Tue

Review of Lecture 1.

- time-average : mean, mean-square, autocorrelation
- ensemble-average : mean, mean-square, covariance
- ergodicity,
- statistical stationarity
- Wiener-Khintchine theorem.
- Fourier Analysis

Today

"Stationarity" = the statistics of a stationary process does not change in time

Rigorously,

A stochastic process is stationary of order k

if the k-th order joint prob. density fn satisfies

$$P(\alpha_1, \alpha_2, \cdots, \alpha_K ; t_1, t_2, \cdots, t_K) = P(\alpha_1, \alpha_2, \ldots, \alpha_K ; t_1+\varepsilon, t_2+\varepsilon, \cdots, t_K+\varepsilon) \; \forall \varepsilon$$

Order 1 $\quad P_1(x ; t_1) = P_1(x ; t_1+\varepsilon)$ for $\forall \varepsilon$.

Order 2 $\quad P_2(x_1, x_2 ; t_1, t_2) = P_2(x_1, x_2 ; t_1+\varepsilon, t_2+\varepsilon)$

⋮

- A process is strictly stationary if it is stationary for any orders $k = 1, 2, \cdots$
- "wide-sense (or weakly) stationary" if the mean value is constant & its autocorrelation fn depends on $\tau = t_2 - t_1$

↓

the autocorrelation fn & the power spectral density fn are a Fourier transform pair [Wiener-Khintchine thm]

"Ergodicity" is very useful if we have only 1 sample fn.

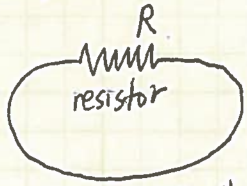
↳ "If the process is ergodic, since time-average = ensemble average

⇒ all statistical info can be derived from only "1" sample fn

(In other words, 1-sample fn represents the entire process

Example

$$\boxed{Q = It}$$



→ $i(t)$ = thermal noise current

$$q(t) = \int_0^t i(t')\,dt$$

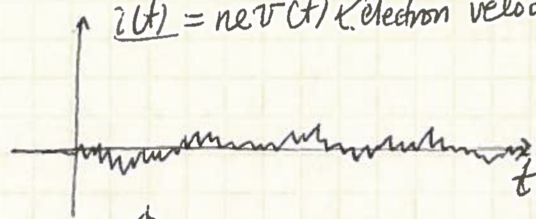
charge → $q(t)$

$i(t')$ ← thermal noise current

"Brownian particle"

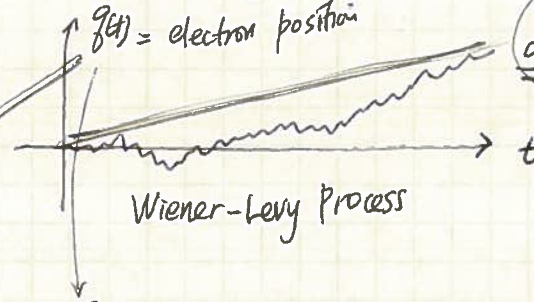
electrons are kicked randomly by vibrating lattice

$i(t) = ne\,v(t)$ ← electron velocity



↑ statiscally-stationary

$q(t)$ = electron position



random walk diffusion

statistically non-stationary

Wiener-Levy Process

↑ dep. on t. $\propto t$

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$\boxed{\text{Power Spectral Density (PSD)}}$

'power' = the average of the signal^2

$x^2(t)$ = the instantaneous power in the signal $x(t)$

The average power of $x_T(t)$

$$\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} x_T^2(t)\,dt = \lim_{T\to\infty} \frac{1}{2\pi}\int_0^{\infty} \frac{2|X_T(i\omega)|^2}{T}\,d\omega$$

~~independent of T~~

if $x(t)$ = statistically stationary, constant & indep. of $\underline{\underline{T}}$

if $x(t)$ = statistically non-stationary, depends on $\underline{\underline{T}}$

∴ cannot take $T\to\infty$ limit.

