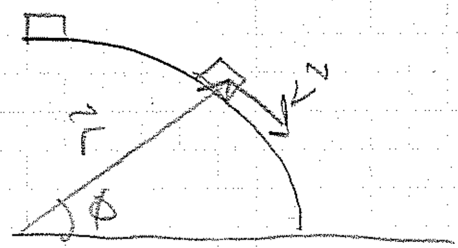
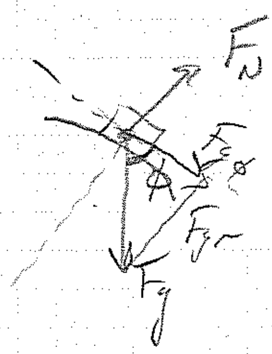
Example - Work-Energy Theorem

for a particle $W = \Delta T$ of the particle.

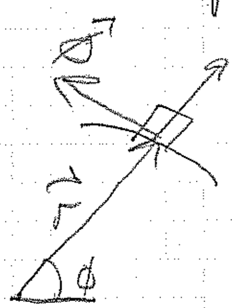
A puck of dry ice slides on a circular ring. At what angle does it leave the ring?



Forces on the puck.



Use polar coordinates.



$\vec{F}_N = F_N \hat{r}$

$\vec{F}_g = -mg\sin\phi\,\hat{r} - mg\cos\phi\,\hat{\phi}$

Newt. 2^{nd} law

$$\vec{F}_{net} = \vec{F}_N + \vec{F}_g = m\ddot{\vec{r}}$$
$$= m(\ddot{r} - r\dot{\phi}^2)\hat{r} + m(r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi}$$

Consider $0 \leq t < t_{\text{leaves the ring}}$

then $\dot{r}$ and $\ddot{r} = 0$ so

$$\vec{F}_N + \vec{F}_g = -mr\dot{\phi}^2\hat{r} + mr\ddot{\phi}\hat{\phi}$$
$$(F_N - mg\sin\phi)\hat{r} - mg\cos\phi\,\hat{\phi} = -mr\dot{\phi}^2\hat{r} + mr\ddot{\phi}\hat{\phi}$$

In $\hat{r}$ dir: $F_N - mg\sin\phi = -mr\dot{\phi}^2$

In $\hat{\phi}$ dir: $-mg\cos\phi = mr\ddot{\phi}$

Puck leaves ring at an angle where $F_N = 0$,

Thus, $-\cancel{m}g\sin\phi = -\cancel{m}r\dot{\phi}^2$

$$\boxed{\dot{\phi}^2 = \frac{g\sin\phi}{r}}$$

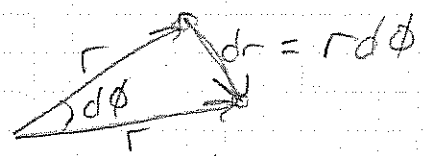
We need $\dot{\phi}$. Use Work-energy Theorem.

F_N does no work and $F_g\hat{r}$ does no work. $F_g\hat{\phi}$ is parallel to displacement so it does work.

$$\Delta T = \int_{\phi=\frac{\pi}{2}}^{\phi_{min}} \vec{F} \cdot \vec{dr} \qquad \vec{dr} = \underbrace{dr}_{0}\,\hat{r} + r\,d\phi\,\hat{\phi}$$

$$\vec{dr} = r\,d\phi\,\hat{\phi}$$

$$= \int_{\phi=\frac{\pi}{2}}^{\phi_{min}} F_\phi \, dr_\phi$$



$$= \int_{\phi=\frac{\pi}{2}}^{\phi_{min}} (-mg\cos\phi)(\,r\,d\phi)$$

$$= \int_{\phi=\frac{\pi}{2}}^{\phi_{min}} -mgr\cos\phi\,d\phi$$

$$= -mgr\sin\phi \Big|_{\frac{\pi}{2}}^{\phi_{min}}$$

$$= -mgr\sin\phi_{min} - (-mgr)$$

$$\Delta T = mgr(1-\sin\phi_{min})$$

$$\Delta T = T_f - T_i = \frac{1}{2}mv_f^2 - \cancel{\frac{1}{2}mv_i^2}^{\,0}$$

$$\vec{v} = \underbrace{\dot{r}}_{0}\,\hat{r} + r\dot{\phi}\,\hat{\phi}$$

$$\vec{v} = r\dot{\phi}\,\hat{\phi} \qquad \Delta T = \frac{1}{2}mv_f^2$$

$$v = r|\dot{\phi}| \qquad = \frac{1}{2}mr^2\dot{\phi}^2$$

$$\frac{1}{2}\cancel{m}r^{\cancel{2}}\dot{\phi}^2 = \cancel{m}g\cancel{r}(1-\sin\phi_{min})$$

$$\boxed{\dot{\phi}^2 = \frac{2g}{r}(1-\sin\phi_{min})}$$

$$\frac{2g}{r}\left(1-\sin\phi_{min}\right) = \frac{g\sin\phi_{min}}{r}$$

$$2 - 2\sin\phi_{min} = \sin\phi_{min}$$

$$2 = 3\sin\phi_{min}$$

$$\sin\phi_{min} = \frac{2}{3}$$

$$\phi_{min} = \sin^{-1}\left(\frac{2}{3}\right) = \boxed{41.8^\circ \approx 42^\circ}$$

This would be $90 - 42 = 48^\circ$ if measured from the $+y$ axis.