QIC 750 Term Paper - Wigner State Tomography

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1 Introduction

Quantum computation relies on high-fidelity state preparation and gate operation. This is well-known as of the publishing of the oft-cited DiVincenzo criteria [2]. However, characterizing the fidelity of state preparation and gate operation is non-trivial. Many metrics have been developed and applied to laboratory set-ups. Density matrix reconstruction, Rabi decay oscillations, Wigner state tomography are just a few of the most popular fidelity characterization methods (for state preparation) used in the literature today. The latter is the focus of this paper. A brief overview of the theory underlying Wigner state tomography will be given then its application to states demonstrated by various groups will be presented.

2 Theory

Consider the phase space representation of a classical system of particles. The experimentalist's ignorance as to the exact configuration of the system results in a probability distribution of the particles in phase space (where the phase space comprises the momentum and position of each particle). Now, the experimentalist's ignorance as to the quantum state of a system of interest, similarly, generates a probability distribution over the Hilbert space of that system. Typically, the representation of this state is given by the density matrix of the system. Briefly, if a quantum state is in any of k possible states (indexed as $|i\rangle$) with probability P(i), then the density matrix, ρ, is given as $\rho = \sum\limits_{i=0}^{k} P(i)|i\rangle\langle i|$. It might be thought that with the proper definition of a statistical function, one can discuss the state of a quantum-mechanical system using the system's density matrix formalism in a way that is strongly analogous to that of the state-space analysis performed in classical mechanical systems. This statistical function that we define is the Wigner function $W(q,p) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{ipx}\left\langle q - x/2\right|\rho\left|q + x/2\right\rangle dx$ [4].

The Wigner function has many useful properties and alternative representations but these are irrelevant for this paper and will be omitted for succinctness. There are only a couple relevant notes with regards to the form of the Wigner function. One, the Wigner function is a probability density function over joint variables which are canonically conjugate (like position and momentum). But, the Wigner function can be, and often is, negative over portions of this phase space. The physical understanding of these negative regions is that the negative regions represent a resource useful for quantum computation; they represent the non-classical portions of the state [4]. Of course, though, this violates the typical understanding of probability density functions as strictly non-negative objects. Additionally, it should be noted that since the Wigner function is constructed using only the density matrix of the system of interest, the determination of the Wigner function determines the density matrix of the system, uniquely (and vice versa).

This is the idea behind Wigner state tomography. By making different measurements of a prepared quantum system one can determine the state of the system uniquely. Consider a quantum mechanical system defined in terms of the canonically conjugate position and momentum operators : $x = \frac{a+a^\dagger}{\sqrt{2}}$ and $p = \frac{a-a^\dagger}{i\sqrt{2}}$. One can then speak of an in-phase operator $x_\phi = p\sin(\phi) + x\cos(\phi)$ and a quadrature operator, $p_\phi = p\cos(\phi) - x\sin(\phi)$. As the Wigner distribution represents a joint probability distribution over $x(x_\phi, p_\phi)$ and $p(x_\phi, p_\phi)$ integration over p_ϕ results in a marginal probability distribution over x_ϕ: $P_\phi(x_\phi) = \int\limits_{-\infty}^{\infty} W\Big(x_\phi\cos(\phi) - p_\phi sin(\phi), x_\phi\sin(\phi) + p_\phi\cos(\phi)\Big)dp_\phi$. Thus, to determine the Wigner function representing the particular state, it is only necessary to obtain the probability distribution over one quadrature (e.g. sweep ϕ and measure x_ϕ) and take the derivative of that distribution with respect to the alternate quadrature (p_ϕ). The physical implications behind sweeping the phase" depends on the particular implementation. The first such successful application of Wigner state tomography to a quantum system was accomplished by Smithey, Beck, Raymer and Faridani. As their platform was an optical one, sweeping the phase corresponded to a physical rotation of reflective apparatuses. Now, this discussion has made the measurement of the Wigner function of some state sound relatively trivial. This is not the case, typically. In order to obtain useful results, one typically needs to apply some more mathematics (like maximal likelihood estimations and filtered back projections); but, the above gives the novice a rough idea as to the mathematical

