Problem 1a

The maximum power available from the source is determined by the condition where the source is conjugately matched to the load (this is when $Z_s = Z_{in}^*$). The load impedance is specified as 100 Ω. Thus, a conjugately matched input impedance would be given by $(100\ \Omega)^*$. This load will divide the voltage of the source. Thus, the power available from the source is given by:

$$\begin{aligned}
P_{avl} &= \Re(V_{in} I_{in}^*) \\
&= \Re(\frac{|V_{in}|^2}{Z_{in}^2} \\
&= \Re\left(\frac{(10\,\mathrm{V})^2}{100\Omega}\right) \\
&= 1\,\mathrm{W}
\end{aligned}$$

Problem 1b

Port 1 Waves

To calculate the port 1 power waves I will use the following relationships:

$$\sqrt{Z_c}\,(a_1 + b_1) = V_s \frac{Z_{in}}{Z_{in} + Z_s} \tag{0.1}$$

$$b_1 = \Gamma_{in} a_1 \tag{0.2}$$

$$\Gamma_{in} = \frac{Z_{in} - Z_c}{Z_{in} + Z_c} \tag{0.3}$$

Combining (0.1) and (0.2) yields:

$$a_1 = \frac{V_s}{\sqrt{Z_c}} \frac{1}{(1 + \Gamma_{in})} \frac{Z_{in}}{Z_{in} + Z_s} \tag{0.4}$$

Rearranging (0.3) for Z_{in} produces $Z_{in} = Z_c \frac{1+\Gamma_{in}}{1-\Gamma_{in}}$. Substituting this into (0.4) yields:

$$\begin{aligned}
a_1 &= \frac{V_s}{\sqrt{Z_c}(1 + \Gamma_{in})} \frac{Z_c \frac{1+\Gamma_{in}}{1-\Gamma_{in}}}{Z_c \frac{1+\Gamma_{in}}{1-\Gamma_{in}} + Z_s} \\
&= \frac{V_s}{\sqrt{Z_c}(1 + \Gamma_{in})} \frac{Z_c\,(1 + \Gamma_{in})}{Z_c\,(1 + \Gamma_{in}) + Z_s\,(1 - \Gamma_{in})} \\
&= V_s \frac{\sqrt{Z_c}}{Z_c\,(1 + \Gamma_{in}) + Z_s\,(1 - \Gamma_{in})} \qquad (0.5)
\end{aligned}$$

b_1 is easily obtained from this using (0.2).

$$b_1 = V_s \frac{\sqrt{Z_c}\Gamma_{in}}{Z_c\,(1 + \Gamma_{in}) + Z_s(1 - \Gamma_{in})} \tag{0.6}$$

Notice that if $\Gamma_{in} = 1$ (an open) that $a_1 = \frac{V_s}{2\sqrt{Z_c}}$ and $b_1 = \frac{V_s}{2\sqrt{Z_c}}$ such that $V_{load} = \sqrt{Z_c}(a_1 + b_1) = V_s$ (reflects completely in phase).

If $\Gamma_{in} = -1$ (a short)

$$a_1 = \frac{V_s\sqrt{Z_c}}{2Z_s}$$
$$b_1 = -\frac{V_s\sqrt{Z_c}}{2Z_s}$$
$$V_{load} = 0$$

If $\Gamma_{in} = 0$ (a matched load)

$$b_1 = 0$$
$$a_1 = \frac{V_s\sqrt{Z_c}}{Z_c + Z_s}$$
$$V_{load} = \frac{V_s Z_c}{Z_c + Z_s}$$

But, of course, in matched conditions $Z_c = Z_L$.

Port 2 Waves

Calculating the port 2 power waves I will use the following relationships:

$$b_2 = S_{21}a_1 + S_{22}a_2 \tag{0.7}$$
$$a_2 = \Gamma_l b_2 \tag{0.8}$$
$$a_1 = V_s \frac{\sqrt{Z_c}}{Z_c\,(1 + \Gamma_{in}) + Z_s\,(1 - \Gamma_{in})} \tag{0.9}$$

Combining (0.7) and (0.8) yield:

$$b_2 = \frac{S_{21}a_1}{1 - S_{22}\Gamma_l} \tag{0.10}$$

Combining (0.10) and (0.9) yields:

$$b_2 = \frac{V_s\sqrt{Z_c}}{Z_c\,(1 + \Gamma_{in}) + Z_s(1 - \Gamma_{in})}\frac{S_{21}}{1 - S_{22}\Gamma_l} \tag{0.11}$$

