Topic 19: Mechanical Waves

Advanced Placement Physics

Dr. Timothy Leung

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Olympiads School
Toronto, Ontario, Canada

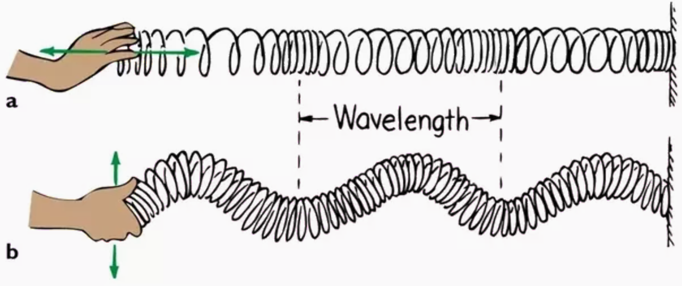
Properties

Mechanical Waves

A **mechanical wave** is a traveling disturbance that transport energy through a medium

- When a disturbance (vibration) causes vibrations in its vicinity, a wave is created
- Does not transport matter
- Examples:
  - Sound wave (medium: air, solids and liquids)
  - Ocean wave (medium: water)
  - Wave on a string (medium: string, rope)
- In contrast, electromagnetic (”EM”) waves do not require a medium

Two Kinds of Waves

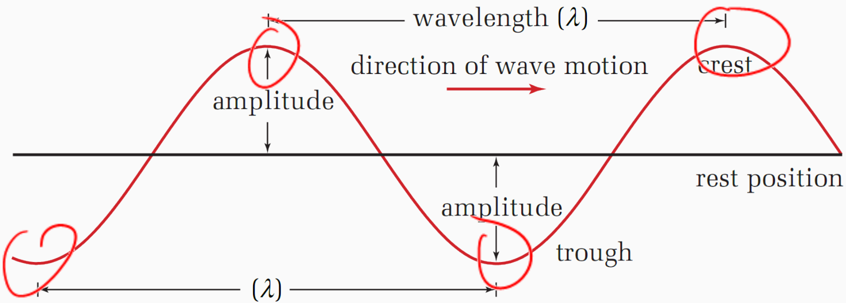


a. **Longitudinal wave**
  - Vibration is parallel to the direction of the motion of the wave
  - Example: sound waves

b. **Transverse wave**
  - Vibrations occur right angles to the direction of the wave
  - Example: electromagnetic waves

Physical Properties of a Wave

- The *highest* point of the wave is called a **crest** or **peak**, while
- The *lowest* point in the wave is called a **trough**.
- The **wavelength** λ is the shortest distance between two points in the medium that are in phase. The easiest way to measure wavelength is from crest to crest, or from trough to trough.



Wave Equation

The Wave Equation

The mechanical wave as we know it is the solution to a second-order partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

subject to initial condition.

Equation of a Traveling Wave

One solution to the wave equation is a **harmonic wave** that can be described as a sinusoidal function that oscillates in both space x and time t:

$$\boxed{u(x,t) = A\sin(kx - \omega t)}$$

Quantity	Symbol	SI Unit
Displacement of the medium	u	m
Amplitude of the wave	A	m
Wave number	k	1/m
Distance from the source	x	m
Time	t	s
Angular frequency	ω	rad/s

$f(x)$

$\downarrow$

$f(ax)$ horizontal compression/expansion of the function

Equation of the Wave

$$\boxed{u(x,t) = A\sin(kx - \omega t)}$$

The angular frequency (angular velocity) is related to the frequency f and period T of the wave by:

$$\omega = \frac{2\pi}{T} = 2\pi f$$

The **wave number** k can be thought of as a the “spatial frequency” of the wave, and is related to the wavelength λ by:

$$k = \frac{2\pi}{\lambda}$$

Speed of a Wave

Using the universal wave equation, we fin that the speed of a wave can be related to the wave number and angular frequency by:

$$v = \frac{\lambda}{T} = \frac{2\pi/k}{2\pi/\omega} = \frac{\omega}{k}$$

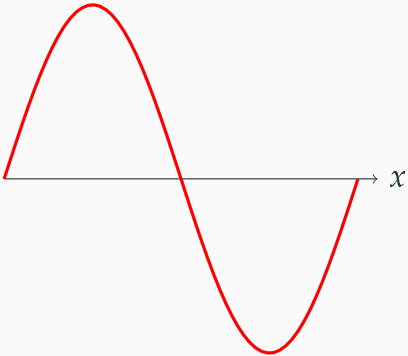
Why Sine and Cosines

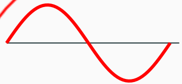
French mathematician Joseph Fourier showed that *all* periodic functions can be represented as an infinite series of $\sin$ and/or $\cos$ functions:

$$\begin{aligned} f(x) = & a_1 \sin(x) + a_2 \sin(2x) + a_3 \sin(3x) + \ldots + \\ & b_1 \cos(x) + b_2 \cos(2x) + b_3 \cos(3x) + \ldots \\ = & \sum_{n=1}^{\infty} a_n \sin(nx) + \sum_{n=1}^{\infty} b_n \cos(nx) \end{aligned}$$

The sum is called the **Fourier series**. Depending on the shape of the wave, some coefficients a_n and b_n are zeros. This part is particularly important to sound waves.

