Physics 760

Assignment 4

Dr. Stefan Kycia

John Rinehart

February 6, 2015

Contents

Problem 1 **3**
a 3
b 4
c 5

Problem 2 **5**
a 5
b 7

Problem 3 **8**
a 8
b 8
c 8

Problem 1

a

As the wave travels through the medium with index of refraction n_2 it will acquire phase relative to the incident wave. Consider a wave that is incident on the slab of material 2 at some angle θ_1 relative to the normal to the surface. It will first travel through a distance of $l_1 = \frac{d}{\cos\theta_2}$. Then it will reflect off the surface where material 3 meets material 2. If we compare the phase of the wave at the point that it entered material 2 with the phase that it has when it reaches a point in its path that intersects the perpendicular to the initial direction of the wave when it entered material 2 we will really be finding the phase difference between adjacent plane waves that are formed by reflections off material 3.

To find this "distance to the perpendicular", consider the wave after the reflection off of material 3. The wave will travel another distance which can be shown to be $l_2 = \frac{d}{\cos\theta_2}\sin(\pi - (\pi/2 + 2\theta_2))$. Summing l_1 with l_2 and using some trigonometric identities, the total distance l traveled through material before the wave reaches the "next plane" can be shown to be $\delta_l = 2d\cos\theta_2$. The phase difference between the two fictitious plane waves (the original plane wave and a time-advanced version of that wave is given by the $\vec{k}\cdot\vec{x}$ in the exponential describing the plane wave. Thus, the phase difference is $\delta_\phi = \frac{2\pi}{\lambda_2}\delta_l = \frac{4\pi}{\lambda_2}d\cos\theta_2$. In the case of normal incidence this expression reduces to $\delta_\phi = \frac{4\pi d}{\lambda_2}$.

The above was necessary in order to perform the folowing steps. See, depending on the indices of refraction n_1, n_2, and n_3 the wave will reflect an infinite number of times between the two surfaces (where material 1 meets material 2 and, also, where material 2 meets material 3. Each time it encounters an interface some portion of the wave will get reflected and some portion will be transmitted. Because of the thickness of material 2, the wave will also acquire a complex phase which will result in one reflected and/or transmitted wave's interference with all of the other instances of reflection or transmission.

Consider a ray of light incident normally to the surface where materal 1 meets material 2. Assign this light a plane wave of the form $E_I(\vec{\ }, t) = \vec{E_0}\exp(i(\vec{k}\cdot\vec{x} - \omega t))$. The first transmitted wave will have the form $\vec{E_{t_1}} = \vec{E_0} * t_{21} * \exp(i(\vec{k_2}\cdot\vec{x} - \omega t)) * \exp(i\phi)$. ϕ in the previous expression accounts for the accumulation of the wave's phase as it travels though material 2 towards material 3. Note, also, that I have introduced a notation that I will use throughout the rest of this problem set. t_{ij} is the scaling factor for waves which are transmitted from region j into region i. Similarly, r_{ij} is the scaling factor for waves which are reflected off material j into material i.
It is sufficient to consider just the amplitude of the wave from this point on. That is, I can safely drop the time component since I am dealing with monochromatic light and the time dependence is the same between all generated rays. The sum of all of the complex amplitudes of the transmitted rays will give me the amplitude of the resultant wave.

Now, I will define a reference for the phase in this problem. The reference phase is that of the first wave immediately after it has exited the 3rd material (or, equivalently, as soon as it encounters the interface). Subsequent transmissions (due to reflections off of the interface between material 2 and material 3) will acquire a phase determined by the total distance traveled through the material. Thus, utilizing this definition and the fact that I can disregard the time component of the wave, I can express the ray that is first transmitted into material 3 as : $\vec{E}_{t_1} = \vec{E_0} * t_{21} * t_{32}$.
Now, part of the incident wave makes it in the "first pass" to material 3. Some of this wave is reflected before any of the wave is transmitted into material 2. Some of the wave is transmitted into material 2 (this is the wave we have just considered). However, after this wave encounters material 3 a portion of this wave may be reflected at this interface. Thus, a new wave will later exit material 2 into material 3 and we must consider this wave's interference with our first wave.