Obtaining a_2 from (0.8) and (**??**) is trivial:

$$a_2 = \frac{V_s\sqrt{Z_c}}{Z_c\,(1 + \Gamma_{in}) + Z_s(1 - \Gamma_{in})}\frac{S_{21}\Gamma_l}{1 - S_{22}\Gamma_l} \tag{0.12}$$

Problem 1c

To determine the power delivered to the load we begin with the definition of real power

$$P_{load} = \frac{1}{2}\Re\left(V_{load} I^*_{load}\right)$$

In this case, $V_{load} = \sqrt{Z_c}\left(a_2 + b_2\right)$ and $I_{load} = \frac{b_2 - a_2}{\sqrt{Z_c}}$. But, $a_2 = b_2\Gamma_l$ so P_{load} can be rewritten as follows:

$$\begin{aligned}
P_{load} &= \frac{1}{2}\Re\Big(\left(a_2 + b_2\right)\left(b_2^* - a_2^*\right)\Big) \\
&= \frac{1}{2}\Re\Big(\left(b_2\left(1 + \Gamma_l\right)\right)\left(b_2^*\left(1 - \Gamma_l^*\right)\right)\Big) \\
&= \frac{|b_2|^2}{2}\Re\left(1 - |\Gamma_l|^2 + 2j\Im(\Gamma_l)\right) \\
&= \frac{|b_2|^2}{2}\left(1 - |\Gamma_l|^2\right)
\end{aligned}$$

To calculate the power reflected to the source I need to take the incident power and subtract the amount that is delivered to the load and the 2-port network.

$$P_{source} = \frac{1}{2}\Re\left(V_{source} I^*_{source}\right)$$

$$V_{source} = V_s\frac{Z_s}{Z_s + Z_{in}}$$

and

$$I_{source} = \frac{V_s}{Z_s + Z_{in}}$$

.

Substituting these two expressions into P_{source} yields:

$$\begin{aligned}
P_{source} &= \frac{1}{2}\Re\left(\frac{V_s Z_s}{Z_s + Z_{in}}\frac{V_s^*}{Z_S^* + Z_{in}^*}\right) \\
&= \frac{|V_S|^2}{2}\Re\left(\frac{Z_s}{|Z_s + Z_{in}|^2}\right) \\
&= \frac{|V_s|^2}{2\left|Z_s + Z_{in}\right|^2}\Re(Z_s)
\end{aligned}$$

We will now plug the following numbers into equations (0.5, 0.6, 0.11,0.12) to obtain a_1, b_1, b_2 and a_2, respectively.

Variable	Value
$Z_l = Z_c$	$50\,\Omega$
Z_s	$100\,\Omega$
$\Gamma_s = \frac{Z_s - Z_c}{Z_s + Z_c}$	$\frac{1}{3}$
$\Gamma_l = \frac{Z_l - Z_c}{Z_l + Z_c}$	0
S_{11}	$.1\underline{/-30^\circ}$
S_{12}	$.4\underline{/-75^\circ}$
S_{21}	$.95\underline{/-45^\circ}$
S_{22}	$.15\underline{/-10^\circ}$
V_s	$20\,\mathrm{V}\underline{/0^\circ}$
$Z_{in} = Z_c \frac{1+\Gamma_{in}}{1-\Gamma_{in}}$	$\approx 56.3\,\Omega$
Γ_{in}	$\approx 59.2 \cdot 10^{-3}$

Plugging these numbers in yields:

Voltage	Voltage
a_1	$\approx 961\,\mathrm{mV}/\sqrt{\Omega}$
b_1	$\approx 57.0\,\mathrm{mV}/\sqrt{\Omega}$
a_2	$0.0\,\mathrm{mV}/\sqrt{\Omega}$ (exactly)
b_2	$\approx 417\,\mathrm{mV}/\sqrt{\Omega}$

