Introduction to Noise Processes
ECE730/QIC890-T33
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Problem Set 2 Suggested Solutions

1. **Wiener-Khintchine Theorem**

The average power of a noisy function $x_T(t)$ is defined by

$$\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} [x_T(t)]^2 dt = \lim_{T\to\infty} \frac{1}{2\pi}\int_0^{\infty} \frac{2|X_T(i\omega)|^2}{T} d\omega,$$

where $x_T(t)$ a gated function is defined by

$$x_T(t) = \{ \begin{array}{ll} x(t) & \text{if } |t| < T/2; \\ 0 & \text{if otherwise} \end{array},$$

T is a measurement time interval and $X_T(i\omega)$ is the Fourier Transform of $x_T(t)$.

(1) If $x_T(t)$ is a statistically stationary process, show that the average power of a noisy function is independent of T and a constant universal quantity.

(Answer: 1 point) In class, we discussed the statistical stationarity, and we limit our discussion to the (wide-sense) statistically stationary process $x_T(t)$ which possess the properties that its mean and autocorrelation functions are time invariant. In other words, the autocorrelation function is only dependent on the relative time difference τ not the absolute time t. The power is the special case of the autocorrelation with $\tau = 0$. Hence, the average power of $x_T(t)$ is a certain function $f(\tau)$ with the parameter $\tau = 0$, namely,

$$\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} [x_T(t)]^2 dt = f(0).$$

Hence, the average power of $x_T(t)$ is a constant quantity, not depending on T.

(2) If $x_T(t)$ is a statistically nonstationary process, show that the average power is dependent on T.

(Answer:2 points) For the nonstaionary process of $x_T(t)$, the average power of the $x_T(t)$ is a function of T, a measurement time interval and depends on the time t in general. Let's think of one classic example of the nonstationary process, the diffusion one defined as $x_T(t) = y_T(t)t$, where $y_T(t)$ is a (wide-sense) statistically stationary process. The integral can be computed with the method of the integration by parts,

$$
\begin{aligned}
\int_{-\infty}^{\infty}[x_T(t)]^2dt &= \int_{-\infty}^{\infty}[y_T(t)]^2t^2dt, \\
&= [y_T(t)]^2\frac{t^3}{3}\Big|_{-T/2}^{T/2} - \int_{-\infty}^{\infty}[y_T(t)]^2 2tdt, \\
&= \Big(y_T^2(T/2)) - y_T^2(-T/2)\Big)\frac{T^3}{8} - 2\Big[y_T^2(t)\frac{t^2}{2}\Big|_{-T/2}^{T/2} - \int_{-\infty}^{\infty}[y_T(t)]^2 2dt\Big], \\
&= \Big(y_T^2(T/2)) - y_T^2(-T/2)\Big)\frac{T^3}{8} - 2\Big(y_T^2(T/2)) - y_T^2(-T/2)\Big)\frac{T^2}{4} \\
&+ 4g(0)T,
\end{aligned}
$$

where $g(0)$ is the average power of $y_T(t)$, which is a constant. Therefore, the average power of the nonstationary process is a function of T explicitly in this particular example.

The concept of the power spectrum for non-stationary process has been an active research topic in the statistics and signal processing fields, and if you are interested in some of the discussions, you may refer the following reference and therein: R. M. Loynes, "On the Concept of the Spectrum for Non-Stationary Processes," Journal of the Royal Statistical Society. Series B (Methodological), Vol. 30, No.1, pp. 1-30 (1968).

For the statistically nonstationary process, we are not allowed to take the limit of $T \to \infty$. In this case, we introduce ensemble averaging which is taken first for many identical gated function $x_T(t)$. Then, the order of $\lim_{T\to\infty}$ and $\int_0^\infty d\omega$ can be interchanged. Now, we can define the unilateral power spectral density $S_x(\omega)$ is defined as

$$S_x(\omega) = \lim_{T\to\infty}\frac{2\langle|X_T(i\omega)|^2\rangle}{T}.$$

(3) Recall the formula in Problem Set 1, 3(2)

$$\int_{-\infty}^{\infty} x_T(t+\tau)x_T(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X_T(i\omega)|^2\exp(i\omega\tau)d\omega \qquad \text{(Eq. 1)}.$$

