Solution of Final Exam 2008

Problem 1)

$$\varphi(x,y,z) = V_0 (x^2+y^2+z^2)$$

Back to Lecture 8, the SE-EM reads:

$$i\hbar \frac{\partial}{\partial t} \Psi(r,t) = \frac{1}{2m}\left(-i\hbar\nabla - eA\right)^2 \Psi(r,t) + e\varphi\, \Psi(r,t)$$

Since the E-field is static, $A=0$, then the TI-SE in the presence of the φ is:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + eV_0(x^2+y^2+z^2) = E\psi$$

where $\Psi(\vec{r},t) = \psi(\vec{r})e^{-iEt/\hbar}$

a)

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y,z) + eV_0(x^2+y^2+z^2) = E\,\psi(x,y,z)$$

b) $$-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2}\psi + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2}{\partial z^2}\psi\right] + eV_0(x^2+y^2+z^2) = E\psi$$

if $\psi(x,y,z) = X(x)\,Y(y)\,Z(z) \Rightarrow$

$$\left(-\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2} + eV_0x^2\right) + \left(-\frac{\hbar^2}{2m}\frac{1}{Y}\frac{d^2Y}{dy^2} + eV_0y^2\right) +$$

$$\left(-\frac{\hbar^2}{2m}\frac{1}{Z}\frac{d^2Z}{dz^2} + eV_0z^2\right) = E$$

$$\begin{cases} -\dfrac{\hbar^2}{2m} \dfrac{d^2X}{dx^2} + eV_0 x^2 X = E_x X \\ -\dfrac{\hbar^2}{2m} \dfrac{d^2Y}{dy^2} + eV_0 y^2 Y = E_y Y \\ -\dfrac{\hbar^2}{2m} \dfrac{d^2Z}{dz^2} + eV_0 z^2 Z = E_z Z \end{cases}$$

where $E_x + E_y + E_z = E$

Note that the above equations are simply 1D harmonic oscillator for each component where

$$\frac{1}{2} m\omega^2 = eV_0 \longrightarrow \omega = \sqrt{\frac{2eV_0}{m}}$$

Therefore:

$$X_n(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \frac{1}{\sqrt{2^{n_x} n_x!}} H_{n_x}\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right) e^{-\frac{mx^2}{2\hbar}}$$

$$Y_n(y) = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \frac{1}{\sqrt{2^{n_y} n_y!}} H_{n_y}\left(\sqrt{\frac{m\omega}{\hbar}}\, y\right) e^{-\frac{my^2}{2\hbar}}$$

$$Z_n(z) = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \frac{1}{\sqrt{2^{n_z} n_z!}} H_{n_z}\left(\sqrt{\frac{m\omega}{\hbar}}\, z\right) e^{-\frac{mz^2}{2\hbar}}$$

$$\psi_n(x,y,z) = X_n(x)\, Y_n(y)\, Z_n(z)$$

$$n_x + n_y + n_z = n$$

c) $E_x = (n_x + 1/2)\hbar\omega$

$n_x, n_y, n_z = 0, 1, 2, \ldots$

$E_y = (n_y + 1/2)\hbar\omega$

$E_z = (n_z + 1/2)\hbar\omega$

$E = E_x + E_y + E_z = (n + 3/2)\hbar\omega \quad n = n_x + n_y + n_z$

or

$$\boxed{E = (n + 3/2)\hbar\sqrt{\frac{2eV_0}{m}}}$$

Problem 2)

a) $H = -\gamma \vec{B}\cdot\vec{S}$ [Lecture 8, eq.(5-59)]

$$H = -\gamma B_0 \sin\omega t\, S_z = \frac{-\gamma B_0 \hbar}{2}\sin\omega t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

b) $\psi(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$ with $\alpha(0) = \beta(0) = \frac{1}{\sqrt{2}}$

$$i\hbar \frac{\partial}{\partial t}\psi(t) = H\psi(t) \implies$$

$$i\hbar \begin{pmatrix} \frac{d\alpha}{dt} \\ \frac{d\beta}{dt} \end{pmatrix} = -\frac{\gamma B_0 \hbar}{2}\sin\omega t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= -\frac{\gamma B_0 \hbar}{2}\sin\omega t \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} \implies$$

$$\begin{cases} i\hbar \frac{d\alpha}{dt} = -\frac{\gamma B_0 \hbar}{2}\sin\omega t\, \alpha(t) \\ i\hbar \frac{d\beta}{dt} = \frac{\gamma B_0 \hbar}{2}\sin\omega t\, \beta(t) \end{cases}$$

