Special Relativity #2

What's "special" about Special Relativity

Einstein first came up with a "theory of relativity" that solved the "speed of light problem" and modified Newton's theory of motion. But Newton's theory of gravity couldn't be accommodated in the theory. It took Einstein about 10 more years to address gravity. The result was General Relativity.

The original theory was then called "special" relativity because it was only a special (no gravity) case of the more general theory.

Special relativity is very simple mathematically, even though the consequences of the theory imply a dramatic change to our notions of space and time. Special relativity could be (and perhaps is?) easily taught in a high school physics class. General relativity is *much* more complicated mathematically, and is typically taught to university physics majors only at a graduate level (if at all).

The postulates of special relativity

1. The speed of light is the same in all inertial frames of reference.
2. There is no experiment that can be done, internal to a given intertial frame of reference, that can distinguish that frame from any other frame.

Or to say it another way:

1. The speed of light is absolute.
2. But, Galilean relativity still holds!

The implications of a fixed speed of light

How is it possible to keep the speed of light fixed? We consider two observers in two different reference frames, who watch a photon (a "particle" of light) bounce off a mirror. Even though the two frames are moving relative to each other, we simply *assume* that both observers see the photon going the same speed. This does *not* produce a logical contradiction, as long as we are willing to concede that *the two observers experience a different duration of time* during the journey of the photon. This is called *time dilation.*

It is important to understand that this is not some sort of illusion, or subjective experience. The duration of time in question is actually different in the two different reference frames.

With a similar argument we showed that lengths in space are different for two different observers. This is known as *length contraction.*

I don't have the pictures we drew on the board, but I've tried to document the sequence of steps so that you can (hopefully) reproduce the calculations at home, if you like.

But here's a key point: *Durations of time and lengths of space are, in a way, like vectors along the axes of an arbitrary coordinate system. You can change their relative magnitude by changing the coordinates.*

Time Dilation

1. Draw the two reference frames, Alice has the mirror. Bob and Alice both agree on: c (the speed of light), v (the magnitude of their relative velocity), and h (the height of the mirror).

2. Alice bounces a photon of light off the mirror. She sees the photon's path as having a length of $2h$ traveled in time t_a. In other words, Alice measures the elapsed time as: $t_a = \frac{2h}{c}$.

3. Bob measures time t_b for the photon go from Alice to the mirror and back. But during that time he sees Alice's entire frame of reference move toward the right at velocity v. So he sees an equilateral triangle with a base of length vt_b.

4. Forming a right triangle from half of that base, with the height h as the other side, then during time $t_b/2$ Bob sees the photon travel the hypotenuse, which has length $\sqrt{h^2 + \frac{1}{4}v^2t_b^2}$.

5. Given that Bob saw the photon travel that same hypotenuse in time $t_b/2$, we conclude that $\frac{1}{2}ct_b = \sqrt{h^2 + \frac{1}{4}v^2t_b^2}$.

6. Solving for t_b

   $\frac{1}{4}c^2t_b^2 = h^2 + \frac{1}{4}v^2t_b^2$ (square both sides)

   $\frac{1}{4}c^2t_b^2 - \frac{1}{4}v^2t_b^2 = h^2$ (subtract term)

   $\frac{1}{4}t_b^2(c^2 - v^2) = h^2$ (factor left side)

   $t_b^2 = \frac{4h^2}{c^2 - v^2}$ (divide)

   $t_b = \frac{2h}{\sqrt{c^2 - v^2}}$ (take positive square root)

7. Since Alice was present at both "events" (explain) her time measurement is considered the *proper time*, noted as t_0, between those events.

8. We are looking for a conversion factor between the proper time and Bob's time: $t_0 \times ? = t_b$. That is:

   $\frac{2h}{c} \times ? = \frac{2h}{\sqrt{c^2 - v^2}}$

9. We can cancel the $2h$ from both sides, and then clearly: $\frac{1}{c}\left(\frac{c}{1}\frac{1}{\sqrt{c^2 - v^2}}\right) = \frac{1}{\sqrt{c^2 - v^2}}$

   Simplifying, we have the conversion factor: $\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

   And the general formula relating a proper time t_0 to the time t of any observer who does not see the proper time.

   $t = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}t_0$

10. What we can conclude from this is that the proper time is always the the shortest time observed. Any other time will appear to be longer (and hence "dilated").

Length Contraction

1. Now put two observers on either end of the ship and one on the ground.
2. The observers on the ship each note the time that they pass the observer on the ground and they calculate t_s as the difference between the two.
3. The observer on the ground notes the time between the passage of the two ends of the ship and calls that t_g.
4. The "events" here are the times that each end of the ship passed the observer on the ground. So that observer sees the proper time: $t_g = t_0$.
5. Hence $t_s = \frac{t_g}{\sqrt{1-\frac{v^2}{c^2}}}$.
6. The observers on the ship calculate $L_s = v \times t_s$ an the one on the ground calculates $L_g = v \times t_g$.
7. You might think that L_g would be the "proper" length because it was calculated with the proper time, but no. The *proper length* is defined as the length of the object measured in *that object's own frame of reference* (which makes sense when you think about it, since this is the only way of defining it uniquely).
8. We want to find the multiplier which will take us from the proper lentgh $L_0 = L_s$ to the other length L_g.
9. So we want: $v\frac{t_g}{\sqrt{1-\frac{v^2}{c^2}}} \times (?) = vt_g$
10. Cancel vt_g from both sides and it's obvious that we have to multiply by $\sqrt{1-\dfrac{v^2}{c^2}}$.
11. So: $L = L_0\sqrt{1-\dfrac{v^2}{c^2}}$.

Summary

1. The *proper time between two events*, t_0, is the time measured by an observer who is present at both events.
2. The *proper length of an object*, L_0, is the length as measured in the object's own frame of reference.
3. $t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}$.
4. $L = L_0\sqrt{1-\dfrac{v^2}{c^2}}$.