Assignment 1

Due: Tuesday, October 13 at 4:00pm

1. Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces, and assume Γ is the alphabet for which $\mathcal{Y} = \mathbb{C}^{\Gamma}$. Suppose further that $\{K_{a,b} : a, b \in \Gamma\} \subset \mathrm{L}(\mathcal{X})$ is a collection of operators, and $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is the map defined by
$$\Phi(X) = \sum_{a,b \in \Gamma} \langle K_{a,b}, X \rangle E_{a,b}$$
for every $X \in \mathrm{L}(\mathcal{X})$.
   (a) Give a necessary and sufficient condition on the collection $\{K_{a,b} : a, b \in \Gamma\}$ for Φ to be completely positive. Your condition should, in principle, be efficiently checkable.
   (b) Give a necessary and sufficient condition on the collection $\{K_{a,b} : a, b \in \Gamma\}$ for Φ to be trace-preserving.
   (c) Suppose $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is given and you wish to find a collection $\{K_{a,b} : a, b \in \Gamma\}$ so that the equation above holds for all $X \in \mathrm{L}(\mathcal{X})$. Give a simple formula that allows you to obtain each operator $K_{a,b}$ from Φ.

2. Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be complex Euclidean spaces and let $\Phi \in \mathrm{C}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ be a channel. Suppose further that there exists a density operator $\rho \in \mathrm{D}(\mathcal{Y})$ such that
$$\mathrm{Tr}_{\mathcal{Z}}(J(\Phi)) = \rho \otimes \mathbb{1}_{\mathcal{X}}.$$
Prove that there exists a complex Euclidean space $\mathcal{W}$, a density operator $\sigma \in \mathrm{D}(\mathcal{Y} \otimes \mathcal{W})$, and a channel $\Psi \in \mathrm{C}(\mathcal{W} \otimes \mathcal{X}, \mathcal{Z})$ so that
$$\Phi(X) = \big(\mathbb{1}_{\mathrm{L}(\mathcal{Y})} \otimes \Psi\big)(\sigma \otimes X)$$
for all $X \in \mathrm{L}(\mathcal{X})$. (Proposition 2.29 in the book may be helpful when answering this problem.)

3. Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces, let Σ be an alphabet, and let $\mu : \Sigma \to \mathrm{Pos}(\mathcal{Y})$ be a measurement. Prove that for every collection of operators $\{X_a : a \in \Sigma\} \subset \mathrm{L}(\mathcal{X})$ it holds that
$$\left\| \sum_{a \in \Sigma} X_a \otimes \mu(a) \right\| \leq \max_{a \in \Sigma} \| X_a \|.$$

4. Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces, let Σ be an alphabet, and let $\eta : \Sigma \to \mathrm{Pos}(\mathcal{X})$ be an ensemble of states. Suppose further that $u \in \mathcal{X} \otimes \mathcal{Y}$ is a vector such that
$$\mathrm{Tr}_{\mathcal{Y}}(uu^*) = \sum_{a \in \Sigma} \eta(a).$$
Prove that there exists a measurement $\mu : \Sigma \to \mathrm{Pos}(\mathcal{Y})$ for which it holds that
$$\eta(a) = \mathrm{Tr}_{\mathcal{Y}}\big((\mathbb{1}_{\mathcal{X}} \otimes \mu(a))uu^*\big)$$
for all $a \in \Sigma$. (Similar to Problem 2 above, Proposition 2.29 in the book may be helpful when answering this problem.)