Class 8: Universal Gravitation

AP Physics (1, 2 and C)

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Fall/Winter 2017

Files to Download

Please download/print the PDF file

If you have not done so already, please download the following files.

- `08-gravitation-print.pdf`–The "print version" of this week's slides. I recommend printing 4 slides per page.
- `08-Homework.pdf` This week's homework.

Today's Plan

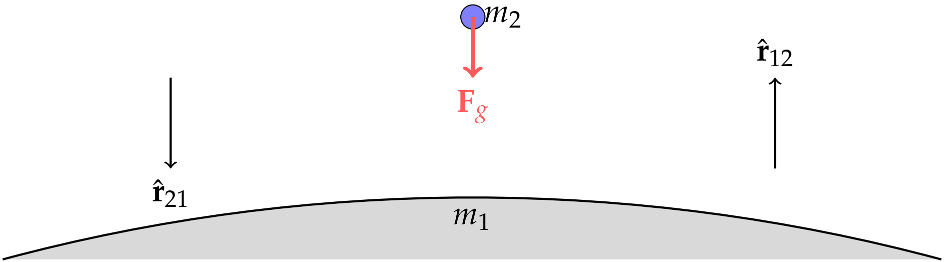
1. Take up (some) questions from Class 7
2. Go over this week's slides (hopefully it won't take too long)

Universal Gravitation

This topics we will discuss in this class are covered in AP Physics 1, 2 and C exams.

- Gravitational force ($\mathbf{F}_g$)
- Gravitational field ($\mathbf{g}$)
- Gravitational potential energy
- Kepler's laws of planetary motion

Newton's Law of Universal Gravitation



$$\boxed{\mathbf{F}_g = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}_{12} = +\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}_{21}}$$

where $\boxed{G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}}$ is the universal gravitation constant, and $\hat{\mathbf{r}}_{12}$ and $\hat{\mathbf{r}}_{21}$ are *unit vectors* (length=1)

Universal Gravitation

- If m_1 exerts a gravitational force $\mathbf{F}_g$ on m_2, then m_2 also exerts a gravitational force $-\mathbf{F}_g$ on m_1.
- The two forces are equal in magnitude and opposite in direction (Newton's 3rd law)
- Assumption: m_1 and m_2 are *point masses* (they do not occupy any space)
- Therefore, for the universal gravitational equation to work:

$$r > (r_1 + r_2)$$

Gravitational Potential Energy

- The gravitational potential energy is defined as:

$$\boxed{U_g = -\frac{Gm_1 m_2}{r}}$$

- It has a very similar form to the the equation for $\mathbf{F}_g$
- $U_g = 0$ at $r = 0$ and *decrease* as r decreases

Relating Gravitational Potential Energy to Force

- If you know vector calculus, you can easily see that gravitational force ($\mathbf{F}_g$) is the negative gradient of the gravitational potential energy (U_g):

$$\mathbf{F}_g = -\nabla U_g = -\frac{\partial U_g}{\partial r}\hat{\mathbf{r}}$$

- Even without using vector calculus, you should still see that, like all conservative force, $\Delta F_g = -\Delta U_g$, as we have seen in Class 3
- The direction of $\mathbf{F}_g$ always points from high to low potential
    - A falling object is always decreasing in U_g
    - “Steepest descent”: the direction of $\mathbf{F}$ is the shortest path to decrease U_g
    - Objects traveling perpendicular to $\mathbf{F}$ has constant U_g

Gravitational Field

A Review

- The concept of gravitational field was studied in Grade 12 Physics, so this *should* be a review

Think Gravitational Field: What is g?

- We generally describe the force of gravity as

$$\mathbf{F}_g = m\mathbf{g}$$

- To find the magnitude of g, we group the variables in Newton's universal gravitation equation:

$$F_g = \underbrace{\left[\frac{Gm_1}{r^2}\right]}_{=g} m_2 = m_2 g$$

- On the surface of Earth, we use use $m_1 = m_{\text{Earth}}$ and $r = r_{\text{Earth}}$ to compute $g = 9.81\,\text{m/s}^2$, or $g = 9.81\,\text{N/kg}$ (both units are equivalent)

Gravitational Field

- The intensity of the **gravitational field** $\mathbf{g}$ generated by a source mass m_s is defined by:

$$\boxed{g(m_s, r) = \frac{G m_s}{r^2}}$$

- Mapping of how m_s influences the gravitational forces on other masses

Quantity	**Symbol**	**SI Unit**
Gravitational field intensity	g	N / kg
Universal gravitational constant	G	N m^2 / kg^2
Mass of source (a point mass)	m_s	kg
Distance from centre of source	r	m

Relating Gravitational Field & Gravitational Force

- $\mathbf{g}$ itself doesn't do anything until there is another mass m. At which point, m experiences a gravitational force related to $\mathbf{g}$ by:

$$\boxed{\mathbf{g} = \frac{\mathbf{F}_g}{m}}$$

- $\mathbf{F}_g$ and $\mathbf{g}$ are vectors in the same direction: toward the centre of the source mass that created the field, therefore all vector operations apply

Quantity	Symbol	SI Unit
Gravitational field	$\mathbf{g}$	N / kg
Gravitational force on a mass	$\mathbf{F}_g$	N
Mass inside the gravitational field	m	kg

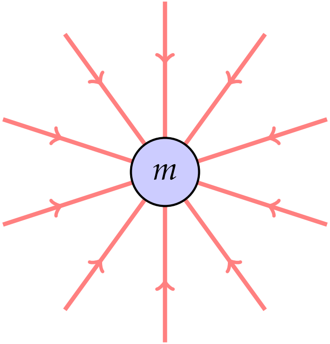
Relating U_g, $\mathbf{F}_g$ and $\mathbf{g}$

