Solution to Problem Set 5

P1)

Referring to section 7_7 in lecture 9, for a single-mo e cavity along the z-axis, we can write the electric field operator as:

a) $\hat{E}_x(z,t) = E_0 (\hat{a} + \hat{a}^\dagger) \sin kz$

$$\rightarrow \langle \hat{E}_x \rangle = \langle n | \hat{E}_x | n \rangle$$

$$= E_0 \sin kz \left\{ \langle n | \hat{a} | n \rangle + \langle n | \hat{a}^\dagger | n \rangle \right\}$$

$$= E_0 \sin kz \left\{ \sqrt{n} \underbrace{\langle n | n-1 \rangle}_{0} + \sqrt{n+1} \underbrace{\langle n | n+1 \rangle}_{0} \right\}$$

$$\Rightarrow \langle \hat{E}_x \rangle = 0.$$

$$\langle \hat{E}_x^2 \rangle = \langle n | \hat{E}_x^2 | n \rangle$$

$$= E_0^2 \sin^2 kz \langle n | \hat{a}^{\dagger 2} + \hat{a}^2 + \hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger | n \rangle$$

$$= E_0^2 \sin^2 kz \langle n | \hat{a}^{\dagger 2} + \hat{a}^2 + 2\hat{a}^\dagger \hat{a} + 1 | n \rangle$$

$$= E_0^2 \sin^2 kz \,(2n+1) \Rightarrow$$

$$\langle \hat{E}_x^2 \rangle = 2E_0^2 \sin^2 kz \,(n + 1/2)$$

$$\sigma_{E_x} = \sqrt{\langle \hat{E}_x^2 \rangle - \langle \hat{E}_x \rangle^2} = \sqrt{2}\, E_0 \sin kz \sqrt{(n + 1/2)}$$

$$\sigma_{E_x} = E_0 \sin kz \sqrt{2n+1}$$

b)

$$[\hat{N}, \hat{E}_x] = \hat{N}\hat{E}_x - \hat{E}_x\hat{N} = E_0 \sin kz \left[\hat{N}(\hat{a}^\dagger + \hat{a}) - (\hat{a}^\dagger + \hat{a})\hat{N}\right]$$

$$= E_0 \sin kz \left[\hat{a}^\dagger \hat{a}(\hat{a}^\dagger + \hat{a}) - (\hat{a}^\dagger + \hat{a})\hat{a}^\dagger \hat{a}\right]$$

$$= E_0 \sin kz \left[\hat{a}^\dagger \hat{a}\hat{a}^\dagger + \hat{a}^\dagger (\hat{a})^2 - (\hat{a}^\dagger)^2 \hat{a} - \hat{a}\hat{a}^\dagger \hat{a}\right]$$

$$= E_0 \sin kz \left\{\hat{a}^\dagger (\underbrace{\hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a}}_{1}) + (\underbrace{\hat{a}^\dagger \hat{a} - \hat{a}\hat{a}^\dagger}_{-1})\hat{a}\right\}$$

$$\boxed{[\hat{N}, \hat{E}_x] = E_0 \sin kz \, (\hat{a}^\dagger - \hat{a})}$$

c) Back to lecture 5 :

$$\boxed{\sigma_N \, \sigma_{E_x} \geq \frac{1}{2} E_0 \, |\sin kz| \, |\langle \hat{a}^\dagger - \hat{a} \rangle|}$$

Remember that if $[\hat{A}, \hat{B}] = \hat{C} \rightarrow \sigma_A \sigma_B \geq \frac{1}{2} |\langle \hat{C} \rangle|$

Problem 2)

a) $|\alpha(0)\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$

$$|\alpha(t)\rangle = e^{-i\frac{\hat{H}t}{\hbar}} |\alpha(0)\rangle = e^{-|\alpha|^2/2} \sum_n \frac{|\alpha|^n}{n!} e^{-i(n+\frac{1}{2})\omega t} |n\rangle$$

$$= e^{-\frac{i\omega t}{2}} e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{|\alpha|^n e^{in(\theta-\omega t)}}{\sqrt{n!}} |n\rangle$$

$$= e^{-\frac{i\omega t}{2}} |\beta(t)\rangle$$

where $|\beta(t)\rangle = |\alpha| e^{i(\theta-\omega t)}$

$|\beta(t)\rangle$ is another coherent state, so $|\alpha(t)\rangle$ is still a coherent state.