Suppose $\tau \neq 0$. One can also divide both sides of Eq. (1) by T, take an ensemble average, and take a limit of $T \to \infty$. Show your steps to reach the following relation,

$$\lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\langle x_T(t+\tau)x_T(t)\rangle dt = \lim_{T\to\infty}\frac{1}{2\pi T}\int_{-\infty}^{\infty}|X_T(i\omega)|^2\cos(\omega\tau)d\omega \qquad \text{(Eq. 2)}.$$

(Answer:2 points) Following the instruction to divide by T, take an ensemble average and take a limit of $\to \infty$, the left hand side becomes

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \langle x_T(t+\tau) x_T(t) \rangle dt,$$

and the right hand side (RHS) is

$$\lim_{T\to\infty} \frac{1}{2\pi T} \int_{-\infty}^{\infty} \langle |X_T(i\omega)|^2 \rangle \exp(i\omega\tau) d\omega.$$

Let us split the integral into the positive and negative regions:

$$\begin{aligned}
\int_{-\infty}^{\infty} \langle |X_T(i\omega)|^2 \rangle \exp(i\omega\tau) d\omega &= \int_{0}^{\infty} \langle |X_T(i\omega)|^2 \rangle \exp(i\omega\tau) d\omega \\
&+ \int_{-\infty}^{0} \langle |X_T(i\omega)|^2 \rangle \exp(i\omega\tau) d\omega.
\end{aligned}$$

Taking the variable, $\omega = -\omega'$, consequently, $d\omega = -d\omega'$, the second term in the RHS can be re-written as

$$\begin{aligned}
\int_{-\infty}^{0} \langle |X_T(i\omega)|^2 \rangle \exp(i\omega\tau) d\omega &= \int_{\infty}^{0} \langle |X_T(-i\omega')|^2 \rangle \exp(-i\omega'\tau)(-d\omega'), \\
&= \int_{0}^{\infty} \langle |X_T(-i\omega')|^2 \rangle \exp(-i\omega'\tau) d\omega', \\
&= \int_{0}^{\infty} \langle |X_T(i\omega)|^2 \rangle \exp(-i\omega\tau) d\omega,
\end{aligned}$$

where the third term is justified if $X_T(i\omega)$ is an even function for the real-valued $x_T(t)$.

The overall RHS becomes :

$$\begin{aligned}
& \int_{0}^{\infty} \langle |X_T(i\omega)|^2 \rangle \exp(i\omega\tau) d\omega + \int_{0}^{\infty} \langle |X_T(i\omega)|^2 \rangle \exp(-i\omega\tau) d\omega, \\
= & \int_{0}^{\infty} \langle |X_T(i\omega)|^2 \rangle (\exp(i\omega\tau) + \exp(-i\omega\tau)) d\omega, \\
= & \int_{0}^{\infty} \langle |X_T(i\omega)|^2 \rangle 2\cos(\omega\tau) d\omega, \\
= & \int_{-\infty}^{\infty} \langle |X_T(i\omega)|^2 \rangle \cos(\omega\tau) d\omega.
\end{aligned}$$

Noe that the equality between the third and the fourth line is valid since cosine function is even.

Thus, we showed that

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \langle x_T(t+\tau) x_T(t) \rangle dt = \lim_{T\to\infty} \frac{1}{2\pi T} \int_{-\infty}^{\infty} |X_T(i\omega)|^2 \cos(\omega\tau) d\omega \qquad \text{(Q.E.D)}.$$

(4) We know that the left-hand side of Eq. (2) is the ensemble averaged autocorrelation function $\langle \phi_x(\tau) \rangle$. Now we obtain the relation of the ensemble averaged autocorrelation $\langle \phi_x(\tau) \rangle$ and the unilateral power spectral density $S_x(\omega)$,

$$\langle \phi_x(\tau) \rangle = \frac{1}{2\pi} \int_0^{\infty} S_x(\omega) \cos(\omega\tau) d\omega \qquad \text{(Eq. 3)}.$$

Show that the inverse relation of Eq. (3) is written as,

$$S_x(\omega) = 4 \int_0^{\infty} \langle \phi_x(\tau) \rangle \cos(\omega\tau) d\tau \qquad \text{(Eq. 4)}.$$

Equations (3) and (4) are known as the Wiener-Khintchine theorem.

