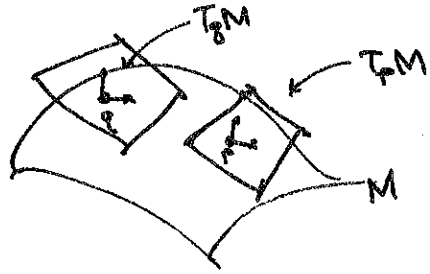
LEC 18: CRASH COURSE on DIFF. FORMS

LAST TIME: FIBER BUNDLE PICTURE



} flat TANGENT SPACES where vectors live

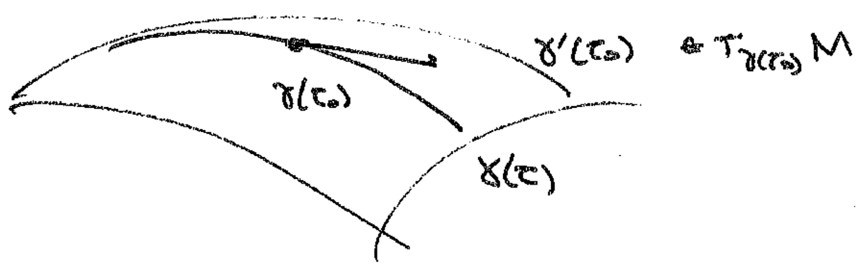
BASE SPACE (eg SPACETIME)

BASIS OF VECTORS: PARTIAL DERIVATIVES

weird: what do DERIVATIVES have to do w/ BASIS VEC?

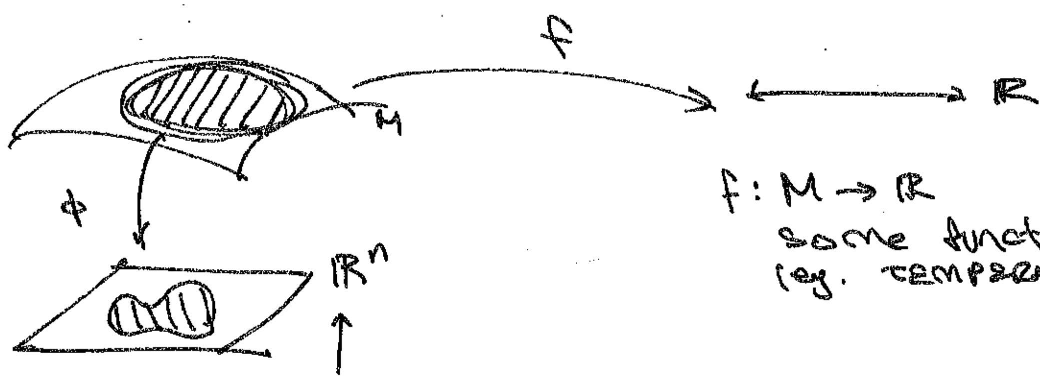
↳ turns out everything: $\boxed{\text{LOCAL DESCRIPTION}}$

what is a tangent vector?



more formally: CHART ϕ that defines coordinates

$\phi: M \to \mathbb{R}^n$ over some PATCH



$f: M \to \mathbb{R}$
some function
(eg. TEMPERATURE)

COORDINATES
ARE n-TUPLES

SO CAN ASK concretely : what is rate of change of $f(\gamma(\tau))$ as observer moves along $\gamma(\tau)$?

$$\frac{d}{d\tau} f \circ \gamma = \frac{d}{d\tau}\left[(f \circ \phi^{-1}) \circ \underbrace{(\phi \circ \gamma)}_{\boxed{x^\mu(\tau)}} \right]$$

$$= \underbrace{\frac{\partial (f \circ \phi^{-1})}{\partial x^\mu}}_{\text{CARTESIAN}} \cdot \frac{d(\phi \circ \gamma)}{d\tau}$$

$$= \frac{dx^\mu}{d\tau} \, \partial_\mu f$$

ARBIT. FUNC.

