19. Mechanical Waves

Advanced Placement Physics

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Olympiads School

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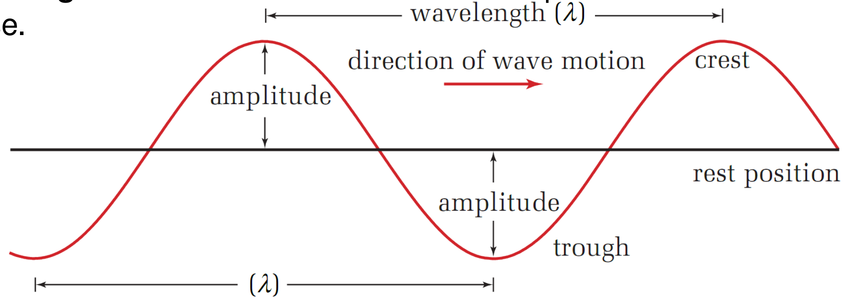
What is a wave?

- When a disturbance (vibration) causes vibrations in its vicinity, a wave is created
- A wave transfers energy through a medium (there is one exception)
  - The medium vibrates and have a net displacement of zero.
  - Each particle vibrates instead of moving horizontally, and the vibration get transferred to the next particle.

Features of a Wave

- **Crest**: Highest point
- **Trough**: Lowest point
- **Wavelength**: Shortest distance between two points in the medium that are in phase.



(The easiest way to measure wavelength is from crest to crest, or from trough to trough.)

Frequency and Speed of A Wave

Frequency of A Wave (f)

- The number of complete wavelengths that pass a point in a given amount of time
- Unit: hertz (Hz)
- Same as the frequency of the disturbance that generated the wave
- **Does not depend on the medium**, only the source that produces the wave.

Speed of A Wave (v)

- The speed at which the wave fronts are moving
- **Depends only on the medium**

Equation

A harmonic wave can be described as a sinusoidal function:

$$\boxed{y(x,t) = A\sin(kx - \omega t)}$$

Quantity	Symbol	SI Unit
Displacement of the medium	y	m (meters)
Wave number	k	/m (per meter)
Distance from the source	x	m (meters)
Time	t	s (seconds)
Angular frequency	ω	/s (per second)

Equation

$$\boxed{y(x,t) = A\sin(kx - \omega t)}$$

If the wave is generated by a mass on a spring, then k is the spring constant of the spring. It is related to the wavelength by:

$$k = \frac{2\pi}{\lambda}$$

The angular frequency (angular velocity) is related to the frequency f and period T of the wave by:

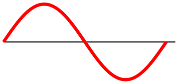
$$\omega = \frac{2\pi}{T} = 2\pi f$$

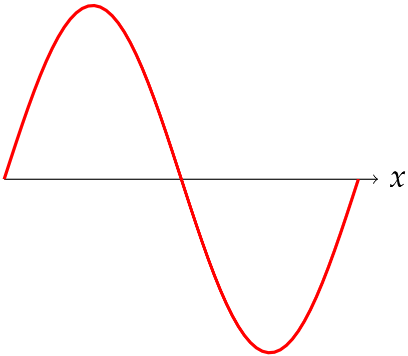
Why Sine and Cosines

French mathematician Joseph Fourier discovered that *all* periodic functions are infinite series of $\sin$ and/or $\cos$ functions:

$$\begin{aligned} f(x) =& a_1 \sin(x) + a_2 \sin(2x) + a_3 \sin(3x) + \ldots + \\ & b_1 \cos(x) + b_2 \cos(2x) + b_3 \cos(3x) + \ldots \\ =& \sum_{n=1}^{\infty} a_n \sin(nx) + \sum_{n=1}^{\infty} b_n \cos(nx) \end{aligned}$$

Making a Square Wave with Sine Waves


$f_1 = \sin(x)$



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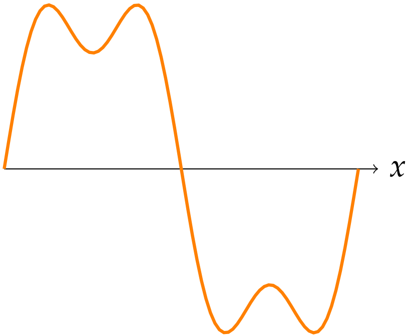

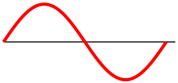

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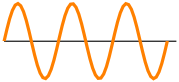
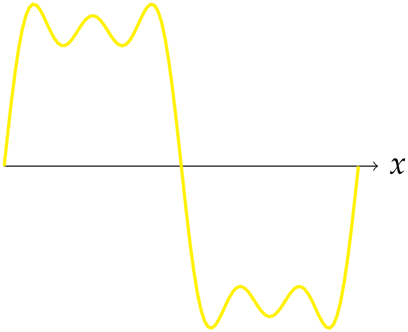
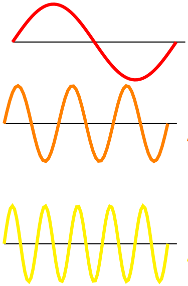


$f_1 = \frac{1}{3}\sin(3x)$

Making a Square Wave with Sine Waves





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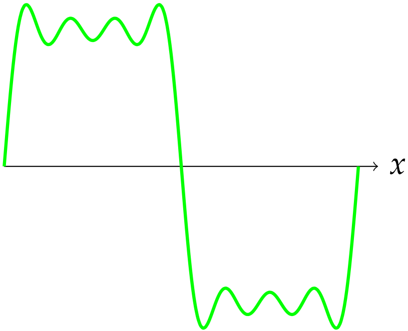

