Review of Basic Kinematics

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Kinematics is a discipline with in mechanics for describing the motion of points, bodies (objects), and systems of bodies (groups of objects). It is the mathematical representation of the relationship between

- Position
- Displacement
- Distance
- Velocity
- Speed
- Acceleration

However, kinematics does *not* deal with what causes motion.

1 Kinematic Quantities

1.1 Position

Position is a vector describing the location of an object in a coordinate system (usually *Cartesian*; can also be *polar*, *cylindrical* or *spherical*). The origin of the coordinate system is called "reference point". In the IJK notation for rectilinear motion, we can express position of an object by its x, y and z components:

$$\mathbf{x}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}$$

The position vector is a function of time t.

1.2 Displacement

Displacement is the change in position from 1 to 2 within the same coordinate system:

$$\Delta\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1 = (x_2 - x_1)\hat{\boldsymbol{\imath}} + (y_2 - y_1)\hat{\boldsymbol{\jmath}} + (z_2 - z_1)\hat{\boldsymbol{k}}$$

It it illustrated in Figure 1.

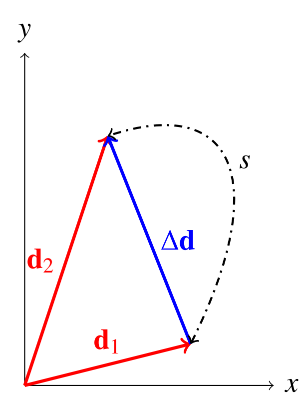

Figure 1: Position, displacement and distance are all related but different quantities.

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- IJK notation makes vector addition and subtraction less prone to errors

Note that since "reference point" is the origin of the coordinate system, i.e. $\mathbf{x}_{\text{ref}} = \mathbf{0}$, any position vector $\mathbf{x}$ is also its displacement from the reference point.

1.3 Distance

Distance s is a quantity that is *similar* (and related) to displacement. It is the *length of the path* taken when an object moves from position $\mathbf{d}_1$ to position $\mathbf{d}_2$. Unlike displacement, however, distance is a scalar quantity that is always positive: $s \geq 0$, i.e. you can never walk a *negative* distance to the store. Because the path is not always a straight line, therefore while the magnitude of the displacement vector is also a scalar, it is not necessarily the same as distance. In general

$$s \geq |\Delta \mathbf{d}|$$

1.4 Instantaneous & Average Velocities

Velocity is a quantity used to describe how *fast* an object is moving. If position $\mathbf{x}(t)$ is differentiable in time t, then its **instantaneous velocity** $\mathbf{v}(t)$ can be found at any time t by differentiating $\mathbf{x}$ with respect to time:

$$\boxed{\mathbf{v}(t) = \frac{d\mathbf{x}(t)}{dt}} \tag{1}$$

Since position $\mathbf{x}$ has x, y and z components in the $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ directions (they are linearly independent), we can take the time derivative of every component to obtain the velocity components v_x, v_y and v_z in those directions:

$$\mathbf{v}(t) = \frac{d\mathbf{x}}{dt} = \frac{dx}{dt}\hat{\boldsymbol{\imath}} + \frac{dy}{dt}\hat{\boldsymbol{\jmath}} + \frac{dz}{dt}\hat{\boldsymbol{k}} = v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}} + v_z\hat{\boldsymbol{k}}$$

By the fundamental theorem of calculus, if instantaneous velocity $\mathbf{v}(t)$ is the rate of change of position $\mathbf{x}(t)$ with respect to time t, then $\mathbf{x}(t)$ is the time integral of $\mathbf{v}(t)$:

$$\boxed{\mathbf{x}(t) = \int \mathbf{v}(t)dt + \mathbf{x}_0} \tag{2}$$

The constant of integration $\mathbf{x}_0 = \mathbf{x}(0)$ is the object's *initial position* at $t = 0$. As was the case in differentiation, we can integrate each component to get $\mathbf{x}$:

$$\mathbf{x}(t) = \left(\int v_x\hat{\boldsymbol{\imath}} + \int v_y\hat{\boldsymbol{\jmath}} + \int v_z\hat{\boldsymbol{k}}\right)dt + \mathbf{x}_0$$

The **average velocity** $(\overline{\mathbf{v}})$[1] of an object is the change in position $\Delta\mathbf{x}$ over a finite time interval Δt:

$$\boxed{\overline{\mathbf{v}} = \frac{\Delta\mathbf{x}}{\Delta t}} \tag{3}$$

Like instantaneous velocity, we can find the x, y and z components of average velocity by separating components in each direction:

$$\overline{\mathbf{v}} = \frac{\Delta x}{\Delta t}\hat{\boldsymbol{\imath}} + \frac{\Delta y}{\Delta t}\hat{\boldsymbol{\jmath}} + \frac{\Delta z}{\Delta t}\hat{\boldsymbol{k}} = \overline{v}_x\hat{\boldsymbol{\imath}} + \overline{v}_y\hat{\boldsymbol{\jmath}} + \overline{v}_z\hat{\boldsymbol{k}}$$

