Quantum Study Group Notes

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Part I

Math Concepts

1 Vectors

Quantum theory is based on *linear algebra.* Linear algebra is a very rich topic, having applications far beyond what we are going to discuss here. But for our purposes we can think of linear algebra as the mathematics of *vectors* and *matrices*. The next few sections will explain what this means.

1.1 What is a vector. Columns and rows

You can think of a vector as a list of numbers. We can have both *column vectors* and *row vectors*. Suppose that a vector contains the numbers 1 and 2. Then we could have either

the column vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$,

or the row vector $\begin{pmatrix} 1 & 2 \end{pmatrix}$.

Note that the first element of the column vector is on top, and the first element of the row vector is on the left. In other words, the elements are numbered either top to bottom or left to right.

Vectors can be added simply by adding the individual elements:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1+3 \\ 2+4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 7 & 9 \end{pmatrix}$$

You can add two row vectors or two columns vectors, but you can't add a row vector to a column vector.

Hopefully it's clear from the example above that vector addition is *commutative* (it doesn't matter in which order you add the vectors).

You can also multiply a vector by a number. In linear algebra the term *scalar* is used to distinguish plain numbers from vectors. So a scalar is just a regular old number. You can

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multiply a scalar and a vector by separately multiplying each of the vector's elements by the scalar. This type of multiplication is commutative:

$$5\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}5 = \begin{pmatrix} 15 \\ 20 \end{pmatrix}$$

Exercise 1.1.1

Add the vectors $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Exercise 1.1.2

Add the vectors $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$.

Exercise 1.1.3

What do you get if you multiply $\begin{pmatrix} 3 & 2 \end{pmatrix}$ by three and then subtract $\begin{pmatrix} 1 & 2 \end{pmatrix}$?

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1.2 Naming things. Dirac (bra-ket) notation

If you've encountered vectors before, say in a math or engineering book, you may have seen them written in either a bold font, or with little arrows above them. They were probably written as row vectors, perhaps with a comma between the elements. Various types of brackets, braces, or parentheses might have been used. For example:

$$\mathbf{v} = (x, y, z)$$

$$\vec{v} = \{5, 22, 17\}$$

$$\mathbf{r} = [r, \phi, \theta]$$

$$\vec{\mathbf{v}} = (v_1, v_2, v_3)$$

Note that, in the last example, *subscripts* are used instead of giving each element of the vector its own name.

In these notes, I am (mostly) going to reserve the use of single letter variable names for scalars: $a = 3$, $x = \sqrt{13}$, $\theta = \pi/2$, and so on. Since the individual elements of vectors are scalars, they will also be represented by letters. Often the elements will have a similar name as the vector, with subscripts to distinguish them.

For the names of vectors I will be using what is called Dirac or Bra-Ket notation. It was invented by Paul Dirac in the early part of the 20th century. This notation consists of the symbol $\langle\ |$ called a "bra", and the symbol $|\ \rangle$ called a "ket." Put them together: $\langle\ |\ \rangle$ and you get a "bra-ket" or bracket. A ket is a column vector and a bra is a row vector:

$$|a\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\langle b| = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}$$

Exercise 1.2.1

Let $|x\rangle = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$, and $|y\rangle = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

Find: $|x\rangle + |y\rangle$, $\langle x| - \langle y|$, $|y\rangle - |x\rangle$, and $2|x\rangle - 3|y\rangle$.

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1.3 Vector multiplication, the Inner Product

Another thing that can be done to a vector is the *transpose* operation. If you start with a row vector and transpose it you get a column vector. If you transpose a column vector you get a row. The transpose operation is indicated by a capital "T" written as a *superscript* on the vector:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^T = \begin{pmatrix} a_1 & a_2 \end{pmatrix}$$

$$\begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}^T = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

So in bra-ket notation: $|v\rangle^T = \langle v|$ and $\langle v|^T = |v\rangle$.

Now let's talk about multiplying vectors. With regular numbers (scalars) there is only one kind of *product*. If $a = 3$ and $b = 4$ then $ab = (3)(4) = 12$. But when it comes to vectors there is more than one kind of product. I'll be talking about other kinds before long, but right now I want to define a kind of vector multiplication called the *inner product*. To take the inner product of two vectors, the first one must be a row vector and the second one a column vector. Then you multiply each element of the first vector with the corresponding element of the second vector add all the results together, like this:

$$\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} = (1)(6) + (2)(5) + (3)(4) = 6 + 10 + 12 = 28$$

Note that the inner product of two vectors is *not a vector*. It's just a number. Some books refer to the inner product as the "scalar product" since the result is a scalar.

In Dirac notation, the inner product is a bra-ket. In other words, the inner product of $\langle a|$ and $|b\rangle$ is simply written as $\langle a|b\rangle$. But when you need to find the actual numeric value then you'll write them out as a row and column to do the multiplication.

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A couple of things to be aware of with regard to the inner product. First of all, the two vectors need to have the same number of elements. Otherwise you can't multiply the "corresponding" elements :-) Also, you might ask why the row vector has to come first. Why isn't this operation commutative? Please just accept for now that (in general) it isn't. The full explanation for why it is done this way will unfold as we get to other kinds of products and to complex numbers.

Exercise 1.3.1

Find the transpose of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Exercise 1.3.2

Find $\begin{pmatrix} a & b & c \end{pmatrix}^T$.

Exercise 1.3.3

Suppose $|x\rangle = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$, $|y\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and$|z\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$.

Find the inner products: $\langle x|y\rangle$, $\langle x|z\rangle$ and $\langle z|z\rangle$.

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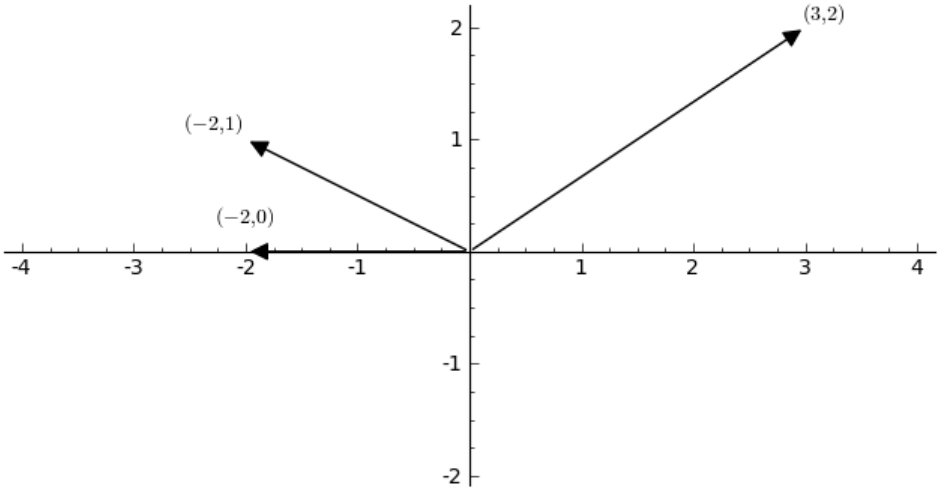
1.4 The absolute value or norm

When you have a vector with two elements, it can be visualized as an arrow on a two dimensional plane. The tail of the arrow is placed at the origin and the first and second elements of the vector give the arrow head's x and y coordinates. For example:



Now let's talk about the term *absolute value.* You might have previously thought about the absolute value of a number as something like "the number with the sign stripped off" or possibly learned a formula like: if x is positive then $|x| = x$ but if x is negative then $|x| = -x$. Now I want you to think about absolute value as the *length* of a vector. If the vector lies along the x axis, like the vector $(-2, 0)$ above, then it should be clear that the length of the vector corresponds to the definition of absolute value for a scalar (just "strip" the sign off the 2). If the vector doesn't lie along one of the axes, then we have to find its length using the "Pythagorean formula" where the length equals the square root of the sums of the squares of the x and y coordinates. So the length of the vector on the right side of the diagram is: $\sqrt{3^2 + 2^2} = \sqrt{13}$. The length of the upper one on the left side is: $\sqrt{(-2)^2 + 1^2} = \sqrt{5}$.

In general, if you have a vector $\langle v| = (v_1, v_2, ...v_n)$ then its length is: $\sqrt{v_1^2 + v_2^2 + ... + v_n^2}$. It doesn't matter whether it's a row or column vector, and this same formula will work even if the vector does happen to lie along one of the axes. For our purposes, the terms *absolute value*, *norm* and *modulus* all mean the same thing. They all refer to the length of a vector.

Exercise 1.4.1

Find the absolute values of: $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$, $\begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$

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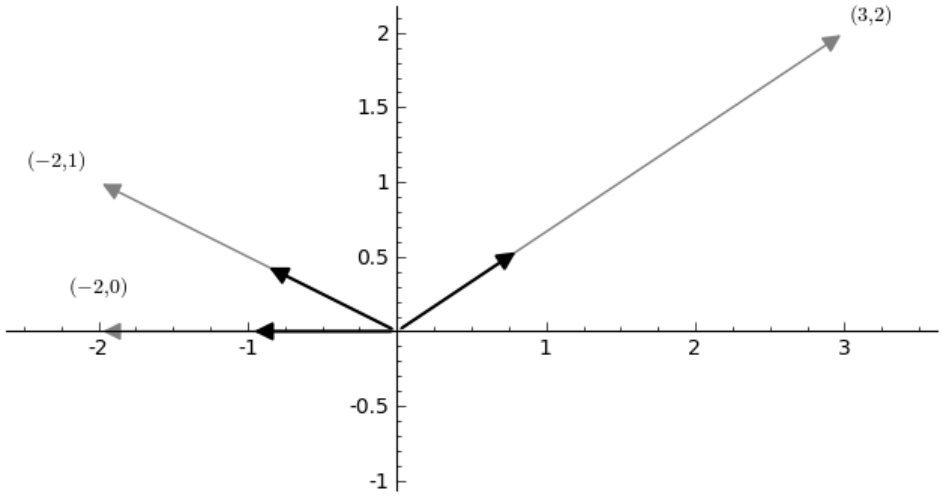
1.5 Normalization

Often we don't care about the length of a vector. We just want to know what direction the vector is pointing. In situations like this, what is needed are directional vectors that all have lengths of 1. To *normalize* a vector is to make its length equal to 1 without changing the direction in which it points. Normalization is accomplished simply by dividing each element of the vector by the vector's length. For example, if you have the vector: $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ then its length is $\sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, and the normalized vector is: $\frac{1}{\sqrt{14}} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

Here are the normalized vectors from the previous page:



Since the length of a vector is often called its norm, the process of dividing by the length is called normalization.

Exercise 1.5.1

Normalize the vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Exercise 1.5.2

Find the numeric values for the three normalized vectors in the diagram above.

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1.6 Expressing the length of a vector using the inner product

The length of a vector can be expressed as the square root of the inner product of the vector with itself:

$$\text{length of } |v\rangle = \sqrt{\langle v|v\rangle}$$

To see why this is true, suppose that:

$$|v\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

Then:

$$\sqrt{\langle v|v\rangle} = \sqrt{\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}} = \sqrt{a^2 + b^2}$$

It follows that we can express the normalized vector $|v\rangle$ as:

$$\frac{|v\rangle}{\sqrt{\langle v|v\rangle}}$$

The formula for the length of a vector using the inner product is not that important right now, but it will become significant when complex numbers are introduced.

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1.7 Vector spaces

A vector *space* is an abstract mathematical space in which all of the vectors we're talking about "live." For example, when we draw the x/y coordinate system on the board, "all the vectors we're talking about" are the arrows that can be drawn (starting from the origin) on the board. The vector space they "live" in is the 2-D plane. When we talk about "regular old" 3 dimensional space, then "all the vectors" are any arrow that you can draw in that space. So, for "real" spaces like this, the vector space is just a mathematical representation of the physical space. Later we will encounter other kinds of vector spaces, but that's enough for now.

1.8 Dimension

A vector space has a dimension, and an individual vector has a dimension. The dimension of a vector is the same as the dimension of the space that it lives in. You could think of the dimension as any of the following:

1. The number of elements in the vector.
2. The number of physical dimensions of the space (if it's a physical space).
3. The number of basis vectors necessary to specify a basis for the space.

All three of the above are equivalent.

