Topic 3: Energy, Momentum and Collisions

Advanced Placement Physics

Dr. Timothy Leung

Olympiads School

Summer 2018

Files for You to Download

- `00-outline.pdf`—The course outline
- `03-momentumEnergy.pdf`—This week's slides
- `03-Homework.pdf` This week's homework

Please download/print the PDF file for the class slides before each class. There is no point copying notes that are already printed out for you. Instead, take notes on things I say that aren't necessarily on the slides.

Work and Energy

We start with some definition at are (unfortunately) not very useful:

- **Energy** is the ability to do work.
- **Work** is the mechanism in which energy is transformed.

Luckily, we can also use equations to define these concepts.

Work

Mechanical work is the integral of the dot product between a force $\mathbf{F}$ and the infinitesimal displacement vector $\mathbf{r}$:

$$\boxed{W = \int_{r_1}^{r_2} \mathbf{F}(r) \cdot d\mathbf{r}}$$

- No work done if the force is perpendicular to displacement (i.e. $\mathbf{F} \cdot \mathbf{r} = 0$)
- No work done if no displacement
- Negative work done is possible if $\mathbf{F}$ is opposite direction to $\mathbf{r}$
- When there are multiple forces acting on an object, we can compute the work done by each *each* force

Kinetic Energy

When we apply a force on an object to accelerate it, and the resulting amount of work done is given by

$$
\begin{aligned}
W &= \int \mathbf{F} \cdot d\mathbf{x} = \int m\mathbf{a} \cdot d\mathbf{x} \\
&= \int m\frac{d\mathbf{v}}{dt} \cdot d\mathbf{x} = \int m d\mathbf{v} \cdot \frac{d\mathbf{x}}{dt} = \int m\mathbf{v} \cdot d\mathbf{v} \\
&= \int_{v_1}^{v_2} mv dv = \Delta\left(\frac{1}{2}mv^2\right) = \Delta K
\end{aligned}
$$

where $K = \frac{1}{2}mv^2$ is the (translational) **kinetic energy**

Work and Kinetic Energy

In fact, the *definition* of kinetic energy came from this integration, in that we want to say that work equals to the change in *something*, and we called that kinetic energy.

$$\boxed{W_{\text{net}} = \Delta K}$$

- ΔK can be positive or negative depending on the dot product
- There may be multiple forces acting on an object; each of the forces can add or take away kinetic energy from an object
- Therefore we use the “net” amount of work done in the above equation

Example

Example 1: A force $F = 4.0x\hat{\imath}$ (in newtons) acts on an object of mass $2.0\,\text{kg}$ as it moves from $x = 0$ to $x = 5.0\,\text{m}$. Given that the object is at rest at $x = 0$,

(a) Calculate the net work

(b) What is the final speed of the object?

Gravitational Force & Potential Energy

The gravitational force (weight) of an object is defined as:

$$\boxed{\mathbf{F}_g = m\mathbf{g}}$$

For objects near the surface of Earth, we assume that $\mathbf{g} = -g\hat{\jmath} = -10\hat{\jmath}$ (in $\mathrm{m/s^2}$) is a constant. The work done to raise an object from height h_1 to h_2 is therefore:

$$W = \int \mathbf{F}_g \cdot d\mathbf{h} = \int_{h_1}^{h_2} -mg\hat{\jmath} \cdot dh\hat{\jmath}$$
$$= -mgh\Big|_{h_1}^{h_2} = -\Delta(mgh) = -\Delta U_g$$

where $U_g = mgh$ is the **gravitational potential energy**

Spring Force & Elastic Potential Energy

The spring force $\mathbf{F}_e$ is the force a compressed or stretched spring exerts onto objects connected to it. It obeys Hooke's Law:

$$\boxed{\mathbf{F}_e = -k\mathbf{x}}$$

When Hooke's law is applied to the work equation, we can find the work done to compress/stretch a spring:

$$W = \int \mathbf{F}_e \cdot d\mathbf{x} = \int -kxdx = -\frac{1}{2}kx^2\Big|_{x_1}^{x_2} = -\Delta\left(\frac{1}{2}kx^2\right)$$
$$= -\Delta U_e$$

where $U_e = \frac{1}{2}kx^2$ is the elastic potential energy

Work and Kinetic Energy

Like kinetic energy, the definition of gravitational and elastic potential energies came from integrating the work equation, in that work equals to the change in *something*, and we called that potential energy.

$$\boxed{W_{\text{net}} = -\Delta U}$$

- ΔU can be positive or negative depending on the direction of the force
- There may be multiple forces acting on an object; each of the forces can add or take away potential energies from an object

The Work-Energy Theorem and Conservation of Energy

Work is equal to the sum of the changing potential and kinetic energy. K is the kinetic energy of objects, and U are all the potential energies.

