Topic 6: Circular Motion

Advanced Placement Physics

Dr. Timothy Leung

December 7, 2019

Olympiads School

Files to Download

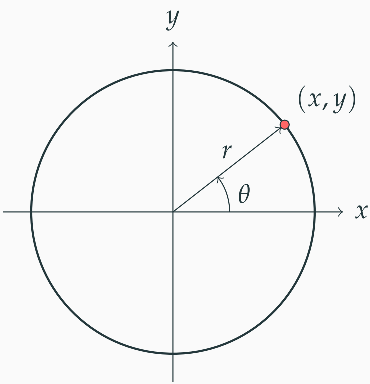
Please download the following files from the school website if you have not already done so:

- `PhysAP-06-rotMotion-print.pdf`—The “print version” of the class slides for this topic.
- `PhysAP-06-Homework.pdf`—Homework problems for this topic.

Please download/print the PDF file for the class slides before each class. There is no point copying notes that are already on the slides. Instead, focus on things that aren’t necessarily on the slides. If you wish to print the slides, we recommend printing 4 slides per page.

Polar Coordinates

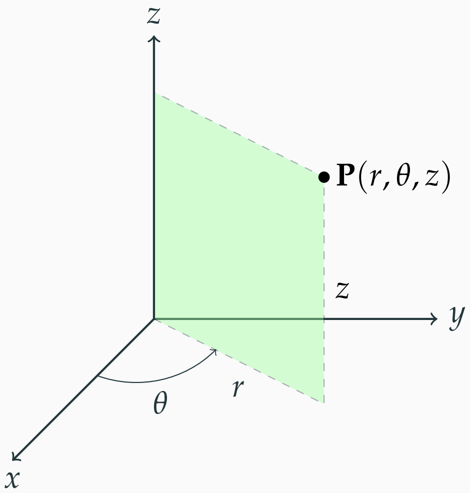
Polar Coordinate System in 2D



- Cartesian coordinate system $\mathbf{x}(x, y)$ is not the only way to describe the position of an object
- For circular motion, **polar coordinates** are better
- Position described by $\mathbf{r}(r, \theta)$
  - r is distance from the origin
  - θ is the standard angle, measured counter-clockwise from the positive x axis
- Cartesian and polar coordinates are related by:

$$x = r \cos \theta$$
$$y = r \sin \theta$$

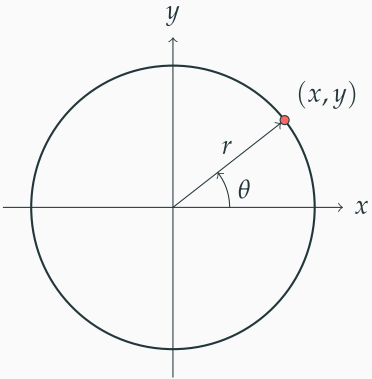
Cylindrical Coordinates in 3D



One way to extend the coordinates coordinate system into 3D. This is called the **cylindrical coordinate system**. Note that the discussions for this topic focuses on xy plane. Since the z-axis is linearly independent of the xy plane, motion along that direction is independent.

Rigid-Body Motion

Angular Position and Angular Velocity



For constant r, **angular position** θ determines an object's position as a function of time:

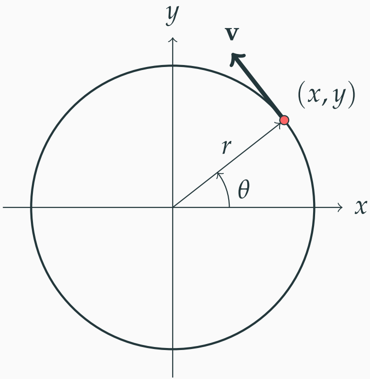
$$\boxed{\theta = \theta(t)}$$

Angular velocity ω (or **angular frequency**) is its time derivative

$$\boxed{\omega(t) = \frac{d\theta(t)}{dt} = \dot{\theta}}$$

θ is measured in radians, and ω in rad/s

Angular Velocity



Velocity $\mathbf{v}$ and angular velocity $\boldsymbol{\omega}$ are related by the cross product with the position vector $\mathbf{r}$:

