EQUALIZER AND COEQUALIZER

ABSTRACT. In this short article basic properties of equalizers and their dual concept of coequalizers are being explored. In particular, equalizers and coequalizers in **Set** are characterized. In the last part it is shown that subspace toplogy constructions correspond to equalizers in **Top**, whereas, dually, quotient topology constructions correspond to coequalizers.

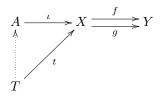
Equalizers (Coequalizers) are limits (colimits) over diagrams



with two objects and two parallel arrows.

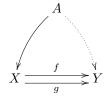
An equalizer of $X \xrightarrow{f \atop g} Y$ is therefore a pair (A, ι) , consisting of an object A and a morphism $A \xrightarrow{\iota} X$, such that

- (1) $f \circ \iota = g \circ \iota$
- (2) and for every other object T with a morphism $T \xrightarrow{t} X$, such that $f \circ \iota = g \circ \iota$, there exists a unique morphism $T \to A$ such that



commutes.

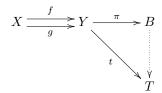
Note that a morphism $A \to Y$ need not be specified since it is already uniquely determined by the demand of commutativity of the following cone:



In the same way the requirement, that $f \circ \iota = g \circ \iota$, also follows directly from the limit property of (A, ι) .

A coequalizer of $X \xrightarrow{f} Y$ is then a pair (B, π) , consisting of an object B and a morphism $Y \xrightarrow{\pi} B$, such that

- (1) $\pi \circ f = \pi \circ g$
- (2) and for every other object T with a morphism $Y \xrightarrow{t} T$, such that $t \circ f = t \circ g$, there exists a unique morphism $B \to T$ such that



commutes.

Per definition the notion of equalizer is *dual* to the notion of coequalizer. From abstract nonsense we also know that any two equalizers resp. coequalizers are uniquely isomorphic.

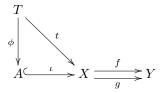
Equalizers and Coequalizers in Set

Given $X \xrightarrow{g} Y$ in **Set** we can easily form the equalizer of this diagram as (A, ι) , where

$$A = \{x \in X \mid f(x) = g(x)\}\$$

and $\iota:A\hookrightarrow X$ is the natural inclusion map from A into X.

Indeed, by definition $f \circ \iota = g \circ \iota$ holds, and given another set T, such that $f \circ t = g \circ t$, we conclude that $t(x) \in A \subset X$ (since f(t(x)) = g(t(x))!) and have the unique map $\phi: T \to A, x \mapsto t(x)$ such that



commutes.

Dually, we can form the coequalizer of $X \xrightarrow{f} Y$ as (B, π) , where $B = Y/\sim$ and π is the canonical projection map

$$Y \xrightarrow{\pi} Y/\sim$$

Here \sim is the equivalence relation that is generated by f(x) = g(x) for all $x \in X$.

Of course $\pi \circ f = \pi \circ g$ holds,

$$\pi(f(x)) = [f(x)] = [g(x)] = \pi(g(x))$$

and given another object T, with map $Y \xrightarrow{t} T$, such that $t \circ f = t \circ g$, we define the map

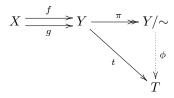
$$\phi: Y/\sim \to T, [y] \mapsto t(y)$$

and conclude that ϕ is well-defined:

Given $y \sim z$, then, by definition, there is a chain $y = a_0, a_1, ..., a_k = z$, such that for i in 1, ..., k

$$a_{i-1} = f(x)$$
 and $a_i = g(x)$, or $a_{i-1} = g(x)$ and $a_i = f(x)$ for x in X

Thus $t(y) = t(a_0) = t(a_1) = \dots = t(a_k) = t(z)$ and t is well-defined. Of course, ϕ is the only map such that



commutes.

Equalizer morphisms are monomorphisms

Let (A, ι) be an equalizer of $X \xrightarrow{f \atop g} Y$, then $\iota : A \to X$ is a monomorphism.

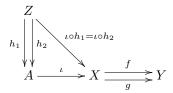
Proof.

Given two morphisms $h_1, h_2: Z \to A$, such that

$$\iota \circ h_1 = \iota \circ h_2$$

we need to show that $h_1 = h_2$.

This, however, follows directly from the universal property of (A, ι) ,



since

$$f \circ (\iota \circ h_1) = f \circ \iota \circ h_1 = g \circ \iota \circ h_1 = g \circ (\iota \circ h_1)$$

holds, there is only *one* morphism $Z \to A$ makes the above diagram commute, it follows that $h_1 = h_2$.