$$\frac{d\alpha}{dt} = i\frac{\gamma B_0}{2}\sin\omega t\, \alpha(t) \implies \frac{d\alpha}{\alpha} = i\frac{\gamma B_0}{2}\sin\omega t\, dt$$

$$\ln\alpha = -\frac{i\gamma B_0}{2\omega}\cos\omega t + cte \implies$$

$$\alpha = A e^{\frac{-i\gamma B_0}{2\omega}\cos\omega t}$$

Since $\alpha(0) = \frac{1}{\sqrt{2}} \Rightarrow \frac{1}{\sqrt{2}} = A e^{-i\frac{\gamma B_0}{2\omega}} \Longrightarrow$

$$A = \frac{e^{\frac{i\gamma B_0}{2\omega}}}{\sqrt{2}} \Rightarrow \alpha(t) = \frac{1}{\sqrt{2}} e^{\frac{i\gamma B_0}{2\omega}(1-\cos\omega t)}$$

or $$\alpha(t) = \frac{1}{\sqrt{2}} e^{i\frac{\gamma B_0}{\omega}\sin^2\frac{\omega t}{2}}$$

$$i\hbar \frac{d\beta}{dt} = \frac{\gamma B_0 \hbar}{2} \sin\omega t \, \beta(t) \rightarrow$$

$$\frac{d\beta}{dt} = -i\frac{\gamma B_0}{2} \sin\omega t \, \beta(t) \rightarrow \frac{d\beta}{\beta} = -i\frac{\gamma B_0}{2} \sin\omega t \, dt$$

$$\ln\beta = \frac{i\gamma B_0}{2\omega}\cos\omega t + cte \rightarrow \beta = B e^{i\frac{\gamma B_0}{2\omega}\cos\omega t}$$

Since $\beta(0) = \frac{1}{\sqrt{2}} \Rightarrow \frac{1}{\sqrt{2}} = B e^{i\frac{\gamma B_0}{2\omega}} \Rightarrow B = \frac{e^{-i\frac{\gamma B_0}{2\omega}}}{\sqrt{2}} \Rightarrow$

$$\beta(t) = \frac{1}{\sqrt{2}} e^{i\frac{\gamma B_0}{2\omega}(\cos\omega t - 1)} = \frac{1}{\sqrt{2}} e^{i\frac{\gamma B_0}{\omega}\cos^2\frac{\omega t}{2}} \Rightarrow$$

$$\beta(t) = \frac{1}{\sqrt{2}} e^{i\frac{\gamma B_0}{\omega}\cos^2\frac{\omega t}{2}}$$

$$\psi(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\frac{\gamma B_0}{\omega}\sin^2\omega t/2} \\ e^{i\frac{\gamma B_0}{\omega}\cos^2\omega t/2} \end{pmatrix}$$

c) $\langle S_y \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{i\gamma B_0}{\omega}\sin^2\frac{\omega t}{2}} & e^{\frac{i\gamma B_0}{\omega}\cos^2\frac{\omega t}{2}} \end{pmatrix} \times$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{i\gamma B_0}{\omega}\sin^2 \omega t/2} \\ e^{\frac{i\gamma B_0}{\omega}\cos^2\frac{\omega t}{2}} \end{pmatrix} =$$

$$\frac{\hbar}{4}\begin{pmatrix} i e^{\frac{i\gamma B_0}{\omega}\cos^2\frac{\omega t}{2}} & -i e^{\frac{\gamma B_0}{\omega}\sin^2\frac{\omega t}{2}} \end{pmatrix} \begin{pmatrix} e^{\frac{i\gamma B_0}{\omega}\sin^2\frac{\omega t}{2}} \\ e^{\frac{i\gamma B_0}{\omega}\cos^2\frac{\omega t}{2}} \end{pmatrix}$$

$$= \frac{\hbar}{4}\left\{ i e^{\frac{i\gamma B_0}{\omega}} - i e^{\frac{\gamma B_0}{\omega}} \right\} = 0 \quad \checkmark.$$

$$\langle S_y \rangle = 0$$

Problem 3)

a) $[\hat{a}, \hat{a}^\dagger] = 1 \Rightarrow \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1 \Longrightarrow$

$\hat{a}^\dagger \hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a}^\dagger \hat{a} = \hat{a}^\dagger \longrightarrow \hat{N}\hat{a}^\dagger - \hat{a}^\dagger \hat{N} = \hat{a}^\dagger$

$\Rightarrow [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$ ✓.

