Topic 1: Kinematics

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Revised: April 7, 2020

Kinematics is a discipline with in mechanics for describing the motion of points, bodies (objects), and systems of bodies (groups of objects). It is the mathematical representation of the relationship between *position*, *displacement*, *distance*, *velocity*, *speed* and *acceleration*. Note that kinematics does *not* deal with what causes motion.

1 Kinematic Quantities

1.1 Position

Position is a vector describing the location of an object in a coordinate system (usually *Cartesian* for rectilinear motion, but for circular motions, the coordinate system can also be *polar* in 2D, or *cylindrical* or *spherical* in 3D). The origin of the coordinate system is called "reference point". In the IJK notation for rectilinear motion, we can express position of an object by its x, y and z components. The SI unit for position (and displacement) is *meters* m:

$$\mathbf{x}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}$$

The position vector is a function of time t.

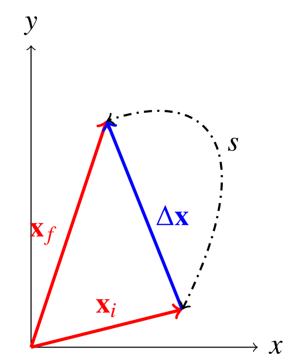

Figure 1: Position, displacement and distance in a Cartesian coordinate system.

1.2 Displacement

Displacement is the change in position from $\mathbf{x}_i$ to $\mathbf{x}_f$ within the same coordinate system. Like position (and not surprisingly), the SI unit for displacement is also *meters* m:

$$\Delta\mathbf{x} = \mathbf{x}_f - \mathbf{x}_i = (x_f - x_i)\hat{\boldsymbol{\imath}} + (y_f - y_i)\hat{\boldsymbol{\jmath}} + (z_f - z_i)\hat{\boldsymbol{k}}$$

It it illustrated in Figure 1.

- IJK notation makes vector addition and subtraction less prone to errors

Note that since "reference point" is the origin of the coordinate system, i.e. $\mathbf{x}_{\text{ref}} = \mathbf{0}$, any position vector $\mathbf{x}$ is also its displacement from the reference point.

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1.3 Distance

Distance s is a quantity that is *similar* (and related) to displacement. It is the *length of the path* taken when an object moves from position $\mathbf{d}_1$ to position $\mathbf{d}_2$. Unlike displacement, however, distance is a scalar quantity that is always positive: $s \geq 0$, i.e. you can never walk a *negative* distance to the store. Because the path is not always a straight line, therefore while the magnitude of the displacement vector is also a scalar, it is not necessarily the same as distance. In general

$$s \geq |\Delta \mathbf{d}|$$

1.4 Instantaneous & Average Velocities

Velocity is a quantity used to describe how *fast* an object is moving. If position $\mathbf{x}(t)$ is differentiable in time t, then its **instantaneous velocity** $\mathbf{v}(t)$ can be found at any time t by differentiating $\mathbf{x}$ with respect to time. The SI unit for velocity is *meters per second* m/s:

$$\boxed{\mathbf{v}(t) = \frac{d\mathbf{x}(t)}{dt}} \tag{1}$$

Since position $\mathbf{x}$ has x, y and z components in the $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ directions (they are linearly independent), we can take the time derivative of every component to obtain the velocity components v_x, v_y and v_z in those directions:

$$\mathbf{v}(t) = \frac{d\mathbf{x}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$$

By the fundamental theorem of calculus, if instantaneous velocity $\mathbf{v}(t)$ is the rate of change of position $\mathbf{x}(t)$ with respect to time t, then $\mathbf{x}(t)$ is the time integral of $\mathbf{v}(t)$:

$$\boxed{\mathbf{x}(t) = \int \mathbf{v}(t)dt + \mathbf{x}_0} \tag{2}$$

The constant of integration $\mathbf{x}_0 = \mathbf{x}(0)$ is the object's *initial position* at $t = 0$. As was the case in differentiation, we can integrate each component to get $\mathbf{x}$:

$$\mathbf{x}(t) = \left(\int v_x\hat{\imath} + \int v_y\hat{\jmath} + \int v_z\hat{k}\right)dt + \mathbf{x}_0$$

