Topic 21: Special Relativity

Advanced Placement Physics

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Introduction

These slides for this topic are an expanded version of the slides used for Grade 12 Physics (with some additional calculus). There are two versions of the slides; both are downloadable from the school website:

- The long version
  - More background information (more than needed for the AP 2 Exam) and derivations and integrations
  - May answer some of your questions about the specifics of the theory
  - `21-relativity-long.pdf`
- The short version
  - More “to the point”
  - The version that is used during class
  - `21-relativity-short.pdf`

There is also a handout on how to solve and interpret the time dilation problem

Frame of Reference

You can think of a **frame of reference** (or “reference frame”, or just “frame”) as a hypothetical mobile “laboratory” an observer uses to make measurements (e.g. mass, lengths, time). At a minimum, it includes:

- Some rulers (i.e. coordinate system) to measure positions and lengths
- A clock to measure the passage of time
- A scale to compare forces
- A balance to measure masses

High-school textbooks often refer to the frame of reference as a “coordinate system”. While it certainly includes that, this definition often makes it difficult to understand special relativity.

Frame of Reference

- We assume that the hypothetical laboratory is *perfect*—all the hypothetical "instruments" have zero errors
- What matters is the *motion* (at rest, uniform motion, acceleration etc) of your laboratory, and how it affects the measurement that you make
- "From the point of view of. . . "

Inertial Frame of Reference

An **inertial frame of reference** is one that is moving in uniform motion (i.e. constant velocity, zero acceleration)

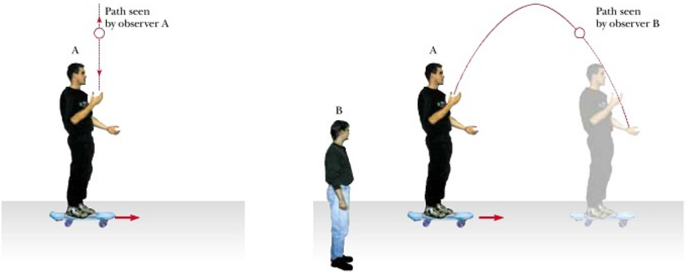
- In an inertial frame, Newton's first and second laws of motion are valid
- Since all uniform motion are treated the same way, you may consider any inertial frames of reference to be *at rest*

The Principle of Relativity

All laws of motion must apply equally in all inertial frames of reference.

Inertial Frame of Reference

Observer A is moving uniformly with the skateboard, while Observer B is standing on the side of the road.



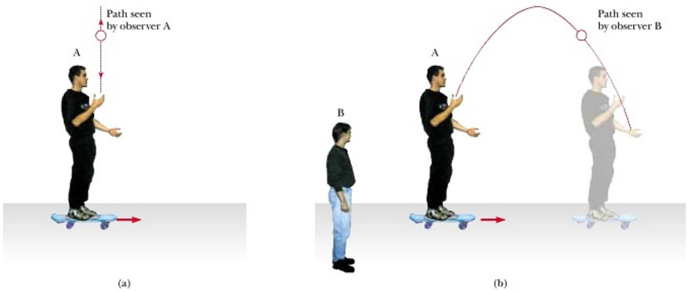
(a) (b)

When Observer A tosses a ball upward:

- A sees only vertical motion, while
- B sees the ball traveling in a parabolic curve, although
- A & B see different motion, they agree on the *equations* that govern the

Inertial Frame of Reference

Observer A sees the same motion (only vertical motion) regardless of whether he is moving uniformly w.r.t. B or not[1].



- Valid for A to conclude that he is at rest, but that B is moving
- Likewise, it is also valid for B to think that he is at rest, but that A is moving

[1] as long as *neither* are accelerating

Newtonian (Classical) Relativity

When studying kinematics and dynamics, we made some assumptions that are obvious: space and time are *absolute*:

- $1\,\mathrm{m}$ is $1\,\mathrm{m}$ no matter where you are, or how you are moving
- $1\,\mathrm{s}$ is $1\,\mathrm{s}$ no matter where you are, or how you are moving
- Measurements of space and time do not depend on motion