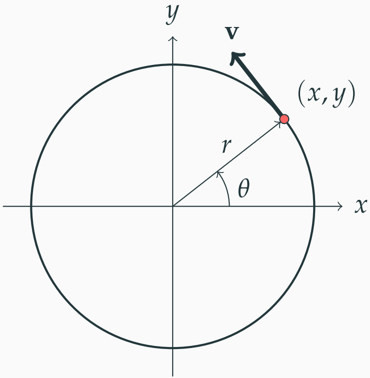
$$\boxed{\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}}$$

If you are uncomfortable with the vector notation, the scalar form may be more useful to you:

$$\boxed{v = r\omega}$$

- $\mathbf{v}$: tangent to circle; perpendicular to $\mathbf{r}$
- $\boldsymbol{\omega}$: *out of the page* if $\mathbf{v}$ is counterclockwise, and *into the page* if $\mathbf{v}$ is clockwise
- Visualizing $\boldsymbol{\omega}$ will take some practice, but it is mathematically rigorous

Period & Frequency of Uniform Circular Motion



For constant angular velocity ω (i.e. uniform circular motion), the motion is **periodic**. Its **frequency** and **period** are given by:

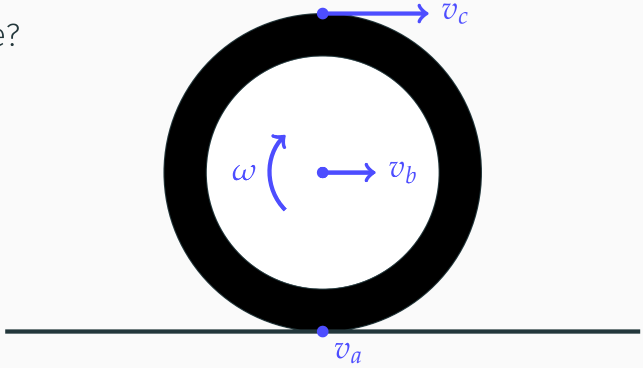
$$\boxed{f = \frac{\omega}{2\pi}} \qquad \boxed{T = \frac{2\pi}{\omega}} \qquad \boxed{f = \frac{1}{T}}$$

T is in **seconds** (s) and f is in **hertz** (Hz)

Rotating Object Without Slipping

A tire with radius r rolls along the road with an angular velocity ω *without slipping*. (This is a very common case for analysis.) What is its velocity v

a. at the contact between the ground and the tire?
b. at the center?
c. at the top of the tire?



Angular Acceleration

The derivative of ω with respect to time gives us **angular acceleration**, which has a unit of rad/s^2:

$$\boxed{\alpha = \dot{\omega} = \ddot{\theta}}$$

Similar to the relationship between velocity and angular velocity, **tangential acceleration** a_θ is related to angular acceleration α by the radius r:

$$\boxed{a_\theta(t) = \dot{v} = r\dot{\omega} = r\alpha}$$

For *uniform* circular motion, ω is constant, and therefore $\alpha_\theta = 0$

Kinematics in the Angular Direction

For constant α, the kinematic equations are just like in linear motion:

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\theta = \theta_0 + \frac{\omega_0 + \omega}{2} t$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

If α is *not* constant, integration will be required.

A Simple Example

Example 1: An object moves in a circle with angular acceleration 3.0 $\mathrm{rad/s^2}$. The radius is 2.0 m and it starts from rest. How long does it take for this object to finish a circle?

Centripetal Acceleration & Centripetal Force

There is also a component of acceleration toward the center of the motion, called the **centripetal acceleration** a_r:

$$\boxed{\mathbf{a}_r = -\frac{v^2}{r}\hat{\mathbf{r}} = -\omega^2 r\hat{\mathbf{r}}}$$

The direction $\hat{\mathbf{r}}$ is the radial outward direction, measured from the center. The force that causes the centripetal acceleration is called the **centripetal force**:

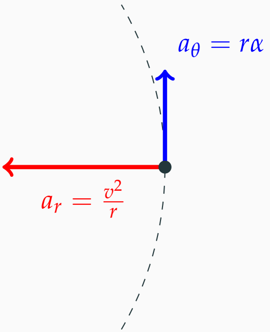
$$\boxed{\mathbf{F}_r = m\mathbf{a}_r = -\frac{mv^2}{r}\hat{\mathbf{r}}}$$