Making a Square Wave with Sine Waves





$f_1 = \sin(x)$

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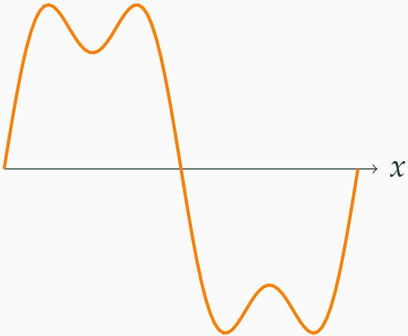

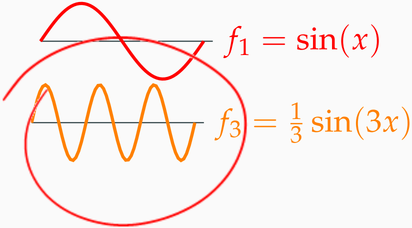

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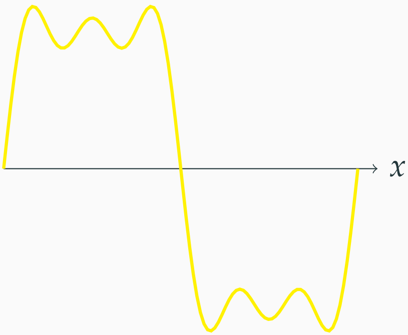

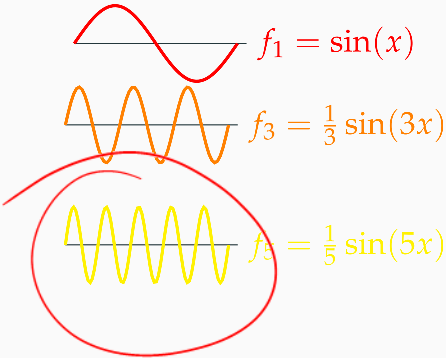

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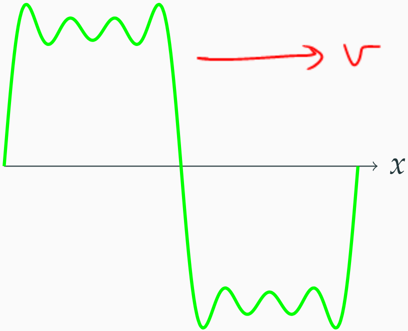
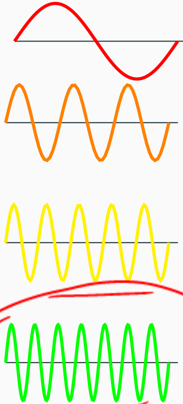




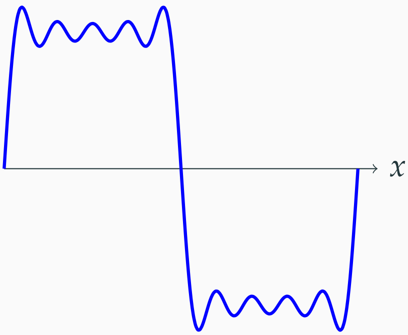
$f_1 = \sin(x)$

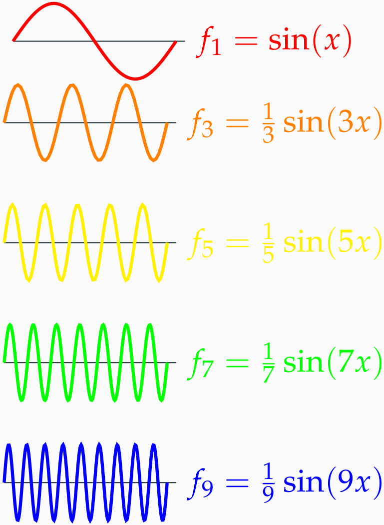
$f_3 = \frac{1}{3}\sin(3x)$

$f_5 = \frac{1}{5}\sin(5x)$

$f_7 = \frac{1}{7}\sin(7x)$

Making a Square Wave with Sine Waves





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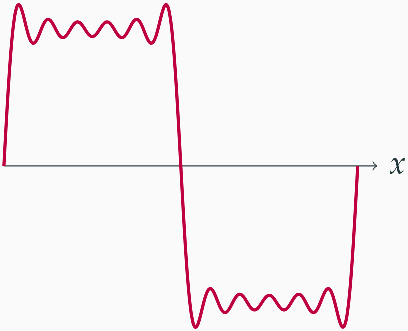

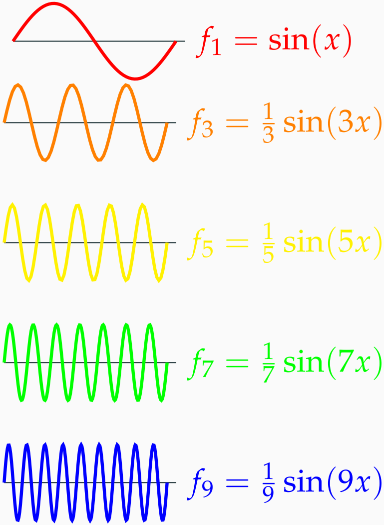

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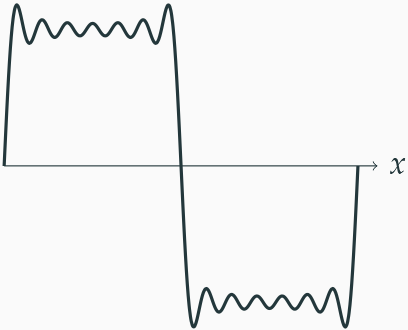

