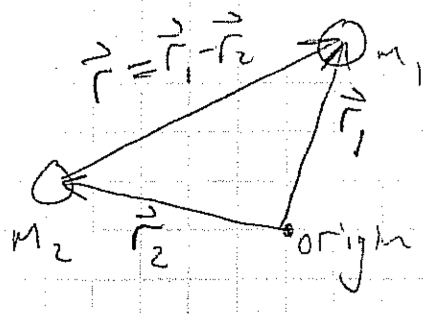
CH08 – Two-body central force problem



$$\vec{F}_{\text{by 2 on 1}} = \frac{-G m_1 m_2}{r^2}\,\hat{r}$$

$\vec{r} = \vec{r}_1 - \vec{r}_2$ is the relative position of m_1

$r = |\vec{r}| = |\vec{r}_1 - \vec{r}_2|$ and $\hat{r} = \dfrac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}$

$$U(\vec{r}_1, \vec{r}_2) = -\int \vec{F}_{\text{by 2 on 1}} \cdot d\vec{r}$$

$$= -\int \frac{-G m_1 m_2}{r^2}\, dr$$

$$= G m_1 m_2 \int \frac{1}{r^2}\, dr = -\frac{G m_1 m_2}{r} + C$$

Choose C so that $U(r=\infty) = 0$, then $C = 0$

$$U = -\frac{G m_1 m_2}{r} \quad \text{where } r = |\vec{r}_1 - \vec{r}_2|$$

So, U is only a function of r. $\xrightarrow{\text{write}}$ $U(r)$.

$$K = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2$$

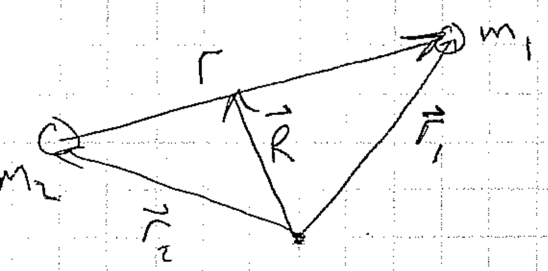
$$K = \tfrac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \tfrac{1}{2} m_2 \dot{\vec{r}}_2^{\,2}$$

The Lagrangian is $L = K - U$

$$= \tfrac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \tfrac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} + \frac{G m_1 m_2}{r}$$

The Lagrangian is a function of six coordinates (3 for $\vec{r}_1$ and 3 for $\vec{r}_2$). It is possible to simplify this (especially for N bodies) by using the <u>CM</u> frame and <u>relative</u> frame.

he Center of mass is



$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \Rightarrow \text{weighted average}$$

$$M = m_1 + m_2$$

$$\begin{aligned}\vec{P}_{sys} &= \vec{P}_1 + \vec{P}_2 \\ &= m_1\vec{V}_1 + m_2\vec{V}_2 \\ &= (m_1+m_2)\dot{\vec{R}} \\ &= M\dot{\vec{R}}\end{aligned}$$

If $\vec{F}_{net} = 0$, then $\vec{P}_{sys}$ is constant and $\dot{\vec{R}}$ is constant.

e Kinetic energy of the system is $K = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2$

$$= \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \qquad \vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$$

$$\dot{\vec{r}} = \dot{\vec{r}}_1 - \dot{\vec{r}}_2 \qquad \dot{\vec{R}} = \frac{m_1}{M}\dot{\vec{r}}_1 + \frac{m_2}{M}\dot{\vec{r}}_2$$

Solve for $\vec{r}_1$ and $\vec{r}_2$ in terms of $\vec{r}$ and $\vec{R}$

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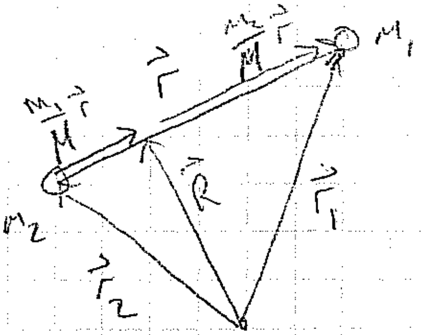
$$\dot{\vec{r}}_1 = \dot{\vec{r}} - \dot{\vec{r}}_2$$
$$= \dot{\vec{r}} - \left(\frac{M\dot{\vec{R}} - m_1\dot{\vec{r}}_1}{m_2}\right)$$
$$\dot{\vec{r}}_1 = \dot{\vec{r}} - \frac{M}{m_2}\dot{\vec{R}} + \frac{m_1}{m_2}\dot{\vec{r}}_1$$
$$\dot{\vec{r}}_1\left(1 - \frac{m_1}{m_2}\right) = \dot{\vec{r}} - \frac{M}{m_2}\dot{\vec{R}}$$
$$\underbrace{\dot{\vec{r}}_1 = \frac{\dot{\vec{r}} - \frac{M\dot{\vec{R}}}{m_2}}{\left(1 - \frac{m_1}{m_2}\right)}}_{\text{messy}}$$