Centripetal Acceleration for Uniform Circular Motion

In uniform circular motion ($\alpha = 0$) problems where period and/or frequency are known, centripetal acceleration can be expressed based on those quantities:

$$\boxed{\mathbf{a}_r = -\frac{4\pi^2 r}{T^2}\hat{\mathbf{r}} = -4\pi^2 r f^2 \hat{\mathbf{r}}}$$

Acceleration: The General Case



- In general circular motion, there are two components of acceleration:
    - Centripetal acceleration a_r depends on radius of curvature r and instantaneous speed v. The direction of the acceleration is toward the center of the circle.
    - Tangential acceleration a_θ depends on radius r and angular acceleration α. The direction of the acceleration is tangent to the circle

Most of the cases in AP Physics are uniform circular motion.

How to Solve Circular Motion Problems

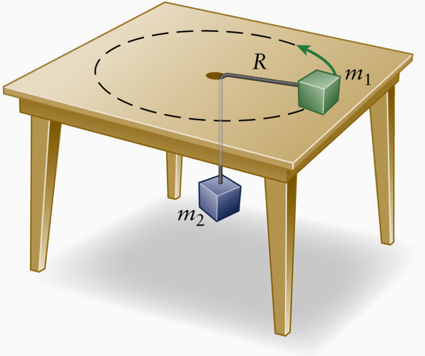
1. Is there any circular motion?
2. If so, the condition for circular motion is:

$$\mathbf{F}_{\text{provided}} = \mathbf{F}_{\text{required}}$$

   - The *provided* force comes from FBD
   - The *required* force comes from the centripetal force equation

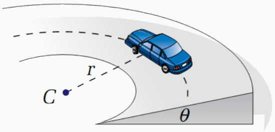
3. If the net force also has a tangential component, then there is also a change in angular velocity

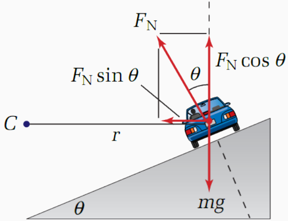
Example: Horizontal Motion



Example 2: In the figure on the left, a mass $m_1 = 3.0\,\mathrm{kg}$ is rolling around a frictionless table with radius $R = 1.0\,\mathrm{m}$. with a speed of $2.0\,\mathrm{m/s}$. What is the mass of the weight m_2?

Banked Curves on Highways and Racetracks





Vertical direction

$$F_N \cos\theta = F_g = mg$$

Horizontal direction (toward C)

$$F_N \sin\theta = F_c = \frac{mv^2}{r}$$

Combine the two equations together:

$$\frac{F_N \sin\theta}{F_N \cos\theta} = \frac{\frac{mv^2}{r}}{mg}$$

Cancel out F_N and m terms:

$$\frac{\sin\theta}{\cos\theta} = \frac{v^2}{rg} \longrightarrow \boxed{\tan\theta = \frac{v^2}{rg}}$$

Another Example: Exit Ramp

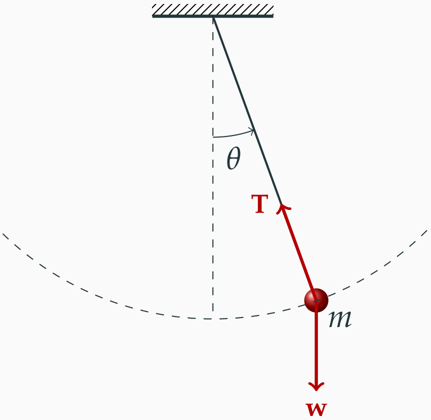
Example 3: A car exits a highway on a ramp that is banked at $15°$ to the horizontal. The exit ramp has a radius of curvature of $65\,\mathrm{m}$. If the conditions are extremely icy and the driver cannot depend on any friction to help make the turn, at what speed should the driver travel so that the car will not skid off the ramp? What if there is friction?

Vertical Circles

- Uniform circular motion with a horizontal path is straightforward
- For vertical motion:
  - Generally not solvable by dynamics
  - We can use conservation of energy to solve for $\mathbf{v}$
  - Then use the equation for centripetal force to find other forces
- **Remember:** If it is impossible to get the required centripetal force, then it could not continue the circular motion

What About a Pendulum?

