P262 : final lecture

3/11/19

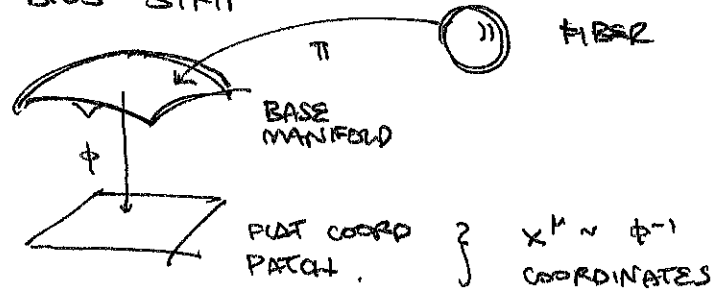
WRAPPING UP : last 3 wks (adv. topics) have been "big picture" — see how advanced ideas fit together.

↳ more systematic refs on website

COURSE EVALS ⟶ support SPECIAL TOPICS CLASSES

Geometric pic so far :

① fiber bundle : MÖBIUS STRIP



② GEOMETRY of MANIFOLDS

↳ K-forms & "CALCULUS" in curvy spaces

GAUGE THEORY: (potential theory)

$$F = dA \quad \longleftarrow \quad F_{\mu\nu}\, dx^\mu \wedge dx^\nu = (\partial_\mu A_\nu(x))\, dx^\mu \wedge dx^\nu$$

↑ (factors of ½ implicit)

GAUGE REDUNDANCY : all physics in $F_{\mu\nu}(x)$

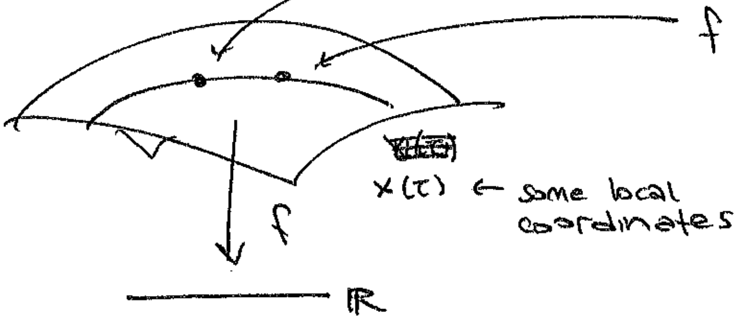
↳ invariant under $A \longrightarrow A + d\alpha$

$\alpha(x)$ IS A FUNCTION: DIFFERENT VALUE @ DIFFERENT SPACETIME POSITIONS.

important: this is <u>very</u> different from rotational invariance!! (GLOBAL vs LOCAL)

a bit more Geometry (to understand Geometry of Groups)
↳ *how do we* <u>*move*</u> *objects along a manifold?*

<u>FUNCTION</u> :



$f(x)$

$$f(x+\Delta x) = f(x) + \underbrace{\Delta\tau \frac{\partial x^\mu}{\partial \tau}\partial_\mu f(x)}_{\text{DIRECTIONAL DERIVATIVE}}$$

$$\uparrow \ \frac{\partial x(\tau)}{\partial \tau} d\tau$$

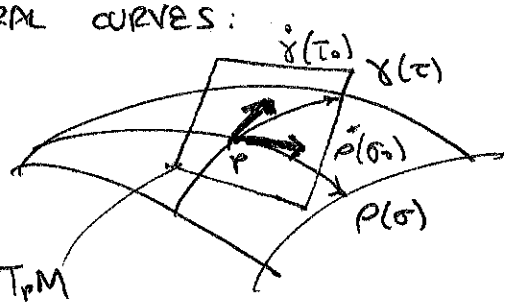
OK. THAT'S JUST CALCULUS.

What about tensors?

