Quantum Study Group - 2nd semester notes

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1 Additional matrix arithmetic

1.1 The determinant

1.2 The inverse

1.3 Eigenvalues and eigenvectors

Given a matrix M and a vector $|v\rangle$, if $M|v\rangle = \lambda|v\rangle$ we say that $|v\rangle$ is an eigenvector of M and λ is the eigenvalue associated with $|v\rangle$. In the situations we will deal with it is typical for a matrix of dimension n to have n eigenvectors, each with an associated eigenvalue.

Given a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and a vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ some algebra will show that, for the vector to be an eigenvector of the matrix, the eigenvalue λ must satisfy the 2nd degree polynomial equation:

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

This is sometimes called the *characteristic equation* of the matrix. This equation can always be solved, either by factoring or by the quadratic formula:

$$\lambda = \frac{1}{2}\left(-B \pm \sqrt{B^2 - 4C}\right), \quad B = -(a+d),\ C = (ad - bc)$$

But the fact that this equation can always be solved for λ does not mean that every 2×2 matrix has valid eigenvectors. for example, it will sometimes it will turn out either that λ is zero, or that the resulting eigenvectors are filled with zeros. In cases like this, we do not have valid eigenvectors.

Once you have found the possible λs (there will typically be two different ones, for a dimension 2 matrix) then you take each one and plug it back in to the original eigenvalue equation. What you will end up with is simply a *ratio between the two components of the vector.* For the matrix and vector above, you will get:

$$\alpha = \frac{-b\beta}{a - \lambda}$$

For matrices with dimension of n greater than 2, the same principle still works, but the characteristic equation will be of the nth degree, and may not solvable. Normally we would go to the computer to find eigenvalues if we're dealing with more than a 'one bit' state.

http://www.wolframalpha.com will accept a query like this:

eigenvalues {{0,1,2,0},{1,0,1,1},{1,1,1,1},{0,0,0,1}}

http://www.sympygamma.com will take something like this:

Matrix([(0,1,2,0),(1,0,1,1),(1,1,1,1),(0,0,0,1)])

1.4 Change of basis operators

1.5 Diagonalizing a matrix

2 Working in Dirac notation

2.1 Review of the three products in vector notation

2.2 Inner products

2.3 Outer product representation of matrices

2.4 Outer products

2.5 Tensor products

2.6 Applying an operator to a state

2.7 Problems

For the problems below, do all the arithmetic in Dirac notation. Do not write out any vectors or matrices explicitly, either while working the problems, or as a final result (unless the problem specifically says to).

For all problems, let $\psi = a|+z\rangle + b|-z\rangle$, and $\phi = c|+z\rangle + d|-z\rangle$.

1. Work out the inner product: $\langle\psi|\phi\rangle$.
2. Work out the outer product: $|\psi\rangle\langle\phi|$. Then go ahead and write it as a matrix at the end.
3. Write each of the three Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$ as outer products.
4. Apply the *not* operator (which is the same as σ_x) to the state $|\psi\rangle$.

2.8 Solutions

1. $(a^*\langle +z| + b^*\langle -z|)\,(c|+z\rangle + d|-z\rangle)$

   $= a^*c\langle +z|+z\rangle + a^*d\langle +z|-z\rangle + b^*c\langle -z|+z\rangle + b^*d\langle -z|-z\rangle$

   $= a^*c(1) + a^*d(0) + b^*c(0) + b^*d(1)$

   $= a^*c + b^*d$

2. $(a|+z\rangle + b|-z\rangle)(\phi = c^*\langle +z| + d^*\langle -z|)$

   $= ac^*|+z\rangle\langle +z| + ad^*|+z\rangle\langle -z| + bc^*|-z\rangle\langle +z| + bd^*|-z\rangle\langle -z|$

   $$= \begin{pmatrix} ac^* & ad^* \\ bc^* & bd^* \end{pmatrix}$$

3. $$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (0)\,|0\rangle\langle 0| + (1)\,|0\rangle\langle 1| + (1)\,|1\rangle\langle 0| + (0)\,|1\rangle\langle 1| = |0\rangle\langle 1| + |1\rangle\langle 0|$$

   $$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = (0)\,|0\rangle\langle 0| + (-i)\,|0\rangle\langle 1| + (i)\,|1\rangle\langle 0| + (0)\,|1\rangle\langle 1| = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$$

   $$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (1)\,|0\rangle\langle 0| + (0)\,|0\rangle\langle 1| + (0)\,|1\rangle\langle 0| + (-1)\,|1\rangle\langle 1| = |0\rangle\langle 0| - |1\rangle\langle 1|$$

