Problem Set 3

QIC750 Winter 2014

Due: Thursday, March 27, 2014

1 The number and phase operators: 10 points

In class, we introduced the Cooper pair number and phase operators, $\hat{n}$ and $\hat{\phi}$. These are conjugate variables and their eigenstates are related by the Fourier transform pair:

$$|\phi\rangle = \sum_{n=-\infty}^{\infty} e^{in\phi}|n\rangle \quad ; \quad |n\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\phi}|\phi\rangle \tag{1}$$

a) Using these relations, show that $\hat{\phi}$ is the generator of number translations, i.e., that

$$e^{i\hat{\phi}}|n\rangle = |n-1\rangle \tag{2}$$

b) Using a), show that we can represent the Josephson Hamiltonian $\hat{H}_J = -E_J \cos\hat{\phi}$ in the number basis as

$$\hat{H}_J = -\frac{E_J}{2}\sum_{n=-\infty}^{\infty}\left[|n\rangle\langle n+1| + |n+1\rangle\langle n|\right] \tag{3}$$

2 Quantum fluctuations in circuits: 15 points

Quantum fluctuations in the environment are the ultimate source of decoherence in qubits. The uncontrolled degrees of freedom in the environment are often modeled as a bath of harmonic oscillators at different frequencies. Accordingly, consider the quantum harmonic oscillator described by the Hamiltonian

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} \tag{4}$$

with the standard representation in creation and annihilation operators

$$\hat{\Phi} = \sqrt{\frac{\hbar Z_0}{2}}(\hat{a}+\hat{a}^\dagger) \quad ; \quad \hat{Q} = -i\sqrt{\frac{\hbar}{2Z_0}}(\hat{a}-\hat{a}^\dagger) \tag{5}$$

where $Z_0 = \sqrt{L/C}$ is the characteristic impedance of the oscillator and $\omega_0 = 1/\sqrt{LC}$ is its frequency.

a) Remembering that the thermal expectation value of an operator is $\langle\hat{A}\rangle = \mathrm{tr}[\hat{A}\exp(-\hat{H}/kT)]/Z$ where $Z = \mathrm{tr}[\exp(-\hat{H}/kT)]$ is the partition function and treating $\hat{\Phi}$ as a Heisenberg operator, show that the correlation function of $\hat{\Phi}$ is

$$\langle\hat{\Phi}(t)\hat{\Phi}(0)\rangle = \frac{\hbar Z_0}{2}\left(\langle\hat{a}^\dagger\hat{a}\rangle e^{i\omega_0 t} + \langle\hat{a}\hat{a}^\dagger\rangle e^{-i\omega_0 t}\right) \tag{6}$$

b) Now, find expressions for $\langle \hat{a}^\dagger \hat{a} \rangle$ and $\langle \hat{a}\hat{a}^\dagger \rangle$ and combine them to get the final expression for $\langle \hat{\Phi}(t)\hat{\Phi}(0) \rangle$.

c) What is the variance of the fluctuations $\langle \hat{\Phi}^2 \rangle = \langle \hat{\Phi}(0)\hat{\Phi}(0) \rangle$? What are the limits of the variance at zero temperature and high temperature?

3 The dispersive Jaynes-Cummings Hamiltonian: 15 points

In class, we introduced the Jaynes-Cummings Hamiltonian

$$\hat{H}_{JC} = -\frac{\hbar\omega_{qb}}{2}\hat{\sigma}_z + \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + \hbar g \left(\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger \right) \tag{7}$$

which describes the quantum interaction of a two-level system and a harmonic oscillator and is an important model in cavity and circuit QED. In the dispersive limit, $(g \ll |\Delta|, \Delta = \omega_{qb} - \omega_0)$, the interaction leads to a shift of the oscillator frequency that depends on the state of the qubit, allowing the measurement of the oscillator frequency to serve as a readout of the qubit state. This effect can be made explicit by applying the unitary transformation

$$\hat{U} = \exp \left[\frac{g}{\Delta} \left(\hat{\sigma}_+ \hat{a} - \hat{\sigma}_- \hat{a}^\dagger \right) \right] \tag{8}$$

to the Hamiltonian.

a) Compute the transformed Hamiltonian $\hat{U}\hat{H}_{JC}\hat{U}^\dagger$ using the Baker-Hausdorf lemma to 2nd order in g/Δ. Argue that the transformed Hamiltonian you calculate has the properties described above.

b) Compute the energy spectrum of your transformed Hamiltonian and confirm that it agrees with the dispersive spectrum presented in class.