Introduction to Noise Processes

ECE730/QIC890-T33

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Problem Set 3

Due: May 31, 2016, 8:30 am

1. **Two Important Theorems of Fourier Transform**

   All physical systems are causal, namely the system output is determined by past and present input signal not by future inputs. Mathematically, the output $y(t_0)$ is expressed by the input values $x(t)$ for $t \leq t_0$.

   Suppose $X(\omega)$ is a Fourier transform of a causal signal $x(t)$.

   (1) **Differentiation Theorem**
   Show the steps explicitly that the Fourier transform of the time derivative of $x(t)$ is written as

   $$\text{F. T.}\Big[\frac{dx(t)}{dt}\Big] = i\omega X(\omega) - x(0),$$

   where F.T means the Fourier Transform.

   (2) **Integration Theorem**
   Show the steps explicitly that the Fourier transform of the time derivative of $x(t)$ is written as

   $$\text{F. T.}\Big[\int_0^t x(t')dt'\Big] = \frac{1}{i\omega}X(\omega) + \frac{X(\omega)}{2}\delta(\omega),$$

   where $\delta(\omega)$ is the Dirac delta function.

2. **Fluctuation-Dissipation Theorem**

   Brownian motion is a classic example of a random process, where a macroscopically small but microscopically large particle is subject only to the collisional forces exerted by the molecules of a surrounding environment.

   Let us limit our discussion to study Fluctuation-Dissipation Theorem using the one-dimensional Brownian motion with a particle whose mass is m, center-of-mass position is $x(t)$, and its associated velocity is $v(t) = d(x(t))/dt$. Note that the system is at the absolute temperature Θ. The system is not completely isolated but it couples with the outside weakly. Two types of forces exist to affect the particle motion: one is the slowly varying external force $\mathcal{F}(t)$, and the other is the rapidly fluctuating function $F(t)$ arising from the interaction to other degrees of freedom in a system. Compared to macroscopic time τ, $F(t)$ varies by the rate of τ^*, which is called as correlation time,

$\tau \gg \tau^*$. For the system with an ensemble of those N particles, the mean value of $F(t)$ at time t is given by the ensemble average, namely,

$$\langle F(t) \rangle = \frac{1}{N} \sum_{k=1}^{N} F^{(k)}(t),$$

where $F^{(k)}(t)$ is the force on the k-th labeled particle.

The equation of motion of individual particle is written as

$$m\frac{dv(t)}{dt} = \mathcal{F}(t) + F(t), \qquad \text{(Eq.1)}$$

and for a system with many particles, we should consider ensembles of particles, and we can describe it statistically.

Reference: Federick Reif, "Fundamentals of Statistical and Thermal Physics", Chapter 15.

(1) Integration

For the time interval τ, which is longer than the correlation time τ^*, integrate (Eq.1) for the duration of τ. Show the detailed steps with the reasons that you get the following equation:

$$m[v(t+\tau) - v(t)] = \mathcal{F}(t)\tau + \int_t^{t+\tau} F(t')dt' \qquad \text{(Eq.2)}.$$

(2) Decomposition of $F(t)$

We can decompose $F(t)$ into two terms, $\langle F(t) \rangle$, the ensemble-averaged mean value, which is slowly varying and $f(t)$, a purely random rapidly fluctuation part, whose average value is 0:

$$F(t) = \langle F(t) \rangle + f(t)$$

Suppose the external force $\mathcal{F}(t)$ is zero. The non-zero interaction $F(t)$ would change the ensemble-averaged velocity value $\langle v(t) \rangle$ from the equilibrium value $\langle v(t) \rangle$=0. Which component of $F(t)$ will make the motion of the particle ultimately return to the equilibrium value of the velocity?

(3) Restoring force Description $F(t)$

What would be the form of such restoring force term?

(4) Ensemble-averaged force - (I)

When the force F acts on the particle, whose velocity is modified, the system energy is changed and computed by the negative of the force change.

In the equilibrium, the heat bath of the system sits at the temperature Θ, and $P_r^{(0)}(t)$ is the probability for the system to be the state r at time t in the system energy E. However, for the non-zero F at time $t+\tau'$ in a longer time internal τ' than the correlation time τ^*, the system energy changes to $E+\Delta E$. We know that the probability to find the system at $t+\tau'$, $P_r(t+\tau')$, the probability to find the system to be $E+\Delta E$ is expressed simply by the Boltzman factor ,

$$P_r(t+\tau') = P_r(t)\exp(\beta\Delta E),$$

with $\beta = 1/k_B\Theta$.
By definition, the ensemble-averaged force is written in terms of the probability function, i.e.

$$\langle F(t)\rangle = \sum_r P_r(t+\tau')F_r.$$

Show that the force can be written as the ensemble-average value with the equilibrium probability function $P_r^{(0)}(t)$,

$$\langle F(t)\rangle = \sum_r P_r(t+\tau')F_r \approx \sum_r P_r^0(t)(1+\beta\Delta E)F_r.$$

Then, since $\sum_r P_r^0(t)F_r = 0$,

$$\langle F(t)\rangle = \sum_r P_r^0(t)\beta\Delta E)F_r \equiv \langle\beta\Delta E F_r\rangle_0,$$

where the symbol of $\langle ...\rangle_0$ is the ensemble average with the probability function $P_r^{(0)}(t)$.

(5) Ensemble-averaged force - (II)
Since the energy ΔE in the time $t'-t$ is the negative of the work done by the force,

$$\Delta E = -\int_t^{t'} v(t'')F(t'')dt''.$$

If the velocity $v(t)$ does not slowly vary over time duration τ, the equation of ΔE becomes

$$\Delta E = -v(t)\int_t^{t'} F(t'')dt''.$$

Plugging the above equation of ΔE to the expression in (4) and taking the average over $v(t)$ first separately, the force term is now expressed as:

$$\langle F(t')\rangle = -\beta\langle v(t)\rangle \int_t^{t'} \langle F(t')F(t'')\rangle_0 dt''. \qquad \text{(Eq. 3)}$$

Note that $\langle v(t) \rangle$ is the mean value of $v(t)$ different from the equilibrium mean value.

Please re-write (Eq.3) using the new parameter $s = t'' - t'$.

(6) Fluctuation-Dissipation Theorem

Taking the ensemble average of (Eq. 2) together with the result in (4), we have now reached the following equation:

$$m\langle v(t+\tau) - v(t) \rangle = \mathcal{F}(t)\tau - \beta\langle v(t) \rangle \int_t^{t+\tau} dt' \int_{t-t'}^{0} ds \langle F(t')F(t'+s) \rangle_0 . \text{(Eq. 3)}$$

Please show that the double integral in (Eq.3) can be approximately,

$$\int_t^{t+\tau} dt' \int_{t-t'}^{0} ds \langle F(t')F(t'+s) \rangle_0 \approx \int_{-\infty}^{\infty} ds \langle F(t')F(t'+s) \rangle_0 .$$

The friction constant α is now introduced to be

$$\alpha \equiv -\frac{1}{2k_B\Theta} \int_{-\infty}^{\infty} ds \langle F(0)F(s) \rangle_0 .$$

This is the example of he Fluctuation-Dissipation Theorem, relating to the interaction force fluctuations with the friction force term.