Topic 17: Mechanical Waves

Advanced Placement Physics

Dr. Timothy Leung

Olympiads School

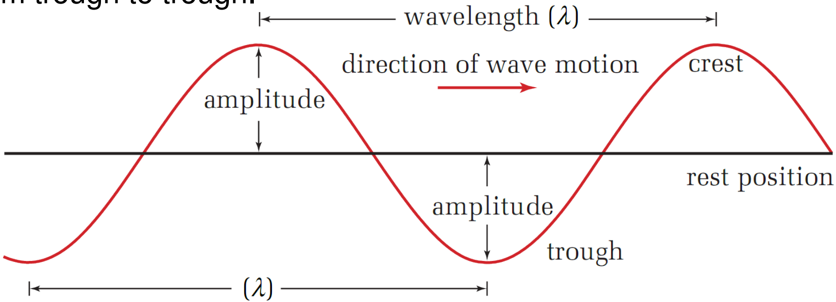
Spring 2019

What is a wave?

- When a disturbance (vibration) causes vibrations in its vicinity, a wave is created
- A wave transfers energy through a medium
  - The medium vibrates and have a net displacement of zero.
  - Each particle vibrates instead of moving horizontally, and the vibration get transferred to the next particle.
  - Electromagnetic ("EM") waves are the only kind that does not require a medium

Features of a Wave

- The *highest* point of the wave is called a **crest** or **peak**, while
- The *lowest* point in the wave is called a **trough**.
- The **wavelength** λ is the shortest distance between two points in the medium that are in phase. The easiest way to measure wavelength is from crest to crest, or from trough to trough.



The Wave Equation

The mechanical wave as we know it is the solution to a second-order partial differential equation[1]:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

subject to initial condition.

[1] The equation is *second order* because it contains second derivatives, and *partial* because it involves partial derivatives with respect to both space x and time t.

Equation of a Traveling Wave

The solution to the wave equation is a **harmonic wave** that can be described as a sinusoidal function that oscillates in both space x and time t:

$$\boxed{u(x,t) = A\sin(kx - \omega t)}$$

Quantity	Symbol	SI Unit
Displacement of the medium	u	m (meters)
Amplitude of the wave	A	m (meters)
Wave number	k	m^{-1} (per meter)
Distance from the source	x	m (meters)
Time	t	s (seconds)
Angular frequency	ω	s^{-1} (per second)

Equation of the Wave

$$\boxed{y(x,t) = A\sin(kx - \omega t)}$$

If the wave is generated by a mass on a spring, then k is the spring constant of the spring. It is related to the wavelength λ by:

$$k = \frac{2\pi}{\lambda}$$

The angular frequency (angular velocity) is related to the frequency f and period T of the wave by:

$$\omega = \frac{2\pi}{T} = 2\pi f$$

Universal Wave Equation

When combining the wave number and angular frequency, we can find that the speed of a wave is the product of the wavelength and the frequency:

$$\boxed{v = f\lambda}$$

Quantity	Symbol	SI Unit
Speed	v	m/s (meters per second)
Frequency	f	Hz (hertz)
Wavelength	λ	m (meters)

The universal wave equation applies to *all* waves. For sound waves, $v = v_{\text{sound}}$; for electromagnetic waves $v = c$.

Why Sine and Cosines

French mathematician Joseph Fourier showed that *all* periodic functions can be represented as an infinite series of $\sin$ and/or $\cos$ functions:

$$
\begin{aligned}
f(x) = & a_1 \sin(x) + a_2 \sin(2x) + a_3 \sin(3x) + \ldots + \\
& b_1 \cos(x) + b_2 \cos(2x) + b_3 \cos(3x) + \ldots \\
= & \sum_{n=1}^{\infty} a_n \sin(nx) + \sum_{n=1}^{\infty} b_n \cos(nx)
\end{aligned}
$$

The sum is called the **Fourier series**. Depending on the shape of the wave, some coefficients a_n and b_n are zeros. This part is particularly important to sound waves.

Frequency and Speed of A Wave

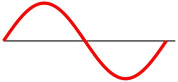
Frequency of A Wave (f)

- The number of complete wavelengths that pass a point in a given amount of time
- Unit: hertz (Hz)
- Same as the frequency of the disturbance that generated the wave
- **Does not depend on the medium**, only the source that produced the wave

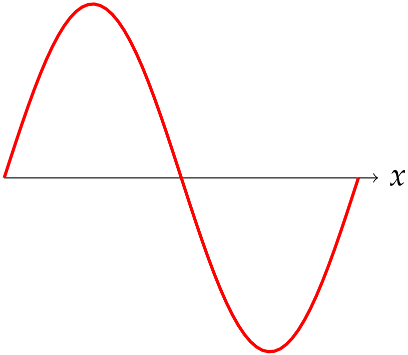
Speed of A Wave (v)

- The speed at which the wave fronts are moving
- **Depends only on the medium**, not the source that produced the wave

