Kepler's Laws of Planetary Motion

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Although the laws were originally obtained by Kepler after careful analysis of empirical data from his teacher Tycho Brahe, the complete mathematical understanding was missing until Newton derived each law as pieces of his orbital mechanics. In his footsteps we will obtain each law in turn, as we consider the orbit of a planet in the gravity of a massive star. In modern form, **Kepler's laws of planetary motion** state that:

1. A planet moves around the Sun in an elliptical path with the Sun as one of the foci.
2. The line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time (Fig. 1).
3. The square of the orbital time period of a planet is proportional to the cube of the semi-major axis of its orbit, i.e. $T^2 \propto r^3$.

To begin, two properties of gravitational force[1] are crucial to the understanding of orbital mechanics. These two properties hold true regardless of the shape of the orbit, and whether an object is even in orbit altogether:

1. Gravity is a *conservative* force, in that
   - Assuming that minor collisions with space dust and other methods of energy dissipation are negligible, then the mechanical energy $E = K + U_g$ is conserved.
   - Work done *by* gravity converts gravitational potential energy U_g into kinetic energy K, while work *against* gravity converts K into U_g.
2. Gravity is a *central* force, in that
   - gravitational force $\mathbf{F}_g$ is always in the $-\hat{\mathbf{r}}$ direction (i.e. $\mathbf{F} \times \mathbf{r} = \mathbf{0}$), therefore gravity does not generate any torque, and as such,
   - Angular momentum $\mathbf{L}$ about the sun is constant
   - The motion of the planet is confined to a two-dimensional plane

For the purpose of the analysis, we also assume that the Sun's mass $M_\odot$ is very large compared to any other object in the solar system (e.g. planets, comets, asteroids), and its motion is essentially unaffected by the gravity from the planets.

[1] Newton's law of universal gravitation: $\mathbf{F}_g = -\dfrac{Gm_1m_2}{r^2}\hat{\mathbf{r}}$

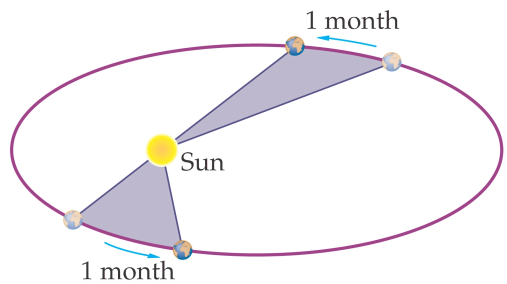

Figure 1: The Earth sweeps out the same shaded area (purple) over the same amount of time regardless of where it is in its orbit.

1 Second law: Equal area in equal time

The second law of planetary motion is the easiest to proof using concepts in rotational motion. The infinitesimal area $d\mathbf{A}$ (Fig. 2) swept out by a planet as it moves in orbit by an infinitesimal

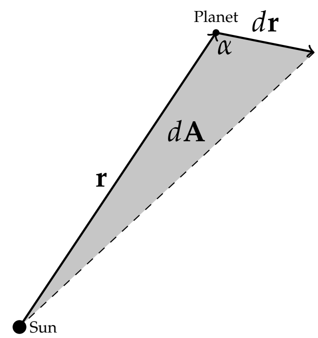

Figure 2: Infinitesimal area $d\mathbf{A}$ swept by a planet as it moves in orbit.

amount $d\mathbf{r}$ can be computed by the area of the triangle:

$$dA = \frac{1}{2} r dr \sin\alpha \tag{1}$$

or in vector form:

$$d\mathbf{A} = \frac{1}{2}\mathbf{r} \times d\mathbf{r} \tag{2}$$

The direction of $d\mathbf{A}$, in this case, points into the page. (The direction of the vector is not important in the calculation.) The time derivative of the area is called the **areal velocity**, which literally means how quickly the area is changing:

$$\frac{d\mathbf{A}}{dt} = \frac{1}{2}\mathbf{r} \times \frac{d\mathbf{r}}{dt} = \frac{1}{2}\mathbf{r} \times \mathbf{v} \tag{3}$$

We can express $\mathbf{r} \times \mathbf{v}$ in terms of angular momentum. Since angular momentum is defined as $\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$, we have $\mathbf{r} \times \mathbf{v} = \dfrac{\mathbf{L}}{m}$. Since gravity is a central force, angular momentum is a constant, i.e:

$$\boxed{\frac{d\mathbf{A}}{dt} = \frac{1}{2}(\mathbf{r} \times \mathbf{v}) = \frac{\mathbf{L}}{2m} = \text{constant}} \tag{4}$$

as predicted by Kepler's second law. The rate that a planet sweeps out the area in orbit is its angular momentum around the sun, divided by twice its mass. It is constant regardless of its location relative to the sun.

