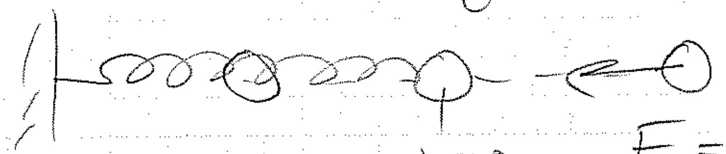
Harmonic Oscillator

mass on a spring



$x=0$
equil

$$F_x = -kx$$

restoring force

$$F_{net_x} = m\ddot{x}$$

$$-kx = m\ddot{x}$$

$$\ddot{x} + \underbrace{\frac{k}{m}}_{\omega^2} x = 0 \qquad \omega^2 = \frac{k}{m} \text{ so } \omega = \sqrt{\frac{k}{m}}$$

$$\boxed{\ddot{x} + \omega^2 x = 0}$$ general form of a harmonic oscillator

One solution is

$$\boxed{x = C_1 e^{i\omega t} + C_2 e^{-i\omega t}}$$ where C_1 and C_2 can be complex numbers that are not time dependent

Verify

$$\dot{x} = i\omega C_1 e^{i\omega t} + -i\omega C_2 e^{-i\omega t}$$

$$\ddot{x} = -\omega^2 C_1 e^{i\omega t} - \omega^2 C_2 e^{-i\omega t}$$

Substitute: $-\omega^2 \cancel{C_1 e^{i\omega t}} - \omega^2 \cancel{C_2 e^{-i\omega t}} + \omega^2 (\cancel{C_1 e^{i\omega t}} + \cancel{C_2 e^{-i\omega t}})$

$\stackrel{?}{=} 0$ yes this $= 0$ ✓

Another solution is

$$x = B_1\cos(\omega t) + B_2\sin(\omega t)$$ B_1 and B_2 are real

Write $x = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$

$$= C_1(\cos\omega t + i\sin\omega t) + C_2(\cos(\omega t) - i\sin(\omega t))$$

$$= \underbrace{(C_1 + C_2)}_{B_1}\cos\omega t + \underbrace{i(C_1 - C_2)}_{B_2}\sin\omega t$$

$$B_1 = C_1 + C_2 \qquad B_2 = i(C_1 - C_2)$$

Find B_1 and B_2 from initial conditions

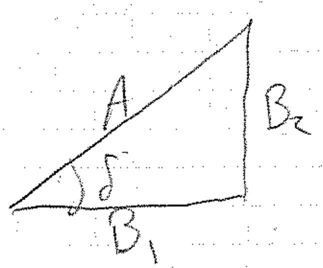
at $t = 0$, $x = x_0$, $\dot{x} = v_{x0}$

$x_0 = B_1$ so $\boxed{B_1 = x_0}$

$$\dot{x} = -B_1\omega\sin(\omega t) + B_2\omega\cos(\omega t)$$

$\dot{x}_0 = v_{x0} = B_2\omega$ so $\boxed{B_2 = \frac{v_{x0}}{\omega}}$

Define A, δ so that



$$A^2 = B_1^2 + B_2^2$$

$$\delta = \tan^{-1}\left(\frac{B_2}{B_1}\right) = \tan^{-1}\left(\frac{v_0}{\omega x_0}\right)$$

Then

$$X(t) = B_1 \cos\omega t + B_2 \sin(\omega t)$$

$$= A\cos\delta\cos\omega t + A\sin\delta\sin(\omega t)$$

$$\boxed{X(t) = A\cos(\omega t - \delta)}$$ using trig identity

Summary

3 ways to write the solution to the harmonic oscillator

$$X = \mathrm{Re}(C_1 e^{i\omega t} + C_2 e^{-i\omega t})$$

$$X = B_1 \cos\omega t + B_2 \sin\omega t$$

$$X = A\cos(\omega t - \delta)$$

All have two constants that depend on X_0 and V_{x0}.

$B_1 = X_0 \quad B_2 = \dfrac{V_{x0}}{\omega}$

$A = (B_1^2 + B_2^2)^{1/2} \qquad \delta = \tan^{-1}\left(\dfrac{V_{x0}}{X_0\omega}\right)$

$B_1 = C_1 + C_2 \qquad B_2 = i(C_1 - C_2)$

solve for C_1 and C_2