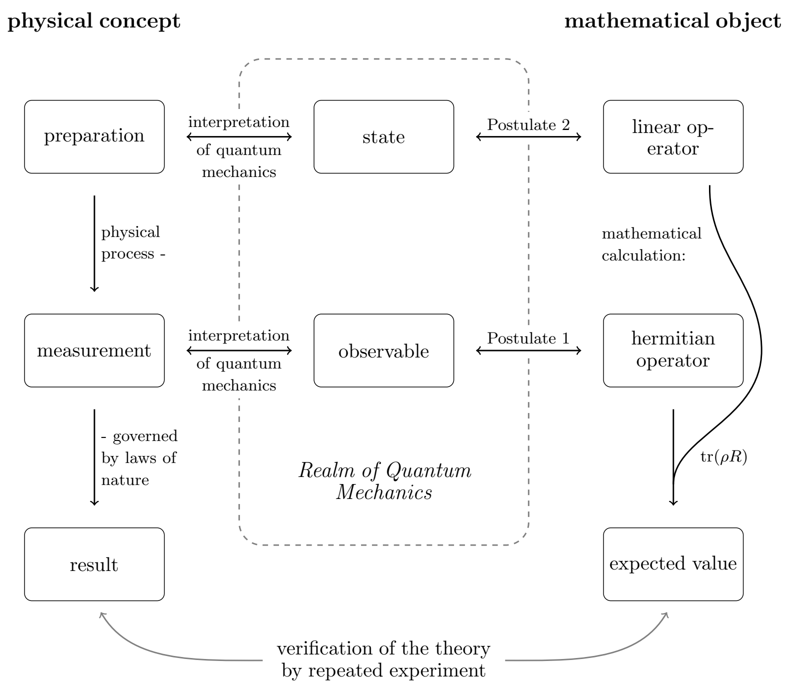
AN OUTLINE OF QUANTUM MECHANICS

Quantum mechanics is, viewed purely as a mathematical theory, a generalisation of probability theory (a *non-commutative* probability theory). Application to a concrete problem however, reduces the theory to a probability theory in the classical sense [1, p. 280]. Quantum Mechanics thus (only) aspires to yield expectation values and probability distributions for the outcomes of statistical experiments.

In such an experiment, two phases that may be distinguished: **Preparation** and **measurement**. This separation comes naturally, the two processes are essentially independent: The same measurement procedure may be used on differently prepared particles, as much as different measurements may be performed on identically prepared particles.



The quantum mechanical concept connected to measurement is that of an **observable**. Since not all abstract quantities inherent to the system might be measurable (the trajectory of an electron is such an example), one declares as an observable only those variables of a system that can in principle be measured [2].

In a classical theory, the range of possible values for such a variable is commonly assumed to be continuous. Considering the energy levels of electrons in atoms however, we know that a proper quantum theory also needs to account for a discrete spectrum of possible values of measurement.

Postulate 1: *The mathematical object assigned to the physical observable shall be a linear operator. It's spectrum shall reflect the possible values of the dynamical variable.*

This makes sense in so far that there are linear operators that have a continuous spectrum and there are such that have a discrete spectrum.

The concept tied to preparation is that of a **state**. The preparation determines the probability distribution for the outcomes of a following measurement. In fact, it is *independent* of the measurement and necessarily determines the probabilities for *any* subsequent measurement. We identify the state with the specification of a probability distribution for every observable.

Postulate 2: *To each state corresponds a unique linear operator* ρ *with* $\mathrm{tr}(\rho) = 1$*. The expected value for the observable* R *in the state* ρ *is given by*

$$\mathrm{tr}(\rho R).$$

Given a physical experiment, we are able to translate it into the realm of quantum mechanics by selecting a state, representing the preparation procedure, and an observable, representing the measurement procedure. Postulates 1 and 2 then correspond these quantum mechanical concepts to mathematical objects. The postulates, however, yield more than just this identification, in the mathematical realm we obtain a *probability theory*. This result is known as the *spectral theorem* and has seen numerous proofs [2, 3].

