

2. Calculus in Physics—Integration

AP Physics

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Olympiads School

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Files for You to Download

- 00-outline.pdf—The course outline
- 01-Calculus-print.pdf—The slides that I used last week
- 01-integration.pdf—The slides that I am using right now
- 01-Homework.pdf—Last/this week's homework assignment

Please download/print the PDF file for the class slides before each class.

On Differential Calculus

A quick review

- Finding out how quickly a physical quantity is changing (“rate of change” of that quantity)
- Math: slopes of functions
- Terminology:
 - A **derivative**: The slope of a function (noun)
 - To **differentiate**: Finding the derivative with respect to a variable (verb)
- Last class: went through the rules and some examples of derivatives

Examples of Derivatives in Physics

- Instantaneous velocity $\mathbf{v}(t)$ is the derivative of position $\mathbf{s}(t)$

$$\mathbf{v}(t) = \frac{d\mathbf{s}}{dt}$$

- Instantaneous acceleration $\mathbf{a}(t)$ is the derivative of velocity $\mathbf{v}(t)$. It's also the “second derivative” (derivative of a derivative) of position \mathbf{s} with respect to time

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{s}}{dt^2}$$

- Instantaneous force $\mathbf{F}(t)$ is the derivative of momentum \mathbf{p} (Newton's second law of motion)

$$\mathbf{F}(t) = \frac{d\mathbf{p}}{dt}$$

They're All Vectors

Resolve them into components

- Notice that position \mathbf{s} , velocity \mathbf{v} , acceleration \mathbf{a} , momentum \mathbf{p} , force \mathbf{F} are all vector quantities with x , y and z components
- In this case, we take the derivative separately in each direction.

$$\mathbf{v}(t) = \frac{d\mathbf{s}}{dt} = \frac{d}{dt} [s_x(t)\hat{\mathbf{i}} + s_y(t)\hat{\mathbf{j}} + s_z(t)\hat{\mathbf{k}}]$$

where s_x , s_y and s_z are the x -, y - and z -components of \mathbf{s}

- In AP or 1st-year physics, s_x , s_y and s_z are functions of time only, but in practical problems in physics and engineering, they are often functions of x , y and z coordinates as well. (This is *multi-variable calculus*. It's a lot of fun!)

Other Derivatives

... Not Always With Respect to Time

- **Electric force** is the derivative of electrical potential energy with respect to radial distance:

$$F_q = -\frac{dU_q}{dr} = -\frac{d}{dr} \left[\frac{kq_1q_2}{r} \right]$$

Gravitational force and gravitational potential energy are related in the same way

- **Electric Field** is the derivative of electric potential difference V with respect to the radial distance

$$E = -\frac{dV}{dr}$$

What the Notation Tells Us

- When we say we that **velocity is the time rate of change of position**

$$\mathbf{v} = \frac{ds}{dt}$$

- We are really asking **what is the (small) change in position ds over an infinitesimal (infinitely small) change in time dt ?**

NOW ON TO INTEGRATION

Integral Calculus

- The opposite of differentiation
- We use it to compute the area under a curve, or
- Summation of many very small terms
- Examples: area under the v - t graph (to calculate displacement), area under the F - t graph (to calculate impulse), area under the F - d graph (to calculate work)

Integration: Area Under the Curve

Let's start with an example

- A car moves with speed $v(t) = 2 + 5t$. What is its displacement at $t = 5$?
- We know that displacement is the area under a v - t graph. How do we find the area? (Pretend that we don't know the area of a trapezoid!)
- Divide the interval from $t = 0$ to 5 into n equal small time intervals Δt
- The displacement in each of these Δt_i is approximately

$$\Delta s_i = v_i \Delta t$$

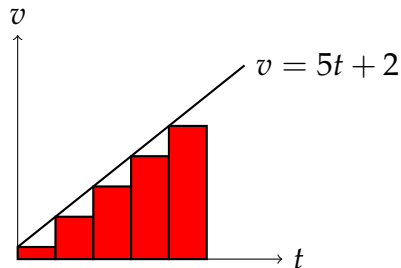
where $v_i = v(t_i)$

- And the total displacement is:

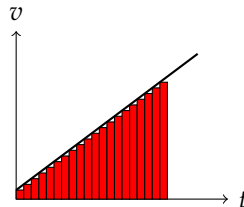
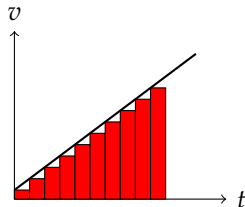
$$\Delta s = \sum_{i=1}^n \Delta s_i = \sum_{i=1}^n v_i \Delta t$$

