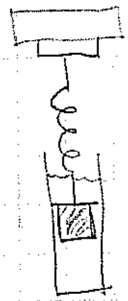
Damped, driven harmonic oscillator



speaker (i.e. driver)

$$F_{net_x} = -kx - bv_x + F_d = m\ddot{x}$$

$F_d \equiv$ driving force

Assume that $F_d = F_0 \cos(\omega t)$ a sinusoidal driving force

$$\ddot{x} = -\frac{k}{m}x - \frac{b}{m}\dot{x} + \frac{F_0}{m}\cos(\omega t)$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos(\omega t)$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0\cos(\omega t) \quad \text{where } f_0 = \frac{F_0}{m}$$

$\omega_0 \equiv$ natural frequency

$$\beta = \frac{b}{2m}$$

It is easiest mathematically to write the solution x as $x = Re(z)$, where z is a complex number. F_d must also be $F = Re(F_0 e^{i\omega t})$.

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$$

Try a solution $z = Ae^{i(\omega t - \delta)}$ } Note that after a long time any transients will dampen out and only sinusoidal solution is left.

$$\dot{z} = Ae^{i(\omega t-\delta)}\, i\omega = i\omega A e^{i(\omega t - \delta)}$$

$$\ddot{z} = i\omega A e^{i(\omega t-\delta)}\, i\omega = -\omega^2 A e^{i(\omega t-\delta)}$$

Now substitute into Diff Eq.

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$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$$

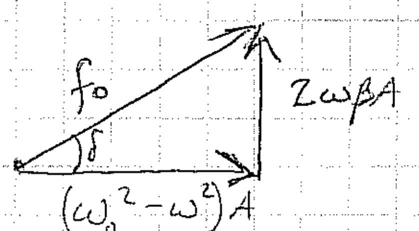
$$-\omega^2 A e^{i(\omega t-\delta)} + 2\beta i\omega A e^{i(\omega t-\delta)} + \omega_0^2 A e^{i(\omega t-\delta)} = f_0 e^{i\omega t}$$

$$\left((\omega_0^2-\omega^2)A + 2i\omega\beta A\right)e^{i(\omega t-\delta)} = f_0 e^{i\omega t}$$

$$\left((\omega_0^2-\omega^2)A + 2i\omega\beta A\right)e^{-i\delta} = f_0$$

$$\underbrace{(\omega_0^2-\omega^2)A}_{\text{real}} + \underbrace{2i\omega\beta A}_{\text{complex}} = f_0 e^{i\delta}$$

You can visualize this geometrically in the complex plane.



Pythagorean Theorem

$$(\omega_0^2-\omega^2)^2A^2 + 4\omega^2\beta^2A^2 = f_0^2$$

$$A^2 = \frac{f_0^2}{(\omega_0^2-\omega^2)^2 + 4\omega^2\beta^2}$$

$$\boxed{A = \frac{f_0}{\left((\omega_0^2-\omega^2)^2 + (2\omega\beta)^2\right)^{1/2}}}$$

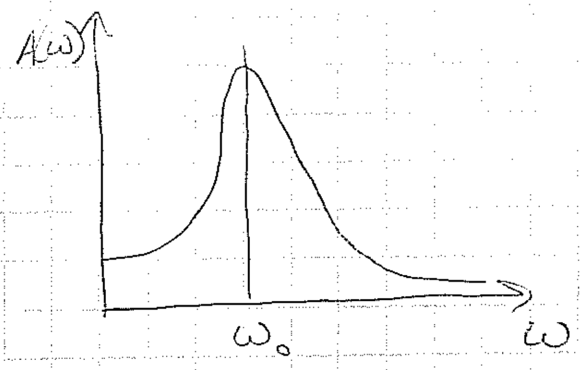
Phase

$$\tan\delta = \frac{2\omega\beta}{\omega_0^2-\omega^2}$$

$$\boxed{\delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2-\omega^2}\right)}$$

Assuming ω is fixed and ω_0 is varied, then...
At $\omega=\omega_0$, A is a max and $\delta = \frac{\pi}{2}$ so oscillator lags driving force by $90°$.

ssed & Understood by me, | Date | Invented by | Date



At ω near ω_0, A is a maximum. Far from ω_0, A is small.

This effect is called <u>resonance</u>.

Max A can be calculated (and ω_{max}).

$$\frac{dA}{d\omega} = 0$$

$$f_0 \, \frac{1}{2}\left[(\omega_0^2-\omega^2)^2+(2\omega\beta)^2\right]^{-1/2}\left[2(\omega_0^2-\omega^2)(-2\omega)+2(2\omega\beta)(2\beta)\right]=0$$

2nd term $=0$

$$(\omega_0^2-\omega^2)(-2\omega)+4\omega\beta^2=0$$

$$(\omega_0^2-\omega^2)\cancel{2\omega}=\cancel{4\omega}\,^{2}\beta^2$$

$$\omega_0^2-\omega^2=2\beta^2$$

$$\omega^2=\omega_0^2-2\beta^2$$

$$\boxed{\omega=(\omega_0^2-2\beta^2)^{1/2}} \equiv \omega \text{ when } A \text{ is a maximum}$$

If $\beta \ll \omega_0$, then $\boxed{\omega \approx \omega_0}$

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If ω is fixed and ω_0 is changed, then

$\frac{dA}{d\omega_0} = 0$ is the max A.

$$= \frac{1}{2}\left[(\omega_0^2-\omega^2)^2+(2\omega\beta)^2\right]^{-1/2} 2(\omega_0^2-\omega^2)(2\omega_0) = 0$$

So $\omega_0^2-\omega^2=0$

$\boxed{\omega_0=\omega}$ max A occurs at exactly the driving frequency.

Transient Solution

The differential equation is

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

The solution is

$$x(t) = x_h(t) + x_p(t)$$

$x_h(t)$ is the homogeneous solution to

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \quad \rightarrow \text{no driving force}$$

$$x_h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} = A_{tr} e^{-\beta t} \cos(\omega_1 t - \delta_{tr}) \text{ for an underdamped oscillator}$$

and it dampens out in time. Thus, it's called the transient solution.

$x_p(t)$ is the particular solution to

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

It is sinusoidal with

$$x_p(t) = A\cos(\omega t - \delta)$$

$$A = \frac{f_0}{\left[(\omega_0^2 - \omega^2)^2 + (2\omega\beta)^2\right]^{1/2}} \qquad \delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right)$$

$$x(t) = A_{tr} e^{-\beta t} \cos(\omega_1 t - \delta_1) + A \cos(\omega t - \delta)$$

for $\beta < \omega_0$

$\frac{b}{2m} < \omega_0$

$b < 2m\omega_0$

$\omega \equiv$ driving frequency , $\delta \equiv$ phase difference from driving force

$A \equiv$ "long-time" amplitude

$A_{tr} \equiv$ transient amplitude

$\omega_1 = \sqrt{\omega_0^2 - \beta^2}$

Example

EJS simulation.

$m = 1$
$k = 1$
$A = 0.1$
$\omega = 0.9$
$b = 0.1$