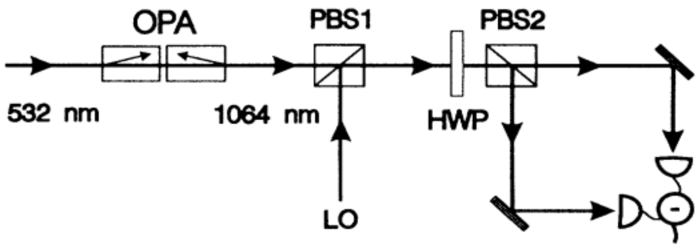

Figure 1: Experimental set-up as reported in [5]. An optical parametric amplifier is driven to upconvert a 532 nanometer signal to a 1064 nanometer signal. In the case of the measurement of squeezed radiation, the output of the parametric amplifier is one of the two inputs to a single port of a 50-50 (balanced) polarizing beam splitter, PBS1 - the other input being the local oscillator (whose association with a piezoelectric crystal determines the phase relationship between the signal and the local oscillator). In the case of the measurement of unsqueezed radiation the output of the parametric amplifier is blocked; then the non-LO input to PBS1 is simply the vacuum state. The polarization of the light into the second polarizing beam splitter, PBS2, adjusted with a $\frac{\lambda}{2}$ plate (HWP). The effect of the half-wave plate is to rotate components of the polarization by 45°. The placement of the last beam splitter produces a superposition of the two incident beams (the vacuum and the combined signal from the first beam splitter). The two superposition beams are detected by the two photodetectors. The two output signals are then processed and analyzed to obtain the quadrature probability distributions, $P_\phi(x_\phi)$.

foundations underlying Wigner representations of quantum states.

3 Smithey et. al.

In 1993 Smithey et. al. reported making an experimental determination of a single-mode of the vacuum state's electrical field [5]. In addition to making a bare measurement of the electrical field they, also, measured a squeezed state of the vacuum electrical field. Note that their approach was optical and that they used a slightly different form of the Wigner function than is proposed by Leonhardt (given earlier) : $W(x,p) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \langle x + x' | \, \rho \, | x - x' \rangle \, e^{-2ipx'} dx'$.

Experimentally, their set-up was as is shown in Figure 1. The phase between the local oscillator and the input signal is controlled by the position of a piezoelectric mirror. The piezoelectric mirror allows for modulation of the beam's path length over fractions of a wave-

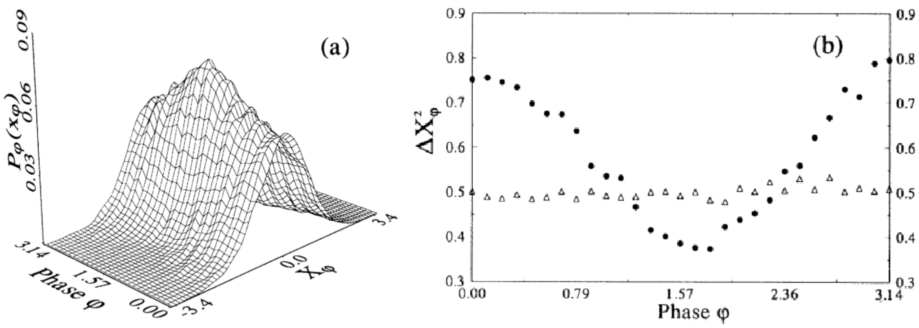

Figure 2: Experimental data obtained by Smithey et. al. Show in a) is the marginal distribution of x_ϕ, of $P_\phi x_\phi, p_\phi$, as a function of the local oscillators' swept phase. Panel b) shows the fluctuations of the position quadrature as a function of the local oscillator phase. The near-constant value of .5 for the triangular markers is indicative of vacuum fluctuations (whose variance is constant over the parameter space formed by x_ϕ and p_ϕ). Similarly, the sinusoidal variation of the fluctuations in x_ϕ, exemplified by the circular markers, is indicative of quadrature squeezing of the vacuum field. Portions of the data which fall significantly below .5 are indicative of the squeezing of the position quadrature. Portions of the data which lie significantly above .5 indicate squeezing of the momentum quadrature.

length. The intensity of the two beams is subtracted and the intensity of the difference is directly proportional to the position operator (x_ϕ) of the signal. Thus, sweeping the position of the piezoelectric mirror over a measurement of the difference of the intensities between the two photodiodes (photodetectors) determines the probability distribution of x_ϕ ($P_\phi(x_\phi)$) The phase (the piezoelectric mirror) was varied over 27 distinct positions (phases) and the intensities measured were histogrammed into 64 bins. This analysis, coupled with the aforedescribed measurement process, yielded the probability distribution explained in Figure 2.

