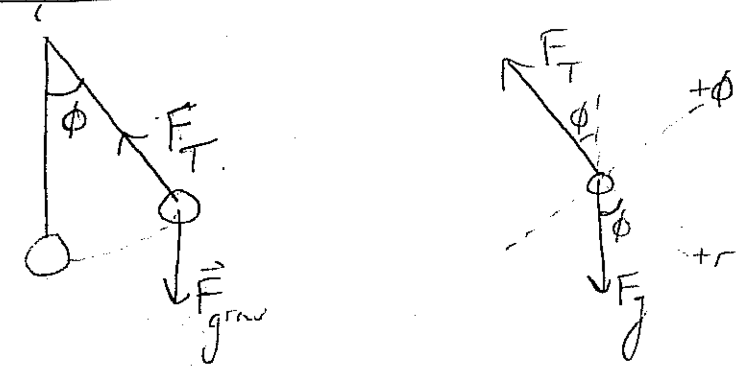
Example: Pendulum




$$F_{gr} = mg\cos\phi$$
$$F_{g\phi} = -mg\sin\phi$$

In rad direction: $F_{net_r} = F_{gr} - F_T = mg\cos\phi - F_T$

Newt 2nd law in polar coord is: $F_r = m(\ddot{r} - r\dot{\phi}^2)$

...ce $r$ is constant, $\dot{r} = 0$ and $\ddot{r} = 0$. Thus,

$$F_{net_r} = -mr\dot{\phi}^2$$

$$mg\cos\phi - F_T = -mr\dot{\phi}^2$$

Not useful since $F_T$ is unknown, so use Newt. 2nd law in $\phi$ dir.

$$F_{net_\phi} = m(\underbrace{2\dot{r}\dot{\phi}}_{0 \text{ since } r \text{ is constant and } \dot{r} = 0} + r\ddot{\phi})$$

$$F_{net_\phi} = mr\ddot{\phi}$$

$$-\cancel{m}g\sin\phi = \cancel{m}r\ddot{\phi}$$

$$\ddot{\phi} + \frac{g}{r}\sin\phi = 0$$

$$\boxed{\ddot{\phi} = -\frac{g}{r}\sin\phi}$$



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Pendulum continued...

$$\ddot{\phi} = -\frac{g}{l} \sin\phi$$

Solve this numerically using EJS.

Analytic solution is possible for small angles.

$\phi \approx \sin\phi$ for small $\phi$ in radians.

$\ddot{\phi} \approx -\frac{g}{l}\phi$ Define $\omega = \sqrt{\frac{g}{l}}$, $\ddot{\phi} = -\omega^2\phi$

one Solution is $\phi = A\cos(\omega t + \theta)$ where $\omega = \sqrt{\frac{g}{l}}$ and $\theta$ is the phase. $A$ is the amplitude

Proof:

$$\dot{\phi} = -A\sin(\omega t + \theta)\,\omega$$
$$= -\omega A \sin(\omega t + \theta)$$

$$\ddot{\phi} = -\omega^2 A \cos(\omega t + \theta)$$

$$\ddot{\phi} \stackrel{?}{=} -\omega^2 \phi$$

$$-\omega^2 A\cos(\omega t + \theta) \stackrel{?}{=} -\omega^2 A \cos(\omega t + \theta)$$

✓

Thus, $\phi$ oscillates sinusoidally. $\theta$ can be determined from initial conditions.

$\dot{\phi} = -\omega A \sin(\omega t + \theta)$ $\phi = A\cos(\omega t + \theta)$

$\dot{\phi}_0 = -\omega A \sin\theta$ $\phi_0 = A\cos\theta$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\dot{\phi}_0/\omega A}{\phi_0/A} = -\frac{1}{\omega}\frac{\dot{\phi}_0}{\phi_0}$$

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Another solution is: $\phi = B\sin(\omega t + \theta)$.

General solution is:

Note: I switched A and B, to be consistent with the book.

$$\phi = A\sin\omega t + B\cos\omega t$$

Two arbitrary constants A and B require knowledge of two constants, the initial conditions $\phi_0$ and $\dot{\phi}_0$

Generally, the solutions of an $n^{th}$ order differential equation contain precisely n constants (because integrating $\ddot{\phi}$ twice, for example, results in two integration constants). These are are determined by initial conditions.

At time $t=0$:

$$\phi_0 = A(0) + B \text{ , so } \boxed{B = \phi_0}$$

[If released from rest, then $\dot{\phi}_0 = 0$]

$$\dot{\phi} = \omega A\cos\omega t - \omega B\sin\omega t$$

$$\dot{\phi}_0 = \omega A$$

$$A = \frac{\dot{\phi}_0}{\omega}$$

$$\boxed{\phi = \frac{\dot{\phi}_0}{\omega}\sin\omega t + \phi_0\cos\omega t}$$

If $\dot{\phi}_0 = 0$, $\boxed{\phi = \phi_0\cos\omega t}$