# Topic 7: Universal Gravitation

## Advanced Placement Physics

Timothy Leung, Ph.D.

Olympiads School

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# Files to Download

## Please download/print the PDF file

If you have not done so already, please download the following files.

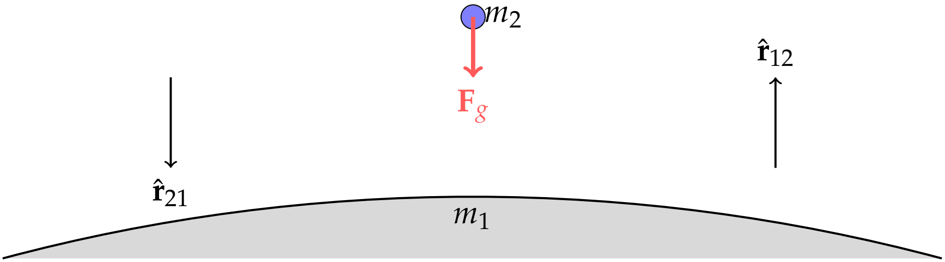
- `08-gravitation.pdf`—This week's slides. I recommend printing 4 slides per page.
- `08-Homework.pdf`—This week's homework. (The file is not ready yet; it will be posted in the next day or two.)

# Universal Gravitation

This topics we will discuss in this class are covered in AP Physics 1 and C exams.

- Gravitational force ($\mathbf{F}_g$)
- Gravitational field ($\mathbf{g}$)
- Gravitational potential energy ($U_g$)
- Kepler's laws of planetary motion

# Newton's Law of Universal Gravitation




$$\boxed{\mathbf{F}_g = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}_{12} = +\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}_{21}}$$

where $\boxed{G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}}$ is the universal gravitation constant, and $\hat{\mathbf{r}}_{12}$ and $\hat{\mathbf{r}}_{21}$ are *unit vectors* (length=1)

# Universal Gravitation

- If $m_1$ exerts a gravitational force $\mathbf{F}_g$ on $m_2$, then $m_2$ also exerts $-\mathbf{F}_g$ on $m_1$.
- The two forces are equal in magnitude and opposite in direction (Newton's 3rd law)
- Assumption: $m_1$ and $m_2$ are *point masses* that do not occupy any space.
- Recall: forces acting on an ensemble of masses is the same as acting at its center of mass
- Therefore, for the universal gravitational equation to work:

$$r > (r_1 + r_2)$$

That is, the two objects hasn't collided into one another

# Think Gravitational Field: What is $g$?

We generally describe the force of gravity as

$$\mathbf{F}_g = m\mathbf{g}$$

To find the magnitude of $g$, we group the variables in Newton's universal gravitation equation:

$$F_g = \underbrace{\left[\frac{Gm_1}{r^2}\right]}_{=g} m_2 = m_2 g$$

On the surface of Earth, we use use $m_1 = m_{\mathrm{Earth}}$ and $r = r_{\mathrm{Earth}}$ to compute $g \approx 10\,\mathrm{m/s^2}$, or $g \approx 10\,\mathrm{N/kg}$ (both units are equivalent)

# Gravitational Field

The **gravitational field g** generated by a source mass $m_s$ is a mapping of how $m_s$ influences the gravitational forces on other masses. It's intensity is defined by:

$$\boxed{g(m_s, r) = \frac{Gm_s}{r^2}}$$

| Quantity | Symbol | SI Unit |
|---|---|---|
| Gravitational field intensity | $g$ | N / kg (newtons per kilogram) |
| Universal gravitational constant | $G$ | N $\mathrm{m}^2$ / $\mathrm{kg}^2$ |
| Mass of source (a point mass) | $m_s$ | kg (kilograms) |
| Distance from center of source | $r$ | m (meters) |

# Relating Gravitational Field & Gravitational Force

$\mathbf{g}$ itself doesn't do anything until there is another mass $m$. At which point, $m$ experiences a gravitational force related to $\mathbf{g}$ by:

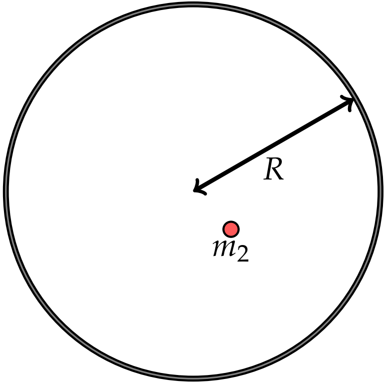
$$\boxed{\mathbf{F} = m\mathbf{g}}$$

$\mathbf{F}_g$ and $\mathbf{g}$ are vectors in the same direction: toward the center of the source mass that created the field, therefore all vector operations apply

| Quantity | Symbol | SI Unit |
|---|---|---|
| Gravitational field | $\mathbf{g}$ | N / kg |
| Gravitational force on a mass | $\mathbf{F}_g$ | N |
| Mass inside the gravitational field | $m$ | kg |

