

Time Dilation

1. Draw the two reference frames, Alice has the mirror. Bob and Alice both agree on: c (the speed of light), v (the magnitude of their relative velocity), and h (the height of the mirror).
2. Alice bounces a photon of light off the mirror. She sees the photon's path as having a length of $2h$ traveled in time t_a . In other words, Alice measures the elapsed time as: $t_a = \frac{2h}{c}$.
3. Bob measures time t_b for the photon go from Alice to the mirror and back. But during that time he sees Alice's entire frame of reference move toward the right at velocity v . So he sees an equilateral triangle with a base of length vt_b .
4. Forming a right triangle from half of that base, with the height h as the other side, then during time $t_b/2$ Bob sees the photon travel the hypotenuse, which has length $\sqrt{h^2 + \frac{1}{4}v^2t_b^2}$.

5. Given that Bob saw the photon travel that same hypotenuse in time $t_b/2$, we conclude that $\frac{1}{2}ct_b = \sqrt{h^2 + \frac{1}{4}v^2t_b^2}$.

6. Solving for t_b

$$\frac{1}{4}c^2t_b^2 = h^2 + \frac{1}{4}v^2t_b^2 \text{ (square both sides)}$$

$$\frac{1}{4}c^2t_b^2 - \frac{1}{4}v^2t_b^2 = h^2 \text{ (subtract term)}$$

$$\frac{1}{4}t_b^2(c^2 - v^2) = h^2 \text{ (factor left side)}$$

$$t_b^2 = \frac{4h^2}{c^2 - v^2} \text{ (divide)}$$

$$t_b = \frac{2h}{\sqrt{c^2 - v^2}} \text{ (take positive square root)}$$

7. Since Alice was present at both "events" (explain) her time measurement is considered the *proper time*, noted as t_0 , between those events.
8. We are looking for a conversion factor between the proper time and Bob's time: $t_0 \times ? = t_b$. That is:

$$\frac{2h}{c} \times ? = \frac{2h}{\sqrt{c^2 - v^2}}$$

9. We can cancel the $2h$ from both sides, and then clearly: $\frac{1}{c} \left(\frac{c}{1} \frac{1}{\sqrt{c^2 - v^2}} \right) = \frac{1}{\sqrt{c^2 - v^2}}$

Simplifying, we have the conversion factor: $\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

And the general formula relating a proper time t_0 to the time t of any observer who does not see the proper time.

$$t = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} t_0$$

10. What we can conclude from this is that the proper time is always the the shortest time observed. Any other time will appear to be longer (and hence "dilated").