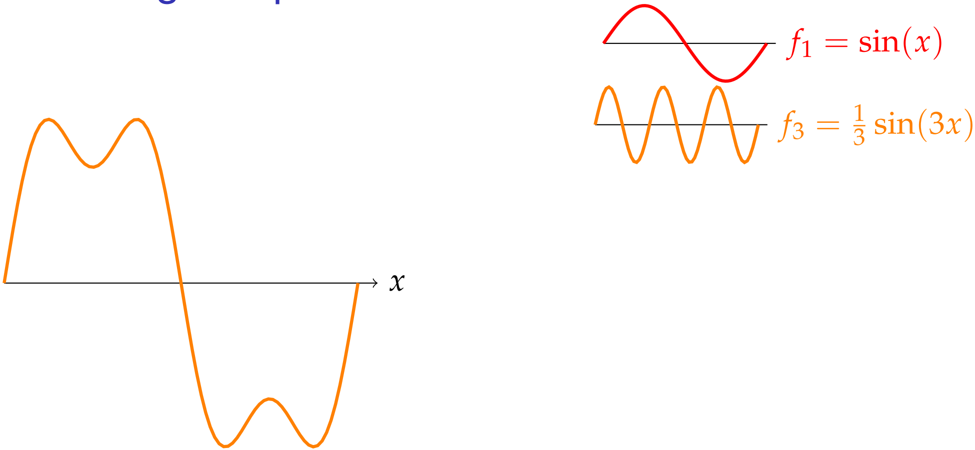
Making a Square Wave with Sine Waves



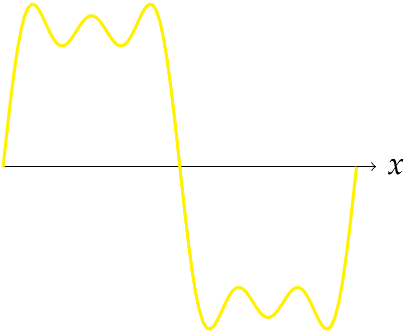
$f_1 = \sin(x)$

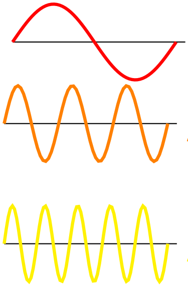


Making a Square Wave with Sine Waves



Making a Square Wave with Sine Waves



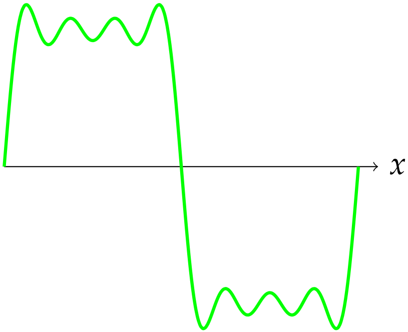


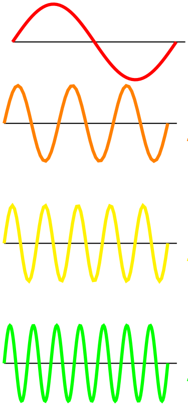
$f_1 = \sin(x)$

$f_3 = \frac{1}{3}\sin(3x)$

$f_5 = \frac{1}{5}\sin(5x)$

Making a Square Wave with Sine Waves





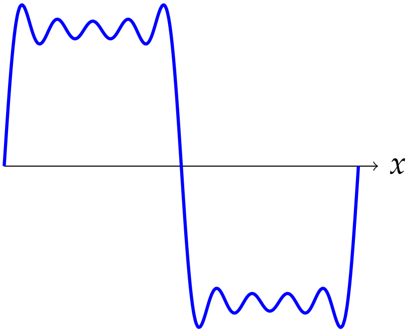
$$f_1 = \sin(x)$$

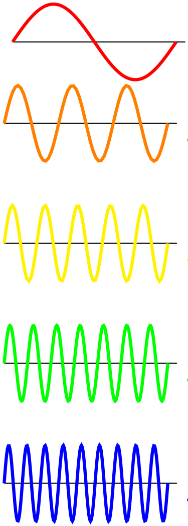
$$f_3 = \tfrac{1}{3}\sin(3x)$$

$$f_5 = \tfrac{1}{5}\sin(5x)$$

$$f_7 = \tfrac{1}{7}\sin(7x)$$

Making a Square Wave with Sine Waves





$f_1 = \sin(x)$

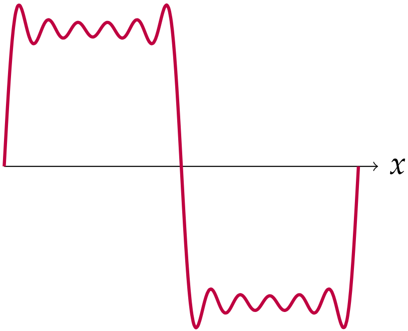
$f_3 = \frac{1}{3}\sin(3x)$

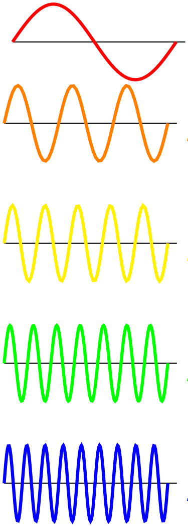
$f_5 = \frac{1}{5}\sin(5x)$

$f_7 = \frac{1}{7}\sin(7x)$

$f_9 = \frac{1}{9}\sin(9x)$

Making a Square Wave with Sine Waves





$f_1 = \sin(x)$

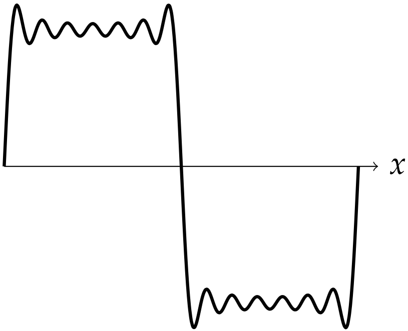
$f_3 = \frac{1}{3}\sin(3x)$

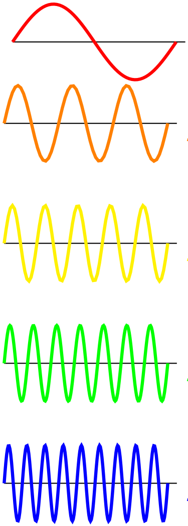
$f_5 = \frac{1}{5}\sin(5x)$

$f_7 = \frac{1}{7}\sin(7x)$

$f_9 = \frac{1}{9}\sin(9x)$

Making a Square Wave with Sine Waves





$$f_1 = \sin(x)$$

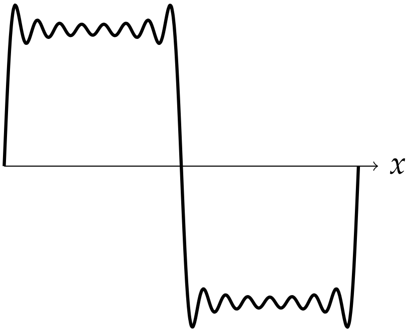
$$f_3 = \frac{1}{3}\sin(3x)$$

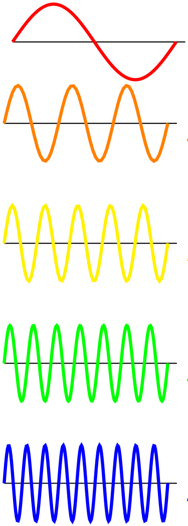
$$f_5 = \frac{1}{5}\sin(5x)$$

$$f_7 = \frac{1}{7}\sin(7x)$$

$$f_9 = \frac{1}{9}\sin(9x)$$

Making a Square Wave with Sine Waves





$$f_1 = \sin(x)$$

