Example: Taylor 4.23

Which of these forces is conservative?

(a) $\vec{F} = k\langle x, 2y, 3z\rangle$

(b) $\vec{F} = k\langle y, x, 0\rangle$

(c) $\vec{F} = k\langle -y, x, 0\rangle$

All forces depend only on position. ✓

$\nabla\times\vec{F} = 0$?

$$\nabla\times\vec{F} = \hat{x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \hat{y}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \hat{z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

(a) $\hat{x}(0-0) - \hat{y}(0-0) + \hat{z}(0-0) = \langle 0, 0, 0\rangle$ ✓

Yes (a) is conservative

(b) $\nabla\times\vec{F} = \hat{x}(0-0) - \hat{y}(0-0) + \hat{z}(1-1)$

$= \langle 0, 0, 0\rangle$ ✓ Yes (b) is cons.

(c) $\nabla\times\vec{F} = \hat{x}(0-0) - \hat{y}(0-0) + \hat{z}(1 - -1)$

$= \langle 0, 0, 2\rangle$.

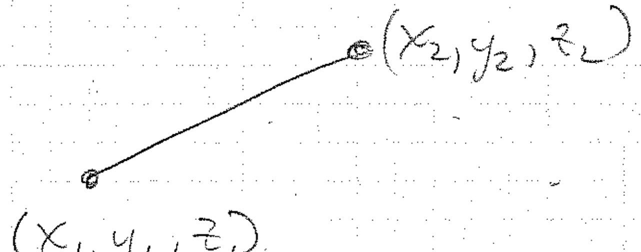
No, (c) is not conservative

Find U and verify that $\vec{F} = -\nabla U$

$$\Delta U = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \quad \text{along any path}$$

(a) $d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

$$\Delta U = -\int \vec{F} \cdot d\vec{r} = -\int_{x_1}^{x_2} kx\,dx - \int_{y_1}^{y_2} 2ky\,dy - \int_{z_1}^{z_2} 3kz\,dz$$



$$\Delta U = -\frac{1}{2}kx^2\Big|_{x_1}^{x_2} - \frac{1}{2}2ky^2\Big|_{y_1}^{y_2} - \frac{1}{2}3kz^2\Big|_{z_1}^{z_2}$$

$$U_2 - U_1 = -\frac{1}{2}kx_2^2 - \left(-\frac{1}{2}kx_1^2\right) - \frac{1}{2}2ky_2^2 - \left(-\frac{1}{2}2ky_1^2\right)$$
$$- \frac{1}{2}3kz_2^2 - \left(-\frac{1}{2}3kz_1^2\right)$$
$$= -\left(\frac{1}{2}kx_2^2 + ky_2^2 + \frac{1}{2}3kz_2^2\right) + \left(\frac{1}{2}kx_1^2 + ky_1^2 + \frac{3}{2}kz_1^2\right)$$

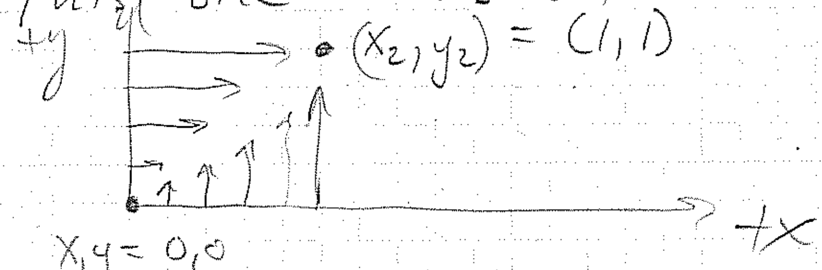
$$U = -\frac{1}{2}kx^2 - ky^2 - \frac{3}{2}kz^2$$

$$\boxed{U = -\frac{1}{2}k\left(x^2 + 2y^2 + 3z^2\right)}$$

$$\vec{F} = -\nabla U$$
$$= \left\langle -\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z} \right\rangle$$
$$= \langle kx, 2ky, 3kz \rangle$$
$$= k\langle x, 2y, 3z \rangle \checkmark$$

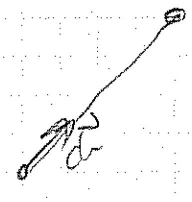
(b) $\vec{F} = k\langle y, x, 0 \rangle$

This one turns out to be quite interesting.



$$\Delta U = -\int_0^{(x_2, y_2)} \vec{F} \cdot d\vec{r}$$

Choose path straight line:



$d\vec{r} = dx\,\hat{x} + dy\,\hat{y}$

$$\Delta U = -\int (F_x dx + F_y dy + F_z dz)$$
$$= -\int F_x dx - \int F_y dy = -\int ky\,dx - \int kx\,dy$$
$$= -kyx\Big|_0^{x_2} - kxy\Big|_0^{y_2}$$
$$= -kyx_2 - kxy_2$$

In first term, what is y?
In second term, what is x?

straight line path $y = mx$ $m = \frac{y_2}{x_2}$

$y = \frac{y_2}{x_2} x$

So at x_2 $y = y_2$ and at y_2, $x = x_2$.

$\Delta U = -k x_2 y_2 - k x_2 y_2$

$U_2 - U_1 = -2k x_2 y_2$

define $U = 0$ at $x = 0, y = 0$. then

$\boxed{U = -2kxy}$

check $\vec{F} = -\nabla U$

$\vec{F} = -\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y} \rangle$

$= -\langle -2ky, -2kx \rangle$

$= \langle 2ky, 2kx \rangle = 2k\langle y, x \rangle$

Problem!!!

