

## Vector Spaces Axioms

Here are the vector space "axioms." Any mathematical entity which follows these rules can be thought of as a vector space. Note that any letters that are not inside kets represent scalars (just regular numbers).

1. Closure with regard to vector addition:  $|v_1\rangle + |v_2\rangle = |v_3\rangle$ .  
(In other words, if you add two vectors you get a vector.)
2. Closure with regard to scalar multiplication:  $a|v_1\rangle = |v_2\rangle$ .
3. Additive identity vector element: There exists a vector  $\mathbf{0}$  such that  $|v\rangle + \mathbf{0} = |v\rangle$ .
4. Additive inverse elements: For every vector  $|v\rangle$  there exists a vector  $|-v\rangle$  such that  $|v\rangle + |-v\rangle = \mathbf{0}$ .
5. Multiplicative identity scalar element: There exists a scalar 1 such that  $1|v\rangle = |v\rangle$ .
6. Vector addition is commutative:  $|v_1\rangle + |v_2\rangle = |v_2\rangle + |v_1\rangle$ .
7. Vector addition is associative:  $|v_1\rangle + (|v_2\rangle + |v_3\rangle) = (|v_1\rangle + |v_2\rangle) + |v_3\rangle$
8. Scalar multiplication is commutative:  $a|v\rangle = |v\rangle a$ .
9. Scalar multiplication is associative:  $a(b|v\rangle) = (ab)|v\rangle$ .
10. Vectors distribute over scalar addition:  $|v\rangle(a + b) = a|v_1\rangle + b|v\rangle$ .
11. Scalar multiplication distributes over vector addition:  $a(|v_1\rangle + |v_2\rangle) = a|v_1\rangle + a|v_2\rangle$ .

In axioms 3 and 4, why did I use the symbol  $\mathbf{0}$  instead of something like  $|0\rangle$ ? The reason is that the expression  $|0\rangle$  typically refers to the a bit with the value of a "classical zero." This is actually the vector with a one and a zero in it. But the zero vector is a vector with all zeros in it:

$$\text{classical zero} = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ classical one} = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ the zero vector} = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Notice that although you can multiply a vector by a scalar, *there is no rule for multiplying two vectors together*. Various different kinds of vector products can be defined, but they are not part of the definition of a vector space.