P262 W2019

LEC 8 : tensor reps 30 Jan

TYING UP LOOSE ENDS

"Real" Reps

A GOOD DISCUSSION: Zee (Group Thy Nut.) 11.4

questions: why do you call it $\mathbb{R}$ if there are i's in the matrices ?!

How is adjoint real...
$\bar{d}(T) = -d(T)^*$
Cartan $\to$ $-$ Cartan
... seems IMAGINARY, if anything!!

OBSERVE: SIMILARITY TRANSFORMS OF A REP ARE EQUIVALENT REPS

if $d(T)$ is an $N\times N$ rep,
$U d(T) U^{-1}$ satisfies same algebra

$$\left(d(T^a)\ d(T^b) - d(T^b)\ d(T^a)\right) = f^{abc}\ d(T^c)$$

insert: U $\quad \mathbb{1} = U^\dagger U$ $\quad U^{-1}$ $\quad U$ $\quad U^{-1}$

the point: even if $\bar{d}(T) \neq d(T)$,
it is real if it is EQUIVALENT to $d(T)$

$\hookrightarrow \exists\ U$ s.t. $\boxed{\bar{d}(T) = U d(T) U^{-1}}$

hw for the adjoint rep: $U = \cancel{\ }$?

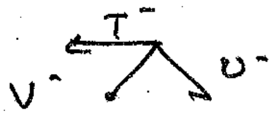
REMARK: ALSO A SENSE OF PSEUDOREAL
we will not say more on this
→ see Zee 11.4, Georgi 21.2
RELATED TO HIGGS YUKAWA couplings $HQd + \tilde{H}Qu$

for $\mathbb{R}$ rep, $\exists$ similarity transform that makes elements real. → Zee 11.4

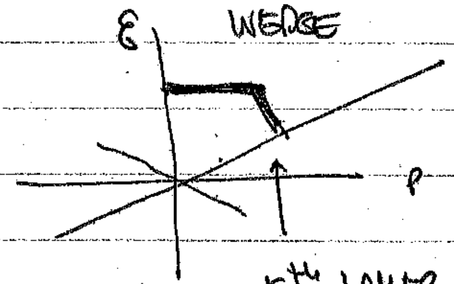
OTHER UNSATISFYING STATEMENTS FROM LAST TIME:
Weights of Hexagonal Diagrams
(GEORGI 10.11 ... Uses root system)

Gutowski 4.1.3

$[T^-, U^-] = -V^-$



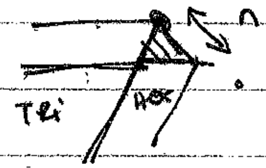
Steps



k^{th} LAYER from MAX

Limiting lowering op: $(U^-)^{n+1}|MAX\rangle$

? n^{th} layer is tri



$k < n+1$

1. in k^{th} HEX LAYER: MULT $\leq k$
   a) SHOW FOR TOP EDGE
      i) $|\text{top edge}, k^{th}\text{ layer}\rangle = \Pi(T^-, U^-)|MAX\rangle$
      ii) commute (U^-)'s to RIGHT, pick up V^-'s

$$|v_k\rangle = \underbrace{\overbrace{(V^-)^{\#} (T^-)^{\#}}^{\text{commute}} (U^-)^{\#}}_{k \text{ of these}} |MAX\rangle$$

   b) SHOW FOR SHORT EDGE.

2. Refl. sym $\Rightarrow$ k^{th} HEX LAYER HAS MULT $= k$
3. also true for 1^{st} TRIANGULAR LAYER

4. inside triangle: mult $\leq n$

same idea: $\Pi(T^-, U^-)|\text{MAX}\rangle$

commute U^-'s to RIGHT

$$\underbrace{(V^-)^\#(T^-)^\#}_{\text{commute (ORDER DOESN'T MATTER)}}\underbrace{(U^-)^\#\ |\text{MAX}\rangle}$$

BUT NOW $(U^-)^{n+1}|\text{MAX}\rangle = 0$

KILLS ADDITIONAL STATES

$\Rightarrow$ only $\leq (n+1)$ states inside tri.

5. ARGUE LIN INDEP OF THOSE $(n+1)$ STATES

I think the intuition is clear.