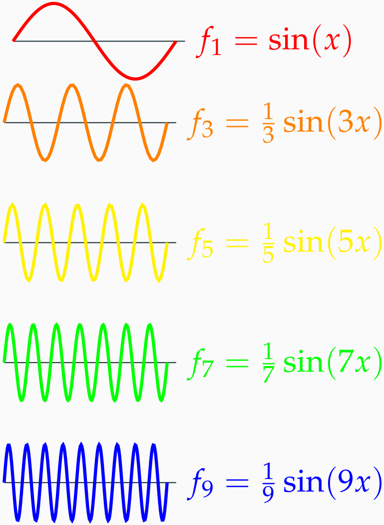

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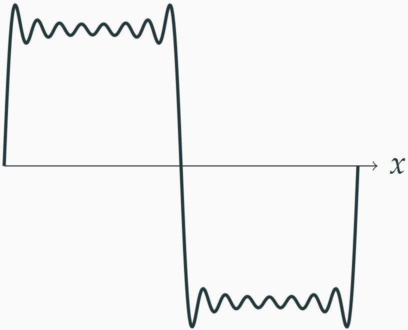
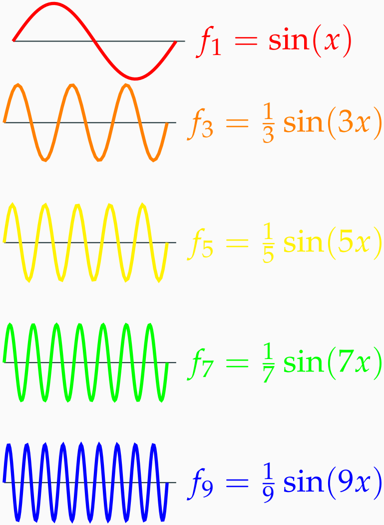
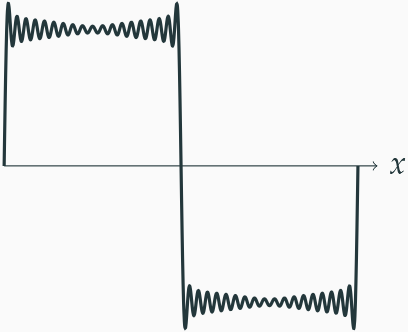
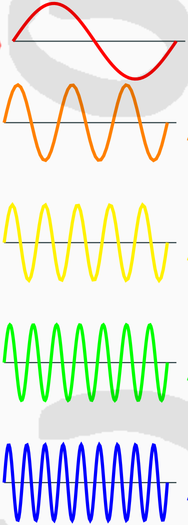




Making a Square Wave with Sine Waves

spatial frequency → "low frequency" →





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$f_7 = \frac{1}{7}\sin(7x)$

$f_9 = \frac{1}{9}\sin(9x)$

high spatial frequency →

Wave Speed

Universal Wave Equation

The **universal wave equation** relates the speed of a mechanical wave to its wavelength, period and frequency of the disturbance:

$$\boxed{v = f\lambda = \frac{\lambda}{T}}$$

Quantity	Symbol	SI Unit
Speed	v	m/s
Frequency	f	Hz
Wavelength	λ	m
Period	T	m

The universal wave equation applies to *all* waves.

Frequency and Speed of A Wave

Frequency (f):

- The number of complete wavelengths that pass a point in a given amount of time
- Same as the frequency of the disturbance that generated the wave
- **Does not depend on the medium**, only the source that produced the wave

Wave speed (v)

- The speed at which the wave fronts are moving
- **Depends only on the medium**, not the source disturbance that produced the wave
- Within the same medium, waves of different wavelengths can travel at different speeds

Speed of Sound in a Gas

The equation for the speed of sound in a gas is given by:

$$\boxed{v_s = \sqrt{\frac{\gamma R T}{M}}}$$

does not depend on the frequency (or wavelength) of sound

Quantity	Symbol	SI Unit
Speed of sound	v_s	m/s
Temperature	T	K
Universal gas constant	R	J/mol K
Molar mass	M	kg/mol
Adiabatic constant	γ	(no units)

For diatomic gases such as air $\gamma = 1.4$, and $M = 29 \times 10^{-3}$ kg/mol. For air near room temperature, the equation can be simplified to: $\boxed{v_s = 331 + 0.59 T_C}$ where T_c is the temperature in *degrees celsius*.

Speed of Sound in Solids and Liquids

Speed of sound in a liquid depends on the "bulk modulus" K and density ρ of the liquid:

$$v = \sqrt{\frac{K}{\rho}}$$

Speed of sound in a solid depends on the "Young's modulus" E of the solid and density ρ

$$v = \sqrt{\frac{E}{\rho}}$$

In general, sound travels fastest in solids, then liquids, then gasses.

