Quantum Electronics and Photonics

Assignment 5

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Contents

Problem 1

(a)

A one-dimensional cavity is shown in figure 1. It's assumed that the electrical field is single mode and it's polarized in x direction. So spatial part of the field operator is simply:

$$u(\mathbf{r}) = \sqrt{\frac{2}{V}} \sin(kz) \qquad k = \frac{m\pi}{L} \tag{1}$$

The boundary condition at $z = L$ yields the allowed frequencies which are connected to k_m by free space disperssion relation. V in (1) is the effective volume of the cavity. Using quantization rules in QED electrical

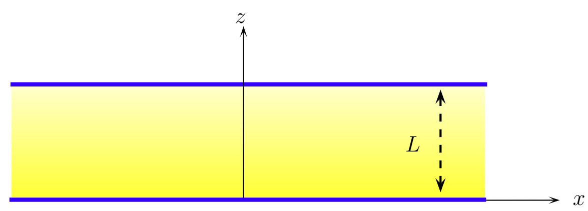

Figure 1: 1D cavity structure

field operator is:

$$\hat{E}_x(z,t) = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}} \left[\hat{a}(t) - \hat{a}^\dagger(t)\right] \sqrt{\frac{2}{V}} \sin(kz) \tag{2}$$

writing down the equation of motion for the ladder operators we can simply conclude that:

$$\hat{a}(t) = \hat{a}(0)\exp(-i\omega t) \qquad \hat{a}^\dagger(t) = \hat{a}^\dagger(0)\exp(i\omega t) \tag{3}$$

To calculate the uncertainty of electric filed we should first consider a specific state. If we choose the photon number state we arrive at:

$$\langle \hat{E}_x(z,t)\rangle = i\sqrt{\frac{\hbar\omega}{\epsilon_0 V}} \langle n|\hat{a}(t) - \hat{a}^\dagger(t)|n\rangle \sin(kz) = 0 \tag{4}$$

$$\begin{aligned}\langle \hat{E}_x^2(z,t)\rangle &= \frac{\hbar\omega}{\epsilon_0 V}\langle n|\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} - \hat{a}^2 - \hat{a}^{\dagger 2}|n\rangle \sin^2(kz) \\ &= \frac{\hbar\omega}{\epsilon_0 V}\langle n|2\hat{N} + 1|n\rangle \sin^2(kz) = \frac{\hbar\omega}{\epsilon_0 V}(2n+1)\sin^2(kz)\end{aligned} \tag{5}$$

where $\hat{N}$ is the number operator. Combining everything we obtain:

$$\langle \Delta E_x(z,t)\rangle_n = \sqrt{\langle E_x^2\rangle - \langle E_x\rangle^2} = \sqrt{\frac{\hbar\omega}{\epsilon_0 V}(2n+1)}\sin(kz) \tag{6}$$

(b)

Using creation and annihilation commutation relation we get:

$$[\hat{N}, \hat{a}] = \left[\hat{a}^\dagger, \hat{a}\right]\hat{a} = -a \tag{7}$$

$$[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger\left[a, \hat{a}^\dagger\right] = a^\dagger \tag{8}$$

The electric field operator is given in (2). We can write:

$$[\hat{E}_x(z,t), \hat{N}] = i\sqrt{\frac{\hbar\omega}{V\epsilon_0}}\sin(kz)\left[\hat{a}(t) - \hat{a}^\dagger(t), \hat{N}\right] = i\sqrt{\frac{\hbar\omega}{V\epsilon_0}}\sin(kz)\left\{\hat{a}(t) + \hat{a}^\dagger(t)\right\} \tag{9}$$

(c)

Using well-known uncertainty inequality between incompatiible observables we have:

$$\Delta E_x \Delta N \geq \frac{1}{2}\left|\langle\left[\hat{E}_x,\hat{N}\right]\rangle\right| \tag{10}$$

Using (9) we can write:

$$\Delta E_x \Delta N \geq \sqrt{\frac{\hbar\omega}{4V\epsilon_0}}\sin(kz)\left\{\langle a\rangle + \langle a^\dagger\rangle\right\} \tag{11}$$

Please note that in fock space $\Delta N = 0$ and $\langle a\rangle = \langle a^\dagger\rangle = 0$.

