

# Topic 10: Electrostatics

## Advanced Placement Physics

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Olympiads School, Toronto, ON, Canada

# Intro

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# Files for You to Download

Download from the school website:

1. **PhysAP-10-Electrostatics.pdf**—This presentation.

Please download/print the PDF file before each class. There is no point copying notes that are already printed out for you. Instead, take notes on things I say that aren't necessarily on the slides.

# Electrostatic Force

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# The Charges Are

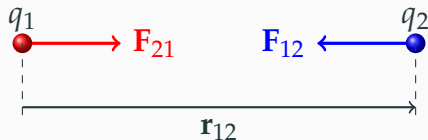
We should already know a bit about charge particles:

- A **proton** carries a **positive** charge
- An **electron** carries a **negative** charge
- A *net charge* of an object means an excess of protons or electrons
- Similar charges repel; opposite charges attract

We start with electrostatics:

- Charges that are not moving relative to one another

# Coulomb's Law for Electrostatic Force



The **electrostatic force** (or **coulomb force**) is a mutually repulsive/attractive force between all charge objects. The force that charge  $q_1$  exerts on  $q_2$  is given by:

$$\mathbf{F}_{12} = \frac{kq_1q_2}{|\mathbf{r}_{12}|^2} \hat{\mathbf{r}}_{12}$$

# Coulomb's Law for Electrostatic Force

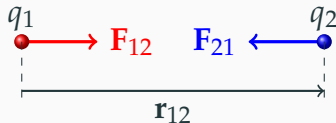
$$\mathbf{F}_{12} = \frac{kq_1q_2}{|\mathbf{r}_{12}|^2} \hat{\mathbf{r}}_{12}$$

Quantity	Symbol	SI Unit
Electrostatic force	$\mathbf{F}_{12}$	N
Coulomb's constant (electrostatic constant)	$k$	$\text{N m}^2/\text{C}^2$
Point charges 1 and 2	$q_1, q_2$	C
Distance between point charges	$ \mathbf{r}_{12} $	m
Unit vector of direction between point charges	$\hat{\mathbf{r}}_{12}$	

Coulomb's constant  $k = \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9 \text{ N m}^2/\text{C}^2$  where

$\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{N m}^2$  is called the “permittivity of free space”

# Coulomb's Law for Electrostatic Force

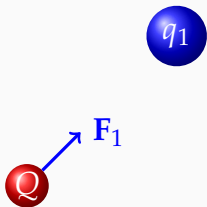


- If  $q_1$  exerts an electrostatic force  $\mathbf{F}_{12}$  on  $q_2$ , then  $q_2$  likewise exerts a force of  $\mathbf{F}_{21} = -\mathbf{F}_{12}$  on  $q_1$ . The two forces are equal in magnitude and opposite in direction (3rd law of motion).
- $q_1$  and  $q_2$  are assumed to be *point charges* that do not occupy any space
- The (more familiar) scalar form is often used as well:

$$F_q = \frac{kq_1q_2}{r^2}$$



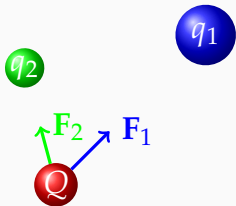
## More Than One Charge



For a charge  $Q$  that is subjected to the influence of multiple discrete point charges  $q_i$ , the total electrostatic force that  $Q$  experiences is the vector sum of all the forces  $\mathbf{F}_i$ :

$$\mathbf{F} = \sum_i \mathbf{F}_i = kQ \left( \sum_{i=1}^N \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i \right)$$

