Magnetism

Special CAP/AP Lecture

Dr. Timothy Leung

November 2019

Olympiads School, Toronto, ON, Canada

Magnetic Field

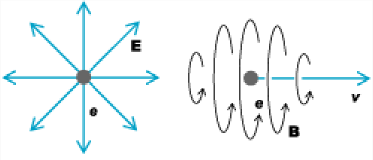
Review of Magnetic Field

- Magnetism is generated by moving charged particles, e.g. a single charge, or an electric current
- It can also be generated by permanent magnets, or Earth

Review of Magnetic Field

- Magnetism affects other *moving* charged particles
- The vector field is called the **magnetic field**
- Magnetic field has unit **tesla**
- Magnetic field lines have no ends—they always run in a loop

Magnetic Field Generated by a Moving Point Charge



A point charge generates an electric field $\mathbf{E}$. When it's moving, it also generates a magnetic field $\mathbf{B}$, given by the equation:

$$\boxed{\mathbf{B} = \frac{\mu_0}{4\pi} \frac{q\mathbf{v} \times \hat{\mathbf{r}}}{r^2}}$$

The direction of $\mathbf{B}$ can be obtained by applying the *right hand rule* if you are not confident with cross products.

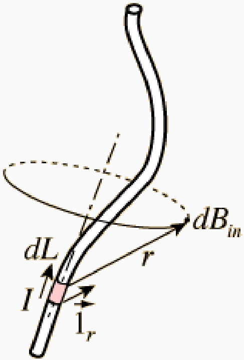
Magnetic Field Generated by a Moving Point Charge

$$\boxed{\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q\mathbf{v} \times \hat{\mathbf{r}}}{r^2}}$$

Quantity	Symbol	SI Unit
Magnetic field	$\mathbf{B}$	T
Charge	q	C
Velocity of the charge	$\mathbf{v}$	m/s
Distance from the moving charge	r	m
Radial outward unit vector from the charge	$\hat{\mathbf{r}}$	no units
Permeability of free space	μ_0	T m/A

Permeability of free space (or **vacuum permeability**) is a constant with a value of $\mu_0 = 4\pi \times 10^{-7}$ T m/A. It is a measure of how well a space can become magnetized.

Magnetic Generated By a Current

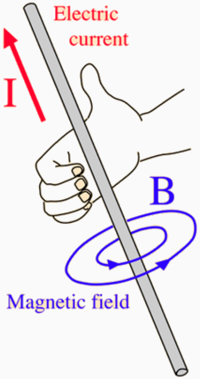


An electric current is really many charges particles moving along a wire; each charge creating its own magnetic field. The total magnetic field in the wire is the integral of the contribution ($d\mathbf{B}$) of the current (I) from each infinitesimal sections ($d\mathbf{L}$) of the wire, given by the **Biot-Savart law**:

$$\boxed{d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I d\mathbf{L} \times \hat{\mathbf{r}}}{r^2}}$$

The magnetic field in the diagram goes *into* the page

Magnetic Field Generated By an Infinitely Long Wire



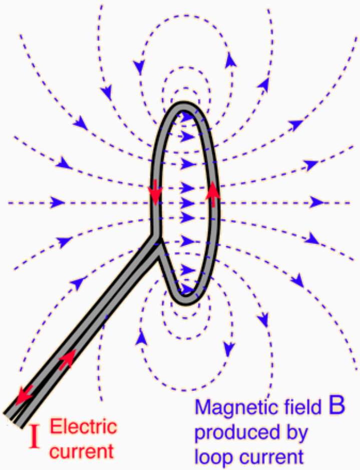
Integrating Biot-Savart law for a point at radial distance r from an *infinitely long wire* gives a simple expression:

$$\boxed{\mathbf{B} = \frac{\mu_0(\mathbf{I} \times \hat{\mathbf{r}})}{2\pi r}} \quad \text{or} \quad \boxed{B = \frac{\mu_0 I}{2\pi r}}$$

The magnitude and direction current "vector" $\mathbf{I}$ is straightforward

Quantity	Symbol	SI Unit
Magnetic field	$\mathbf{B}$	T
Current	$\mathbf{I}$	A
Radial direction from the wire	$\hat{\mathbf{r}}$	(no units)
Radial distance from the wire	r	m

Current-Carrying Wire Loop

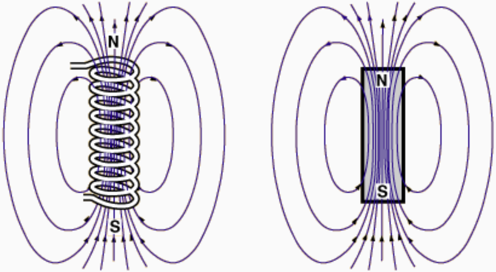


When we shape the current-carrying wire into a loop, we can (again) use the Biot-Savart law to find the magnetic field away from it.

