

Solution of Final Exam 2010

Problem 1)

$$\Psi(x, t=0) = A \sum_{n=0}^{\infty} c^n \psi_n(x)$$

$$a) \int |\Psi(x, t=0)|^2 dx = 1 \Rightarrow \int |A|^2 \sum_{n=0}^{\infty} |c|^{2n} |\psi_n(x)|^2 dx = 1$$

$$\text{since } \int |\psi_n(x)|^2 dx = 1 \Rightarrow |A|^2 \sum_{n=0}^{\infty} |c|^{2n} = 1$$

$$|A|^2 \frac{1}{1 - |c|^2} = 1 \Rightarrow A = \pm \sqrt{1 - |c|^2}$$

A is known up to a constant phase.

b)

$$\Psi(x, t) = \psi(x) e^{-i \frac{E}{\hbar} t} = A e^{-i \frac{\omega t}{2}} \sum_{n=0}^{\infty} c^n e^{-i n \omega t} \psi_n(x)$$

$$\left\{ \begin{array}{l} E = (n + \frac{1}{2}) \hbar \omega \end{array} \right.$$

$$c) \langle \Psi(0) | \Psi(t) \rangle = |A|^2 e^{-i \omega t / 2} \sum_{n=0}^{\infty} |c|^{2n} e^{-i n \omega t}$$

$$= e^{-i \omega t / 2} \frac{1 - |c|^2}{1 - |c|^2 e^{-i \omega t}}$$

$$p(t) = |\langle \psi(0) | \psi(t) \rangle|^2 = \frac{(1 - |c|^2)^2}{1 - |c|^2 e^{i\omega t} - |c|^2 e^{-i\omega t} + |c|^4}$$

$$p(t) = \frac{(1 - |c|^2)^2}{1 + |c|^4 - 2|c|^2 \cos \omega t}$$

Further Simplification, $\cos \omega t = 1 - 2 \sin^2 \frac{\omega t}{2} \Rightarrow$

$$1 + |c|^4 - 2|c|^2 \cos \omega t = 1 + |c|^4 - 2|c|^2 + 4|c|^2 \sin^2 \frac{\omega t}{2}$$

$$= (1 - |c|^2)^2 + 4|c|^2 \sin^2 \frac{\omega t}{2}$$

there fore $p(t) = \left[1 + \frac{4|c|^2 \sin^2 \omega t / 2}{(1 - |c|^2)^2} \right]^{-1}$

$$\begin{aligned} \text{d) } \langle \hat{H} \rangle &= \langle \psi | \hat{H} | \psi \rangle = |A|^2 \sum_{n=0}^{\infty} E_n |c|^{2n} \\ &= |A|^2 \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) \hbar \omega |c|^{2n} \\ &= \frac{\hbar \omega}{2} (1 - |c|^2) \sum_{n=0}^{\infty} (2n |c|^{2n} + |c|^{2n}) \\ &= \frac{\hbar \omega}{2} \left[1 + (1 - |c|^2) \sum_{n=0}^{\infty} 2n |c|^{2n} \right] \end{aligned}$$

Since:

$$\frac{d}{d|c|} \sum_{n=0}^{\infty} |c|^{2n} = \frac{d}{d|c|} \frac{1}{1 - |c|^2} = \sum_{n=0}^{\infty} 2n |c|^{2n-1}$$

$$\Rightarrow \langle \hat{H} \rangle = \frac{\hbar \omega}{2} \frac{1 + |c|^2}{1 - |c|^2}$$

