Simple Harmonic Motion

Special AP/CAP Lecture

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Fall 2019

Hooke's Law

Hooke's law for an ideal spring relates the force exerted by a compressed (or stretched) spring onto another object (**spring force** $\mathbf{F}_s$) to the stiffness of the spring (**spring constant** k) and spring displacement $\mathbf{x}$:

$$\boxed{\mathbf{F}_s = -k\mathbf{x}}$$

Quantity	Symbol	SI Unit
Spring force	$\mathbf{F}_s$	N
Spring constant	k	N / m
Amount of extension/compression	$\mathbf{x}$	m

Spring constant is also called **Hooke's constant** or **force constant**.

Elastic Potential Energy

Applying Hooke's law in the work equation gives the amount of **elastic potential energy** stored in the spring when it is compressed or stretched:

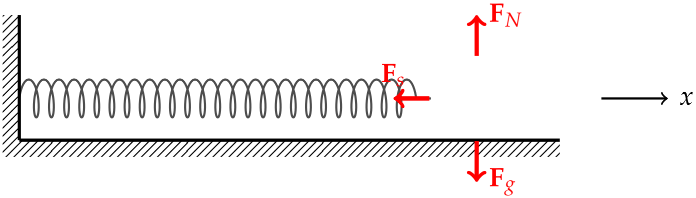
$$W = \int_{x_1}^{x_2} \mathbf{F}_e \cdot d\mathbf{x} = -\int_{x_1}^{x_2} kx\,dx = -\frac{1}{2}kx^2\Big|_{x_1}^{x_2} = -\Delta U_e$$

where elastic potential energy is defined as

$$\boxed{U_e = \frac{1}{2}kx^2}$$

Mass on a Spring

Consider the forces acting on a mass connected horizontally to a spring



$\mathbf{F}_g$ and $\mathbf{F}_N$ cancel out, so net force is due only to spring force $\mathbf{F}_s = -k\mathbf{x}$. This is true both when the spring is in compression or extension. (In the diagram above, the spring is in extension.)

Mass on a Spring

Applying Newton's 2nd law in the x-direction:

$$\sum F = F_s = ma \quad \longrightarrow \quad -kx = m\frac{d^2x}{dt^2}$$

The equation above is generally written in the standard form

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

This equation is called a *second-order ordinary differential equation with constant coefficients*.

Mass on a Spring

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

- To solve the above equation, we look for a function $x(t)$ where the second derivative $x''(t)$ looks like $x(t)$ itself but with a negative sign
- The obvious choice are the two trigonometric functions: $\sin(t)$ and $\cos(t)$
- We start with this general form

$$x(t) = A\sin(\omega t + \phi)$$

Mass on a Spring

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

Starting with the general form, we can take the time derivatives to obtain the velocity and acceleration of the mass as a function of time:

$$x(t) = A\sin(\omega t + \phi)$$
$$v(t) = A\omega\cos(\omega t + \phi)$$
$$a(t) = -A\omega^2\sin(\omega t + \phi) = -\omega^2 x$$

where ω is the angular frequency, and A is the amplitude of the oscillation and ϕ is a phase shift that depends on the initial condition

Mass on a Spring

Angular Frequency

Substituting expressions of $x(t)$ and $a(t) = x''(t)$ into the ODE, we find that the solution is satisfied if angular frequency is related to the spring constant and mass by:

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

The angular frequency for the undamped oscillator is called the **natural frequency**.

Mass on a Spring

Frequency and Period

The period T and frequency f of the motion are given by:

$$\boxed{f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}} \qquad \boxed{T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}}$$

Angular frequency, frequency and period do not depend on amplitude A

Mass on a Spring

Side Note #1

Side Note #1: The solution to any ODE is the linear combination of *all* possible solutions, i.e.:

$$x(t) = c_1 \sin(\omega t) + c_2 \cos(\omega t)$$

Where c_1 and c_2 are constant coefficients based on the initial conditions. The solution form that was shown in previous slides are in fact identical to this solution.

