Topic 22: Wave Particle Duality
Advanced Placement Physics

Timothy M. Leung*

Olympiads School
Toronto, Ontario, Canada

April 21, 2020

Anyone who is not shocked by the quantum theory has not understood it.

- Niels Bohr

The final topic in the AP Physics course at Olympiads School concerns one of the most significant advancements in physics at the turn of the twentieth century: the developement of the quantum theory. Unlike the development of the theory of relativity (both special and general), which owes much to the singular work by Albert Einstein, quantum mechanics was developed by many leading scientists at the time. The material covered in this course deals particularly the early development of the quantum theory, called **wave mechanics**.

Earlier in this course, the difference between *particles* and *waves* were discussed. However,

1 Quantization of Energy

1.1 Blackbody Radiation

The **blackbody** is a concept was coined by Gustav Kirchhoff in 1860. It is an idealized object that absorbs all incident EM radiation, regardless of frequency or angle of incidence. It is often modeled as a box (a "cavity") with a mirror inside, and a hole where light (EM radiation) is allowed in, as shown in Fig.1. Some of the light reflects inside the cavity, and some gets absorbed by the blackbody itself. Eventually, all the light inside the cavity is absorbed.

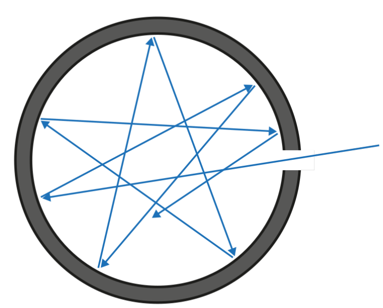


Figure 1: A blackbody

A blackbody is in thermodynamic equilibrium; all of the absorbed energy is then immediately radiated back as EM radiation, and the spectral distribution depends only on temperature. A blackbody at room temperature appears black, as most of the radiative energy is infrared and cannot be perceived by the human eye. Although blackbodies do not exist in the universe, thermal radiation spontaneously emitted by many ordinary objects can be approximated as blackbody radiation.

*Ph.D. (Toronto), M.A.Sc. (Toronto), B.A.Sc. (British Columbia), `tleung@olympiadsmail.ca`

1.2 Rayleigh-Jeans Law and the Ultraviolet Catastrophe

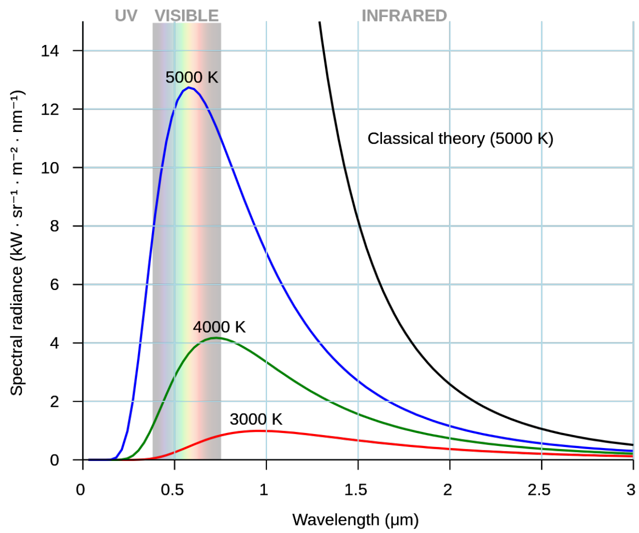
We can see that the peak of the emission spectrum shifts towards shorter wavelengths as temperature increases. **Wien's displacement law**[1] states that the spectral radiance of blackbody radiation peaks at the wavelength λ_{peak}, given by:

$$\lambda_{\text{peak}} = \frac{b}{T} \tag{1}$$

where T is the thermodynamic (absolute) temperature in, and b is called Wien's displacement constant, with a value of 2.898×10^{-3} m K.

Based on classical thermodynamics the power emitted by a blackbody should be given by the **Rayleigh-Jean law**:

$$P(\lambda, T) = 8\pi k T \lambda^{-4} \tag{2}$$



where k is Boltzman's constant. This equation agrees with experimental results for long wavelengths, but strongly disagrees for short wavelengths. As $\lambda \to 0$, $P \to \infty$. This is known as the **"ultra-violet catastrophe"**.

1.3 Energy Quanta

Planck first made a strange modification in the classical calculations, and derived a function of $P(\lambda, T)$ that agreed with experimental data for all wavelengths. Then he searched for a way to modify the usual calculations.