Problem 2a: Single Open Stub Tuner

The goal is to match a load impedance to a transmission line by placing a stub of a certain length l_s a certain distance l_l away from a load. To be matched means that the input impedance looks like the characteristic impedance Z_c. To the source, the two paths (the stub and the load) will seem to be connected in parallel. Thus, quite generally:

$$Z_{in} = Z_{stub}||Z_l'$$

where Z_l' is the impedance of the load transformed by a certain length l_l down the line. A load of impedance Z_l looks the following impedance when we're a length "l" down the line:

$$Z_l'(l) = Z_c \frac{Z_l + jZ_c \tan \beta l}{Z_c + jZ_l \tan \beta l}$$

However, because we are combining parallel impedances it will be easier to deal with admittances:

$$Y_{in} = Y_{stub} + Y_l'$$

where

$$Y_l'(l_l) = Y_c \frac{Y_l + jY_c \tan \beta l_l}{Y_c + jY_l \tan \beta l_l}$$

Our goal is to make $\Re(Y_{in}) = Z_c$ and $\Im(Y_{in}) = 0$. We have an expression for Y_l' already. We'd like an expression for Y_{stub}. However, Y_{stub} is just a infinite impedance (zero admittance) load. Thus, we can use the same equation as before before:

$$\begin{aligned} Y_l'(l_s) &= Y_c \frac{Y_s + jY_c \tan \beta l_s}{Y_c + jY_s \tan \beta l_s} \\ &= Y_c \frac{0 + jY_c \tan \beta l_s}{Y_c + j0 \tan \beta l_s} \\ &= jY_c \tan \beta l_s \end{aligned}$$

Unfortunately, the load can not, in general, be simplified any further. Thus, the input admittance is:

$$Y_{in} = Y_c \left(\frac{Y_l + jY_c \tan \beta l_l}{Y_c + jY_l \tan \beta l_l} + j \tan \beta l_s \right)$$

In accordance with our specifications given earlier we have the following two equations by considering the imaginary and real parts of Y_{in} separately.

$$\begin{aligned} &\Im(Y_{in}) = 0 \\ &\quad = \Im\left((Y_l + jY_c \tan \beta l_l)(Y_c - jY_l \tan \beta l_l) + j \tan \beta l_s \left(Y_c^2 + Y_l^2 \tan^2 \beta l_l\right) \right) \end{aligned}$$

This yields the following equation:

$$0 = Y_c^2 \tan \beta l_l - Y_l^2 \tan \beta l_l + \tan \beta l_s \left(Y_c^2 + Y_l^2 \tan^2 \beta l_l\right)$$

Considering, separately the real part of the input admittance.

$$\Re(Y_{in}) = Y_c = \Re\left(Y_c \left(\frac{Y_l + jY_c \tan \beta l_l}{Y_c + jY_l \tan \beta l_l} + j \tan \beta l_s \right) \right)$$

Re-arranged and simplified a bit:

$$1 = \Re\left(\frac{(Y_l + jY_c \tan \beta l_l)(Y_c - jY_l \tan \beta l_l)}{Y_c^2 + Y_l^2 \tan^2 \beta l_l} + j \tan \beta l_s \right)$$

Taking the real part:

$$1 = \frac{Y_l Y_c + Y_l Y_c \tan^2 \beta l_l}{Y_c^2 + Y_l^2 \tan^2 \beta l_l}$$

These two equations can be solved simultaneously for pairs of l_l and l_s that satisfy them. Given the transcendental nature of these two solutions it is not, in general, possible to find an analytic solution. However, we can reduce the complexity of these two equations, somewhat, if we rewrite $Y_l = \alpha Y_c$ and, instead, for known α solve the two equations for l_l and l_s. Rewriting these two equations in terms of $Y_l = \alpha Y_c$ yields:

$$\begin{aligned}
0 &= Y_c^2 \tan \beta l_l - \alpha^2 Y_c^2 \tan \beta l_l + \tan \beta l_s \left(Y_c^2 + \alpha^2 Y_c^2 \tan^2 \beta l_l \right) \\
1 &= \frac{\alpha Y_c^2 + \alpha Y_c^2 \tan^2 \beta l_l}{Y_c^2 + \alpha^2 Y_c^2 \tan^2 \beta l_l}
\end{aligned}$$

These can be immediately reduced to:

$$\begin{aligned}
0 &= \tan \beta l_l \left(1 - \alpha^2 \right) + \tan \beta l_s \left(1 + \alpha^2 \tan^2 \beta l_l \right) \\
1 &= \frac{\alpha \left(1 + \tan^2 \beta l_l \right)}{1 + \alpha^2 \tan^2 \beta l_l}
\end{aligned}$$

