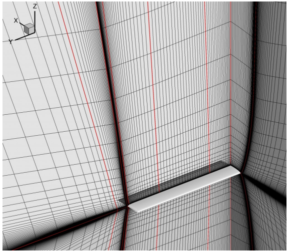
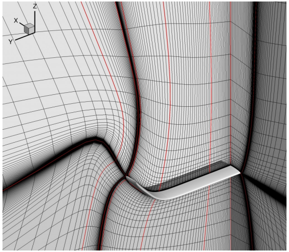
WELCOME TO AP PHYSICS

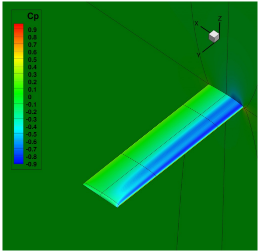
Hi, My Name is Tim

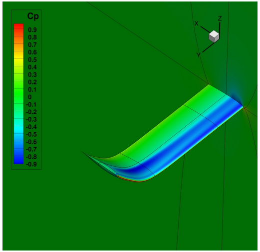
- B.A.Sc. in Engineering Physics (UBC)
  - Won the Roy Nodwell Prize for my design of a solar car
- M.A.Sc. and Ph.D. in Aerospace Engineering (UTIAS)
  - "Computational Fluid Dynamics" (CFD)
  - "Aerodynamic shape optimization"
  - Aircraft design
- Also spent a year in Vancouver as a professional violinist. . .

Tim's Past Research Work









Classroom Rules

- Treat me and each other with respect, and I'll treat you like an adult
- If you need to go to the bathroom, you don't need my permission
- Raise your hands if you have a question. Don't wait too long
- *"There is no such thing as a stupid question"*
- E-mail me at `tim@timleungjr.ca` for any questions related to physics and math and engineering
- Do ***not*** try to find me on social media

1. Calculus in Physics

AP Physics

Dr. Timothy Leung

Olympiads School

Fall 2017

Files for You to Download

- `01-Calculus-2x2.pdf`–The slides that I am using right now
- `01-Homework.pdf`–This week's homework assignment

Please download/print the PDF file for the class slides before each class. There is no point copying notes that are already printed out for you. Instead, take notes on things I say that aren't necessarily on the slides.

Invented for Physics

Thanks, Issac Newton

- We cannot learn physics properly without calculus (you have got away with it long enough in Grade 11 and 12 Physics classes. . .)
- Differential calculus was "invented" so that we can understand motion, especially on non-constant velocity and acceleration.
- If you are taking calculus, you may have noticed that a lot of the word problems are really physics problems

Differentiation and Integration

- **Differential Calculus**
  - Finding how quickly something is changing ("rate of change" of a quantity)
  - Math: slopes of functions
  - Physics: how quickly a physical quantity is changing in time and/or space
  - Examples: velocity (how quickly position changes), acceleration (how quickly velocity changes), power (how quickly work is done)
- **Integral Calculus**
  - The opposite of differentiation
  - We use it to compute the area under a curve, or
  - Summation of many small terms
  - Examples: area under the $\mathbf{v}$-t graph (to calculate displacement), area under the $\mathbf{F}$-t graph (to calculate impulse), area under the F-d graph (to calculate work)

FIRST, WE LOOK AT DIFFERENTIATION

Velocity, Time Derivative of Displacement

Suppose the motion of a car is governed by the equation:

$$s(t) = 3t^2$$

where s is the car's position along a straight path at time t. **What is its velocity at $t = 2$?**

(At the moment it's not important what *units* we use. May be s is in metres and t in seconds, or s in kilometres and t in hours. The principle still holds regardless.)

Instantaneous Velocity

- We can use $s(t)$ to find the average velocity between $t = 2$ and 3:

$$v_{\text{ave}} = \frac{s(3) - s(2)}{3 - 2} = \frac{27 - 12}{1} = 15$$

- Or the average velocity between $t = 2$ and $t = 2.5$
- Or the average velocity between $t = 2$ and $t = 2.1$
- But I cannot just plug in $t = 2$ into $s(t)$ and expect to get the instantaneous velocity, because average velocity needs two specific time values.
- Perhaps I can find the average velocity between $t = 2$ and...

