Homework 1: **Representations of SU(2) and SU(3)**

Course: Physics 262, *Group Theory for Physicists* (Fall 2019)
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Due by: Be ready to discuss on Friday, Feb 8

1 The spin-1 representation of SU(2)

The spin-1 representation of SU(2) is the three dimensional representation with highest weight $j = 1$ and states $|1\rangle$, $|0\rangle$, and $|-1\rangle$. A vector in this space is:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\,|1\rangle + b\,|0\rangle + c\,|-1\rangle \ . \tag{1.1}$$

1.1 Normalizations

We chose a normalization of our states so that:

$$T^- |m\rangle = N_m\,|m-1\rangle \tag{1.2}$$
$$T^+ |m-1\rangle = N_m\,|m\rangle \ , \tag{1.3}$$

where $|n\rangle$ is orthonormal. Write out N_m for each of the weights $m = \pm 1, 0$.

1.2 Raising, lowering, Cartan

Write out the explicit form of the generators in this representation: $d(T^+)$, $d(T^-)$, $d(T^3)$.

1.3 Generators in the 'usual' basis

Write out the explicit form of the generators in this representation: $d(T^1)$, $d(T^2)$, $d(T^3)$. Recall that:

$$d(T^\pm) = d(T^1) \pm i d(T^2) \ . \tag{1.4}$$

2 The adjoint representation of SU(2)

The **adjoint** representation is one where the generators themselves are states. The action of a generator (as a matrix) on a generator (as a state) is given by the **structure constant**, f^{abc}. Recall that the structure constant is defined by

$$[T^a, T^b] = i f^{abc} T^c \ . \tag{2.1}$$

For SU(2) $f^{abc} = \varepsilon^{abc}$, the totally antisymmetric tensor with $\varepsilon 123 = 1$. The matrices of the adjoint representation are:

$$\mathrm{ad}(T^a)^{bc} = f^{abc} \ . \tag{2.2}$$

In other words, the action of a generator in the adjoint representation $\mathrm{ad}(T^a)$ on a state $\left|T^b\right\rangle$ is

$$\mathrm{ad}(T^a)\left|T^b\right\rangle = f^{abc}\left|T^c\right\rangle = \left|-i[T^a, T^b]\right\rangle \ . \tag{2.3}$$

2.1 Dimension of the adjoint representation

What is the dimension of the adjoint representation? (How many basis states are there?) Answer: three. (You may want to write one sentence explaining why the answer is three. If the answer is more than one sentence, then it's probably wrong.)

2.2 Dimension of the adjoint representation

Write out the explicit matrix form of $\mathrm{ad}(T^3)$ and confirm that it matches $d(T^3)$ in the spin-1 representation. Write out the explicit matrix forms of $ad(T^1)$ and $\mathrm{ad}(T^2)$. Observe that this is *not* quite the same as $d(T^1)$ and $d(T^2)$ of the spin-1 repesentation.

2.3 What gives?

Confirm that $\mathrm{ad}(T^1)$, $\mathrm{ad}(T^2)$, and $\mathrm{ad}(T^3)$ satisfy the SU(2) commutation relations:

$$[\mathrm{ad}(T^a), \mathrm{ad}(T^b)] = \varepsilon^{abc}\mathrm{ad}(T^c) \ .$$

Confirm that if some set of matrices $d(T)$ are a representation—that is, they satisfy the algebra's commutation relations—then another set of matrices that differ by a unitary transformation, $\widetilde{d}(T) \equiv U d(T) U^\dagger$, is also a representation. Write down the matrix U that transforms the spin-1 representation into the adjoint representation. This proves that the spin-1 and adjoint representations are, in fact, the same. The difference between them is purely "cosmetic." Hint: https://physics.stackexchange.com/q/279880/166736. The difference has to do with the spherical basis of rotations.

3 SU(3) with different Cartan bases

3.1 A weird basis

Suppose we chose a weird basis for the Cartan subalgebra of SU(3):

$$H_1 = \frac{1}{2}\mathrm{diag}(0, 1, -1) \qquad\qquad H_2 = \frac{1}{2\sqrt{3}}\mathrm{diag}(-2, 1, 1) \ . \tag{3.1}$$

These generators have "'quantum numbers" (r, s):

$$H_1\left|r, s\right\rangle = r\left|r, s\right\rangle \qquad\qquad H_2\left|r, s\right\rangle = s\left|r, s\right\rangle \ . \tag{3.2}$$

For each set of raising and lowering operators, $X^\pm$, work out $[X^+, X^-]$, $[H_1, X^\pm]$, and $[H_2, X^\pm]$. Using what we know about SU(2), identify the conditions on $(r,s)_{\text{max}}$, the "highest weight" of the representation. Draw the weight diagram for the **fundamental representation** in (r,s) weight space. How is this relsssssated to the (p,q) weight space we introduced in Lecture 4?

3.2 A stupid basis

Here's a stupid basis for the Cartan subalgebra of SU(3). It's close to the basis we used in class, but improperly normalized.

$$H_1 = \frac{1}{2}\text{diag}(1,-1,) \qquad\qquad H_2 = \frac{1}{2}\text{diag}(1,1,-2)\ . \tag{3.3}$$

What are the consequences on the weight diagram? Draw the *correct* weight diagram for the fundamental representation (what we did in class) and overlay it with the *stupid* weight diagram from this improperly normalized set of Cartan generators.

3.3 Another stupid basis

Here's another stupid basis for the Cartan subalgebra of SU(3).

$$H_1 = \frac{1}{2}\text{diag}(1,-1,0) \qquad\qquad H_2 = \frac{1}{2\sqrt{3}}\text{diag}(-2,1,1)\ . \tag{3.4}$$

What's goes wrong with this basis? HINT: notice that these generators are not 'orthogonal' with respect to the Killing form, $\kappa(T^a, T^b) = \text{Tr}(T^a T^b)$. That is: $\kappa(H_1, H_2) \neq 0$. What implications does this have when drawing the weight diagram?

4 Properties of Algebras

Show that:

- If a matrix Lie group is defined to be **special** (unit determinant), then the algebra is made up of traceless matrices.
- If a matrix Lie group is defined to be **unitary**, then the algebra is made up of Hermitian matrices.

5 Adjoint representation of a group, algebra

Let G be a matrix Lie group. Let T be an element of its associated algebra. That is: we can write $g = \exp -i\theta^a T^a$ for some $g \in G$ sufficiently close to the origin. The adjoint representation of the *group* is the action of G on elements of its algebra:

$$\text{Ad}(g)\,|X\rangle = gXg^{-1}\ , \tag{5.1}$$

where $X = \theta^a T^a$ is an element of the algebra. Show that if $g = \exp -i\epsilon^a T^a$ for small angles ϵ^a, the adjoint action of the *group* reduces to the adjoint action of the *algebra*:

$$Ad(g)\,|X\rangle = \left(1 + c^a \text{ad}(T^a) + \mathcal{O}(\epsilon^2)\right) X \ . \tag{5.2}$$

Determine what c^a is. HINT: See Gutowski sections 2.14 - 2.26.

Extra Credit

These problems are for your own edification. You are encouraged to explore them according to your own personal and research interests. **Relevant**: `https://youtu.be/OobMRztklqU`.

BCH

Derive the Baker-Campbell-Hausdorff formula. Start by looking up what the Baker-Campbell-Hausdorff formula is. I don't have anything deep to say about this, but going through the derivation once gives you a feel for how to think of the group and algebra as a manifold and tangent space.