Figure 1: Transmission coefficient with $n_2 = 1.5$ (green), 1 (blue), 1.2 (orange)

Utilizing the prior disussion regarding the acquired phase. I may write that the complex amplitude of the wave due to the transmission into material 1, reflections off of the two interfaces, transmission into material 3 and total distance traveled through material 2 as : $\vec{E}_{t_2} = \vec{E_0} * t_{21} * r_{23} * r_{21} * t_{32} * exp(i\frac{4\pi d}{\lambda_2}) = \vec{E}_{t_1} * r_{23} * r_{21} * exp(i\phi)$. I will omit the vector arrow above E_0 for brevity. I still maintain that it is a complex vector quantity. I will also allow the phase acquired due to the thickness of the plate to be designated as ϕ.

Although we have discovered the amplitude of the second ray to penetrate material 3 we must consider that some of this ray reflected at the interface between material 2 and 3 and thus, there is a 3rd ray that will exit material 2 into material 3. Its amplitude is described by: $E_0 * t_{21} * r_{23} * r_{21} * r_{23} * r_{21} * t_{23} * \exp(2i\phi) = E_{t_2} * r_{23} * r_{21} * exp(i\phi) = E_{t_1} * (r_{23} * r_{21} * exp(i\phi))^2$.

It is clear that this trend will continue ad infinitum and that the amplitude of the nth transmitted wave can be described as $E_{t_n} = E_0 * t_{21} * t_{32} * (r_{23} * r_{21} * exp(i * \phi))^n$. Thus, the net wave will have an amplitude $E_t = \sum_{n=0}^{\infty} E_0 t_{21} t_{32} (r_{23} r_{21} \exp(i\phi))^n$. This is a simple geometric series with solution $E_t = E_0 t_{21} t_{32} \frac{1}{1 - r_{23} r_{21} \exp(i\phi)}$. To find the transmission coefficient T I will first normalize the transmitted wave amplitude by the incident wave. Then, I will multiply the wave amplitude by its complex conjugate. This is not complete, though. The transmission coeffecient is defined as the ratio of the transmited power per unit area and the incident power per unit area. The power per unit area is given by $\frac{1}{2}\sqrt{\epsilon}\mu|E_0|^2$. This can be expressed in terms of n (the index of refraction of the material) as $\frac{1}{2}\sqrt{n}c\mu|E_0^2$. Thus, the intensity of light in the third medium divided by the intensity of light in the first medium would be given by $T = \frac{n_3}{n_1}\frac{|E_t|^2}{|E_I^2|}$.
Using the previous expression and explicitly avoiding typing a lot of tedious algebra:

$$\boxed{T = \frac{n_3}{n_1}|\frac{E_t}{E_0}|^2 = \frac{n_3}{n_1}\frac{(t_{21}t_{32})^2}{1 - (r_{23}r_{21})^2 - 2r_{23}r_{21}\cos\phi}}$$

Since $\phi = \frac{4\pi}{\lambda_2}d$. ϕ in terms of the wavelength in vacuum is $\phi = \frac{4\pi n_2}{\lambda_0}d$.
I have plotted this transmission coefficient as n_2 takes on different values.n_1 and n_3 are 1 and 1.5, respectively.

b

The analysis for this part of the problem will be very similar to the previous problem. The zeroth wave that is reflected will have amplitude $E_{r_0} = E_0 * r_{12}$. The first wave will be related to the incident wave by $E_{r_1} = E_0 * t_{21} * r_{23} * t_{12} * \exp(i\phi)$. Here, ϕ has the same form it did before. The second wave will

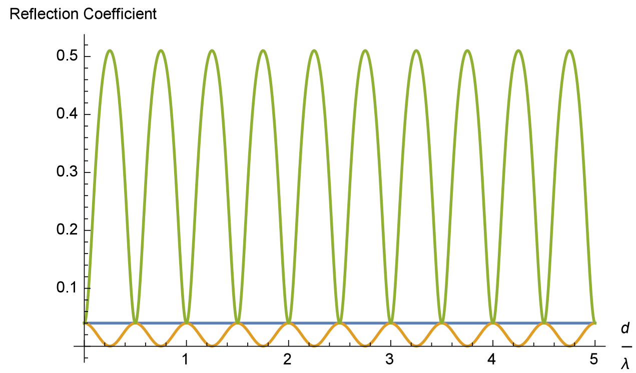

Figure 2: Transmission coefficient with $n_2 = 1.5$ (green), 1 (blue), 1.2 (orange)

