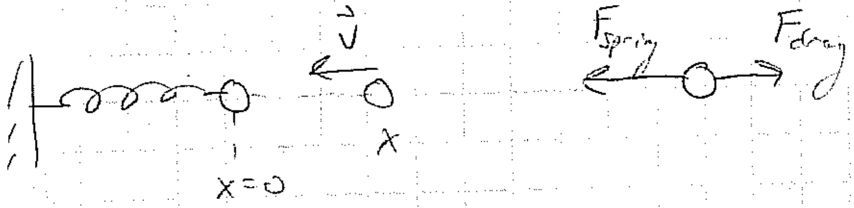
CH5: Damped harmonic oscillator



$$\vec{F}_{net} = \vec{F}_{spring} + \vec{F}_{drag}$$

Assume linear drag: $\vec{F}_{drag} = -b|\vec{v}|\hat{v}$

$$F_{net_x} = -kx + -bv_x = m\ddot{x}$$

$$-kx - b\dot{x} = m\ddot{x}$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

Define: $\frac{b}{m} = 2\beta$ and $\omega_0 = \sqrt{\frac{k}{m}} \equiv$ Natural Frequency

then $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$

The auxiliary equation is

$$r^2 + 2\beta r + \omega_0^2 = 0$$

$$r = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega_0^2}}{2}$$

$$\boxed{r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}}$$

Then, $x = C_1 e^{r_1 t} + C_2^{+r_2 t}$ is a solution

where $r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}$ and $r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$

and C_1 and C_2 are determined by the initial conditions.

Thus, the general solution is

$$x = C_1 e^{\left(-\beta+\sqrt{\beta^2-\omega_0^2}\right)t} + C_2 e^{\left(-\beta-\sqrt{\beta^2-\omega_0^2}\right)t}$$

$$= e^{-\beta t}\left[C_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right]$$

Undamped oscillation: $\beta=0$, then

$$x = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$$ just as expected.

Weak damping (or "underdamped")

$\beta < \omega_0$ then $\sqrt{\beta^2-\omega_0^2}$ is imaginary

$$\sqrt{\beta^2-\omega_0^2} = i\sqrt{\omega_0^2-\beta^2} = i\omega_1 \text{ where } \omega_1 = \sqrt{\omega_0^2-\beta^2}$$

then $x = e^{-\beta t}\underbrace{\left[C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right]}$

can be written as $A_0\cos(\omega_1 t-\delta)$ as done before, so

$$\boxed{x = \underbrace{e^{-\beta t}A_0}_{\text{decaying amplitude}}\underbrace{\cos(\omega_1 t-\delta)}_{\text{oscillatory}}}$$

$A(t) = e^{-\beta t}A_0$

β is the decay constant.

$\frac{1}{\beta} = \tau$ is time constant

$= \Delta t$ for amplitude to decay to $\frac{1}{e}A_0$

β is a rate that A dies out

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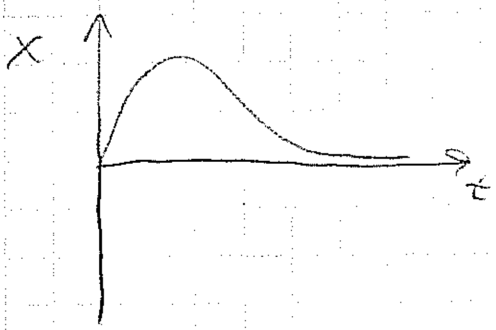
Strong Damping ("overdamped")

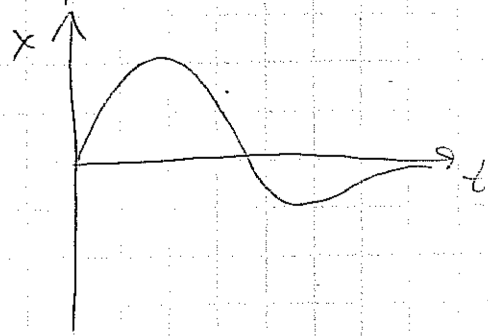
$\beta > \omega_0$ then $\sqrt{\beta^2 - \omega_0^2}$ is real

$$x(t) = C_1 e^{\left(-\beta + \sqrt{\beta^2-\omega_0^2}\right)t} + C_2 e^{\left(-\beta - \sqrt{\beta^2-\omega_0^2}\right)t}$$

$$= C_1 e^{-\left(\beta - \sqrt{\beta^2-\omega_0^2}\right)t} + C_2 e^{-\left(\beta + \sqrt{\beta^2-\omega_0^2}\right)t}$$

Both functions are exponential decays with the second term decaying more quickly.





It doesn't oscillate, so to speak.

Critical Damping

$\beta = \omega_0$, then $\beta^2 = \omega_0^2$ and $\sqrt{\beta^2 - \omega_0^2} = 0$

The auxiliary equation is $r = -\beta$ which gives

$$x = C_1 e^{-\beta t}$$

However we must have two solutions:

$$x = C_2 t e^{-\beta t}$$

is also a solution.

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The general solution is

$$X = C_1 e^{-\beta t} + C_2 t e^{-\beta t}$$

In this case, the oscillation dampens at the quickest rate. Often, one wants oscillations to dampen out quickly (such as for a magnet in a liquid filled compass or a needle in a bathroom scale.)

$\beta = \omega_0$ gives quickest damping.