use geometric approach



$$\vec{r}_2 + \frac{m_1}{m_2}\vec{r} = \vec{R} \qquad\qquad \vec{R} + \frac{m_2}{m_1}\vec{r} = \vec{r}_1$$
$$\boxed{\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}} \qquad\qquad \boxed{\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}}$$

$$K = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2}$$
$$= \frac{1}{2}m_1\left(\dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}}\right)^2$$
$$= \frac{1}{2}m_1\left[\dot{\vec{R}}^2 + \frac{m_2^2}{M^2}\dot{\vec{r}}^2 + \underbrace{2\dot{\vec{R}}\dot{\vec{r}}\frac{m_2}{M}}\right] + \frac{1}{2}m_2\left[\dot{\vec{R}}^2 + \frac{m_1^2}{M^2}\dot{\vec{r}}^2 \underbrace{- \dot{\vec{R}}\dot{\vec{r}}\frac{m_1}{M}}\right]$$

Cancel

$$= \frac{1}{2}m_1\dot{\vec{R}}^2 + \frac{1}{2}m_1\frac{m_2^2}{M^2}\dot{\vec{r}}^2 + \frac{1}{2}m_2\dot{\vec{R}}^2 + \frac{1}{2}m_2\frac{m_1^2}{M^2}\dot{\vec{r}}^2$$
$$= \frac{1}{2}(m_1 + m_2)\dot{\vec{R}}^2 + \frac{1}{2}\frac{(m_1+m_2)(m_1m_2)}{M^2}\dot{\vec{r}}^2$$

$$\boxed{K = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\underbrace{\left(\frac{m_1m_2}{m_1+m_2}\right)}\dot{\vec{r}}^2}$$

has units of mass, called reduced mass

$$\boxed{\mu = \frac{m_1m_2}{m_1+m_2}}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{m_1(1+\frac{m_2}{m_1})} = \frac{m_2}{1+\frac{m_2}{m_1}} \overset{\text{or}}{=} \frac{m_1}{1+\frac{m_1}{m_2}}$$

if $m_2 > m_1$, $\mu < m_2$
if $m_1 > m_2$, $\mu < m_1$
if $m_1 = m_2$, $\mu = \frac{m_1^2}{2m_1} = \frac{m_1}{2}$

$\mu < m_1$ and $\mu < m_2$ regardless of the masses.

This is why it is called the <u>reduced</u> <u>mass</u>

$K_{trans} = \frac{1}{2} M \dot{\vec{R}}^2 \qquad K_{rel} = \frac{1}{2}\mu \dot{\vec{r}}^2$

If $K_{trans} = 0$ (center-of mass position is constant) then

$K = K_{rel} = \frac{1}{2}\mu\dot{\vec{r}}^2$. Then...

we can treat the system as if it's a single particle of mass μ with position $\vec{r}$ (relative to $\vec{r}_2^{(1)}$) and velocity $\dot{\vec{r}}$.

The Lagrangian is $L = K - U$

$$L = \underbrace{\frac{1}{2}M\dot{\vec{R}}^2}_{L_{trans}} + \underbrace{\left(\frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)\right)}_{L_{rel}}$$

The translational term has no associated potential energy because there are no external forces. The relative term is the Lagrangian for a single particle moving in a central force field.

Generalized coordinates are $\vec{R}$ and $\vec{r}$.

$\vec{R}$: $\frac{\partial L}{\partial R_x} = \frac{d}{dt}\frac{\partial R}{\partial \dot{R}_x}$

$0 = \frac{d}{dt}(M\dot{R}_x) = M\ddot{R}_x$ also $0 = M\ddot{R}_y$ and $0 = M\ddot{R}_z$

so $0 = M\ddot{\vec{R}}$ and $\dot{\vec{R}}$ = constant

Newt. 2nd law tells us the same thing since the net external force is zero.

In other words, the momentum is <u>conserved</u>.

Use the CM reference frame

For the CM frame, $\dot{\vec{R}} = 0$. Then,

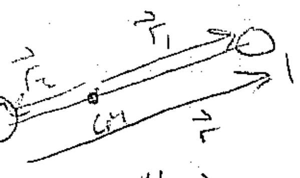
$$L = L_{rel} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$$

The system can be thought of as a single particle, mass μ and velocity $\dot{\vec{r}}$ and position $\vec{r}$.