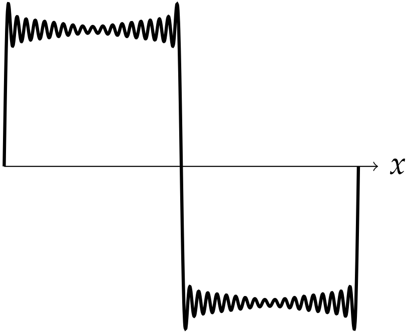
$$f_3 = \frac{1}{3}\sin(3x)$$

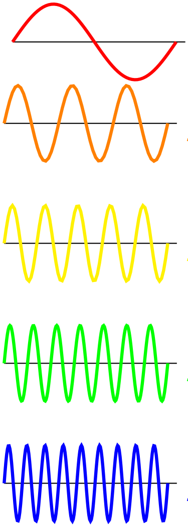
$$f_5 = \frac{1}{5}\sin(5x)$$

$$f_7 = \frac{1}{7}\sin(7x)$$

$$f_9 = \frac{1}{9}\sin(9x)$$

Making a Square Wave with Sine Waves





$$f_1 = \sin(x)$$

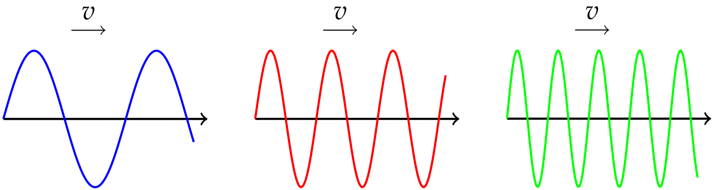
$$f_3 = \frac{1}{3}\sin(3x)$$

$$f_5 = \frac{1}{5}\sin(5x)$$

$$f_7 = \frac{1}{7}\sin(7x)$$

$$f_9 = \frac{1}{9}\sin(9x)$$

Fourier Series and Harmonic Frequencies



- The first wave—with the longest wavelength and lowest frequency—is called the **fundamental frequency**, or **first harmonic**
- The second term has half the wavelength and twice the frequency. It's called the **second harmonic**, the **first overtone**
- Also, third, fourth, fifth. . . harmonics

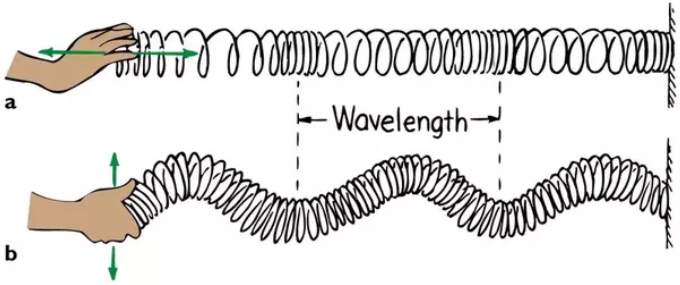
Harmonic Frequencies

Every whole-number multiples of the fundamental frequency f_1 is its harmonic frequency, i.e. the n-th harmonic is:

$$\boxed{f_n = n f_1} \quad n = 1, 2, 3, \ldots$$

For sound waves, when a musical instrument produces a sound that has, the frequency that is "heard" is the fundamental frequency

Two Kinds of Waves



a. **Longitudinal wave**
   - Vibration is parallel to the direction of the motion of the wave
   - Example: sound waves

b. **Transverse wave**
   - Vibrations occur right angles to the direction of the wave
   - Example: electromagnetic waves

Wave Simulation

A helpful simulation can be found on the PhET website at University of Colorado.

