LEC 12 : structure of LIE ALG 11 FEB '19

see last few pages of LEC 11
→ WEIGHTS OF so(4)

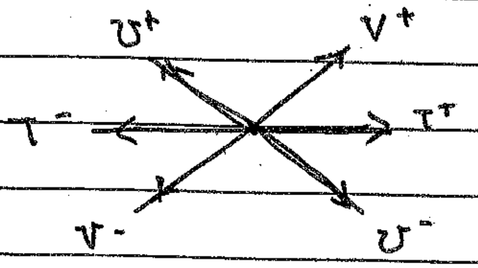
following : KEIR BLUM'S PARTICLES I tutorial (2008)
LIKE CLIFFS NOTES of GEORGI CH. 6, 7, 8

Generators

CARTAN : $\{H_i\}$ max set of commuting generators

ROOTS ← "weights of adjoint rep"
Eg. associated w/ RAISING/LOWERING op

call these $\{E_\alpha\}$
↳ ROOT VECTORS



$\alpha_V = (\frac{1}{2}, \sqrt{3}/2)$

$\alpha_T = (1, 0)$

$\alpha_U = (\frac{1}{2}, -\sqrt{3}/2)$

ROOTS

eg T^3, T^8 $\qquad$ $(0, 8)$

$$H_i |E_\alpha\rangle = \alpha_i |E_\alpha\rangle$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha$$

TAKE H.C. OF THIS:

$$(H_i E_\alpha - E_\alpha H_i)^\dagger = \alpha_i E_\alpha^\dagger$$

$$\shortparallel$$

$$-[H_i, E_\alpha^\dagger]$$

$$\Rightarrow \boxed{E_\alpha^\dagger = E_{-\alpha}}$$

raising & lowering are H.C. of each other.

... as we know

also: E_α not HERMITIAN

... as we know

definitely still a raising/lowering

$$H_i E_{\pm\alpha} |\mu\rangle = ([H_i, E_{\pm\alpha}] + E_{\pm\alpha} H_i) |\mu\rangle$$
$$= (\alpha_i \pm \mu_i) E_{\pm\alpha} |\mu\rangle$$

REDISCOVERING OUR SU(2) LADDERS:

$$\boxed{[E_\alpha, E_{-\alpha}] = \alpha \cdot H} \leftarrow \sum_i \alpha_i H_i$$

w/ appropriate normalization

RAISING & LOWERING

for each $\pm\alpha$ PAIR, we have an SU(2)

$$E^{\pm} = \frac{E_{\pm\alpha}}{|\alpha|} \qquad E^{3} = \frac{\alpha \cdot H}{\alpha^{2}}$$

gives $[E^3, E_\pm] = \pm E_\pm$
$[E_+, E_-] = E_3$ ← note: no factor of 2

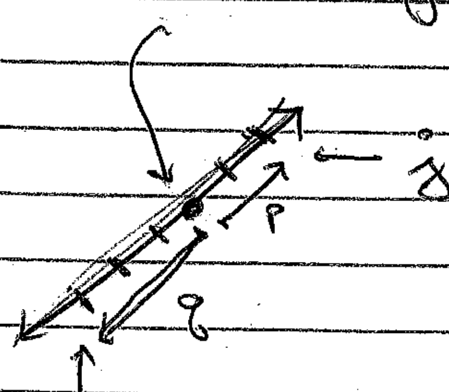
NOW CONSIDER A STATE $|\mu\rangle$
with weight vector $\vec{\mu}$

& consider some SU(2) AXIS, $\pm\alpha$

μ HAS WEIGHT ALONG THIS SU(2)

$$\boxed{\frac{\alpha \cdot \mu}{\alpha^{2}}} \leftarrow \text{from } E^{3} = \frac{\alpha \cdot H}{\alpha^{2}}$$

m along the $\underline{\alpha}$ AXIS



j HIGHEST WEIGHT is $|\mu + p\alpha\rangle$

$-j$ LOWEST WEIGHT IS $|\mu - q\alpha\rangle$

THE NORMALIZATION $\longleftrightarrow$ HIGHEST / LOWEST WEIGHT of THE REP

we know how this all pans out
$\Rightarrow$ half integer highest weight

weight along α $\longrightarrow$ $m + p = j$

$m - q = -j$

$2m + p - q = 0$

$$\Rightarrow \boxed{m = \frac{q-p}{2}}$$

$$\boxed{\frac{\alpha \cdot \mu}{\alpha^2} = \frac{q-p}{2}}$$ "master formula"

$\forall$ rep

IN THE ADJOINT, the ROOTS ARE THE NONZERO WEIGHTS.

α's!

