Class 12: Circuits Analysis

AP Physics

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Olympiads School

February 2018

Notice: Tim Will Be Away Next Week

Tim will be away for a concert in Ottawa next Saturday (February 17). There will be a supply teacher for next week. Please be kind! Tim will return to Olympiads on Sunday, February 18,, and will resume teaching this class on the 24th (class 14).

Files for You to Download

Download from the school website:

1. `12-Circuits_print.pdf`—The print version of this presentation. If you want to print on paper, I recommend printing 4 pages per side.
2. `13-Homework.pdf`—Homework assignment for this class and next class. The file will be posted along Class 13 slides.

Please download/print the PDF file before each class. There is no point copying notes that are already printed out for you. Instead, take notes on things I say that aren't necessarily on the slides.

Current

We define conventional current as the rate at which positive charges flows through a wire, i.e.:

$$\boxed{I = \frac{dQ}{dt}}$$

We can also think of the current as the flux of "charge density" flowing past a surface, i.e.:

$$\boxed{I = \int \mathbf{J} \cdot d\mathbf{A}}$$

$d\mathbf{A}$ is defined the same way as Gauss's law, with its magnitude being the infinitesimal area dA and the direction being the outward normal from the surface. This expression is especially important when dealing with plasma etc.

Current

- We now know that in a wire, instead of positive charges flowing in one direction, we have, in fact, electrons (negative charges) flowing in the opposite direction
- We are treating current as a scalar quantity, but later we will have to treat it as a vector
- As charges move through a conductor, the will lose potential energy
- How much energy it loses depends on the resistance of the material

Resistivity

The resistivity of a material is related to the electric field and current density inside

$$\boxed{\mathbf{E} = \rho \mathbf{J}} \qquad \text{in scalar: } \boxed{\rho = \left| \frac{E}{J} \right|}$$

Quantity	Symbol	SI Unit
Electric field	$\mathbf{E}$	N/C (newtons per coulomb)
Charge density	$\mathbf{J}$	$\mathrm{A/m^2}$ (amperes per metre square)
Resistivity	ρ	Ω/m (ohms per metre)

Resistivity

$$\boxed{\mathbf{E} = \rho\mathbf{J}} \qquad \text{in scalar: } \boxed{\rho = \left|\frac{E}{J}\right|}$$

- In a conductor, because the electrons are free to move, the electric field tend to be very small, and the resistivity is low.
- In a dielectric (non conducting material), electrons cannot move easily–only polarize themselves–the electric field are generally strong, and the resistivity is higher.

Resistance of a Conductor

The resistance of a conductor is proportional to the resistivity ρ and its length L, and inversely proportional to the cross-sectional area A:

$$\boxed{R = \rho \frac{L}{A}}$$

Quantity	Symbol	SI Unit
Resistance	R	Ω (ohms)
Resistivity	ρ	Ω m (ohm metres)
Length of conductor	L	m (metres)
Cross-sectional area	A	m^2 (square metres)

Resistance of a Conductor

$$\boxed{R = \rho \frac{L}{A}}$$

Gauge	Diameter (mm)	R/L ($10^{-3}\,\Omega/\text{m}$)
0	9.35	0.31
10	2.59	2.20
14	1.63	8.54
18	1.02	21.90
22	0.64	51.70

Material	Resistivity ρ (Ω m)
silver	1.6×10^{-8}
copper	1.7×10^{-8}
aluminum	2.7×10^{-8}
tungsten	5.6×10^{-8}
Nichrome	100×10^{-8}
carbon	3500×10^{-8}
germanium	0.46
glass	10^{10} to 10^{14}

Ohm's Law

The electric potential difference V across a "load" (resistor) equals the product of the current I through the load and the resistance R of the load.

$$\boxed{V = IR}$$

Quantity	Symbol	SI Unit
Potential difference	V	V (volt)
Current	I	A (ampere)
Resistance	R	Ω (ohm)

A resistor is considered "ohmic" if it obeys Ohm's law

Power Dissipated by a Resistor

Power is the rate at which work W is done, and from electrostatics, the change in electric potential energy ΔE_q (i.e. the work done!) is proportional to the amount of charge q and the voltage V. This gives a very simple expression for power through a resistor:

$$P = \frac{dW}{dt} = \frac{d(qV)}{dt} = \left(\frac{dq}{dt}\right) V = \boxed{IV}$$

