2. Calculus in Physics–Integration

AP Physics

Dr. Timothy Leung

Olympiads School

Fall 2017

Files for You to Download

- `00-outline.pdf`—The course outline
- `01-Calculus-print.pdf`—The slides that I used last week
- `01-integration.pdf`—The slides that I am using right now
- `01-Homework.pdf`—Last/this week's homework assignment

Please download/print the PDF file for the class slides before each class.

On Differential Calculus

A quick review

- Finding out how quickly a physical quantity is changing (“rate of change” of that quantity)
- Math: slopes of functions
- Terminology:
  - A ***derivative***: The slope of a function (noun)
  - To ***differentiate***: Finding the derivative with respect to a variable (verb)
- Last class: went through the rules and some examples of derivatives

Examples of Derivatives in Physics

- Instantaneous velocity $\mathbf{v}(t)$ is the derivative of position $\mathbf{s}(t)$

$$\mathbf{v}(t) = \frac{d\mathbf{s}}{dt}$$

- Instantaneous acceleration $\mathbf{a}(t)$ is the derivative of velocity $\mathbf{v}(t)$. It's also the "second derivative" (derivative of a derivative) of position $\mathbf{s}$ with respect to time

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{s}}{dt^2}$$

- Instantaneous force $\mathbf{F}(t)$ is the derivative of momentum $\mathbf{p}$ (Newton's second law of motion)

$$\mathbf{F}(t) = \frac{d\mathbf{p}}{dt}$$

They're All Vectors

Resolve them into components

- Notice that position $\mathbf{s}$, velocity $\mathbf{v}$, acceleration $\mathbf{a}$, momentum $\mathbf{p}$, force $\mathbf{F}$ are all vector quantities with x, y and z components
- In this case, we take the derivative separately in each direction.

$$\mathbf{v}(t) = \frac{d\mathbf{s}}{dt} = \frac{d}{dt}\left[s_x(t)\hat{\boldsymbol{\imath}} + s_y(t)\hat{\boldsymbol{\jmath}} + s_z(t)\hat{\boldsymbol{k}}\right]$$

where s_x, s_y and s_z are the x-, y- and z-components of $\mathbf{s}$

- In AP or 1st-year physics, s_x, s_y and s_z are functions of time only, but in practical problems in physics and engineering, they are often functions of x, y and z coordinates as well. (This is *multi-variable calculus*. It's a lot of fun!)

Other Derivatives

…Not Always With Respect to Time

- **Electric force** is the derivative of electrical potential energy with respect to radial distance:
$$F_q = -\frac{dU_q}{dr} = -\frac{d}{dr}\left[\frac{kq_1q_2}{r}\right]$$
Gravitational force and gravitational potential energy are related in the same way
- **Electric Field** is the derivative of electric potential difference V with respect to the radial distance
$$E = -\frac{dV}{dr}$$

What the Notation Tells Us

- When we say we that **velocity is the time rate of change of position**

$$\mathbf{v} = \frac{d\mathbf{s}}{dt}$$

- We are really asking **what is the (small) change in position ds over an infinitesimal (infinitely small) change in time dt?**

NOW ON TO INTEGRATION

Integral Calculus

- The opposite of differentiation
- We use it to compute the area under a curve, or
- Summation of many very small terms
- Examples: area under the $\mathbf{v}$-t graph (to calculate displacement), area under the $\mathbf{F}$-t graph (to calculate impulse), area under the F-d graph (to calculate work)

Integration: Area Under the Curve

Let's start with an example

- A car moves with speed $v(t) = 2 + 5t$. What is its displacement at $t = 5$?
- We know that displacement is the area under a v-t graph. How do we find the area? (Pretend that we don't know the area of a trapezoid!)
- Divide the interval from $t = 0$ to 5 into n equal small time intervals Δt
- The displacement in each of these Δt_i is approximately

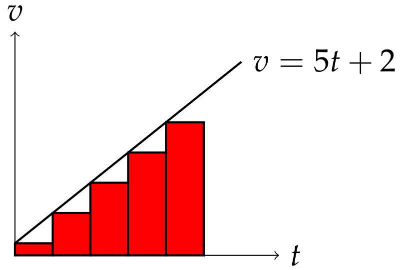
$$\Delta s_i = v_i \Delta t$$

  where $v_i = v(t_i)$
- And the total displacement is:

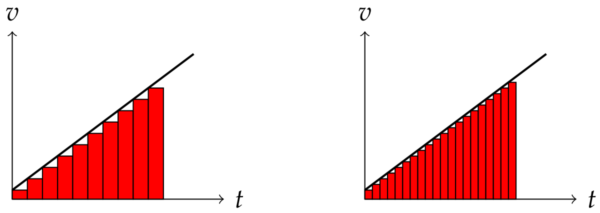
$$\Delta s = \sum_{i=1}^{n} \Delta s_i = \sum_{i=1}^{n} v_i \Delta t$$

