Topic 8: Universal Gravitation

Advanced Placement Physics

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Olympiads School

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Files to Download

If you have not done so already, please download the following files.

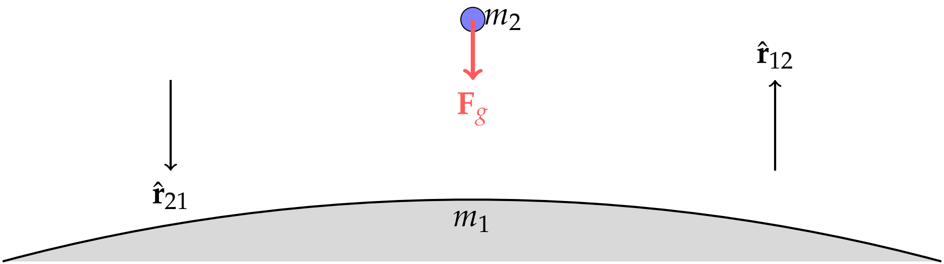
- `Phys08-Gravity.pdf`—The class slides for this topic.

Universal Gravitation

This topics we will discuss in this class are covered in AP Physics 1 and C exams.

- Gravitational force ($\mathbf{F}_g$)
- Gravitational field ($\mathbf{g}$)
- Gravitational potential energy (U_g)
- Kepler's laws of planetary motion

Newton's Law of Universal Gravitation



$$\boxed{\mathbf{F}_g = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}_{12} = +\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}_{21}}$$

where $\boxed{G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}}$ is the universal gravitation constant, and $\hat{\mathbf{r}}_{12}$ and $\hat{\mathbf{r}}_{21}$ are *unit vectors* (length=1)

Universal Gravitation

- If m_1 exerts a gravitational force $\mathbf{F}_g$ on m_2, then m_2 also exerts $-\mathbf{F}_g$ on m_1.
- The two forces are equal in magnitude and opposite in direction (Newton's 3rd law)
- Assumption: m_1 and m_2 are *point masses* that do not occupy any space.
- Recall: forces acting on an ensemble of masses is the same as acting at its center of mass
- Therefore, for the universal gravitational equation to work:

$$r > (r_1 + r_2)$$

That is, the two objects hasn't collided into one another

Think Gravitational Field: What is g?

We generally describe the force of gravity as

$$\mathbf{F}_g = m\mathbf{g}$$

To find the magnitude of g, we group the variables in Newton's universal gravitation equation:

$$F_g = \underbrace{\left[\frac{Gm_1}{r^2}\right]}_{=g} m_2 = m_2 g$$

On the surface of Earth, we use use $m_1 = m_{\text{Earth}}$ and $r = r_{\text{Earth}}$ to compute $g \approx 10\,\text{m/s}^2$, or $g \approx 10\,\text{N/kg}$ (both units are equivalent)

Gravitational Field

The **gravitational field** $\mathbf{g}$ generated by a source mass m_s shows how it influences the gravitational forces on other masses. It has a magnitude of:

$$\boxed{g(m_s, r) = \frac{Gm_s}{r^2}}$$

Quantity	**Symbol**	**SI Unit**
Gravitational field intensity	g	N/kg
Universal gravitational constant	G	$\mathrm{Nm}^2/\mathrm{kg}^2$
Mass of source (a point mass)	m_s	kg
Distance from source mass	r	m

The *direction* of the gravitational field is toward m_s.

Relating Gravitational Field & Gravitational Force

$\mathbf{g}$ itself doesn't *do* anything until there is another mass m. At which point, m experiences a gravitational force due to m_s, and it is related to $\mathbf{g}$ by:

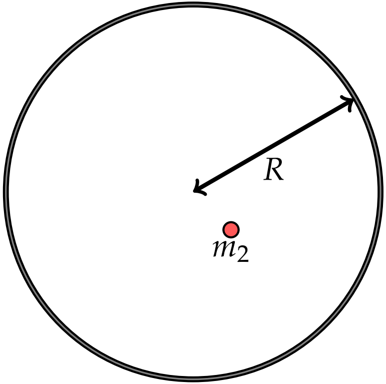
$$\boxed{\mathbf{F} = m\mathbf{g}}$$

$\mathbf{F}_g$ and $\mathbf{g}$ are vectors in the same direction: toward the center of the source mass.

