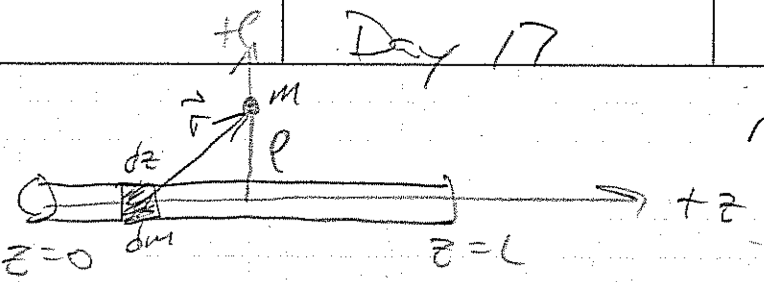
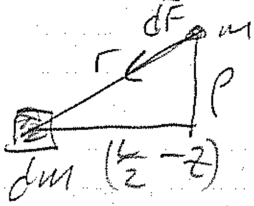
(2)



rod: M, L



$$r^2 = \rho^2 + \left(\frac{L}{2} - z\right)^2$$

$$d\vec{F} = \frac{Gm(dm)}{r^2}(-\hat{r}) \qquad \hat{r} = \frac{\vec{r}}{r}$$

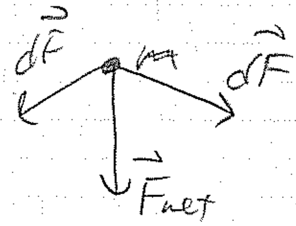
$$= \frac{\langle \rho, 0, \frac{L}{2} - z\rangle}{r}$$

$$d\vec{F} = -\frac{Gm(dm)\vec{r}}{r^3}$$

In cylindrical coord.

$$= \frac{-Gm(dm)\langle \rho, 0, \frac{L}{2} - z\rangle}{\left(\rho^2 + \left(\frac{L}{2} - z\right)^2\right)^{3/2}}$$

$$\vec{F} = \int_{z=0}^{z=L} d\vec{F}$$



- Due to symmetry, $\vec{F}_{net}$ is in $-\hat{\rho}$ direction.
- Only integrate over $\hat{\rho}$ component.

$$\vec{F} = \int \frac{-Gm(dm)\,\rho}{\left(\rho^2 + \left(\frac{L}{2} - z\right)^2\right)^{3/2}}\hat{\rho}$$

Uniform density $\mu = \frac{M}{L} = \frac{dm}{dz}$

So $dm = \mu\, dz$

$$\vec{F} = \int_0^L \frac{-Gm\rho\mu}{\left(\rho^2 + \left(\frac{L}{2} - z\right)^2\right)^{3/2}}\, dz$$

$$= -Gm\rho\mu \left[\frac{2z - L}{\rho^2\left((L-2z)^2 + 4\rho^2\right)^{1/2}} \right]_0^L$$

$$= -\frac{Gm\cancel{\rho}\mu}{\rho^{\cancel{2}}} \left[\frac{L}{\left(L^2 + 4\rho^2\right)^{1/2}} - \frac{-L}{\left(L^2 + 4\rho^2\right)^{1/2}} \right]$$

$$= -\frac{Gm\mu}{\rho} \left(\frac{2L}{\left(L^2 + 4\rho^2\right)^{1/2}} \right)$$

$$= -\frac{Gm\mu}{\rho} \left(\frac{\cancel{2}L}{\cancel{2}\left(\left(\frac{L}{2}\right)^2 + \rho^2\right)^{1/2}} \right)$$

(a) $$\boxed{F_\rho = -\frac{Gm\mu}{\rho} \frac{L}{\left(\left(\frac{L}{2}\right)^2 + \rho^2\right)^{1/2}}}$$

