P262 · $\boxed{\text{LEC 6}}$ SU(3) WEIGHT SPACE 1/25/19

Rem: ① HW CORRECTIONS (thanks Lexi)
② Gerardo: GEORGI CH. 2.4
(nb Georgi's T^a is our ad T^a)
→ IAN + GER. ARE RIGHT. KILLING FORM IS REALLY ABOUT ADJOINT
(prop of Algebra w/o "MAKING A NEW VEC SPACE")

$$\text{Georgi: } \underset{\text{DIFF. BASIS}}{\widetilde{ad}\, T^a} \equiv \underset{\text{LINEAR COMB}}{L^{ab}}\, ad\, T^b$$

$$\Rightarrow Tr(\widetilde{ad}\,T^a\ \widetilde{ad}\,T^b) = L^{ac} L^{bd}\, Tr(ad\,T^c\, ad\,T^d)$$

for SOME L

$$= K^a \delta_{ab} \quad (\text{no sum})$$

NORM OF T^a (or $\widetilde{ad}\,T^a$) GIVES MAGNITUDE of K^a
→ CAN NORMALIZE

$\boxed{\text{BUT SIGN IS INVT.}}$

$\boxed{\text{COMPACT} \leftrightarrow K > 0}$

tells you about the GROUP.

ANALOG: the easiest group: U(1) ← why not SU(1)?
generator: 1 (OR −1, or π, or −2.17 ...)
FINITE REPHASE: $e^{-i\theta\cdot 1}$
$K(1,1) = 1$ ← $K(-1,-1), K(\pi,\pi), \dots > 0$
VS: generator: $\boxed{i}$ (or $-i$, or $i\pi$, ...)
FINITE TRANSF: $e^{-i\theta\cdot i} = \boxed{e^{\theta}}$ ← RESCALING
→ non compact!! ← USE YOUR FAV. DEF.

we will study noncompact groups
↓
LORENTZ & POINCARE
... SHOULD WE FOCUS ON THIS!

Remark on HW

PROB 2: $\mathrm{ad}(T^a)^{bc} = (-i) f^{abc}$ $\qquad f^{abc}T^c = -i[T^a,T^b]$

$\mathrm{ad}(T^a)^{bc} = -i f^{abc} |T^c\rangle = -|[T^a,T^b]\rangle$

commutator
so WHAT?

PROB 4: it's kind of weird that a state transforms by a commutator.

BUT: comes from a very intuitive finite transformation law:

If X is a state (Lie Algebra valued)

$\quad X = x^a T^a$

DEFINE FINITE TRANSF

$$X \to e^{-i\theta^a T^a} X e^{i\theta^a T^a}$$

$$\overset{\theta^a \ll 1}{=} X + (\pm)\underbrace{[T, X]} + \mathcal{O}(\theta^2)$$

the infinitesimal transf is the ADJOINT.

SO WHAT: CAN WRITE INVARIANTS ... $H^\dagger \Phi H \mapsto H^\dagger U^\dagger U \Phi U^\dagger U H$

*BTW: USUALLY CARE MORE ABOUT INV. than COVARIANCE

ASK IAN.

LIGHTNING REVIEW AGAIN: SU(3)

ALGEBRA → 3 RAISE/LOW T, V, U

→ 2 CARTAN

$\boxed{\text{CARTAN}}$ $\mathrm{Tr}(H^2) = \frac{1}{2}$ ← very useful norm. (OTHERWISE STRETCH WEIGHT DIAG)

$$T^3 = \frac{1}{2}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \qquad T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

<u>SPECIAL COMBOS</u>:

$$T^3 \leftarrow J^z \text{ of } T$$
$$\tfrac{1}{2}T^3 + \tfrac{\sqrt{3}}{2}T^8 \leftarrow J^z \text{ of } V$$
$$-\tfrac{1}{2}T^3 + \tfrac{\sqrt{3}}{2}T^8 \leftarrow J^z \text{ of } U$$

s.t. for $X^\pm = \{T^\pm, V^\pm, U^\pm\}$

$$[X^+, X^-] = 2X^3 \leftarrow X^3 \text{ is } J^z \text{ of } X$$
$$[X^3, X^\pm] = \pm X^\pm$$

eg. $(\frac{1}{2}T^3 + \frac{\sqrt{3}}{2}T^8)$ for V

get this from $[T^3, X^\pm]$, $[T^8, X^\pm]$

other obs:

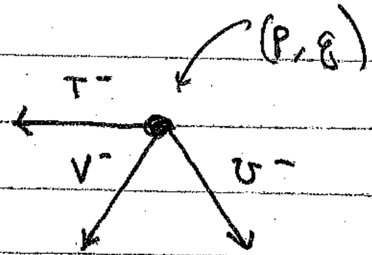
$$[T^\pm, V^\mp] = \mp U^\mp$$
$$[T^\pm, U^\pm] = \pm V^\pm$$
$$[V^\pm, U^\mp] = \pm T^\pm$$

typos in Lec 4!

for EACH SU(2): HEIGHEST/LOWEST WEIGHT
& INCREMENT BY 1 ALONG
THAT AXIS IN WEIGHT SPACE

(p, q) ← T^3, T^8 eigv.

$$T^{\pm} \rightarrow \pm(1,0)$$
$$V^{\pm} \rightarrow \pm\left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right)$$
$$U^{\pm} \rightarrow \pm\left(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right)$$

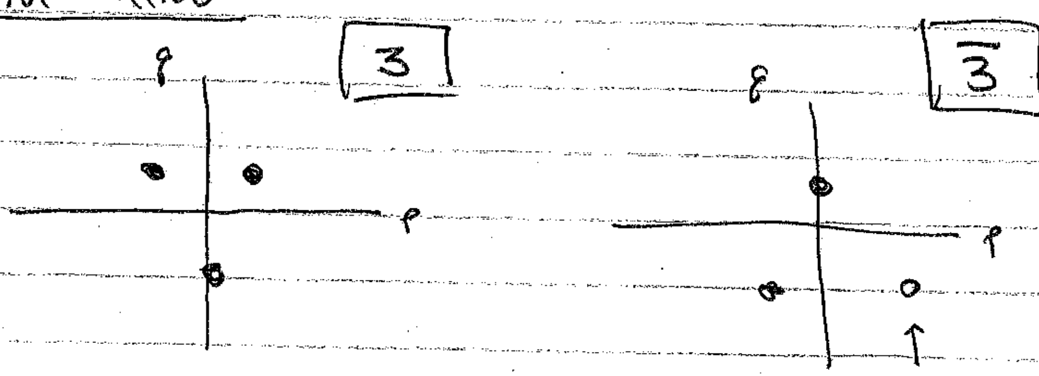


from normaliz
& SU(2) ladder.

USING "$2 j_X$ is a counting #" for $X = T, V, U$

$$(p, q)_{max} = \left(n_p/2, \; n_q/2\sqrt{3} \right)$$

LAST TIME:



BY THE WAY: THIS STATE HAS $p >$ "p_{max}"

SO WHAT: given "∵"
you know what each generator does to a given • = $|p,q\rangle$

EACH • IS A COMPONENT OF THE VECTOR THAT IS ROTATING IN THIS REPRESENTATION.

$$\begin{pmatrix} x \\ y \\ z \\ \vdots \end{pmatrix} = x\,|p_1,q_1\rangle + \cdots$$

$$d(T^a)\begin{pmatrix} x \\ y \\ z \\ \vdots \end{pmatrix} = x\,\boxed{T^a|p_1,q_1\rangle} + \cdots$$

WANT THIS MATRIX
(eg for $e^{-i\theta^a d(T^a)}$)

you know this.
T^a is some lin comb of $X^\pm, H_i$

So if you know what a lin. transf does to the basis, you know what it ~~does to~~ looks like as a matrix.

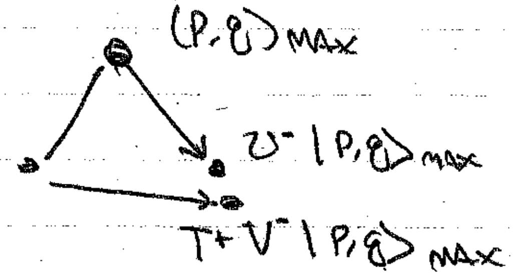
STRATEGY:
1. HIGHEST WEIGHT
2. LOWER in all possible ways
3. NORM. WILL CAUSE THIS TO TERMINATE

   ($j_x \geq m_x \geq -j_x$ for $x = T, U, V$

Questions

- what do more complicated reps look like?
- are all of these states unique?

