QIC 890 Problem Set 1. Due Feb 2, 2016.

The assignment mark will be determined from answers to a subset, possibly all, of the following questions.

1.1 (3)

a) Prove $D^\dagger(\alpha)aD(\alpha)$=$a+\alpha$, where D is the displacement operator.
b) Prove $S^\dagger(\xi)aS(\xi)$=$a\cosh r - a^\dagger e^{i\theta}\sinh r$, where S is the (single mode) squeezing operator and the squeezing parameter $\xi = re^{i\theta}$

1.2 (3)

Prove $[a, D(\alpha)] = \alpha D(\alpha)$, where D is the displacement operator. Use this commutator and the fact that $a|0\rangle = 0$ and the definition of a coherent state $a|\alpha\rangle = \alpha|\alpha\rangle$ to prove that $D(\alpha)|0\rangle$ is the coherent state $|\alpha\rangle$.

1.3 (3)

Calculate Δq and Δp for the thermal state of temperature, T. When is this state minimum uncertainty?

1.4 (3)

Prove $\int d^2\alpha|\alpha\rangle\langle\alpha| = \pi\mathbb{1}$.

1.5 (3)

For a pure state, $|\psi\rangle_{AB}$, the entropy of entanglement is defined as the von Neuman entropy $S(\rho_A) = -\mathrm{Tr}(\rho_A \log_2 \rho_A)$, where ρ_A is the reduced density matrix of

$|\psi\rangle$. Calculate the entropy of entanglement for the two-mode squeezed state as a function of the squeezing parameter and plot it on a graph. For what magnitude of the squeezing parameter $|\xi|$ and 'dB' does the entropy of entanglement exceed that of a Bell state $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

1.6 (2)

Find the average number of photons in a squeezed beam with 10dB squeezing, roughly the current experimental record. Would you characterize this beam as "bright"?

1.7 (4)

The goal of this question is to show that the normal ordered form of the thermal operator,

$$T(\theta) = \mathrm{sech}\theta \exp(a^\dagger \widetilde{a}^\dagger \tanh\theta) \exp[(a^\dagger a + \widetilde{a}^\dagger \widetilde{a}) \ln \mathrm{sech}\theta] \exp(-a\widetilde{a}\tanh\theta), \quad (1.1)$$

is equivalent to the definition $T(\theta) = \exp[\theta(a^\dagger \widetilde{a}^\dagger - a\widetilde{a})]$. There are several methods for doing so in the literature, the following is based on the approach in Ref. [1]. The essential idea there is that the normal ordered form is determined entirely by the commutation properties of the operators. If we can find a different set of operators with the same commutation properties the form will not change, but the factorization could be easier to show.

a) Define operators $A^\dagger = a^\dagger \widetilde{a}^\dagger$, $A = a\widetilde{a}$, $B = a^\dagger a + \widetilde{a}^\dagger \widetilde{a} + 1$ and calculate the commutators $[A, B]$, $[A^\dagger, B]$, and $[A, A^\dagger]$.

b) Make the following associations $A^\dagger \leftrightarrow \sigma_+$, $A \leftrightarrow -\sigma_-$ and $B \leftrightarrow \sigma_z$, where σ_z is the Pauli matrix and $\sigma_\pm = (\sigma_x + i\sigma_y)/2$. Show that these operators have the same commutation relations.

c) Using the 2×2 matrix representations for the Pauli operators, show that

$$e^{\theta(\sigma_+ - (-\sigma_-))} = e^{\sigma_+ \tanh\theta} e^{\sigma_z \ln \mathrm{sech}\theta} e^{-(-\sigma_-)\tanh\theta}. \quad (1.2)$$

d) Use the fact that the commutation properties of these sets of operators are the same to show that the thermal operator and the normal ordered form above are equivalent.

1.8 (4)

The overcompleteness of the basis of coherent states of coherent states has some unusual properties. For example, any pure state can be written as [2, p. 89],

$$|\psi\rangle = \int_C d\alpha b(\alpha) \exp(|\alpha|^2/2)|\alpha\rangle \tag{1.3}$$

where the integral is a line integral around a suitably chosen contour, C, in the complex plane. As a simpler example, show that the vacuum state $|0\rangle$ can be expressed as such a contour integral with the amplitude $b(\alpha) = (2\pi i\alpha)^{-1}$ and the contour is any that encloses the origin. (Bonus points if you can prove the general case!)