1.9 Linear Combination

A *linear combination* of vectors is something like this:

$$a|v_1\rangle + b|v_2\rangle + c|v_3\rangle$$

where a, b, and c are *scalars* (which means they're just numbers, not vectors). In other words, a linear combination of vectors is a bunch of vectors multiplied by something and added together. The only reason you need to know this is because the term "linear combination" is used all over the place.

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1.10 Linear Independence

A vector $|v_1\rangle$ is *linearly independent* of the vectors $|v_2\rangle$ and $|v_3\rangle$ if there is no possible linear combination such that:

$$|v_1\rangle = a|v_2\rangle + b|v_3\rangle$$

A couple of examples:

1. Draw the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ on a piece of paper. (The arrow from the origin the the point 1 on the x-axis.) Then draw the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (The arrow going to the point x=2, y=3.) The vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is linearly independent of the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ because there is nothing you can multiply $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by to get $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

2. Now add the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to the drawing (the arrow going straight up the y-axis). The vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is *not* linearly independent of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ together because you can add them together in a linear combination to get $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$:

   $$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

1.11 Bases and Basis Vectors

A *basis* for a given vector space is a *set* of vectors which can be used, in a linear combination, to make up *any* vector in the space. In other words, there are no vectors in the space which are linearly independent of the (full set of) basis vectors.

Typically, a vector space will have an infinite number of different *bases* that you can choose from. There is often one "obvious" basis, such as the two vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the 2-D plane.

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1.12 Orthonormality

Two vectors are *orthogonal* to one another if their inner product is zero.
That is, $|v\rangle$ and $|w\rangle$ are orthogonal if:

$$\langle v|w\rangle = \langle w|v\rangle = 0$$

In a physical space, two orthogonal vectors have a 90° angle between them.

A vector can be called either *normal* or *normalized* if its length is 1.

A basis is *orthonormal* if all the basis vectors are normalized and each basis vector is orthogonal to every other basis vector.

1.13 Projection

A vector can be thought of as a list of *components*. For example the row vector $\begin{pmatrix} a & b & c \end{pmatrix}$ has the three components: "a", "b", and "c." Each component is simply a multiplier for one of the basis vectors.

For example, take the vector: $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ on the 2-D plane.

This vector is a linear combination of the basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The "2" is just the multiplier for the first basis vector and the "3" is the multiplier for the 2nd. If you "carry out the operations" from the formula above, it will go like this:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Another way to look at this is that the "2" is the vector's *projection* on to the x-axis and the "3" is the projection on to the y-axis. In this case "projection" just means drawing a line straight over from the head of the vector to the axis of interest.

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If you have a vector $|v\rangle$ and a basis vector $|b\rangle$, then you can obtain the projection of $|v\rangle$ onto $|b\rangle$ by using the inner product, like this:

$$(\text{the projection}) = \langle b|v\rangle$$

The projection gives you one of the components of $|v\rangle$.

Let's try this with the basis vectors (1 0) and (0 1) and the vector (2 3). As you may have noticed, it's not always convenient to create column vectors when you're typing. So I'm just specifying the vectors in row format. But if I put them in a ket, then they're a column and if I put them in a bra, then they're a row. So, let's proceed:

The *basis* consists of two vectors:

$$b_1 = (1\ 0) \text{ and } b_2 = (0\ 1)$$

And the vector of interest is:

$$v = (2\ 3)$$

We find the first *component* of v by projecting it onto the first basis vector:

$$\langle b_1|v\rangle = 2$$

And the 2nd component by projecting it onto the 2nd basis vector:

$$\langle b_2|v\rangle = 3$$

Obviously, I'm not actually writing out all the steps here, so *please actually work out the two inner products above.*

1.14 Change of Basis

The inner products in the last section were really trivial, and might have seemed pointless. But this becomes more interesting when you want to change from one basis to another. I'm not going to try to write about this in detail here, but I'll give a short explanation. Try it out, if you can. In any event, we'll discuss it in detail on the board next Sunday.

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Another possible basis for the 2-D plane is the set of vectors:

$b_1 = (1/\sqrt{2},\ \ 1/\sqrt{2})$ and $b_2 = (1/\sqrt{2},\ \ -1/\sqrt{2})$.
(I put in the commas here just to make things more clear.)

In case you don't recognize them, these were the "blue" basis vectors from last Sunday. Suppose that we want to "change to the blue basis." We can get the two components of the vector $v = (2\ 3)$ in the new basis by doing the two projections:

$\langle b_1|v\rangle = 5/\sqrt{2}$
$\langle b_2|v\rangle = -1/\sqrt{2}$

This means that, in the "blue" basis, the vector v would be

$$\left(\frac{5}{\sqrt{2}},\ \ \frac{-1}{\sqrt{2}}\right).$$

Or, equivalently, it means that we know how to construct v out of the "blue" basis vectors by doing the following linear combination:

$$\begin{aligned} v &= \frac{5}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \ \frac{1}{\sqrt{2}}\right) + \frac{-1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \ \frac{-1}{\sqrt{2}}\right) \\ &= \left(\frac{5}{2},\ \ \frac{5}{2}\right) + \left(\frac{-1}{2},\ \ \frac{1}{2}\right) \\ &= \left(\frac{4}{2},\ \ \frac{6}{2}\right) \\ &= \begin{pmatrix} 2 & 3 \end{pmatrix} \end{aligned}$$

Here is one more thing. The general formula for the calculation of an arbitrary vector's components in an arbitrary basis. Given that we have the basis: $\{b_n\}$ (meaning that we have a basis made out of the set of basis vectors: b_1, b_2, ... and so on), then the nth component of v is:

$$v_n = \langle b_n|v\rangle$$

and the entire vector v is simply the sum of all those components times the corresponding basis vectors:

$$v = \sum_n \langle b_n|v\rangle|b_n\rangle$$

This formula is one that you will see often in textbooks.

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2 Matrices

This is just a *very* brief summary of what we've covered in person.

2.1 What is a matrix?

A matrix consists of *rows* and *columns*. The following matrix is named M and it has 2 rows and three columns:

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

You could think of row 1 as the row vector $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and you could think of column 3 as the column vector $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$.

When we specify an individual element of a matrix, the first index always refers to the row and the second index to the column. So, in the matrix above:

$M_{12} = 2$
$M_{23} = 6$

We will be interested in the following matrix concepts:

1. Addition
2. Subtraction
3. Multiplication
4. Transpose
5. Identity
6. Inverse

Addition and subtraction are easy. You simply add (or subtract) corresponding elements (just like we did with vectors).

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

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2.2 Matrix Multiplication

To multiply two matrices, you take the inner product of every row in the first matrix with every column in the second matrix. In other words, if you have three matrices A, B, and C:

$$A\,B = C$$

Then element C_{jk} is equal to the inner product of row j of A and column k of C.

Matrix multiplication is *not* commutative.

2.3 The Transpose of a Matrix

The transpose operation of a matrix M is denoted by M^T. Transposing a matrix turns row n into column n and column n into row n. When we transpose M the element M_{jk} becomes the element M_{kj}:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

2.4 The Identity Matrix

The *identity matrix* is a square matrix with ones in the main diagonal and zeros everywhere else:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The identity matrix is the matrix version of the number 1. If you multiply any matrix by the identity matrix it doesn't change. The identity matrix is denoted by the letter I.

2.5 The Inverse of a Matrix

The *inverse* of a matrix is denoted by M^{-1}. Not all matrices have an inverse. If a matrix *does* have an inverse, then the matrix times the inverse equals the identity matrix:

$$M\,M^{-1} = I$$

This doesn't tell you how to *find* the inverse of a matrix, but it will enable you to test whether a given matrix is the inverse of another one.

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2.6 Multiplying a vector by a matrix. The Operator concept

Matrices are used for many different purposes, in different branches of mathematics. Our chief use of matrices will be as *operators* on vectors. When we multiply a matrix times a column vector, we are using the matrix as an operator on the vector:

output vector = matrix × input vector

The following are some examples in the 2D vector space, where it's easy to visualize what's happening. (You should draw a picture of each of these!)

"Minus" Operator: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}$

The minus operator gives the negative of the input vector, the same as multiplying it by -1.

X Reflection Operator: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}$

Reflects the vector about the x-axis.

Y Reflection Operator: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ b \end{pmatrix}$

Reflects the vector about the y-axis.

Rotation Operator: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$

Rotates the vector by θ degrees. (If you don't know trigonometry don't worry about this one. We'll discuss it in more detail if we ever end up using it.)

Identity Operator: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

Naturally, if we multiply a vector by the Identity Matrix, it doesn't change at all (which is why it's called the Identity Matrix).

Here's one last one, and this is one that we *will* be using:

The Not Operator: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$

This operator exchanges the 2 components of the vector. I'll explain later why this exchange of components is called a "Not" operation.

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2.7 Eigenvalues and Eigenvectors.

Sometimes when we operate on a vector with a matrix, what we get back is simply a multiple of the original vector. In other words: $M\vec{v} = k\vec{v}$ where k is some scalar. Here's an example where $M\vec{v} = (3)\vec{v}$. (That is, $k = 3$.)

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix} = (3) \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Here's another one where $M\vec{v} = (-1)\vec{v}$. (So, $k = -1$):

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

When the pattern $M\vec{v} = k\vec{v}$ occurs, then we say that $\vec{v}$ is an *eigenvector* of the matrix M and k is the associated *eigenvalue*.

2.8 The Outer Product

As you may remember, the inner product is a row vector times a column vector. You can also think of this as multiplying a matrix consisting of a single row with a matrix consisting of a single column. The inner product yields a scalar, which is an object whose "rank" is one less than the objects that were multiplied together.

$$\text{Let } a = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$\langle a|b\rangle = \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = 8 + 15 = 23$$

The outer product is a ket times a bra, which would be a column times a row. What does this mean? If you think of it as a single column matrix times a single row matrix, you will see that the result is a matrix. An object whose rank is one *greater* than the objects that were multiplied together.

$$|a\rangle\langle b| = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \begin{pmatrix} 4 & 5 \end{pmatrix} = \begin{pmatrix} (2)(4) & (2)(5) \\ (3)(4) & (3)(5) \end{pmatrix} = \begin{pmatrix} 8 & 10 \\ 12 & 15 \end{pmatrix}$$

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3 Complex Numbers

3.1 The number i

We say that "the number i" is the square root of negative one, or $i = \sqrt{-1}$.

There is nothing deep here. It's not as if we've somehow *discovered* what the square root of minus one is, and it turned out to be i. We have simply made a *definition*. We define a number. We name it i. Then we make a rule that when you square it you get minus one. $i^2 = -1$.

Note that i is not a *variable* like x or y. It is a *constant*. It names a specific number. It just happens to be a number that is not on "the number line."

The numbers on the number line are called the *real* numbers. Therefore, i is called an *imaginary* number. The terms "real" and "imaginary" are unfortunate. Cows and horses are real animals. Griffins and unicorns (so far as I know) are imaginary animals. It's not like that with "real" and "imaginary" numbers. They are all equally real in the normal English sense of the word. But we're stuck with these names.

3.2 Basic Definitions

1. Any number on the number line is a *real number.*

   Examples are: $-100,\ -2.4,\ 0,\ 1,\ \sqrt{2},\ e,\ \pi,\ \dfrac{927}{13}$.

2. An *imaginary number* is i multiplied by any real number.

   For example: $2i$, $\dfrac{1}{2}i$, $(\sqrt{2})i$, and $\dfrac{3i}{7}$ are all imaginary numbers.

3. A *complex number* is a number that has a real part and an imaginary part.
   Here are some examples:

   $2 + 3i$ (the real part is 2 and the imaginary part is $3i$)

   $\sqrt{2}$ (the real part is $\sqrt{2}$ and the imaginary part is 0)

   i (the real part is 0 and the imaginary part is i)

As you can see, the complex numbers include the real and imaginary numbers. So, when we talk about "complex numbers" we are talking about real numbers, imaginary numbers, and numbers that are made up out of both real and imaginary parts.

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It is typical to refer to a complex number by the variable name z and we say that: $z = a + b\,i$, where a and b are real. This is the same thing as definition 3 above.

