Quantum Computing Notes

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Part I

Basic Math Concepts

1 Vectors

Quantum computing is based on *linear algebra*. Linear algebra is a very rich topic, having applications far beyond what we are going to discuss here. But for our purposes we can think of linear algebra as the mathematics of *vectors* and *matrices*. The next few sections will explain what this means.

1.1 What is a vector. Columns and rows

You can think of a vector as a list of numbers. We can have both *column vectors* and *row vectors*. Suppose that a vector contains the numbers 1 and 2. Then we could have either

the column vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$,

or the row vector $\begin{pmatrix} 1 & 2 \end{pmatrix}$.

Note that the first element of the column vector is on top, and the first element of the row vector is on the left. In other words, the elements are numbered either top to bottom or left to right.

Vectors can be added simply by adding the individual elements:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1+3 \\ 2+4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 7 & 9 \end{pmatrix}$$

You can add two row vectors or two columns vectors, but you can't add a row vector to a column vector.

Hopefully it's clear from the example above that vector addition is *commutative* (it doesn't matter in which order you add the vectors).

You can also multiply a vector by a number. In linear algebra the term *scalar* is used to distinguish plain numbers from vectors. So a scalar is just a regular old number. You can

multiply a scalar and a vector by separately multiplying each of the vectors elements by the scalar. This type of multiplication is commutative:

$$5\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}5 = \begin{pmatrix} 15 \\ 20 \end{pmatrix}$$

Exercise 1.1.1

Add the vectors $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Exercise 1.1.2

Add the vectors $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$.

Exercise 1.1.3

What do you get if you multiply $\begin{pmatrix} 3 & 2 \end{pmatrix}$ by three and then subtract $\begin{pmatrix} 1 & 2 \end{pmatrix}$?

1.2 Naming things. Dirac (bra-ket) notation

It you've encountered vectors before, say in a math or engineering book, you may have seen them written in either a bold font, or with little arrows above them. They were probably written as row vectors, perhaps with a comma between the elements. Various types of brackets, braces, or parentheses might have been used. For example:

$\mathbf{v} = (x, y, z)$

$\vec{v} = \{5, 22, 17\}$

$\mathbf{r} = [r, \phi, \theta]$

$\vec{\mathbf{v}} = (v_1, v_2, v_3)$

Note that, in the last example, *subscripts* are used instead of giving each element of the vector its own name.

In these notes, I am (mostly) going to reserve the use of single letter variable names for scalars: $a = 3$, $x = \sqrt{13}$, $\theta = \pi/2$, and so on. Since the individual elements of vectors are scalars, they will also be represented by letters. Often the elements will have a similar name as the vector, with a subscripts to distinguish them.

For the names of vectors I will be using what is called Dirac or Bra-Ket notation. It was invented by by Paul Dirac in the early part of the 20th century. This notation consists of the symbol $\langle\ |$ called a "bra", and the symbol $|\ \rangle$ called a "ket." Put them together: $\langle\ |\ \rangle$ and you get a "bra-ket" or bracket. A ket is a column vector and a bra is a row vector:

$$|a\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\langle b| = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}$$

Exercise 1.2.1

Let $|x\rangle = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$, and $|y\rangle = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

Find: $|x\rangle + |y\rangle$, $\langle x| - \langle y|$, $|y\rangle - |x\rangle$, and $2|x\rangle - 3|y\rangle$.

1.3 Vector multiplication, the Inner Product

Another thing that can be done to a vector is the *transpose* operation. If you start with a row vector and transpose it you get a column vector. If you transpose a column vector you get a row. The transpose operation is indicated by a capital "T" written as a *superscript* on the vector:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^T = \begin{pmatrix} a_1 & a_2 \end{pmatrix}$$

$$\begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}^T = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

So in bra-ket notation: $|v\rangle^T = \langle v|$ and $\langle v|^T = |v\rangle$.

Now let's talk about multiplying vectors. With regular numbers (scalars) there is only one kind of *product*. If $a = 3$ and $b = 4$ then $ab = (3)(4) = 12$. But when it comes to vectors there is more than one kind of product. I'll be talking about other kinds before long, but right now I want to define a kind of vector multiplication called the *inner product*. To take the inner product of two vectors, the first one must be a row vector and the second one a column vector. Then you multiply each element of the first vector with the corresponding element of the second vector add all the results together, like this:

$$\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} = (1)(6) + (2)(5) + (3)(4) = 6 + 10 + 12 = 28$$

Note that the inner product of two vectors is *not a vector*. It's just a number. Some books refer to the inner product as the "scalar product" since the result is a scalar.

In Dirac notation, the inner product is a bra-ket. In other words, the inner product of $\langle a|$ and $|b\rangle$ is simply written as $\langle a|b\rangle$. But when you need to find the actual numeric value then you'll write them out as a row and column to do the multiplication.

A couple of things to be aware of with regard to the inner product. First of all, the two vectors need to have the same number of elements. Otherwise you can't multiply the "corresponding" elements :-) Also, you might ask why the row vector has to come first. Why isn't this operation commutative? Please just accept for now that (in general) it isn't. The full explanation for why it is done this way will unfold as we get to other kinds of products and to complex numbers.

Exercise 1.3.1

Find the transpose of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Exercise 1.3.2

Find $\begin{pmatrix} a & b & c \end{pmatrix}^T$.

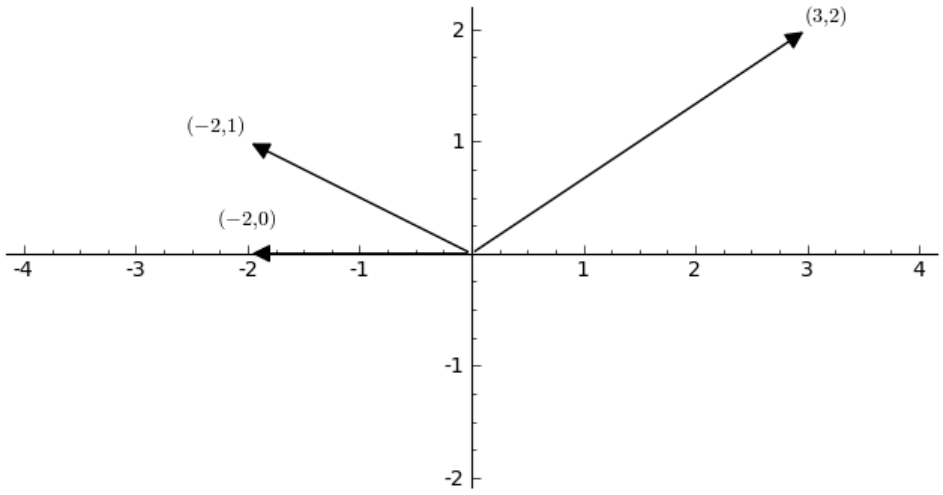
Exercise 1.3.3

Suppose $|x\rangle = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$, $|y\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and$|z\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$.

Find the inner products: $\langle x|y\rangle$, $\langle x|z\rangle$ and $\langle z|z\rangle$.

1.4 The absolute value or norm

When you have a vector with two elements, it can be visualized as an arrow on a two dimensional plane. The tail of the arrow is placed at the origin and the first and second elements of the vector give the arrow head's x and y coordinates. For example:



Now let's talk about the term *absolute value.* You might have previously thought about the absolute value of a number as something like "the number with the sign stripped off" or possibly learned a formula like: if x is positive then $|x| = x$ but if x is negative then $|x| = -x$. Now I want you to think about absolute value as the *length* of a vector. If the vector lies along the x axis, like the vector $(-2, 0)$ above, then it should be clear that the length of the vector corresponds to the definition of absolute value for a scalar (just "strip" the sign off the 2). If the vector doesn't lie along one of the axes, then we have to find its length using the "Pythagorean formula" where the length equals the square root of the sums of the squares of the x and y coordinates. So the length of the vector on the right side of the diagram is: $\sqrt{3^2 + 2^2} = \sqrt{13}$. The length of the upper one on the left side is: $\sqrt{(-2)^2 + 1^2} = \sqrt{5}$.