The **average velocity** ($\overline{\mathbf{v}}$)[1] of an object is the change in position $\Delta\mathbf{x}$ over a finite time interval Δt:

$$\boxed{\overline{\mathbf{v}} = \frac{\Delta\mathbf{x}}{\Delta t}} \tag{3}$$

Like instantaneous velocity, we can find the x, y and z components of average velocity by separating components in each direction:

$$\overline{\mathbf{v}} = \frac{\Delta x}{\Delta t}\hat{\imath} + \frac{\Delta y}{\Delta t}\hat{\jmath} + \frac{\Delta z}{\Delta t}\hat{k} = \overline{v}_x\hat{\imath} + \overline{v}_y\hat{\jmath} + \overline{v}_z\hat{k}$$

[1] For *time averages*, the convention is to write a bar over the quantity, as we have done here. In contrast, for *ensemble averages*, e.g. the average speeds of many particles, we use the notation $\langle v\rangle$. (See thermodynamics slides later in the course)

1.5 Instantaneous & Average Speed

Instantaneous speed $v(t)$ is the rate of change of *distance* with respect to time.[2] Like velocity, the unit for speed is also m/s:

$$\boxed{v(t) = \frac{ds}{dt}}$$

Since distance is a scalar quantity, so too is speed. As distance of any path must always be positive $s > 0$, instantaneous speed must also be positive. Instantaneous speed v is the magnitude of the instantaneous velocity vector $\mathbf{v}$. Likewise, **average speed** $\overline{v}$ is similar to average velocity: it is the distance travelled over a finite time interval.

$$\boxed{\overline{v} = \frac{s}{\Delta t}}$$

1.6 Path

Sometimes instead of explicitly describing the position $x = x(t)$ and $y = y(t)$, the path of an object can be given in terms of x coordinate $y = y(x)$, while giving the x (or y) coordinate as a function of time.

- In this case, substitute the expression for $x(t)$ into $y = y(x)$ to get an expression of $y = y(t)$
- Take derivative using chain rule to get $v_y = v_y(t)$

1.7 Instantaneous and Average Acceleration

In the same way that velocity is the rate of change in position with respect to time, **instantaneous acceleration** $\mathbf{a}(t)$ is the rate of change in velocity with respect to time, and the second time derivative of position. The SI unit for acceleration is *meters per second squared* m/s^2:

$$\boxed{\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{x}(t)}{dt^2}} \tag{4}$$

Although in Grades 11 and 12, students deal almost exclusively with constant acceleration, in AP Physics, it must be understood that acceleration can also vary with time, and that calculus must be used in many cases. Again, by the fundamental theorem of calculus, instantaneous velocity $\mathbf{v}(t)$ is the time integral of instantaneous acceleration $\mathbf{a}(t)$:

$$\boxed{\mathbf{v}(t) = \int \mathbf{a}(t)dt + \mathbf{v}_0} = \left(\int a_x\hat{\boldsymbol{\imath}} + \int a_y\hat{\boldsymbol{\jmath}} + \int a_z\hat{\boldsymbol{k}}\right)dt + \mathbf{v}_0 \tag{5}$$

1.8 Expressing Acceleration as Functions of Position and Velocity

Since, according to Newton's second law of motion, acceleration is related to the net force, therefore there are instances where acceleration is often expressed as functions of other motion quantities rather than time. For example,

[2] It is regrettable that both velocity and speed use the symbol v, but *c'est la vie*.

- **Spring force** is proportional to displacement, which means that acceleration is best expressed as:

$$a(x) = -bx \quad \text{or} \quad \frac{d^2x}{dt^2} = -bx$$

The solution to this *second-order ordinary differential equation with constant coefficient* is a sinusoidal function, i.e. $x(t) = A\sin(\omega t + \phi)$, and the motion is a *simple harmonic motion* that will be studied in a later topic.

