Projectile Motion

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Revised: April 6, 2020

1 Projectile Motion

2 Symmetric Trajectory

A **symmetric trajectory** is a special case of projectile motion where an object is launched at an angle of θ (between 0° and 90°) above the horizontal[1] with an initial speed $\mathbf{v}_0$, and then lands at the same height, as shown below in Fig. 1. Examples may include hitting a golf ball toward the hole, or shooting a bullet toward a horizontal target[2]. The equations for symmetric trajectory are *not* included in the AP Exam equation sheet; if you need these equations during the exams, you will need to derive them yourself. Thankfully, the derivation is quite striaght forward. To derive the equations, we use the x-axis for the horizontal direction and y-axis for the vertical.

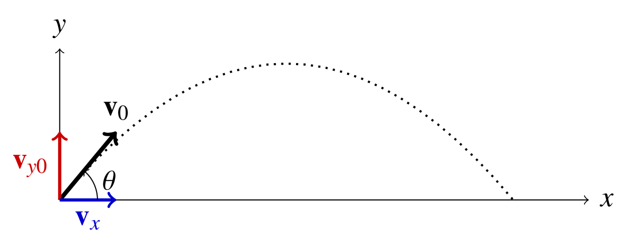

Figure 1: Symmetric project trajectory

The initial velocity $\mathbf{v}_0$ can be resolved into its $\hat{\imath}$ and $\hat{\jmath}$ components, also shown in Fig. 1:

$$\mathbf{v}_0 = v_x\hat{\imath} + v_{y0}\hat{\jmath} = v_0\cos\theta\hat{\imath} + v_0\sin\theta\hat{\jmath} \tag{1}$$

The x component of velocity $\mathbf{v}_x$ remains constant during the motion, as there are no forces acting in the x direction (as long as air resistance can be ignored), and therefore no acceleration. In the y direction, acceleration is due to gravity alone, $a_y = -g$.

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[1] This may be obvious, but angles *below* the horizontal will never have a symmetric trajectory.

[2] Shooting a bullet toward a horizontal target always require an upward angle because of gravity.

Maximum height H: Apply the kinematic equation in the y-direction. Recognizing that at maximum height $H = y - y_0$, the vertical component of velocity is zero $v_y = 0$:

$$\begin{aligned} v_y^2 &= v_{y0}^2 + 2a_y(y - y_0) \\ 0 &= (v_0 \sin\theta)^2 - 2gH \end{aligned}$$

Solving for H, we get the maximum height equation:

$$\boxed{H = \frac{v_0^2 \sin^2\theta}{2g}} \tag{2}$$

Total time of flight $t_{\max}$: We apply the kinematic equation in the y direction. When the object lands at the same height, the final velocity is the same in magnitude and opposite in direction as the initial velocity, i.e. $v_{y2} = -v_{y1} = -v_0 \sin\theta$:

$$\begin{aligned} v_y &= v_{y0} + a_y t \\ -v_0 \sin\theta &= v_0 \sin\theta - g t_{\max} \end{aligned}$$

Solving for $t_{\max}$ we have:

$$\boxed{t_{\max} = \frac{2v_0 \sin\theta}{g}} \tag{3}$$

Range R: We substitute the expression for $t_{\max}$ from Eq. 3 into the t term, then apply the kinematic equation in the x-direction to compute $R = x - x_0$ for any given launch angle and initial speed:

$$\begin{aligned} x &= x_0 + v_x t \\ R &= v_0 \cos\theta \left(\frac{2v_0 \sin\theta}{g}\right) \end{aligned}$$

Using the trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$, we simplify the equation to:

$$\boxed{R = \frac{v_0^2 \sin(2\theta)}{g}} \tag{4}$$

It is obvious that for any given initial speed v_0, the maximum range $R_{\max}$ occurs at an angle where $\sin(2\theta) = 1$ (i.e. $\theta = \pi/4$), with a value of

$$\boxed{R_{\max} = \frac{v_0^2}{g}} \tag{5}$$

Also, for a known initial speed v_0 and range R we can compute the launch angle θ:

$$\theta_1 = \frac{1}{2}\sin^{-1}\left(\frac{gR}{v_0^2}\right)$$

This angle is labelled θ_1 because it is *not* the only angle that can reach this range. Recall that for any angle $0 < \phi < \pi/2$, there is also another angle ϕ_2 where $\sin(\phi)$ value is the same:

$$\sin\phi = \sin(\pi - \phi)$$

Which means that for any θ_1, there is also another angle θ_2 where $2\theta_2 = \pi - 2\theta_1$, or quite simply:

$$\theta_2 = \frac{\pi}{2} - \theta_1$$