5/31/2016 Tue. ECE 730 Lecture 5.

① Johnson–Nyquist Noise (= Thermal Noise)

J.B. Johnson "Thermal agitation of electricity in conductors" Nature 119, 50 (1927)

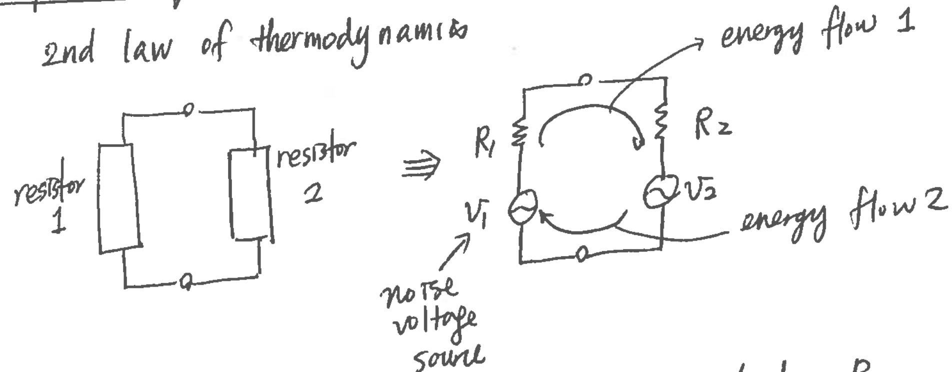
J.B Johnson "Thermal agitation of electricity in conductors" Phys. Rev. 32, 97 (1928)

H. Nyquist "Thermal agitation of electric charge in conductors" Phys. Rev. 32, 110 (1928)

W. Schottky. (1918) Ann. d. Phys 57. 541. "tube noise" in vacuum tube amplifiers

macroscopic Theory of thermal Noise

2nd law of thermodynamics



Energy flow 1 : power generated in R_1 & absorbed in R_2

$$\frac{R_2}{(R_1+R_2)^2}\langle v_1^2\rangle$$

Energy flow 2

$$\frac{R_1}{(R_1+R_2)^2}\langle v_2^2\rangle$$

If the 2 resistors are at the same Θ temp & $R_1=R_2=R$,

the energy flows cancel out.

otherwise, the 2nd law of thermodynamics is violated. → It's impossible to take heat from a reservoir at an eq. temp. and convert it to work. The entropy of the universe is only increasing

This must hold not only for the total energy flow

but also for the energy flow in any frequency bands.

The power spectral density $\langle S_v(\omega)\rangle$ of voltage fluctuation's

The power spectral density of the detailed structure & the material of the resistor

① should be indep. of the detailed structure & the material of the resistor

② should be a universal fn of R, Θ & ω.

In general $R_1 \neq R_2$ $\quad \frac{R_2}{\ }\langle S_{v_1}(\omega)\rangle = \frac{R_1}{\ }\langle S_{v_2}(\omega)\rangle$ $\quad \therefore \boxed{\langle S_v(\omega)\rangle \propto R.}$

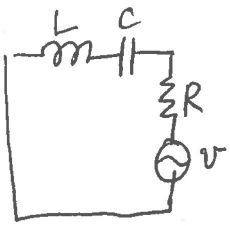
If $\theta_1 \neq \theta_2$, the total energy flows do not cancel.

There should be the net heat flow proportional to the temp. difference

$$\frac{1}{4R}\left(\langle S_1(\omega)\rangle - \langle S_{V_2}(\omega)\rangle\right) \propto \theta_1 - \theta_2$$

$$\therefore \boxed{S_V(\omega) \propto \theta}$$

Suppose



magnetic energy

$$\frac{1}{2}L\langle I^2\rangle = \frac{1}{2}L\int_0^\infty \langle S_I(\omega)\rangle \frac{d\omega}{2\pi} \quad \text{Parseval theorem.}$$

$\uparrow$ circuit current fluctuations

$$\langle S_I(\omega)\rangle = \frac{1}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\langle S_V(\omega)\rangle \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

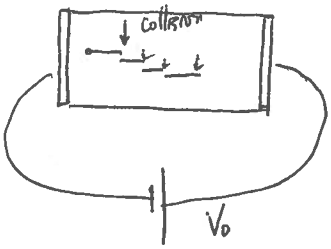
$$= \frac{\langle S_V(\omega)\rangle}{(2L)^2\left[(\omega-\omega_0)^2 + \left(\frac{R}{2L}\right)^2\right]}$$