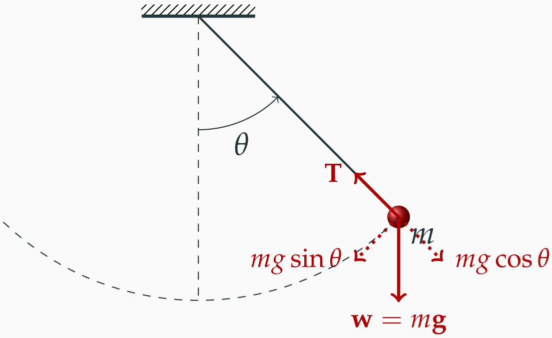
The motion of a pendulum is also like a vertical circular motion problem. There are two forces act on the pendulum: weight $\mathbf{w} = m\mathbf{g}$, and tension $\mathbf{T}$



- Speed of the pendulum at any height is found using conservation of energy
  - Tension $\mathbf{T}$ is always $\perp$ to motion, therefore it doesn't do any work
  - Work is done by gravity (a conservative force) alone
- The velocity vector is tangent to the path
- Tangential and centripetal acceleration are based on the net force along the angular and radial directions

What About a Pendulum?

At the top of the swing, velocity is zero, therefore



Centripetal acceleration is zero:

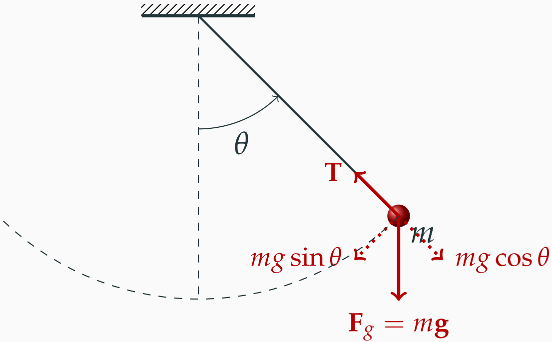
$$a_r = \frac{v^2}{r} = 0$$

which means that net force along the radial direction $\hat{\mathbf{r}}$ is zero. Therefore the tension force is:

$$T = mg\cos\theta$$

At the highest point, tension is the lowest.

What About a Pendulum?



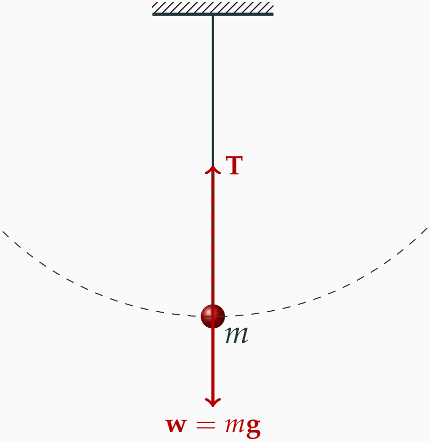
There is still a net force of $mg\sin\theta$ along the tangential direction, therefore there is an acceleration in that direction, with a magnitude of:

$$a_\theta = g\sin\theta$$

This is the same acceleration as an object sliding down a frictionless ramp at an angle of θ.

What About a Pendulum?

At the bottom of the swing, the velocity is at maximum,



- Maximum centripetal acceleration:

$$a_r = \frac{v^2}{r}$$

- No tangential acceleration:

$$a_\theta = 0$$

- At the lowest point, tension is the highest:

$$T = w + F_r = m\left(g + \frac{v^2}{r}\right)$$

Example Problem

Example 4: You are playing with a yo-yo with a mass of 225 g. The full length of the string is 1.2 m. You decide to see how slowly you can swing it in a vertical circle while keeping the string fully extended, even when the yo-yo is at the top of its swing.

A. Calculate the minimum speed at which you can swing the yo-yo while keeping it on a circular path.
B. Find the tension in the string when the yo-yo is at the side and at the bottom of its swing.

Example Problem

Example 5: A cord is tied to a pail of water, and the pail is swung in a vertical circle of 1.0 m. What must be the minimum velocity of the pail be at its highest point so that no water spills out?