I should also mention the functions Re() and Im(). You will see these functions (or variations on them) in various books. They will certainly be implemented in any programming languages that support complex numbers. If you have a complex variable $z = a + b\,i$. Then the expression Re(z) returns a, and Im(z) returns b. Note that Re(z) returns the real part of z. But Im(z) does not return what I am calling the imaginary part. It returns only the *real number associated with the imaginary part.*

3.3 Operations on Complex Numbers

Complex numbers have all the operations that real numbers do, including addition, subtraction, multiplication, division, absolute value, powers, roots, and so on. In addition, they have one more operation called *conjugation.*

For the time being, we will only be concerned with:

1. Addition
2. Subtraction
3. Multiplication
4. Conjugation
5. Absolute Value (or Modulus)
6. Division

Addition and subtraction are easy. You just add (or subtract) the corresponding parts of the number:

$$(2 + 3i) + (1 + 2i) = (2 + 1) + (3i + 2i) = 3 + 5i$$
$$(2 + 3i) - (1 + 2i) = (2 - 1) + (3i - 2i) = 1 + i$$

And, in general:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

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3.4 Multiplication

Multiplying complex numbers is just like multiplying polynomials:

$$
\begin{aligned}
&(a_1 + b_1 i)(a_2 + b_2 i) \\
&= a_1 a_2 + a_1 b_2 i + b_1 i a_2 + b_1 i\, b_2 i \quad \text{(multiply out all the terms)} \\
&= a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2 i^2 \quad \text{(rearrange things a bit)} \\
&= a_1 a_2 + a_1 b_2 i + a_2 b_1 i - b_1 b_2 \quad \text{(i squared is -1)} \\
&= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i \quad \text{(gather real and imag parts together)}
\end{aligned}
$$

Here are some examples. See if you can get the same results.

$$(1 + i)(2 + 3i) = -1 + 5i$$

$$(2 + 3i)(3 + 2i) = 13i$$

$$\left(\frac{1}{2} i\right)(1 - i) = \frac{1}{2} + \frac{i}{2}$$

3.5 Conjugation

If we have a complex number: $a + bi$ then the *complex conjugate* of that number is: $a - bi$. In other words, we simply "flip the sign" of the imaginary part. Complex conjugation is denoted in some books by a * and in other books by a bar over the number. (Either a star or a bar.)

Let $z = a + bi$, then:

$$z^* = (a + bi)^* = a - bi$$

$$\bar{z} = \overline{a + bi} = a - bi$$

I will most likely use the *. Here are some examples:

$$z = 1 + i, \quad z^* = 1 - i$$

$$z = -5 + \sqrt{2}\, i, \quad z^* = -5 - \sqrt{2}\, i$$

$$z = -5i, \quad z^* = 5i$$

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3.6 Absolute Value or Modulus

The *absolute value* of a complex number is "the square root of the number times its conjugate."

$|z| = \sqrt{zz^*}$

If $z = a + bi$, then: $|z| = |a + bi| = \sqrt{(a + bi)(a - bi)} = \sqrt{a^2 + b^2}$ (work it out!)

Just like the absolute value of a real number or a vector, *the absolute value of a complex number is always a positive real number.*

Terminology note: As the absolute value of a vector is somtimes called a "norm" the absolute value of a complex number is sometimes called a "modulus". The only reason you need to know about the terms "norm" and "modulus" is that you may run into them in books or papers. It's OK just to think of all these things as "absolute values."

3.7 Division

The best way to deal with division of complex numbers is to "rationalize" the denominator. This can be done by multiplying both the numerator and denominator by the conjugate of the denominator.

If I tell you to divide $3 + 2i$ by $1 + i$, then that means you have the fraction:

$$\frac{3 + 2i}{1 + i}$$

The denominator is $1 + i$ and its conjugate is $1 - i$. So, you multiply both the numerator and denominator by $1 - i$.

$$\frac{3 + 2i}{1 + i}\,\frac{1 - i}{1 - i}$$

Note that, since you are actually multiplying the original fraction by 1, you are not changing its value. You're just putting it in a different form.

$$\frac{3 + 2i}{1 + i}\,\frac{1 - i}{1 - i} = \frac{(3 + 2i)(1 - i)}{(1 + i)(1 - i)} = \frac{5 - i}{2} = \frac{5}{2} - \frac{1}{2}i$$

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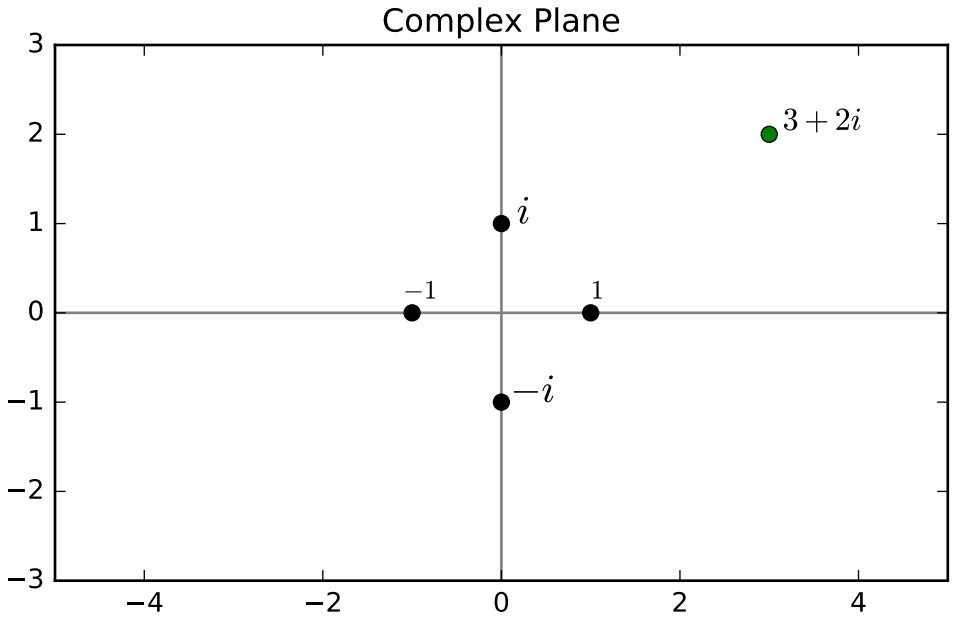
3.8 Plotting numbers on the complex plane

Complex numbers can be depicted graphically by plotting them on a *complex plane* (sometimes referred to as an *Argand diagram*). Plots like this can help to get a better feeling for what complex numbers are all about.

Consider the following diagram:



The horizontal line is the *real axis* and the vertical line is the *imaginary axis*. We have plotted the four points 1, -1, i, and $-i$ in black. Note that each of those four points is one unit away from the origin.

In order to plot a complex number you simply use its real part as the coordinate on the real axis, and its imaginary part as its coordinate on the imaginary axis. The point $3 + 2i$ has been plotted in green.

A complex number's absolute value (or modulus) is just its distance from the origin on the complex plane. The absolute value of $3 + 2i$ is $(3 + 2i)(3 - 2i) = \sqrt{13}$ or roughly 3.6. That's how far the green point is from the origin.

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3.9 Complex number problems

Exercise 3.9.1

Given: $a = i,\ b = 5,\ c = 1 + i,\ d = 3 - 2i,\ e = 2i - 1,$ Find the following:

(a) $a + b$

(b) $c + d$

(c) $c - d$

(d) $a\,c$

(e) $b\,c$

(f) $c\,d$

(g) The complex conjugate of a

(h) b^*

(i) c^*

(j) e^*

(k) $c\,c^*$

(l) $|d|$

(m) The modulus of e

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4 Complex Vector Spaces

A complex vector space is one where the vectors (and associated matrices) contain complex numbers as their elements. So the numbers inside a vector can be real numbers, imaginary numbers, or they can have both real and imaginary parts. Here's a dimension three vector in a complex space:

$$\begin{pmatrix} 2+3i \\ 10 \\ i\sqrt{3} \end{pmatrix}$$

4.1 The Adjoint Operator

In complex spaces, the *adjoint operator* becomes very important. The adjoint is denoted by a "dagger" (and is sometimes called the "dagger operator").

The adjoint of M is $M^\dagger$

The adjoint is a combination of the transpose and the complex conjugate operators. Whether we are taking the adjoint of a vector or a matrix, we transpose it and we take the complex conjugate of each of its elements. It doesn't matter whether we transpose or conjugate first.

The adjoint of a column vector:

$$\begin{pmatrix} a \\ b \end{pmatrix}^\dagger = \begin{pmatrix} a^* & b^* \end{pmatrix}$$

The adjoint of a row vector:

$$\begin{pmatrix} a & b \end{pmatrix}^\dagger = \begin{pmatrix} a^* \\ b^* \end{pmatrix}$$

The adjoint of a matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

Examples:

$$\begin{pmatrix} 2 & i \end{pmatrix}^\dagger = \begin{pmatrix} 2 \\ -i \end{pmatrix}, \qquad \begin{pmatrix} 2-3i \\ 1+i \end{pmatrix}^\dagger = \begin{pmatrix} 2+3i & 1-i \end{pmatrix}$$

$$\begin{pmatrix} 2 & i \\ 2-i & 1+2i \end{pmatrix}^\dagger = \begin{pmatrix} 2 & 2+i \\ -i & 1-2i \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

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4.2 The adjoint operator changes some things!

The introduction of the adjoint operator in a complex vector space means we have to make a few changes.

We previously defined a bra as being the transpose of a ket. This was sufficient for real vector spaces. But in a complex vector space, the bra is actually the adjoint of a ket.

$$\text{If } |v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \text{then } \langle v| = \begin{pmatrix} v_1^* & v_2^* \end{pmatrix}$$

This enhanced definition of a bra changes the way the inner product works. The calculations we have previously done with the inner product are correct, but only because all the vector elements were real. With the addition of complex numbers, the inner product becomes:

$$\langle a|b\rangle = \begin{pmatrix} a_1^* & a_2^* \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1^* b_1 + a_2^* b_2$$

Notice that the inner product is no longer commutative:

$$\langle b|a\rangle = \begin{pmatrix} b_1^* & b_2^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 b_1^* + a_2 b_2^* \neq \langle a|b\rangle$$

The calculation of the length of a vector also changes. At this point it is better to calculate the length (or norm) of a vector using the inner product:

$$\text{length of } |v\rangle = \sqrt{\langle v|v\rangle}$$

This formula will yield the correct length even when the components are complex. For example, take the vector:

$$|v\rangle = \begin{pmatrix} 1+i \\ 1-2i \end{pmatrix}$$

Then:

$$\text{length of } |v\rangle = \sqrt{\langle v|v\rangle} = \sqrt{\begin{pmatrix} 1-i & 1+2i \end{pmatrix} \begin{pmatrix} 1+i \\ 1-2i \end{pmatrix}}$$

$$= \sqrt{(1-i)(1+i) + (1+2i)(1-2i)} = \sqrt{(1+1) + (1+4)} = \sqrt{7}$$

So we get a length which is a positive real number, as we must.

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4.3 The Hermitian Matrix

A *Hermitian* matrix H is one where $H = H^\dagger$. In other words, *A Hermitian matrix is equal to its own adjoint.*

Examples:

(1) $\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ is Hermitian.

When you transpose it you get $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$.

Then, when you conjugate it you get $\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$.

Which is equal to the original matrix.

(2) $\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ is not Hermitian.

When you transpose it you get $\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$.

Then, when you conjugate it you get $\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$.

Which is not equal to the original matrix.

(3) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is Hermitian.

When you transpose it you still have $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

And conjugating it doesn't change it.

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4.4 The Unitary Matrix

A *unitary* matrix is one where $U^\dagger = U^{-1}$. In other words *The adjoint of a unitary matrix is also its inverse.*

(It would be nice to find a couple of examples that are unitary but *not* Hermitian.)

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Part II

Quantum Theory

5 The Postulates

In quantum theory, there are four basic "rules of the game." These are the same postulates or axioms that you will see in many quantum mechanics books. Here they have been simplified to apply to the situations we are going to study.

1. The state of a quantum system is described by a column vector with complex components, for example: $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.
2. As long as we don't measure a system, a change in its state can be modeled by multiplying the current state by a unitary matrix.
3. The measurement of a system is represented using a Hermitian matrix. The result of the measurement is one of the eigenvalues of the matrix, and the resulting state is one of the eigenvectors of the matrix.
4. If a system is in the state $\alpha|a\rangle + \beta|b\rangle$, the probability of a measurement resulting in state a is $|\alpha|^2$ and the probability of a measurement resulting in state b is $|\beta|^2$. If a system is in the state $|\psi\rangle$, the probability of a measurement resulting in the state $|\phi\rangle$ is $|\langle\phi|\psi\rangle|^2$.

