Rigid-Body Rotational Motion

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1 Pure Rolling of Rigid Body on Flat Surface

The case of the rolling ball is a standard example of combining the dynamics of translational and rotational motions of a rigid body. In this example, a perfectly smooth ball rolls on a perfectly smooth surface without slipping (rolling without slippage is called **pure rolling**). The force diagram is shown in Fig. 1. Notice that *there is no friction between the ball and the surface*. In theory, the steel ball bearing should be able to roll along the surface forever, as there is neither a net force nor a net torque acting on the ball.

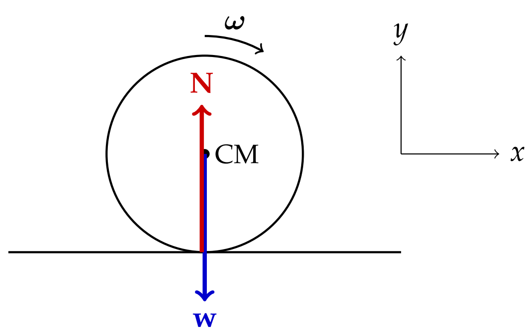

Figure 1: Force diagram on a steel ball bearing rolling on a smooth flat surface without slipping.

But any casual observer will notice that the steel ball will, in reality, slow down and eventually stop. So what is causing this?

Firstly, we should recognize neither the ball bearing nor the rail are perfectly smooth. When the ball rolls over a bump, the surface roughness means that the normal force does not necessarily points towards the centre of the ball. This means that unlike in Fig. A, there is a net force and net torque that will slow down the motion of the ball.

Secondly, we should recognize that there is no such thing as a perfectly rigid body. Both the ball and the surface deform as they make contact. A perfect illustration is how a tire flattens when it makes contact with the ground.

2 Pure Rolling on an Inclined Surface

But what if the ball rolls without slippage down a ramp instead? The free-body diagram for a ball rolling down a ramp of angle θ is shown in Fig. 2. This time, there is a static friction $\mathbf{f}_s$ acting up the ramp. The

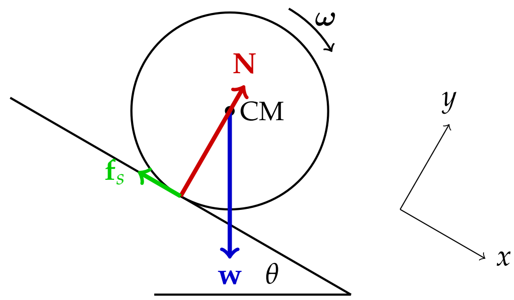

Figure 2: Force diagram on a steel ball bearing rolling on a smooth ramp without slipping. The ball travels distance d to the bottom of the ramp.

weight of the ball acts at the centre of gravity, while the normal force acts at the point of contact. Neither forces generate any torque about the CM, therefore, without friction, the ball will just *slide* down the ramp without any rotation. To solve this problem, we have three dynamic equations along the three axes:

$$\sum F_x = mg\sin\theta - f_s = ma \tag{1}$$
$$\sum F_y = N - mg\cos\theta = 0 \tag{2}$$
$$\sum \tau = rf_s = I_z\alpha \tag{3}$$

At this stage, the actual static friction force is not known and is a quantity that needs to be solved. Knowledge of the coefficient of static friction μ_s may not be useful, because it only tells you the *maximum* static friction force, not the actual fritcion force that exists. However, we will use it to double check to see if the answer makes sense.

Inserting the expression for the moment of inertia of the ball[1] and recognizing that for pure rolling, $\alpha = \frac{a}{r}$, we can use Eq. 3 to express static friction in terms of a:

$$f_s = \frac{I_z\alpha}{r} = \tag{4}$$

Substituting the expression in Eq. 4 into Eq. 2, the force equation in the x-direction becomes:

$$mg\sin\theta - f_s = ma \tag{5}$$

Cancelling the mass terms and solving for acceleration, we find a value of:

$$a = \tag{6}$$

As weight, normal force and friction are all time independent, acceleration is constant.

[1] Note that if instead of using a solid ball, we will have to use the moment of inertia of those objects:
Hollow sphere
Solid cylinder
Hollow cylinder

Compare the results in Eq. 6 to that of an object sliding without friction down the same ramp, the acceleration for the sliding block is $a = g\sin\theta$ which is higher than the pure rolling case. The simplest explanation is that some of the gravitational potential energy is converted to both translational and rotational kinetic energies, while for the sliding case, all of the potential energy is converted into translational kinetic energy.

There is, of course, one check that we must make, that is to make sure that the friction calculated in Eq. 4 has not exceeded the maximum static friction, given by

$$\max f_s = \mu_s N \tag{7}$$

If this is indeed the case, it means that the ball will actually slip while rolling down the ramp, and the friction at the contact point is in fact kinetic friction.

Since acceleration is constant, we can use the basic kinematic equation to compute the speed of the ball when it reaches the bottom of the ramp, a distance d away. For simplicity, we assume that the ball starts from rest:

$$v = \sqrt{2ad} = \tag{8}$$

Of course, there is a much simpler way to find v, that is, by using the conservation of energy.

$$\begin{aligned}
\Delta U_g &= K_t + K_r \\
mg\Delta h &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \\
mgd\sin\theta &= \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{3}mr^2\right)\left(\frac{v}{r}\right)^2 \\
&= \frac{2}{3}mv^2
\end{aligned}$$

Now cancelling mass terms on both sides, and solving for v, we arrive at the same expression as using dynamics and kinematics equations:

This is pretty significant for the novice leave because while there is indeed friction, this friction didn't do any work. This should not be a surprise, because in order for a force to do any work, it has to actually *move* something. This is clearly not the case if friction is *static*. To be clear, if the ball slips, then there is definitely work done by kinetic friction.