Problem 2

(a)

Strating with

$$[\hat{a},\hat{a}^\dagger] = 1$$

we arrive at:

$$[\hat{N},\hat{a}^\dagger] = [\hat{a}^\dagger\hat{a},\hat{a}^\dagger] = \hat{a}^\dagger[\hat{a},\hat{a}^\dagger] + [\hat{a}^\dagger,\hat{a}^\dagger]\hat{a} = \hat{a}^\dagger \tag{1}$$

Note that in (1) we have used the following identity:

$$[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B} \tag{2}$$

In the same line of reasoning we get:

$$[\hat{N},\hat{a}] = [\hat{a}^\dagger\hat{a},\hat{a}] = \hat{a}^\dagger[\hat{a},\hat{a}] + [\hat{a}^\dagger,\hat{a}]\hat{a} = -\hat{a} \tag{3}$$

(b)

In the prsence of nonlinearity the Hamiltonian is:

$$\hat{H} = \hbar\omega_0\hat{a}^\dagger\hat{a} - \frac{\hbar\kappa}{2}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} \tag{4}$$

hence:

$$[\hat{N},\hat{H}] = [\hat{a}^\dagger\hat{a},\hbar\omega_0\hat{a}^\dagger\hat{a} - \frac{\hbar\kappa}{2}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}] = -\frac{\hbar\kappa}{2}[\hat{N},\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}] \tag{5}$$

Successive application of (2) we can write:

$$[\hat{N},\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}] = \hat{a}^\dagger[\hat{N},\hat{a}^\dagger\hat{a}\hat{a}] + [\hat{N},\hat{a}^\dagger]\hat{a}^\dagger\hat{a}\hat{a} \tag{6}$$

$$[\hat{N},\hat{a}^\dagger\hat{a}\hat{a}] = \hat{a}^\dagger[\hat{N},\hat{a}\hat{a}] + [\hat{N},\hat{a}^\dagger]\hat{a}\hat{a} \tag{7}$$

$$[\hat{N},\hat{a}\hat{a}] = \hat{a}[\hat{N},\hat{a}] + [\hat{N},\hat{a}]\hat{a} \tag{8}$$

So we have:

$$[\hat{N},\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}] = -2\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + 2\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} = 0 \tag{9}$$

combining all we get:

$$[\hat{N},\hat{H}] = 0 \tag{10}$$

We can employ a simpler analysis as well. If we write the Hamiltonian in the following form:

$$\hat{H} = \hbar\omega_0\hat{N} - \frac{\hbar\kappa}{2}\hat{a}^\dagger\hat{N}\hat{a}$$

Problem 2 continued on next page...

using (1) we can write:

$$\hat{H} = \hbar\omega_0\hat{N} - \frac{\hbar\kappa}{2}\left(\hat{N}\hat{a}^\dagger - \hat{a}^\dagger\right)\hat{a} = \hbar\omega_0\hat{N} - \frac{\hbar\kappa}{2}\left(\hat{N}^2 - \hat{N}\right) \tag{11}$$

We can readily see that the Hamiltonian is a function number operator and equation (10) holds.

