QIC 890 - Intro. to Noise Processes

Problem Set 2

Instructor: Dr. Na Young Kim

John Rinehart

Monday, May 17, 2016

Contents

Two Important Theorems of the Fourier Transform

Differentiation Theorem

Explicitly, the Fourier transform of the derivative of a function is

$$\int_{-\infty}^{\infty} \frac{dx(t)}{dt} \exp(-i\omega t)dt \ .$$

Integrating this by parts yields

$$\int_{-\infty}^{\infty} \frac{dx(t)}{dt} \exp(-i\omega t)dt = x(t)\exp(-i\omega t)\Big|_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} x(t)\exp(-i\omega t)dt \ .$$

Now, the second integral is clearly $X(\omega)$, the Fourier transform of $x(t)$. The first term, however, must be zero in order that $x(t)$ possess a valid Fourier transform (that is, it's in L^2). So, finally,

$$\int_{-\infty}^{\infty} \frac{dx(t)}{dt} \exp(-i\omega t)dt = i\omega X(\omega) \ .$$

This is not the desired result. However, if I change the lower bound to 0 instead of $-\infty$ I would obtain the desired result.

However, the problem does not state that $x(t < 0) = 0$. It merely says that $x(t)$ is related to some function $y(t)$ in that $y(t_0)$ is related to $x(t)$ only for $t \leq t_0$. This does not impose that $x(t < 0) = 0$.

Integration Theorem

The previous result can be expressed in the following way,

$$\mathcal{F}(dx(t)dt) = i\omega \mathcal{F}(x(t)) \ .$$

Allow, $y(t) = \frac{dx(t)}{dt}$ such that $x(t) = \int_{-\infty}^{t} y(t')dt'$. Then,

$$\mathcal{F}(y(t)) = i\omega \mathcal{F}\Big(\int_{-\infty}^{t} y(t')dt' + C\Big) \ .$$

C, above, is some constant that's included to account for the fact that if we consider two "versions" of $x(t)$, $x_1(t)$ and $x_2(t)$, which only differ in that $x_2(t) - x_1(t) = k$, k being some constant, their derivatives would be equal, $\frac{dx_1(t)}{dt} = \frac{dx_2(t)}{dt}$. The value of the integration constant is $C = x(t) - \int_{-\infty}^{t} y(t')dt'$. By linearity of the Fourier transform, however, we can break the above up into two parts and rearrange for what we want:

$$\mathcal{F}\Big(\int_{-\infty}^{t} y(t')dt'\Big) = \frac{\mathcal{F}(y(t))}{i\omega} - \mathcal{F}(C) \ .$$

But, we know that the Fourier transform of a constant C is just $2\pi\delta(\omega)C$. So, the end result is that

$$\mathcal{F}\left(\int_{-\infty}^{t} y(t')dt'\right) = \frac{\mathcal{F}(y(t))}{i\omega} - 2\pi\delta(\omega)C \ .$$

I'm not sure how to relate the constant of integration to $X(0)$. Hopefully, that will be something the official solutions will teach me.

Fluctuation-Dissipation Theorem

$$m\frac{dv(t)}{dt} = \mathcal{F}(t) + F(t) \tag{1}$$

Integration

I'll start by writing the integral of Eq. **??**.

$$\begin{aligned} \int_{t'}^{t'+\tau} m\frac{dv(t)}{dt}dt &= \int_{t'}^{t'+\tau} \big(\mathcal{F}(t) + F(t)\big)dt \\ m\int_{t'}^{t'+\tau} \frac{dv(t)}{dt}dt &= \int_{t'}^{t'+\tau} \big(\mathcal{F}(t) + F(t)\big)dt \end{aligned}$$

Then, by the second fundamental theorem of calculus, the left hand side of the above can be expressed as $v(t)$ evaluated at the bounds of the integral.

$$m\Big(v(t'+\tau) - v(t')\Big) = \int_{t'}^{t'+\tau} \big(\mathcal{F}(t) + F(t)\big)dt$$

Now, over a macroscopic (large) time interval τ, the external force $\mathcal{F}(t)$ does not vary appreciably; we will assume that it does not change at all over this time scale. Thus, $\mathcal{F}(t)$ can be pulled out of the integral. However, we can not do the same with $F(t)$, only because it is assumed to be fluctuating much over τ. Thus, we obtain the desired form, below.

$$m\Big(v(t'+\tau) - v(t')\Big) = \mathcal{F}(t')\tau + \int_{t'}^{t'+\tau} F(t)dt$$

Decomposition of $F(t)$

The component that will make the particle ultimately return to the equilibrium value of the velocity is $f(t)$. It must be negative such that the velocity at some later time is less than that of some earlier time. Consider the recently derived expression in the context of an absent external force.