Max Planck

Planck postulated that the walls of a blackbody are composed of subatomic electric oscillators[2], which he called "resonators"), whose exact nature were unknown to Planck. In this case, a blackbody has billions of resonators vibrating at different frequencies, and therefore emitting radiation at those frequencies.[3] In classical physics, the resonators can have any value of energy, and change its amplitude continuously. However, in order to agree with experimental results for the blackbody, Planck had to assume that energy emitted the resonator must be *discrete*. That is, when energy is emitted from the resonator, it drops to the next lower energy level.

The total energy E of *any* harmonic oscillator can only be integral multiples of hf:

$$\boxed{E_{\text{res}} = nhf} \tag{3}$$

where n is the enegy level of the oscillator, f is its frequency, and h is a constant that we now call **Planck's constant**, which is experimentally determined to be $h = 6.626 \times 10^{-34}$ J s. This is shown graphically in Fig. 2.

As for his formula, it's called **Planck's law**:

$$\boxed{P(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1}} \tag{4}$$

[1]The law is obtained by considering the adiabatic expansion of the cavity in thermodynamic equilibrium

[2]Think of oscillators as particles that are in harmonic motion, e.g. mass on a spring, or a pendulum

[3]Remember that for a wave, the frequency of disturbance at the source determines the frequency of the wave.

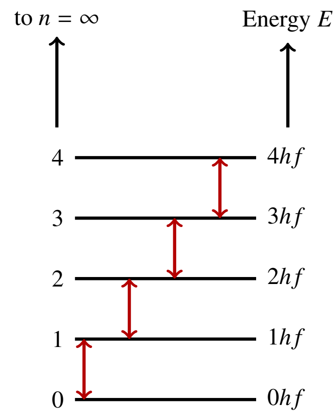
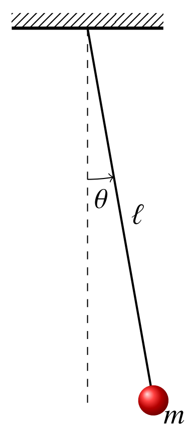


Figure 2: Energy levels of a harmonic oscillator

1.4 Classical vs. Quantum Oscillator

This "quantum" behavior exists even for the simple pendulum that we studied in mechanics topics in harmonic motion and circular motion. The natural frequency for an undamped pendulum with $\ell = 1$ m is



$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}} \approx 0.50\,\mathrm{Hz}$$

According to Eq. 3, each energy level corresponds to

$$\Delta E = hf \approx 3.3 \times 10^{-34}\,\mathrm{J}$$

For a pendulum with m =100 g and a maximum deflection of θ =10°, the total energy can be calculated at the maximum deflection, when it consists entirely of gravitational potential energy:

$$\frac{\Delta E}{E} = \frac{\Delta E}{mg\ell(1 - \cos\theta)} \approx 2.2 \times 10^{-32}$$

No wonder we can't observe it in a marcoscopic level!

2 Waves as Particles

Recall Maxwell's equation from earlier in the course:

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu_o \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}$$

Solving the equations shows that disturbances in electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ travel as an EM wave with a definite speed of

$$c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 299\,792\,458\,\mathrm{m/s} \tag{5}$$

in a vacuum. At that time, the speed of light was measured to be within one percent of this value.[4] To prove that light is an EM wave, German physicist Heinrich Hertz devised a "spark gap experiment" to generate frequencies in the range of 10^{14} Hz.[5]

2.1 Photoelectric Effect

At the end of Hertz's experimental results, he left a terse remark:

> *"It is essential that the pole surfaces of the spark gap should be frequently repolished to ensure reliable operation of the spark.*

This is now known as the **photoelectric effect** that was caused by ultra-violet radiation. Hertz and other physicists who repeated his experiments did not have a good explanation.

When EM waves (e.g. light) strike certain metals, electrons are knocked off the surface, shown graphically in Fig. 3. When observing this **photoelectric effect**, physicists discovered that:

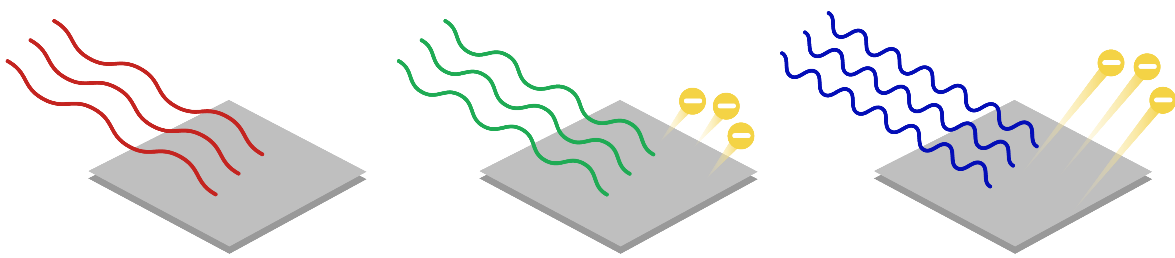


Figure 3: Photoelectric effect

- Increasing intensity of light knocked off more electrons, but doesn't change the maximum kinetic energy of the electrons, but
- Changing the frequency of the light did change K though, although
- Below a certain frequency, *no* electrons were emitted

[4]French physicist Léon Foucault measured the speed of light to be $298\,000 \pm 500\,\mathrm{km/s}$ using a rotating mirror experiment in 1862, around the time when Maxwell published his work. That measurement is within 0.60 % of the correct value.

[5]The wavelenghts of light has been known as early as 1801, after Thomas Young's double-slit experiments to show the wave interference behavior in light. Calculating the expected frequency of light only requires using the equation $v = f\lambda$ and then solving for f using $v = c_0$.

In his 1905 paper *On a Heuristic Viewpoint Concerning the Production and Transformation of Light*, Einstein postulated that light is not continuous wave, but a collection of discrete energy packets (photons), each with energy $E = hf$, in agreement with Planck (Eq. 3). In this case, the maximum kinetic energy $K_{\max}$ of a "photoelectron" is a simple equation:

Albert Einstein in 1905

$$\boxed{K_{\max} = \begin{cases} hf - \varphi & \text{if } hf > \varphi \\ 0 & \text{otherwise} \end{cases}} \tag{6}$$

where f is frequency of the the incident EM radiation, h is Planck's constant, and φ (measured in J or more commonly, in eV) is the **work function** that is specific to the metal.[6] It is the minimum energy required to remove an electron from a solid to a point immediately outside the solid surface. The minimum frequency at which electrons will be ejected is called the **threshold frequency** f_0. At this frequency, $K = 0$, and we can solve for f_0

$$f_0 = \frac{\varphi}{h} \tag{7}$$

The slope the graph in Eq. 6 is always h, indepedent of the metal. The work functions of common metals are shown in Table 1.

Metal	φ (eV)
Aluminum	4.28
Calcium	2.87
Cesium	2.14
Copper	4.65
Iron	4.50
Lead	4.25
Lithium	2.90
Nickel	5.15
Platinum	5.65
Potassium	2.30
Tin	4.42
Tingsten	4.55
Zinc	4.33

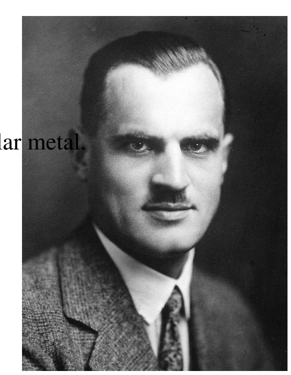
Table 1: Work functions of common materials

Fig. 4 shows the kinetic energy of the photoelectrons for zinc. The threshold frequency occurs at $f_0 = 10.4 \times 10^{14}$ Hz, which is in the ultra-violet range. The y-intercept of the graph is (not surprisingly) $-\varphi = -4.33$ eV.

2.2 Compton Scattering

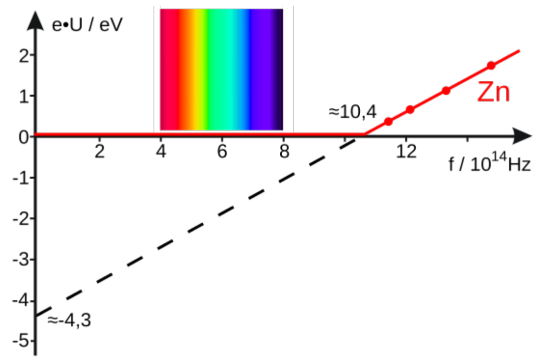
American physicist Arthur Holly Compton[7] studied x-ray scattering by free electrons. **MORE INFORMATION REQUIRED FOR COMPTON SCATTERING**.

Using Einstein invariant[8] and applied to a massless particle, the momentum carried by



Arthur Holly

[6]Although φ is called the work *function*, it is merely an experimentally-determined constant for a particular metal.
[7]1892–1962; Nobel Prize winner in 1927
[8]The Einstein invariant is defined as $E^2 = p^2c^2 + m^2c^4$. For massless particles, it simplifies to $E = pc$.