Problem 2)

$$\hat{\rho} = \frac{1}{Z} \exp\left(-\frac{\hat{H}}{k_B T}\right) \quad \& \quad Z = \text{Tr}\left[\exp\left(-\frac{\hat{H}}{k_B T}\right)\right]$$

a) Since $\hat{H} = \hbar\omega\left(n + \frac{1}{2}\right) \Rightarrow$

$$\begin{aligned} Z &= \text{Tr}\left[\exp\left(-\frac{\hbar\omega}{k_B T}\left(n + \frac{1}{2}\right)\right)\right] = \sum_{n=0}^{\infty} \langle n | \exp\left(-\frac{\hbar\omega}{k_B T}\left(n + \frac{1}{2}\right)\right) | n \rangle \\ &= \exp\left(-\frac{\hbar\omega}{2k_B T}\right) \sum_{n=0}^{\infty} \exp\left(-\frac{\hbar\omega}{k_B T} n\right) \end{aligned}$$

Since $\exp\left(-\frac{\hbar\omega}{k_B T}\right) < 1 \Rightarrow$ The sum is a geometric series,

$$Z = \exp\left(-\frac{\hbar\omega}{2k_B T}\right) \frac{1}{1 - \exp\left(-\frac{\hbar\omega}{k_B T}\right)}$$

b)

$$P_n = \langle n | \hat{\rho} | n \rangle = \frac{1}{Z} \exp\left(-\frac{\hbar\omega}{k_B T}\left(n + \frac{1}{2}\right)\right)$$

c)

$$\hat{\rho} = \sum_{n=0}^{\infty} P_n |n\rangle \langle n|$$

d)

$$\begin{aligned} \bar{n} = \langle n \rangle &= \text{Tr}(\hat{n} \hat{\rho}) = \sum_{n=0}^{\infty} \langle n | \hat{n} \hat{\rho} | n \rangle \\ &= \sum_{n=0}^{\infty} n P_n = \frac{\exp\left(-\frac{\hbar\omega}{2k_B T}\right)}{Z} \sum_{n=0}^{\infty} n \exp\left(-\frac{\hbar\omega}{k_B T} n\right) \quad (3) \end{aligned}$$

Note that $\sum_{n=0}^{\infty} n e^{-nx} = -\frac{d}{dx} \left(\sum_{n=0}^{\infty} e^{-nx} \right) = -\frac{e^{-x}}{(1-e^{-x})^2}$

Thus: $\bar{n} = \frac{\exp(-\hbar\omega/k_B T)}{1 - \exp(-\hbar\omega/k_B T)} = \frac{1}{\exp(\hbar\omega/k_B T) - 1}$

e)

$$\text{Tr}(\hat{p}^2) = (1 - e^{-\hbar\omega/k_B T})^2 \sum_{n=0}^{\infty} e^{-2n\hbar\omega/k_B T}$$

$$\text{Tr}(\hat{p}^2) = \frac{(1 - e^{-\hbar\omega/k_B T})^2}{1 - e^{-2\hbar\omega/k_B T}} \quad \text{Tr}(\hat{p}^2) < 1.$$

Note that $e^{-\hbar\omega/k_B T} = \frac{\bar{n}}{1 + \bar{n}} \Rightarrow \text{Tr}(\hat{p}^2) = \frac{1}{1 + 2\bar{n}} < 1$

$$\text{Tr}(\hat{p}^2) < 1.$$

Problem 3)

a)

$$\hat{H} = -\gamma \vec{B} \cdot \vec{S} = -\gamma (B_x S_x + B_y S_y + B_z S_z)$$

$$= -\gamma \frac{\hbar}{2} (B_x \sigma_x + B_y \sigma_y + B_z \sigma_z)$$

$$= -\gamma \frac{\hbar}{2} \left[B_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + B_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + B_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$$

$$= -\gamma \frac{\hbar}{2} \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$$

$$\hat{H} = -\frac{\gamma \hbar}{2} \begin{pmatrix} B_0 & B_\omega e^{i\omega t} \\ B_\omega e^{-i\omega t} & -B_0 \end{pmatrix}$$

b)

$$i\hbar \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle \Rightarrow \text{if } \frac{d}{dt} \rightarrow 0$$

$$i\hbar \begin{pmatrix} a^0 \\ b^0 \end{pmatrix} = -\frac{\gamma \hbar}{2} \begin{pmatrix} B_0 & B_\omega e^{i\omega t} \\ B_\omega e^{-i\omega t} & -B_0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow$$