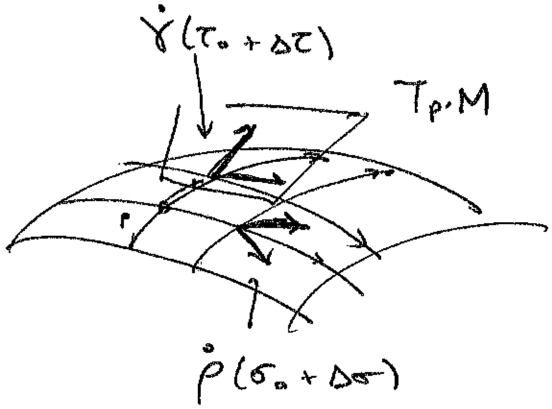
EG: vectors live in <u>tangent space @ a point</u>

NEED TO COMPARE T_pM & $T_{p+\varepsilon}M$

USE INTEGRAL CURVES:



but how do we compare vectors (TENSORS) on different spaces?



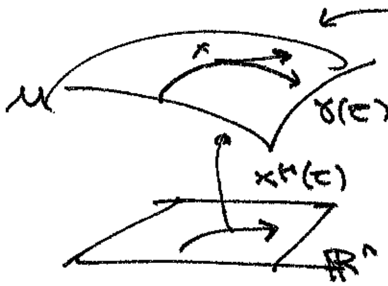
NEED ADDITIONAL STRUCTURE: connection.

GR: if you have a metric, there is a systematic way to write metric compatible connection

(COVARIANT DERIV.)

2 → we will define a different kind of derivative that is ALSO IMPORTANT.

Lie DERIVATIVE (in contrast to covariant deriv.)



$$\underline{V}(x) = \underbrace{V^\mu(x)}_{\frac{dx^\mu(\tau)}{d\tau}}\,\partial/\partial x^\mu \qquad \text{VEC. FIELD}$$

can think of subsequent points on $x^\mu(\tau)$ ← or $\gamma(\tau)$
as exponentiation of $(d/d\tau)$ operator

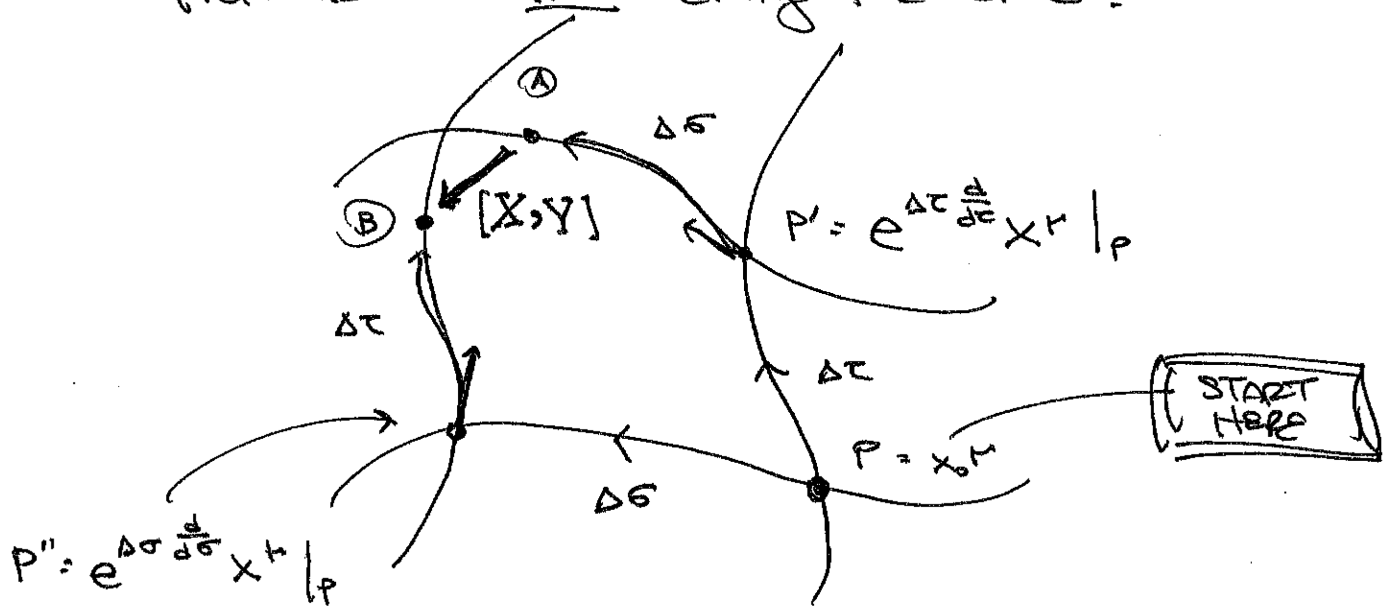
$$x^\mu(\tau_0+\Delta\tau) = x^\mu(\tau_0) + \Delta\tau \left.\frac{dx^\mu}{d\tau}\right|_{\tau_0} + \dots$$
$$= e^{\Delta\tau(d/d\tau)}\, x^\mu \big|_{\tau_0}$$