One loop isn't very interesting (except when you're integrating Biot-Savart law) but what if we have many loops

Wounding Wires Into a Coil

- A **solenoid** is when you wound a wire into a coil
- You create a magnet very similar to a bar magnet, with an effective north pole and a south pole
- Magnetic field inside the solenoid is uniform
- Magnetic field strength can be increased by the addition of an iron core

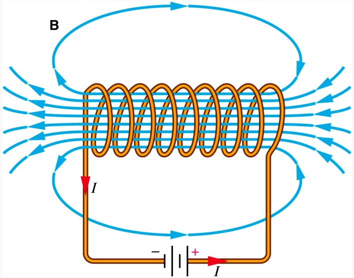


A Practical Solenoid

A practical solenoid usually has hundreds or thousands of turns:

This “air core” coil is used for high school and university experiments. It has approximately 600 turns of copper wire wound around a plastic core.

Magnetic Field Inside a Solenoid



The magnetic field **inside** a solenoid is uniform, with its strength given by:

$$\boxed{B = \frac{\mu N I}{L}}$$

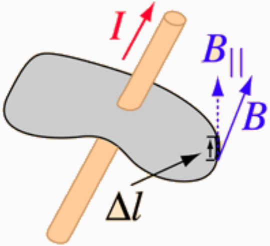
Direction of $\mathbf{B}$ determined by **right hand rule**

Quantity	Symbol	SI Unit
Magnetic field intensity	B	T
Number of coils	N	
Lenght of the solenoid	L	m
Current	I	A
Effective permeability	μ	T m/A

Ampère's Law

Ampère’s Law

Like Gauss’s law is used to calculate electric fields, **Ampère’s law** is used to calculate the magnetic field for symmetric configurations:

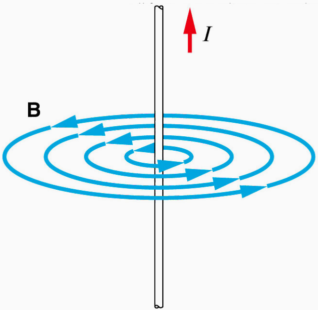


$$\boxed{\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_C}$$

where

- C is a closed curve around a current (“Amperian loop”)
- $d\boldsymbol{\ell}$ is an infinitesimal length along the closed curve
- I_c is the net current that penetrates the area bounded by C

Application of Ampère's Law: Infinitely Long Wire



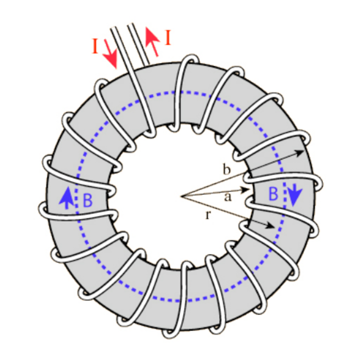
An *infinitely* long wire must generate a magnetic field that only depend on radial distance. We place an Amperian loop as a circle of radius r inside the toroid. Ampère's law reduces to:

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_C \;\rightarrow\; B(2\pi r) = \mu_0 I$$

From this, we get our expression of the magnetic field from an infinitely long wire:

$$B = \frac{\mu_0 I}{2\pi r}$$

Toroid



A toroid consists of a current-carrying wire wound around a donut-shaped core

Another application of Ampère's law is the **toroid**. This time, we put our loop at $a < r < b$ inside the toroid. Once again, because of symmetry, Ampère's law reduces to:

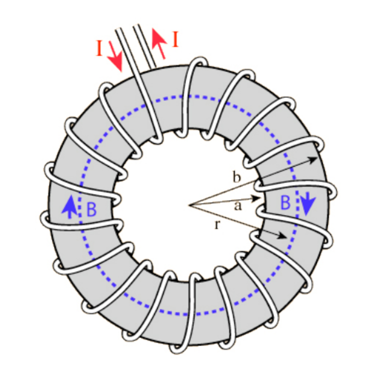
$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_C$$

$$B(2\pi r) = \mu_0 N I$$

$$B = \frac{\mu_0 N I}{2\pi r}$$

where N is the number of times the wire is wound around the core

Toroid



When the loop is placed at $r < a$, there is no enclosed current, and therefore the magnetic field is zero:

$$B = 0 \quad \text{for} \quad r < a$$

When the loop is placed at $r > b$, the amount of current penetrating the loop is the same in both direction, i.e. $I_c = 0$, and

$$B = 0 \quad \text{for} \quad r > b$$

In fact, magnetic field *only* exists inside the core, between a and b.