# What If You Are Inside Another Mass?

## Case 1: A Spherical Shell of Radius $R$



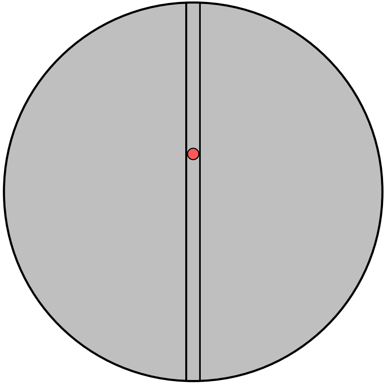

- If a mass $m_2$ is *inside* a spherical shell of mass $m_1$, the force of gravity it experiences is **zero**!

$$\boxed{\mathbf{F}_g = \begin{cases} \mathbf{0} & \text{if } \; r < R \\ Gm_1 m_2 / r^2 \hat{\mathbf{r}} & \text{otherwise} \end{cases}}$$

- It also means that gravitational field is also **zero**
- This is very similar to a charged conducting sphere, where the electric field inside is zero.

# What If You Are Inside Another Mass?

## Case 2: Uniform Mass of Radius $R$



Suppose you could drill a hole through the Earth and then drop into it. How long would it take you to pop up on the other side of the Earth?

# Falling To the Center of the Earth

Your initial gravitational force on the surface is:

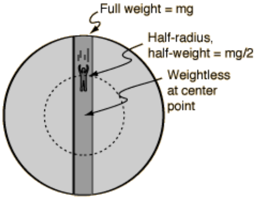
$$F_g = mg_{\text{surface}} \qquad g_{\text{surface}} \approx 10\,\text{N}/\text{kg}$$

$F_g$ becomes smaller as you approached the center, and when you reach the center, $F_g = 0$.

Let's assume that the Earth is uniform density. We will also neglect air friction and other factors. Let's look at the value of $g$ as the person falls through Earth ($r < R$):

$$g(r) = \frac{GM(r)}{r^2} \quad M(r) = \frac{4}{3}\rho\pi r^3 \quad \rho = \frac{3M_{\text{Earth}}}{4\pi R^3}$$

# Falling To the Center of the Earth



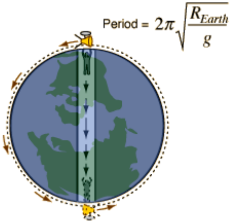

Now we have an expression for the gravitational field strength inside "Earth":

$$g(r) = \frac{GM_{\text{Earth}}r}{R^3} = g_{\text{surface}}\frac{r}{R}$$

Gravitational field strength decreases linearly with $r$. At the center ($r = 0$), $g = 0$. Here's the interesting part, the gravitational force inside Earth is

$$F_g = -mg(r) = -\underbrace{\left[\frac{mg_{\text{surface}}}{R}\right]}_{\text{constant}} r = -kr$$

# Falling To the Center of the Earth




A satellite at the Earth's radius would have the same period as one falling through the Earth.

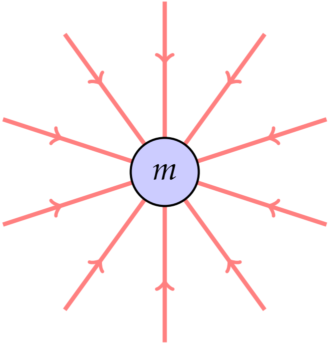
So this poor guy is going to oscillate through Earth with a harmonic motion with a period of:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

For Earth, is $T = 5068\,\mathrm{s}$. He would pop up on the opposite side after about $42\,\mathrm{min}$.

Suppose a satellite is in a circular orbit just above the surface, and passes overhead just above the traveler as he popped up out of the hole. The period of such an orbit would be the same as oscillating traveler.