Mass on a Spring

Side Note #2

Side Note #2: When searching for a solution to the ODE, one may "guess" a solution that is the exponential function, where its derivatives have the same form as the function itself, i.e.:

$$x(t) = e^{\omega t}$$

However, the second derivative of $e^{\omega t}$ does not have the negative sign that is needed. But if the exponential function is complex:

$$x(t) = e^{i\omega t}$$

then the 2nd derivative *will* in fact have the negative sign:

$$x'(t) = i\omega e^{i\omega t} \quad \rightarrow \quad x''(t) = i^2\omega^2 e^{i\omega t} = -\omega^2 e^{i\omega t}$$

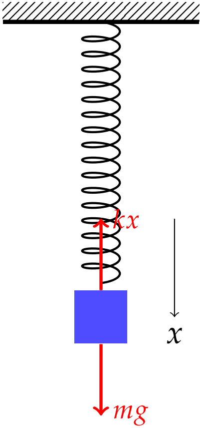
Mass on a Spring

Side Note #2

This should not come as a surprise, since the complex exponential function and the sinusoidal functions are related:

$$e^{it} = \sin(t) + i\cos(t)$$

Vertical Spring-Mass System



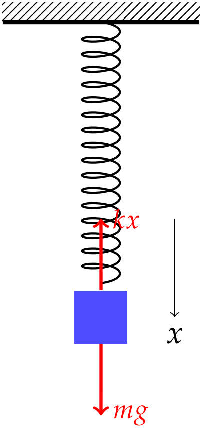
For a vertical spring-mass system, the analysis is *slightly* more complicated (we have to consider weight as well):

$$mg - kx = m\frac{d^2x}{dt^2}$$

But since mg is a constant, the only difference is the addition of a constant B in our expression of $x(t)$:

$$x(t) = A\sin(\omega t + \phi) + B$$
$$v(t) = A\omega\cos(\omega t + \phi)$$
$$a(t) = -A\omega^2\sin(\omega t + \phi)$$

Vertical Spring-Mass System



Substituting $x(t)$ and $x''(t)$ into the differential equation gives

$$B = \frac{mg}{k}$$

which is just the stretching of the spring due to its weight

Angular frequency is the same as the horizontal case:

$$\omega = \sqrt{\frac{k}{m}}$$

Conservation of Energy in a Spring-Mass System

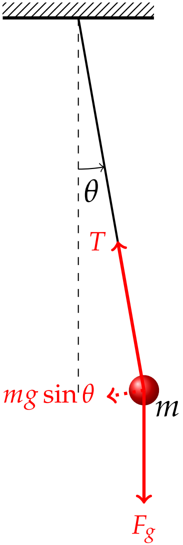
In the spring-mass systems, if there are no frictional losses, then the only forces doing work are the spring force (horizontal and vertical) and gravity (vertical). Both forces are *conservative*, therefore the total mechanical energy is conserved:

$$K_1 + U_{e,1} + U_{g,1} = K_2 + U_{e,2} + U_{g,2}$$

For the horizontal spring-mass system, the total energy is:

$$E_T = \frac{1}{2}kA^2$$

What About a Pendulum?

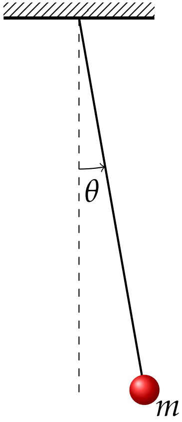


- Pendulums also exhibit oscillatory motion
- For a pendulum, there are two forces acting on the mass: weight $F_g = mg$ and tension T
- When the mass is deflected by an angle θ, it's easy to show (using polar coordinates) that the force in the angular direction is $F_\theta = -mg\sin\theta$
- We needn't worry about the radial direction because it doesn't have anything to do with the restoring force

The Pendulum

Substitute F_θ into Newton's second law, and cancelling mass term, we get:

$$F_\theta = ma_\theta \quad \longrightarrow \quad -g\sin\theta = L\frac{d^2\theta}{dt^2}$$



For small angles, $\sin\theta \approx \theta$, and we get the ODE for the pendulum:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

This ODE has the same form as the spring-mass system!