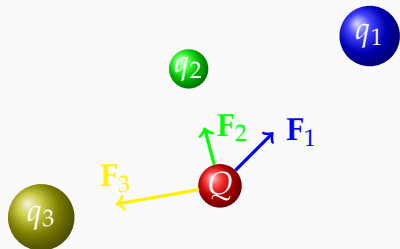
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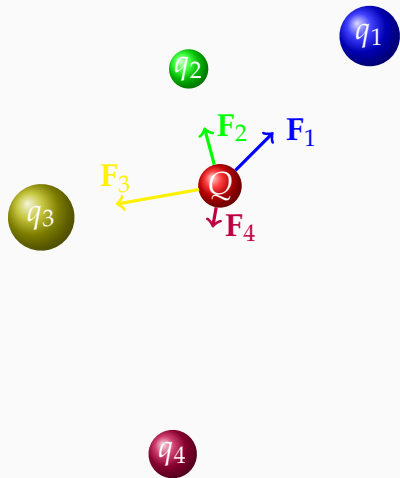
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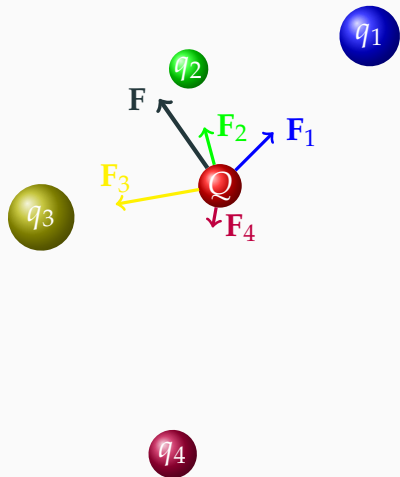
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# More Than One Charge



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$$\mathbf{F} = \sum_i \mathbf{F}_i = kQ \left( \sum_{i=1}^N \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i \right)$$

# Continuous Distribution of Mass

As  $N \rightarrow \infty$ , the summation becomes an integral, and can now be used to describe the force from charges with *spatial extend* i.e. charges that take up physical space (e.g. a continuous distribution of charge):

$$\mathbf{F} = \int d\mathbf{F} = kQ \int \frac{dq}{r^2} \hat{\mathbf{r}}$$

# Electric Field

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# Electric Field

The expression for **electric field** is obtained by repeating the same procedure as with gravitational field, by grouping the variables in Coulomb's law:

$$F_q = \underbrace{\left[ \frac{kq_1}{|\mathbf{r}_{12}|^2} \hat{\mathbf{r}} \right]}_{\mathbf{E}} q_2$$

The electric field **E** created by  $q_1$  is a vector function (called a **vector field**) that shows how it influences other charged particles around it.



# Electric Field Intensity Near a Point Charge

The electric field a distance  $r$  away from a point charge  $q$  is given by:

$$\mathbf{E}(q, \mathbf{r}) = \frac{kq}{|\mathbf{r}|^2} \hat{\mathbf{r}}$$

Quantity	Symbol	SI Unit
Electric field intensity	$\mathbf{E}$	N/C
Coulomb's constant	$k$	$\text{N m}^2/\text{C}^2$
Source charge	$q$	C
Distance from source charge	$ \mathbf{r} $	m
Outward unit vector from point source	$\hat{\mathbf{r}}$	

The direction of  $\mathbf{E}$  is radially outward from a positive point charge and radially inward towards a negative charge.

## More Than One Charge

When multiple point charges are present, the total electric field at any position  $\mathbf{r}$  is the vector sum of all the fields  $\mathbf{E}_i$ :

$$\mathbf{E} = \sum_i \mathbf{E}_i = k \left( \sum_i \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i \right)$$

## More Than One Charge

As  $N \rightarrow \infty$ , the summation becomes an integral, and can now be used to describe the electric field generated by charges with *spatial extend*:

$$\mathbf{E} = \int d\mathbf{E} = E \int \frac{dq}{r^2} \hat{\mathbf{r}}$$

This integral may be difficult to compute if the geometry of is complicated, but in general, there are usually symmetry that can be exploited.

# Think Electric Field

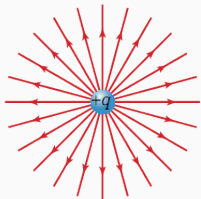
$\mathbf{E}$  itself *doesn't do anything* until another charge interacts with it. And when there is a charge  $q$ , the electrostatic force  $\mathbf{F}_q$  that the charge experiences is proportional to  $q$  and  $\mathbf{E}$ , regardless of how the electric field is generated:

$$\mathbf{F}_q = \mathbf{E}q$$

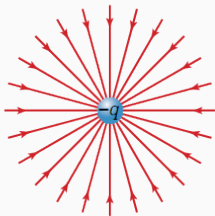
A positive charge in the electric field experiences a electrostatic force  $\mathbf{F}$  in the same direction as  $\mathbf{E}$ .

