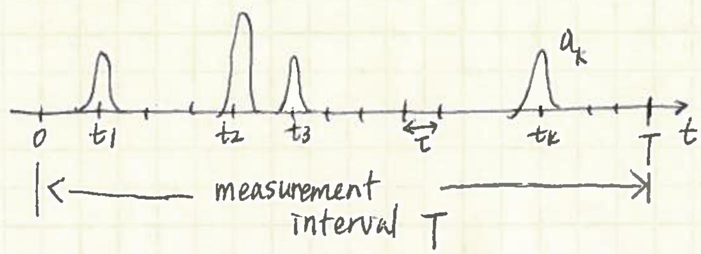
Tue.

· <u>Random Pulse Trains</u> a noisy waveform $x(t)$

Ref. M. J. Buckingham. "Noise in Electronic Devices and Systems"

A. vander Ziel "Noise in Solid State Devices and Circuits"



① τ = divided time slot $= \frac{T}{N} \gg \tau_m$ = memory time of the system.

② If the probability for more than two pulses per time slot τ is negligible

⇒ each pulse emission occurs "<u>independently</u>"

③ $$x(t) = \sum_{k=1}^{K} a_k f(t-t_k)$$

random variables: pulse amplitude a_k, t_k = pulse arrival time

$f(t-t_k)$: pulse shape fn often <u>fixed</u>, given by the systems

e.g. relaxation time of a system / transit time of a carrier

<u>Q</u> What is the probability of finding K events in the measurement time interval T out of N trials?

→ <u>Binomial</u> distribution

$$W_N(K) = \frac{N!}{K!(N-K)!} p^K (1-p)^{N-K}$$

K events; # of trial

p = probability of pulse emission in each time slot.

suppose ν = the average rate of pulse emission per second.

$$\boxed{p = \nu\tau}$$

Normalization condition $\sum^{N} W_N(K) = (p + (1-p))^N = 1$

$(1-p)$: probability of no pulse emission in each time slot

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· Bernoulli Process = a discrete time stochastic process that takes only 2 values

$$P_x(x_0) = \begin{cases} 1-p \,; & x_0 = 0 \checkmark \ \text{(failure)} \\ p \,, & x_0 = 1 \checkmark \ \text{(success)} \\ 0 \,, & \text{otherwise} \end{cases}$$

PMF to describe the Bernoulli Process

x = random variable

if $x=1 \Rightarrow \exists_a$ pulse emission

if $x=0 \Rightarrow \exists_{no}$ pulse emission

Using the z-transform,

transform

zero emission

one emission

$$P_x^T(z) \equiv \sum_{x_0=0}^{\infty} z^{x_0} P_x(x_0) = z^0 P_x(0) + z^1 P_x(1)$$

$$= 1\,(1-p) + z\,p = 1-p+zp$$

random variable

Bernouli $x_0=0$ or $x_0=1$

therefore $\langle x \rangle = \left[\frac{d}{dz} P_x^T(z)\right]_{z=1} = p$

$$\langle x \rangle = \sum_{x_0} x_0 P_x(x_0)$$

$$= 0 \times P_x(0) + 1\,P_x(1) = p$$

$$\langle x^2 \rangle = \left[\frac{d^2}{dz^2} P_x^T(z) + \frac{d}{dz} P_x^T(z)\right]_{z=1} = \left[0 + p\right]_{z=1} = p$$

$$\langle x^2 \rangle = \sum_{x_0} x_0^2 P_x(x_0)$$

$$= 0 \times P_x(0) + 1 \cdot P_x(1)$$

$$= p$$

variance $\sigma_x^2 \equiv \langle x^2 \rangle - \langle x \rangle^2 = p - p^2 = p(1-p)$

$\boxed{\text{Bernoulli process} \quad \langle x \rangle = p \,, \ \langle x^2 \rangle = p \,, \ \sigma_x^2 = p(1-p)}$

② Binomial distribution

↳ a series of independent Bernoulli trials w/ the same success probability.

