

Tue

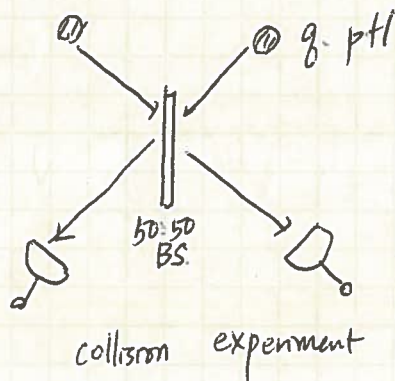
2 most important assumptions of the Q. Thy

- ① Symmetrization postulate \Rightarrow "statistical properties of an pti ensemble @ thermal eq"
- ② non-commutability of conjugate observables \Rightarrow "uncertainty product"

I. Symmetrization Postulate

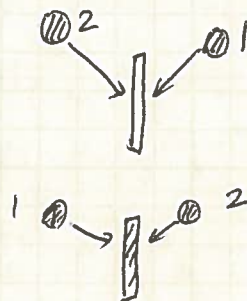
Quantum Indistinguishability

Suppose an ideal 50%-50% pti beam splitter. There are two identical q. ptls incident upon the BS, and we measure the output.



Input states $|1, R; 2, L\rangle$ pti 1 at the right input
pti 2 at the left input

or

 $|1, L; 2, R\rangle$ 

Most general input state

$C_0 |1, R; 2, L\rangle + C_1 |1, L; 2, R\rangle$ where $|C_0|^2 + |C_1|^2 = 1$ if two states are orthogonal

\therefore experimental results dep. on C_0 & C_1 , $C_0, C_1 \in \mathbb{C}$

inherent uncertainty/ambiguity to predict the exp. result

\therefore Need a fundamental assumption, symmetrization postulate

Statement of the postulate

a physical state of a real system consisting of identical q. ptls

is either symmetric
or anti-symmetric

w.r.t. the permutation of any 2 ptls

symmetric: Boson

anti-sys.: fermion

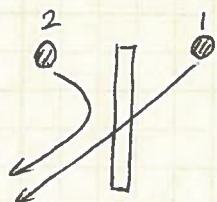
e.g.
$$\left. \begin{aligned} &\frac{1}{\sqrt{2}} [|1, R; 2, L\rangle + |1, L; 2, R\rangle] \\ &\frac{1}{\sqrt{2}} [|1, R; 2, L\rangle - |1, L; 2, R\rangle] \end{aligned} \right\}$$

if two states are orthogonal

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Let's calculate the probability of finding the 2 pths in the same output port (L)

2 possibilities



direct term

or



exchange term

BS scattering matrix $\hat{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ in the basis of 2 transmission modes $\begin{bmatrix} |k_R \rightarrow L\rangle \\ |k_L \rightarrow R\rangle \end{bmatrix}$

$$\therefore |\langle 1, L; 2, L | \hat{U} \frac{1}{\sqrt{2}} [|1, R; 2, L\rangle \pm |1, L; 2, R\rangle]|^2$$

$$= \left| \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \right) \pm \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \right) \right|^2 = \begin{cases} \frac{1}{2} & \text{boson : constructive interfere} \\ 0 & \text{fermion : destructive interfere} \end{cases}$$

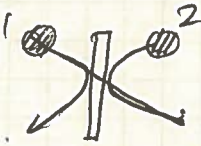
$$|\langle 1, R; 2, R | \hat{U} \frac{1}{\sqrt{2}} [|1, R; 2, L\rangle \pm |1, L; 2, R\rangle]|^2 = \begin{cases} \frac{1}{2} & \text{boson bunching} \\ 0 & \text{fermion} \end{cases}$$

How about the probability of finding $|1, L; 2, R\rangle$?