$$\begin{cases} a^0 = i \frac{\gamma}{2} (B_0 a + B_\omega e^{i\omega t} b) = \frac{i}{2} (\Omega e^{-i\omega t} b + \omega_L a) \\ b^0 = -i \frac{\gamma}{2} (B_0 b - B_\omega e^{-i\omega t} a) = \frac{i}{2} (\Omega e^{-i\omega t} a - \omega_L b) \end{cases}$$

c)

$$a(t) = \left\{ a_0 \cos\left(\frac{\omega' t}{2}\right) + \frac{i}{\omega'} [a_0 (\omega_L - \omega) + b_0 \Omega] \sin\left(\frac{\omega' t}{2}\right) \right\} e^{i\omega t/2}$$

$$b(t) = \left\{ b_0 \cos\left(\frac{\omega' t}{2}\right) + \frac{i}{\omega'} [b_0 (\omega - \omega_L) + a_0 \Omega] \sin\left(\frac{\omega' t}{2}\right) \right\} e^{-i\omega t/2}$$

d)

$$\text{If } a_0 = 1 \text{ \& } b_0 = 0 \Rightarrow b(t) = i \frac{\Omega}{\omega'} \sin\left(\frac{\omega' t}{2}\right) e^{-i\omega t/2}$$

$$P(t) = |b(t)|^2 = \left(\frac{\Omega}{\omega'}\right)^2 \sin^2\left(\frac{\omega' t}{2}\right)$$

Problem 4)

a)

$$\hat{A}(r, t) = \sum_{k\lambda} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k}} \left[\hat{a}_k(t) \vec{u}_{k\lambda}(r) + \hat{a}_k^\dagger(t) \vec{u}_{k\lambda}^*(r) \right]$$

$$\hat{E}(r, t) = -\frac{\partial}{\partial t} \hat{A} = \sum_{k\lambda} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k}} \left[\hat{a}_k^\circ(t) \vec{u}_{k\lambda} + \hat{a}_k^{\circ\dagger}(t) \vec{u}_{k\lambda}^*(r) \right]$$

$$\hat{H}(r, t) = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A} = \frac{1}{\mu_0} \sum_{k\lambda} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k}} \left[\hat{a}_k(t) (\vec{\nabla} \times \vec{u}_{k\lambda}(r)) + \hat{a}_k^\dagger(t) (\vec{\nabla} \times \vec{u}_{k\lambda}^*(r)) \right]$$

b)

$$\frac{\partial}{\partial t} \langle \hat{E} \rangle = \frac{1}{i\hbar} \langle [\hat{E}, \hat{H}_{em}] \rangle$$

$$\text{where } \hat{H}_{em} = \sum_{k\lambda} \hbar \omega_k \left(\hat{a}_{k\lambda}^\dagger \hat{a}_{k\lambda} + \frac{1}{2} \right)$$

$$[\hat{E}, \hat{H}_{em}] = \sum_{k\lambda} \sum_{k'\lambda'} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k}} \hbar \omega_{k'} \left[[\hat{a}_{k\lambda}^\circ, \hat{a}_{k'\lambda'}^\dagger, \hat{a}_{k'\lambda'}] u_{k\lambda}(r) + [\hat{a}_{k\lambda}^{\circ\dagger}, \hat{a}_{k'\lambda'}^\dagger, \hat{a}_{k'\lambda'}] u_{k\lambda}^*(r) \right]$$

Note that \hat{E} commutes with constant $\frac{\hbar \omega_k}{2}$.