Using this theory, we write the observable R as

$$R = \int \lambda \, dE_R(\lambda),$$

where the map $E_R(\cdot)$, which assigns a linear operator to every measurable subset of $\mathbb{R}$, is called the *projection valued measure* of R. Then the probability of the measurement outcome lying in an interval B of $\mathbb{R}$, for a system prepared in the state ρ, is given by

$$\Pr(B) = \mathrm{tr}(\rho \, E_R(B)).$$

Having transfered the physical experiment to a probability theory, we are now able to test our quantum theory by repeating the experiment and comparing the results with the expected values, calculated from by the probability theory.

It should be noted, that, while we utilize the intricate connection between probability and frequency to verify our theory, the frequentist view on probability [4] might not be too helpful when trying for an interpretation of quantum mechanics. Leslie Ballentine advises to adopt the propensity interpretation instead [5, p. 32], while John Baez suggests that the Bayesian interpretation might be preferable [6].

The two postulates given are, however, not sufficient to account for all aspects of quantum mechanics. As opposed to the classical case, measurement in quantum mechanics strongly influences the outcomes of subsequent measurements. The principle of *state reduction* has been axiomatized by von Neumann [7]:

Postulate 3: *After obtaining eigenvalue r of the observable R as a result of measurement of a system prepared in the state ρ, the new state of the system is given by*

$$\rho' = \frac{P_r\,\rho\,P_r}{\mathrm{tr}(P_r\,\rho)},$$

where P_r is the projection operator onto the eigenspace of r.

Dynamics

So far we didn't really do justice to the "mechanics-part" in quantum mechanics. Dynamics enter the picture, when we look at the time evolution of a given system. As the corresponding states and observables already completely determine the (quantum mechanical) behaviour of this system, only a time evolution of those can account for dynamical properties. In the *Schrödinger picture* the states evolve in time, in the *Heisenberg picture*, it is the observables that depend on a time parameter. Whether time dependence of observables or states is causing dynamics is only philosophically interesting at best, mathematically the formulations are shown to be equivalent.

Time evolution in either case is implemented by a *time evolution operator* $U(t, t_0)$. The operator should evolve states ρ via

$$\rho(t) = U^\dagger(t, t_0)\,\rho(t_0)\,U(t, t_0),$$

and do this continuously so. Furthermore, to guarantee probability conversation, U needs to be unitary. Is H the Hamilton operator for a given system, the time evolution operator satisfies

$$i\hbar\partial_t U(t, t_0) = HU(t, t_0),$$

the *Schrödinger equation*. At this point quantum mechanics stops being a purely mathematical formulation and becomes a proper physical theory: to obtain results that can be verified by experiment, the Hamiltonian H must be chosen in such a way, that it represents the given physical system.

References

[1] M. Schürmann et al. *Quantum Potential Theory*. Lecture Notes in Mathematics. Springer, 2008.

[2] Stefan Waldmann. *Operator-Algebraic Methods in Quantum Mechanics*. 2010. URL: https://perswww.kuleuven.be/~u0077076/Lectures/Darstellung1011/main.pdf.

[3] Michael Reed and Barry Simon. *I: Functional Analysis, Volume 1 (Methods of Modern Mathematical Physics) (vol 1)*. 1st ed. Academic Press, Jan. 1981.

[4] Alan Hájek. "Interpretations of Probability". In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Summer 2003. 2003. URL: http://plato.stanford.edu/archives/sum2003/entries/probability-interpret/.

[5] Leslie E Ballentine. *Quantum Mechanics: A Modern Development*. World Scientific, Singapore, 2003.

[6] John Baez. *Bayesian Probability Theory and Quantum Mechanics*. 2003. URL: http://math.ucr.edu/home/baez/bayes.html.

[7] John Von Neumann. *Mathematical Foundations of Quantum Mechanics*. Princeton Landmarks in Mathematics and Physics. Princeton University Press, 1996.