Symmetric Projectile Trajectory

A **symmetric trajectory** is a special case of projectile motion where an object is launched at an angle of θ (between 0° and 90°) above the horizontal[1] with an initial speed v_i, and then lands at the same height, as shown below in Figure 1. Examples may include hitting a golf ball towards the hole, or shooting a bullet towards a horizontal target[2]. The equations for symmetric trajectory is *not* included in the AP Exam equation sheet; if you need these equations during the exams, you will need to derive them during the exam. To derive the equations, we use the x-axis for the horizontal direction and y-axis for the vertical.

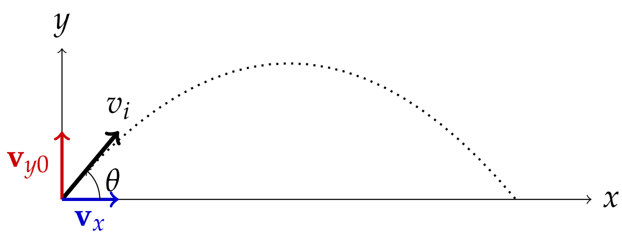

Figure 1: Symmetric project trajectory

The initial velocity $\mathbf{v}_i$ can be resolved into its components, also shown in Figure 1:

$$\mathbf{v}_x = v_i \cos\theta \hat{\imath}$$
$$\mathbf{v}_{y0} = v_i \sin\theta \hat{\jmath}$$

v_x remains constant during the motion, as there are no forces acting in the x direction, and therefore no acceleration. In the y direction, there is an acceleration due to gravity $a_y = -g$.

Maximum height H: Apply the kinematic equation in the y-direction. Recognizing that at maximum height $H = y - y_0$, the vertical component of velocity is zero $v_y = 0$:

$$v_y^2 = v_{y0}^2 + 2a_y(y - y_0)$$
$$0 = (v_i \sin\theta)^2 - 2gH$$

Solving for H, we get the maximum height equation:

$$\boxed{H = \frac{v_i^2 \sin^2\theta}{2g}} \tag{1}$$

[1]This may be obvious, but any angles *below* the horizontal will never have a symmetric trajectory.
[2]Shooting a bullet towards a horizontal target always require an upward angle because of gravity.

Total time of flight $t_{\max}$: We apply the kinematic equation in the y direction. When the object lands at the same height, the final velocity is the same in magnitude and opposite in direction as the initial velocity, i.e. $v_{y2} = -v_{y1} = -v_i \sin\theta$:

$$v_y = v_{y0} + a_y t$$
$$-v_i \sin\theta = v_i \sin\theta - g t_{\max}$$

Solving for $t_{\max}$ we have:

$$\boxed{t_{\max} = \frac{2v_i \sin\theta}{g}} \tag{2}$$

Range R: We substitute the expression for $t_{\max}$ from Eq. 2 into the t term, then apply the kinematic equation in the x-direction to compute $R = x - x_0$ for any given launch angle and initial speed:

$$x = x_0 + v_x t$$
$$R = v_i \cos\theta \left(\frac{2v_i \sin\theta}{g}\right)$$

Using the trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$, we simplify the equation to:

$$\boxed{R = \frac{v_i^2 \sin(2\theta)}{g}} \tag{3}$$

It is obvious that for any given initial speed v_i, the maximum range $R_{\max}$ occurs at an angle where $\sin(2\theta) = 1$ (i.e. $\theta = 45^\circ$), with a value of

$$\boxed{R_{\max} = \frac{v_i^2}{g}} \tag{4}$$

Also, for a known initial speed v_i and range R we can compute the launch angle θ:

$$\theta_1 = \frac{1}{2}\sin^{-1}\left(\frac{gR}{v_i^2}\right)$$

This angle is labelled θ_1 because it is *not* the only angle that can reach this range. Recall that for any angle $0^\circ < \phi < 90^\circ$, there is also another angle where the $\sin$ are equal:

$$\sin\phi = \sin(180^\circ - \phi)$$

Which means that for any θ_1, there is also another angle θ_2 where $2\theta_2 = 180^\circ - 2\theta_1$, or quite simply:

$$\theta_2 = 90^\circ - \theta_1$$