Physics 760

Assignment 4

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Problem 1

a

As the wave travels through the medium with index of refraction n_2 it will acquire phase relative to the incident wave. Consider a wave that is incident on the slab of material 2 at some angle θ_1 relative to the normal to the surface. It will first travel through a distance of $l_1 = \frac{d}{\cos\theta_2}$. Then it will reflect off the surface where material 3 meets material 2. If we compare the phase of the wave at the point that it entered material 2 with the phase that it has when it reaches a point in its path that intersects the perpendicular to the initial direction of the wave when it entered material 2 we will really be finding the phase difference between adjacent plane waves that are formed by reflections off material 3.

To find this "distance to the perpendicular", consider the wave after the reflection off of material 3. The wave will travel another distance which can be shown to be $l_2 = \frac{d}{\cos\theta_2}\sin(\pi - (\pi/2 + 2\theta_2))$. Summing l_1 with l_2 and using some trigonometric identities, the total distance l traveled through material before the wave reaches the "next plane" can be shown to be $\delta_l = 2d\cos\theta_2$. The phase difference between the two fictitious plane waves (the original plane wave and a time-advanced version of that wave is given by the $\vec{k}\cdot\vec{x}$ in the exponential describing the plane wave. Thus, the phase difference is $\delta_\phi = \frac{2\pi}{\lambda_2}\delta_l = \frac{4\pi}{\lambda_2}d\cos\theta_2$. In the case of normal incidence this expression reduces to $\delta_\phi = \frac{4\pi d}{\lambda_2}$.

The above was necessary in order to perform the folowing steps. See, depending on the indices of refraction n_1, n_2, and n_3 the wave will reflect an infinite number of times between the two surfaces (where material 1 meets material 2 and, also, where material 2 meets material 3. Each time it encounters an interface some portion of the wave will get reflected and some portion will be transmitted. Because of the thickness of material 2, the wave will also acquire a complex phase which will result in one reflected and/or transmitted wave's interference with all of the other instances of reflection or transmission.

Consider a ray of light incident normally to the surface where materal 1 meets material 2. Assign this light a plane wave of the form $E_I(\vec{,}t) = \vec{E_0}\exp(i(\vec{k}\cdot\vec{x} - \omega t))$. The first transmitted wave will have the form $\vec{E_{t_1}} = \vec{E_0} * t_{21} * \exp(i(\vec{k_2}\cdot\vec{x} - \omega t)) * \exp(i\phi)$. ϕ in the previous expression accounts for the accumulation of the wave's phase as it travels though material 2 towards material 3. Note, also, that I have introduced a notation that I will use throughout the rest of this problem set. t_{ij} is the scaling factor for waves which are transmitted from region j into region i. Similarly, r_{ij} is the scaling factor for waves which are reflected off material j into material i.

It is sufficient to consider just the amplitude of the wave from this point on. That is, I can safely drop the time component since I am dealing with monochromatic light and the time dependence is the same between all generated rays. The sum of all of the complex amplitudes of the transmitted rays will give me the amplitude of the resultant wave.

Now, I will define a reference for the phase in this problem. The reference phase is that of the first wave immediately after it has exited the 3rd material (or, equivalently, as soon as it encounters the interface). Subsequent transmissions (due to reflections off of the interface between material 2 and material 3) will acquire a phase determined by the total distance traveled through the material. Thus, utilizing this definition and the fact that I can disregard the time component of the wave, I can express the ray that is first transmitted into material 3 as : $\vec{E}_{t_1} = \vec{E_0} * t_{21} * t_{32}$.
Now, part of the incident wave makes it in the "first pass" to material 3. Some of this wave is reflected before any of the wave is transmitted into material 2. Some of the wave is transmitted into material 2 (this is the wave we have just considered). However, after this wave encounters material 3 a portion of this wave may be reflected at this interface. Thus, a new wave will later exit material 2 into material 3 and we must consider

this wave's interference with our first wave.

Utilizing the prior disussion regarding the acquired phase. I may write that the complex amplitude of the wave due to the transmission into material 1, reflections off of the two interfaces, transmission into material 3 and total distance traveled through material 2 as : $\vec{E}_{t_2} = \vec{E}_0 * t_{21} * r_{23} * r_{21} * t_{32} * exp(i\frac{4\pi d}{\lambda_2}) = \vec{E}_{t_1} * r_{23} * r_{21} * exp(i\phi)$. I will omit the vector arrow above E_0 for brevity. I still maintain that it is a complex vector quantity. I will also allow the phase acquired due to the thickness of the plate to be designated as ϕ.

