Lec 11 : Roots & Stuff

8 FEB 2019

HW REVIEW

1. from SU(2) notes :

$$T^- |m\rangle = N_m |m-1\rangle$$
$$T^+ |m+1\rangle = N_m |m\rangle$$

$$N_m = \sqrt{(j-m+1)(j+m)}$$

for spin-1 : $j = 1$

$N_m = \sqrt{2}$ for $m = 1, 0$
$N_{-1} = 0$ (bottom of ladder)

BASIS: $|1\rangle, |0\rangle, |-1\rangle$

$$d(T^-) \begin{pmatrix} v^+ \\ v^0 \\ v^- \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2}\, v^+ \\ \sqrt{2}\, v^0 \end{pmatrix}$$

$$d(T^+) \begin{pmatrix} v^+ \\ v^0 \\ v^- \end{pmatrix} = \begin{pmatrix} \sqrt{2}\, v^0 \\ \sqrt{2}\, v^- \\ 0 \end{pmatrix}$$

$$d(T^3) \begin{pmatrix} v^+ \\ v^0 \\ v^- \end{pmatrix} = \begin{pmatrix} v^+ \\ 0 \\ -v^- \end{pmatrix}$$

$$d(T^-) = \begin{pmatrix} 0 & & \\ \sqrt{2} & 0 & \\ & \sqrt{2} & 0 \end{pmatrix}$$

$$d(T^+) = \begin{pmatrix} 0 & \sqrt{2} & \\ & 0 & \sqrt{2} \\ & & 0 \end{pmatrix}$$

$$d(T^3) = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$d(T^1) = \frac{1}{\sqrt{2}} \begin{pmatrix} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{pmatrix}$$

$$T^{\pm} = T^1 \pm i T^2$$
$$T^1 = \frac{1}{2}(T^+ + T^-)$$
$$T^2 = \frac{1}{2i}(T^+ - T^-)$$

$$d(T^2) = \frac{-i}{\sqrt{2}} \begin{pmatrix} & +1 & \\ -1 & 0 & +1 \\ & -1 & \end{pmatrix}$$

2. ADJOINT : careful w/ matrix elements.

$$(\text{ad } T^a)^{bc} \equiv -i f^{abc}$$

$$\underbrace{|T^c\rangle\langle T^c|}_{\mathbb{1}}(\text{ad } T^a)|T^b\rangle = |T^c\rangle(-i)f^{acb}$$
$$= i f^{abc}|T^c\rangle$$
$$= |[T^a, T^b]\rangle$$

nb f^{abc} only for NORMALIZED HERMITIAN GENS

$$(\text{ad } T^+)\begin{pmatrix} v^+ \\ v^0 \\ v^- \end{pmatrix} = \begin{pmatrix} -v^0 \\ 2v^- \\ 0 \end{pmatrix}$$

$$(\text{ad } T^-)\begin{pmatrix} v^+ \\ v^0 \\ v^- \end{pmatrix} = \begin{pmatrix} 0 \\ -2v^+ \\ v^0 \end{pmatrix}$$

$$(\text{ad } T^3)\begin{pmatrix} v^+ \\ v^0 \\ v^- \end{pmatrix} = \begin{pmatrix} v^+ \\ 0 \\ -v^- \end{pmatrix}$$

USING: $[T^3, T^\pm] = \pm T^\pm$
$[T^+, T^-] = 2T^3$

$$(\mathrm{ad}\ T^+) = \begin{pmatrix} & -1 & \\ 0 & 0 & 2 \\ & 0 & \end{pmatrix}$$

$$(\mathrm{ad}\ T^-) = \begin{pmatrix} & 0 & \\ -2 & 0 & 0 \\ & 1 & \end{pmatrix}$$

$$(\mathrm{ad}\ T^3) = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$(\mathrm{ad}\ T^1) = \begin{pmatrix} & -1/2 & \\ -1 & 0 & 1 \\ & 1/2 & \end{pmatrix}$$

$$(\mathrm{ad}\ T^2) = \begin{pmatrix} & i/2 & \\ -i & 0 & -i \\ & i/2 & \end{pmatrix}$$

comm. relations work.