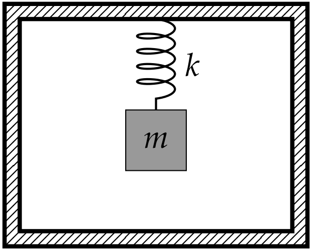
$$\boxed{W = \Delta U + \Delta K}$$

For an *isolated system*, the net amount of work done to the system by its surrounding must be zero, i.e. $W = 0$, therefore energy in the system must be conserved

$$\boxed{U + K = U' + K'}$$

Isolated Systems and the Conservation of Energy

- An isolated system is a system of objects that does not interact with its surroundings
- “Interaction” can be in the form of
    - Friction
    - Exchange of heat
    - Sound emission
- Think of an isolated system as a bunch of objects inside an insulated box

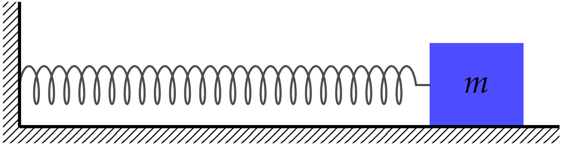


Isolated Systems and Conservation of Energy

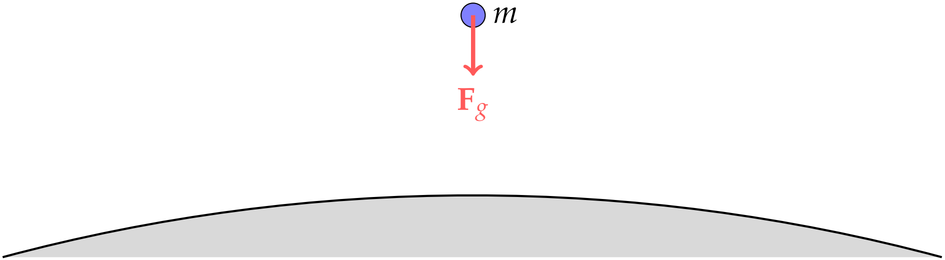
- Since the system is isolated from the surrounding environment, the environment can't do any work on it, by definition!
- Likewise, the energy inside the system cannot escape either
- Therefore energy of the system is conserved
- There are *internal* force inside the system that is doing work, but the work only converts kinetic energy into potential energies, and vice versa.

Example: Mass sliding on a spring

- Assuming that there is no friction in any part of the system
- The isolated system consists of the mass and the spring
- Energies:
  - Kinetic energy of the mass
  - Elastic potential energy stored in the spring

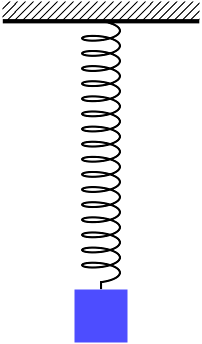


Example: Gravity



- The isolated system consists only of the mass and Earth.
- Assuming no friction
- Energies:
  - Kinetic energy of the mass
  - Gravitational potential energy of the mass

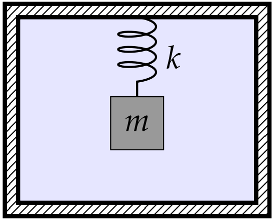
Example: A vertical spring-mass system



- The system consists of a mass, a spring and Earth
- Energies:
  - Kinetic energy of the mass
  - Gravitational potential energy of the mass
  - Elastic potential energy stored in the spring
- The total energy of the system is conserved if there is no friction

What if there is friction?

