Lecture 6: The Josephson effect

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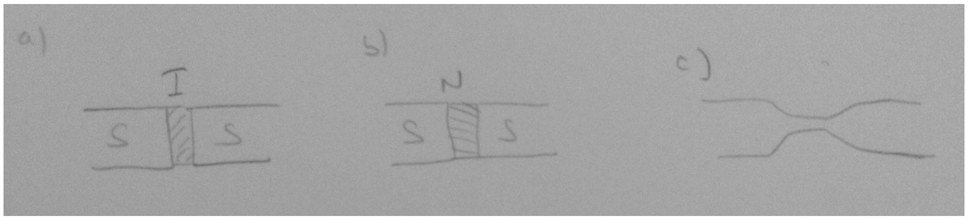

FIG. 1. Three different types of superconducting weak links: a) superconducting-insulator-superconducting , b)superconducting-normal-superconducting, and c)microbridges.

I. INTRODUCTION

In 1962 [1] Josephson made a set of predictions for the properties of a structure of the type shown in Fig. 1a, which is a set of two superconductors separated by a thin tunnel barrier (which is typically an oxide layer):

- a dissipationless current I_s can flow through the junction, which is a function of the phase difference $\Delta\phi$ between the two superconductors:

$$I_s = I_c \sin \Delta\phi, \tag{1}$$

  with I_c a parameter characteristic to the junction, called the *critical current.*

- the voltage V across the junction is related to the time variation of the phase difference through

$$V = \frac{h}{2|e|} \frac{d\Delta\phi}{dt} \tag{2}$$

His discovery had very far reaching consequences. It shed new light on fundamental concepts in superconductivity. In addition, most applications of superconductors rely on the use of the Josephson effect.

After the discovery of the effects in a SIS (superconductor-insulator-superconductor) type junction, it became clear that similar phenomena occur for other types of *weak links* between superconductors. Examples are superconducting-normal-superconducting (SNS) junctions (Fig. 1b) and microbridges (Fig. 1c).

II. DERIVATION OF THE HAMILTONIAN OF A SIS JUNCTION

A. Quasi-degenerate state 2nd order perturbation theory (follows Gottfried & Yan)

We will need to derive the effective Hamiltonian for a set of states coupled through other levels, well separated in energy. The treatment follows closely the one in the book [2]..

We consider a system with the following Hamiltonian:

$$H = H_0 + \lambda H_1, \tag{3}$$

with H_0 a easy Hamiltonian (known spectrum) and λH_1 a small perturbation.

We focus on a manifold $\mathfrak{D} = \{|\alpha\rangle\}$ of eigenstates of H_0:

FIG. 2.

We have

$$H_0|\alpha\rangle = E_\alpha|\alpha\rangle \tag{4}$$

The energies E_α are well separated from the other (E_μ) energies and the perturbation λH_1 is small:

$$|\langle\alpha|\lambda H_1|\mu\rangle| \ll |E_\alpha - E_\mu|. \tag{5}$$

With perturbation λH_1 taken into account, consider a state $|a\rangle$ with energy E_a in the range of the E_α values in the $\mathfrak{D}$ manifold. We have

$$|a\rangle = \sum_{\alpha\in\mathfrak{D}} c_\alpha|\alpha\rangle + \sum_{\mu\notin\mathfrak{D}} d_\mu|\mu\rangle \tag{6}$$

with $c_\alpha \sim O(1)$ and $d_\mu \sim O(\lambda)$. Using

$$(H - E_a)|a\rangle = 0 \tag{7}$$

we obtain

$$\sum_{\alpha\in\mathfrak{D}}(E_\alpha - E_a + \lambda H_1)c_\alpha|\alpha\rangle + \sum_{\mu\notin\mathfrak{D}}(E_\mu - E_a + \lambda H_1)d_\mu|\mu\rangle = 0 \tag{8}$$

Taking scalar product of Eq. (8) with $|\beta\rangle \in \mathfrak{D}$:

$$c_\beta(E_\beta - E_a) + \lambda\sum_{\alpha\in\mathfrak{D}} c_\alpha\langle\beta|H_1|\alpha\rangle + \lambda\sum_{\mu\notin\mathfrak{D}} d_\mu\langle\beta|H_1|\mu\rangle = 0 \tag{9}$$