LC Matching Network

The goal in constructing a matching network for the load using an LC network is going to be similar as that that was performed in the previous section. Namely, our goal is going to be to make $Z_{in} = Z_c$ such that $\Re\left(Z_{in}\right) = Z_c$ and $\Im\left(Z_{in}\right) = 0$. It is easy, in this case to construct the input impedance as

$$Z_{in} = Z_A + Z_B || Z_L$$

The only knowledge we have, currently, regarding Z_A and Z_B is that both impedances are purely imaginary. We know that Z_L is purely real ($100\,\Omega$). Thus, we start by considering $\Re\left(Z_{in}\right)$.

$$\begin{aligned}
\Re(Z_{in}) = Z_c &= \Re\left(\frac{Z_B Z_L}{Z_B + Z_L}\right) \\
&= \Re\left(\frac{j X_B R_L}{j X_B + R_L}\right) \\
&= \frac{X_B^2 R_L}{X_B^2 + R_L^2}
\end{aligned}$$

Considering the imaginary part of Z_{in}

$$\Im(Z_{in}) = 0 = \Im\left(jX_A + \frac{jX_B R_L}{jX_B + R_L}\right) \tag{0.13}$$

$$= \Im\left(jX_A + \frac{jX_B R_L\left(R_L - jX_B\right)}{X_B^2 + R_L^2}\right) \tag{0.14}$$

$$= X_A + \frac{X_B R_L^2}{X_B^2 + R_L^2} \tag{0.15}$$

The real part of Z_{in} only involves Z_B. So, we can easily solve for Z_B that way. Doing so yields:

$$X_B^2 = \frac{R_L^2 Z_c}{R_L - Z_c}$$

Since we have constrained X_B to be real (so that the impedance Z_B is purely reactive) we know that we must take the positive root. We also know that this only works if $R > Z_c$. If the load is smaller than the characteristic impedance than an LC matching network will not work. You must add some resistance.

Now, X_A is easily determined:

$$\begin{aligned}
X_A &= -\frac{X_B R_L^2}{X_B^2 + R_L^2} \\
&= -\frac{X_B}{\frac{Z_c}{R_L - Z_c} + 1} \\
&= -\frac{X_B\left(R_L - Z_c\right)}{Z_c + R_L - Z_c} \\
&= -\frac{X_B\left(R_L - Z_c\right)}{R_L} \\
&= -\sqrt{\frac{Z_c}{R_L - Z_c}}\left(R_L - Z_c\right) \\
&= -\sqrt{Z_c\left(R_L - Z_c\right)}
\end{aligned}$$

If one allows, again, $R_L = \alpha Z_c$ then

$$X_B^2 = \frac{\alpha^2 Z_c^2}{\alpha - 1}$$

and

$$X_A = -Z_c\sqrt{\alpha - 1}$$

The reflection coefficient for the load and the matching network can be found by considering $\Gamma_{in} = \frac{Z_{in} - Z_c}{Z_{in} + Z_c}$. In this case, Γ_{in} is a function of frequency. Plotting it as a function of frequency yields the following:

Problem 3: Lossy LC matching network

The first thing to do is to find the amount of loss in the matching network. This loss will be due to two non-ideal conditions for the circuit elements

1. The inductor is lossy with series lossy element R_s
2. The capacitor is lossy with parallel lossy element R_p

Thus, the loss in R_s can be found by driving the now-matched input network with a voltage V_{in} and finding the following:

$$P_s = \frac{1}{2}\Re\left(V_s I_s^*\right)$$

$V_s = V_{in}\frac{R_s}{Z_{in}}$ and $I_s = \frac{V_{in}}{Z_{in}}$
The power delivered to the series resistor is now

$$P_s = \frac{1}{2}\frac{\left|V_{in}\right|^2 R_s}{\left|Z_{in}\right|^2}$$

The power delivered to the parallel resistor is a bit more tricky. We can consider the voltage across the parallel branch (with R_p, R_L and Z_C). The voltage across that branch is

$$V_p = V_{in}\frac{Z_c||R_L||R_p}{Z_{in}}$$

The current through R_p, I_p, is:

$$I_p = \frac{V_p}{R_p}$$

Thus, we can write the power delivered to this resistor as:

$$P_p = \frac{1}{2}\frac{\left|V_p\right|^2}{R_p}$$

And the total power delivered to these two circuits elements is just

$$P_s + P_p = \frac{1}{2}\frac{\left|V_{in}\right|^2 R_s}{\left|Z_{in}\right|^2} + \frac{1}{2}\frac{\left|V_{in}\right|^2\left|Z_c||R_L||R_p\right|^2}{R_p\left|Z_{in}\right|^2} \tag{0.16}$$