Magnetic Force

So What Does the Magnetic Field Do?

Gravitational Field $\mathbf{g}$

- Generated by objects with mass
- Affects objects with mass

Electric Field $\mathbf{E}$

- Generated by charged particles
- Affects charged particles

Magnetic Field $\mathbf{B}$

- Generated by *moving* charged particles
- Affects moving charged particles

Lorentz Force Law

Since a moving charge or current create both electric and magnetic fields, another moving charge is therefore affected by both $\mathbf{E}$ and $\mathbf{B}$. The total effect is given by the **Lorentz force law**:

$$\boxed{\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

$\mathbf{F}_q = q\mathbf{E}$ is the electrostatic force, and $\mathbf{F}_M = q\mathbf{v} \times \mathbf{B}$ is the magnetic force.

Quantity	Symbol	SI Unit
Total force on the moving charge	$\mathbf{F}$	N
Charge	q	C
Velocity of the charge	$\mathbf{v}$	m/s
Magnetic field	$\mathbf{B}$	T
Electric field	$\mathbf{E}$	N/C

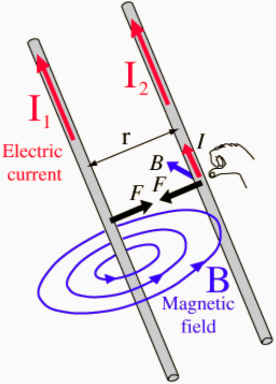
Force on a Current-Carrying Conductor in a Magnetic Field

Likewise, $\mathbf{B}$ exerts a force on another current-carrying conductor.

$$\boxed{F_M = \mathbf{I}l \times \mathbf{B}}$$

Quantity	Symbol	SI Unit
Magnetic force on the conductor	$\mathbf{F}_M$	N
Electric current in the conductor	$\mathbf{I}$	A
Length of the conductor	l	m
Magnetic field	$\mathbf{B}$	T

Magnetic Force on Two Current-Carrying Wires



Two parallel current-carrying wires of length L are at a distance r apart. Magnetic field at wire 2 from current I_1 has constant strength along the wire, given by:

$$B = \frac{\mu_0 I_1}{2\pi r}$$

The force of B on I_2 is:

$$F = I_2 L B = \frac{\mu_0 I_1 I_2 L}{2\pi r} \rightarrow \boxed{\frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi r}}$$

I_1 also exerts the same force on I_2, pulling the wires towards each other. (We should expect this because of Newton's third law of motion.)

Circular Motion Caused by a Magnetic Field

When a charged particle enters a magnetic field at right angle…

- Magnetic force $\mathbf{F}_M$ perpendicular to both velocity $\mathbf{v}$ and magnetic field $\mathbf{B}$.
- Results in circular motion

Centripetal force $\mathbf{F}_c$ is provided by the magnetic force $\mathbf{F}_M$. Equating the two expressions:

$$\frac{mv^2}{r} = qvB$$

We can solve for r get the radius for a charge with a known mass, or solve for mass m of a charged particle based on its radius:

$$r = \frac{mv}{qB} \qquad m = \frac{qrB}{v}$$

Faraday's Law

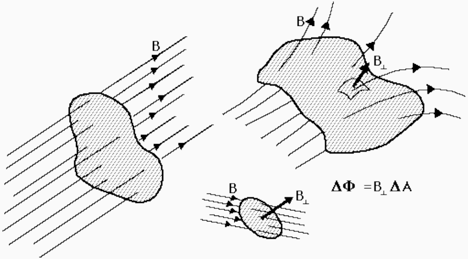
Magnetic Flux

Question: If a current-carrying wire can generate a magnetic field, can a magnetic field affect the current in a wire?

Answer: Yes, sort of...

To understand how to *induce* a curent by a magnetic field, we need to look at fluxes again.

