Topic 5: Circular Motion

Advanced Placement Physics

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Olympiads School

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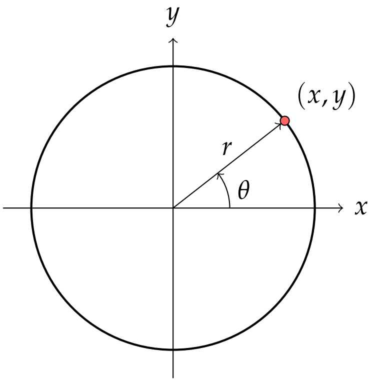
Polar Coordinate System in 2D

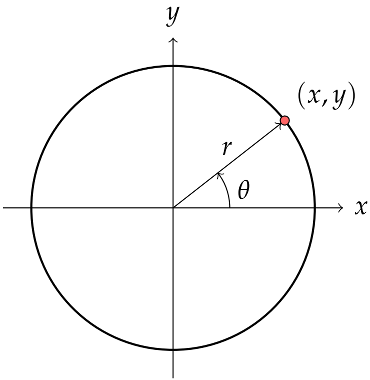


- Cartesian system $\mathbf{x}(x, y)$ is not the only option!
- For circular motion, **polar coordinates** are better
- Position described by $\mathbf{r}(r, \theta)$
    - r is distance from the origin
    - θ standard angle
- Cartesian coordinates are related to the polar coordinates by:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Rigid Body Motion

Angular Position and Angular Velocity



- For constant r, **angular position** θ determines an object's position as a function of time:

$$\boxed{\theta = \theta(t)}$$

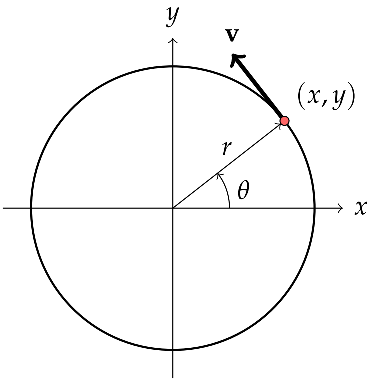
- **Angular velocity** ω (or **angular frequency**) is its time derivative

$$\boxed{\omega = \frac{d\theta(t)}{dt}}$$

- θ is measured in radians, and ω in rad/s
- e.g. an object rotating at one revolution per second has an angular velocity of $\omega = 2\pi$ /s

Rigid Body Motion

Angular Velocity



- Speed v and ω are related simply by:

$$\boxed{v = r\omega}$$

- Or in vector form:

$$\boxed{\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}}$$

- $\mathbf{v}$ is always tangent to circle (perpendicular to $\mathbf{r}$)
- We can also relate ω to **frequency** and **period** of the rotation:

$$\boxed{f = \frac{\omega}{2\pi}} \quad \boxed{T = \frac{2\pi}{\omega}} \quad \boxed{f = \frac{1}{T}}$$

T is in seconds (s) and f is in hertz (Hz)

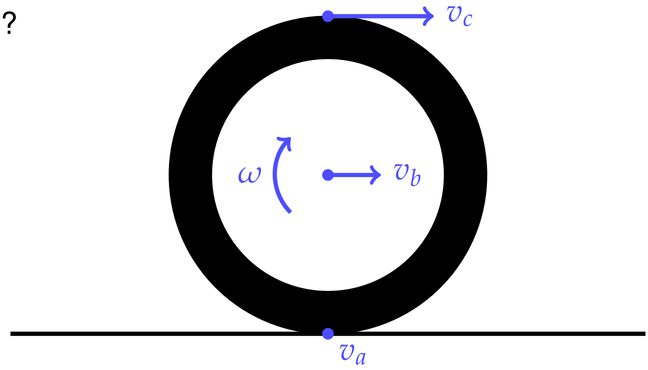
Rotating Object Without Slipping

A tire with radius r rolls along the road with an angular velocity ω *without slipping*. (This is a very common case for analysis.) What is its velocity v

a. at the contact between the ground and the tire?

b. at the center?

c. at the top of the tire?