Energy is always conserved as long as your system is defined properly



- The system consists of a mass, a spring, Earth and all the air particles inside the box
- As the mass vibrates, friction with air slows it down
- While the mass loses energy, the temperature of the air rises due to friction
- Energies:
    - Kinetic and gravitational potential energies of the mass
    - Elastic potential energy stored in the spring
    - Kinetic energy of the vibration of the air molecules
- Total energy is conserved even as the mass stops moving

Conservative Forces

- Gravitational force and spring force (and also electrostatic force) are called **conservative forces**
- A conservative force has the property that the work done in moving a particle between two points is independent of the path taken. This force is related to a potential energy by:

$$\boxed{F_x = -\frac{dU}{dx}}$$

- The direction of the force decreases the potential energy

Conservation of Energy

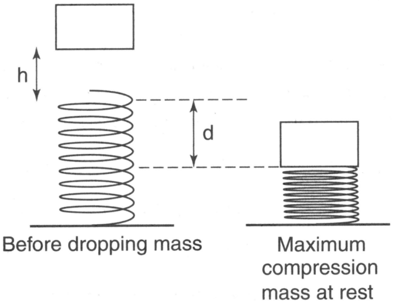
- If there are only conservative forces, energy is always conserved
- When there are non-conservative forces, instead of *trying* to isolate the system, we can calculate the work done by them $W_{\text{non-conservative}}$ and subtract it from the total energy of the system

$$\boxed{K_+U + W_{\text{non-conservative}} = K' + U'}$$

- Work done by $W_{\text{non-conservative}}$ are usually friction forces, convert mechanical energy in the system into sound and heat

Example

Example 2: A mass m is dropped from a height of h above the equilibrium position of a spring. Set up the equation that determines the spring's compression d when the object is instantaneously at rest.



Example

Example 3: A mass m is pulled a distance d up an incline (angle of elevation θ) at constant speed using a rope that is parallel to the incline. The coefficient of friction is μ_k.

(a) What is the magnitude of the tension force in the rope?

(b) What is the magnitude of the normal force?

(c) What is the work done by the normal force?

(d) What is the work done by friction?

(e) What is the work done by the tension force?

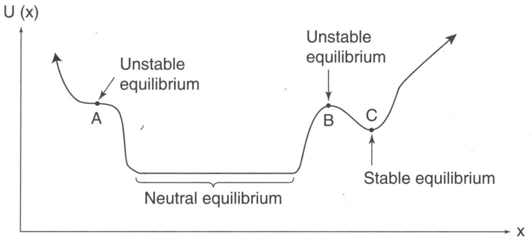
(f) What is the net work?

(g) What is the change in total mechanical energy?

(h) Show that $\Delta E_{\text{mech}} = W_{\text{non-conservative}}$.

Energy Diagrams

- Plots of potential energy (U) vs. position for a conservative force



- If more than one conservative force, they can be combined into one graph
- Where slope is zero means no force acting on it: it is in a state of **equilibrium**
- An object placed at an equilibrium point with $K = 0$ it will remain there

Linear Momentum

Linear momentum is directly proportional to the object's **mass** and its **velocity**.

$$\boxed{\mathbf{p} = m\mathbf{v}}$$

Quantity	Symbol	SI Unit
Momentum	$\mathbf{p}$	kg m/s (kilogram meters per second)
Mass	m	kg (kilograms)
Velocity	$\mathbf{v}$	m/s (meters per second)

- Momentum $\mathbf{p}$ is a vector in the same direction as velocity
- Like all vectors, $\mathbf{p}$ obeys *superposition*

Newton's Second Law of Motion

Start with our "standard form" of Newton's second law of motion with constant m, we can find out how $\Delta\mathbf{p}$ relates to $\mathbf{F}$:

$$\sum \mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt}$$

- $\mathbf{F} = \mathbf{p}'(t)$ is the general form, $\mathbf{F} = m\mathbf{a}$ is a special case
- Momentum is conserved (i.e. $\sum \mathbf{p}$ constant) when the net force on an object or a system of objects is zero.
- Internal forces do not contribute to net force, in that case:

$$\sum \mathbf{p}(t_1) = \sum \mathbf{p}(t_2)$$

Impulse

Let's get this by looking at Newton's 2nd law again. If we rearrange the variables:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \rightarrow \mathbf{F}dt = d\mathbf{p}$$

We can integrate both sides to get the **impulse-momentum theorem**.

$$\boxed{\mathbf{J} = \int_{t_1}^{t_2} \mathbf{F}dt = \int d\mathbf{p} = \Delta\mathbf{p}}$$

The quantity $\mathbf{J}$ is called the impulse.

