}_H(0)}{m\omega}\sin\omega t \tag{22}$$

$$\hat{p}_H(t) = \hat{p}_H(0)\cos\omega_0 t - m\omega(\hat{x}_H(0) - x_0)\sin\omega t \tag{23}$$

(e)

Equations (17) and (22) show how the position operator evolves in Dirac and Heisenberg pictures respectively. Clearly both picture coincide when $x_0 = 0$. In fact in the Driac picture the opearor is evolved by just the main portion of the Hamiltonian and the state vector evaloves by the perturbing potential. In the Heisenberg picture $\hat{x}_H(t)$ oscilates around the equilibrium point x_0 launched by the initial momnetum $\hat{p}_D(t = 0)$. This evolution is the reminiscent of classical oscilation around the equilibrium point. In the Heisenberg picture states are *frozen* and just the operators evolve. In Dirac picture according to (17) opearor $\hat{x}_D$ oscilates by the unperubed Hamiltonian and the state vectors are also time varying by the perturbing potential.

Problem 4

(a)

The Hamiltonian describing time evolution of SHO is:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(t)x^2 \tag{1}$$

Like an ordinary SHO we can define ladder operators as:

$$\hat{a} = \sqrt{\frac{m\omega(t)}{2\hbar}}\left(x + \frac{ip}{m\omega(t)}\right) \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega(t)}{2\hbar}}\left(x - \frac{ip}{m\omega(t)}\right) \tag{2}$$

Problem 4 continued on next page. . .

In this problem we use Heisenberg picture in which operator evolve with time. The equations of motion for creation and anihilation operators are:

$$\frac{d\hat{a}}{dt} = \frac{1}{i\hbar}[\hat{a}, H] + \frac{\partial \hat{a}}{\partial t} = \frac{\hbar\omega(t)}{i\hbar}\left[\hat{a}, \hat{a}^\dagger\hat{a} + \frac{1}{2}\right] + \frac{\partial \hat{a}}{\partial t}$$
$$= -i\omega\hat{a} + \frac{\dot{\omega}}{2\omega}\hat{a} - \sqrt{\frac{m\omega}{2\hbar}}i\frac{p}{m\omega}\frac{\dot{\omega}}{\omega}$$
$$= -i\omega\hat{a} + \frac{\dot{\omega}}{2\omega}\hat{a}^\dagger \quad (3)$$

The second equation can be simply derived by applying complex conjugate operator on the both sides of (3). The coupled equations of motion are:

$$\dot{\hat{a}} = -i\omega\hat{a} + \frac{\dot{\omega}}{2\omega}\hat{a}^\dagger \quad (4)$$
$$\dot{\hat{a}}^\dagger = +i\omega\hat{a}^\dagger + \frac{\dot{\omega}}{2\omega}\hat{a} \quad (5)$$
$$(6)$$

(b)

Bogoliubov transformation relates $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ to their initial values as:

$$\hat{a} = e^{-i\alpha(t)}\hat{a}(0)\cosh\beta(t) + e^{i\gamma(t)}\hat{a}^\dagger(0)\sinh\beta(t) \quad (7)$$
$$\hat{a}^\dagger = e^{-i\gamma(t)}\hat{a}(0)\sinh\beta(t) + e^{i\alpha(t)}\hat{a}^\dagger(0)\cosh\beta(t) \quad (8)$$

The hamiltonian in the parametric SHO is:

$$H = \hbar\omega(t)\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) =$$
$$\hbar\omega(t)\left\{\frac{1}{2} + \xi_1(t)\hat{a}(0)\hat{a}(0) + \xi_2(t)\hat{a}(0)\hat{a}^\dagger(0) + \xi_3(t)\hat{a}^\dagger(0)\hat{a}(0) + \xi_4(t)\hat{a}^\dagger(0)\hat{a}^\dagger(0)\right\} \quad (9)$$

