P262, WIN 2019: SU(N) INDICES 5 FEB 2019

first: naive notation: all indices lower.

GOOD REF: ZEE IV.4

$$\psi_i \longrightarrow \underbrace{\boxed{U_{ij}}}_{= \exp(-i\theta d(T))} \psi_j$$

NOW TAKE COMPLEX CONJUGATE of THIS:

$$\psi^*_i \longmapsto U^*_{ij} \psi_j$$

$$U^*_{ij} = \exp\left(+i\theta d(T)^*\right)$$

$$= \exp\left(-i\theta \underbrace{\left(-d(T)^*\right)}_{\bar{d}(T)}\right)$$

$\boxed{\text{SO}}$: OUR DEFINITION OF CONJUGATE REP MAKES SENSE

TAKE TRANSPOSE:

$$\psi^*_i \longmapsto (U^\dagger)_{ji} \psi_j = \psi_j (U^\dagger)_{ji}$$

$$U^\dagger = U^{-1}$$

THE POINT: CONJUGATE TRANSFORMS w/ U^{-1}, "APPLIED FROM THE RIGHT"

VECTOR SPACES - reminder (from p231)

T_pM VECTOR: $v^i |e_{(i)}\rangle$ coord has upper index

T^*_pM DUAL VEC: $w_j \langle \tilde{e}^{(j)}|$ coord has lower index

$\underbrace{w_j}_{\text{just arrays of \#'s}} \; \underbrace{\langle \tilde{e}^{(j)}|}_{\text{carry vector-ness}}$

DUAL VEC: linear map
takes vector, spits out #

SIMILARLY: vec is a linear map
takes dual vec, spits out #

$$\langle \underline{w} | \underline{v} \rangle = \left(w_j \langle \tilde{e}^{(j)}|\right)\left(v^i |e_{(i)}\rangle\right)$$

$$= w_j v^i \underbrace{\langle \tilde{e}^{(j)} | e_{(i)} \rangle}_{\delta^j_i \text{ if orthonormal basis}}$$

$$= \boxed{w_i v^i} \leftarrow \text{a \#}$$

"MATRIX": linear map, vector $\longrightarrow$ vector

$$\boxed{\underline{\underline{M}} = m^i{}_j |e_{(i)}\rangle\langle \tilde{e}^{(j)}|}$$

clearly: $\underline{\underline{M}}|\underline{v}\rangle = \left(m^i{}_j |e_{(i)}\rangle\langle\tilde{e}^{(j)}|\right)\left(v^k |e_{(k)}\rangle\right)$

$$= m^i{}_j v^k |e_{(i)}\rangle \underbrace{\langle\tilde{e}^{(j)}| e_{(k)}\rangle}_{\delta^j{}_k}$$

$$= \underbrace{m^i{}_j v^j |e_{(i)}\rangle}_{\text{a vector.}}$$

MATRIX has an upper & a lower index.
⇒ easy to see how more general TENSORS work.

Ok - so let's get back to our SU(N) tensors.

NEW NOTATION

VECTORS (fundamental): $\psi^i \mapsto \boxed{U^i{}_j}\psi^j$ ← a unitary NxN matrix

nb: index structure had to be this
BTW: this means $U = \exp(-i\theta\, a(T))$ ALSO HAS SAME INDEX STRUCTURE

$\psi^*_i \longrightarrow \psi_j (U^\dagger)^j{}_i$

RULE: ONLY UPPER & LOWER INDICES MAY BE CONTRACTED

INVARIANT : $\langle \psi, \chi \rangle = \psi_i \underbrace{(U^\dagger)^i{}_j (U)^j{}_k}_{\delta^i{}_k} \chi^k$

in general

$$(\psi)^{i_1 \dots}{}_{j_1 \dots} \mapsto [(U)^{i_1}{}_{k_1} \cdots][(U^\dagger)^{l_1}{}_{j_1} \cdots] \psi^{k_1 \dots}{}_{l_1 \dots}$$

ANALOG : similarity transform of a matrix :

$$\boxed{U^\dagger M U}$$

is because $M = M^i{}_j$

[could have written a 2-index tensor N_{ij} that transforms as $U^\dagger U^\dagger N$

