Symmetric Projectile Trajectory

A **symmetric trajectory** is a special case of projectile motion where an object is launched at an angle of θ (between $0°$ and $90°$) above the horizontal[1] with an initial speed v_i, and then lands at the same height, as shown below in Figure 1. Examples may include hitting a golf ball towards the hole, or shooting a bullet towards ahorizontal target[2]. The equations for symmetric trajectory is *not* included in the AP Exam equation sheet. If you need these equations during the exams, you will need to derive them during the exam. To derive the equations, we use the x-axis for the horizontal direction and y-axis for the vertical.

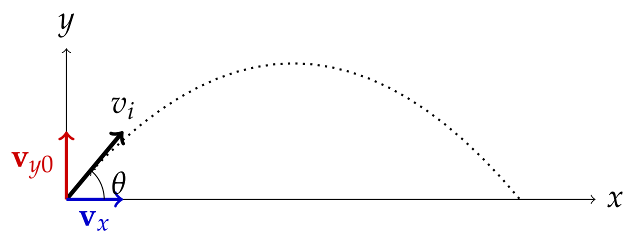

Figure 1: Symmetric project trajectory

The initial velocity $\mathbf{v}_i$ can be resolved into its components, also shown in Figure 1:

$$\mathbf{v}_x = v_i \cos\theta \hat{\imath}$$
$$\mathbf{v}_{y0} = v_i \sin\theta \hat{\jmath}$$

v_x remains constant during the motion, as there are no forces acting in the x direction, and therefore no acceleration. In the y direction, there is an acceleration due to gravity $a_y = -g$.

Maximum height H: Apply the kinematic equation in the y-direction. Recognizing that at maximum height $H = y - y_0$, the vertical component of velocity is zero $v_y = 0$:

$$v_y^2 = v_{y0}^2 + 2a_y(y - y_0)$$
$$0 = (v_i \sin\theta)^2 - 2gH$$

Solving for H, we get the maximum height equation:

$$\boxed{H = \frac{v_i^2 \sin^2\theta}{2g}} \tag{1}$$

[1]This may be obvious, but any angles *below* the horizontal will never have a symmetric trajectory.

[2]Shooting a bullet towards a horizontal target always require an upward angle because of gravity

Total time of flight $t_{\max}$: We apply the kinematic equation in the y direction. When the object lands at the same height, the final velocity is the same in magnitude and opposite in direction as the initial velocity, i.e. $v_{y2} = -v_{y1} = -v_i \sin\theta$:

$$v_y = v_{y0} + a_y t$$
$$-v_i \sin\theta = v_i \sin\theta - g t_{\max}$$

Solving for $t_{\max}$ we have:

$$\boxed{t_{\max} = \frac{2v_i \sin\theta}{g}} \tag{2}$$

Range R: We substitute the expression for $t_{\max}$ from Eq. 2 into the t term, then apply the kinematic equation in the x-direction to compute $R = x - x_0$ for any given launch angle and initial speed:

$$x = x_0 + v_x t$$
$$R = v_i \cos\theta \left(\frac{2v_i \sin\theta}{g}\right)$$

Using the trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$, we simplify the equation to:

$$\boxed{R = \frac{v_i^2 \sin(2\theta)}{g}} \tag{3}$$

It is obvious that for any given initial speed v_i, the maximum range $R_{\max}$ occurs at an angle where $\sin(2\theta) = 1$ (i.e. $\theta = 45^\circ$), with a value of

$$\boxed{R_{\max} = \frac{v_i^2}{g}} \tag{4}$$

Also, for a known initial speed v_i and range R we can compute the launch angle θ:

$$\theta_1 = \frac{1}{2}\sin^{-1}\left(\frac{gR}{v_i^2}\right)$$

This angle is labelled θ_1 because it is *not* the only angle that can reach this range. Recall that for any angle $0^\circ < \phi < 90^\circ$, there is also another angle where the $\sin$ are equal:

$$\sin\phi = \sin(180^\circ - \phi)$$

Which means that for any θ_1, there is also another angle θ_2 where $2\theta_2 = 180^\circ - 2\theta_1$, or quite simply:

$$\theta_2 = 90^\circ - \theta_1$$