1.9 (4)

The Susskind-Glogower phase operators are defined [3, 4],

$$\hat{E} \equiv (\hat{n}+1)^{-1/2} a \tag{1.4}$$
$$\hat{E}^\dagger \equiv a^\dagger (\hat{n}+1)^{-1/2} \tag{1.5}$$

(where the use of the letter E relates to exponential not electric field).
a) Show that this phase operator can be written $\hat{E} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|$. Show that $\hat{E}$ is not unitary and therefore cannot be viewed as an exponentiated Hermitian phase operator (similar to the case discussed in class).
b) Show that

$$|\phi\rangle = \sum_n e^{in\phi}|n\rangle, \tag{1.6}$$

are eigenstates of $\hat{E}$, but these states are not normalizable nor orthogonal for different values of ϕ.
c) Nevertheless, we can define a phase distribution of a state, ρ

$$P(\phi) = \frac{1}{2\pi}\langle\phi|\rho|\phi\rangle. \tag{1.7}$$

Calculate the phase distribution for a Fock state, $|n\rangle$, and its uncertainty. Show that the phase is completely uncertain. Calculate the phase uncertainty scale for coherent states as a function of the average photon number and estimate the asymptotic scaling (i.e., does the phase uncertainty scale as $n = |\alpha|^2$, n^2, etc.)?

1.10 (4)

Robertson's Uncertainty Relation [5] (sometimes called the Generalized Uncertainty Relation) between two observables A and B is,

$$\Delta A^2 \Delta B^2 \geq \left(\frac{1}{2i}\langle[A,B]\rangle\right)^2. \tag{1.8}$$

a) Prove the following inequality:

$$\Delta A^2 \Delta B^2 \geq \left(\frac{1}{2}\langle\{A,B\}\rangle - \langle A\rangle\langle B\rangle\right)^2 + \left(\frac{1}{2i}\langle[A,B]\rangle\right)^2, \tag{1.9}$$

where $\{A,B\} = AB + BA$ is the *anti-commutator* of A and B. This inequality was derived by Schroedinger in 1930 (see arXiv:quant-ph/9903100 for an english translation and this particular form of the inequality). The additional term over Robertson's inequality is the *covariance* of A and B. (hint: see for example, D. Griffiths Intro. to Quantum Mechanics for the proof of Robertson's Relation; the extension to the Schroedinger Relation requires a straightforward modification)
b) In class, we showed that coherent states are minimum uncertainty states for the quadrature, q and p, using the Robertson relation. Show that they are also minimum uncertainty states for Schroedinger's relation.
c) Show that *some* single mode squeezed states are minimum uncertainty states for the q and p quadratures using Robertson's relation; what is the condition on ξ for this to be the case?
d) Show that single mode squeezed states are minimum uncertainty for Schroedinger's relation for all ξ.

Unmarked problems

For our quantization procedure, we used a mode expansion in a cubic volume and assumed periodic boundary conditions. Repeat the calculation assuming perfectly reflecting boundaries where the electric field goes to zero at the boundaries. Does this different condition change the density of states? (hint: follow the approach used in Loudon [6])

In PRD 29, 1110 (1984) the following identity was written $D(\alpha)S(z) = S(z)D(\gamma)$, where D is the displacement operator and S is the squeezing operator. Find γ in terms of α and z.

Calculate the expectation value of $\hat{n}^2$ for a thermal state directly and using the properties of the thermal operator (i.e., the Bogliubov transformation) and show that they yield the same result.

Find an analogous eigenvalue expression for

$$\begin{aligned} S(\xi)aS^\dagger(\xi)S(\xi)|0\rangle &= 0 \quad (1.10) \\ (a\cosh r + a^\dagger e^{i\theta}\sinh r)|\xi\rangle &= 0 \quad (1.11) \end{aligned}$$

for the displaced squeezed state $D(\alpha)S(\xi)|0\rangle$.

Prove the Baker-Campbell-Hausdorff lemma:

$$e^A B e^{-A} = A + [A, B] + \frac{1}{2}[A, [A, B]] + ... + \frac{1}{n!}[A, [A, [A, ...[A, B]]]...] + ... \quad (1.12)$$

where A and B are operators. (hint: use proof by induction).

Bibliography

[1] B. L. Schumaker and C. M. Caves, Physical Review A **31**, 3093 (1985).

[2] S. M. Barnett and P. M. Radmore, *Methods in Theoretical Quantum Optics* (Clarendon Press, Oxford, 1997).

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[5] H. P. Robertson, Physical Review **34**, 163 (1929).

[6] R. Loudon, *The Quantum Theory of Light* (Clarendon Press, Oxford, 1983), 2nd ed.