Don't worry if these four paragraphs don't make much sense. We'll be investigating them in detail in the next few sections. In fact, one way to look at it is that learning about quantum theory is primarily about fully grasping the consequences of the postulates.

So later on, when you are confronted by a problem that you have no idea how to work, go back and ask: what do the postulates have to say about this situation?

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6 Electron Spin

Our interest in electron spin stems partially from the fact that it can be used to model (or perhaps even implement) a single quantum bit, and it is simple enough that we can more or less visualize it.

When we talk about "spin" we are, in a sense, picturing the electron as a little ball spinning around. Of course, a real electron is nothing like this. This picture is simply a way of constructing a physical model we can visualize. This will help get a handle on the mathematical model, which otherwise would be pretty abstract. *Also please note that the "electron spin" we are talking about here is somewhat simplified from the true spin model you might study in a regular physics course, as we're ignoring elements that aren't necessary for our purposes here.*

So, think of an electron as a ball spinning around in three dimensional space. Suppose that you are above the electron looking down at it. Suppose that you see it spinning in a counter clockwise direction, from your perspective. Then we say the spin is "up" (that is toward you). If you see it spinning clockwise, then the spin is "down."

Another way to say this is that the "spin vector" obeys the so called "right hand rule." If you take an x-y-z coordinate system, and you curl the fingers of your right hand in the direction the ball is spinning, then if you stick out your thumb it points in the direction of the spin vector. So, for example, if your fingers go from the x axis toward the y axis, then the spin points up along the z axis.

When we write down the "state" of the electron spin, we do it like this:

$|+z\rangle$ means the spin vector points along the positive z axis (up)

$|-z\rangle$ means the spin vector points along the negative z axis (down)

$|+x\rangle$ the spin is along the positive x axis

$|-y\rangle$ the spin is along the negative y axis

and so on ...

Now, what happens when we *measure* the spin? Well, if the electron were really a little ball, then it would go like this: We measure the spin along, say, the x axis. But the ball isn't spinning *exactly* in that direction. So our measurement might show a spin 30 degrees off the x axis. But with a real electron this never happens. No matter what direction we measure the spin, the result we get is always exactly in that direction, or exactly in the opposite direction. So, if we measure spin along the x axis, we either get $|+x\rangle$ or $|-x\rangle$.

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This shows that an electron is not really a little ball spinning around. An electron is a quantum object, which has some attribute which *reminds us* of spin, but it is really an attribute that we can't comprehend in classical terms. It follows quantum laws.

When we measure the spin of an electron the **observable** which is the **eigenvalue** of the measurement **operator** will either be $+1$ or -1, depending on whether the spin turns out to be toward or away from the direction we choose to measure. Here is a summary for the directions along the three primary axes:

Operator	Matrix	Eigenvalue	Eigenstate	Eigenvalue	Eigenstate
σ_x	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$+1$	$\lvert +x\rangle$	-1	$\lvert -x\rangle$
σ_y	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$+1$	$\lvert +y\rangle$	-1	$\lvert -y\rangle$
σ_z	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$+1$	$\lvert +z\rangle$	-1	$\lvert -z\rangle$

And here are the column vector representations of each state:

$$|+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$|+y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |-y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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Now that we've got some idea how spin works, let's see how the postulates can be adapted to talk specifically about electron spin.

The postulates adapted for the spin of a single electron

1. The spin state of an electron is described by a two element vector with complex components and (in the z direction for example) may be written as: $\alpha|+z\rangle + \beta|-z\rangle$.
2. As long as we don't measure the spin, a change in its state can be modeled by multiplying the current state by a unitary matrix.
3. The measurement of the spin is represented using a Hermitian matrix. This will typically be one of the Pauli matrices described above. The result of the measurement is one of the eigenvalues of the matrix, and the resulting state is one of the eigenvectors of the matrix.
4. If an electron has the spin state $\alpha|+z\rangle + \beta|-z\rangle$, the probability of a measurement resulting in a +z spin is $|\alpha|^2$ and the probability of a measurement resulting in a -z spin is $|\beta|^2$. If a bit is in the state $|\psi\rangle$, the probability of a measurement resulting in the state $|\phi\rangle$ is $|\langle\phi|\psi\rangle|^2$.

The state vector (postulate 1)

The mathematical description of all the state vectors for the three primary directions (x, y, and z) are given in the table on the previous page.

Operating on the spin of an electron (postulate 2)

We won't go into this here. We'll talk about operating on a quantum bit in the next chapter.

Measuring the spin of an electron (postulates 3 and 4)

The thing to notice here is that the states, for example $|+z\rangle$ and $|-z\rangle$, are all eigenvectors of their respective measurement matrices (σ_x, σ_y, and σ_z). The eigenvalues are always $+1$ for the "plus" state and -1 for the "minus" state. The mathematical model has been constructed specifically so that things come out this way.

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We are going to be working with two different types of electron spin problems:

(1) X, Y, Z Spin Problems

We are given an electron in some superposition of the x, y, or z spin basis states. Our job is to find out the probabilities for getting a + or - spin for some specified measurement also in x, y, or z. The measurement basis specified may or may not be the same as the one in which the original spin was given.

(2) Arbitrary Spin Problems

We are given an electron whose spin is in some arbitrary physical direction. Then we are asked to find the probabilities of + or - for a measurement done in some other arbitrary direction. This is a more complicated problem, because we first have to calculate the "mathematical model" spin states that represent the physical directions given.

6.1 Working with X, Y, Z Spin

We are given an electron in the state ψ which is some superposition in one of the three bases x, y, or z. We are then asked to calculate the probabilities of measuring a + or - spin in one of those three same bases. The given basis and the target basis may or may not be the same. There are three steps to solving the problem:

1. Write out the state in the z basis
2. Do a change of basis to the desired measurement basis
3. Calculate the probabilities

We'll go through these steps in detail. As an example, say that we were given the following electron spin state in the y basis:

$$\psi_y = |-y\rangle$$

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Step 1: Write out the state in the z basis

First we need to write out the given state as a column vector. We do this simply by substituting in the values of the basis states given, multiplying them by the given probability moduli, and adding it all together. Note that the resulting column vector will automatically be in the z basis.

$$\psi_z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Step 2: Do a change of basis

Now we need to do a change of basis into the specified target basis (in this case, the x basis). We find the two components of ψ in the x basis by projecting ψ_z on to the two x basis vectors. Suppose we call the two components of ψ in the x basis by the names α and β. Then:

$$\alpha = \langle +x|\psi_z\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{2} + \frac{i}{2}$$

$$\beta = \langle -x|\psi_z\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{2} - \frac{i}{2}$$

And the column vector in the x basis is: $\begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ \frac{1}{2} - \frac{i}{2} \end{bmatrix}$

Step 3: Calculate the probabilities

We can write this in Dirac notation in the x basis by recognizing that the components of the vector are the same as thing as the probability moduli.

$$\psi_x = \left(\frac{1}{2} + \frac{i}{2}\right)|+x\rangle + \left(\frac{1}{2} - \frac{i}{2}\right)|-x\rangle$$

And then calculate the probabilities:

$$P(+) = \left|\frac{1}{2} + \frac{i}{2}\right|^2 = \left(\frac{1}{2} + \frac{i}{2}\right)\left(\frac{1}{2} - \frac{i}{2}\right) = \frac{1}{2}$$

$$P(-) = \left|\frac{1}{2} - \frac{i}{2}\right|^2 = \left(\frac{1}{2} - \frac{i}{2}\right)\left(\frac{1}{2} + \frac{i}{2}\right) = \frac{1}{2}$$

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X, Y, Z Electron Spin Problems

1. Given the spin state: $\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$

   If you do a measurement in the z basis.

   What is the probability of getting $|+z\rangle$

2. Given the spin state: $\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$

   If you do the observation associated with the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

   What is the probability of getting the $+1$ eigenvalue

3. Given the spin state: $-\frac{4}{5}|+x\rangle - \frac{3}{5}|-x\rangle$

   If you do a measurement in the x basis.

   What is the probability of getting $|+x\rangle$

4. Given the spin state: $\frac{3}{\sqrt{10}}|+y\rangle + \frac{1}{\sqrt{10}}|-y\rangle$

   If you do the observation associated with the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

   What is the probability of seeing the 1 eigenvalue

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5. Given the spin state: $-\frac{5+4i}{\sqrt{51}}|+x\rangle + \frac{3+i}{\sqrt{51}}|-x\rangle$

   If you do a measurement in the z basis.

   What is the probability of getting $|-z\rangle$

6. Given the spin state: $-\frac{1+2i}{\sqrt{10}}|+z\rangle + \frac{1+2i}{\sqrt{10}}|-z\rangle$

   If you do a measurement in the y basis.

   What is the probability of getting $|-y\rangle$

7. Given the spin state: $\frac{3i}{\sqrt{35}}|+z\rangle - \frac{1+5i}{\sqrt{35}}|-z\rangle$

   If you do the observation associated with the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

   What is the probability of seeing the -1 eigenvalue

8. Given the spin state: $\frac{4+3i}{5\sqrt{2}}|+z\rangle - \frac{3+4i}{5\sqrt{2}}|-z\rangle$

   If you do a measurement in the x basis.

   What is the probability of getting $|-x\rangle$

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Solutions for the X, Y, Z Electron Spin Problems

1. $\frac{1}{2}$ or 0.5

2. 1 or 1.0

3. $\frac{16}{25}$ or 0.64

4. $\frac{4}{5}$ or 0.8

5. $\frac{89}{102}$ or approximately 0.8725

6. $\frac{1}{2}$ or 0.5

7. $\frac{26}{35}$ or approximately 0.7429

8. $\frac{49}{50}$ or 0.98

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6.2 Working with arbitrary spin

In an arbitrary spin problem we are given two *physical* spin vectors, s_1 and s_2. These are *not* state vectors. They are vectors in physical space and have three real components. The components represent the projection of the vector along the physical x, y, and z axes. So, for example, the *physical* spin vector for an electron with its spin oriented along the positive z axis would be: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ where the *state vector* would be $|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

To work an arbitrary spin problem, we do the following steps:

1. Find the two measurement operators O_1, O_2 which correspond to the two given spin vectors s_1, s_2.
2. Use the measurement operators to find the the state vectors: $|+s_1\rangle, |+s_2\rangle, |-s_2\rangle$. The state vectors are the eigenvectors of the measurement operators. You will normally use the computer to calculate them.
3. Project the given state $|+s_1\rangle$ onto the basis states for the direction to be measured: $|+s_2\rangle, |-s_2\rangle$. In other words, change basis from s_1 to s_2.
4. You now have the given state in the s_2 basis, from which you can calculate the probabilities of getting $+s_2$ or $-s_2$.

Here is a detailed example. *In this example, as in most arbitrary spin problems, we will work in decimal notation, approximating to three decimal digits.*

Suppose we are given an electron with its spin oriented along the positive x axis, and told to measure it along an axis exactly 45° between x and z. This means that we are given the following spin vectors:

$$s_1 = \begin{bmatrix} 1.0 \\ 0 \\ 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 0.707 \\ 0 \\ 0.707 \end{bmatrix}$$

We want to know: Given an electron in the state $|+s_1\rangle$.
What are the probabilities of measuring $|+s_2\rangle$ or $|-s_2\rangle$

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Step 1:

Calculate the measurement operators for both spin vectors using the formula:

$$O = s_x\sigma_x + s_y\sigma_y + s_z\sigma_z$$

Calculating the measurement operator for s_1 ...

$$O_1 = 1.0\begin{bmatrix} 0 & 1.0 \\ 1.0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & -1.0i \\ 1.0i & 0 \end{bmatrix} + 0\begin{bmatrix} 1.0 & 0 \\ 0 & -1.0 \end{bmatrix} = \begin{bmatrix} 0 & 1.0 \\ 1.0 & 0 \end{bmatrix}$$

Calculating the measurement operator for s_2 ...

$$O_2 = 0.707\begin{bmatrix} 0 & 1.0 \\ 1.0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & -1.0i \\ 1.0i & 0 \end{bmatrix} + 0.707\begin{bmatrix} 1.0 & 0 \\ 0 & -1.0 \end{bmatrix} = \begin{bmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{bmatrix}$$

Step 2:

The computer will find the eigenvalues and eigenvectors

Eigenvalues for O_1 are: -1.0 and 1.0

Eigenvectors for O_1 are: $\begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix}$ and $\begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$

Eigenvalues for O_2 are: -1.0 and 1.0

Eigenvectors for O_2 are: $\begin{bmatrix} -0.383 \\ 0.924 \end{bmatrix}$ and $\begin{bmatrix} 0.924 \\ 0.383 \end{bmatrix}$

The eigenvalues tell us which vectors represent the "plus" states and which represent the "minus" states

$$|+s_1\rangle = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$$

$$|+s_2\rangle = \begin{bmatrix} 0.924 \\ 0.383 \end{bmatrix}, \quad |-s_2\rangle = \begin{bmatrix} -0.383 \\ 0.924 \end{bmatrix}$$

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Step 3:

Now that we know what the s_2 state vectors are, we want to project our given state, $|+s_1\rangle$, onto to the s_2 basis. This will give us an expression such as: $\alpha|+s_2\rangle + \beta|-s_2\rangle$ from which we can calculate the probabilities.