Quantity	**Symbol**	**SI Unit**
Power through a resistor	P	W (watt)
Current through a resistor	I	A (ampere)
Voltage across the resistor	V	V (volt)

Other Equations for Power

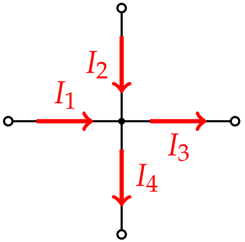
When we combine Ohm's Law ($V = IR$) with power equation, we get two additional expressions for power through a resistor:

$$\boxed{P = \frac{V^2}{R}} \qquad \boxed{P = I^2 R}$$

Quantity	Symbol	SI Unit
Power	P	W (watts)
Voltage	V	V (volts)
Resistance	R	Ω (ohms)
Current	I	A (amperes)

Kirchkoff's Current Law

The electric current that flows into any junction in an electric circuit must be equal to the current which flows out.



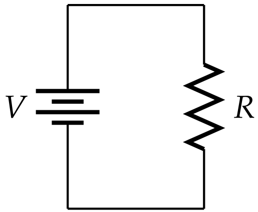
e.g. if there are 4 paths to the junction at the center, with I_1 and I_2 going into the junction, and I_3 and I_4 coming out, then the current law says that

$$I_1 + I_2 - I_3 - I_4 = 0$$

Basically, it means that there cannot be any accumulation of charges anywhere in the circuit. The law is a consequence of conservation of energy.

Kirchkoff's Voltage Law

The voltage changes around any closed loop in the circuit must sum to zero, no matter what path you take through an electric circuit.



Assume that the current flows clockwise and we draw a clockwise loop, we get

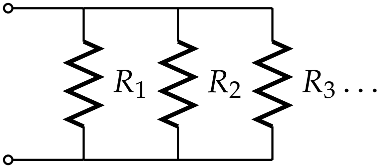
$$V - V_R = 0 \quad \rightarrow \quad V - IR = 0$$

If I incorrectly guess that I flows counterclockwise, I will still have a similar expression

$$-V_R - V = 0 \quad \rightarrow \quad -V - IR = 0$$

When solving for I, we get a negative number, indicating that my guess was in the wrong direction.

Resistors in Parallel



From the current law, we know that the total current is the current through all the resistors, which we can rewrite in terms of voltage and resistance using Ohm's law:

$$I = I_1 + I_2 + I_3 \cdots = \frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3} \cdots$$

But since we also know that $V_1 = V_2 = V_3 = \cdots = V$ from the voltage law, we can re-write as

$$I = \frac{V}{R_{\text{eq}}} = V\left(\frac{1}{R_1} + \frac{1}{R_1} + \frac{1}{R_1} \cdots\right)$$

Resistors in Parallel

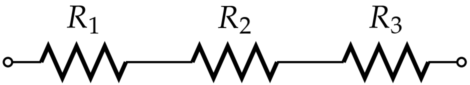
Equivalent Resistance

Through applying Ohm's Law and Kirchkoff's laws, we find the equivalent resistance of a parallel circuit: **The inverse of the equivalent resistance for resistors connected in parallel is the sum of the inverses of the individual resistances.**

$$\boxed{\frac{1}{R_{\mathrm{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}}$$

Quantity	Symbol	SI Unit
Equivalent resistance	R_{eq}	Ω (ohm)
Resistance of individual loads	$R_{1,2,3,\cdots,N}$	Ω (ohm)

Resistors in Series



The analysis for resistors in series is similar (but easier). From the current law, the current through each resistor is the same:

$$I_1 = I_2 = I_3 = \cdots = I$$

And the total voltage drop across all resistor is therefore:

$$V = V_1 + V_2 + V_3 + \cdot = I(R_1 + R_2 + R_3 + \cdots)$$

Equivalent Resistance

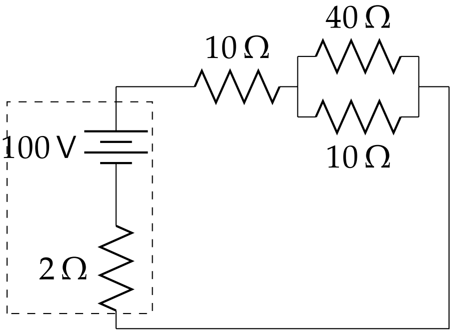
Again, through applying Ohm's Law and Kirchkoff's laws, we find that when resistors are connected in series: **the equivalent resistance of loads is the sum of the resistances of the individual loads.**

$$\boxed{R_{\text{eq}} = R_1 + R_2 + \cdots + R_N}$$

Quantity	Symbol	SI Unit
Equivalent resistance	R_{eq}	Ω (ohm)
Resistance of individual loads	$R_{1,2,3,\cdots,N}$	Ω (ohm)