have amplitude: $E_{r_2} = E_0 * t_{21} * r_{23} * r_{21} * r_{23} * t_{12} * \exp(2i\phi) = E_{r_1} * r_{21} * r_{23} * \exp(i\phi)$. The third wave will bear the same relationship with the second wave as the second wave has with the first $E_{r_3} = E_{r_2} * r_{21} * r_{23} * \exp(i\phi) = E_{r_1} * (r_{21} * r_{23})^2 * \exp(2i\phi)$. Thus, it seems that the $E_{n>0}$ wave can be expressd as $E_n = E_{r_1}(r_{21} * r_{23} * exp(i\phi))^(n-1)$.

Thus, the reflected amplitude is $E_r = E_{r_0} + \sum_{n=1}^{\infty} E_{r_1}(r_{21} * r_{23} * \exp(i\phi))^(n-1)$. Recognizing the geometric series, again, and simplifying the resulting expression yields: a. Taking the magnitude squared of this expression yields the reflection coefficient $R = |\frac{E_r}{E_0}|^2 = (\frac{\alpha^2 + 2\alpha r_{12}\cos\delta\phi - 2\alpha r_{12} r_{21} r_{23}}{1 + (r_{21} r_{23})^2 - 2r_{21} r_{23}\cos\delta\phi} + r_{12}^2)$. $\alpha = t_{21} t_{12} r_{23}$: Substituting this into my previous expression, factoring and simplifying yields:

$$\boxed{R = \frac{r_{12}^2 + r_{23}^2 + 2r_{12} r_{23}\cos\delta\phi}{1 + r_{23}^2 r_{21}^2 - 2r_{23} r_{21}\cos\delta\phi}}$$

I have plotted this in the second figure for the same $n_1, n_2 \,\text{and}\, n_3$ values as before for the transmission coefficient. Note that the sum of each of the corresponding plots yields 1 at any choice of the thickness of the plates.

c

An obvious application of this device is that of filtering a particular frequency of light. One could tune the distance of separation between material 1 and 3 to result in total interference in the transmission. Another application could be in using a variable distance (variable d) apparatus to experimentally determine the particular wavelength of light being generated by a monochromatic source.

Problem 2

a

The trick to this problem is to understand that a finite (not infinity) conductivity σ has the physical effect of introducing an imaginary component into the permittiviy $\epsilon = \epsilon_d + j\frac{\sigma}{\omega}$. All of the other equations remain the same. As ϵ is now a complex quantity let us write it as $\epsilon = A' \exp(i2\theta)$. This will have advantages over the polar form as will become apparent shortly.

Light reflecting normally off of a surface can be shown to change amplitude as : $\frac{E_r}{E_0} = \frac{n_1 - n_2}{n_2 + n_1}$ where n_2 is the index of refraction of the material upon which the light impinges. If this index of refraction is larger than n_1 then the wave will experience a sign reversal. Now, $n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$. Assuming the material has a magnetic permeability close to that of vacuum (a reasonably assumption for optical materials). $n \approx \sqrt{\epsilon/\epsilon_0}$. But, I will not make this approximation (it's not necessary). Using my expression for ϵ in conjunction with my expression for n allows

$$\begin{aligned} n^2 &= \frac{\mu A' exp(i2\theta)}{\mu_0\epsilon_0} \\ &= A^2 \exp(2i\theta) \\ A^2 &= \mu/(\mu_0\epsilon_0)\sqrt{\epsilon_d^2 + (\sigma/\omega)^2} \end{aligned}$$

Substituting this expression for n into the prior expression for the ratio of the electric field amplitudes allows me to write:

$$\begin{aligned} \frac{E_r}{E_I} &= \frac{1 - A\exp(i\theta)}{1 + A\exp(i\theta)} \\ &= \frac{1 - A^2 - 2i\sin(\theta)}{1 + A^2 + 2A\cos\theta} \end{aligned}$$

Now, I must consider expansions of $A(\sigma)$ and $\theta(\sigma)$.