4. (tbd)

3 Summation notation

3.1 Introduction and examples

3.2 Identity matrix

3.3 Matrix full of ones

3.4 Matrix with specific values

3.5 Problems

1. For each summation first (i) expand the series, then (ii) calculate the result.

   Example: $\sum_{n=1}^{6} n+1$, Solution: (i) (1+1)+(2+1)+(3+1)+(4+1)+(5+1)+(6+1), (ii) 27

   Note that I grouped the individual terms in parentheses. Do that if it will make things clearer.

   (a) $\sum_{n=1}^{5} n$

   (b) $\sum_{m=1}^{3} 2m + 1$

   (c) $\sum_{x=0}^{2} x^2 - x$

2. For each summation, simply expand the series

   Example: $\sum_{n=1}^{4} a^n + bn$, Solution: $(a + b) + (a^2 + 2b) + (a^3 + 3b) + (a^4 + 4b)$

   (a) $\sum_{n=1}^{5} \frac{n}{m}$

   (b) $\sum_{m=1}^{3} \frac{m^k}{2m}$

   (c) $\sum_{x=0}^{2} xA^{2x} + xB^x + xC$

3. For the following summations, let $\{z_n\}$, $\{x_n\}$, and $\{y_n\}$ be the x, y, and z basis vectors. Let $\psi = \alpha|+z\rangle + \beta|-z\rangle$.

   Example, project ψ onto the y basis: $\sum_n \langle y_n|\psi\rangle|y_n\rangle$

   Solution: $\langle y_1|\psi\rangle|y_1\rangle + \langle y_2|\psi\rangle|y_2\rangle$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}|y_1\rangle + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}|y_2\rangle$$

$$= \frac{\alpha - i\beta}{\sqrt{2}}|+y\rangle + \frac{\alpha + i\beta}{\sqrt{2}}|-y\rangle$$

   Notice that I did not "expand" the y basis vectors in the kets on the right, but rather just gave them the names we usually call them by. This is a judgement call. But since the point of the summation was to project the state ψ onto the y basis, that seemed like the right way to leave the answer. Note that, if I had replaced $|+y\rangle$ and $|-y\rangle$ with actual vectors and then added it all up, I would have ended up with $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ which is just ψ back in the z basis.

   (a) $\sum_n \langle x_n|\psi\rangle|x_n\rangle$

   (b) $\sum_n |z_n\rangle\langle z_n|$

   (c) $\sum_n |x_n\rangle\langle x_n|$

   (d) $\sum_n |y_n\rangle\langle y_n|$

3.6 Solutions

1. (a) (i) $(1) + (2) + (3)$, (ii) 6

   (b) (i) $(2 + 1) + (4 + 1) + (6 + 1)$, (ii) 15

   (c) (i) $(0 - 0) + (1 - 1) + (2^2 - 2)$, (ii) 2

2. (a) $\frac{1}{m} + \frac{2}{m} + \frac{3}{m} + \frac{4}{m} + \frac{5}{m}$

   (b) $\frac{1}{2} + \frac{2^k}{4} + \frac{3^k}{6}$

   (c) $(0 + 0 + 0) + (A^2 + B + C) + (2A^4 + 2B^2 + 2C)$

3. (a) $(\alpha + \beta)|+x\rangle + (\alpha - \beta)|-x\rangle$

   (b) 1

   (c) 1

   (d) 1

4 The Density Matrix

We now turn to the study of the density operator (or density matrix as it is typically called). This, like the operators that represent observables, is an object that doesn't actually operate on a state, but rather *represents* something.

