Examples of Rigid-Body Rotational Motion

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The case of a rolling sphere is a standard example of combining the dynamics of translational and rotational motions of a rigid body. In this handout, two typical examples are be presented for a non-slip (i.e. pure rolling) case, while one example is presented for a case with slippage.

1 Pure Rolling of Rigid Body on Flat Surface

In the first case, a smooth solid sphere of constant density rolls on a smooth surface without slipping (rolling without slippage is called pure rolling). We assume that the sphere and the surface are both perfectly rigid, in that they cannot be deformed. We also assume that the sphere and the surface are both perfectly smooth without defects even at the atomic level. The free-body diagram is shown in Fig. 1. Notice that there is no friction between the sphere and the surface. Since there is neither a net force nor a net torque acting on the sphere, neither the translational nor rotational state of the ball changes over time. In theory—if our assumptions are correct—a the sphere with translational velocity $\mathbf{v}$ and angular velocity $\boldsymbol{\omega}$ will roll along forever.

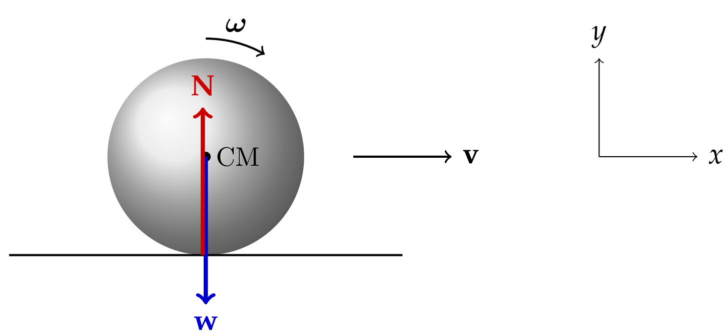

Figure 1: Force diagram on a uniform density solid sphere rolling on a smooth flat surface without slipping.

But of course, we are very observant. Even a casual observer will notice that in reality, a ball will slow down and eventually come to a stop. A steel ball bearing rolling on a track rolls over a much longer distance and much longer time than a soccer ball on a grassy field, but neither will roll forever. So what causes this? Specifically, what is missing in the free-body diagram in Fig. 1?

Our assumptions aren't quite correct. There are two major oversights in our initial assumptions (perfectly smooth surfaces, rigid bodies).

1. Nothing is perfectly smooth. Firstly, our original assumptions mean that the contact area is infinitesimal small, and the normal force, by basic geometry, points straight toward the CM. However, we should recognize that neither the ball bearing nor the rail are perfectly smooth. When a non-smooth ball rolls over a non-smooth surface, their surface roughness means that the contact point is finite in size, and that the normal force does not necessarily point toward the CM of the ball. This means that unlike in Fig. A, there is a net force and net torque that will slow down the motion of the ball.

2. Nothing is perfectly rigid. Secondly, we must recognize that there is no such thing as a perfectly rigid body[1]. Both the ball and the surface deform as they make contact. A perfect illustration is how a tire flattens when it makes contact with the ground, shown in Fig. 2. The normal force is large in magnitude on the front side is large in magnitude than on the other, and therefore exerts both a resistive force to slow down the wheel, as well as negative torque to slow down the tire. Also, the normal force does not point toward the CM. This is called "rolling resistance".

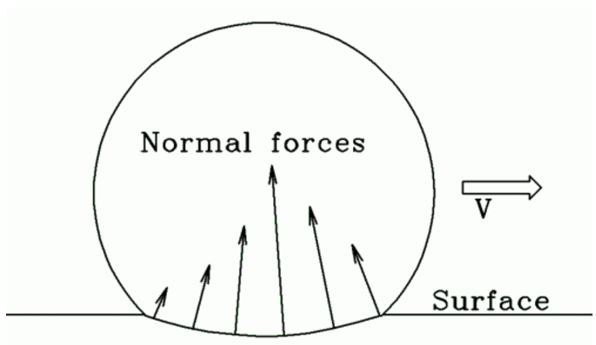

Figure 2: Deformation of a tire under load as it rolls over a surface without slipping.

Now that we have understood the basic problem, we can tackle the next problem that involves friction.

2 Pure Rolling on an Inclined Surface

But what if the sphere rolls without slippage down a ramp of angle θ instead? The free-body diagram for that sphere shown in Fig. 3. The radius of the sphere is R. This time, there is also a static friction $\mathbf{f}_s$ acting up the ramp at at the point of contact between the sphere and the surface.