Expanding $A^2(\sigma)$ about zero σ yields : $\frac{\mu\epsilon_d}{\mu_0\epsilon_0} + \frac{\mu\sigma^2}{2\mu_0\omega^2\epsilon_0\epsilon_d} + O\left(\sigma^3\right)$. Keeping only first order in σ simplifies this expression to:

$$A^2(\sigma) \approx \frac{\mu\epsilon_d}{\mu_0\epsilon_0}$$

Expanding $A(\sigma)$ about zero σ and retaining small orders yields $\sqrt{\frac{\mu\epsilon_d}{\mu_0\epsilon_0}} + \frac{\sigma^2\sqrt{\frac{\mu}{\mu_0\epsilon_0}}}{4\omega^2\epsilon_d^{3/2}} + O\left(\sigma^3\right)$: However, I'm only required to retain orders up to 1st in σ. Thus,

$$A(\sigma) \approx \sqrt{\frac{\mu\epsilon_d}{\mu_0\epsilon_0}}$$

.

Finally, expanding $\theta(\sigma)$ similarly yields:

$$\frac{\sigma}{\omega\epsilon_d} + O\left(\sigma^3\right) \approx \frac{\sigma}{\omega\epsilon_d}$$

.

You may notice that the second order term in the expansion for A^2 and the second order term in A are not related through a square root. We might expect this to be the case as, in general, for some function f, its power series expansions (denoted by $P(f)$) does exhibit the following property: $(P(f))^2 = P(f^2)$. However, this is only true for both expansions. While executing $P(f) * P(f)$ to determine $P(f^2)$ terms of different order will combine to form the terms generated natively in expanding $P(f^2)$. This is why it is important to expand A^2 and A separately. This idea can be summarized more concretely by considering the power series expansion of $\sin(x)$ whose preliminary expansion about $x = 0$ is $x - \frac{x^3}{3!}$ and $\sin^2(x)$ whose first preliminary terms are $x^2 - \frac{x^4}{3}$. But, $(\frac{x^3}{3})^2 \neq \frac{x^4}{3}$. Notice that $2 * x * \frac{x^3}{3!} = \frac{x^4}{3}$ which is exactly the term that appears in the power series expansion of $\sin^2(x)$.

I will also utilize the fact that $\sin\theta$ expands to first order in θ as θ and that $\cos\theta$ expands to first order in θ as 1. More properly, I should separately expand $\cos(\theta(\sigma))$ and $\sin(\theta(\sigma))$. However, this approximation turns out to be the same. So, ignoring this slight breach of protocol: the ratio of my field amplitudes is now (letting $\frac{\mu\epsilon_d}{\mu_0\epsilon_0} = B^2$): $\frac{1-B^2-2iB\frac{\sigma}{\omega\epsilon_d}}{1+B^2+2B\frac{\sigma}{\omega\epsilon_d}}$.

The phase of the reflected wave can be obtained by taking the arctangent of the imaginary and the real portions of the previous expression.

$$\boxed{\Phi = \arctan(2\frac{B\frac{\sigma}{\omega\epsilon_d}}{B^2-1})}$$

I could express this in terms of more fundamental variables (by substituting my expression for B), but this is not particularly illuminating. This solves the problem. Note that in the limit of no conductivity the reflected phase is zero.

Now, to determine the intensity of the reflected wave I must find the squared magnitude of $\frac{E_r}{E_I}$. Utilizing my prowess in solving for squared magnitudes: $|\frac{E_r}{E_I}|^2 = \frac{(1-A^2)^2+4\sin^2\theta}{(1+A^2+2A\cos\theta)^2}$. Substituting my approximations of θ and A I obtain:

$$\boxed{\frac{(1-B^2)^2+4(\frac{\sigma}{\omega\epsilon_d})^2}{(1+B)^2}}$$

Don't complain that my expansions of $\sin\theta$ and $\cos\theta$ were premature or ill-formed. They are satisfactory in the limit of sufficiently low σ. There may exist a simpler expression than this. I could substitute $B = \sqrt{\frac{\mu\epsilon_d}{\mu_0\epsilon_0}}$ but that would only make this solution uglier. This is a solution to the problem.

b

This problem requires me using knowledge of the skin depth of a material. The next few lines are pretty much copy-pasted from 5.18 (a) of Jackson's Electrodynamics where he discusses this exact phenomenon.