(b) If $L \gg \rho$, then $\left(\left(\frac{L}{2}\right)^2 + \rho^2\right)^{1/2} \simeq \left(\left(\frac{L}{2}\right)^2\right)^{1/2} \simeq \frac{L}{2}$

$$F_\rho \simeq -\frac{Gm\mu}{\rho} \frac{\cancel{L}}{\frac{\cancel{L}}{2}}$$

$$\boxed{F_\rho \simeq -\frac{2Gm\mu}{\rho}}$$

(c) In cylindrical coordinates,

$$\nabla = \frac{\partial}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{z}$$

$\vec{F}$ is only in $\hat{\rho}$ dir so F_ϕ and $F_z = 0$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{1}{\rho}\frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & 0 & 0 \end{vmatrix}$$

$$= \hat{\rho}(0) - \hat{\phi}\left(-\frac{\partial F_\rho}{\partial z}\right) + \hat{z}\left(-\frac{1}{\rho}\frac{\partial F_\rho}{\partial \phi}\right)$$

Since F_ρ does not depend on z or ϕ, then

$$\frac{\partial F_\rho}{\partial z} = \frac{\partial F_\rho}{\partial \phi} = 0.$$

So,

$\nabla \times \vec{F} = 0$ ✓

(d) $\vec{F} = \frac{-2Gm\mu}{\rho}\hat{\rho}$

where in Cart. Coord.: $\rho^2 = x^2+y^2$

and $\vec{\rho} = x\hat{x} + y\hat{y}$

$\hat{\rho} = \frac{\vec{\rho}}{\rho}$ so

$$\vec{F} = \frac{-2Gm\mu}{\rho^2}\vec{\rho} = \frac{-2Gm\mu}{\rho^2}(x\hat{x}+y\hat{y})$$

$$F_x = \frac{-2Gm\mu}{(x^2+y^2)}x$$

$$F_y = \frac{-2Gm\mu}{x^2+y^2}y$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & 0 \end{vmatrix}$$

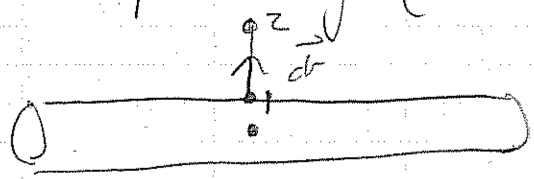
$$= \hat{x}\underbrace{\left(\frac{-\partial F_y}{\partial z}\right)}_{0} - \hat{y}\underbrace{\left(\frac{-\partial F_x}{\partial z}\right)}_{0} + \hat{z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$\left.\begin{aligned} \frac{\partial F_y}{\partial x} &= -2GM\mu\left(\frac{-y}{(x^2+y^2)^2}(2x)\right) = 2GM\mu\frac{2xy}{(x^2+y^2)^2} \\ \frac{\partial F_x}{\partial y} &= -2GM\mu\left(\frac{-x}{(x^2+y^2)^2}2y\right) = 2GM\mu\frac{2xy}{(x^2+y^2)^2} \end{aligned}\right\}\text{ same}$$

$$(\nabla\times F)_z = \hat{z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0 \checkmark$$

$$\Delta U = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} \qquad \vec{F} = F_\rho \hat{\rho}$$

Choose path along $\hat{\rho}$



$d\vec{r} = d\rho\,\hat{\rho}$

At pt. 1, $\rho = R$ of rod.

Define $U=0$ at $\rho = R$.

$$\Delta U = U_2 - \underbrace{\cancel{U_1}}_{0} = -\int_R^{\rho} F_\rho \, d\rho$$

$$U = -\int_R^{\rho} \left(\frac{-2Gm\mu}{\rho}\right) d\rho$$

$$= 2Gm\mu \int_R^{\rho} \frac{1}{\rho}\, d\rho = 2Gm\mu \ln\rho \Big|_R^{\rho}$$

$$= 2Gm\mu\left[\ln(\rho) - \ln(R)\right]$$

$$\boxed{U = 2Gm\mu \ln\left(\frac{\rho}{R}\right)}$$

with $U=0$ at $\rho = R$.