Suppose n indep. Bernoulli processes are to be performed.

k = the # of successes in n trials = random variable

Note

$$\underbrace{\left[P_x^T(z)\right]}_{\text{single Bernoulli}}{}^n = [1-p+zp]^{\overset{\text{Bernoulli}}{n}} = \sum_{k=0}^{n}\binom{n}{k}(zp)^k(1-p)^{n-k}$$

n-trials

note $(a+b)^n = \sum_{l=0}^{n}\binom{n}{l}a^l b^{n-l}$

$$= P_k(0) + zP_k(1) + z^2P_k(2) + \cdots$$

zero emission, 1 emission, 2 emissions

k = # of success

binomial theorm

$$P_k(k_0) = \binom{\overset{\text{\# of trials}}{n}}{k_0} p^{k_0}(1-p)^{n-k_0}, \quad k_0 = 0, 1, 2, \cdots$$

$$\boxed{P_k(k_0) = \frac{n!}{(n-k_0)!\,k_0!}p^{k_0}(1-p)^{n-k_0}}$$

binomial PMF

$$\langle x\rangle = \left[\frac{d}{dz}\left(P_x^T(z)\right)^n\right] = \frac{d}{dz}\left[zp+1-p\right]^n\Big|_{z=1}$$

$$= n(zp+1-p)^{n-1}p\Big|_{z=1} = np(\cancel{p}+1-\cancel{p})^{n-1} = \boxed{np}$$

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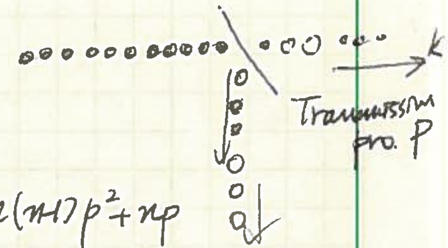
$$\langle x^2\rangle = \left[\frac{d^2}{dz^2}P_x^T(z) + \frac{d}{dz}P_z^T(z)\right]_{z=1}$$



$$= n(n-1)(zp+1-p)^{n-2}p^2 + n(zp+1-p)^{n-1}p\Big|_{z=1} = n(n-1)p^2+np$$

$$= n^2p^2 - n^2p^2 + np$$

$$= (np)^2 - np(np-1)$$

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2$$

$$= n(n-1)p^2 + np - n^2p^2 = -np^2 + np = \boxed{np(1-p)}$$

$$= npq = n\sigma^2$$

(f)

K = the # of successful pulses, binomial distribution

$$\boxed{\langle K\rangle} = \sum_{K=0}^{N} K\frac{N!}{K!(N-K)!}p^K(1-p)^{N-K} \qquad 1-p = q$$

$$= \sum_{K=0}^{N}\frac{N!}{K!(N-K)!}\left[p\frac{\partial}{\partial p}p^K\right]q^{N-K} \qquad \left(p\frac{\partial}{\partial p}\right)$$

$$= p\frac{\partial}{\partial p}(p+q)^N = p\,N\underbrace{(p+q)^{N-1}}_{=1} = \boxed{Np}$$

Mean-square

$$p\frac{\partial}{\partial p}p^K = pKp^{K-1}$$

$$\boxed{\langle K^2\rangle} = \sum_{K=0}^{N}K^2W_N(K) = \sum_{K=0}^{N}\frac{N!}{K!(N-K)!}\left[\left(p\frac{\partial}{\partial p}\right)^2p^K\right]q^{N-K}$$

$$\sum_{K=0}^{N}W_N(K) = 1$$

$$= \left(p\frac{\partial}{\partial p}\right)^2(p+q)^N = p\frac{\partial}{\partial p}N(p+q)^{N-1} \quad p\frac{\partial}{\partial p}\left[pN(N-1)(p+q)^{N-2}\right]$$

