Topic 23: Special Relativity

Advanced Placement Physics

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Olympiads School

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Introduction

These slides are an expanded version of the slides used for Grade 12 Physics (with some additional calculus). Two versions of the slides are downloadable from the school website:

- The long version
  - More background information (more than needed for the AP 2 Exam) and derivations and integrations
  - May answer some of your questions about the specifics of the theory
  - `23a-relativity_long.pdf`
- The short version
  - More "to the point"
  - The version that is used during class
  - `23a-relativity_short.pdf`

There is also a handout on how to solve and interpret the time dilation problem

Frame of Reference

A Quick Review

A **frame of reference** (or “reference frame”, or just “frame”) is a hypothetical mobile “laboratory” an observer uses to make measurements (e.g. mass, lengths, time). At a minimum, it includes:

- A ruler to measure lengths
- A clock to measure the passage of time
- A scale to compare forces
- A balance to measure masses

High-school textbooks often refer to the frame of reference as a “coordinate system”. While it certainly includes that, this definition often makes it difficult to understand special relativity.

Frame of Reference

A Quick Review

- We assume that the hypothetical laboratory is *perfect*—the hypothetical "instruments" have zero errors
- What matters is the *motion* (at rest, uniform motion, acceleration etc) of your laboratory, and how it affects the measurement that you make
- "From the point of view of. . . "

Newtonian (Classical) Relativity

In Newtonian physics, space and time are *absolute*:

- $1\,\mathrm{m}$ is $1\,\mathrm{m}$ no matter where you are in the universe
- $1\,\mathrm{s}$ is $1\,\mathrm{s}$ no matter where you are in the universe
- Measurements of space and time do not depend on motion

If space and time are absolute, then *all* velocities are relative

- Measured velocities depend on the motion of the observer

An **inertial frame of reference** is moving in uniform motion (constant velocity, zero acceleration)

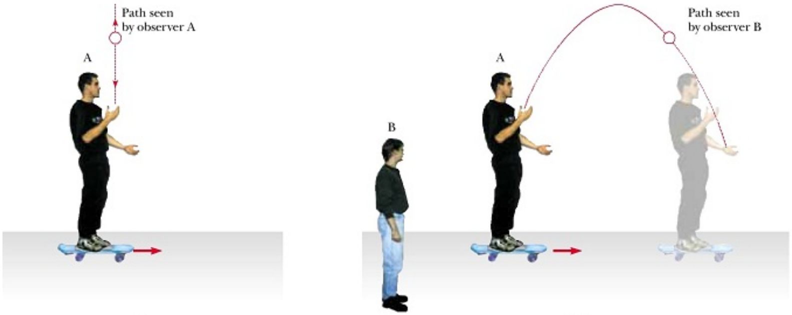
The Principle of Relativity

All laws of physics must apply equally in all inertial frames of reference.

Inertial Frame of Reference

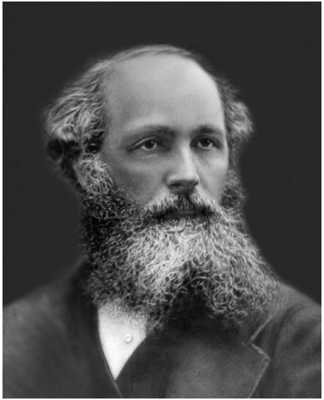
A Quick Review

- Observer A is moving with the skateboard at a constant velocity
- Observer B is standing on the side of the road
- Both observers see different motion, but they agree on the equations that govern the motion



(a) (b)

New Physics: Maxwell's Equations



James Clerk Maxwell

- Classical laws of electrodynamics
- Published in 1861 and 1862
- Explains the relationship between
  - Electricity
  - Electric Circuits
  - Magnetism
  - Optics
- Previously these disciplines are thought to be separate and not related

Maxwell's Equations in a Vacuum

Everything Comes Back to This

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Disturbances in $\mathbf{E}$ and $\mathbf{B}$ travel as an "electromagnetic wave", with a speed:

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 299\,792\,458\,\mathrm{m/s}$$

Peculiar features of Maxwell's equation

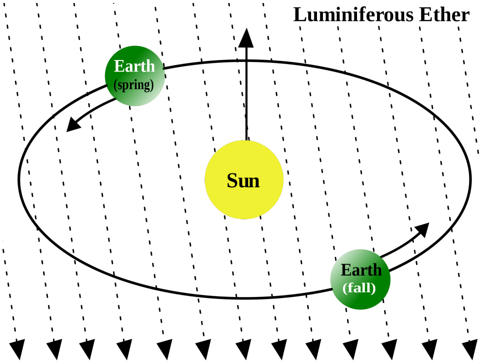
- Does not mention the *medium* in which EM waves travels
- When applying *Galilean transformation* (classical equation for adding velocities) to Maxwell's equations, asymmetry is introduced
- Gauss's law for magnetism break down: magnetic field lines appear to have beginnings/ends
- In *some* inertial frames of reference, Maxwell's equations are simple and elegant, but in another inertial frame of reference, they are ugly and complex
- Physicists at the time theorized that—perhaps—there is/are actually *preferred* inertial frame(s) of references
- This violate the *principle of relativity*

The Illusive Aether

- Maxwell's hypothesis: the speed of light c_0 is relative to a hypothetical "luminiferous aether"
- In order for this "aether" (or "ether") to exist, it must have some fantastic (as in, a fantasy, too good to be true!) properties:
  - *All* space is filled with aether
  - Massless
  - Zero viscosity
  - Non-dispersive
  - Incompressible
  - Continuous at a very small (sub-atomic) scale

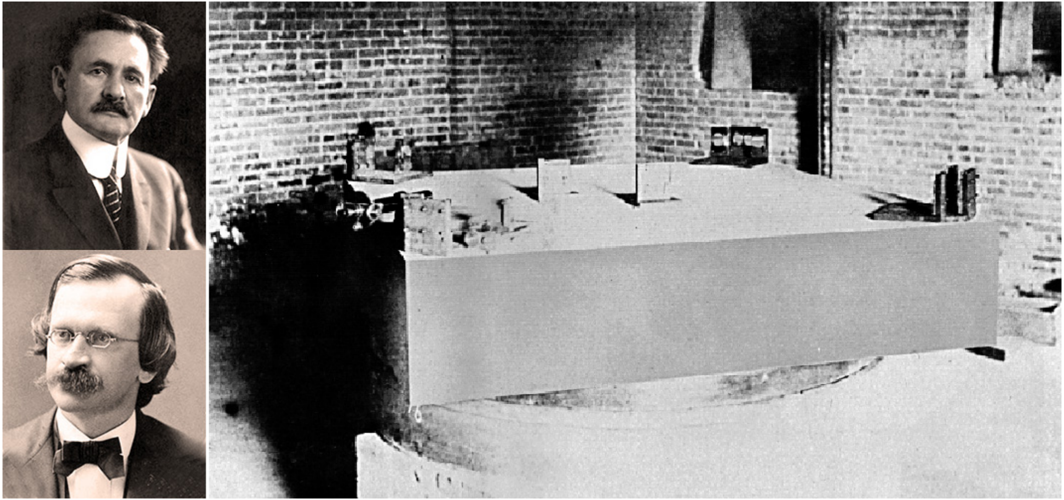
The Michelson-Morley Experiment

If ether exists, then at different times of the year, the Earth will have a different relative velocity with respect to it:



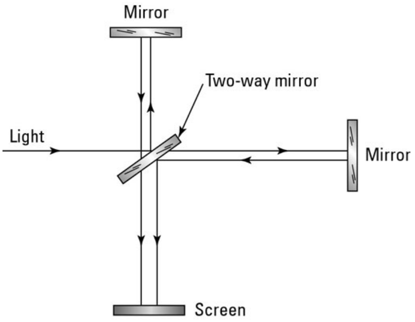
And it will cause light to either speed up, or slow down.