The projection onto s_2 goes like this:

$$\alpha = \langle +s_2| + s_1\rangle = \begin{bmatrix} 0.924 & 0.383 \end{bmatrix} \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix} = 0.924$$

$$\beta = \langle -s_2| + s_1\rangle = \begin{bmatrix} -0.383 & 0.924 \end{bmatrix} \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix} = 0.383$$

The state written in the s_2 basis is:

$$(0.924)|+s_2\rangle + (0.383)|-s_2\rangle$$

Step 4: Calculate the probabilities

$$P(|+s_2\rangle) = |\alpha|^2 = \alpha\alpha^* = (0.924)(0.924) = 0.854$$

$$P(|-s_2\rangle) = |\beta|^2 = \beta\beta^* = (0.383)(0.383) = 0.146$$

Additional Note:

The angle between s_1 and s_2 is: $\theta = 0.785$. So the probabilities are equivalent to:

$$\cos^2(\theta/2) = 0.854 \quad \text{and} \quad \sin^2(\theta/2) = 0.146$$

I'm not giving you any other arbitrary spin problems to do. But everyone should work the the example above carefully. In this example s_2 turns out to be the 45° basis: $|+45\rangle, |-45\rangle$ which we will use extensively later on. Be sure that you understand it.

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7 A single quantum bit

To introduce the subject of quantum information, first we'll talk about a single quantum bit. A quantum bit is very different from the bits that are found in today's computers.

I'm going to assume that you know the basics of computers, bits, bytes, and related things. You know that a bit can represent the numbers 0 or 1, that a byte is 8 bits and can represent a number between 0 and 255, and so on. If there are any questions about these basic computer topics, ask now and we'll clear them up.

In these notes, I'll generally refer to the kind of a bit that you would find in current (early 21st century) computers as a *classical bit*. We're primarily concerned with the bits in a quantum computer. I'll use the term *quantum bit* initially just to highlight that point. But after we get warmed up you can assume the "bits" being discussed are quantum bits. If we need to talk about a classical bit, I'll call them out as such.

The postulates adapted for a single qbit

1. The state of a quantum bit is described by a two element vector with complex components, and may be written as: $\alpha|0\rangle + \beta|1\rangle$.
2. As long as we don't measure a bit, a change in its state (i.e., sending the bit through a "quantum gate") can be modeled by multiplying the current state by a unitary matrix.
3. The measurement of a bit is represented using a Hermitian matrix. The result of the measurement is one of the eigenvalues of the matrix, and the resulting state is one of the eigenvectors of the matrix.
4. If a bit is in the state $\alpha|0\rangle + \beta|1\rangle$, the probability of a measurement resulting in a 0 is $|\alpha|^2$ and the probability of a measurement resulting in a 1 is $|\beta|^2$. If a bit is in the state $|\psi\rangle$, the probability of a measurement resulting in the state $|\phi\rangle$ is $|\langle\phi|\psi\rangle|^2$.

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7.1 The State Vector

The state of a quantum bit is described by a two element vector with complex components, and may be written as: $\alpha|0\rangle + \beta|1\rangle$.

Let's break this statement down. The "state" of a quantum bit. What does that mean? The "state" of a *classical* bit is either 0 or 1. Apparently a quantum bit is a lot more complicated. To describe its state we need not only a vector, but one containing complex numbers.

Now, by convention, the *classical* bit states 0 and 1 are described by the kets $|0\rangle$ and $|1\rangle$, which are equal to the following column vectors:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These two vectors above are sometimes referred to as the *classical basis* because a measurement using them as basis vectors will result in either a zero or one, just like a classical bit.

So, bearing in mind that α and β are any complex numbers, the state of a bit can be any combination of:

$$\alpha|0\rangle + \beta|1\rangle = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Looking at the column vector on the right you can see that the state is, ultimately, a single vector with two complex components. This by the way is sometimes referred to as the *state vector.*

Since the quantum bit is equal to α times a classical zero plus β times a classical one, in a certain sense the quantum bit is a combination of the two classical states. If $\alpha = 0$ and $\beta = 1$ then the bit is 1. If $\alpha = 1$ and $\beta = 0$ then the bit is 0. If both α and β are nonzero then the qbit is in what's called a *superposition* of a classical zero and one.

7.2 Operating on a bit

Postulate 2

7.3 Measuring a bit

Postulates 3 and 4

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8 Putting bits together

8.1 The Tensor Product

In order to "put bits together" we need a new kind of product, called the *tensor product.* The tensor product can be applied to either vectors or matrices. The symbol used for the tensor product is a circle with a slanted cross in it, like this: $\otimes$.

If we have two vectors:

$$v_1 = \begin{pmatrix} a \\ b \end{pmatrix}, \text{ and } v_2 = \begin{pmatrix} x \\ y \end{pmatrix},$$

then the tensor product of v_1 and v_2 is:

$$v_1 \otimes v_2 = \begin{pmatrix} a\, v_2 \\ b\, v_2 \end{pmatrix} = \begin{pmatrix} a \begin{pmatrix} x \\ y \end{pmatrix} \\ b \begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ax \\ ay \\ bx \\ by \end{pmatrix}$$

If we have two matrices:

$$M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ and } M_1 = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

then the tensor product of M_1 and M_2 is:

$$M_1 \otimes M_2 = \begin{pmatrix} a\, M_2 & b\, M_2 \\ c\, M_2 & d\, M_2 \end{pmatrix} = \begin{pmatrix} a \begin{pmatrix} w & x \\ y & z \end{pmatrix} & b \begin{pmatrix} w & x \\ y & z \end{pmatrix} \\ c \begin{pmatrix} w & x \\ y & z \end{pmatrix} & d \begin{pmatrix} w & x \\ y & z \end{pmatrix} \end{pmatrix} = \begin{pmatrix} aw & ax & bw & bx \\ ay & az & by & bz \\ cw & cx & dw & dx \\ cy & cz & dy & dz \end{pmatrix}$$

Note that the tensor product, just like the inner and matrix products, is *not* commutative. For example, if we switch the order of the vectors above:

$$v_2 \otimes v_1 = \begin{pmatrix} x\, v_1 \\ y\, v_1 \end{pmatrix} = \begin{pmatrix} x \begin{pmatrix} a \\ b \end{pmatrix} \\ y \begin{pmatrix} a \\ b \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ax \\ bx \\ ay \\ by \end{pmatrix}$$

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8.2 The tensor product in Dirac notation

Say we have two vectors: $|a\rangle$ and $|b\rangle$. Then the "correct" way of writing the tensor product of the two would be: $|a\rangle \otimes |b\rangle$. But since there is no other valid operation for multiplying a "ket times a ket" the tensor product can be abbreviated as $|a\rangle|b\rangle$. In certain contexts, we simply write $|ab\rangle$.

8.3 The classical basis for 2 bits

The *classical basis* is the basis used to represent ones and zeros. It was the "default" basis that we already used for the representation of a single bit. We get the basis for two bits by taking the tensor product of the one bit basis vectors. Of course, if you're familiar with computers, this may have been obvious to you already. This is just like stringing bits together in the "base 2" numbering system.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |0\rangle \otimes |0\rangle = |00\rangle = 0$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |0\rangle \otimes |1\rangle = |01\rangle = 1$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |1\rangle \otimes |0\rangle = |10\rangle = 2$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |1\rangle \otimes |1\rangle = |11\rangle = 3$$

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8.4 Product States

A two bit *product state* is a state that can be created by multiplying two individual bits together. First, let's look at the general tensor product in bra-ket notation. Say we have the following two one-bit states:

$a|0\rangle + b|1\rangle$ and $x|0\rangle + y|1\rangle$

We can multiply them together by treating them just like polynomials:

$$\begin{aligned}
&(a|0\rangle + b|1\rangle) (x|0\rangle + y|1\rangle) \\
&= a|0\rangle x|0\rangle + a|0\rangle y|1\rangle + b|1\rangle x|0\rangle + b|1\rangle y|1\rangle \\
&= ax|0\rangle|0\rangle + ay|0\rangle|1\rangle + bx|1\rangle|0\rangle + by|1\rangle|1\rangle \\
&= ax|00\rangle + ay|01\rangle + bx|10\rangle + by|11\rangle
\end{aligned}$$

It's important to understand that the result above can be written in vector form, and that it is the *same* vector that we got when we previously did the tensor product in vector notation:

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ ay \\ bx \\ by \end{pmatrix}$$

Now, suppose we have the state:

$$\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

and we want to know whether or not it's a product state. If it *is* a product state, then we can find some a, b, x, y such that:

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ ay \\ bx \\ by \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} a \times x &= 1/2 \\ a \times y &= 1/2 \\ b \times x &= 1/2 \\ b \times y &= 1/2 \end{aligned}$$

There are actually an infinite number of solutions to the four equations on the right. But once you normalize any of them, you will get one unique solution:

a, b, x, y are all equal to $\frac{1}{\sqrt{2}}$, and

$$\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right) \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right)$$

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8.5 Entangled States

A two-bit *entangled* state is a state that can *not* be factored into two individual one-bit states. For example, consider the state:

$$\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

If we try to factor this state, as we did previously, into the two single-bit states $a|0\rangle + b|1\rangle$ and $x|0\rangle + y|0\rangle$, then we get:

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ ay \\ bx \\ by \end{pmatrix} = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{l} a \times x = 0 \\ a \times y = 1/\sqrt{2} \\ b \times x = 1/\sqrt{2} \\ b \times y = 0 \end{array}$$

From the equations on the right, you can see that:

1. a cannot be zero if $a \times y$ is to equal $1/\sqrt{2}$.
2. x cannot be zero if $b \times x$ is to equal $1/\sqrt{2}$.
3. But since both a and x are non-zero, $a \times x$ cannot be zero.

The conclusion is that there are *no values of* a, b, x, y *for which these equations are true.*

Stop for a moment and consider this. We have a perfectly good 2-bit state. A pair of quantum bits can "be in" this state. But it is not a "combination" of two individual bit states. This is entanglement. *Many people consider entanglement to be the fundamental reason for the weirdness of quantum systems.*

8.6 Cartesian vs Tensor product spaces

(TBD)

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8.7 Questions for discussion

Alice and Bob each have an electron. We can represent the *pair* of electrons in a ket, by specifying Alice's state first and then Bob's. In other words: $|+z,-z\rangle$ means that Alice's electron has a $+z$ spin and Bob's has a $-z$ spin.

Now, suppose that all we know is the *joint* state of the two electrons:

$$\frac{1}{2\sqrt{2}}|+z,+z\rangle + \frac{1}{2\sqrt{2}}|+z,-z\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|-z,+z\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|-z,-z\rangle$$

Question 1: Assuming that both Alice and Bob do a "z measurement" on their electrons, what is the probability that they will both get a $-z$ result?

Question 2: Without trying to factor the state, can you determine the probability of Alice seeing a $+z$ result?

Question 3: Is this a product state or an entangled state? If it is a product state, then what are the two individual electron states?

