
Topic 1: Projectile Motion

1 Projectile Motion

For projectile motion, there is no acceleration in the horizontal (\hat{i}) direction, i.e. $a_x = 0$. Therefore the kinematic equations is reduced to a single equation

$$x - x_0 = v_x t \quad (1)$$

It is customary to define the \hat{i} direction to be the forward horizontal direction.

The only acceleration is due to gravity in the vertical (\hat{j}) direction. In the standard Cartesian coordinate system, \hat{j} is direction is therefore usually *up*, and the acceleration due to gravity is

$$a_y = -g \quad (2)$$

The kinematic equation are reduced to:

$$v_y = v_{y0} - gt \quad (3)$$

$$y = y_0 + v_{y0}t - \frac{1}{2}gt^2 \quad (4)$$

$$v_y^2 = v_{y0}^2 - 2g(y - y_0) \quad (5)$$

In general, only Eq. 4 is needed to solve projectile motion problems.

2 Symmetric Trajectory

A **symmetric trajectory** is a special case of projectile motion where an object is launched at an angle of θ (between 0° and 90°) above the horizontal¹ with an initial velocity \mathbf{v}_0 , and then lands at the same height, as shown below in Fig. 1. Examples may include hitting a golf ball toward the hole, or shooting a bullet toward a horizontal target². The equations for symmetric trajectory are *not* included in the AP Exam equation sheet; if you need these equations during the exams, you will need to derive them yourself. Thankfully, the derivation is quite straight forward. To derive the equations, we use the x -axis for the horizontal direction and y -axis for the vertical.

¹This may be obvious, but angles *below* the horizontal will never have a symmetric trajectory.

²Shooting a bullet toward a horizontal target always require an upward angle because of gravity.

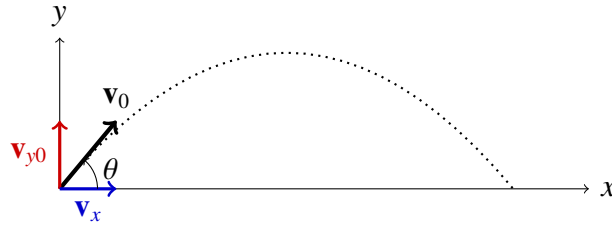


Figure 1: Symmetric project trajectory

The initial velocity \mathbf{v}_0 can be resolved into its \hat{i} and \hat{j} components, also shown in Fig. 1:

$$\mathbf{v}_0 = v_x \hat{i} + v_{y0} \hat{j} = v_0 \cos \theta \hat{i} + v_0 \sin \theta \hat{j} \quad (6)$$

The x component of velocity \mathbf{v}_x remains constant during the motion, as there are no forces acting in the x direction (as long as air resistance can be ignored), and therefore no acceleration. In the y direction, acceleration is due to gravity alone, $a_y = -g$.

Maximum height H : Apply the kinematic equation in the y -direction. Recognizing that at maximum height $H = y - y_0$, the vertical component of velocity is zero $v_y = 0$:

$$\begin{aligned} v_y^2 &= v_{y0}^2 + 2a_y(y - y_0) \\ 0 &= (v_0 \sin \theta)^2 - 2gH \end{aligned}$$

Solving for H , we get the maximum height equation:

$$H = \frac{v_0^2 \sin^2 \theta}{2g} \quad (7)$$

Total time of flight t_{\max} : We apply the kinematic equation in the y direction. When the object lands at the same height, the final velocity is the same in magnitude and opposite in direction as the initial velocity, i.e. $v_{y2} = -v_{y1} = -v_0 \sin \theta$:

$$\begin{aligned} v_y &= v_{y0} + a_y t \\ -v_0 \sin \theta &= v_0 \sin \theta - g t_{\max} \end{aligned}$$

Solving for t_{\max} we have:

$$t_{\max} = \frac{2v_0 \sin \theta}{g} \quad (8)$$

Range R : We substitute the expression for t_{\max} from Eq. 8 into the t term, then apply the kinematic equation in the x -direction to compute $R = x - x_0$ for any given launch angle and initial speed:

$$\begin{aligned} x &= x_0 + v_x t \\ R &= v_0 \cos \theta \left(\frac{2v_0 \sin \theta}{g} \right) \end{aligned}$$

Using the trigonometric identity $\sin(2\theta) = 2 \sin \theta \cos \theta$, we simplify the equation to:

$$R = \frac{v_0^2 \sin(2\theta)}{g} \quad (9)$$

It is obvious that for any given initial speed v_0 , the maximum range R_{\max} occurs at an angle where $\sin(2\theta) = 1$ (i.e. $\theta = \pi/4$), with a value of

$$R_{\max} = \frac{v_0^2}{g} \quad (10)$$

Also, for a known initial speed v_0 and range R we can compute the launch angle θ :

$$\theta_1 = \frac{1}{2} \sin^{-1} \left(\frac{gR}{v_0^2} \right)$$

This angle is labelled θ_1 because it is *not* the only angle that can reach this range. Recall that for any angle $0 < \phi < \pi/2$, there is also another angle ϕ_2 where $\sin(\phi)$ value is the same:

$$\sin \phi = \sin(\pi - \phi)$$

Which means that for any θ_1 , there is also another angle θ_2 where $2\theta_2 = \pi - 2\theta_1$, or quite simply:

$$\theta_2 = \frac{\pi}{2} - \theta_1$$