If space and time are absolute, then *all* velocities are relative

- Measured velocities depend on the motion of the observer
- Galilean velocity addition rule:

$$\boxed{\mathbf{v}_{AC} = \mathbf{v}_{AB} + \mathbf{v}_{BC}}$$

Maxwell's Equations

Maxwell's equations on electrodynamics in a vacuum (studied previously):

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Disturbances in $\mathbf{E}$ and $\mathbf{B}$ travel as an *electromagnetic wave* with a speed c:

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 299\,792\,458\ \mathrm{m/s}$$

Peculiar features of Maxwell's equation

- Does not mention the *medium* in which EM waves travels
- When applying *Galilean transformation* (from which we obtain the velocity addition rule) to Maxwell's equations, asymmetry is introduced
- Gauss's law for magnetism break down: magnetic field lines appear to have beginnings/ends
- In *some* inertial frames of reference, Maxwell's equations are simple and elegant, but transform the equation into another inertial frame, the equations are ugly and complex!
- Physicists at the time began to theorize that (perhaps) there is an actual *preferred* inertial frame of references
- This violate the *principle of relativity*

The Illusive Aether

Maxwell's hypothesis: the speed of light c must be relative to a hypothetical subtance called **luminiferous aether** (or just **ether**) that permeates the universe. Ether must have some fantastic properties:

- *All* space is filled with aether
- Massless
- Zero viscosity
- Non-dispersive
- Incompressible
- Continuous at a very small (sub-atomic) scale

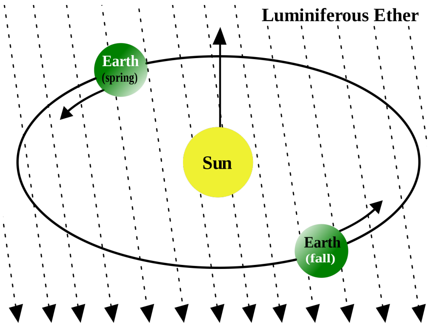
It was thought that the preferred inertial reference frame is that of the ether.

Spoiler Alert

Spoiler alert: Aether doesn't exist.

The Michelson-Morley Experiment

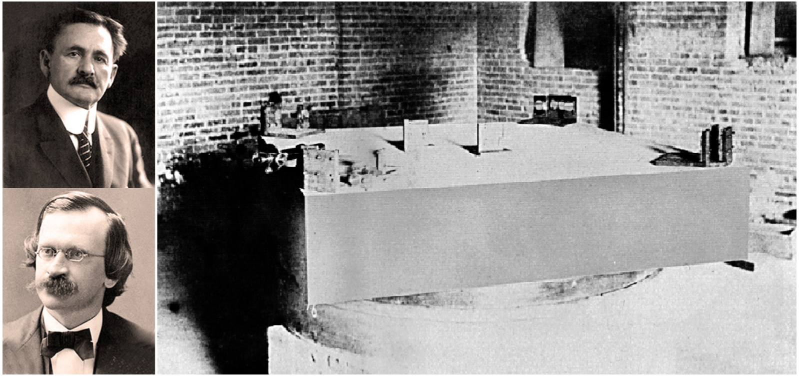
If ether exists, then at different times of the year, the Earth will have a different relative velocity with respect to it:



And it will cause light to either speed up, or slow down. By measuring and comparing the speed of light at various times of the year, we should be able to determine to flow of ether relative to Earth.

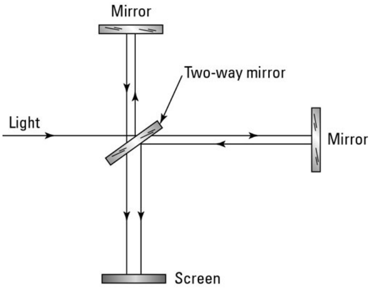
Michelson-Morley Experiment

American phycisists Albert Michelson[2] and Edward Morley designed an ingenious but very difficult experiment to detect ether using an **interferometer** designed by Michelson



[2]Nobel Prize in Physics, 1907

The Michelson Interferometer



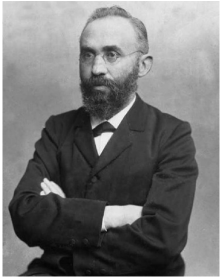
- A beam of light is split into two using a two-way (half-silvered) mirror
- The two beams are reflected off mirrors and finally arriving at the screen where interference patterns are observed
- The two paths are the same length, so if the *speed* of the light changes, we should see an interference pattern
- **Except none were ever found!** The interference patterns that could be observed were well within experimental errors, and far below expected values

What To Do with "Null Result"

The Michelson-Morley experiment failed to detect the flow of ether, even after many refinements. What does this mean?