direct term

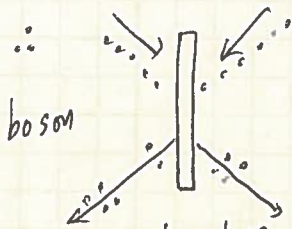
or



exchange term

$$|\langle 1, L; 2, R | \hat{U} \frac{1}{\sqrt{2}} [|1, R; 2, L\rangle \pm |1, L; 2, R\rangle]|^2 = \begin{cases} 0 & \text{boson : destructive} \\ \frac{1}{2} & \text{fermion : constructive anti-bunching} \end{cases}$$

$$|\langle 1, R; 2, L | \hat{U} \frac{1}{\sqrt{2}} [|1, R; 2, L\rangle \pm |1, L; 2, R\rangle]|^2 = \begin{cases} 0 & \text{boson} \\ \frac{1}{2} & \text{fermion} \end{cases}$$



boson

bunching

"Bose-Einstein condensation"

fermion



anti-bunching, "Pauli exclusion principle"

$$|\psi\rangle_{\text{out}} = \frac{1}{\sqrt{2}} [|1, R; 2, R\rangle - |1, L; 2, L\rangle]$$

$$|\psi\rangle_{\text{out}} = \frac{1}{\sqrt{2}} [|1, R; 2, L\rangle - |1, L; 2, R\rangle]$$

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Symmetrization postulate for spin $\frac{1}{2}$ particles

Suppose 2 spin- $\frac{1}{2}$ p'tls are incident upon the 50-50 BS.

Assume that the scattering matrix of the BS is assumed to be spin-independent.

If the 2-p'tls are in spin triplet (symmetric spin states),
the orbital states are

$\left\{ \begin{array}{l} \text{symmetric for bosons} \\ \text{antisym. for fermions} \end{array} \right.$ for the overall-state symmetrization

$$\text{Boson: } \frac{1}{\sqrt{2}} [|R\rangle_1 |L\rangle_2 + |L\rangle_1 |R\rangle_2] \otimes \left\{ \begin{array}{l} |\uparrow\rangle_1 |\uparrow\rangle_2 \\ |\downarrow\rangle_1 |\downarrow\rangle_2 \\ \frac{1}{\sqrt{2}} [|\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2] \end{array} \right.$$

\Downarrow
 short notation
 for $|1, R; 2, L\rangle$

$$\text{fermions: } \frac{1}{\sqrt{2}} [|R\rangle_1 |L\rangle_2 - |L\rangle_1 |R\rangle_2] \otimes \left\{ \begin{array}{l} |\uparrow\rangle_1 |\uparrow\rangle_2 \\ |\downarrow\rangle_1 |\downarrow\rangle_2 \\ \frac{1}{\sqrt{2}} [|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2] \end{array} \right.$$

The outputs are same as the spinless cases

If 2 p'tl are in spin-singlet (anti-symmetric spin. state)

$$\text{Boson: } \frac{1}{\sqrt{2}} [|R\rangle_1 |L\rangle_2 - |L\rangle_1 |R\rangle_2] \otimes \frac{1}{\sqrt{2}} [|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2]$$

$$\text{fermion: } \frac{1}{\sqrt{2}} [|R\rangle_1 |L\rangle_2 + |L\rangle_1 |R\rangle_2] \otimes \frac{1}{\sqrt{2}} [|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2]$$

Now the outputs are.

bosons feature (1, 1) output

fermions (0, 2) or (2, 0) output.

This means that 2 spin- $\frac{1}{2}$ fermions can occupy the same state.

Conversely, if 2-identical fermions occupy the same orbital state,
the spin state has to be always a spin-singlet!

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II. Non-Commutability Postulate of Quantum Mechanics

fundamental postulate
 • Heisenberg uncertainty principle.

in QM, a pair of conjugate observables must satisfy the following commutation relation

$$[\hat{q}, \hat{p}] = \hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar \quad \leftarrow p. \& q \text{ do not commute.}$$

$\uparrow \quad \uparrow$
 position momentum

cf) in classical mechanics $[q, p] = 0$.

Introducing the fluctuation operators

$$\Delta q = \hat{q} - \langle q \rangle, \quad \Delta p = \hat{p} - \langle p \rangle$$

\nwarrow ensemble-averaged values
 $\langle q \rangle, \langle p \rangle \in \mathbb{R}$.
 $\therefore q \& p$: observables

$$[\Delta q, \Delta p] = [\hat{q} - \langle q \rangle, \hat{p} - \langle p \rangle] = [\hat{q}, \hat{p}] = i\hbar$$

Let's calculate the uncertainty product for Δq and Δp .

