Solution of Final Exam 2010

Problem 1)

$$\Psi(x,t=0) = A \sum_{n=0}^{\infty} c^n \psi_n(x)$$

a) $$\int |\Psi(x,t=0)|^2 dx = 1 \Rightarrow \int |A|^2 \sum_{n=0}^{\infty} |c|^{2n} |\psi_n(x)|^2 dx = 1$$

since $$\int |\psi_n(x)|^2 dx = 1 \Rightarrow |A|^2 \sum_{n=0}^{\infty} |c|^{2n} = 1$$

$$|A|^2 \frac{1}{1-|c|^2} = 1 \Rightarrow \boxed{A = \pm\sqrt{1-|c|^2}}$$

A is known up to a constant phase.

b)

$$\Psi(x,t) = \psi(x)\, e^{-i\frac{E}{\hbar}t} = A e^{-\frac{i\omega t}{2}} \sum_{n=0}^{\infty} c^n e^{-in\omega t} \psi_n(x)$$

$$\left\{ E = (n+\tfrac{1}{2})\hbar\omega \right.$$

c) $$\langle \Psi(0) | \Psi(t) \rangle = |A|^2 e^{-i\omega t/2} \sum_{n=0}^{\infty} |c|^{2n} e^{-i\omega n t}$$

$$= e^{-i\omega t/2} \frac{1-|c|^2}{1-|c|^2 e^{-i\omega t}}$$

$$P(t) = |\langle \Psi(0) | \Psi(t) \rangle|^2 = \frac{(1-|C|^2)^2}{1 - |C|^2 e^{i\omega t} - |C|^2 e^{-i\omega t} + |C|^4}$$

$$p(t) = \frac{(1-|C|^2)^2}{1+|C|^4 - 2|C|^2 \cos\omega t}$$

Further Simplification, $\cos\omega t = 1 - 2\sin^2\frac{\omega t}{2} \Rightarrow$

$$1+|C|^4 - 2|C|^2\cos\omega t = 1 + |C|^4 - 2|C|^2 + 4|C|^2 \sin^2\frac{\omega t}{2}$$

$$= (1-|C|^2)^2 + 4|C|^2\sin^2\frac{\omega t}{2}$$

therefore $$p(t) = \left[1 + \frac{4|C|^2\sin^2 \omega t/2}{(1-|C|^2)^2}\right]^{-1}$$

d) $$\langle \hat{H} \rangle = \langle \Psi | \hat{H} | \Psi \rangle = |A|^2 \sum_{n=0}^{\infty} E_n |C|^{2n}$$

$$= |A|^2 \sum_{n=0}^{\infty} (n+\tfrac{1}{2})\hbar\omega\, |C|^{2n}$$

$$= \frac{\hbar\omega}{2}(1-|C|^2) \sum_{n=0}^{\infty} \left(2n|C|^{2n} + |C|^{2n}\right)$$

$$= \frac{\hbar\omega}{2}\left[1 + (1-|C|^2)\sum_{n=0} 2n|C|^{2n}\right]$$

Since:
$$\frac{d}{d|C|}\sum_{n=0}^{\infty}|C|^{2n} = \frac{d}{d|C|}\frac{1}{1-|C|^2} = \sum_{n=0}^{\infty} 2n|C|^{2n-1}$$

$$\Rightarrow \langle \hat{H} \rangle = \frac{\hbar\omega}{2}\,\frac{1+|C|^2}{1-|C|^2}$$

Problem 2)

$$\hat{\rho} = \frac{1}{Z} \exp\left(-\frac{\hat{H}}{k_B T}\right) \quad \& \quad Z = Tr\left[\exp\left(-\frac{\hat{H}}{k_B T}\right)\right]$$

a) Since $\hat{H} = \hbar\omega\left(n+\frac{1}{2}\right) \Rightarrow$

$$Z = Tr\left[\exp\left(-\frac{\hbar\omega}{k_B T}\left(n+\frac{1}{2}\right)\right)\right] = \sum_{n=0}^{\infty} \langle n| \exp\left(-\frac{\hbar\omega}{k_B T}\left(n+\frac{1}{2}\right)\right)|n\rangle$$

$$= \exp\left(-\frac{\hbar\omega}{2k_B T}\right) \sum_{n=0}^{\infty} \exp\left(-\frac{\hbar\omega n}{k_B T}\right)$$