(a) 3.1 m/s
(b) 5.6 m/s
(c) 20.7 m/s
(d) 100.5 m/s

Example: Roller Coaster

Example 6: A roller coaster car is on a track that forms a circular loop, of radius R, in the vertical plane. If the car is to maintain contact with the track at the top of the loop (generally considered to be a good thing), what is the minimum speed that the car must have at the bottom of the loop. Ignore air resistance and rolling friction.

(a) $\sqrt{2gR}$

(b) $\sqrt{3gR}$

(c) $\sqrt{4gR}$

(d) $\sqrt{5gR}$

Example

Example 7: A stone of mass m is attached to a light strong string and whirled in a *vertical* circle of radius r. At the exact bottom of the path, the tension of the string is three times the weight of the stone. The stone's speed at that point is given by:

(a) $2\sqrt{gR}$

(b) $\sqrt{2gR}$

(c) $\sqrt{3gR}$

(d) $4gR$

Torque

Let's consider this question:

> Two people stand on a board of uniform density. One person has a mass of 50 kg and stands 10 m away from the fulcrum (pivot). The second person has a mass of 65 kg. How far away from the fulcrum would the second person have to stand for the system to have to be in equilibrium?

Equation of Motion

Recall Newton's second law of motion for objects with constant mass:

$$\mathbf{F}_{\text{net}} = m\mathbf{a}$$

- Is it also true for *circular* motion?
- If a net force $\mathbf{F}_{\text{net}}$ causes a mass to accelerate (linearly), what causes a mass to go into circular motion?

Answer: We need to introduce a few concepts first...

Torque

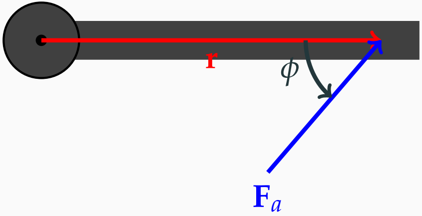
I have a rod on a table, and with my fingers, I push the two ends of the rod with equal force F. **What happens?**

$\mathbf{F}_{\text{net}} = \mathbf{0}$, therefore $\mathbf{a} = \mathbf{0}$. But (obviously) it won't stay still either!

What is Torque?

Torque (or **moment**) is the tendency for a force to change the rotational motion of a body.

- A force $\mathbf{F}_a$ acting at a point some distance $\mathbf{r}$ (called the **moment arm**) from a **fulcrum** (or **pivot**) at an angle ϕ between $\mathbf{F}_a$ and $\mathbf{r}$
- e.g. the force to twist a screw



Torque

In scalar form, we can express torque $\boldsymbol{\tau}$ as the force $\mathbf{F}_a$, the **moment arm** $\mathbf{r}$ and the angle ϕ between $\mathbf{F}_a$ and $\mathbf{r}$:

$$\boxed{\tau = rF_a \sin\phi}$$

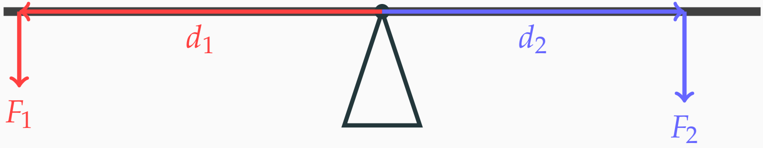
In vector form, we use the cross-product:

$$\boxed{\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}_a}$$

Quantity	Symbol	SI Unit
Torque	$\boldsymbol{\tau}$	N m
Applied force	$\mathbf{F}_a$	N
Moment arm (from fulcrum to force)	$\mathbf{r}$	m
Angle between force and moment arm	ϕ	(no units)

Torque

Going back to the example question:



- F_1 will rotate the board counter clockwise
- F_2 will rotate the board clockwise
- The beam will remain static (in equilibrium) if

$$F_1 d_1 = F_2 d_2$$

Rotational Equilibrium

An object is in **translational equilibrium** is when the force acting it is zero:

$$\mathbf{F} = \mathbf{0}$$

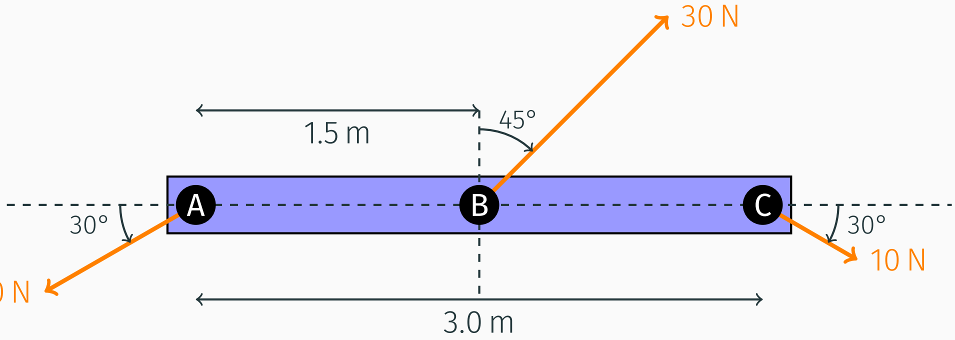
Likewise, an object is in **rotational equilibrium** when the net torque acting on it is zero:

$$\boldsymbol{\tau} = \mathbf{0}$$

Note that it does *not* mean that the object has no rotational motion, it just means that the object's overall *rotational state* is not changing, i.e. $\alpha = 0$

Example Problem

Example 8a: Find the net torque on point C.



Example 8b: Now find the net torque on A.

Angular Momentum

Angular Momentum

Consider a mass m connected to a massless beam rotates with speed v at a distance r from the center (shown on the right). It has an **angular momentum** (L) of

$$\boxed{\mathbf{L} = \mathbf{r} \times \mathbf{p}}$$

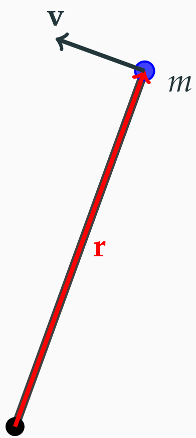
Expanding the terms in the definition:

$$\mathbf{L} = \mathbf{r} \times (m\mathbf{v}) = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = mr^2\boldsymbol{\omega}$$

Which gives us:

$$\boxed{\mathbf{L} = I\boldsymbol{\omega}}$$

The quantity I is called the **moment of inertia**.



Moment of Inertia

A single particle:

$$\boxed{I = r^2 m}$$

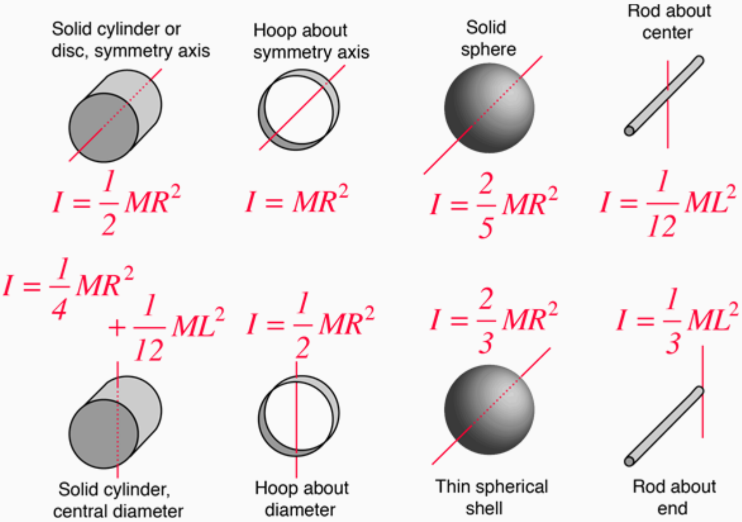
A collection of particles:

$$\boxed{I = \sum r_i^2 m_i}$$

Continuous distribution of mass:

$$\boxed{I = \int r^2 dm}$$

Moment of Inertia



Angular Momentum and Moment of Inertia

- Linear and angular momentum have very similar expressions

$$\mathbf{p} = m\mathbf{v} \qquad\qquad \mathbf{L} = I\boldsymbol{\omega}$$

- Just as $\mathbf{p}$ describes the overall *translational* state of a physical system, $\mathbf{L}$ describes its overall *rotational* state
- Momentum of inertia I can be considered to be an object's "rotational mass"

Newton's Second Law of Motion

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \frac{d(\mathbf{r} \times \mathbf{p})}{dt} \longrightarrow \boxed{\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}}$$

- If the net torque on a system is zero, then the rate of change of angular momentum is zero, and we say that the angular momentum is conserved.
- e.g. When an ice skater starts to spin and draws his arms inward. Since angular momentum is conserved, a decrease in r means an increase in ω.