Suppose $|\varphi\rangle = \Delta q |\psi\rangle, |\chi\rangle = \Delta p |\psi\rangle$.

Using the Schwartz inequality,

$$\langle \varphi | \varphi \rangle \langle \chi | \chi \rangle \geq |\langle \varphi | \chi \rangle|^2 \quad \text{where } |\psi\rangle \text{ is a ket vector representing a quantum state of a given pt. sy.}$$

\nwarrow
 the equality holds if and only if $|\varphi\rangle = c|\chi\rangle$ where $c \in \mathbb{C}$

$$\langle \Delta q^2 \rangle \langle \Delta p^2 \rangle \geq |\langle \Delta q \Delta p \rangle|^2 \quad \text{where,}$$

$$\begin{aligned} \Delta q \Delta p &= \frac{1}{2} (\Delta q \Delta p + \Delta p \Delta q) + \frac{1}{2} (\Delta q \Delta p - \Delta p \Delta q) \\ &= \frac{1}{2} (\Delta q \Delta p + \Delta p \Delta q) + \frac{i}{2} \hbar \end{aligned}$$

$$\langle \Delta q^2 \rangle \langle \Delta p^2 \rangle \geq \frac{1}{4} |\langle \Delta q \Delta p + \Delta p \Delta q \rangle + i\hbar|^2$$

\Downarrow
 real number \therefore it's an ensemble-averaged value

$$\therefore \langle \Delta q^2 \rangle \langle \Delta p^2 \rangle \geq \frac{\hbar^2}{4}$$

Heisenberg uncertainty principle.

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Minimum uncertainty wavepacket

For the equality $\langle \Delta q^2 \rangle \langle \Delta p^2 \rangle = \frac{\hbar^2}{4}$

$$\Delta q |\psi\rangle = c_1 \Delta p |\psi\rangle \quad \left. \begin{array}{l} |\psi\rangle \text{ has to satisfy these conditions} \\ \text{simultaneously.} \end{array} \right\}$$

$$\langle \psi | \Delta q \Delta p + \Delta p \Delta q | \psi \rangle$$

$$\langle \psi | \Delta q \Delta q | \psi \rangle = \langle \psi | G^* c_1 \Delta p^2 | \psi \rangle$$

$$\langle \psi | G^* \Delta p^2 | \psi \rangle + \langle \psi | G \Delta p^2 | \psi \rangle = \langle \psi | \Delta p^2 | \psi \rangle (G + G^*) \stackrel{?}{=} 0$$

If $|\psi\rangle$ is not an eigenstate of \hat{p} , $\langle \Delta p^2 \rangle \neq 0$ so that $\underline{c_1}$ must be a pure imag. #.

If $c_1 = -ic_2$, $c_2 \in \mathbb{R}$,

$$(\hat{q} - \langle q \rangle) |\psi\rangle = -ic_2 (\hat{p} - \langle p \rangle) |\psi\rangle$$

If we project an eigen-bra $\langle q' |$,

$$\langle q' | \psi \rangle = \psi(q') \text{ wavefn.}$$

$$(\hat{q}' - \langle q \rangle) \langle q' | \psi \rangle = (\hat{q}' - \langle q \rangle) \psi(q') = -ic_2 \left(\frac{\hbar}{i} \frac{\partial}{\partial q'} - \langle p \rangle \right) \psi(q')$$

$$\uparrow$$

$$p = \frac{\hbar}{i} \frac{\partial}{\partial x}$$

The solution is given by

$$\psi(q') = \underset{\substack{\uparrow \\ \text{a constant of integration}}}{C_3} \exp \left[\frac{i}{\hbar} \langle p \rangle q' - \frac{1}{2\hbar c_2} (q' - \langle q \rangle)^2 \right]$$

$$\int_{-\infty}^{\infty} |\psi(q')|^2 dq' = 1$$

$$\int_{-\infty}^{\infty} (q' - \langle q \rangle)^2 |\psi(q')|^2 dq' = \langle \Delta q^2 \rangle$$

$$\therefore c_2 = \frac{2\langle \Delta q^2 \rangle}{\hbar}, \quad |C_3|^2 = \frac{1}{\sqrt{2\pi\langle \Delta q^2 \rangle}}$$

$$\therefore \text{w/o loss of generality, } \psi(q') = (2\pi\langle \Delta q^2 \rangle)^{-\frac{1}{4}} \exp \left[\frac{i}{\hbar} \langle p \rangle q' - \frac{(q' - \langle q \rangle)^2}{4\langle \Delta q^2 \rangle} \right]$$

