QIC 890 - Intro. to Noise Processes

Problem Set 2

Instructor: Dr. Na Young Kim

John Rinehart

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Problem 1: Wiener-Khintchine Theorem

The average power of a noisy function $x_T(t)$ is defined by

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} [x_T(t)]^2 \, dt = \lim_{T\to\infty} \frac{1}{2\pi} \int_0^{\infty} \frac{2|X_T(i\omega)|^2}{T} d\omega$$

where $x_T(t)$ a gated function is defined by

$$x_T(t) = \begin{cases} x(t), & \text{if } |t| < T/2; \\ 0 & \text{if otherwise.} \end{cases}$$

T is a measurement time interval and $X_T(i\omega)$ is the Fourier transform of $x_T(t)$.

Stationary Process

If $x_T(t)$ is a statistically stationary process, show that the average power of a noisy function is independent of T and is a constant universal quantity.

I don't have a good answer for this question. I have thought a lot about it. Ultimately, the answer to this question boils down to the following logic: If $x_T(t)$ is a stationary process, then, by definition, the process has the property that all statistical quantities related to this process are time independent. So, by virtue of being stationary, the process satisfies the requirement that the average power of a noisy function, as defined above, is independent of T and is a constant universal quantity.

Nonstationary Process

If $x_T(t)$ is as statistically stationary process, show that the average power is dependent on T.
For the statistically nonstationary process, we are not allowed to take the limit of $T \to \infty$. In this case, we introduce ensemble averaging which is taken first for many identical gated functions $x_T(t)$. Then, the order of $\lim_{T\to\infty}$ and $\int_0^\infty d\omega$ can be interchanged. Now, we can define the unilateral power spectral density $S_x(\omega)$

$$S_x(\omega) = \lim_{T\to\infty} \frac{2\langle |X_T(i\omega)|^2 \rangle}{T}$$

My answer to this question is very similar to the previous question. To be a statistically non-stationary process, it means that we can not take the limit of statistical quantities as $T \to \infty$. This procedure of taking the limit doesn't make sense for non-stationary processes. The limit doesn't exist because the statistical quantities related to the stochastic process change as a function of t. Thus, the average power is dependent on T.

Problem 1 continued on next page. . .

Autocorrelation in Terms of Spectral Density

Recall the formula in Problem Set 1, 3(2)

$$\int_{-\infty}^{\infty} x_T(t+\tau)x_T(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(i\omega)|^2 \exp(i\omega\tau)d\omega \tag{1}$$

Suppose $\tau \neq 0$. One can also divide both sides of Eq. 1 by T, take an ensemble average, and take a limit of $T \to \infty$. Show your steps to reach the following relation,

$$\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} \langle x_T(t+\tau)x_T(t)\rangle dt = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(i\omega)|^2 \cos(\omega\tau)d\omega \tag{2}$$

Let's do as the problem statement indicates and, first, divide the left hand side of Eq. 1 by T and then take the limit as $T \to \infty$. Doing so, yields

$$\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} \langle x_T(t+\tau)x_T(t)\rangle dt = \lim_{T\to\infty} \frac{1}{2\pi T}\int_{-\infty}^{\infty} |X_T(i\omega)|^2 \exp(i\omega\tau)d\omega$$

Now, this can be split into two terms.

$$= \lim_{T\to\infty} \frac{1}{2\pi T}\int_{-\infty}^{\infty} |X_T(i\omega)|^2 \cos(\omega\tau)d\omega + i \lim_{T\to\infty} \frac{1}{2\pi T}\int_{-\infty}^{\infty} |X_T(i\omega)|^2 \sin(\omega\tau)d\omega$$

Now, in general, the second integral can not be said to go to zero. However, if the process is purely real ($x_T(t) \in \mathbb{R},\ \forall t$) then the imaginary part is identically zero. Additionally, if the bounds are chosen to be symmetric about ω then, since $|X_T(i\omega)|^2$ is even about $\omega = 0$ (shown in next problem), then that integral can be made to disappear. Thus, we can say, for either or both of these two cases:

$$= \lim_{T\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|X_T(i\omega)|^2}{T} \cos(\omega\tau)d\omega$$

