Class 11: Gauss's Law and Other Wonderful Topics

AP Physics

Dr. Timothy Leung

Olympiads School

February 2018

Files for You to Download

Download from the school website:

1. `11-Gauss.pdf`—This presentation. If you want to print on paper, I recommend printing 4 pages per side.
2. `11-Homework.pdf`—Homework assignment for this class.

Please download/print the PDF file before each class. There is no point copying notes that are already printed out for you. Instead, take notes on things I say that aren't necessarily on the slides.

Electric Field from Charge Distributions

From last class (and Physics 12), we know that the electric field from a point charge q is given by:

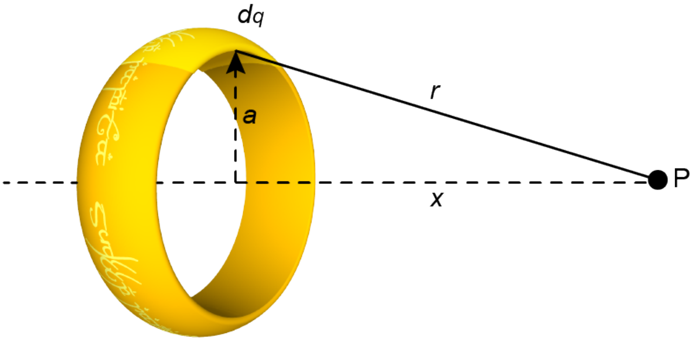
$$\mathbf{E} = \frac{kq}{r^2}\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is the outward radial direction from the charge. The total field at a point P from a distribution of charge is found by integrating through the entire volume of the charge:

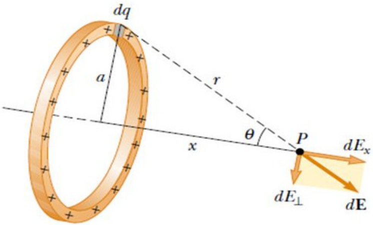
$$\boxed{\mathbf{E} = \int_V \frac{k\,dq}{r^2}\hat{\mathbf{r}}}$$

Lord of the Ring Charge

You're Given The One Ring To Rule Them All. . . what is its electric field at point P along its axis?



Electric Field Along Axis of a Ring Charge



- We can separate the electric field $d\mathbf{E}$ from charge dq into axial (dE_x) and radial ($dE_\perp$) components
- From symmetry, $dE_\perp$ doesn't do contribute to anything; but dE_x is pretty easy to find:

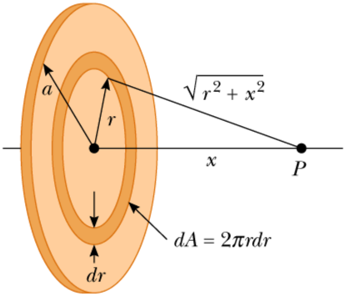
$$dE_x = \frac{kdq}{r^2}\cos\theta = \frac{kdq}{r^2}\frac{x}{r} = \frac{kxdq}{(x^2+a^2)^{3/2}}$$

Integrating this over all charges dq, we have:

$$E_x = \frac{kx}{(x^2+a^2)^{3/2}}\int dq = \boxed{\frac{kQx}{(x^2+a^2)^{3/2}}}$$

Electric Field Along Axis of a Uniformly Charged Disk

Let's extend what we know to a disk of radius R and charge density σ



- We start with the solution from the ring problem, and replace Q with $dq = 2\pi\sigma a da$:

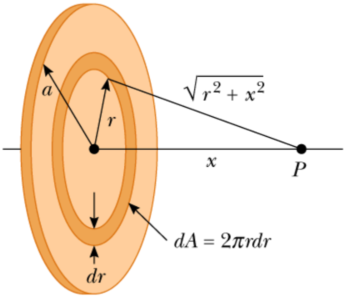
$$dE_x = \frac{2\pi k x \sigma a da}{(x^2 + a^2)^{3/2}}$$

- Integrating over the entire disk:

$$E_x = \pi k x \sigma \int \frac{2a da}{(x^2 + a^2)^{3/2}}$$

This is not an easy integral!