Quantity	Symbol	SI Unit
Gravitational field	$\mathbf{g}$	N/kg
Gravitational force on a mass	$\mathbf{F}_g$	N
Mass inside the gravitational field	m	kg

What If You Are Inside Another Mass?

Case 1: A Spherical Shell of Radius R



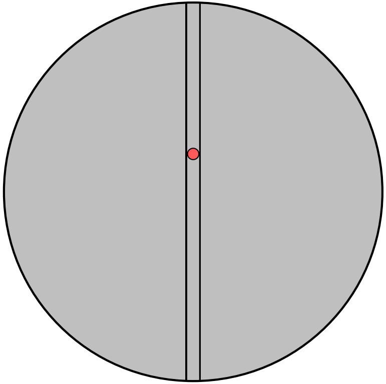
- If a mass m_2 is *inside* a spherical shell of mass m_1, the force of gravity it experiences is **zero**!

$$\boxed{\mathbf{F}_g = \begin{cases} \mathbf{0} & \text{if } r < R \\ Gm_1m_2/r^2\hat{\mathbf{r}} & \text{otherwise} \end{cases}}$$

- It also means that gravitational field is also **zero**
- This is very similar to a charged conducting sphere, where the electric field inside is zero.

What If You Are Inside Another Mass?

Case 2: Uniform Mass of Radius R



Suppose you could drill a hole through the Earth and then drop into it. How long would it take you to pop up on the other side of the Earth?

Falling To the Center of the Earth

Initial gravitational force on the surface is:

$$F_g = mg_{\text{surf}} \qquad g_{\text{surf}} \approx 10\,\text{N}/\text{kg}$$

F_g becomes smaller as you approached the center, and when you reach the center, $F_g = 0$.

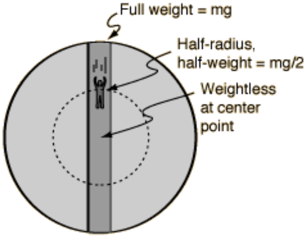
Assume that the density of Earth is uniform, and neglect air resistance and other factors. The value of g as the person falls through Earth ($r < R$) is given by:

$$g(r) = \frac{GM(r)}{r^2} \quad M(r) = \frac{4}{3}\rho\pi r^3 \quad \rho = \frac{3M_{\text{Earth}}}{4\pi R^3}$$

Falling To the Center of the Earth

Now we have an expression for the gravitational field strength inside "Earth":

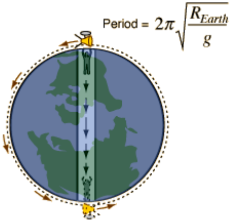
$$g(r) = \frac{GM_{\text{Earth}}r}{R^3} = g_{\text{surface}}\frac{r}{R}$$



Gravitational field strength decreases linearly with r. At the center ($r = 0$), $g = 0$. Here's the interesting part, the gravitational force inside Earth is

$$F_g = -mg(r) = -\underbrace{\left[\frac{mg_{\text{surface}}}{R}\right]}_{\text{constant}} r = -kr$$

Falling To the Center of the Earth



A satellite at the Earth's radius would have the same period as one falling through the Earth.

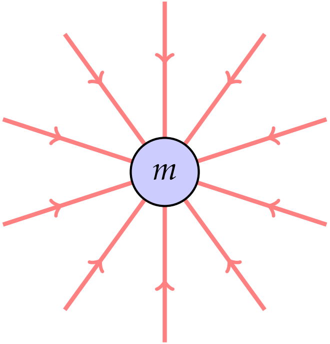
So this poor guy is going to oscillate through Earth with a harmonic motion with a period of:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

For Earth, $T = 5068\,\mathrm{s}$. He would pop up on the opposite side after about $42\,\mathrm{min}$.