Angular Momentum

$\vec{l} = \vec{r}\times\vec{p}$ so $\vec{L}_{tot} = \vec{l}_1 + \vec{l}_2 = \vec{r}_1\times\vec{p}_1 + \vec{r}_2\times\vec{p}_2$

$$= \vec{r}_1\times\dot{\vec{r}}_1 m_1 + \vec{r}_2\times\dot{\vec{r}}_2 m_2$$



$$= \frac{m_2}{M}\vec{r}\times\frac{m_1 m_2}{M}\dot{\vec{r}} + \left(\frac{-m_1}{M}\right)\vec{r}\times\left(\frac{-m_1 m_2}{M}\right)\dot{\vec{r}}$$

$\vec{r}_1 = \frac{m_2}{M}\vec{r}$

$\vec{r}_2 = \frac{-m_1}{M}\vec{r}$

$$= \frac{m_1 m_2}{M^2}\left(m_2\vec{r}\times\dot{\vec{r}} + m_1\vec{r}\times\dot{\vec{r}}\right)$$

$$= \frac{m_1 m_2}{M^{\cancel{2}}}\left(\cancel{m_2 + m_1}\right)\vec{r}\times\dot{\vec{r}}$$

$\vec{L}_{tot} = \mu\vec{r}\times\dot{\vec{r}} = \boxed{\vec{r}\times\mu\dot{\vec{r}}}$ Same as particle of mass μ, velocity $\dot{\vec{r}}$.

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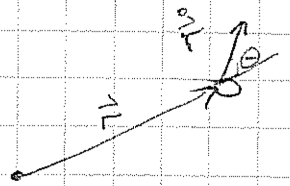
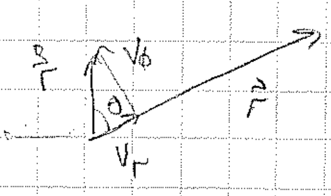
$|\vec{L}| = |\vec{r}|\mu|\dot{\vec{r}}|\sin\theta$ $\quad \dot{\vec{r}} = \langle V_r, V_\phi \rangle$ (polar coordinates)

so $|\dot{\vec{r}}|^2 = V_r^2 + V_\phi^2$
$= \dot{r}^2 + (r\dot{\phi})^2$





$\sin\phi = \frac{V_\phi}{|\dot{\vec{r}}|}$ $\quad$ so $|\vec{L}| = r\mu V_\phi$
$= r\mu r\dot{\phi}$

$\boxed{|\vec{L}| = r^2\mu\dot{\phi}} \equiv l$ for reduced mass particle in our model

The Lagrangian is

$$L = L_{rel} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U$$

$$= \frac{1}{2}\mu\left(\dot{r}^2 + (r\dot{\phi})^2\right) - U(r) = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 - U(r)$$

ϕ: $\frac{\partial L}{\partial \phi} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}}$

$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}}$

So $\frac{\partial L}{\partial \dot{\phi}} = \text{constant}$. $\Rightarrow$ $\frac{1}{2}\mu\, 2r^2\dot{\phi} = \text{const}$

$$\boxed{\mu r^2\dot{\phi} = \text{const}}$$

This means that $l = \mu r^2\dot{\phi}$ is constant, exactly as expected since $\vec{\tau}_{net} = 0$.

r: $\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}}$

$\mu r\dot{\phi}^2 - \frac{\partial U}{\partial r} = \frac{d}{dt}(\mu\dot{r})$

$\mu r\dot{\phi}^2 - \underbrace{\frac{\partial U}{\partial r}}_{\nabla U = -F_r} = \mu\ddot{r}$ $\Rightarrow$ $\boxed{\mu\ddot{r} = \underbrace{-\nabla U}_{\text{due to } F_{grav}(r)} + \underbrace{\mu r\dot{\phi}^2}_{\text{outward radial force}}}$

call it a "centrifugal force"

$F_{cf} = \mu r\dot{\phi}^2$

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The centrifugal force term is

$$F_{cf} = \mu r \dot{\phi}^2 \left(\frac{r}{r}\right) = \mu r^2 \dot{\phi}\left(\frac{\dot{\phi}}{r}\right) = \frac{\ell \dot{\phi}}{r}\left(\frac{r^2\mu}{r^2\mu}\right) = \frac{\ell^2}{\mu r^3}$$

Write a potential energy function using

$$U_{cf} = -\int F_{cf}\, dr = -\int \frac{\ell^2}{\mu r^3}\, dr = \frac{\ell^2}{\mu}\,\frac{r^{-2}}{2} = \frac{\ell^2}{2\mu r^2}$$

Thus, $U = U(r) + U_{cf}(r)$

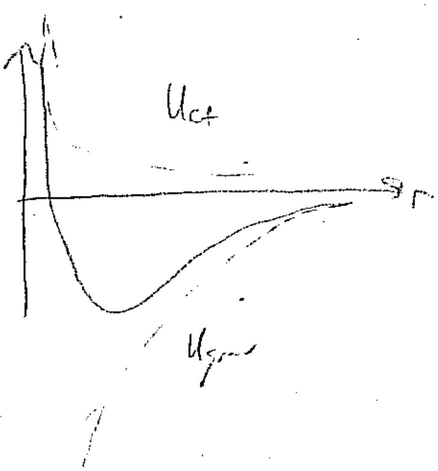
Then, $\mu \ddot{r} = \frac{-dU}{dr} + \frac{-dU_{cf}}{dr}$

$$\mu \ddot{r} = \frac{-d}{dr}\left(U + U_{cf}\right)$$

$$\mu \ddot{r} = \frac{-dU_{eff}}{dr}$$

It's a 1-D problem with $U_{eff} = U + U_{cf}$

$$U_{eff} = \frac{-Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$$



CH 8 Continued . . .