CLAIM: $\exists\, U$ s.t. $ad(T^a) = U\, d^{(1)}(T^a)\, U^{-1} \quad \forall a$

the useful thing to notice is

$$ad(T^3) = d^{(1)}(T^3)$$

$\Rightarrow$ so let's try diagonal U's

EASIEST TO WORK @ $d^{(1)}(T^\pm)$

$$\hookrightarrow U = \begin{pmatrix} -1/\sqrt{2} & & \\ & 1 & \\ & & 1/\sqrt{2} \end{pmatrix}$$

EASY TO CHECK (I guess via trial & error) in Mathematica.

SO WHAT: transformation of ADJOINT INDEX IS THE SAME AS SPIN-1 REP.

3. PROPERTIES of ALG FROM GROUP.

$\underline{\det M = 1 \rightarrow Tr(T) = 0}$

see lec 4.

$M = U\hat{M}U^{\dagger}$ for U unitary, $\hat{M}$ Diagonal

nb: $U^{-1} M U = \hat{M}$ ← DIAGONAL

$M = 1 - i\theta T + \cdots \qquad \hat{M} = 1 - i\theta\hat{T}$

so ORDER BY ORDER IN θ:

$U^{-1} T U = \hat{T}$

ie U also diagonalizes generators

$$\det M = \det \hat{M} = \det(1 - i\theta\hat{T} + \cdots) = (1 - i\theta t_1 \cdots)(1 - i\theta t_2 \cdots)\cdots$$

$$\underbrace{\frac{d}{d\theta} \det M\Big|_{\theta=0}}_{=0} = -i\sum_i t_i = -i\, Tr\,\hat{T}$$

$$\Rightarrow Tr\,\hat{T} = 0 \Leftrightarrow Tr(U\hat{T}U^{-1}) = \boxed{Tr(T) = 0}$$

EASIER $M = e^{-i\theta T}$

$$\frac{d}{d\theta}(M^\dagger M) = \frac{d}{d\theta}(\mathbb{1}) = 0$$

$$= \dot{M}^\dagger M + M^\dagger \dot{M}$$

$M = 1 - i\theta T \cdots$ $\quad \dot{M} = -iTM = -iMT$

$M^\dagger = 1 + i\theta T^\dagger \cdots$ $\quad \dot{M}^\dagger = iT^\dagger M^\dagger = iM^\dagger T^\dagger$

can eval @ $\theta = 0$

$$\Rightarrow \boxed{iT^\dagger - iT = 0}$$

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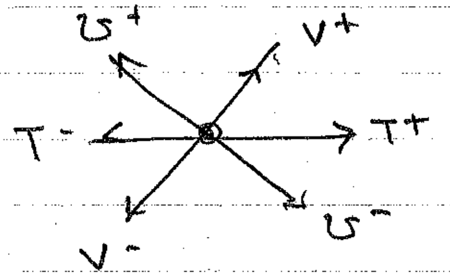
hw for next time

$\frac{1}{2} \otimes \frac{1}{2} = 3 \oplus 1$

the traceless part of $\frac{1}{2} \otimes \frac{1}{2}$
transforms like the adjoint

Roots & stuff



ROOT DIAGRAM:
all of the allowed "single moves"

can read off zero commutators

$\hookrightarrow [V^+, T^+] = ?$

if this were a root, then it needs to move the weight by

$\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right) \leftarrow$ no such root!

so it must be zero.

in contrast: $[V^+, U^-]$ would move a weight by $(1, 0)$
... that's precisely what T^+ does.

IRREPS: take highest available weight & lower in all allowed unique ways.

what makes a weight "high"?

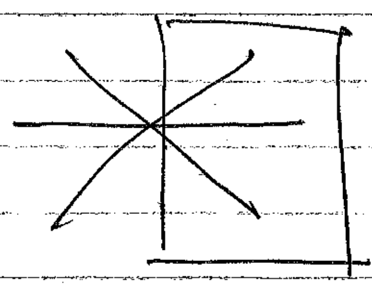
NEED DEF of POSITIVE

PICK: prioritize first component of root.

