Lecture 12: Two qubit gates - fixed coupling schemes

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I. CAPACITIVELY COUPLED PHASE QUBITS

We discuss the circuit shown in Fig. 1 used in the experiments in [1]. The derivation of

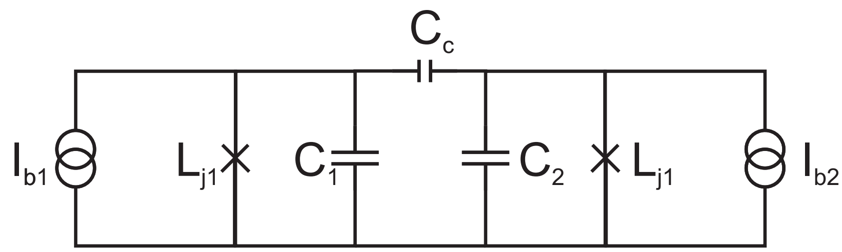

FIG. 1. Capacitively coupled phase qubits.

the circuit Hamiltonian starts with the Kirchhoff's circuit laws (two current relations are written below, with voltage relations and branch relations already taken into account):

$$I_{b1} = I_{c1}\sin\gamma_1 + \phi_0 C_1\ddot{\gamma}_1 + \phi_0 C(\ddot{\gamma}_1 - \ddot{\gamma}_2) \tag{1}$$

$$I_{b2} = I_{c2}\sin\gamma_1 + \phi_0 C_2\ddot{\gamma}_2 - \phi_0 C(\ddot{\gamma}_1 - \ddot{\gamma}_2) \tag{2}$$

The Lagrange function satisfying the above is

$$\mathcal{L} = \mathcal{T} - \mathcal{U}. \tag{3}$$

Here the "kinetic"energy

$$\mathcal{T} = \frac{\phi_0^2}{2}\dot{\gamma}\mathbf{C}\dot{\gamma}^T, \tag{4}$$

where the capacitance matrix

$$\mathbf{C} = \begin{pmatrix} C_1 + C & -C \\ -C & C_2 + C \end{pmatrix} \tag{5}$$

and

$$\gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \end{pmatrix}. \tag{6}$$

The potential energy is

$$\mathcal{U} = -\phi_0 I_{c1}\cos\gamma_1 - \phi_0 I_{b1}\gamma_1 - \phi_0 I_{c2}\cos\gamma_2 - \phi_0 I_{b2}\gamma_2. \tag{7}$$

The Hamiltonian operator is

$$\mathcal{T} = \frac{1}{2\phi_0^2}\mathbf{p}_\gamma \mathbf{C}^{-1} \mathbf{p}_\gamma{}^T + \mathcal{U}. \tag{8}$$

The addition of the coupling capacitance C has two effects:

1. a coupling term arises, of the form $p_{\gamma_1} p_{\gamma_2}$
2. a small change is induced in the effective capacitance corresponding to each junction seen as a separate quantum system (this is a small effect if $C \ll C_1, C_2$).

When the coupling term is expressed in the Hilbert space pertaining to the qubit, the net result is

$$H_{int} = \frac{S}{2}\sigma_y^1 \sigma_y^2. \tag{9}$$

. We next apply the usual rotating frame which unwinds the evolution of each qubit, namely it cancels the uncoupled Hamiltonian

$$H_0 = -\frac{\hbar\omega_{01}^2}{2}\sigma_z^1 - \frac{\hbar\omega_{01}^2}{2}\sigma_z^2 \tag{10}$$

(where we took the minus sign so that the state $|0\rangle$ is the ground state). The frame transformation is

$$|\tilde{\psi}(t)\rangle = U_f(t)|\psi(t)\rangle \tag{11}$$

with

$$U_f(t) = e^{-i\frac{\omega_{01}^1}{2}t\sigma_z^1} e^{-i\frac{\omega_{01}^2}{2}t\sigma_z^2}. \tag{12}$$

The Hamiltonian in the rotating frame is

$$H_{int,f} = \frac{S}{2}\left(\cos(\omega_{01}^1 t)\sigma_y^1 - \sin(\omega_{01}^1 t)\sigma_x^1\right)\left(\cos(\omega_{01}^2 t)\sigma_y^2 - \sin(\omega_{01}^2 t)\sigma_x^2\right). \tag{13}$$

