Lecture 1: Superconductors, phenomenology - part 1

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I. (BRIEF) HISTORY OF SUPERCONDUCTIVITY

Superconductivity was discovered in 1911 by Kamerlingh Onnes and collaborators, at the University of Leiden. Onnes and collaborators were interested in characterizing the dependence of resistivity of metals with temperature. At about 4K, they observed a sudden drop in resistivity of mercury, of many orders of magnitude. Over the next decades, many metals and compounds were found to be superconducting. It is important to note that the discovery of superconductivity in Leiden was enabled by the liquefaction of 4He, a technique which allowed reaching temperatures of a few Kelvins. In fact, the fields of low temperature physics and superconducitivity have been closely inter-connected throughout the history of modern superconductivity: discoveries in low temperature physics enabled advancements in superconductivity research and vice versa.

Most metals and intermetallic compounds have superconducting transition temperatures in the 1-10 K range. Most importantly, at low temperature they share not only the property of perfect conductivity, but also other generic properties (ie field exclusion, specific heat suppresion at low temperatures, etc). These properties are well described by the theory of superconductivity discovered by Bardeen, Cooper and Schrieffer (BCS) in 1956 [1]. This theory explains superconductivity as being a new thermodynamic phase of the electrons in a metal which appears below a metal/compound specific transition temperature. The attractive interaction mediated by phonons plays a major role.

In 1986, Bednorz and Müller discovered another class of superconductors [2]. These materials have typically much higher transition temperatures, for which reason they are called "high-Tc" superconductors. While many phenomena exhibited by this new class are similar to those of BCS superconductors, there is no consensus on the microscopic theory. Many high-Tc superconductors have a critical temperature which exceeds 77 K, the liquefaction temperature of liquid nitrogen, which makes them very interesting for applications. In recent years, other types of superconductors have been discovered. These include iron superconductors [3] and magnesium diboride [4].

The focus in this course is on the study of conventional superconductors. In quantum electrical circuits, conventional superconductors are almost exclusively used[1]. While their transition temperature is an inconvenience, they have advantages which are essential:

[1] There are experimental efforts in progress which address the use of high-Tc superconductors as quantum devices, see *eg* [5].

- their physical properties are much better understood
- microwave losses are low
- microfabrication is much better established (more expertise accumulated over time, but also intrinsic factors)
- a long existing tradition in superconducting electronics. This is the field concerned with applications of superconductors based on classical devices (examples include SQUID magnetometers, voltage standards, superconducting magnets, digital electronics)

II. SIGNATURES OF SUPERCONDUCTIVITY

In this section we review the properties of metals in the superconducting state. As it turns out, these properties are generic in all conventional superconductors. The only major difference in physical properties arises between the so-called type-I and -II superconductors (this distinction will be discussed below).

A. Perfect conductivity

This is the main signature of the superconducting state, and a quite intriguing property when looked at in contrast with the case of a metal in normal state. This is one of the most relevant properties for practical application (*eg* it enables superconducting magnets). . It is found that the conductivity at finite frequencies is in fact finite and it increases as one approaches a frequency corresponding to the superconducting energy excitation gap - a quantity to be introduced later.

B. Magnetic response

At a macroscopic scale, it is found that superconductors fully expell an applied magnetic field (See Fig. 1). This perfect diamagnetic behaviour disappears when the magnetic field exceeds the critical field. The latter has a value specific to each superconductor. Beyond the critical field there is no magnetic response. This type of behaviour is characteristic to