In above equation $\xi_i(t)$ are simply mutiplication of time dependent mutipliers in Bogoliubov transformation. The expectation value of H in the initial eigenstates is:

$$\langle n|H(t)|n\rangle = \hbar\omega(t)\langle n|\frac{1}{2} + \xi_1(t)\hat{a}(0)\hat{a}(0) + \xi_2(t)\hat{a}(0)\hat{a}^\dagger(0) + \xi_3(t)\hat{a}^\dagger(0)\hat{a}(0) + \xi_4(t)\hat{a}^\dagger(0)\hat{a}^\dagger(0)|n\rangle \quad (10)$$

Since

$$\langle n|\hat{a}(0)\hat{a}(0)|n\rangle = \sqrt{n(n-1)}\langle n-2|n\rangle = 0 \quad (11)$$
$$\langle n|\hat{a}(0)\hat{a}^\dagger(0)|n\rangle = (n+1)\langle n+1|n+1\rangle = n+1 \quad (12)$$
$$\langle n|\hat{a}^\dagger(0)\hat{a}(0)|n\rangle = n\langle n-1|n-1\rangle = n \quad (13)$$
$$\langle n|\hat{a}^\dagger(0)\hat{a}^\dagger(0)|n\rangle = \sqrt{n(n-1)}\langle n|n-2\rangle = 0 \quad (14)$$

So we arrive at:

$$\langle n|H(t)|n\rangle = \hbar\omega(t)\left(\frac{1}{2} + (n+1)\xi_2(t) + n\xi_3(t)\right) \quad (15)$$

where

$$\xi_2(t) = \sinh^2\beta(t)$$

and

$$\xi_3(t) = \cosh^2\beta(t) = 1 + \sinh^2\beta(t)$$

so we get:

$$\langle n|H(t)|n\rangle = \hbar\omega(t)\left(n + \frac{1}{2}\right)(2\sinh^2\beta + 1) \tag{16}$$

Therefore $f(t)$ is:

$$f(t) = 2\sinh^2\beta(t) + 1 = \cosh(2\beta) \tag{17}$$

(c)

If we substitute $\hat{a}$ and $\hat{a}^\dagger$ from (7) and (8) into the coupled differential equations derived in part (a) we get:

$$\begin{aligned}
\dot{\hat{a}}(t) &= -i\dot{\alpha}e^{-i\alpha(t)}\hat{a}(0)\cosh\beta(t) + \dot{\beta}e^{-i\alpha(t)}\hat{a}(0)\sinh\beta(t) + i\dot{\gamma}e^{i\gamma}\hat{a}^\dagger(0)\sinh\beta(t) + \dot{\beta}e^{i\gamma(t)}\hat{a}^\dagger(0)\cosh\beta \\
&= -i\omega(t)\left\{e^{-i\alpha(t)}\cosh\beta(t)\hat{a}(0) + e^{i\gamma(t)}\sinh\beta(t)\hat{a}^\dagger(0)\right\} \\
&\quad + \frac{\dot{\omega}}{2\omega(t)}\left\{e^{-i\gamma(t)}\sinh\beta(t)\hat{a}(0) + e^{i\alpha(t)}\cosh\beta(t)\hat{a}^\dagger(0)\right\}
\end{aligned} \tag{18}$$

This equation may be expressed more neatly by defining two new funcations:

$$\Gamma_1(\alpha,\dot{\alpha},\beta,\dot{\beta},\gamma,\dot{\gamma},\omega,\dot{\omega}) = -i\dot{\alpha}e^{-i\alpha}\cosh\beta + \dot{\beta}e^{-i\alpha}\sinh\beta + i\omega e^{-i\alpha}\cosh\beta - \frac{\dot{\omega}}{2\omega}e^{-i\gamma}\sinh\beta \tag{19}$$

$$\Gamma_2(\alpha,\dot{\alpha},\beta,\dot{\beta},\gamma,\dot{\gamma},\omega,\dot{\omega}) = i\dot{\gamma}e^{i\gamma}\sinh\beta + \dot{\beta}e^{i\gamma}\cosh\beta + i\omega e^{i\gamma}\sinh\beta - \frac{\dot{\omega}}{2\omega}e^{i\alpha}\cosh\beta \tag{20}$$