POSITIVE: FIRST non-zero element is >0

(so subsequent elements are tie breakers)

... yes: this totally depends on the arbitrary order we picked for the cartan basis!



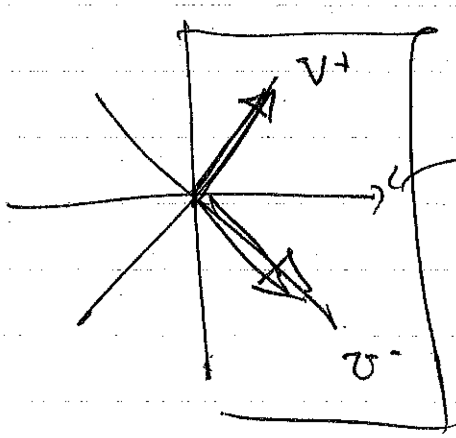
POSITIVE ROOTS.

to understand structure of lie group, want to understand the way in which there are "too many" roots given the rank.

↳ eg: 2D weight space
(RANK 2 ⟷ 2 elem in H)
BUT: 3 raising-lowering PAIRS!

want: subset of positive roots that could be a good basis. ("good" not yet well defined)

SIMPLE ROOT: subset of positive roots for which:
ANY POSITIVE ROOT can be written as a "positive sum" of SIMPLE roots.
non-neg. coeff.

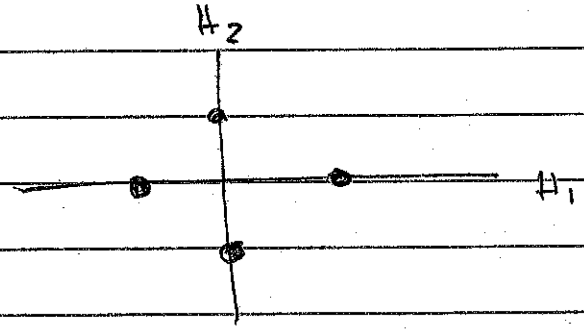


$\vec{T}^+ = \vec{V}^+ + \vec{v}^-$

Weights of the fundamental of $so(4)$

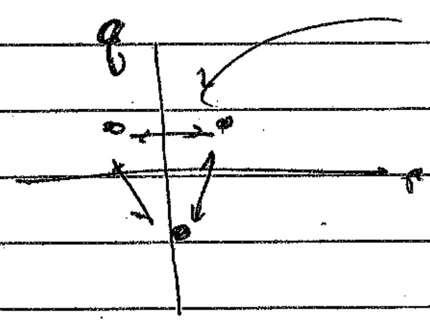
BASIS: $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ $\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

WEIGHT: $\begin{array}{c} H_1: \\ H_2: \end{array}$ $\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}-1\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix}$ $\begin{pmatrix}0\\-1\end{pmatrix}$



$\underline{4}$ of $so(4)$

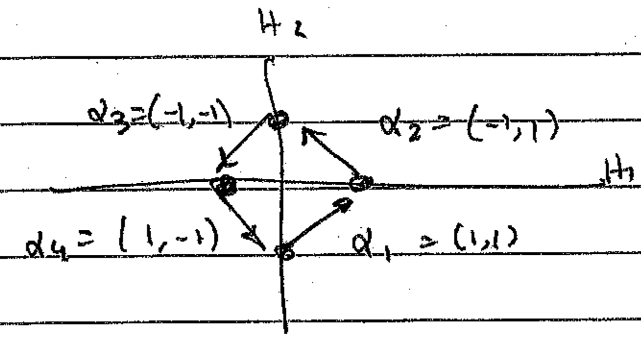
for $su(3)$:

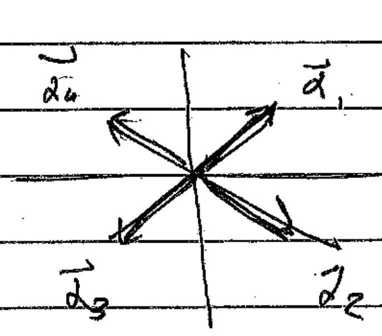


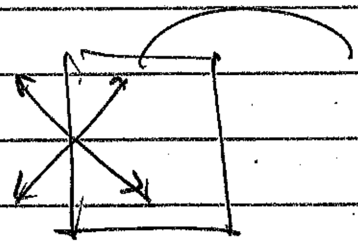
weights related by root vectors

ROOT DIAGRAM

$so(4)$







POSITIVE ROOTS

ALSO SIMPLE

SINCE $\vec{\alpha}_1+\vec{\alpha}_2$ not a root $\boxed{[T_{\alpha_1}, T_{\alpha_2}] = 0}$

eg. $SO(5)$

$$i\begin{pmatrix} 0 & \times & \times & \times & \bullet \\ & 0 & \times & \times & \bullet \\ & & 0 & \times & \bullet \\ & & & 0 & \bullet \\ & & & & 0 \end{pmatrix}$$

4 ADDITIONAL GENERATORS

what is RANK?

↳ no room for another rotation that leaves J_{12}, J_{34} alone.

$\boxed{\text{STILL RANK-2}}$ ← 2D WEIGHT SPACE

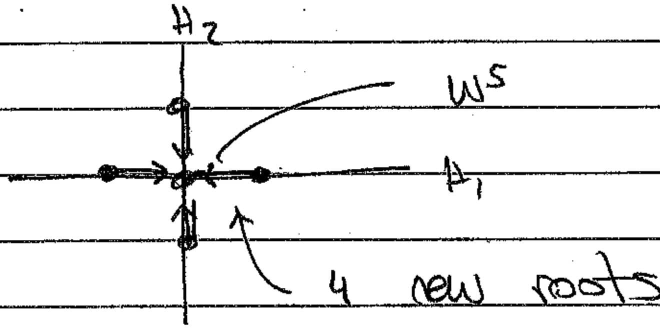
Weights in the fundamental:

NOW THERE'S A FIFTH STATE

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \leftarrow \begin{pmatrix} I \\ 0 \end{pmatrix}$$

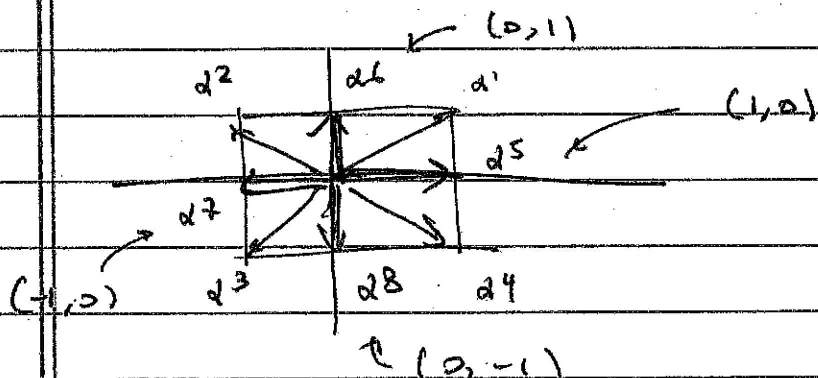
PREVIOUS ~~WEIGHTS~~ BASIS GET ADDITIONAL "0"

$$\begin{array}{c} H_1 \\ H_2 \end{array} \boxed{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} \leftarrow \text{origin of weight space}$$



HOW TO ROTATE OTHER BASIS ELEMENTS INTO NEW BASIS ELEMENT.

UPDATED ROOT DIAGRAM



new

POSITIVE ROOTS : $\alpha^1, \alpha^4, \boxed{\alpha^5, \alpha^6}$

observe: lengths of roots are different!

LONG & SHORT ROOTS.

~~all positive roots~~

SIMPLE ROOTS $\boxed{\alpha^4, \alpha^6}$