Material	**Speed** (m/s)
Gases (0 °C, 101 kPa)	
Carbon dioxide	259
Oxygen	316
Air	331
Helium	965
Liquids (20 °C)	
Ethanol	1162
Fresh water	1482
Seawater	1440-1500
Solids	
Copper	5010
Glass	5640
Steel	5960

Wave on a String

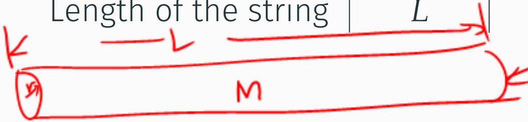
The speed of a traveling wave on a stretched string is determined by:

$$\boxed{v = \sqrt{\frac{F_T}{\mu}}} \quad \text{where} \quad \boxed{\mu = \frac{m}{L}}$$

NOT density $\rho = \frac{m}{V}$

Quantity	Symbol	SI Unit
Wave speed	v	m/s
Tension	F_T	N
Linear mass density	μ	kg/m
Mass of the string	m	kg
Length of the string	L	m

$\mu = \frac{M}{L}$



$\rho = \frac{M}{V} = \frac{M}{\pi r^2 L}$

Speed of an Surface Ocean Wave

The speed of a surface wave in deep ocean is given by:

$$\boxed{v = \sqrt{\frac{\lambda g}{2\pi}}}$$

Quantity	Symbol	SI Unit
Wave speed	v	m/s
Wavelength	λ	m
Acceleration due to gravity	g	m/s^2

In this case, in an ocean wave, the higher frequency (shorter wavelength) travel faster than the lower frequency waves (longer wavelength). This is called **dispersion**.

Power Transmitted by a Harmonic Wave

The total power $\bar{P}$ transmitted by a harmonic wave is given by:

$$\boxed{\bar{P} = \frac{1}{2}\mu\omega^2 A^2 v}$$

wave on a string

for all waves

$P \propto A^2$

amplitude

Quantity	Symbol	SI Unit
Averge power	$\bar{P}$	W
Linear mass density	μ	kg/m
Angular frequency	ω	rad/s
Wave amplitude	A	m
Wave speed	v	m/s

Intensity of a 3D Wave

For a three dimension waves (e.g. sound waves, ripple in pond) where the wave front expands as the wave travels, it makes more sense to describe the power transmitted by its **intensity** I:

$$\boxed{I = \frac{\bar{P}}{A}}$$

area

For a spherical wave (e.g. sound emitted from a stationary source), the area that the wavefront passes through is $A = 4\pi r^2$, where r is the distance from the source.

The Decibel

The **decibel** is defined as by the intensity of sound I compared to the **threshold of hearing** I_o (defined as $1 \times 10^{-12}\,\mathrm{W/m^2}$):

Sound intensity

$$\boxed{\beta = 10 \log_{10} \left[\frac{I}{I_0} \right]}$$

reference level.

Quantity	Symbol	SI Unit
Decibel	β	dB
Sound intensity	I	$\mathrm{W/m^2}$
Threshold of hearing	I_0	$\mathrm{W/m^2}$

- The threshold of hearing is 0 dB, while the **threshold of pain** is 120 dB.
- Humans perceive a doubling of loudness when intensity is increases by a factor of 10 (an increase of 10 dB)

$1\,\mathrm{W/m^2}$

Mach Number

When dealing with sound waves, it is often useful to express speed in terms of its ratio to the speed of sound. This ratio is called the **mach number**:

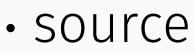
$$\boxed{M = \frac{v}{v_s}}$$

Quantity	Symbol	SI Unit
Mach number	M	(no units)
Speed of the object	v	m/s
Local speed of sound	v_s	m/s

- Speeds lower than $M < 1$ is called *subsonic*
- Speeds higher than $M > 1$ is called *supersonic*

Sound from a Stationary Source

When a sound is emitted from a stationary point source, the sound wave moves radially outward from the origin:



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- All points hear the same wavelength (and frequency) of sound

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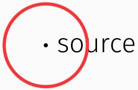

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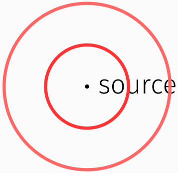

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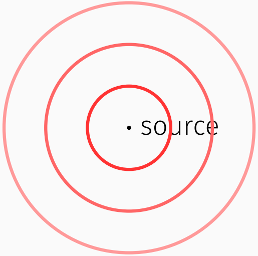

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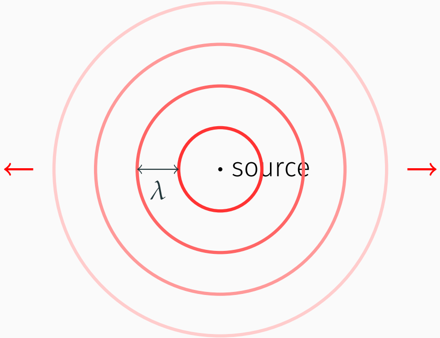

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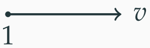
Sound from a Moving Source

When sound is emitted from a *moving* source, the diagram looks different. In this case, the sound source is moving to the right, from 1 to 4:



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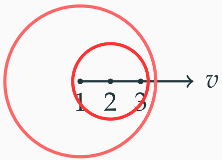

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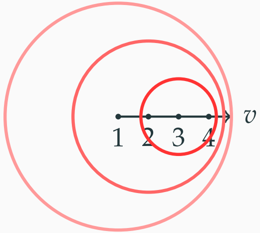

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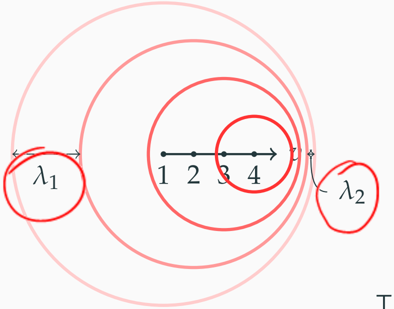


- When the source is moving *toward you*, the wavelength λ_2 decreases, and the apparent frequency increases.
- When the source is moving *away from you*, the wavelength λ_1 increases, and the aparent frequency decreases.