Ordinary Differential Equation for the Pendulum

- The solution for $\theta(t)$ is very similar to the spring-mass system:

$$\boxed{\theta(t) = \theta_{\max} \sin(\omega t + \phi)}$$

where ω is given by

$$\boxed{\omega = \sqrt{\frac{g}{L}}}$$

and ϕ is a phase shift based on the initial condition of the pendulum.

- Remember that this equation is only valid for small angles, i.e. $\theta_{\max} < 15^\circ$

A Pendulum Example

Example: A bucket full of water is attached to a rope and allowed to swing back and forth as a pendulum from a fixed support. The bucket has a hole in its bottom that allows water to leak out. How does the period of motion change with the loss of water?

(a) The period does not change.

(b) The period continuously decreases.

(c) The period continuously increases.

(d) The period increases to some maximum and then decreases again.

Think About g

Example: A little girl is playing with a toy pendulum while riding in an elevator. Being an astute and educated young lass, she notes that the period of the pendulum is $T = 0.5\,\mathrm{s}$. Suddenly the cables supporting the elevator break and all of the brakes and safety features fail simultaneously. The elevator plunges into free fall. The young girl is astonished to discover that the pendulum has:

(a) continued oscillating with a period of $0.5\,\mathrm{s}$.

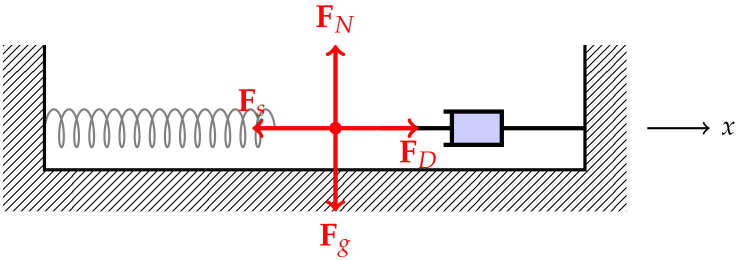
(b) stopped oscillating entirely.

(c) decreased its rate of oscillation to have a longer period.

(d) increased its rate of oscillation to have a lesser period.

It's Never Perfect

In reality, there are friction, or drag, or other damping forces present in the spring-mass system, represented schematically like the shock absorber:



The damping force is typically related to velocity, in the opposite direction:

$$\mathbf{F}_D = -b\mathbf{v}^n$$

In the simplest case is to use $n = 1$ to represent viscous effects.

Damped Oscillator

The 2nd-order ODE is obtained by applying Newton's second law of motion:

$$\sum F = F_s + F_D = ma \quad \rightarrow \quad -kx - b\frac{dx}{dt} = \frac{d^2x}{dt^2}$$

Arranging into standard form:

$$\boxed{\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0}$$

The solution to this ODE is still relatively straightforward.

Damped Oscillator

The solution to the damped oscillator is a standard problem in calculus class. It is the product of a exponential decay and a sinusoidal function:

$$x(t) = A_0 e^{-\frac{b}{2m}t} \sin(\omega' t + \phi)$$

where A_0 is the initial amplitude of the damped oscillator, and the angular frequency is now given by:

$$\boxed{\omega' = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}}$$

The angular frequency ω' of the damped oscillator differs from the undamped case ω_0 depending on the damping factor b.

Critical Damping

Critical damping occurs when the angular frequency ω' term is zero:

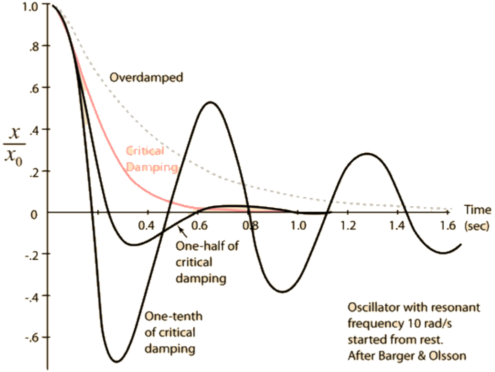
$$\omega' = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} = 0$$

which occurs when the damping constant is:

$$\boxed{b_{\text{critical}} = 2m\omega_0}$$

- A critically damped system returns to its equilibrium position in the shortest time with *no* oscillation
- When $b > b_{\text{critical}}$, the system is **over-damped**
- Critical or near-critical damping is desired in many engineering designs
  - Example: shock absorbers on car suspensions

Comparing Damped System



The motion of the damped oscillator is not strictly periodic.