Eclectic Field Along Axis of a Uniformly Charged Disk



- Luckily for us, the integral is in the form of $\int u^n du$, with $u = x^2 + a^2$ and $n = \frac{-3}{2}$.
- You can find the integral in any math textbook:

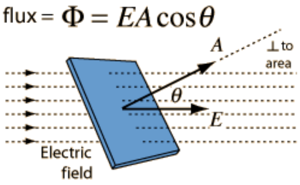
$$\boxed{E_x = 2\pi k\sigma \left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right)}$$

Flux

Flux is an important concept in many disciplines in physics. The flux of a vector quantity $\mathbf{X}$ is the amount of that quantity flowing through a surface. In integral form:

$$\Phi = \int \mathbf{X} \cdot d\mathbf{A}$$

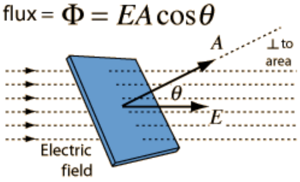
The direction of the infinitesimal area $d\mathbf{A}$ is **outward normal** to the surface.



Flux

Φ can be something very physical, like water, or bananas, or something abstract, like electric field (which is what we are interested in). We can compute a flux as long as there is a vector field i.e. $\mathbf{X} = \mathbf{X}(x, y, z)$. In the case of **electric flux**, the quantity $\mathbf{X}$ is just the electric field, i.e.:

$$\Phi_{\text{electric}} = \int \mathbf{E} \cdot d\mathbf{A}$$



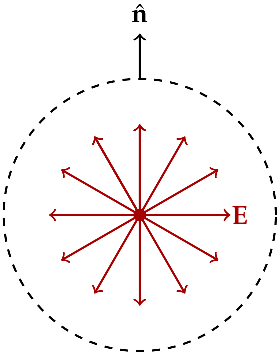
Electric Flux and Gauss's Law

Gauss's law tells us that if we have a closed surface (think of the surface of a balloon), the total electric flux is very well defined:

$$\boxed{\Phi_{\text{total}} = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{encl}}}{\epsilon_0}}$$

where Q_{encl} is the charge enclosed by the surface, and $\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C}^2/\mathrm{N\,m}^2$ is the permittivity of free space. That closed surface is called a **Gaussian surface**.

Electric Field from a Positive Point Charge



By symmetry, electric field lines are radially outward from the charge, so the integral reduces to:

$$\Phi = \oint \mathbf{E} \cdot d\mathbf{A} = EA = \frac{q}{\epsilon_0}$$

And since area of the sphere is just $A = 4\pi r^2$, we recover Coulomb's law:

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} = \frac{kq}{r^2}$$

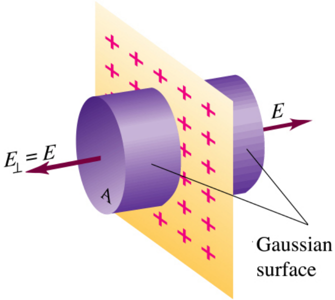
In fact, it was through studying point charges that Gauss's law was discovered, so it should not be a surprise that they agree.

In Your Homework This Week

In the homework questions this week, you will be asked to find the electric field strength inside and outside of a few common configurations:

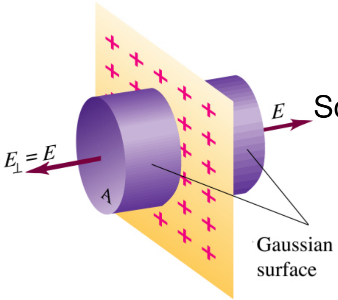
- Inside & outside of a spherical shell of charge
- Inside & outside of a uniformly charged solid sphere
- Near an infinite line charge
- Inside & outside an infinitely long solid cylinder of charge
- Inside & outside a cylindrical shell of charge

Electric Field Near an Infinite Plane of Charge



- Charge density (charge per unit area) σ
- By symmetry, $\mathbf{E}$ must be perpendicular to the plane
- Our Gaussian surface is a cylinder shown in the left with an area A; the height of the cylinder is unimportant
- We can see that nothing "flows out" of the side of the cylinder, only at the ends.
- The total flux is $\Phi = E(2A)$
- The enclosed charge is $Q_{\mathrm{encl}} = \sigma A$.

Electric Field Near an Infinite Plane of Charge



Gauss's law simplifies to:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{encl}}}{\epsilon_0} \quad \rightarrow \quad E(2A) = \frac{\sigma A}{\epsilon_0}$$

Solving for E, we get:

$$\boxed{E = \frac{\sigma}{2\epsilon_0}}$$

- E is a constant
- Independent of distance from the plane
- Both sides of the plane are the same

Electric Field Between Parallel Charged Plates

Remember being told that the electric field between two charged plates is constant?