Taking scalar product of Eq. (8) with $|\nu\rangle \notin \mathfrak{D}$

$$\lambda\sum_{\alpha\in\mathfrak{D}}\langle\nu|H_1|\alpha\rangle c_\alpha + d_\nu(E_\nu - E_a) + \sum_{\mu\notin\mathfrak{D}}\lambda\langle\nu|H_1|\mu\rangle d_\mu = 0 \tag{10}$$

In the last equation, the last sum is $O(\lambda^2)$. We have:

$$d_\nu \cong \lambda\frac{\sum_{\alpha\in\mathfrak{D}}\langle\nu|H_1|\alpha\rangle c_\alpha}{E_a - E_\nu}. \tag{11}$$

Inserting this in Eq. (8):

$$(E_\beta - E_a)c_\beta + \lambda\sum_{\alpha\in\mathfrak{D}}\langle\beta|H_1|\alpha\rangle c_\alpha + \lambda^2\sum_{\alpha\in\mathfrak{D}}\sum_{\mu\notin\mathfrak{D}}\frac{\langle\beta|H_1|\mu\rangle\langle\mu|H_1|\alpha\rangle}{E_a - E_\mu}c_\alpha = 0. \tag{12}$$

This equation looks like an eigenvalue equation (eigenvalue is E_a) except for the fact that the matrix depends on E_a, which is to be determined. But E_a is $\sim E_\alpha$, very different from E_μ. We can finally write, if we replace all E_a by some average energy $\bar{E}_a$, that the effective Hamiltonian H_{eff} is given by

$$\langle\beta|H_{\text{eff}}|\alpha\rangle = \langle\beta|H_0|\alpha\rangle + \lambda\langle\beta|H_1|\alpha\rangle + \lambda^2\sum_{\mu\notin\mathfrak{D}}\frac{\langle\beta|H_1|\mu\rangle\langle\mu|H_1|\alpha\rangle}{\bar{E}_a - E_\mu}. \tag{13}$$

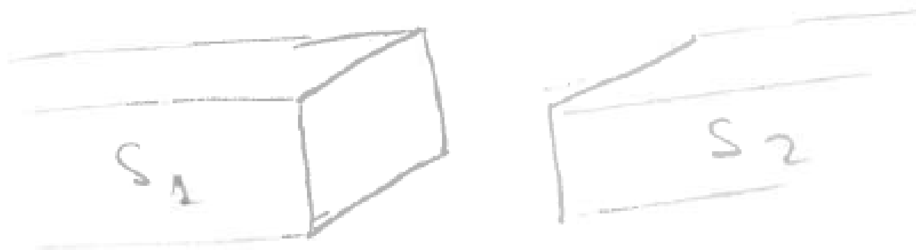

FIG. 3.

B. Effective Hamiltonian for a Josephson junction

To derive the Hamiltonian of a Josephson junction, we start by considering two isolated superconductors. The low energy states are:

$$\text{State:} \quad |\psi_{GS}^{(1)}(N_1)\rangle \otimes |\psi_{GS}^{(2)}(N_2)\rangle \tag{14}$$

$$\text{Energy:} \quad E_{GS}^{(1)}(N_1) + E_{GS}^{(2)}(N_2) \tag{15}$$

The GS index stands for the BCS superconducting ground state wave function.

Other states can be built by transferring n Cooper pairs, or equivalently $2n$ electrons, from superconductor 1 to superconductor 2:

$$\text{“2n” state:} \quad |\psi^{(0)}(n)\rangle \equiv |\psi_{GS}^{(1)}(N_1 - 2n)\rangle \otimes |\psi_{GS}^{(2)}(N_2 + 2n)\rangle \tag{16}$$

$$\text{Energy:} \quad E_{GS}(n) = E_{GS}^{(1)}(N_1 - 2n) + E_{GS}^{(2)}(N_2 + 2n) \tag{17}$$

The energy $E_{GS}(n)$ can be calculated as follows:

- for small n, and when interaction between S_1 and S_2 is neglected:

$$E_{GS}(n) \cong E_{GS}(0) - 2n\frac{\partial E_{GS}^{(1)}}{\partial N}(N_1) + 2n\frac{\partial E_{GS}^{(2)}}{\partial N}(N_2) = E_{GS}(0) + 2n(\mu_2 - \mu_1) \tag{18}$$

  μ: chemical potential.