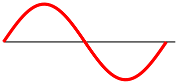

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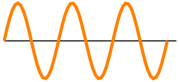

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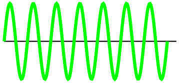

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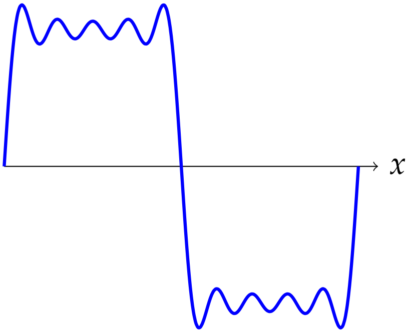
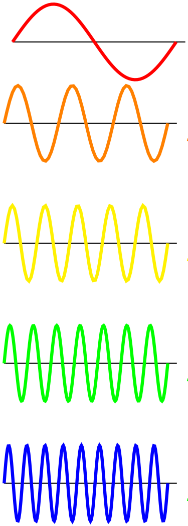
Making a Square Wave with Sine Waves





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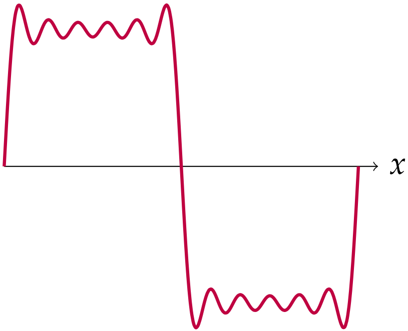
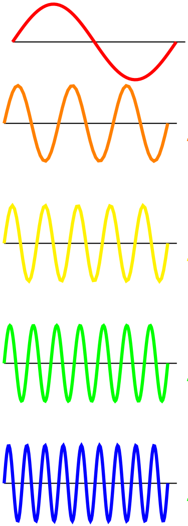
Making a Square Wave with Sine Waves





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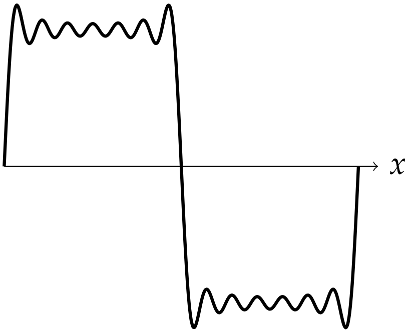

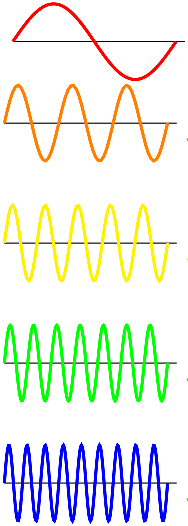

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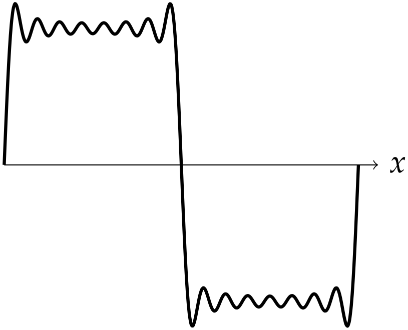

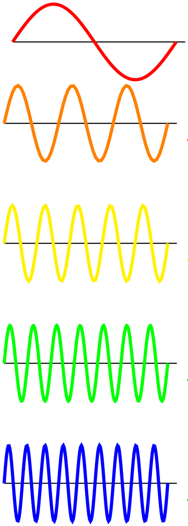

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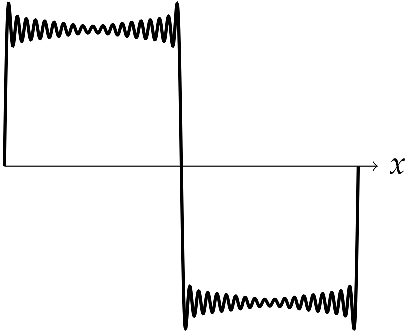

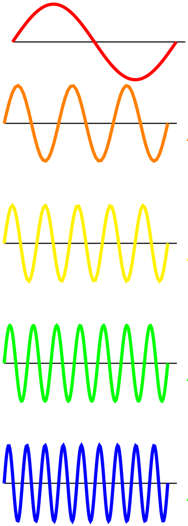

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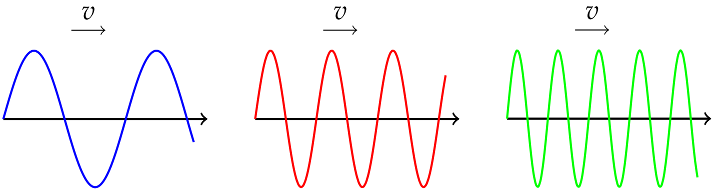
Fourier Series and Harmonic Frequencies



- The first wave—with the longest wavelength and lowest frequency—is called the **fundamental frequency**, or **first harmonic**
- The second term has half the wavelength and twice the frequency. It's called the **second harmonic**, the **first overtone**
- Also, third, fourth, fifth. . . harmonics

Harmonic Frequencies

- When a musical instrument produces a sound, the frequency that is “heard” is the fundamental frequency
- Every whole-number multiples of the fundamental frequency f_1 is its harmonic frequency, i.e. the n-th harmonic is:

$$\boxed{f_{\mathrm{harm},n} = n f_1} \quad \text{where} \quad n \geq 1$$

Universal Wave Equation

When combining the wave number and angular frequency, we can find that the speed of a wave is the product of the wavelength and the frequency:

$$\boxed{v = f\lambda}$$

Quantity	Symbol	SI Unit
Speed	v	m/s (meters per second)
Frequency	f	Hz (hertz)
Wavelength	λ	m (meters)