In general, if you have a vector $\langle v| = (v_1, v_2, ...v_n)$ then its length is: $\sqrt{v_1^2 + v_2^2 + ... + v_n^2}$. It doesn't matter whether it's a row or column vector, and this same formula will work even if the vector does happen to lie along one of the axes. For our purposes, the terms *absolute value*, *norm* and *modulus* all mean the same thing. They all refer to the length of a vector.

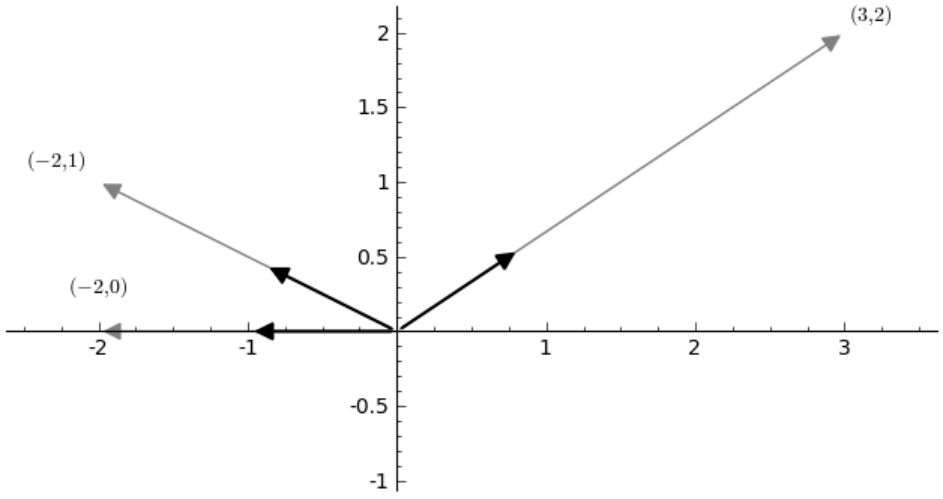
Exercise 1.4.1

Find the absolute values of: $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$, $\begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$

1.5 Normalization

Often we don't care about the length of a vector. We just want to know what direction the vector is pointing. In situations like this, what is needed are directional vectors that all have lengths of 1. To *normalize* a vector is to make its length equal to 1 without changing the direction in which it points. Normalization is accomplished simply by dividing each element of the vector by the vector's length. For example, if you have the vector: $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ then its length is $\sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, and the normalized vector is: $\frac{1}{\sqrt{14}} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

Here are the normalized vectors from the previous page:



Since the length of a vector is often called its norm, the process of dividing by the length is called normalization.

Exercise 1.5.1

Normalize the vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Exercise 1.5.2

Find the numeric values for the three normalized vectors in the diagram above.

1.6 Expressing the length of a vector using the inner product

The length of a vector can be expressed as the square root of inner product of the vector with itself:

$$\text{length of } |v\rangle = \sqrt{\langle v|v\rangle}$$

To see why this is true, suppose that:

$$|v\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

Then:

$$\sqrt{\langle v|v\rangle} = \sqrt{\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}} = \sqrt{a^2 + b^2}$$

It follows that we can express the normalized vector $|v\rangle$ as:

$$\frac{|v\rangle}{\sqrt{\langle v|v\rangle}}$$

The formula for the length of a vector using the inner product is not that important right now, but it will become significant when complex numbers are introduced.

1.7 Vector spaces

A vector *space* is "an abstract mathematical space in which all of the vectors we're talking about live." For example, when we draw the x/y coordinate system on the board, the "all the vectors we're talking about" are the arrows that can be drawn (starting from the origin) on the board. The vector space they "live" in is the 2-D plane. When we talk about "regular old" 3 dimensional space, then "all the vectors" are any arrow that you can draw in that space. So, for "real" spaces like this, the vector space is just a mathematical representation of the physical space. *Later* we will encounter other kinds of vector spaces, but that's enough for now.

1.8 Dimension

A vector space has a dimension, and an individual vector has a dimension. The dimension of a vector is the *same* as the dimension of the space that it lives in. You could think of the dimension as any of the following:

1. The number of elements in the vector.
2. The number of physical dimensions of the space (if it's a physical space).
3. The number of basis vectors necessary to specify a basis for the space.

All three of the above are equivalent.

1.9 Linear Combination

A *linear combination* of vectors is simply something like this:

$$a|v_1\rangle + b|v_2\rangle + c|v_3\rangle$$

where a, b, and c are *scalars* (which means they're just numbers, not vectors). In other words, a linear combination of vectors is just a bunch of vectors multiplied by something and added together. The only reason you need to know this is because the term "linear combination" is used all over the place.

1.10 Linear Independence

A vector $|v_1\rangle$ is *linearly independent* of the vectors $|v_2\rangle$ and $|v_3\rangle$ if there is no possible linear combination such that:

$$|v_1\rangle = a|v_2\rangle + b|v_3\rangle$$

A couple of examples:

1. Draw the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ on a piece of paper. (The arrow from the origin the the point 1 on the x-axis.) Then draw the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (The arrow going to the point x=2, y=3.) The vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is linearly independent of the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ because there is nothing you can multipy $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by to get $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

2. Now add the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to the drawing (the arrow going straight up the y-axis). The vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is *not* linearly independent of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ together because you can add them together in a linear combination to get $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

1.11 Bases and Basis Vectors

A *basis* for a given vector space is a *set* of vectors which can be used, in a linear combination, to make up *any* vector in the space. In other words, there are no vectors in the space which are linearly independent of the (full set of) basis vectors.

Typically, a vector space will have an infinite number of different *bases* that you can choose from. There is often one "obvious" basis, such as the two vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the 2-D plane.

1.12 Orthonormality

Two vectors are *orthogonal* to one another if their inner product is zero.
That is, $|v\rangle$ and $|w\rangle$ are orthogonal if:

$$\langle v|w\rangle = \langle w|v\rangle = 0$$

In a physical space, two orthogonal vectors have a 90° angle between them.

A vector can be called either *normal* or *normalized* if its length is 1.

A basis is *orthonormal* if all the basis vectors are normalized and each basis vector is orthogonal to every other basis vector.

1.13 Projection

A vector can be thought of as a list of *components*. For example the row vector $\begin{pmatrix} a & b & c \end{pmatrix}$ has the three components: "a", "b", and "c." Each component is simply a multiplier for one of the basis vectors.

For example, take the vector: $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ on the 2-D plane.
This vector is a linear combination of the basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The "2" is just the multiplier for the first basis vector and the "3" is the multiplier for the 2nd. If you "carry out the operations" from the formula above, it will go like this:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Another way to look at this is that the "2" is the vector's *projection* on to the x-axis and the "3" is the projection on to the y-axis. In this case "projection" just means drawing a line straight over from the head of the vector to the axis of interest.

If you have a vector $|v\rangle$ and a basis vector $|b\rangle$, then you can obtain the projection of $|v\rangle$ onto $|b\rangle$ by using the inner product, like this:

(the projection) = $\langle b|v\rangle$

The projection gives you one of the components of $|v\rangle$.

