

# Topic 14: Maxwell's Equations

## Advanced Placement Physics

---

Dr. Timothy Leung

March 2020

Olympiads School, Toronto, ON, Canada

# Intro

---

# Files for You to Download

Download from the school website:

1. **PhysAP-14-maxwellsEquations.pdf**—This presentation. If you want to print the slides on paper, I recommend printing 4 slides per page.

Please download/print the PDF file before each class. When you are taking notes, pay particular attention to things I say that aren't necessarily on the slides.

# Making Ampère's Law Better

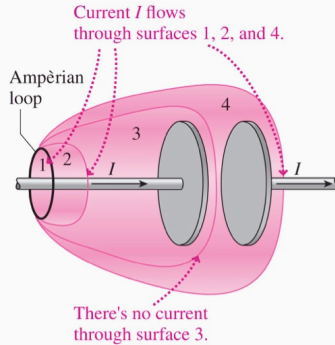
Ampère's law, as we know it, only applied to *steady* currents  $I_c$ :

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_c$$

However,

- Current are usually not steady in RC, RL, LC or RLC circuits
- Applying Ampère's law at a charging/discharging capacitor gives an ambiguous answer

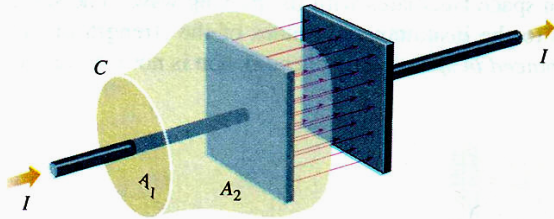
# Modifying Ampère's Law for Unsteady Current



Four surfaces bounded by the same circular Amperian loop (think blowing a soap bubble). Surfaces 1, 2 and 4 have currents penetrating through them, but surface 3 does not.

# Modifying Ampère's Law

This might give a better view of what the “soap bubble” looks like



There is no current through the surface  $A_2$  (same as surface 3 in the last slide), but there is definitely a changing *electric flux*

# Maxwell's Modification to Ampère's Law

James Clerk Maxwell, in 1860, proposed a modification to Ampère's Law to make it work with unsteady current as well

$$\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 \left( I + \varepsilon_0 \frac{d\Phi_E}{dt} \right)$$

Maxwell called the correction term  $\varepsilon_0 \frac{d\Phi_E}{dt}$  **displacement current**.

- The word “displacement” has historical roots, but no physical meaning
- However, “current” means that the effect of changing the electric flux is indistinguishable from real currents in producing magnetic field

# Maxwell's Equations

- Maxwell recognized the relationship between electricity and magnetism in **Gauss's law**, **Faraday's law** and **Ampère's law**
- Combined them into a unified set of equations, now known as **Maxwell's equations** for electrodynamics



# Maxwell's Equations in Integral Form

Maxwell's equations can be expressed in its integral form, which is how we have studied the equations in the first place:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\epsilon_0} \quad (\text{Gauss, for } \mathbf{E})$$

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0 \quad (\text{Gauss, for } \mathbf{B})$$

$$\oint \mathbf{E} \cdot d\boldsymbol{\ell} = \frac{d\Phi_M}{dt} \quad (\text{Faraday})$$

$$\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \quad (\text{Ampère, with Maxwell's mod})$$

# Maxwell's Equations in Vacuum

$$\oint \mathbf{E} \cdot d\mathbf{A} = 0$$

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0$$

$$\oint \mathbf{E} \cdot d\boldsymbol{\ell} = \frac{d\Phi_B}{dt}$$

$$\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0\epsilon_0 \frac{d\Phi_E}{dt}$$

In vacuum, we can remove all references to matter in the equation, and Maxwell's equations simplifies.