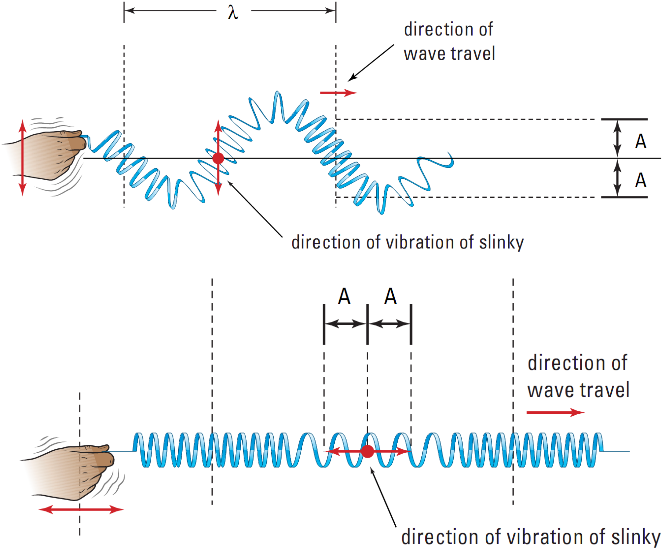
The universal wave equation applies to *all* waves. For sound waves, $v = v_{\text{sound}}$; for electromagnetic waves $v = c$.

Two Types of Wave

There are two types of waves

- **Transverse waves**:
  - Particles of a medium vibrate at right angles to the direction of the motion.
  - e.g.: ocean waves, electromagnetic waves
- **Longitudinal waves:**
  - Particles of a medium vibrate parallel to the direction of the motion of the wave
  - e.g.: sound waves

Transverse Wave vs. Longitudinal Wave



Wave Simulation

A helpful simulation can be found on the PhET website at University of Colorado.

Click for external link:
wave on a string simulation

Wave on a String

The speed of a travelling wave on a stretched string is given by:

$$\boxed{v = \sqrt{\frac{F_T}{\mu}}} \quad \text{where} \quad \boxed{\mu = \frac{m}{L}}$$

Quantity	**Symbol**	**SI Unit**
Wave speed	v	m/s (meters per second)
Tension	F_T	N (newtons)
Linear mass density	μ	kg/m (kilograms per meter)
Mass of the string	m	kg (kilograms)
Length of the string	L	m (meters)

Power Transmitted by a Harmonic Wave

Then the power transmitted by a harmonic wave is through a travelling wave on a string is determined by the linear mass density μ, the angular frequency ω, amplitude A and wave speed v:

$$\boxed{P = \frac{1}{2}\mu\omega^2 A^2 v}$$

The Decibel

The decibel is defined as by the intensity of sound I compared to the *threshold of hearing* intensity I_0:

$$\boxed{\beta = 10\log_{10}\left[\frac{I}{I_0}\right]} \quad \text{where} \quad I_0 = 10^{-12}\,\mathrm{W/m^2} \quad \text{and} \quad \boxed{I = \frac{P_{\text{ave}}}{4\pi r^2}}$$

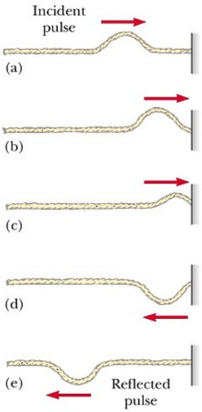
Quantity	Symbol	SI Unit
Intensity of sound	β	dB (decibels)
Intensity of sound	I	$\mathrm{W/m^2}$ (watts per square meters)
Threshold intensity	I_0	$\mathrm{W/m^2}$ (Watts per square meters)
Average power of the source	P_{ave}	W (watts)
Distance from the source	r	m (meters)

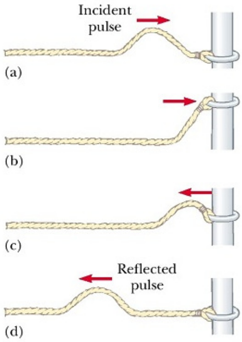
The *threshold of pain* for human ears is defined at $120\,\mathrm{dB}$.

Reflection of Wave at a Boundary

When a wave on a string reflects at a boundary, how the reflected wave looks depends on the type of boundary

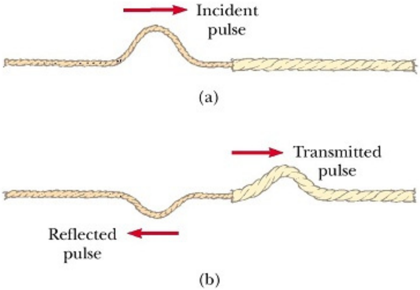
- At a *fixed end* (left), the reflected wave is *inverted*, i.e. a crest becomes a trough
- At a *free end* (right), the reflected wave is upright





Transmission of Waves: Fast to Slow Medium

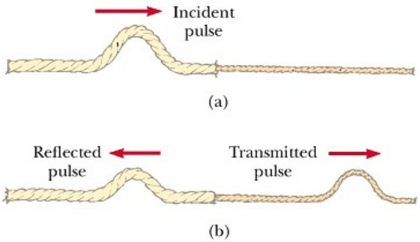
- Reflected wave:
  - Inverted, like a fixed end
  - Same frequency and wavelength as the incoming wave
  - The amplitude is decreased
- Transmitted wave:
  - Upright
  - Same frequency as incoming wave, but has a shorter wavelength because the wave slowed down



Transmission of Waves: Slow to Fast Medium

- Reflected wave:
  - Upright, like a free end
  - Same frequency and wavelength as the incoming wave
  - The amplitude is decreased
- Transmitted wave:
  - Upright
  - Same frequency as incoming wave, but has a longer wavelength because the wave sped up

Note that the transmitted wave is *always* upright.