- for large n, an interaction energy appears. It is expressed as

$$E^{\text{int}}(n) = \frac{(2ne)^2}{2C}, \tag{19}$$

  with C the capacitance of the junction.

We neglect the charging energy to start with. When S_1 and S_2 are brought together, tunneling takes place. The complete Hamiltonian is given by

$$H = H^{(1)} + H^{(2)} + H_T = H_0 + H_T \tag{20}$$

We describe tunneling using the *transfer Hamiltonian* formalism:

$$H_T = \sum_{k,l;\sigma} T_{k,l} c^{(1)\dagger}_{k;\sigma} c^{(2)}_{l;\sigma} + h.c., \tag{21}$$

where $c^{(1)}$ and $c^{(2)}$ are annihilation operators corresponding to superconductors 1 and 2 respectively. The indices k and l correspond to momentum and σ is the spin index. Tunneling is spin-conserving. We also assume time-reversal symmetry:

$$T_{-k,-l} = T^*_{k,l}. \tag{22}$$

The key result we will obtain is the following: in second order perturbation theory, terms arise which couple states with different numbers of Cooper pairs.

$$\langle \psi^{(0)}(n_1)|H_c|\psi^{(0)}(n_2)\rangle = -K_0\delta_{n_1,n_2} - K_1(\delta_{n_1,n_2-1} + \delta_{n_1,n_2+1}) \tag{23}$$

We only derive below the K_1 term (the derivation for K_0 is similar, and it does not have a consequence on the structure of states).

We consider a process of type:

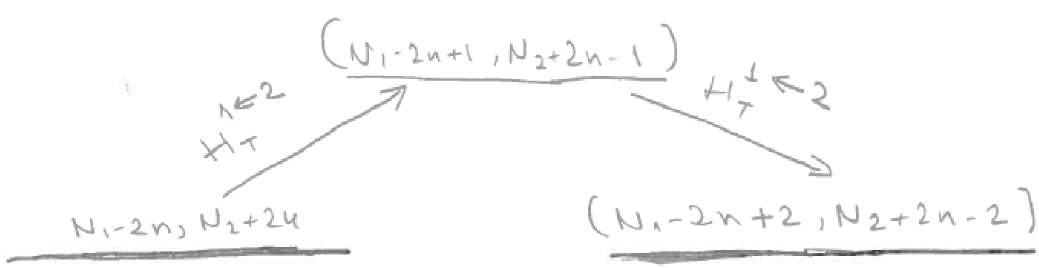

FIG. 4.

We have

$$H_T = H_T^{1\leftarrow 2} + H_T^{1\rightarrow 2} \tag{24}$$

with $H_T^{1\leftarrow 2}(H_T^{1\rightarrow 2})$ the coupling term moving single electrons from S_2 to S_1 (S_1 to S_2). We have

$$H_T^{1\leftarrow 2} = \sum_{k,l;\sigma} T_{k,l} c^{(1)\dagger}_{k;\sigma} c^{(2)}_{l;\sigma} \tag{25}$$

and

$$H_T^{1\to 2} = (H_T^{1\leftarrow 2})^\dagger \tag{26}$$

We assume S_1 and S_2 are identical (equal gap, $\Delta_1 = \Delta_2 = \Delta$) which simplifies some of the calculations. We write down again $H_T^{1\leftarrow 2}$, as a function of the Bogoliubov operators γ :

$$\begin{aligned} H_T^{1\leftarrow 2} = \sum_{k,l} T_{k,l} \Big\{ &(u_k^1 \gamma_{k0}^{1\dagger} + v_k^{1*}\gamma_{k1}^1)(u_l^{2*}\gamma_{l0}^2 + v_l^2 \gamma_{l1}^{2\dagger}) \\ &+(-v_{-k}^{1*}\gamma_{-k0}^1 + u_{-k}^1 \gamma_{-k1}^{1\dagger})(-v_{-l}^2 \gamma_{-l0}^{2\dagger} + u_{-l}^{2*}\gamma_{-l1}^2)\Big\} . \end{aligned} \tag{27}$$

We have:

1. Intermediate state $k0, l1$, energy $\epsilon_k + \epsilon_l$

$$\frac{T_{kl} u_k^1 v_l^{2*} \times T_{-k-l}(-v_k^{1*} u_l^{2*})(-1)}{0 - (\epsilon_k + \epsilon_l)} \tag{28}$$