$$= Np(N-1)(p+q)^{N-2} + p^2N(N-1)(N-2)(p+q)^{N-3}$$

$$\boxed{\sigma_K^2 = Npq}$$

$\boxed{\text{partition noise}}$

$$\left(p\frac{\partial}{\partial p}\right)\left(p\frac{\partial}{\partial p}\right)(p+q)^N$$

$$p\frac{\partial}{\partial p}\left[pN(p+q)^{N-1}\right]$$

$$\frac{\partial}{\partial p}pN(p+q)^{N-1} = N(p+q)^{N-1} + pN(N-1)(p+q)^{N-2}$$

$$= Np(N-1) + Np^2(N-1)(N-2)$$

$$= Np(N-1)\left[p(N-2)+1\right]$$

$$= p\left[N(p+q)^{N-1} + pN(N-1)(p+q)^{N-2}\right] = pN + p^2N(N-1)$$

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③ Geometric distribution

Suppose l_1 is a discrete random variable,

= the # of Bernoulli trials (after) any pulse emission and before the next pulse emission

\# of Bernoulli trials up to the 1st emission

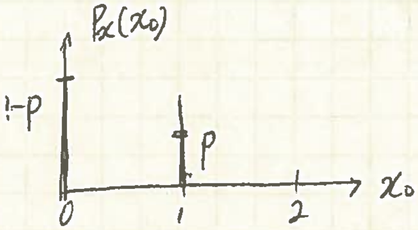
the 1st-order interarrival time ~~of ...~~ $l_1 = 1, 2, 3, \dots$

Let's determine $P_{l_1}(l)$

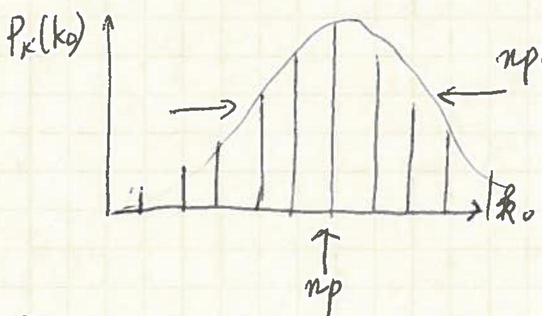
$$P_{l_1}(l) = \underbrace{p}_{\text{first success}} \underbrace{(1-p)^{l-1}}_{l-1 \text{ successive failure}} \qquad l = 1, 2, \dots$$

$$\therefore P_{l_1}^T(z) = \frac{zp}{1-z(1-p)} \longrightarrow \overline{l_1} = \frac{1}{p} \qquad \overline{l_1^2} = \frac{2-p}{p^2}, \quad \sigma_{l_1}^2 = \frac{1-p}{p^2}$$

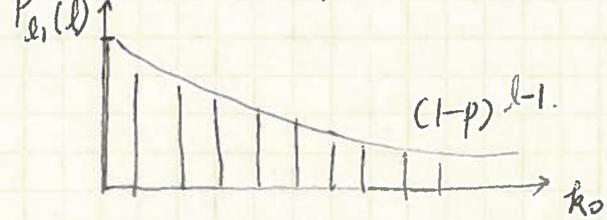
Comparison



Bernoulli trial



Binomial distribution



Geometric distribution

· Poisson Process

$\updownarrow$ a probabilistic description of the behavior of arrivals of successesses at points on a <u>continuous</u> line

cf) Bernoulli process = a particular probabilistic description of the "arrivals" of successes in a series of indep. identical discrete trials

→ the limit $\Delta t \to 0$ of a series of identical indep. Bernoulli trials at Δt w/ $\underbrace{p = \lambda \Delta t}_{\text{success prob.}}$

<u>Suppose</u> exactly <u>k</u> arrivals during any interval of duration t

→ $\underbrace{p(k,t)}_{\text{PMF}}$: probability to have exactly <u>k</u> arrivals during t.