$\hat{a}\hat{a}^\dagger \hat{a} - \hat{a}^\dagger \hat{a}\hat{a} = \hat{a} \Rightarrow \hat{a}\hat{N} - \hat{N}\hat{a} = \hat{a}$

$[\hat{a}, \hat{N}] = -\hat{a}$ ✓.

b) $[\hat{N}, \hat{H}] = [\hat{N}, \hbar\omega_0 \hat{N} - \frac{k}{2}\hat{a}^\dagger \hat{N}\hat{a}]$

$[\hat{N}, \hbar\omega \hat{N}] - \frac{k}{2}[\hat{N}, \hat{a}^\dagger \hat{N}\hat{a}] = 0 - \frac{k}{2}[\hat{N}, \hat{a}^\dagger \hat{N}\hat{a}]$

We need to calculate $[\hat{N}, \hat{a}^\dagger \hat{N}\hat{a}] = 0$

$[\hat{N}, \hat{a}^\dagger \hat{N}\hat{a}] = \hat{N}\hat{a}^\dagger \hat{N}\hat{a} - \hat{a}^\dagger \hat{N}\hat{a}\hat{N}$

From $[\hat{a}^\dagger, \hat{N}] = \hat{a}^\dagger \rightarrow \hat{a}^\dagger \hat{N} - \hat{N}\hat{a}^\dagger = \hat{a}^\dagger \Longrightarrow$

$\hat{a}^\dagger N \hat{N}\hat{a} - \underline{\hat{N}\hat{a}^\dagger \hat{N}\hat{a}} = \hat{a}^\dagger \hat{N}\hat{a}$ (a)

From $[\hat{a}, \hat{N}] = \hat{a}$

$\underline{\hat{a}^\dagger \hat{N}\hat{a}\hat{N}} - \hat{a}^\dagger \hat{N} N a = \hat{a}^\dagger \hat{N}\hat{a}$ (b)

$(-a) - b \Rightarrow \hat{N}\hat{a}^\dagger \hat{N}\hat{a} - \hat{a}^\dagger \hat{N}\hat{a}\hat{N} = 0 \Longrightarrow$

$$[\hat{N}, \hat{a}^\dagger \hat{N} \hat{a}] = 0 \Rightarrow [\hat{N}, \hat{H}] = 0. \checkmark.$$

c) Since $\hat{H}$ & $\hat{N}$ commute, then they share the same eigenstates & eigenvalues, thus:

$$\hat{H}|n\rangle = \left[\hbar\omega_0 \hat{N} - \frac{\hbar k}{2} \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a}\right]|n\rangle$$

$$= \hbar\omega_0 \hat{N}|n\rangle - \frac{\hbar k}{2} \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a}|n\rangle$$

$$= \hbar\omega_0 n|n\rangle - \frac{\hbar k}{2} \hat{a}^\dagger \hat{a}^\dagger \hat{a} \sqrt{n}\,|n-1\rangle$$

$$= \hbar\omega_0 n|n\rangle - \hbar\frac{k}{2}\sqrt{n}\, \hat{a}^\dagger \hat{a}^\dagger \hat{a}|n-1\rangle \rightarrow \sqrt{n-1}\,|n-2\rangle$$

$$= n\hbar\omega_0|n\rangle - \frac{\hbar k}{2}\sqrt{n(n-1)}\, \hat{a}^\dagger \hat{a}^\dagger |n-2\rangle$$

$$= n\hbar\omega_0|n\rangle - \frac{\hbar k}{2}\sqrt{n}\,(n-1)\, \hat{a}^\dagger|n-1\rangle$$

$$= n\hbar\omega_0|n\rangle - \frac{\hbar k}{2} n(n-1)|n\rangle$$

$$\boxed{\hat{H}|n\rangle = \left\{n\hbar\omega_0 - \frac{\hbar k}{2} n(n-1)\right\}|n\rangle}$$

Eigenstates are $|n\rangle$ with eigenvalues $n\hbar\omega_0 - \hbar\frac{k}{2} n(n-1)$.

d) $i\hbar \frac{d}{dt} \hat{N}(t) = [\hat{N}(t), \hat{H}(t)] = 0 \Rightarrow$

$\hat{N}(t) = \hat{N}$. Conservation of photon number!