Example Problem (Simple)

A simple circuit analysis problem will involve one voltage source and resistors connected, some in parallel, and some in series. Below is a typical example:



Two $10\,\Omega$ resistors and a $40\,\Omega$ resistor are connected as shown to a $100\,\mathrm{V}$ emf source with internal resistance $2\,\Omega$. How much power is dissipated by the $40\,\Omega$ resistor?

(A) $160\,\mathrm{W}$

(B) $40\,\mathrm{W}$

(C) $400\,\mathrm{W}$

(D) $5\,\mathrm{W}$

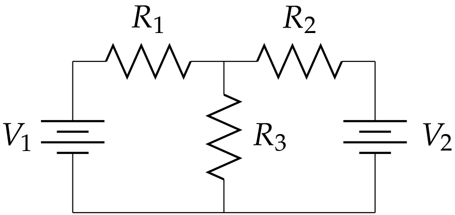
(E) $500\,\mathrm{W}$

Tips for Solving "Simple" Circuit Problems

1. Identify groups of resistors that are in parallel or in series, and find their equivalent resistance.
2. Gradually reduce the entire circuit to one voltage source and one resistor.
3. Using Ohm's law, find the current out of the battery.
4. Uing Kirchkoff's laws, find the current through each of the resistors.

Circuits Aren't Always Simple

Some of these problems require you to solve a system of linear equations. The following is a simple example with two voltage sources:



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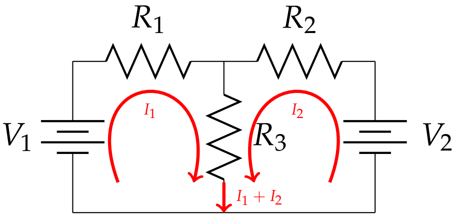
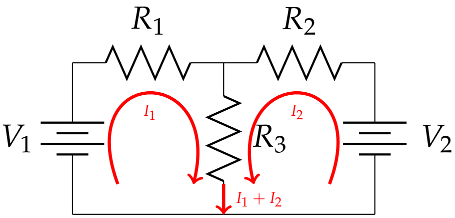


In this case, we have to draw two loops of current.

A More Difficult Example



We split the circuit into two loops, and apply Kirchkoff's voltage in both:

$$V_1 - I_1 R_1 - (I_1 + I_2) R_3 = 0$$
$$V_2 - I_2 R_2 - (I_1 + I_2) R_3 = 0$$

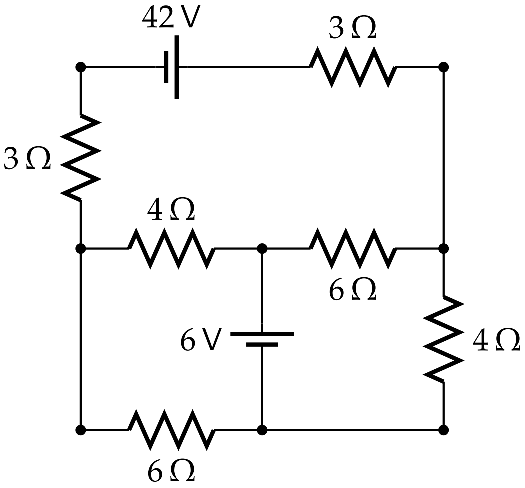
Two equations, two unknowns (I_1 and I_2). We can subtract (2) from (1), then solve for I_1 and I_2:

$$I_1 = \frac{V_1 - I_2 R_3}{R_1 + R_3} \qquad I_2 = \frac{\left[V_2 - \frac{(V_1 - V_2) R_3}{R_1}\right]}{\left[R_2 + \frac{(R_1 + R_2) R_3}{R_1}\right]}$$

(Try this at home as an exercise.)