b)

if $\hat{E}_x(r,t) = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \left[\hat{a}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} - \hat{a}^{\dagger} e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \right]$

$$\langle\alpha|\hat{E}_x|\alpha\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \left[\alpha\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} - \alpha^* e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \right]$$

since $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ & $\langle\alpha|\hat{a}^{\dagger} = \alpha^*\langle\alpha|$.

Let $\alpha = |\alpha| e^{i\theta} \Longrightarrow$

$$\langle\alpha|\hat{E}_x|\alpha\rangle = 2|\alpha|\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \sin(\omega t - \vec{k}\cdot\vec{r} - \theta)$$

c)

$$\Delta E_x = \sqrt{\langle (\Delta \hat{E}_x)^2 \rangle - \langle \Delta E_x \rangle^2} = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}$$

Since $\langle \alpha | \hat{E}_x^2(r,t) | \alpha \rangle = \frac{\hbar\omega}{2\epsilon_0 V}\left(1 + 4|\alpha|^2 \sin^2(\omega t - \vec{k}\cdot\vec{r} - \theta)\right)$

$$\Delta E_x = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}$$

d)

Since $\hat{N} = \hat{a}^\dagger \hat{a} \longrightarrow \langle \hat{N} \rangle = \langle \alpha | \hat{N} | \alpha \rangle = |\alpha|^2$

$$\langle \alpha | \hat{N}^2 | \alpha \rangle = \langle \alpha | \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} | \alpha \rangle$$

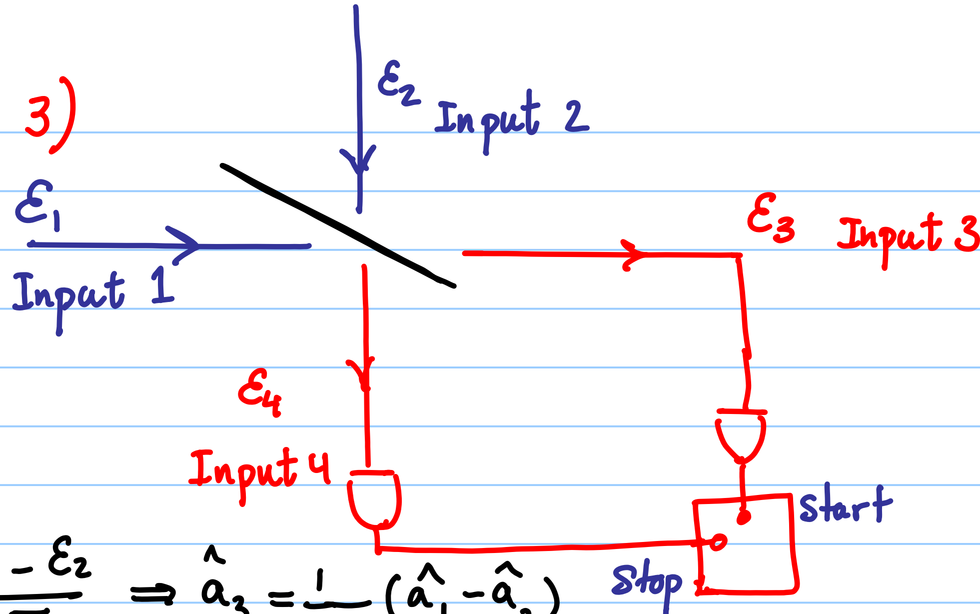
$$= \langle \alpha | \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} + \hat{a}^\dagger \hat{a} | \alpha \rangle$$

$$= |\alpha|^4 + |\alpha|^2 = \langle \hat{N} \rangle^2 + \langle \hat{N} \rangle$$

$$\Delta N = \sqrt{\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2} = \sqrt{\langle \hat{N} \rangle}$$

$$\rightarrow \frac{\Delta N}{\langle \hat{N} \rangle} = \frac{1}{\sqrt{\langle \hat{N} \rangle}} = \frac{1}{|\alpha|}$$