- **Damping force** is usually proportional to velocity, which leads to an expression for acceleration

$$a(v) = -cv \quad \text{or} \quad \frac{dv}{dt} = -cv$$

This time, the equation is a *first-order ordinary differential equation* that can be solved by separating the dt term and v terms and then integrating, and the expression for velocity is an exponential function:

$$\frac{dv}{dt} = -cv \quad \rightarrow \quad \int \frac{dv}{v} = -\int c dt \quad \rightarrow \quad \ln(v) = -ct + C \quad \rightarrow \quad v(t) = v_0 e^{-ct}$$

Once the velocity expression is obtained, the expression for $x(t)$ can also easily be obtained by integrating.

- **Aerodynamic forces** such as drag and lift are proportional for the square of the velocity, therefore the acceleration is

$$a(v) = -kv^2 \quad \text{or} \quad \frac{dv}{dt} = -kv^2$$

Not surprisingly, the process of solving the problem is similar to that of the damping function, but this time, the solution is a hyperbolic function:

$$\frac{dv}{dt} = -kv^2 \quad \rightarrow \quad \int \frac{dv}{v^2} = -\int k dt \quad \rightarrow \quad -\frac{1}{v} = -kt + C \quad \rightarrow \quad v(t) = \frac{1}{ct + C}$$

In practice, multiple forces may act on an object, and each of them will be functions of other motion quantities, and therefore the solution may require solving complex differential equations (although this is highly unlikely in AP Physics).

1.9 Special Notation When Differentiating With Time

Physicists and engineers often use a special notation when the derivative is taken with respect to *time* (and not spatial derivatives), by writing a dot above the variable for *first* derivative, and *two* dots for *second* derivative, etc. For example, velocity is $\mathbf{v}(t) = \dot{\mathbf{x}}$ while acceleration is $\mathbf{a}(t) = \dot{\mathbf{v}} = \ddot{\mathbf{x}}$. This notation will be used occasionally in this course when it is convenient to do so.

1.10 Higher Derivatives of Position (For Those Who Are Curious)

For those who are curious about higher derivatives, the time derivative of acceleration is called **jerk** $\mathbf{j}(t)$ with a unit of m/s^3:

$$\mathbf{j}(t) = \frac{d\mathbf{a}}{dt} = \frac{d^2\mathbf{v}}{dt^2} = \frac{d^3\mathbf{x}}{dt^3} \tag{6}$$

Jerk is used in many sensors, for example, in accelerometers in airbags to determine if the acceleration of a car is under normal operation (small j value) or if a crash is in progress (high j value). The time derivative of jerk is **jounce**, or **snap**, with unit of m/s^4:

$$\mathbf{s}(t) = \frac{d\mathbf{j}}{dt} = \frac{d^2\mathbf{a}}{dt^2} = \frac{d^3\mathbf{v}}{dt^3} = \frac{d^4\mathbf{x}}{dt^4} \tag{7}$$

The next two derivatives of snap is facetiously called **crackle** and **pop**[3], but these higher derivatives are rarely used, and will *not* be used in in AP Physics.

2 Kinematic Equations

Although kinematic problems in AP Physics often require calculus, these basic kinematic equations for constant acceleration are still a very powerful tool. For constant acceleration **a**, velocity can be obtained by integrating in time:

$$\boxed{\mathbf{v}(t) = \int \mathbf{a}dt = \mathbf{v}_0 + \mathbf{a}t} \tag{8}$$

where $\mathbf{v}_0$ is the initial velocity at $t = 0$. Integrating again for the position vector:

$$\boxed{\mathbf{x}(t) = \int \mathbf{v}dt = \mathbf{x}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2} \tag{9}$$

where **x** is the initial position at $t = 0$. The position vector is quadratic in time.

The derivation of the last equation is slightly more laborious. From Eqs. 8 and 9, if acceleration is constant, then both the the velocity and position vectors are continuously differentiable. For simplicity, a one-dimensional problem is presented.

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$$
$$a = v\frac{dv}{dx}$$

Multiplying both sides by dx and integrating, we have:

$$\int_{x_0}^{x} a dx = \int_{v_0}^{v} v dv$$
$$a(x - x_0) = \frac{1}{2}\left(v^2 - v_0^2\right)$$

or in the more familiar form:

$$\boxed{v^2 = v_0^2 + 2a(x - x_0)} \tag{10}$$

Eqs. 8, 9 and 10 are provided in the AP Exam equation sheet.

[3] As in the cartoon mascots for Kellogg's rice crispies