Rigid Body Motion

Angular Acceleration

- The derivative of ω with respect to time gives us **angular acceleration**:

$$\boxed{\alpha = \frac{d\omega(t)}{dt} = \frac{d^2\theta(t)}{dt^2}}$$

  α has the unit of $\mathrm{rad}/\sec^2$.
- For *uniform* circular motion, ω is constant, and $\alpha = 0$
- Not surprisingly, **tangential acceleration** is related to angular acceleration by the radius r

$$\boxed{a_\theta = \frac{dv}{dt} = r\frac{d\omega(t)}{dt} = r\alpha}$$

Kinematics in the Angular Direction

These Should Look Familiar

For constant α, the kinematic equations are just like in linear motion:

$$\Delta\theta = \omega_1 t + \frac{1}{2}\alpha t^2$$

$$\Delta\theta = \omega_2 t - \frac{1}{2}\alpha t^2$$

$$\Delta\theta = \frac{\omega_1 + \omega_2}{2} t$$

$$\omega_2 = \omega_1 + \alpha t$$

$$\omega_2^2 = \omega_1^2 + 2\alpha\Delta\theta$$

Of course, if α is *not* constant, we will have to integrate

A Simple Example

Example 1: An object moves in a circle with angular acceleration $3.0\,\mathrm{rad/s^2}$. The radius is $2.0\,\mathrm{m}$ and it starts from rest. How long does it take for this object to finish a circle?

Nothing is Ever *That* Simple

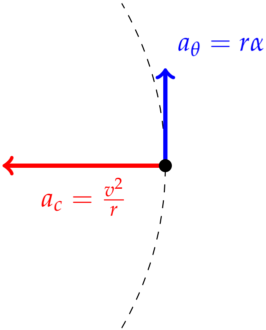
Remember, even when $\alpha = 0$, we still have a centripetal acceleration a_r for any circular motion

$$\boxed{a_r = \frac{v^2}{r} = \omega^2 r = \frac{4\pi^2 r}{T^2} = 4\pi^2 r f^2}$$

And with the centripetal acceleration, there is also a **centripetal force**

$$\boxed{F_r = m a_r = \frac{m v^2}{r}}$$

Acceleration: The General Case



- In general circular motion, there are two components of acceleration:
  - **Centripetal acceleration** a_c depends on radius of curvature r and instantaneous speed v
  - **Tangential acceleration** a_θ depends on radius r and angular acceleration α
- Most of the cases in AP Physics are uniform circular motion

How to Solve Circular Motion Problems

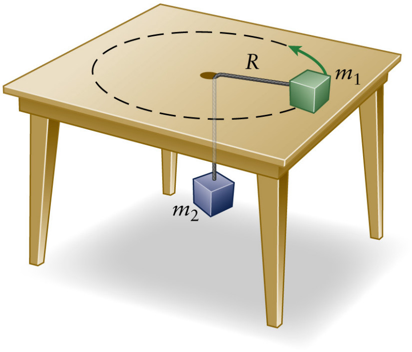
A two-step process:

1. Is there any circular motion?
2. If so, the condition for circular motion is:

$$\mathbf{F}_{\text{provided}} = \mathbf{F}_{\text{required}}$$

- The *provided* force comes from FBD
- The *required* force comes from the centripetal equation we have

Example: Horizontal Motion



Example 2: In the figure on the left, a mass $m_1 = 3.0\,\text{kg}$ is rolling around a frictionless table with radius $R = 1.0\,\text{m}$. with a speed of $2.0\,\text{m/s}$. What is the mass of the weight m_2?

Another Example: Exit Ramp

Example 3: A car exits a highway on a ramp that is banked at 15° to the horizontal. The exit ramp has a radius of curvature of $65\,\mathrm{m}$. If the conditions are extremely icy and the driver cannot depend on any friction to help make the turn, at what speed should the driver travel so that the car will not skid off the ramp? What if there is friction?

Vertical Circles

- Uniform circular motion with a horizontal path is straightforward
- For vertical motion:
  - Generally not solvable by dynamics
  - We can use conservation of energy to solve for $\mathbf{v}$
  - Then use the equation for centripetal force to find other forces
- **Remember:** If it is impossible to get the required centripetal force, then it could not continue the circular motion

Example

Example 4: A cord is tied to a pail of water, and the pail is swung in a vertical circle of $1.0\,\mathrm{m}$. What must be the minimum velocity of the pail be at its highest point so that no water spills out?

(a) $3.1\,\mathrm{m/s}$
(b) $5.6\,\mathrm{m/s}$
(c) $20.7\,\mathrm{m/s}$
(d) $100.5\,\mathrm{m/s}$

Example: Roller Coaster

Example 5: A roller coaster car is on a track that forms a circular loop, of radius R, in the vertical plane. If the car is to maintain contact with the track at the top of the loop (generally considered to be a good thing), what is the minimum speed that the car must have at the bottom of the loop. Ignore air resistance and rolling friction.

