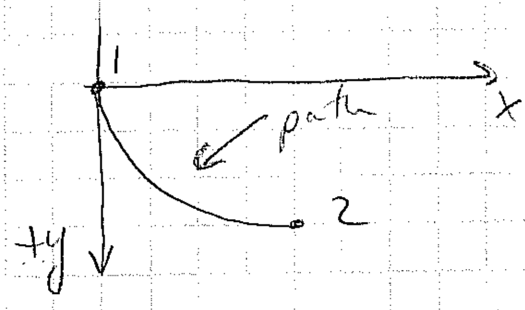
Brachistochrone Problem

Given two points at different elevations, how should we construct a track so that a frictionless roller coaster will travel from 1 to 2 in the least time?



Put point 1 at the origin.

Release from rest, so $v_1 = 0$.

Define +y downward.

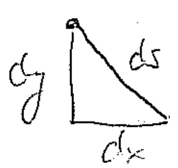
Since the track is frictionless, Emech is conserved, so

$$\Delta K + \Delta U = 0$$

$$\frac{1}{2} m v_2^2 + -mgh = 0$$

$$\frac{1}{2} m v^2 - mgy = 0$$

$$v = \sqrt{2gy}$$



$$v = \frac{ds}{dt} \quad \text{so} \quad dt = \frac{ds}{v}$$

$$dt = \frac{ds}{\sqrt{2gy}}$$

$$\Delta t = \int_1^2 dt$$ We want to minimize Δt

$$\Delta t = \int_1^2 \frac{ds}{\sqrt{2gy}}$$ where $ds = (dx^2 + dy^2)^{1/2}$

Write in terms of dy: $ds = dy\left(\left(\frac{dx}{dy}\right)^2 + 1\right)^{1/2}$

In this case $\underline{y}$ is the $\underline{\text{independent}}$ variable and x is the dependent variable. Note that this changes the Euler–Lagrange equation to

$$S = \int_{y_1}^{y_2} f\left(x(y), x'(y), y\right) dy$$ and S is a

stationary point if $\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = 0$.

$$\Rightarrow \Delta t = \int_1^2 \frac{(x'^2+1)^{1/2}}{\sqrt{2gy}} dy = \frac{1}{\sqrt{2g}} \int_1^2 \frac{(x'^2+1)^{1/2}}{y^{1/2}} dy$$

$$f = \frac{(x'^2+1)^{1/2}}{y^{1/2}}$$

$\frac{\partial f}{\partial x} = 0$ since f is not a function of x

Thus Eul.–Lag Eq gives $\frac{d}{dy}\frac{\partial f}{\partial x'} = 0$

so $\frac{\partial f}{\partial x'} = \text{const}$.

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$$f = \frac{(x'^2+1)^{1/2}}{y^{1/2}}$$

$$\frac{\partial f}{\partial x'} = \text{const}$$

$$= \frac{\frac{1}{2}(x'^2+1)^{-1/2}(2x')}{y^{1/2}} = \text{const.}$$

So squaring it is still a constant.

$$\frac{x'^2}{(1+x'^2)y} = \text{const.}$$

Call this $\frac{1}{2a}$ with "experience"

$$\frac{x'^2}{(1+x'^2)y} = \frac{1}{2a}$$

$$2ax'^2 = y(1+x'^2)$$

$$= y + yx'^2$$

$$x'^2(2a-y) = y$$

$$x' = \left(\frac{y}{2a-y}\right)^{1/2}$$

$$\frac{dx}{dy} = \left(\frac{y}{2a-y}\right)^{1/2}$$

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$$X = \int_{x_1}^{x_2} dx = \int_{y_1}^{y_2} \left(\frac{y}{2a-y}\right)^{1/2} dy$$

$$X = \int_{y=0}^{y=y} \left(\frac{y}{2a-y}\right)^{1/2} dy$$

Substitute $y = a(1-\cos\theta)$

$dy = a\sin\theta \, d\theta$

$$X = \int \left(\frac{a(1-\cos\theta)}{2a - a + a\cos\theta}\right)^{1/2} a\sin\theta \, d\theta$$

$$= \int \left(\frac{a(1-\cos\theta)}{a(1+\cos\theta)}\right)^{1/2} a\sin\theta \, d\theta$$

$$= \int \left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{1/2} a\sin\theta \, d\theta$$

$$\left(\frac{1-\cos\theta}{1+\cos\theta}\right)\left(\frac{1-\cos\theta}{1-\cos\theta}\right) = \frac{(1-\cos\theta)^2}{1-\cos^2\theta} = \frac{(1-\cos\theta)^2}{\sin^2\theta}$$

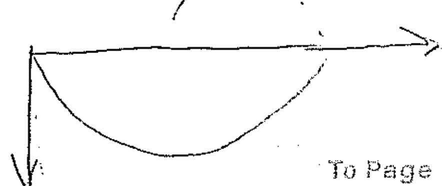
$$= \int \frac{(1-\cos\theta)}{\cancel{\sin\theta}} a \cancel{\sin\theta} \, d\theta$$

$$= \int a(1-\cos\theta)\, d\theta$$

$$X = a(\theta - \sin\theta) + C^{\to 0 \text{ if } x=0 \text{ at } \theta=0}$$

$$= a(1-\cos\theta)$$

parametric equations of a cycloid.



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