Kepler's Laws of Planetary Motion

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Johannes Kepler (1571–1630) formulated his **laws of planetary motion** between 1609 to 1619, by carefully interpreting the empirical planetary motion data from his teacher, Tycho Brahe. It is an improvement over the heliocentric theory of Nicolaus Copernicus. It should be noted that the complete mathematical understanding was missing until Newton derived each law as pieces of his orbital mechanics. In this handout we seek to obtain each law in turn, as we consider the orbit of a small planet in the gravitational field of a much more massive star. In modern form, Kepler's laws of planetary motion state that:

1. A planet moves around the Sun in an elliptical path with the Sun as one of the foci.
2. The line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
3. The square of the orbital time period of a planet is proportional to the cube of the semi-major axis of its orbit, i.e. $T^2 \propto r^3$.

To begin, we recognize that two properties of gravitational force, or Newton's **law of universal gravitation**:

$$\mathbf{F}_g = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} \tag{1}$$

are crucial to the understanding of orbital mechanics. These two properties hold true regardless of the shape of the orbit, and whether an object is even in orbit altogether:

1. Gravity is a *conservative* force, in that
   - assuming that minor collisions with space dust and other methods of energy dissipation are negligible, then the mechanical energy $E = K + U_g$ is conserved.
   - work done *by* gravity (positive work) converts gravitational potential energy U_g into kinetic energy K, while work *against* gravity (negative work) converts K into U_g.
2. Gravity is a *central* force, in that
   - gravitational force $\mathbf{F}_g$ is always in the $-\hat{\mathbf{r}}$ direction (i.e. $\mathbf{F} \times \mathbf{r} = \mathbf{0}$), shown in Fig. 1, therefore gravity does not generate any torque, and as such,
   - angular momentum $\mathbf{L}$ about the sun is constant
   - the motion of the planet is confined to a two-dimensional plane

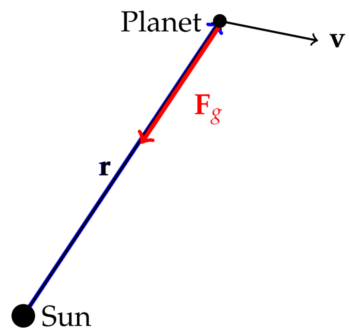

Figure 1: A central force such as gravity do not generate a torque (moment).

For the purpose of the analysis, we also assume that the Sun's mass $M_\odot$ is very large compared to any other object in the solar system (e.g. planets, comets, asteroids), and its motion is essentially unaffected by the gravity from the planets.

1 Second law: Equal area in equal time

The second law of planetary motion, sometimes called the **law of equal areas**, states (in modern terms) that line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time (Fig. 2). The second law of planetary motion is the easiest to proof using concepts in rotational motion.

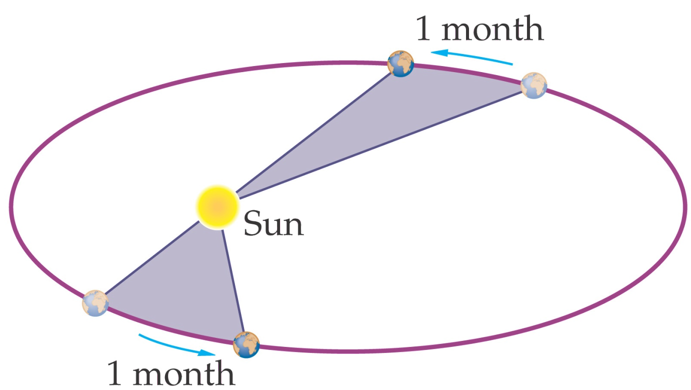

Figure 2: The Earth sweeps out the same shaded area (purple) over the same amount of time regardless of where it is in its orbit.