(a) $\sqrt{2gR}$

(b) $\sqrt{3gR}$

(c) $\sqrt{4gR}$

(d) $\sqrt{5gR}$

Example

Example 6: A stone of mass m is attached to a light strong string and whirled in a *vertical* circle of radius r. At the exact bottom of the path, the tension of the string is three times the weight of the stone. The stone's speed at that point is given by:

(a) $2\sqrt{gR}$

(b) $\sqrt{2gR}$

(c) $\sqrt{3gR}$

(d) $4gR$

Torque and Rotational Equilibrium

Let's consider this question:

> Two people stand on a board of uniform density. One person has a mass of $50\,\mathrm{kg}$ and stands $10\,\mathrm{m}$ away from the fulcrum (pivot). The second person has a mass of $65\,\mathrm{kg}$. How far away from the fulcrum would the second person have to stand for the system to have to be in equilibrium?

Equation of Motion

Newton's Second Law

- Newton's 2nd law of motion:

$$\mathbf{F}_{\text{net}} = m\mathbf{a}$$

- Is it also true for *circular* motion?
- If a net force $\mathbf{F}_{\text{net}}$ causes a mass to accelerate (linearly), what causes a mass to go into circular motion?

Equation of Motion

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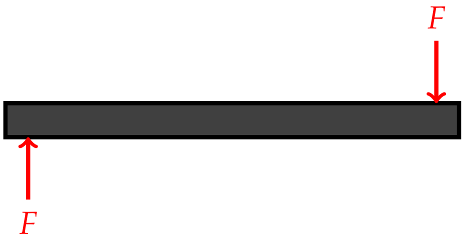
Answer: We need to introduce a few concepts first. . .

Torque (Moment)

I have a pencil sitting on a table, and with my fingers, I push the two ends of the pencil with equal force F. **What happens?**



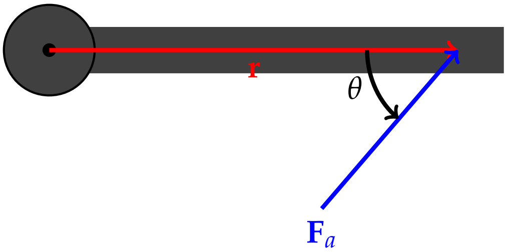
$\mathbf{F}_{\text{net}} = \mathbf{0}$, therefore $\mathbf{a} = \mathbf{0}$. But (obviously) it won't stay still either!

What is Torque?

- The tendency of a force to cause or change the rotational motion of a body.
- A force acting at a point some distance from a fulcrum (or pivot, CG etc...)
- Also referred to as *moment*
- e.g. the force to twist a screw

$$\boxed{\tau = rF_a \sin\theta}$$

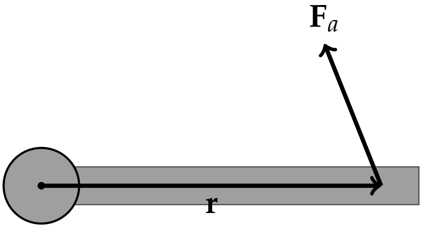
- τ = torque
- r = "moment arm"
- F_a = applied force
- θ = angle between $\mathbf{F}_a$ and $\mathbf{r}$



Vector Notation

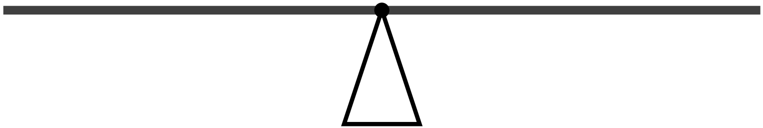
In vector form, torque $\boldsymbol{\tau}$ is the cross product of the moment arm $\mathbf{r}$ and the applied force $\mathbf{F}_a$.

$$\boxed{\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}_a}$$



Torque (Moment)

Going back to the example question:



Torque (Moment)

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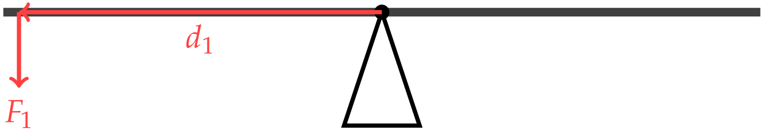

- F_1 will rotate the board counter clockwise

Torque (Moment)