For a material with finite conductivity, imagine an H field (oriented in the $\hat{x}$ direction) is given by $H_x(z,t) = h(z)\exp(-i\omega t)$. By manipulating Maxwell's equations ($\nabla x H = J + \frac{\partial D}{\partial t}$, $\nabla x E = -\frac{\mu \partial H}{\partial t}$ and $J = \sigma E$) we can show that $\nabla^2 B = \mu\sigma\frac{\partial B}{\partial t} + \mu\epsilon\frac{\partial^2 B}{\partial t^2}$ (where I have assumed a linear medium ($\mu B = H$). I have assumed a linear medium in this case. Assuming a wave incident on the medium as described previously: Substituting this into Maxwell's equations we can find that $k^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$. The complex part of this solution accounts for an attenuation into the medium ($\exp(i(\text{real} + i * \text{imaginary})) = \exp(i * \text{real}) * \exp(-\text{imaginary})$). Thus, after being transmitted into the material, the wave encounters an exponential attenuation due to the finite conductivity. This length scale depends on $1/Im(k) = \left(2/(\mu\epsilon\omega^2)(\sqrt{1+(\sigma/(\epsilon\omega))^2} - 1)\right)^{-.5}$. This is the so-called "skin depth" of the material. By making the assumption that σ is small relative to the other quantities in the problem I obtain an approximate expression for this skin depth: In this limit, $\delta \approx 2\sqrt{\frac{\epsilon}{\mu\sigma^2}}$: the high-frequency or low conductivity limit of the skin depth of this material.

Thus, consider the amplitude of the transmitted wave. If the incident wave has amplitude E_0 the transmitted wave will have intensity $E_0 * n_1/(n_1+n_2)$, where n_1 describes the index of refraction of the "host" material. Thus, using our knowledge of the skin depth we can postulate that the amplitude of the wave a distance "d" into the surface will be of the strength $E_0\exp(-d/\delta)/(1+n_2)$, where n_2 can be obtained from part a) of this problem. Thus, the intensity of the wave is $|E_0|^2|1/(1+n_2)|^2\exp(-2d/\delta)$ (the magnitude squared of the amplitude). The expression for $|1/(1+n_2)|^2$ is not particularly interesting but can be shown to be $1/(1+A^2-2A\ cos\phi)$, where $A = \sqrt{\epsilon_d^2+(\sigma/\omega)^2}$ and $\phi = \arctan(\sigma/\omega\epsilon_d)$. Expanding this expression about small σ yields

$$1/(1+A^2-2A\cos\phi) \approx \frac{1}{\frac{\mu\epsilon_d}{\mu_0\epsilon_0}+2\sqrt{\frac{\mu\epsilon_d}{\mu_0\epsilon_0}}+1} + \frac{\mu^2\sigma^2\left(2\omega^2\epsilon_d^2-\omega\epsilon_d^2\left(\sqrt{\frac{\mu\epsilon_d}{\mu_0\epsilon_0}}+1\right)\right)}{2\omega^2\omega\epsilon_d^2\left(\frac{\mu\epsilon_d}{\mu_0\epsilon_0}\right)^{3/2}\left(\mu\epsilon_d+\mu_0\epsilon_0\left(2\sqrt{\frac{\mu\epsilon_d}{\mu_0\epsilon_0}}+1\right)\right)^2} + O\left(\sigma^3\right).$$

The first term looks just like $1/(1+n_2)^2$ for n_2 being completely real.

Admittedly, the second term in my solution is a little ugly. However, the problem explicitly asks for an expression for the transmitted intensity given the "leading order" in the approximation of σ. Thus, my final solution for problem 2 part b) is as follows:

$$\boxed{\begin{aligned} &|E_0|^2|1/(1+n_2)|^2\exp(-2d/\delta) \\ &\qquad = |E_0|^2\Big(1/(1+A^2-2A\cos\phi)\Big) \\ &\qquad \approx |E_0|^2\frac{1}{(1+\sqrt{\frac{\mu\epsilon_d}{\mu_0\epsilon_0}})^2} \\ &\qquad + \frac{\mu^2\sigma^2\left(2\omega^2\epsilon_d^2-\omega\epsilon_d^2\left(\sqrt{\frac{\mu\epsilon_d}{\mu_0\epsilon_0}}+1\right)\right)}{2\omega^2\omega\epsilon_d^2\left(\frac{\mu\epsilon_d}{\mu_0\epsilon_0}\right)^{3/2}\left(\mu\epsilon_d+\mu_0\epsilon_0\left(2\sqrt{\frac{\mu\epsilon_d}{\mu_0\epsilon_0}}+1\right)\right)^2} + O\left(\sigma^3\right)\Big)\exp\Big(-d\sqrt{\frac{\mu\sigma^2}{\epsilon}}\Big)\end{aligned}}$$

