

# Topic 4: Center of Mass

## Advanced Placement Physics

Dr. Timothy Leung

Olympiads School

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# Center of Mass

Finding an object's center of mass is important, because

- Newton's laws of motion are formulated by treating an objects as point masses (for real-life objects, we let the forces apply to the center of mass)
- objects can have *rotational* motion in addition to *translational* motion as well (we will examine that a bit more next week)

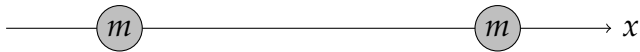
# Start with a Definition

The **center of mass** (“CM”), or **center of gravity** (“CG”), is the *weighted average of the masses in a system*. The “system” may be:

- A collection of individual particles (use summation to compute CM)
- A continuous distribution of mass with constant density (use integration to compute CM); in this case, CM is also the geometric center of the object (*centroid*)
- A continuous distribution of mass with varying density (use integral to compute CM)

# Simple Example

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The answer is really simple: it's at the half way point between the two masses!

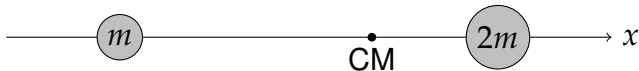
## But Things Aren't Always That Example

- What if one of the masses are increased to  $2m$ ?
- This is still not a terribly difficult problem; you can still *guess* the right answer without know the equation for center of mass.



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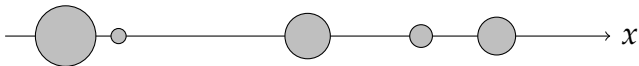


- The answer is still simple. The CM is no longer at the half way point between the two masses, but now  $\frac{1}{3}$  the total distance from the larger masses.

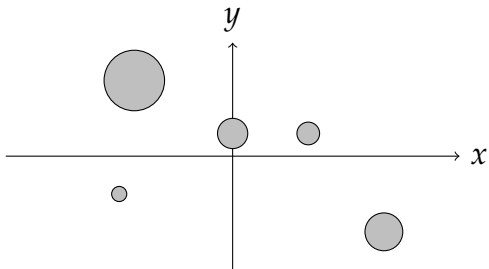
# Complicating Things Further

## Many Point Masses

If we increase the number of point masses along the  $x$ -axis, our problem can become much more complicated (although still not devastatingly so)



Difficulties really arises when there are many masses in the system in 2D or 3D:





## An Equation Helps

The center of mass is defined as:

$$\mathbf{x}_{\text{CM}} = \frac{\sum \mathbf{x}_i m_i}{\sum m_i}$$

Quantity	Symbol	SI Unit
Position of center of mass (vector)	$\mathbf{x}_{\text{CM}}$	m (meters)
Position of point mass $i$ (vector)	$\mathbf{x}_i$	m (meters)
Point mass $i$	$m_i$	kg (kilograms)
Total mass	$\sum m_i$	kg (kilograms)

# Breaking Down Into Components

$$\mathbf{x}_{\text{CM}} = \frac{\sum \mathbf{x}_i m_i}{\sum m_i}$$

- Position vectors have  $x$ ,  $y$  and  $z$  components:  $\mathbf{x} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$
- We can deal with each component individually, i.e.:

$$x_{\text{CM}} = \frac{\sum x_i m_i}{\sum m_i} \quad y_{\text{CM}} = \frac{\sum y_i m_i}{\sum m_i} \quad z_{\text{CM}} = \frac{\sum z_i m_i}{\sum m_i}$$

## Let's Do An Example

**Example 1:** Consider the following masses and their coordinates which make up a “discrete mass” rigid body”

$$m_1 = 5.0 \text{ kg}$$

$$\mathbf{x}_1 = 3\hat{i} - 2\hat{k}$$

$$m_2 = 10.0 \text{ kg}$$

$$\mathbf{x}_2 = -4\hat{i} + 2\hat{j} + 7\hat{k}$$

$$m_3 = 1.0 \text{ kg}$$

$$\mathbf{x}_3 = 10\hat{i} - 17\hat{j} + 10\hat{k}$$

What are the coordinates for the center of mass of this system?

## Continuous Mass Distribution

In general, objects are not a discrete collection of point masses, but a continuous distribution of mass. Therefore, we take the limit of when the number of masses approaches  $\infty$ :

$$\mathbf{x}_{\text{CM}} = \lim_{n \rightarrow \infty} \left( \frac{\sum_{i=1}^n \mathbf{x}_i m_i}{\sum_{i=1}^n m_i} \right)$$

This gives us an integral form of our equation:

$$\mathbf{x}_{\text{CM}} = \frac{\int \mathbf{x} dm}{\int dm}$$

# Densities

- Linear density (for 1D problems)

$$\gamma = \frac{m}{L}$$

- Surface area density (for 2D problems)