2 First Law: The Law of Ellipses

In order to proof Kepler's first law, we must show that all orbital motion must agree with the equations of an ellipse.
The longest dimension from the origin is the **semi-major axis** a, while the shortest dimension is the **semi-minor axis** b. The two lengths are related through the **eccentricity** of the ellipse e:

$$b^2 = a^2(1-e^2) \quad \text{where} \quad 0 \leq e < 1 \tag{5}$$

The minimum value of eccentricity is 0, where $a = b = r$ which is a circle. When eccentricity is 1, the shape becomes a parabola, and is no longer an ellipse. For all values between 0 and 1, the area A of an ellipse is given by:

$$A = \pi ab = \pi a^2 \sqrt{1-e^2} \tag{6}$$

Figure 3: An ellipse

which is in the form:

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \tag{7}$$

where e is called the eccentricity of the ellipse.

$$\frac{d^2r}{dt^2} - r\frac{d\omega}{dt}^2 = -GM\frac{1}{r^2} \tag{8}$$

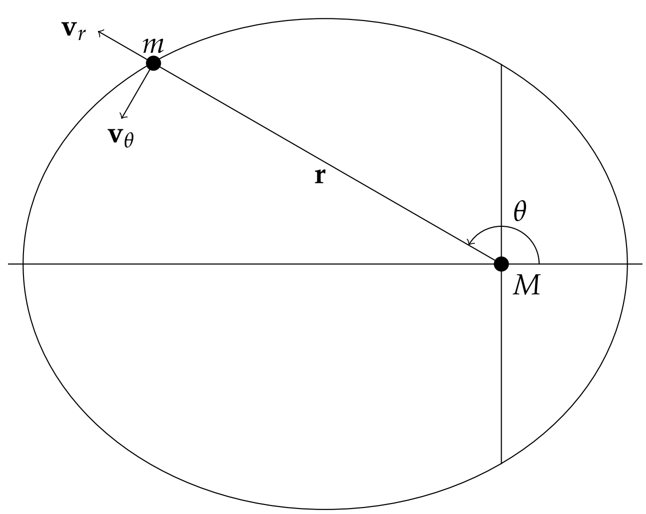

Figure 4: Elliptical orbit of a small mass m around a large mass M.

As shown in Fig. 4, there are two components of velocity when a planet orbits a star:

- **Angular velocity $\mathbf{v}_\theta$.** The presence of $\mathbf{v}_\theta$ means a centripetal acceleration toward M:

$$a = -r\omega^2 \hat{\mathbf{r}} \tag{9}$$

- **Radial velocity $\mathbf{v}_r$.** If this velocity changes with time (which is the case for elliptical orbits but not circular orbits), then there is an acceleration, also in the radial direction:

$$a = \frac{dr}{dt}\hat{\mathbf{r}} \tag{10}$$

Both components of acceleration are due entirely to gravitational force. Applying Newton's second law of motion gives the differential equation:

$$\frac{d^2r}{dt^2} - r\omega^2 = -\frac{GM}{r^2} \tag{11}$$

The $(+)$ direction is radially outward from M. In the circular motion case, where $d^2r/dt^2 = 0$, we are left with only the centripetal force.
The differential equation, in its current form, is difficult to solve, and the equation for an ellipse depends on θ but not on t. However, the differential equation is easier to solve by making a variable substitution that may not be immediately obvious, by introducing a new variable u which is the inverse of the radius r:

$$u = \frac{1}{r} \tag{12}$$

We can then use the fact that angular momentum L is constant to relate derivatives in time t to derivatives in angle θ:

$$L = mr^2\frac{d\theta}{dt} = \frac{m}{u^2}\frac{d\theta}{dt} \quad \longrightarrow \quad \frac{d}{d\theta} = \frac{Lu^2}{m}\frac{d}{dt} \tag{13}$$

The derivatives $\dot{r}$ and $\ddot{r}$ can now be expressed in terms of derivatives of θ instead of time t:

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u}\frac{du}{dt} \tag{14}$$

$$= -\frac{L}{m}\frac{du}{d\theta} \tag{15}$$

$$= -u^{-2}\frac{du}{d\theta}\frac{Lu^2}{m} \tag{16}$$

$$= -\frac{L}{m}\frac{du}{d\theta} \tag{17}$$

Repeating the same process for the second derivative, we have

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(-\frac{L}{m}\frac{du}{d\theta}\right) \tag{18}$$

$$= -\frac{L}{m}\frac{d}{d\theta}\frac{du}{d\theta} \tag{19}$$

$$= -\left(\frac{L^2u^2}{m^2}\right)\frac{d^2u}{d\theta^2} \tag{20}$$

With this identity in hand, our original differential equation (Eq. 8) becomes

$$-\left(\frac{L^2u^2}{m^2}\right)\frac{d^2u}{d\theta^2} - \left(\frac{L}{m}\right)^2 u^3 = -GMu^2 \tag{21}$$

which can be simplified to:

$$\frac{d^2u}{d\theta^2} + u = \frac{GMm^2}{L^2} \tag{22}$$

Anyone experienced with calculus can recognize that Eq. 22 is a second-order ordinary differential equation with constant coefficients and a constant forcing function. The solution to this type of equation is in the form

$$u = A\cos\theta + \frac{GMm^2}{L^2} \tag{23}$$

where the coefficient A is determined by initial conditions. Solving for $r = 1/u$, we have

$$r(\theta) = \frac{1}{A\cos\theta + \frac{GMm^2}{L^2}} = \left(\frac{L^2}{GMm^2}\right)\frac{1}{1 + e\cos\theta} \tag{24}$$

where e is a constant:

$$e = \frac{AL^2}{GMm^2} \tag{25}$$

From this form of r, it is clear that the maximum and a minumum values of r, in fact represent the *aphelion* and *perihelion* of the ellipes (points of furthest and closest distance to the focus):

$$r_{\text{max}} = \left(\frac{L^2}{GMm^2}\right)\frac{1}{1-e} \qquad r_{\text{min}} = \left(\frac{L^2}{GMm^2}\right)\frac{1}{1+e} \tag{26}$$

This the semi-major axis r is the average of the two values:

$$a = \frac{1}{2}\left(r_{\text{min}} + r_{\text{max}}\right) = \left(\frac{L^2}{GMm^2}\right)\frac{1}{1-e^2} \tag{27}$$

More importantly, Eq. 27 allows us to relate the eccentricity e and the semi-major axis a to the numerator in the r expression:

$$a(1-e^2) = \frac{L^2}{GMm^2} \tag{28}$$

We see that the orbit is given by an ellipse as Kepler found from Brahe's dataset. Moreover, since r_{min} and r_{max} are distances from the Sun, we see that the Sun is at one focus of the orbit. Thus, we have derived Kepler's first law.

3 Third law: Period of motion

The total area A swept by the planet through one orbital period is the areal velocity (which is constant, as shown in Eq. 4) integrated by time, from $t = 0$ to T, the orbital period of the planet:

$$A = \int dA = \int_0^T \frac{dA}{dt}dt = \frac{L}{2m}\int_0^T dt = \frac{L}{2m}T \tag{29}$$

From proofing Kepler's first law in the previous section, we know that this area is an ellipse, given in Eq. 6. where a is the semi-major axis, $b = a\sqrt{1-e^2}$ is the semi-minor axis. Equating the expressions for area in Eq. 29 and Eq. 6, then squaring both sides, give this expression:

$$T^2 = \frac{m^2}{L^2}4\pi^2a^4(1-e^2) \tag{30}$$

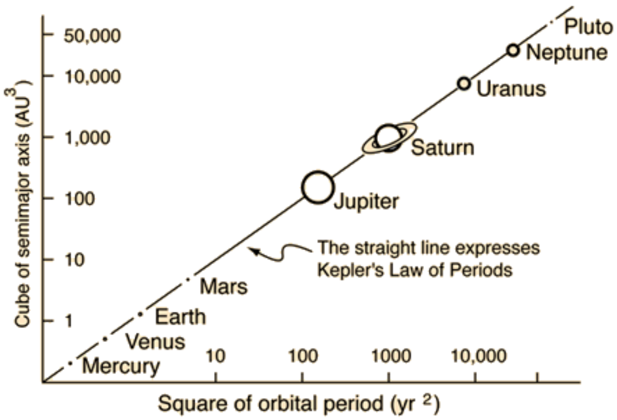

Figure 5: The pl

Substituting Eq. 28, shown below for reference:

$$a(1-e^2) = \frac{L^2}{GMm^2}$$

into Eq. 30 above, and after some simple algebra, we arrive at this expression:

$$\boxed{T^2 = \left[\frac{4\pi^2}{GM}\right] a^3} \tag{31}$$

which is Kepler's third law. Note that this law holds for all elliptical orbits, regardless of their eccentricities.