INVARIANT TENSORS

$\mathbb{1}$ = Kronecker δ

$$\delta^i_j \mapsto \underbrace{(U)^i{}_k \, (U^+)^l{}_j}\, \delta^k{}_l$$

$$U^i{}_k \, (U^+)^k{}_j = \delta^i_j \quad \checkmark$$

N-component Levi-Civita
where N is the number in SU(N)

$$\varepsilon^{i_1 \cdots i_N} = \varepsilon_{i_1 \cdots i_N} = \begin{cases} 1 & \text{EVN PERMUTE} \\ -1 & \text{ODD PERMUTE} \\ 0 & \text{OTHERWISE} \end{cases}$$

$$\varepsilon^{i_1 \cdots} \mapsto \underbrace{\left(U^{i_1}{}_{j_1} \cdots\right) \varepsilon^{j_1 \cdots j_N}}$$

$$\equiv \det U \;\; \varepsilon^{i_1 \cdots i_N}$$

eg: $\det U = \left(U^1{}_{i_1} U^2{}_{i_2} U^3{}_{i_3} \cdots\right) \varepsilon^{i_1 \cdots i_N}$

(HW)

other invariants

the generators (in, say, fundamental)

$$(T^a)^i{}_j$$

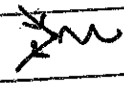
this is a "converter" of indices.

eg. $\underbrace{\underline{3} \otimes \bar{3}}_{i \; j} = \underbrace{\underline{8}}_{(T^a)^i{}_j} \oplus \underbrace{1}_{\delta^i{}_j}$ ← trace

adjoint index

in general: $\underline{N} \otimes \bar{N} = \underline{(N-1)} \oplus \underline{1}$
(SU(N))

ADJOINT TRACE

think of $(T^a)^i{}_j$ as an <u>index</u> <u>converter</u>.

↳ there are similar converters for every reducible tensor rep → irrep (clebsch - Gordan)

IS IT INVARIANT? I think so

most refs just say $(T^a)^i{}_j$ DOESN'T transform → just index converter for things that do transform

BUT I THINK IT IS INVT ⇒ I COULD BE WRONG!
(punchline: DO NOT TRANSFORM INDICES OF INVARIANTS, TRANSFORM THE INDICES OF PHYSICAL OBJECTS)

intuition: ADJOINT INDEX: $\square^a \mapsto U^{-1} \square^a U$

fund. INDICES: $\square^i \to U \square^i$

$\square_j \to \square_j U^{-1}$

So together: $(\square^a)^i{}_j \to U U^{-1} \square U^{-1} U$

but the ADJOINT INDEX IS TRICKY

eg. how does v^i transform?

$$\underline{v} = v^i |e_{(i)}\rangle \to \boxed{|e_{(i)}\rangle U^{-1\,i}{}_k}\boxed{U^k{}_j v^j}$$

invariant if you're transforming everything (kind of a silly thing to do)

simultaneous ACTIVE AND PASSIVE

what about adjoint index?

$$\underline{A} = a^a |T^a\rangle \mapsto a^a V^{ab} (V^{-1})^{bc} |T^c\rangle$$

transform "oppositely"

So we said that infinitesimally

$$|T^b\rangle \xrightarrow{T^a} |[T^a, T^b]\rangle$$

So presumably a^a transforms as $-[T^a, T^b]$

$\boxed{\text{then}}$: write $(T^a)^i{}_j = \Theta^{a\,i}{}_j$ $\quad x \; -i\theta$

T^b

UNDER ADJOINT INDEX: $\Theta^{a\,i}{}_j \to -[T^b, \Theta^{a\,i}{}_j]$

UNDER FUND INDEX: $\Theta^{a\,i}{}_j \to T^b \Theta^{a\,i}{}_j$

UNDER ANTIFUN INDEX: $\Theta^{a\,i}{}_j \to -\Theta^{a\,i}{}_j T^b$

$\Theta U^\dagger$

so $(T^a)^i{}_j$ is invariant.