Suppose a satellite is in a circular orbit just above the surface, and passes overhead just above the traveler as he popped up out of the hole. The period of such an orbit would be the same as oscillating traveler.

Gravitational Field Lines



- The direction of $\mathbf{g}$ is toward the center of the object that created it
- Field lines do not tell the intensity (i.e. magnitude) of $\mathbf{g}$, only the direction

Gravitational Potential Energy

Gravitational potential energy is found by integrating the work equation and using universal gravitation:

$$W = \int \mathbf{F}_g \cdot d\mathbf{r} = -\int_{r_1}^{r_2} \frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} \cdot d\mathbf{r}$$

$$= -\int_{r_1}^{r_2} \frac{Gm_1m_2}{r^2} dr = \left.\frac{Gm_1m_2}{r}\right|_{r_1}^{r_2} = -\Delta U_g$$

where

$$\boxed{U_g = -\frac{Gm_1m_2}{r}}$$

- U_g is the work required to move two objects from r to ∞
- $U_g = 0$ at $r = \infty$ and *decrease* as r decreases

Relating Gravitational Potential Energy to Force

Vector calculus (actually fundamental theorem of calculus) shows that gravitational force ($\mathbf{F}_g$) is the negative gradient of the gravitational potential energy (U_g):

$$\mathbf{F}_g = -\nabla U_g = -\frac{\partial U_g}{\partial r}\hat{\mathbf{r}}$$

The direction of $\mathbf{F}_g$ is always from high to low potential energy

- A falling object is always decreasing in U_g
- "Steepest descent": the direction of $\mathbf{F}$ is the shortest path to decrease U_g
- Objects traveling perpendicular to $\mathbf{F}_g$ has constant U_g

Relating U_g, $\mathbf{F}_g$ and $\mathbf{g}$

Knowing that $\mathbf{F}_g$ and $\mathbf{g}$ only differ by a constant, we can also relate gravitational field to U_g by the gradient operator:

$$\mathbf{g} = \frac{\mathbf{F}_g}{m} = -\nabla\left(\frac{U_g}{m}\right) = -\frac{\partial}{\partial r}\left(\frac{U_g}{m}\right)\hat{\mathbf{r}}$$

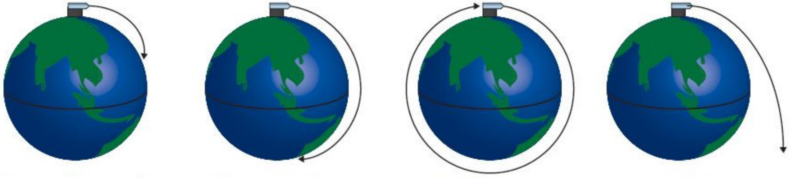
We already know that the direction of $\mathbf{g}$ is the same as $\mathbf{F}_g$, i.e.

- The direction of $\mathbf{g}$ is the shortest path to decrease U_g
- Objects traveling perpendicular to $\mathbf{g}$ has constant U_g

Orbital Speed

Newton's Thought Experiment

Newton theorized that if the initial velocity of the cannonball is fast enough, it will never fall down. So how fast is fast enough?



And how fast is too fast that it never comes back?

Orbital Speed

Relating Gravitational and Centripetal Force

Assuming a small mass m in circular orbit around a much larger mass M. The required centripetal force is provided by the gravitational force:

$$F_g = F_c \quad \longrightarrow \quad \frac{GMm}{r^2} = \frac{mv^2}{r}$$

Solving for v, we get the **orbital speed** v_{orbit} (or **orbital velocity**):

$$\boxed{v_{\text{orbit}} = \sqrt{\frac{GM}{r}}}$$

v_{orbit} does not depend on the mass of the smaller object in orbit

Escape Speed

An object can leave the surface of Earth at any speed. But when all the kinetic energy of that object is converted to gravitational potential energy, it will return back to the surface of the earth. There is, however, a *minimum* velocity at which the object *would not* fall back to Earth.

