EQUALIZER AND COEQUALIZER

Abstract. In this short article basic properties of equalizers and their dual concept of coequalizers are being explored. In particular, equalizers and coequalizers in **Set** are characterized. In the last part it is shown that subspace toplogy constructions correspond to equalizers in **Top**, whereas, dually, quotient topology constructions correspond to coequalizers.

Equalizers (Coequalizers) are limits (colimits) over diagrams

$$\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}$$

with two objects and two parallel arrows.

An equalizer of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is therefore a pair (A, ι), consisting of an object A and a morphism $A \xrightarrow{\iota} X$, such that

(1) $f \circ \iota = g \circ \iota$

(2) and for every other object T with a morphism $T \xrightarrow{t} X$, such that $f \circ \iota = g \circ \iota$, there exists a unique morphism $T \to A$ such that

$$\begin{tikzcd} A \arrow[r, "\iota"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ T \arrow[u, dotted] \arrow[ur, "t"'] & & \end{tikzcd}$$

commutes.

Note that a morphism $A \to Y$ need not be specified since it is already uniquely determined by the demand of commutativity of the following cone:

$$\begin{tikzcd} & A \arrow[dl, bend right] \arrow[dr, dotted, bend left] & \\ X \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & Y \end{tikzcd}$$

In the same way the requirement, that $f \circ \iota = g \circ \iota$, also follows directly from the limit property of (A, ι).

A coequalizer of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is then a pair (B, π), consisting of an object B and a morphism $Y \xrightarrow{\pi} B$, such that

(1) $\pi \circ f = \pi \circ g$

(2) and for every other object T with a morphism $Y \xrightarrow{t} T$, such that $t \circ f = t \circ g$, there exists a unique morphism $B \to T$ such that

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\pi"] \arrow[dr, "t"'] & B \arrow[d, dotted] \\ & & T \end{tikzcd}$$

commutes.

Per definition the notion of equalizer is *dual* to the notion of coequalizer. From abstract nonsense we also know that any two equalizers resp. coequalizers are uniquely isomorphic.

Equalizers and Coequalizers in Set

Given $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in **Set** we can easily form the equalizer of this diagram as (A, ι), where

$$A = \{x \in X \mid f(x) = g(x)\}$$

and $\iota : A \hookrightarrow X$ is the natural inclusion map from A into X.

Indeed, by definition $f \circ \iota = g \circ \iota$ holds, and given another set T, such that $f \circ t = g \circ t$, we conclude that $t(x) \in A \subset X$ (since $f(t(x)) = g(t(x))$!) and have the unique map $\phi : T \to A, x \mapsto t(x)$ such that

$$\begin{tikzcd} T \arrow[d, "\phi"'] \arrow[dr, "t"] & & \\ A \arrow[r, hook, "\iota"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

commutes.

Dually, we can form the coequalizer of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ as (B, π), where $B = Y/\!\sim$ and π is the canonical projection map

$$Y \xrightarrow{\pi} Y/{\sim}$$

Here $\sim$ is the equivalence relation that is generated by $f(x) = g(x)$ for all $x \in X$.

Of course $\pi \circ f = \pi \circ g$ holds,

$$\pi(f(x)) = [f(x)] = [g(x)] = \pi(g(x))$$

and given another object T, with map $Y \xrightarrow{t} T$, such that $t \circ f = t \circ g$, we define the map

$$\phi : Y/{\sim} \to T, [y] \mapsto t(y)$$

and conclude that ϕ is well-defined:

Given $y \sim z$, then, by definition, there is a chain $y = a_0, a_1, ..., a_k = z$, such that for i in $1, ..., k$

$$\begin{aligned} &a_{i-1} = f(x) \text{ and } a_i = g(x), \text{ or} \\ &a_{i-1} = g(x) \text{ and } a_i = f(x) \text{ for } x \text{ in } X \end{aligned}$$

Thus $t(y) = t(a_0) = t(a_1) = ... = t(a_k) = t(z)$ and t is well-defined. Of course, ϕ is the only map such that

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, two heads, "\pi"] \arrow[dr, "t"'] & Y/{\sim} \arrow[d, dotted, "\phi"] \\ & & T \end{tikzcd}$$

commutes.