# Gravitational Field Lines




- The direction of $\mathbf{g}$ is toward the center of the object that created it
- Field lines do not tell the intensity (i.e. magnitude) of $\mathbf{g}$, only the direction

# Gravitational Potential Energy

The gravitational potential energy is defined as:

$$\boxed{U_g = -\frac{Gm_1m_2}{r}}$$

obtained by integrating $\mathbf{F}_g$ by a distance $r$ to find the work done

- $U_g$ is the work required to move two objects from $r$ to $\infty$
- $U_g = 0$ at $r = \infty$ and *decrease* as $r$ decreases

# Relating Gravitational Potential Energy to Force

Vector calculus shows that gravitational force ($\mathbf{F}_g$) is the negative gradient of the gravitational potential energy ($U_g$):

$$\mathbf{F}_g = -\nabla U_g = -\frac{\partial U_g}{\partial r}\hat{\mathbf{r}}$$

The direction of $\mathbf{F}_g$ always points from high to low potential

- A falling object is always decreasing in $U_g$
- "Steepest descent": the direction of $\mathbf{F}$ is the shortest path to decrease $U_g$
- Objects traveling perpendicular to $\mathbf{F}$ has constant $U_g$

# Relating $U_g$, $\mathbf{F}_g$ and $\mathbf{g}$

Knowing that $\mathbf{F}_g$ and $\mathbf{g}$ only differ by a constant, we can also relate gravitational field to $U_g$ by the gradient operator:

$$\mathbf{g} = \frac{\mathbf{F}_g}{m} = -\nabla\left(\frac{U_g}{m}\right) = -\frac{\partial}{\partial r}\left(\frac{U_g}{m}\right)\hat{\mathbf{r}}$$

We already know that the direction of $\mathbf{g}$ is the same as $\mathbf{F}_g$, i.e.

- The direction of $\mathbf{g}$ is the shortest path to decrease $U_g$
- Objects traveling perpendicular to $\mathbf{g}$ has constant $U_g$

# Orbital Mechanics & Kepler's Laws of Planetary Motion

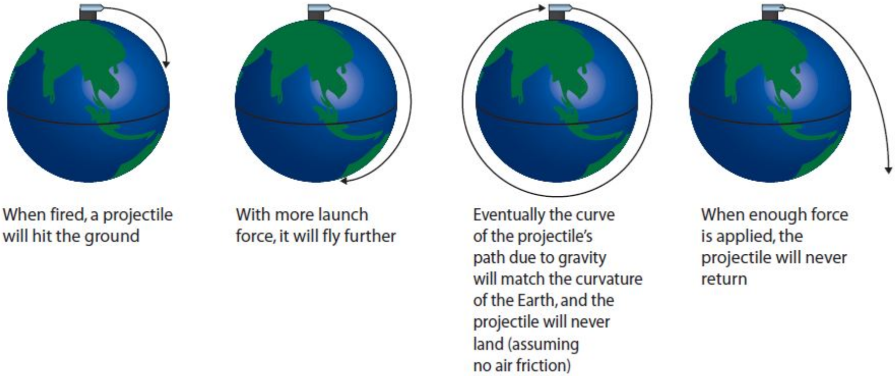
In Physics 12, you studied briefly at the orbits of satellites around Earth, or the orbit of Earth and other planets around the Sun.

- Used your understanding of centripetal motion and gravity
- Assumed a circular orbit

Let's review some of those ideas.

# Orbital Speed

## Newton's Thought Experiment




So how fast is fast enough?

# Orbital Speed

## Relating Gravitational and Centripetal Force

Assuming a circular orbit, where the centripetal force is supplied by the gravitational force:

$$\underbrace{\frac{GMm}{r^2}}_{\mathrm{F_g}} = \underbrace{\frac{mv^2}{r}}_{\mathrm{F_c}}$$

Solving for $v$, we get the **orbital speed** (aka orbital velocity), which does not depend on the mass of the object in orbit:

$$\boxed{v_{\mathrm{orbit}} = \sqrt{\frac{GM}{r}}}$$

# Escape Speed

An object can leave the surface of Earth at any speed. But when all the kinetic energy of that object is converted to gravitational potential, it'll return back to the surface of the earth. There is, however, a *minimum* velocity at which the object *would not* fall back to Earth because of gravity.

# Escape Speed

- The gravitational potential energy of an object with mass $m$ on the surface of a planet (with mass $M$ and radius $r$) is

$$U_g = -\frac{GMm}{r}$$

- The most amount of work that you can do is to bring it to the other side of the universe $r = \infty$, where $U_g = 0$.
- If you start with *more* kinetic energy than required to do all the work, then the object has *escaped* the gravitational pull of the planet.

