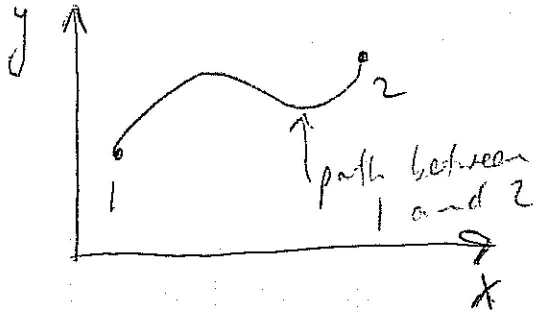
CH6 — Calculus of Variations



y is vertical displacement
x is horizontal displacement

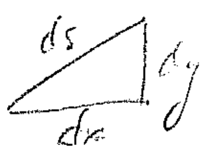
What is the shortest path between the two points?

This is a minimization problem — you want to minimize the distance between the two points in this case.

The distance, s is

$$s = \int_1^2 ds \Rightarrow \text{the sum of the length } ds \text{ along the path}$$

The shortest path is the path for which the integral of ds is a minimum.



$$ds^2 = dx^2 + dy^2$$

$$ds = \left(dx^2 + dy^2\right)^{1/2}$$

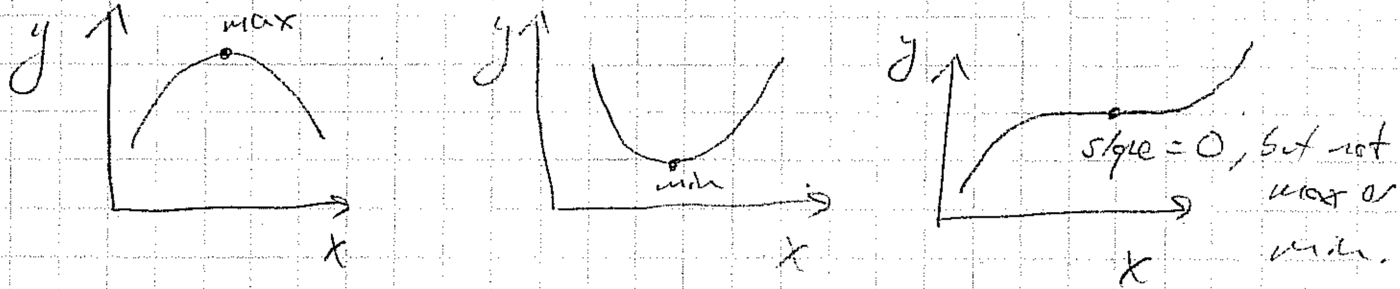
$$= dx\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2}$$

Write $\frac{dy}{dx} = y'(x)$

$$S = \int_1^2 (1 + y'^2)\,dx$$

The goal is to minimize this integral.

Note that you've found minima or maxima before.



At max or min (or zero slope), $\frac{dy}{dx} = 0$.
These are called stationary points.

In general, if

$$S = \int_{x_1}^{x_2} f(y(x), y'(x), x)\,dx$$

Note that y and x can represent any variables, not just 2-D space

is a stationary point if

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \quad \text{(Euler-Lagrange Equation)}$$

where x is the independent variable and y is the dependent va

Back to the shortest path...

distance between 1 and 2 is

$$S = \int_{x_1}^{x_2} \left(1 + y'(x)\right)^{1/2} dx$$

Euler-Lagrange eq:

$$f = (1 + y')^{1/2}$$ y is dependent variable and x is independent variable

so $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$ for shortest path.

$\frac{\partial f}{\partial y} = 0$ since f is not a function of y.

$$\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'}\left[(1+y')^{1/2}\right] = \frac{1}{2}(1+y')^{-1/2} =$$

Since $\frac{\partial f}{\partial y'}$ is not a function of x, then

$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ and $\frac{\partial f}{\partial y'}$ is a constant

$$\frac{\partial f}{\partial y'} = \text{const}$$

So, $\frac{1}{2}(1+y')^{-1/2} = \text{const.}$

thus, $y' = \text{const}$ (though a different constant.)

Gee, let's call the constant m (for slope), then

$y' = m$

$\frac{dy}{dx} = m$

$\int dy = \int m\,dx = m\int dx = mx + C$ ← integration constant

8) $\boxed{y = mx + b}$ where b is the integration constant.

This is the equation of a straight line, so a straight line is the shortest distance between two points.