The Michelson Interferometer



The experiment is ingenious but very difficult. . .

The Michelson Interferometer



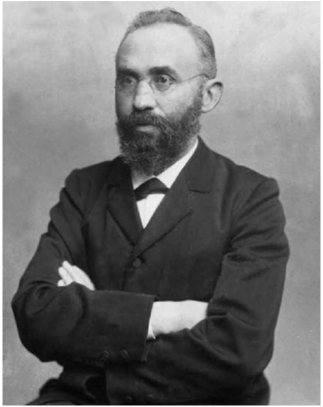
- A beam of light is split into two using a two-way (half-silvered) mirror
- The two beams are reflected off mirrors and finally arriving at the screen where interference patterns are observed
- The two paths are the same length, so if the *speed* of the light changes, we should see an interference pattern
- **Except none were ever found!**

What To Do with “Null Result”

The Michelson-Morley experiment failed to detect the illusive ether, even after many refinements. What does this mean?

- Majority view
  - **The experiment was flawed!**
  - Keep improving the experiment (or design a better experiment) and the ether will eventually be found
- Minority view:
  - **The hypothesis is wrong!**
  - The experiment showed it for what it is: ether cannot be found
- A few physicists: The must be **another explanation** that saves both experiment and theory

Hendrik Lorentz



Hendrik Antoon Lorentz

- Considered the Michelson-Morley experiment to be significant
- Objects travelling in the direction of ether contracts in length, nullifying the experimental results
- Lorentz Factor:

$$\boxed{\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}$$

- *No known physical phenomenon* can cause anything to contract
- Lorentz was on to something, but his thinking was wrong

Making Maxwell’s Equations Work

Albert Einstein in 1905, Age 26

Albert Einstein

- Einstein was 26 years old working as a patent clerk in Switzerland
  - believed in the principle of relativity, and therefore
  - rejected the concept of a preferred frame of reference
- The failure of the Michelson-Morley experiment to find the flow of ether proves that it does not exist
- In order to make Maxwell’s equations to work again, Einstein revisited two most fundamental concepts in physics: *space* and *time*

Special Relativity

Published in the journal *Annalen der Physik* on September 26, 1905 in the article *On the Electrodynamics of Moving Bodies*

- Submitted on June 30, 1905 and passed for publication
- Einstein's third paper that year
- Mentions the names of only five other scientists: Newton, Maxwell, Hertz, Doppler and Lorentz
- Does not have any references to any publications
- Ignored by most physicists at first, until Max Planck took interests
- Called "Special relativity" because it describes a "special case" without effects of gravity

Postulates of Special Relativity

The Principle of Relativity

All laws of physics must apply equally in all inertial frames of reference.

- Reaffirms the principle in which all physics is based on
- Extend the principle to include electrodynamics

The Principle of Invariant Light Speed

As measured in any inertial frame of reference, light is always propagated in empty space with a definite velocity c that is independent of the state of motion of the emitting body.

- Reaffirms the results from Michelson-Morley experiment
- Discounts the existances of the hypothetical ether

Postulates of Special Relativity

The two postulates are unremarkable by themselves, but Einstein is able to show that when combined, the consequences are profound

What's so Special About Special Relativity?

Classical (Newtonian) relativity:

- Space and time are absolute (invariant), therefore
- The speed of light must be relative to the observer

Einstein's special relativity:

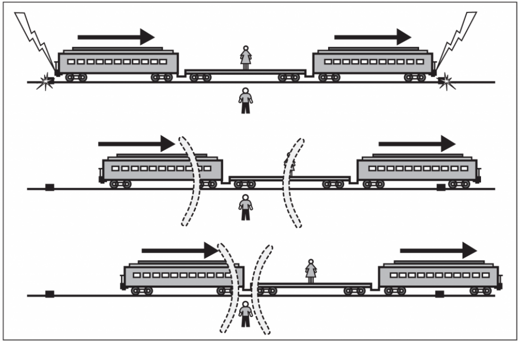
- The speed of light is absolute (invariant), therefore
- Space and time must be relative to the observer