# Electric Field Lines

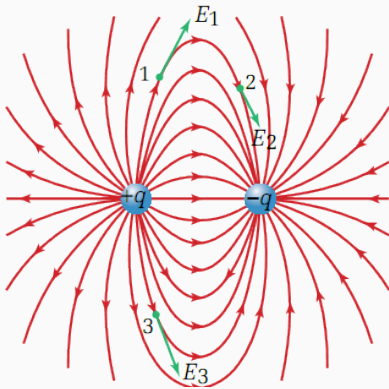
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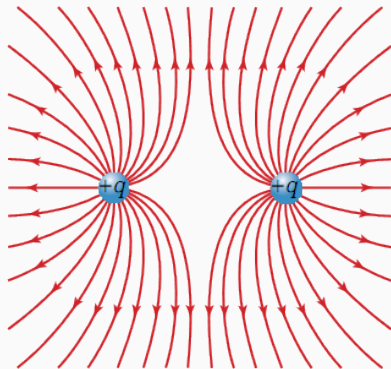
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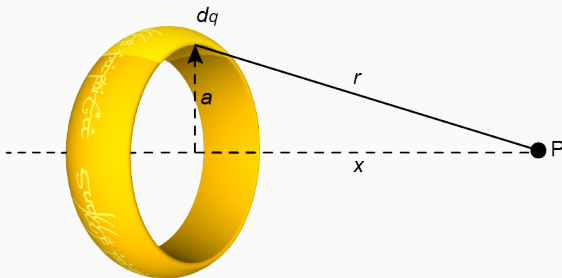


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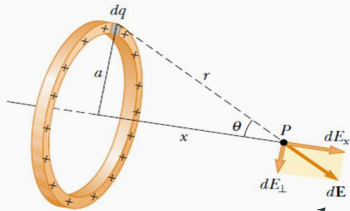
# Lord of the Ring Charge

Suppose you have been given *The One Ring To Rule Them All*, and you found out that it is charged! What is its electric field at point  $P$  along its axis?



Note that calculating the electric field away from the axis is very difficult.

# Electric Field Along Axis of a Ring Charge



- We can separate the electric field  $d\mathbf{E}$  from charge  $dq$  into axial ( $dE_x$ ) and radial ( $dE_{\perp}$ ) components
- Based on symmetry,  $dE_{\perp}$  doesn't contribute to anything; but  $dE_x$  is pretty easy to find:

$$dE_x = \frac{k dq}{r^2} \cos \theta = \frac{k dq}{r^2} \frac{x}{r} = \frac{k x dq}{(x^2 + a^2)^{3/2}}$$

Integrating this over all charges  $dq$ , we have:

$$E_x = \frac{kx}{(x^2 + a^2)^{3/2}} \int dq = \boxed{\frac{kQx}{(x^2 + a^2)^{3/2}}}$$

# Electric Field Along Axis of a Uniformly Charged Disk

Let's extend what we know to a disk of radius  $a$  and charge density  $\sigma$

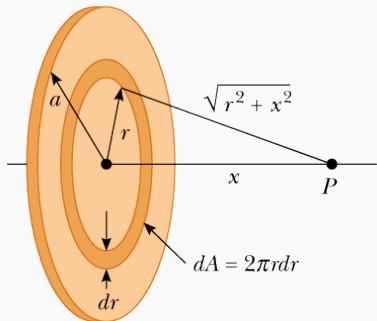
We start with the solution from the ring problem, and replace  $Q$  with  $dq = 2\pi\sigma a da$ :

$$dE_x = \frac{2\pi k x \sigma a da}{(x^2 + a^2)^{3/2}}$$

Integrating over the entire disk:

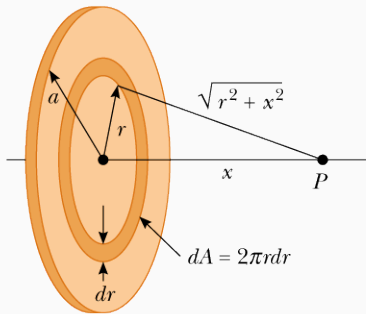
$$E_x = \pi k x \sigma \int \frac{2a da}{(x^2 + a^2)^{3/2}}$$

This is not an easy integral!