# Escape Speed from Circular Orbits

Set $K$ to equal to $-U_g$:

$$\frac{1}{2}mv^2 = \frac{GMm}{r}$$

We can then solve for escape speed $v = v_{\text{esc}}$:

$$\boxed{v_{\text{esc}} = \sqrt{\frac{2GM}{r}}}$$

There is a simple relationship between orbital speed and escape speed:

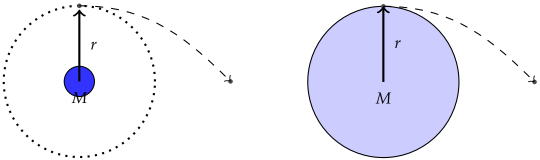
$$\boxed{v_{\text{esc}} = \sqrt{2}v_{\text{orbit}}}$$

# Example Problem

**Example:** Determine the escape velocity and energy for a $1.60 \times 10^4$ kg rocket leaving the surface of Earth.

# What if I'm not escaping from the surface?

These two objects have the same escape velocity:




Both objects are escaping from distance $r$ from the centre of the planet. The difference is that the object in orbit (left) already has orbital speed $v_{\text{orbit}}$, so escaping from that orbit requires an additional speed of

$$\Delta v = v_{\text{esc}} - v_{\text{orbit}} = (\sqrt{2} - 1)v_{\text{orbit}}$$

# Orbital Energies

We can obtain the **orbital kinetic energy** by using the orbital speed in our expression of kinetic energy:

$$K_{\text{orbit}} = \frac{1}{2}mv_{\text{orbit}}^2 = \frac{1}{2}m\left(\sqrt{\frac{GM}{r}}\right)^2 = \boxed{\frac{GMm}{2r}}$$

We already have an expression for **gravitational potential energy**:

$$U_g = -\frac{GMm}{r} = -2K_{\text{orbit}}$$

Therefore the **total orbital energy** is the sum of $K$ and $U_g$:

$$E_T = K + U_g = -\frac{GMm}{2r} = -K_{\text{orbit}}$$

# Houston, We Have a Problem!

As always, this understanding isn't completely correct!

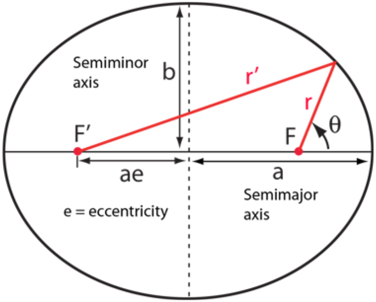
- Centripetal motion is based on rotation around a fixed point, but this is *not* the case for celestial mechanics!
- Just as Earth experiences a gravitational force by the Sun, the Sun also experiences a gravitational force from Earth
- This problem is especially important when the two objects orbiting each other has similar masses (e.g. a binary star system)

# Kepler's Laws of Planetary Motion

1. **Law of Ellipses:** The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. **Law of Equal Areas:** A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time
3. **Law of Periods:** The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

# Kepler's Law of Planetary Motion

To fully understand Kepler's laws, we have to first understand the ellipse, at least a little bit.




$$r' + r = 2a$$

- The area of the ellipse is $A = \pi ab$
- For an ellipse, the relationship between $r$ and $\theta$ given by:

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \quad \text{where} \quad 0 \leq e < 1$$

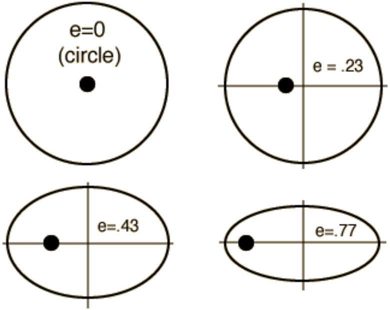
- when $e = 0$ it's a circle: $r = a$
- When $e = 1$ it's no longer an ellipse

# Kepler's Law of Planetary Motion

Most of the planets have very small eccentricity, so their orbits are fairly close to being circular, but comets are much more eccentric

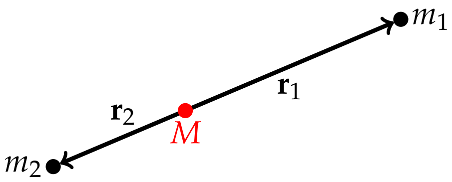
| **Object** | $e$ |
|---|---|
| Mercury | 0.206 |
| Venus | 0.0068 |
| Earth | 0.0167 |
| Mars | 0.0934 |
| Jupiter | 0.0485 |
| Saturn | 0.0556 |
| Uranus | 0.0472 |
| Neptune | 0.0086 |
| Pluto | 0.25 |
| Halley's Comet | 0.9671 |
| Comet Hale-Bopp | 0.9951 |
| Comet Ikeya-Seki | 0.9999 |