This is called the **Doppler effect**.

Doppler Effect

We all experience Doppler effect every time an ambulance speeds by us with its sirens on.

When it is moving toward us, the pitch of the siren is high, but the moment it passes us, the pitch decreases.

Doppler Effect

When a wave source is moving at a speed v_{src} and an observing is moving at observer v_{ob}, the perceived frequency is shifted:

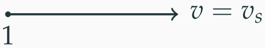
$$\boxed{f' = \frac{v_s \pm v_{\text{ob}}}{v_s \mp v_{\text{src}}} f}$$

Quantity	Symbol	SI Unit
Apparent frequency	f'	Hz
Actual frequency	f	Hz
Speed of sound	v_s	m/s
Speed of source	v_{src}	m/s
Speed of observer	v_{ob}	m/s

the
~~Te~~ $+$ sign is used if the source and observer are approaching each other, while $-$ is when they are receding

Sound Source at Sonic Speed

Doppler effect is even more interesting is when sound source is moving at the speed of sound ($M = 1$):



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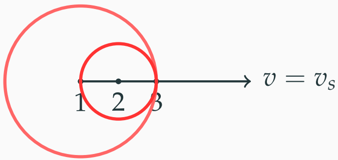

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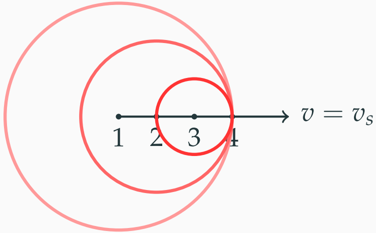

Sound Source at Sonic Speed

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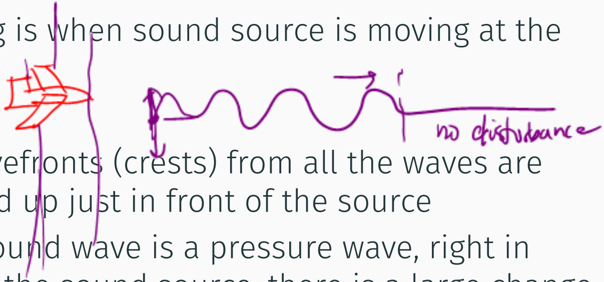
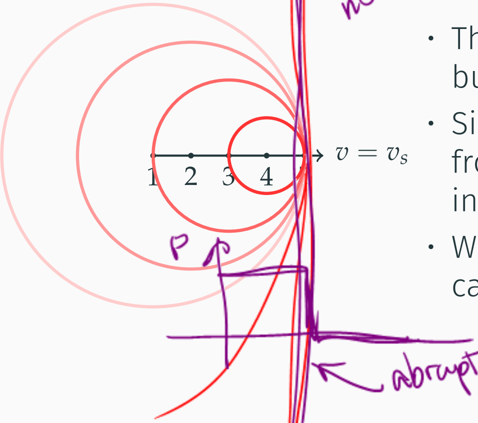
no disturbance





- The wavefronts (crests) from all the waves are bunched up just in front of the source
- Since sound wave is a pressure wave, right in front of the sound source, there is a large change in pressure (called a shock wave)
- When the shock passes an observer, an loud bang can be heard (aka **sonic boom**)

abrupt change in wave properties

Sound from a Supersonic Source

When sound source is moving at $M > 1$, it out runs the sound that it makes:

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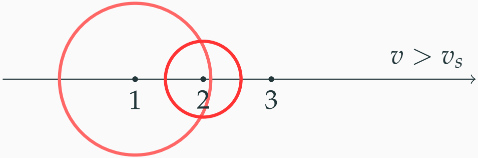

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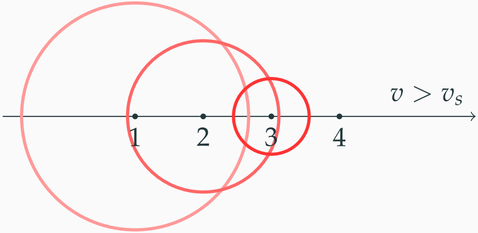

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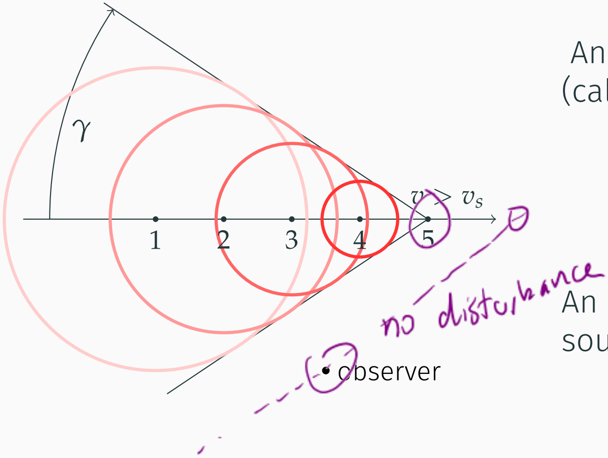