  OR ARE THERE MULTIPLICITIES?



are these the same or different?!

CHECK:
$$T^+V^- = [T^+, V^-] + V^-T^+$$
$$= -U^- + V^-T^+$$

so
$$T^+V^-|MAX\rangle = \underbrace{-U^-|MAX\rangle}_{\text{SAME!}} + V^-\underbrace{T^+|MAX\rangle}_{=0}$$

I HAVE nothing DEEP TO SAY ABOUT THIS SIGN

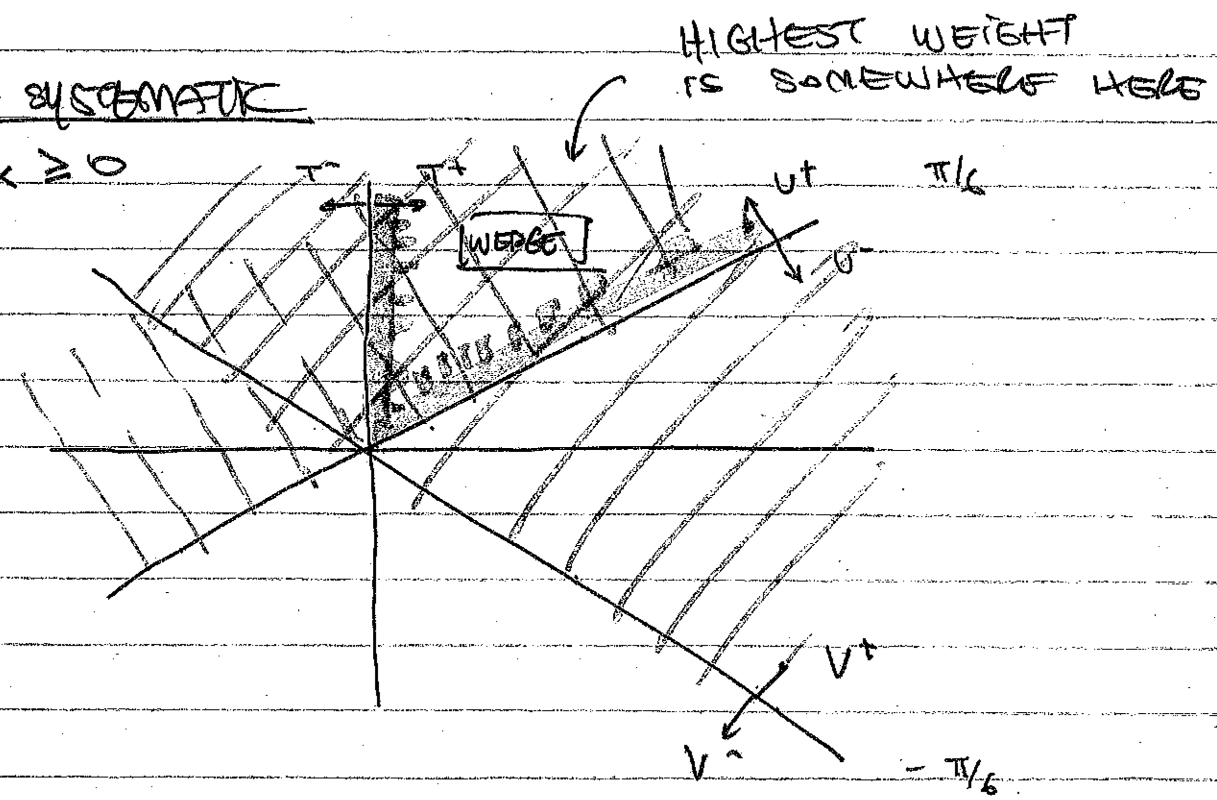
CAN SEE how to generalize: same game as before

→ USE $X^+|MAX\rangle = 0$ & check if prod of lowerings ARE EQUIVALENT.