- Knowing that $\mathbf{F}_g$ and $\mathbf{g}$ only differ by a constant, we can also relate gravitational field to U_g by the gradient operator:

$$\mathbf{g} = \frac{\mathbf{F}_g}{m} = -\nabla\left(\frac{U_g}{m}\right) = -\frac{\partial}{\partial r}\left(\frac{U_g}{m}\right)\hat{\mathbf{r}}$$

- We already know that the direction of $\mathbf{g}$ is the same as $\mathbf{F}_g$, i.e.
  - The direction of $\mathbf{g}$ is the shortest path to decrease U_g
  - Objects traveling perpendicular to $\mathbf{g}$ has constant U_g

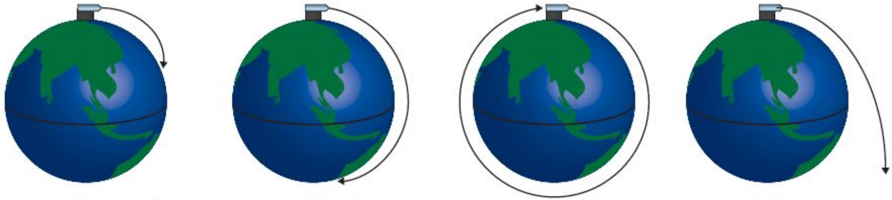
Gravitational Field Lines



- The direction of $\mathbf{g}$ is towards the centre of the object that created it
- Field lines do not tell the intensity (i.e. magnitude) of $\mathbf{g}$, only the direction

Orbital Speed

Newton's Thought Experiment



When fired, a projectile will hit the ground

With more launch force, it will fly further

Eventually the curve of the projectile's path due to gravity will match the curvature of the Earth, and the projectile will never land (assuming no air friction)

When enough force is applied, the projectile will never return

So how fast is fast enough?

Orbital Speed

Relating Gravitational and Centripetal Force

- Assume a circular orbit (since we know it best)
- The centripetal force (required force) is equal to the gravitational force (supplied force):

$$\underbrace{\frac{GMm}{r^2}}_{\mathrm{F_g}} = \underbrace{\frac{mv^2}{r}}_{\mathrm{F_c}}$$

where r is the distance between the centres of the two objects

Orbital Speed

- Cancelling m, and r in both sides of the equation, and solving for v, we get:

$$\boxed{v_{\text{orbit}} = \sqrt{\frac{GM}{r}}}$$

- Orbital speed v_{orbit} is sometimes also called orbital velocity
- Does not depend on the mass of the object in orbit
- A 1.5×10^{-13} kg speck of cosmic dust and the $419\,600$ kg International Space Station both have the same v_{orbit} around Earth if they are at the same altitude

Escape Speed

An object can leave the surface of Earth at any speed. But when all the kinetic energy of that object is converted to gravitational potential, it'll return back to the surface of the earth. There is, however, a *minimum* velocity at which the object *would not* fall back to Earth because of gravity.

Escape Speed

- The gravitational potential energy of an object with mass m on the surface of a planet (with mass M and radius r) is

$$U_g = -\frac{GMm}{r}$$

- The most amount of work that you can do is to bring it to the other side of the universe $r = \infty$, where $U_g = 0$.
- If you start with *more* kinetic energy than required to do all the work, then the object has *escaped* the gravitational pull of the planet.

Escape Speed

- Set K to equal to $-U_g$:

$$\frac{1}{2}mv^2 = \frac{GMm}{r}$$

- We can then solve for escape speed $v = v_{\text{esc}}$:

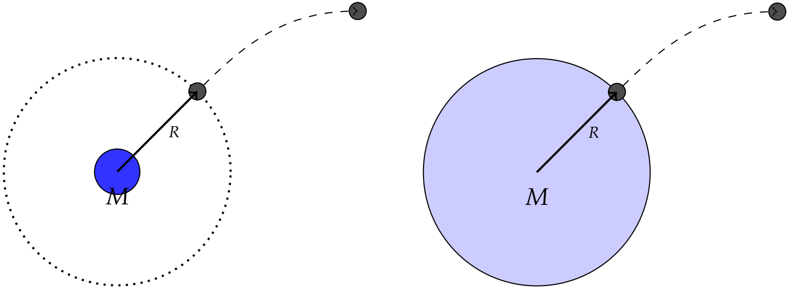
$$\boxed{v_{\text{esc}} = \sqrt{\frac{2GM}{r}}}$$

- There is a simple relationship between orbital speed and escape speed:

$$\boxed{v_{\text{esc}} = \sqrt{2}v_{\text{orbit}}}$$

What if I'm not escaping from the surface?

- These two situations have the same escape speed:



- The only difference is that an object already in orbit (left figure) already has orbital speed v_{orbit}, so escaping from that orbit requires an additional speed of

$$\Delta v = v_{\text{esc}} - v_{\text{orbit}}$$

Orbital Energies

- We can obtain the **orbital kinetic energy** by using the orbital speed in our expression of kinetic energy:

$$K_{\text{orbit}} = \frac{1}{2}mv_{\text{orbit}}^2 = \frac{1}{2}m\left(\sqrt{\frac{GM}{r}}\right)^2 = \boxed{\frac{GMm}{2r}}$$

- And we already have an expression for **gravitational potential energy**:

$$U_g = -\frac{GMm}{r} = -2K_{\text{orbit}}$$

- **Total orbital energy** is the sum of K and U_g!

$$E_T = K + U_g = -\frac{GMm}{2r} = -K_{\text{orbit}}$$

Kepler's Law of Planetary Motion