Note that due to the evenness of $|X_T(i\omega)|^2$ we can re-express the above integral.

$$= \lim_{T\to\infty} \frac{1}{2\pi}\int_{0}^{\infty} \frac{2|X_T(i\omega)|^2}{T} \cos(\omega\tau)d\omega$$

The last step to yield Eq. 2 would be to take an ensemble average of both sides. This would yield:

$$\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} \langle x_T(t+\tau)x_T(t)\rangle dt = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle|X_T(i\omega)|^2\rangle \cos(\omega\tau)d\omega$$

Note that no assumptions needed to be made about the stationarity/ergodicity of the process in order to take the ensemble average.

Problem 1 continued on next page. . .

Spectral Density in Terms of Autocorrelation

We know that the left-hand side of Eq. 2 is the ensemble averaged autocorrelation function $\phi_x(\tau)$. Now we obtain the relation of the ensemble averaged autocorrelation $\phi_x(\tau)$ and the unilateral power spectral density $S_x(\omega)$,

$$\phi_x(\tau) = \frac{1}{2\pi}\int_0^\infty S_x(\omega)\cos(\omega\tau)d\omega. \tag{3}$$

Show that the inverse relation of Eq. 3 is written as

$$S_x(\omega) = 4\int_0^\infty \phi_x(\tau)\cos(\omega\tau)d\tau. \tag{4}$$

Equations 3 and 4 are known as the Wiener-Khintchine theorem.

The solution to this problem is similar to that of a Fourier transform. To establish the desired equality I will integrate the autocorrelation function over all non-negative frequencies.

$$\int_0^\infty \phi_x(\tau)\cos(\omega\tau)d\tau = \frac{1}{2\pi}\int_0^\infty \left(\int_0^\infty S_x(\omega')\cos(\omega'\tau)d\omega'\right)\cos(\omega\tau)d\tau$$

Assuming $S_x(\omega')$ is mathematically well-behaved, I exchange the order of integration.

$$= \frac{1}{2\pi}\int_0^\infty S_x(\omega')d\omega'\left(\int_0^\infty \cos(\omega'\tau)\cos(\omega\tau)d\tau\right)$$

As shown in the appendix, the integral over τ results in $\frac{\pi}{2}\delta(\omega'-\omega) + \frac{\pi}{2}\delta(\omega+\omega')$.

$$= \frac{1}{4}\int_0^\infty S_x(\omega')\left(\delta(\omega'-\omega) + \delta(\omega+\omega')\right)d\omega'$$

Now, because we are only integrating over non-negative frequencies, one of the delta distributions will not be integrated over. The other one will be integrated over. Thus, only one contributes. Without loss of generality, assume that $w > 0$. Then, the answer to the integral is

$$= \frac{S_x(\omega)}{4}$$

Note, that in the case that $\omega < 0$, the answer would be

$$= \frac{S_x(-\omega)}{4}$$

But, by construction, $S_x(\omega)$ is an even function of ω. Observe the following for proof that $S_x(\omega)$ is even in ω about $\omega = 0$.

$$\begin{aligned} S_x(\omega) &= \lim_{T\to\infty}\frac{2\langle|X_T(i\omega)|^2\rangle}{T} = \lim_{T\to\infty}\frac{2\langle|X_T(i\omega)|^2\rangle}{T} \\ &= \lim_{T\to\infty}\frac{2\langle X_T(i\omega)X_T(-i\omega)\rangle}{T} \end{aligned}$$

Since this function has the property that $f(-a) = f(a)$ then it is even. So, $S_x(\omega) = S_x(-\omega)$ and the result is that

$$S_x(\omega) = 4\int_0^\infty \phi_x(\tau)\cos(\omega\tau)d\tau$$

Problem 2: Unilateral Power Spectral Density

In class, we examined one example with a noisy waveform $x(t)$, which is a wide-sense statistically stationary. The autocorrelation function $\phi_x(\tau)$ has a form of

$$\phi_x(\tau) = \phi_x(0)\exp(-\frac{|\tau|}{\tau_1})$$

where τ_1 is a relaxation time constant. Compute the unilateral power spectral density $S_x(\omega)$ using the Wiener-Khintchine theorem.