looks familiar?
exp. of infinitesimal → finite transf.

IN A PATCH U:
SUPPOSE WE HAVE 2 VECTOR FIELDS (X, Y) THAT ARE
not degenerate w/ each other

↳ & suppose they have integral curves $\gamma(\tau)$ & $\rho(\sigma)$

Goal: want to define a derivative of one vector
field as we flow along the other.



can compare what happens if we flow in τ then σ vs. σ then τ.

$$A = e^{\Delta\sigma \frac{d}{d\sigma}} e^{\Delta\tau \frac{d}{d\tau}} x^\mu \big|_p$$

$$B = e^{\Delta\tau \frac{d}{d\tau}} e^{\Delta\sigma \frac{d}{d\sigma}} x^\mu \big|_p$$

$$\underbrace{A - B}_{\text{compare points on manifold}} = \left[e^{\Delta\sigma \frac{d}{d\sigma}}, \underbrace{e^{\Delta\tau \frac{d}{d\tau}}}_{1 + \Delta\sigma \frac{d}{d\sigma} + \mathcal{O}(\Delta\sigma^2) + \dots} \right] x^\mu \big|_p$$

$$= \Delta\sigma \Delta\tau \left[\frac{d}{d\sigma}, \frac{d}{d\tau} \right] x^\mu \big|_p \qquad \frac{d}{d\tau} x^\mu \big|_p = X^\mu \big|_p$$

$$= \Delta\sigma \Delta\tau \left(\frac{d}{d\sigma} X^\mu - \frac{d}{d\tau} Y^\mu \right) \big|_p$$

$$= - \Delta\sigma \Delta\tau \underbrace{[X, Y]^\mu}$$

LIE BRACKET: COMPARE TANGENT VECTORS
THIS COMPARISON IS WELL DEFINED FROM THE PERSPECTIVE OF INTEGRAL CURVES

OPERATOR PICTURE (QM-esque) (acts on test fn.)

$$XY = (X^\mu \partial_\mu)(Y^\nu \partial_\nu) = \underbrace{X^\mu (\partial_\mu Y^\nu)}_{\text{coeff}} \underset{\uparrow \atop e_{(\nu)}}{\partial_\nu} + X^\mu Y^\nu \underbrace{\partial_\mu \partial_\nu}_{?!}$$

2ND DERIV?? DOES NOT TRANSFORM NICELY.

BUT THAT TERM CANCELS IN $[X, Y]$

$$[X, Y] = \left(X^\mu (\partial_\mu Y^\nu) - Y^\mu (\partial_\mu X^\nu) \right) \partial_\nu$$

bona fide vector field.

BIG ASSUMPTION: indep. vector fields are integrable

↳ integral curves γ, ρ form coordinates

e.g. SUFFICIENT: $[X, Y] = 0$

s.t. X int curves have const. Y int curve coords

NECESSARY: $\boxed{[X_{(I)}, X_{(J)}] = c_{IJ}{}^{K} X_{(K)}}$

↑ involutive

Pf. called FROBENIUS THM.

↳ "integrability" is a big topic in formal physics.

RECAP: LIE DERIVATIVE

$$\mathcal{L}_X f \equiv X f \qquad \text{DIR. DERIV.}$$

$$\mathcal{L}_X Y = [X, Y]$$

$$\mathcal{L}_X \boxtimes = \cdots \qquad \text{Def. by LEIBNIZ RULE.}$$

$\omega = \omega_\mu(x)\, dx^\mu$ func.