- Majority view
  - **The experiment was flawed!** It is actually a reasonable guess, since the experiment is known to be a difficult one, errors can be introduced
  - Keep improving the experiment (or design a better experiment) and Earth's motion relative to ether will eventually be found
- Minority view:
  - **The ether hypothesis is wrong!**
  - The experiment showed it for what it is: ether either cannot be detected or it doesn't exist
- A few physicists: The must be **another explanation** that saves both experiment and theory

Hendrik Lorentz[3]



Hendrik Lorentz

- Lorentz considered the findings of Michelson-Morley experiment to be significant
- His hypothesis: objects traveling in the direction of ether must contract in length, nullifying the experimental results

$$\boxed{\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}$$

- *No known physical phenomenon* causes an object to contract

[3]Nobel Prize in Physics, 1902

Strange Behavior in Absolute Space Time

French mathematician Henri Poincaré also hypothesized that ether affects the flow of time the direction of flow. The equation is similar to the hypothesis by Lorentz:

$$\boxed{t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}$$

But *no known physical phenomenon* can alter the flow of time!

Both Poincaré and Lorentz depended their hypothesis on

- Absolute time and space
- Existence of aether

Making Maxwell's Equations Work

Albert Einstein in 1905, Age 26

Albert Einstein

- Einstein was working as a patent clerk in Switzerland while completing his Ph.D.
  - Believed in the principle of relativity, and therefore
  - Rejected the concept of a "preferred" frame of reference
- The failure of the Michelson-Morley experiment to find the flow of ether proves that it does not exist
- In order to make Maxwell's equations to work again, Einstein revisited two most fundamental concepts in physics: *space* and *time*

Special Relativity

Published in *Annalen der Physik* on September 26, 1905 in the article *On the Electrodynamics of Moving Bodies*

- Submitted on June 30, 1905 and passed for publication by a referee
- Einstein's third paper (of four) that year
- Mentions only five other scientists by name: Newton, Maxwell, Hertz, Doppler and Lorentz, but does not contain references to any publications
- Ignored by most physicists at first, until Max Planck took interests
- Called "special relativity" because it describes a "special case" without effects of gravity

Postulates of Special Relativity

The Principle of Relativity

All laws of physics must apply equally in *all* inertial frames of reference.

- Reaffirms the principle in which physics is based on
- Extends the principle to include electrodynamics (Maxwell's equations)

The Principle of Invariant Light Speed

As measured in any inertial frame of reference, light always propagates in empty space with a definite velocity c, independent of the state of motion of the emitting body.

- Reaffirms the results from Michelson-Morley experiment
- Discounts the existances of the hypothetical ether

Postulates of Special Relativity

The two postulates are unremarkable by themselves, but Einstein is able to show that when combined, the consequences are profound

What's so Special About Special Relativity?

Classical (Newtonian) relativity:

- Space and time are absolute (invariant), therefore
- The speed of light must be relative to the observer

Einstein's special relativity:

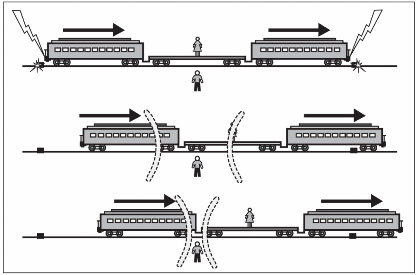
- The speed of light is absolute (invariant), therefore
- Space and time must be relative to the observer

We must modify our traditional concepts:

- Measurement of space (our concept of the coordinate system)
- Measurement of time (our clocks)
- Concept of simultaneity (whether two events happens at the same time)

The Relativity of Simultaneity

This *thought experiment* is similar to the one that Einstein presented. Suppose lightning bolt strikes the two ends of a high-speed moving train. Did it happen simultaneously?