So two coupled equations of motion in the terms of Bogoliubov transformation parameters are:

$$\Gamma_1(\alpha,\dot{\alpha},\beta,\dot{\beta},\gamma,\dot{\gamma},\omega,\dot{\omega})\hat{a}(0) + \Gamma_2(\alpha,\dot{\alpha},\beta,\dot{\beta},\gamma,\dot{\gamma},\omega,\dot{\omega})\hat{a}^\dagger(0) = 0 \tag{21}$$

$$\Gamma_2^*(\alpha,\dot{\alpha},\beta,\dot{\beta},\gamma,\dot{\gamma},\omega,\dot{\omega})\hat{a}(0) + \Gamma_1^*(\alpha,\dot{\alpha},\beta,\dot{\beta},\gamma,\dot{\gamma},\omega,\dot{\omega})\hat{a}^\dagger(0) = 0 \tag{22}$$

The second equation has been simply derived by applying the conjugate transpose opertor on the both sides of (18).

(d)

Time-dependent uncertainty for position operator can be calulated through the use of creation and annihilation operators in the Heisenberg picture:

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2m\omega(t)}}\left(\hat{a}(t) + \hat{a}^\dagger(t)\right) \tag{23}$$

Uncertainty in a specific state is defined by:

$$(\Delta x(t))^2 = \langle n|\hat{x}^2(t)|n\rangle - \langle n|\hat{x}(t)|n\rangle^2 \tag{24}$$

Using (7) and (8) we are able to calculate $\hat{x}(t)$ in terms of initial ladder operators:

$$\hat{x}(t) = \Lambda(t)\hat{a}(0) + \Lambda^*(t)\hat{a}^\dagger(0) \tag{25}$$

where $\Lambda(t)$ is defined as:

$$\Lambda_1(t) = \sqrt{\frac{\hbar}{2m\omega(t)}}\left[\cosh\beta(t)e^{-i\alpha(t)} + \sinh\beta(t)e^{-i\gamma(t)}\right]$$

We obtain:

$$\langle n|\hat{x}(t)|n\rangle = \langle n|\Lambda_1(t)\hat{a}(0) + \Lambda_1^*(t)\hat{a}^\dagger(0)|n\rangle = \sqrt{n}\Lambda_1(t)\langle n|n-1\rangle + \sqrt{n+1}\Lambda_1^*(t)\langle n|n+1\rangle = 0 \tag{26}$$

$$\langle n|\hat{x}^2(t)|n\rangle = |\Lambda_1(t)|^2 \langle n|\hat{a}^\dagger(0)\hat{a}(0)|n\rangle + |\Lambda_1(t)|^2 \langle n|\hat{a}(0)\hat{a}^\dagger(0)|n\rangle = |\Lambda_1(t)|^2 (2n+1) \tag{27}$$

By inserting (26) and (27) into (24) we arraive at:

$$\Delta x_n(t) = |\Lambda_1(t)| \sqrt{2n+1} \tag{28}$$

Using the same line of reasoning we can evaluate $\Delta p(t)$. We use the following definition:

$$\Lambda_2(t) = \sqrt{\frac{1}{2}m\omega(t)\hbar}\left[\cosh\beta(t)e^{-i\alpha(t)} - \sinh\beta(t)e^{-i\gamma(t)}\right]$$

$\hat{p}(t)$ may be expanded as

$$\hat{p}(t) = -i\left(\Lambda_2(t)\hat{a}(0) + \Lambda_2^*(t)\hat{a}^\dagger(0)\right) \tag{29}$$

So time dependent expectation value of $\hat{p}(t)$ and $\hat{p}^2(t)$ in the eigenstates of the initial Hamiltonian are:

$$\langle n|\hat{p}(t)|n\rangle = -i\langle n|\Lambda_2(t)\hat{a}(0) + \Lambda_2^*(t)\hat{a}^\dagger(0)|n\rangle = \sqrt{n}\Lambda_2(t)\langle n|n-1\rangle + \sqrt{n+1}\Lambda_2^*(t)\langle n|n+1\rangle = 0 \tag{30}$$

