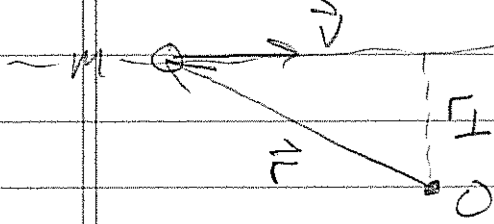
Angular Momentum

Particle

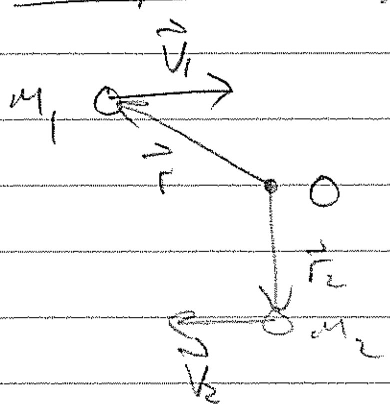


$\vec{l} = \vec{r} \times \vec{p}$ about point O

$\vec{l}$ is $\perp$ to plane of $\vec{r}$ and $\vec{p}$.

$|\vec{l}| = |\vec{r}||\vec{p}| \sin\theta = r_\perp |\vec{p}|$

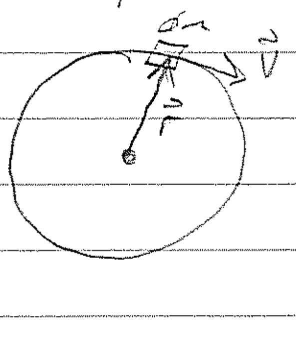
Multi-particle system



$$\vec{L} = \sum_{\alpha=1}^{N} \vec{l}_\alpha$$
$$= \vec{l}_1 + \vec{l}_2 + \cdots$$
$$= m_1 \vec{r}_1 \times \vec{p}_1 + m_2 \vec{r}_2 \times \vec{p}_2 + \cdots$$

Rigid Body

Example: hoop radius R, mass M, origin at CM, rotating about axle through center.



$$d\vec{l} = \vec{r} \times (dm\,\vec{v})$$

$$\vec{L} = \sum_{\substack{\text{around} \\ \text{hoop}}} d\vec{l} = \int \vec{r} \times dm\,\vec{v}$$

$$\vec{L} = \int (\vec{r} \times \vec{v})\, dm$$

density: $\lambda = \frac{M}{\text{circumference}} = \frac{M}{2\pi R}$

arclength $ds = R\, d\theta$

$$\frac{M}{2\pi R} = \frac{dm}{ds}$$

so $dm = \frac{M}{2\pi R} ds$

$$dm = \frac{M}{2\pi R}(R\, d\theta) = \frac{M}{2\pi} d\theta$$

$|\vec{v}| = v$ for all points on hoop and $\vec{r} \perp \vec{v}$ at all points on hoop.

Thus,

$$|\vec{L}| = \int R v\, dm = \int R v \frac{M}{2\pi} d\theta$$

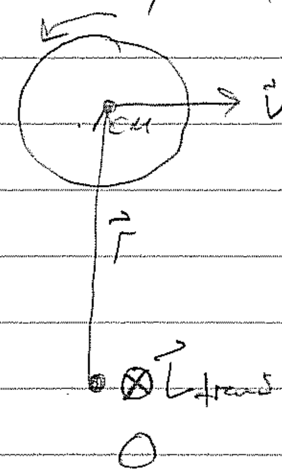
$$= \frac{MRv}{2\pi} \int_0^{2\pi} d\theta = \frac{MRv}{2\pi}(2\pi)$$

$$= MRv$$

$v = R\omega$ so $|\vec{L}| = \underbrace{MR^2}_{I_{cm}}\omega$

In general $|\vec{L}| = I_{cm}\omega$ where I is moment of inertia about the CM of the body.

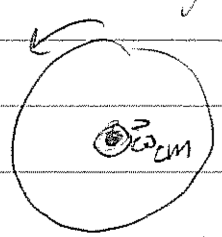
Rigid Body w/ Translation and Rotation



$\vec{L}_{trans}$ is $\vec{L}$ of CM as if it is a particle located at CM.

$$\vec{L}_{trans} = \vec{r}_{cm} \times \vec{p}_{cm}$$

$\vec{L}_{rot}$ is $\vec{L}$ of rigid body about an axis through the C.M.



$$\vec{L}_{rot} = I_{cm}\vec{\omega}_{cm}$$

$$\vec{L}_{tot} = \vec{L}_{trans} + \vec{L}_{rot}$$

$$= \vec{r}_{cm} \times \vec{p}_{cm} + I_{cm}\vec{\omega}_{cm}$$