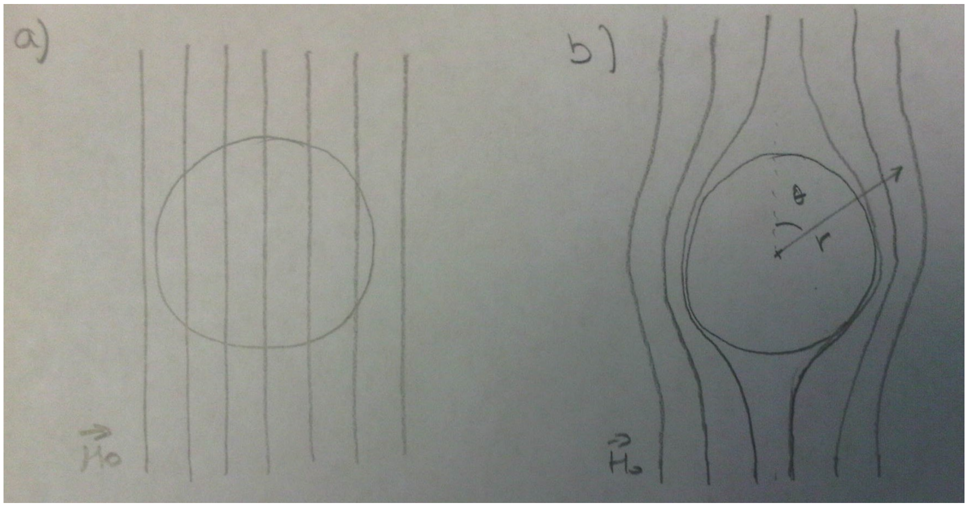

FIG. 1. Distribution of magnetic field around a metallic sphere: (a) below the transition temperature, and (b) above the transition temperature. In (b) the polar coordinate system is indicated.

type-I superconductors. For *type-II* superconductors, there is in addition a region of fields over which the field penetrates gradually through tubes of magnetic flux called vortices.

C. Specific heat

The specific heat of a normal metal at low temperature varies proportionally with temperature. For a superconductor, the specific heat is found to be suppressed at low temperatures. This is a signature of the presence of a superconducting gap.

D. Energy gap

Excitations in a superconductor are characterized by a minimum energy. The magnitude of this gap is (up to a constant of the order unity) proportional to $k_B T_c$.

E. Flux quantization

In non simply connected superconducting specimens it is fund that the magnetic flux through a hole is quantized in units of the flux quantum $\Phi_0 = h/2e$.

F. Weak link phenomena

Superconductors with Josephson junctions display nonlinear effects, which are key in applications in superconducting electronics, as well as in superconducting qubits.

III. ELECTRODYNAMICS OF SUPERCONDUCTORS

A. London equation

In 1935, London and London found a simple equation which relates the current in a superconductor and the vector potential of an applied magnetic field [6]. Further developments showed that the London equation has a very limited domain of applicability: it describes the response of superconductors in weak magnetic fields, for slow variations of the field. Moreover, the London equation represent a particular instance of weak-field response; it is a local equation which is only consistent with the more general non-local theory when the coherence length is shorter than the penetration depth (for the so-called type-II superconductors). With these limitations in mind, we provide a possible derivation of this equation and also discuss a simple consequence - the diamagnetic response and the existence of a finite magnetic field penetration depth.

The London equation is

$$\boldsymbol{j} = -\frac{1}{\Lambda}\boldsymbol{A}, \tag{1}$$

where $\boldsymbol{j}$ is the current density, and $\boldsymbol{A}$ is the vector potential corresponding to a magnetic field $\boldsymbol{B}$. The London equation can be connected with the zero dissipation property of a superconductor. Indeed, we assume that conduction is due to electrons of charge e (this will be reexamined later, when we will see that Cooper pairs are the "elementary carriers" of charge), mass m, and density n_s moving under the action of an applied field $\boldsymbol{E}$. We have

$$m\dot{\boldsymbol{v}}_s = e\boldsymbol{E} \tag{2}$$

which can be rewritten as a function of the current density

$$\boldsymbol{j} = n_s e \boldsymbol{v}_s \tag{3}$$

as

$$\boldsymbol{E} = \frac{d}{dt}\left(\Lambda \boldsymbol{j}\right) \tag{4}$$

where we introduced

$$\Lambda = \frac{m}{n_s e^2}. \tag{5}$$

We compare 4 with the equation defining the electric field as a function of the vector $\boldsymbol{A}$ and scalar ϕ_e potentials:

$$\boldsymbol{E} = -\frac{d\boldsymbol{A}}{dt} - \nabla \phi_e \tag{6}$$

Assuming the lack of an electrostatic potential in the above and comparing with 4 we can deduce indeed 1. The above argument cannot be considered a proper derivation of London equation. It does show however that the equation is consistent with the assumption of perfect conductivity for a superconductor (but see discussion below).