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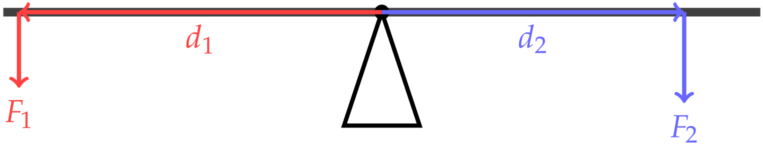
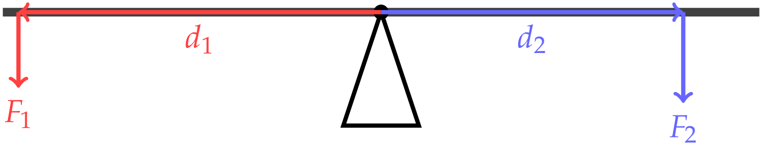


- F_1 will rotate the board counter clockwise
- F_2 will rotate the board clockwise

Torque (Moment)

Going back to the example question:



- F_1 will rotate the board counter clockwise
- F_2 will rotate the board clockwise
- The beam will remain static (in equilibrium) if

$$F_1 d_1 = F_2 d_2$$

Rotational Equilibrium

Just like **translational equilibrium** is when the force acting on an object is zero:

$$\mathbf{F} = \mathbf{0}$$

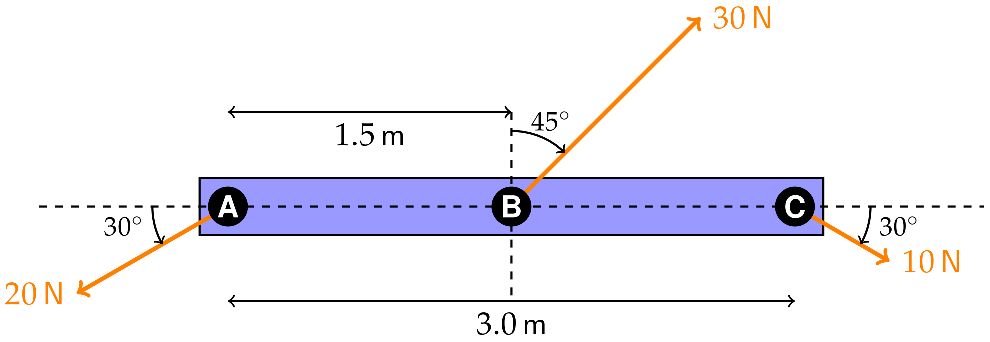
An object is in **rotational equilibrium** when the net torque acting on it is zero:

$$\boldsymbol{\tau} = \mathbf{0}$$

Note that it doesn't mean that the object isn't rotating, it just means that the object's rotational state isn't changing, i.e. $\alpha = 0$

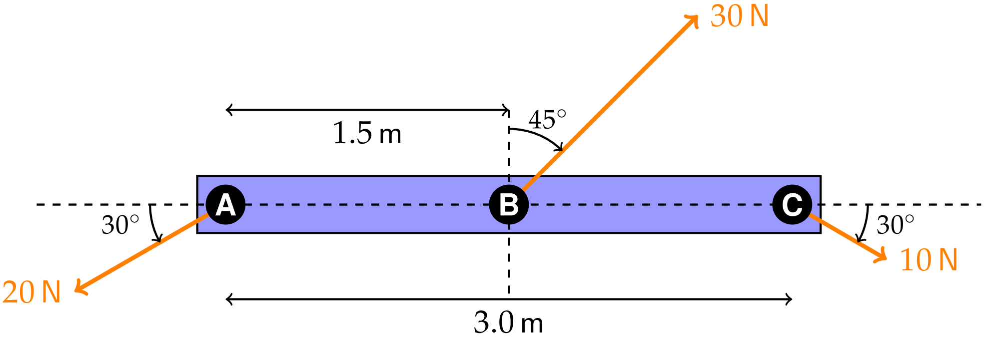
Example Problem

Example 7a: Find the net torque on point C.



Example Problem

Example 7a: Find the net torque on point C.



Example 7b: Now find the net torque on A.

Angular Momentum

Consider a mass m connected to a massless beam rotates with speed v at a distance r from the center (shown on the right). It has an **angular momentum** (L) of

$$\boxed{\mathbf{L} = \mathbf{r} \times \mathbf{p}}$$

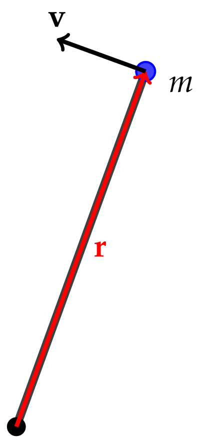
Expanding the terms in the definition:

$$\mathbf{L} = \mathbf{r} \times (m\mathbf{v}) = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = mr^2\boldsymbol{\omega}$$

Which gives us:

$$\boxed{\mathbf{L} = I\boldsymbol{\omega}}$$

The quantity I is called the **moment of inertia**.



