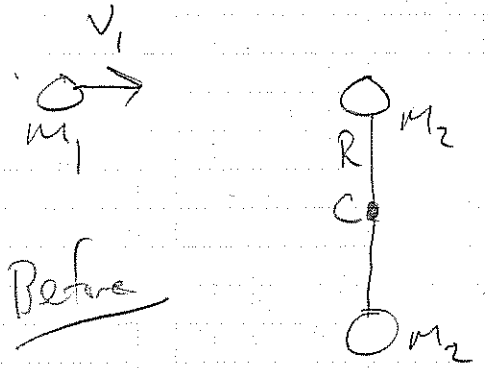
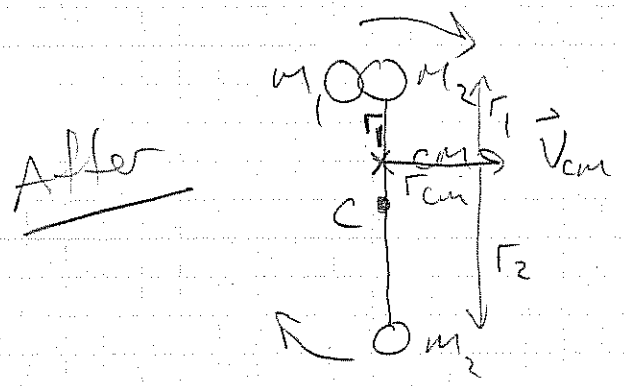
Example – Sticky Collision



$\omega = 0$

What is ω after collision?

What is ΔK?



$\vec{\omega}$ in $-z$ dir.

Newt. 2nd law: $\vec{F}_{net} = 0$ so $\vec{p}$ is constant.

$$\vec{P}_{i_{sys}} = \vec{P}_{f_{sys}}$$ System is m_1 + barbell

$$m_1 \vec{v}_1 = (m_1 + 2m_2)\vec{v}_{cm}$$

$$\boxed{\vec{v}_{cm} = \left(\frac{m_1}{m_1 + 2m_2}\right)\vec{v}_1}$$

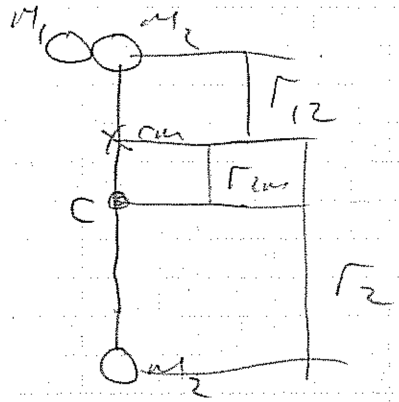
Angular Momentum: $\vec{\tau}_{net} = \frac{d\vec{L}}{dt} = 0$ so $\vec{L}$ is constant.

Choose pt. C. $\vec{L}_{i_C} = \vec{L}_{f_C}$

z: $-m_1 v_1 R = L_{trans\,f_z} + L_{rot\,f_z}$

$$-m_1 v_1 R = -(m_1 + 2m_2) r_{cm} v_{cm} - I_{12}\omega - I_2 \omega$$

Need to calculate r_{cm}, r_1, and r_2 relative to C.



$$r_{12} + r_2 = 2R$$

If $y_{cm} = 0$, then

$$y_{cm} = \frac{(m_1+m_2)r_{12} - m_2(r_2)}{m_1+m_2+m_2} = 0$$

$$(m_1+m_2)r_{12} = m_2 r_2$$

$$\boxed{r_{12} = \left(\frac{m_2}{m_1+m_2}\right) r_2}$$

$$r_{12} + r_2 = 2R$$

$$\left(\frac{m_2}{m_1+m_2}\right) r_2 + r_2 = 2R$$

$$r_2\left(1 + \frac{m_2}{m_1+m_2}\right) = 2R$$

$$r_2\left(\frac{m_1+2m_2}{m_1+m_2}\right) = 2R$$

$$\boxed{r_2 = \left(\frac{2(m_1+m_2)}{m_1+2m_2}\right) R}$$

$$r_{12} = 2R - r_{12} = 2R - 2R\left(\frac{m_1+m_2}{m_1+2m_2}\right)$$

$$= 2R\left(1 - \frac{(m_1+m_2)}{m_1+2m_2}\right) = 2R\left(\frac{\cancel{m_1} + \overset{1 m_2}{\cancel{2m_2}} - \cancel{m_1} - \cancel{m_2}}{m_1+2m_2}\right)$$

