ECE 770-T14/ QIC 885: Quantum Electronics & Photonics

Problem Set 3, University of Waterloo, Winter 2013

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Problem 1- Example of a Simple Harmonic Oscillator

Consider a linear time-independent electrical LC circuit.

a) Write down the Hamiltonian for the system in terms of the voltage of the capacitor, $v(t)$ and current of the inductor $i(t)$.

b) Compare the Hamiltonian with the simple harmonic oscillator and construct the analogy between dynamical variables (x, p) and (v, i).

c) If v is chosen to be one of the dynamical variables of the system what is its conjugate variable? Write down this conjugate variable in terms of the current.

d) We would like to quantize the LC circuit, therefore we are treating the voltage and current as operators. Check that they satisfy the commutation relationship similar to $\hat{x}$ and $\hat{p}$ in SHO.

e) Develop creation and annihilation operators for the Hamiltonian of the LC circuit.

f) Determine the voltage and current operators in terms of the ladder operators that you have found in section e).

g) Find the quantized energy level in terms of L and C.

h) Find the uncertainty relationship between the current and voltage operators and interpret at what frequency range the quantum effects are more important.

i) Consider a pico Farad capacitance and a nano Henry inductor and find the current and voltage quantum fluctuation levels.

j) At what temperatures the quantized energy levels of the LC circuit starting from ground state can be measured?

k) Write down the Schrodinger equation for voltage wave function.

l) Write down the ground state voltage wave function.

Problem 2- Driven Simple Harmonic Oscillator

Consider a driven simple harmonic oscillator that is driven by a general time-dependent force. Example of such a system is driven LC circuit by a voltage or current source.

Consider the following Hamiltonian for a forced or driven simple harmonic oscillator as
$H = \dfrac{p^2}{2m} + \dfrac{1}{2}m\omega^2 x^2 - xF(t) - pG(t)$.

a) Find the classical equation of motion for $x(t)$.

b) Consider the quantization of a driven simple harmonic oscillator by finding the corresponding hamiltonian operator in terms of ladder operators.

c) Find the time-dependent ladder operators for this system using the Heisenberg picture.

d) Given that $F(t)$ and $G(t)$ vanish outside the interval $0 < t < T$, compute the probability $p(n)$ that the system is in the state of n quanta level for $t > T$ if it was originally in the ground state of the Hamiltonian $H(t)$ at $t < 0$.

Problem 3- Displaced Simple Harmonic Oscillator

Consider a simple harmonic oscillator that is suddenly displaced from its equilibrium point, namely $x = 0$ to $x = x_o$ e.g. much faster than the oscillation period. An example of such system can be an electron in a simple harmonic oscillator subject to a constant electric field.

a) Write down the Hamiltonian of the displaced harmonic oscillator.

b) Use the Dirac picture to find $|\Psi_D(t)\rangle$ and position operator $\hat{x}_D(t)$.

c) Use the Heisenberg picture and find the equations of motion for position and momentum operators.

Hint: You might need to use that if $[\hat{A}, \hat{B}] = \lambda \hat{A}$ then $\hat{A}e^{\hat{B}} = e^{\lambda}e^{\hat{B}}\hat{A}$.

d) Solve the equations of motion found in part c).

e) Compare and interpret your results based on Dirac and Heisenberg pictures.

Problem 4- Parametric Simple Harmonic Oscillator

A parametric simple harmonic oscillator can be modeled by a time-varying angular frequency, $\omega(t)$. Parametric oscillators exhibit various interesting applications in making amplifiers, resonators, oscillators and mixers. Mechanical example of such a system can be considered by a time-dependent spring coefficient, $k(t)$. Electrical examples include a time-dependent capacitor such as varactor diode or a time-dependent inductor such as Josephson junction in a parametric oscillator/amplifier/mixer/resonator circuits. In the context of electromagnetics, time-dependent dielectric constant (or conductivity) in a simple medium can be considered as a parametric medium for oscillation, amplification or mixing. Linear regime of optically-pumped medium, Photoconductivity in semiconductors or photoresistivity in superconductors are examples of such parametric medium.

We would like to consider the quantization of parametric simple harmonic oscillators. By defining the time-dependent ladder operators, $\hat{a}(t), \hat{a}^{\dagger}(t)$ similar to the simple harmonic oscillator

a) Find the equations of motion, namely coupled differential equations for ladder operators.

b) The operators $\hat{a}(t), \hat{a}^{\dagger}(t)$ can be related to their initial values, $\hat{a}(0), \hat{a}^{\dagger}(0)$ through **Bogoliubov transformation** with three real functions $\alpha(t), \beta(t), \gamma(t)$ as:

$$\begin{aligned}\hat{a} &= e^{-i\alpha(t)}\hat{a}(0)\cosh\beta(t) + e^{i\gamma(t)}\hat{a}^{\dagger}(0)\sinh\beta(t)\\ \hat{a}^{\dagger} &= e^{-i\gamma(t)}\hat{a}(0)\sinh\beta(t) + e^{i\alpha(t)}\hat{a}^{\dagger}(0)\cosh\beta(t)\end{aligned}$$

Now by rewriting the Hamiltonian of the system in terms of new real functions, $\alpha(t), \beta(t), \gamma(t)$, and using the fact that the Hamiltonian of the system at initial value is $\hat{H}(0)|n\rangle = (n+\frac{1}{2})\hbar\omega(0)|n\rangle$, show that

$$\langle n|\hat{H}(t)|n\rangle = (n+\frac{1}{2})\hbar\omega(t)f(t)$$

and find $f(t)$.

c) Using the Bogoliubov transformation, rewrite the coupled differential equations obtained in part a) in terms of $\alpha(t), \beta(t), \gamma(t)$. Now given the initial values of these parameters, namely $\alpha(0), \beta(0), \gamma(0)$ and the explicit time dependence of $\omega(t)$ you are able to find the time evolution of the ladder operators.

d) Find the time-dependent uncertainty relation for position and momentum, $\Delta x(t).\Delta p(t)$ in terms of the parameters of Bogoliubov transformation. Try to interpret your result.

Hint: Start solving this problem by using the Heisenberg picture.

Problem 5- Coupled Simple Harmonic Oscillators

Consider two identical simple harmonic oscillators that are coupled together. The interaction potential is given by $H_{int} = \eta x_1 x_2$ where x_1 and x_2 are the positions of each SHO.

a) What is the physics of the interaction potential?

b) Write down the Hamiltonian operator of such a coupled system.

c) Find the quantized energy levels.

d) Determine the quantized energy levels to the first order in $\frac{\eta}{k}$, where $\eta << k$, where k is the spring coefficient.

Note- You will deal with **nonlinear simple harmonic oscillator** in upcoming homeworks.

Due: Wednesday March 13, 2013. (before starting the class)