Let's try this with the basis vectors (1 0) and (0 1) and the vector (2 3). As you may have noticed, it's not always convenient to create column vectors when you're typing. So I'm just specifying the vectors in row format. But if I put them in a ket, then they're a column and if I put them in a bra, then they're a row. So, let's proceed:

The *basis* consists of two vectors:

$b_1 = (1\ 0)$ and $b_2 = (0\ 1)$

And the vector of interest is:

$v = (2\ 3)$

We find the first *component* of v by projecting it onto the first basis vector:

$\langle b_1|v\rangle = 2$

And the 2nd component by projecting it onto the 2nd basis vector:

$\langle b_2|v\rangle = 3$

Obviously, I'm not actually writing out all the steps here, so *please actually work out the two inner products above.*

1.14 Change of Basis

The inner products in the last section were really trivial, and might have seemed pointless. But this becomes more interesting when you want to change from one basis to another. I'm not going to try to write about this in detail here, but I'll give a short explanation. Try it out, if you can. In any event, we'll discuss it in detail on the board next Sunday.

Another possible basis for the 2-D plane is the set of vectors:

$b_1 = (1/\sqrt{2},\ \ 1/\sqrt{2})$ and $b_2 = (1/\sqrt{2},\ \ -1/\sqrt{2})$.
(I put in the commas here just to make things more clear.)

In case you don't recognize them, these were the "blue" basis vectors from last Sunday. Suppose that we want to "change to the blue basis." We can get the two components of the vector $v = (2\ 3)$ in the new basis by doing the two projections:

$\langle b_1|v\rangle = 5/\sqrt{2}$
$\langle b_2|v\rangle = -1/\sqrt{2}$

This means that, in the "blue" basis, the vector v would be

$$\left(\frac{5}{\sqrt{2}},\ \ \frac{-1}{\sqrt{2}}\right).$$

Or, equivalently, it means that we know how to construct v out of the "blue" basis vectors by doing the following linear combination:

$$v = \frac{5}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \ \frac{1}{\sqrt{2}}\right) + \frac{-1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \ \frac{-1}{\sqrt{2}}\right)$$
$$= \left(\frac{5}{2},\ \ \frac{5}{2}\right) + \left(\frac{-1}{2},\ \ \frac{1}{2}\right)$$
$$= \left(\frac{4}{2},\ \ \frac{6}{2}\right)$$
$$= \begin{pmatrix} 2 & 3 \end{pmatrix}$$

Here is one more thing. The general formula for the calculation of an arbitrary vector's components in an arbitrary basis. Given that we have the basis: $\{b_n\}$ (meaning that we have a basis made out of the set of basis vectors: b_1, b_2, ... and so on), then the nth component of v is:

$$v_n = \langle b_n|v\rangle$$

and the entire vector v is simply the sum of all those components times the corresponding basis vectors:

$$v = \sum_n \langle b_n|v\rangle|b_n\rangle$$

This formula is one that you will see often in, for example, quantum computing textbooks.

2 Matrices

This is just a *very* brief summary of what we've covered in person.

2.1 What is a matrix?

A matrix consists of *rows* and *columns*. The followin matrix is named M and it has 2 rows and three columns:

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

You could think of row 1 as the row vector $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and you could think of column 3 as the column vector $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$.

When we specify an individual element of a matrix, the first index always refers to the row and the second index to the column. So, in the matrix above:

$M_{12} = 2$
$M_{23} = 6$

We will be interested in the following matrix concepts:

1. Addition
2. Subtraction
3. Multiplication
4. Transpose
5. Identity
6. Inverse

Addition and subtraction are easy. You simply add (or subtract) corresponding elements (just like we did with vectors).

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

2.2 Matrix Multiplication

To multiply two matrices, you take the inner product of every row in the first matrix with every column in the second matrix. In other words, if you have three matrices A, B, and C:

$$A\,B = C$$

Then element C_{jk} is equal to the inner product of row j of A and column k of C.

Matrix multiplication is *not* commutative.

2.3 The Transpose of a Matrix

The transpose operation of a matrix M is denoted by M^T. Transposing a matrix turns row n into column n and column n into row n. When we transpose M the element M_{jk} becomes the element M_{kj}:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

2.4 The Identity Matrix

The *identity matrix* is a square matrix with ones in the main diagonal and zeros everywhere else:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The identity matrix is the matrix version of the number 1. If you multiply any matrix by the identity matrix it doesn't change. The identity matrix is denoted by the letter I.

2.5 The Inverse of a Matrix

The *inverse* of a matrix is denoted by M^{-1}. Not all matrices have an inverse. If a matrix *does* have an inverse, then the matrix times the inverse equals the identity matrix:

$$M\,M^{-1} = I$$

This doesn't tell you how to *find* the inverse of a matrix, but it will enable you to test whether a given matrix is the inverse of another one.

2.6 Multiplying a vector by a matrix. The Operator concept

Matrices are used for many different purposes, in different branches of mathematics. Our chief use of matrices will be as *operators* on vectors. When we multiply a matrix times a column vector, we are using the matrix as an operator on the vector:

output vector = matrix × input vector

The following are some examples in the 2D vector space, where it's easy to visualize what's happening. (You should draw a picture of each of these!)

"Minus" Operator: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}$

The minus operator gives the negative of the input vector, the same as multiplying it by -1.

X Reflection Operator: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}$

Reflects the vector about the x-axis.

Y Reflection Operator: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ b \end{pmatrix}$

Reflects the vector about the y-axis.

Rotation Operator: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$

Rotates the vector by θ degrees. (If you don't know trigonometry don't worry about this one. We'll discuss it in more detail if we ever end up using it.)

Identity Operator: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

Naturally, if we multiply a vector by the Identity Matrix, it doesn't change at all (which is why it's called the Identity Matrix).

Here's one last one, and this is one that we *will* be using:

The Not or "Swap" Operator: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$

This operator exchanges the 2 components of the vector. I'll explain later why this exchange of components is called a "Not" operation.

2.7 Eigenvalues and Eigenvectors.

Sometimes when we operate on a vector with a matrix, what we get back is simply a multiple of the original vector. In other words: $M\vec{v} = k\vec{v}$ where k is some scalar. Here's an example where $M\vec{v} = (3)\vec{v}$. (That is, $k = 3$.)

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix} = (3) \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Here's another one where $M\vec{v} = (-1)\vec{v}$. (So, $k = -1$):

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

When the pattern $M\vec{v} = k\vec{v}$ occurs, then we say that $\vec{v}$ is an *eigenvector* of the matrix M and k is the associated *eigenvalue*.

2.8 The Outer Product

As you may remember, the inner product is a row vector times a column vector. You can also think of this as multiplying a matrix consisting of a single row with a matrix consisting of a single column. The inner product yields a scalar, which is an object whose "rank" is one less than the objects that were multiplied together.

$$\text{Let } a = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$\langle a|b\rangle = \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = 8 + 15 = 23$$

The outer product is a ket times a bra, which would be a column times a row. What does this mean? If you think of it as a single column matrix times a single row matrix, you will see that the result is a matrix. An object whose rank is one *greater* than the objects that were multiplied together.

$$|a\rangle\langle b| = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \begin{pmatrix} 4 & 5 \end{pmatrix} = \begin{pmatrix} (2)(4) & (2)(5) \\ (3)(4) & (3)(5) \end{pmatrix} = \begin{pmatrix} 8 & 10 \\ 12 & 15 \end{pmatrix}$$

3 Complex Numbers

3.1 The number i

We say that "the number i" is the square root of negative one, or $i = \sqrt{-1}$.