Gutowski ch. 4.1

MORE SYSTEMATIC

$j_x \geq 0$



further: SU(2) LADDER is symmetric W/RT $M=0$.

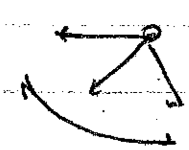
$\rightarrow$ SUFFICIENT TO UNDERSTAND STATES IN THE WEDGE.

UNIQUENESS OF $|\text{MAX}\rangle$:

ALL STATES COME FROM LOWERING:

$$\underbrace{\Pi\left(T^-, V^-, U^-\right)}_{\text{"some product of } T^-, V^-, U^-\text{"}} |\text{MAX}\rangle$$

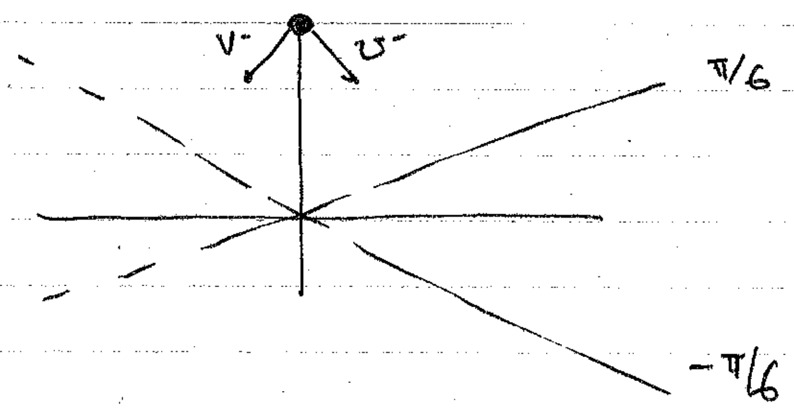
So ALL STATES ARE in this wedge relative to $|\text{MAX}\rangle$



$\rightarrow$ CANNOT have 2 DIFF. MAX STATES (COMP W/ SYM)

<u>THREE TYPES OF DIAGRAMS</u>: Δ, ∇, ⬡ (hex)

Δ ← antifundamental & cousins
↳ HIGHEST WEIGHT IS $\left(0, \frac{q}{2\sqrt{3}}\right)$



end up w/ 
(when to stop? $-j_x \leq m_x \leq j_x$
but symmetry axes ($\pm\pi/6$) tell us this.

Q: are all states unique, or is there a multiplicity?

⊙ ← two dots for A GIVEN (p,q)

THIS IS WHERE WE (*finally*) USE THE CROSS RAISE/LOW COMMUTATORS:

$$\boxed{[T^-, U^-] = -V^-}$$

So game is: commute T^- to the RIGHT!

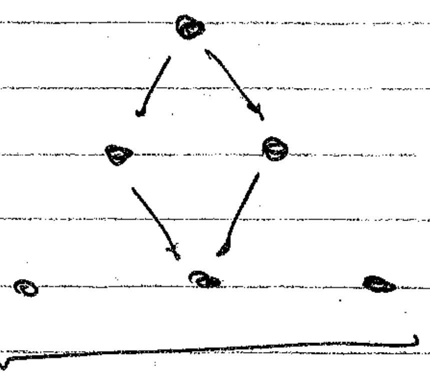
WE KNOW $T^-|MAX\rangle = 0$ b/c $j_T = 0$ ($p_{MAX} = 0$)

ALSO IMPORTANT

$$[T^-, V^-] = 0$$
$$[U^-, V^-] = 0$$

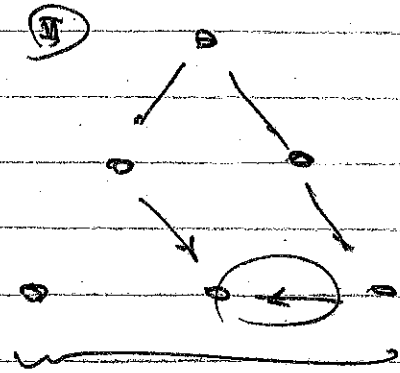
POSSIBLE MULTIPLICITIES

(I)



involving only U^-, V^-

(II)



involving T^-

what about involving X^+ for some X?
→ doesn't give a new state
(from "DESCENDING LADDER" constr. on $SU(2)$)
← can pr. w/ $[\cdot,\cdot]$.
→ think about this if need be!