$$\frac{d}{d\tau} = \underbrace{\frac{dx^\mu}{d\tau}}_{} \partial_\mu$$

DIRECTIONAL DERIVATIVES

tangent vector

what DIRECTION (on TANGENT SPACE)

can be glib about $\mathbb{R}^n$ & $T_p M$ b/c $T_p \mathbb{R}^n = \mathbb{R}^n$.

this is part of a deep connection of PDEs & GEOMETRY & lies @ the heart of GEOMETRICAL MECHANICS.

BUT not our main focus. INSTEAD: if ∂_μ is A BASIS for VEC SPACE:

$\underline{V} = V^\mu \partial_\mu$ ... what is basis of <u>dual</u> space?

$\boxed{\text{DUAL}}$: LINEAR functions that take vectors $\longrightarrow$ #

$T_p M^* : T_p M \to \mathbb{R}$

BASIS of T_pM^* : $\boxed{dx^\mu}$ DIFFERENTIAL FORMS

like the infinitesimal? yes.

IN PARTICULAR: $\boxed{\text{1-form}}$ (vs. k-form)

$$\boxed{dx^\mu(\partial_\nu) = \delta^\mu{}_\nu}$$

where do these dx^μ's come from?

DIFFERENTIAL or (EXTERIOR DERIVATIVE) d : k-form $\rightarrow$ (k+1) form

eg. given a 0-form $f(x)$ — just a function on M

$$\text{1-form:} \quad df = \left(\frac{\partial f}{\partial x^\mu}\right) \underbrace{\boxed{dx^\mu}}_{\tilde{e}^{(\mu)}}$$

$$df(V) = \frac{\partial f}{\partial x^\mu} V^\nu \underbrace{dx^\mu(\partial_\nu)}_{\delta^\mu{}_\nu} = \boxed{V^\mu \frac{\partial f}{\partial x^\mu}}$$

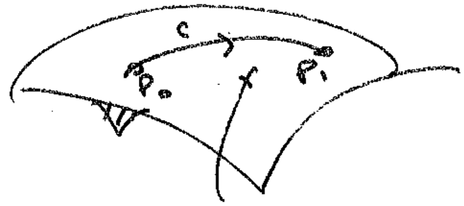
some vector

DIREC. DERIV of f ALONG velocity V

the key thing about differential forms:
BORN to BE INTEGRATED.

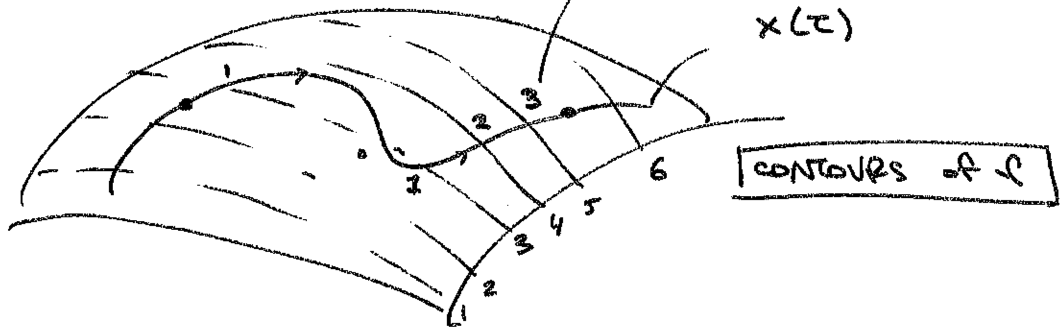
CALCULUS : $\int_c df = f(P_1) - f(P_0)$

$$= \int_0^1 \frac{df(x)}{d\tau} d\tau = \int \frac{\partial f}{\partial x} \frac{\partial x}{\partial \tau} d\tau$$



$$x(\tau) \text{ s.t. } \begin{cases} x(0) = P_0 \\ x(1) = P_1 \end{cases}$$

Geometric :



$\int_c df$ = # times we go up a contour − ———"——— down —"—

important: ORIENTED PATH

$\int_c df = f\big|_{\partial c}$ ← BOUNDARY OF PATH

2-FORMS : when forming 'tensors' of DIFF. FORMS, you only take antisym. comb.