Click for external link:
wave on a string simulation

Wave on a String

The speed of a traveling wave on a stretched string is determined by:

$$\boxed{v = \sqrt{\frac{F_T}{\mu}}} \quad \text{where} \quad \boxed{\mu = \frac{m}{L}}$$

Quantity	Symbol	SI Unit
Wave speed	v	m/s (meters per second)
Tension	F_T	N (newtons)
Linear mass density	μ	kg/m (kilograms per meter)
Mass of the string	m	kg (kilograms)
Length of the string	L	m (meters)

Power Transmitted by a Harmonic Wave

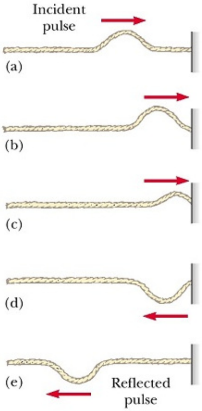
Then the power transmitted by a harmonic wave is through a traveling wave on a string is determined by the linear mass density μ, the angular frequency ω, amplitude A and wave speed v:

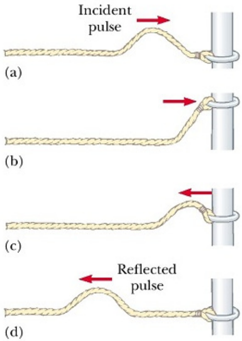
$$\boxed{P = \frac{1}{2}\mu\omega^2 A^2 v}$$

Reflection of a Wave at a Boundary

When a wave on a string reflects at a boundary, how the reflected wave looks depends on the type of boundary

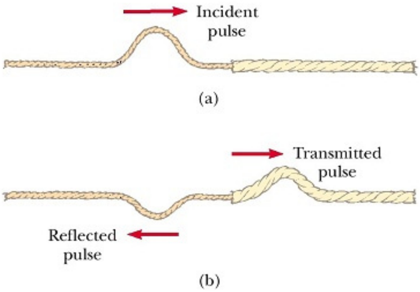
- At a *fixed end* (left), the reflected wave is *inverted*, i.e. a crest becomes a trough
- At a *free end* (right), the reflected wave is upright





Transmission of Waves: Fast to Slow Medium

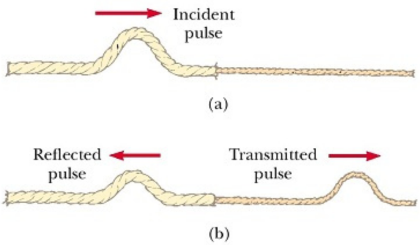
- Reflected wave:
  - Inverted, like a fixed end
  - Same frequency and wavelength as the incoming wave
  - The amplitude is decreased
- Transmitted wave:
  - Upright
  - Same frequency as incoming wave, but has a shorter wavelength because the wave slowed down



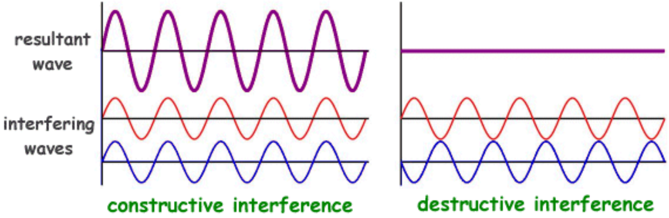
Transmission of Waves: Slow to Fast Medium

- Reflected wave:
  - Upright, like a free end
  - Same frequency and wavelength as the incoming wave
  - The amplitude is decreased
- Transmitted wave:
  - Upright
  - Same frequency as incoming wave, but has a longer wavelength because the wave sped up

Note that the transmitted wave is *always* upright.

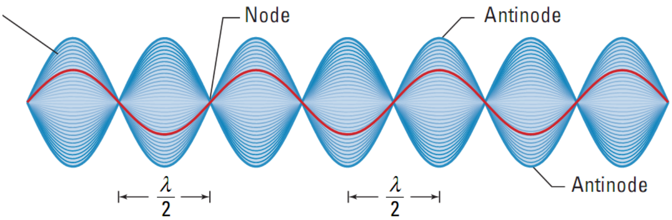


Superposition of Waves



- **Principle of Superposition:** When multiple waves pass through the same point, the resultant wave is the *sum* of the waves
  - A fancy way of saying that waves add together
- The consequence of the principle of superposition is *interference of waves*. There are two kinds of interference:
  - **Constructive interference:** Two wave fronts (crests) passing through creates a wave front with greater amplitude
  - **Destructive interference:** A crest and trough will cancel each other

Standing Waves

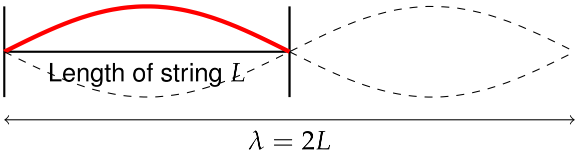


If two waves of the same frequency meet up under the right conditions, they may appear to be "standing still". This is called a standing wave

- Node: A point that never moves
- Anti-node: A point which moves/vibrates maximally

Standing Waves On a String Length L

- A "vibrating" string is actually a standing wave on a string
- Both ends of the string are nodes
- **Resonance frequencies** are frequencies where standing waves can be created. The lowest resonance frequency is called the **fundamental frequency** f_1, which occurs when $\lambda = 2L$:



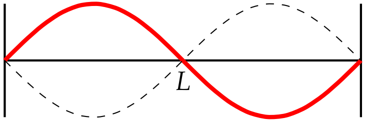
Fundamental Frequency of a Wave on a String

Using the universal wave equation we can relate the fundamental frequency f_1 to the speed of a traveling wave along the string v_{str}:

$$\boxed{f_1 = \frac{v_{\text{str}}}{\lambda} = \frac{v_{\text{str}}}{2L}}$$

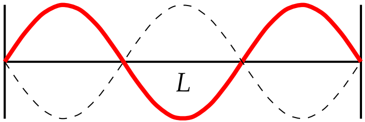
Standing Waves On a String Length L

A second resonance frequency occurs when $L = \lambda$:



$$f_2 = \frac{v_{\text{str}}}{\lambda} = \frac{v_{\text{str}}}{L} = 2f_1$$

A third resonance frequency occurs at $L = \frac{3}{2}\lambda$:



$$f_3 = \frac{3v_{\text{str}}}{2L} = 3f_1$$

Standing Waves On a String Length L

The n-th resonance frequency of a wave on string is just whole-number multiples of the fundamental frequency:

$$\boxed{f_n = n f_1} \quad n = 1, 2, 3, \ldots$$

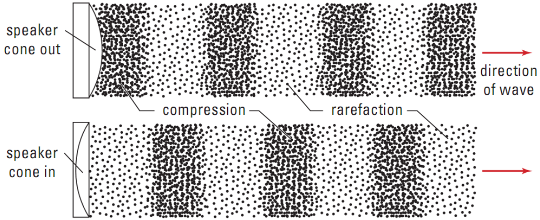
This equation is *identical* to the harmonic frequencies, meaning that every harmonic is a resonance frequency. Standing waves on a string have a *full set of harmonics*.

The wavelengths corresponding to the resonance frequencies are:

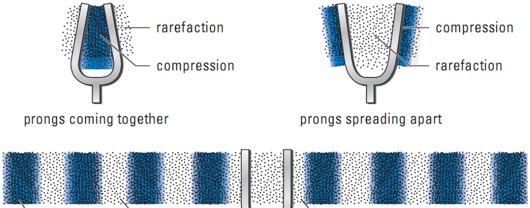
$$\boxed{\lambda = \frac{2L}{n}} \quad n = 1, 2, 3, \ldots$$

Transfer of Sound Wave in Air

Sound wave is a *longitudinal wave*, caused by the compression and rarefaction (expansion) of the air molecules. Example: speakers



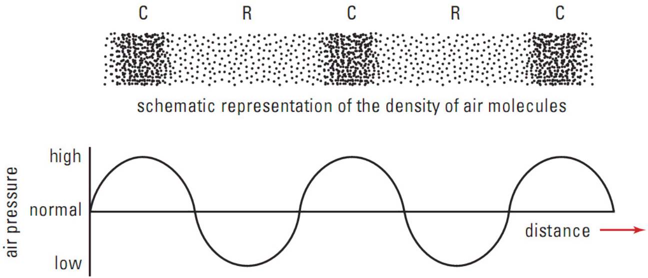
Example: tuning fork



Propagation of Sound Wave

Schematic Diagram vs. Wave Graph

We can also express the amplitude of the sound wave by plotting the change in *air pressure*:



The sinusoidal shape of the pressure (or density) distribution is how sound waves are visualized.

Speed of Sound in a Gas

The equation for the speed of sound in a gas (e.g. air) is given by:

$$\boxed{v_s = \sqrt{\frac{\gamma R T}{M}}}$$

Quantity	**Symbol**	**SI Unit**
Speed of sound	v_s	m/s
Temperature	T	K
Universal gas constant	R	J/mol K
Molar mass	M	kg/mol
Adiabatic constant	γ	(no units)

For diatomic gases such as air $\gamma = 1.4$, and $M = 29 \times 10^{-3}$ kg/mol.

Speed of Sound in Air

We can *linearize* the speed of sound equation near room temperature (about $300\,\mathrm{K}$) using a method called *least-square approximation*, which gives us a handy equation for the speed of sound:

$$v_s = 331 + 0.59 T_C$$

Quantity	**Symbol**	**SI Unit**
Speed of sound	v_s	m/s
Temperature of air in Celsius	T_c	

Note that this approximate equation only works with air at/near room temperature.

Speed of Sound in Different Media

Speed of sound in a liquid depends on the “bulk modulus” K and density ρ of the liquid:

$$v = \sqrt{\frac{K}{\rho}}$$

Speed of sound in a solid depends on the “Young’s modulus” E of the solid and density ρ

$$v = \sqrt{\frac{E}{\rho}}$$

Material	**Speed** (m / s)
Gases ($0\,^\circ$C, $101\,$kPa)	
Carbon dioxide	259
Oxygen	316
Air	331
Helium	965
Liquids ($20\,^\circ$C)	
Ethanol	1162
Fresh water	1482
Seawater	1440-1500
Solids	
Copper	5010
Glass	5640
Steel	5960