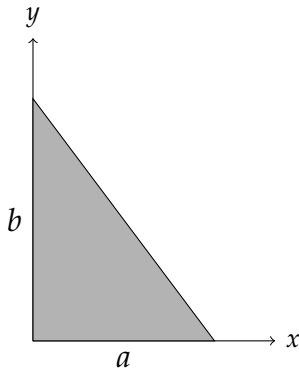
$$\sigma = \frac{m}{A}$$

- Volume density (for 3D problems)

$$\rho = \frac{m}{V}$$

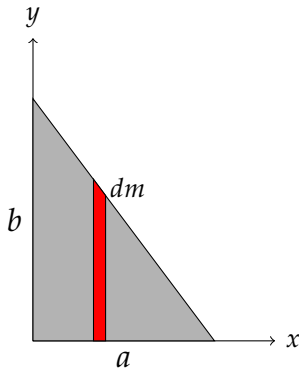
## An Example with Integrals

**Example 2:** A triangular plate is placed in a Cartesian coordinate system with two of its edges along the  $x$  and  $y$ -axis. The length of the edges along the axes are  $a$  and  $b$  respectively. Assuming that the surface area density  $\sigma$  is uniform, determine the coordinate of its center of mass.



## An Example with Integrals

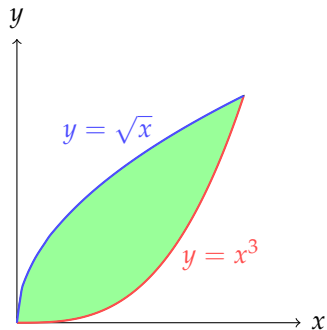
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# A Difficult Example to Try at Home

This is not typically an AP-level problem, but this example shows how we can use integral to find the center of mass for something very complicated.

**Example 3:** Find the  $x$ -coordinate of the center of mass in the shape bound by the two functions shown on the right.





# Symmetry

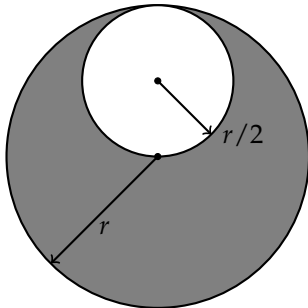
There are always shortcuts!

- Any plane of symmetry, mirror line, axis of rotation, point of inversion *must* contain the center of mass.
- Caveat: only works if the density distribution is also symmetric
- Again: if density is uniform, CM is also geometric center (centroid)

# “Negative Mass”

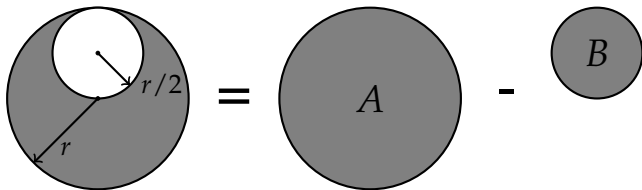
A mathematical trick for complicated geometries

- Where there is a “hole” in the geometry, treat it as having negative mass density  $-\sigma$  in that region.
- Negative masses don’t exist, so this is really just a trick.
- **Example:** What is the center of mass of this shape?



## Negative Mass Example

- This is how we would think of it:

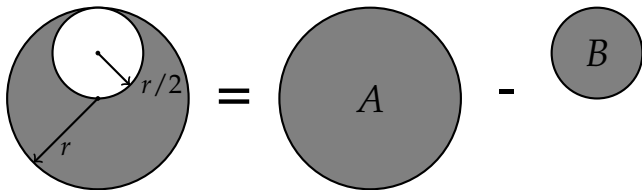


- Let the origin of the coordinate system to be located at the center of  $A$
- Based on symmetry:  $x_{\text{CM}} = 0$ ; only have to find  $y$ -coordinate.
- Sum our weighted average:

$$y_{\text{CM}} = \frac{\sum y_i m_i}{\sum m_i} = \frac{m_A(0) + m_B(r/2)}{m_A + m_B} = \frac{-\sigma\pi (r/2)^2 (r/2)}{\sigma\pi r^2 - \sigma\pi (r/2)^2}$$

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## Velocity, Acceleration and Momentum

We can take the time derivative of the equation for  $\mathbf{x}_{\text{CM}}$  to get the velocity of the CM:

$$\mathbf{v}_{\text{CM}} = \frac{d\mathbf{x}_{\text{CM}}}{dt} = \frac{1}{m} \frac{d}{dt} \left( \int \mathbf{x} dm \right) = \frac{1}{m} \int \frac{d\mathbf{x}}{dt} dm = \frac{\int \mathbf{v} dm}{m}$$

The integral in the numerator is the sum of the momentum of all the masses in the system ( $\mathbf{p}_{\text{net}}$ ) which means that we have

$$\mathbf{p}_{\text{net}} = m\mathbf{v}_{\text{CM}}$$

Taking the derivative of  $\mathbf{p}_{\text{net}}$  allows us to relate force and acceleration at the CM as well:

$$\mathbf{F}_{\text{net}} = \frac{d\mathbf{p}_{\text{net}}}{dt} = m \frac{d\mathbf{v}_{\text{CM}}}{dt} = m\mathbf{a}_{\text{CM}}$$