Escape Speed

The gravitational potential energy of an object with mass m on the surface of a planet (with mass M and radius r) is

$$U_g = -\frac{GMm}{r}$$

- The most amount of work that you can do is to bring it to the other side of the universe $r = \infty$, where $U_g = 0$.
- The work done by gravity converts kinetic energy into gravitational potential energy.
- If you start with *more* kinetic energy than required to do all the work, then the object has *escaped* the gravitational pull of the planet.

Escape Speed from Circular Orbits

Setting K to equal to $-U_g$:

$$\frac{1}{2}mv^2 = \frac{GMm}{r}$$

We can then solve for **escape speed** $v = v_{\text{esc}}$ (or **escape velocity**):

$$\boxed{v_{\text{esc}} = \sqrt{\frac{2GM}{r}}}$$

There is a simple relationship between orbital speed and escape speed:

$$\boxed{v_{\text{esc}} = \sqrt{2}v_{\text{orbit}}}$$

Example Problem

Example: Determine the escape velocity and energy for a 1.60×10^4 kg rocket leaving the surface of Earth.

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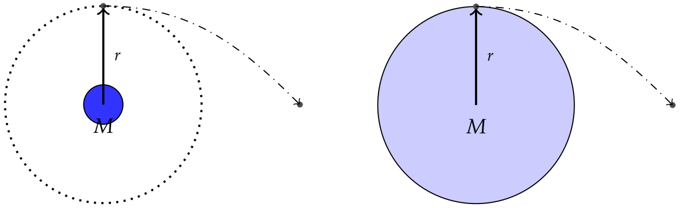
Note: The equation for the escape speed is based on the object have a *constant* mass, which is *not* the case for a rocket going into space.

What if I'm not escaping from the surface?

Both objects have the same escape velocity:



The difference is that the object in orbit (left) already has orbital speed v_{orbit}, so escaping from that orbit requires only an additional speed of

$$\Delta v = v_{\text{esc}} - v_{\text{orbit}} = (\sqrt{2} - 1)v_{\text{orbit}}$$

What If?

What if an object is in an orbit with a speed $v_{\text{orbit}} < v < v_{\text{esc}}$?

What if an object has a speed $v < v_{\text{orbit}}$?

Orbital Energies

We can obtain the **orbital kinetic energy** by using the orbital speed in our expression of kinetic energy:

$$K_{\text{orbit}} = \frac{1}{2}mv_{\text{orbit}}^2 = \frac{1}{2}m\left(\sqrt{\frac{GM}{r}}\right)^2 = \boxed{\frac{GMm}{2r}}$$

We already have an expression for gravitational potential energy:

$$U_g = -\frac{GMm}{r} = -2K_{\text{orbit}}$$

The **total orbital energy** is the sum of K and U_g:

$$E_T = K_{\text{orbit}} + U_g = -\frac{GMm}{2r} = -K_{\text{orbit}}$$

Houston, We Have a Problem!

As always, this understanding isn't completely correct!

- Centripetal motion is based on rotation around a fixed point, but this is *not* the case for orbital mechanics!
- Just as Earth experiences a gravitational force by the Sun, the Sun also experiences a gravitational force from Earth
- The smaller mass m does not actually orbit about the center of the larger mass M, but rather, the center of mass between M and m.
- This problem is especially important when the two objects orbiting each other has similar masses (e.g. a binary star system)

A Central Force

Gravity is called a **central force** in that

- Gravitational force $\mathbf{F}_g$ is always in the $-\hat{\mathbf{r}}$ direction, i.e. $\mathbf{F} \times \mathbf{r} = \mathbf{0}$
- Therefore gravity doesn't generate any torque
- And therefore angular momentum $\mathbf{L}$ is constant
- True regardless of whether the orbit is circular or elliptical