Lagrangian: $L = L_{rel}$ (in CM frame $L_{trans} = 0$)

$$L_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 - U(r)$$

ϕ: $\boxed{\mu r^2\dot{\phi} = \text{const}}$

r: $\mu\ddot{r} = -\nabla U + \mu r\dot{\phi}^2$

$\boxed{\mu\ddot{r} = -\nabla U_{eff}(r)}$ where $U_{eff} = U + U_{cf}$

$$= -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$$

$\mu\ddot{r}\dot{r} = \frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right)$ (verify this!)

$= 2\left(\frac{1}{2}\right)\mu\ddot{r} = \mu\ddot{r}$ ✓

so $\mu\ddot{r}\dot{r} = -\frac{d}{dr}U_{eff}(r)\,\dot{r}$

$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right) = -\frac{dU_{eff}}{dt}$

$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2 + U_{eff}\right) = 0$

$\frac{1}{2}\mu\dot{r}^2 + U_{eff}$ IS constant!

...ll this the total energy $\boxed{E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}}$ of our single particle model.

...ummary

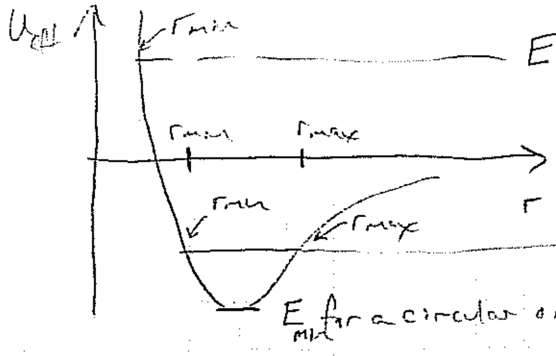
Treat system as a single particle with one generalized coord.

$L = \frac{1}{2}\mu\dot{r}^2 - U_{eff}$, $U_{eff} = -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$

constant

... changes according to $\mu\ddot{r} = -\nabla U_{eff}$, $\ell = r^2\mu\dot{\phi}$ = constant, $E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}$.

$$E = \tfrac{1}{2}\mu\dot{r}^2 - U_{eff}$$

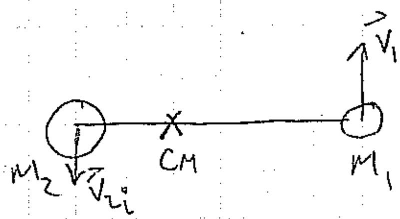


$E > 0$, then orbit is unbound

$E < 0$, then orbit is bound

E_{min} for a circular orbit

Example



$M_1 = 2e30$ kg

$m_2 = 2m_1$

$\vec{V}_{1i} = \langle 0,\ 2\times10^4,\ 0\rangle \frac{m}{s}$

$\vec{V}_{2i} = -\frac{m_1}{m_2}\vec{V}_{1i}$ so that $\vec{V}_{cm} = 0$

$\vec{r}_1 = \langle +1.5e11, 0, 0\rangle$

$\vec{r}_2 = -\frac{m_1}{m_2}\vec{r}_1$ so $\vec{R}_c$

$\vec{l}_1 = \vec{r}_{1i}\times\vec{p}_{1i}$

$l_1 = r_{1i} m_1 v_{1i}$

$\vec{l}_2 = \vec{r}_{2i}\times\vec{p}_{2i}$

$\vec{l}_2 = r_{2i} m_2 v_{2i}$

$l = r_{1i}M_1V_{1i} + r_{2i}M_2V_{2i}$

$\dot{\phi} = \frac{l}{r^2\mu}$

$\phi = \frac{d\dot{\phi}}{dt}$

$\mu\ddot{r} = \frac{-dU_{eff}}{dr}$ $\quad U_{eff} = -\frac{Gm_1m_2}{r^2} + \frac{l^2}{2\mu r^2}$ $\quad K = \frac{1}{2}\mu\dot{r}^2$

$\mu\ddot{r} = \frac{-Gm_1m_2}{r^2} + \frac{l^2}{\mu r^3}$

$x = r\cos\phi$

$y = r\sin\phi$