This is As Difficult As It'll Get

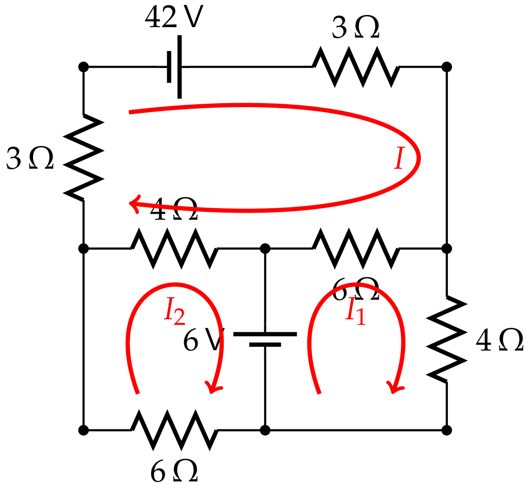
And This Is Your Homework



- To solve this problem, we define a few "loops" around the circuit: one on top, one on bottom left, and one on bottom right.

This is As Difficult As It'll Get

And This Is Your Homework

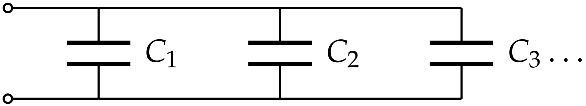


- To solve this problem, we define a few "loops" around the circuit: one on top, one on bottom left, and one on bottom right.
- Apply the voltage law in the loops. For example, in the lower left:

$$4(I - I_2) - 6 - 6I_2 = 0$$

- Solve the linear system to find the current. If the current that you worked out is negative, it means that you have the direction wrong.

Capacitors in Parallel



From the voltage law, we know that the voltage across all the capacitors are the same, i.e. $V_1 = V_2 = V_3 - \cdots = V$. We can express the total charge Q_{tot} stored across all the capacitors of capacitance and this common voltage V:

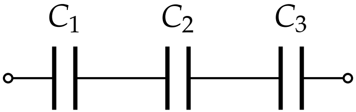
$$Q_{\text{tot}} = Q_1 + Q_2 + Q_3 + \cdots = C_1 V + C_2 V + C_3 V + \cdots$$

Factoring out V from each term gives us the equivalent capacitance:

$$\boxed{C_{\text{eq}} = C_1 + C_2 + \cdots + C_N}$$

Capacitors in Series

Likewise, we can do a similar analysis to capacitors connected in series.



The total voltage across these capacitors are the sum of the voltages across each of them, i.e. $V_{\text{tot}} = V_1 + V_2 + V_3 + \cdots$

We recognize that the charge stored on all the capacitors must be the same! We can then write the total voltage in terms of capacitance and charge:

$$V_{\text{tot}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} + \cdots$$

Capacitors in Series

Equivalent Capacitance

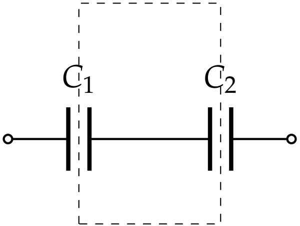
The inverse of the equivalent capacitance for N capacitors connected in series is the sum of the inverses of the individual capacitance.

$$\boxed{\frac{1}{C_{\mathrm{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}}$$

Make sure we don't confuse ourselves with resistors.

How Do We Know That Charges Are The Same?

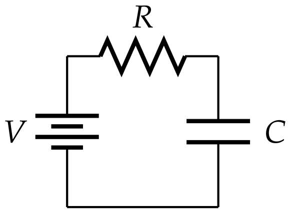
It's actually quite easy to show that the charges across all the capacitors are the same.



The capacitor plates and the wire connecting them are really one piece of conductor. There is nowhere for the charges to leave the conductor, therefore when charges are accumulating on C_1, C_2 must also have the same charge because of conservation of charges.

Circuits with Resistors and Capacitors

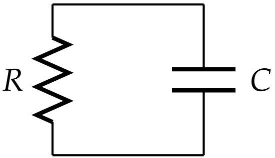
Now that we have seen how resistors and capacitors behave in a circuit, we can look into combining them in to an “R-C circuit”.



The simplest form is a resistor and capacitor connected in series, and then connect to a voltage source. Because of the nature of capacitors, the current through the circuit will not be steady as were the case with only resistors.