Problem 3) (20)



a)

$$\mathcal{E}_3 = \frac{\mathcal{E}_1 - \mathcal{E}_2}{\sqrt{2}} \Rightarrow \hat{a}_3 = \frac{1}{\sqrt{2}}(\hat{a}_1 - \hat{a}_2)$$

$$\mathcal{E}_4 = \frac{\mathcal{E}_1 + \mathcal{E}_2}{\sqrt{2}} \Rightarrow \hat{a}_4 = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2)$$

Input State to HBT is $|\Psi\rangle = |\psi_1, 0_2\rangle = |\psi_1\rangle|0\rangle_2$

where $|\psi_1\rangle$ is an arbitrary input state to port 1

and $|0\rangle_2$ is vacuum state input to port 2.

Therefore $$g^{(2)}(0) = \frac{\langle \hat{N}_3(t)\hat{N}_4(t)\rangle}{\langle \hat{N}_3(t)\rangle\langle \hat{N}_4(t)\rangle}$$

$$g^{(2)}(0) = \frac{\langle \hat{a}_3^\dagger \hat{a}_3 \hat{a}_4^\dagger \hat{a}\rangle}{\langle \hat{a}_3^\dagger \hat{a}_3\rangle\langle \hat{a}_4^\dagger \hat{a}_4\rangle}$$

We need to calculate three terms, so

$$\langle \hat{a}_3^\dagger \hat{a}_3\rangle = \langle \Psi | \hat{a}_3^\dagger \hat{a}_3 | \Psi\rangle =$$

$$= \langle \psi_1 | \langle 0_2 | \hat{a}_3^\dagger \hat{a}_3 | \psi_1\rangle |0_2\rangle$$

$$= \langle \psi_1 | \langle 0_2 | \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 | \psi_1 \rangle | 0 \rangle_2$$

$$= \langle \psi_1 | \hat{a}_1^\dagger \hat{a}_1 | \psi_1 \rangle$$

$$\boxed{\langle \hat{a}_3^\dagger \hat{a}_3 \rangle = \langle \psi_1 | \hat{N}_1 | \psi_1 \rangle}$$

Note $\hat{a}_2 | 0_2 \rangle = 0$

like wise

$$\boxed{\langle \hat{a}_4^\dagger \hat{a}_4 \rangle = \langle \psi_1 | \hat{N}_1 | \psi_1 \rangle}$$

The numerator:

$$\langle \hat{a}_3^\dagger \hat{a}_3 \hat{a}_4^\dagger \hat{a}_4 \rangle = \langle \Psi | \hat{a}_3^\dagger \hat{a}_3 \hat{a}_4^\dagger \hat{a}_4 | \Psi \rangle$$

$$= \langle \Psi | (\hat{a}_1^\dagger - \hat{a}_2^\dagger)(\hat{a}_1 - \hat{a}_2)(\hat{a}_1^\dagger + \hat{a}_2^\dagger)(\hat{a}_1 + \hat{a}_2) | \Psi \rangle$$

This has 16 terms but most of them are zero.

$$\langle \hat{a}_3^\dagger \hat{a}_3 \hat{a}_4^\dagger \hat{a}_4 \rangle = \langle \psi_1 | \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 | \psi_1 \rangle$$

$$= \frac{1}{4} \langle \psi_1 | \hat{N}_1 (\hat{N}_1 - 1) | \psi_1 \rangle$$

$$\boxed{g^{(2)}(0) = \frac{\langle \hat{N}(\hat{N}-1) \rangle}{\langle \hat{N} \rangle^2}}$$

b) if $\begin{cases} \hat{N} | N \rangle = n | N \rangle \\ | \psi_1 \rangle = | N \rangle \end{cases} \Rightarrow \boxed{g^{(2)}(0) = 1 - \frac{1}{n}}$