(Answer:2 points) Plugging (Eq. 3) into the integrand in the RHS of (Eq. 4),

$$\begin{aligned}
4 \int_0^{\infty} \langle \phi_x(\tau) \rangle \cos(\omega\tau) d\tau &= 4 \int_0^{\infty} \Big(\frac{1}{2\pi} \int_0^{\infty} S_x(\omega') \cos(\omega'\tau) d\omega' \Big) \cos(\omega\tau) d\tau, \\
&= 4 \frac{1}{2\pi} \int_0^{\infty} d\omega' S_x(\omega') \int_0^{\infty} \cos(\omega'\tau) \cos(\omega\tau) d\tau, \\
&= \frac{2}{\pi} \int_0^{\infty} d\omega' S_x(\omega') \int_0^{\infty} d\tau \frac{\cos(\omega + \omega'\tau) + \cos(\omega - \omega'\tau)}{2}, \\
&= \int_0^{\infty} d\omega' S_x(\omega') [\delta(\omega + \omega') + \delta(\omega - \omega')], \\
&= S_x(\omega) \qquad \text{(Q.E.D)}.
\end{aligned}$$

Remember the Fourier Transform of the cosine function,

$$\int_{-\infty}^{\infty} \cos(\omega t) \exp(-i\omega' t) dt = \pi[\delta(\omega' - \omega) + \delta(\omega' + \omega)].$$

2. **Unilateral power spectral density** $S_x(\omega)$

In class, we examined one example with a noisy waveform $x(t)$, which is a wide-sense statistically stationary. The autocorrelation function $\phi_x(\tau)$ has a form of

$$\phi_x(\tau) = \phi_x(0) \exp(-\frac{|\tau|}{\tau_1}),$$

where τ_1 is a relaxation time constant.

Compute the unilateral power spectral density $S_x(\omega)$ using the Wiener-Khintchine theorem.

(Answer:2 points) Plugging into the expression of $\phi_x(\tau)$ into the RHS in the Eq. (4) and expressing the cosine function with the exponential functions,

$$\begin{aligned}
S_x(\omega) &= 4\int_0^\infty \phi_x(0)\exp(-\frac{\tau}{\tau_1})\cos(\omega\tau)d\tau, \\
&= 4\int_0^\infty \phi_x(0)\exp(-\frac{\tau}{\tau_1})\frac{\exp(i\omega\tau)+\exp(-i\omega\tau)}{2}d\tau, \\
&= 2\phi_x(0)\int_0^\infty d\tau\exp((i\omega-\frac{1}{\tau_1})\tau)+\exp((i\omega+\frac{1}{\tau_1})\tau), \\
&= 2\phi_x(0)\Big[\frac{\exp((i\omega-\frac{1}{\tau_1})\tau)}{i\omega-\frac{1}{\tau_1}}\Big|_0^\infty+\frac{\exp(-(i\omega-\frac{1}{\tau_1})\tau)}{-(i\omega+\frac{1}{\tau_1})}\Big|_0^\infty\Big], \\
&= 2\phi_x(0)\Big[\frac{-1}{i\omega-\frac{1}{\tau_1}}+\frac{1}{i\omega+\frac{1}{\tau_1}}\Big], \\
&= \frac{4\phi_x(0)\tau_1}{(\omega\tau_1)^2+1},
\end{aligned}$$

we reach that the unilateral power spectral density $S_x(\omega)$ has the form of a Lorentzian function.

3. **Mathematical Identity**

Show that

$$\lim_{a\to 0}\int_0^\infty \frac{1-\cos(\omega t)}{\omega^2+a^2}d\omega = \frac{\pi}{2}t.$$

(Answer : 1 point)

Introducing $\frac{t}{2}\omega=\omega'$ with $\frac{t}{2}d\omega=d\omega'$,

$$\begin{aligned}
\lim_{a\to 0}\int_0^\infty \frac{1-\cos(\omega t)}{\omega^2+a^2}d\omega &= \lim_{a\to 0}\int_0^\infty \frac{2\sin^2(\omega t/2)}{\omega^2+a^2}d\omega, \\
&= \lim_{a\to 0}\int_0^\infty \frac{2\sin^2(\omega')}{\frac{4}{t^2}[\omega'^2+\frac{(ta)^2}{4}]}\frac{2}{t}d\omega', \\
&= \lim_{a\to 0}\int_0^\infty \frac{4t\sin^2(\omega')}{\omega'^2+\frac{(ta)^2}{4}}d\omega', \\
&= t\int_0^\infty \frac{\sin^2(\omega')}{\omega'^2}d\omega', \\
&= \frac{\pi}{2}t. \qquad \text{(Q.E.D.)}
\end{aligned}$$

The last line is reached using the fact that

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$