There is nothing "deep" here. It's not as if we've somehow *discovered* what the square root of minus one is, and it turned out to be i. We have simply made a *definition*. We define a number. We name it i. Then we make a rule that when you square it you get minus one. $i^2 = -1$.

Note that i is not a *variable* like x or y. It is a *constant*. It names a specific number. It just happens to be a number that is not on "the number line."

The numbers on the number line are called the *real* numbers. Therefore, i is called an *imaginary* number. The terms "real" and "imaginary" are unfortunate. Cows and horses are real animals. Griffins and unicorns (so far as I know) are imaginary animals. It's not like that with "real" and "imaginary" numbers. They are all equally real in the normal english sense of the word. But we're stuck with these names.

3.2 Basic Definitions

1. Any number on the number line is a *real number.*

   Examples are: $-100,\ -2.4,\ 0,\ 1,\ \sqrt{2},\ e,\ \pi,\ \dfrac{927}{13}$.

2. *imaginary number* is i multiplied by any real number.

   For example: $2i,\ \dfrac{1}{2}i,\ (\sqrt{2})\,i$ and $\dfrac{3i}{7}$ are all imaginary numbers.

3. A *complex number* is a number that has a real part and an imaginary part.
   Here are some examples:

   $2 + 3i$ (the real part is 2 and the imaginary part is $3i$)

   $\sqrt{2}$ (the real part is $\sqrt{2}$ and the imaginary part is 0)

   i (the real part is 0 and the imaginary part is i)

As you can see, the complex numbers include the real and imaginary numbers. So, when we talk about "complex numbers" we are talking about real numbers, imaginary numbers, and numbers that are made up out of both real and imaginary parts.

It is typical to refer to a complex number by the variable name z and we say that: $z = a + b\,i$, where a and b are real. This is the same thing as definition 3 above.

I should also mention the functions Re() and Im(). You will see these functions (or variations on them) in various books. They will certainly be implemented in any programming languages that support complex numbers. If you have a complex variable $z = a + b\,i$. Then the expression Re(z) returns a, and Im(z) returns b. Note that Re(z) returns the real part of z. But Im(z) does not return what I am calling the imaginary part. It returns only the *real number associated with the imaginary part.*

3.3 Operations on Complex Numbers

Complex numbers have all the operations that real numbers do, including addition, subtraction, multiplication, division, absolute value, powers, roots, and so on. In addition, they have one more operation called *conjugation.*

For the time being, we will only be concerned with:

1. Addition
2. Subtraction
3. Multiplication
4. Conjugation
5. Absolute Value (or Modulus)
6. Division

Addition and subtraction are easy. You just add (or subtract) the corresponding parts of the number:

$$(2 + 3i) + (1 + 2i) = (2 + 1) + (3i + 2i) = 3 + 5i$$
$$(2 + 3i) - (1 + 2i) = (2 - 1) + (3i - 2i) = 1 + i$$

And, in general:

$$(a + bi) + (c + di) = (a + b) + (c + d)i$$

3.4 Multiplication

Multiplying complex numbers is just like multiplying polynomials:

$$
\begin{aligned}
&(a_1 + b_1 i)(a_2 + b_2 i) \\
&= a_1 a_2 + a_1 b_2 i + b_1 i a_2 + b_1 i\, b_2 i \quad \text{(multiply out all the terms)} \\
&= a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2 i^2 \quad \text{(rearrange things a bit)} \\
&= a_1 a_2 + a_1 b_2 i + a_2 b_1 i - b_1 b_2 \quad \text{(i squared is -1)} \\
&= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i \quad \text{(gather real and imag parts together)}
\end{aligned}
$$

Here are some examples. See if you can get the same results.

$$(1 + i)(2 + 3i) = -1 + 5i$$

$$(2 + 3i)(3 + 2i) = 13i$$

$$\left(\frac{1}{2} i\right)(1 - i) = \frac{1}{2} + \frac{i}{2}$$

3.5 Conjugation

If we have a complex number: $a + bi$ then the *complex conjugate* of that number is: $a - bi$. In other words, we simply "flip the sign" of the imaginary part. Complex conjugation is denoted in some books by a "*" and in other books by a bar over the number. (Either a star or a bar.)

Let $z = a + bi$, then:

$$z^* = (a + bi)^* = a - bi$$

$$\bar{z} = \overline{a + bi} = a - bi$$

I will most likely use the *. Here are some examples:

$$z = 1 + i, \quad z^* = 1 - i$$

$$z = -5 + \sqrt{2}\, i, \quad z^* = -5 - \sqrt{2}\, i$$

$$z = -5i, \quad z^* = 5i$$

3.6 Absolute Value or Modulus

The *absolute value* of a complex number is "the square root of the number times its conjugate."

$|z| = \sqrt{zz^*}$

If $z = a + bi$, then: $|z| = |a + bi| = \sqrt{(a + bi)(a - bi)} = \sqrt{a^2 + b^2}$ (work it out!)

Just like the absolute value of a real number or a vector, *the absolute value of a complex number is always a postitive real number.*

Terminology note: As the absolute value of a vector is somtimes called a "norm" the absolute value of a complex number is sometimes called a "modulus". The only reason you need to know about the terms "norm" and "modulus" is that you may run into them in books or papers. It's OK just to think of all these things as "absolute values."

3.7 Division

The best way to deal with division of complex numbers is to "rationalize" the denominator. This can be done by multiplying both the numerator and denominator by the conjugate of the denominator.

If I tell you to divide $3 + 2i$ by $1 + i$, then that means you have the fraction:

$$\frac{3 + 2i}{1 + i}$$

The denominator is $1 + i$ and its conjugate is $1 - i$. So, you multiply both the numerator and denominator by $1 - i$.

$$\frac{3 + 2i}{1 + i}\frac{1 - i}{1 - i}$$

Note that, since you are actually multiplying the original fraction by 1, you are not changing its value. You're just putting it in a different form.

$$\frac{3 + 2i}{1 + i}\frac{1 - i}{1 - i} = \frac{(3 + 2i)(1 - i)}{(1 + i)(1 - i)} = \frac{5 - i}{2} = \frac{5}{2} - \frac{1}{2}i$$

3.8 Argand diagrams (plotting numbers on the complex plane)

3.9 Complex number problems

Exercise 3.9.1

Given: $a = i,\ b = 5,\ c = 1 + i,\ d = 3 - 2i,\ e = 2i - 1,$ Find the following:

(a) $a + b$

(b) $c + d$

(c) $c - d$

(d) $a\,c$

(e) $b\,c$

(f) $c\,d$

(g) The complex conjugate of a

(h) b^*

(i) c^*

(j) e^*

(k) $c\,c^*$

(l) $|d|$

(m) The modulus of e

4 Complex Vector Spaces

A complex vector space is one where the vectors (an associated matrices) contain complex numbers as their elements. So the numbers inside a vector can be real numbers, imaginary numbers, or they can have both real and imaginary parts. Here's a dimension three vector in a complex space:

$$\begin{pmatrix} 2+3i \\ 10 \\ i\sqrt{3} \end{pmatrix}$$

4.1 The Adjoint Operator

In complex spaces, the *adjoint operator* becomes very important. The adjoint is denoted by a "dagger" (and is sometimes called the "dagger operator").

The adjoint of M is $M^\dagger$

The adjoint is a combination of the transpose and the complex conjugate operators. Whether we are taking the adjoint of a vector or a matrix, we transpose it and we take the complex conjugate of each of its elements. It doesn't matter whether we transpose or conjugate first.