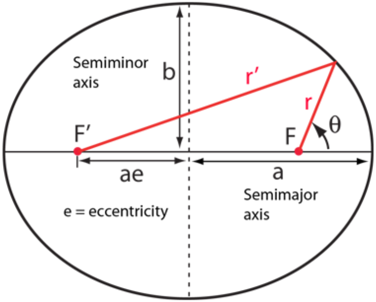
Kepler's Laws of Planetary Motion

1. **Law of ellipses:** The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. **Law of equal areas:** A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time
3. **Law of periods:** The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

Johannes Kepler (1571–1630) formulated these laws between 1609 to 1619 based on interpreting planetary motion data from his teacher, Tycho Brahe. It is an improvement over the heliocentric theory of Nicolaus Copernicus.

Kepler's Law of Planetary Motion

To fully understand Kepler's laws, we have to first understand the ellipse, at least a little bit.



$$r' + r = 2a$$

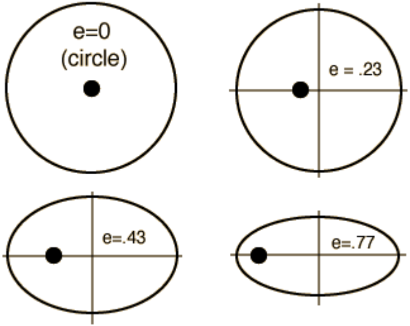
- The area of the ellipse is $A = \pi ab$
- For an ellipse, the relationship between r and θ given by:

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \quad \text{where} \quad 0 \leq e < 1$$

- when $e = 0$ it's a circle: $r = a$
- When $e = 1$ it's no longer an ellipse

Kepler's Law of Planetary Motion

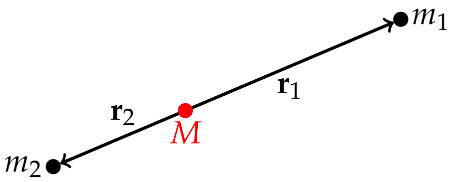
Most of the planets have very small eccentricity, so their orbits are fairly close to being circular, but comets are much more eccentric



Object	e
Mercury	0.206
Venus	0.0068
Earth	0.0167
Mars	0.0934
Jupiter	0.0485
Saturn	0.0556
Uranus	0.0472
Neptune	0.0086
Pluto	0.25
Halley's Comet	0.9671
Comet Hale-Bopp	0.9951
Comet Ikeya-Seki	0.9999

Reduced Mass

When two objects are orbiting each other, the center of mass of the system is along the line between them:



The vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are relative to the center of mass, and the relative position $\mathbf{r}$ velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ between m_1 and m_2 are therefore:

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \quad \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 \quad \mathbf{a} = \mathbf{a}_2 - \mathbf{a}_1$$

Reduced Mass

From Newton's third law, we know that the gravitational force exerted by m_1 on m_2 is opposite the force exerted by m_2 on m_1:

$$m_1\mathbf{a}_1 + m_2\mathbf{a}_2 = 0$$

Substituting the expression for relative acceleration, we have:

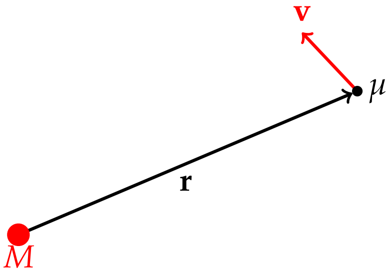
$$\mathbf{a} = \mathbf{F}_g\left[\frac{1}{m_1} + \frac{1}{m_2}\right] = \mathbf{F}_g\left[\frac{m_1 + m_2}{m_1 m_2}\right] = \frac{\mathbf{F}_g}{\mu}$$

We can now define a new concept called **reduced mass**:

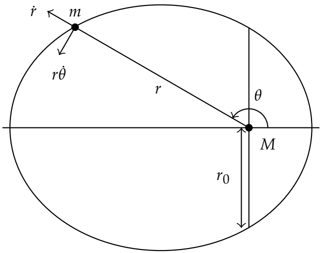
$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

An Equivalent System

The orbit of one of the masses in a binary system is equivalent to the motion of the reduced mass orbiting around a point at relative distance r where the total mass M is placed. The magnitude of r is the same as the relative distance r in the development above.