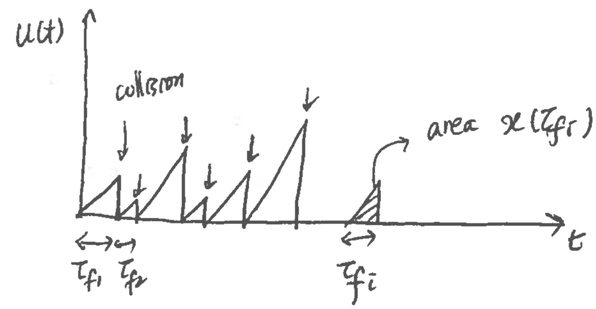
$$\frac{1}{2}L\int_0^\infty \langle S_I(\omega)\rangle \frac{d\omega}{2\pi} = \frac{L}{2}\cdot\frac{\langle S_V(\omega)\rangle}{(2L)^2}\int_0^\infty \frac{1}{(\omega-\omega_0)^2 + \left(\frac{R}{2L}\right)^2}\cdot\frac{d\omega}{2\pi}$$

$$= \frac{L}{2}\cdot\frac{\langle S_V(\omega)\rangle}{4L^2}\cdot\frac{1}{\frac{R}{L}} = \frac{1}{2}k_B\theta$$

$$\therefore \langle S_V(\omega)\rangle = 4k_B\theta R.$$

Microscopic Theory of Thermal Noise





electron drift velocity $u(t) = \frac{qE}{m} t$

displacement btw 2 collision $x(\tau_f) = \frac{1}{2} a \tau_f^2 = \frac{qE}{2m} \tau_f^2$ $\qquad \tau_f$ = free time

$\Big\downarrow$ after K collisions (stochastic process)

mean drift velocity : ensemble average

$$\langle u(t) \rangle \equiv \frac{\text{total displacement}}{\text{total time}} = \frac{\left(\frac{qE}{2m}\right)\langle \tau_f^2 \rangle K}{\langle \tau_f \rangle K} = \underbrace{\frac{q \langle \tau_f^2 \rangle}{2m \langle \tau_f \rangle}}_{\Downarrow \ \text{mobility definition}} E$$

$\langle \tau_f \rangle$: mean free time

$\langle \tau_f^2 \rangle$: mean-square free time

$\Big\downarrow$ if $\langle \tau_f^2 \rangle = 2 \langle \tau_f \rangle^2$

$$\boxed{\mu = \frac{q}{m} \langle \tau_f \rangle}$$

Suppose $P_i(m, \tau)$ = Probability that i-th electron experiences "m" collisions in a time interval τ.

if. ν_i = the mean rate of collision per second, indep. of an electron velocity.

each collision occurs independently in a continuous variable (time) t

poisson $\rightarrow$ $\therefore P_i(m, \tau) = \frac{(\nu_i \tau)^m}{m!} e^{-\nu_i \tau}$ Poisson distribution.

$g_i(\tau_{fi}) d\tau_i$ = the probability that a free time τ_{fi} btw collisions is btw τ_{fi} & $\tau_{fi} + d\tau_{fi}$

$$= \underbrace{P_i(0, \tau_{fi})}_{e^{-\nu_i \tau_{fi}}} \ \underbrace{P_i(1, d\tau_{fi})}_{(\nu_i d\tau_{fi}) \underbrace{e^{-\nu_i d\tau_{fi}}}_{\approx 1 \text{ for } d\tau_{fi}}} \leftarrow \text{joint probability} \quad \Rightarrow \nu_i e^{-\nu_i \tau_{fi}} d\tau_{fi}$$

$\uparrow$ exponential

$$\langle \tau_{fi} \rangle = \int_0^\infty \tau_{fi}\, \gamma_i e^{-\gamma_i \tau_{fi}} d\tau_{fi} = \frac{1}{\gamma_i}$$

$$\gamma_i \int_0^\infty \tau_{fi} e^{-\gamma_i \tau_{fi}} d\tau_{fi}$$
$$= \gamma_i \left[-\frac{e^{-\gamma_i \tau_{fi}}}{\gamma_i} \tau_{fi} \right]_0^\infty - \frac{1}{\gamma_i}\int_0^\infty -e^{-\gamma_i \tau_{fi}} d\tau_{fi}$$
$$= -\gamma_i \left[+\frac{1}{\gamma_i^2} \right] e^{-\gamma_i \tau_{fi}} = \frac{1}{\gamma_i}$$

$$\langle \tau_{fi}^2 \rangle = \int_0^\infty \tau_{fi}^2 \gamma_i e^{-\gamma_i \tau_{fi}} d\tau_{fi} = \frac{2}{\gamma_i^2} = 2\langle \tau_{fi} \rangle^2$$