(carries over from triangle diagrams

... just replace $|\text{MAX}\rangle$ w/ $(U^-)^n|\text{MAX}\rangle$.

The details are not illuminating for our goals

ANOTHER Q: HOW is $\underline{3}$ of $SU(3)$ DIFF FROM $\underline{3}$ of $SU(2)$?

IR VS. R. EVEN JUST CARTAN: DIFF REPRESENTATIONS.

Tensor Product Reps

"Addition of Angular Momentum"

GUTOWSKI P.55 § 3.4

REPS d_1 & d_2 of $SU(2)$... combine them.

$$\begin{pmatrix} \cdot \\ \cdot \\ \vdots \end{pmatrix} \begin{matrix} \leftarrow \text{spin } j_1 \\ \leftarrow \text{spin } j_1 - 1 \end{matrix} \qquad \begin{pmatrix} \cdot \\ \cdot \\ \vdots \end{pmatrix} \begin{matrix} \leftarrow \text{spin } j_2 \\ \leftarrow \text{spin } j_2 - 1 \end{matrix}$$

how many states?
$(2j_1+1) \times (2j_2+1)$

act w $d_1(T)$

or $|\downarrow\rangle \otimes |\uparrow\rangle$

act w $d_2(T)$

eg. $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$

composite of 2 states

SU(2) GENERATORS IN COMBINED REP:

$$d(T^a) = d_1(T^a) \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes d_2(T^a)$$

do nothing on second — nothing on 1st

→ nb: this is like chain rule for DERIVATIVES... B/C THIS IS the TANGENT SPACE.

$d(T^a)\,|m\rangle \otimes |n\rangle$

$$= \left(d_1(T^a)|m\rangle\right) \otimes |n\rangle + |m\rangle \otimes \left(d_2(T^a)|n\rangle\right)$$

Q. WHAT IS THE <u>WEIGHT</u> of $|m\rangle \otimes |n\rangle$?

$$d(T^3)\,|m\rangle \otimes |n\rangle = d_1(T^3)|m\rangle \otimes |n\rangle + |m\rangle \otimes d_2(T^3)|n\rangle$$
$$= \underline{(m+n)}\,|m\rangle \otimes |n\rangle$$

WEIGHTS <u>ADD</u>

Q. WHAT IS WEIGHT OF $d(T^-)\,|m\rangle \otimes |n\rangle$?

$$d(T^-)|m\rangle \otimes |n\rangle = d_1(T^-)|m\rangle \otimes |n\rangle + |m\rangle \otimes d_2(T^-)|n\rangle$$

$$= |m-1\rangle \otimes |n\rangle + |m\rangle \otimes |n-1\rangle$$

weight is $(m+n-1)$
state is now a linear comb
... what about orthog comb?

what if you annihilate one
of the products?

LET US CONSIDER $\boxed{\text{SPIN}-1 \otimes \text{SPIN}-\tfrac{1}{2}}$

what do we want? <u>REPRESENTATION</u>
(when do we want it? now!)

I mean: BASIS OF STATES THAT ROTATE INTO EACH OTHER.

WE HAVE THE ALGORITHM: highest weight. lowering op.

HIGHEST WEIGHT:

I DON'T WANT TO THINK ABOUT NORMS

$$|3/2\rangle = |1\rangle \otimes |\tfrac{1}{2}\rangle$$

$$d(T^-)\,|3/2\rangle = d_1(T^-)|1\rangle \otimes |\tfrac{1}{2}\rangle + |1\rangle \otimes d_1(T^-)\,|\tfrac{1}{2}\rangle$$

$\rangle\!\rangle$ means: not normalized

$$|1/2\rangle\!\rangle \equiv |0\rangle\!\rangle \otimes |\tfrac{1}{2}\rangle\!\rangle + |1\rangle\!\rangle \otimes |-\tfrac{1}{2}\rangle\!\rangle$$

$$|-1/2\rangle\!\rangle = |-1\rangle\!\rangle \otimes |\tfrac{1}{2}\rangle\!\rangle + |0\rangle\!\rangle \otimes |-1/2\rangle + |0\rangle\!\rangle \otimes |-1/2\rangle\!\rangle + 0$$

$\{ d_2(T^-)|-\tfrac{1}{2}\rangle = 0$ (fact!)

is this the end? <u>no</u>. state is not annihilated.