We assume the two qubits are on resonance. Neglecting fast rotating terms and defining new operators σ_+ and σ_- using relations

$$\sigma_x = \sigma_+ + \sigma_- \tag{14}$$

and

$$\sigma_y = i(\sigma_+ - \sigma_-) \tag{15}$$

we obtain

$$H_{int,f} = \frac{S}{2}(\sigma_+^1\sigma_-^2 + \sigma_+^2\sigma_-^1) \tag{16}$$

which can also be expressed as

$$H_{int,f} = \frac{S}{2}(|01 >< 10| + |10 >< 01|). \tag{17}$$

Assume the two qubits are prepared in state $|01\rangle$. This can be done in the following way: qubit two is detuned and and a microwave pulse is applied to it. Then this qubit is brought fast enough in resonance with the first qubit. The evolution in the rotating frame is

$$|\psi(t)\rangle = \frac{1}{2}(|01\rangle + |10\rangle)e^{iSt/2\hbar} + \frac{1}{2}(|01\rangle - |10\rangle)e^{-iSt/2\hbar} \tag{18}$$

$$= \cos{(S\hbar t/2)}|01\rangle + i\sin{(S\hbar t/2)}|10\rangle. \tag{19}$$

At $t = h/2S$, state $|10>$ evolves to $i|01>$. The transformation at this time is

$$U_{eff}(h/2S) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

which is a iSWAP gate. Stopping the gate at half the time of that necessary for a iSWAP produces the $\sqrt{\text{iSWAP}}$ gate. It can be shown (see [2]) that the $\sqrt{\text{iSWAP}}$ gate can be used to construct a CNOT gate, so in this sense it is a universal gate for quantum computing.

II. INDUCTIVELY COUPLED FLUX QUBITS

We will work out the circuit details for the case of two inductively coupled RF-SQUIDS. The relevant system, from an experimental point of view, is that of two coupled PCQs; we use the calculation for coupled RF-SQUIDs as a guide, however we do not deal with the full calculation for coupled PCQs which is significantly more complicated.

The circuit for two coupled RF-SQUIDS is shown in Fig. 2. The two rings have geometric inductances L_1 and L_2 respectively and are coupled by a mutual inductance M.

It can be shown that the Hamiltonian for the coupled system is given by

$$H = \frac{1}{2}\frac{1}{\phi_0^2 C_1}p_1^2 + \frac{1}{2}\frac{1}{\phi_0^2 C_2}p_2^2 + U(\gamma_1, \gamma_2) \tag{21}$$

with

$$U(\gamma_1, \gamma_2) = \frac{\phi_0^2}{2}(\boldsymbol{\gamma} + 2\pi\boldsymbol{f})L^{-1}(\boldsymbol{\gamma}^T + 2\pi\boldsymbol{f}^T) \tag{22}$$

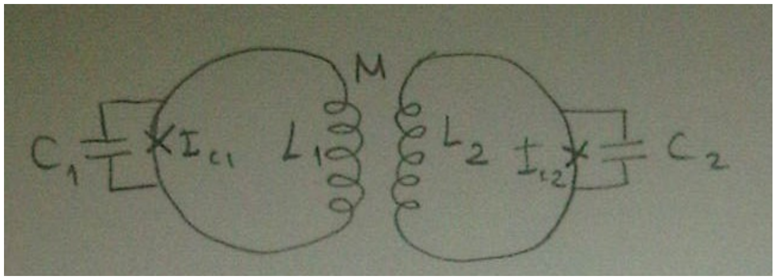

FIG. 2. Coupled RF SQUIDs.

where we defined the vector

$$\boldsymbol{\gamma} + 2\pi \boldsymbol{f} = (\gamma_1 + 2\pi f_1, \gamma_2 + 2\pi f_2) \tag{23}$$

and the inductance matrix

$$L = \begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix}. \tag{24}$$

γ_1 and γ_2 are the phases across the two junctions and f_1 and f_2 are the frustrations applied to each ring.