SO NOW WE HAVE A TOOLKIT for SU(N) INDICES TO REDUCE TENSORS

① SYMMETRIZE / ANTISYMMETRIZE INDICES OF SAME HEIGHT

② CONTRACT W/ INVARIANTS

$$\varepsilon^{\cdots}, \quad \varepsilon_{\cdots}, \quad \delta, \quad T$$

trace over upper/lower

convert indices

simple examples

$$\psi^{ij} = \psi^{(ij)} \oplus \psi^{[ij]}$$

↑ ANTISYM ... CAN HIT W ε ...

$$\psi^{[ij]}\varepsilon_{ijk} = (\cdots)_k$$

$$= \underline{6} \oplus \bar{\underline{3}}$$

$$\psi^i{}_j = \psi^i{}_j\,\delta^j{}_i \oplus \psi^i{}_j\,(T^a)^j{}_i$$

$$\underline{1} \quad \oplus \quad \underline{8}$$

HW: SU(2) IS PSEUDOREAL.

↳ $\exists$ S s.t. $\boxed{S\, d(T^a)\, S^{-1} = \bar{d}(T^a)}$

why? in SU(2) $\exists$ ε^{ij}, ε_{ij}
that converts upper ⟷ lower
w/o complex conjugation.

~~A NEAT TRICK~~
~~there are all sorts~~

RECALL: ULTIMATELY WE WANT COMBINATIONS
OF OBJECTS THAT ARE INVARIANT

TOWARD A GENERAL THY of LIE GROUPS

[Zee]

ex $\boxed{SU(3)}$

→ CARTAN : DIAGONAL ⟷ COMMUTING

$\boxed{\text{dim} = \text{RANK of group}}$ $\begin{cases} T^3 \\ T^8 \Longleftarrow \frac{2}{\sqrt{3}} T^8 \equiv Y \end{cases}$

convenient

why "Y"? reminiscent of hypercharge in standard model

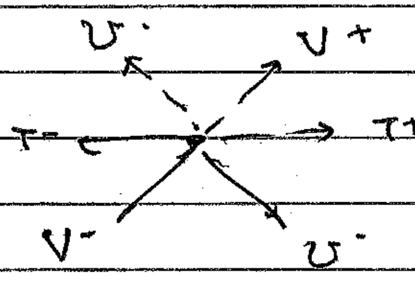
Raising/Lowering : ROOT VECTORS

Y

$T^{\pm}$ moves $(\pm 1, 0)$

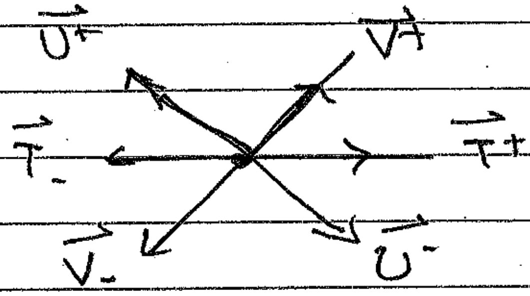
$U^{\pm}$ moves $(\mp \frac{1}{2}, \pm 1)$

$V^{\pm}$ moves $(\pm \frac{1}{2}, \pm 1)$



} ROOT DIAGRAM

READING OFF COMMUTATION RELATIONS



these vectors are "moves" in weight space

ROOT DIAGRAM IS SPACE OF POSSIBLE $\Delta(p,q)$'s.

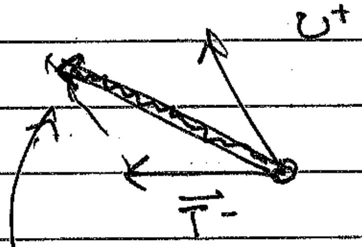
question: is $[\underbrace{T^-, U^+}] = 0$ or nonzero?

this is some raise/lower. if it is in the root diagram then that's what it is.

if $[T^-, U^-]$ is not in the root diagram, then it must vanish.

$$T^- U^+ |(p,q)\rangle = |(p,q) + \underset{\left(-\frac{1}{2}, +\frac{\sqrt{3}}{2}\right)}{\underset{\uparrow}{\vec{U}^+}} + \underset{(-1,0)}{\underset{\uparrow}{\vec{T}^-}}\rangle$$