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Solution: Differentiate!

- The premise of differential calculus (as applied to our example) is that if we can find the average velocity between $t = 2$ and $t = 2 + h$, where h is a *very* small positive number, we have actually found the *instantaneous* velocity at $t = 2$

$$\begin{aligned} v &= \frac{s(2+h) - s(2)}{(2+h) - 2} = \frac{3(2+h)^2 - 3(2)^2}{h} \\ &= \frac{3(4+4h+h^2) - 12}{h} = \frac{4h+h^2}{h} = 4 + h \end{aligned}$$

- Since we know that h is a very very small number, we have $v = 4$!

Instantaneous Velocity

Time Derivative of Displacement

- In fact, this is the very *definition* of a derivative.
- For any arbitrary function $f(x)$, the derivative with respect to x is defined by:

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}$$

- The "limit as h approaches 0" is the mathematical way of making h a very small number

Instantaneous Velocity

Time Derivative of Displacement

- So what we have now is that the instantaneous velocity of an object is the time derivative of its position:

$$\boxed{v(t) = s'(t) = \frac{ds}{dt}}$$

- In physics, we *usually* use the prime notation (e.g. v') to indicate the derivative is the rate of change with respect to time, and use the d/dx notation to indicate rate of change with respect to spatial coordinates (x, y or z). But it's not always the case.

You Don't Have to Apply The Definition All The Time

Ways To Save Time

It's tedious to use the definition of the derivative every time I want to compute the velocity of an object. Thankfully mathematicians have recognized some patterns:

- The derivative of a constant ("C") is zero:

$$\frac{d}{dt}C = 0$$

  This shouldn't be surprising, as the slope of the function $f(x) = C$ is always zero

- A constant multiple of any function can be factored outside the derivative:

$$\frac{d}{dt}\left[af(x)\right] = a\frac{d}{dx}f(x)$$

Time-Saving Rules for Differentiation

- The derivative of a sum is the sum of a derivative:

$$\frac{d}{dt}\left(f(t)+g(t)\right)=\frac{df}{dt}+\frac{dg}{dt}$$

- Power Rule:

$$\frac{d}{dt}\left(t^n\right)=nt^{n-1} \quad \text{for all} \quad n \neq 0$$

FYI: if $n=0$ we really just have a constant.

- Try these examples:

$$\frac{d}{dt}\left(3t^2\right)= \qquad \frac{d}{dt}\left(t^3+t+4\right)= \qquad \frac{d}{dt}\left(\frac{1}{t}\right)=$$

Time-Saving Rules for Differentiation

- Sines and cosines:

$$\frac{d}{dt}\sin t = \cos t \qquad \frac{d}{dt}\cos t = \sin t$$

- For AP-level physics, you will not need to be an expert in all things differential, but helps to have a lot of experience before tackling difficult problems

Instantaneous Acceleration

In the same way that velocity is the time derivative of displacement, **acceleration is the time derivative of velocity**, i.e.:

$$\boxed{a(t) = v'(t) = s''(t)}$$

- Acceleration is the second derivative of position, i.e.
  1. Take derivative of $s(t)$ to get $v(t) = s'(t)$
  2. Take derivative again of $v(t)$ to get $a(t) = v'(t)$

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  2. Take derivative again of $v(t)$ to get $a(t) = v'(t)$
- **Example:** If the position of an object is given by $s(t) = 3t^5$, what is
  - the velocity at $t = 1$ and
  - the acceleration at $t = 1$?

Newton's Second Law of Motion

- You may be familiar with Newton's second law written as:

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = m\frac{d^2\mathbf{s}}{dt}$$

- That's a *special case* where mass m remains constant, and $\mathbf{F}$ is only related to acceleration
- In Grade 11 and 12 physics (no calculus!), we only deal with cases where $\mathbf{F}$ is a constant (acceleration $\mathbf{a}$ is constant)
- With differential calculus, however, we can relate an acceleration that is time depending (i.e. changes with time) with a time-depend force

Newton's Second Law of Motion

General Form

- The general form of Newton's second law is actually this:

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt}$$

  the vector quantity $\mathbf{p} = m\mathbf{v}$ is an object's momentum

- In this case, we do not require either the mass or velocity to be constant; both can vary with time.