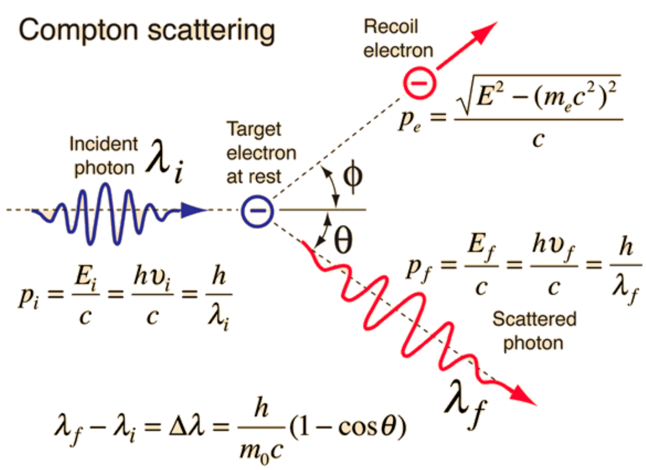
THIS FIGURE MUST BE REPLACED BY MY OWN

Figure 4: Photoelectric equation for zinc

light can be expressed as:

$$\boxed{p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda}} \tag{8}$$

This is a very odd expression, which treats photon both as a particle (with momentum) and a wave (with a wavelength λ). If x-ray is treated as a photon (a particle!) with momentum



THIS FIGURE MUST BE REPLACED WITH MY OWN

Figure 5: Compton scattering

$p = h/\lambda$ then the interaction with the electron is merely an elastic collision, where both momentum and energy are conserved after the collision i.e.

$$\mathbf{p} = \mathbf{p}_e + \mathbf{p}' \tag{9}$$

where $\mathbf{p}$ and $\mathbf{p}'$ are the momenta of the x-ray photon before and after the interaction, and $\mathbf{p}_e$ is the momentum of the recoil electron after the interaction. We can relate the magnitude of the momentum vectors by the angle of the scattered photon:[9]

$$p^2 = p'^2 + p_e^2 - 2p' p_e \cos\theta \tag{10}$$

Unfortunately, in this case, while the momentum of the photon is known, the momentum of the recoil electron is not, so an extra equation is needed: conservation of energy.

[9]This is literally using the cosine law to solve the vector problem.

3 Particles as Waves

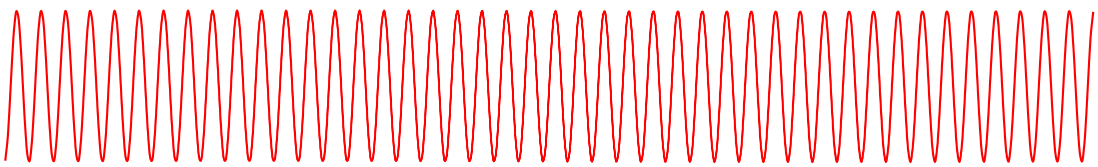
Louis De Broglie

If electromagnetic waves are really particles of energy, then are particles (e.g. electrons) a wave of some sort? Louis De Broglie[10], while completing his PhD in 1924, proposed a hypothesis: a particle can also have a wavelength. If matter is also a wave, then its wavelength can be obtained by solving the momentum equation for λ, and replacing by the (non-relativistic) expression for momentum into the equation, i.e.:

$$\boxed{\lambda = \frac{h}{p} = \frac{h}{mv}} \tag{11}$$

4 Uncertainty Principle

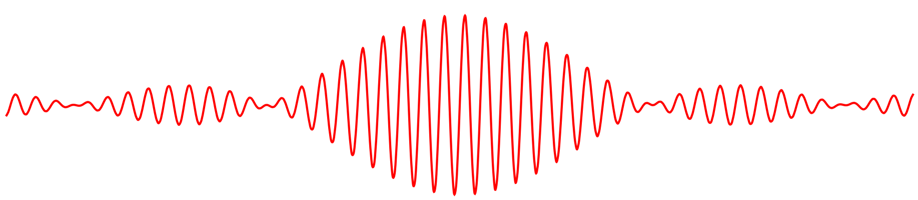
If a particle is a wave, as shown in Fig. 6, how do you tell where it is?



$\cos(400x)$

Figure 6: A wave with a well-defined wavelength

In the figure, the particle (expressed as a wave) has a single wavelength λ (therefore a single value of momentum p, from Eq. 8), but it has no distinguishing features that can tell you its location x. The bottom line is that **when we have precise knowledge of a particle wave's momentum, then we have no knowledge of *where* it is.** However, if a particle is defined as waves with small variations of wavelengths, when we add up the different waves together, we begin to see a **wave packet** emerge, as shown in Fig. 7.



