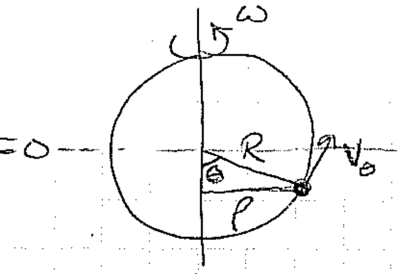
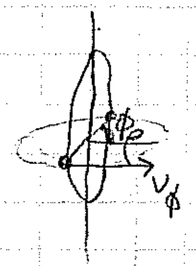
Example: Bead on a spinning wire loop



$\rho = R\sin\theta$ is the radius of the path of the bead at any instant

$\omega$ = rate of rotation of hoop.



angle of plane of hoop with respect to x,y plane is $\phi$

$\omega = \dot\phi \equiv$ constant.

$\vec V$ has two components: $V_\theta$ is tangent to the wire in $\hat\theta$ direction
$\vec V = \langle V_\theta, V_\phi \rangle$ $\quad$ $V_\phi$ is perp to plane of hoop in $\hat\phi$ direction.

$$V_\theta = R\dot\theta \qquad V_\phi = \rho\dot\phi = \rho\omega = R\sin\theta\,\dot\phi$$

$$K = \tfrac{1}{2}mV^2$$
$$= \tfrac{1}{2}m\left(V_\theta^2 + V_\phi^2\right) = \tfrac{1}{2}m\left(R^2\dot\theta^2 + R^2\sin^2\theta\,\dot\phi^2\right)$$

$U_{grav} = mgy = -mgR\cos\theta$ $\quad$ (for $\theta > 90°$, $\cos\theta$ is $-$ and $U$ is $+$ ✓)

$$L = K - U_{grav} = \tfrac{1}{2}m\left(R^2\dot\theta^2 + R^2\sin^2\theta\,\dot\phi^2\right) + Rmg\cos\theta$$

Note that $\dot\phi = \omega$ is constant and is not a generalized coord.

$$q = \theta \qquad \frac{\partial L}{\partial\theta} = \frac{d}{dt}\frac{\partial L}{\partial\dot\theta}$$

$$\frac{\partial L}{\partial\theta} = \tfrac{1}{2}mR^2\omega^2\,2\sin\theta\cos\theta + -mg\sin\theta R$$
$$= mR^2\omega^2\sin\theta\cos\theta - mg\sin\theta R$$

$$\frac{\partial L}{\partial \dot\theta} = mR^2\dot\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = mR^2\ddot\theta$$

So, $mR^2\omega^2\sin\theta\cos\theta - mg\sin\theta R = mR^2\ddot\theta$

$$\omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta = \ddot\theta$$

$$\boxed{\ddot\theta = \sin\theta\left(\omega^2\cos\theta - \frac{g}{R}\right)}$$

Where will the bead be at rest in equilibrium?

$\ddot\theta = 0$ and $\dot\theta = 0$ so set $\dot\theta_0 = 0$ and then it will remain 0.

$$\ddot\theta = \sin\theta\left(\omega^2\cos\theta - \frac{g}{R}\right) = 0$$

$$\text{so } \omega^2\cos\theta - \frac{g}{R} = 0$$

$$\omega^2\cos\theta = \frac{g}{R}$$

$$\cos\theta = \frac{g}{\omega^2 R}$$

$$\theta = \cos^{-1}\left(\frac{g}{\omega^2 R}\right)$$

Suppose $\omega = 1\,\text{rev} = 2\pi$
$R = 1/2$
$g = 9.8$

$\theta = 0.879$ rad $= 60°$