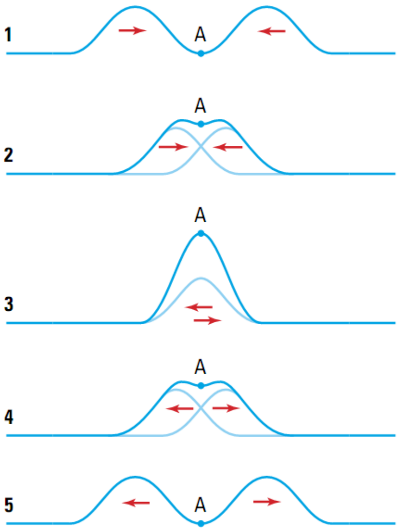
Superposition of Waves

- **Principle of Superposition:** When multiple waves pass through the same point, the resultant wave is the *sum* of the waves
  - A fancy way of saying that waves add together
- The consequence of the principle of superposition is *interference of waves*. There are two kinds of interference:
  - **Constructive interference:** Two wave fronts (crests) passing through creates a wave front with greater amplitude
  - **Destructive interference:** A crest and trough will cancel each other

Superposition of Waves

Constructive interference: In-phase wave fronts sum together

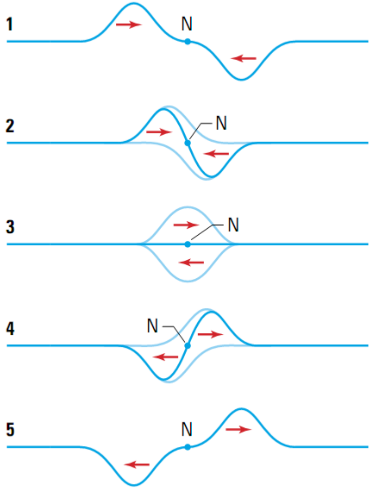
- In this example, two identical pulses move towards each other
- Their crests pass through A at the same time
- The amplitude at A when the waves pass through is higher



Superposition of Waves

Destructive interference: Out-of-phase wave fronts shows the difference of the wave fronts

- Two pulses move towards each other, one a crest, the other a trough
- They both pass through A at the same time
- Two waves cancel each other at A



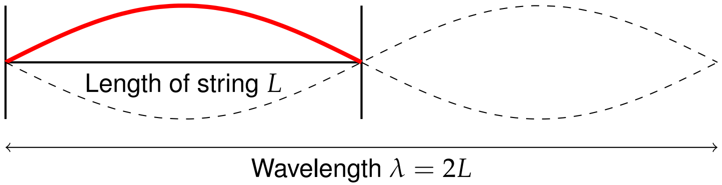
Standing Waves

If two waves of the same frequency meet up under the right conditions, they may appear to be "standing still". This is called a standing wave

- Node: A point that never moves
- Anti-node: A point which moves/vibrates maximally

Standing Waves On a String Length L

- A "vibrating" string is actually a standing wave on a string
- Both ends of the string are nodes
- **Resonance frequency** is a frequency that allows a standing wave to be created on the string. The first resonance (fundamental) frequency at occurs when $\lambda = 2L$:

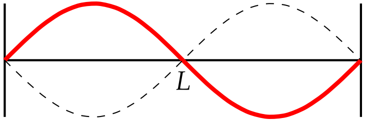


- Fundamental frequency is based on the speed of the travelling wave along the string v_{str}:

$$\boxed{f_1 = \frac{v_{\text{str}}}{\lambda} = \frac{v_{\text{str}}}{2L}}$$

Standing Waves On a String Length L

A second resonance frequency happens when $L = \lambda$:



$$f_{\text{res},2} = \frac{v_{\text{str}}}{\lambda} = \frac{v_{\text{str}}}{L} = 2f_1$$

Standing Waves On a String Length L

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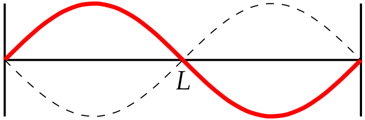
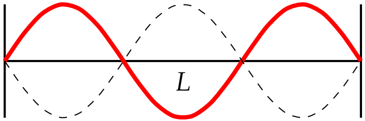


$$f_{\text{res},2} = \frac{v_{\text{str}}}{\lambda} = \frac{v_{\text{str}}}{L} = 2f_1$$

And again, a third resonance frequency occurs at $L = \frac{3}{2}\lambda$:



$$f_{\text{res},3} = \frac{3v_{\text{str}}}{2L} = 3f_1$$

Standing Waves On a String Length L

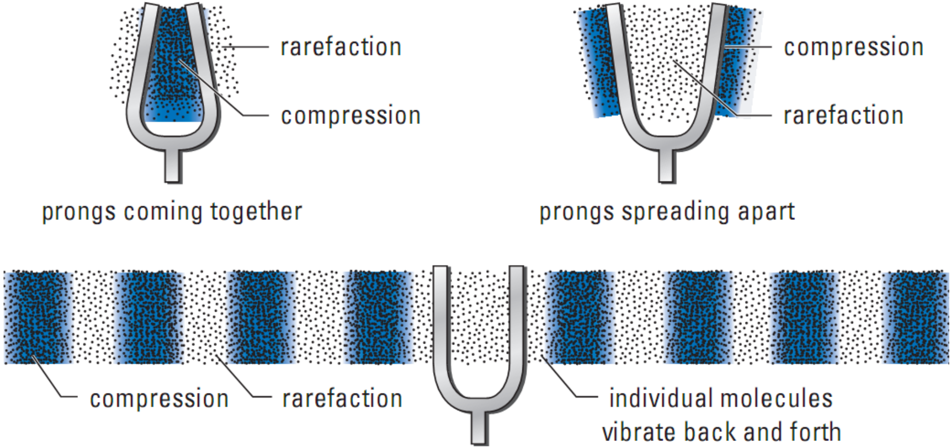
In fact, the n-th resonance frequency of a wave on string is just:

$$\boxed{f_{\text{res},n} = n f_1} \qquad \text{(standing wave on string)}$$

- f_1 is the fundamental frequency, and n is a whole-number multiple
- This equation is *identical* to the equation for harmonic frequencies, meaning that every harmonic is a resonance frequency
- It has a "full set of harmonics"

Transfer of Sound Wave

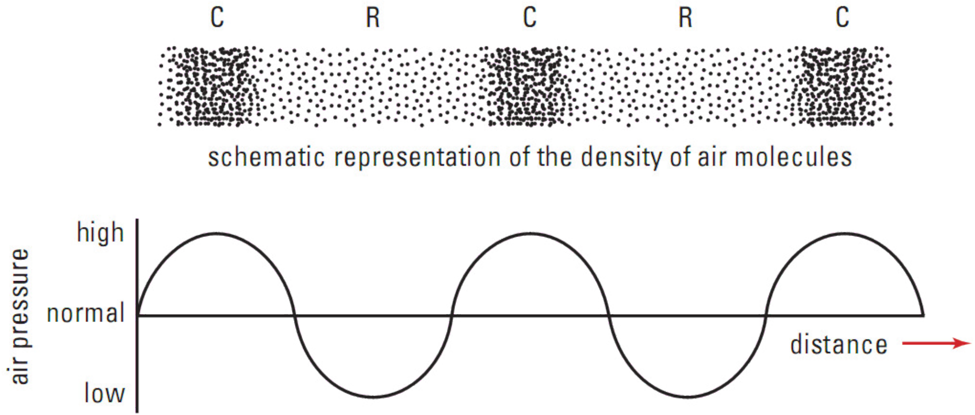
Example: tuning fork



Transfer of Sound Wave

Schematic Diagram vs. Wave Graph

We can also express the amplitude of the sound wave by plotting the change in *air pressure*:



Speed of Sound in a Gas

The equation for the speed of sound in a gas (e.g. air) is given by:

$$\boxed{v_s = \sqrt{\frac{\gamma R T}{M}}}$$

Quantity	Symbol	SI Unit
Speed of sound	v_s	m / s (meters per second)
Temperature	T	K (kelvin)
Universal gas constant	R	J / mol K (joule per mol per kelvin)
Molar mass	M	kg / mol (kilograms per mol)
Adiabatic constant	γ	(no units)

For air $\gamma = 1.4$, and $M = 29 \times 10^{-3}$ kg / mol.

Mach Number

When working with sound, it is useful to express speed in terms of its ratio to the speed of sound. This is called the **Mach number**:

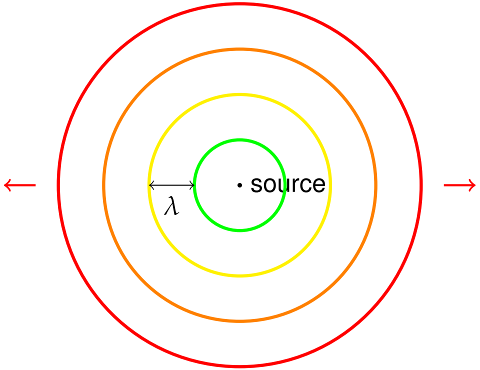
$$\boxed{M = \frac{v}{v_s}}$$

Quantity	Symbol	SI Unit
Mach Number	M	no units
Speed of the object	v	m/s (meters per second)
Local speed of sound	v_s	m/s (meters per second)

- When an object is travelling at $M < 1$, it is travelling at a *subsonic* speed
- When an object is travelling at $M > 1$, it is travelling at a *supersonic* speed

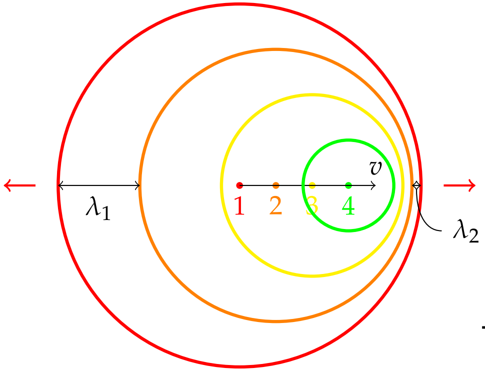
Sound from a Moving Source

When a sound is emitted from a point source, the sound wave moves radially outward from the point of origin. In this diagram, the source is stationary:



Sound from a Moving Source

But when sound is emitted from a *moving* source, the diagram looks different. In this case, the sound source is moving to the right, from 1 to 4:



- When the sound source is moving *towards you*, the wavelength λ_2 decreases, and the frequency increases.
- When the sound source is moving *away from you*, the wavelength λ_1 increases, and the frequency decreases.

This is called the **Doppler Effect**.