Since $\exp\left(-\frac{\hbar\omega}{k_B T}\right) < 1 \Rightarrow$ The sum is a geometric series,

$$Z = \exp\left(-\frac{\hbar\omega}{2k_B T}\right) \frac{1}{1-\exp\left(-\frac{\hbar\omega}{k_B T}\right)}$$

b)

$$P_n = \langle n|\hat{\rho}|n\rangle = \frac{1}{Z} \exp\left(-\frac{\hbar\omega}{k_B T}\left(n+\frac{1}{2}\right)\right)$$

c)

$$\hat{\rho} = \sum_{n=0}^{\infty} P_n |n\rangle\langle n|$$

d)

$$\bar{n} = \langle n\rangle = Tr(\hat{n}\hat{\rho}) = \sum_{n=0}^{\infty} \langle n|\hat{n}\hat{\rho}|n\rangle$$

$$= \sum_{n=0}^{\infty} n P_n = \frac{\exp(-\hbar\omega/2k_B T)}{Z} \sum_{n=0}^{\infty} n \exp\left(-\frac{\hbar\omega n}{k_B T}\right)$$

Note that $\sum_{n=0}^{\infty} n e^{-nx} = -\frac{d}{dx}\left(\sum_{n=0}^{\infty} e^{-nx}\right) = -\frac{e^{-x}}{(1-e^{-x})^2}$

Thus : $\boxed{\bar{n} = \frac{\exp(-\hbar\omega/k_BT)}{1-\exp(-\frac{\hbar\omega}{k_BT})} = \frac{1}{\exp(\frac{\hbar\omega}{k_BT})-1}}$

e)

$$Tr(\hat{\rho}^2) = \left(1-e^{-\frac{\hbar\omega}{k_BT}}\right)^2 \sum_{n=0}^{\infty} e^{-2n\hbar\omega/k_BT}$$

$$\boxed{Tr(\hat{\rho}^2) = \frac{(1-e^{-\hbar\omega/k_BT})^2}{1-e^{-2\hbar\omega/k_BT}}} \qquad Tr(\hat{\rho}^2) < 1.$$

Note that $e^{-\hbar\omega/k_BT} = \frac{\bar{n}}{1+\bar{n}} \Rightarrow Tr(\hat{\rho}^2) = \frac{1}{1+2\bar{n}} < 1$

$\boxed{Tr(\hat{\rho}^2) < 1.}$

Problem 3)

a)

$$\hat{H} = -\gamma \vec{B}\cdot\vec{S} = -\gamma\left(B_x S_x + B_y S_y + B_z S_z\right)$$

$$= -\gamma \frac{\hbar}{2}\left(B_x \sigma_x + B_y \sigma_y + B_z \sigma_z\right)$$

$$= -\gamma \frac{\hbar}{2}\left[B_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + B_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + B_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

$$= -\gamma \frac{\hbar}{2} \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$$

$$\hat{H} = -\frac{\gamma\hbar}{2} \begin{pmatrix} B_0 & B_\omega e^{i\omega t} \\ B_\omega e^{-i\omega t} & -B_0 \end{pmatrix}$$

b)

$$i\hbar \frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle \Rightarrow \qquad \text{if } \frac{d}{dt} \longrightarrow {}^{\circ}$$

$$i\hbar \begin{pmatrix} a^\circ \\ b^\circ \end{pmatrix} = -\frac{\gamma\hbar}{2} \begin{pmatrix} B_0 & B_\omega e^{i\omega t} \\ +B_\omega e^{-i\omega t} & -B_0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow$$