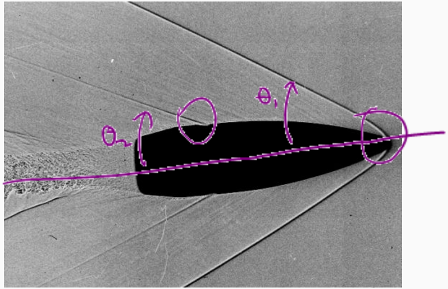
An *oblique shock* is formed at an angle (called the **Mach angle**) given by:

$$\gamma = \sin^{-1}\left(\frac{1}{M}\right)$$

An observer does not hear the sound source until it has gone past!

Bullet in Supersonic Flight

Generating a shock doesn't require an actual sound source. Any object moving through air creates a pressure disturbance. Below is a NATO bullet in supersonic flight:

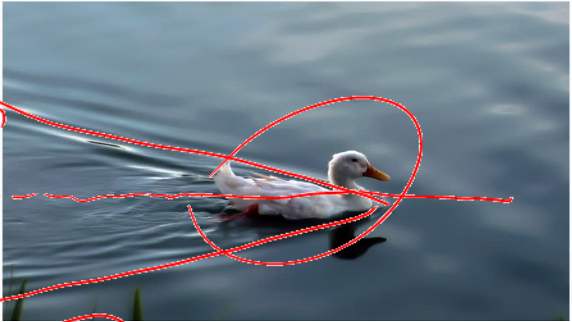


The flow around this bullet is taken inside a *shock tube* that generates a short burst of supersonic flow. A high-speed camera is used to take the photo.

Duck in Water

A similar shock behavior is observed when the duck swims in water, because the duck swims faster than the speed of the water wave, it also creates a cone shape.

γ



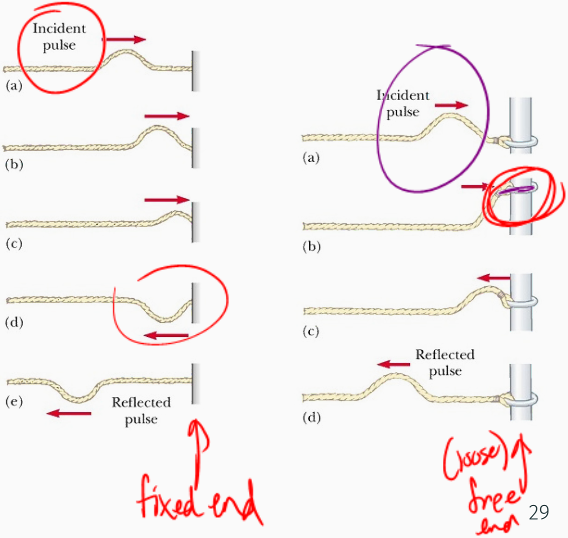
$v_{duck} > v_{wave}$

Reflection and Transmission

Reflection of a Wave at a Boundary

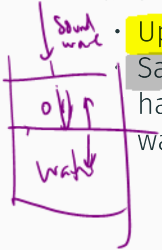
When a wave on a string reflects at a boundary, how the reflected wave looks depends on the type of boundary

- At a *fixed end* (left):
  - the reflected wave is *inverted*
  - i.e. a phase shift of π
  - i.e. a crest becomes a trough
- At a *free end* (right)
  - the reflected wave is upright
  - No phase shifts



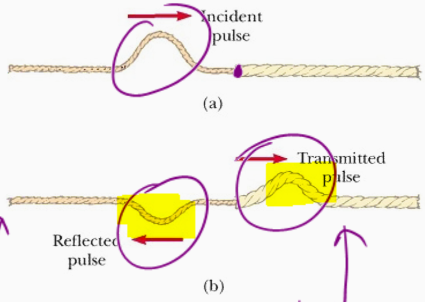
Transmission of Waves: Fast to Slow Medium

- Reflected wave:
  - Inverted, like a fixed end
  - Same frequency and wavelength as the incoming wave
  - The amplitude is decreased because energy is split between the reflected and transmitted waves
- Transmitted wave:
  - Upright
  - Same frequency as incoming wave, but has a shorter wavelength because the wave slowed down



$v = f\lambda$

lower   same   lower



light rope (fast medium)

heavy rope (slow)

Transmission of Waves: Slow to Fast Medium

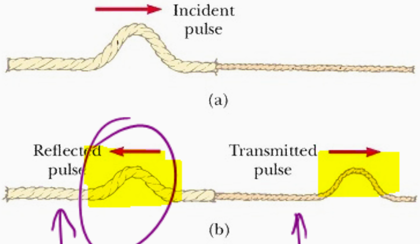
- Reflected wave:
  - Upright, like a free end
  - Same frequency and wavelength as the incoming wave
  - The amplitude is decreased because energy is split between the reflected and transmitted waves
- Transmitted wave:
  - Upright
  - Same frequency as incoming wave, but has a longer wavelength because the wave sped up

same

$v = f\lambda$

higher higher

Note that the transmitted wave is *always* upright.