Now, suppose that we have two bits, in the joint state:

$$\frac{1}{4}|00\rangle + \sqrt{\frac{7}{8}}|11\rangle + \frac{1}{4}|01\rangle$$

Question 4: What is the probability that a joint measurement will result in $|10\rangle$?

Question 5: Without trying to factor the state, can you determine the probability, for each of the individual bits, of getting a 1?

Question 6: Is this a product state or an entangled state? If it is a product state, then what are the two individual electron states?

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9 Entanglement correlations and statistics

9.1 The Bell States

This assignment is about the so called Bell States:

$$(1)\ |\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \frac{1}{\sqrt{2}}|+z,+z\rangle + \frac{1}{\sqrt{2}}|-z,-z\rangle$$

$$(2)\ |\phi^-\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle = \frac{1}{\sqrt{2}}|+z,+z\rangle - \frac{1}{\sqrt{2}}|-z,-z\rangle$$

$$(3)\ |\psi^+\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle = \frac{1}{\sqrt{2}}|+z,-z\rangle + \frac{1}{\sqrt{2}}|-z,+z\rangle$$

$$(4)\ |\psi^-\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle = \frac{1}{\sqrt{2}}|+z,-z\rangle - \frac{1}{\sqrt{2}}|-z,+z\rangle$$

A few things to note:

1. I've written out each state first in its "information theory" form (using ones and zeros) and then in its "electron spin" form (using plus and minus spin on the z axis). In both cases the math is exactly the same, since $|+z\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|-z\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
2. The pluses and minuses in the names of the Bell states have nothing to do with spin or eigenvalues. The plus in $|\phi^+\rangle$ is a reminder of the " in the middle of the state. The same for the minus in $|\phi^-\rangle$, and so on.
3. I gave each state both a name and a number. So, for example, "Bell 3" and ψ^+ refer to the same state. The main reason for using ϕ and ψ is that it appears to be pretty common practice. Take a look at the Wikipedia entry for "Bell States" and make sure that you can see the equivalence between their notation and what's above.

For everything that follows, we are going to assume that each of the Bell states represents an entangled pair of either (quantum) bits or electrons. In either case, the bit or electron on the left belongs to Alice and the one on the right belongs to Bob. So, for example, the state $|01\rangle$ would mean that Alice has a zero and Bob has a one.

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9.2 Bell State problems

Problem 1:

For each of the Bell states, if Alice does a measurement resulting in a zero, then what is the probability of Bob also getting a zero? If Bob measures a -z, then what is Alice's probability of getting a +z?

Problem 2:

Express each of the four Bell states as a column vector. You may be able to recognize the "pattern" of the kets, but if all else fails you can take the tensor products of all the individual ones and zeros and add everything up. For example, Bell 1 should come out to:

$$\text{Either: } \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \text{or: } \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Problem 3:

Take the four column vectors, from the last problem, and change them to the *x basis*. In order to do this, you will have to project each of the vectors on to all four of the x basis vectors. If you don't know what the four "two bit" x basis vectors are, you can find them by taking the appropriate tensor products of the two "1 bit" x basis vectors: $|+x\rangle$ and $|-x\rangle$.

Problem 4:

Write out all four Bell states, in the x basis, in ket form. Here is the first one:

$$|\phi^+\rangle \;=\; \frac{1}{\sqrt{2}}|+z,+z\rangle + \frac{1}{\sqrt{2}}|-z,-z\rangle \;=\; \frac{1}{\sqrt{2}}|+x,+x\rangle + \frac{1}{\sqrt{2}}|-x,-x\rangle;$$

So for Bell 1 the pattern doesn't change at all when you switch to the x basis.

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Problem 5:

For all four states, discuss the correlations between the measurements, and how the correlations differ depending on whether you do a z or an x measurement. For example, we just saw that when we changed ϕ^+ from the z to the x basis the correlations didn't change at all. Alice and Bob's measurement were positively correlated in both z and x. What I mean by positively correlated is that Alice and Bob are guaranteed to get the same result, if they do the same type of measurement. In the (z basis) ψ states, however, the measurement are negatively correlated. Alice and Bob are guaranteed to get different results for the same type of measurement. So your job here is to figure out what type of correlations there are for all the states in both the x and z bases, and find out when a change of basis results in a change in correlations.

Note:

It should be possible to do the problems above by hand. But for the remaining problems you will need to use the computer, if for no other reason than to get the eigenvectors for the 45° measurement matrix.

Problem 6:

Find the 1-bit basis: $|+45\rangle, |-45\rangle$ associated with the spin vector that is exactly half way between the x and z axes. If you can't figure out how to do this, see the "Electron Spin" notebook.

Problem 7:

Construct the 2-bit 45° basis by taking the appropriate tensor products of the 1-bit vectors. Remember you can use the computer, and you can use decimal approximations.

Problem 8:

Construct all four of the Bell states in the 45° basis, using same procedure that worked for the x basis in problem 3.

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Problem 9:

Write out all the 45° states as kets like we did for the x states in problem 4.

Problem 10:

Analyze the correlations for the 45° basis, like we did for the x basis in problem 5.

9.3 Entanglement statistics

For the purposes of this exercise we're going to use the the "singlet" state. This is the state that we have been calling $|\psi^-\rangle$ or Bell 4. The singlet state is also used for this purpose in Leonard Susskind's Quantum Entanglement lecture #5, and this state produces the so-called "EPR correlations."

Here is the singlet state in the three bases we're using:

In the Z basis: $\frac{1}{\sqrt{2}}|+z,-z\rangle - \frac{1}{\sqrt{2}}|-z,+z\rangle$

In the X basis: $-\frac{1}{\sqrt{2}}|+x,-x\rangle + \frac{1}{\sqrt{2}}|-x,+x\rangle$

In the 45° basis: $\frac{1}{\sqrt{2}}|+45,-45\rangle - \frac{1}{\sqrt{2}}|-45,+45\rangle$

And this is a summary of all the basis vectors we'll need:

$$|+z\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |-z\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$|+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}.707\\.707\end{pmatrix}, \quad |-x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}.707\\-.707\end{pmatrix}$$

$$|+45\rangle = \begin{pmatrix}.924\\.383\end{pmatrix}, \quad |-45\rangle = \begin{pmatrix}-.383\\.924\end{pmatrix}$$

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How to analyze measurements on entangled states in multiple bases.

We have reason to be suspicious of statements like "Alice's measurement changes the state of Bob's electron." However, when we are analyzing measurements on an entangled state, we'll simply follow this "superficial" account. It will yield the correct results:

1. The 1st of the two parties does their measurement. This "breaks" the entanglement, meaning that both electrons are now in separate one-bit states.
2. The one-bit state of the 2nd electron is the one that follows from the 1st party's measurement, in the basis used by the 1st party.
3. The basis of the 2nd electron can then be changed to that of the measurement made by the 2nd party.

Example using the singlet state:

1. Alice measures in the Z basis and gets the one-bit state $|+z\rangle$. This breaks the entanglement.
2. Bob's electron (in the Z basis) is therefore in the one-bit $|-z\rangle$ state.
3. If Bob now does a measurement in the 45° basis, we simply do a change of basis:

$$\langle +45|-z\rangle = \begin{pmatrix} .924 & .383 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = .383$$

$$\langle -45|-z\rangle = \begin{pmatrix} -.383 & .924 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = .924$$

$$|-z\rangle = .383\,|+45\rangle + .924\,|-45\rangle$$

Bob's chances of measuring +45 are $.383^2 \approx .15$, and so on.

There is an important point which should be noted here. You probably understand by now that (due to the relativity of simultaneity) we can't really say, in general, that one of the two parties "goes first." This means that there must be a certain kind of symmetry between the possible measurements. Everything has to "work out the same" no matter whether we assume that Alice or Bob goes first. This *does work out*, although I'm at a loss to come up with a simple formula for why. Try a few examples!

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The following table of statistics for the singlet state was produced by one of our IPython notebooks. See if you can reproduce (at least part of) it by hand, using the technique outlined above.

Bob		+z	-z	+x	-x	+45	-45
Alice	+z	0.000	1.000	0.500	0.500	0.146	0.854
	-z	1.000	0.000	0.500	0.500	0.854	0.146
	+x	0.500	0.500	0.000	1.000	0.146	0.854
	-x	0.500	0.500	1.000	0.000	0.854	0.146
	+45	0.146	0.854	0.146	0.854	0.000	1.000
	-45	0.854	0.146	0.854	0.146	1.000	0.000

(Generated by: Entanglement_Statistics.ipynb, RCS Rev 1.3, 30 December 2014)

Now, the question before us is the *meaning* of a superposition. If we have a quantum coin in the state $\frac{1}{\sqrt{2}}|H\rangle + \frac{1}{\sqrt{2}}|T\rangle$, is the coin *really* either heads or tails but we just don't know which? Or is the superposition actually something different? Something that can not be explained classically.

It would be easy to explain this particular example classically. I could just keep flipping a coin and covering it up. When you uncover the coin, you're doing a "measurement." You would find that you couldn't predict any particular measurement, but that the "statistics" (50% heads and 50% tails) would work out.

Now suppose I wanted to prepare a pair "entangled" electrons *classically* and then hand them to Alice and Bob, who would then do individual measurements (in the Z basis).

To mimic the singlet state, all I would have to do would be to flip a coin. If I get heads, I set Alice's electron to $|+z\rangle$ and Bob's to $|-z\rangle$. If I get tails then I set Alice's electron to $|-z\rangle$ and Bob's to $|+z\rangle$.

If I do this process over and over again, Alice and Bob will both measure 50% plus and 50% minus and they will always get opposite results. So far, this would look just like an entangled state.

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When we studied special relativity we concluded that it was highly doubtful that the explanation of entanglement involved communication between the two electrons. The other possibility was that the electrons really are in one state or another. Here we are investigating whether this is possible or not. Can a pair of electrons be in specific states (before they are measured) which can reproduce all of the statistics in the table above, after the measurements happen.

We may not know exactly what happens when entanglement occurs. But if we can come up with a procedure that *could* create such an electron pair, then we'll know that it is classically *possible* for them to exist. It turns out that this *can't* be done. In what follows, I'll try provide a sense of why this is so.

We already saw above that it's easy enough to reproduce the results if all the measurements are done in the Z basis. In order to address other bases, we will need a separate "variable" for each basis. This variable will hold the value to be displayed if a measurement is done in that basis.

By the way, these are what are called *hidden variables*. They're "hidden" because (according to quantum theory) there is no way to find out the value of one of them until we do a measurement in the related basis, and once we do that all of the *other* variables are lost.

Here's an algorithm (a procedure) that will reproduce the first two rows of the table:

1. Assign Alice's Z variable plus or minus both with probability .5
   Make Bob's Z variable the opposite of Alice's.
2. If Alice got +z, then assign Bob +45 with probability .146 and -45 with probability .854
   If Alice got -z, then assign Bob +45 with probability .854 and -45 with probability .146
3. Assign Bob's X variable plus or minus both with probability .5

Think about this for a while, and see if you are convinced that, as long as Alice just does Z measurements, that Bob can do all three types of measurements, and in the long run they will see the right statistics.

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Can this idea be extended to cover Alice doing an X measurement? If we just add one line to the bottom of the algorithm, it will cover Alice measuring in X as long as Bob measures in Z or X:

1. Assign Alice's Z variable plus or minus both with probability .5
   Make Bob's Z variable the opposite of Alice's.
2. If Alice got +z, then assign Bob +45 with probability .146 and -45 with probability .854
   If Alice got -z, then assign Bob +45 with probability .854 and -45 with probability .146
3. Assign Bob's X variable plus or minus both with probability .5
   Make Alice's X variable the opposite of Bob's.

Basically, this works because the Z and X measurements are not correlated.

However, we run into trouble trying to finish the two rows where Alice measures X. Remember, when the entangled pair is "created" we don't have any way of knowing *which* type of measurements will be done. So, Alice and Bob *might* be going to do any pair of measurements. It might seem that, since plus and minus 45 are correlated the same way with Z and X, that we could set them successfully. But now the fact that Z and X are not correlated with each other ruins things.

There will be times (actually 50% of the time) when Alice's Z variable must be set the opposite of her X variable. We don't know whether to correlate 45 with Z or X. Whichever one we choose, we'll be wrong half the time!

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Bell's Inequality and Bell's Theorem.