   (-1) factor: order of fermionic operators, that is $\gamma_{k0}^1 \gamma_{l1}^2 \gamma_{k0}^{1\dagger}\gamma_{l1}^{2\dagger} = -\gamma_{l1}^2 \gamma_{k0}^1 \gamma_{k0}^{1\dagger}\gamma_{l1}^{2\dagger}$

   Final result:

$$-\sum_{kl} |T_{kl}|^2 u_k^1 v_k^{1*} u_l^{2*} v_l^{2*}/(\epsilon_k + \epsilon_l) \tag{29}$$

2. Intermediate state: $-k1, -l0$, energy $\epsilon_k + \epsilon_l$

$$\frac{T_{kl} u_{-k}^1 (-v_{-l}^2) \times T_{-k,-l} v_{-k}^{1*} u_{-l}^{2*})(-1)}{-(\epsilon_k + \epsilon_l)} \tag{30}$$

   (-1) factor: order of fermionic operators

   Final result:

$$-\sum_{kl} |T_{kl}|^2 u_k^1 v_k^{1*} u_l^{2*} v_l^{2}/(\epsilon_k + \epsilon_l) \tag{31}$$

Combining 1 & 2:

$$K_1 = -2\sum_{kl} |T_{kl}|^2 u_k^1 v_k^{1*} u_l^{2*} v_l^{2}/(\epsilon_k + \epsilon_l) \tag{32}$$

We next calculate K_1 explicitly. We assume u_k/v_k ratios are real and use:

$$u_k v_k = \frac{\Delta_k}{2\epsilon_k} \tag{33}$$

$$u_l v_l = \frac{\Delta_l}{2\epsilon_l} \tag{34}$$

Also, $|T_{kl}|$ is taken outside the integral and assigned an average value $\overline{|T|^2}$. Finally, we calculate the integral over energy:

$$-K_1 = -2\overline{|T|^2} N_1(0) N_2(0) \frac{1}{4} \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \frac{\Delta}{\epsilon_1}\frac{\Delta}{\epsilon_2}\frac{1}{\epsilon_1 + \epsilon_2} \tag{35}$$

with $N_{1,2}(\xi)$ the densities of states expressed as $\xi = E - \mu$ (as in lecture on BCS).

III. DIAGONALIZATION OF EFFECTIVE HAMILTONIAN

JOSEPHSON RELATIONS

$$\langle\psi^{(0)}(n_1)|H_{\text{eff}}|\psi^{(0)}(n_2)\rangle = -K_0\delta_{n_1,n_2} - K_1(\delta_{n_1,n_2-1} + \delta_{n_1-1,n_2}) \tag{36}$$

We neglect K_0 (energy shift).

We show that

$$|\tilde{\psi}(\varphi)\rangle = \sum_n e^{i\varphi n}|\psi^{(0)}(n)\rangle \tag{37}$$

are eigenstates. Indeed

$$H_{\text{eff}}|\tilde{\psi}(\varphi)\rangle = -K_1\sum_n e^{i\varphi n}(|\psi^{(0)}(n-1)\rangle + |\psi^{(0)}(n+1)\rangle) = -K_1(e^{i\varphi} + e^{-i\varphi})|\tilde{\psi}(\varphi)\rangle \tag{38}$$

The energy is

$$E(\varphi) = -2K_1\cos\varphi \tag{39}$$

We denote

$$2K_1 \equiv E_J \quad \text{— Josephson energy} \tag{40}$$

We also introduce the charging energy

$$E_C = \frac{(2e)^2}{C} \tag{41}$$

The final expression for the junction Hamiltonian is:

$$H_J = -E_J\cos\hat{\varphi} + \frac{E_C}{2}\hat{n}^2 \tag{42}$$

The Cooper pair number operator can be expressed using the phase representation

$$\hat{n}|\tilde{\psi}(\varphi)\rangle = \sum_n e^{i\varphi n} n|\psi^{(0)}(n)\rangle \tag{43}$$

$$-i\frac{\partial}{\partial\varphi}|\tilde{\psi}(\varphi)\rangle = \sum_n -i\frac{\partial}{\partial\varphi}(e^{i\varphi n})|\psi^{(0)}(n)\rangle = \sum_n e^{i\varphi n} n|\psi^{(0)}(n)\rangle \tag{44}$$