Using the Wiener-Khintchine theorem, we have:

$$\begin{aligned} S_x(\omega) &= 4\int_0^\infty \phi_x(0)\exp(-\frac{|\tau|}{\tau_1})\cos(\omega\tau)d\tau \\ &= 4\phi_x(0)\int_0^\infty \exp(-\frac{|\tau|}{\tau_1})\cos(\omega\tau)d\tau \end{aligned}$$

Now, substituting $\cos(\omega\tau) = .5\left(\exp(i\omega\tau) + \exp(-i\omega\tau)\right)$.

$$\begin{aligned} &= 2\phi_x(0)\int_0^\infty \exp(-\frac{|\tau|}{\tau_1})\left(\exp(i\omega\tau) + \exp(-i\omega\tau)\right)d\tau \\ &= 2\phi_x(0)\int_0^\infty \exp(-\frac{|\tau|}{\tau_1})\exp(i\omega\tau)d\tau + 2\phi_x(0)\int_0^\infty \exp(-\frac{|\tau|}{\tau_1})\exp(-i\omega\tau)d\tau \end{aligned}$$

Because the integral bounds are only over positive τ, the absolute values signs are redundant. They will now be dropped.

$$\begin{aligned} &= 2\phi_x(0)\int_0^\infty \exp(\frac{-\tau}{\tau_1} + i\omega\tau)d\tau + 2\phi_x(0)\int_0^\infty \exp(-\frac{\tau}{\tau_1} - i\omega\tau)d\tau \\ &= 2\phi_x(0)\frac{\exp(\frac{-\tau}{\tau_1} + i\omega\tau)}{\frac{-1}{\tau_1} + i\omega}\Bigg|_{\tau=0}^{\infty} + 2\phi_x(0)\frac{\exp(\frac{-\tau}{\tau_1} - i\omega\tau)}{\frac{-1}{\tau_1} - i\omega}\Bigg|_{\tau=0}^{\infty} \end{aligned}$$

The upper bounds for both integrals clearly yields 0. All that remains is the lower bound.

$$\begin{aligned} &= 2\phi_x(0)\frac{\tau_1}{1 - i\omega\tau_1} + 2\phi_x(0)\frac{\tau_1}{1 + i\omega\tau_1} \\ &= 2\phi_x(0)\frac{\tau_1\left(1 + i\omega\tau_1\right) + \tau_1\left(1 - i\omega\tau_1\right)}{1 + \omega^2\tau_1^2} \\ &= 4\phi_x(0)\frac{\tau_1}{1 + \omega^2\tau_1^2} \end{aligned}$$

This is a Lorentzian-type function of τ_1.

Problem 3: Mathematical Identity

Show that

$$\lim_{a\to 0}\int_0^\infty \frac{1-\cos(\omega t)}{\omega^2+a^2}d\omega = \frac{\pi}{2}t$$

Well, the way the limit has been written, there is only a problem at $\omega = 0$, for the integrand. However, the limit exists at $\omega = 0$. This can be shown by realizing that $1-\cos(\omega t) = 2\sin^2(\omega t/2)$. This allows for the left hand side of the problem to be rewritten as follows, assuming $u = \omega t/2$:

$$2\lim_{a\to 0}\int_0^\infty \frac{\sin^2(\omega t/2)}{\omega^2+a^2}d\omega = 2\lim_{a\to 0}\int_0^\infty \frac{\sin^2(u)}{\left(\frac{2u}{t}\right)^2+a^2}\left(\frac{2}{t}\right)du = t\lim_{a\to 0}\int_0^\infty \frac{\sin^2(u)}{u^2+a^2}du$$

Now, let's consider the following limit.

$$\lim_{u\to 0}\frac{f(u)}{g(u)} = \lim_{u\to 0}\frac{\sin^2(u)}{u^2} \to \frac{0}{0}$$

