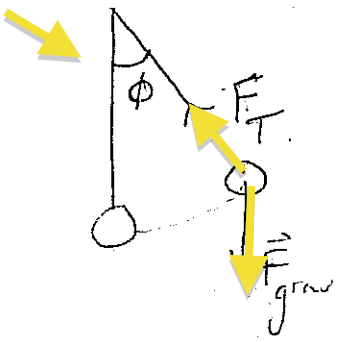
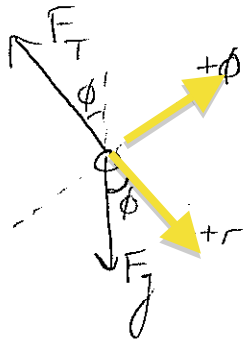
Example: Pendulum





$F_{gr} = mg\cos\phi$

$F_{g\phi} = -mg\sin\phi$

In rad direction: $F_{net_r} = F_{gr} - F_T = mg\cos\phi - F_T$

Newt 2nd law in polar coord is: $F_r = m(\ddot{r} - r\dot{\phi}^2)$

Since r is constant, $\dot{r} = 0$ and $\ddot{r} = 0$. Thus,

$F_{net_r} = -mr\dot{\phi}^2$

$mg\cos\phi - F_T = -mr\dot{\phi}^2$

Not useful since F_T is unknown, so use Newt. 2nd law in ϕ dir.

$F_{net_\phi} = m(2\dot{r}\dot{\phi} + r\ddot{\phi})$

$= 0$ since r is constant and $\dot{r} = 0$

$F_{net_\phi} = mr\ddot{\phi}$

$-mg\sin\phi = mr\ddot{\phi}$

$\ddot{\phi} + \frac{g}{r}\sin\phi = 0$

$$\boxed{\ddot{\phi} = -\frac{g}{r}\sin\phi}$$

To Page No. ____

From Page No. ____

Pendulum continued...

$$\ddot{\phi} = -\frac{g}{L}\sin\phi$$

Solve this numerically using EJS.

Analytic solution is possible for small angles.

$\phi \approx \sin\phi$ for small ϕ in radians.

$\ddot{\phi} \approx -\frac{g}{L}\phi$ Define $\omega = \sqrt{\frac{g}{L}}$, $\ddot{\phi} = -\omega^2\phi$

One Solution is $\phi = A\cos(\omega t + \theta)$ where $\omega = \sqrt{\frac{g}{L}}$ and

θ is the phase.

A is the amplitude

Proof:

$$\dot{\phi} = -A\sin(\omega t + \theta)\,\omega$$
$$= -\omega A\sin(\omega t + \theta)$$

$$\ddot{\phi} = -\omega^2 A\cos(\omega t + \theta)$$

$$\ddot{\phi} \stackrel{?}{=} -\omega^2\phi$$

$$-\omega^2 A\cos(\omega t + \theta) \stackrel{?}{=} -\omega^2 A\cos(\omega t + \theta)$$

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Thus ϕ oscillates sinusoidally. θ can be determined from initial conditions.

$\dot{\phi} = -\omega A\sin(\omega t + \theta)$ $\quad \phi = A\cos(\omega t + \theta)$

$\dot{\phi}_0 = -\omega A\sin\theta$ $\quad \phi_0 = A\cos\theta$

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\dot{\phi}_0/\omega A}{\phi_0/A} = \frac{-1}{\omega}\frac{\dot{\phi}_0}{\phi_0}$$

To Page N

Another solution is: $\phi = B\sin(\omega t + \theta)$.

General solution is:

$$\phi = A\sin\omega t + B\cos\omega t$$

Note: I switched A and B, to be consistent with the book.

Two arbitrary constants A and B require knowledge of two constants, the initial conditions ϕ_0 and $\dot{\phi}_0$

Generally, the solutions of an n^{th} order differential equation contain precisely n constants. (because integrating $\ddot{\phi}$ twice, for example, results in two integration constants). These are are determined by initial conditions.

At the $t = 0$:

$$\phi_0 = A(0) + B, \text{ so } \boxed{B = \phi_0}$$

If released from rest, then $\dot{\phi}_0 = 0$

$$\dot{\phi} = \omega A\cos\omega t - \omega B\sin\omega t$$

$$\dot{\phi}_0 = \omega A$$

$$A = \frac{\dot{\phi}_0}{\omega}$$

$$\phi = \frac{\dot{\phi}_0}{\omega}\sin\omega t + \phi_0\cos\omega t$$

If $\dot{\phi}_0 = 0$, $\boxed{\phi = \phi_0\cos\omega t}$

To Page No. ____