The adjoint of a column vector:

$$\begin{pmatrix} a \\ b \end{pmatrix}^\dagger = \begin{pmatrix} a^* & b^* \end{pmatrix}$$

The adjoint of a row vector:

$$\begin{pmatrix} a & b \end{pmatrix}^\dagger = \begin{pmatrix} a^* \\ b^* \end{pmatrix}$$

The adjoint of a matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

Examples:

$$\begin{pmatrix} 2 & i \end{pmatrix}^\dagger = \begin{pmatrix} 2 \\ -i \end{pmatrix}, \qquad \begin{pmatrix} 2-3i \\ 1+i \end{pmatrix}^\dagger = \begin{pmatrix} 2+3i & 1-i \end{pmatrix}$$

$$\begin{pmatrix} 2 & i \\ 2-i & 1+2i \end{pmatrix}^\dagger = \begin{pmatrix} 2 & 2+i \\ -i & 1-2i \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

4.2 The adjoint operator changes some things!

The introduction of the adjoint operator in a complex vector space means we have to make a few changes.

We previously defined a bra as being the transpose of a ket. This was sufficient for real vector spaces. But in a complex vector space, the bra is actually the adjoint of a ket.

$$\text{If } |v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \text{then } \langle v| = \begin{pmatrix} v_1^* & v_2^* \end{pmatrix}$$

This "enhanced" definition of a bra changes the way the inner product works. The calculations we have previously done with the inner product are correct, but only because all the vector elements were real. With the addition of complex numbers, the inner product becomes:

$$\langle a|b\rangle = \begin{pmatrix} a_1^* & a_2^* \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1^* b_1 + a_2^* b_2$$

Notice that the inner product is no longer commutative:

$$\langle b|a\rangle = \begin{pmatrix} b_1^* & b_2^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 b_1^* + a_2 b_2^* \neq \langle a|b\rangle$$

The calculation of the length of a vector also changes. At this point it is better to calculate the length (or norm) of a vector using the inner product:

$$\text{length of } |v\rangle = \sqrt{\langle v|v\rangle}$$

This formula will yield the correct length even when the components are complex. For example, take the vector:

$$|v\rangle = \begin{pmatrix} 1+i \\ 1-2i \end{pmatrix}$$

Then:

$$\text{length of } |v\rangle = \sqrt{\langle v|v\rangle} = \sqrt{\begin{pmatrix} 1-i & 1+2i \end{pmatrix} \begin{pmatrix} 1+i \\ 1-2i \end{pmatrix}}$$

$$= \sqrt{(1-i)(1+i) + (1+2i)(1-2i)} = \sqrt{(1+1) + (1+4)} = \sqrt{7}$$

So we get a length with is a positive real number, as we must.

4.3 The Hermitian Matrix

A *Hermitian* matrix H is one where $H = H^\dagger$. In other words, *A Hermitian matrix is equal to its own adjoint.*

Examples:

(1) $\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ is Hermitian.

When you transpose it you get $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$.

Then, when you conjugate it you get $\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$.

Which is equal to the original matrix.

(2) $\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ is not Hermitian.

When you transpose it you get $\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$.

Then, when you conjugate it you get $\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$.

Which is not equal to the original matrix.

(3) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is Hermitian.

When you transpose it you still have $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

And conjugating it doesn't change it.

4.4 The Unitary Matrix

A *unitary* matrix is one where $U^\dagger = U^{-1}$. In other words *The adjoint of a unitary matrix is also its inverse.*

(It would be nice to find a couple of examples that are unitary but *not* Hermitian.)

Part II

Quantum Theory

5 The Postulates

In quantum theory, there are four basic "rules of the games." These are the same postulates or axioms that you will see in many quantum mechanics books. Here they have been simplified to apply to the situations are going to study.

1. The state of a quantum system is described by a column vector with complex components, for example: $\alpha|a\rangle + \beta|b\rangle$.

2. As long as we don't measure a system, the evolution of its state can be modeled by multiplying the current state by a unitary matrix.

3. The measurement of a system is represented using a Hermitian matrix. The result of the measurement is one of the eigenvalues of the matrix, and the resulting state is one of the eigenvectors of the matrix.

4. If a system is in the state $\alpha|a\rangle + \beta|b\rangle$, the probability of a measurement resulting in state a is $|\alpha|^2$ and the probability of a measurement resulting in state b is $|\beta|^2$. If a bit is in the state $|\psi\rangle$, the probability of a measurement resulting in the state $|\phi\rangle$ is $|\langle\phi|\psi\rangle|^2$.

Don't worry if these four paragraphs don't make much sense. We'll be investigating them in detail in the next few sections. In fact, one way to look at it is that learning about quantum theory is primarily about fully grasping the consequences of the postulates.

So later on, when you are confronted by a problem that you have no idea how to work, go back and ask: what do the posulates have to say about this situation?

6 Electron Spin

We'll spend quite a while talking about electron spin. This is because the "spin state" of an electron can be used as a model of a single quantum bit.

Then we talk about "spin" we are, in a sense, picturing the electron as a little ball spinning around. Of course, a real electron is nothing like this. This "picture" is simply a way of constructing a physical model we can visualize. This will help get a handle on the mathematical model, which otherwise would be pretty abstract. *Also please note that the "electron spin" we are talking about here is somewhat simplified from true spin model as you might study in a regular physics course, as we're ignoring elements that aren't necessary for our purposes here.*

So, think of an electron as a ball spinning around in three dimensional space. Suppose that you are above the electron looking down at it. Suppose that you see it spinning in a counter clockwise direction, from your perspective. Then we say the spin is "up" (that is toward you). If you see it spinning clockwise, then the spin is "down."

Another way to say this is that the "spin vector" obeys the so called "right hand rule." If you take an x-y-z coordinate system, and you curl the fingers of your right hand in the direction the ball is spinning, then if you stick out your thumb it points in the direction of the spin vector. So, for example, if your fingers go from the x axis toward the y axis, then the spin points up along the z axis.

When we write down the "state" of the electron spin, we do it like this:

$|+z\rangle$ means the spin vector points along the positive z axis (up)
$|-z\rangle$ means the spin vector points along the negative z axis (down)
$|+x\rangle$ the spin is along the positive x axis
$|-y\rangle$ the spin is along the negative y axis
and so on ...

Now, what happens when we *measure* the spin? Well, if the electron were really a little ball, then it would go like this: We measure the spin along, say, the x axis. But the ball isn't spinning *exactly* in that direction. So our measurement might show a spin 30 degrees off the x axis. But with a real electron this never happens. No matter what direction we measure the spin, the result we get is always exactly in that direction, or exactly in the opposite direction. So, if we measure spin along the x axis, we either get $|+x\rangle$ or $|-x\rangle$.

This shows that an electron is not really a little ball spinning around. An electron is a quantum object, which has some attribute which *reminds us* of spin, but it is really an attribute that we can't comprehend in classical terms. If follows quantum laws.

Our interest in electron spin stems from the fact that it can be used to model (or even implement) a single quantum bit, and it is simple enough that we can more or less visualize it.

When we measure the spin of an electron the **observable** which is the **eigenvalue** of the measurement **operator** will either be $+1$ or -1, depending on whether the spin turns out to be toward or away from the direction we choose to measure. Here is a summary for the directions along the three primary axes:

Operator	Matrix	Eigenvalue	Eigenstate	Eigenvalue	Eigenstate
σ_x	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$+1$	$\lvert +x\rangle$	-1	$\lvert -x\rangle$
σ_y	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$+1$	$\lvert +y\rangle$	-1	$\lvert -y\rangle$
σ_z	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$+1$	$\lvert +z\rangle$	-1	$\lvert -z\rangle$

And here are the column vector representations of each state:

$$|+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$|+y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |-y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

6.1 Some electron spin problems (week 9 homework)

1. Given the spin state: $\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$

   If you do a measurement in the z basis.