# Kepler's Law of Planetary Motion



# Reduced Mass

Consider two objects orbiting each other. We know that the center of mass of the system is along the line between the two objects:




The vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are relative to the center of mass, and the relative position $\mathbf{r}$ velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ between $m_1$ and $m_2$ are therefore:

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \quad \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 \quad \mathbf{a} = \mathbf{a}_2 - \mathbf{a}_1$$

# Reduced Mass

From Newton's third law, we know that the gravitational force exerted by $m_1$ on $m_2$ is opposite the force exerted by $m_2$ on $m_1$:

$$m_1\mathbf{a}_1 + m_2\mathbf{a}_2 = 0$$

Substituting the expression for relative acceleration, we have:

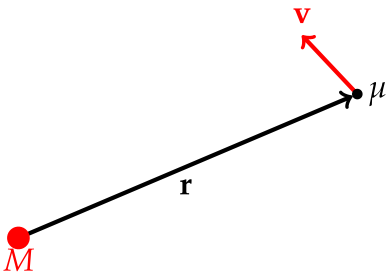
$$\mathbf{a} = \mathbf{F}_g\left[\frac{1}{m_1} + \frac{1}{m_2}\right] = \mathbf{F}_g\left[\frac{m_1+m_2}{m_1 m_2}\right] = \frac{\mathbf{F}_g}{\mu}$$

We can now define a new concept called *reduced mass*:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

# An Equivalent System

The orbit of one of the masses in a binary system is equivalent to the motion of the reduced mass orbiting around a point at relative distance $r$ where the total mass $M$ is placed. The magnitude of $r$ is the same as the relative distance $r$ in the development above.




Both systems have the same total angular momentum.

# Gravity is a Central Force

Gravity is called a **central force** in that

- Gravitational force $\mathbf{F}_g$ is always in the $-\hat{\mathbf{r}}$ direction, i.e. $\mathbf{F} \times \mathbf{r} = \mathbf{0}$
- Therefore gravity doesn't add torque
- And therefore angular momentum $\mathbf{L}$ is constant
- True regardless of whether the orbit is circular or elliptical

# Now The Complicated Math

To show that the orbit is an ellipse (or it can be an ellipse), we need to show that the relationship between $r$ and $\theta$ is the one that we described earlier:

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

The following steps will be shown *only once*, and it's not important to be able to follow/remember everything here.

# Complicated Calculations

The acceleration of the reduced mass:

$$\mathbf{a} = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

To find the expression for $r$, we do a cross product of $\mathbf{a}$ with $\mathbf{L}$:

$$\mathbf{a} \times \mathbf{L} = -\frac{GM}{r^2}\hat{\mathbf{r}} \times \left(\mu r^2 \hat{\mathbf{r}} \times \frac{d\hat{\mathbf{r}}}{dt}\right) = GM\mu\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \frac{d\hat{\mathbf{r}}}{dt}\right)$$

We then apply the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$ and the expression magically reduces to:

$$\mathbf{a} \times \mathbf{L} = GM\mu\frac{d\hat{\mathbf{r}}}{dt}$$

# Confused Yet?

Now we move on to another expression that is closer to where we need to be:

$$\frac{d}{dt}(\mathbf{v} \times \mathbf{L}) = \frac{d\mathbf{v}}{dt} \times \mathbf{L} + \mathbf{v} \times \cancel{\frac{d\mathbf{L}}{dt}} = \mathbf{a} \times \mathbf{L} = \frac{d}{dt}(GM\mu\hat{\mathbf{r}})$$

Integrating both sides with respect to $t$ and we get:

$$\mathbf{v} \times \mathbf{L} = GM\mu\hat{\mathbf{r}} + \mathbf{D}$$

where $\mathbf{D}$ is some constant of integration. Now we take the dot product with $\mathbf{r}$

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{L}) = GM\mu r + \mathbf{r} \cdot \mathbf{D}$$

# Just One More Slide!

We once again apply some vector identity: $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ and we get:

$$(\mathbf{r} \times \mathbf{v}) \cdot \mathbf{L} = GM\mu r + rD\cos\theta$$

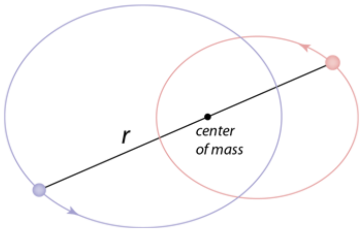
The final step is to recognize that if we define $e = D/GM\mu$, then we can express $r$ as:

$$r = \frac{L^2/\mu^2}{GM(1 + e\cos\theta)}$$

This is an expression for an ellipse!