→ Then, we need to introduce $\boxed{\text{"ensemble average"}}$ to define T-dep. average power

<u>Suppose</u> take "ensemble averaging" of many identical $x_T(t)$

→ then the order of $\lim_{T\to\infty}$ & $\int_0^{\infty}$ can be interchangeable in the right side

$$\therefore \lim_{T\to\infty} \frac{1}{2\pi}\int_0^{\infty} \frac{2\langle|X_T(i\omega)|^2\rangle}{T}\,d\omega = \int_0^{\infty} \frac{1}{2\pi} \lim_{T\to\infty} \frac{2\langle|X_T(i\omega)|^2\rangle}{T}\,d\omega$$

Define
$$\boxed{S_x(\omega) \equiv \lim_{T\to\infty} \frac{2\langle|X_T(i\omega)|^2\rangle}{T}}$$
unilateral power spectral density

"ensembled averaged quantity

<u>Note</u> $S_x(\omega)$ of a stationary process is different from $S_x(\omega)$ of a non-stationary process

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Wiener-Khintchine Theorem

↳ relationship btw "ensemble-averaged" autocorrelation fn & "power spectral density"

① statistically - stationary case

$$\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} \langle x_T(t+\tau)\, x_T(t)\rangle\, dt = \lim_{T\to\infty} \frac{1}{2\pi}\int_0^{\infty} \frac{2|X_T(i\omega)|^2}{T} \cos(\omega\tau)\, d\omega$$

the ensemble-averaged autocorrelation fn

$$\phi_x(\tau) = \frac{1}{2\pi}\int_0^{\infty} \lim_{T\to\infty} \frac{2|X_T(i\omega)|^2}{T} \cos\omega\tau\, d\omega$$

$$\equiv \frac{1}{2\pi}\int_0^{\infty} \underbrace{S_x(\omega)}_{\text{definition}} \cos\omega\tau\, d\omega \qquad S_x(\omega) = \lim_{T\to\infty} \frac{2\langle |X_T(i\omega)|^2\rangle}{T}$$

② statistically non-stationary case

$$\frac{1}{T}\int_{-\infty}^{\infty} \langle x_T(t+\tau)\, x_T(t)\rangle = \frac{1}{2\pi}\int_0^{\infty} \frac{2|X_T(i\omega)|^2}{T} \cos(\omega\tau)\, d\omega$$

ensemble average

W-K theorem

$$\underbrace{\langle \phi_x(\tau)\rangle}_{\text{ensemble averaged autocorrelation}} = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} \langle x_T(x+\tau)\, x_T(t)\rangle\, dt = \lim_{T\to\infty} \frac{1}{2\pi}\int_0^{\infty} \frac{2\langle |X(i\omega)|\rangle^2}{T} \cos\omega\tau\, d\omega$$

$$\equiv \frac{1}{2\pi}\int_0^{\infty} S_x(\omega) \cos\omega\tau\, d\omega$$

here $\boxed{S_x(\omega) = \lim_{T\to\infty} \frac{2\langle |X(i\omega)|\rangle^2}{T}}$ power spectral density

$$\boxed{\begin{aligned} \langle \phi_x(\tau)\rangle &= \frac{1}{2\pi}\int_0^{\infty} S_x(\omega) \cos(\omega\tau)\, d\omega \\ S_x(\omega) &= 4\int_0^{\infty} \langle \phi_x(\tau)\rangle \cos(\omega\tau)\, d\tau \end{aligned}}$$

F.T. pairs

$$2\langle \phi_x(\tau)\rangle \longleftrightarrow S_x(\omega)$$

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Example 1

$x(t)$: a noisy waveform → t "wide-sense statistically stationary"

