Topic 6: Simple Harmonic Motion

Advanced Placement Physics

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Olympiads School

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Quick Review of Hooke's Law

- Hooke's Law for an ideal spring

$$\boxed{\mathbf{F}_s = -k\mathbf{x}}$$

- Spring force $\mathbf{F}_s$ is the force that a compressed/stretched spring exerts on another object
- Spring constant k (or *Hooke's constant*, *force constant*) describes the stiffness of the spring

Review of Elastic Potential Energy

Applying Hooke's law in the work equation, we can find the work done when compressing/stretching a spring:

$$W = \int_{x_1}^{x_2} F dx = -\int_{x_1}^{x_2} kx dx = -\frac{1}{2}kx^2 \Big|_{x_1}^{x_2}$$

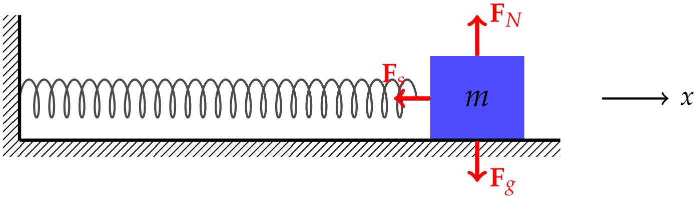
$$= -\Delta\left(\frac{1}{2}kx^2\right) = -\Delta U_e$$

where elastic potential energy is defined as

$$\boxed{U_e = \frac{1}{2}kx^2}$$

Mass on a Spring

- Consider the forces acting on a mass connected horizontally to a spring



- $\mathbf{F}_g$ and $\mathbf{F}_N$ cancel out, so net force is due only to spring force $\mathbf{F}_s = -k\mathbf{x}$. This is true both when the spring is in compression or extension. (In the diagram above, the spring is in extension.)
- Applying Newton's 2nd law in the x-direction, we get

$$F_s = ma \quad \longrightarrow \quad -kx = m\frac{d^2x}{dt^2}$$

Mass on a Spring

$$-kx = m\frac{d^2x}{dt^2} \quad \text{or} \quad \boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

- To solve the above equation, we look for a function $x(t)$ where the second derivative $x''(t)$ looks like $x(t)$ itself but with a negative sign
- The obvious choice are two trigonometric functions: $\sin(t)$ and $\cos(t)$
- We start with this general form

$$x(t) = A\sin(\omega t + \phi)$$

We usually prefer using $\sin$ over $\cos$, but the two functions only differ in ϕ

It's Very Ordinary

- The governing equation for simple harmonic equation is a *second-order ordinary differential equation*:

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

- Starting with our general form, we can take the 1st and 2nd derivatives:

$$\begin{aligned} x(t) &= A\sin(\omega t + \phi) \\ x'(t) &= A\omega\cos(\omega t + \phi) \\ x''(t) &= -A\omega^2\sin(\omega t + \phi) = -\omega^2 x \end{aligned}$$

ω is the angular velocity, and A is the amplitude, and ϕ is a phase shift that depends on the initial condition

Mass on a Spring

Angular Velocity

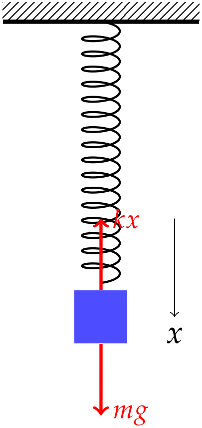
- Substituting expressions of $x(t)$ and $x''(t)$ into the ODE, we find our equation is satisfied if angular velocity (angular frequency) is:

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

- Angular velocity ω does not depend on amplitude A
- The period T and frequency f of the motion are given by:

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}} \qquad \boxed{T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}}$$

Vertical Spring-Mass System



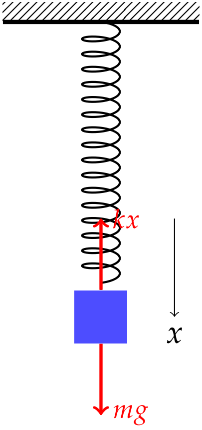
- For a vertical spring-mass system, the analysis is *slightly* more complicated (we have to consider weight as well):

$$mg - kx = m\frac{d^2x}{dt^2}$$

- But since mg is a constant, the only difference is the addition of a constant B in our expression of $x(t)$:

$$\begin{aligned} x(t) &= A\sin(\omega t + \phi) + B \\ x''(t) &= -A\omega^2\sin(\omega t + \phi) \end{aligned}$$