Dually, of course, coequalizer morphisms are epimorphisms.

In **Set** the converse holds:

Proposition 1.

If $A \xrightarrow{\iota} X$ is a monomorphism, (A, ι) is an equalizer.

Proof.

To see this consider X/\sim , where \sim is the induced equivalence relation on X by $\iota(a_1) = \iota(a_2)$ for all $a_1, a_2 \in A$.

Assuming that $A \neq \emptyset$, denote * as the equivalence class of any $a \in \operatorname{im} A$.

Then (A, ι) is an equalizer of $X \xrightarrow{\pi} X/\sim$, where π is the canonical projection and * is the *constant* function that maps every element of X onto * in X/\sim .

Indeed, $\pi \circ \iota = * \circ \iota$ holds, and given another object T with morphism $T \xrightarrow{t} X$, such that $\pi \circ t = * \circ t$, we contend that $t(x) \in \operatorname{im} A$ for all x in X.

For this assume $t(x) \notin \operatorname{im} A$, but then, by definition of \sim , $\pi(t(x)) = [t(x)] \neq *$, which contradicts the definition of t.

Thus we can define the map $\phi: T \to A, x \mapsto \iota^{-1}(t(x))$ and immediately verify that it satisfies the commutativity requirement. Also, ϕ is unique by the assumption that ι is a monomorphism.

If A is empty, (A, ι) , where ι is in this case the empty function, is the equalizer of $X \xrightarrow{f \atop g} X \times \{0,1\}$, where f(x) = (x,0) and g(x) = (x,1).

There is a similar construction for coequalizers:

Proposition 2.

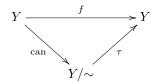
If $Y \xrightarrow{\pi} B$ is an epimorphism in **Set**, (B, π) is a coequalizer.

Proof.

We contend that (B,π) is a coequalizer of $Y \xrightarrow{\operatorname{id}_Y} Y$, where id_Y is the identity map and f is defined by $f(y) = \tau([y])$, where τ is any map

$$\tau: Y/\sim \to Y$$

such that $[\tau([y])] = [y]$.



Here, \sim is the equivalence relation induced by π ,

$$y_1 \sim y_2 \Leftrightarrow \pi(y_1) = \pi(y_2).$$

The map f therefore maps two elements y_1, y_2 in Y to the same $f(y_1) = f(y_2)$, if and only if $\pi(y_1) = \pi(y_2)$ and we have the identity

$$\pi(f(y)) = \pi(y).$$

The assertion is now easily verified. Firstly the equation above is exactly the demanded identity of $\pi \circ f = \pi \circ \mathrm{id}_Y$ and for the second property consider a set T and map $t: Y \to T$ such that t(f(y)) = t(y).

Since π is surjective and by the requirement to commute with π and t, the map $B \to T$ is already uniquely determined by $\pi(y) \mapsto t(y)$, and

it is well-defined: Given y_1, y_2 in Y with $\pi(y_1) = \pi(y_2)$, it follows that $f(y_1) = f(y_2)$ and thus

$$t(y_1) = t(f(y_1)) = t(f(y_2)) = t(y_2).$$

Equalizer and coequalier in Top

Equalizers in **Top** correspond to *subspace topology* constructions, whereas coequalizers correspond to *quotient topology* constructions.

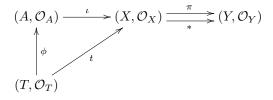
First consider equalizers. We will show that to every subspace topology construction we can find a diagram that the subspace topology is an equalizer of. Then, conversely, we verify that every equalizer in **Top** is already a subspace topology construction.

We start with a topological space (X, \mathcal{O}_X) and a subset A of X. Then the pair $((A, \mathcal{O}_A), \iota)$, consisting of the topological space (A, \mathcal{O}_A) and the map $\iota : A \to X$, where ι is the natural inclusion map of A in X and $\mathcal{O}_A = \{\iota^{-1}(U) \mid U \in \mathcal{O} : X\}$ is the subspace topology on A, is an equalizer of $(X, \mathcal{O}_X) \xrightarrow{\pi} (Y, \mathcal{O}_Y)$.

Here $Y = X/\sim$, π and * are constructed as in Proposition 1 and $\mathcal{O}_Y = \{U \subseteq Y \mid \pi^{-1} \in \mathcal{O}_X\}$ is the quotient topology on Y.

