Time Dilation

1. Draw the two reference frames, Alice has the mirror. Bob and Alice both agree on: c (the speed of light), v (the magnitude of their relative velocity), and h (the height of the mirror).

2. Alice bounces a photon of light off the mirror. She sees the photon's path as having a length of $2h$ traveled in time t_a. In other words, Alice measures the elapsed time as: $t_a = \frac{2h}{c}$.

3. Bob measures time t_b for the photon go from Alice to the mirror and back. But during that time he sees Alice's entire frame of reference move toward the right at velocity v. So he sees an equilateral triangle with a base of length vt_b.

4. Forming a right triangle from half of that base, with the height h as the other side, then during time $t_b/2$ Bob sees the photon travel the hypotenuse, which has length $\sqrt{h^2 + \frac{1}{4}v^2t_b^2}$.

5. Given that Bob saw the photon travel that same hypotenuse in time $t_b/2$, we conclude that $\frac{1}{2}ct_b = \sqrt{h^2 + \frac{1}{4}v^2t_b^2}$.

6. Solving for t_b

   $\frac{1}{4}c^2t_b^2 = h^2 + \frac{1}{4}v^2t_b^2$ (square both sides)

   $\frac{1}{4}c^2t_b^2 - \frac{1}{4}v^2t_b^2 = h^2$ (subtract term)

   $\frac{1}{4}t_b^2(c^2 - v^2) = h^2$ (factor left side)

   $t_b^2 = \frac{4h^2}{c^2 - v^2}$ (divide)

   $t_b = \frac{2h}{\sqrt{c^2 - v^2}}$ (take positive square root)

7. Since Alice was present at both "events" (explain) her time measurement is considered the *proper time*, noted as t_0, between those events.

8. We are looking for a conversion factor between the proper time and Bob's time: $t_0 \times ? = t_b$. That is:

   $\frac{2h}{c} \times ? = \frac{2h}{\sqrt{c^2 - v^2}}$

9. We can cancel the $2h$ from both sides, and then clearly: $\frac{1}{c}\left(\frac{c}{1}\frac{1}{\sqrt{c^2 - v^2}}\right) = \frac{1}{\sqrt{c^2 - v^2}}$

   Simplifying, we have the conversion factor: $\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

   And the general formula relating a proper time t_0 to the time t of any observer who does not see the proper time.

   $t = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}t_0$

10. What we can conclude from this is that the proper time is always the the shortest time observed. Any other time will appear to be longer (and hence "dilated").