slow

fast

Interference

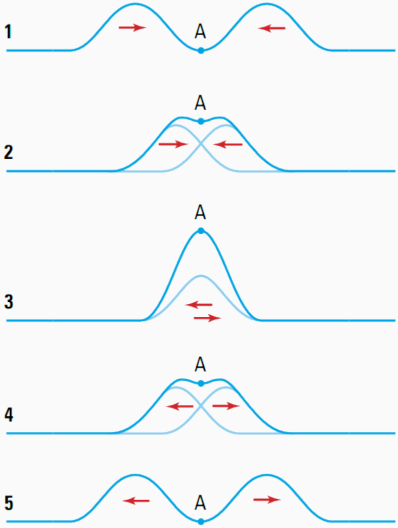
Superposition of Waves

- **Principle of Superposition:** When multiple waves pass through the same point, the resultant wave is the *sum* of the waves
    - A fancy way of saying that waves add together
- The consequence of the principle of superposition is **interference of waves**. There are two kinds of interference:
    - **Constructive interference:** Two wave fronts (crests) passing through creates a wave front with greater amplitude
    - **Destructive interference:** A crest and trough will cancel each other

Superposition of Waves

Constructive interference: In-phase wave fronts sum together

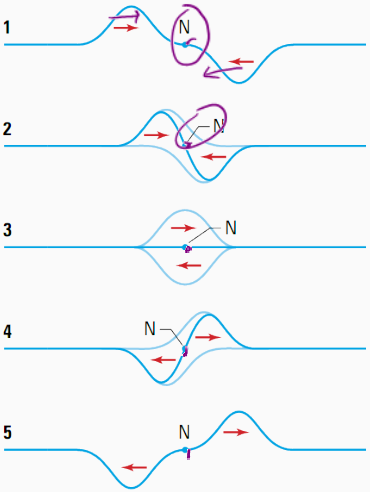
- In this example, two identical pulses move towards each other
- Their crests pass through A at the same time
- The amplitude at A when the waves pass through is higher



Superposition of Waves

Destructive interference: Out-of-phase wave fronts shows the difference of the wave fronts

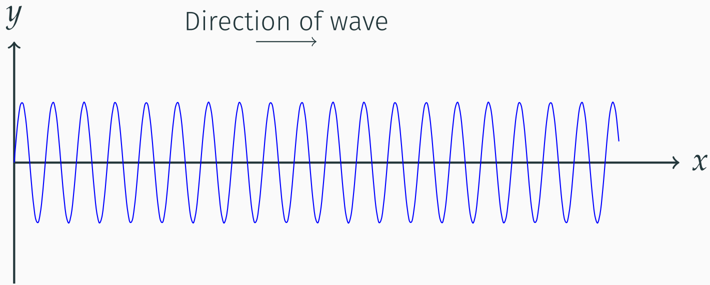
- Two pulses move towards each other, one a crest, the other a trough
- They both pass through A at the same time
- Two waves cancel each other at A



Beat Frequency

When waves (e.g. sound waves) of two different frequencies are added together, there is both constructive and destructive interference because of the principle of superposition

- Plotting two functions representing two waves with equal magnitude and wave speed v_s: $y = \sin(x)$



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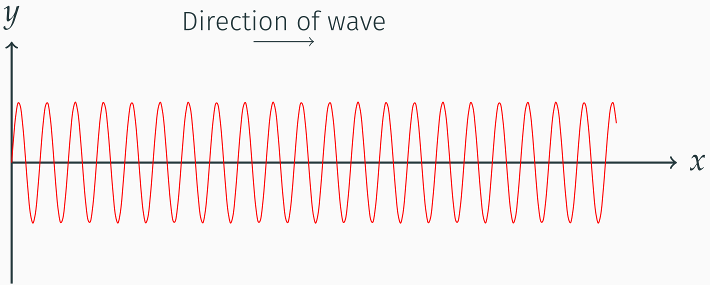

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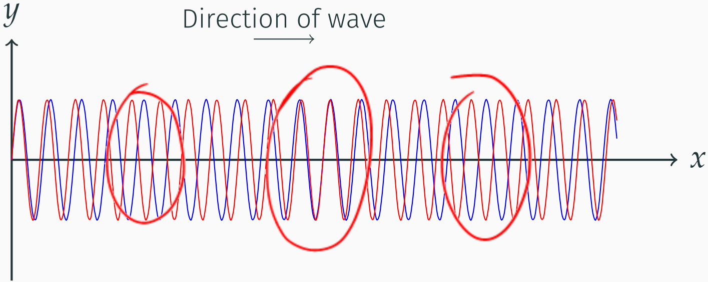

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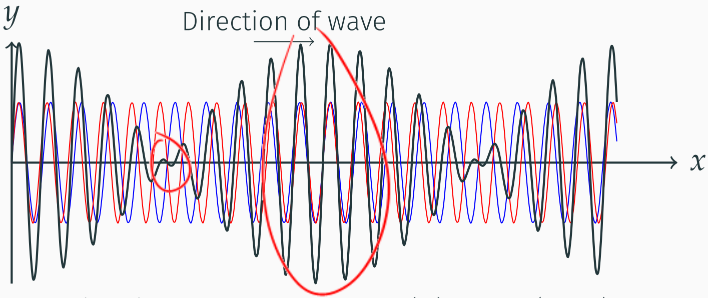