HW

$$|-3/2\rangle\!\rangle = |-1\rangle\!\rangle \otimes |-1/2\rangle\!\rangle$$

normalizations work out.

so: we have a $\boxed{j = 3/2}$ REP!

we have 4 states that rotate into each other.

IS THERE A WAY TO ROTATE OUT OF THESE 4 STATES? NO.
it's a bona fide rep.

we should start calling it an <u>IRREDUCIBLE</u> REP

What's missing?
spin-3/2 rep we found has 4 states

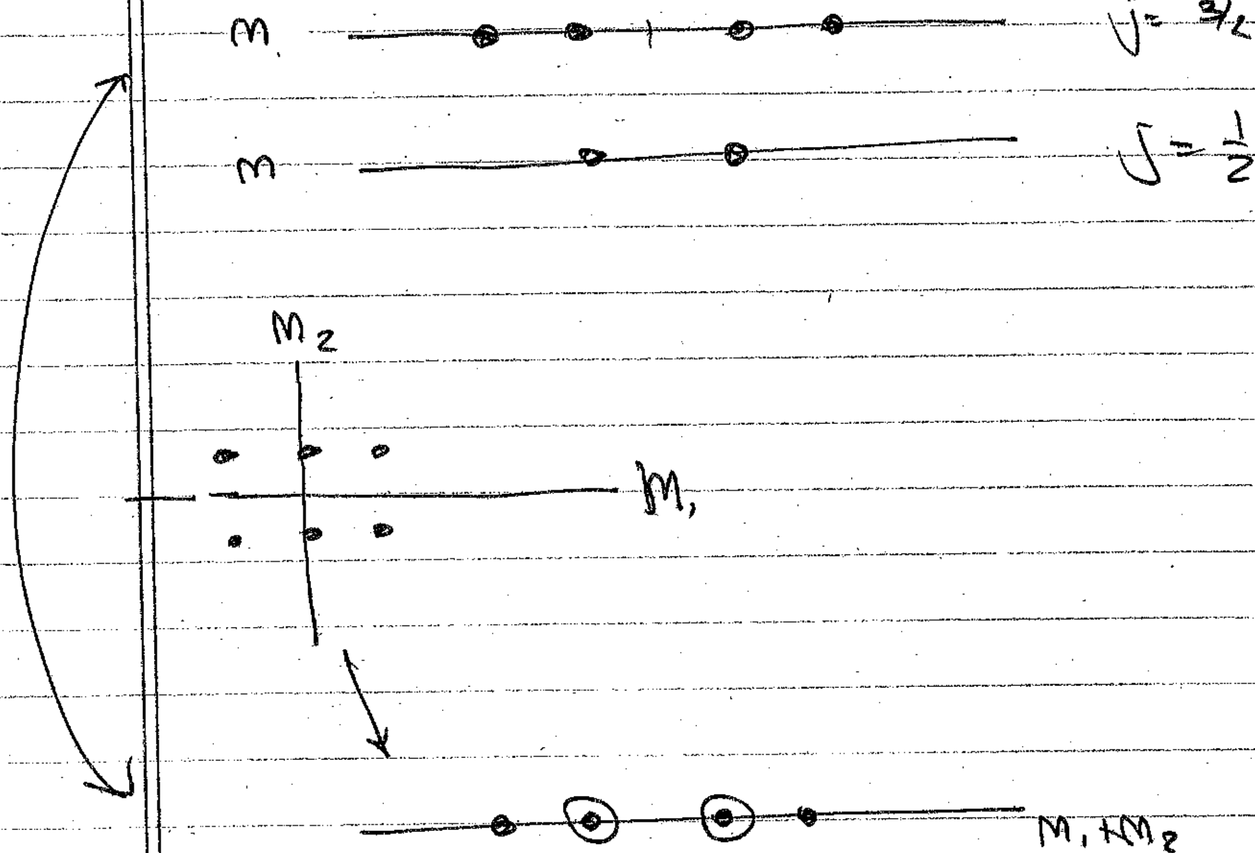
but spin 1 ⊗ spin ½ has $3 \times 2 = 6$!