Doppler Effect

When a wave source is moving at a speed v_{src} and the observer is moving at v_{ob}, the frequency perceived by the observer is shifted to f':

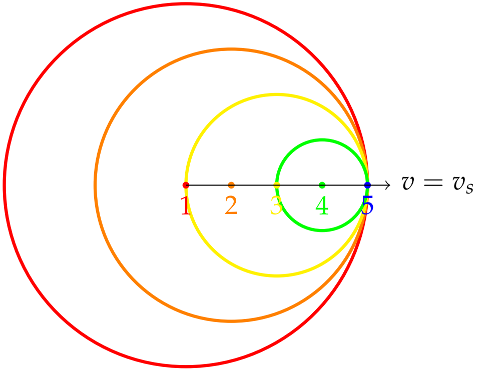
$$\boxed{f' = \frac{v_s + v_{\text{ob}}}{v_s - v_{\text{src}}} f}$$

Quantity	Symbol	SI Unit
Apparent frequency	f'	hertz (hertz)
Actual frequency	f	hertz (hertz)
Speed of sound	v_s	m/s (meters per second)
Speed of source	v_{src}	m/s (meters per second)
Speed of observer	v_{ob}	m/s (meters per second)

The Doppler effect equation works for all types of waves, including sound waves *and* electromagnetic waves.

Sound from a Source Moving At Sonic Speed

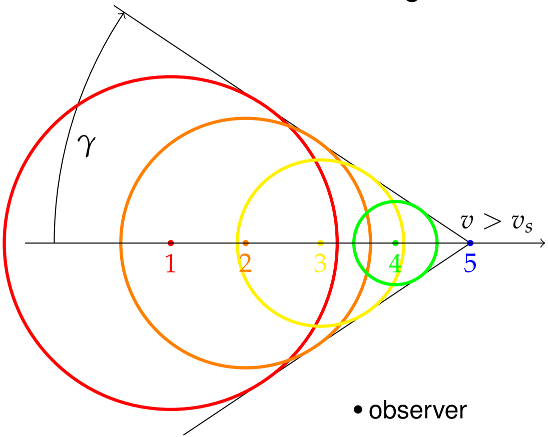
Doppler effect is more interesting is when sound source is moving at $M = 1$, the speed of sound:



- The wave fronts (crests) from all the waves are bunched up just in front of the source
- Since sound wave is a pressure wave, right in front of the sound source, there is a large change in pressure (called a shock wave)
- When the shock passes an observer, an loud bang can be heard (aka **sonic boom**)

Sound from a Supersonic Source

When sound source is moving at $M > 1$:



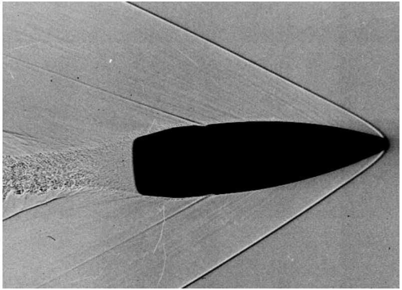
An *oblique shock* is formed at an angle (called the **Mach angle**) given by:

$$\gamma = \sin^{-1}\left(\frac{1}{M}\right)$$

A stationary observer does not hear the sound source coming until it has gone past!

Bullet in Supersonic Flight

Generating a shock doesn't require an actual sound source. Any object moving through air creates a pressure disturbance. This is a $7.62\,\mathrm{mm}$ NATO bullet in supersonic flight.



This bullet was not fired from a gun. Instead, it was placed in a shock tube that generates a short burst of supersonic flow, and a high-speed camera is then used to take the photo.

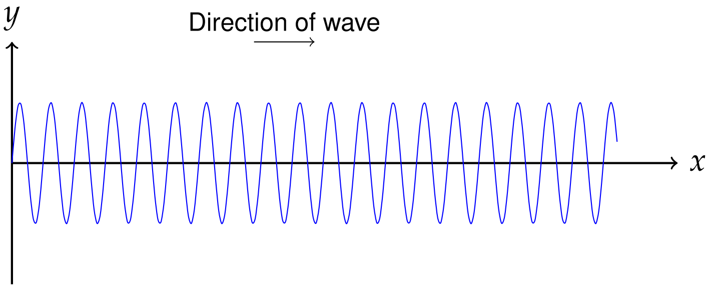
Duck in Water

No sonic booms here (just a duck swimming), but a similar shock behaviour is observed. The duck swims faster than the speed of the water wave, and it also creates a cone shape.

Beat Frequency

When waves of two different frequencies are added together, there is both constructive and destructive interference

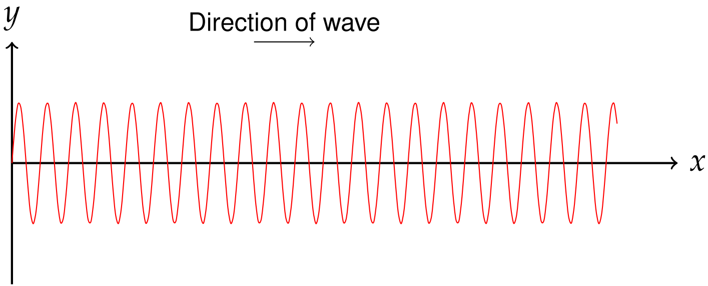
- Plotting two functions representing two waves with equal magnitude and wave speed v_s: $y = \sin(x)$



Beat Frequency

When waves of two different frequencies are added together, there is both constructive and destructive interference

- Plotting two functions representing two waves with equal magnitude and wave speed v_s: $y = \sin(x)$ and $y = \sin(1.1x)$