(I) because $[U^-, V^-] = 0$, $(U^-)^n (V^-)^m |max\rangle$ is ~~unique~~ the same as any rearrangement of the U^- & V^- ops.

VI $[T^-, U^-] = -V^-$
$[T^-, V^-] = 0$

if we commute all T^-'s to the right, we can annihilate $|MAX\rangle$

$$\ldots T^- (V^-)^m (U^-)^n |MAX\rangle$$
$$= \ldots (V^-)^m T^- (U^-)^n |MAX\rangle \quad \leftarrow [T^-, V^-] = 0$$
$$= \ldots (V^-)^m [T^-, U^-] (U^-)^{n-1} |MAX\rangle$$
$$+ \ldots (V^-)^m U^- T^- (U^-)^{n-1} |MAX\rangle$$

$$= - \ldots (V^-)^{m+1} (U^-)^{n-1} |MAX\rangle \quad \leftarrow$$ A STATE WE ALREADY REACHED W/ JUST V^- & U^-
$$+ \ldots (V^-)^m U^- \underbrace{T^- (U^-)^{n-1} |MAX\rangle}_{\text{continue commuting right}}$$

$$= \underbrace{\ldots}_{\text{states w/ just } V^- \,\&\, U^-} \cancel{\text{[illegible]}} + \underbrace{(\ldots)(U^-) T^- |MAX\rangle}_{=0}$$

(WE ALREADY REACHED THEM)

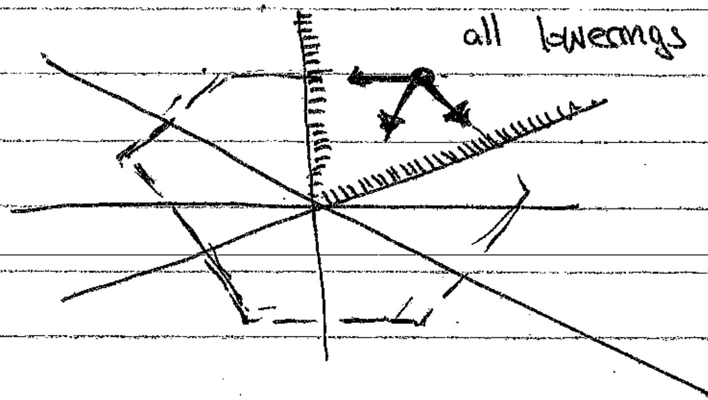
$\boxed{\text{SO}}$: any states we reach w T^- ARE NOT unique states from those we reach w/ U^-, V^-.

$\Rightarrow$ EACH WEIGHT IN Δ DIAGRAMS IS UNIQUE.

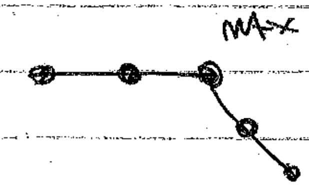
★ <u>for HOMEWORK</u>: show that each state in ▽ diagram is unique (multiplicity one)

<u>MORE SUBTLE</u>: hexagonal
<u>inside</u> the wedge of allowed $|MAX\rangle$, not on the edge

all lowerings allowed on $|MAX\rangle$

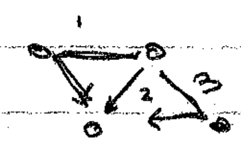


CAN SEE:



these two edges have multiplicity 1 (only one way to reach them w/ lowering ops)

symmetry: whole outer layer has multiplicity 1
... BUT NEXT LAYER HAS HIGHER MULTIPLICITY!



1. $U^-T^-|MAX\rangle$
2. $V^-|MAX\rangle$
3. $T^-U^-|MAX\rangle = \underbrace{\overbrace{[T^-,U^-]}^{-V^-}|MAX\rangle + U^-T^-|MAX\rangle}_{\text{lin comb of 1 \& 2}}$