Class 4. Center of Mass

AP Physics

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Files for You to Download

Please download/print the PDF file

- `04-CM-print.pdf`—The “print version” of this week’s slides. I recommend that you print 4 slides per page.
- `04-Homework.pdf` (Available after Dec. 2) This week’s homework. We are taking up questions from Class 3 today, but please hand in Classes 3 & 4 homework together next week.

Today's Plan

1. Go over today's slides on center of mass
2. Take up homework questions from last week's class (do not hand them in yet; wait another week)
3. Time permitting: take up questions from Class 1 and 2 (calculus)

Center of Mass

Finding an object's center of mass[1] is important, because

- Newton's laws of motion are formulated by treating an objects as point masses (for real-life objects, we let the forces apply to the center of mass, or *center of gravity*)
- objects can have *rotational* motion in addition to *translational* motion as well (we will examine that a bit more next week)

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[1] **Pro-tip:** Since the AP Physics exam is *American*, the spelling convention we will use in class will be American as well, as in *center* instead of *centre*, *labor* instead of *labour*.

Start with a Definition

The center of mass (“CM”), or center of gravity (“CG”), is the *weighted average of the masses in a system.* The “system” may be:

- A collection of individual particles (use summation to compute CM)
- A continuous distribution of mass with constant density (use integration to compute CM); in this case, CM is also the geometric center of the object (*centroid*)
- A continuous distribution of mass with varying density (use integral to compute CM)

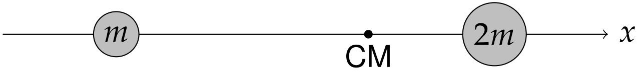
Simple Example

- We start with a very simple example: there are two equal masses along the x-axis. What is the center of mass of the system?

- The answer is really simple: it's at the half way point between the two!

But Things Aren't Always That Example

- What if one of the masses are increased to $2m$?
- This is still not a terribly difficult problem; you can still *guess* the right answer without know the equation for center of mass.

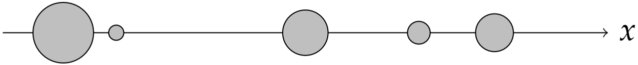


- The answer is still simple. The CM is no longer at the half way point between the two masses, but now $\frac{1}{3}$ the total distance from the larger masses.

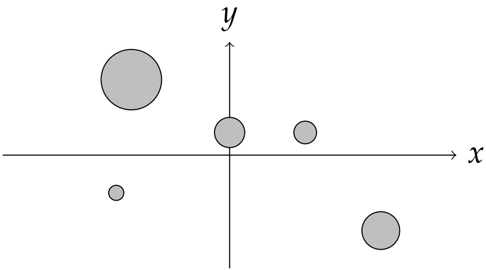
Complicating Things Further

Many Point Masses

- If we increase the number of point masses along the x-axis, our problem can become much more complicated (although still not devastatingly so)



- Difficulties really arises when there are many masses in the system in 2D or 3D:



An Equation Helps

The center of mass is defined as:

$$\boxed{\mathbf{x}_{\mathrm{CM}} = \frac{\sum \mathbf{x}_i m_i}{\sum m_i}}$$

Quantity	Symbol	SI Unit
Position of center of mass (vector)	$\mathbf{x}_{\mathrm{CM}}$	m (meters)
Position of point mass i (vector)	$\mathbf{x}_i$	m (meters)
Point mass i	m_i	kg (kilograms)
Total mass	$\sum m_i$	kg (kilograms)

Breaking Down Into Components

$$\mathbf{x}_{\mathrm{CM}} = \frac{\sum \mathbf{x}_i m_i}{\sum m_i}$$

- Position vectors have x, y and z components: $\mathbf{x} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$
- We can deal with each component individually. For example, in the y-direction:

$$y_{\mathrm{CM}} = \frac{\sum y_i m_i}{\sum m_i}$$

where y_{CM} is the y-coordinate of the CM, and y_i are the y-coordinates of the individual point masses

Let's Do An Example

Example 1: Consider the following masses and their coordinates which make up a "discrete mass" rigid body"

$$
\begin{aligned}
m_1 &= 5.0\,\mathrm{kg} & \mathbf{x}_1 &= 3\hat{\imath} - 2\hat{k} \\
m_2 &= 10.0\,\mathrm{kg} & \mathbf{x}_2 &= -4\hat{\imath} + 2\hat{\jmath} + 7\hat{k} \\
m_3 &= 10.0\,\mathrm{kg} & \mathbf{x}_3 &= 10\hat{\imath} - 17\hat{\jmath} + 10\hat{k}
\end{aligned}
$$

What are the coordinates for the center of mass of this system?