Their experiment is notable due to its historical weight. Smithey et. al. were the first to perform such a Wigner characterization of a quantum mechanical state. At the time, this was fairly novel. It did not gain much popularity and did not see considerable development for 10 years (until Banaszek et. al. performed their direct Wigner tomography).

4 Deléglise et. al.

In 2008, Deléglise et. al. [1] performed markedly different state preparation and characterization compared to that done by Smithey et. al [5]. Their approach involved the use of a cavity quantum electrodynamics (QED) set-up. To prepare the initial state, a highly reflective mirror is brought to superconducting temperatures. The purpose of the mirror is to trap incident photons. One of the purposes of the low temperature for the mirror is to suppress spurious photons generated by the vacuum (which occurs with higher probablility - read: frequency - at higher temperatures. Their scheme purportedly traps the incident light between the mirrors for long times ($130ms$). A stream of rubidium atoms are prepared such that the state of the atoms (rather the valence electrons) occupy the $n = 50$ spherical orbital. The adjacent transition in this atom is the $n = 51$ spherical orbital. These serve as the ground and first excited states.

The effect of the cavity field, C in figure 3, on the stream of rubidium atoms is determined by a measurement of the field after the atom is determined to be in the ground or excited state (at D in figure 3). Then, after the state of the atoms (whose state has been modified by the various fields) has been measured a sufficient number of times (millions) the density matrix for the field can be determined. The density matrix obtained for a variety of prepared states is show in figure 4. The focus of this paper was on the preparation of Schrödinger cat states, which are of the form : $|A|(|\alpha\rangle \pm |{-\alpha}\rangle)$, where $|A|$ is the normalization constant and $|\alpha\rangle$ is the conventional notation for a coherent state of light. The reader is reminded that a coherent state of light is one that is expressed in the Fock (number) basis as : $|a\rangle = \sum_{n=0}^{\infty} e^{-\frac{1}{2}\alpha\alpha^*} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = e^{-\frac{1}{2}\alpha\alpha^*} e^{\alpha\hat{a}^\dagger} |0\rangle$. The even cat states and the odd cat states are physically interesting in that an even cat state contains only Fock states of even value while an odd cat state contains only Fock states of odd value.

Now, while the state preparation is not explained in great detail, the authors of the Deléglise paper indicate that the state of the atom upon exiting the third cavity is a linear combination of an even photonic cat state and odd cat state. The even cat state is entangled with the ground state of the rubidium atom and the odd cat state is entangled with the excited state of the rubidium atom. Thus, detector D, which discriminates between excited and ground states of the rubidium atom, projects the state of the field onto one of the Schrödinger cat states. If the atom is detected without using detector D to measure the excitation of the atom, the state of

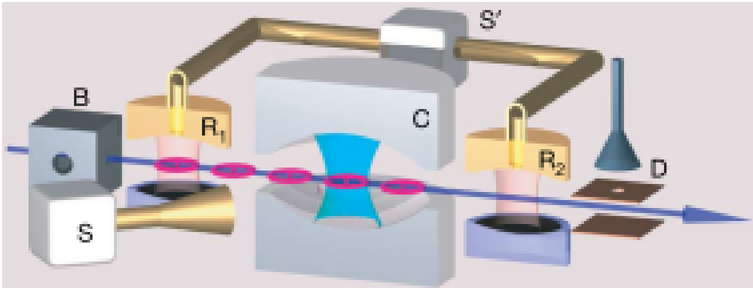

Figure 3: Cartoon of the experimental set-up used by Deléglise et. al. The Rydberg atoms (rubidium) are reported as being prepared in 'Box B'. These are collimated from the exit port of Box B and projected through the cavity resonator at a speed of 250 $\frac{m}{s}$. Note that, though, there are two cavities external to the main central cavity, C. These cavities R_1 and R_2 are both used along with an external electromagnetic source (microwave) to prepare the Rydberg atoms in superposition states $\frac{1}{\sqrt{2}}\Big(\left|0\right\rangle + e^{i\phi}\left|1\right\rangle\Big)$. The state of the atoms once they have traveled through the three cavity resonators (and having been irradiated three separate times) is determined by the atomic detector, D. Note that the insets have been generated by taking the experimental nonlinearaties of the apparatuses into account.