Although we have discovered the amplitude of the second ray to penetrate material 3 we must consider that some of this ray reflected at the interface between material 2 and 3 and thus, there is a 3rd ray that will exit material 2 into material 3. Its amplitude is described by: $E_0 * t_{21} * r_{23} * r_{21} * r_{23} * r_{21} * t_{23} * \exp(2i\phi) = E_{t_2} * r_{23} * r_{21} * exp(i\phi) = E_{t_1} * (r_{23} * r_{21} * exp(i\phi))^2$.

It is clear that this trend will continue ad infinitum and that the amplitude of the nth transmitted wave can be described as $E_{t_n} = E_0 * t_{21} * t_{32} * (r_{23} * r_{21} * exp(i * \phi))^n$. Thus, the net wave will have an amplitude $E_t = \sum_{n=0}^{\infty} E_0 t_{21} t_{32} (r_{23} r_{21} \exp(i\phi))^n$. This is a simple geometric series with solution $E_t = E_0 t_{21} t_{32} \frac{1}{1 - r_{23} r_{21} \exp(i\phi)}$. To find the transmission coefficient T I will first normalize the transmitted wave amplitude by the incident wave. Then, I will multiply the wave amplitude by its complex conjugate. Avoiding tying a lot of tedious algebra: $T = |\frac{E_t}{E_0}|^2 = \frac{(t_{21} t_{32})^2}{1 - (r_{23} r_{21})^2 - 2 r_{23} r_{21} \cos\phi}$. Since $\phi = \frac{4\pi}{\lambda_2} d$. ϕ in terms of the wavelength in vacuum is $\phi = \frac{4\pi n_2}{\lambda_0} d$.

b

The analysis for this part of the problem will be very similar to the previous problem. The zeroth wave that is reflected will have amplitude $E_{r_0} = E_0 * r_{12}$. The first wave will be related to the incident wave by $E_{r_1} = E_0 * t_{21} * r_{23} * t_{12} * \exp(i\phi)$. Here, ϕ has the same form it did before. The second wave will have amplitude: $E_{r_2} = E_0 * t_{21} * r_{23} * r_{21} * r_{23} * t_{12} * \exp(2i\phi) = E_{r_1} * r_{21} * r_{23} * \exp(i\phi)$. The third wave will bear the same relationship with the second wave as the second wave has with the first $E_{r_3} = E_{r_2} * r_{21} * r_{23} * \exp(i\phi) = E_{r_1} * (r_{21} * r_{23})^2 * \exp(2i\phi)$. Thus, it seems that the $E_{n>0}$ wave can be expressd as $E_n = E_{r_1}(r_{21} * r_{23} * exp(i\phi))^(n-1)$.
Thus, the reflected amplitude is $E_r = E_{r_0} + \sum_{n=1}^{\infty} E_{r_1}(r_{21} * r_{23} * \exp(i\phi))^(n-1)$. Recognizing the geometric series, again, and simplifying the resulting expression yields: a. Taking the magnitude squared of this expression yields the reflection coefficient $R =$.

c

An obvious application of this device is that of filtering a particular frequency of light. One could tune the distance of separation between material 1 and 3 to result in total interference in the transmission. Another application could be in using a variable distance (variable d) apparatus to experimentally determine the particular wavelength of light being generated by a monochromatic source.

Problem 2

a

The trick to this problem is to understand that a finite (not infinity) conductivity σ has the physical effect of introducing an imaginary component into the permittiviy $\epsilon = \epsilon_{real} + j\frac{\sigma}{\omega}$. All of the other equations remain the same. As ϵ is now a complex quantity let us write it as $\epsilon = A\exp(i\theta)$. This will have advantages over the polar form as will become apparent shortly.