Be careful what forces are acting on it. The weight of the sphere acts at the center of gravity, while the normal force acts at the point of contact. Neither forces generate any torque about the CM, therefore,

[1]This should be obvious, but in the pursuit of learning physics, this is a detail that may be lost. When two objects collide in any collision, it takes a finite amount of time for either objects to accelerate to the new velocities. If both objects are perfectly rigid, then the collision will occur over an infinitely small time interval, with infinitely large forces acting on them.

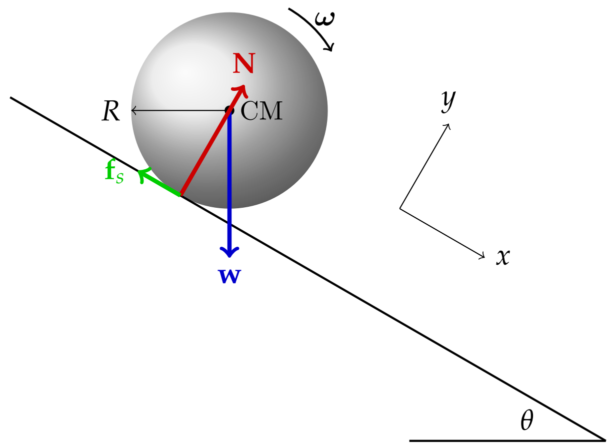

Figure 3: Force diagram on a smooth solid sphere of radius R rolling down a smooth ramp without slipping. The ball travels distance d to the bottom of the ramp.

without friction, the sphere will just slide down the ramp without rotation. To solve this problem, we have three dynamic equations along the three axes[2]:

$$\sum F_x = mg\sin\theta - f_s = ma \tag{1}$$

$$\sum F_y = N - mg\cos\theta = 0 \tag{2}$$

$$\sum \tau_z = rf_s = I_z\alpha \tag{3}$$

Don't be so sure about what μ_s tells us. At this stage, the actual static friction force is not known and is a quantity that needs to be solved. Knowledge of the coefficient of static friction μ_s may not be useful, because it only tells you the maximum static friction, not the actual friction that exists. However, we can use it to double check to see if the answer makes sense.

Relating rotational and translational motions. Inserting the expression for the moment of inertia of the solid sphere $I_z = \frac{2}{5}mR^2$ and recognizing that for pure rolling, $\alpha = \frac{a}{R}$, we can use Eq. 3 to express static friction in terms of linear acceleration a:

$$f_s = \frac{I_z\alpha}{R} = \left(\frac{2}{5}mR^2\right)\left(\frac{a}{R}\right)\left(\frac{1}{R}\right) = \frac{2}{5}ma \tag{4}$$

Substituting the expression in Eq. 4 into Eq. 2, the force equation in the x-direction becomes:

$$mg\sin\theta - \frac{2}{5}ma = ma \tag{5}$$

Cancelling the mass terms and solving for acceleration, we find a constant acceleration of :

$$a = \frac{5}{7}g\sin\theta \tag{6}$$

Compare the results in Eq. 6 to that of an object sliding without friction down the same ramp, the acceleration for the sliding block is $a = g\sin\theta$ which is higher than the pure rolling case. The simplest

[2] the $\hat{\boldsymbol{k}}$ axis points out of the page. Counter-clockwise rotational motion is positive, while clockwise rotational motion is negative

explanation is that some of the gravitational potential energy is converted to both translational and rotational kinetic energies, while for the sliding case, all of the potential energy is converted into translational kinetic energy.

There is, of course, one “sanity check” that must be done, that is to make sure that the friction calculated in Eq. 4 has not exceeded the maximum static friction, given by

$$\max f_s = \mu_s N \tag{7}$$

If this is indeed the case, it means that the sphere will actually slip while rolling down the ramp, and the friction at the contact point is in fact kinetic friction.

Since acceleration is constant, kinematic equation can be used to compute the speed of the sphere when it reaches the bottom of the ramp, a distance d away. For simplicity, we assume that the sphere starts from rest:

$$v = \sqrt{2ad} = \sqrt{\frac{10}{7} gd \sin\theta} \tag{8}$$

Energy is conserved, of course. There is a much simpler way to find v, that is, by using the conservation of energy. In this case, kinetic energy is split between translational component K_t and rotational component K_r:

$$\begin{aligned}
\Delta U_g &= K_t + K_r \\
mg\Delta h &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \\
mgd\sin\theta &= \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\left(\frac{v}{r}\right)^2 = \frac{1}{2}mv^2 + \frac{1}{5}mv^2 \\
&= \frac{7}{10}mv^2
\end{aligned}$$

Now cancelling mass terms on both sides, and solving for v, we arrive at the same expression as using dynamics and kinematics equations:

$$v = \sqrt{\frac{10}{7} gd \sin\theta} \tag{9}$$

Friction doesn’t do work. That energy is conserved even when there is friction should be a significant insight for the novice physics student. However, this should not come as a surprise either, because in order for a force to do any work (conservative or not), it has to actually move something. This is clearly not the case if friction is static. Since no non-conservative work is done by friction, energy is conserved. To be clear, if the sphere slips, then there will definitely be non-conservative work done by kinetic friction.

