ECE770-T14/QIC 885: Quantum Electronics & Photonics

Problem Set 4, Winter 2014, Instructor: A. Hamed Majedi

Problem 1- A beam of spin-half object is sent through a Stern-Gerlach spin polarizer when a magnetic field is in z direction. One of the resulting beam is filtered out and the other beam is sent through another Stern-Gerlach spin polarizer, where the magnetic field has an inclination angle θ with respect to the z direction.

Calculate the relative number of objects that appear in the two beams leaving the second polarizer.

Problem 2- Consider a system of fermionic or bosonic objects created by the field $\hat{\psi}^{\dagger}(\mathbf{r})$ interacting under the following potential

$$V(\mathbf{r}) = \begin{cases} U & \text{if } \mathbf{r} < \mathbf{R} \\ 0 & \text{if } \mathbf{r} > \mathbf{R} \end{cases} \tag{1}$$

a) Write down this interaction in the second quantization form.

b) Switch to the momentum basis, where $\hat{\psi}(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k}\hat{c}_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{r}}$ and write the interaction in this new basis.

Problem 3- Macroscopic Quantum Model of Superconductivity

If the wave function of a system of particles with electric charge q is given by $\psi(\mathbf{r},t) = \sqrt{n(\mathbf{r},t)}e^{i\theta(\mathbf{r},t)}$,

a) By using the Schrodinger equation in the presence of EM field, show that the electric current density vector is

$$\mathbf{J} = qn(\mathbf{r},t)\left(\frac{\hbar}{m}\nabla\theta(\mathbf{r},t) - \frac{q}{m}\mathbf{A}(\mathbf{r},t)\right) \tag{2}$$

b) Using the imaginary portion of Schrodinger equation and the result of part (a), show that the following continuity equation must hold.

$$\nabla.\mathbf{J} + \frac{\partial\rho}{\partial t} = 0 \tag{3}$$

where $\rho(\mathbf{r},t) = qn(\mathbf{r},t)$.

c) Based on part (b), what is the physical interpretation of $|\psi(\mathbf{r},t)|^2$?

Generally, it is extremely difficult, if not impossible, to obtain the overall wave function of a system of many particles. For example, in metals, there are in the order of 10^{22} electrons per cubic centimeter. For sufficiently pure metals, the wave function of a single electron can be taken as a plane wave $e^{i\mathbf{k}.\mathbf{r}}$. So the overall wave function is a superposition of many plane waves with different wave vectors $\mathbf{k}$. Nevertheless, if all of the electrons had the same plane wave eigenfunction, the overall wavefunction of the system would be $\psi = Ae^{i\mathbf{k}.\mathbf{r}}$, with A some constant proportional to the number of the electrons in the sample. On its face, it sounds contrary to the Pauli's exclusion principle, that prohibits the electrons to share the same quantum state. Therefore, they must possess different wave functions under normal circumstances; and the wave function ψ introduced above seems invalid. The simplest way to overcome this restriction, is to assume that every two electrons bond together to form electron pairs, namely superelectrons. Superelectrons are exempt from the Pauli's exclusion principle, and thus the wave function ψ is quantum mechanically valid to describe a system of superelectrons with charge $q^* = 2e$ and $m^* = 2m$.

In fact, the situation explained above, is the simplified description of superconductors. However, it looks somehow strange that how two electrons with repulsive Coulomb force are able to form a bond. The qualitative answer is easy; there is an attractive force between the electrons and the positively charged lattice, which at low temperatures overcompensate the Coulomb repulsive force of the two electrons. Rigorous treatment of superconductivity is of course beyond the scope of this course; however, superconductors are one of the best manifestations of laws of quantum mechanics over macroscopic scales, i.e. we can measure them directly in the lab. In the next part, we explore some of the basic properties of superconductors based on their macroscopic wave function and the Schrodinger equation.

d) A sample of superconductor is represented by its wavefunction $\psi = \sqrt{n^*}e^{i\theta(\mathbf{r},t)}$, where n^* is the number density of the superelectrons and is assumed to be a constant over the space and time.

e) Using the result of the previous problem show that

$$\nabla \times (\Lambda \mathbf{J}) = -\mathbf{B} \tag{4}$$

where $\Lambda \equiv \frac{m^*}{(q^*)^2 n^*}$. This is called the second London's equation and is one of the two constitutive relations governing the electrodynamics of superconductors, without which Maxwell's equations are inadequate to account for the electrodynamics of superconductors.

f) Using Maxwell's equations and the second London's equation prove the first London's equation as

$$\frac{\partial}{\partial t}(\Lambda \mathbf{J}) = \mathbf{E} \tag{5}$$

It is easy to see that (5) implies that it is possible to have constant current in superconductors in the absence of electric field. This is in contrast to Ohm's law which characterizes conductors with their conductivity σ, whereas superconductors are characterized by Λ which is called London constant.

g) By integrating (2) over a closed contour and using the requirement that the wavefunction ψ is single-valued in the space, show that

$$\oint_C (\Lambda \mathbf{J}).d\mathbf{l} + \int_S \mathbf{B}.d\mathbf{S} = n\Phi_0 \tag{6}$$

where $\Phi_0 \equiv \frac{h}{2e}$ and is called flux quantum and n is an integer. Equation (6) implies that magnetic flux is quantized in superconductors.

We have already seen, in the lectures, that electrons can tunnel through a potential barrier. One of the physical realizations of this phenomenon is when two conductors are separated by a thin oxide layer. For a junction made of similar normal metals on both sides under no external bias, the probability of tunneling of electrons from left to right equals to that of right to left and hence no net current flows across the junction. Now consider that rather than normal metals we have superconductors on both sides. These types of junctions with very thin oxide layers are called Josephson junctions and the tunneling associated with the superelectrons is called Josephson tunneling.

h) Consider two superconducting samples represented by $\psi_L = \sqrt{n^*}e^{i\theta_L(\mathbf{r},t)}$ and $\psi_R = \sqrt{n^*}e^{i\theta_R(\mathbf{r},t)}$, respectively. If we bring the two samples into close proximity in order to make a Josephson junction the wavefunctions of the superconductors couple together due to tunneling of superelectrons. One can assume that the two Schrodinger equations governing the two superconductors are

$$i\hbar\frac{\partial \psi_L}{\partial t} = E_L\psi_L + K\psi_R \tag{7}$$

$$i\hbar\frac{\partial\psi_R}{\partial t} = E_R\psi_R + K\psi_L \tag{8}$$

where $E_{L,R}$ are the energy eigenvalues of the schrodinger for corresponding isolated superconductor and K is the coupling factor.

i) Show that the current and voltage across the junction obey the basic Josephson junction equations

$$J = J_c \sin(\theta) \tag{9}$$

$$\frac{\partial\theta}{\partial t} = \frac{eV}{\hbar} \tag{10}$$

where $\theta \equiv \theta_R - \theta_L$, $E_L - E_R \equiv eV$ and J_c is a constant proportional to K.

j) What is the current flowing through the junction if there is no external voltage? (d.c. Josephson effect)

k) What is the current flowing through the junction if the external voltage is constant $V = V_0$? (a.c. Josephson effect)

Due: Wednesday March 26, 2014. (before starting the class)