- Two *independent* events: lightning striking the front, and lightning striking the back of the train
- The man on the ground sees the lightning bolt striking at the same time
- The woman on the moving train sees the lightning bolt on the front first

Relativity of Simultaneity

From the man's perspective:

- He is stationary, but the train is moving
- When the lightnings strike, he is at an equal distance from the front and the back of the train
- The flash from the two lightning bolts arrive at his eyes at the same time
- Since the speed of light is a constant regardless of motion

Therefore, his conclusions are:

- The two lightnings must have happened at the same time
- The woman in the train made the wrong observation: she only *thinks* that the lightning struck the front first because she is moving toward the light from the front

Relativity of Simultaneity

From the woman's perspective:

- *She* is stationary, but the man and the rest of the world are moving
- When the lightnings strike, she is at an equal distance from the two ends of the train
- The flash from the front arrive first, then the back
- Since the speed of light is a constant regardless of motion

Therefore, her conclusions are:

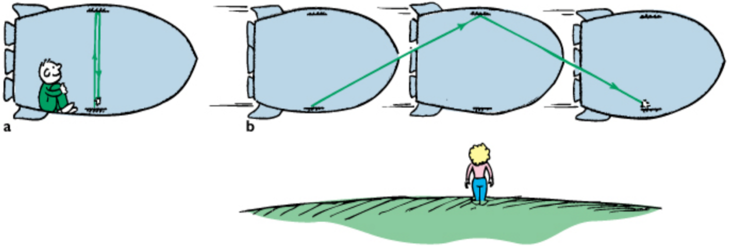
- Lightnings must have struck the front first
- The man on the road made the wrong observation: he only *thinks* that the lightning struck at the same time because he's moving toward the light from the back

Relativity of Simultaneity

- The two observers disagree on the result, but
  - Neither person is wrong
  - Neither person is misinformed
- Both observers are valid *inertial* frames of reference, and therefore both can consider themselves at rest
- This means that *simultaneity depends on your motion*

Relativity of Simultaneity: Events that are simultaneous in one inertial frame of reference are not simultaneous in another.

Relativity of Time

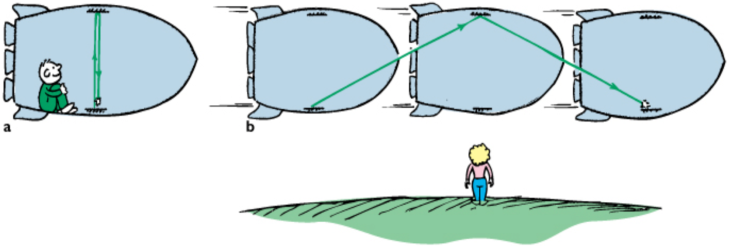


While on a spaceship traveling through space, and I shine a light from A (the bottom of my ship) to B (at the top of my ship). The distance between A and B is:

$$|AB| = ct$$

Knowing the speed of light c, and how long it takes for the light pulse to reach B, I can calculate the dimensions of my spaceship.

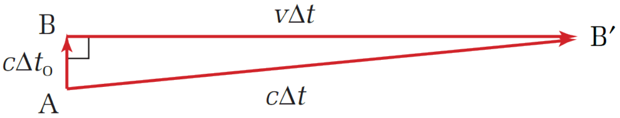
Relativity of Time



Meanwhile, you are on a small planet watching my spaceship go past at speed v. You see that same beam of light travel from A to B' instead.

Relativity of Time

We can relate the time interval of the beam of light observed by me (on the spaceship) and you (on the planet) using pythagorean theorem:



$$c^2 t'^2 = v^2 t'^2 + c^2 t^2$$

$$\left(c^2 - v^2\right) t'^2 = c^2 t^2$$

$$\left(1 - \frac{v^2}{c^2}\right) t'^2 = t^2$$

$$t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

Relativity of Time

The relationship between observer is given by:

$$\boxed{t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}$$

- t is called the **proper time**. It is the time measured by a person at rest relative to the object or event.
- t' is called the **ordinary time** (aka **expanded time** or **dilated time**). It is the time measured by a moving observer in another inertial frame of reference. Since $v < c$, t' is always greater than t.

