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07 - Lagrange's Equations

Suppose that you wish to minimize the integral

$$S = \int f(y', y, x)\, dx$$

then this integral is stationary if

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

But what if x is a function of u and y is a function of u and

$$S = \int f(x'(u), x(u), y'(u), y(u), u)\, du$$

In this case,

$$\frac{\partial f}{\partial y} - \frac{d}{du}\frac{\partial f}{\partial y'} = 0 \quad \text{and} \quad \frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'} = 0$$

For each variable, x, you get an Euler-Lagrange equation.

Consider a particle moving in 3-dimensions subject to a conservative force $\vec{F}$.

The particle's kinetic energy is $K = \frac{1}{2}mv^2 = \frac{1}{2}m(V_x^2 + V_y^2 + V_z^2)$

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

The potential energy associated with this force and particle is

$$U = -\int F_x\,dx + F_y\,dy + F_z\,dz, \quad \text{in general } U(x, y, z)$$

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The Lagrangian (function) is

$$L = K - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x,y,z)$$

Consider the following derivatives:

$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = F_x$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p_x$$

Similarly the derivative of L with respect to y and z give F_y and F_z, etc.

Now take,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}(p_x) = \dot{p}_x \Rightarrow \text{Newton's } 2^{nd} \text{ law.}$$

Then $\frac{\partial L}{\partial x} = F_x$ and $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = F_x$

and thus, $\boxed{\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}}$ (and similarly for the other variables)

This is an Euler-Lagrange equation! And it means that

$$S = \int L(\dot{x}, x, \dot{y}, y, \dot{z}, z, t)\,dt$$

is stationary for the path followed by the particle. This integral is called the <u>action</u> and so Lagrange's equations give the path of <u>least action</u>

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Hamilton's Principle

The actual path which a particle follows between two points 1 and 2 in a given time interval t_1 to t_2 is such that the action integral

$$S = \int_{t_1}^{t_2} L\,dt$$

is stationary when taken along the actual path.

Suppose that you wish to use other coordinates besides (x, y, z).

If $\vec{r}$ specifies a unique value of (q_1, q_2, q_3) then $\vec{r} = \vec{r}(q_1, q_2, q_3)$ and likewise $q_i = f_i(x, y, z)$, then

$$L = L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t)\,dt$$

These are called <u>generalized</u> coordinates.

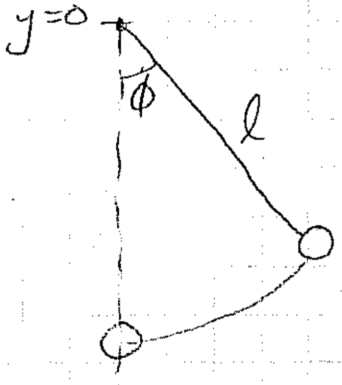
If there are N particles, then there are <u>at most</u> $3N$ generalized coordinates (3 for each particle).

The Lagrange equations are

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \qquad [i = 1 \ldots 3N]$$

If a system is *constrained*, then the number of generalized coordinates needed to describe the position of a particle is less than $3N$.

For example, a pendulum



In $z=0$ plane, $\vec{r} = \langle x, y, 0 \rangle$

$x = l\sin\phi$, $y = -l\cos\phi$.

$\vec{r} = l\langle \sin\phi, -\cos\phi, 0 \rangle$

Now only one variable ϕ is needed to describe the position of the pendulum.

If $\vec{r} = \vec{r}(q_i, \dot{q}_i)$ and is independent of time then the generalized coordinates are *natural*.

The *degrees of freedom* is the number of coordinates that can be independently varied in a small displacement — the number of independent *directions* in which the system can move from any given initial configuration.

The pendulum constrained to a plane has 2 degrees of freedom; however, it can be written in terms of 1 generalized coordinate. If the number of generalized coordinates is less than or equal to the degrees of freedom, then it is a *holonomic* system.