Energy in a Damped System

The damping force (non-conservative) dissipates energy from the oscillator at a rate of:

$$P = \frac{dE}{dt} = \mathbf{F}_D \cdot \mathbf{v} = -bv^2$$

As velocity relate to energy by: $(v_{\mathrm{av}})^2 = E/m$, power dissipation is a first-order linear ODE:

$$\frac{dE}{dt} = -\frac{b}{m}E$$

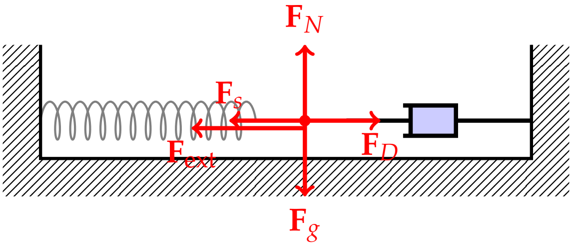
The solution to the ODE shows the total amount of energy as a function of time:

$$\boxed{E(t) = E_0 e^{-\frac{b}{m}t} = E_0 e^{-\frac{t}{\tau}}}$$

Forced Harmonic Motion

To keep a damped system going, energy must be added into the system. Assuming that the system is subjected to an external force that is harmonic with time:

$$F_{\text{ext}} = F_0 \sin(\omega t)$$



Forced Harmonic Motion

Again, the second-order ordinary differential equation is obtained by applying Newton's second law of motion:

$$\sum F = -kx - bv + F_0 \sin(\omega t) = ma$$

Rearranging the terms gives a similar ODE to the damped case, but with the additional external force term on the right-hand side:

$$\boxed{m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \sin(\omega t)}$$

Forced Harmonic Motion

$$\boxed{m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \sin(\omega t)}$$

The solution to this ODE has two components:

- A **transient solution** that is identical to that of the damped oscillator
  - Obtained by setting the external force term to zero
  - Depends on the initial condition
  - Solution becomes negligible over time because of exponential-decay
- A **steady-state solution** which does not depend on the initial condition

Forced Harmonic Motion

Solving for the steady-state solution will be left as a (straightforward but not simple) exercise in calculus, but it can be shown that the solution is a harmonic motion at the frequency ω of the external force:

$$\boxed{x(t) = A\sin(\omega t + \phi)}$$

where the amplitude of the oscillation A and phase constant ϕ are given by:

$$\boxed{A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}}}$$

$$\boxed{\tan\phi = \frac{b\omega}{m(\omega_0^2 - \omega^2)}}$$

Resonance

Resonance is caused by in-phase excitation at natural frequency. This means that:

- The frequency of the driving force is same as the natural frequency of the oscillator

$$\omega = \omega_0 \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

- The driving force follows the motion of the oscillator.

Resonance

Looking at the expression for amplitude of the oscillation:

$$\boxed{A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}}}$$

Amplitude is at maximum when the frequency of the driving force ω is equal to the natural frequency ω_0, with a maximum value of:

$$A_{\max} = \frac{F_0}{b\omega}$$

Resonance

$$\boxed{\tan\phi = \frac{b\omega}{m(\omega_0^2 - \omega^2)}}$$

When $\omega = \omega_0$ is substituted into the phase constant expression, the right-hand side becomes undefined. From this, we obtain a phase constant of $\phi = \pi/2$. Taking derivative of $x(t)$ for velocity $v(t)$, and substituting $\phi = \pi/2$:

$$v(t) = \frac{dx}{dt} = A\omega\cos(\omega t + \frac{\pi}{2}) = A\omega\sin(\omega t)$$

Resonance

At resonance, the object is always moving in the same direction as the driving force:

$$v(t) = A\omega \sin(\omega t)$$
$$F_{\text{ext}}(t) = F_0 \sin(\omega t)$$