↳ this is the first hint that T_pM & T_pM^* are very different.

INTRODUCE WEDGE PRODUCT

$$dx^\mu \wedge dx^\nu = \underset{\substack{\uparrow \\ 1^{st}\ ARG}}{dx^\mu} \otimes \underset{\substack{\uparrow \\ 2^{ND}\ ARG}}{dx^\nu} - dx^\nu \otimes dx^\mu$$

eg. $\omega = \frac{1}{2!} \omega_{\mu\nu} \, dx^\mu \wedge dx^\nu$

$$= \tfrac{1}{2} \omega_{xy} \, dx \wedge dy + \tfrac{1}{2} \omega_{yx} \, dy \wedge dx$$

$$= \tfrac{1}{2} (\omega_{xy} - \omega_{yx}) \, dx \wedge dy$$

$$= \boxed{\omega_{xy} \, dx \wedge dy}$$

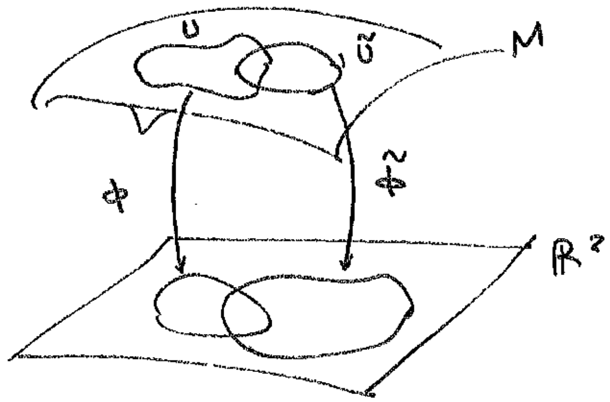
takes 2 vectors
& spits out antisym prod. of
components.

$$dx \wedge dy (\underline{V}, \underline{W}) = V^1 W^2 - V^2 W^1$$

} gives AREA (ORIENTED)

this generalizes: 3-form is a volume.
↳ makes sense that this is what
is born to be integrated.

DIFFERENTIAL GEOMETRY



MANIFOLD:
LOCALLY (PATCHWISE)
DIFFEOMORPHIC w/ $\mathbb{R}^n$

s.t. MAPS $\phi, \tilde{\phi}$ AGREE
ON OVERLAP $U \cap \tilde{U}$

why: GIVES POWER TO DESCRIBE
THINGS THAT CANNOT BE
DESCRIBED BY ONE PATCH!

eg S^2

CALCULUS :

$$\int_V \overbrace{d\omega}^{(k+1)\text{ form}} = \int_{\partial V} \omega$$

(k+1) dim manif — k form (primitive)

BOUNDARY of V

(1D) $\int_c df = \int_{\partial c} f = f(P_1) - f(P_0)$

(2D) $\omega = A_i dx^i$ over $i = 1,2,3$

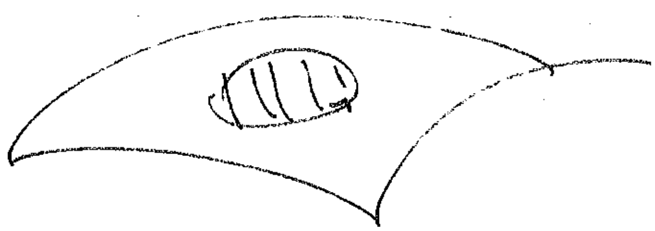
$$d\omega = \partial_x A_x \, \cancel{dx \wedge dx} + \partial_y A_x \, dy \wedge dx + \partial_z A_x \, dz \wedge dx$$
$$+ \partial_x A_y \, dx \wedge dy + \partial_y A_y \, \cancel{dy \wedge dy} + \partial_z A_y \, dz \wedge dy$$
$$+ \partial_x A_z \, dx \wedge dz + \partial_y A_z \, dy \wedge dz + \partial_z A_z \, \cancel{dz \wedge dz}$$

