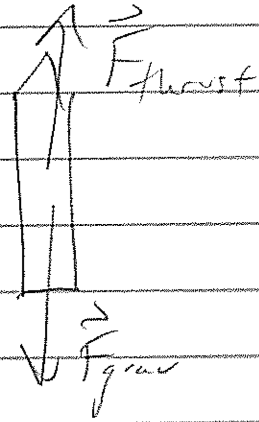
Rocket Motion with gravity

system = rocket + fuel
(but not exhaust)



$$\vec{F}_{net} = \vec{F}_{thrust} + \vec{F}_{grav}$$

$$m\dot{v}_y = F_{thrust\,y} - mg$$

$$m\dot{v}_y = -\dot{m}v_{exhaust} - mg$$

Note that $\dot{m}$ is negative, so F_{thrust} is in +y dir.

Find v_y as a function of mass.

$$m\frac{dv_y}{dt} = -\dot{m}\,v_{exhaust} - mg$$

$$dv_y = -\frac{\dot{m}}{m}\,dt\,v_{exhaust} - g\,dt$$

$$dv_y = -\frac{dm}{m}\,v_{exhaust} - g\,dt$$

write $\dot{m} = \frac{dm}{dt}$ and $dt = \frac{dm}{\dot{m}}$

$$dv_y = -\frac{dm}{m}\,v_{exhaust} - g\left(\frac{dm}{\dot{m}}\right)$$

$$\int_{v_y}^{v_y} dv_y = v_y - v_{y0} = \int_{m=m_0}^{m} \frac{-dm}{m}\,v_{exhaust} - \int_{m=m_0}^{m} \frac{g}{\dot{m}}\,dm$$

$$v_y - v_{y_0} = -v_{exhaust} \ln(m) \Big|_{m_0}^{m} - \frac{g}{\dot{m}}(m - m_0)$$

$$= -v_{exhaust} \underbrace{\left(\ln(m) - \ln(m_0)\right)} - \frac{g}{\dot{m}}(m - m_0)$$

$$\ln\left(\frac{m}{m_0}\right) = -\ln\left(\frac{m_0}{m}\right)$$

$$\boxed{v_y = v_{y_0} + v_{exhaust} \ln\left(\frac{m_0}{m}\right) - \frac{g}{\dot{m}}(m - m_0)}$$

Suppose the rocket starts from rest. Then

$$\boxed{v_y = v_{exhaust} \ln\left(\frac{m_0}{m}\right) - \frac{g}{\dot{m}}(m - m_0)}$$

Find $v_y(t)$.

$$\dot{m} = \frac{dm}{dt} \quad \text{so} \quad dm = \dot{m}\,dt$$

$$\int_{m_0}^{m} dm = \int_{t=0}^{t} \dot{m}\,dt$$

$$m - m_0 = \dot{m}t \quad \text{and} \quad m = m_0 + \dot{m}t$$

$$\boxed{v_y = v_{exhaust} \ln\left(\frac{m_0}{m_0 + \dot{m}t}\right) - gt}$$

$$V_y = -V_{exhaust} \ln\left(\frac{m_0 + \dot{m}t}{m_0}\right) - gt$$

$$\boxed{V_y = -V_{exhaust} \ln\left(1 + \frac{\dot{m}}{m_0}t\right) - gt}$$