Problem 3

a

This exact problem is solved in any electrodynamics text. The approach here is borrowed heavily from John Jackson's Electrodynamics. I refer the interested reader to Section 7.3 of this text.
To solve this problem I must first establish the boundary conditions on the electric and magnetic fields of the plane wave. For an incident wave of complex magnitude E_0 it will reflect with amplitude $E_0^{''}$ and transmit with amplitude $E_0^{'}$. Maxwell's equations impose the following four boundary conditions on the E field (the boundary conditions on the magnetic field have been imposed on the E field by the relationship between E and B for a plane wave):

$$\begin{aligned} \left[\epsilon(E_0+E_0'')-\epsilon' E_0'\right]\cdot\hat{n} &= 0 \\ \left[\vec{k}\times E_0+\vec{k''}\times E_0'-\vec{k'}\times E_0''\right]\cdot\hat{n} &= 0 \\ \left(E_0+E_0''-E_0'\right)\times\hat{n} &= 0 \\ \left[\frac{1}{\mu}(\vec{k}\times E_0+\vec{k''}\times E_0'')-\frac{1}{\mu'}(\vec{k'}\times E_0')\right]\times\hat{n} &= 0 \end{aligned}$$

For an E-field with its polarization perpendicular to the plane of incidence the third and fourth relationships yield:

$$\begin{aligned} E_0+E_0''-E_0' &= 0 \\ \sqrt{\frac{\epsilon}{\mu}}(E_0-E_0'')\cos\theta_i-\sqrt{\frac{\epsilon'}{\mu'}}E_0'\cos\theta_r &= 0 \end{aligned}$$

Above are two equations with three unknowns. Identifying $\frac{E_0'}{E_0}$ as one variable and $\frac{E_0''}{E_0}$ as another variable we can see that these two equations only involve these two variables. Next, we can utilize our skills in algebraic arithmetic to yield the following expressions:

$$\boxed{\begin{aligned}\frac{E_0'}{E_0} &= \frac{2n\cos\theta_i}{n\cos\theta_i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta_i}} \\ \frac{E_0''}{E_0} &= \frac{n\cos\theta_i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin\theta_i}}{n\cos\theta_i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta_i}}\end{aligned}}$$

b

The critical angle is the angle at which $\sin(\theta_2) = \pi/2$ in Snell's law - $n_1 \sin\theta_1 = n_2 \sin\theta_2$. If n_1 is air (which its claimed to be in this problem) then the critical incidence angle is related to the incident angle θ_I through $\sin\theta_I = n_2$. Since $\theta_I \approx \pi/2$ $\sin\theta_I = \sin(\pi/2 - \phi_I) = \cos(\phi_I)$. Now, ϕ_I is pretty small according to the problem statement. Thus, $1 + \phi_I^2/2 \approx n_2 = 1 - \delta$ and, trivially,

$$\boxed{\sqrt{2\delta} \approx \phi_I}$$

c

To derive the reflectivity $R_\perp$ I need to consider the expression $\frac{n\cos\theta_i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta_i}}{n\cos\theta_i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta_i}}$ which relates the amplitude of the incident wave of that to the reflected wave. Now, consider that $\theta_i = \pi/2 - \epsilon$ where $\epsilon << 1$. So, $\cos\theta_i = \sin\phi_i \approx \phi_i$. Furthermore:

$$\begin{aligned}\sqrt{n'^2 - \sin^2\theta_i} &= \sqrt{(1-\delta)^2 - \sin^2\theta_i} \\ &= \sqrt{\cos^2\theta_i - 2\delta} \\ &\approx \sqrt{\phi_i^2 - 2\delta} \\ &\approx \sqrt{\phi_i^2 - \phi_c^2}\end{aligned}$$

The final step before expressing the final answer is to realize that for optical (and above) frequencies it is typical for most materials that $\mu = \mu'$. That is, most materials have the same permeability as vacuum. These results contract the expression for the reflected amplitude to:

$$\boxed{R_\perp = \frac{\phi_i - \sqrt{\phi_i^2 - \phi_c^2}}{\phi_i + \sqrt{\phi_i^2 - \phi_c^2}}}$$