$$= (\partial_x A_y - \partial_y A_x) \, dx \wedge dy$$
$$+ (\partial_y A_z - \partial_z A_y) \, dy \wedge dz$$
$$+ (\partial_z A_x - \partial_x A_z) \, dz \wedge dx$$

DIFFERENTIAL AREAS

familiar coefficients : $\vec{\nabla} \times \underline{A}$

$$\int_{AREA} d\omega = \int_{AREA} \vec{\nabla} \times \underline{A} \, dA = \oint_{PERIM} A_i \, dx^i = \oint \vec{A} \cdot d\ell$$



<u>other avatars of vector calculus</u>

$\boxed{d^2 = 0} \longleftrightarrow$ boundary of a boundary $= 0$

e.g. if $\omega =$ 0-form

$d\omega$ is 1-form $\quad \partial_x f\, dx + \cdots = \nabla f \cdot d\vec{x}$

$d^2\omega$ is 2-form $\quad (\underbrace{\nabla \times (\vec{\nabla} f)}_{\vec{\nabla}\times\vec{\nabla} = 0 \text{ (curl} \circ \text{grad)}})_x \, dy \wedge dz \cdots$

eg if $\omega =$ 1 form $: f_x\, dx + \cdots$

$d\omega$ is 2 form $\quad (\vec{\nabla} \times \vec{f})_z\, dx \wedge dy + \cdots$

$d^2\omega$ is 3 form $\quad \underbrace{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{f})}_{\vec{\nabla}\cdot\vec{\nabla}\times = 0 \text{ (div} \circ \text{curl)}} \, dx \wedge dy \wedge dz$

<u>Jargon</u>: if $\omega = dA$, ω is <u>EXACT</u>

physics: ω comes from <u>POTENTIAL</u> A

if $d\omega = 0$, ω is <u>CLOSED</u>

<u>Poincaré lemma</u>: EXACT $\Rightarrow$ CLOSED obviously

but closed $\rightarrow$ EXACT
for nice, contractible spaces

POTENTIAL THY : E&M → $\boxed{\text{MAXWELL}}$

$F_{\mu\nu}$ ← MAXWELL FIELD STRENGTH

EASY MAXWELL EQS:

$$\left.\begin{aligned} \nabla\times \underline{E} + \dot{\underline{B}} &= 0 \\ \nabla\cdot \underline{B} &= 0 \end{aligned}\right] \longleftrightarrow \boxed{dF = 0}$$

Poincaré : [for nice spacetime] ⇒ $\boxed{F = dA}$

A_μ : GAUGE POT.

(OR:) the moment we SAY $F = dA$ → ½ of MAXWELL RELATIONS ARE TRIVIAL

(2b) ⇒ A_μ is convenient DESCRIPTION
→ it is LORENTZ COVARIANT.

Rest of MAXWELL: $\boxed{\partial_\mu F^{\mu\nu} = 4\pi J^\mu}$

comes from VARIATION of ACTION.