Example Problem

Example 1a: Kim is riding a rocket that speeds past an asteroid at $v = 0.600c$. If Kim sees $10.0\,\mathrm{s}$ pass on her watch, how long would that time interval be as seen by Jim, an observer on the asteroid?

$$t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{10.0}{\sqrt{1 - 0.600^2}} = 16.7\,\mathrm{s}$$

Example Problem

Example 1a: Kim is riding a rocket that speeds past an asteroid at $v = 0.600c$. If Kim sees $10.0\,\mathrm{s}$ pass on her watch, how long would that time interval be as seen by Jim, an observer on the asteroid?

$$t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{10.0}{\sqrt{1 - 0.600^2}} = 16.7\,\mathrm{s}$$

- Jim observes that in the time it took Kim's clock to run $10.0\,\mathrm{s}$, his watch has already gone $16.7\,\mathrm{s}$, therefore
- Jim concludes that Kim's watch must be running slow

Relativity of Time: A moving clock appears to run slow.

Example Problem

Example 1b: Kim is riding a rocket that speeds past an asteroid at $0.600c$. If Jim, an observer in the *asteroid*, sees $10.0\,\mathrm{s}$ pass on his watch, how long would that time interval be as seen by Kim?

Example Problem

Example 1b: Kim is riding a rocket that speeds past an asteroid at $0.600c$. If Jim, an observer in the *asteroid*, sees $10.0\,\mathrm{s}$ pass on his watch, how long would that time interval be as seen by Kim?

$$t' = \frac{t}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{10.0}{\sqrt{1-0.600^2}} = 16.7\,\mathrm{s}$$

- This problem is exactly the same as the last one!
- Kim observes that in the time it took Jim's clock to run $10.0\,\mathrm{s}$, her watch has already gone $16.7\,\mathrm{s}$, therefore
- Kim concludes that Jim's watch must be running slow

How can that be?

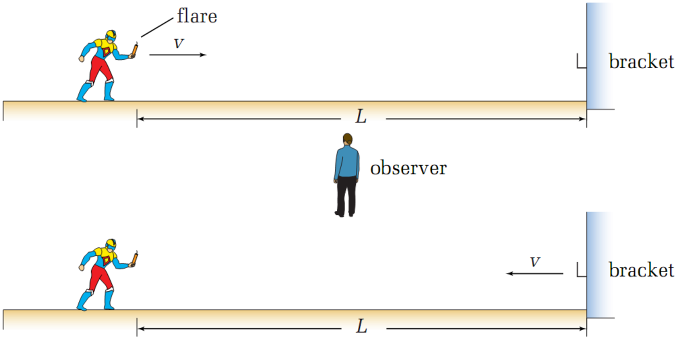
How can the observer in the asteroid sees time in the rocket runs slowly, while the observer in the rocket *also* sees time in the asteroid runs slowly?

Answer: the relativty of simultaneity. The clocks on the asteroid and on the rocket are *not* synchronized.

- In example 1a, when Kim (on the rocket) starts measuring a $10.0\,\mathrm{s}$ time interval, in order for Jim to compare that interval to *his* watch, he has to start and end at the same time (simultaeously!) as Kim.
- But simultaneity is only relative. In Kim's reference frame, Jim never got the timing right!
- This problem reverses itself when Kim tries to synchronize her watch to Jim's $10.0\,\mathrm{s}$ interval.

Relativity of Space

Captain Quick is a comic book hero who can run at nearly the speed of light. In his hand, he is carrying a bomb set to explode in $1.5\,\mu s$. The bomb must be placed into its bracket before this happens. The distance (L) between the flare and the bracket is $402\,\mathrm{m}$.



Relativity of Space

Suppose Captain Quick runs at 2.00×10^8 m/s, according to classical mechanics, he will not make it in time:

$$t = \frac{L}{v} = \frac{402\,\mathrm{m}}{2.00 \times 10^8\,\mathrm{m/s}} = 2.01 \times 10^{-6}\,\mathrm{s} = 2.01\,\mu\mathrm{s}$$

But according to relativistic mechanics, he makes it just in time...