Moment of Inertia

- For a single particle:

$$\boxed{I = r^2 m}$$

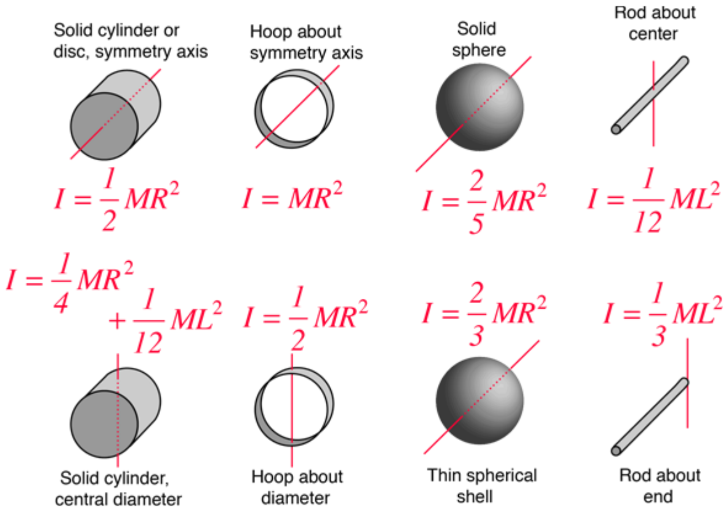
- A collection of particles:

$$\boxed{I = \sum r_i^2 m_i}$$

- Continuous distribution of mass

$$\boxed{I = \int r^2 dm}$$

Moment of Inertia



Angular Momentum and Moment of Inertia

- Linear and angular momentum have very similar expressions

$$\mathbf{p} = m\mathbf{v} \qquad \mathbf{L} = I\boldsymbol{\omega}$$

- Just like $\mathbf{p}$ describes the overall translational state of a physical system, $\mathbf{L}$ describes its overall rotational state
- In that case, momentum of inertia I can be considered an object's "rotational mass"

Conservation of Angular Momentum

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \frac{d(\mathbf{r} \times \mathbf{p})}{dt} \longrightarrow \boxed{\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}}$$

- If the net torque on a system is zero, then the rate of change of angular momentum is zero, and we say that the angular momentum is conserved.
- e.g. When an ice skater starts to spin and draws his arms inward. Since angular momentum is conserved, a decrease in r means an increase in ω.
- If moment of inertia I is constant in time, the net torque is given by:

$$\boxed{\boldsymbol{\tau} = I\boldsymbol{\alpha}}$$

Example Problem

Example 8: A skater extends her arms (both arms!), holding a $2.0\,\mathrm{kg}$ mass in each hand. She is rotating about a vertical axis at a given rate. She brings her arms inward towards her body in such a way that the distance of each mass from the axis changes from $1.0\,\mathrm{m}$ to $0.50\,\mathrm{m}$. Her rate of rotation (neglecting her own mass) will?

Last Example

Example 9: A $1.0\,\mathrm{kg}$ mass swings in a vertical circle after having been released from a horizontal position with zero initial velocity. The mass is attached to a massless rigid rod of length $1.5\,\mathrm{m}$. What is the angular momentum of the mass, when it is in its lowest position?

Rotational Kinetic Energy

To find the kinetic energy of a rotating system of particles (discrete number of particles, or continuous mass distribution), we sum (or integrate) the kinetic energy of the individual particles:

$$K = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

$$K = \int \frac{1}{2} v^2 dm = \frac{1}{2}\left(\int r^2 dm\right)\omega^2$$

It's no surprise that in both case, rotational kinetic energy is given by:

$$\boxed{K = \frac{1}{2} I\omega^2}$$

Kinetic Energy of a Rotating System

The total kinetic energy of a rotating system is the sum of its translational and rotational kinetic energies at its center of mass:

$$\boxed{K = \frac{1}{2}mv_{\mathrm{CM}}^2 + \frac{1}{2}I_{\mathrm{CM}}\omega^2}$$

In this case, I_{CM} is calculated at the center of mass. For simple problems, we only need to compute rotational kinetic energy at the pivot:

$$\boxed{K = \frac{1}{2}I_{\mathrm{P}}\omega^2}$$

In this case, the I_{P} is calculated at the pivot. **IMPORTANT:** $I_{\mathrm{CM}} \neq I_{\mathrm{P}}$