Indeed, from the construction in Proposition 1 we already know that (A, ι) is the set-theoretical equalizer of $(X, \mathcal{O}_X) \xrightarrow{\pi} (Y, \mathcal{O}_Y)$, so we just need to check the additional requirements in **Top**. The maps π and * are indeed continuous (and therefore proper morphisms in **Top**), since the canonical surjection π is continuous by definition of \mathcal{O}_Y and constant maps such as * are always continuous.

Given another topological space (T, \mathcal{O}_T) with a continuous map $T \xrightarrow{t} X$, such that $\pi \circ t = * \circ t$, we know from Proposition 1, that there is a unique map $\phi : T \to A$ such that



commutes. But since $\iota \circ \phi = t$ is continuous, from the characteristic property of \mathcal{O}_A it follows that $\phi : (T, \mathcal{O}_T) \to (A, \mathcal{O}_A)$ is continuous and thus a morphism in **Top**.

For the converse direction, let $((A, \mathcal{O}_A), \iota)$ be an equalizer of a diagram

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$$

in **Top**. We contend that the subspace topology construction $(\iota(A), \mathcal{O}_{\iota(A)})$, with the subset $\iota(A)$ in X, is already homeomorphic to (A, \mathcal{O}_A) .

This is easily verified by using the universal property of $((A, \mathcal{O}_A), \iota)$: We show that $((\iota(A), \mathcal{O}_{\iota(A)}), j)$, where j is the natural inclusion map, is itself an equalizer. Set theoretically, we see that $\iota': A \to \iota(A), a \mapsto \iota(a)$ is the unique bijection (ι') is injective) between A and $\iota(A)$ that makes

$$(\iota(A), \mathcal{O}_{\iota(A)}) \xrightarrow{j} (X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$$

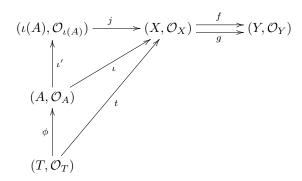
$$\downarrow^{\iota'} \qquad \qquad \downarrow^{\iota'} \qquad$$

commute. From

$$f(j(\iota'(x)) = f(\iota(x)) = g(\iota(x)) = g(j(\iota'(x)))$$

we conclude that $f \circ j = g \circ j$ holds.

Then, given any other topological space (T, \mathcal{O}_T) with continuous map $T \xrightarrow{t} X$, such that $f \circ t = g \circ t$ we get the unique continuous map $\phi : T \to A$ from the universal property of (A, \mathcal{O}_A) and thus the unique map $\iota' \circ \phi : T \to \iota(A)$ such that



commutes. Since $j \circ (\iota' \circ \phi) = t$ is continuous, and j is continuous by the definition of $\mathcal{O}_{\iota A}$, the map $\iota' \circ \phi$ is continuous by the characteristic property of the subspace topology.

Thus $(\iota(A), \mathcal{O}_{\iota(A)}, j)$ is itself an equalizer of

$$(X, \mathcal{O}_X) \xrightarrow{f \atop g} (Y, \mathcal{O}_Y)$$

and hence have a homeomorphism between (A, \mathcal{O}_A) and $(\iota(A), \mathcal{O}_{\iota(A)})$. The equalizer (A, \mathcal{O}_A) may therefore be viewed as a subspace topology construction.

In the dual case of coequalizers we proceed analoguously. First consider a quotient topology construction $(B, \mathcal{O}_{\text{Quot}(\pi)})$ to a surjective map $\pi: Y \to B$ and a topology (Y, \mathcal{O}_Y) on Y.

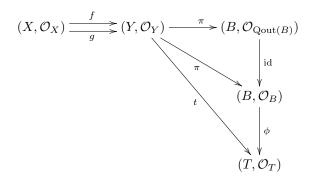
From Proposition 2, we obtain a map f such that $(B, \mathcal{O}_{\text{Quot}(\pi)})$ is a coequalizer of the diagram

$$(Y, \mathcal{O}_Y) \xrightarrow{\operatorname{id}_Y} (Y, \mathcal{O}_Y)$$

in **Set**, and contend that this is also true in **Top**. Again, $\pi \circ f = \pi \circ id_Y$ of course still holds and, as in the case of equalizers, the continuity of the unique map $B \to T$ follows from the characteristic property of the quotient topology.

Conversely, given a coequalizer $((B, \mathcal{O}_B), \pi)$ of $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ we show that $(B, \mathcal{O}_{\text{Quot}(\pi)})$ is a coequalizer of the same diagram.

Here $\pi \circ f = \pi \circ g$ holds by definition for $(B, \mathcal{O}_{Quot(\pi)})$ and the universal property is obtained from the diagram



and the characteristic property of the quotient topology.