Vertical Spring-Mass System



- Substituting $x(t)$ and $x''(t)$ into the differential equation gives

$$B = \frac{mg}{k}$$

- B is just the stretching of the spring due to its weight
- Angular velocity is the same as the horizontal case! It is still given by:

$$\omega = \sqrt{\frac{k}{m}}$$

Conservation of Energy in a Spring-Mass System

- In both the horizontal and vertical spring-mass systems, there is no friction, and so the mass and the spring form an isolated system
- The forces by the spring on the mass (and by the mass on the spring) are *internal* forces, and energy in the system is conserved:

$$K_1 + U_{e,1} + U_{g,1} = K_2 + U_{e,2} + U_{g,2}$$

- For the horizontal spring-mass system, the total energy is:

$$E_{\text{total}} = \frac{1}{2}kA^2$$

Simple Example

Example 2: A mass suspended from a spring is oscillating up and down. Consider the following two statements:

1. At some point during the oscillation, the mass has zero velocity but it is accelerating
2. At some point during the oscillation, the mass has zero velocity and zero acceleration.

(a) Both occur at some time during the oscillation

(b) Neither occurs during the oscillation

(c) Only (1) occurs

(d) Only (2) occurs

Another Example

Example 3: An object of mass $5\,\mathrm{kg}$ hangs from a spring and oscillates with a period of $0.5\,\mathrm{s}$. By how much will the equilibrium length of the spring be shortened when the object is removed.

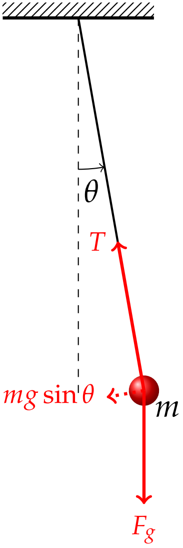
(a) $0.75\,\mathrm{cm}$

(b) $1.50\,\mathrm{cm}$

(c) $3.13\,\mathrm{cm}$

(d) $6.20\,\mathrm{cm}$

What About a Pendulum?



- Pendulums also exhibit oscillatory motion
- For a pendulum, there are two forces acting on the mass: weight $F_g = mg$ and tension T
- When the mass is deflected by an angle θ, it's easy to show (using polar coordinates) that the force in the angular direction is $F_\theta = -mg\sin\theta$
- We needn't worry about the radial direction because it doesn't have anything to do with the restoring force

The Pendulum

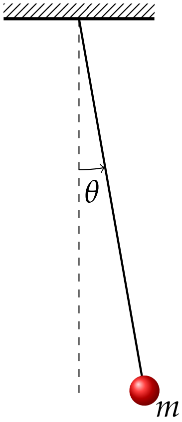
- Substitute F_θ into Newton's second law, and cancelling mass term, we get:

$$F_\theta = ma_\theta \quad \longrightarrow \quad -g\sin\theta = L\frac{d^2\theta}{dt^2}$$

- For small angles, $\sin\theta \approx \theta$, and we get the ODE for the pendulum:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

- This ODE has the same form as the mass-spring system!



Ordinary Differential Equation for the Pendulum

- The solution for $\theta(t)$ is very similar to the spring-mass system:

$$\boxed{\theta(t) = \theta_{\text{max}} \sin(\omega t + \phi)}$$

where angular velocity ω is given by

$$\boxed{\omega = \sqrt{\frac{g}{L}}}$$

and ϕ is a phase shift based on the initial condition of the pendulum.

- Remember that this equation is only valid for small angles, i.e. $\theta_{\text{max}} < 15^\circ$

Pendulum Example Problem

Example 4: A simple pendulum consists of a mass m attached to a light string of length l. If the system is oscillating through small angles, which of the following is true

(a) The frequency is independent of the acceleration due to gravity, g.

(b) The period depends on the amplitude of the oscillation.

(c) The period is independent of the mass m.

(d) The period is independent of the length l.

Another Pendulum Example

Example 5: A bucket full of water is attached to a rope and allowed to swing back and forth as a pendulum from a fixed support. The bucket has a hole in its bottom that allows water to leak out. How does the period of motion change with the loss of water?

(a) The period does not change.

(b) The period continuously decreases.

(c) The period continuously increases.

(d) The period increases to some maximum and then decreases again.

Think About g

Example 6: A little girl is playing with a toy pendulum while riding in an elevator. Being an astute and educated young lass, she notes that the period of the pendulum is $T = 0.5\,\mathrm{s}$. Suddenly the cables supporting the elevator break and all of the brakes and safety features fail simultaneously. The elevator plunges into free fall. The young girl is astonished to discover that the pendulum has:

(a) continued oscillating with a period of $0.5\,\mathrm{s}$.

(b) stopped oscillating entirely.

(c) decreased its rate of oscillation to have a longer period.

(d) increased its rate of oscillation to have a lesser period.