- The thick black line is the sum: $y = \sin(x) + \sin(1.1x)$

Beats

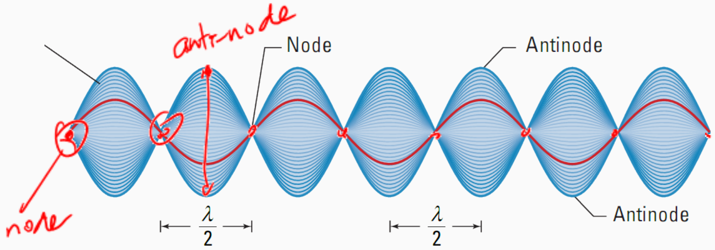
The **beat frequency** is the absolute difference of the frequencies of the two component waves:

$$\boxed{f_{\text{beat}} = |f_1 - f_2|}$$

Quantity	Symbol	SI Unit
Beat frequency	f_{beat}	Hz
Frequency of 1st component wave	f_1	Hz
Frequency of 2nd component wave	f_2	Hz

For sound waves, they sound like a pulsating “whoomf”. Musicians often use the beat frequencies to determine whether someone is play in tune or out of tune.

Standing Waves

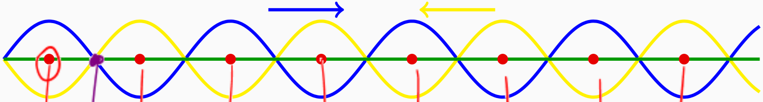


If two waves of the same frequency meet up under the right conditions, they may appear to be “standing still”. This is called a standing wave

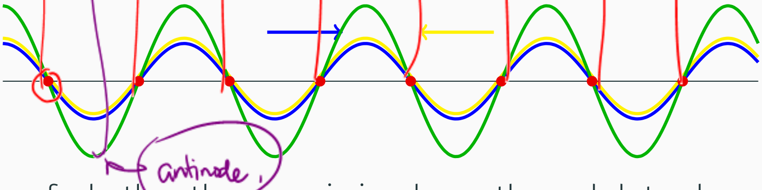
- Node: A point that never moves
- Anti-node: A point which moves/vibrates maximally

Standing Waves

Two identical waves (blue & yellow) move in opposite directions in the same medium. At some moment in time, they are out of phase, resulting in destructive interference (green):



A quarter of a wavelength later, the 2 waves are in phase (constructive interference):



But regardless of whether the wave is in phase, the red dots always have zero displacement. They are called **nodes**.

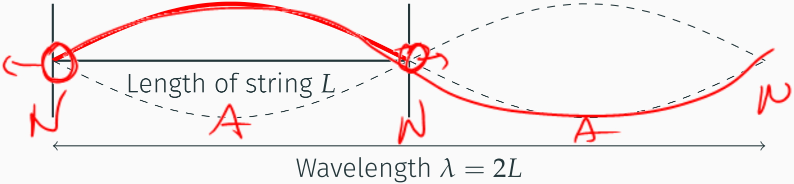
Standing Waves

Standing Waves On a String

- A “vibrating” string is actually a standing wave on a string
- Both ends of the string are nodes
- As the string vibrates, the air around it vibrates at the same frequency
- The vibration travels as a sound wave toward your ears
- Examples:
    - Plucking a guitar or violin string
    - Hitting a key on a piano/harpsichord

Standing Waves On a String of Length L

Resonance frequencies are frequencies where a standing wave can be created. The first resonance (fundamental) frequency at occurs when $\lambda = 2L$:



The fundamental frequency is based on the speed of the traveling wave along the string v_{str}:

$$\boxed{f_{r,1} = \frac{v_{\text{str}}}{\lambda} = \frac{v_{\text{str}}}{2L}}$$

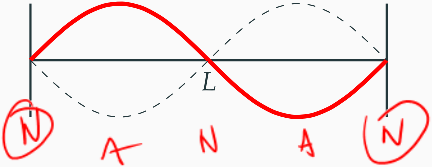
$v = \sqrt{\frac{F_T}{\mu}}$

wavelength = twice the length of the string.

$v = f\lambda \rightarrow f = \frac{v}{\lambda}$

Standing Waves On a String of Length L

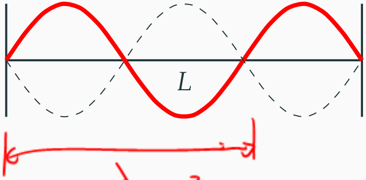
A second resonance frequency occurs when $L = \lambda$:



$v = f\lambda$

$$f_{r,2} = \frac{v_{\text{str}}}{\lambda} = \frac{v_{\text{str}}}{L} = 2f_{r,1}$$

And a third resonance frequency occurs at $L = \dfrac{3}{2}\lambda$:



$\lambda = \frac{2}{3}L$

$$f_{r,3} = \frac{3v_{\text{str}}}{2L} = 3f_{r,1}$$

$f_{r,3}$

Standing Waves On a String of Length L

The n-th resonance frequency of a wave on string is given by:

$$\boxed{f_n = n f_1} \quad \text{where} \quad \boxed{f_1 = \frac{v_{\mathrm{str}}}{2L}}$$

- n is a whole-number multiple
- This equation is *identical* to the equation for harmonic frequencies, meaning that on a string, every harmonic is a resonance frequency
- A vibrating string is said to have a "full set of harmonics"