Free Response Question 4: A steel ball is dropped from a point with (x, y) coordinate of $(8\,\mathrm{m}, 16\,\mathrm{m})$. At the same time, another ball is launched from the origin with a speed of 20 m/s at an angle of 30°.

1. Find the minimum distance of separation occur of the two balls.
2. At what time does this separation occur?
3. Give the coordinates of the two balls for the minimum separation.

There are two ways to solve the problem. The first method *looks* easy on first glance, but will require a lot of calculus. The second method, on the other hand, is a straightforward geometry problem that requires a bit of ingenuity.

Method 1 (not recommended)

The most straightforward approach is to express the distance between the steel balls as a function of time, and then take the derivative with respect to time to find out when it occurs t and minimum value of d.

Let's call the steel ball being dropped A, and the ball that is launched B. Their respective position in the coordinate system are expressed as functions of time using kinematic equations[1], and using $g = 10\,\mathrm{m/s^2}$ for both cases[2] for simplicity.

$$\mathbf{x}_A = 8\hat{\imath} + (16 - 5t^2)\hat{\jmath} \tag{1}$$

$$\mathbf{x}_B = 20\cos 30^\circ t\hat{\imath} + (20\sin 30^\circ t - 5t^2)\hat{\jmath} \tag{2}$$

The "displacement" vector between A and B can be expressed as:

$$\Delta\mathbf{x} = \mathbf{x}_A - \mathbf{x}_B = (8 - 20\cos 30^\circ t)\hat{\imath} + (16 - 20\sin 30^\circ t)\hat{\jmath} \tag{3}$$

Not surprisingly, the gravitational acceleration term $\frac{1}{2}gt^2 = 5t^2$ term cancels, because both A and B are free-falling objects with the same downward acceleration. The square of the *distance* between A and B are expressed as:

$$d^2 = (8 - 20\cos 30^\circ t)^2 + (16 - 20\sin 30^\circ t)^2 \tag{4}$$

What we will need to do now is to take the time derivative of d^2 with respect to time, and to find d and t. This process is laborious and tedious (and prone to error for someone uncomfortable with using chain rule) and therefore generally not recommended. We will instead try a completely different approach.

Method 2

However, we have already noted that in Equation 3, acceleration due to gravity terms cancel, which means that the minimum separation distance d does not depend on g! We instead consider an observer who is falling alongside steel ball A (which is equivalent to effectively treating $g = 0$.) In this case, the observer sees that A remains stationary while B travels in a straight line instead of the parabolic path of a projectile. The observer's point of view is shown in Fig. 1. The minimum distance of separation occurs at C in this frame of reference, with a value of d. Using basic geometry, we can find distance DE and AE:

$$DE = 8\tan 30^\circ = 4.6\,\mathrm{m}$$

$$AE = 16 - DE = 11.4\,\mathrm{m}$$

[1] For the x direction, $x = v_x t$ and for the y direction, $y = y_0 + v_{y0}t - \frac{1}{2}gt^2$.

[2] which is acceptable for all AP exams

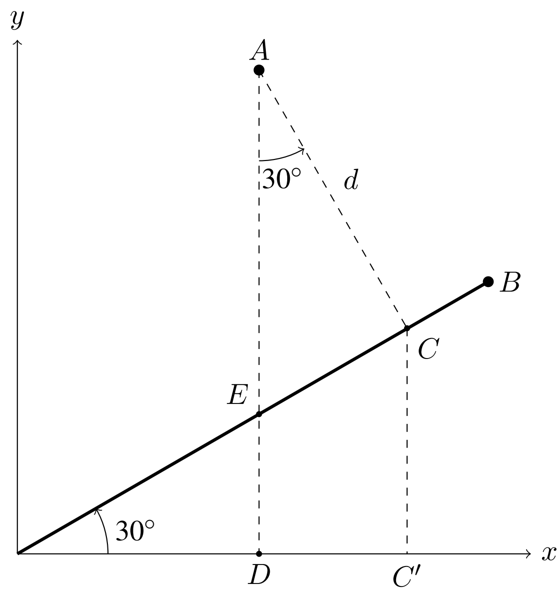


Figure 1: Observing the steel balls while free-falling

Using basic trigonometry again, we can now find the minimum distance of separation:

$$d = AE\cos 30^\circ = \boxed{9.9\,\mathrm{m}}$$

The second part is slightly trickier, because we have (so far) ignored acceleration due to gravity. However, there is no acceleration in the x direction, i.e. the horizontal distance that B travels is the same. We need to compute DC' which is just