∴ in any interval of length t, $t \geq 0$,

<u>Possible definitions</u>

$$\sum_{k=0}^{\infty} p(k,t) = \underline{1}$$

① Any events defined on "<u>non overlapping time intervals</u>" = mutually independent.

② If $\Delta t \to 0$,
$$p(k,\Delta t) = \begin{cases} 1-\lambda\Delta t & k=0 \\ \lambda \Delta t & k=1 \\ 0 & k>1 \end{cases}$$
no-memory

only 1 pulse per small Δt

If $\Delta t \ll 0$, we consider only the possibility of zero or 1 arrivals btw t & $t+\Delta t$.

$$\underline{p(k, t+\Delta t)} = p(k,t)\, p(0,\Delta t) + p(k-1,t)\, p(1,\Delta t)$$
$$= p(k,t)\,(1-\lambda\Delta t) + p(k-1,t)\,\lambda\Delta t$$

$$\therefore \frac{d}{dt} p(k,t) + \lambda\, p(k,t) = \lambda\, p(k-1,t)$$

<u>Iterative</u> solution for $k=0$, $k=1$, $k=2$, $\vdots$

<u>initial</u> $p(k,0) = \begin{cases} 1 & k=0 \\ 0 & k\neq 0 \end{cases}$

$$\boxed{p(k,t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}} \quad t \geq 0,\ k=1,2\ldots$$

Poisson <u>PMF</u>

$\mu = \lambda t$ familiar form

$$\boxed{P_k(k_0) = \frac{\mu^{k_0} e^{-\mu}}{k_0!} \quad k_0 = 0,1,2,\ldots}$$

$\boxed{\text{mean} = \text{variance}}$

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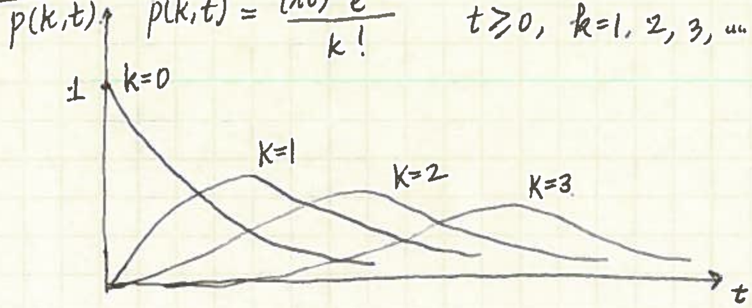
Poisson

$$p(k,t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad t \geq 0,\ k = 1, 2, 3, \ldots$$



$$P(0,t) = e^{-\lambda t}$$
$$P(1,t) = \frac{(\lambda t)\, e^{-\lambda t}}{1!}$$
$$P(2,t) = \frac{(\lambda t)^2 e^{-\lambda t}}{2!}$$

$\Rightarrow$ converting to the familiar form

setting $\mu = \lambda t$

$$\boxed{P_k(k_0) = \frac{\mu^{k_0} e^{-\mu}}{k_0!}, \quad \mu = \lambda t,\ k_0 = 0, 1, 2, \ldots}$$

Poisson PMF.

We derived the Poisson PMF by considering the # of arrivals in an interval of length t but "ubiquitously" present in any case.

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ taylor series

$1 + x + x^2 + \cdots$

z-transform

$$P_K^T(z) = \sum_{k_0=0}^{\infty} P_K(k_0) z^{k_0} = e^{-\mu} \underbrace{\sum_{k_0=0}^{\infty} \frac{(\mu z)^{k_0}}{k_0!}} = e^{\mu z} e^{-\mu} = e^{\mu(z-1)}$$

random variable

$$\langle k \rangle = \frac{d}{dz} P_K^T(z) \Big|_{z=1} = \mu \qquad \boxed{\mu e^{\mu(z-1)}}\Big|_{z=1} = \mu$$