(c)

Since $\hat{N}$ and $\hat{H}$ commutate , they can be simultaneously diagonalize. Eigenstates of the number operator are not degenerate so we can use number operator basis function as the eigenstates of the Hamiltonian:

$$\hat{H}|n\rangle = \hbar\omega_0 n|n\rangle - \frac{\hbar\kappa}{2}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|n\rangle \tag{12}$$

Using

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

we obtain:

$$\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|n\rangle = \sqrt{n(n-1)(n-1)n}|n\rangle = n(n-1)|n\rangle \tag{13}$$

So we have:

$$\hat{H}|n\rangle = \left\{\hbar\omega_0 n - \frac{\hbar\kappa}{2}n(n-1)\right\}|n\rangle \tag{14}$$

we could also use (11) which leads to the same result:

$$\hat{H}|n\rangle = \hbar\omega_0\hat{N}|n\rangle - \frac{\hbar\kappa}{2}\left(\hat{N}^2 - \hat{N}\right)|n\rangle = \left\{\hbar\omega_0 n - \frac{\hbar\kappa}{2}n(n-1)\right\}|n\rangle \tag{15}$$

(d)

We have already proved that $\hat{N}$ and $\hat{H}$ commute. So we can immediately conclude that the number operator is a constant of motion. The Hesenberg time evolution for number operator is:

$$\frac{d\hat{N}}{dt} = \frac{1}{i\hbar}[\hat{N}, \hat{H}] = 0 \quad \Longrightarrow \quad \hat{N} = \hat{N}(0) \tag{16}$$

Since $\hat{N}$ is a constant motion the number photons inside the cavity remains constant during time evolution.

(e)

The Heisenberg time evolution equation for the annihilation operator is:

$$\frac{d\hat{a}}{dt} = \frac{1}{i\hbar}[\hat{a}, \hat{H}] = -i[\hat{a}, \omega\hat{N} - \frac{\kappa}{2}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}] = -i\omega_0\hat{a} + \frac{i\kappa}{2}[\hat{a}, \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}] \tag{17}$$

Using (2) identity we obtain:

$$[\hat{a}, \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}] = \hat{a}^\dagger\hat{a}^\dagger[\hat{a}, \hat{a}\hat{a}] + [\hat{a}, \hat{a}^\dagger\hat{a}^\dagger]\hat{a}\hat{a} = 2\hat{a}^\dagger\hat{a}\hat{a} = 2\hat{N}\hat{a} \tag{18}$$

Since $\hat{N}$ is a constant of motion we can write:

$$\frac{d\hat{a}(t)}{dt} = -i\omega_0\hat{a}(t) + i\kappa\hat{N}(0)\hat{a}(t) \tag{19}$$

Problem 2 continued on next page. . .

To solve this first order differential equation we define am auxiliary operator:

$$\hat{\zeta}(t) = \hat{a}(t)\exp(i\omega_0 t) \tag{20}$$

Inserting in (19) we obtain:

$$\frac{d\hat{\zeta}(t)}{dt} = i\kappa\hat{N}(0)\hat{\zeta}(t) \tag{21}$$

So we arrive at:

$$\hat{\zeta}(t) = e^{i\kappa\hat{N}(0)t}\zeta(\hat{0}) \tag{22}$$

finally we get:

$$\hat{a}(t) = \exp\left[-i\omega_0 t + i\kappa\hat{N}(0)t\right]\hat{a}(0) \tag{23}$$

(f)

Applying transpose conjugate operator on both sides of (23) we obtain:

$$\hat{a}^\dagger(t) = \hat{a}^\dagger(0)\exp\left[i\omega_0 t - i\kappa\hat{N}^\dagger(0)t\right] = \hat{a}^\dagger(0)\exp\left[i\omega_0 t - i\kappa\hat{N}(0)t\right] \tag{24}$$

From operator algebra we know that :

$$[A, B] = \lambda A \quad \Longrightarrow \quad Ae^B = e^\lambda e^B A \tag{25}$$

Since $[\hat{a}^\dagger(0), \hat{N}(0)] = -\hat{a}^\dagger(0)$ then:

$$\hat{a}^\dagger(t) = \exp\left[i\kappa t + i\omega_0 t - i\kappa\hat{N}(0)t\right]\hat{a}^\dagger(0) \tag{26}$$

(g)