$$DC' = d\sin 30^\circ = 4.9\,\mathrm{m}$$

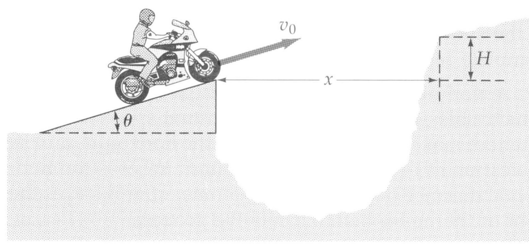
which means that in the time to reach minimum separation distance, the steel ball B has travelled $8 + 4.9 = 12.9\,\mathrm{m}$ horizontally. We can now use the kinematic equation in the x-direction to compute t:

$$t = \frac{\Delta x}{v_x} = \frac{12.9}{20\cos 30^\circ} = \boxed{0.75\,\mathrm{s}}$$

Finally, we substitute t back into the *actual* position of A and B to compute their *actual* location when the minimum separation occurs:

$$\mathbf{x}_A = 8\hat{\imath} + (16 - 5t^2)\hat{\jmath} = \boxed{8\hat{\imath} + 13.2\hat{\jmath}}$$
$$\mathbf{x}_B = 20\cos 30^\circ t\hat{\imath} + (20\sin 30^\circ t - 5t^2)\hat{\jmath} = \boxed{12.9\hat{\imath} + 1.7\hat{\jmath}}$$

Free Response Question 6: A trail bike take off from a ramp with velocity $\mathbf{v}_0$ at angle θ to clear a ditch of width x and land on the other side, which is elevated at a height H.



1. For a given angle θ and distance x, what is the upper limit for H such that the bike has an chance of making the jump?
2. For H less than this upper limit, what is the minimum take-off speed v_0 necessary for a successful jump?

Neglect the size of the trail bike, and assume that covering a horizontal distance x and a vertical distance H is sufficient to clear the ditch.

To solve this problem, we must break down the initial velocity $\mathbf{v}_0$ into its horizontal ($\hat{\imath}$) and vertical ($\hat{\jmath}$) directions, i.e.:

$$\mathbf{v}_0 = v_o \cos\theta \hat{\imath} + v_0 \sin\theta \hat{\jmath}$$

Ignoring air resistance, the only acceleration will be due to gravity (which is constant), in the $\hat{\jmath}$ direction. Also assuming that H is the maximum height that the dirt bike reaches. We can use the kinematic equations to obtain an expression for H in terms of v_0 and θ:

$$v_y^2 = v_{y0}^2 - 2gH \quad \longrightarrow \quad 0 = v_o^2 \sin^2\theta - 2gH \quad \longrightarrow \quad \boxed{H = \frac{v_o^2 \sin^2\theta}{2g}} \tag{5}$$

Note that this is the same calculation we used to obtain the maximum height for a symmetric trajectory projectile motion. The time it takes to reach this height is also straightforward to obtain:

$$v_y = v_{y0} - gt \quad \longrightarrow \quad 0 = v_0 \sin\theta - gt \quad \longrightarrow \quad \boxed{t = \frac{v_0 \sin\theta}{g}} \tag{6}$$

Likewise, this the exactly *half* of the value for a symmetric projectile. In the $\hat{\imath}$ direction, there is no acceleration, and the width of the ditch x can be related to the initial velocity as:

$$x = v_x t = v_0 \cos\theta \left(\frac{v_0 \sin\theta}{g}\right) = \boxed{\frac{v_0^2 \sin\theta \cos\theta}{g}} \tag{7}$$

For completeness, Eq. 7 is also *half* the range of a symmetric projectile. Substituting the expression for H from Eq. 5 into Eq. 7, and solving for H obtains the answer to the first part of the question:

$$x = \frac{2H \cos\theta}{\sin\theta} \quad \longrightarrow \quad \boxed{H = \frac{1}{2} x \tan\theta} \tag{8}$$

For the second part of the question, we assume that the width of the ditch x and the angle of the ramp θ are constant, and that H is the only thing that will change. Therefore, we want to express v_0 needed to clear H in terms of x and θ. Then any v_0 less than this value will clear a height less than H. Equating the expression for H in Eq. 8 to Eq. 5, and rearranging terms:

$$\frac{v_0^2 \sin^2 \theta}{2g} = \frac{1}{2} x \tan \theta = \frac{1}{2} x \frac{\sin \theta}{\cos \theta} \tag{9}$$

$$v_0^2 = \frac{gx}{\sin \theta \cos \theta} = \frac{2gx}{\sin(2\theta)} \tag{10}$$

Finally, solving for v_0 we have the expression for the velocity to clear

$$\boxed{v_0 \geq \sqrt{\frac{2gx}{\sin(2\theta)}}} \tag{11}$$