This is the perfect place to apply L'Hospital's rule.

$$\begin{aligned} &= \lim_{u\to 0}\frac{f'(u)}{g'(u)} = \lim_{u\to 0}\frac{2\sin(u)\cos(u)}{2u} \to \frac{0}{0} \\ &= \lim_{u\to 0}\frac{f''(u)}{g''(u)} = \lim_{u\to 0}\frac{\cos^2(u)-\sin^2(u)}{1} \to \frac{1}{1} = 1 \end{aligned}$$

So, there is no problem allowing $a = 0$ in the above integral, since this limit is well defined. Now, all that is left to solve is the above integral with $a = 0$.

$$t\lim_{a\to 0}\int_0^\infty \frac{\sin^2(u)}{u^2+a^2}du = t\int_0^\infty \frac{\sin^2(u)}{u^2}du$$

Now, this is a famous integral. This is the $\text{sinc}^2(u)$ integral. It is derived in the appendix. But, it's value is $\pi/2$.

$$= \frac{\pi}{2}t$$

Appendices

Appendix A: Integral of Cosine

Consider the following integral:

$$\int_0^\infty \cos(\omega'\tau)\cos(\omega\tau)d\tau.$$

By using the following identity and replacing $\cos(a)$ with $.5\big(\exp(ia)+\exp(-ia)\big)$ we can evaluate the integral:

$$\cos(a)\cos(b) = \frac{\cos(a+b)}{2} + \frac{\cos(a-b)}{2}$$

This allows the integral to be solved as follows.

$$\begin{aligned}
\int_0^\infty \cos(\omega'\tau)\cos(\omega\tau)d\tau &= \int_0^\infty \frac{\cos((\omega'-\omega)\,\tau)}{2}d\tau + \int_0^\infty \frac{\cos((\omega'+\omega)\,\tau)}{2}d\tau \\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{\cos((\omega'-\omega)\,\tau)}{2}d\tau + \frac{1}{2}\int_{-\infty}^\infty \frac{\cos((\omega'+\omega)\,\tau)}{2}d\tau \\
&= \frac{1}{8}\Big(\int_{-\infty}^\infty \exp(i\,(\omega'-\omega)\,\tau)d\tau + \int_{-\infty}^\infty \exp(-i\,(\omega'-\omega)\,\tau)d\tau + \\
&\quad \int_{-\infty}^\infty \exp(i\,(\omega'+\omega)\,\tau)d\tau + \int_{-\infty}^\infty \exp(-i\,(\omega'+\omega)\,\tau)d\tau\Big) \\
&= \frac{2\pi}{8}\Big(\delta(\omega'-\omega)+\delta(\omega-\omega')+\delta(\omega'+\omega)+\delta(-\omega'-\omega)\Big)
\end{aligned}$$

But, $\delta(x-a)=\delta(a-x)$.

$$= \frac{\pi}{4}\big(2\delta(\omega'-\omega)+2\delta(\omega'+\omega)\big) = \frac{\pi}{2}\left(\delta(\omega'-\omega)+\delta(\omega'+\omega)\right)$$

Appendix B: Integral of Sinc Squared

Consider the following integral:

$$\int_0^\infty \frac{\sin^2(x)}{x^2}dx$$

This function can be integrated by parts as follows:

$$\begin{aligned}
\int_0^\infty \frac{\sin^2(x)}{x^2}dx &= -\frac{1}{x}\sin^2(x) - \int_0^\infty \frac{2\sin(x)\cos(x)}{-x}dx \\
&= -\frac{1}{x}\sin^2(x)|_{x=0}^\infty - \int_0^\infty \frac{\sin(2x)}{-x}dx \\
&= -0+0+\int_0^\infty \frac{\sin(u)}{u}du
\end{aligned}$$

The first integral is zero since the limit as $x\to\infty$ of that expression is zero and the value of that expression at $x=0$ is zero. The second integral is an infamous integral called the sine integral. It won't be computer here, but the value of that integral is $\pi/2$.

$$= \frac{\pi}{2}$$