If just one photon is lost from the cavity total energy of the system inside the cacity would change. We can simply calculate the energy difference in the number state. Before $t = T$ the quantum state of the field is given by $|n\rangle$ and by photon annihilation the state in the nuumber state would be $|n-1\rangle$ so we have:

$$\hbar\omega_{ph} = \Delta E = E_1 - E_2 = \langle n|\hat{H}|n\rangle - \langle n-1|\hat{H}|n-1\rangle \tag{27}$$

Using (15) we can write:

$$\begin{aligned}
\Delta E &= \left\{\hbar\omega_0 n - \frac{\hbar\kappa}{2}n(n-1)\right\} - \left\{\hbar\omega_0(n-1) - \frac{\hbar\kappa}{2}(n-1)(n-2)\right\} \\
&= \hbar\omega_0 - \hbar\kappa(n-1)
\end{aligned} \tag{28}$$

So the spectrometer measures:

$$\omega_{ph} = \frac{\Delta E}{\hbar} = \omega_0 - \kappa(n-1) \tag{29}$$

Note that we can measure the frequency with certainty.

Problem 3

(a)

Cherent photon state can be derived by applying generalized displacement operator on the vacume state:

$$|\alpha\rangle = D(\alpha)|0\rangle = \exp[\alpha\hat{a}^\dagger - \alpha^*\hat{a}]|0\rangle \tag{1}$$

According to the lecture notes, two diffetrent coherent states characterized by α and β are corrolated as:

$$\langle\alpha|\beta\rangle = \exp\left[-\frac{1}{2}(|\alpha|^2 + |\beta|^2) + \alpha^*\beta\right] \tag{2}$$

The state $|\psi\rangle$ is defined by:

$$|\psi\rangle = A(|\alpha\rangle + |-\alpha\rangle) \tag{3}$$

So we get:

$$\langle\psi|\psi\rangle = |A|^2 \{\langle\alpha|\alpha\rangle + \langle-\alpha|-\alpha\rangle + 2\Re\langle-\alpha|\alpha\rangle\} \tag{4}$$

Using (2) we arrive at:

$$\langle\psi|\psi\rangle = |A|^2 \left\{2 + 2e^{-2|\alpha|^2}\right\} \implies A = \frac{1}{\sqrt{2 + 2e^{-2|\alpha|^2}}}$$

(b)

If α is very large normalization costant is:

$$\alpha \gg 1 \qquad \implies \qquad A \approx \lim_{\alpha\to\infty} \frac{1}{\sqrt{2 + 2e^{-2|\alpha|^2}}} = \frac{1}{\sqrt{2}} \tag{5}$$

this means that we would have two uncorrolated states.

(c)

According to the lecture notes the coherent state $|\alpha\rangle$ can be expanded in number states basis:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{6}$$

So $|\psi\rangle$ is:

$$|\psi\rangle = \frac{2e^{-|\alpha|^2/2}}{\sqrt{2 + 2e^{-2|\alpha|^2}}} \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{\sqrt{(2k)!}}|2k\rangle \tag{7}$$

If α is too large then:

$$|\psi\rangle \approx \sqrt{2}e^{-|\alpha|^2/2} \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{\sqrt{(2k)!}}|2k\rangle \tag{8}$$

So photon number probability distribution is:

$$p(n) = \begin{cases} \frac{2e^{-|\alpha|^2}}{1+e^{-2|\alpha|^2}} \frac{\alpha^{2n}}{n!} \approx 2e^{-|\alpha|^2} \frac{\alpha^{2n}}{n!} & \text{n even} \\ 0 & \text{n odd} \end{cases} \tag{9}$$

This looks like Poisson statistics.

Problem 3 continued on next page. . .