↳ e.g. $\mathcal{L}_X \omega$ is s.t. $\mathcal{L}_X \boxed{\omega(V)} = X[\omega(V)]$

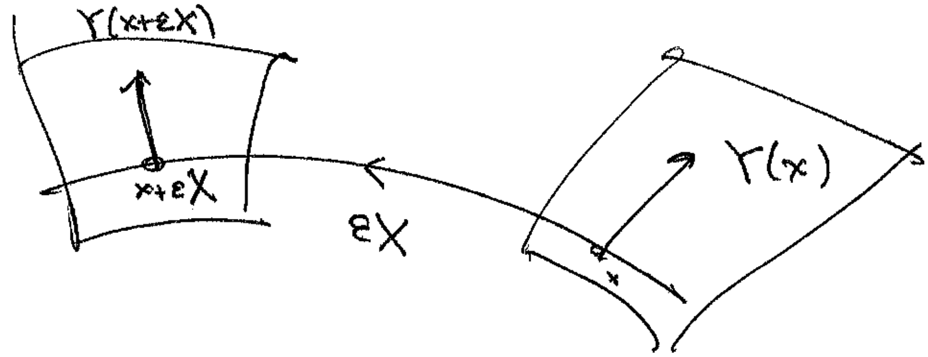
$$\mathcal{L}_X \omega(V) = (\mathcal{L}_X \omega) V + \omega(\underbrace{\mathcal{L}_X V}_{[X,V]})$$

$$\Rightarrow \underline{(\mathcal{L}_X \omega) V} = X[\omega(V)] - \omega([X,V])$$

$$= \underline{\left(X^\nu (\partial_\nu \omega_\mu) + \omega_\nu (\partial_\mu X^\nu)\right)} V^\mu$$

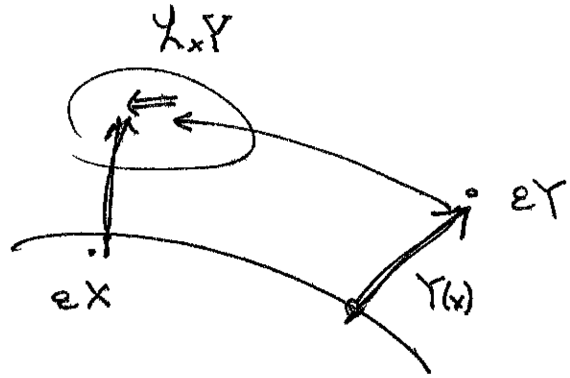
this is the natural derivative:

$$\text{"} \underbrace{Y(x+\varepsilon X)}_{\in T_{x+\varepsilon X}M} - \underbrace{Y(x)}_{\in T_x M} \text{"}$$



need to compare @ same tangent space

think of X, Y as infinitesimal flow



application: ISOMETRIES: $\mathcal{L}_X g = 0$

metric is const. along X flow

↳ symmetry of spacetime.

[eg sym of AdS_5]

LIE GROUP → is a group → group multiplic. $G \times G \to G$

↓

is a manifold
(continuous param)

eg $g(\theta) = \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \in SO(2)$

$\underbrace{\text{tangent vectors @ identity}}_{T_e M}$: LIE ALGEBRA

$e = \mathbb{1}$

$g(\theta)$ is a curve in $SO(2)$. $\underbrace{\frac{d}{d\theta} g(\theta)\Big|_{\theta=0}}_{\in \text{ ALGEBRA}}$

GROUP MULTIPLICATION : gives a way to define translation

let $a, g \in G$.