- The equations show “symmetry”
- Magnetic and electric fields are on equal footing
- In a vacuum where charges and currents are absent, the only source of either field is a change in the other field

# Maxwell's Equations in Differential Form

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

- Maxwell's equations are usually expressed in *differential* form, which is obtained using vector calculus. Follow [\[this link\]](#) if you want to see how it's done.
- The differential form shows how the *time derivatives* of  $\mathbf{E}$  and  $\mathbf{B}$  are related to the *spatial derivatives* of the other field
- The last two equations (Faraday's and Ampère's laws) together represent two set of second order partial differential equations (one for each field), the solution of which represents a traveling wave

# Electromagnetic (EM) Wave

Maxwell's equations show that an “electromagnetic wave” must exist. In a simple case where electric and magnetic fields only vary in  $x$  and time  $t$  only, i.e.  $\mathbf{E} = E(x, t)$  and  $\mathbf{B} = B(x, t)$ , Faraday's and Ampère's laws reduce to:

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} \quad \frac{\partial B}{\partial x} = -\mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

Taking the spatial derivative of  $E$  with respect to  $x$  on both side of Faraday's law, and switch the order of differentiation, we get:

$$\frac{\partial}{\partial x} \left( \frac{\partial E}{\partial x} \right) = -\frac{\partial}{\partial x} \left( \frac{\partial B}{\partial t} \right) \rightarrow \frac{\partial^2 E}{\partial x^2} = -\frac{\partial}{\partial t} \left( \frac{\partial B}{\partial x} \right)$$

# Electromagnetic (EM) Wave

But we already have an expression for  $\partial B / \partial x$  from Ampère's law:

$$\frac{\partial^2 E}{\partial x^2} = -\frac{\partial}{\partial t} \left( \frac{\partial B}{\partial x} \right) = -\frac{\partial}{\partial t} \left( -\mu_0 \epsilon_0 \frac{\partial E}{\partial t} \right)$$

Rearranging the terms on the right hand side, we get

$$\frac{\partial^2 E}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

This is the standard form of the 1D wave equation (a 2nd-order partial differential equation):

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2}$$

# Electromagnetic (EM) Wave

- “Second-order” means that the equation deals with second derivatives, in this case, in  $x$  and in  $t$ .
- “Partial” means the equation involves partial derivatives (i.e. when a function has more than one variables, and you only differentiate against one variable)
- We can also repeat the exercise by first differentiating Ampère’s law to get

$$\frac{\partial^2 B}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}$$

# Electromagnetic (EM) Wave

- The wave equation shows that disturbances in electric and magnetic fields propagate as an electromagnetic wave with a universal speed

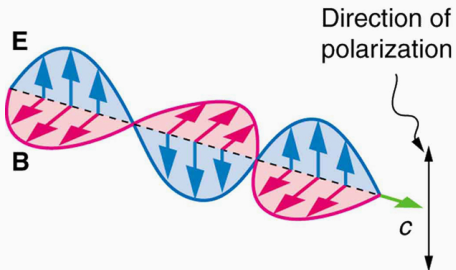
$$v = c_0 = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 299\,792\,458 \text{ m/s}$$

generally referred to as the **speed of light**.

- Our simple exercise can't show (because we have effectively ignored the cross-product) that **E** and **B** are actually perpendicular to each other

# Electromagnetic (EM) Wave

A EM wave is considered to be **polarized** if both  $\mathbf{E}$  (and therefore)  $\mathbf{B}$  of the wave are confined to a single plane. The direction of the polarization is the direction of  $\mathbf{E}$ .





# “Failure” of Maxwell’s Equation

A peculiar feature of Maxwell’s equation:

- When applying *Galilean transformation* (our classical equation for *relative motion*) to Maxwell’s equations, they seem to “fail”
- Gauss’s law for magnetism break down: magnetic field lines appear to have beginnings/ends
- So does that mean that in *some* inertial frames of reference, Maxwell’s equations are valid, but in others, they are not?
- Physicists theorized that, perhaps, there is/are actually *preferred* inertial frame(s) of references
- This violate the long-standing *principle of relativity*, which says that *the laws of physics are equal in all inertial frames of reference*

# Making The Equations Work Again

Maxwell's equations didn't "fail"; it was our understanding of space and time that needed to change

- Albert Einstein believed in the principle of relativity, and rejected the concept of a preferred frame of reference
- In Maxwell's equations, the speed of an electromagnetic wave (speed of light) is independent of the frame of reference
- In order to make the equations to work again, Einstein revisited the most basic concepts involved in our understanding of physics: space and time

# Einstein and the Principle of Relativity

Einstein's Postulates of Special Relativity:

1. All laws of physics must apply equally in all inertial frames of reference.
2. As measured in any inertial frame of reference, light always propagated empty space with a definite velocity  $c$  that is independent of the state of motion of the emitting body.

Published in 1905 in the article *On the Electrodynamics of Moving Bodies* when Einstein was 26 years old working as a patent clerk in Switzerland