   What is the probability of getting $|+z\rangle$

2. Given the spin state: $\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$

   If you do the observation associated with the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

   What is the probability of getting the $+1$ eigenvalue

3. Given the spin state: $-\frac{4}{5}|+x\rangle - \frac{3}{5}|-x\rangle$

   If you do a measurement in the x basis.

   What is the probability of getting $|+x\rangle$

4. Given the spin state: $\frac{3}{\sqrt{10}}|+y\rangle + \frac{1}{\sqrt{10}}|-y\rangle$

   If you do the observation associated with the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

   What is the probability of seeing the 1 eigenvalue

5. Given the spin state: $-\frac{5+4i}{\sqrt{51}}|+x\rangle + \frac{3+i}{\sqrt{51}}|-x\rangle$

   If you do a measurement in the z basis.

   What is the probability of getting $|-z\rangle$

6. Given the spin state: $-\frac{1+2i}{\sqrt{10}}|+z\rangle + \frac{1+2i}{\sqrt{10}}|-z\rangle$

   If you do a measurement in the y basis.

   What is the probability of getting $|-y\rangle$

7. Given the spin state: $\frac{3i}{\sqrt{35}}|+z\rangle - \frac{1+5i}{\sqrt{35}}|-z\rangle$

   If you do the observation associated with the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

   What is the probability of seeing the -1 eigenvalue

8. Given the spin state: $\frac{4+3i}{5\sqrt{2}}|+z\rangle - \frac{3+4i}{5\sqrt{2}}|-z\rangle$

   If you do a measurement in the x basis.

   What is the probability of getting $|-x\rangle$

Solutions for week 9 problems

1. $\frac{1}{2}$ or 0.5

2. 1 or 1.0

3. $\frac{16}{25}$ or 0.64

4. $\frac{4}{5}$ or 0.8

5. $\frac{89}{102}$ or approximately 0.8725

6. $\frac{1}{2}$ or 0.5

7. $\frac{26}{35}$ or approximately 0.7429

8. $\frac{49}{50}$ or 0.98

7 A single quantum bit

To introduce the subject of quantum information, first we'll talk about a single quantum bit. A quantum bit is very different from the bits that are found in today's computers.

First of all, since these notes are largely aimed at programmers, I'm going to assume that you know the basics of computers, bits, bytes, and related things. You know that a bit can represent the numbers 0 or 1, that a byte is 8 bits and can represent a number between 0 and 255, and so on.

In these notes, I'll generally refer to the kind of a bit that you would find in current (early 21st century) computers as a *classical bit*. We're primarily concerned with the bits in a quantum computer. I'll use the term *quantum bit* initially just to highlight that point. But after we get warmed up you can assume the "bits" being discussed are quantum bits. If we need to talk about a classical bit, I'll call them out as such.

The postulates adapted for a single qbit

1. The state of a quantum bit is described by a two-element vector with complex components, and may be written as: $\alpha|0\rangle + \beta|1\rangle$.
2. As long as we don't measure a bit, the evolution of its state (i.e., sending the bit through a "quantum gate") can be modeled by multiplying the current state by a unitary matrix.
3. The measurement of a bit is represented using a Hermitian matrix. The result of the measurement is one of the eigenvalues of the matrix, and the resulting state is one of the eigenvectors of the matrix.
4. If a bit is in the state $\alpha|0\rangle + \beta|1\rangle$, the probability of a measurement resulting in a "0" is $|\alpha|^2$ and the probability of a measurement resulting in a "1" is $|\beta|^2$. If a bit is in the state $|\psi\rangle$, the probability of a measurement resulting in the state $|\phi\rangle$ is $|\langle\phi|\psi\rangle|^2$.

7.1 The State Vector

The state of a quantum bit is described by a two-element vector with complex components, and may be written as: $\alpha|0\rangle + \beta|1\rangle$.

Let's break this statement down. The "state" of a quantum bit. What does that mean? The "state" of a *classical* bit is either 0 or 1. Apparently a quantum bit is a lot more complicated. To describe its state we need not only a vector, but one containing complex numbers.

Now, by convention, the *classical* bit states 0 and 1 are described by the kets $|0\rangle$ and $|1\rangle$, which are equal to the following column vectors:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These two vectors above are sometimes referred to as the *classical basis* because a measurement using them as basis vectors will result in either a zero or one, just like a classical bit.

So, bearing in mind that α and β are any complex numbers, the state of a bit can be any combination of:

$$\alpha|0\rangle + \beta|1\rangle = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Looking at the column vector on the right you can see that the state is, ultimatly, a single vector with two complex components. This by the way is sometimes referred to as the *state vector.*

Since the quantum bit is equal to α times a classical zero plus β times a classical one, in a certain sense the quantum bit is a combination of the two classical states. If $\alpha = 0$ and $\beta = 1$ then the bit is 1. If $\alpha = 1$ and $\beta = 0$ then the bit is 0. If both α and β are nonzero then the qbit is in what's called a *superposition* of a classical zero and one.

8 Putting bits together

8.1 The Tensor Product

In order to "put bits together" we need a new kind of product, called the *tensor product.* The tensor product can be applied to either vectors or matrices. The symbol used for the tensor product is a circle with a slanted cross in it, like this: $\otimes$.

If we have two vectors:

$$v_1 = \begin{pmatrix} a \\ b \end{pmatrix}, \text{ and } v_2 = \begin{pmatrix} x \\ y \end{pmatrix},$$

then the tensor product of v_1 and v_2 is:

$$v_1 \otimes v_2 = \begin{pmatrix} a\, v_2 \\ b\, v_2 \end{pmatrix} = \begin{pmatrix} a \begin{pmatrix} x \\ y \end{pmatrix} \\ b \begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ax \\ ay \\ bx \\ by \end{pmatrix}$$

If we have two matrices:

$$M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ and } M_1 = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

then the tensor product of M_1 and M_2 is:

$$M_1 \otimes M_2 = \begin{pmatrix} a\, M_2 & b\, M_2 \\ c\, M_2 & d\, M_2 \end{pmatrix} = \begin{pmatrix} a \begin{pmatrix} w & x \\ y & z \end{pmatrix} & b \begin{pmatrix} w & x \\ y & z \end{pmatrix} \\ c \begin{pmatrix} w & x \\ y & z \end{pmatrix} & d \begin{pmatrix} w & x \\ y & z \end{pmatrix} \end{pmatrix} = \begin{pmatrix} aw & ax & bw & bx \\ ay & az & by & bz \\ cw & cx & dw & dx \\ cy & cz & dy & dz \end{pmatrix}$$

Note that the tensor product, just like the inner and matrix products, is *not* commutative. For example, if we switch the order of the vectors above:

$$v_2 \otimes v_1 = \begin{pmatrix} x\, v_1 \\ y\, v_1 \end{pmatrix} = \begin{pmatrix} x \begin{pmatrix} a \\ b \end{pmatrix} \\ y \begin{pmatrix} a \\ b \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ax \\ bx \\ ay \\ by \end{pmatrix}$$

8.2 The tensor product in bra-ket notation

Say we have two vectors bra-ket notation: $|a\rangle$ and $|b\rangle$. Then the "correct" way of writing the tensor product of the two would be: $|a\rangle \otimes |b\rangle$. But since there is no other valid operation for multiplying a "ket times a ket" the tensor product can be abbreviated as $|a\rangle|b\rangle$. In certain contexts, we simply write $|ab\rangle$.