Impulse

- $\mathbf{F}$, $\mathbf{p}$ and $\mathbf{J}$ are all vectors, so the integral can be evaluated in each of the x, y and z axis, e.g.

$$J_x = \int_{t_1}^{t_2} F_x dt = \int dp_x = \Delta p_x$$

for the x direction.

- In Physics 12, we often used the "average force" to compute impulse. In reality, the average force really is a "force" that gets the same impulse as the integral on the last slide, i.e.

$$\mathbf{F}_{\text{ave}} = \frac{\int_{t_1}^{t_2} \mathbf{F} dt}{t_2 - t_1} = \frac{\mathbf{J}}{\Delta t}$$

- Note that impulse does not depend on whether the object moves

Impulse: An Example

Example 4: Jim pushes a box with mass $1.0\,\mathrm{kg}$ with a $5.0\,\mathrm{N}$ force for $10\,\mathrm{s}$ while the box stays on the same place. Find the impulse of the pushing force, friction force, the gravitational force, and the net force.

Impulse: Another Example

Example 5: Two balls of the same mass are dropped from the same height onto the floor. The first ball bounces upwards from the floor elastically. The second ball sticks to the floor. The first applies an impulse to the floor of I_1 and the second applies an impulse I_2. The two impulses obey:

(a) $I_2 = 2I_1$

(b) $I_2 = I_1/2$

(c) $I_2 = 4I_1$

(d) $I_2 = I_1/4$

Conservation of Momentum

- From Newton's third law, we know that the action and reaction forces are always equal in magnitude and in opposite direction. Thus, their total impulse would be zero.
- Suppose there is no external force, the momentum of the total system would always be constant. We saw that a few slides ago:

$$\sum \mathbf{p}(t_1) = \sum \mathbf{p}(t_2)$$

How to Solve Conservation of Momentum Problem

1. Check whether the condition for the conservation of momentum is satisfied.
2. If so, write out expressions for initial momentum and final momentum, and equate the two. You will get 1 to 3 equations (one for each direction).
3. Solve these equations, find the quantity you need to find.

Two Remarks

- Sometimes, the external force *does* exist, but are too small, or the time interval of the external force is very short. In these cases, we can still regard the total momentum as conserved.
- Remember that momentum is a vector. If there is no external force component in some direction, then the momentum component in this direction is still conserved.

Example

Example 6: Two blocks A and B, both have mass $1.0\,\mathrm{kg}$. Block A has velocity $3.0\,\mathrm{m/s}$ and block B is at rest. Their distance is $1.0\,\mathrm{m}$. The surface is has dynamic friction coefficient 0.02. After they collide, they move together, what would be the final velocity of these two blocks? How far can they go after the collision?

Before We Dive Into Some Exercises

- The most typical applications of momentum conservation are collision and explosions
- **Collision: object A hits object B**. Regardless of whether they move together or not afterwards, momentum is conserved.
  - Head-on collisions are usually 1D
  - Glancing collisions are usually 2D or 3D.
- **Explosion: A explodes and becomes B and C (and D and E. . .)**. Total momentum of B and C (and D and E. . .) is the same as A in the beginning.

Collision Problem

Example 7: Two objects with equal mass are heading towards each other with equal speeds, undergo a head-on collision. Which one of the following statement is correct?

(a) Their final velocities are zero

(b) Their final velocities may be zero

(c) Each must have a final velocity equal to the other's initial velocity

(d) Their velocities must be reduced in magnitude

Conservation of Momentum Example

Example 8: Two astronauts, each of mass $75\,\mathrm{kg}$, are floating next to each other in space, outside the space shuttle. One of them pushes the other through a distance of $1.0\,\mathrm{m}$ (about an arm's length) with a force of $300\,\mathrm{N}$. What is the final relative velocity of the two?

(a) $2.0\,\mathrm{m/s}$

(b) $2.83\,\mathrm{m/s}$

(c) $4.0\,\mathrm{m/s}$

(d) $16.0\,\mathrm{m/s}$

Continuous Problems in the Application of Momentum

Example 10: A rocket generates a thrust force by ejecting hot gases from an engine. If it takes $1\,\mathrm{ms}$ to combust $1.0\,\mathrm{kg}$ of fuel, ejecting it at a speed of $1000\,\mathrm{m/s}$, what thrust is generated?