The density matrix is used in formulating measures of entanglement, and it is a key component in the study of decoherence and 'open' quantum systems.

4.1 Pure and mixed states

Up until now, when we have discussed the state of a quantum system, we have assumed that the system in question is in some very specific state. The states which we have been dealing with are called *pure* states. There is a very different kind of state, called a *mixed* state, which represents a 'classical mixture' of different states. Hopefully an example will help to clarify this.

Experiment 1: Pure states

A stream of electrons are emerging from a Stern-Gerlach device which was oriented to do an x measurement, and we are using only the ones that come out in the '+' direction. So we know that their state is $|+x\rangle$. However, we intend to then measure them in the z direction. So in our basis of measurement, their states are $\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$.

Note that there is no ambiguity about the states. We only know the probabilities for the result of our upcoming measurements, but we know exactly what the states are before we measure them. These are called pure states.

Experiment 2: Mixed states

Again we capture electrons coming out of a Stern-Gerlach device. But this time the device is oriented in the z direction. This time, we take all the electrons coming out. So, half of them are $|+z\rangle$ and half of them are $|-z\rangle$. These are called mixed states.

Consider the probabilities for getting a plus or minus z measurement in experiment 1 and experiment 2. They are identical. In both cases, we have a 50/50 chance of seeing plus or minus. But the reasons are very different in the two cases. In the first case, we are measuring a superposition of $+z$ and $-z$. In the second case, we are measuring *either* $+z$ *or* $-z$, we simply don't know which.

In understanding the difference between measuring pure and mixed states, it is crucial to remember how we showed that a superposition does *not* have a predetermined value prior to the measurement. A mixed state does, in fact, have a predetermined value. So in the case of a mixed state the probabilities really do reflect our ignorance about what the state is.

In a pure state, we know exactly what the state is. The uncertainty about further measurement is a quantum effect. In a mixed state, there is 'classical uncertainty' about the value of the state.

Finally, it is possible in principle to construct a measurement that would pass all electrons in any single pure state. But since the mixed state electrons are actually in *different* states, no single measurement could pass all of them.

4.2 The density matrix for pure states

The density matrix for a pure state is simply the outer product of the state with itself:

$$\rho = |\psi\rangle\langle\psi|$$

Some examples:

$$\rho_{(+z)} = |+z\rangle\langle +z| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\rho_{(+x)} = |+x\rangle\langle +x| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\rho_{(-y)} = |-y\rangle\langle -y| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix} = \begin{pmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{pmatrix}$$

Here are some facts about the three density matrices above. As usual, all of the arithmetic is being done in the z basis. So all three matrices have their components in z. The main diagonal of each matrix gives the probabilities of getting $+z$ and $-z$. So for $\rho_{(+z)}$ we have certainty of getting $+z$, as expected. For both $\rho_{(+x)}$ and $\rho_{(-y)}$ we have a 50/50 chance of getting plus or minus z, which are also the values that you would calculate.

The off diagonal terms, which are sometimes called *interference terms*, do not come into the picture in terms of calculating probabilities. But they do, in some sense, signify the 'quantumness' of any prospective z measurement.

4.3 The density matrix for mixed states

4.4 Purity

4.5 Entropy

4.6 The partial trace operation

4.7 Quantifying entanglement

Apply the partial trace to some specific states. Include product states, maximally entangled states, non-maximally entangled states, and the three bit states encountered in quantum teleportation.

5 Continuous (unitary) evolution

5.1 Schroedinger's Equations

A very brief introduction. Just look at the equation and probably some animated waveforms that result.

5.2 A very simplified equation

Try to come up with something.

6 The Measurement Problem

6.1 Unitary evolution vs measurement

6.2 The problem of outcomes

6.3 The preferred basis problem

7 Brief look at Interpretations of quantum theory

The measurement problem is not by any means the only problem (eg. entangled states) and various interpretations may be aimed at one or more issues in QT. Here we'll just briefly look at how three specific interpretations address the measurement problem.

7.1 Copenhagen

7.2 de Broglie-Bohm

7.3 Everett

8 Decoherence