We must modify our traditional concepts:

- Measurement of space (our ruler in the x-, y- and z-directions)
- Measurement of time (our clock)
- Concept of simultaneity (whether two events happens at the same time)

Simultaneity: A Thought Experiment

Lightning bolt strikes the ends of a moving train



- The man on the ground sees the lightning bolt striking at the same time
- The woman on the moving train sees the lightning bolt on the right first

Simultaneity: A Thought Experiment

From the man's perspective:

- He is stationary, but the train is moving
- When the lightnings strike, he is at an equal distance from the front and the back of the train
- The flash from the two lightning bolts arrive at his eyes at the same time

Therefore, his conclusions are:

- The two lightnings must have happened at the same time
- The woman in the train made the wrong observation: she only thinks that the lightning struck the front first because she's moving towards the light from the front

Simultaneity: A Thought Experiment

Now, from the woman's perspective:

- *She* and the train are stationary, but the man and the rest of the world are moving
- When the lightnings strike, she is at an equal distance from the two ends of the train
- The flash from the front arrive first, then the back

Therefore, her conclusions are:

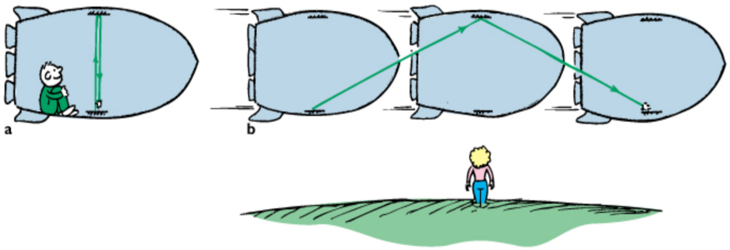
- Lightnings must have struck the front first
- The man on the road made the wrong observation: he only thinks that the lightning struck at the same time because he's moving towards the light from the back

Simultaneity: A Thought Experiment

- The two observers disagree on the result, but
  - Neither person is wrong
  - Neither person is misinformed
- Both observers are valid *inertial* frames of reference
- This means that simultaneity depends on your motion

Events that are simultaneous in one inertial frame of reference are not simultaneous in another.

Time Dilation: A Thought Experiment

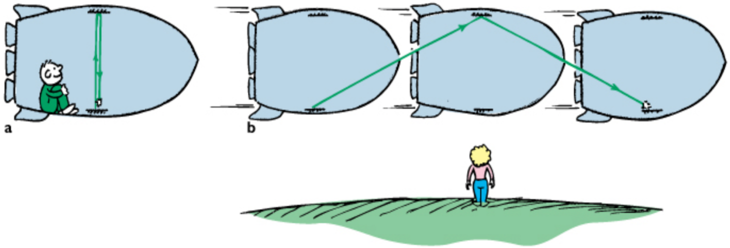


I'm on a spaceship traveling in deep space, and I shine a light from A to B. The distance between A and B is really just:

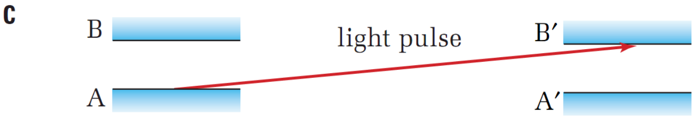
$$|AB| = ct$$

I know the speed of light c, and I know how long it took for the light pulse to reach B.

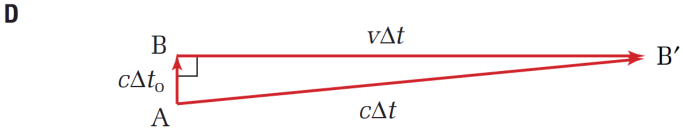
Time Dilation: A Thought Experiment



You are in space station watching my spaceship go past you at speed v. You would see that same beam of light travel from A to B' instead.