Equalizer morphisms are monomorphisms

Let (A, ι) be an equalizer of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, then $\iota : A \to X$ is a monomorphism.

Proof.

Given two morphisms $h_1, h_2 : Z \to A$, such that

$$\iota \circ h_1 = \iota \circ h_2$$

we need to show that $h_1 = h_2$.

This, however, follows directly from the universal property of (A, ι),

$$\begin{tikzcd} Z \arrow[d, shift right, "h_1"'] \arrow[d, shift left, "h_2"] \arrow[dr, "\iota \circ h_1 = \iota \circ h_2"] & & \\ A \arrow[r, "\iota"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

since

$$f \circ (\iota \circ h_1) = f \circ \iota \circ h_1 = g \circ \iota \circ h_1 = g \circ (\iota \circ h_1)$$

holds, there is only *one* morphism $Z \to A$ makes the above diagram commute, it follows that $h_1 = h_2$. $\square$

Dually, of course, *coequalizer morphisms are epimorphisms.*

In **Set** the converse holds:

Proposition 1.

If $A \xrightarrow{\iota} X$ is a monomorphism, (A, ι) is an equalizer.

Proof.

To see this consider $X/\sim$, where $\sim$ is the induced equivalence relation on X by $\iota(a_1) = \iota(a_2)$ for all $a_1, a_2 \in A$.

Assuming that $A \neq \emptyset$, denote $*$ as the equivalence class of any $a \in \operatorname{im} A$.

Then (A, ι) is an equalizer of $X \overset{\pi}{\underset{*}{\rightrightarrows}} X/\sim$, where π is the canonical projection and $*$ is the *constant* function that maps every element of X onto $*$ in $X/\sim$.

Indeed, $\pi \circ \iota = * \circ \iota$ holds, and given another object T with morphism $T \xrightarrow{t} X$, such that $\pi \circ t = * \circ t$, we contend that $t(x) \in \operatorname{im} A$ for all x in X.

For this assume $t(x) \notin \operatorname{im} A$, but then, by definition of $\sim$, $\pi(t(x)) = [t(x)] \neq *$, which contradicts the definition of t.

Thus we can define the map $\phi : T \to A, x \mapsto \iota^{-1}(t(x))$ and immediately verify that it satisfies the commutativity requirement. Also, ϕ is unique by the assumption that ι is a monomorphism.

If A is empty, (A, ι), where ι is in this case the empty function, is the equalizer of $X \underset{g}{\overset{f}{\rightrightarrows}} X \times \{0, 1\}$, where $f(x) = (x, 0)$ and $g(x) = (x, 1)$.

$\square$

There is a similar construction for coequalizers:

Proposition 2.

If $Y \xrightarrow{\pi} B$ *is an epimorphism in* ***Set****,* (B, π) *is a coequalizer.*

Proof.

We contend that (B, π) is a coequalizer of $Y \underset{f}{\overset{\mathrm{id}_Y}{\rightrightarrows}} Y$, where id_Y is the identity map and f is defined by $f(y) = \tau([y])$, where τ is any map

$$\tau : Y/\sim \to Y,$$

such that $[\tau([y])] = [y]$.

$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr, "\mathrm{can}"'] & & Y \\
& Y/\sim \arrow[ur, "\tau"'] &
\end{tikzcd}$$

Here, $\sim$ is the equivalence relation induced by π,

$$y_1 \sim y_2 \Leftrightarrow \pi(y_1) = \pi(y_2).$$

The map f therefore maps two elements y_1, y_2 in Y to the same $f(y_1) = f(y_2)$, if and only if $\pi(y_1) = \pi(y_2)$ and we have the identity

$$\pi(f(y)) = \pi(y).$$

The assertion is now easily verfied. Firstly the equation above is exactly the demanded identity of $\pi \circ f = \pi \circ \mathrm{id}_Y$ and for the second property consider a set T and map $t : Y \to T$ such that $t(f(y)) = t(y)$.