$$\boxed{r_{12} = 2R\left(\frac{m_2}{m_1+2m_2}\right)}$$

$$r_{12} + r_{cm} = R$$

$$r_{cm} = R - r_{12}$$

$$= R - \left(\frac{2R\, m_2}{m_1 + 2m_2} \right)$$

$$= R\left(1 - \frac{2m_2}{m_1 + 2m_2}\right) = R\left(\frac{m_1 + 2m_2 - 2m_2}{m_1 + 2m_2}\right)$$

$$\boxed{r_{cm} = R\left(\frac{m_1}{m_1 + 2m_2}\right)}$$

$$I_{12} = (m_1 + m_2) r_{12}^2 = (m_1 + m_2)(2R)^2 \left(\frac{m_2}{m_1 + 2m_2}\right)^2$$

$$\boxed{I_{12} = 4R^2 (m_1 + m_2)\left(\frac{m_2}{m_1 + 2m_2}\right)^2}$$

$$I_2 = m_2 r_2^2 = m_2 \left(\frac{2(m_1 + m_2)}{m_1 + 2m_2}\right)^2 R^2$$

$$\boxed{I_2 = 4R^2 m_2 \left(\frac{m_1 + m_2}{m_1 + 2m_2}\right)^2}$$

$$L_{iz} = L_{fz}$$

$$-m_1 v_1 R = -(m_1 + 2m_2) r_{cm} v_{cm} - I_{12}\omega - I_2 \omega$$

$$-m_1 v_1 R = -(m_1 + 2m_2) R \left(\frac{m_1}{m_1 + 2m_2}\right) v_{cm} - (I_{12} + I_2)\omega$$

$$-m_1 v_1 R = -(m_1 + 2m_2) R \left(\frac{m_1}{m_1 + 2m_2}\right)\left(\frac{m_1}{m_1 + 2m_2}\right) v_1 - (I_{12} + I_2)\omega$$

$$m_1 v_1 R = \frac{m_1^2}{(m_1+2m_2)} R v_1 + \left(I_{12}+I_2\right)\omega$$

$$\omega = \left(m_1 v_1 R - \left(\frac{m_1^2}{m_1+2m_2}\right) R v_1\right)\left(\frac{1}{I_{12}+I_2}\right)$$

$$= m_1 v_1 R\left(1-\frac{m_1}{m_1+2m_2}\right)\left(\frac{1}{I_{12}+I_2}\right)$$

$$= m_1 v_1 R\left(\frac{2m_2}{m_1+2m_2}\right)\left(\frac{1}{I_{12}+I_2}\right)$$

$$= 2\left(\frac{m_1 m_2}{m_1+2m_2}\right)\left(\frac{1}{I_{12}+I_2}\right) v_1 R$$

$$= 2\left(\frac{m_1 m_2}{m_1+2m_2}\right)\left(\frac{v_1 R}{4R^2(m_1+m_2)\left(\frac{m_2}{m_1+2m_2}\right)^2 + 4R^2 m_2\left(\frac{m_1+m_2}{m_1+2m_2}\right)^2}\right)$$

$$= \cancel{2}\left(\frac{m_1 m_2}{\cancel{m_1+2m_2}}\right)\frac{v_1 \cancel{R}}{\frac{{}_2\cancel{4}R^{\cancel{2}}}{(m_1+2m_2)^{\cancel{2}}}\left(m_2^2(m_1+m_2)+m_2(m_1+m_2)^2\right)}$$