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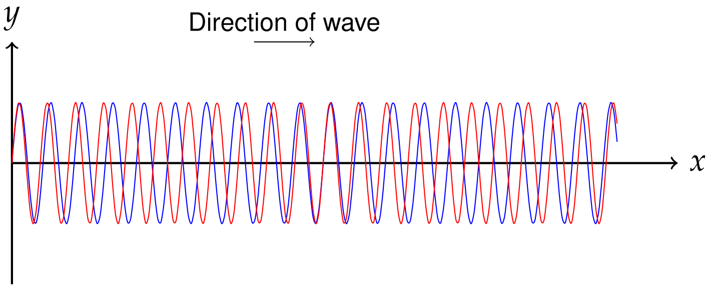

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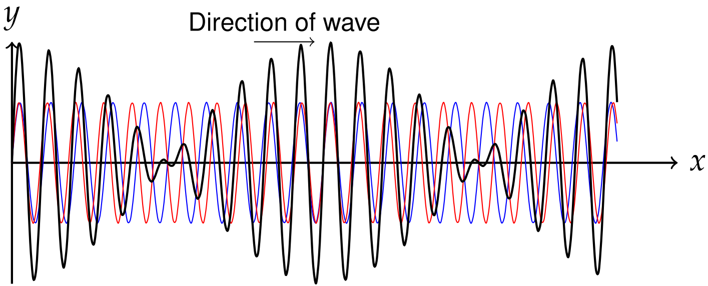


- The thick black line is the sum: $y = \sin(x) + \sin(1.1x)$

Beat Frequency

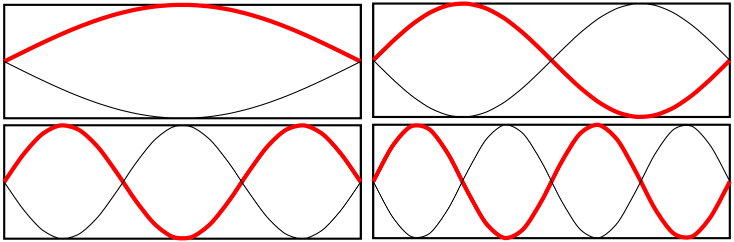
The *beat frequency* is the absolute value of the difference of the frequencies of the two component waves:

$$\boxed{f_{\text{beat}} = |f_1 - f_2|}$$

Quantity	Symbol	SI Unit
Beat frequency	f_{beat}	Hz (hertz)
Frequency of 1st component wave	f_1	Hz (hertz)
Frequency of 2nd component wave	f_2	Hz (hertz)

Standing Waves in a Closed Pipe

A standing-wave patterns can be found on pipes that have both ends closed:



Standing Waves in Closed Pipes

Like strings, pipes that are *closed at both ends* also have a full set of harmonics. The n-th resonance frequency is given by:

$$\boxed{f_{\mathrm{res},n} = n f_1} \qquad \text{(closed pipe)}$$

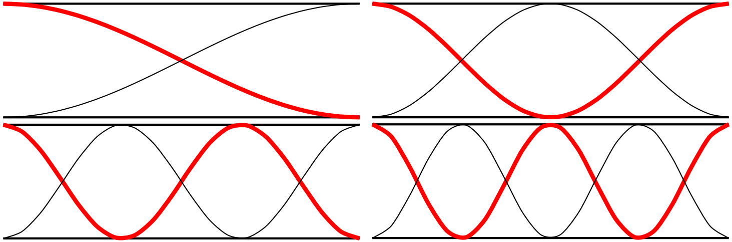
where n is a whole-number multiple of the fundamental frequency f_1:

$$\boxed{f_1 = \frac{v_s}{2L}}$$

The difference between a closed pipe and a string is that the wave speed is now the speed of sound v_s inside the pipe.

Standing Waves in Open Pipes

- Example: Some organ pipes, flute
- Both ends of the pipes are anti-nodes



First resonance at $\lambda = 2L$

$$f_1 = \frac{v_s}{\lambda} = \frac{v_s}{2L}$$

Second resonance at $\lambda = L$

$$f_2 = \frac{v_s}{\lambda} = \frac{v_s}{L} = 2f_1$$

Standing Waves in Open Pipes

Open pipes also have a “full set of harmonics”. The n-th resonance frequency is given by:

$$\boxed{f_{\text{res},n} = n f_1} \qquad \text{(open pipe)}$$

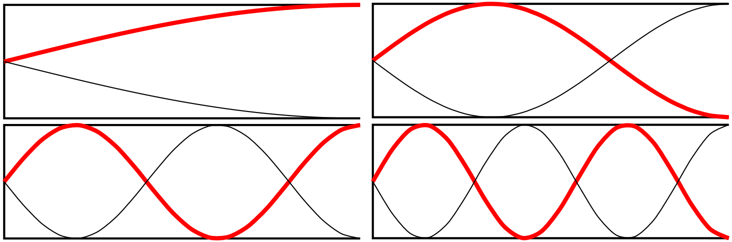
where n is a whole-number multiple of fundamental frequency f_1:

$$\boxed{f_1 = \frac{v_s}{2L}}$$

Standing Waves in Semi-Open Pipes

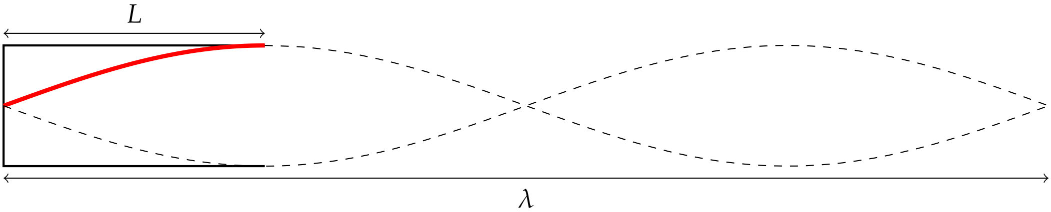
This is when things get a bit more interesting...