We can write

$$H_J = \frac{E_C}{2}(-i\frac{\partial}{\partial\varphi})^2 - E_J\cos\varphi. \tag{45}$$

For an operator $\hat{A}$, the time dependence in the Heisenberg representation is given by

$$\frac{dA}{dt} = \frac{i}{\hbar}[H, A]. \tag{46}$$

First Josephson relation

$$I = 2e\frac{dn}{dt} \tag{47}$$

$$\begin{aligned}\frac{dn}{dt} &= \frac{i}{\hbar}[H, n] &(48)\\ &= \frac{i}{\hbar}[-E_J\cos\varphi + \frac{E_C}{2}n^2, n] &(49)\\ &= -\frac{iE_J}{\hbar}[\cos\varphi, -i\frac{\partial}{\partial\varphi}] &(50)\\ &= -\frac{E_J}{\hbar}\sin\varphi &(51)\end{aligned}$$

$$I_c \equiv -\frac{2eE_J}{\hbar} \quad \text{(positive): critical current} \tag{52}$$

$$\boxed{I = I_c \sin\varphi} \tag{53}$$

Second Josephson relation

$$\begin{aligned}\frac{d\varphi}{dt} &= \frac{i}{\hbar}[H, \varphi] = \frac{i}{\hbar}(\frac{E_C}{2})[-\frac{\partial^2}{\partial\varphi^2}, \varphi] = \frac{-i}{\hbar}\frac{E_C}{2}2\frac{\partial}{\partial\varphi}\\ &= \frac{1}{\hbar}E_C\hat{n} = \frac{1}{\hbar}\frac{-2e}{C}(-2en) = \frac{2|e|}{\hbar}V &(54)\end{aligned}$$

$$\boxed{V = \frac{\hbar}{2|e|}\frac{d\varphi}{dt}} \tag{55}$$

IV. THE AMBEGAOKAR-BARATOFF RELATION

From Eq. (35) and (40)

$$\frac{E_J}{2} = 2\overline{|T|^2}N_1(0)N_2(0)\frac{1}{4}\int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \frac{\Delta}{\epsilon_1}\frac{\Delta}{\epsilon_2}\frac{1}{\epsilon_1+\epsilon_2} \tag{56}$$

Take $x_1 = \frac{\xi_1}{\Delta}$, $x_2 = \frac{\xi_2}{\Delta}$.

$$\frac{E_J}{2} = \frac{1}{2}\overline{|T|^2}N_1(0)N_2(0)\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \frac{1}{\sqrt{1+x_1^2}}\frac{1}{\sqrt{1+x_2^2}}\frac{1}{\sqrt{1+x_1^2}+\sqrt{1+x_2^2}} \tag{57}$$

$$I_c = \frac{1}{\varphi_0}\overline{|T|^2}N_1(0)N_2(0)\pi^2\Delta \tag{58}$$

We can relate the relation above to the normal state conduction of the junction.

FIG. 5.

$$I = e\frac{2\pi}{\hbar}\sum_{k,q;\xi_k>0;\xi_q<eV}|T_{kq}^2|\delta(\xi_k-\xi_q) \quad \text{Fermi golden rule} \tag{59}$$

$$I = 2e\frac{2\pi}{\hbar}\overline{|T|^2}eVN_1(0)N_2(0) \tag{60}$$

$$\frac{1}{R_n} = \frac{I}{V} = \frac{4\pi e^2}{\hbar}\overline{|T|^2}N_1(0)N_2(0) \tag{61}$$

$$\boxed{I_cR_n = \frac{\pi}{2}\frac{\Delta}{e}} \tag{62}$$

This is an elementary derivation of the Ambegaokar-Baratoff relation [4].

AB considered the general case in which Δ_1 is different of Δ_2, and also at finite temperature.

AB is very useful: with Δ known, E_C/E_J can be predicted from a measurement of R_n at room temperature.

[1] B. Josephson, Phys. Lett. **1**, 251 (1962).

[2] K. Gottfried and T. Yan, *Quantum mechanics: fundamentals* (Springer Verlag, 2003).

[3] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions: Basic Processes and Applications* (Wiley, 2004).

[4] V. Ambegaokar and A. Baratoff, Phys. Rev. Lett. **10**, 486 (1963).