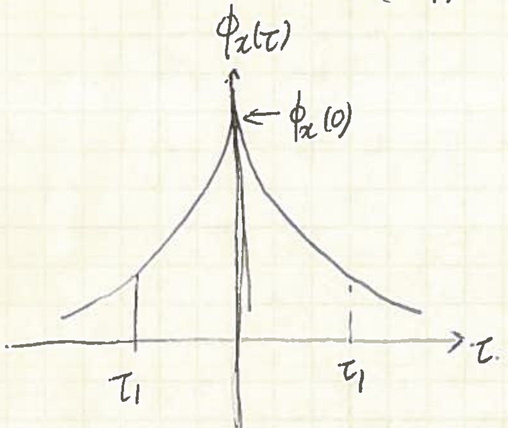
autocorrelation fn. $\phi_x(\tau) = \phi_x(0)\exp\left(-\frac{|\tau|}{\tau_1}\right)$ where τ_1 is a relaxation time const. (a system's memory time)

$\phi_x(0) = \underline{\langle x^2 \rangle}$ by definition

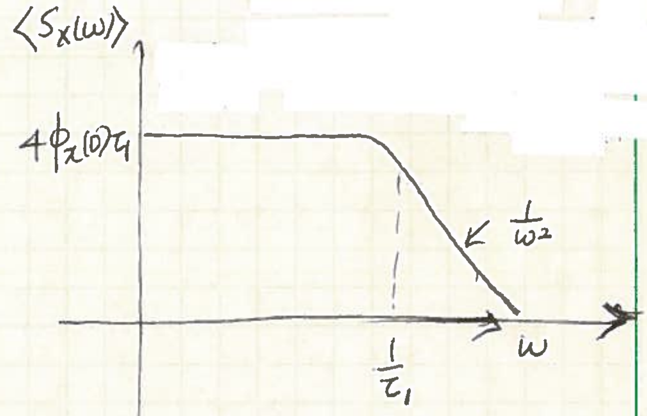
Q. what is the unilateral power spectral density ?

$$\langle S_x(\omega) \rangle = 4\int_0^\infty \phi_x(\tau)\cos(\omega\tau)\,d\tau = 4\int_0^\infty \phi_x(0)\exp\left(-\frac{|\tau|}{\tau_1}\right)\cos(\omega\tau)\,d\tau$$

$$\overset{HW}{=} \frac{4\phi_x(0)\tau_1}{1+(\omega\tau_1)^2}$$



auto correlation



power spectral density

if $|t_1 - t_2| \gg \tau_1$, ($|t_1-t_2| = \tau$)

no correlation btw $x(t_1)$ and $x(t_2)$.

⇓

$x(t)$ consists of indep. random processes in a time scale $\underline{\tau \gg \tau_1}$

if $\omega \ll \frac{1}{\tau_1}$,

$\langle S_x(\omega) \rangle$ is constant, freq.-indep. (white) noise spectral density

⇓

if $\omega \gg \frac{1}{\tau_1}$ ⇒

[the noisy waveform has no fluctuation component]

?

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Example 2

Consider a random walk diffusion process $x(t)$

A time-integrated ftn $y(t) = \int_0^t x(t')\,dt'$

where $x(t')$ is a statistically-stationary process.

$y(t)$ = a statistically-non-stationary process

Suppose $x(t)$ has an infinitisimally short correlation time τ_1 ($\tau_1 \Rightarrow 0$)

Then $y(t)$, a time-integrated waveform, is called a "Wiener-Levy process"

a classic example of a statistically non-stationary process

$\therefore$ Let's define a gated ftn by
$$y(t) = \begin{cases} \int_0^t x(t')\,dt' & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

Consider
$$\langle y(t)\, y(t+\tau)\rangle = \int_0^t \int_0^{t+\tau} \underbrace{\langle x(t')\, x(t'')\rangle}_{\text{covariance}} dt'dt''$$

if $x(t)$ is ergodic, covariance can be replaced by the Auto correlation ftn

$\downarrow$ W.K

using "$S_x(\omega)$ power spectral density

$$\stackrel{?}{=} \int_0^t \int_0^{t+\tau} dt'dt'' \frac{1}{2\pi}\int_0^\infty S_x(\omega)\cos(\omega\tau)\,d\omega$$

$$= \frac{1}{2\pi}\int_0^\infty S_x(\omega) \int_0^t \int_0^{t+\tau} \cos\omega(t'-t'')\,dt'dt''$$

$$= \frac{1}{2\pi}\int_0^\infty S_x(\omega)\frac{1}{\omega^2}\left[1+\cos(\omega\tau) - \cos(\omega t) - \cos(\omega(t+\tau))\right]d\omega.$$