Continuous Mass Distribution

Most real-life masses are not a discrete collection of point masses, so for continuous mass distributions, we take the limit of when the number of masses approaches ∞:

$$\mathbf{x}_{\mathrm{CM}} = \lim_{n\to\infty}\left(\frac{\sum_{i=1}^{n}\mathbf{x}_i m_i}{\sum_{i=1}^{n} m_i}\right)$$

This, in fact, gives us an integral version of our equation:

$$\boxed{\mathbf{x}_{\mathrm{CM}} = \frac{\int \mathbf{x}\,dm}{\int dm}}$$

Densities

- Linear density (for 1D problems)

$$\gamma = \frac{m}{L}$$

- Surface area density (for 2D problems)

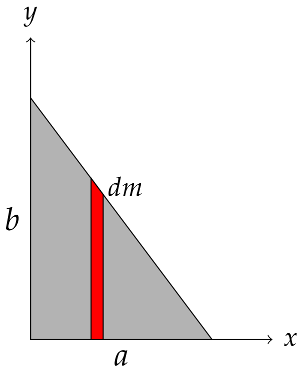
$$\sigma = \frac{m}{A}$$

- Volume density (for 3D problems)

$$\rho = \frac{m}{V}$$

An Example with Integrals

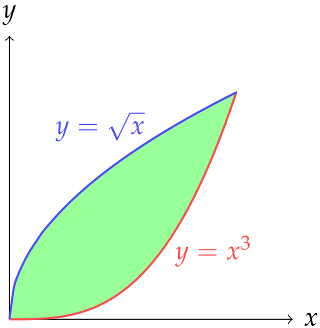
Example 2: A triangular plate is placed in a Cartesian coordinate system with two of its edges along the x and y-axis. The length of the edges along the axes are a and b respectively. Assuming that the surface area density σ is uniform, determine the coordinate of its center of mass.



A Difficult Example to Try at Home

This is not typically an AP-level problem, but this example shows how we can use integral to find the center of mass for something very complicated.

Example 3: Find the x-coordinate of the center of mass in the shape bound by the two functions shown on the right.



Symmetry

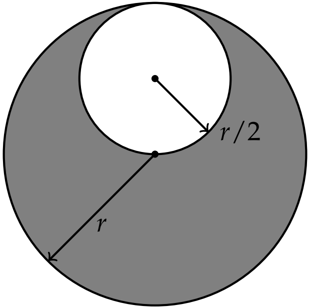
There are always shortcuts!

- Any plane of symmetry, mirror line, axis of rotation, point of inversion *must* contain the center of mass.
- Caveat: only works if the density distribution is also symmetric
- Again: if density is uniform, CM is also geometric center (centroid)

“Negative Mass”

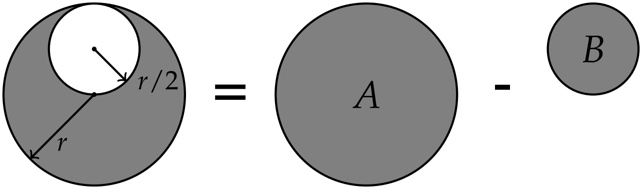
A mathematical trick for complicated geometries

- Where there is a “hole” in the geometry, treat it as having negative mass density $-\rho$ in that region.
- Negative masses don’t exist, so this is really just a trick.
- **Example:** What is the center of mass of this shape?



Negative Mass Example

- This is how we would think of it:



- Let the origin of the coordinate system to located at the center of A
- Based on symmetry: $x_{\text{CM}} = 0$; only have to find y-coordinate.
- Sum our weighted average:

$$y_{\text{CM}} = \frac{\sum y_i m_i}{\sum m_i} = \frac{m_A(0) + m_B(r/2)}{m_A + m_B} = \frac{-\rho\pi\left(r/2\right)^2\left(r/2\right)}{\rho\pi r^2 - \rho\pi\left(r/2\right)^2} = \frac{-r}{6}$$

Velocity, Acceleration and Momentum

- We can take the time derivative of the equation for $\mathbf{x}_{\mathrm{CM}}$ to get a (very similar) expression for the velocity of the CM:

$$\mathbf{v}_{\mathrm{CM}} = \frac{d\mathbf{x}_{\mathrm{CM}}}{dt} = \frac{1}{m}\frac{d}{dt}\left(\int \mathbf{r}dm\right) = \frac{1}{m}\int \frac{d\mathbf{r}}{dt}dm = \frac{\int \mathbf{v}dm}{m}$$

- The integral in the numerator is the sum of the momentum of all the masses in the system ($\mathbf{p}_{\mathrm{net}}$) which means that we have

$$\mathbf{p}_{\mathrm{net}} = m\mathbf{v}_{\mathrm{CM}}$$

- Taking the derivative of $\mathbf{p}_{\mathrm{net}}$ allows us to relate force and acceleration at the CM as well:

$$\mathbf{F}_{\mathrm{net}} = \frac{d\mathbf{p}_{\mathrm{net}}}{dt} = m\frac{d\mathbf{v}_{\mathrm{CM}}}{dt} = m\mathbf{a}_{\mathrm{CM}}$$