To begin, recognize that for a planet with a velocity vector $\mathbf{v}$ that is not co-linear with the displacement vector $\mathbf{r}$ will sweep out an infinitesimal area $d\mathbf{A}$ (Fig. 3) as it moves in orbit by an infinitesimal amount $d\mathbf{r}$, which can be computed by the area of the triangle:

$$dA = \frac{1}{2} r dr \sin\alpha \tag{2}$$

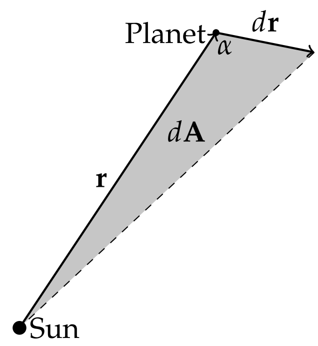

Figure 3: Infinitesimal area $d\mathbf{A}$ swept by a planet as it moves $d\mathbf{r}$ in orbit.

or in vector form:

$$d\mathbf{A} = \frac{1}{2}\mathbf{r} \times d\mathbf{r} \tag{3}$$

The direction of $d\mathbf{A}$, in this case, points into the page.[1] The time derivative of the area is called the **areal velocity**, which literally means how quickly the area is changing:

$$\frac{d\mathbf{A}}{dt} = \frac{1}{2}\mathbf{r} \times \frac{d\mathbf{r}}{dt} = \frac{1}{2}\mathbf{r} \times \mathbf{v} \tag{4}$$

We can express $\mathbf{r} \times \mathbf{v}$ in terms of angular momentum. Since angular momentum is defined as $\mathbf{L} = m(\mathbf{r} \times \mathbf{v}) \;\rightarrow\; \mathbf{r} \times \mathbf{v} = \mathbf{L}/m$. Since gravity is a central force, angular momentum is a constant, i.e:

$$\boxed{\frac{d\mathbf{A}}{dt} = \frac{1}{2}(\mathbf{r} \times \mathbf{v}) = \frac{\mathbf{L}}{2m} = \text{constant}} \tag{5}$$

as predicted by Kepler's second law. The rate that a planet sweeps out the area in orbit is its angular momentum around the sun, divided by twice its mass. It is constant regardless of its location relative to the sun.

2 First Law: The Law of Ellipses

Kepler's first law is more difficult to proof, an in order to proof the law, we must show that all orbital motion must agree with the equations of an ellipse. An ellipse is shown in Fig. 4 with the origin of the coordinate system located at one of the two foci.

The longest dimension from the origin is the **semi-major axis** a, while the shortest dimension is the **semi-minor axis** b. The two lengths are related by the parameter e, generally called the **eccentricity** of the ellipse:

$$b^2 = a^2(1 - e^2) \quad \text{where} \quad 0 \leq e < 1 \tag{6}$$

When the eccentricity of a planet's orbit is zero, the orbit is perfectly circular, and $a = b = r$. As e approaches one, the orbit is stretched out into more elongated elliptical trajectories. When $e = 1$, the shape becomes a parabola, and is no longer an ellipse.

[1]The direction of the vector is not important in the calculation, but is only used to formalize the vector operations.

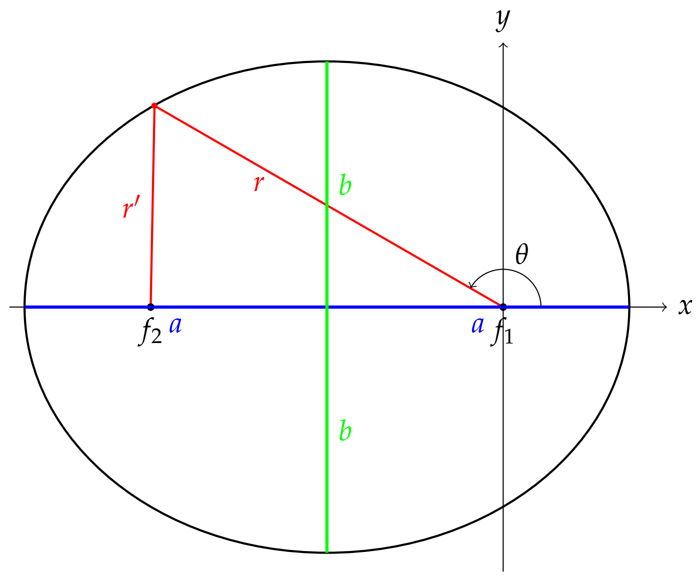

Figure 4: A basic ellipse with the origin of the coordinate system placed at one of the focus.

The area A of an ellipse is given by the semi-major and semi-minor axes:

$$A = \pi ab = \pi a^2 \sqrt{1 - e^2} \tag{7}$$

When the origin of the coordinate system placed one of the two foci, an ellipse can be expressed using polar coordinates:

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \tag{8}$$

As shown in Fig. 5, there are two components of velocity when a planet orbits a star.