the atom entangled to the coherent light in the resonator is, in this case, a statistical mixture of an even and odd cat state. Note that the measured quantity is the radiation escaping from cavity C (not the state of the atoms). The atoms are just used for projective purposes.

The tomography of the states is done in such a way as to, initially, obtain the density matrix of the system. Then, the Wigner function over the phase space (the real and imaginary parts of the field amplitude α) is given by $W(\alpha) = \frac{2}{\pi}Tr[D(-\alpha)\rho D(\alpha)e^{i\pi N}]$, where the N in the exponential is the standard bosonic number operator $N = a^{\dagger}a$. For obvious reasons, this exponentiated operator is often called the parity operator and has eigenvalues $(-1)^n$ (where n is the nth Fock state). The authors make note that the Wigner function would be experimentally determined if the "atom-field phase shift" was linear. I understand this to mean than nonlinearities in the mechanisms used to correlate the state of the atom (qubit) to that of the light make a direct determination of the Wigner function impossible. The results of measurement, post-processed, are shown in figure 4. The density matrices are not shown out of their irrelevance to this paper.

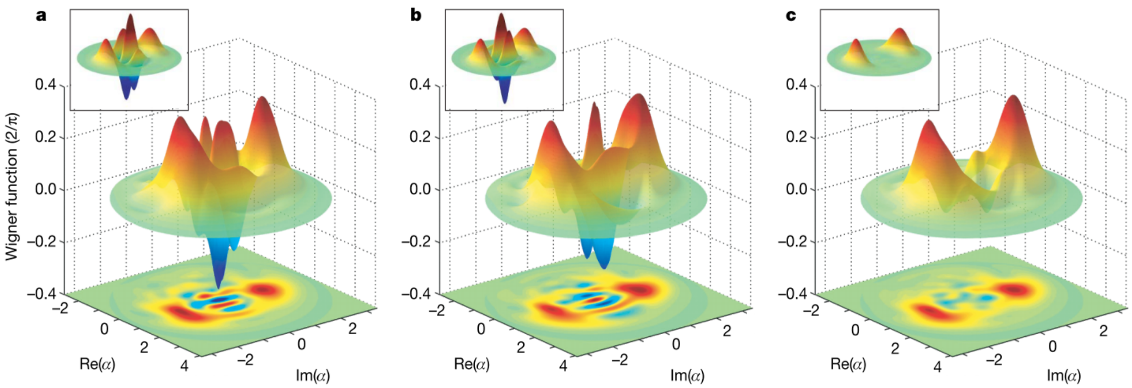

Figure 4: Shown in panels a), b) and c) are Wigner function reconstructions for three different states: a) An even Schrödinger cat state, b) an odd Schrödinger cat stat and c) a symmetric mixture of the even and odd cat states. The theoretically-predicted Wigner function for each respective state is given in the inset of each panel.

5 Banaszek et. al.

In 1999, Banaszek et. al. made direct optical measurements on a number of optical quantum states. Their experimental set-up was fairly simple and the post-processing on their acquired data was relatively trivial. The set-up for their experiment is shown in figure 5. Typically, the method to determine the Wigner function for a quantum state is to, first, determine the density matrix representation of the state and, second, using the equivalence between the density matrix and the Wigner function, transform the data into the Wigner state-space representation. There are a number of reasons this is typically done. However, the inversion that is needed to go from the discrete density matrix representation to the continuous state-space representation is non-trivial. It typically involves heavy statistics (maximum likelihood estimation theorems to determine which state the data represents) and awkward filtered back-projection schemes analogous to those used in the medical field to render images of the human body using tomography machines (PET, CT, MRI, etc.). There is no utility to this process since the density matrix is the fundamental data set and has just as much information as the Wigner representation.