Based on London equation we can explain the diamagnetic response of a superconductor. Take the curl of one of Maxwell's equation, that is

$$\nabla \times \boldsymbol{H} = \boldsymbol{j}. \tag{7}$$

and use London equation 1. We have

$$\nabla \times (\nabla \times \boldsymbol{H}) = -\frac{1}{\Lambda} \nabla \times \boldsymbol{A}. \tag{8}$$

We then use

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} \tag{9}$$

(note this relation correspond to a microscopic viewpoint, no averaging of currents) and

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{10}$$

to find

$$\nabla \times (\nabla \times \boldsymbol{B}) = -\frac{\mu_0}{\Lambda} \boldsymbol{B}. \tag{11}$$

We use the identity

$$\nabla \times (\nabla \times \boldsymbol{B}) = \nabla (\nabla \boldsymbol{B}) - \nabla^2 \boldsymbol{B} \tag{12}$$

to find

$$\nabla^2 \boldsymbol{B} = \frac{1}{\lambda_L^2} \boldsymbol{B}, \tag{13}$$

where we introduced the *London penetration length*

$$\lambda_L = \sqrt{\frac{\Lambda}{\mu_0}} = \sqrt{\frac{m}{\mu_0 n_s e^2}}. \tag{14}$$

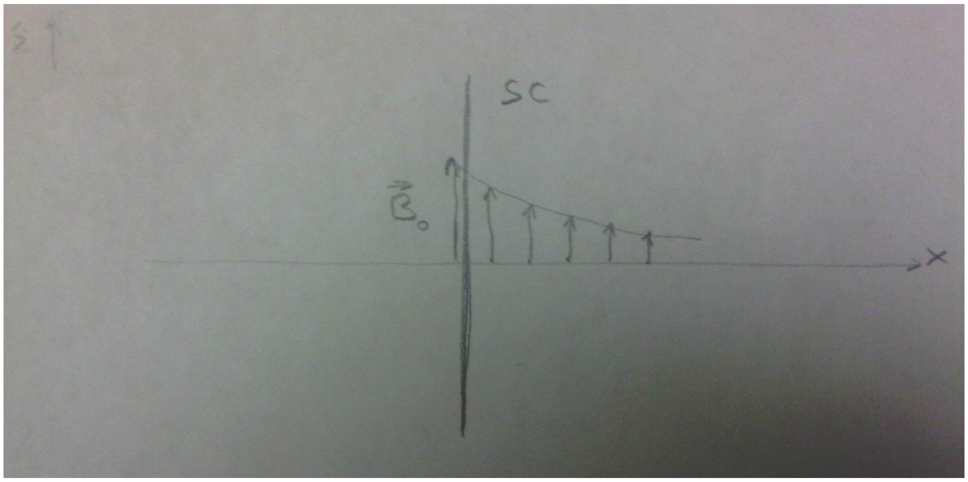

FIG. 2. Penetration of a magnetic field into a superconductor, when the field is parallel to the surface of the superconductor, which occupies the semispace $x > 0$.

Assume the situation represented in Fig. 2. A superconductor occupies the semispace $x > 0$. Assuming that the field at the surface is oriented parallel to the surface, we take its direction to be the $\hat{\boldsymbol{z}}$ direction and denote its amplitude by B_0. The field is continuous at the interface. Inside the superconductor we assume the field to remain parallel to the surface, and of the form $\boldsymbol{B}_{inside} = B_z(x)\hat{\boldsymbol{z}}$. This field automatically fulfills

$$\nabla \boldsymbol{B} = 0. \tag{15}$$

Using 13:

$$B_z(x) = B_0 e^{-x/\lambda_L}. \tag{16}$$

. We find that the magnetic field inside the superconductor is screened deep inside the superconductor. However, there is a layer of thickness λ_L over which the field penetrates. Eq. 16 implies the existence of screening current:

$$\boldsymbol{j} = \frac{1}{\mu_0} \nabla \times \boldsymbol{B} = -\frac{1}{\mu_0} \frac{\partial}{\partial x} B_z \hat{\boldsymbol{y}} = \frac{1}{\mu_0 \lambda_L} B_0 e^{-x/\lambda_L} \hat{\boldsymbol{y}}. \tag{17}$$

As the magnetic field, the screening current runs in a region of thickness $\approx \lambda_L$ at the surface.