The Hamiltonian 21 is very similar to the coupling Hamiltonian 8. In this case, the coupling terms appear as off-diagonal terms in the phase instead of the conjugate momenta. To find out the form of the coupling we need to consider the representation of the operator $\gamma + 2\pi f$ in the basis formed by the computational states (the two lowest energy states) of the RF-SQUID. The second term (that is $2\pi f$) is simply the identity operator. For the γ term, it is simpler to consider expectation values of the form $\langle \text{cw flux}|\gamma|\text{cw flux}\rangle$, $\langle \text{cw flux}|\gamma|\text{acw flux}\rangle$, .., where cw/acw stand for clockwise/anticlockwise values of the flux. A representation of the relevant expectation values to calculate is given in Fig. 3. Based on this, the matrix representation in this basis is given by

$$\gamma \rightarrow \begin{pmatrix} \overline{\gamma} & 0 \\ 0 & -\overline{\gamma} \end{pmatrix} = \overline{\gamma}\sigma_z \tag{25}$$

where we denoted by $\overline{\gamma}$ the absolute values of the expectation value of the operator γ in the clockwise/anticlockwise state (these two are equal by symmetry).

The coupled system Lagrangian for two coupled flux qubits is similar to the case of two

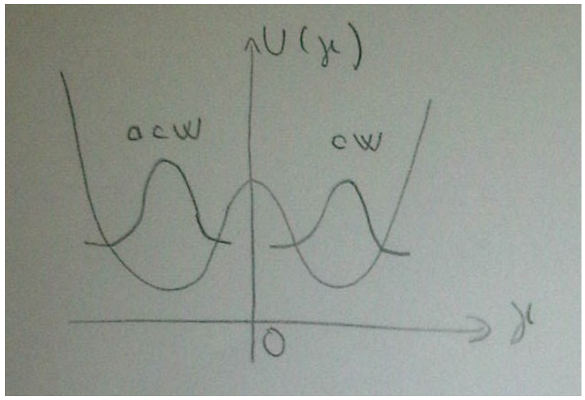

FIG. 3. Schematic representation of the wavefunctions for the clockwise and anticlockwise current states in a RF-SQUID.

RF-SQUIDS (no proof is given here for this). The coupled system Hamiltonian is given by

$$H = H_1 + H_2 + H_{int} \text{ ,with} \tag{26}$$
$$H_1 = -\frac{1}{2}\Delta_1\sigma_x^1 - \frac{1}{2}\epsilon_1\sigma_z^1 \tag{27}$$
$$H_2 = -\frac{1}{2}\Delta_2\sigma_x^2 - \frac{1}{2}\epsilon_2\sigma_z^2 \tag{28}$$
$$\hat{H}_{int} = J\sigma_z^1\sigma_z^2, \tag{29}$$

where Δ_1 and Δ_2 are design parameters and ϵ_1 and ϵ_2 are tunable parameters (see the section on the PCQ).

We consider the limit $\epsilon_i \gg \Delta_i$. In this limit the Hamiltonian

$$H \approx -\frac{1}{2}\epsilon_1\sigma_z^1 - \frac{1}{2}\epsilon_2\sigma_z^2 + J\sigma_z^1\sigma_z^2 \tag{30}$$

The spectrum is given in Fig. 4. Since each qubit Hamiltonian commutes with the interaction Hamiltonian, the eigenstates are written simply as product states. The coupling is such that the energy level splitting between 00 and 01 is different of the energy level splitting between 10 and 11.This means that if microwaves are applied which match the transition 10 $<->$ 11, no transition takes place between the states 00 and 01. This provides a mechanism for a CNOT gate. . This was proven experimentally by Plantenberg *et al* [3].