def: Left Translation : $L_a : G \to G$

$L_a(g) = (ag) \in G$

REMARK : if we have a map between manifolds

$\varphi : M \to N$

then we can define a push forward map

$\varphi_* : TM \to TN$

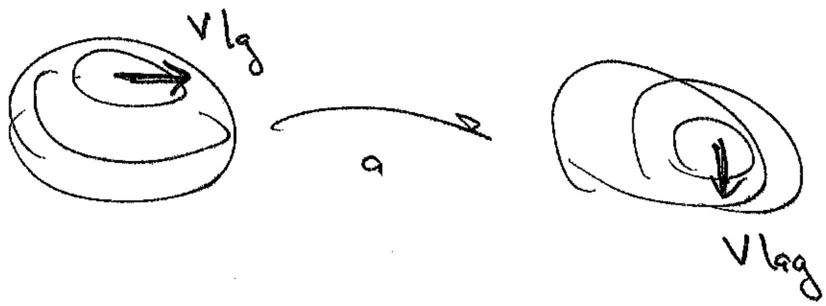
IDEA : given $\gamma(\tau)$ on M s.t. $\dot{\gamma}(\tau)$ is a vector @ $T_{\gamma(\tau)}M$,
can define $\tilde{\gamma}(\tau)$ on N by $\tilde{\gamma}(\tau) = \varphi(\gamma(\tau))$
then $\dot{\tilde{\gamma}}(\tau)$ is a vector @ $T_{\tilde{\gamma}(\tau)}N$.

L_a is precisely a map between manifolds

eg maps ORIGIN to a

<u>Def</u>: A VECTOR field is <u>LEFT INVARIANT</u> if

$$(L_a)_* V|_g = V|_{ag}$$



<u>So what?</u> We can construct LEFT-INVT. VEC FIELDS by PUSHING ELEMENTS of T_eG = LIE ALGEBRA

LET: $v \in T_eG$, algebra.

then: $V(g) \equiv L_{g*} v \qquad \forall\ g \in G$

vector @ g coming from pushing $v \in T_eG$

$\Rightarrow$ PUSH A COPY OF v TO EVERY TANGENT SPACE T_gG $\forall$ g

ie: we've mapped $\boxed{T_eG \to TG}$

the only thing we need is T_eG.

LIE BRACKET : CAME FROM GEOMETRY — wanted a derivative for vectors.

is PRECISELY the COMMUTATOR that we started with on wk. 1.

REQUIRED : $(L_a)_* [X,Y]|_g = [X,Y]_{ag}$

this is true, but not obvious

(noncommutivity @ any g is same as @ $\mathbb{1}$).

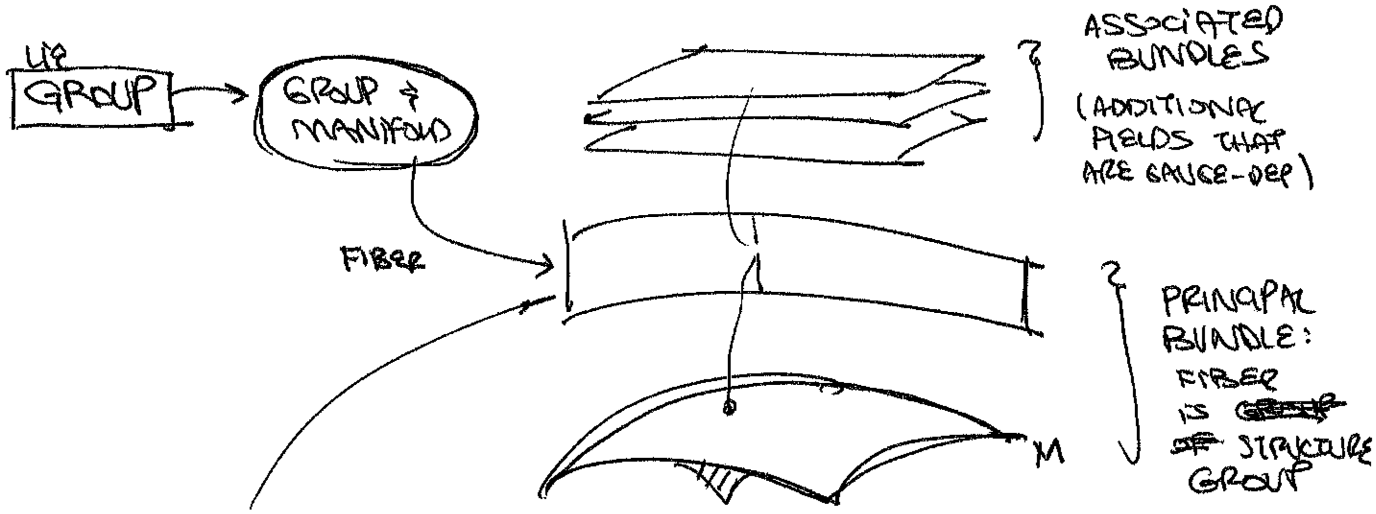
So: T_eG is special.

elements are Generators.