Could we apply a perpendicular field? Assume we had $\boldsymbol{B} = B_x(x)\hat{\boldsymbol{x}}$. Then 13 would predict an exponential decay, which is inconsistent with 15.

B. Sphere in an applied magnetic field; demagnetization factor

We consider the response of a superconducting sphere of radius a to an applied expternal field (see Fig.1) $H_0\hat{\boldsymbol{z}}$. We start considering the field $\boldsymbol{H}_{ext}$ outside the sphere. We have

$$\nabla \times \boldsymbol{H}_{ext} = 0 \tag{18}$$

as there are no sources (assumed to be far away) and

$$\nabla \boldsymbol{H}_{ext} = 0 \tag{19}$$

which follows from $\boldsymbol{B}_{ext} = \mu_0 \boldsymbol{H}_{ext}$ and $\nabla \boldsymbol{B}_{ext} = 0$. Eq. 18 justifies the introduction of a scalar potential Φ_{ext} such that

$$\boldsymbol{H}_{ext} = -\nabla \Phi_{ext}. \tag{20}$$

Use 20 in 19. We have

$$\triangle \Phi_{ext} = 0 \tag{21}$$

For a problem with azimuthal symmetry the general solution of the Laplace equation 21 is [7]

$$\Phi_{ext}(r,\theta) = \sum_{l>0} (A_l r^l + B_l r^{-(l+1)}) P_l(\cos\theta), \tag{22}$$

where P_l is the Legendre polynomial of order l and we are using spherical coordinates - radius r and polar angle θ. We use first the boundary condition for $r \to \infty$: $\boldsymbol{H}(r \to \infty) \to H_0\hat{\boldsymbol{z}}$. Looking at the radial component ($H_r = -\frac{\partial}{\partial r}\Phi_{ext}$), $\boldsymbol{H}_r(r \to \infty) \to H_0 \cos\theta$. This means that among the terms $A_l r^l$, only the terms with $l = 0$ and $l = 1$ remain. Taking $P_l(\cos\theta) = \cos\theta$, we find $A_1 = -H_0$. Then we use the boundary condition at the surface of the sphere: $\boldsymbol{H}_r(r = a) = 0$ (the normal component vanishes). This implies $-\frac{\partial}{\partial r}\left(A_l r^l + B_l r^{-(l+1)}\right)$ which together with the conditions for the A_l implies that all the B_l coefficients vanish, except for $B_1 = A_1 \frac{a^3}{2}$. The final solution for the field *outside* the sphere is

$$H(r,\theta) = H_0\left(1 - \frac{a^3}{r^3}\right)\cos\theta\hat{\boldsymbol{r}} - H_0\left(1 + \frac{a^3}{2r^3}\right)\sin\theta\hat{\boldsymbol{\theta}}. \tag{23}$$

The tangential contribution of the field at the surface of the sphere reaches a maximum in the equatorial plane $\theta = \pi/2$, where it equals $3H_0/2$. Given this result, one can distinguish three cases:

- $H_0 < 2/3H_c$ with H_c the critical field. The tangential field at all positions is smaller than the critical field, so the sphere remains superconducting.

- $2/3H_c < H_0 < H_c$. The tangential field exceeds the critical field H_c over part of the sphere. The sphere is in the so-called mixed state

- $H_c < H_0$. The sphere is in the normal state

The example here serves to introduce one important notion - the demagnetization factor. For an arbitrary specimen, the demagnetization factor n is defined as $1-\frac{H_0}{H_{max}}$, where H_{max} is the maximum field at the surface of the superconductor (orientation of the specimen matters). For $H_0/(1-n) < H_c$ the specimen remains superconducting. For a sphere, $n = 1/3$. For a long cylinder the demagnetization factor $n \to 0$ (Strictly speaking the field over most of the surface is nearly equal to the applied field, however there might be more significant differences over a relatively unimportant region at the ends of the cylinder). This situation simplifies the analysis of the phase transition in an applied field and is used in the next section.

