Problem Set 2

QIC750 Winter 2014

Due: Monday, February 24, 2014

1 Pauli Matrix Identity

Prove the following vector identity involving the Pauli matrix vector, $\vec{\sigma}$:

$$(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i\vec{\sigma} \cdot (\vec{a} \times \vec{b}) \tag{1}$$

Hint: Use tensor notation and the (anti-)commutators of the Pauli matrices

2 The density matrix

In class we saw that the density matrix of a state, $\hat{\rho}$, can be written in terms of a so-called Bloch vector $\vec{a}$ as:

$$\hat{\rho} = \frac{1}{2}\left(\hat{1} + \vec{\sigma} \cdot \vec{a}\right) \tag{2}$$

where $|\vec{a}| \leq 1$.

a) Show that the eigenvalues of $\hat{\rho}$ are $\frac{1}{2}(1 \pm |\vec{a}|)$

b) The *purity* of a state is sometimes defined as $P = \mathrm{tr}(\hat{\rho}^2)$. Show that $P = \frac{1}{2}(1 + |\vec{a}|^2)$. Why is it called the purity?

3 The J-coupled spectrum

In class, we learned that the so called J-coupling term between two spins is

$$\hat{H}_J = \frac{\hbar J}{4}\hat{\sigma}_z^{(1)}\hat{\sigma}_z^{(2)} \tag{3}$$

In the measured magnetization spectrum of the spins, this leads to the splitting of the individual spin lines by frequency J, which we'll now derive. We can write the magnetization of spin 1 as:

$$\langle \hat{M}_1(t) \rangle = \mathrm{tr}(\hat{\rho}(t)\hat{\sigma}_+^{(1)}) = \mathrm{tr}\left[\hat{\rho}(t)\hat{\sigma}_+^{(1)} \otimes (\hat{e}_\uparrow^{(2)} + \hat{e}_\downarrow^{(2)})\right] \tag{4}$$

where all operators were defined in class. Assuming $\rho(t)$ evolves according to $\hat{U}(t) = \exp(-i\hat{H}_J t/\hbar)$, show that

$$\langle \hat{M}_1(t) \rangle = e^{iJt/2}\mathrm{tr}\left[\hat{\rho}(0)\hat{\sigma}_+^{(1)}\hat{e}_\uparrow^{(2)}\right] + e^{-iJt/2}\mathrm{tr}\left[\hat{\rho}(0)\hat{\sigma}_+^{(1)}\hat{e}_\downarrow^{(2)}\right] \tag{5}$$

4 Refocusing

In class, we learned that the NMR Hamiltonian of two coupled spins is:

$$\hat{H} = \frac{\hbar\omega_1}{2}\hat{\sigma}_z^{(1)} + \frac{\hbar\omega_2}{2}\hat{\sigma}_z^{(2)} + \frac{\hbar J}{4}\hat{\sigma}_z^{(1)}\hat{\sigma}_z^{(2)} \tag{6}$$

The J-coupling term can be used to perform 2 qubit gates, but has the problem of not being able to be turned off. However, it can effectively be turned off using the technique of refocusing discussed in class. Here we will explore refocusing. In refocusing, we use resonant pulses to apply two π-rotations (180 degree) rotations to one of the spins, say spin 1. To understand the effect, start by showing that

$$\hat{D}_x^{(1)}(\pi)\exp\left(\frac{-i\omega_1 t}{2}\hat{\sigma}_z^{(1)}\right)\hat{D}_x^{(1)}(\pi) = \exp\left(\frac{i\omega_1 t}{2}\hat{\sigma}_z^{(1)}\right) \tag{7}$$

where $D_x^{(1)}(\phi) = \exp(-i\hat{\sigma}_x^{(1)}\phi/2)$ is the x-rotation operator for spin 1. (You can show this either using operator techniques or using the explicit matrix representations of the Pauli matrices.) This expresses mathematically that two 180-degree rotations have the effect of reversing the free dynamics of spin 1. Using this, now show that

$$\exp\left(\frac{-i\hat{H}t}{\hbar}\right)\hat{D}_x^{(1)}(\pi)\exp\left(\frac{-i\hat{H}t}{\hbar}\right)\hat{D}_x^{(1)}(\pi) = \exp\left(-i\omega_2 t\hat{\sigma}_z^{(2)}\right) \tag{8}$$

This expresses, in operator language, that the pulse sequence removes the effect of the coupling on spin 2 for the time $2t$, that is, it evolves according to its free Hamiltonian.