Magnetic Flux



Magnetic flux is defined as:

$$\boxed{\Phi_M = \int \mathbf{B} \cdot d\mathbf{A}}$$

where $\mathbf{B}$ is the magnetic field, and $d\mathbf{A}$ is the infinitesimal area pointing **outwards**. Note that magnetic flux can also be expressed as:

$$\boxed{\Phi_M = \int \mathbf{B} \cdot \hat{\mathbf{n}} dA}$$

where $\hat{\mathbf{n}}$ is the outward normal direction

Magnetic Flux Over a Closed Surface

The unit for magnetic flux is a “weber” (Wb), in honor of German physicist Wilhelm Weber, who invented the electromagnetic telegraph with Carl Gauss. The unit is defined as:

$$1\,\mathrm{Wb} = 1\,\mathrm{T\,m^2}$$

The magnetic flux over a closed surface is always zero:

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{A} = 0}$$

Since magnetic field lines only exist as a loop, that means there should be equal amount of “flux” flowing out of a closed surface as entering the surface.

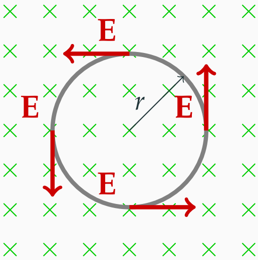
Changing Magnetic Flux

Changes to magnetic flux can be due to a number of reasons:

1. **Changing magnetic field**...if the magnetic field is created by a time-dependent source (e.g. alternating current)
2. **Changing orientation of magnetic field** either because the surface area is moving relative to the magnetic field.
3. **Changing area** the surface area from which the flux is calculated is changing.

When Magnetic Flux is Changing

- When the magnetic flux Φ_M is changing, an electromotive force (*emf*, $\mathcal{E}$) is created in the wire.
- Unlike in a circuit, where the *emf* is concentrated at the terminals of the battery, the induced *emf* is spread across the entire wire.



- Since *emf* is work per unit charge, that means that there is an electric field inside the wire to move the charges.
- In this example:
  - Magnetic field $\mathbf{B}$ into the page
  - The direction of the electric field $\mathbf{E}$ corresponds to an *increase* in magnetic flux

Faraday's Law

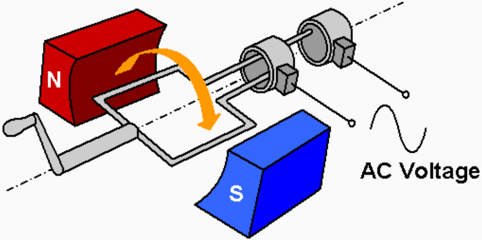
Faraday's law states that the rate of change of magnetic flux produces an electromotive force:

$$\boxed{\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}}$$

The negative sign highlighted in red is the result of Lenz's law, which is related to the conservation energy

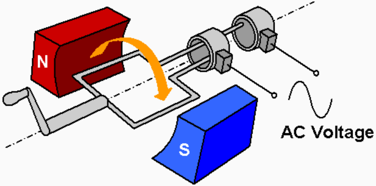
AC Generators

A simple AC (alternating current) generator makes use of the fact that a coil rotating against a fixed magnetic field has a changing flux.



Let's say the permanent magnets produce a uniform magnetic field B, and the coil between them has N turns, and an area A. Now let's say that the coil is rotating with an angular frequency ω.

AC Generators



When the coil is turning, the angle between the coil and the magnetic field is:

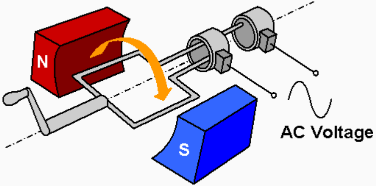
$$\theta = \omega t + \delta$$

where δ is the initial angle. The magnetic flux through the coil is given by:

$$\begin{aligned}\Phi &= NBA\cos\theta \\ &= NBA\cos(\omega t + \delta)\end{aligned}$$

as the coil turns.

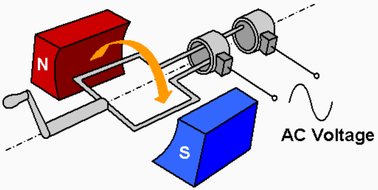
AC Generators



The electromotive force *emf* produced is therefore the rate of change of the magnetic flux:

$$\begin{aligned}\mathcal{E} &= -\frac{d\Phi_M}{dt} \\ &= NBA\omega \sin(\omega t + \delta)\end{aligned}$$