Newton's Second Law of Motion

Newton's second law of motion for rotational motion has a very similar form to translational motion:

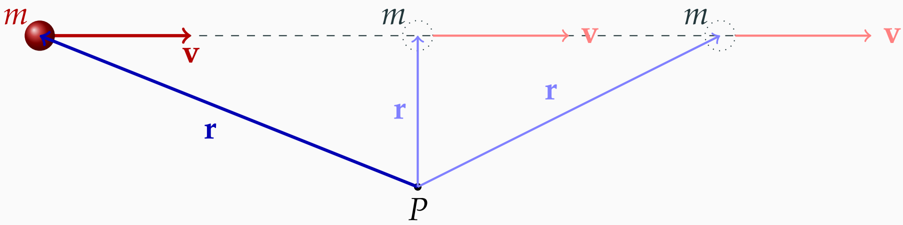
$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \qquad \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$

For objects with constant mass (translational motion) or constant moment of inertia (rotational motion), Newton's second law reduces to:

$$\mathbf{F} = m\mathbf{a} \qquad \boldsymbol{\tau} = I\boldsymbol{\alpha}$$

But there is no rotational motion, is there?

Even when there is no apparent rotational motion, it does not mean that angular momentum is zero! In this case, mass m travels along a straight path at constant velocity (uniform motion), but the angular momentum around point P is not zero:



Since there is no force and no torque acting on the object, both the linear momentum ($\mathbf{p} = m\mathbf{v}$) and angular momentum ($\mathbf{L} = \mathbf{r} \times \mathbf{v}$) are constant.

Example Problem

Example 9: A skater extends her arms (both arms!), holding a 2.0 kg mass in each hand. She is rotating about a vertical axis at a given rate. She brings her arms inward toward her body in such a way that the distance of each mass from the axis changes from 1.0 m to 0.50 m. Her rate of rotation (neglecting her own mass) will?

Last Example

Example 10: A 1.0 kg mass swings in a vertical circle after having been released from a horizontal position with zero initial velocity. The mass is attached to a massless rigid rod of length 1.5 m. What is the angular momentum of the mass, when it is in its lowest position?

Rotational Kinetic Energy

Rotational Kinetic Energy

To find the kinetic energy of a rotating system of particles (discrete number of particles, or continuous mass distribution), we sum (or integrate) the kinetic energy of the individual particles:

$$K = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2$$

$$K = \int \frac{1}{2} v^2 dm = \frac{1}{2} \left(\int r^2 dm \right) \omega^2$$

It's no surprise that in both case, rotational kinetic energy is given by:

$$\boxed{K = \frac{1}{2} I \omega^2}$$

Kinetic Energy of a Rotating System

The total kinetic energy of a rotating system is the sum of its translational and rotational kinetic energies at its center of mass:

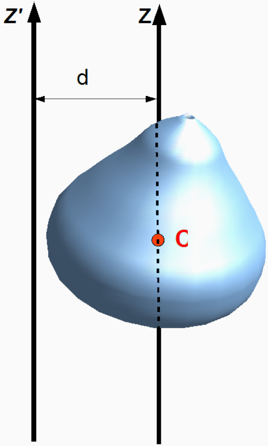
$$\boxed{K = \frac{1}{2}mv_{\mathrm{CM}}^2 + \frac{1}{2}I_{\mathrm{CM}}\omega^2}$$

In this case, I_{CM} is calculated at the center of mass. For simple problems, we only need to compute rotational kinetic energy at the pivot:

$$\boxed{K = \frac{1}{2}I_{\mathrm{P}}\omega^2}$$

In this case, the I_{P} is calculated at the pivot. **IMPORTANT:** $I_{\mathrm{CM}} \neq I_{\mathrm{P}}$

Parallel Axis Theorem



The **parallel axis theorem** relates the moment of inertia of an object along two different but parallel axis by:

$$\boxed{I = I_{\mathrm{CM}} + md^2}$$