8.3 The classical basis for 2 bits

The *classical basis* is the basis used to represent ones and zeros. It was the "default" basis that we already used for the representation of a single bit. We get the basis for two bits by taking the tensor product of the one bit basis vectors. Of course, if you're familiar with computers, this may have been obvious to you already. This is just like stringing bits together in the "base 2" numbering system.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |0\rangle \otimes |0\rangle = |00\rangle = 0$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |0\rangle \otimes |1\rangle = |01\rangle = 1$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |1\rangle \otimes |0\rangle = |10\rangle = 2$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |1\rangle \otimes |1\rangle = |11\rangle = 3$$

8.4 Product States

A two bit *product state* is a state that can be created by multiplying two individual bits together. First, let's look at the general tensor product in bra-ket notation. Say we have the following two one-bit states:

$a|0\rangle + b|1\rangle$ and $x|0\rangle + y|1\rangle$

We can multiply them together by treating them just like polynomials:

$$\begin{aligned}
&(a|0\rangle + b|1\rangle) (x|0\rangle + y|1\rangle) \\
&= a|0\rangle x|0\rangle + a|0\rangle y|1\rangle + b|1\rangle x|0\rangle + b|1\rangle y|1\rangle \\
&= ax|0\rangle|0\rangle + ay|0\rangle|1\rangle + bx|1\rangle|0\rangle + by|1\rangle|1\rangle \\
&= ax|00\rangle + ay|01\rangle + bx|10\rangle + by|11\rangle
\end{aligned}$$

It's important to understand that the result above can be written in vector form, and that it is the *same* vector that we got when we previously did the tensor product in vector notation:

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ ay \\ bx \\ by \end{pmatrix}$$

Now, suppose we have the state:

$$\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

and we want to know whether or not it's a product state. If it *is* a product state, then we can find some a, b, x, y such that:

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ ay \\ bx \\ by \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} a \times x &= 1/2 \\ a \times y &= 1/2 \\ b \times x &= 1/2 \\ b \times y &= 1/2 \end{aligned}$$

There are actually an infinite number of solutions to the four equations on the right. But once you normalize any of them, you will get one unique solution:

a, b, x, y are all equal to $\frac{1}{\sqrt{2}}$, and

$$\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right) \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right)$$

8.5 Entangled States

A two-bit *entangled* state is a state that can *not* be factored into two individual one-bit states. For example, consider the state:

$$\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

If we try to factor this state, as we did previously, into the two single-bit states $a|0\rangle + b|1\rangle$ and $x|0\rangle + y|0\rangle$, then we get:

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ ay \\ bx \\ by \end{pmatrix} = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{l} a \times x = 0 \\ a \times y = 1/\sqrt{2} \\ b \times x = 1/\sqrt{2} \\ b \times y = 0 \end{array}$$

From the equations on the right, you can see that:

1. a cannot be zero if $a \times y$ is to equal $1/\sqrt{2}$.
2. x cannot be zero if $b \times x$ is to equal $1/\sqrt{2}$.
3. But since both a and x are non-zero, $a \times x$ cannot be zero.

The conclusion is that there are *no values of* a, b, x, y *for which these equations are true.*

Stop for a moment and consider this. We have a perfectly good 2-bit state. A pair of quantum bits can "be in" this state. But it is not a "combination" of two individual bit states. This is entanglement. *Many people consider entanglement to be the fundamental reason for the weirdness of quantum systems.*

8.6 Cartesian vs Tensor product spaces

(TBD)

8.7 Questions for discussion

Alice and Bob each have an electron. We can represent the *pair* of electrons in a ket, by specifying Alice's state first and then Bob's. In other words: $|+z, -z\rangle$ means that Alices's electron has a $+z$ spin and Bob's has a $-z$ spin.

Now, suppose that all we know is the *joint* state of the two electrons:

$$\frac{1}{2\sqrt{2}}|+z,+z\rangle + \frac{1}{2\sqrt{2}}|+z,-z\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|-z,+z\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|-z,-z\rangle$$

Question 1: Assuming that both Alice and Bob do a "z measurement" on their electrons, what is the probability that they will both get a $-z$ result?

Question 2: Without trying to factor the state, can you determine the probability of Alice seeing a $+z$ result?

Question 3: Is this a product state or an entangled state? If it is a product state, then what are the two individual electron states?

Now, suppose that we have two bits, in the joint state:

$$\frac{1}{4}|00\rangle + \sqrt{\frac{7}{8}}|11\rangle + \frac{1}{4}|01\rangle$$

Question 4: What is the probability that a joint measurement will result in $|10\rangle$?

Question 5: Without trying to factor the state, can you determine the probability, for each of the individual bits, of getting a 1?

Question 6: Is this a product state or an entangled state? If it is a product state, then what are the two individual electron states?

9 Multi-bit Operations

9.1 What are operators?

A *Unitary operator* changes the state of a system (postuate 2).
if U is the operator then:

$U|state1\rangle = |state2\rangle$
(which is often written as)
$U|\psi\rangle = |\phi\rangle$

For us, operators are matrices and states are vectors. So this ends up being:

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} u_{11}\psi_1 + u_{12}\psi_2 \\ u_{21}\psi_1 + u_{22}\psi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

So a unitary operator actually does "operate" on a state, and it does so by multiplying the matrix representing the operator with the vector representing the state, to yield a new state vector.

A *Hermitian operator* does not actually "operate" on a state. Rather, it *represents* a specific measurement. This is described in postulate 3. It might be good to review that postulate right now.

9.2 Review of single bit operations

We very briefly touched on the concept of matrices as operators back in section 2.6. There wasn't a lot to say because there are not a lot of interesting one-bit operations. But once you have two bits (or two electrons) there are a lot more possible operations, and a lot more to say about how operators are "constructed."

But first let's just review the three interesting one-bit operators: I, NOT, and H.

The I (or Identity) operator is: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

It's called the "identity" operator because it doesn't change the state:

$I|\psi\rangle = |\psi\rangle$

The NOT operator is: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The one-bit NOT operator switches the components of the vector it operates on:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

As a result of switching the components, NOT will change a one into a zero and a zero into a one.

$$NOT|1\rangle = |0\rangle,\ NOT|0\rangle = |1\rangle$$

This is the same effect as a "NOT gate" in classical computer terminology, which is why this operator is called "NOT."

Notice that the NOT operator is the same matrix as σ_x, the measurement operator for electron spin on the x axis. This matrix is sometimes denoted simply by the letter X.

The H (or Hadamard) operator is: $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

The Hadamard operator is named after the mathematician of the same name. It is, arguably, the most important operator in quantum computation. Basically, the Hadamard operator *places a bit into an equal superpostition:*

$$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Much of the "magic" of quantum computing comes from working on bits in superposition. So the Hadamard operation is fundamental.

Both H and NOT are "square roots" of the Identity operator:

$$H * H = NOT * NOT = I$$

So if you take either a Hadamard or a NOT operator and apply it twice, you will get back the same state that you started with.

9.3 Putting one-bit operators together to form two-bit operators

In section 8.1 we discussed how to use the tensor product to put both vectors and matrices together to form higher dimension objects. Please go back and review that section now.