↓ Gaussian wavepacket

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p' -representation of the Schrödinger wavefn by the F.T.

$$\begin{aligned}\varphi(p') &\equiv \langle p' | \psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp(-\frac{i}{\hbar} p' q') \psi(q') dq' \\ &= (2\pi \langle \Delta p^2 \rangle)^{-\frac{1}{4}} \exp \left[-\frac{i}{\hbar} \langle q \rangle (p' - \langle p \rangle) - \frac{(p' - \langle p \rangle)^2}{4 \langle \Delta p^2 \rangle} \right] \\ \text{Where } \langle \Delta p^2 \rangle &= \frac{\hbar^2}{4 \langle \Delta q^2 \rangle}\end{aligned}$$

If we rewrite $(\hat{q} - \langle q \rangle) | \psi \rangle = -i c_2 (\hat{p} - \langle p \rangle) | \psi \rangle$

$$(\hat{q} + i c_2 \hat{p}) | \psi \rangle = (\langle q \rangle + i c_2 \langle p \rangle) | \psi \rangle$$

$$(e^r \hat{q} + i e^{-r} \hat{p}) | \psi \rangle = (e^r \langle q \rangle + i e^{-r} \langle p \rangle) | \psi \rangle$$

where a new parameter is defined by $c_2 = e^{-2r}$.

The min. $| \psi \rangle$ is an eigenstate of a "non-Hermitian" operator,
uncertainty state

$$\boxed{e^r \hat{q} + i e^{-r} \hat{p}} \text{ w/ a c-# eigenvalue } \boxed{e^r \langle q \rangle + i e^{-r} \langle p \rangle}$$

Coherent state and squeezed state

Consider a mechanical harmonic oscillator.

$$H = \frac{p^2}{2m} + \frac{1}{2} k q^2 \quad \leftarrow \text{total energy of the system}$$

$$\omega = \sqrt{\frac{k}{m}} \quad \leftarrow \text{oscillation freq.}$$

the minimum uncertainty state \Leftarrow the eigenstate of the non-Hermitian operator

$$\boxed{e^r \hat{q} + i e^{-r} \hat{p}}$$

$$\therefore \langle \Delta q^2 \rangle = \frac{\hbar}{2} e^{-2r}$$

$$\langle \Delta p^2 \rangle = \frac{\hbar}{2} e^{2r}$$

$$\text{pg 5. } \langle \Delta q^2 \rangle = \frac{\hbar}{2} c_2 = \frac{\hbar}{2} e^{-2r} \checkmark$$

$$\langle \Delta p^2 \rangle = \frac{\hbar^2}{4 \langle \Delta q^2 \rangle} = \frac{\hbar^2}{4 \cdot \frac{\hbar}{2} e^{-2r}} = \frac{\hbar}{2} e^{2r} \checkmark$$

"r" determines the noise distribution between q and p w/n $\langle \Delta q^2 \rangle \langle \Delta p^2 \rangle$
↑ "squeezing parameter"

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Expressing \hat{p} , \hat{q} in terms of the annihilation and creation operators \hat{a} & \hat{a}^\dagger .

$$\hat{q} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger)$$

$$\hat{p} = \frac{1}{i} \sqrt{\frac{\hbar m\omega}{2}} (\hat{a} - \hat{a}^\dagger)$$

\therefore We introduce 2 quadrature ^{amplitude} parameters a_1 & a_2

$$a_1 = \frac{1}{2} (\hat{a} + \hat{a}^\dagger) = \sqrt{\frac{m\omega}{2\hbar}} \hat{q}$$

$$a_2 = \frac{1}{2i} (\hat{a} - \hat{a}^\dagger) = \sqrt{\frac{1}{2\hbar m\omega}} \hat{p}$$