At this point there are a whole lot of details that may not seem completely clear. You might not be convinced that just because we weren't able to make this work, that *some* way might not possibly exist. In fact, we haven't actually proven anything. What we've done has (I hope) just helped you see that it really doesn't *seem like* an entangled pair can exist *with pre-defined variables* that will produce the right statistics.

What John Bell is famous for is providing a rigorous argument that there is a certain kind of mathematical inequality which must hold for *any* independent *classical* variables. It covers variables like our Z, X, and 45 above and much more general situations. It turns out that entanglement statistics violate this inequality, and hence Bell "proved" that an entangled state can't be produced by classical means (without communication between the two systems that are entangled).

There's a *lot* more ground to cover to learn about Bell's work in detail (which we are not going to do, at least at this time). But here are a couple of related articles:

- John Stewart Bell
- Bell's theorem

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10 Multi-bit Operations

10.1 What are operators?

A *Unitary operator* changes the state of a system (postulate 2).
If U is the operator then:

$U|SomeState\rangle = |AnotherState\rangle$
(which is often written as)
$U|\psi\rangle = |\phi\rangle$

For us, operators are matrices and states are vectors. So this ends up being:

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} u_{11}\psi_1 + u_{12}\psi_2 \\ u_{21}\psi_1 + u_{22}\psi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

So a unitary operator actually does "operate" on a state, and it does so by multiplying the matrix representing the operator with the vector representing the state, to yield a new state vector.

A *Hermitian operator* does not actually "operate" on a state. Rather, it *represents* a specific measurement. This is described in postulate 3. It might be good to review that postulate right now.

10.2 Review of single bit operations

We very briefly touched on the concept of matrices as operators back in section 2.6. There wasn't a lot to say because there are not a lot of interesting one-bit operations. But once you have two bits (or two electrons) there are a lot more possible operations, and a lot more to say about how operators are "constructed."

But first let's just review the three interesting one-bit operators: I, NOT, and H.

The I (or Identity) operator is: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

It's called the "identity" operator because it doesn't change the state:

$I|\psi\rangle = |\psi\rangle$

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The NOT operator is: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The one-bit NOT operator switches the components of the vector it operates on:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

As a result of switching the components, NOT will change a one into a zero and a zero into a one.

$$NOT|1\rangle = |0\rangle,\ NOT|0\rangle = |1\rangle$$

This is the same effect as a "NOT gate" in classical computer terminology, which is why this operator is called "NOT."

Notice that the NOT operator is the same matrix as σ_x, the measurement operator for electron spin on the x axis. This matrix is sometimes denoted simply by the letter X.

The H (or Hadamard) operator is: $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

The Hadamard operator is named after the mathematician of the same name. It is, arguably, the most important operator in quantum computation. Basically, the Hadamard operator *places a bit into an equal superposition:*

$$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Much of the "magic" of quantum computing comes from working on bits in superposition. So the Hadamard operation is fundamental.

Both H and NOT are "square roots" of the Identity operator:

$$H * H = NOT * NOT = I$$

So if you take either a Hadamard or a NOT operator and apply it twice, you will get back the same state that you started with.

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10.3 Putting one-bit operators together to form two-bit operators

In section 8.1 we discussed how to use the tensor product to put both vectors and matrices together to form higher dimension objects. Please go back and review that section now.

In some cases, you can construct the two-bit operation you want by using the tensor product to put together two one-bit operators. Say you have a two bit state $|01\rangle$ and you want to flip both the bits. In this case you simply tensor two one-bit NOT operators:

$$NOT2 = NOT \otimes NOT = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$NOT2\,|01\rangle = |10\rangle$
$NOT2\,|10\rangle = |01\rangle$
$NOT2\,|00\rangle = |11\rangle$

Now suppose that you only wanted to flip the first bit and leave the second one alone. This would be like applying a NOT to the first bit and applying the Identity operator to the second bit. So you tensor those two:

$$NOT \otimes I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$NOT \otimes I\,|01\rangle = |11\rangle$
$NOT \otimes I\,|10\rangle = |00\rangle$

Similarly, you can construct an operator to flip only the second bit by doing the tensor product in the reverse order:

$I \otimes NOT\,|01\rangle = |00\rangle$
$I \otimes NOT\,|10\rangle = |11\rangle$

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10.4 Other two-bit operators

Of course there are more multi-bit operators than the ones which can be made by putting together single-bit operators. We'll look at two interesting ones here: SWAP and CNOT.

The SWAP operator exchanges (swaps) the two bits.

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{SWAP}\,|00\rangle = |00\rangle$$
$$\text{SWAP}\,|01\rangle = |10\rangle$$
$$\text{SWAP}\,|10\rangle = |01\rangle$$
$$\text{SWAP}\,|11\rangle = |11\rangle$$

CNOT stands for "controlled not." In this two-bit operator, the first bit of the state is the control bit and the second bit of the state is the target bit. So, for example, in the state $|01\rangle$ the control bit is zero and the target bit is one.

Here's how CNOT works. If the control bit is zero, then nothing happens. If the control bit is one, then a NOT operation is executed on the target bit. In other words, the target bit gets flipped only if the control bit is one.

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\text{CNOT}\,|00\rangle = |00\rangle$$
$$\text{CNOT}\,|01\rangle = |01\rangle$$
$$\text{CNOT}\,|10\rangle = |11\rangle$$
$$\text{CNOT}\,|11\rangle = |10\rangle$$

CNOT is one of the most important operators in quantum theory. We'll be using it shortly in the description of quantum teleportation.

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11 Linearity and the No Cloning Theorem

11.1 What is linearity?

I briefly mentioned a linear combination in sec 1.9, but didn't really define what "linearity" was, per se. It goes like this: If you have two vectors $|v_1\rangle$ and $|v_2\rangle$, and two scalars α and β, then L is a linear operator if:

$$L\ (\ \alpha|\psi\rangle + \beta|\phi\rangle\ ) = \alpha L|\psi\rangle + \beta L|\phi\rangle$$

It should be relatively easy to see that that multiplication by a matrix is a linear operation:

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} (\ a|\psi\rangle + b|\phi\rangle\ ) =$$

$$a \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} |\psi\rangle + b \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} |\phi\rangle$$

11.2 Why you can't copy a quantum bit

Suppose you had a "copy" operator. If it was to faithfully produce a copy of a qbit, then it would have to work like this when you applied it to a zero or one:

$$\text{COPY}\ |0\rangle = |00\rangle$$
$$\text{COPY}\ |1\rangle = |11\rangle$$

Because all quantum operators are linear, you would get the following result if you applied the operator to an arbitrary state:

$$\text{COPY}\ |\psi\rangle = \text{COPY}(\ \alpha|0\rangle + \beta|1\rangle\ ) = \alpha\,\text{COPY}\,|0\rangle + \beta\,\text{COPY}\,|1\rangle = \alpha|00\rangle + \beta|11\rangle$$

However, if the operator actually copied the arbitrary state, you would have:

$$\text{COPY}\ |\psi\rangle = |\psi,\psi\rangle$$
$$= (\ \alpha|0\rangle + \beta|1\rangle\ ) \otimes (\ \alpha|0\rangle + \beta|1\rangle\ )$$
$$= \alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle$$

This contradiction, between two things that a COPY operator *must* do, means that no such operator exists. This result is known as the *no cloning theorem.* You can't copy (or "clone") a quantum bit. (Or any quantum state, as it turns out.)

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12 Quantum Teleportation

12.1 General Description

Suppose that Alice and Bob both have laboratories in which they do research on various types of quantum states. Perhaps Alice is in Mumbai and Bob is in Helsinki. However Alice and Bob do share an entangled pair of electrons, in the joint state: $\frac{1}{\sqrt{2}}|+z+z\rangle + \frac{1}{\sqrt{2}}|-z-z\rangle$. These electrons have been very carefully entangled and then transported to the two labs, where they have been kept isolated from any interactions that could affect their entanglement.

In addition to her entangled electron, Alice has a special electron in an *unknown* spin state which we'll call $|\psi\rangle$. When Bob hears about the new state $|\psi\rangle$, he tells Alice that he would really like to do an experiment on it in his lab.

Given the no-cloning theorem, we know that Alice can't simply make a copy of the unknown state. So it would appear that the only way for her to get this state to Bob would be to package up and send him the actual electron. But it turns out that by using *quantum teleportation* she can effectively send him the state without either physically moving the special electron or making a copy of its state.

The quantum teleportation algorithm consists of Alice manipulating her special electron in such a way as to entangle its state with the entangled pair that she and Bob already share. Then she is able to send Bob enough *classical information* so that he can operate on his previously entangled electron to put it into the special state $|\psi\rangle$.

First I'll provide a summary of the algorithm, and then a detailed "blow by blow" account. In these descriptions I'll be working with *bits* rather than spin states. Remember that a spin state of $|+z\rangle$ is the same thing as a bit state of $|0\rangle$, and $|-z\rangle$ is the same thing as $|1\rangle$. So the entangled state that Alice and Bob share is: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$. The spin state and the bit state are just *two different descriptions of the same thing.* Make sure you understand the relationship between the spin state and the bit state descriptions before going on.

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12.2 Summary of the algorithm

Just skim over this summary on first reading, to get an idea where we're going. Then go on to the detailed description. After you understand the details of how each step works, then come back to the summary, to get a picture of the procedure as a whole.

We'll break the teleportation algorithm down into five steps:

1. Alice has an unknown state: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

   Alice and Bob share the state: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

   So their initial joint state is: $\frac{1}{\sqrt{2}}(\ \alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle\ )$.

2. Alice does a CNOT on her two bits, resulting in:

   $\frac{1}{\sqrt{2}}(\ \alpha|000\rangle + \alpha|011\rangle + \beta|101\rangle + \beta|110\rangle\ )$

3. Alice does a Hadamard on her first bit:

   $\frac{1}{2}(\ \alpha|000\rangle + \alpha|011\rangle + \alpha|100\rangle + \alpha|111\rangle + \beta|001\rangle + \beta|010\rangle - \beta|101\rangle - \beta|110\rangle\ )$

   which can be rewritten as:

   $\frac{1}{2}|00\rangle\otimes(\alpha|0\rangle+\beta|1\rangle) + \frac{1}{2}|01\rangle\otimes(\alpha|1\rangle+\beta|0\rangle) + \frac{1}{2}|10\rangle\otimes(\alpha|0\rangle-\beta|1\rangle) + \frac{1}{2}|11\rangle\otimes(\alpha|1\rangle-\beta|0\rangle)$

4. Alice measures her two bits, obtaining the result $|m_1, m_2\rangle$.

5. Alice sends the two bits of her result m_1, m_2 to Bob. This tells Bob which of the four possibilities above actually happened.

6. Bob adjusts his bit to equal $|\psi\rangle$ by doing: $Z^{m_1}X^{m_2}|\text{Bob's bit}\rangle$.

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12.3 Detailed description of the algorithm

12.3.1 Setup

We have three quantum bits, which we'll call: b_1, b_2, and b_3.

- b_1 is Alice's special state $|\psi\rangle$, which she wants to transmit to Bob.
- b_2 is Alice's half of the entangled pair.
- b_3 is Bob's half of the entangled pair.

We will write Alice's special state $|\psi\rangle$ as: $\alpha|0\rangle + \beta|1\rangle$. Remember, we don't know what this state is, but we can still write it this way. It must be *some* superposition of zero and one. We just don't know what the values of α and β are. Or course one of those values *could* be zero.

To summarize:

- $b_1 = \alpha|0\rangle + \beta|1\rangle$
- $|b_2, b_3\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

And we can write the joint state of all three bits as

- $|b_1, b_2, b_3\rangle = \frac{1}{\sqrt{2}}\,(\ \alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle\ )$

12.3.2 Alice does a CNOT on her two bits

The first thing Alice does is a CNOT operation on her two bits b_1 and b_2. The unknown state in the bit b_1 is the control bit and b_2 is the bit that gets changed. There are two ways to think about how this operation works.