Mach Number

When dealing with sound waves, it is often useful to express speed in terms of its ratio to the speed of sound. This ratio is called the **Mach number**[2]:

$$\boxed{M = \frac{v}{v_s}}$$

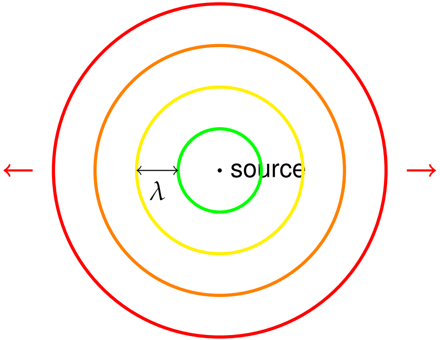
Quantity	Symbol	SI Unit
Mach Number	M	no units
Speed of the object	v	m/s
Local speed of sound	v_s	m/s

- When an object is traveling at $M < 1$, it is traveling at a *subsonic* speed
- When an object is traveling at $M > 1$, it is traveling at a *supersonic* speed

[2] named after German engineer Ernst Mach

Sound from a Moving Source

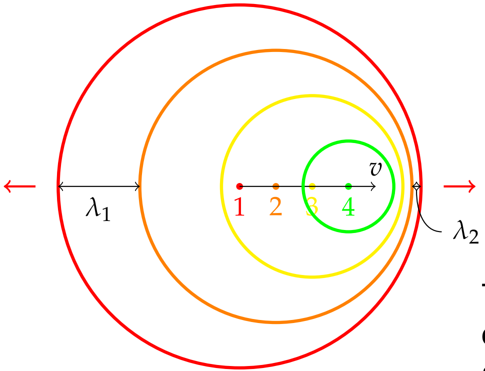
When a sound is emitted from a point source, the sound wave moves radially outward from the point of origin. In this diagram, the source is stationary:



As the wave front moves away from the point source, its area increases (think 3D), therefore the sound is not as loud.

Sound from a Moving Source

But when sound is emitted from a *moving* source, the diagram looks different. In this case, the sound source is moving to the right, from 1 to 4:



- When the source is moving *toward you*, the wavelength λ_2 decreases, and the frequency increases.
- When the source is moving *away from you*, the wavelength λ_1 increases, and the frequency decreases.

This is called the **Doppler effect**. The most common example is of an ambulance/police siren as the vehicle approaches you.

Doppler Effect

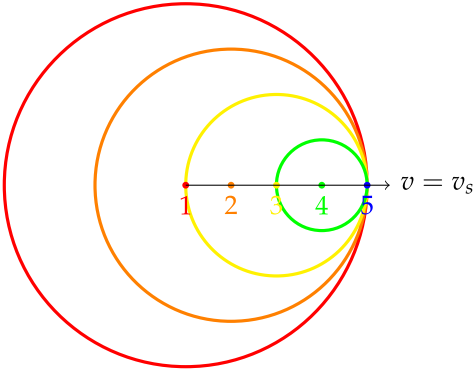
When a wave source is moving at a speed v_{src} and the observer is moving at v_{ob}, the frequency perceived by the observer is shifted to f':

$$\boxed{f' = \frac{v_s + v_{\text{ob}}}{v_s - v_{\text{src}}} f}$$

Quantity	Symbol	SI Unit
Apparent frequency	f'	hertz (hertz)
Actual frequency	f	hertz (hertz)
Speed of sound	v_s	m/s (meters per second)
Speed of source	v_{src}	m/s (meters per second)
Speed of observer	v_{ob}	m/s (meters per second)

Sound from a Source Moving At Sonic Speed

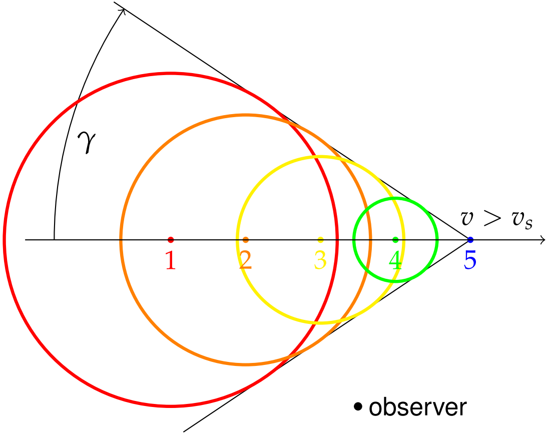
Doppler effect is more interesting is when the sound source is moving at the speed of sound ($M = 1$):



- The wave fronts (crests) from all the waves are bunched up just in front of the source
- Since sound wave is a pressure wave, right in front of the sound source, there is a large change in pressure (called a shock wave)
- When the shock passes an observer, an loud bang can be heard (aka **sonic boom**)

Sound from a Supersonic Source

When the sound source is moving at $M > 1$:



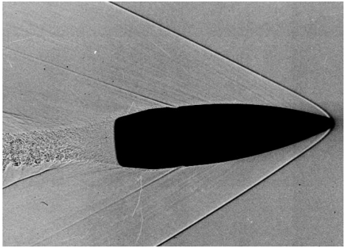
An *oblique shock* is formed at an angle (called the **Mach angle**) given by:

$$\gamma = \sin^{-1}\left(\frac{1}{M}\right)$$

A stationary observer does not hear the sound source coming until it has gone past!