$$\begin{cases} a^\circ = i\frac{\gamma}{2}\left(B_0 a + B_\omega e^{i\omega t} b\right) = \frac{i}{2}\left(\Omega e^{-i\omega t} b + \omega_L a\right) \\ b^\circ = -i\frac{\gamma}{2}\left(B_0 b - B_\omega e^{-i\omega t} a\right) = \frac{i}{2}\left(\Omega e^{-i\omega t} a - \omega_L b\right) \end{cases}$$

c)

$$a(t) = \left\{a_0 \cos\left(\frac{\omega' t}{2}\right) + \frac{i}{\omega'}\left[a_0(\omega_L - \omega) + b_0 \Omega\right] \sin\left(\frac{\omega' t}{2}\right)\right\} e^{i\omega t/2}$$

$$b(t) = \left\{b_0 \cos\left(\omega' \frac{t}{2}\right) + \frac{i}{\omega'}\left[b_0(\omega - \omega_L) + a_0 \Omega\right] \sin\left(\frac{\omega' t}{2}\right)\right\} e^{-\frac{i\omega t}{2}}$$

d)

If $a_0 = 1$ & $b_0 = 0$ $\Rightarrow$ $b(t) = i\,\frac{\Omega}{\omega'} \sin\left(\frac{\omega' t}{2}\right) e^{-i\omega t/2}$

$$P(t) = |b(t)|^2 = \left(\frac{\Omega}{\omega'}\right)^2 \sin^2\left(\frac{\omega' t}{2}\right)$$

Problem 4)

a)

$$\hat{A}(r,t) = \sum_{k\lambda} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k}} \left[\hat{a}_k(t)\, \vec{u}_{k\lambda}(r) + \hat{a}_k^{\dagger}(t)\, \vec{u}^{*}_{k\lambda}(r) \right]$$

$$\hat{E}(r,t) = -\frac{\partial}{\partial t} \hat{A} = \sum_{k\lambda} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k}} \left[\overset{\circ}{a}_k(t)\, u_{k\lambda} + \overset{\circ}{\hat{a}}{}_k^{\dagger}(t)\, u^{*}_{k\lambda}(r) \right]$$

$$\hat{H}(r,t) = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A} = \frac{1}{\mu_0} \sum_{k\lambda} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k}} \left[\hat{a}_k(t)\, (\vec{\nabla} \times \vec{u}_{k\lambda}(r)) + \hat{a}_k^{\dagger}(t)\, (\vec{\nabla} \times u^{*}_{k\lambda}(r)) \right]$$

b)

$$\frac{\partial}{\partial t} \langle \hat{E} \rangle = \frac{1}{i\hbar} \langle [\hat{E}, \hat{\mathcal{H}}_{em}] \rangle$$

where $\hat{H}_{em} = \sum_{k\lambda} \hbar\omega_k (\hat{a}^{\dagger}_{k\lambda} \hat{a}_{k\lambda} + \frac{1}{2})$

$$[\hat{E}, \hat{H}_{em}] = \sum_{k\lambda} \sum_{k'\lambda'} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k}}\, \hbar\omega_{k'} \Big[[\hat{a}^{\circ}_{k\lambda}, a^{\dagger}_{k'\lambda'} \hat{a}_{k'\lambda'}]\, u_{k\lambda}(r)$$

$$+ [\hat{a}^{\circ\dagger}_{k\lambda}, \hat{a}_{k'\lambda'} \hat{a}^{\dagger}_{k'\lambda'}]\, u^{*}_{k\lambda}(r) \Big]$$

Note that $\hat{E}$ commutes with constant $\frac{\hbar\omega_k}{2}$.