(d)

The density operator associated with this state is:

$$\hat{\rho} = |\psi\rangle\langle\psi| = |A|^2 \{|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle\alpha| + |\alpha\rangle\langle-\alpha| + |-\alpha\rangle\langle-\alpha|\} \tag{10}$$

This density operator can be also written in terms of orthogonal number states. Using (7) we obtain:

$$\hat{\rho} = \frac{2e^{-|\alpha|^2}}{1+e^{-2|\alpha|^2}} \sum_{nm} \frac{\alpha^{2n}\alpha^{*2m}}{\sqrt{(2n)!(2m)!}} |2n\rangle\langle 2m| \tag{11}$$

(e)

Te probe state is another coherent state expressed by $|\mu\rangle$, Q function can be calculated as follows:

$$Q(\mu) = \frac{1}{2\pi}\langle\mu|\hat{\rho}|\mu\rangle \tag{12}$$

Using (10) we can write:

$$Q(\mu) = \frac{1}{2\pi}|A|^2 \left\{ |\langle\mu|\alpha\rangle|^2 + |\langle-\mu|\alpha\rangle|^2 + 2\Re\langle\mu|\alpha\rangle\langle-\alpha|\mu\rangle \right\} \tag{13}$$

More explicitly:

$$Q(\mu) = \frac{1}{2\pi}\frac{1}{2+2e^{-2|\alpha|^2}} \left\{ \exp(-|\mu-\alpha|^2) + \exp(-|\mu+\alpha|^2) + 2\Re\exp(-|\mu|^2 - |\alpha|^2 + \mu^*\alpha - \alpha^*\mu) \right\} \tag{14}$$

This expression can be simplifyed more as:

$$Q(\mu) = \frac{1}{4\pi(1+e^{-2|\alpha|^2})} \left\{ \exp(-|\mu-\alpha|^2) + \exp(-|\mu+\alpha|^2) + 2\Re\exp(-|\mu|^2 - |\alpha|^2)\cos[\Im(\mu^*\alpha)] \right\} \tag{15}$$

Problem 4

(a)

The expreimental setup has been shown in figure 2. As shown in the figure the original beam splitter accepts two input beams and and produces two other beams which are a linear combination of the inputs. According to [1] the classical EM field in the output braches can be expressed as below:

$$\mathbf{E}_1' = \frac{\mathbf{E}_1 - \mathbf{E}_2}{\sqrt{2}} \tag{1}$$

$$\mathbf{E}_2' = \frac{\mathbf{E}_1 + \mathbf{E}_2}{\sqrt{2}} \tag{2}$$

These transformtion can be applied to the quantized version of the field. To do this we shoukd just use the same linear combinations for annihilation and creation operators. If we use the same notaions to signify these operators we can write:

$$a_1' = \frac{a_1 - a_2}{\sqrt{2}} \tag{3}$$

$$a_2' = \frac{a_1 + a_2}{\sqrt{2}} \tag{4}$$

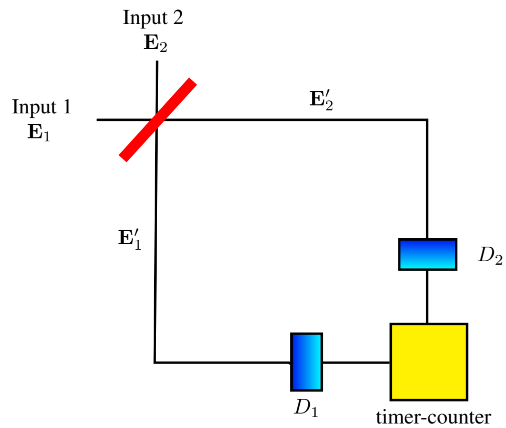

Figure 2: HBT experiment

the second-order correlation function is then obtained from:

$$g^{(2)}(\tau) = \frac{\langle n'_1(t)n'_2(t+\tau)\rangle}{\langle n'_1(t)\rangle\langle n'_2(t+\tau)\rangle} \tag{5}$$