In some cases, you can construct the two-bit operation you want by using the tensor product to put together two one-bit operators. Say you have a two bit state $|01\rangle$ and you want to flip both the bits. In this case you simply tensor two one-bit NOT operators:

$$NOT2 = NOT \otimes NOT = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$NOT2\,|01\rangle = |10\rangle$
$NOT2\,|10\rangle = |01\rangle$
$NOT2\,|00\rangle = |11\rangle$

Now suppose that you only wanted to flip the first bit and leave the second one alone. This would be like applying a NOT to the first bit and applying the Identity operator to the second bit. So you tensor those two:

$$NOT \otimes I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$NOT \otimes I\,|01\rangle = |11\rangle$
$NOT \otimes I\,|10\rangle = |00\rangle$

Similarly, you can construct an operator to flip only the second bit by doing the tensor product in the reverse order:

$I \otimes NOT\,|01\rangle = |00\rangle$
$I \otimes NOT\,|10\rangle = |11\rangle$

9.4 Other two-bit operators

Of course there are more multi-bit operators than the ones which can be made by putting together single-bit operators. We'll look at two interesting ones here: SWAP and CNOT.

The SWAP operator exchanges (swaps) the two bits.

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{SWAP}\,|00\rangle = |00\rangle$$
$$\text{SWAP}\,|01\rangle = |10\rangle$$
$$\text{SWAP}\,|10\rangle = |01\rangle$$
$$\text{SWAP}\,|11\rangle = |11\rangle$$

CNOT stands for "controlled not." In this two-bit operator, the first bit of the state is the control bit and the second bit of the state is the target bit. So, for example, in the state $|01\rangle$ the control bit is zero and the target bit is one.

Here's how CNOT works. If the control bit is zero, then nothing happens. If the control bit is one, then a NOT operation is executed on the target bit. In other words, the target bit gets flipped only if the control bit is one.

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\text{CNOT}\,|00\rangle = |00\rangle$$
$$\text{CNOT}\,|01\rangle = |01\rangle$$
$$\text{CNOT}\,|10\rangle = |11\rangle$$
$$\text{CNOT}\,|11\rangle = |10\rangle$$

CNOT is one of the most important operators in quantum computing. It will also be used before too long in the description of "quantum teleportation."

10 Linearity and the No Cloning Theorem

10.1 What is linearity?

I briefly mentioned a linear combination in sec 1.9, but didn't really define what "linearity" was, per se. It goes like this: If you have two vectors $|v_1\rangle$ and $|v_2\rangle$, and two scalars α and β, then L is a linear operator if:

$$L\ (\ \alpha|\psi\rangle + \beta|\phi\rangle\ ) = \alpha L|\psi\rangle + \beta L|\phi\rangle$$

It should be relatively easy to see that that multiplication by a matrix is a linear operation:

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} (\ a|\psi\rangle + b|\phi\rangle\ ) =$$

$$a \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} |\psi\rangle + b \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} |\phi\rangle$$

10.2 Why you can't copy a quantum bit

Suppose you had a "copy" operator. If it was to faithfully produce a copy of a qbit, then it would have to work like this when you applied it to a zero or one:

$$\text{COPY}\ |0\rangle = |00\rangle$$
$$\text{COPY}\ |1\rangle = |11\rangle$$

Because all quantum operators are linear, you would get the following result if you applied the operator to an arbitrary state:

$$\text{COPY}\ |\psi\rangle = \text{COPY}(\ \alpha|0\rangle + \beta|1\rangle\ ) = \alpha\,\text{COPY}\,|0\rangle + \beta\,\text{COPY}\,|1\rangle = \alpha|00\rangle + \beta|11\rangle$$

However, if the operator actually copied the arbitrary state, you would have:

$$\text{COPY}\ |\psi\rangle = |\psi,\psi\rangle$$
$$= (\ \alpha|0\rangle + \beta|1\rangle\ ) \otimes (\ \alpha|0\rangle + \beta|1\rangle\ )$$
$$= \alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle$$

This contradiction, between two things that a COPY operator *must* do, means that no such operator exists. This result is known as the *no cloning theorem.* You can't copy (or "clone") a quantum bit. (Or any quantum state, as it turns out.)

11 Solutions

Exercise 1.1.1

$$\begin{pmatrix} 5 \\ 10 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$$

Exercise 1.1.2

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Exercise 1.1.3

$$3\begin{pmatrix} 3 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 6 \end{pmatrix} - \begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 4 \end{pmatrix}$$

Exercise 1.2.1

$$|x\rangle + |y\rangle = \begin{pmatrix} 3 \\ 7 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$$

$$\langle x| - \langle y| = \begin{pmatrix} 3 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \end{pmatrix}$$

$$|y\rangle - |x\rangle = \begin{pmatrix} 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$2|x\rangle - 3|y\rangle = 2\begin{pmatrix} 3 \\ 7 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \end{pmatrix} - \begin{pmatrix} 6 \\ 15 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Exercise 1.3.1

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}^T = \begin{pmatrix} 2 & 1 \end{pmatrix}$$

Exercise 1.3.2

$$\begin{pmatrix} a & b & c \end{pmatrix}^T = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Exercise 1.3.3

$$\langle x|y\rangle = \begin{pmatrix} 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 5 + 6 = 11$$

$$\langle x|z\rangle = \begin{pmatrix} 5 & 6 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 5z_1 + 6z_2$$

$$\langle z|z\rangle = \begin{pmatrix} z_1 & z_1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (z_1)^2 + (z_2)^2$$

Exercise 1.4.1

For $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$: $\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

For $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$: $\sqrt{1^2 + 2^2 + 3^3} = \sqrt{1 + 4 + 9} = \sqrt{14}$

For $\begin{pmatrix} a \\ b \end{pmatrix}$: $\sqrt{a^2 + b^2}$

For $\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$: $\sqrt{v_1^2 + v_2^2 + v_3^2}$

Exercise 1.5.1

The length of the vector is: $\sqrt{x^2 + y^2 + z^2}$, so the normalized vector is: $\dfrac{1}{\sqrt{x^2 + y^2 + z^2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Exercise 1.5.2

For the vector $(-2, 0)$ the normalization is $\dfrac{1}{2}(-2, 0) = (-1, 0)$.

For the vector $(-2, 1)$ the normalization is $\dfrac{1}{\sqrt{5}}(-2, 1)$

For the vector $(3, 2)$ the normalization is $\dfrac{1}{\sqrt{13}}(3, 2)$

Exercise 3.9.1

(a) $a + b = i + 5 = 5 + i$

(b) $c + d = (1 + i) + (3 - 2i) = 1 + 3 + i - 2i = 4 - i$

(c) $c - d = (1 + i) - (3 - 2i) = 1 - 3 + i + 2i = -2 + 3i$

(d) $a\,c = i(1 + i) = i + i^2 = i - 1 = -1 + i$

(e) $b\,c = 5(1 + i) = 5 + 5i$

(f) $c\,d = (1+i)(3-2i) = (1)(3)+(1)(-2i)+(i)(3)+(i)(-2i) = 3-2i+3i-2i^2 = 3 - 2i + 3i + 2 = 5 + i$

(g) The complex conjugate of $a = a^* = -i$

(h) $b^* = 5$

(i) $c^* = (1 + i)^* = (1 - i)$

(j) $e^* = (2i - 1)^* = -2i - 1 = -1 - 2i$

(k) $c\,c^* = (1 + i)(1 - i) = 1 - i + 1 - i^2 = 1 + 1 = 2$

(l) $|d| = \sqrt{dd^*} = \sqrt{(3 - 2i)(3 + 2i)} = \sqrt{9 + 4} = \sqrt{13}$

(m) The modulus of $e = |e| = \sqrt{ee^*} = \sqrt{(-1 + 2i)(-1 - 2i)} = \sqrt{1 + 4} = \sqrt{5}$