Kepler's First Law

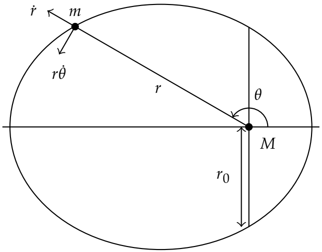


The total energy of the object m (reduced mass) in orbit around M (total mass):

$$E_T = K + U_g = \frac{1}{2}mv^2 - \frac{GMm}{r^2}$$
$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r^2}$$

- $v_r = \dot{r} = dr/dt$ is the radial component of velocity
- $v_\theta = r\omega = r\dot{\theta}$ is the angular component
- The two velocity components are orthogonal $\therefore$ $v^2 = v_r^2 + v_\theta^2$

Kepler's First Law



The angular momentum (a constant!) of m about M is:

$$L = mr^2\dot{\theta} \quad \rightarrow \quad \dot{\theta} = \frac{L}{mr^2}$$

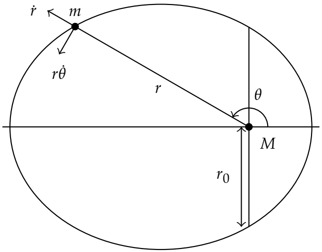
We make a subsitution of $\rho = 1/r$, and then integrating with time:

$$\theta = \int \frac{L}{m}\rho^2 dt = \int \frac{L}{m}\rho^2 \frac{dt}{d\rho} d\rho$$

But since $\dot{r} = \frac{dr}{dt} = \frac{-1}{\rho^2}\frac{d\rho}{dt}$, the integral reduces to:

$$\theta = -\int \frac{L}{m\dot{r}} d\rho$$

Kepler's First Law



Combining the equations for angular momentum and tatal energy, and solving for $\dot{r}$:

$$\dot{r}^2 = \frac{2E}{m} + 2GM\rho - \frac{L^2}{m^2}\rho^2$$

Now the crazy manipulation (totally *not* obvious) by this substitution (both are constants):

$$r_0 = \frac{L^2}{GMm^2}$$
$$e^2 = 1 + \frac{2Er_0}{GMm}$$

This way, when we solve for r, we have an expression that is immediately recognizable as an ellipse.

Kepler's First Law

After our manipulation, we get (after a bit of algebra):

$$\dot{r} = \frac{L}{m}\left[\frac{e^2}{r_0^2} - \left(\rho - \frac{1}{r_0}\right)\right]^{1/2}$$

which we will substitute into the expression for $\dot{\theta}$, and get:

$$\theta = -\int \frac{1}{\sqrt{(e/r_0)^2 - (\rho - 1/r_0)^2}} d\rho$$

$$= \cos^{-1}\left(\frac{\rho - 1/r_0}{e/r_0}\right)$$

Kepler's First Law

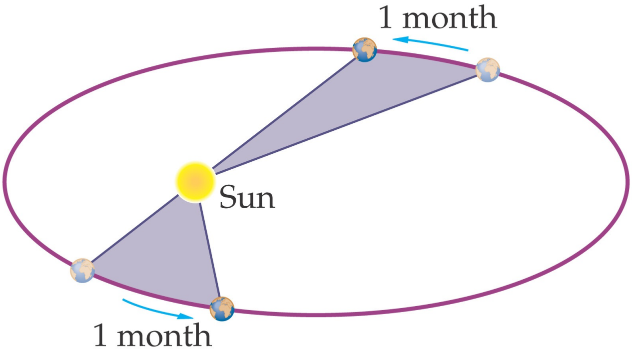
Most importantly, once we untangle the last equation, we have a simple expression of:

$$r = \frac{r_0}{1 + e\cos\theta}$$

which is the equation for an ellipse. We can now say that e is the eccentricity of the ellipse and $r_0 = a(1 - e^2)$ is the semi-latus rectum of the ellipse.