# Eclectic Field Along Axis of a Uniformly Charged Disk



- Luckily for us, the integral is in the form of  $\int u^n du$ , with  $u = x^2 + a^2$  and  $n = \frac{-3}{2}$ .
- You can find the integral in any math textbook:

$$E_x = 2\pi k\sigma \left( 1 - \frac{x}{\sqrt{x^2 + R^2}} \right)$$

# Gauss's Law

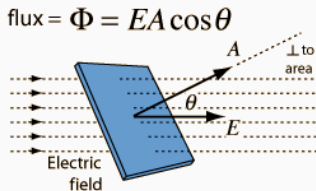
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# Flux

**Flux** is an important concept in many disciplines in physics. The flux of a vector quantity  $\mathbf{X}$  is the amount of that quantity flowing through a surface. In integral form:

$$\Phi = \int \mathbf{X} \cdot d\mathbf{A} \quad \text{or} \quad \Phi = \int (\mathbf{X} \cdot \hat{\mathbf{n}}) dA$$

The direction of the infinitesimal area  $d\mathbf{A}$  is **outward normal** to the surface.

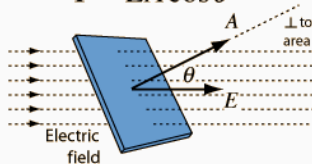


# Flux

$\Phi$  can be something physical, like water, or bananas, or something abstract, like electric field (which is what we are interested in). We can compute a flux as long as there is a vector field i.e.  $\mathbf{X} = \mathbf{X}(x, y, z)$ . In the case of **electric flux**, the quantity  $\mathbf{X}$  is just the electric field, i.e.:

$$\Phi_q = \int \mathbf{E} \cdot d\mathbf{A}$$

$$\text{flux} = \Phi = EA \cos \theta$$



# Electric Flux and Gauss's Law

**Gauss's law** tells us that if we have a closed surface (think of the surface of a balloon), the total electric flux is very well defined:

$$\Phi_q = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{encl}}}{\epsilon_0}$$

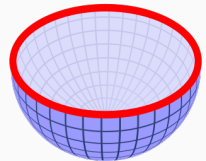
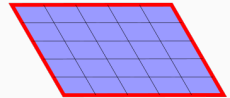
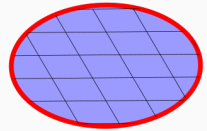
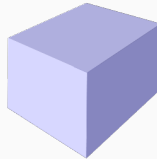
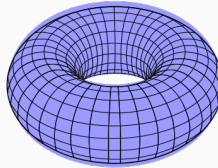
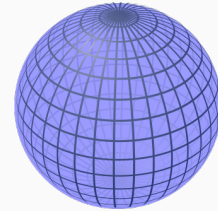
where

- $Q_{\text{encl}}$  is the charge enclosed by the surface
- $\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{N m}^2$  is the permittivity of free space

That closed surface is called a **Gaussian surface**

# Closed Surfaces

A **closed surface** is one that does not have a boundary, like the sphere, toroid, and cube on the left.



