Vectors and Calculus That You Need to Know

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The AP Physics C exams are calculus based, and will use vectors extensively. Students should be familiar with the material in this handout, however, it is likely that the calculus and vector operations used in the exams will be much simpler. If these concepts are difficult, this should be a good time to grab a calculus textbook and review them.

1 Vectors

Vectors are used extensively in physics. They are an integral part of a larger discipline within mathematics called **linear algebra**. For the purpose of AP Physics, it is sufficient to think of vectors as "a number with a direction".

1.1 Notation

In keeping with the convention used in *most* technical journals and university-level textbooks, vectors are *printed* (e.g. on the slides and handouts) using a bold face font:

$$\mathbf{v} \quad \mathbf{F}_g \quad \mathbf{p} \quad \mathbf{I}$$

while the "arrow on top" notation is used when *writing* (e.g. on the blackboard)[1]:

$$\vec{v} \quad \vec{F}_g \quad \vec{p} \quad \vec{I}$$

The magnitude of vectors are expressed either with the absolute-value symbol:

$$|\mathbf{v}| \quad |\mathbf{F}_g| \quad |\mathbf{p}| \quad |\mathbf{I}|$$

or as a scalar quantity (afterall, the magnitude of a vector is indeed a scalar with a positive value):

$$v \quad F_g \quad p \quad I$$

1.2 Writing Vectors

In Grades 11 and 12 Physics, vectors are usually written by separating the magnitude from the direction. For example, a velocity vector are usually written as:

$$\mathbf{v} = 4.5\,\text{m/s [N } 55^\circ \text{ E]}$$

[1]Although this format is still used in *some* introductory level physics textbooks in universities

This approach is based on using the **polar coordinate system**, which is the preferred coordinate system for circular motion. In general, polar coordinate system is very intuitive for describing *one* vector in two dimensions (that's why it is used extensively in high-school level physics courses), but it is more complicated when extended into 3D; the coordinate system needs to be extended to **spherical coordinate system** or the **cylindrical coordinate system**. Moreover, it is difficult to perform vector arithmetic for *rectilinear* motion.

Intead, for rectilinear motion, vectors in 2D/3D Cartesian space are generally written in their x, y & z components using the **IJK notation**:

$$\mathbf{A} = A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} + A_z\hat{\boldsymbol{k}}$$

The vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ are **basis vectors** indicating the directions of the x, y and z axes. Basis vectors are **unit vectors** (i.e. length 1). Note that the IJK notation does not give the magnitude of the vector, which needs to be calculated:

$$A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

1.3 Vector Addition and Subtraction

Adding and subtracting vectors is straightforward:

$$\mathbf{A} \pm \mathbf{B} = (A_x \pm B_x)\hat{\boldsymbol{\imath}} + (A_y \pm B_y)\hat{\boldsymbol{\jmath}} + (A_z \pm B_z)\hat{\boldsymbol{k}}$$

1.4 Dot Product

The vector **dot product** (or **inner product** for general vectors) is the *scalar* multiplication of two vectors. This is a vector operation that have been used throughout Grades 11 and 12 Physics courses (although without explicitly using this notation), for example, when calculating mechanical work. It is determined by the magnitude of the two vectors and the cosine of the angle θ between them:

$$C = \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} = |\mathbf{A}||\mathbf{B}| \cos\theta$$

In the cross product, C is the *projection* of the vector **A** onto **B**, or the component of **A** along **B**. Note that $\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} = 1$, $\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}} = 1$, and $\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = 1$. For general vectors written in IJK notation, where the magnitude and direction of vectors are not immediately known, so intead we sum the product of individual components of **A** and **B**:

$$C = \mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

1.5 Cross Products

The vector **cross product** is the vector multiplication of two vectors:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

The magnitude of the cross product is determined by the magnitude of **A** and **B** and the angle θ between them:

$$C = AB \sin\theta$$

The cross product **C** is perpendicular to *both* **A** and **B**; its direction given by the right hand rule. Cross products are used extensively in rotational motion and in electromagnetism Note that unlike the cross product, the order of the cross product is important. (This is why you have to get the right hand rule correctly.)

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

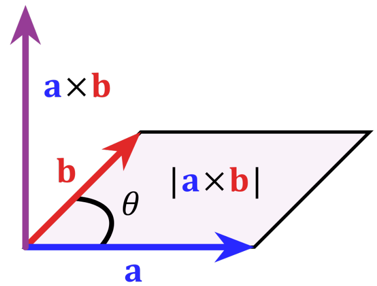

Figure 1: Vector cross product.

In general, the cross product of any two vectors in 3D space is the determinant of this 3×3 matrix:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z - A_zB_y)\hat{\imath} + (A_zB_x - A_xB_z)\hat{\jmath} + (A_zB_y - B_yA_x)\hat{k}$$

although it is extremely rare that such notation will ever be used in any Physics C exams. Most cross product applications in AP Physics C are much simpler, so we only have to remember the circle shown in Figure 2.