Integration: Area Under a Curve

- Shown graphically:



- This is not very inspiring, but we can do better if we increase n:



Integration: Area Under a Curve

- If we increase n to ∞, we will have the *actual* displacement:

$$\Delta s = \lim_{n \to \infty} \sum_{i=1}^{n} v_i \Delta t$$

- In fact, this limit is called the integral:

$$\Delta s = \int_{t_1}^{t_2} v(t) dt$$

- As $n \to \infty$, the time interval Δt becomes infinitesimally small, i.e. "dt"

Integration: Area Under a Curve

- In our example, we have this particular integral:

$$\Delta s = \int_{t_1}^{t_2} v(t)dt = \int_0^5 (5t + 2)\ dt$$

- How do we compute it?

The Antiderivative

- If: $v(t)$ is the derivative of $s(t)$,
- Then: $s(t)$ is the "antiderivative" of $v(t)$
- In general, if $F(x)$ is the antiderivative of $f(x)$, they are related this way:

$$\frac{d}{dx}F(x) = f(x) \quad \longrightarrow \quad F(x) = \int f(x)dx$$

- If we want to integrate $f(x)$, we are actually asking "what function $F(x)$ has a derivative equal to $f(x)$"?
- A simple example with $f(x) = t$ and $F(x) = \frac{1}{2}t^2$:

$$\frac{d}{dt}\left(\frac{1}{2}t^2\right) = t \quad \longrightarrow \quad \int tdt = \frac{1}{2}t^2$$

Commonly Used Integrals in Physics

Calculating an integral can be a very daunting task. But these few known examples should help in most cases:

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

We can “ignore” (i.e. cancel) the constants C if it is a definite integral.

Definite vs. Indefinite Integral

- An integral can be either “indefinite” or “definite”
- An “indefinite” integral gives us another function.
- For example, given velocity $v(t)$, we can find position $s(t)$ as a function of time:

$$s(t) = \int v(t)dt = \cdots + C$$

- A constant C is added to the anti-derivative of $v(t)$. The exact value of C is obtained through applying an “initial condition” to the problem.
- Note: If $s(t) = [\text{something}] + C$ and we take the derivative of s to find $v(t)$, the constant term disappears regardless of what value it has.

Definite Integral

- An integral can also be **definite**, with lower and upper bounds.
- For example: given $v(t)$, find displacement between $t = 3$ and $t = 5$:

$$\Delta s = \int_3^5 v(t)dt$$

- Once we have computed the integral, we have to evaluate between the limits:

$$\Delta s = s(t)\Big|_3^5 = s(5) - s(3)$$

- In this case we do not have to bother with the constant C, since it'll cancel out when we evaluate the bounds.

Back to Our Example

- The integral in our example is a *definite* integral because it has limits 0 and 5
- We can evaluate it by:

$$\begin{aligned}
\Delta s &= \int_0^5 (5t+2)\,dt = \int_0^5 5t dt + \int_0^5 2dt \\
&= \frac{5}{2}t^2\Big|_0^5 + 2t\Big|_0^5 = \frac{125}{2} + 10 \\
&= \boxed{\frac{145}{2}}
\end{aligned}$$

- Each part of the sum can be integrated separately

Area Under A Curve

A Math Problem

What is the area under the curve

$$f(x) = 2x^2 + 3x + 1 \quad \text{between} \quad x = 1 \text{ and } x = 5$$

Our integration works like this:

$$\begin{aligned} A &= \int_1^5 \left(2x^2 + 3x + 1\right) dt \\ &= \left(\frac{2}{3}x^3 + \frac{3}{2}x^2 + x\right) \Big|_1^5 \\ &= 24 + \frac{196}{3} \end{aligned}$$

Remember Our Kinematic Equations?

- In Physics 11 and 12, you were introduced to some kinematic equations for constant acceleration.
- Now that we know something about integration, we can understand these equations a little bit better
- Start with a constant acceleration a. The velocity is the integral:

$$v(t) = \int a\,dt = at + C$$

- At $t = 0$, $v = v_0$ ("initial value"). Substituting those values to find $C = v_0$, and therefore

$$\boxed{v(t) = v_0 + at}$$

Remember Our Kinematic Equations?

- Now we integrate $v(t)$ again to get position $s(t)$:

$$s(t) = \int v(t)dt = \int (v_0 + at)dt = v_o t + \frac{1}{2}at^2 + C$$

- Again, we take advantage of know our initial position, so $C = s_o$, and we have:

$$\boxed{s(t) = s_0 + v_o t + \frac{1}{2}at^2}$$

- You may be more familiar with this expression, where we use *displacement* $\Delta s(t) = s(t) - s_0$ instead of position s:

$$\Delta s(t) = v_o t + \frac{1}{2}at^2$$

Remember Our Kinematic Equations?