Since π is surjective and by the requirement to commute with π and t, the map $B \to T$ is already uniquely determined by $\pi(y) \mapsto t(y)$, and it is well-defined: Given y_1, y_2 in Y with $\pi(y_1) = \pi(y_2)$, it follows that $f(y_1) = f(y_2)$ and thus

$$t(y_1) = t(f(y_1)) = t(f(y_2)) = t(y_2).$$

$\square$

Equalizer and coequalier in Top

Equalizers in **Top** correspond to *subspace topology* constructions, whereas coequalizers correspond to *quotient topology* constructions.

First consider equalizers. We will show that to every subspace topology construction we can find a diagram that the subspace topology is an equalizer of. Then, conversely, we verify that every equalizer in **Top** is already a subspace topology construction.

We start with a topological space $(X, \mathcal{O}_X)$ and a subset A of X. Then the pair $((A, \mathcal{O}_A), \iota)$, consisting of the topological space $(A, \mathcal{O}_A)$ and the map $\iota : A \to X$, where ι is the natural inclusion map of A in X and $\mathcal{O}_A = \{\iota^{-1}(U) \mid U \in \mathcal{O} : X\}$ is the subspace topology on A, is an *equalizer* of $(X, \mathcal{O}_X) \overset{\pi}{\underset{*}{\rightrightarrows}} (Y, \mathcal{O}_Y)$.

Here $Y = X/\!\sim$, π and $*$ are constructed as in Proposition 1 and $\mathcal{O}_Y = \{U \subseteq Y \mid \pi^{-1} \in \mathcal{O}_X\}$ is the quotient topology on Y.

Indeed, from the construction in Proposition 1 we already know that (A, ι) is the set-theoretical equalizer of $(X, \mathcal{O}_X) \overset{\pi}{\underset{*}{\rightrightarrows}} (Y, \mathcal{O}_Y)$, so we just need to check the additional requirements in **Top**. The maps π and $*$ are indeed continuous (and therefore proper morphisms in **Top**), since the canonical surjection π is continuous by definition of $\mathcal{O}_Y$ and constant maps such as $*$ are always continuous.

Given another topological space $(T, \mathcal{O}_T)$ with a continuous map $T \xrightarrow{t} X$, such that $\pi \circ t = * \circ t$, we know from Proposition 1, that there is a unique map $\phi : T \to A$ such that

$$\begin{tikzcd}
(A, \mathcal{O}_A) \arrow[r, "\iota"] & (X, \mathcal{O}_X) \arrow[r, shift left, "\pi"] \arrow[r, shift right, "*"'] & (Y, \mathcal{O}_Y) \\
(T, \mathcal{O}_T) \arrow[u, "\phi"] \arrow[ur, "t"'] & &
\end{tikzcd}$$

commutes. But since $\iota \circ \phi = t$ is continuous, from the characteristic property of $\mathcal{O}_A$ it follows that $\phi : (T, \mathcal{O}_T) \to (A, \mathcal{O}_A)$ is continuous and thus a morphism in **Top**.

For the converse direction, let $((A, \mathcal{O}_A), \iota)$ be an equalizer of a diagram

$$(X, \mathcal{O}_X) \overset{f}{\underset{g}{\rightrightarrows}} (Y, \mathcal{O}_Y)$$

in **Top**. We contend that the subspace topology construction $(\iota(A), \mathcal{O}_{\iota(A)})$, with the subset $\iota(A)$ in X, is already homeomorphic to $(A, \mathcal{O}_A)$.