Time Dilation: A Thought Experiment



$$c^2 t'^2 = v^2 t'^2 + c^2 t^2$$

$$\left(c^2 - v^2\right) t'^2 = c^2 t^2$$

$$\left(1 - \frac{v^2}{c^2}\right) t'^2 = t^2$$

$$t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

Time Dilation: A Thought Experiment

$$\boxed{t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}$$

- t is called the **proper time**. It is the time measured by a person at rest relative to the object or event.
- t' is called the expanded time or **dilated time**. It is the time measured by a moving observer in another inertial frame of reference. t' is always greater than t.

Example Problem

Example 1a: A rocket speeds past an asteroid at $0.600c$. If an observer in the rocket sees $10.0\,\mathrm{s}$ pass on her watch, how long would that time interval be as seen by an observer on the asteroid?

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Example 1b: A rocket speeds past an asteroid at $0.600c$. If an observer in the *asteroid* sees $10.0\,\mathrm{s}$ pass on his watch, how long would that time interval be as seen by an observer on the *rocket*?

Example Problem

Example 1a: A rocket speeds past an asteroid at $0.600c$. If an observer in the rocket sees $10.0\,\mathrm{s}$ pass on her watch, how long would that time interval be as seen by an observer on the asteroid?

Example 1b: A rocket speeds past an asteroid at $0.600c$. If an observer in the *asteroid* sees $10.0\,\mathrm{s}$ pass on his watch, how long would that time interval be as seen by an observer on the *rocket*?

How can that be?!

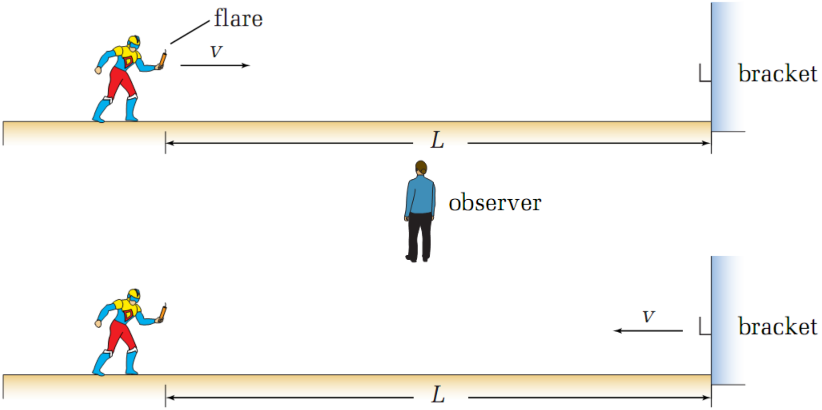
Abandoning Concept of Absolute Space: Length Contraction

Another Example

Captain Quick is a comic book hero who can run at nearly the speed of light. In his hand, he is carrying a flare with a lit fuse set to explode in $1.5\,\mu s$. The flare must be placed into its bracket before this happens. The distance (L) between the flare and the bracket is $402\,m$.

Abandoning Concept of Absolute Space: Length Contraction

Another Example



Abandoning Concept of Absolute Space: Length Contraction

Another Example

If Captain Quick runs at $2.00 \times 10^8\ \mathrm{m/s}$, according to classical mechanics, he will not make it in time:

$$t = \frac{L}{v} = \frac{402\ \mathrm{m}}{2.00 \times 10^8\ \mathrm{m/s}} = 2.01 \times 10^{-6}\ \mathrm{s} = 2.01\ \mu\mathrm{s}$$

But according to relativistic mechanics, he makes it just in time...

Abandoning Concept of Absolute Space: Length Contraction

Another Example

To a stationary observer, the time on the flare is slowed:

$$t' = \frac{t}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1.5 \times 10^{-6}\,\mathrm{s}}{\sqrt{1-\left(\frac{2}{3}\right)^2}} = \frac{1.5 \times 10^{-6}\,\mathrm{s}}{0.7454} = 2.01 \times 10^{-6}\,\mathrm{s}$$

The stationary observer sees a passage of time of $t' = 2.01\,\mu\mathrm{s}$, but Captain Quick, who is in the same reference frame as the flare, experiences a passage of time of $t = 1.50\,\mu\mathrm{s}$, precisely the time for the flare to explode.