Integration: Area Under a Curve

- Shown graphically:



- This is not very inspiring, but we can do better if we increase n :



Integration: Area Under a Curve

- If we increase n to ∞ , we will have the *actual* displacement:

$$\Delta s = \lim_{n \rightarrow \infty} \sum_{i=1}^n v_i \Delta t$$

- In fact, this limit is called the integral:

$$\Delta s = \int_{t_1}^{t_2} v(t) dt$$

- As $n \rightarrow \infty$, the time interval Δt becomes infinitesimally small, i.e. “ dt ”

Integration: Area Under a Curve

- In our example, we have this particular integral:

$$\Delta s = \int_{t_1}^{t_2} v(t) dt = \int_0^5 (5t + 2) dt$$

- How do we compute it?

The Antiderivative

- If: $v(t)$ is the derivative of $s(t)$,
- Then: $s(t)$ is the “antiderivative” of $v(t)$
- In general, if $F(x)$ is the antiderivative of $f(x)$, they are related this way:

$$\frac{d}{dx}F(x) = f(x) \quad \longrightarrow \quad F(x) = \int f(x)dx$$

- If we want to integrate $f(x)$, we are actually asking “what function $F(x)$ has a derivative equal to $f(x)$ ”?
- A simple example with $f(x) = t$ and $F(x) = \frac{1}{2}t^2$:

$$\frac{d}{dt} \left(\frac{1}{2}t^2 \right) = t \quad \longrightarrow \quad \int t dt = \frac{1}{2}t^2$$

Commonly Integrals in Physics

Integrating a function can be a very daunting task, but most examples in AP Physics will be straightforward. These examples should help in most cases:

- Power Rule in reverse:

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

We can “ignore” (i.e. cancel) the constants C for definite integrals.

Definite vs. Indefinite Integral

- An integral can be either “indefinite” or “definite”
- An “indefinite” integral gives us another function.
- For example, given velocity $v(t)$, we can find position $s(t)$ as a function of time:

$$s(t) = \int v(t)dt = \dots + C$$

- A constant C is added to the anti-derivative of $v(t)$. The exact value of C is obtained through applying an “initial condition” to the problem.
- Note: If $s(t) = [\text{something}] + C$ and we take the derivative of s to find $v(t)$, the constant term disappears regardless of what value it has.

Definite Integral

- An integral can also be **definite**, with lower and upper bounds.
- For example: given $v(t)$, find displacement between $t = 3$ and $t = 5$:

$$\Delta s = \int_3^5 v(t) dt$$

- Once we have computed the integral, we have to evaluate between the limits:

$$\Delta s = s(t) \Big|_3^5 = s(5) - s(3)$$

- In this case we do not have to bother with the constant C , since it'll cancel out when we evaluate the bounds.

Back to Our Example

- The integral in our example is a *definite* integral because it has limits 0 and 5
- We can evaluate it by:

$$\begin{aligned}\Delta s &= \int_0^5 (5t + 2) dt = \int_0^5 5t dt + \int_0^5 2 dt \\ &= \left. \frac{5}{2}t^2 \right|_0^5 + \left. 2t \right|_0^5 = \frac{125}{2} + 10 \\ &= \boxed{\frac{145}{2}}\end{aligned}$$

- Each part of the sum can be integrated separately

Area Under A Curve

A Math Problem

What is the area under the curve

$$f(x) = 2x^2 + 3x + 1 \quad \text{between} \quad x = 1 \text{ and } x = 5$$

Our integration works like this:

$$\begin{aligned} A &= \int_1^5 (2x^2 + 3x + 1) dt \\ &= \left(\frac{2}{3}x^3 + \frac{3}{2}x^2 + x \right) \Big|_1^5 \\ &= 24 + \frac{196}{3} \end{aligned}$$

Remember Our Kinematic Equations?

- In Physics 11 and 12, you were introduced to some kinematic equations for constant acceleration.
- Now that we know something about integration, we can understand these equations a little bit better
- Start with a constant acceleration a . The velocity is the integral:

$$v(t) = \int a dt = at + C$$

- At $t = 0$, $v = v_0$ (“initial value”). Substituting those values to find $C = v_0$, and therefore

$$v(t) = v_0 + at$$

Remember Our Kinematic Equations?

- Now we integrate $v(t)$ again to get position $s(t)$:

$$s(t) = \int v(t)dt = \int (v_0 + at)dt = v_0t + \frac{1}{2}at^2 + C$$

- Again, we take advantage of know our initial position, so $C = s_0$, and we have:

$$s(t) = s_0 + v_0t + \frac{1}{2}at^2$$

- You may be more familiar with this expression, where we use *displacement* $\Delta s(t) = s(t) - s_0$ instead of position s :

$$\Delta s(t) = v_0t + \frac{1}{2}at^2$$

Remember Our Kinematic Equations?