So $$\frac{\alpha \cdot \alpha'}{|\alpha|^2} = \frac{m}{2}$$

$$\frac{\alpha' \cdot \alpha}{|\alpha'|^2} = \frac{m'}{2}$$

integer.

$$\frac{(\alpha \cdot \alpha')^2}{|\alpha|^2 |\alpha'|^2} = \boxed{\cos^2\theta = \frac{mm'}{4}}$$

only finite options: $\cos^2\theta \leq 1 \Rightarrow mm' \leq 4$

nothing to do w/ each other. (su(4))

SU(3)

mm'	θ	
0	$90°$	
1	$60°$	or $120°$
2	$45°$	or 135
3	$30°$	or $150°$
4	$180°$	

ORIENTATION OF ROOTS

SIMPLE ROOT:

POSITIVE ROOT,
CANNOT BE WRITTEN AS [POS] SUM
OF OTHER POSITIVE ROOTS

① if α, β ARE SIMPLE ROOTS
$\longrightarrow \gamma = (\alpha - \beta)$ IS NOT A ROOT

OTHERWISE: $\alpha = \beta + \gamma$ OR
$\beta = \alpha \underbrace{- \gamma}_{\text{one of these is positive}}$

② $\Rightarrow \underbrace{E_{-\alpha}|E_\beta\rangle}_{\substack{\text{bottom of} \\ \alpha \text{ ladder}}} = \underbrace{E_{-\beta}|E_\alpha\rangle}_{\substack{\text{bottom of} \\ \beta \text{ ladder}}} = 0$

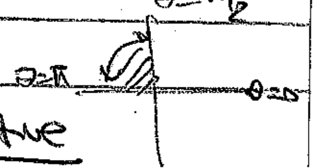
MASTER: $\boxed{\dfrac{\alpha \cdot \beta}{\alpha^2} = \dfrac{\cancel{q} - p}{2}}$ $\boxed{\dfrac{\beta \cdot \alpha}{\beta^2} = \dfrac{\cancel{q'} - p'}{2}}$

$\boxed{\dfrac{\beta^2}{\alpha^2} = \dfrac{p}{p'}}$ relative lengths of roots

$\theta = 90$
120
135
180

$\cos\theta = -\dfrac{p}{2} \dfrac{\alpha^2}{|\alpha||\beta|}$ $= |\alpha|/|\beta| = \sqrt{p'/p}$

$\boxed{\cos\theta = \dfrac{-\sqrt{pp'}}{2}}$ ← $\cos\theta$ IS negative



relative len & orientation

positive, "indep"

CLAIM: given the p's of the SIMPLE ROOTS, can construct whole algebra

eg. SU(3)

SIMPLE ROOTS: $\alpha^{(v)} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ ← v^+

$\alpha^{(u)} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ ← u^-

$$\boxed{\begin{array}{l} |\alpha^{(i)}| = 1 \\ \alpha^{(v)} \cdot \alpha^{(u)} = -\frac{1}{2} \end{array}} \longrightarrow \boxed{p^{(v)} = p^{(u)} = 1}$$

BUT p is just "how many steps to highest weight of this SU(2)?"

SO can RAISE $|E_{\alpha^{(v)}}\rangle$ BY $E_{\alpha^{(u)}}$ once & vice versa, GIVES A POSITIVE ROOT

OF COURSE THIS IS $E_{\alpha^{(t)}}$

t $\alpha^{(t)} = \alpha^{(v)} + \alpha^{(u)}$

ALL ROOTS: $\boxed{\pm\alpha^{(i)}}$ ← & now you know the whole algebra.

HW CASIMIRS

BUILDING IRREPS

HIGHEST WEIGHT $|\mu\rangle \longleftrightarrow \mu + \alpha^{(i)} \neq$ weight

$$\frac{\alpha^{(i)}\cdot\mu}{|\alpha^{(i)}|^2} = \frac{q_i - \cancel{p_i}}{2} \leftarrow \text{HIGHEST}$$

$\boxed{\text{SIMPLE ROOTS}}$

dynkin indices of rep

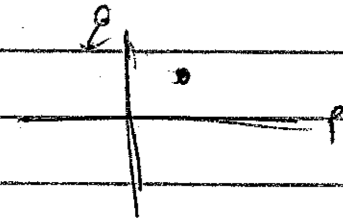
FUNDAMENTAL : $\mu^{(i)}$: $q_i = \delta_{ij}$

eg. $SU(3)$: $q = (1,0)$ or $(0,1)$

$$\frac{\alpha^{(1)}\cdot\mu^{(1)}}{|\alpha^{(1)}|^2} = \frac{1}{2} \qquad \frac{\alpha^{(2)}\cdot\mu^{(1)}}{|\alpha^{(2)}|^2} = 0$$

$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $\qquad |\alpha^{(1)}|^2 = 1$ $\qquad \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ $\qquad |\alpha^{(2)}|^2 = 1$

$$\left.\begin{aligned} \tfrac{1}{2}(\mu^{(1)})^1 + \tfrac{\sqrt{3}}{2}(\mu^{(1)})^2 &= 1/2 \\ \tfrac{1}{2}(\mu^{(1)})^1 - \tfrac{\sqrt{3}}{2}(\mu^{(1)})^2 &= 0 \end{aligned}\right\} \quad \mu^{(1)} = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$$

 $\longleftrightarrow \underline{3}$

SIMILARLY w/ $\bar{3}$

HW → $\underline{6}$ of SU(3)

HOW TO ENCODE THIS INFO:

Dynkin diagram

 ← SIMPLE ROOT

connect simple roots by lines

 90° (orthogonal) ← so(4)

 120° SU(3)

 135°

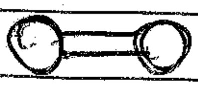
 150

ADDITIONAL RULES WHEN LENGTHS ARE DIFFERENT

eg  ← $T^{\pm}$ for SU(2)

 for SU(3)

$V^{\pm}$ $U^{\pm}$

 SO(5)

 SO(4)