[1]For *time averages*, the convention is to write a bar over the quantity, as we have done here. In contrast, for *ensemble averages*, e.g. the average speeds of many particles, we use the notation $\langle v \rangle$. (See thermodynamics slides later in the course)

1.5 Instantaneous & Average Speed

Intantaneous speed $v(t)$ is the rate of change of *distance* with respect to time[2]:

$$\boxed{v(t) = \frac{ds}{dt}}$$

Since distance is a scalar quantity, so too is speed. As distance of any path must always be positive $s > 0$, instantaneous speed must also be positive. Instantaneous speed v is the magnitude of the instantaneous velocity vector $\mathbf{v}$. Likewise, **average speed** $\overline{v}$ is similar to average velocity: it is the distance travelled over a finite time interval.

$$\boxed{\overline{v} = \frac{s}{\Delta t}}$$

1.6 Path

Sometimes instead of explicitly describing the position $x = x(t)$ and $y = y(t)$, the path of an object can be given in terms of x coordinate $y = y(x)$, while giving the x (or y) coordinate as a function of time.

- In this case, substitute the expression for $x(t)$ into $y = y(x)$ to get an expression of $y = y(t)$
- Take derivative using chain rule to get $v_y = v_y(t)$

1.7 Instantaneous and Average Acceleration

In the same way that velocity is the rate of change in position with respect to time, **instantaneous acceleration** $\mathbf{a}(t)$ is the rate of change in velocity with respect to time, and the second time derivative of position, i.e.:

$$\boxed{\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{x}(t)}{dt^2}} \tag{4}$$

Although in Grades 11 and 12, students deal almost exclusively with constant acceleration, in AP Physics, it must be understood that acceleration can also vary with time, and that calculus must be used in many cases. Again, by the fundamental theorem of calculus, instantaneous velocity $\mathbf{v}(t)$ is the time integral of instantaneous acceleration $\mathbf{a}(t)$:

$$\boxed{\mathbf{v}(t) = \int \mathbf{a}(t)dt + \mathbf{v}_0} = \left(\int a_x\hat{\imath} + \int a_y\hat{\jmath} + \int a_z\hat{\boldsymbol{k}}\right)dt + \mathbf{v}_0 \tag{5}$$

1.8 Special Notation When Differentiating With Time

Physicists and engineers often use a special notation when the derivative is taken with respect to *time* (and not spatial derivatives), by writing a dot above the variable for *first* derivative, and *two* dots for *second* derivative, etc. For example, velocity is $\mathbf{v}(t) = \dot{\mathbf{x}}$ while acceleration is $\mathbf{a}(t) = \dot{\mathbf{v}} = \ddot{\mathbf{x}}$. This notation will be used occasionally in this course when it is convenient to do so.

[2]It is regrettable that both velocity and speed use the symbol v, but *c'est la vie*.

1.9 Higher Derivatives of Position (For Those Who Are Curious)

For those who are curious about higher derivatives, the time derivative of acceleration is called **jerk** $\mathbf{j}(t)$ with a unit of m/s^3:

$$\mathbf{j}(t) = \frac{d\mathbf{a}}{dt} = \frac{d^2\mathbf{v}}{dt^2} = \frac{d^3\mathbf{x}}{dt^3}$$

Jerk is used in many sensors, for example, in accelerometers in airbags to determine if the acceleration of a car is under normal operation (small j value) or if a crash is in progress (high j value).

The time derivative of jerk is **jounce**, or **snap**, with a unit of m/s^4:

$$\mathbf{s}(t) = \frac{d\mathbf{j}}{dt} = \frac{d^2\mathbf{a}}{dt^2} = \frac{d^3\mathbf{v}}{dt^3} = \frac{d^4\mathbf{x}}{dt^4}$$

The next two derivatives of snap is facetiously called **crackle** and **pop**[3], but these higher derivatives are rarely used, and will *not*be used in in AP Physics.

2 Kinematic Equations

Although kinematic problems in AP Physics often require calculus, these basic kinematic equations for constant acceleration are still a very powerful tool.

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$$
$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}t$$
$$v^2 = v_0^2 + 2a(x - x_0)$$

If acceleration is nonconsta, differentiation and integration will be necessary.

[3]As in the cartoon mascots for Kellogg's rice crispies