Relativity of Space

To a stationary observer, the time on the flare is slowed:

$$t' = \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{1.5 \times 10^{-6}}{\sqrt{1 - \left(\frac{2.00}{3.00}\right)^2}} = 2.01 \times 10^{-6}\,\text{s}$$

The stationary observer sees a passage of time of $t' = 2.01$ µs, but Captain Quick, who is in the same reference frame as the flare, experiences a passage of time of $t = 1.50$ µs, precisely the time for the flare to explode.

Relativity of Space

If Captain Quick sees only $t = 1.50\,\mu\text{s}$, then how far did he travel?

- Both Captain Quick and the observer on the side of the road agree that he is traveling at $v = 2.00 \times 10^8\,\text{m/s}$
- The only possibility is that *the distance actually got shorter* in Captain Quick's frame of reference, by this amount:

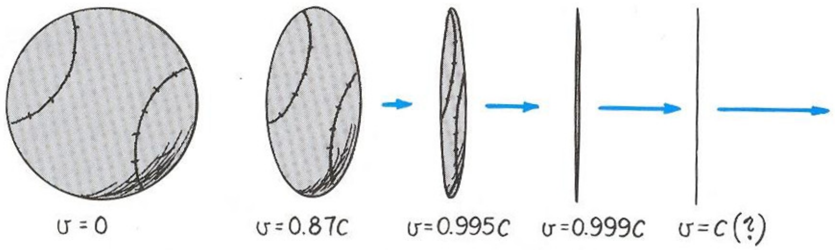
$$\boxed{L' = L\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

For this example:

$$L' = L\sqrt{1 - \left(\frac{v}{c}\right)^2} = 402\sqrt{1 - \left(\frac{2.00}{3.00}\right)^2} = 300\,\text{m}$$

Length Contraction

Length contraction only occurs in the direction of motion



Example Problem

Example 2: A spacecraft passes Earth at a speed of $2.00 \times 10^8\,\mathrm{m/s}$. If observers on Earth measure the length of the spacecraft to be $554\,\mathrm{m}$, how long would it be according to its passengers?

Lorentz Factor

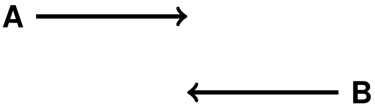
The **Lorentz factor** γ is a short-hand for writing length contraction, time dilation and relativistic mass:

$$\boxed{\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}$$

Then time dilation and length contraction can be written simply as:

$$\boxed{t' = \gamma t} \quad \boxed{L' = \frac{L}{\gamma}}$$

Summary



If observers A and B are moving at constant velocity relative to one another (doesn't matter if they're moving toward, or away from each other)

- They cannot agree whether any events happens at the same time or not
- Each sees the other's clock running slow
- Each sees the other "contracted" in length along the direction of motion

Lorentz Transformation

Time dilation and length contraction only tell part of the story. To account for the loss of simultaneity from one inertial frame to another, we need to use the **Lorentz transformation:**

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

The Lorentz transformation "solves" many paradoxes (e.g. the twin paradox) from the time-dilation and length-contraction equations, but aren't really there.

Lorentz Transformation

For slow speeds $v \ll c$, Lorentz transformation reduces to the Galilean transformation from classical mechanics, from which the velocity addition rule is formulated:

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t'$$

Relative Velocity

Unlike in classical mechanics, velocities (speeds) do not simply add. We have to account for time dilation and length contraction, which are included in the Lorentz transformation

Einstein velocity addition rule:

$$\boxed{\mathbf{v}_{AC} = \frac{\mathbf{v}_{AB} + \mathbf{v}_{BC}}{1 + \frac{\mathbf{v}_{AB} \cdot \mathbf{v}_{BC}}{c^2}}}$$