AC Generators



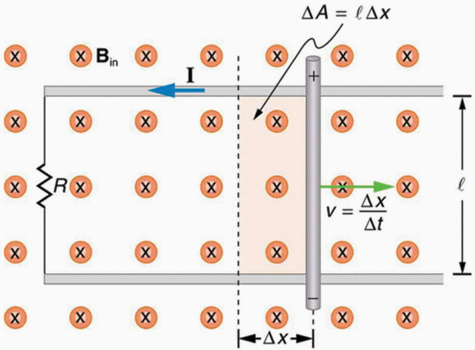
We commonly write it this way instead:

$$\mathcal{E} = \mathcal{E}_{\max} \sin(\omega t + \delta)$$

where

$$\mathcal{E}_{\max} = NBA\omega$$

Motional EMF



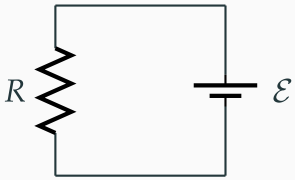
When sliding the rod to the right with speed v, the magnetic flux through the loop (and its rate of change) is:

$$\Phi = BA = B\ell x$$

$$\frac{d\Phi}{dt} = B\ell \frac{dx}{dt} = \boxed{B\ell v = \mathcal{E}}$$

We can use the Lorentz force law on the charges on the rod to find the direction of the current I.

Motional EMF

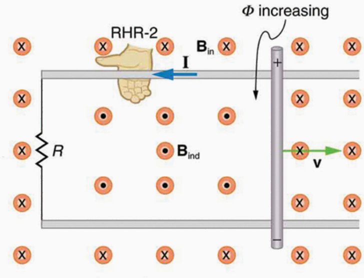


- An equivalent circuit is shown on the left
- The amount of current can be found using Ohm's law
- Note that the "motional emf" produced is spread over the entire circuit

Lenz’s Law

Lenz's Law

Something very interesting happens when the current starts running on the wire.



It produces an "induced magnetic field" out of the page, in the opposite direction as the field that generated the current in the first place.

Lenz's Law

LENZ'S LAW

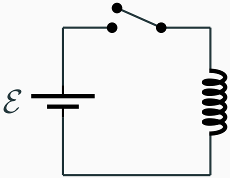
The induced *emf* and induced current are in such are direction as to oppose the change that produces them

So basically, the conservation of energy

Inductance

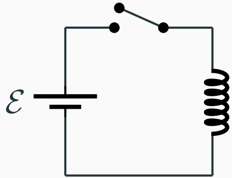
Back *emf*

Consider a very simple circuit consisting of a voltage source and a coil



- When the switch is closed and current begins to flow, the coil begins to generate a magnetic flux inside
- As the current changes (initially increasing with time), it self-induces a "back *emf*" that opposes the change in current
- A current can't jump from zero to some value (or from some value to zero) instantaneously

Back *emf*



- Breaking the circuit causes the magnetic flux to change very rapidly
- The rapid change of Φ_M creates a large induced back *emf* that is proportional to $d\Phi_M/dt$
- The back *emf* creates a large voltage drop across the switch
- Large voltage across two metal contact produces a very strong electric field–strong enough to tear electrons away from air molecules ("dielectric breakdown")
- Air conducts electricity in the form of a "spark"

Self Inductance

A solenoid carrying a current generates a magnetic field; its strength given by Biot-Savart law (or Ampère's law):

$$B = \frac{\mu_0 N I}{L}$$

Since $\mathbf{B} \propto I$, the magnetic flux through the solenoid (really $\Phi = NBA$ where A is the cross-sectional area of the solenoid and N is the number of coils) is therefore also proportional to I, i.e.

$$\boxed{\Phi_M = LI}$$

where L is the called the **self inductance** of the coil.

Self Inductance

For a solenoid, we can see that the self inductance is given by:

$$\boxed{L = \frac{\Phi_M}{I} = \mu_0 n^2 A l}$$

where μ_0 is the magnetic permeability of free space, n is the number of coil turns per unit length, and A and l are the cross-section and length of the solenoid. (i.e. Al is the enclosed volume.)

Self Inductance and Induced EMF

If the current changes, the magnetic flux changes as well, therefore inducing an electromotive force in the circuit! According Faraday's law:

$$\boxed{\mathcal{E} = -\frac{d\Phi_M}{dt} = -L\frac{dI}{dt}}$$

The self-induced *emf* is proportional to the rate of change of current

Magnetic Energy

Just as a capacitor stores energy in its electric field, an inductor coil carrying a current I stores energy in its magnetic field, given by:

$$\boxed{U_M = \frac{1}{2}LI^2}$$

We can also define a **magnetic energy density**:

$$\boxed{\eta_M = \frac{B^2}{2\mu_0}}$$