\Rightarrow New commutator relation

$$\boxed{[a_1, a_2] = \frac{i}{2}}$$

$$\langle \Delta a_1^2 \rangle \langle \Delta a_2^2 \rangle \geq \frac{1}{16}$$

& the min. uncertainty state = the eigenstate of the operator

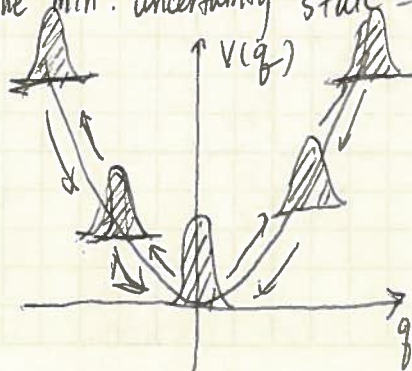
$$e^{r\hat{a}_1} + ie^{-r\hat{a}_2} \quad \&$$

$$\langle \Delta \hat{a}_1^2 \rangle = \frac{1}{4} e^{-2r}, \quad \langle \Delta \hat{a}_2^2 \rangle = \frac{1}{4} e^{2r}$$

$$\textcircled{1} r=0. \quad \langle \Delta \hat{a}_1^2 \rangle = \frac{1}{4} \quad \langle \Delta \hat{a}_2^2 \rangle = \frac{1}{4}$$

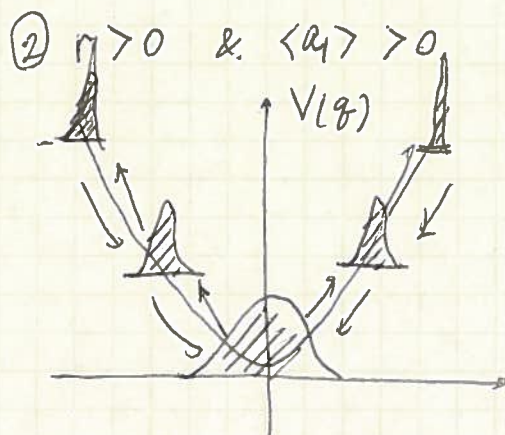
$$\rightarrow e^{r\hat{a}_1} + ie^{-r\hat{a}_2} = \hat{a}_1 + i\hat{a}_2 = \frac{1}{2} (\hat{a} + \hat{a}^\dagger) + \frac{i}{2i} (\hat{a} - \hat{a}^\dagger) = \hat{a}$$

\rightarrow the min. uncertainty state = coherent state



same-width wave packets

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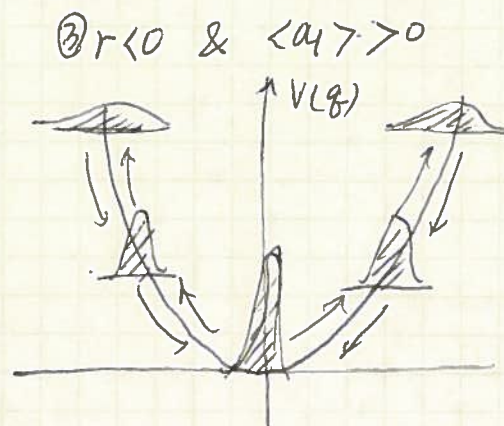


amplitude squeezed state



the quantum uncertainty is min.

when an amplitude is measured



phase-squeezed state



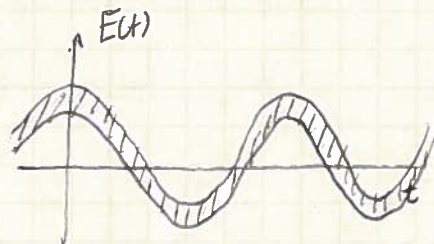
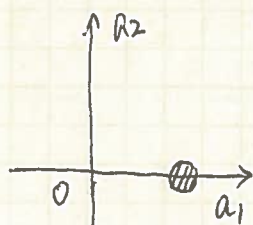
the quantum uncertainty is min.