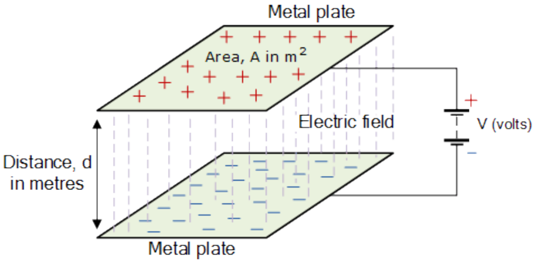
- Two plates, each producing an electric field pointing in the same direction
- The total electric field is twice the value that we found on the last slide

$$\boxed{E = \frac{\sigma}{\epsilon_0}}$$

which is what we already know!

Capacitors

Capacitors stores energy in a circuit. The simplest form of a capacitor is a set of closely spaced parallel plates:



When the plates are connected to a battery, the battery transfer charges to the plates until the potential difference (voltage) V equals the battery terminals. At that time, the plates has charge $+Q$ on one side, and $-Q$ on the other.

Parallel Capacitors

Since we know what $E = \sigma/\epsilon_0$ between the plates, and the relationship $E = V/d$, we can relate V to the amount of charge Q stored between the plates:

$$V = Ed = \frac{\sigma}{\epsilon_0} d = \frac{Qd}{\epsilon_0 A}$$

The ratio between charge Q and voltage V is defined as the **capacitance** C:

$$\boxed{C = \frac{Q}{V}}$$

for parallel plates: $C = \dfrac{\epsilon_0 A}{d}$ (constant!)

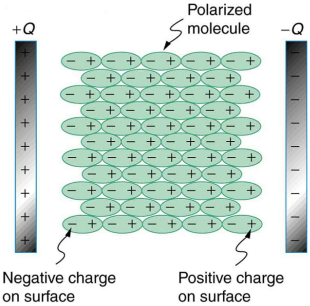
Capacitors

Capacitance C is defined as the ratio between charge Q and potential difference (voltage) V:

$$\boxed{C = \frac{Q}{V}}$$

Quantity	Symbol	SI Unit
Capacitance	C	F (farads)
Charge	Q	C (coulombs)
Voltage across the plates	V	V (volts)

Real Capacitors



- Parallel-plate capacitors are very common in electric circuits, but a vacuum between the plates is not very effective
- Instead, a **dielectric** (nonconducting) material is inserted between the plates
- When the plates are charged, the electric field of the plates polarizes the dielectric.
- The dielectric now produces an electric field that opposes the field from the plates, therefore reduces the effective voltage, and increasing the capacitance

Dielectric Constant

If electric field without dielectric is E_0, then E in the dielectric is reduced by κ, the **dielectric constant**:

$$\boxed{E = \frac{E_0}{\kappa}}$$

The capacitance in a dielectric is now amplified:

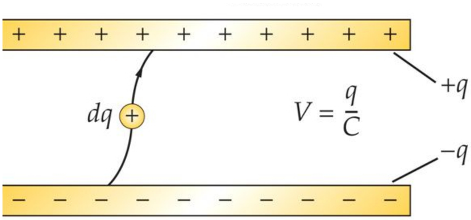
$$\boxed{C = \kappa C_0}$$

We can also view the dielectric as something that increases the effective permittivity:

$$\boxed{\epsilon = \kappa \epsilon_0}$$

Material	κ
Air	1.000 59
Bakelite	4.9
Pyrex glass	5.6
Neoprene	6.9
Plexiglas	3.4
Polystyrene	2.55
Water (20 °C)	80

Storage of Electrical Energy



- When charging up a capacitor, imagine positive charges moving from the negatively charged plate to the positively charged plate
- Initially neither plates are charged, so moving the first charge takes very little work; as the electric field builds, more and more work needs to be done

Storage of Electrical Energy

Starting from the beginning, if we move an infinitesimally charge dq across the plate, the infinitesimal work done dU is related to the capacitance:

$$dU = V dq = \frac{q}{C} dq$$

To fully charge the plates, the total work U is the integral:

$$U = \int dU = \int_0^Q \frac{q}{C} dq = \frac{1}{2} \frac{Q^2}{C}$$

There are different ways to express U using definition of capacitance:

$$\boxed{U = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} QV = \frac{1}{2} CV^2}$$