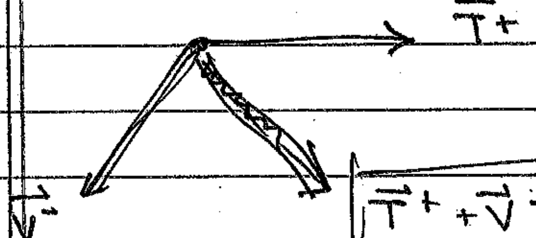
$$U^+ T^- |(p,q)\rangle = |(p,q) + (-1,0) + \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\rangle$$



not in the root diagram

↳ $[T^-, U^+] = 0$

by contrast:



$\boxed{\vec{T}^+ + \vec{V}^- = \vec{U}^-}$

so $\boxed{[T^+, V^-] \propto U^-}$

observe: T^+ is an operator
$\vec{T}^+$ is a $\Delta(p,q)$ 2-vector on weight space

POSITIVITY

OUR ALGORITHM : $\boxed{\text{highest weight}} \longrightarrow$ lower.

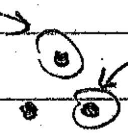
what does "highest" mean?

MOST POSITIVE ; eg. $j_{max} \geq M$ for $su(2)$

highest

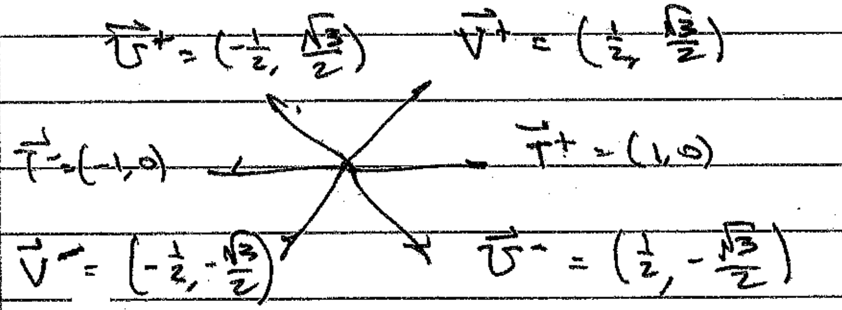
"more positive" p!

... BUT



pick a convention

A ROOT is POSITIVE if its first non-zero element is positive



POSITIVE ROOTS

by arbitrary def

favor T^3 over T^8 eigenval.

this is just the "charged" stuff in ad!

<u>POSITIVE ROOTS</u> : 3 of them

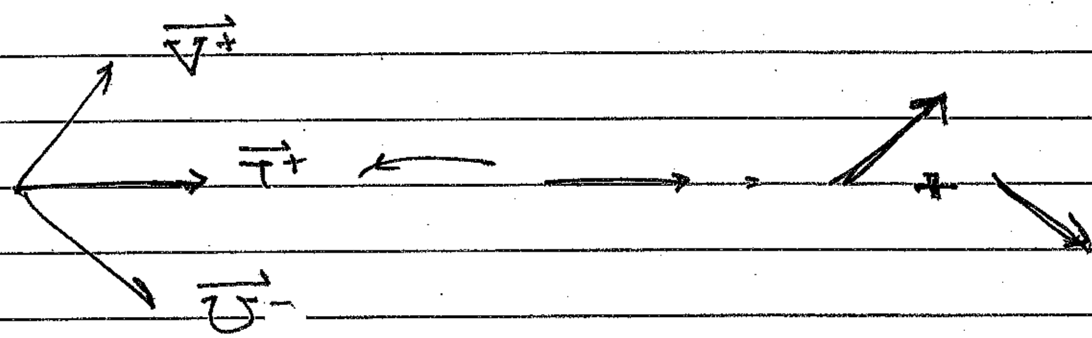
not lin. indep.

<u>SIMPLE ROOTS</u> : subset of <u>positive</u> roots

set for which : ANY POSITIVE ROOT

CAN BE WRITTEN AS SUM OF SIMPLE ROOTS

W/ POSITIVE COEFFICIENTS



So $\boxed{\vec{V}^+ \ \& \ \vec{U}^-}$ are <u>simple roots</u>

$\boxed{SO(4)}$ special, orthog, 4x4

$\mathbb{R}$ version of SU(4)

→ traceless, ~~Hermitian~~ ⊕ 4x4

symmetric $\mathbb{R}$ 4x4

$\boxed{6 \text{ generators}}$

~~Analogs of~~

$\boxed{\text{Cartan}}$ ← max set of commuting gens

$$\left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & \\ & & & \end{array}\right), \quad \left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right)$$