Abandoning Concept of Absolute Space: Length Contraction

Another Example

- So, if Captain Quick sees only $t = 1.50\,\mu\text{s}$, then how far did he travel?
- He isn't travelling any faster, so he only other possibility is that **the distance actually got shorter** (in his frame of reference).
- How much did the distance contract?

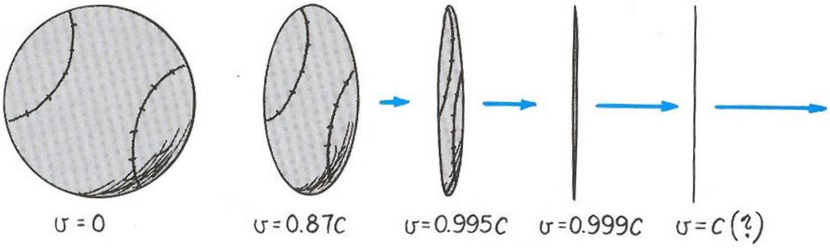
$$\boxed{L' = L\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

For this example:

$$L' = L\sqrt{1 - \left(\frac{v}{c}\right)^2} = (402\,\text{m}) \cdot \sqrt{1 - \left(\frac{2}{3}\right)^2} = 300\,\text{m}$$

Length Contraction

Length contraction only occurs in the direction of motion



Lorentz Factor

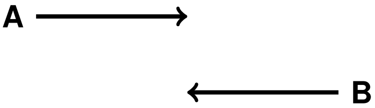
The **Lorentz factor** γ is a short-hand for writing length contraction, time dilation and relativistic mass:

$$\boxed{\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}}$$

Then time dilation and length contraction can be written simply as:

$$\boxed{t' = \gamma t} \quad \boxed{L' = \frac{L}{\gamma}}$$

Let’s Summarize



If Person A and Person B are moving at constant velocity relative to one another (doesn’t matter if they’re moving towards, or away from each other)

- They cannot agree whether any events happens at the same time or not
- Each sees the other’s clock running slow
- Each sees the other “contracted” in length along the direction of motion

Example Problem

Example 2: A spacecraft passes Earth at a speed of $2.00 \times 10^8\ \mathrm{m/s}$. If observers on Earth measure the length of the spacecraft to be $554\ \mathrm{m}$, how long would it be according to its passengers?

Lorentz Transformation

The equations for time dilation and length contraction only tell part of the story. In order to account for the loss of simultaneity from one frame to another, we need to use the **Lorentz transformation:**

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

The Lorentz transformation "solves" many paradoxes (e.g. the twin paradox) from the time-dilation and length-contraction equations, but aren't really there.

Lorentz Transformation

For slow speeds $v \ll c$, Lorentz transformation reduces to the Galilean transformation from classical mechanics.

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t'$$

Relative Velocity

- Unlike in classical mechanics, velocities (speeds) do not simply add
- If the observer in frame S measures an object moving along the x-axis at velocity u, then the observer in the referenfe frame S that is moving at velocity v in the x-direction with respect to S, will measure the object moving with velocity u:

$$u' = \frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - vdx/c^2)} = \frac{(dx/dt) - v}{1 - (v/c^2)(dx/dt)} = 1$$

- The other frame S will measure:

$$u = \frac{dx}{dt} = \frac{\gamma(dx' + vdt')}{\gamma(dt' + vdx'/c^2)} = \frac{(dx'/dt') + v}{1 + (v/c^2)(dx'/dt')} =$$

Relativistic Momentum

In Physics 12, you were taught that momentum is mass times velocity. And back in Physics 11, you were taught that velocity is displacement over time. **These definitions have not changed.**

$$\mathbf{p} = m\frac{d\mathbf{x}}{dt}$$

But now that you know $d\mathbf{x}$ and dt are relativistic quantities that depend on motion, we can find a new expression for "relativistic momentum":

$$\mathbf{p} = m\frac{d\mathbf{x}}{dt} = \frac{m d\mathbf{x}}{\sqrt{1-\left(\frac{v}{c}\right)^2}\, dt} = \frac{m\mathbf{v}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