Langevin Equation

$$m\frac{du}{dt} = \mathcal{F}(t) + F(t)$$

$u = \langle u \rangle + u'$
↑
ensemble averaged velocity.

$F(t) = \langle F(t) \rangle + F'(t)$

$= -\alpha \langle u(t) \rangle + F'(t)$

restoring force to push back to the equilibrium

if $\mathcal{F}(t) = 0$ $\quad m\frac{d\langle u \rangle}{dt} = -\alpha \langle u \rangle + F'(t)$ → fluctuations

↓

$$m\frac{du}{dt} = -\alpha u + F'(t)$$

$u = \dot{x} \quad \frac{du}{dt} = \frac{d\dot{x}}{dt}$

key $\langle \dot{x}^2 \rangle$

$$mx\frac{d\dot{x}}{dt} = m\left[\frac{d}{dt}(x\dot{x}) - \dot{x}^2\right] = -\alpha x\dot{x} + xF'(t)$$

↓ ensemble average $\langle xF'(t) \rangle = 0$.

$\frac{1}{2}m\langle v^2 \rangle = \frac{1}{2}m\langle \dot{x}^2 \rangle = \frac{1}{2}k_B\Theta$ equipartition thm

$$m\frac{d}{dt}\langle x\dot{x} \rangle = m\langle \dot{x}^2 \rangle - \alpha \langle x\dot{x} \rangle$$
$$= k_B\Theta - \alpha\langle x\dot{x} \rangle$$

$$\frac{d}{dt}\frac{1}{2}xx = \frac{1}{2}\left(\frac{dx}{dt} + x\frac{dx}{dt}\right) = \langle x\dot{x} \rangle$$

↓ solving

$$\langle x\dot{x} \rangle = Ce^{-\frac{\alpha}{m}t} + \frac{k_B\Theta}{\alpha} \equiv \frac{1}{2}\frac{d}{dt}\langle x^2 \rangle$$

initially $x=0$ at $t=0$ $\Rightarrow$ $\quad C + \frac{k_B\Theta}{\alpha} = 0 \quad \therefore \boxed{C = -\frac{k_B\Theta}{\alpha}}$

$$\frac{1}{2}\frac{d}{dt}\langle x^2\rangle = - \frac{k_B\theta}{\alpha} e^{-\frac{\alpha}{m}t} + \frac{k_B\theta}{\alpha}$$

$$\therefore \frac{d}{dt}\langle x^2\rangle = \frac{2k_B\theta}{\alpha}\left(1-e^{-\frac{\alpha}{m}t}\right)$$

$$\int_0^t e^{-\frac{\alpha}{m}t} = -\frac{m}{\alpha} e^{-\frac{\alpha}{m}t}\Big|_0^t$$

$$= -\frac{m}{\alpha} e^{-\frac{\alpha}{m}t} + \frac{m}{\alpha}$$

$$= \frac{m}{\alpha}\left(1-e^{-\frac{\alpha}{m}t}\right)$$

$$\downarrow$$

$$\langle x^2\rangle = \frac{2k_B\theta}{\alpha}\left(t - \frac{m}{\alpha}\left(1-e^{-\frac{\alpha}{m}t}\right)\right)$$

When $t \ll \frac{m}{\alpha} \approx \tau_c$ characteristic time $\quad e^{-\frac{\alpha}{m}t} \approx 1 - \frac{\alpha}{m}t + \frac{1}{2}\left(\frac{\alpha}{m}\right)^2 t^2 + \cdots$

$$1-e^{-\frac{\alpha}{m}t} = \frac{\alpha}{m}t - \frac{1}{2}\left(\frac{\alpha}{m}\right)^2 t^2 \cdots$$

$$\langle x^2\rangle = \frac{2k_B\theta}{\alpha}\left(t - t + \frac{1}{2}\left(\frac{\alpha}{m}\right)^2 t^2\right)$$

$$= \boxed{\frac{k_B\theta\,\alpha}{m}}\,t^2$$

particle behaves like a free ptl w/ thermal velocity $\sqrt{\frac{k_B\theta}{m}}$