One way is to actually construct an eight by eight matrix which will do the job, and then use it to multiply the eight element column vector that represents Alice and Bob's three bit joint state. Since b_3 (Bob's bit) is not being operated on, you would construct the matrix by tensoring a two-bit CNOT with a one-bit identity matrix:

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$$(CNOT \otimes I) \,|\, b_1, b_2, b_3 \,\rangle$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \left(\, \alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle \, \right)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha/\sqrt{2} \\ 0 \\ 0 \\ \alpha/\sqrt{2} \\ \beta/\sqrt{2} \\ 0 \\ 0 \\ \beta/\sqrt{2} \end{pmatrix} = \begin{pmatrix} \alpha/\sqrt{2} \\ 0 \\ 0 \\ \alpha/\sqrt{2} \\ 0 \\ \beta/\sqrt{2} \\ \beta/\sqrt{2} \\ 0 \end{pmatrix}$$

This may look daunting based on the size of the matrix, but notice that there is only a single "one" bit in each row. So the operation it's not as bad as it looks.

Using the matrix form like this can provide a good double check, if you're not sure how the operation is going to work, and it's easy to do if you have a computer handy. But it's tedious (and error prone) to do by hand, and it doesn't give much insight into the operation.

The second way is the way we will use from now on. All you need to do, in order to do the calculations in Dirac (ket) notation, is to remember that a CNOT, like *all* quantum operators, is linear.

We apply the CNOT to the three bits (I'm not bothering to tensor in the identity matrix here. We understand that the CNOT is just operating on the first two bits).

$$CNOT \frac{1}{\sqrt{2}} \left(\, \alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle \, \right)$$

Because the operation is linear the scalars "move through" it, and we can distribute it over each of the kets.

$$\frac{1}{\sqrt{2}} \left(\, \alpha \, CNOT \, |000\rangle + \alpha \, CNOT \, |011\rangle + \beta \, CNOT \, |100\rangle + \beta \, CNOT \, |111\rangle \, \right)$$

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Then we actually perform the CNOT on each ket. If the first bit (the control bit) is zero we do nothing. If the first bit is one, we flip the second bit.

$$\frac{1}{\sqrt{2}}\left(\ \alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle\ \right)$$

Finally, we simply reorder the last two kets, so they are in the standard order.

$$\frac{1}{\sqrt{2}}\left(\ \alpha|000\rangle + \alpha|011\rangle + \beta|101\rangle + \beta|110\rangle\ \right)$$

At this point you should be able to see that our result is equivalent to the one obtained above by matrix multiplication.

12.3.3 Alice does a Hadamard on the her first bit

Next Alice applies a Hadamard operation to her first bit. This is the bit which originally contained the unknown state $|\psi\rangle$.

$$H\frac{1}{\sqrt{2}}\left(\ \alpha|000\rangle + \alpha|011\rangle + \beta|101\rangle + \beta|110\rangle\ \right)$$

We use linearity to get:

$$\frac{1}{\sqrt{2}}\left(\ \alpha H|000\rangle + \alpha H|011\rangle + \beta H|101\rangle + \beta H|110\rangle\ \right)$$

Then we apply the Hadamard to the first bit in each term, putting those bits into equal superpositions. (See section 10.2 to review how a Hadamard operation works.)

$$\frac{1}{\sqrt{2}}\left(\alpha\frac{1}{\sqrt{2}}(\ |0\rangle + |1\rangle\ )\,|00\rangle + \alpha\frac{1}{\sqrt{2}}(\ |0\rangle + |1\rangle\ )\,|11\rangle + \beta\frac{1}{\sqrt{2}}(\ |0\rangle - |1\rangle\ )\,|01\rangle + \beta\frac{1}{\sqrt{2}}(\ |0\rangle - |1\rangle\ )\,|10\rangle\right)$$

Collect the individual $1/\sqrt{2}$ factors in front:

$$\frac{1}{2}\left(\ \alpha(\ |0\rangle + |1\rangle\ )\,|00\rangle + \alpha(\ |0\rangle + |1\rangle\ )\,|11\rangle + \beta(\ |0\rangle - |1\rangle\ )\,|01\rangle + \beta(\ |0\rangle - |1\rangle\ )\,|10\rangle\ \right)$$

Put Alice's bits together:

$$\frac{1}{2}\left(\ \alpha(\ |00\rangle + |10\rangle\ )\,|0\rangle + \alpha(\ |01\rangle + |11\rangle\ )\,|1\rangle + \beta(\ |00\rangle - |10\rangle\ )\,|1\rangle + \beta(\ |01\rangle - |11\rangle\ )\,|0\rangle\ \right)$$

And we can now *factor out* Bob's bit:

$$\frac{1}{2}|00\rangle\otimes(\alpha|0\rangle+\beta|1\rangle) + \frac{1}{2}|01\rangle\otimes(\alpha|1\rangle+\beta|0\rangle) + \frac{1}{2}|10\rangle\otimes(\alpha|0\rangle-\beta|1\rangle) + \frac{1}{2}|11\rangle\otimes(\alpha|1\rangle-\beta|0\rangle)$$

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12.3.4 Alice measures her two bits

Alice now does a measurement on her two bits. As you can see from the three bit state, she has an equal chance of getting any of the possible two-bit results: $|00\rangle, |01\rangle, |10\rangle$, or $|11\rangle$

Notice that, although we don't know the values of α and β, once we know the result of Alice's measurement, we know exactly what Bob's state is *in terms of* α *and* β.

We will refer to Alice's result as $|m_1, m_2\rangle$, and the two bits individually simply as m_1 and m_2. So, for example, if $m_1 = 1$ and $m_2 = 0$ then Alice got $|10\rangle$ and Bob's state is now $(\alpha|0\rangle - \beta|1\rangle)$

12.3.5 Alice sends her 2-bit result to Bob

Now Alice sends Bob the two bits m_1 and m_2. Note that this is two bits of *classical* information. She can send it to him any way that we normally transmit classical information. She could send him a letter, an email, call him on the phone, or whatever. But however she does it, it will take some amount of time (in *all* frames of reference). So (for all observers) Bob will receive the information *after* Alice sends it.

12.3.6 Bob uses the classical information to operate on his bit

When Bob gets m_1 and m_2 he then knows exactly what state his bit is in. This is mapped out explicitly in the following table:

	$m_2 = 0$	$m_2 = 1$
$m_1 = 0$	$\alpha\|0\rangle + \beta\|1\rangle$	$\alpha\|1\rangle + \beta\|0\rangle$
$m_1 = 1$	$\alpha\|0\rangle - \beta\|1\rangle$	$\alpha\|1\rangle - \beta\|0\rangle$

Compare this table to the state before Alice's measurement:

$$\frac{1}{2}|00\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) + \frac{1}{2}|01\rangle \otimes (\alpha|1\rangle + \beta|0\rangle) + \frac{1}{2}|10\rangle \otimes (\alpha|0\rangle - \beta|1\rangle) + \frac{1}{2}|11\rangle \otimes (\alpha|1\rangle - \beta|0\rangle)$$

and make sure you understand the relationship between them.

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Now all Bob has to do is adjust his state so that it matches the original $\alpha|0\rangle + \beta|1\rangle$. It turns out that this can be done by using the X and Z operators. (Remember that X and Z are the same as σ_x and σ_z. We just use those names when we're using them as actual operators rather than matrices that represent measurements.)

(1) If $m_1 = 0$ and $m_2 = 0$ then Bob already has the state $\alpha|0\rangle + \beta|1\rangle$ and so he doean't have to do anything.

(2) If $m_1 = 0$ and $m_2 = 1$ then Bob has the state $\beta|0\rangle + \alpha|1\rangle$. He needs to switch the probability amplitudes. This can be done with the X (NOT) operator: $X(\beta|0\rangle+\alpha|1\rangle) = \alpha|0\rangle+\beta|1\rangle$.

(3) If $m_1 = 1$ and $m_2 = 0$ then Bob has the state $\alpha|0\rangle - \beta|1\rangle$. He needs to flip the sign on the second amplitude. The Z operator will do this: $Z(\alpha|0\rangle - \beta|1\rangle) = \alpha|0\rangle + \beta|1\rangle$.

(4) If $m_1 = 1$ and $m_2 = 1$ then Bob has the state $-\beta|0\rangle + \alpha|1\rangle$. So first he needs to switch and then to flip:

$X(-\beta|0\rangle + \alpha|1\rangle) = \alpha|0\rangle - \beta|1\rangle$

$Z(\alpha|0\rangle - \beta|1\rangle) = \alpha|0\rangle + \beta|1\rangle$

In a textbook you may see the four possibilities above summarized as:

$$Z^{m_1}X^{m_2}\,|\text{Bob}\rangle$$

Let me "decode" this expression. X and Z are matrices. Exponents on matrices work like exponents on regular old numbers, in the following sense: When you take a matrix to the first power it doesn't change it, and when you take a matrix to the zero power you get a "one" (the Identity matrix). So what the expression above is saying is that you actually apply either X or Z (or both) if the corresponding m bit is one.

12.4 Discussion

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13 Solutions

Exercise 1.1.1

$$\begin{pmatrix} 5 \\ 10 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$$

Exercise 1.1.2

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Exercise 1.1.3

$$3\begin{pmatrix} 3 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 6 \end{pmatrix} - \begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 4 \end{pmatrix}$$

Exercise 1.2.1

$$|x\rangle + |y\rangle = \begin{pmatrix} 3 \\ 7 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$$

$$\langle x| - \langle y| = \begin{pmatrix} 3 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \end{pmatrix}$$

$$|y\rangle - |x\rangle = \begin{pmatrix} 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$2|x\rangle - 3|y\rangle = 2\begin{pmatrix} 3 \\ 7 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \end{pmatrix} - \begin{pmatrix} 6 \\ 15 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Exercise 1.3.1

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}^T = \begin{pmatrix} 2 & 1 \end{pmatrix}$$

Exercise 1.3.2

$$\begin{pmatrix} a & b & c \end{pmatrix}^T = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

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Exercise 1.3.3

$$\langle x|y\rangle = \begin{pmatrix} 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 5 + 6 = 11$$

$$\langle x|z\rangle = \begin{pmatrix} 5 & 6 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 5z_1 + 6z_2$$

$$\langle z|z\rangle = \begin{pmatrix} z_1 & z_1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (z_1)^2 + (z_2)^2$$

Exercise 1.4.1

For $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$: $\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

For $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$: $\sqrt{1^2 + 2^2 + 3^3} = \sqrt{1 + 4 + 9} = \sqrt{14}$

For $\begin{pmatrix} a \\ b \end{pmatrix}$: $\sqrt{a^2 + b^2}$

For $\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$: $\sqrt{v_1^2 + v_2^2 + v_3^2}$

Exercise 1.5.1

The length of the vector is: $\sqrt{x^2 + y^2 + z^2}$, so the normalized vector is: $\dfrac{1}{\sqrt{x^2 + y^2 + z^2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Exercise 1.5.2

For the vector $(-2, 0)$ the normalization is $\dfrac{1}{2}(-2, 0) = (-1, 0)$.

For the vector $(-2, 1)$ the normalization is $\dfrac{1}{\sqrt{5}}(-2, 1)$

For the vector $(3, 2)$ the normalization is $\dfrac{1}{\sqrt{13}}(3, 2)$

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Exercise 3.9.1

(a) $a + b = i + 5 = 5 + i$

(b) $c + d = (1 + i) + (3 - 2i) = 1 + 3 + i - 2i = 4 - i$

(c) $c - d = (1 + i) - (3 - 2i) = 1 - 3 + i + 2i = -2 + 3i$

(d) $a\,c = i(1 + i) = i + i^2 = i - 1 = -1 + i$

(e) $b\,c = 5(1 + i) = 5 + 5i$

(f) $c\,d = (1+i)(3-2i) = (1)(3)+(1)(-2i)+(i)(3)+(i)(-2i) = 3-2i+3i-2i^2 = 3 - 2i + 3i + 2 = 5 + i$

(g) The complex conjugate of $a = a^* = -i$

(h) $b^* = 5$

(i) $c^* = (1 + i)^* = (1 - i)$

(j) $e^* = (2i - 1)^* = -2i - 1 = -1 - 2i$

(k) $c\,c^* = (1 + i)(1 - i) = 1 - i + 1 - i^2 = 1 + 1 = 2$

(l) $|d| = \sqrt{dd^*} = \sqrt{(3 - 2i)(3 + 2i)} = \sqrt{9 + 4} = \sqrt{13}$

(m) The modulus of $e = |e| = \sqrt{ee^*} = \sqrt{(-1 + 2i)(-1 - 2i)} = \sqrt{1 + 4} = \sqrt{5}$

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