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \quad \text{if} \quad L = \mu\sqrt{GMa(1 - e^2)}$$

# Kepler's First Law: Law of Orbits

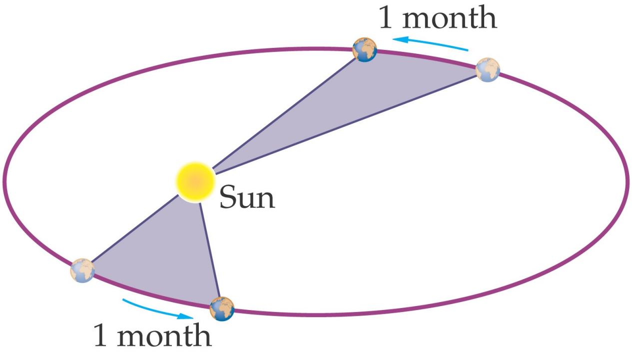



For a bound binary orbit, each object will follow an elliptical orbit about the center of mass of the system. The center of mass will be at one focus of each ellipse

- This is of course *slightly* different from Kepler's conclusion!
- But if one of the mass is very large compared to the other one, i.e. $m_1 \gg m_2$, then the center of gravity is essentially at $m_1$ and $m_2$ is orbiting around $m_1$ in an ellipse

# Kepler's 2nd Law: The Law of Equal Areas

**Second Law: A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time**



# Kepler's 2nd Law: The Law of Equal Areas

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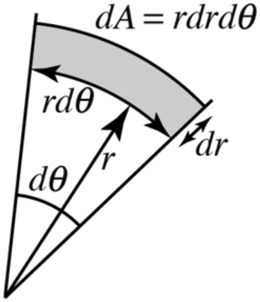


- The infinitesimal area swept out by an object in orbit is given by

$$dA = \int_0^r r dr d\theta = \frac{1}{2} r^2 d\theta$$

- Take the time derivative of both sides, and we have an expression for *areal velocity* (yes, that's a real word in celestial mechanics):

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \omega$$

# Kepler's 2nd Law: The Law of Equal Areas

But hurray! We know that the angular component of velocity ($\mathbf{v}_\theta$) is related to the angular velocity $\omega$, so:

$$v_\theta = r\omega \quad \longrightarrow \quad \frac{dA}{dt} = \frac{1}{2}r(r\omega) = \frac{1}{2}rv_\theta$$

Hurray again! Because we know that

$$rv_\theta = |\mathbf{r} \times \mathbf{v}_\theta| = \left|\frac{\mathbf{L}}{\mu}\right| = \frac{L}{\mu} \quad \longrightarrow \quad \boxed{\frac{dA}{dt} = \frac{L}{2\mu}}$$

Since we know that angular momentum is conserved (i.e. $L$ is constant), areal velocity is a constant, therefore it sweeps out equal areas in equal intervals of time

# Kepler's Third Law: The Law of Periods

**Law of Periods: The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.**

# Kepler's Third Law: The Law of Periods

The area swept by the planet through one orbital period is the areal velocity (constant!) integrated by time, from $t = 0$ to $t = T$:

$$A = \int dA = \int_0^T \frac{dA}{dt} dt = \frac{L}{2\mu} \int_0^T dt = \frac{L}{2\mu} T$$

This area is an ellipse, given by the equation:

$$A = \pi ab = \pi a^2 \sqrt{1 - e^2}$$

Combining the two equations above give this expression:

$$T^2 = \frac{\mu^2}{L^2} 4\pi^2 a^4 (1 - e^2)$$

There is still the variables $L$ and $e$ to eliminate

# Kepler's Third Law: The Law of Periods

After some algebra, we end up with this equation

$$\boxed{T^2 = \underbrace{\left[\frac{4\pi^2}{GM}\right]}_{\text{constant}} a^3}$$

Remember that $M$ is the total mass of the two objects. For the case where one object (e.g. sun) is much more massive than the other (e.g. Earth), i.e. $m_1 \gg m_2$, we can just use $M \approx m_1$

# Kepler's Third Law: The Law of Periods