IV. THERMODYNAMICS OF SUPERCONDUCTORS

A superconductor expels magnetic field as long as the field is below its critical field. In the previous section we discussed the connection between the diamagnetic response and the property of perfect conductivity. Consider the following experiment: a superconductor is cooled in zero applied field, through the critical temperature. If a magnetic field is applied the superconductor will repel the field. This behaviour can be seen as a consequence of the property of infinite conductivity: currents are generated which completely screen the applied field and these currents do not decay.. One may also bring the superconductor in the final configuration with a temperature below the transition temperature and an applied field by first applying the field and then cooling the specimen. The field is expelled as well, however in this case one cannot explain this based on the property of infinite conductivity. The state with the metal in the SC state arises for both cooldwon/field application sequences is in fact the *thermodynamically stable* state of the superconductor.

A. Relation between the critical field the free energy

We can use thermodynamics to extract useful conclusions on quantities related to the experimental state. To avoid complications related to a finite demagnetization factor, we assume a cylindrical specimen aligned with the applied field H. Field repulsion occurs for $H < H_c(T)$, where $H_c(T)$ is the temperature dependent critical field. Lets consider the setup shown in figure 3[2]. A superconducting cylindrical specimen of length L and radius r_0 is placed inside a solenoid of equal length and radius r_1 and N turns. The coil is supplied by a current I which generates a field $B = \mu_0 N/LI$ in the absence of the superconductor. We calculate the total energy of the superconductor and the field in two situations

- the metal is in the normal state (the field occupies the entire space inside the solenoid)

$$\mathcal{F}_n = \pi r_0^2 L F_n + \pi r_1^2 L \frac{1}{2\mu_0} B^2 \tag{24}$$

- the metal is in the superconducting state (the field occupies the space inside the solenoid but outside the superconductor)

$$\mathcal{F}_s = \pi r_0^2 L F_s + \pi (r_1^2 - r_0^2) L \frac{1}{2\mu_0} B^2 \tag{25}$$

When the current is increased and it reaches a value such that $B = \mu H_c(T)$ the specimen undergoes the transition to the normal state. We have

$$\mathcal{F}_n - \mathcal{F}_s = \pi r_0^2 L (F_n - F_s) + \pi r_0^2 L \frac{1}{2\mu_0} B^2 \tag{26}$$

The free energy increases, which is not what is expected for a phase transition ($F_n - F_s$ is positive since the superconducting state is more stable; this difference does not depend on magnetic field). This can be explained by the fact that the source which maintains the current I does work on the system (superconductor + field). The source develops a voltage V_s to counterbalance the induced electromotive source $-d\Phi/dt$ with Φ the magnetic flux inside the solenoid. The work is thus given by

$$W = \int_{t_0}^{t_1} dt V_s I = I(\Phi_2 - \Phi_1) \tag{27}$$

[2] We follow the discussion in section 2.1 of [8].

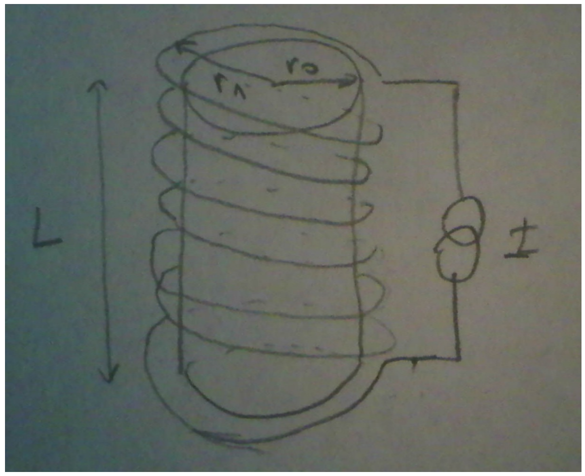

FIG. 3. A superconducting cylinder of radius r_0 and length L is inserted inside a N turns solenoid, of equal length, radius r_1, and supplied by a current source I

With t_1 and t_2 the times at the beginning/end of the transition and Φ_1 and Φ_2 are the magnetic flux before and after the transition respectively. We have

$$W = \mathcal{F}_n - \mathcal{F}_s \tag{28}$$

resulting in

$$\frac{B}{\mu \frac{N}{L}} N \pi r_0^2 B = \pi r_0^2 L (F_n - F_s) + \pi r_0^2 L \frac{1}{2\mu_0} B^2 \tag{29}$$

which finally gives (with $B = \mu_0 H_c(T)$)

$$F_n(T) - F_s(T) = \frac{1}{2} \mu_0 H_c(T)^2 \tag{30}$$

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