$$\langle n|\hat{x}^2(t)|n\rangle = |\Lambda_2(t)|^2 \langle n|\hat{a}^\dagger(0)\hat{a}(0)|n\rangle + |\Lambda_2(t)|^2 \langle n|\hat{a}(0)\hat{a}^\dagger(0)|n\rangle = |\Lambda_2(t)|^2 (2n+1) \tag{31}$$

therefore

$$\Delta p_n(t) = |\Lambda_2(t)| \sqrt{2n+1} \tag{32}$$

and uncertainty in x-p is:

$$\Delta x_n(t)\Delta p_n(t) = |\Lambda_1(t)\Lambda_2(t)|(2n+1) = \frac{\hbar}{2}(2n+1)\left|\cosh^2\beta e^{-2i\alpha} - \sinh^2\beta e^{-2i\gamma}\right| \tag{33}$$

$x-p$ uncertainty in the parametrix SHO is composed of two parts. One part is similar to an ordinary SHO but the second motulating part is time dependent and it reflects the time varying nature of the equivalent spring constant.

Problem 5

(a)

Two linearly coupled quantum-mechanical simple harmonic oscillators, e.g., two masses on springs, connected by a third spring, constitute an ideally simple model of quantum/classical coupling. The simplest version of the two-oscillator system consists of two identical oscillators, with equal masses, spring constants, and frequencies, plus a connecting spring with its own spring constant. Figure 2 shows this simple model.

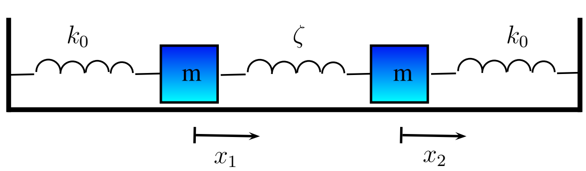

Figure 2: classical interpretation of coupled SHOs

In this simple classical example total energy of the system is:

$$E = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}k_0^2x_1^2 + \frac{1}{2}k_0x_2^2 + \frac{1}{2}(x_2-x_1)^2 = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}(k_0+\zeta)x_1^2 + \frac{1}{2}(k_0+\zeta)x_2^2 - \zeta x_1x_2 \tag{1}$$

The last term is the interaction term reprsenting the coupling between two SHOs.

Problem 5 continued on next page...

(b)

The Hamiltonian is simply total energy of the system:

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \eta x_1 x_2 \tag{2}$$

In above equation k is the spring constant in the individual oscilators.

(c)

To find the quantized energy levels we first try to employ a proper canonical transformation two contruct a new *Kamiltonian* which can be treated as two decoupled equivalent SHOs. We use $F = F_2(x_i, P_i, t)$ as the generator of the transformation [2]:

$$F = F_2(x_i, P_i, t) = P_1\frac{x_1 + x_2}{\sqrt{2}} + P_2\frac{x_1 - x_2}{\sqrt{2}} \tag{3}$$

Please note that (X_i, P_i) are transformed canonical varibles. So we have [2]:

$$p_1 = \frac{\partial F_2}{\partial x_1} = \frac{P_1 + P_2}{\sqrt{2}} \tag{4}$$
$$p_2 = \frac{\partial F_2}{\partial x_1} = \frac{P_1 - P_2}{\sqrt{2}} \tag{5}$$
$$X_1 = \frac{\partial F_2}{\partial P_1} = \frac{x_1 + x_2}{\sqrt{2}} \tag{6}$$
$$X_2 = \frac{\partial F_2}{\partial P_2} = \frac{x_1 - x_2}{\sqrt{2}} \tag{7}$$
$$K = H + \frac{\partial F_2}{\partial t} \tag{8}$$

Finally we get:

$$p_1 = \frac{P_1 + P_2}{\sqrt{2}} \tag{9}$$
$$p_2 = \frac{P_1 - P_2}{\sqrt{2}} \tag{10}$$
$$x_1 = \frac{X_1 + X_2}{\sqrt{2}} \tag{11}$$
$$x_2 = \frac{X_1 - X_2}{\sqrt{2}} \tag{12}$$
$$K = H_{new} = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + \frac{1}{2}(k+\eta)X_1^2 + \frac{1}{2}(k-\eta)X_2^2 \tag{13}$$

Now we can apply Dirac's quantization rules to investigate quantized energy levels. we have:

$$[\hat{X}_j, \hat{P}_k] = i\hbar\delta_{jk} \tag{14}$$
$$[\hat{X}_j, \hat{X}_k] = 0 \tag{15}$$
$$[\hat{P}_j, \hat{P}_k] = 0 \tag{16}$$
$$\tag{17}$$

Problem 5 continued on next page. . .