Using number operator we can write:

$$g^{(2)}(0) = \frac{\langle a_1'^{\dagger}a'_1a_2'^{\dagger}a'_2\rangle}{\langle a_1'^{\dagger}a'_1\rangle\langle a_2'^{\dagger}a'_2\rangle} \tag{6}$$

We can rewrite this equation in terms of unprimed ladder operators. Actually we have to use this new representation because we know about the number state of the input braches. It's assumed that there is no photon in one of the inputs and the second one is in $|n\rangle$ state. If $|\phi\rangle$ describes the number state of the inputs we can write:

$$|\phi\rangle = |n, 0\rangle \tag{7}$$

Using (3) and (4) we have:

$$g^{(2)}(0) = \frac{\langle(a_1^\dagger - a_2^\dagger)(a_1 - a_2)(a_1^\dagger + a_2^\dagger)(a_1 + a_2)\rangle}{\langle(a_1^\dagger - a_2^\dagger)(a_1 - a_2)\rangle\langle(a_1^\dagger + a_2^\dagger)(a_1 + a_2)\rangle} \tag{8}$$

Using the fact that

$$a_2|\phi\rangle = 0$$

we can extremely simplify all calculation:

$$\langle\phi|(a_1^\dagger - a_2^\dagger)(a_1 - a_2)|\phi\rangle = \langle\phi|a_1^\dagger a_1|\phi\rangle = \langle\phi|\hat{N}_1|\phi\rangle \tag{9}$$

$$\langle\phi|(a_1^\dagger + a_2^\dagger)(a_1 + a_2)|\phi\rangle = \langle\phi|a_1^\dagger a_1|\phi\rangle = \langle n|\hat{N}_1|n\rangle \tag{10}$$

For the third term we have to work more:

$$\langle\phi|(a_1^\dagger - a_2^\dagger)(a_1 - a_2)(a_1^\dagger + a_2^\dagger)(a_1 + a_2)|\phi\rangle = \langle\phi|a_1^\dagger(a_1 - a_2)(a_1^\dagger + a_2^\dagger)a_1|\phi\rangle \tag{11}$$

If we expand the left hand side of (11) we will encounter to four terms. Using commuation relations we can simplify every term on the right hand side of above equation:

$$\langle\phi|a_1^\dagger a_1 a_1^\dagger a_1|\phi\rangle = \langle\phi|\hat{N}_1^2|\phi\rangle \tag{12}$$

$$\langle\phi|a_1^\dagger a_1 a_2^\dagger a_1|\phi\rangle = \langle\phi|a_2^\dagger a_1^\dagger a_1 a_1|\phi\rangle = 0 \tag{13}$$

$$\langle\phi|a_1^\dagger a_2 a_1^\dagger a_1|\phi\rangle = \langle\phi|a_1^\dagger a_1^\dagger a_1 a_2|\phi\rangle = 0 \tag{14}$$

$$\langle\phi|a_1^\dagger a_2 a_2^\dagger a_1|\phi\rangle = \langle\phi|a_1^\dagger a_2^\dagger a_2 a_1|\phi\rangle + \langle\phi|a_1^\dagger[a_2, a_2^\dagger]a_1|\phi\rangle = -\langle\phi|\hat{N}_1|\phi\rangle \tag{15}$$

In all expression we have use the fact that $\hat{a}_1$ and $\hat{a}_2$ commute. At the end of the day we arrive at the following equation:

$$g^{(2)}(0) = \frac{\langle \hat{N}_1^2 - \hat{N}_1 \rangle}{\langle \hat{N}_1 \rangle^2} \tag{16}$$

(b)

If the input is in photon number state $|n\rangle$ we have:

$$g^{(2)}(0) = \frac{\langle \hat{N}_1^2 - \hat{N}_1 \rangle}{\langle \hat{N}_1 \rangle^2} = \frac{n^2 - n}{n^2} = \frac{n-1}{n} \tag{17}$$

References

[1] M. Fox, *Quantum Optics*. Axford University Press. 2006.