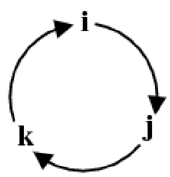

Figure 2: Cross product circle that you will likely see in Physics C exams

The direction of the arrow gives the index of the cross product (e.g. $\hat{\imath} \times \hat{\jmath} = \hat{k}$); going against the direction of the arrow gives the negative of the next index (e.g. $\hat{k} \times \hat{\jmath} = -\hat{\imath}$)

2 Calculus

We cannot learn physics properly without calculus (you got away with it for long enough in grades 11 and 12?!)

- **Differential Calculus**
  - How quickly something is changing ("rate of change" of a quantity)
  - Math: slopes of functions
  - Physics: how quickly a physical quantity is changing in time and/or space
  - Examples: velocity (how quickly position changes with time), acceleration (how quickly velocity changes with time), power (how quickly work is done), electric fields (how electric potential changes in space)
- **Integral Calculus**
  - The opposite of differentiation
  - We use it to compute the area under a curve, or

- Summation of many small terms
- Examples: area under the **v**-t graph (displacement), area under the **F**-t graph (impulse), area under the **F**-d graph (work)

2.1 Derivative

For any arbitrary function $f(x)$, the derivative with respect to x is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The "limit as h approaches 0" is the mathematical way of making h a very small non-zero number.

2.2 Basic Rules for Differentiation

The derivative of a constant C with respect to any variable is zero. This should be obvious, since the slope of any function $f(x) = C$ is zero.

$$\frac{dC}{dx} = 0$$

A constant multiple a of any function f can be factored outside the derivative:

$$\frac{d}{dx}(af) = a\frac{df}{dx}$$

The derivative of a sum of two functions is the sum of the derivatives of the functions:

$$\frac{d}{dt}\left(f(t) + g(t)\right) = \frac{df}{dt} + \frac{dg}{dt}$$

Power Rule:

$$\frac{d}{dt}\left(t^n\right) = nt^{n-1} \quad \text{for} \quad n \neq 0$$

Product Rule:

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

Chain Rule:

$$\frac{d}{dx}f\left(g(x)\right) = f'(g(x))g'(x)$$

Quotient Rule is rarely used in physics tests in AP or first-year university, but you should remember it anyway:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

2.3 Elementary Derivatives

When studying **harmonic motion** and **circular motion**, trigonometric and exponential functions are often used. We will also find out the relationship between complex exponential functions and sine/cosine functions.

$$\frac{d}{dt}\sin t = \cos t$$
$$\frac{d}{dt}\cos t = -\sin t$$

And the exponential function:

$$\frac{d}{dt}e^{at} = ae^{at}$$

2.4 Partial Derivatives

Some functions have many variables (multi-variable function). For example, gravitational potential energy U_g has three variables: masses m_1 and m_2 and the distance r between them:

$$U_g(m_1, m_2, r) = -\frac{Gm_1m_2}{r}$$

Differentiating with respect to one variable while holding others constant gives its **partial derivative**. (We use the ∂ symbol). For example, the partial derivative of U_g with respect to r is

$$\frac{\partial U_g}{\partial r} = \frac{Gm_1m_2}{r^2}$$

In case you have not noticed: the derivative is the is the relationship between gravitational potential energy U_g and the magnitude of the gravitational force F_g.

2.5 Integration

If $F(x)$ is the anti-derivative of $f(x)$, they are related this way:

$$\frac{d}{dx}F(x) = f(x) \quad \longrightarrow \quad F(x) = \int f(x)dx$$

The mathematical proof is the **fundamental theorem of calculus**.

2.6 Common Integrals in Physics

Integration, while often necessary, can be very daunting, but integrals in AP Physics C are generally straightforward. These rules should help in most cases.

Power rule in reverse:

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C$$

Natural logarithm:

$$\int \frac{1}{x}dx = \ln|x| + C$$

Sines and cosines:

$$\int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

2.7 Definite and Indefinite Integrals

- Integrals can be either **indefinite** or **definite**
- An "indefinite" integral is another function, e.g. position $\mathbf{x}(t)$ as a function of time is found by integrating velocity $\mathbf{v}(t)$:

$$\mathbf{x}(t) = \int \mathbf{v}(t)dt = \cdots + \mathbf{C}$$

- A **constant of integration C** is added to the integral $\mathbf{x}(t)$. It is obtained through applying "initial condition" to the problem.

2.8 Definite Integrals

A **definite integral** has lower and upper bounds. e.g. given $\mathbf{v}(t)$, the displacement between t_1 and t_2 can be found:

$$\Delta\mathbf{x} = \int_{t_0}^{t_1} \mathbf{v}(t)dt$$

Once we have computed the integral, we evaluate the limits:

$$\Delta\mathbf{x} = \mathbf{x}(t)\Big|_{t_0}^{t_1} = \mathbf{x}(t_1) - \mathbf{x}(t_0) = \mathbf{x}_1 - \mathbf{x}_0$$

The constant of integration **C** cancels when we evaluate the upper and lower bounds.