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If $\tau=0$,

$$\underbrace{\langle y^2(t)\rangle}_{\text{mean-square}} = \frac{1}{\pi}\int_0^\infty S_x(\omega)\cdot\frac{1}{\omega^2}\left[1-\cos(\omega t)\right]d\omega$$

For the infinitesimally short correlation time process, (= memoryless noisy process)

$S_x(\omega)$ is indep. of ω. (i.e. white noise)

$$\therefore \langle y^2(t)\rangle = \frac{S_x(\omega=0)}{\pi}\int_0^\infty \frac{1}{\omega^2}\left[1-\cos(\omega t)\right]d\omega = \frac{S_x(\omega=0)}{2}t \equiv 2D_y t$$

↑ Using Mathematical identity

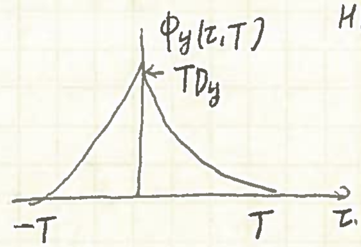
$$\lim_{a\to 0}\int_0^\infty \frac{1-\cos(\omega t)}{\omega^2+a^2}d\omega = \frac{\pi}{2}t$$

↑ HW.

$$D_y = \frac{S_x(\omega=0)}{4}$$ Diffusion constant of the Wiener-Levy process

The autocorrelation function for y is

$$\underset{\substack{\uparrow\\ \text{measurement}\\ \text{time window}}}{\phi_y(\tau, T)} = T\left(1-\frac{|\tau|}{T}\right)^2 D_y.$$

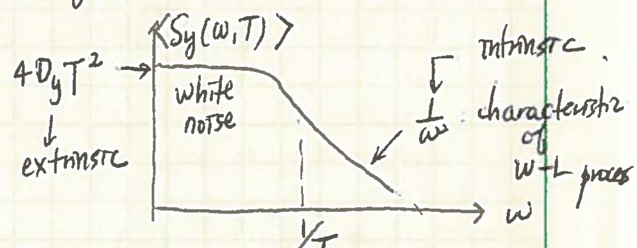


Note that $y(t)$ is a cumulative process of a memoryless process $x(t)$ ($\tau_c\to 0$)

$$\therefore \langle y(t+\tau)y(t)\rangle = \langle (y(t)+\Delta y(\tau))\,y(t)\rangle = \langle y(t)^2\rangle = 2D_y t$$

↑ assuming $\langle y(t)\Delta y(\tau)\rangle = 0$ due to zero-correlation

$$\therefore S_y(\omega, T) = \frac{8D_y}{\omega^2}\left[1-\frac{\sin(\omega T)}{\omega T}\right]$$



now "correlation time" is proportional to the measurement time window T

This finite time window "T" prevents the divergence of the power spectral density at $\omega=0$.

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Cross-correlation

↳ correlation between 2 noisy waveform $x(t)$, $y(t)$

The cross-correlation function

$$\phi_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \langle x(t+\tau)\, y(t)\rangle\, dt$$

The cross-spectral density

$$\langle S_{xy}(\omega)\rangle = \lim_{T\to\infty} \frac{2\langle X(i\omega)Y^*(i\omega)\rangle}{T} = \langle S_{yx}^*(\omega)\rangle$$

Using the Parseval theorem

$$\left.\begin{aligned} \phi_{xy}(\tau) &= \frac{1}{4\pi}\int_{-\infty}^{\infty} \langle S_{xy}(\omega)\rangle e^{i\omega\tau}\, d\omega \\ S_{xy}(\omega) &= 2\int_{-\infty}^{\infty} \phi_{xy}(\tau) e^{-i\omega\tau}\, dt \end{aligned}\right\} \text{generalized W-K theorem}$$