e) $i\hbar \frac{d}{dt} \hat{a}(t) = [\hat{a}(t), \hat{H}(t)]$

$$i\hbar \frac{d}{dt} \hat{a}(t) = \hbar\omega_0 \hat{a}(t) - \hbar k \hat{a}^\dagger \hat{a} \hat{a}$$

$$\frac{d}{dt} \hat{a}(t) = -i\omega_0 \hat{a} + ik \hat{N}(t) \hat{a}(t) \Rightarrow$$

$$\frac{d\hat{a}}{\hat{a}} = (-i\omega + ik\hat{N})\, dt \Rightarrow$$

$$\hat{a}(t) = \hat{a} \exp[-i\omega_0 t + ikt\hat{N}]$$

f) $\hat{N} = \hat{a}^\dagger(t)\, \hat{a}(t) \Rightarrow$

$$\hat{a}^\dagger(t) = \hat{a}^\dagger \exp[i\omega_0 t - ikt\hat{N}]$$

g) Energy lost from the cavity is

$$n\hbar\omega_0 - \frac{\hbar k}{2} n(n-1) - \left[(n-1)\hbar\omega_0 - \frac{\hbar k}{2}(n-1)(n-2)\right]$$

$$= \hbar\omega_0 - \hbar k(n-1) = \hbar(\omega_0 - k(n-1))$$

The frequency of the photon is $\omega_0 - k(n-1)$

& measurement probability is 1.

h) A coherent state is a superposition of number states with probability $\frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$.
Spectrometer measures $\omega_0 - k(n-1)$ with probability $\frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$ for $n = 1, 2, \ldots$. For a single photon will be $|\alpha|^2 e^{-|\alpha|^2}$. ✓.

Problem 4)

a) $\psi_i(z) = \sqrt{\frac{2}{d}} \cos\left(\frac{\pi}{z} L\right)$

b) $\psi_f(z) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} z\right) & n \text{ is even.} \\ \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi}{L} z\right) & n \text{ is odd.} \end{cases}$

c) $H'_{if} = \langle \psi_i | eE_0 z | \psi_f \rangle = eE_0 \langle \psi_i | z | \psi_f \rangle$

$$H'_{if} = \begin{cases} \int_{-d/2}^{d/2} z \frac{2}{\sqrt{Ld}} \cos\left(\frac{\pi}{d} z\right) \sin\left(\frac{n\pi}{L} z\right) dz \neq 0 \\ \int_{-d/2}^{d/2} z \frac{2}{\sqrt{Ld}} \cos\left(\frac{\pi}{d} z\right) \cos\left(\frac{n\pi}{L} z\right) dz = 0 \end{cases}$$

$$H'_{if} = 2\int_0^{d/2} \frac{2}{\sqrt{Ld}} z \cos\left(\frac{\pi}{d} z\right) \sin\left(\frac{n\pi}{L} z\right) dz$$

$$= \frac{4}{\sqrt{Ld}} \int_0^{d} z \cos\left(\frac{\pi}{d} z\right) \sin\left(\frac{n\pi}{L} z\right) dz$$

$$= \frac{2 d^{3/2}}{\sqrt{L}} F(L, d)$$

where $F(L,d) = \frac{L^2}{\pi(L^2 - n^2 d^2)} \left(\sin\frac{n\pi d}{2L} - \frac{4ndL}{L^2 - n^2 d^2} \cos\frac{n\pi d}{2L} \right)$

d) Considering the density of state for 1D,

$$g_{1D} = \frac{\sqrt{2m}}{\pi\hbar} E^{-1/2} L$$ is the DOS, the # of energy state per unit energy.

$$R = \frac{2\pi}{\hbar} |H'_{if}|^2 \, g_{1D}(E_f)$$

$$R = \frac{2\cancel{\pi}}{\hbar} \, 4 \frac{d^3}{\cancel{L}} F^2 \cdot \frac{\sqrt{2m}}{\cancel{\pi}\hbar} \frac{1}{\sqrt{E_F}} \cdot \cancel{L}$$

$$\boxed{R = \frac{8\sqrt{2m}}{\hbar^2} d^3 \frac{F^2}{\sqrt{E_f}}}$$

e)

Note that since $\frac{n\pi}{L} = n k_f \rightarrow$

$E_f \simeq \frac{\hbar^2 k_f^2}{2m}$ since electrons are free to move in continuum, therefore

$$F(E_f) = \frac{\pi}{\pi^2 - d^2 k_f^2} \left(\sin \frac{k_f d}{2} - \frac{4 k_f d}{\pi^2 - d^2 k_f^2} \cos \frac{k_f d}{2} \right)$$

This means R is independent from L.

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