When $t \gg \frac{m}{\alpha}$, $\quad e^{-\frac{\alpha}{m}t} \approx 0$

$$\langle x^2\rangle = \frac{2k_B\theta}{\alpha}t \quad \leftarrow \text{diffusion constant}$$

$$\equiv 2Dt$$

$$\boxed{D = \frac{k_B\theta}{\alpha}} \quad \text{diffusion coeff}$$

$$\boxed{\mu \equiv \frac{q}{\alpha} = \frac{q}{k_B\theta}D} \quad \text{mobility}$$

thermal equilibrium

$$\frac{d}{dt}u(t) = -\frac{u(t)}{\langle \tau_f \rangle} + \frac{F'(t)}{m}$$

statistically stationary

F.T. $\quad \leftarrow \boxed{\mu = \frac{q}{m}\langle \tau_f \rangle}$

$$\boxed{U_T(i\omega) = \frac{(\mu/q)}{1 + i\omega\langle \tau_f \rangle} F_T'(i\omega)}$$

F.T. of $F'(t)$

$$i\omega U_T(i\omega) = -\frac{1}{\langle \tau_f \rangle} U_T(i\omega) + \frac{1}{m} F_T'(i\omega)$$

$$\left(i\omega + \frac{1}{\langle \tau_f \rangle} \right) U_T(i\omega) = \frac{F_T'(i\omega)}{m}$$

$$U_T(i\omega) = \frac{\frac{1}{m} F_T'(i\omega)}{i\omega + \frac{1}{\langle \tau_f \rangle}} = \frac{\frac{\langle \tau_f \rangle}{m} F_T'(i\omega)}{1 + i\omega\langle \tau_f \rangle}$$

$$\langle S_u(\omega) \rangle = 2\frac{1}{\langle \tau_f \rangle} \langle F_T(i\omega)^2 \rangle \frac{(\mu/q)^2}{1 + \omega^2\langle \tau_f \rangle^2}$$ Carson's theorem.

Mean kinetic energy

$$\langle KE \rangle = \frac{1}{2}m\langle u^2 \rangle = \frac{m}{2}\int_0^\infty \langle S_u(\omega) \rangle \frac{d\omega}{2\pi}$$ Parseval theorem

$$\equiv \frac{1}{2}k_B \Theta$$

Langevin noise spectrum

$$\langle F_T'(i\omega)^2 \rangle = 2m k_B \Theta$$

$$\therefore \langle S_u(\omega) \rangle = 2\frac{1}{\langle \tau_f \rangle} \frac{(\mu/q)^2 \cdot 2m k_B \Theta}{1 + \omega^2\langle \tau_f \rangle^2} = \frac{4 k_B \Theta / \alpha}{1 + \omega^2\langle \tau_f \rangle^2}$$

velocity spectrum

$$\frac{1}{\langle \tau_f \rangle} \cdot \left(\frac{1}{\alpha}\right)^2 \cdot m = \left(\frac{1}{\alpha}\right)^2 \cdot \alpha = \boxed{\frac{1}{\alpha}}$$

$$\alpha = \frac{m}{\tau_f}$$

W-K. thm $\quad \langle \Phi_u(\tau) \rangle = \frac{1}{2\pi}\int^\infty \langle S_u(\omega) \rangle \cos(\omega\tau) d\omega = \frac{k_B \Theta \mu}{q\langle \tau_f \rangle} \cdot e^{-|\tau|/\tau_f}$

if resistor is length L,

$$i(t) = \frac{q\,u(t)}{L}$$. short-circuit fluctuation

$q = It$.
$I = \frac{q}{t}$

$$\langle S_i(\omega)\rangle = \langle S_u(\omega)\rangle \frac{q^2}{L^2} \cdot A L \cdot n$$

electron density

length

cross-sectional area.

} additive energy

$$\boxed{\langle S_i(\omega)\rangle = \frac{4k_B T/R}{1+\omega^2\langle \tau_f^2\rangle}}$$

$\because R = \rho \frac{L}{A} = \frac{L}{nq\mu A}$

resistivity w/ mobility