Light reflecting normally off of a surface can be shown to change amplitude as : $\frac{E_r}{E_0} = \frac{n_1 - n_2}{n_2 + n_1}$ where n_2 is the index of refraction of the material upon which the light impinges. If this index of refraction is larger than n_1 then the wave will experience a sign reversal. Now, $n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$. Assuming the material has a magnetic permeability close to that of vacuum (a reasonably assumption for optical materials). $n \approx \sqrt{\epsilon/\epsilon_0}$.
Substituting this expression for n into the prior expression for the ratio of the electric field amplitudes allows me to write: $\frac{E_r}{E_I} = \frac{1-\sqrt{\epsilon/\epsilon_0}}{1+\sqrt{\epsilon/\epsilon_0}} = \frac{\sqrt{\epsilon_0}-\sqrt{\epsilon}}{\sqrt{\epsilon_0}+\sqrt{\epsilon}}$. Expressing ϵ as the complex quantity given earlier yields : $\frac{E_r}{E_I} = \frac{\sqrt{\epsilon_0}-\sqrt{A}\exp(i\theta/2)}{\sqrt{\epsilon_0}+\sqrt{A}\exp(i\theta/2)}$. I can manipulate this algebraically to obtain: $\frac{E_r}{E_I} = \frac{\epsilon_0 - A - 2i\sqrt{\epsilon_0 A}\, sin(\theta/2)}{\epsilon_0 + A + 2\sqrt{\epsilon_0 A}\cos(\theta/2)}$. The phase of the reflected wave can be obtained by taking the arctangent of the imaginary and the real portions of the previous expression. $\Phi = \arctan(-\frac{2\sqrt{\epsilon_0 A}\sin(\theta/2)}{\epsilon_0 - A})$. Now, $\theta(\sigma)$ and $\hat(\sigma)$ so let's consider their expansions to determine an expression for the phase that only contains to first order in σ. $A = \sqrt{(\epsilon_{real}^2 + (\sigma/\omega)^2}$. Expanding $A(\sigma)$ about $\sigma = 0$ yields $\epsilon_{real} + \frac{\sigma^2}{2\epsilon_{real}\omega^2} + O(\sigma^4)$. Since the assignment requires an expansion of the reflected phase to first order in σ I can safely neglect the second term and keep the 0th order term ϵ_{real}. Performing a similar analysis of $\theta(\sigma)$, expanding about zero, yields $\sigma/(\omega\epsilon_{real}) + O(\sigma^3)$. Thus, I am only required to keep the first order term and the reflecte phase, to first order is $\Phi = \arctan(-2\sqrt{\epsilon_0\epsilon_{real}}\frac{\sigma}{\omega\epsilon_{real}}/(\epsilon_0 - \epsilon_{real}))$.
Now, to determine the intensity of the reflected wave I must find the squared magnitude of $\frac{E_r}{E_I}$. Utilizing my prowess in solving for squared magnitudes: $|\frac{E_r}{E_I}|^2 = \frac{\sqrt{(\epsilon_0-\epsilon_{real})^2 + 4\epsilon_0\epsilon_{real}\sin^2(\theta/2)}}{(\epsilon_0+\epsilon_{real}+2\sqrt{\epsilon_0\epsilon_{real}}\cos\theta/2)^2}$. There may exist a simpler expression than this. But, I'm not going to try and find it. This is a solution to the problem.

b

This problem requires me using knowledge of the skin depth of a material. The next few lines are pretty much copy-pasted from 5.18 (a) of Jackson's Electrodynamics where he discusses this exact phenomenon.

For a material with finite conductivity, imagine an H field (oriented in the $\hat{x}$ direction) is given by $H_x(z,t) = h(z)\exp(-i\omega T)$. By manipulating Maxwell's equations ($\nabla x H = J$ and $J = \sigma E$ - knowing $B = \nabla x A$) we can show that $\nabla^2 A = \mu\sigma\frac{\partial A}{\partial t}$. In this particular situation $\left(\frac{d^2}{dz^2} + i\mu\sigma\omega\right)h(z) = 0$. A solution to this equation is given by $H_x(z,t) = A\exp(-z/\delta)\exp(i(z/\delta - \omega t)) + B\exp(z/\delta)\exp(-i(z/\delta + \omega t))$. Here, $\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$: the "skin depth" of the material. In order to conserve energy it must be the case that $B = 0$. For a free wave $H = \hat{n} x E \frac{\epsilon_0}{\mu_0}$. Thus, we should expect the same qualitative behavior for $E(z,t)$. In this case, E is oriented in the $-\hat{y}$ direction (in order to make the cross product return the right direction for H).
Thus, if we assume the amplitude of the wave upon encountering the surface is E_0 then after a distance d into the surface the amplitude is $E_0\exp(-d/\delta)\exp(i(d/\delta - \omega t))$. Thus, the intensity of the wave is $E_0^2\exp(-2d/\delta)$ (the magnitude squared of the amplitude).

Problem 3

a

Read Jackson.

b

The critical angle is the angle at which $\sin(\theta_2) = \pi/2$ in Snell's law - $n_1\sin\theta_1 = n_2\sin\theta_2$. If n_1 is air (which its claimed to be in this problem) then the critical incidence angle is related to the incident angle θ_I through

$\sin\theta_I = n_2$. Since $\theta_I \approx \pi/2$ $\sin\theta_I = \sin(\pi/2 - \phi_I) = \cos(\phi_I)$. Now, ϕ_I is pretty small according to the problem statement. Thus, $1 + \phi_I^2/2 \approx n_2 = 1 - \delta$ and, trivially, $\sqrt{2\delta} \approx \phi_I$.