- In practical situations, acceleration is *not* constant, and we generally have to differentiate or integrate to find your answers.

Other Integrals

- **Impulse** (The components of the $\mathbf{F}$ can be integrated separately)

$$\mathbf{J} = \int_{t_1}^{t_2} \mathbf{F} dt$$

- **Work** by non-constant force

$$W = \int_{x_1}^{x_2} \mathbf{F}(x) \cdot d\mathbf{s}$$

This integral is straight forward if $\mathbf{F}$ is expressed as a function of position $\mathbf{s}$, but if it is written as a function of time, i.e. $\mathbf{F}(t)$, then we have to express $\mathbf{s}$ as a function of time as well

One Last Example

Work by Non-Constant Force

A force of $F(t) = 5t$ N is applied on an object $m = 1$ kg initially at rest, there is no friction force. What would be the displacement and work done on this object after 3 s?

1. Apply Newton's second law to find acceleration:

$$a(t) = \frac{F(t)}{m} = 5t$$

2. Then we integrate $a(t)$ with respect with time to get velocity $v(t)$:

$$v(t) = \int a(t)dt = \frac{5}{2}t^2$$

We already know that $v_0 = 0$, so we don't have to add C after the integral.

One Last Example

Work by Non-Constant Force

A force of $F(t) = 5t$ N is applied on an object $m = 1\,\text{kg}$ initially at rest, there is no friction force. What would be the displacement and work done on this object after $3\,\text{s}$?

3. We integrate again to get an expression for position $s(t)$:

$$\Delta s(t) = \int_0^3 v(t)dt = \frac{5}{6}t^3\Big|_0^3 = \frac{5}{6}\left(3^3 - 0\right) = \frac{45}{2}\text{ m}$$

One Last Example

Work by Non-Constant Force

A force of $F(t) = 5t$ N is applied on an object $m = 1$ kg initially at rest, there is no friction force. What would be the displacement and work done on this object after 3 s?

4. We know from our differential calculus that

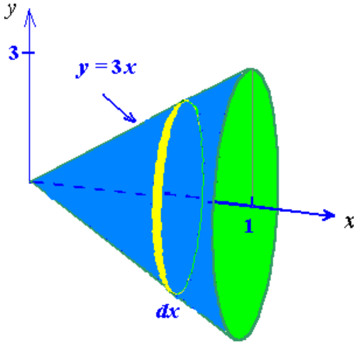
$$v(t) = \frac{ds}{dt} \quad \longrightarrow \quad ds = v(t)dt$$

5. The last step is to integrate force with velocity to find work done. (This works because we know F as a function of time.)

$$W = \int F ds = \int F(t)v(t)dt = \int_0^3 \frac{25}{2}t^3 dt = \frac{25}{8}t^4\Big|_0^3 = \frac{2025}{8} = \boxed{253 \text{ J}}$$

Integration to Find Volume

- Interested in finding the volume when we rotate *any* function about the x axis
- There are many applications, e.g. finding the CG or centroid of shapes



- Each yellow disk has a volume of $\pi r^2 dx$, where $r = f(x)$, so the infinitesimal volume dV of each disk is in fact:

$$dV = \pi f(x)^2 dx$$

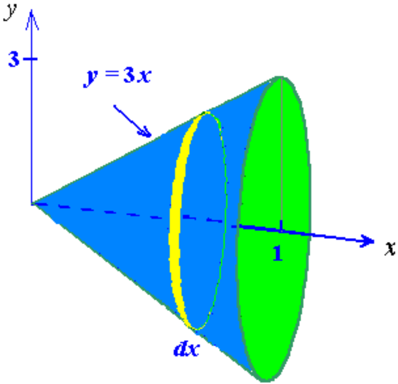
- "summing" them together gives us the integral:

$$\boxed{V = \int_{x_1}^{x_2} dV = \int_{x_1}^{x_2} \pi f(x)^2 dx}$$

Integration to Find Volume

Example: Find the volume of the following shape:

- In this question, $f(x) = 3x$, and we are integrating from $x_1 = 0$ to $x_2 = 1$



We use the formula from before:

$$
\begin{aligned}
V &= \int_{x_1}^{x_2} \pi f(x)^2 dx \\
&= \int_0^1 \pi 9x^2 dx \\
&= 9\pi \int_0^1 x^2 dx \\
&= 3\pi x^3 \Big|_0^1 \\
&= 3\pi
\end{aligned}
$$