can see what is missing:

spin ½ REP.
$$\begin{cases} |\tfrac{1}{2}\rangle'' \sim |0\rangle \otimes |\tfrac{1}{2}\rangle - |1\rangle \otimes |-\tfrac{1}{2}\rangle \\ |-\tfrac{1}{2}\rangle'' \sim |-1\rangle \otimes |\tfrac{1}{2}\rangle - |0\rangle \otimes |-\tfrac{1}{2}\rangle \end{cases}$$

CAN I LOWER AGAIN?
no: will annihilate

now we have two ladders



took highest wt: extracted ladder. then took next highest wt not already in ladder, and made a ladder out of that.

↳ will be a full irred. rep.

$\underline{3} \otimes \underline{2} = \underline{4} \oplus \underline{2}$

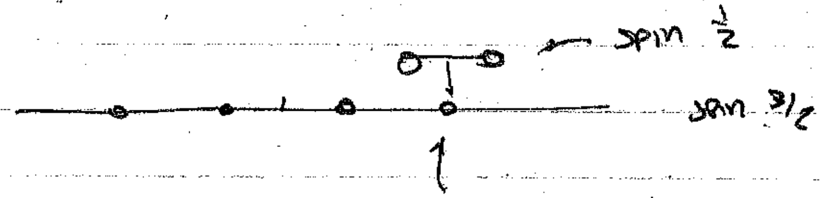
spin 1   spin-$\frac{1}{2}$

never mix into each other.

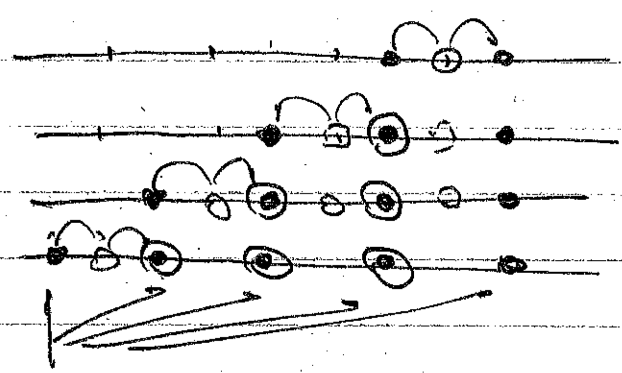
HW

YOU CAN WORK OUT ALL THE NORMALIZATIONS
↳ related to Clebsch-Gordan coeff.

LETS TRY SPIN-$\frac{3}{2}$ $\otimes$ spin-$\frac{1}{2}$



on each one, overlay
a spin-$\frac{1}{2}$



8 states

can see the ladders: spin 2 ($\underline{5}$)
spin 1 ($\underline{3}$)

$\underline{4} \otimes \underline{2} = \underline{5} \oplus \underline{3}$

<u>one more</u> : spin $\frac{1}{2}$ $\otimes$ spin $\frac{1}{2}$

$\underline{2} \times \underline{2} = \underline{3} \oplus \underline{1}$

VERY SPECIAL

the singlet rep does not transform.

"there is a combination of 2 spin-$\frac{1}{2}$ objects that is <u>INVARIANT</u>"

vs. <u>CO</u> VARIAN.

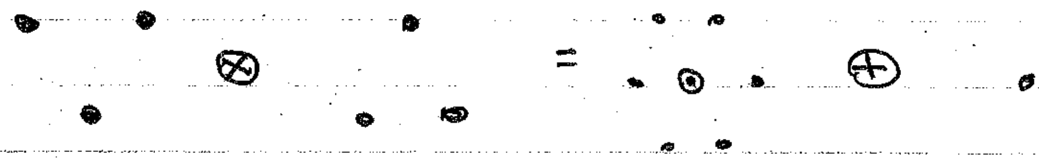
there is also a combo that is <u>not</u> invariant... I care less about that.