However, Banaszek et. al. implemented a direct measurement of the Wigner function of the quantum state of optical light. The key to sidestepping all the statistics is to realize that the Wigner representation of the state of light (expressed in terms of the displacement of the amplitude of light in phase space) is given by: $W(\alpha) = \frac{2}{\pi} Tr[D(-\alpha)\rho D(\alpha) e^{i\pi N}]$ [1] which

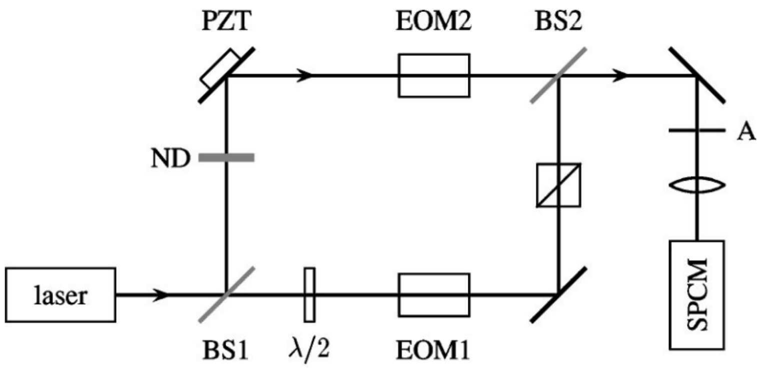

Figure 5: Set-up of the optical experiment conducted by Banaszek et. al. in 1999.

Banaszek et. al. express, equivalently, as $W(\alpha) = \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n D(\alpha)\left|n\right\rangle\left\langle n\right| D^{\dagger}(\alpha)$. Thus, by sweeping α (the phase space parameter), and projecting the resulting quantum state onto the Fock basis, one can determine $W(\alpha)$. Physically, Banaszek et. al. implemented a sweep of $\alpha = |\alpha|e^{i\theta}$ through the use the two branches of their Mach-Zehnder interferometer: a half-wave plate, a Pockels cell (which is a tunable wave plate, called EOM2) and a polarizer control the magnitude of $\alpha =, |\alpha|$, while the upper branch, consisting of a neutral density filter (ND) and an electro-optic phase modulator (EOM2) controls the angle of α, θ, in phase space; see figure 5. The measurement of the light is done with a single photon counting module (SPCM). This projects the state of light onto the Fock basis. The piezoelectric translator (piezoelectric mirror, called PZT) is used only for one data set. For this data set, the translator is modulated with a high-frequncy drive, altering the phase of α over very short time scales. In a similar manner as Smithey et. al. [5], Banaszek et. al. adjusted the amplitude and phase of α in discrete amounts: they used 20 amplitudes and 40 phases of α. Many trials are conducted at a particular amplitude and phase, α_i to obtain a distribution over Fock states. Then, $W(\alpha_i)$ is determined. Then, the phase and/or amplitude is adjusted and another multitude of measurements is made. The results of their experiment are shown in figure 6.

6 Hofheinz et. al.

As a small, final example of an experimental determination of the Wigner function of quatnum states, I present the paper of Hofheinz et. al., 2009 [3].

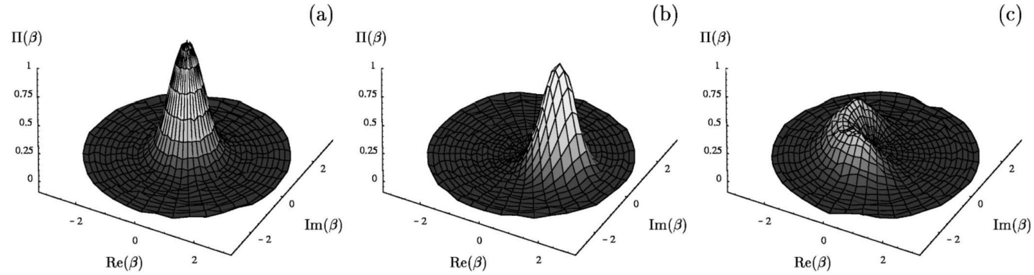

Figure 6: Measured results of three quantum states prepared by Banaszek et. al. Panel a) corresponds to measurements conducted on the vacuum state. Panel b) corresponds to measurements of a weakly displaced vacuum state. Panel c) corresponds to the coherent state whose phase was modulated using the piezoelectric mirror. $\beta = e^{i\theta}\sqrt{n_{vac}}$. Note that β is simply a scaled version of α.