If $v_{AB} \ll c$ and $v_{BC} \ll c$, we recover Galilean velocity addition rule

Relativistic Momentum

In Grade 12 Physics, you were taught that momentum is mass times velocity. And in Grade 11 Physics, you were taught that velocity is displacement over time. *These definitions have not changed.*

$$\mathbf{p} = m\frac{d\mathbf{x}}{dt}$$

But now that you know $d\mathbf{x}$ and dt are relativistic quantities that depend on motion, we can find a new expression for "relativistic momentum":

$$\mathbf{p} = m\frac{d\mathbf{x}}{dt} = \frac{m\,d\mathbf{x}}{\sqrt{1-\left(\frac{v}{c}\right)^2}\,dt} = \boxed{\frac{m\mathbf{v}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}} = \gamma m\mathbf{v}$$

Relativistic Mass

From the relativistic momentum expression, we see the relativistic aspect to mass as well. The **apparent mass** (or **relativistic mass**) m' as measured by a moving observer is related to its **rest mass** (or **intrinsic mass** or **invariant mass**) m by the Lorentz factor:

$$\boxed{m' = \frac{m}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma m}$$

The intrinsic mass of a moving object does not change, but a moving observer will observe that it behaves as if it is more massive. As $v \to c$, $m' \to \infty$.

Work and Energy

Einstein published a fourth paper in *Annalen der Physik* on November 21, 1905 (received Sept. 27) titled “Does the Inertia of a Body Depend Upon Its Energy Content?” (In German: Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?)

- Einstein deduced the most famous of equations: $E = mc^2$

Work and Energy

In Grade 12 Physics, you were taught that force is the rate of change of momentum with respect to time. *This definition has not changed.*

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

and that work is the integral of the dot product between force and displacement vectors. *This definition has not changed either.*

$$W = \int \mathbf{F} \cdot d\mathbf{x} = \int \frac{d\mathbf{p}}{dt} \cdot \mathbf{dx}$$

Since we now have a relativistic expression for momentum, we substitute that new expression into the expression for force, and then integrate.

Work and Energy

For 1D motion (for simplicity), we can rearrange the terms in the integral:

$$W = \int F dx = \int \frac{dp}{dt} dx = \int v dp$$

Assuming that both v and p are continuous in time, we can apply the chain rule to find the infinitesimal change in momentum (dp) with respect to γ and v:

$$p = \gamma m v \quad \rightarrow \quad dp = \gamma dv + v d\gamma$$

Substituting that back into the integral, we have:

$$W = \int v dp = \int m v (\gamma dv + v d\gamma) = \int m \left(\gamma v dv + v^2 d\gamma \right)$$

Work and Energy

One of the integral is with respect to γ, so we express v and dv in terms of γ using its definition:

$$v^2 = c^2\left[1 - \left(\frac{1}{\gamma}\right)^2\right] \quad \rightarrow \quad dv = \frac{c^2}{\gamma^3 v}d\gamma$$

Work and Energy

Putting everything together, we have

$$W = \int m(\gamma v dv + v^2 d\gamma) = \int m \left[\frac{c^2}{\gamma^2} + c^2\left(1 - \frac{1}{\gamma^2}\right)\right] d\gamma$$

This is a surprisingly simple integral:

$$W = \int_1^\gamma mc^2 d\gamma$$

The limit of the integral is from 1 because at $v = 0$, $\gamma = 1$

Work and Kinetic Energy

The integral gives us this expression:

$$W = \gamma mc^2 - mc^2$$

We know from the work-kinetic energy theorem that the work W done is equal to the change in kinetic energy K, therefore

$$\boxed{K = m'c^2 - mc^2}$$

Variable	Symbol	SI Unit
Kinetic energy of an object	K	J
Apparent mass (measured in moving frame)	m'	kg
Rest mass (measured in stationary frame)	m	kg
Speed of light	c	m/s

Relativistic Energy

What This All Means

$$\boxed{K = m'c^2 - mc^2}$$

The minimal energy that any object has, regardless of it's motion (or lack of) is its **rest energy**:

$$E_0 = mc^2$$

The **total energy** of an object has is

$$E_T = m'c^2 = \gamma mc^2$$

The difference between total energy and rest energy is the kinetic energy:

$$K = E_T - E_0$$

Relativistic Energy

What This All Means

$$\boxed{E = mc^2}$$

Mass-energy equivalence:

- Whenever there is a change of energy, there is also a change of mass
- "Conservation of mass" and "conservation of energy" must be combined into a single concept of **conservation of mass-energy**
- Mass-energy equivalence doesn't merely mean that mass can be converted into energy, and vice versa (although this is true), but rather, one can be converted into the other *because they are fundamentally the same thing*

Example Problem

Example 3: An electron has a rest mass of 9.11×10^{-31} kg. In a detector, it behaves as if it has a mass of 12.55×10^{-31} kg. How fast is that electron moving relative to the detector?