$$\text{Note that } \begin{cases} [\hat{a}_{k\lambda}, \hat{a}_{k\lambda}^\dagger \hat{a}_{k\lambda}] = \hat{a}_{k\lambda} \\ [\hat{a}_{k\lambda}^\dagger, \hat{a}_{k\lambda} \hat{a}_{k\lambda}^\dagger] = -\hat{a}_{k\lambda}^\dagger \\ \begin{cases} [\hat{a}_{k\lambda}, \hat{a}_{k'\lambda'}^\dagger, \hat{a}_{k'\lambda'}] = \delta_{k\lambda} \delta_{k'\lambda'} \\ [\hat{a}_{k\lambda}^\dagger, \hat{a}_{k'\lambda'}^\dagger, \hat{a}_{k'\lambda'}] = \delta_{k\lambda} \delta_{k'\lambda'} \end{cases} \end{cases}$$

$$[\hat{E}, \hat{H}_{em}] = \sum_{k\lambda} \sqrt{\frac{\hbar}{2\epsilon_0\omega_k}} \hbar\omega_k \left(\hat{a}_k^0 u_{k\lambda}(r) - \hat{a}_k^{\dagger 0} u_{k\lambda}^*(r) \right)$$

$$\text{So } \frac{\partial}{\partial t} \langle \hat{E} \rangle = \frac{1}{i\hbar} \sum_{k\lambda} \sqrt{\frac{\hbar}{2\epsilon_0\omega_k}} \hbar\omega_k \left(\langle \hat{a}_k^0 \rangle u_{k\lambda}(r) - \langle \hat{a}_k^{\dagger 0} \rangle u_{k\lambda}^*(r) \right)$$

If you compare this with $\vec{\nabla} \times \langle \hat{H} \rangle$, we have

$$\epsilon_0 \frac{\partial}{\partial t} \langle \hat{E} \rangle = \vec{\nabla} \times \langle \hat{H} \rangle$$

With the same token,

$$\frac{\partial}{\partial t} \langle \hat{H} \rangle = \frac{1}{i\hbar} \sum_{k\lambda} \sqrt{\frac{\hbar}{2\epsilon_0\omega_k}} \hbar\omega_k \left(\langle \hat{a}_k \rangle (\vec{\nabla} \times u_{k\lambda}) - \langle \hat{a}_k^{\dagger} \rangle (\vec{\nabla} \times u_{k\lambda}^*) \right)$$

$$\text{and } \mu_0 \frac{\partial}{\partial t} \langle \hat{H} \rangle = -\vec{\nabla} \times \langle \hat{E} \rangle$$

Problem 5)

a)

$$|n\rangle = \sqrt{\frac{2}{L}} \sin k_n z \quad k_n = n \frac{\pi}{L} \quad \& \quad E_n = \frac{\hbar^2 k_n^2}{2m_e}$$

$$\omega_{21} = \frac{1}{\hbar} (E_2 - E_1) = \frac{1}{\hbar} \cdot \frac{\hbar^2}{2m_e} \left(\frac{4\pi^2}{L^2} - \frac{\pi^2}{L^2} \right) = \frac{3\hbar\pi^2}{2m_e L^2}$$

$$\omega_{21} = \frac{3\hbar\pi^2}{2m_e L^2}$$

b)

$$P_{12} = \frac{e^2 |E_0|^2}{\hbar^2} \langle 2|3|1 \rangle^2 \left| \int_0^t e^{i\omega_{21}t' - \frac{t'}{\tau}} dt' \right|^2$$

$$\langle 2|3|1 \rangle = \frac{2}{L} \int_0^L z \sin(k_2 z) \sin(k_1 z) dz = -\frac{16L}{9\pi^2}$$

$$\left| \int_0^t e^{(i\omega_{21} - \frac{1}{\tau})t'} dt' \right|^2 = \frac{1 + e^{-\frac{2t}{\tau}} - 2e^{-\frac{t}{\tau}} \cos(\omega_{21}t)}{\omega_{21}^2 + \frac{1}{\tau^2}}$$

$$P_{12} = \left(\frac{16e|E_0|^2 L}{9\pi^2 \hbar} \right)^2 \left(\frac{1 + e^{-\frac{2t}{\tau}} - 2e^{-\frac{t}{\tau}} \cos(\omega_{21}t)}{\omega_{21}^2 + \frac{1}{\tau^2}} \right)$$