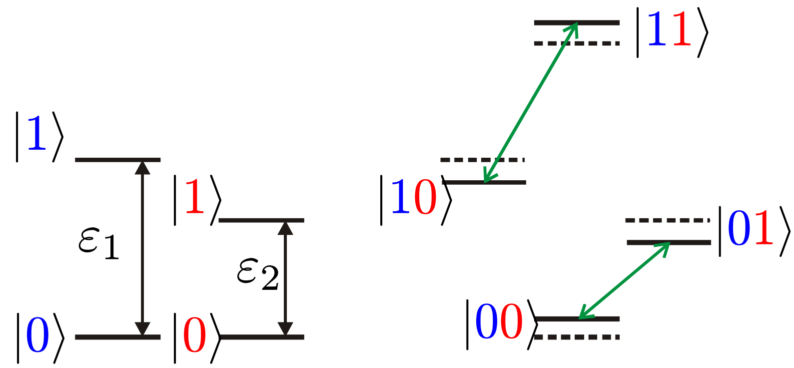

FIG. 4. Energy levels of coupled PCQs away from the symmetry point: without interaction (left) and with the coupling switched on (right).

III. TRANSMONS IN CQED

We consider the coupling of transmons in circuit quantum electrodynamics. Two transmons are coupled to a superconducting resonator. In this case the coupling is of the form

$$H_{2T} = \sum \left(|(i-1)_1(j+1)_2\rangle\langle i_1, j_2| + |(i+1)_1(j-1)_2\rangle\langle i_1, j_2| \right) \tag{31}$$

where $|i_1\rangle$ and $|j_2\rangle$ are eigenstates of qubits 1 and 2 respectively, with i and j equal to 0,1,2,.. (in increasing order of energy). The way in which this coupling arises [4] will be discussed in more detail in the lectures of circuit quantum electrodynamics.. Note that we include specifically all the levels, *ie* i and j are 0,1,2, ..

If the multilevel structure is neglected and we take the qubits to be far off resonant to each other, we have

$$E^{2qb}_{00,11} \cong E^{qb1}_{0,1} + E^{qb2}_{0,1}. \tag{32}$$

The presence of the higher levels plays however an important role. Even though the two qubits have different transition frequencies ($E^{qb1}_{0,1} \neq E^{qb2}_{0,1}$) the state $|11\rangle$ may become quasiresonant with state $|02\rangle$. This leads to the energy of $|11\rangle$ being pushed down, and therefore to $E^{2qb}_{00,11} - E^{2qb}_{00,01} - E^{2qb}_{00,10}$ being a negative number. The unitary evolution which is implemented

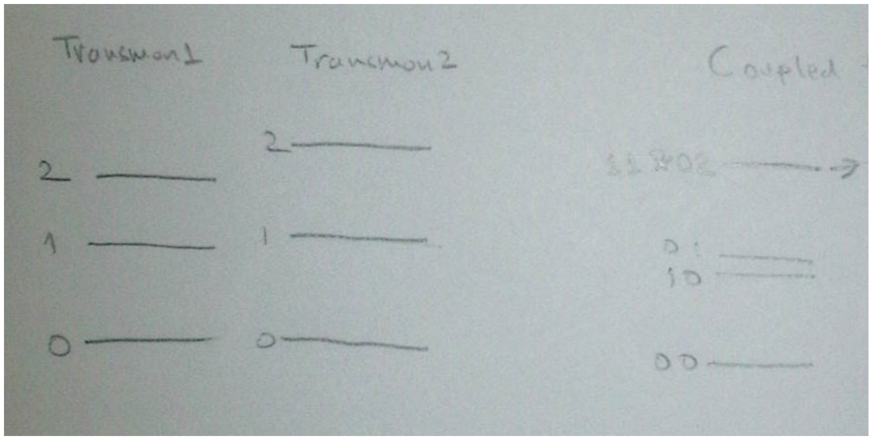

FIG. 5. Energy levels for coupled transmons.

is

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\phi_{01}} & 0 & 0 \\ 0 & 0 & e^{i\phi_{10}} & 0 \\ 0 & 0 & 0 & e^{i\phi_{11}}. \end{pmatrix} \tag{33}$$

This is equivalent to

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i(\phi_{11}-\phi_{01}-\phi_{10})} \end{pmatrix} \tag{34}$$

with the phase accumulated for state 11 proportional to time. This can be made equal to π, which provides a C-Phase gate. This scheme was used in [5] to implement a simple quantum protocol

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