The new Hamiltonian is:

$$\hat{H}_1 = \frac{\hat{P}_1^2}{2m} + \frac{1}{2}(k+\eta)\hat{X}_1^2 = \frac{\hat{P}_1^2}{2m} + \frac{1}{2}m\Omega_1^2 X_1^2 \tag{18}$$

$$\hat{H}_2 = \frac{\hat{P}_2^2}{2m} + \frac{1}{2}(k-\eta)\hat{X}_2^2 = \frac{\hat{P}_2^2}{2m} + \frac{1}{2}m\Omega_2^2 X_2^2 \tag{19}$$

$$\hat{H}_{new} = \hat{H}_1 + \hat{H}_2 \tag{20}$$

In obove equations Ω_1 and Ω_2 are the natural frequencies of two imaginary decoupled SHOs:

$$\Omega_1 = \sqrt{\frac{k+\eta}{m}} \tag{21}$$

$$\Omega_2 = \sqrt{\frac{k-\eta}{m}} \tag{22}$$

Assume the we have solved the eigenvalue problem assocoated with $\hat{H}_1$ and $\hat{H}_2$ and we represent the energy eigenstates of the first Hamiltonian as $|n_1\rangle$ and for the second Hamiltonian $|n_2\rangle$:

$$\hat{H}_1|n_1\rangle = \hbar\Omega_1\left(n_1 + \frac{1}{2}\right)|n_1\rangle \tag{23}$$

$$\hat{H}_2|n_2\rangle = \hbar\Omega_2\left(n_2 + \frac{1}{2}\right)|n_1\rangle \tag{24}$$

Overal state of the whole system can be expressed as the tensor product of the eigenstates of the individual SHOs:

$$|n_1, n_2\rangle = |n_1\rangle \otimes |n_2\rangle$$

And $\hat{H}_{new}$ can be more regiriously expressed as:

$$\hat{H}_{new} = \hat{H}_1 \otimes \mathbf{1} + \mathbf{1} \otimes \hat{H}_2$$

So we energy eigenvalues associated with each state is:

$$\hat{H}_{new}|n_1, n_2\rangle = \hbar\left\{\Omega_1\left(n_1 + \frac{1}{2}\right) + \Omega_2\left(n_2 + \frac{1}{2}\right)\right\}|n_1, n_2\rangle \tag{25}$$

(d)

First order energy shift can be calculated based on fisrt order Taylor expansion of E_{n_1,n_2}:

$$E_{n_1,n_2} = \hbar\left\{\left(n_1 + \frac{1}{2}\right)\sqrt{\frac{k}{m}(1+\gamma)} + \left(n_2 + \frac{1}{2}\right)\sqrt{\frac{k}{m}(1-\gamma)}\right\} \approx$$
$$\hbar\omega_0\left(n_1 + \frac{1}{2}\right)\left(1 + \frac{\gamma}{2}\right) + \hbar\omega_0\left(n_2 + \frac{1}{2}\right)\left(1 - \frac{\gamma}{2}\right) = (n_1 + n_2 + 1)\,\hbar\omega_0 + \frac{1}{2}(n_1 - n_2)\,\hbar\omega_0\gamma \tag{26}$$

where γ is defined as $\frac{\eta}{k}$ and ω_0 is $\sqrt{k/m}$.

References

[1] J. J.Sakurai, *Modern Quantum Mechanics.* Addison Wesley, Massachusetts, Revised Edition.

[2] H.Goldstein *Classical Mechanics.* Third Edition, Addison-Wesley Pub. Co., 1980s,