J_{12} $\quad$ J_{34}

analogs of $J_3 \sim \left(\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & \\ \hline & & 0 \end{array}\right)$

CAN DIAGONALIZE THESE

$$J_{12} \sim \left(\begin{array}{cc|cc} 1 & & & \\ & -1 & & \\ \hline & & 0 & \\ & & & 0 \end{array}\right) \qquad J_{34} \sim \left(\begin{array}{cc|cc} 0 & & & \\ & 0 & & \\ \hline & & 1 & \\ & & & -1 \end{array}\right)$$

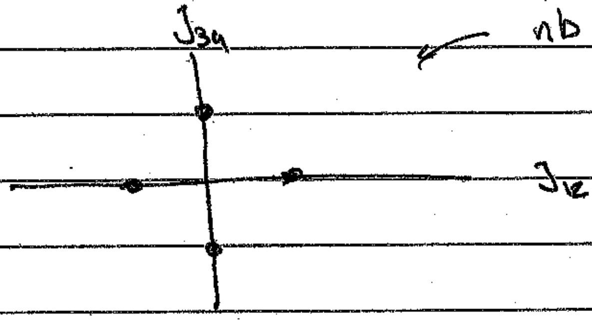
WEIGHTS OF FUNDAMENTAL

vector space on which
$d(T^a) = T^a$

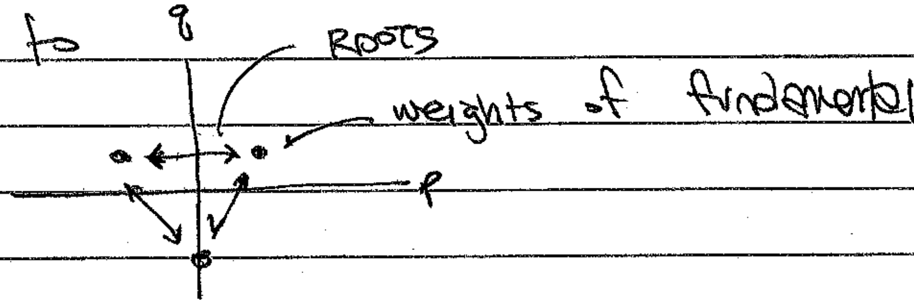
BASIS: $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

WEIGHT: $\begin{pmatrix}1\\0\end{pmatrix} \quad \begin{pmatrix}-1\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\end{pmatrix} \quad \begin{pmatrix}0\\-1\end{pmatrix} \begin{matrix}\leftarrow J_{12}\\ \leftarrow J_{34}\end{matrix}$

no REAL REP.

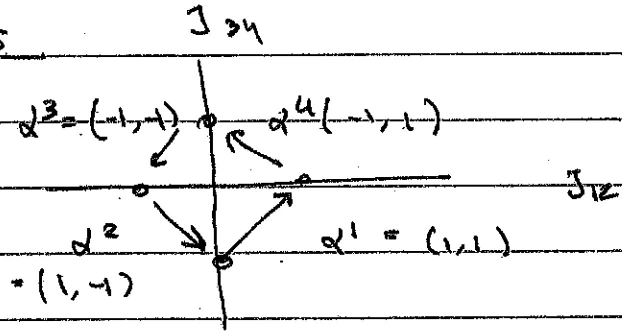


compare to

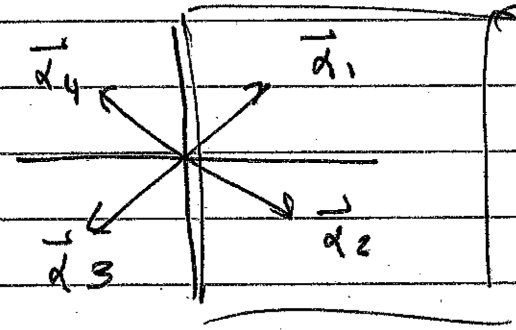


ROOT VECTORS

FUNDAMENTAL



ROOT DIAGRAM



positive roots

... also simple

observe: $\vec{\alpha}_1 + \vec{\alpha}_2$ is not a root.

$$[T_{\alpha_1}, T_{\alpha_2}] = 0$$

RAISING/LOWERING OPS