- Examples: Most organ pipes, clarinet, oboes, brass instruments
- Closed end: node (like in the closed pipes)
- Open end: anti-node (like in the open pipes)



Standing Waves in Semi-Open Pipes

Again starting with the fundamental frequency (lowest frequency where a standing wave can form inside the pipe). This occurs at $\lambda = 4L$:

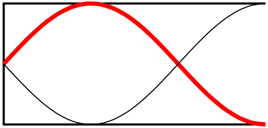


Fundamental frequency f_1 differs from the open-pipe and closed-pipe configurations by a factor of 2:

$$\boxed{f_1 = \frac{v_s}{\lambda} = \frac{v_s}{4L}}$$

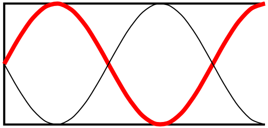
Standing Waves in Semi-Open Pipes

Likewise, second resonance can be found at $\lambda = \frac{4}{3}L$:



$$f_{\text{res},2} = \frac{v_s}{\lambda} = \frac{3v_s}{4L} = 3f_1$$

And then, third resonance at $\lambda = \frac{4}{5}L$:



$$f_{\text{res},3} = \frac{v_s}{\lambda} = \frac{5v_s}{4L} = 5f_1$$

We can repeat that for 4th, 5th. . . resonances.

Standing Waves in Semi-Open Pipes

Only **odd-number multiples** of the fundamental frequency are resonance frequencies in a semi-open pipe. We say that semi-open pipes have an *odd* set of harmonics.

$$\boxed{f_{\mathrm{res},n} = (2n-1) f_1} \qquad \text{(semi-open pipes)}$$

Because fundamental frequency f_1 is lower than open-pipe and closed-pipe configurations by a factor of 2 for the same length L, it has advantages when designing an organ pipe.

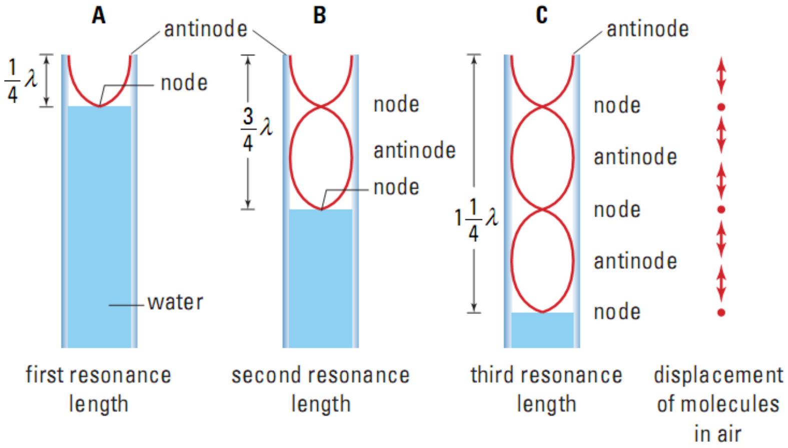
$$\boxed{f_1 = \frac{v_s}{\lambda} = \frac{v_s}{4L}}$$

Resonance *Length* in a Semi-Open Pipe

- Now that we have looked at resonance *frequencies*, we'll look at resonance *lengths*
- We produce a single frequency in the pipe, and vary the length of the pipe until we have resonance

Resonance Length in a Semi-Open Pipe

Let's submerge a part of this pipe in water…



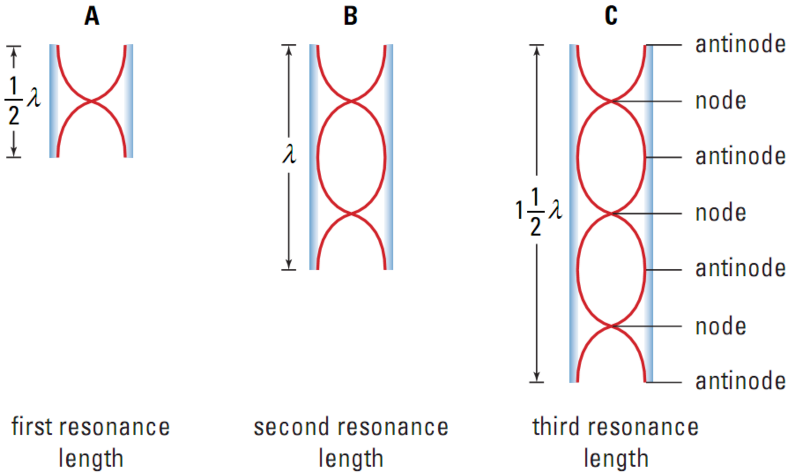
Resonance Length in a Semi-Open Pipe

The resonance lengths are **odd whole-number multiples** of the first resonance length L_1:

$$\boxed{L_{\text{res},n} = (2n-1)L_1} \quad \text{where} \quad \boxed{L_1 = \frac{\lambda}{4}}$$

Resonance in an Open Pipe

We can also repeat this with pipes that are open on both ends.



Resonance in an Open Pipe

Resonance lengths of an open pipe are **whole-number multiples** of the first resonance length L_1:

$$\boxed{L_{\text{res},n} = nL_1} \qquad \text{(open pipe)}$$

where first resonance length is given by:

$$\boxed{L_1 = \frac{\lambda}{2}}$$

Be careful! This equation looks a lot like the resonance frequency equation!