$\cos(380x) + \cos(382.5x) + \cdots + \cos(400x) + \cos(402.5x) + \cdots + \cos(420x)$

Figure 7: A wave with a spread of wavelengths

The bottom line is that **in order to gain knowledge about the location of a particle, we *must* give up information about its momentum.** Because of the wave properties of particles, an observer can never be completely certain of the relationship between an object's momentum p and position x. The more you know about an object's position, the less you know about its momentum, and vice versa. The limitation, discovered by Werner

[10] 1892–1987; Nobel Prize winner in 1929

Heisenberg[11] is called the **Heisenberg uncertainty principle**:

$$\boxed{\sigma_p \sigma_x \geq \frac{\hbar}{2}} \tag{12}$$

The quantity $\hbar = \dfrac{h}{2\pi}$ is called the **reduced Planck's constant**.

5 Atomic Model

The "orbital" model of electrons does not work, because as the electron orbits (accelerates around) the nucleus, it radiates electromagnetic radiation, and therefore lose energy. The orbit will eventually collapse. A young Danish physicist Niels Bohr postulated that electrons can move in certain "non-radiating" orbits, corresponding to energy levels:

It provides the first model that incorporates physicists' new knowledge of quantum mechanics into the atomic model. Bohr began by assuming that a single electron (with elementary charge e) is in a perfectly circular orbit around a nucleus with Z protons (with charge Ze, where Z is called the **atomic number**). The electrostatic force (coulomb force) therefore provides the centripetal force keeping the electron in orbit:

$$F_c = F_q = \frac{kZe^2}{r^2} \tag{13}$$

From studying the motion of planets in circular orbits using gravity as the centripetal force, the potential energy U, kinetic energy K and total energy E for the atom in orbit can be expressed as:

$$U = -\frac{kZe^2}{r}; \quad K = -\frac{kZe^2}{2r} = -\frac{1}{2}U; \quad \boxed{E = \frac{kZe^2}{2r} = \frac{1}{2}U} \tag{14}$$

The total energy is one half of the potential energy, in agreement with the study with gravity. (Thus far, the derivation is completely *classical*.) Bohr then applied the energy quantitation energy of Planck (Eq. 3)

$$\boxed{E_n = -\frac{k^2 e^4 m}{2\hbar^2} \frac{Z^2}{n^2}} \tag{15}$$

From the wave-particle duality perspective, the "orbits" correspond more to a standing wave around the nucleus (a standing wave does not lose energy)

Successful in describing the behaviour of the hydrogen atom—but fails for heavier atoms—although it still relies on

- Quantization of energy (new physics!)

De Broglie's hypothesis gives us a glimpse of what Bohr is missing

- The "orbits" correspond to a standing wave around the nucleus
- A standing wave does not lose energy

[11] 1901–1976; Nobel prize winner in 1932

6 Hydrogen Emission

7 Schrodinger Equation

7.1 Particle in a Box

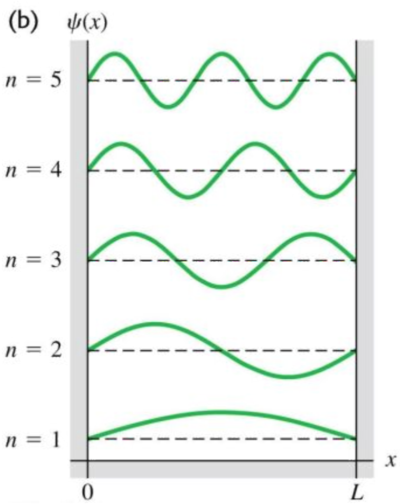


Figure 8: Energy states of the particle in a box.

A particle in a 1D box has to behave like a standing wave. The resonance modes (frequencies where a stable standing wave exists) correspond to the wavelengths:

$$\lambda = \frac{2L}{n} \quad n = 1, 2, 3 \ldots \tag{16}$$

and the momentum of the particle is:

$$p = \frac{h}{\lambda} = \frac{nh}{2L} \tag{17}$$

The kinetic energy K of the particle can be expressed in terms of momentum p:

$$K_n = \frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{n^2h^2}{8mL^2} \tag{18}$$

If a particle is a standing wave, then kinetic energy of the particle can never be zero (as long as it is confined inside the box), therefore

- It cannot have zero velocity
- The lowest energy level ($n = 1$) is called the **zero-point energy**