$[T_i, T_j] = c_{ij}{}^k T_k$

(linear, involutive)

RAPID ZOOM OUT TO BIG PICTURE



GAUGE REDUNDANCY

↳ manifestly covariant description of physics that is not obviously so

e.g. $\underline{E}$ & $\underline{B}$ are covariant... but you wouldn't have guessed it from indices.

generalize $\quad A \to A + \boxed{d\alpha}$

GAUGE

$$\psi \to e^{i\alpha}\psi$$

ON ASSOCIATED BUNDLE FIELDS

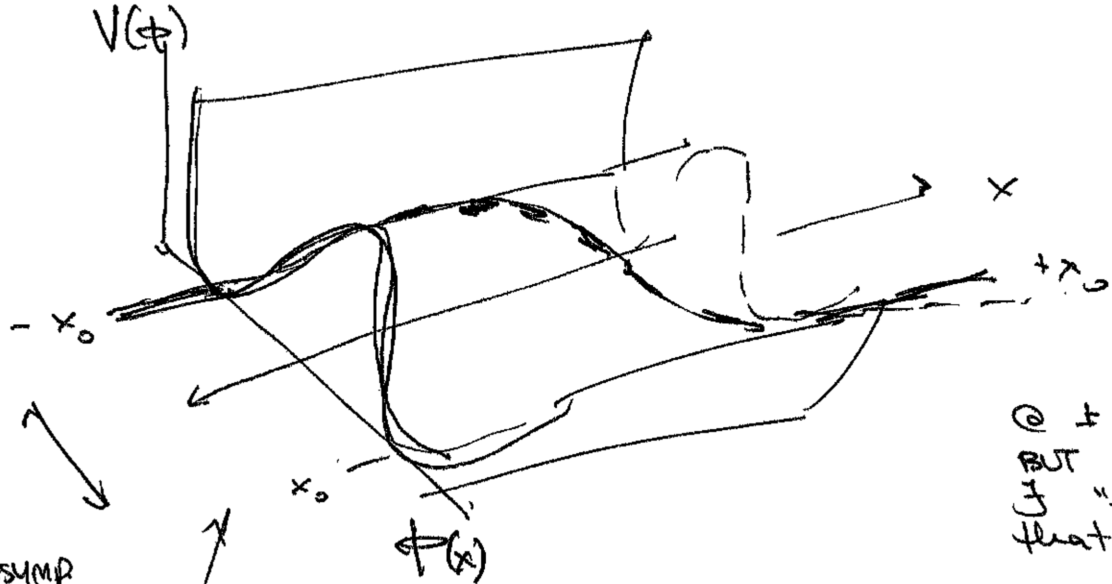
$$D\psi \to e^{i\alpha} D\psi$$

COVARIANT DERIVATIVE def to transf. covariantly

every fundamental force that we know can be described this way.

Topology of Gauge field theories

CONSIDER SCALAR FIELD in 1+1 DIM w/ DOUBLE WELL POT.



Z_2 sym in ASYMP VACUUM.

@ ±∞, E is zero
BUT in middle,
∃ "excitation"
that is stable.

this classical solution is a <u>kink</u>
BEHAVES A LOT LIKE A PARTICLE
↳ LOCAL LUMP OF ENERGY.

<u>Derrek's thm</u>: no generalization to higher DIM.

<u>huge exception</u>: GAUGE FIELDS

why? YOU CAN BE <u>ZERO ENERGY</u> in many different
ways ... GAUGE EQUIV. to MINIMUM config.

end up w/ local lumps of energy

- monopole
- ~~[illegible]~~ cosmic string
- domain wall
- instanton / sphaleron