The degree of cross-correlation between $x(t)$ and $y(t)$ is given by the coherence ftn defined by

$$\Gamma_{xy}(\omega) = \frac{S_{xy}(\omega)}{\sqrt{S_{xx}(\omega)\, S_{yy}(\omega)}}$$

: "C-#"

∴ correlation amplitude
relative phase

the power-spectral density of $x(t)$ and $y(t)$ respectively

cf) $g^{(1)}(\tau) = \dfrac{\langle E^*(t)\, E(t+\tau)\rangle}{\langle |E(t)|^2\rangle}$

quantum optics.

$$g^{(1)}(\vec{r}_1, t_1; \vec{r}_2, t_2) = \frac{\langle E^*(r_1, t_1)\, E(r_2, t_2)\rangle}{\sqrt{\langle |E(\vec{r}_1, t_1)|^2\rangle \langle |E(\vec{r}_2, t_2)|^2\rangle}}$$

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· Basic Stochastic Processes

a noisy waveform $x(t) \Leftarrow$ represented by a very large # of random & discrete pulses

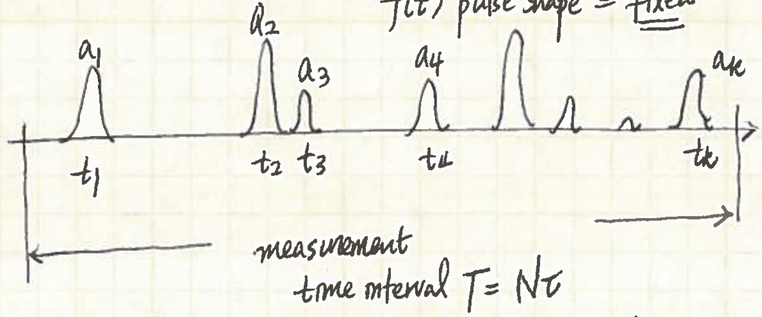
$$\therefore \quad x(t) = \sum_{k=1}^{K} a_k f(t - t_k)$$

$K \leftarrow$ large #

a_k ↑ pulse amplitude; t_k ↑ the time of pulse emission event t_k

assume : a_k, t_k = random variables.

$f(t)$ pulse shape = fixed



f(t) (pulse-shape) examples in a real physical situation

e.g. the relaxation time of a circuit; the transit time of a charged carrier } inherent property of the system determines $f(t)$

· Convenient Concept

① Probability mass fn $P(x)$

For a discrete random variable x,

the discrete

discrete transform

$$P_x^T(z) \equiv \sum_{x=0}^{\infty} z^x P(x) \quad z\text{-transform}$$

"moment generating fn"

mean values of any order moment

$$\langle x \rangle = \frac{d}{dz} P_x^T(z)\Big|_{z=1} \quad \text{1st-moment}$$

$$\langle x^2 \rangle = \frac{d^2}{dz^2} P_x^T(z) + \frac{d}{dz} P_x^T(z)\Big|_{z=1} \quad \text{2nd moment}$$

$$\boxed{\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2}$$

variance

$\vdots$

$$\frac{d^n}{dz^n} P_x^T(z)\Big|_{z=1} = \sum_{x=0}^{\infty} x(x-1)\cdots(x-n+1)P(x) = \langle x^n \rangle + \cdots$$

② Probability density fn $f(x)$

For a continuous random variable x

$$f_x^T(s) = \int_{-\infty}^{\infty} dx\, e^{-sx} f(x) \quad s\text{-transform}$$

$$\langle x \rangle = -\frac{d}{ds} f_x^T(s)\Big|_{s=0}$$

$$\langle x^2 \rangle = -\frac{d^2}{ds^2} f_x^T(s)\Big|_{s=0}$$

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Examples of Basic Stochastic Processes

Bernoulli Process
- Bernoulli distribution
- Binomial distribution
- Geometric distribution (X)

Poisson Process
- Poisson distribution
- Erlang distribution (X)
- Addition and random deletion of Poisson Processes (X)

Gaussian Process (continuous)
- Gaussian PDF
- Gaussian s-transform (X)