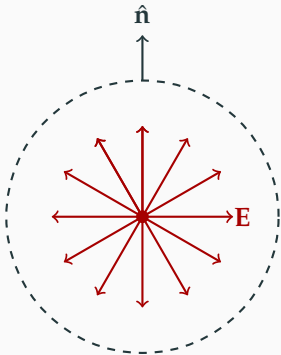
# Electric Field from a Positive Point Charge

By symmetry, electric field lines must be radially outward from the charge, so the integral reduces to:

$$\Phi_q = \oint \mathbf{E} \cdot d\mathbf{A} = EA = \frac{q}{\epsilon_0}$$

Since area of a sphere is  $A = 4\pi r^2$ , we recover Coulomb's law and the magnitude of the electric field from a point charge:

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} = \frac{kq}{r^2}$$



# Electric Potential & Potential Energy

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# Electrical Potential Energy

The work done by the electrostatic force is given by:

$$W = \int \mathbf{F}_q \cdot d\mathbf{r} = kq_1q_2 \int_{r_1}^{r_2} \frac{dr}{r^2} = -\frac{kq_1q_2}{r} \Big|_{r_1}^{r_2} = -\Delta U_q$$

where

$$U_q = \frac{kq_1q_2}{r}$$

is the **electrical potential energy**.  $U_q$  can be (+) or (−), because charges can be either (+) or (−).

## How it Differs from Gravitational Potential Energy

Two positive charges:

$$U_q > 0$$

Two negative charges:

$$U_q > 0$$

One positive and one  
negative charge:

$$U_q < 0$$

- $U_q > 0$  means positive work is done to bring two charges together from  $r = \infty$  to  $r$  (both charges of the same sign)
- $U_q < 0$  means negative work (the charges are opposite signs)
- For gravitational potential  $U_g$  is always  $< 0$

# Electric Potential

When I move an object of mass  $m$  against a gravitational force from one point to another, the work that I do is directly proportional to  $m$ , i.e. there is a “constant” in that scales with *any* mass, as long as they move between those same two points:

$$W = \Delta U_g = Km$$

In the trivial case (small changes in height, no change in  $g$ ), this constant is just

$$\frac{\Delta U_g}{m} = g\Delta h$$

# Electric Potential

This is also true for moving a charged particle  $q$  against an electric field created by  $q_s$ , and the “constant” is called the **electric potential**. For a point charge, it is defined as:

$$V = \frac{U_q}{q} = \frac{kq_s}{r}$$

The unit for electric potential is a *volt* which is *one joule per coulomb*:

$$1 \text{ V} = 1 \text{ J/C}$$

We can easily the relationship between  $V$  and  $\mathbf{E}$ :

$$\Delta V = \int \mathbf{E} \cdot d\mathbf{r}$$

# Potential Difference (Voltage)

The change in electric potential is called the **electric potential difference** or **voltage**:

$$\boxed{\Delta V = \frac{\Delta U_q}{q}} \quad \text{and} \quad \boxed{dV = \frac{dU_q}{q}}$$

Here, we can relate  $\Delta V$  to an equation that we knew from Grade 11 Physics, which related to the energy dissipated in a resistor in a circuit  $\Delta U$  to the voltage drop  $\Delta V$ :

$$\boxed{\Delta U_q = q\Delta V}$$

Electric potential difference also has the unit volts (V)

# Getting Those Names Right

Remember that these three scalar quantities, as opposed to electrostatic force  $\mathbf{F}_q$  and electric field  $\mathbf{E}$  which are vectors

- Electric potential energy:

$$U = \frac{kq_1q_2}{r}$$

- Electric potential:

$$V = \frac{kq}{r}$$

- Electric potential difference (voltage):

$$\Delta V = \frac{\Delta U_q}{q}$$

## Relating $U_q$ , $F_q$ and $E$

From the fundamental theorem calculus, we can relate electrostatic force ( $\mathbf{F}_q$ ) to electric potential energy ( $U_q$ ) by the gradient operator, and electric field ( $\mathbf{E}$ ) to the electric potential ( $V$ ) the same way:

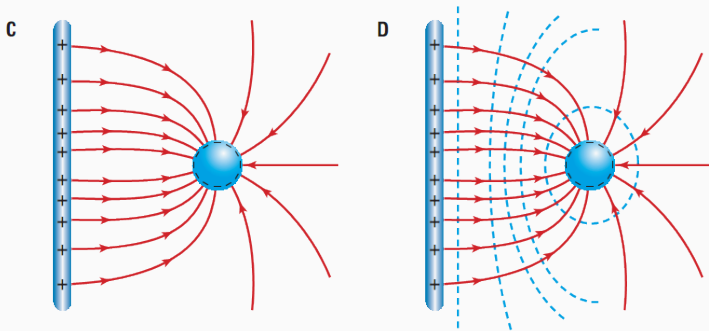
$$\mathbf{F}_q = -\nabla U_q = -\frac{\partial U_q}{\partial r} \hat{\mathbf{r}} \quad \mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r} \hat{\mathbf{r}}$$

- Electrostatic force  $\mathbf{F}_q$  always points from high to low potential energy (steepest descent direction)
- Electric field can also be expressed as the change of electric potential per unit distance, which has the unit

$$1 \text{ N/C} = 1 \text{ V/m}$$

- Electric field is also called “potential gradient”

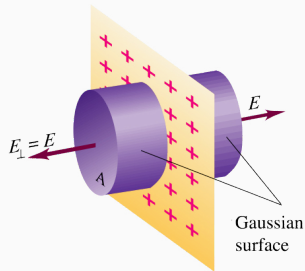
# Equipotential Lines



The dotted blue lines are called **equipotential lines**. They are always *perpendicular* to the electric field lines. Charges moving in the direction of the equipotential lines have constant electric potential



# Electric Field Near an Infinite Plane of Charge



- Charge density (charge per unit area)  $\sigma$
- By symmetry,  $\mathbf{E}$  must be perpendicular to the plane
- Our Gaussian surface is a cylinder shown in the left with an area  $A$ ; the height of the cylinder is unimportant
- Nothing “flows out” of the side of the cylinder, nly at the ends
- The total flux is  $\Phi_q = E(2A)$
- The enclosed charge is  $Q_{\text{encl}} = \sigma A$

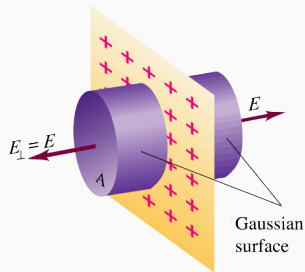
# Electric Field Near an Infinite Plane of Charge

Gauss's law simplifies to:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{encl}}}{\epsilon_0} \rightarrow E(2A) = \frac{\sigma A}{\epsilon_0}$$

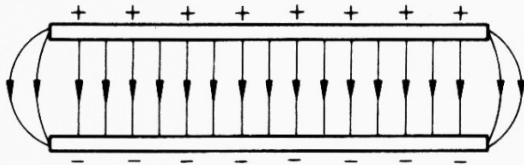
Solving for  $E$ , we get:

$$E = \frac{\sigma}{2\epsilon_0}$$



- $E$  is a constant
- Independent of distance from the plane
- Both sides of the plane are the same

# Electric Field Between Parallel Charged Plates



- Two plates, each producing an electric field pointing in the same direction
- The total electric field is twice the value of *one* infinite plane, pointing from the positively charged plate towards the negatively charged plate

$$E = \frac{\sigma}{\epsilon_0}$$

- **E** outside the plates is very low (close to zero), except for fringe effects at the edges of the plates

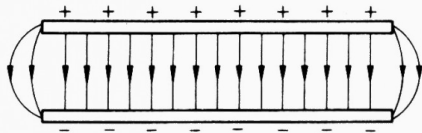
# Electric Field and Electric Potential Difference

Recall the relationship between electric field ( $\mathbf{E}$ ) and electric potential difference ( $V$ ):

$$\mathbf{E} = -\frac{\partial V}{\partial r}\hat{\mathbf{r}}$$

This relationship holds regardless of the charge configuration.

# Electric Field and Electric Potential Difference



In the case of two parallel plates, the electric field is uniform, and the relationship simplifies to:

$$E = \frac{\Delta V}{d}$$

Quantity	Symbol	SI Unit
Electric field intensity	$E$	N/C
Electric potential difference between plates	$\Delta V$	V
Distance between plates	$d$	m