(a) $1000\,\mathrm{N}$

(b) $10\,000\,\mathrm{N}$

(c) $100\,000\,\mathrm{N}$

(d) $1\,000\,000\,\mathrm{N}$

Another Space Example

Example 11: A rocket for mining the asteroid belt is designed like a large scoop. It is approaching asteroids at a velocity of $10^4\ \mathrm{m/s}$. The asteroids are much smaller than the rocket. If the rocket scoops asteroids at at rate of $100\ \mathrm{kg/s}$, what thrust (force) must the rocket's engine provide in order for the rocket to maintain constant velocity? Ignore any variation in the rocket's mass due to the burning fuel.

(a) 10^3 N

(b) 10^6 N

(c) 10^9 N

(d) 10^{12} N

Example

Example 12: A ball is dropped from a height h. It hits the ground and bounces up with a momentum loss of 10% due to the impact. The maximum height it will reach is:

(a) $0.90h$

(b) $0.81h$

(c) $0.949h$

(d) $0.3h$

Conservation of Energy Example

Example 13: A simple pendulum has a bob of mass $2\,\mathrm{kg}$ hanging on a cord of length $1\,\mathrm{m}$. Suppose the pendulum is raised until it is horizontal (and angular displacement of 90°) and then released. What is the speed of the bob at the bottom of its swing?

(a) $9.91\,\mathrm{m/s}$

(b) $19.6\,\mathrm{m/s}$

(c) $3.13\,\mathrm{m/s}$

(d) $4.43\,\mathrm{m/s}$

Conservation of Energy Example

Example 14: A toy firing a ball vertically consists of a vertical spring which is compressed by $0.10\,\mathrm{m}$. A force of $10.0\,\mathrm{N}$ is needed to hold the spring at that compression. If a ball of mass $0.050\,\mathrm{kg}$ is placed on the compressed spring and the spring is released, the ball will reach a height (above its initial position) of:

(a) $1.0\,\mathrm{m}$

(b) $1.2\,\mathrm{m}$

(c) $1.4\,\mathrm{m}$

(d) $1.6\,\mathrm{m}$

Classifications of Collisions

- Elastic Collision:
  - Total kinetic energy is conserved
  - Momentum is conserved
- Inelastic collision:
  - Kinetic energy is **not** conserved
  - Momentum is conserved
- Completely inelastic collision:
  - "Perfectly inelastic collision"
  - The objects move together after the collision
  - Kinetic energy is **not** conserved
  - Momentum is conserved

Elastic Collision

If two objects 1 and 2 of mass m_1 and m_2 and initial velocities $v_{1,i}$ and $v_{2,i}$ collide elastically, their final velocities will be:

$$v_{1,f} = \frac{v_{1,i}(m_1 - m_2) + 2m_2 v_{2,i}}{m_1 + m_2}$$

$$v_{2,f} = \frac{v_{2,i}(m_2 - m_1) + 2m_1 v_{1,i}}{m_1 + m_2}$$

Elastic Collision Example

Example 15: Blocks A and B have the same mass; A hits B with a speed of $5.0\,\mathrm{m/s}$ while B is initially at rest. If the collision is elastic, what would be the final speed of these two objects?

Elastic Collision Example

Example 16: Blocks A and B with the same mass; A has a velocity $3.0\,\mathrm{m/s}$ to the east while B has $2\,\mathrm{m/s}$ to the west. If the collision is elastic, after the collision, what would the velocity of the two blocks be?

Elastic Collision Example

Example 17: Throw a ball to a really big wall, when the ball reaches the wall, it has a velocity $10\,\mathrm{m/s}$ towards the wall. If the collision is elastic, what would the final velocity of the ball be?

Elastic Collision Example

Example 18: Throw a ball with a velocity $4.0\,\mathrm{m/s}$ toward a train with a velocity $40\,\mathrm{m/s}$ towards the ball. If the collision is elastic, what would the final velocity of the ball be?

Inelastic Collision: Calculating Energy Loss

Example 19: Two blocks A and B with mass $2.0\,\mathrm{kg}$, block A hits B with velocity $4.0\,\mathrm{m/s}$ while B is at rest.

(a) Suppose the collision is completely inelastic, what would the final velocity of A and B be?

(b) What is the loss of energy?