Bullet in Supersonic Flight

Generating a shock wave does not require an actual sound source. *Any* object moving through air creates a pressure disturbance. For example, this is a bullet in supersonic flight:



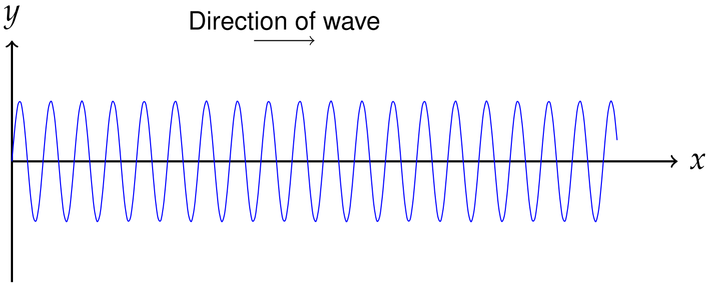
Duck in Water

Shock waves are not confined to sound waves only. In this example, a duck swims faster than the speed of the water wave, and it also creates a cone shape.

Beat Frequency

When waves (e.g. sound waves) of two different frequencies are added together, there is both constructive and destructive interference because of the principle of superposition

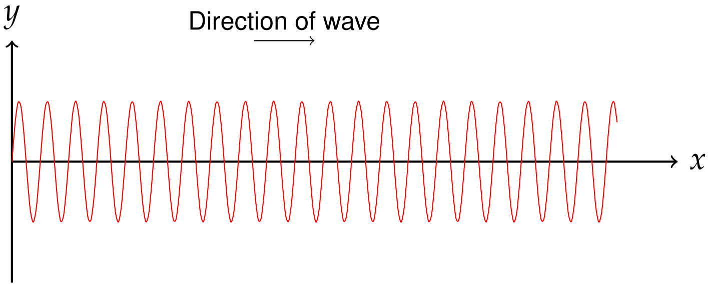
- Plotting two functions representing two waves with equal magnitude and wave speed v_s: $y = \sin(x)$



Beat Frequency

When waves (e.g. sound waves) of two different frequencies are added together, there is both constructive and destructive interference because of the principle of superposition

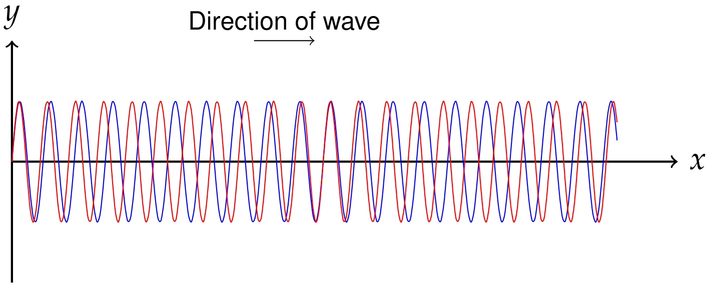
- Plotting two functions representing two waves with equal magnitude and wave speed v_s: $y = \sin(x)$ and $y = \sin(1.1x)$



Beat Frequency

When waves (e.g. sound waves) of two different frequencies are added together, there is both constructive and destructive interference because of the principle of superposition

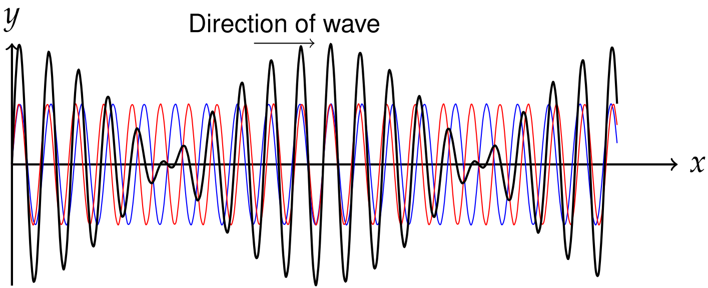
- Plotting two functions representing two waves with equal magnitude and wave speed v_s: $y = \sin(x)$ and $y = \sin(1.1x)$



Beat Frequency

When waves (e.g. sound waves) of two different frequencies are added together, there is both constructive and destructive interference because of the principle of superposition

- Plotting two functions representing two waves with equal magnitude and wave speed v_s: $y = \sin(x)$ and $y = \sin(1.1x)$



- The thick black line is the sum: $y = \sin(x) + \sin(1.1x)$

Beat Frequency

The **beat frequency** is the absolute difference of the frequencies of the two component waves:

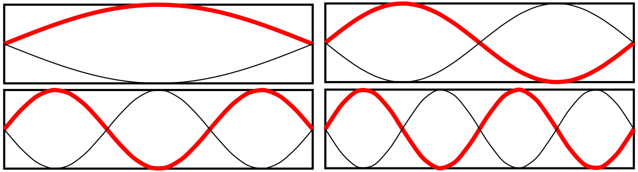
$$\boxed{f_{\text{beat}} = |f_1 - f_2|}$$

Quantity	Symbol	SI Unit
Beat frequency	f_{beat}	Hz (hertz)
Frequency of 1st component wave	f_1	Hz (hertz)
Frequency of 2nd component wave	f_2	Hz (hertz)

For sound waves, they sound like a pulsating “whoomf”. Musicians often use the beat frequencies to determine whether someone is play in tune or out of tune.

Standing Waves in a Closed Pipe

We have already studied standing-wave patterns on a“vibrating string”. Standing-wave patterns of sound waves can be found on pipes that have both ends closed:



The air molecules at the end of the pipe cannot vibrate along the direction of wave motion, therefore they have to be nodes. This pattern is identical to that of the vibrating string.