when a phase is measured

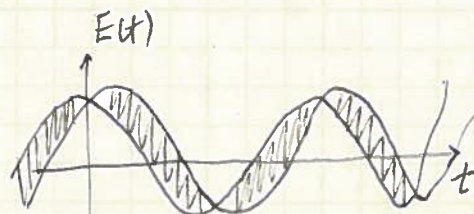
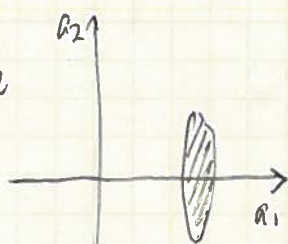
The pulsating uncertainties of the normalized position

$$\langle \Delta a_1^2 \rangle = \frac{m\omega}{2\hbar} \langle \Delta q^2 \rangle$$

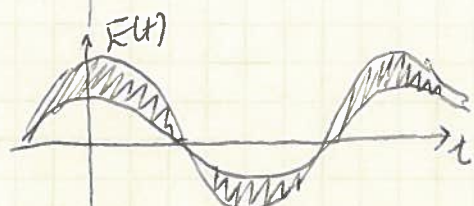
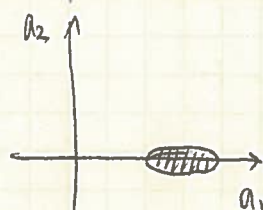
Coherent state



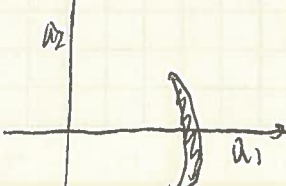
squeezed amplitude state



phase squeezed state



number-phase squeezed state



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Quantum Noise & thermal noise of a simple HO (harmonic oscillator)

Suppose a HO is at thermal eq.

According to the equipartition thm,

the energy associated with the position q and momentum p are independently given by $\frac{1}{2}k_B\Theta$

Taking into the quantization effect, namely, the energy is quantized in $\hbar\omega$, the thermal energy $E_t = \hbar\omega\bar{n} = \hbar\omega \cdot \frac{1}{e^{\hbar\omega/k_B\Theta} - 1}$

\bar{n} = the eq. photon statistics at a temp. Θ —

① When $\hbar\omega \ll k_B\Theta \Rightarrow$ quantization effect is not important \rightarrow thermal noise

$$E_t \rightarrow k_B\Theta \quad \hbar\omega \cdot \frac{1}{1 + \frac{\hbar\omega}{k_B\Theta} - 1} \approx k_B\Theta$$

$$\text{expected result} = \frac{k_B\Theta}{2} + \frac{k_B\Theta}{2} = k_B\Theta$$

② When $\hbar\omega \gg k_B\Theta$ (zero-temp. limit) \rightarrow "quantum noise".

$$E_t \rightarrow 0 \quad \text{is this correct? } \underline{\text{NO}}$$

even at $\Theta = 0$, the uncertainty principle requires the zero-point fluctuation vacuum fluctuation in \hat{p} & \hat{q}

For the ground state $|0\rangle$,

$$\hat{a}|0\rangle = 0 \Rightarrow \therefore |0\rangle \text{ is a coherent state w/ an eigenvalue of } 0 = 0 \cdot |0\rangle$$

$$\therefore E_g = \frac{1}{2m} \langle \Delta p^2 \rangle + \frac{1}{2}k \langle \Delta q^2 \rangle = \hbar\omega (\langle \hat{a}^2 \rangle + \langle \hat{a}^{\dagger 2} \rangle) = \frac{1}{2} \hbar\omega$$

$$\therefore E_{\text{Total}} = E_t + E_g = \underbrace{\hbar\omega \left(\frac{1}{e^{\hbar\omega/k_B\Theta} - 1} \right)}_{\text{thermal noise}} + \underbrace{\frac{1}{2} \hbar\omega}_{\text{quantum noise}}$$

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Quantum Noise of a lossless LC circuit

Consider a lossless LC circuit w/ $\omega_0 = \frac{1}{\sqrt{LC}}$

$i(t)$: the current flowing in L.

