Homework 2: Tensor Stuff

Course: Physics 262, *Group Theory for Physicists* (Fall 2019)
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Due by: Be ready to discuss on Mon, Feb 22

1 Weight Diagram Example

Draw the SU(3) weight diagram for the SU(3) representation with highest weight $(3/2, \sqrt{3}/2)$. What is the dimension of this representation? It may be helpful to use the triangular graph paper available on the course website.

2 Tensor Representation

What is the decomposition of $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$ into SU(3) irreducible representations? Is there a combination that is invariant?

3 Interpretation: SU(3) Color Symmetry

3.1 The Δ^{++}

The Δ^{++} is a composite fermion that is a cousin of the proton. One puzzle from the early days of modern particle physics was why the Δ^{++} existed. We suspected that the Δ^{++} was composed of three identical fermions, each with charge 2/3 (we now know this as the up quark). However, identical fermion wavefunctions must be antisymmetrized. Thus the Δ^{++} could not be made up of identical fermions. Explain why this was indirect motivation for SU(3) color symmetry for which the up quark is in the fundamental representation. Hint: your explanation should make use of a SU(3) invariant tensor.

3.2 Not SU(2)

Argue that in order to account for the Δ^{++}, the new symmetry group could not have been SU(2). There are many ways to argue this, but for this problem, argue based on tensor representations of SU(2).

3.3 Not SU(4)

Argue that in order to account for the Δ^{++}, the new symmetry group could not have been SU(4). There are many ways to argue this, but for this problem, argue based on invariant tensors.

3.4 "Technigluons"

Suppose I have a new theory based on SU(N) symmetry. I have the following particles:

1. A techni-quark in the fundamental representation.
2. A techni-tensor with $(N-1)$ fundamental (upper) indices.
3. A techni-gluon in the adjoint representation.

Particles are allowed to interact if I can write down an invariant combination of their tensor product. Argue that the three particles above may interact with one another.

4 Adjoint vs. Fundamental–Anti-fundamental

We saw that the tensor product of a fundamental and an anti-fundamental of SU(2) decomposes as

$$\mathbf{2} \otimes \bar{\mathbf{2}} = \mathbf{3} \oplus \mathbf{1}.$$

One way to understand the $\mathbf{3}$ on the right-hand side is to see how it transforms as a spin-1 representation versus as a tensor product.

4.1 Tensor Product Transformation

Recall that SU(N) indices transform as follows:

$$\psi^i \to U^i{}_j \psi^j \qquad\qquad \chi_i \to \chi_j (U^\dagger)^j{}_i \ , \tag{4.1}$$

where U is a finite SU(N) transformation.

1. Write the infinitesimal transformation of $\Psi^i{}_j$ under an arbitrary, small SU(2) transformation.
2. Let $\mathbf{v} = (v^+, v^0, v^-)$ be a state in the $\mathbf{3}$ of the aforementioned tensor product. With respect to the fundamental and the antifundamental indices, $\mathbf{v}$ may be written
$$\mathbf{v} = v^+ (T^+)^i{}_j + v^0 (T^3)^i{}_j + v^- (T^3)^i{}_j \ . \tag{4.2}$$
. Write the components of $\mathbf{v}$ after performing an infinitesimal SU(2) transformation in the T^1 direction according to the transformation rule you wrote above.

4.2 Triplet Transformation

Alternatively, we may think of $\mathbf{v}$ as an object with only a triplet index:

$$\mathbf{v} = v^+ |m=1\rangle + v^0 |m=0\rangle + v^- |m=-1\rangle \ . \tag{4.3}$$

1. Write the infinitesimal transformation of v^a under an arbitrary, small SU(2) transformation, where a is a triplet (spin-1) index.
2. Write the components of $\mathbf{v}$ after performing an infinitesimal SU(2) transformation in the T^1 direction according to the transformation rule you wrote above.

Discussion: In this way, the tensor T is used to convert between different kinds of indices. Observe that the transformation on the components in $\mathbf{v}$ are the same whichever picture you use.