Let's Try To Do an Example

Suppose there is a small cart moving along an icy road with no friction. The cart has mass 5 kg and a constant velocity 5 m/s. Suddenly it begins to rain and rain water is collected in the bed of the cart. Now the cart's mass changes as

$$m(t) = 5 + 0.01t \text{ kg}$$

That is, every second the mass of the cart increases by 0.01 kg. **If the cart wants to have the same velocity, what would be the force needed?**

Solving the Example Without Calculus

Not Recommended

- Before the rain started, the net force on the truck is zero. Once the rain started though,
- After 1 s, the cart gains 0.01 kg of water in the bed
- That means we need to accelerate this water from rest to 5 m/s in 1 s, i.e. $a = 1\mathrm{m/s}^2$
- This 1 s worth of water will require a force of

$$F = ma = 0.05\mathrm{N}$$

Much Easier

- Apply Newton's second law of motion:

$$F = \frac{d(mv)}{dt} = \frac{d}{dt}(5 + 0.01t)(5) = 5\frac{d}{dt}(5 + 0.01t) = 0.05 \text{ N}$$

- We can see that in this case, although v is constant, because mass m changes with time, there is a net force applied to the cart.

A Familiar Problem

Suppose there is a small cart moving along an icy road with no friction. The cart has mass 5 kg and a constant velocity 5 m/s. Suddenly it begins to snow and the wet snow is collected in the bed of the cart. Now the cart's mass changes as

$$m(t) = 5 + 0.01t \text{ kg}$$

If the cart wants to have the velocity of

$$v(t) = 5 + 0.1t$$

what would be the force needed?

Newton's Second Law

- We can apply Newton's second law again:

$$F(t) = \frac{d(mv)}{dt} = \frac{d}{dt}\left[(5 + 0.01t)(5 + 0.1t)\right]$$
$$= \frac{d}{dt}\left(25 + 0.55t + 0.001t^2\right) = 0.55 + 0.002t$$

- Since both mass and velocity are changing with time, force is not a constant, but it's also a function of time

Another Way of Looking At the Problem

- Think of this as a two part problem:
  - Force F_1 is used to provide the acceleration for the existing mass $(5+0.01t)\times 0.1$
  - Force F_2 is used to accelerate new water into the speed $0.01\times(5+0.1t)$
  - In all, $F=F_1+F_2$
- In fact, this is actually the "product rule" in calculus:

$$\frac{d}{dx}\left(f(x)g(x)\right)=f'(x)g(x)+f(x)g'(x)$$

- Applying this to our example, we really have

$$\begin{aligned} F(t)=\frac{d(mv)}{dt}&=m(t)v'(t)+m'(t)v(t)\\ &=(5+0.01t)\frac{d}{dt}(5+0.1t)+(5+0.1t)\frac{d}{dt}(5+0.01t)\\ &=(5+0.01t)(0.1)+(5+0.1t)(0.01)=0.55+0.002t\end{aligned}$$

NOW ON TO INTEGRATION

Integration: Area Under the Curve

- Let's to an example: A car is moving with speed $v(t) = 5t$. What is its displacement at $t = 5$?
- We know that if on a v-t graph, and the area under that curve is the displacement. So how do we find the area?
- If we divide 5 into many small time intervals:

$$\Delta t_1,\ \Delta t_2,\ \Delta t_3,\ \Delta t_4, \ldots,\ \Delta t_n$$

We can find the displacement in teach of these Δt_i, and

- In this example, the total displacement would be

$$d(5) = \sum_{i=1}^{n} v(t_i)\Delta t_i = \int_{t_1}^{t_2} v(t)dt = \int_{t=0}^{5} 5t\ dt = \frac{5}{2}t^2\Big|_0^5 = \frac{125}{2}$$

Integration: Differentiation in Reverse

$$\frac{d}{dt}\left(t^2\right) = \frac{1}{2}t \qquad \longrightarrow \qquad \int \frac{1}{2}t dt = t^2$$

Commonly Used Integrals in Physics

Calculating an integral can be a very daunting task. But these few rules should help:

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

$$\int \frac{1}{x} = \ln x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

Area Under A Curve

What is the area under the curve

$$f(x) = 2x^2 + 3x + 1 \quad \text{between} \quad x = 1 \text{ and } x = 5$$

Our integration works like this:

$$\begin{aligned} A &= \int_1^5 \left(2x^2 + 3x + 1\right) dt \\ &= \left(\frac{2}{3}x^3 + \frac{3}{2}x^2 + x\right) |_3^5 \\ &= 24 + \frac{196}{3} \end{aligned}$$

Kinematic Equations

- Remember this equation:

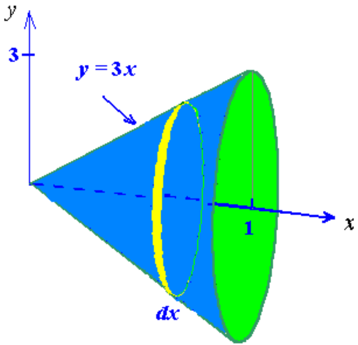
$$s(t) = s_0 + v_0 t + \frac{1}{2}at^2$$

(the notation that you used may be a little bit different, but it's the same equation)

- We actually obtained this by integrating a constant acceleration

Integration to Find Volume

- Interested in finding the volume when we rotate *any* function about the x axis
- Many applications in physics, e.g. finding the centre of mass or centroid of shapes



- Each circular disk the yellow has a volume of $\pi r^2 dx$, where $r = f(x)$, so the volume of each disk is in fact:

$$dV = \pi f(x)^2 dx$$

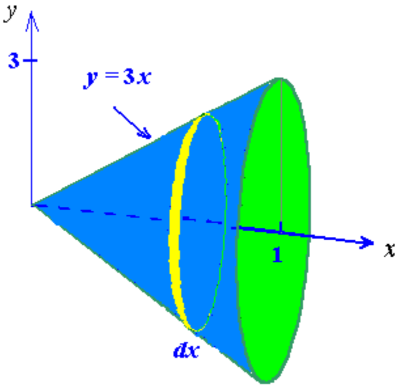
- "summing" them together gives us the integral:

$$\boxed{V = \int_{x_1}^{x_2} dV = \int_{x_1}^{x_2} \pi f(x)^2 dx}$$

Integration to Find Volume

Example: Find the volume of the following shape:

- In this question, $f(x) = 3x$, and we are integrating from $x_1 = 0$ to $x_2 = 1$



We use the formula from before:

$$
\begin{aligned}
V &= \int_{x_1}^{x_2} \pi f(x)^2 dx \\
&= \int_0^1 \pi 9x^2 dx \\
&= 9\pi \int_0^1 x^2 dx \\
&= 3\pi x^3 \Big|_0^1 \\
&= 3\pi
\end{aligned}
$$

One Last Example

Using Integration to calculate work done by non-constant force

A force of $F(t) = 5t$N is applied on an object $m = 1\,\text{kg}$ at rest, there is no friction force. What would be the displacement and work done on this object at $t = 3\,\text{s}$?

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1. Apply Newton's second law to find acceleration: $a(t) = \dfrac{F}{m} = 5t$

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1. Apply Newton's second law to find acceleration: $a(t) = \dfrac{F}{m} = 5t$
2. Then we integrate to get velocity: $v(t) = \int a(t) = \dfrac{5}{2}t^2$

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2. Then we integrate to get velocity: $v(t) = \displaystyle\int a(t) = \frac{5}{2}t^2$
3. And finally, displacement: $s(t) = \displaystyle\int v(t) = \frac{5}{6}t^3 \quad \longrightarrow \quad$ at $t = 3$, $\boxed{d = \dfrac{45}{2}\text{m}}$

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3. And finally, displacement: $s(t) = \int v(t) = \dfrac{5}{6}t^3 \longrightarrow$ at $t = 3$, $\boxed{d = \dfrac{45}{2}\text{m}}$
4. Integrate force with velocity to find work done:

$$W = \int F(t)v(t)dt = \int \frac{25}{2}t^3 dt = \frac{25}{8}t^4 \longrightarrow \text{at } t = 3, \boxed{W = \frac{2025}{8}\text{J}}$$