Standing Waves in Closed Pipes

Like strings, pipes that are *closed at both ends* have resonance frequencies that are whole-number multiple of the fundamental frequency f_1:

$$\boxed{f_n = n f_1 = n\frac{v_s}{2L}} \quad n = 1, 2, 3, \ldots$$

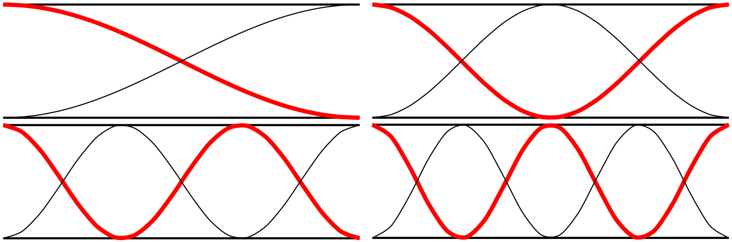
And the wavelengths corresponding to the resonance frequencies are:

$$\boxed{\lambda = \frac{2L}{n}} \quad n = 1, 2, 3, \ldots$$

The difference between a closed pipe and a string is that the wave speed is now the speed of sound v_s inside the pipe.

Standing Waves in Open Pipes

Many types of organ pipes, as well as the flute, can be modelled as pipes that are open on both ends. The standing-wave patterns for open pipes are similar to strings and closed pipes, but with nodes and anti-nodes reversed:



The air molecules at the ends of the pipe have maximum vibrations, and are anti-nodes in the standing wave.

Standing Waves in Open Pipes

Not surprisingly, like strings and closed pipes, open pipes have resonance frequencies that are whole-number multiple of the fundamental frequency f_1:

$$\boxed{f_n = n f_1 = n\frac{v_s}{2L}} \quad n = 1, 2, 3, \ldots$$

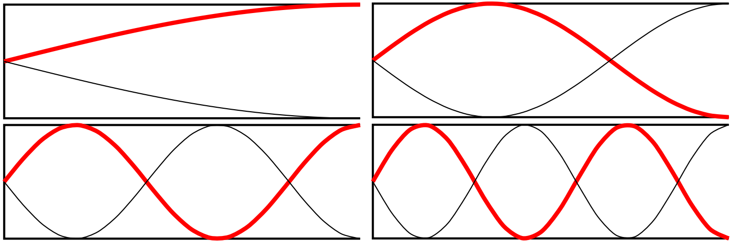
And the wavelengths corresponding to the resonance frequencies are also identical to that of the closed pipes.

$$\boxed{\lambda = \frac{2L}{n}} \quad n = 1, 2, 3, \ldots$$

Strings, closed pipes and open pipes are all said to have “full set of harmonics” because every harmonic frequency is also a resonance frequency.

Standing Waves in Semi-Open Pipes

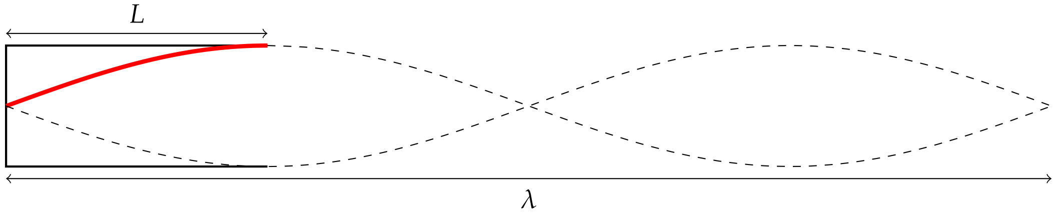
However, most organ pipes, woodwind and brass instruments are in fact modelled as pipes that are *closed at one end and open at the other*



The closed end is a node (like in the closed pipes), while the open end is an anti-node (like in the open pipes).

Standing Waves in Semi-Open Pipes

Again starting with the fundamental frequency (lowest frequency where a standing wave can form inside the pipe). This occurs at $\lambda = 4L$:

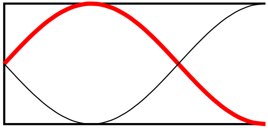


Fundamental frequency f_1 differs from the open-pipe and closed-pipe configurations by a factor of 2:

$$\boxed{f_1 = \frac{v_s}{\lambda} = \frac{v_s}{4L}}$$

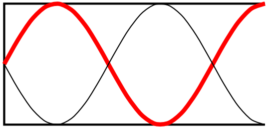
Standing Waves in Semi-Open Pipes

Likewise, second resonance can be found at $\lambda = \frac{4}{3}L$:



$$f_{\text{res},2} = \frac{v_s}{\lambda} = \frac{3v_s}{4L} = 3f_1$$

And then, third resonance at $\lambda = \frac{4}{5}L$:



$$f_{\text{res},3} = \frac{v_s}{\lambda} = \frac{5v_s}{4L} = 5f_1$$

Standing Waves in Semi-Open Pipes

Only **odd-number multiples** of the fundamental frequency are resonance frequencies in a semi-open pipe. We say that semi-open pipes have an *odd* set of harmonics.

$$\boxed{f_n = (2n-1)f_1 = \frac{(2n-1)v_s}{4L}} \qquad n = 1, 2, 3, \ldots$$

Because fundamental frequency f_1 is lower than the open-pipe configuration by a factor of 2 for the same length L, it has advantages when designing an organ pipe.