Relativistic Mass

From the relativistic momentum expression, we see the relativistic aspect to mass as well. The apparent mass m' as measured by a moving observer is related to its rest mass (intrinsic mass, invariant mass) m by the Lorentz factor:

$$\boxed{m' = \frac{m}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma m}$$

The intrinsic mass has not increased, but a moving observer will note that the object behaves as if it is more massive. As $v \to c$, $m \to \infty$.

Work and Energy

Einstein published a fourth paper in *Annalen der Physik* on November 21, 1905 (received Sept. 27) titled “Does the Inertia of a Body Depend Upon Its Energy Content?” (In German: Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?)

- Einstein deduced the most famous of equations: $E = mc^2$

Work and Energy

In Physics 12, you were taught that force is the rate of change of momentum with respect to time. **This definition has not changed.**

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

and that work is the integral of the dot product between force and displacement vectors. **This definition has not changed either.**

$$W = \int \mathbf{F} \cdot d\mathbf{x} = \int \frac{d\mathbf{p}}{dt} \cdot \mathbf{dx}$$

Since we now have a relativistic expression for momentum, we substitute that new expression into the expression for force, and then integrate.

Work and Energy

For 1D motion, we can rearrange the terms in the integral:

$$W = \int F dx = \int \frac{dp}{dt} dx = \int v dp$$

We assume that both velocity and momentum are continuous in time. Since momentum is a function of both γ and v, we apply the chain rule to find the infinitesimal change in momentum (dp) with respect to $d\gamma$ and dv:

$$p = \gamma m v \quad \rightarrow \quad dp = \gamma dv + v d\gamma$$

Substituting that into the integral, we have:

$$W = \int v dp = \int m v(\gamma dv + v d\gamma) = \int m \left(\gamma v dv + v^2 d\gamma\right)$$

Work and Energy

We want to integrate with respect to γ, so we need to express v and dv in terms of γ using its definition:

$$v^2 = c^2 \left[1 - \left(\frac{1}{\gamma}\right)^2\right] \qquad dv = \frac{c^2}{\gamma^3 v} d\gamma$$

Work and Energy

Putting everything together, we have

$$W = \int m(\gamma v dv + v^2 d\gamma) = \int m \left[\frac{c^2}{\gamma^2} + c^2 \left(1 - \frac{1}{\gamma^2} \right) \right] d\gamma$$

We end up with a surprisingly simple integral:

$$W = \int_1^\gamma mc^2 d\gamma$$

The limit of the integral is from 1 because at $v = 0$, $\gamma = 1$

Work and Kinetic Energy

The integral gives us this expression:

$$W = \gamma mc^2 - mc^2 = K$$

We know from the work-kinetic energy theorem that the work W done is equal to the change in kinetic energy K, therefore

$$\boxed{K = m'c^2 - mc^2}$$

Variable	Symbol	SI Unit
Kinetic energy of an object	K	J
Relativistic mass (measured in moving frame)	m'	kg
Rest mass (measured in stationary frame)	m	kg
Speed of light	c_0	m/s

Relativistic Energy

What This All Means

$$\boxed{K = m'c^2 - mc^2}$$

The minimal energy that any object has, regardless of it's motion (or lack of) is its **rest energy**:

$$E_0 = mc^2$$

The **total energy** of an object has is

$$E_T = m'c^2 = \gamma mc^2$$

The difference between total energy and rest energy is the kinetic energy:

$$K = E_T - E_0$$

Relativistic Energy

What This All Means

$$\boxed{E = mc^2}$$

Mass-energy equivalence:

- Whenever there is a change of energy, there is also a change of mass
- “Conservation of mass” and “conservation of energy” must be combined into a single concept of **conservation of mass-energy**
- Mass-energy equivalence doesn’t merely mean that mass can be converted into energy, and vice versa (although this is true), but rather, one can be converted into the other **because they are fundamentally the same thing**

Example Problem

Example 3: An electron has a rest mass of 9.11×10^{-31} kg. In a detector, it behaves as if it has a mass of 12.55×10^{-31} kg. How fast is that electron moving relative to the detector?