This architecture is different still than the optical platforms already discussed. It involves the use of superconducting circuits. But, the implementation follows the same math, so the results should be the same (excluding environmental differences between the two types of schemes). In not so many words, (because there is not room for many more), microwave pulses are applied to the qubit (see figure 7) to excite the qubit into an arbitrary superposition of $|0\rangle$ and $|1\rangle$. Initially, the qubit transition frequency is detuned (off-resonance) from the resonator frequency. This deters unwanted, spontaneous energy transfer from the qubit to the resonator. Once the state has been prepared in the qubit, the qubit is brought on resonance with the resonator and the exchange of a photon from the qubit to the resonator takes place. By repeatedly doing this and keeping track of the phase of old photons, one can generate arbitrary Fock states in the resonator. Then, the resonator can be probed and analyzed over many instantiations of the state of interest. This allows full tomography to be conducted on the state and the Wigner function to be determined. The results of their measurements are shown in figure 8.

7 Conclusion

In conclusion, numerous researchers have published papers highlighting the feasibility of quantum state tomography using the Wigner representation of the quantum state. However, all of this being said, Wigner state representations have little experimental relevance. Most quan-

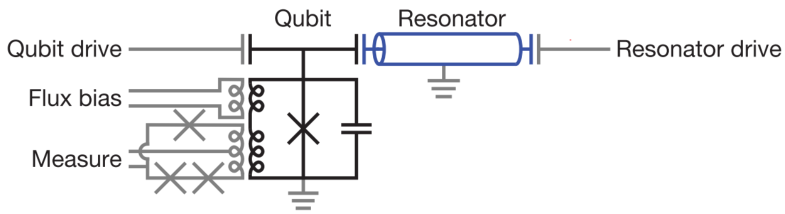

Figure 7: Circuit schematic of the superconducting circuit used in the Hofheinz et. al. paper of 2009.

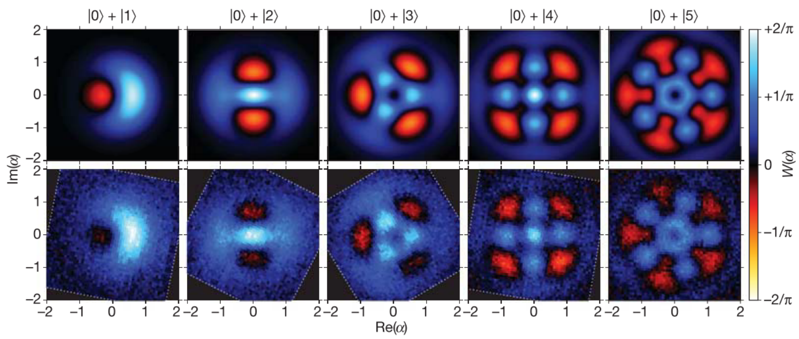

Figure 8: Results reported from the 2009 Hofheinz paper.

tum computational engineers characterize their states using the density matrix representation, which is equivalent. But, considering the case of a quantum system whose states are continuous under the parameter of interest (like coherent states under the parameter α) a density matrix representation may not be as direct as a continuous state-space representation as provided by the Wigner function.

Experimentalism aside, though, Wigner functions have been receiving a lot of attention by theoretical physicists within the last decade in that they lend themselves nicely to understanding how much "quantum resource" exists in a certain state [6]. Coherent states of light (the appropriate description of light in the classical regime) are entirely positive in phase space. Fock states, however, must, and are always, negative over portions of the phase space. This motivates correlating the amount of "quantum resource" with the amount of negativity of the state's Wigner representation in phase space. So, while the Wigner representation may not be consistently practical to obtain experimentally, it is theoretically useful and is seeing much use in this domain, today (especially in the field of quantum optics).

References

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