$$\langle k^2 \rangle = \frac{d^2}{dz^2} P_K^T(z) + \frac{d}{dz} P_K^T(z) \Big|_{z=1} = \mu^2 + \mu \quad \frac{d}{dz}\left(\mu e^{\mu(z-1)}\right) = \mu^2 e^{\mu(z-1)} + \mu e^{\mu(z-1)} = \mu^2 + \mu$$

$$\sigma_k^2 = \langle k^2 \rangle - \langle k \rangle = \mu.$$

$$\therefore \boxed{\text{mean} = \text{variance} = \mu}$$

Addition and random deletion of Poisson processes

Consider ω = random variable (discrete)

$= x+y$ for x, y = 2 independent Poisson random variables

$\langle x \rangle, \langle y \rangle$ = the mean

What is $P_\omega(\omega_0)$? Is this a Poisson PMF?

Way 1 $P_x^T(z) = e^{\langle x \rangle (z-1)}$ $P_y^T(z) = e^{\langle y \rangle (z-1)}$

$$\therefore P_\omega^T(z) = e^{\langle \omega \rangle (z-1)} = e^{(\langle x \rangle + \langle y \rangle)(z-1)} = P_x^T(z)\, P_y^T(z)$$

$$\Rightarrow P_\omega(\omega_0) = \frac{\langle \omega \rangle^{\omega_0} e^{-\langle \omega \rangle}}{\omega_0 !} = \frac{(\langle x \rangle + \langle y \rangle)^{\omega_0} e^{-(\langle x \rangle + \langle y \rangle)}}{\omega_0 !}, \quad \omega_0 = 0, 1, 2, \ldots$$

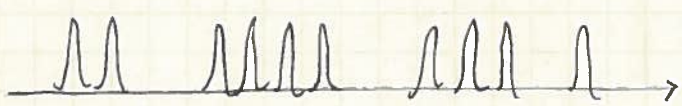
the arrival process representing all the arrivals in several indep. Poisson processes = Poisson

Suppose [$\exists$ an original process a new arrival process has probability p to appear as "an arrival" at t_1

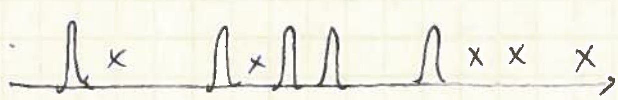
↳ $1-p$ probability not to appear at t_1

"independent random erasures"

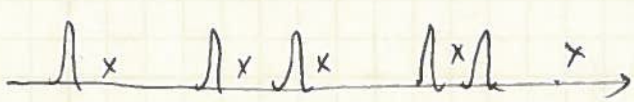
then the new process = Poisson "no memory" "independent"



$\to$ P.P



random deletion $\Rightarrow$ Poisson



every-other pulse erased $\Rightarrow$ no Poisson

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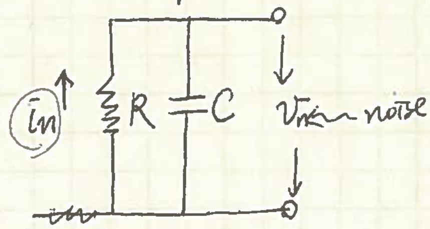
Example a lumped RC circuit



parallel

$$Z(\omega) = \frac{R}{1 + i\omega CR}$$

$$v_n(t) = i_n(t)\, Z$$

impedance

$$\downarrow F(\omega)$$

$$V_n(\omega) = I_n(\omega) Z(\omega)$$

$$= I_n(\omega) \frac{R}{1 + i\omega CR}$$

$$V_n^2(\omega) = \frac{R^2}{1 + (\omega CR)^2} I_n^2(\omega)$$

$V_n(\omega) V_n^*(\omega)$

$$\therefore \quad S_V(\omega) = \frac{R^2}{1 + (\omega CR)^2} S_I(\omega)$$