3 Rolling on Flat Surface with Slippage

We return to the flat-surface problem, but this time, we allow slippage at the point of contact between the sphere and the surface. In this case, because there is relative motion between them, there is kinetic friction $f_k = \mu_k N$ at the point of contact, as shown in Fig. 4. At that point, the sphere slides to the left relative to the surface, and therefore the force of friction is toward the right.

There is now a net force in the $\hat{\boldsymbol{\imath}}$ direction, and a positive net torque in the $\hat{\boldsymbol{k}}$ direction (i.e. net torque is counter-clockwise). The consequences are that:

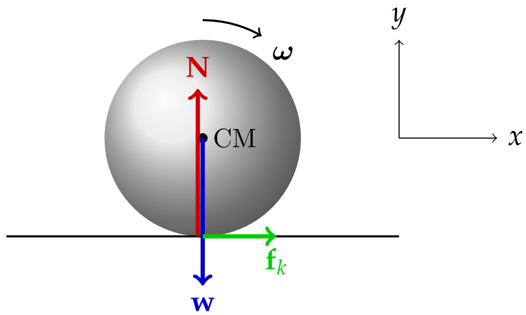

Figure 4: Force diagram on a smooth solid sphere rolling on a flat surface with slippage.

1. The net force toward the right causes the sphere to accelerate in the positive $\hat{\imath}$ direction. Since the kinetic friction is a constant, the acceleration is also constant in time as well (for as long as the sphere slips). At first glance, this may seem counter intuitive. However, we know that a car with its tires spinning on ice will still have a small acceleration.
2. The net torque in the positive $\hat{\boldsymbol{k}}$ (counter clockwise) direction causes the angular velocity ω to decrease over time.

It is important to note that, unlike the previous no-slip cases where we can relate angular acceleration α with linear acceleration a by the radius of the sphere, for the slippage case, there is no relationship between α and a. The velocity of the sphere $\mathbf{v}$ toward the right can be expressed with a simple kinetic equation:

$$v_x = v_0 + at \tag{10}$$

while the angular velocity of the sphere is given by:

$$\omega = \omega_0 + \alpha t \tag{11}$$

Note that ω_0 is negative, since the rotation is clockwise. At some time t there will be a point in time where $v = \omega r$. When this happens, the sphere stops slipping, and the problem returns to the no-slip case that was discussed in Section 1.

4 How To Solve Rotational Problems

When solving for rotational problems like the ones described in the previous sections, it is imperative to carefully draw a free-body diagram to account for all the forces and torques acting on an object, as we have done in the previous examples. A few things to keep in mind:

- The direction of friction force is not always obvious.
- The magnitude of any static friction force cannot be assumed to be at maximum.
- If the object is to change its rotational state, there must be a net torque causing it.

Once the free-body diagram is complete, we can breaks down the forces into $\hat{\imath}$, $\hat{\jmath}$ and $\hat{\boldsymbol{k}}$ components. We have now a set of three equations from the second law of motion:

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \sum F_z = ma_z$$

It is likely that only one direction will have acceleration.[3] In the problems that are presented in this handout, there are no forces in the $\hat{\boldsymbol{k}}$ direction. We have only needed to use the $\hat{\boldsymbol{\jmath}}$ direction in the third (with slippage) problem to calculate the normal force, so that the kinetic friction f_s can be calculated.

Because the motion is rotational in nature, we will also have to sum the net torque along those same three axes as well:

$$\sum \tau_x = I_x \alpha_x \qquad \sum \tau_y = I_y \alpha_y \qquad \sum \tau_z = I_z \alpha_z$$

In simpler cases like the ones presented here, net torque will only be along the $\hat{\boldsymbol{k}}$ direction, and there were no torque by any of the forces along the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ directions (although for more complicated problems, there can be net torque in all three directions). Note that the moments of inertia are not equal ($I_x \neq I_y \neq I_z$) if the rolling object is not a sphere.

Depending on whether an object rolls with or without slipping, there may be no relationship between angular velocity ω and translational velocity $\mathbf{v}$, or between angular acceleration α and translational acceleration $\mathbf{a}$. But in any case, you will be left with a system of equations with equal number of unknowns to be solved. Use whatever method that you are comfortable with to solve for the answers.

[3] In fact, whenever possible, it is a good practice to orient the Cartesian coordinate system such that acceleration only occurs in one direction.