E&CE 770-T14: Quantum Electronics & Photonics

E&CE Dept., University of Waterloo

Instructor: A. Hamed Majedi, **Final Exam**, Apr. 10, 2010 , Duration: 3 hours

Problem 1 (20 points):

A simple harmonic oscillator (SHO) is initially (at time $t = 0$) in a state with wavefunction, $\Psi(x, t = 0) = A \sum_{n=0}^{\infty} c^n \psi_n(x)$ where $\psi_n(x)$ are the SHO energy eigenfunctions, c is a complex number and $|c| < 1$.

a) Determine the normalization constant A.

b) Find the wavefunction of the system at a later time $t > 0$, i.e. $\Psi(x, t)$.

c) Compute the probability of finding the system again in the initial state at a later time, $t > 0$.

d) Calculate the expectation value of the total energy of the system.

Hint 1: You need to use $\sum_{n=0}^{n=\infty} x^n = \frac{1}{1-|x|}$, $|x| < 1$ and its derivative with respect to x.

Problem 2 (20 points):

Consider a density operator describing a single-mode thermal radiation field $\hat{\rho} = \frac{1}{Z}\exp(-\frac{\hat{H}}{k_B T})$, where Z=Tr$[\exp(-\frac{\hat{H}}{k_B T})]$ is the partition function, $\hat{H}$ is the Hamiltonian operator for a simple harmonic oscillator, k_B is the Boltzmann constant and T is the temperature.

a) Calculate the partition function, Z.

b) Determine the probability P_n that the mode is thermally excited in the n^{th} level.

c) Using the previous part, rewrite the density operator as $\hat{\rho} = \sum_{n=0}^{\infty} P_n |n\rangle\langle n|$.

d) Compute the average photon number, $\bar{n}$.

e) Calculate Tr$[\hat{\rho}^2]$ in terms of $\bar{n}$. Is your result is smaller than unity?

Hint 2: You might need Hint 1!

Problem 3 (20 points):

An electron with gyromagnetic ratio γ at rest in a static magnetic field $B_0\mathbf{z}$ precesses at the Larmor frequency $\omega_L = \gamma B_0$. The time-varying small transverse magnetic field $B_\omega[\mathbf{x}\cos\omega t - \mathbf{y}\sin\omega t]$ is applied to the electron, ($\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are unit vectors).

a) Construct the 2×2 Hamiltonian matrix for this system.

b) If $|\psi\rangle = \binom{a(t)}{b(t)}$ is the spin state at time t, by using the Schrodinger equation construct the first order coupled differential equations for $a(t)$ and $b(t)$ in terms of ω, ω_L and $\Omega = \gamma B_\omega$.

c) If $a(t) = \Big[a_o \cos(\omega' t/2) + \frac{i}{\omega'}[a_o(\omega_L - \omega) + b_o\Omega]\sin(\omega' t/2)\Big]e^{i\omega t/2}$ is the solution of the coupled differential equations, find $b(t)$, where $\omega' = \sqrt{(\omega - \omega_L)^2 + \Omega^2}$ and $a_o = a(t=0)$ and $b_o = b(t=0)$.

d) If the electron starts with spin up, i.e. $a_o = 1, b_o = 0$, find the probability of a transition to spin down as a function time t.

Problem 4 (20 points):

Consider the vector magnetic potential operator in the following form

$$\hat{\mathbf{A}}(\mathbf{r},t) = \sum_{k\lambda}\sqrt{\frac{\hbar}{2\epsilon_o\omega_k}}\Big[\hat{a}_k(t)\mathbf{u}_{\mathbf{k}\lambda}(\mathbf{r}) + \hat{a}_k^\dagger(t)\mathbf{u}_{\mathbf{k}\lambda}^*(\mathbf{r})\Big]$$

a) Write down the electric field operator, $\hat{\mathbf{E}}(\mathbf{r},t)$ and magnetic flux operator $\hat{\mathbf{B}}(\mathbf{r},t)$, using the Maxwell's equations along with Coulomb gauge.

b) By using the Heisenberg equation of motion find the time evolution of the expectation values of both electric field and magnetic field operators.

Problem 5 (20 points):

An electron with mass m_e is initially (at $t = 0$) in the ground state of a one-dimensional infinite square quantum well for which $V(z) = 0$ for $0 < z < L$ and $V(z) = \infty$ elsewhere. At time $t > 0$ the system is subject to a perturbation $V(z,t) = -e|\mathbf{E_o}|ze^{-t/\tau}$, where τ is a time constant and $\mathbf{E_o}$ is the maximum of the applied electric field.

a) Determine the transition frequency between the first excited state and the ground state, namely ω_{12}.

b) Compute the transition probability for the electron that will be found in the first excited state as a function of time.

Hint:

$$\int z\sin(az)\sin(bz)dz = \frac{1}{2}\Big[\frac{1}{(a-b)^2}\cos(a-b)z + \frac{z\sin(a-b)z}{a-b} - \frac{1}{(a+b)^2}\cos(a+b)z - \frac{z\sin(a+b)z}{a+b}\Big]$$