Extra factor of 2.

Why?

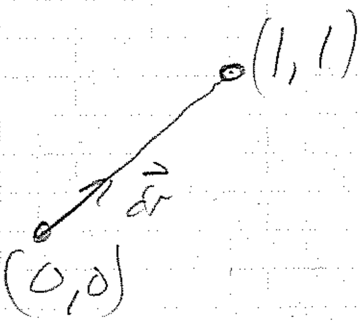
But ΔU is path independent!

Put numbers to the points.

$x_1, y_1 = 0, 0$

$x_2, y_2 = 1, 1$

Straight path



$$-\int \vec{F} \cdot \vec{dr} = -\int F_x dx - \int F_y dy$$

$$= -\int_0^1 ky\,dx - \int_0^1 kx\,dy$$

y depends on x and x depends on y.

$y = mx$ where $m = 1$. So $y = x$

$$= -\int_0^1 kx\,dx - \int_0^1 ky\,dy$$

$$= -k\frac{x^2}{2}\Big|_0^1 - k\frac{y^2}{2}\Big|_0^1$$

$$= -\frac{k}{2} - \frac{k}{2}$$

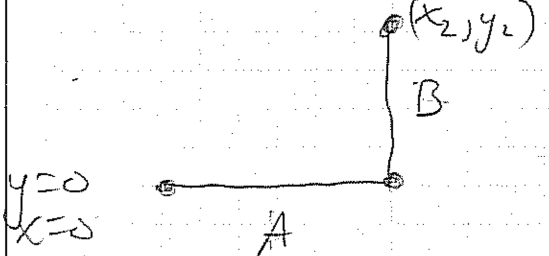
$$= -k$$

If $U = -kxy$ then $\Delta U = -k(1)(1) - k(0)(0)$

$$= -k$$

It works!

Select another path.



Along part A, $\vec{dr} = dx\,\hat{x}$ and $y = 0$ so $F_x = 0$

$$\Delta U = -\int F_x\,dx = 0$$

Along part B $\vec{dr} = dy\,\hat{y}$ and $F_y = kx_2$

$$\Delta U = -\int_0^{y_2} F_y\,dy = -\int_0^{y_2} kx_2\,dy = -kx_2 y_2$$

$$U_2 - U_1 = -kx_2 y_2$$

Since U_1 at $x=0, y=0$ is defined to be 0,

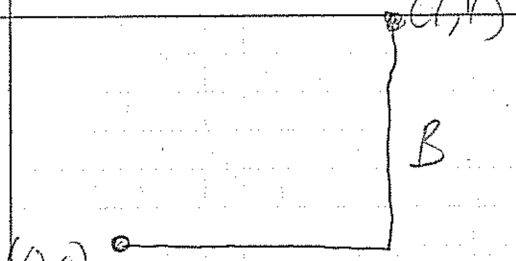
then $\boxed{U = -kxy}$ Now check that

$$\vec{F} = -\nabla U.$$

$$= -\left\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y} \right\rangle$$

$$= -\langle -ky, -kx \rangle$$

$$= k\langle y, x\rangle \checkmark$$ It works



Path along axes.

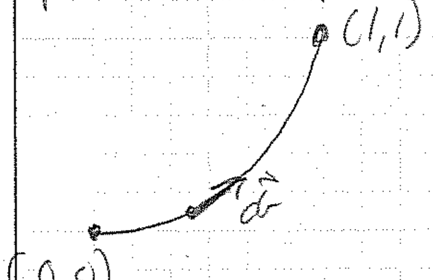
$$-\int \vec{F}\cdot d\vec{r} = -\int_A \cancel{F_x}\, dx - \int_B F_y\, dy$$

$$= -\int_{y=0}^{y=1} 2x\,dy \quad \text{at } x=1$$

$$= -2(1)\,y\Big|_0^1$$

$$= -2$$ ✓ this agrees!

quadratic path



$$y = x^2$$
$$dy = 2x\,dx$$
$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y}$$

$$-\int_0^1 \vec{F}\cdot d\vec{r} = -\int_0^1 (F_x dx + F_y dy)$$

$$= -\int_0^1 2y\,dx - \int_0^1 2x\,dy$$

$$= -2\int_0^1 x^2 dx - 2\int_0^1 \sqrt{y}\,dy$$

$$= -k\frac{x^3}{3}\Big|_0^1 - k\frac{y^{3/2}}{\frac{3}{2}}\Big|_0^1$$

$$= -k\frac{1}{3} - \frac{2}{3}ky^{3/2}\Big|_0^1$$

$$= -\frac{k}{3} - \frac{2}{3}k$$

$$= -k$$ ✓ It agrees!

Note that $y = mx$. $\Delta U = -\int (F_x dx + F_y dy)$

$$\int -ky\,dx = \int -kmx\,dx = -km\frac{x^2}{2}$$

$$\int -kx\,dy = \int -\frac{k}{m} y\,dy = -\frac{k}{m}\frac{y^2}{2}$$

$$\Delta U = -km\frac{x^2}{2} - \frac{k}{m}\frac{y^2}{2}$$

$$= -k\frac{yx}{2} - k\frac{xy}{2}$$

$$= -kxy \checkmark$$ Now it works.