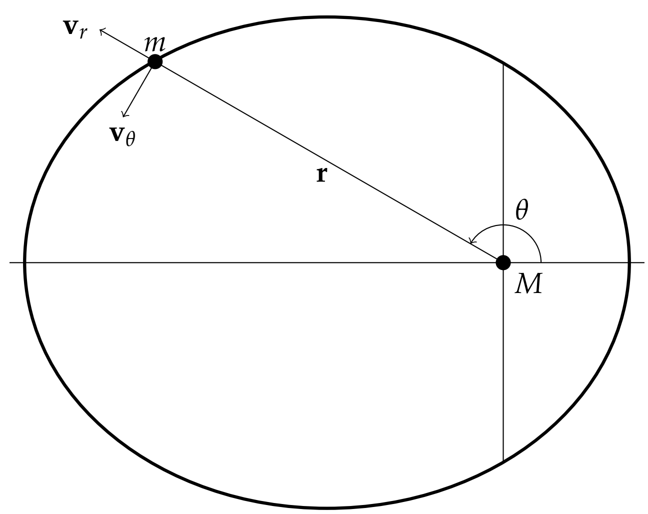

Figure 5: Elliptical orbit of a small mass m around a large mass M.

- **Angular velocity** $\mathbf{v}_\theta$. The presence of $\mathbf{v}_\theta$ means a centripetal acceleration a_c toward M. The direction of the acceleration is perpendicular to the direction of $\mathbf{v}_\theta$:

$$a_c = -r\omega^2\hat{\mathbf{r}} \tag{9}$$

- **Radial velocity** $\mathbf{v}_r$. If this velocity is non-zero and changes with time (which is the case for elliptical orbits but not for circular orbits), then there is an acceleration a_r, also in the radial direction:

$$a_r = \frac{dv_r}{dt}\hat{\mathbf{r}} = \frac{d^2r}{dt^2}\hat{\mathbf{r}} \tag{10}$$

Both components of acceleration are due entirely to gravitational force. Applying Newton's second law of motion ($\sum F = ma$), and dividing all sides by the mass of the planet m, we arrive at a differential equation:

$$\frac{d^2r}{dt^2} - r\omega^2 = -\frac{GM}{r^2} \tag{11}$$

In this equation, the $(+)$ direction is radially outward from M. In the circular motion case, where $\ddot{r} = 0$, we are left with only the centripetal acceleration $a_c = r\omega^2$ which is expected in uniform circular motion. The differential equation, in its present form, is difficult to solve. Moreover, the equation that for an ellipse (Eq. 8) depends on θ but not on t. However, the differential equation is easier to solve by making a variable substitution[2], by introducing a new variable u which is the inverse of the radius r:

$$u = \frac{1}{r} \tag{12}$$

We can then use the fact that angular momentum L is constant to relate derivatives in time t to derivatives in angle θ:

$$L = mr^2\frac{d\theta}{dt} = \frac{m}{u^2}\frac{d\theta}{dt} \quad \longrightarrow \quad \frac{d}{d\theta} = \frac{Lu^2}{m}\frac{d}{dt} \tag{13}$$

The derivative $\dot{r}$ can now be expressed in terms of derivatives of θ instead of time t:

$$\begin{aligned}
\frac{dr}{dt} &= \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u}\frac{du}{dt} \\
&= -\frac{L}{m}\frac{du}{d\theta} \\
&= -u^{-2}\frac{du}{d\theta}\frac{Lu^2}{m} \\
\frac{dr}{dt} &= -\frac{L}{m}\frac{du}{d\theta} \qquad (14)
\end{aligned}$$

Repeating the same process for the second derivative $\ddot{r}$, we have

$$\begin{aligned}
\frac{d^2r}{dt^2} &= \frac{d}{dt}\left(-\frac{L}{m}\frac{du}{d\theta}\right) \\
&= -\frac{L}{m}\frac{d}{d\theta}\frac{du}{d\theta} \\
\frac{d^2r}{dt^2} &= -\left(\frac{L^2u^2}{m^2}\right)\frac{d^2u}{d\theta^2} \qquad (15)
\end{aligned}$$

[2]this variable may not be immediately obvious without some experience with solving ODEs