- In practical situations, acceleration is *not* constant, and we generally have to differentiate or integrate to find your answers.

Other Integrals

- **Impulse** (The components of the \mathbf{F} can be integrated separately)

$$\mathbf{J} = \int_{t_1}^{t_2} \mathbf{F} dt$$

- **Work** by non-constant force

$$W = \int_{x_1}^{x_2} \mathbf{F}(x) \cdot d\mathbf{s}$$

This integral is straight forward if \mathbf{F} is expressed as a function of position \mathbf{s} , but if it is written as a function of time, i.e. $\mathbf{F}(t)$, then we have to express \mathbf{s} as a function of time as well

One Last Example

Work by Non-Constant Force

A force of $F(t) = 5t$ N is applied on an object $m = 1$ kg initially at rest, there is no friction force. What would be the displacement and work done on this object after 3 s?

1. Apply Newton's second law to find acceleration:

$$a(t) = \frac{F(t)}{m} = 5t$$

2. Then we integrate $a(t)$ with respect with time to get velocity $v(t)$:

$$v(t) = \int a(t)dt = \frac{5}{2}t^2$$

We already know that $v_0 = 0$, so we don't have to add C after the integral.

One Last Example

Work by Non-Constant Force

A force of $F(t) = 5t$ N is applied on an object $m = 1$ kg initially at rest, there is no friction force. What would be the displacement and work done on this object after 3 s?

3. We integrate again to get an expression for position $s(t)$:

$$\Delta s(t) = \int_0^3 v(t) dt = \left. \frac{5}{6} t^3 \right|_0^3 = \frac{5}{6} (3^3 - 0) = \frac{45}{2} \text{ m}$$

One Last Example

Work by Non-Constant Force

A force of $F(t) = 5t$ N is applied on an object $m = 1$ kg initially at rest, there is no friction force. What would be the displacement and work done on this object after 3 s?

4. We know from our differential calculus that

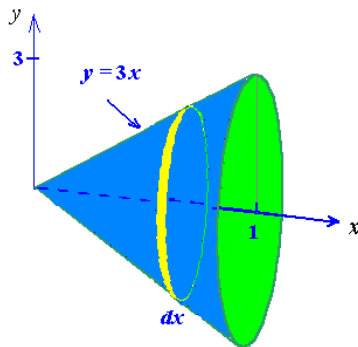
$$v(t) = \frac{ds}{dt} \quad \longrightarrow \quad ds = v(t)dt$$

5. The last step is to integrate force with velocity to find work done. (This works because we know F as a function of time.)

$$W = \int Fds = \int F(t)v(t)dt = \int_0^3 \frac{25}{2}t^3dt = \frac{25}{8}t^4 \Big|_0^3 = \frac{2025}{8} = \boxed{253 \text{ J}}$$

Integration to Find Volume

- Interested in finding the volume when we rotate *any* function about the x axis
- There are many applications, e.g. finding the CG or centroid of shapes



- Each yellow disk has a volume of $\pi r^2 dx$, where $r = f(x)$, so the infinitesimal volume dV of each disk is in fact:

$$dV = \pi f(x)^2 dx$$

- “summing” them together gives us the integral:

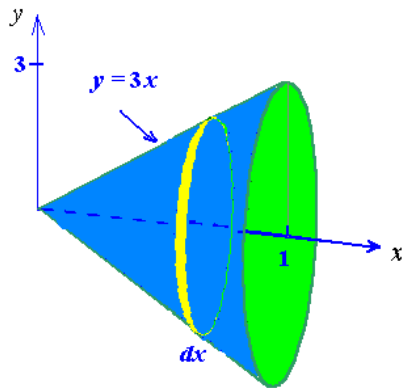
$$V = \int_{x_1}^{x_2} dV = \int_{x_1}^{x_2} \pi f(x)^2 dx$$

Integration to Find Volume

Example: Find the volume of the following shape:

- In this question, $f(x) = 3x$, and we are integrating from $x_1 = 0$ to $x_2 = 1$

We use the formula from before:



$$\begin{aligned} V &= \int_{x_1}^{x_2} \pi f(x)^2 dx \\ &= \int_0^1 \pi 9x^2 dx \\ &= 9\pi \int_0^1 x^2 dx \\ &= 3\pi x^3 \Big|_0^1 \\ &= 3\pi \end{aligned}$$