Energy-Momentum Relation

The **energy-momentum relation** relates an object's rest (intrinsic) mass m, total energy E, and momentum p:

$$\boxed{E^2 = p^2c^2 + m^2c^4}$$

Quantity	Symbol	SI Unit
Total energy	E	J (joules)
Momentum	p	kgm/s (kilogram meters per second)
Rest mass	m	kg (kilogram)
Speed of light	c	m/s (meter per second)

Energy-Momentum Relation

This equation is derived using the expression for relativistic momentum:

$$p = \gamma m v = \frac{mv}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

If we square both sides of the equation, we get:

$$p^2 = \gamma^2 m^2 v^2 = \frac{m^2 v^2}{1 - \left(\frac{v}{c}\right)^2}$$

Energy-Momentum Relation

Solving for v^2 and substituting it back into the Lorentz factor, we obtain its alternative form in terms of momentum and mass:

$$\gamma = \sqrt{1 + \left(\frac{p}{mc}\right)^2}$$

Inserting this form of the Lorentz factor into the energy equation, we have

$$E = mc^2\sqrt{1 + \left(\frac{p}{mc}\right)^2}$$

Which is the same equation as in the last slide.

Energy-Momentum Relation

In the **stationary frame of reference**, (rest frame, center-of-momentum frame) the momentum is zero, so the equation simplifies to

$$E = mc^2$$

where m is the rest mass of the object.

If the object is **massless**, as is the case for a **photon**, then the equation reduces to

$$E = pc$$

Kinetic Energy–Classical vs. Relativistic

Relativistic:

$$K = \frac{mc^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} - mc^2$$

Newtonian:

$$K = \frac{1}{2}mv^2$$

But are they really that different?

- If space and time are indeed relative quantities, then the relativistic equation for K must apply to all velocities
- But we know that when $v \ll c$, the Newtonian expression works perfectly
- i.e. The Newtonian expression for K must be a very good approximation for the relativistic expression for K for $v \ll c$

Binomial Series Expansion

The **binomial series** is the Maclaurin series for the function $f(x) = (1+x)^\alpha$, given by:

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots$$

In the case of relativistic kinetic energy, we use:

$$x = -\left(\frac{v}{c}\right)^2 \quad \text{and} \quad \alpha = -\frac{1}{2}$$

Binomial Series Expansion

Substituting these terms into the equation:

$$K = mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right) - mc^2$$
$$\approx \frac{1}{2}mv^2 + \frac{3}{4}m\frac{v^4}{c^2} + \cdots$$

For $v \ll c$, we can ignore the high-order terms. The leading term reduces to the Newtonian expression

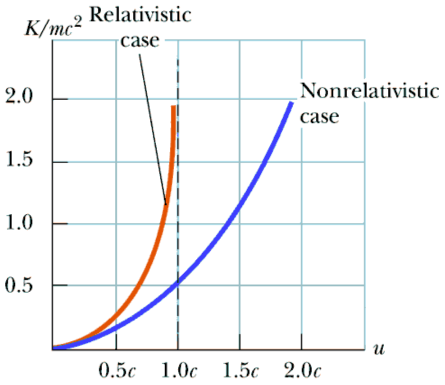
Comparing Classical and Relativistic Energy

In classical mechanics:

$$K = \frac{1}{2}mv^2$$

In relativistic mechanics:

$$K = \gamma mc^2 - mc^2$$



The classical expression is accurate for speeds up to $v \approx 0.3c$.

Example Problem

Example 4: A rocket car with a mass of 2.00×10^3 kg is accelerated from rest to 1.00×10^8 m/s. Calculate its kinetic energy:

1. Using the classical equation
2. Using the relativistic equation