Note that
$$\begin{cases} [\hat{a}_{k\lambda}, \hat{a}^{\dagger}_{k\lambda} a_{k\lambda}] = \hat{a}_{k\lambda} \\ [\hat{a}^{\dagger}_{k\lambda}, a_{k\lambda} \hat{a}^{\dagger}_{k\lambda}] = -\hat{a}^{\dagger}_{k\lambda} \end{cases}$$

$$\begin{cases} [\hat{a}_{k\lambda}, \hat{a}^{\dagger}_{k'\lambda'} \hat{a}_{k'\lambda'}] = \delta_{k\lambda} \delta_{k'\lambda'} \\ [\hat{a}^{\dagger}_{k\lambda}, \hat{a}^{\dagger}_{k'\lambda'} \hat{a}_{k'\lambda'}] = \delta_{k\lambda} \delta_{k'\lambda'} \end{cases}$$

$$[\hat{E}, \hat{H}_{em}] = \sum_{k\lambda} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k}}\, \hbar\omega_k \left(\hat{a}_k^{\circ}\, u_{k\lambda}(r) - \hat{a}_k^{\circ\dagger}\, u_{k\lambda}^{*}(r) \right)$$

So $$\boxed{\frac{\partial}{\partial t} \langle \hat{E} \rangle = \frac{1}{i\hbar} \sum_{k\lambda} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k}}\, \hbar\omega_k \left(\langle \hat{a}_k^{\circ} \rangle u_{k\lambda}(r) - \langle \hat{a}_k^{\dagger\circ} \rangle u_{k\lambda}^{*} \right)}$$

If you compare this with $\vec{\nabla} \times \langle \hat{H} \rangle$, we have

$$\epsilon_0 \frac{\partial}{\partial t} \langle \hat{E} \rangle = \vec{\nabla} \times \langle \hat{H} \rangle$$

With the same token,

$$\frac{\partial}{\partial t} \langle \hat{H} \rangle = \frac{1}{i\hbar} \sum_{k\lambda} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k}}\, \hbar\omega_k \left(\langle a_k \rangle (\nabla \times u_{k\lambda}) - \langle \hat{a}_k^{\dagger} \rangle \cdot (\nabla \times \vec{u}_{k\lambda}^{*}) \right)$$

and $$\mu_0 \frac{\partial}{\partial t} \langle \hat{H} \rangle = -\vec{\nabla} \times \langle \hat{E} \rangle$$

Problem 5)

a)

$$|n\rangle = \sqrt{\frac{2}{L}}\, \sin k_n z \qquad k_n = n\frac{\pi}{L} \quad \& \quad E_n = \frac{\hbar^2 k_n^2}{2m_e}$$

$$\omega_{21} = \frac{1}{\hbar}\left(E_2 - E_1\right) = \frac{1}{\hbar}\cdot\frac{\hbar^2}{2m_e}\left(\frac{4\pi^2}{L^2} - \frac{\pi^2}{L^2}\right) = \frac{3\hbar\pi^2}{2m_eL^2}$$

$$\omega_{21} = \frac{3\hbar\pi^2}{2m_eL^2}$$

b)

$$P_{12} = \frac{e^2|E_0|^2}{\hbar^2}\langle 2|z|1\rangle^2 \left|\int_0^t e^{i\omega_{21}t' - \frac{t'}{\tau}}\, dt'\right|^2$$

$$\langle 2|z|1\rangle = \frac{2}{L}\int_0^L z \sin(k_2 z)\sin(k_1 z)\, dz = \frac{-16L}{9\pi^2}$$

$$\left|\int_0^t e^{(i\omega_{12}-\frac{1}{\tau})t'}\, dt'\right|^2 = \frac{1 + e^{-\frac{2t}{\tau}} - 2e^{-t/\tau}\cos(\omega_{21}t)}{\omega_{21}^2 + \frac{1}{\tau^2}}$$

$$P_{12} = \left(\frac{16e|E_0|^2L}{9\pi^2\hbar}\right)^2\left(\frac{1 + e^{-2t/\tau} - 2e^{-t/\tau}\cos(\omega_{21}t)}{\omega_{21}^2 + \frac{1}{\tau^2}}\right)$$