Kepler's 2nd Law: The Law of Equal Areas

Second Law: A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time



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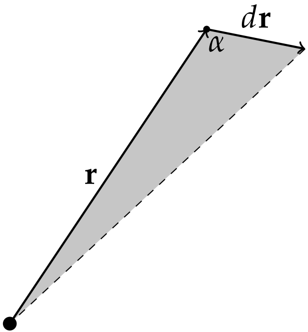


The infinitesimal area swept out by an object in orbit is given by

$$dA = \frac{1}{2} r\,dr \sin\alpha \quad \rightarrow \quad d\mathbf{A} = \frac{1}{2}\mathbf{r} \times d\mathbf{r}$$

The time derivative gives an expression for *areal velocity* (yes, that's a real word):

$$\dot{\mathbf{A}} = \frac{d\mathbf{A}}{dt} = \frac{1}{2}\mathbf{r} \times \frac{d\mathbf{r}}{dt} = \frac{1}{2}\mathbf{r} \times \dot{\mathbf{r}} = \frac{1}{2}(\mathbf{r} \times \mathbf{v})$$

Kepler's Second Law

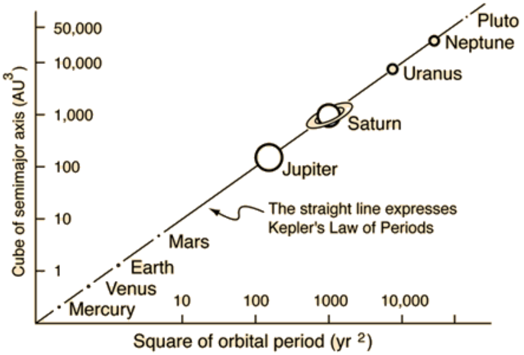
We can express $\mathbf{r} \times \mathbf{v}$ in terms of angular momentum, $\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$. But in motion under any central force (such as gravity), angular momentum is a constant, and therefore:

$$\dot{\mathbf{A}} = \frac{1}{2}(\mathbf{r} \times \mathbf{v}) = \frac{\mathbf{L}}{2m} = \text{constant}$$

As predicted by Kepler's second law.

Kepler's Third Law: The Law of Periods

Law of Periods: The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.



Kepler's Third Law: The Law of Periods

The area swept by the planet through one orbital period is the areal velocity (constant!) integrated by time, from $t = 0$ to $t = T$:

$$A = \int dA = \int_0^T \frac{dA}{dt} dt = \frac{L}{2m} \int_0^T dt = \frac{L}{2m} T$$

From Kepler's first law, this area is an ellipse, given by the equation based on a (the semi-major axis), b (the semi-minor axis):

$$A = \pi ab = \pi a^2 \sqrt{1 - e^2}$$

Combining the two equations above give this expression:

$$T^2 = \frac{m^2}{L^2} 4\pi^2 a^4 (1 - e^2)$$

Kepler's Third Law: The Law of Periods

But we also (from proving the first law) have:

$$r_0 = a(1-e^2) = \frac{L^2}{GMm^2}$$

Substituting this expression into the equation for the period, and after some more algebra, we end up with this expression:

$$\boxed{T^2 = \left[\frac{4\pi^2}{GM}\right] a^3}$$

In cases where $m_2 \ll m_1$ (e.g. Earth orbiting the sun), $M = m_1 + m_2 \approx m_1$.