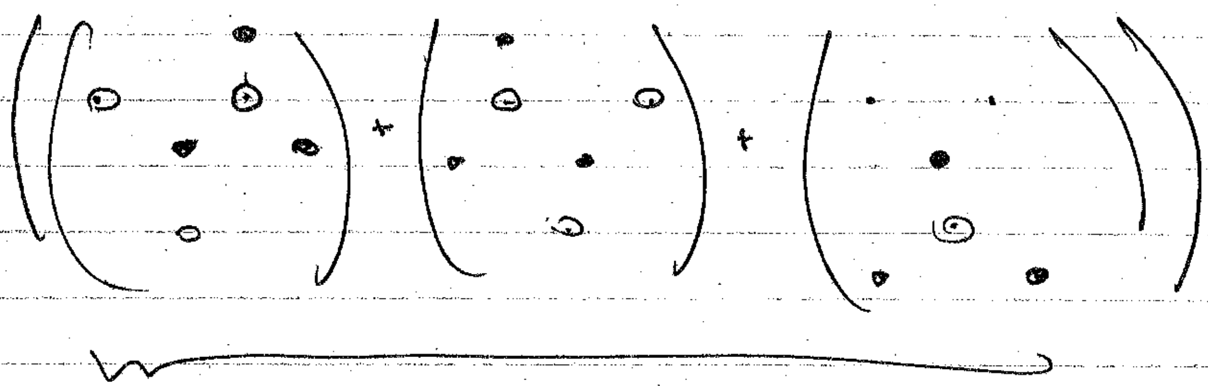
INVARIANT $\longrightarrow$ $\boxed{\text{SYMMETRIC}}$

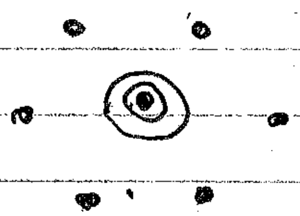
I can write this in a Hamiltonian & the hamiltonian will be SU(2) invariant!

THE SAME THING HOLDS for SU(3)

$$\underline{3} \otimes \bar{\underline{3}} = \underline{8} \oplus \underline{1}$$







$$= \underline{8} \oplus \underline{1}$$

the $\underline{1}$: $q = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $\begin{array}{l} \leftarrow R \text{ eg } |\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle \\ \leftarrow G \text{ eg } |-\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle \\ \leftarrow B \text{ eg } |0, \frac{1}{\sqrt{3}}\rangle \end{array}$

$$q \to Uq \quad \leftarrow U = e^{-i\theta^a d(T^a)}$$

claim: $\bar{q} \to \bar{q}\, U^{+}$ s.t. $\bar{q}q \to \bar{q}q$

$\uparrow$ as a ROW VECTOR

<u>INDICES</u> start w/ all indices upper.

$$q^i \to U^{ij} q^j$$

$$U^{ij} = \exp\left(-i\theta^a\, d(T^a)^{ij}\right)$$

$$\bar{q}^i \to \bar{U}^{ij}\, \bar{q}^j$$

$$\exp\left(-i\theta^a\, \bar{d}(T^a)^{ij}\right)$$
$$= \exp\left(i\theta^a\, d(T^a)^{ij*}\right)$$

$$= \exp\left(i\theta^a\, d(T^a)^{ji}\right) \equiv (U^{-1})^{ji}$$

order doesn't matter, but this is ordinary MATRIX MULT.

$$\bar{q}^i \to \bar{q}^j (U^{-1})^{ji}$$

$$\bar{q}^i q^i \to \bar{q}^j \underbrace{(U^{-1})^{ji}\,(U)^{ik}}_{(U^{-1}U)^{jk} = \delta^{jk}} q^k = \bar{q}^k q^k \checkmark$$

So let's make up a notation

~~LOWER~~

UPPER INDEX: vector
transforms as $v^i \to (Uv)^i$

LOWER INDEX: dual vector
transforms as $w_j \to (U^{-1}w)_j$

another useful rule:
upper & lower indices automatically contract $\Big\{$ vec & dual vec are linear ops on each other.

$v^i w_i \equiv \sum_i v^i w_i$

$\Rightarrow$ U has indices $U^i{}_j$
U^{-1} has indices $(U^{-1})^i{}_j$ $\leftarrow$ st $U^{-1}U = \mathbb{1}^i{}_j$
$\hookrightarrow$ but $w_i \to w_j U^{-1\,j}{}_i$