Discharging a Capacitor

We start the analysis with something even simpler. There is no voltage source, and the capacitor is already charged to $V_c = Q_{\text{tot}}/C$. What happens when the current begin to flow?



As current starts to flow, the charge on the capacitor decrease. Over time the current decreases, until the capacitor is fully discharged, and current stops flowing.

Apply the voltage law for the circuit, and substitute the definition of current $I = dQ/dt$ and the voltage across a capacitor $V_c = Q/R$:

$$V_c - IR = 0 \quad \rightarrow \quad \frac{Q}{C} - R\frac{dQ}{dt} = 0$$

Discharging a Capacitor

Separating the Q terms on the left side of the equation, and leaving everything else on the right side, we get:

$$\frac{dQ}{Q} = \frac{-dt}{RC}$$

which we can now integrate and "exponentiate":

$$\int \frac{dQ}{Q} = \int \frac{dt}{RC} \rightarrow \ln Q = \frac{-t}{RC} + K \rightarrow Q = e^K e^{-t/RC}$$

the constant of integration K is related to initial charge on the capacitor Q_{tot}:

$$e^K = Q_{\text{tot}}$$

Discharging a Capacitor

The expression of charge across the capacitor is time-dependent:

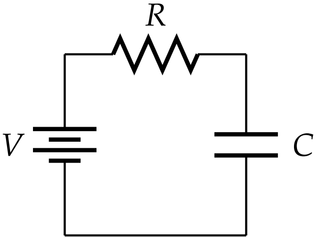
$$\boxed{Q(t) = Q_0 e^{-t/\tau}}$$

where Q_{tot} is the initial charge on the capacitor, and $\tau = RC$ is called the **time constant**. Taking the time derivative of $Q(t)$ gives us the current through the circuit:

$$\boxed{I(t) = \frac{dQ}{dt} = I_0 e^{-t/\tau}}$$

where the initially current at $t = 0$ is given by $I_0 = Q_{\text{tot}}/\tau = Q_{\text{tot}}/RC$.

Charging a Capacitor



In charging up the capacitor, we go back to our original circuit, and apply the voltage law, then substitute the expression for current and voltage across the capacitor:

$$V - R\frac{dQ}{dt} - \frac{Q}{C} = 0$$

Again, separating variables, and integrating, we get:

$$\int \frac{dQ}{CV - Q} = \int \frac{dt}{RC} \quad \rightarrow \quad \ln(CV - Q) = \frac{-t}{RC} + K$$

Charging a Capacitor

"Exponentiating" both sides, we have

$$CV - Q = e^K e^{-t/RC}$$

To find the constant of integration K, we note that at $t = 0$, the charge across the capacitor is 0, and we get

$$e^K = CV = Q_{\text{tot}}$$

which is the charge stored in the capacitor at the end. Substitute this back into the equation:

$$\boxed{Q(t) = Q_{\text{tot}}(1 - e^{-t/RC})}$$

Capacitors

$$\boxed{Q(t) = Q_{\text{tot}}(1 - e^{-t/\tau})}$$

Charging a capacitor has the same time constant $\tau = RC$ as during discharge. We can also differentiate to find the current through the circuit; it is identical to the equation for discharge:

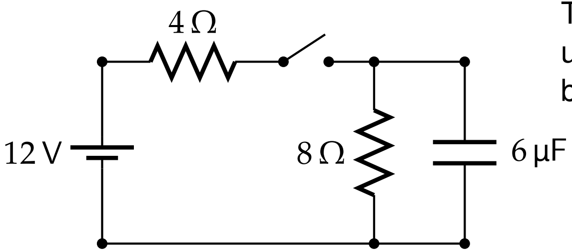
$$\boxed{I(t) = \frac{dQ}{dt} = I_0 e^{-t/\tau}}$$

where the initial current is given by $I_o = Q_{\text{tot}}/\tau = V/R$. This makes sense because $V_C(t = 0) = 0$, and all of the energy must be dissipated through the resistor. Similarly at $I(t = \infty) = 0$.

Two Small Notes

1. When a capacitor is uncharged, there is no voltage across the plate, it acts like a short circuit.
2. When a capacitor is charged, there is a voltage across it, but no current flows *through* it. Effectively it acts like an open circuit.

A More Difficult Problem



The capacitor in the circuit is initially uncharged. Find the current through the battery

1. Immediately after the switch is closed
2. A long time after the switch is closed