With this identity in hand, our original differential equation (Eq. 11) becomes

$$-\left(\frac{L^2u^2}{m^2}\right)\frac{d^2u}{d\theta^2}-\left(\frac{L}{m}\right)^2u^3=-GMu^2 \tag{16}$$

which can be simplified to:

$$\frac{d^2u}{d\theta^2}+u=\frac{GMm^2}{L^2} \tag{17}$$

Anyone experienced with calculus can recognize that Eq. 17 is a *second-order ordinary differential equation with constant coefficients and a constant forcing function*. The solution to this type of equation is in the form

$$u=A\cos\theta+\frac{GMm^2}{L^2} \tag{18}$$

where the coefficient A is determined by initial conditions. Solving for $r=1/u$, we have

$$r(\theta)=\frac{1}{u}=\frac{1}{A\cos\theta+\frac{GMm^2}{L^2}}=\left(\frac{L^2}{GMm^2}\right)\frac{1}{1+e\cos\theta} \tag{19}$$

where e is a constant:

$$e=\frac{AL^2}{GMm^2} \tag{20}$$

It should become obvious that this e is, in fact, the eccentricity of the elliptical orbit. From this form of r, it is clear that the maximum and a minimum values of r represent the *aphelion* and *perihelion* of an ellipse (points of furthest and closest distance to the focus):

$$r_{\text{max}}=\left(\frac{L^2}{GMm^2}\right)\frac{1}{1-e}\qquad r_{\text{min}}=\left(\frac{L^2}{GMm^2}\right)\frac{1}{1+e} \tag{21}$$

This the semi-major axis r is the average of the two values:

$$a=\frac{1}{2}\left(r_{\text{min}}+r_{\text{max}}\right)=\left(\frac{L^2}{GMm^2}\right)\frac{1}{1-e^2} \tag{22}$$

More importantly, Eq. 22 allows us to relate the eccentricity e and the semi-major axis a to the numerator in the r expression:

$$a(1-e^2)=\frac{L^2}{GMm^2} \tag{23}$$

We see that the orbit is given by an ellipse as Kepler found from Brahe's data. Moreover, since r_{min} and r_{max} are distances from the Sun, we see that the Sun is at one focus of the orbit. Thus, we have derived Kepler's first law.

3 Third law: Period of motion

The total area A swept by the planet through one orbital period is the areal velocity (which is constant, as shown in Eq. 5) integrated by time, from $t=0$ to T, the orbital period of the planet:

$$A=\int dA=\int_0^T\frac{dA}{dt}dt=\frac{L}{2m}\int_0^T dt=\frac{LT}{2m} \tag{24}$$

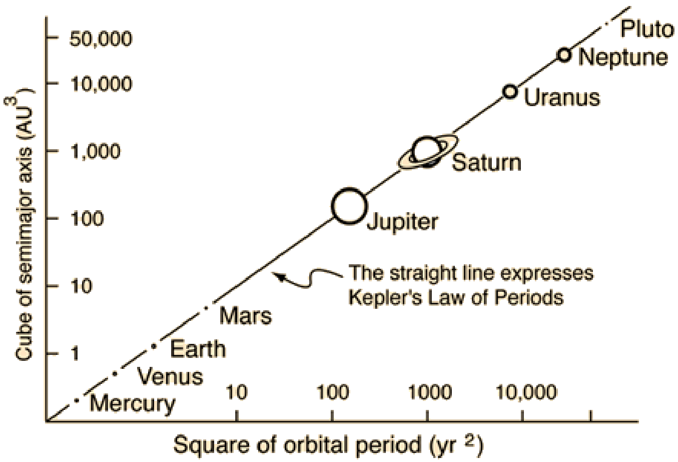

Figure 6: The orbits of all the planets in the solar system obey the third law of planetary motion.

From proofing Kepler's first law in the previous section, we know that the area A is in the form of an ellipse, given in Eq. 7. Solving for period T in Eq. 24, then substituting the expression for area from Eq. 7, and then squaring both sides, we arrive at this expression:

$$T^2 = \frac{m^2}{L^2} 4\pi^2 a^4 (1 - e^2) \tag{25}$$

Substituting Eq. 23 into Eq. 25 above, and after some simple algebra, we arrive at this expression:

$$\boxed{T^2 = \left[\frac{4\pi^2}{GM}\right] a^3} \tag{26}$$

which is Kepler's third law. Note that this law holds for all elliptical orbits, regardless of their eccentricities.