$$\omega = \frac{m_1 \cancel{m_2}\, v_1}{2R\,\frac{\cancel{m_2}(m_1+m_2)}{\cancel{(m_1+2m_2)}}\left(\cancel{m_2+m_1+m_2}\right)} = \frac{m_1}{m_1+m_2} v_1$$

$$\boxed{\omega = \left(\frac{m_1}{m_1+m_2}\right)\frac{v_1}{2R}}$$

$$\boxed{\omega_z = -\left(\frac{m_1}{m_1+m_2}\right)\frac{v_1}{2R}}$$

$$\Delta K = ?$$

$$= K_f - K_i$$

$$= \left(K_{trans_f} + K_{rot_f}\right) - K_i$$

$$= \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2 - \frac{1}{2} m_1 v_1^2$$

$$= \frac{1}{2}(m_1 + 2m_2)\left(\frac{m_1}{m_1 + 2m_2}\right)^2 v_1^2 + \frac{1}{2}(I_{12} + I_2)\omega^2 - \frac{1}{2} m_1 v_1^2$$

$$= \frac{1}{2} \frac{m_1^2}{(m_1 + 2m_2)} v_1^2 + \frac{1}{2}(I_{12} + I_2)\left(\frac{m_1}{m_1 + m_2}\right)^2 \frac{v_1^2}{4R^2} - \frac{1}{2} m_1 v_1^2$$

$$I_{12} + I_2 = I_{sys,cm}$$

$$I_{sys,cm} = 4R^2(m_1 + m_2)\left(\frac{m_2}{m_1 + 2m_2}\right)^2 + 4R^2 m_2 \left(\frac{m_1 + m_2}{m_1 + 2m_2}\right)^2$$

$$I_{cm} = \frac{4R^2 m_2 (m_1 + m_2)}{(m_1 + 2m_2)^{\cancel{2}}} \cancel{\left(m_2 + m_1 + m_2\right)}$$

$$= \frac{4R^2 m_2 (m_1 + m_2)}{m_1 + 2m_2}$$

$$\Delta K = \frac{1}{2} \frac{m_1^2}{(m_1 + 2m_2)} v_1^2 + \frac{1}{2} \frac{\cancel{4R^2} m_2 \cancel{(m_1 + m_2)}}{(m_1 + 2m_2)} \frac{m_1^2}{(m_1 + m_2)^{\cancel{2}}} \frac{v_1^2}{\cancel{4R^2}} - \frac{1}{2} m_1 v_1^2$$

$$= \frac{1}{2} \frac{m_1^2}{(m_1 + 2m_2)} v_1^2 + \frac{1}{2} \frac{m_2 m_1^2}{(m_1 + 2m_2)(m_1 + m_2)} v_1^2 - \frac{1}{2} m_1 v_1^2$$

$$= \frac{1}{2} \frac{m_1^2 v_1^2}{(m_1 + 2m_2)}\left(1 + \frac{m_2}{m_1 + m_2}\right) - \frac{1}{2} m_1 v_1^2$$

$$\Delta K = \frac{1}{2}\frac{m_1^2 v_1^2}{\cancel{(m_1+2m_2)}}\left(\frac{\cancel{m_1+m_2+m_2}}{m_1+m_2}\right) - \frac{1}{2}m_1 v_1^2$$

$$= \frac{1}{2}\left(\frac{m_1^2}{m_1+m_2}\right)v_1^2 - \frac{1}{2}m_1 v_1^2$$

$$= \frac{1}{2}m_1 v_1^2\left(\frac{m_1}{m_1+m_2} - 1\right)$$

$$= \frac{1}{2}m_1 v_1^2\left(\frac{m_1-(m_1+m_2)}{m_1+m_2}\right)$$

$$= \frac{1}{2}m_1 v_1^2\left(-\frac{m_2}{m_1+m_2}\right)$$

$$\boxed{\Delta K = -\frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)v_1^2}$$

K is lost. Applying conservation of energy gives

$$\Delta K + \Delta U = W_{nc}$$

So

$$W_{nc} = \Delta K = -\frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)v_1^2$$

As with any inelastic collision, K is lost and thermal energy is gained.