This is easily verified by using the universal property of $((A, \mathcal{O}_A), \iota)$: We show that $((\iota(A), \mathcal{O}_{\iota(A)}), j)$, where j is the natural inclusion map, is itself an equalizer. Set theoretically, we see that $\iota' : A \to \iota(A)$, $a \mapsto \iota(a)$ is the unique bijection (ι' is injective) between A and $\iota(A)$ that makes

$$\begin{tikzcd}
(\iota(A), \mathcal{O}_{\iota(A)}) \arrow[r, "j"] & (X, \mathcal{O}_X) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, \mathcal{O}_Y) \\
(A, \mathcal{O}_A) \arrow[u, "\iota'"] \arrow[ur, "\iota"'] & &
\end{tikzcd}$$

commute. From

$$f(j(\iota'(x)) = f(\iota(x)) = g(\iota(x)) = g(j(\iota'(x)))$$

we conclude that $f \circ j = g \circ j$ holds.

Then, given any other topological space $(T, \mathcal{O}_T)$ with continuous map $T \xrightarrow{t} X$, such that $f \circ t = g \circ t$ we get the unique continuous map $\phi : T \to A$ from the universal property of $(A, \mathcal{O}_A)$ and thus the unique map $\iota' \circ \phi : T \to \iota(A)$ such that

$$\begin{tikzcd}
(\iota(A), \mathcal{O}_{\iota(A)}) \arrow[r, "j"] & (X, \mathcal{O}_X) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, \mathcal{O}_Y) \\
(A, \mathcal{O}_A) \arrow[u, "\iota'"] \arrow[ur, "\iota"'] & & \\
(T, \mathcal{O}_T) \arrow[u, "\phi"] \arrow[uur, "t"'] & &
\end{tikzcd}$$

commutes. Since $j \circ (\iota' \circ \phi) = t$ is continuous, and j is continuous by the definition of $\mathcal{O}_{\iota A}$, the map $\iota' \circ \phi$ is continuous by the characteristic property of the subspace topology.

Thus $(\iota(A), \mathcal{O}_{\iota(A)}, j)$ is itself an equalizer of

$$\begin{tikzcd}
(X, \mathcal{O}_X) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, \mathcal{O}_Y)
\end{tikzcd}$$

and hence have a homeomorphism between $(A, \mathcal{O}_A)$ and $(\iota(A), \mathcal{O}_{\iota(A)})$. The equalizer $(A, \mathcal{O}_A)$ may therefore be viewed as a subspace topology construction.

In the *dual* case of coequalizers we proceed analoguously. First consider a quotient topology construction $(B, \mathcal{O}_{\mathrm{Quot}(\pi)})$ to a surjective map $\pi : Y \to B$ and a topology $(Y, \mathcal{O}_Y)$ on Y.

From Proposition 2, we obtain a map f such that $(B, \mathcal{O}_{\mathrm{Quot}(\pi)})$ is a coequalizer of the diagram

$$\begin{tikzcd} (Y, \mathcal{O}_Y) \arrow[r, shift left, "\mathrm{id}_Y"] \arrow[r, shift right, "f"'] & (Y, \mathcal{O}_Y) \end{tikzcd}$$

in **Set**, and contend that this is also true in **Top**. Again, $\pi \circ f = \pi \circ \mathrm{id}_Y$ of course still holds and, as in the case of equalizers, the continuity of the unique map $B \to T$ follows from the characteristic property of the quotient topology.

Conversely, given a coequalizer $((B, \mathcal{O}_B), \pi)$ of $\begin{tikzcd} (X, \mathcal{O}_X) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, \mathcal{O}_Y) \end{tikzcd}$ we show that $(B, \mathcal{O}_{\mathrm{Quot}(\pi)})$ is a coequalizer of the same diagram.

Here $\pi \circ f = \pi \circ g$ holds by definition for $(B, \mathcal{O}_{\mathrm{Quot}(\pi)})$ and the universal property is obtained from the diagram

$$\begin{tikzcd}
(X, \mathcal{O}_X) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, \mathcal{O}_Y) \arrow[r, "\pi"] \arrow[dr, "\pi"] \arrow[ddr, "t"'] & (B, \mathcal{O}_{\mathrm{Qout}(B)}) \arrow[d, "\mathrm{id}"] \\
& & (B, \mathcal{O}_B) \arrow[d, "\phi"] \\
& & (T, \mathcal{O}_T)
\end{tikzcd}$$

and the characteristic property of the quotient topology.

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