GAUGE REDUNDANCY

$A \to A + d\alpha$ leaves $F \to dA + d^2\alpha = dA$ invariant.

a $\mathbb{R}$ # @ every point in spacetime

FIBER BUNDLE

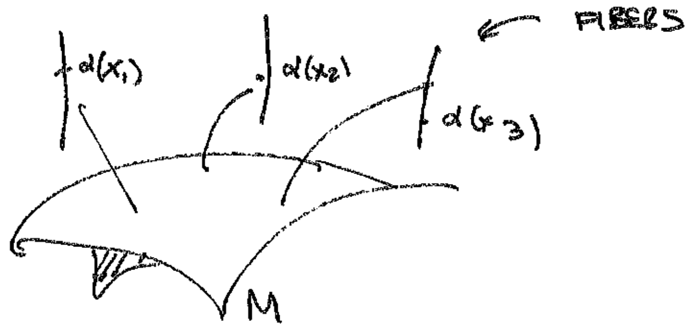
the GAUGE choice $\alpha(x)$ is a true REDUNDANCY of the theory.

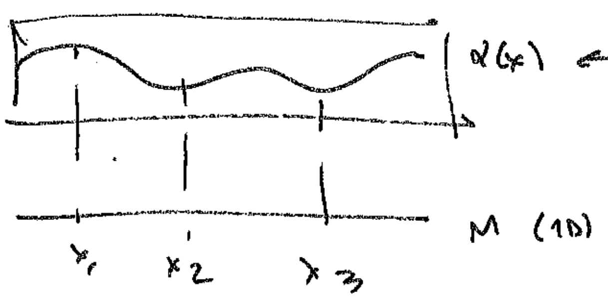
↳ when you quantize, YOUR PATH INTEGRAL GOES CRAZY IF YOU DON'T MOD OUT BY THIS!!

(get fake stationary points)

the α is AN S^1 FIBER OVER SPACETIME.

↑ $\alpha(x)$ is the PHASE @ $x \in M$

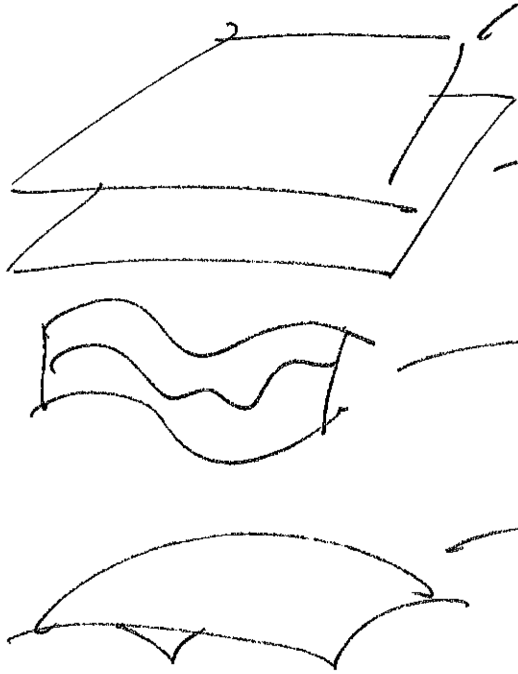




To "DO CALCULUS" ($ E&M is all about this)
then need to be able to compare fibers @ different spacetime points to account for this "MODDING OUT" by $\alpha(x)$.

↳ leads to COVARIANT DERIVATIVE in the GR-sense.

BUT ACTING W/RT "REPS" OF $\boxed{U(1)}$ phase sym.



SPACETIME DEP.

$$\psi \to e^{ieq_\psi \theta(x)} \psi$$ UNDER GAUGE TRANSF

$$A \to A - \partial\theta(x)$$

INVARIANCE WRT GAUGE/LOCAL TRANS:

$$\partial_\mu \xmapsto[\text{PROMOTE}]{} \partial_\mu - i e q_\psi A_\mu \equiv D_\mu$$

$$D_\mu \psi = \partial\psi - ieqA\psi$$

$$e^{iq\theta}\partial\psi \underline{+ iq\partial\theta\, e^{iq\theta}\psi} \qquad -ieq(A - \underline{\partial\theta})e^{iq\theta}\psi$$

cancel

$$\to \boxed{e^{iq\theta} D_\mu \psi}$$ transforms COVARIANTLY UNDER U(1)