Energy-Momentum Relation

The **energy-momentum relation** relates an object's rest (intrinsic) mass m, total energy E, and momentum p:

$$\boxed{E^2 = p^2c^2 + m^2c^4}$$

Quantity	Symbol	SI Unit
Total energy	E	J
Momentum	p	kg m / s
Rest mass	m	kg
Speed of light	c	m / s

Energy-Momentum Relation

This equation is derived using the expression for relativistic momentum:

$$p = \gamma m v = \frac{mv}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

If we square both sides of the equation, we get:

$$p^2 = \gamma^2 m^2 v^2 = \frac{m^2 v^2}{1 - \left(\frac{v}{c}\right)^2}$$

Energy-Momentum Relation

Solving for v^2 and substituting it back into the Lorentz factor, we obtain its alternative form in terms of momentum and mass:

$$\gamma = \sqrt{1 + \left(\frac{p}{mc}\right)^2}$$

Inserting this form of the Lorentz factor into the energy equation, we have

$$E = mc^2\sqrt{1 + \left(\frac{p}{mc}\right)^2}$$

Which is the same equation as in the last slide.

Energy-Momentum Relation

In the **stationary frame of reference**, (rest frame, center-of-momentum frame) the momentum is zero, so the equation simplifies to

$$E = mc^2$$

where m is the rest mass of the object.

If the object is **massless**, as is the case for a **photon**, then the equation reduces to

$$E = pc$$

Kinetic Energy–Classical vs. Relativistic

Relativistic:

$$K = \frac{mc^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} - mc^2$$

Newtonian:

$$K = \frac{1}{2}mv^2$$

But are they really that different?

- If space and time are indeed relative quantities, then the relativistic equation for K must apply to all velocities
- But we know that when $v \ll c$, the Newtonian expression works perfectly
- i.e. The Newtonian expression for K must be a very good approximation for the relativistic expression for K for $v \ll c$

Binomial Series Expansion

The **binomial series** is the Maclaurin series for the function $f(x) = (1+x)^{\alpha}$, given by:

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots$$

In the case of relativistic kinetic energy, we use:

$$x = -\left(\frac{v}{c}\right)^2 \quad \text{and} \quad \alpha = -\frac{1}{2}$$

Binomial Series Expansion

Substituting these terms into the equation:

$$K = mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right) - mc^2$$
$$\approx \frac{1}{2}mv^2 + \frac{3}{4}m\frac{v^4}{c^2} + \cdots$$

For $v \ll c$, we can ignore the high-order terms. The leading term reduces to the Newtonian expression

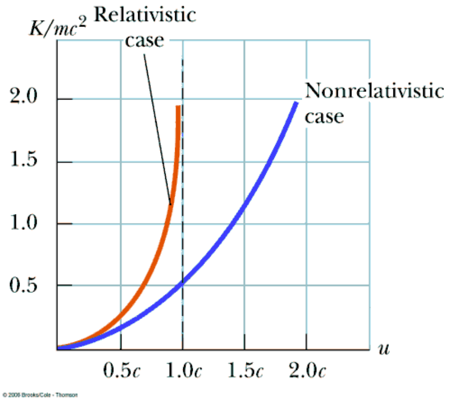
Comparing Classical and Relativistic Energy

In classical mechanics:

$$K = \frac{1}{2}mv^2$$

In relativistic mechanics:

$$K = \gamma mc^2 - mc^2$$



The classical expression is accurate for speeds up to $v \approx 0.3c$.

Example Problem

Example 4: A rocket car with a mass of $2.00 \times 10^3\,\mathrm{kg}$ is accelerated from rest to $1.00 \times 10^8\,\mathrm{m/s}$. Calculate its kinetic energy:

1. Using the classical equation
2. Using the relativistic equation