$v(t)$: the voltage across C.

$$\Rightarrow C \frac{dv(t)}{dt} = i(t) \quad L \frac{di(t)}{dt} = -v(t)$$

Introducing the normalized voltage & normalized current

$$q(t) = C v(t) \quad p(t) = L i(t)$$

$$\frac{dq(t)}{dt} = \frac{p(t)}{L} \quad \frac{dp(t)}{dt} = -\frac{q(t)}{C}$$

The total energy stored in the LC circuit is

$$H = \frac{1}{2} L i^2 + \frac{1}{2} C v^2 = \frac{p^2}{2L} + \frac{q^2}{2C}$$

Taking the analogy w/ a mechanical HD.

$$L \leftrightarrow m$$

$$C \leftrightarrow \frac{1}{k}$$

$$\therefore \frac{dq}{dt} = \frac{\partial H}{\partial p} = \frac{p}{L}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\frac{1}{C} q \quad \leftarrow \text{from classical Hamiltonian}$$

$\therefore q, p$ are a pair of conjugate observables

QM. $\begin{cases} [\hat{q}, \hat{p}] = i\hbar \\ \hat{q}, \hat{p} : \text{q. operators} \end{cases}$

$$\left[\langle \Delta q^2 \rangle \langle \Delta p^2 \rangle \geq \frac{\hbar^2}{4} \right]$$

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w/ the non-Hermitian creation (\hat{a}^\dagger) and annihilation (\hat{a}) operators for a circuit photon,

$$\hat{q} = \frac{1}{\sqrt{2\hbar\omega_0 L}} (\omega_0 L \hat{q} + i\hat{p})$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega_0 L}} (\omega_0 L \hat{q} - i\hat{p})$$

$$H = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \quad [\hat{a}, \hat{a}^\dagger] = 1$$

\exists a vacuum state = an excited LC circuit's QM state

$$\hat{a}|0\rangle = 0 \quad \langle 0|\hat{a}^\dagger = 0, \quad \hat{a}^\dagger|0\rangle = |1\rangle$$

notation for the vacuum state \rightarrow

$$\bar{q} \equiv \langle 0|\hat{q}|0\rangle = 0 \quad \hat{q} = \sqrt{\frac{\hbar\omega_0 C}{2}} (\hat{a}^\dagger + \hat{a}) = 0$$

$$\bar{q}^2 \equiv \langle 0|\hat{q}^2|0\rangle = \langle 0|\frac{\hbar\omega_0 C}{2} (\hat{a}^{\dagger 2} + \hat{a}^2 + \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger)|0\rangle$$

$$= \frac{\hbar\omega_0 C}{2}$$

$$\bar{p} \equiv \langle 0|\hat{p}|0\rangle = 0 \quad \hat{p} = i\sqrt{\frac{\hbar\omega_0 L}{2}} (\hat{a}^\dagger - \hat{a})$$

$$\bar{p}^2 \equiv \langle 0|\hat{p}^2|0\rangle = \langle 0|\frac{\hbar\omega_0 L}{2} (-\hat{a}^{\dagger 2} - \hat{a}^2 + \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger)|0\rangle = \frac{\hbar\omega_0 L}{2}$$

$$\Delta q^2 = \bar{q}^2 - \bar{q}^2 = \frac{\hbar\omega_0 C}{2} \quad \Delta q^2 \Delta p^2 = \frac{\hbar^2}{4} \quad \text{min. uncertainty}$$

$$\Delta p^2 = \bar{p}^2 - \bar{p}^2 = \frac{\hbar\omega_0 L}{2}$$

$$\langle 0|H|0\rangle = \frac{\hbar\omega_0}{2} \leftarrow \text{zero-point fluctuation}$$

full quantum mechanical expression for an open-circuit voltage power spectral density

$$S_V(\omega) = 2\hbar\omega R \coth\left(\frac{\hbar\omega}{2k_B\theta}\right)$$

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$$S_V(\omega) = 2\hbar\omega R \coth\left(\frac{\hbar\omega}{2k_B\Theta}\right) \text{ generalized Nyquist formula.}$$

① $\hbar\omega \ll k_B\Theta$ high temp. limit

$$S_V(\omega) \rightarrow 4k_B\Theta R$$

② $\hbar\omega \gg k_B\Theta$ low temp limit.

$$S_V(\omega) \rightarrow 2\hbar\omega R$$