$$\begin{aligned} m\Big(v(t'+\tau) - v(t')\Big) &= \mathcal{F}(t')\tau + \int_{t'}^{t'+\tau} F(t)dt \\ &= \int_{t'}^{t'+\tau} \Big(\langle F(t)\rangle + f(t)\Big)dt \end{aligned}$$

But, $\langle F(t)\rangle$ is 0, by construction.

$$= \int_{t'}^{t'+\tau} f(t)dt$$

The particles must slow down since there is no external force supplying energy to the system of particles. In order that the particles slow down, they must have acceleration that is oppositely signed relative to their velocity. Another way to say this is that the velocity at some greater time must be less than that at some later time so that $v(t' + \tau) - v(t') < 0$. Without loss of generality, we can assume that the velocity is in the positive direction. Then, the right hand side must be negative, which means that $f(t)$ must be somewhere (if not everywhere) negative over t' to $t' + \tau$. If one breaks $f(t)$ into a slowly varying component and a quickly varying component, the slowly varying component would carry the information that returns the particles to the equilibrium velocity.

Restoring force Description $F(t)$

The typical form a restoring force is one which is negatively signed with respect to the coordinate of the thing that is moving. For example, in the case of a spring, the restoring force has the following form

$$F_{\text{spring}} = -kx$$

, where k represents the stiffness of the spring and x represents the position of the end of the spring relative to its unstretched position (stretched being in the positive direction and compressed being in the negative direction). In this case, however, it is not the position of the spring that we are considering but the velocity of a particle. Thus, the force would have the following velocity dependence,

$$F(v) = -av$$

for some non-zero, positive a. The size of a determines the rate at which the particle returns to its equilibrium velocity (like the spring constant determines the rate at which the spring returns to its equilibrium position).

Ensemble-averaged Force - (I)

We have

$$\langle F(t)\rangle = \sum_r P_r(t + \tau')F_r \ \ .$$

But, we can write $P_r(t + \tau') = P_r(t)\exp(\beta\Delta E)$. So, now,

$$\langle F(t)\rangle = \sum_r P_r(t)\exp(\beta\Delta E)F_r \ \ .$$

But, $\exp(x) = 1 + x + x^2 + x^3 + \mathcal{O}(x^4)$. So, finally, to first order in $\beta\Delta E$,

$$\langle F(t)\rangle \approx \sum_r P_r(t)\Big(1 + \beta\Delta E\Big)F_r \ \ .$$

As mentioned in the problem statement $\sum_r P_r(t + \tau')F_r = 0$. So,

$$\langle F(t)\rangle \approx \sum_r P_r(t)\beta\Delta E F_r \ \ .$$

Ensemble-averaged Force - (II)

The integral is a function of t''. So, $t \to t - t'$ and $t' \to 0$. Also, $dt'' \to ds''$. Thus the integral should be written as

$$-\beta\langle v(t)\rangle \int_{t-t'}^{0} \langle F(t')F(t' + s)\rangle ds''$$

Fluctuation-Dissipation Theorem

The argument that the double integral can be approximated by the single integral above is somewhat complicated. First, the limits of the integral form a domain that is integrated over. Remembering that s is a function of t', one can discover that the lower limit of the s integral is over a line through the st' plane: $s(t') = t - t'$. The upper bound is the line $s = 0$. The domain of integration over s is between these two lines. The lower limit of the t' integral is t (which corresponds to $s(t) = 0$, for the lower bound of the s integral) and the upper limit of the t' integral is $t + \tau$ (which corresponds to $s(t + \tau) = -\tau$, for the lower bound of the s integral). Thus, the region of integration over the st' plane is a triangle with points $(0, t)$, $(0, t + \tau)$ and $(t + \tau, -\tau)$.

Realizing that $\langle F(t')F(t' + s)\rangle_0$ is not dependent on t', but only on s, it would be great to integrate over t' first and then over s. So, we exchange the order of integration. Taking the above result and casting the points into the $t's$ plane instead of the st' plane yields

$$\int_{-\tau}^{0} \langle F(t')F(t' + s)\rangle_0 ds \int_{t-s}^{t+\tau} dt' = \int_{-\tau}^{0} \langle F(t')F(t' + s)\, (\tau + s)\, ds \ .$$

Now, it's important to consider the scale of the quantities in the problem. τ is the macroscopic time scale, which is very large. The correlation function drops off very quickly for large s. This means that the integrand contributes very little when $|s| \geq \tau$. However, when $|s| \leq \tau$, it can be ignored in comparison to τ itself. So, it's now reasonable to turn $\tau + s$ into τ in the previous integral. Since the correlation function drops off very quickly for large s beyond τ it is reasonable, also, to allow the lower limit to extend to $-\infty$. Finally, since the correlation function is even, we can extend the integral to $+\infty$ if we take half of that integral. So, we have

$$\frac{\tau}{2} \int_{-\infty}^{\infty} \langle F(t')F(t' + s)\rangle ds$$