GROUP THEORY for Physicists

1/7/19

GOALS: understand the structure of continuous symmetries in physics.

no finite groups

(QUANTUM) FIELD THY

↳ Generalize ROTATIONS

SO(3)

QM: RAISING / LOWERING OPERATORS

REPRESENTATION { vectors, tensors, ⋮

SPINORS ??

what are these, anyway?

reducible irreducible

(CLEBSCH-GORDAN)

flip.tanedo @ ucr

tanedo.github.io/Physics 262 - 2019

Some physics

$\boxed{\text{Symmetry}}$ ← **big** idea in theoretical physics

→ eg DEGENERACY of SYSTEMS
REDUCE DIMENSIONALITY of MATHEMATICAL DESCRIPTIONS

⋮

Really interesting connection to Geometry

Goldstones ↔ LOW ENERGY (IR) EXCITATIONS

⋮

"EFFECTIVE THEORY"

GAUGE THEORY ↔ forces in nature
... topology in some systems

here's the key observation:

Some symmetry transformations do not commute.

- There is nothing *QUANTUM* about this observation.

example: rotate a book.

EASY TO DISTINGUISH DIFFERENT FACES (spine, front, back, ...)

① $\pi/2$ ROT ABOUT $\hat{x}$ then $\pi/2$ ROT ABOUT $\hat{y}$

② ————"———— $\hat{y}$ then ————"———— $\hat{x}$

these two sequences of transformations are not the same.

- We could choose any rotation angle.

  There are a continuum of rotations. *INFINITE* # transformations that are "rotations"

our ~~effective~~ def's.
PLS. REFER TO ANY TEXT FOR FORMAL DEF.

Some "definitions"

A group is a collection (SET) of abstract symmetry transformations.

eg: $\{$ ROT ABOUT $\hat{x}$ BY 5°,
— " — $\hat{y}$ BY 23.4°,
— " — $\hat{z}$ BY $\sqrt{11}$ °, ... $\}$

U(1) (unitary 1×1 matrices)

~~EXAMPLES~~:
~~MULTIPLICATION: an element of a group can~~

- RULE: you can combine elements of a group by multiplication (meaning of multiplication is part of group definition)

$\underbrace{(\text{ROT ABOUT } \hat{x} \text{ BY } 5°) \cdot (\text{ROT ABOUT } \hat{x} \text{ BY } 6°)}$

happens to equal (ROT ABOUT $\hat{x}$ BY 11°)

- ASSOCIATIVITY: if $a, b, c \in G$.

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

• CLOSURE: if $\overbrace{a, b \in G}^{\text{if } a \,\&\, b \text{ are elements of group } G}$, then $(ab) \in G$

eg: $G = \{ e^{i\pi/4}, e^{i\pi/2} \}$?

phase rotations

multiplication: $e^{ix} e^{iy} \equiv e^{i(x+y)}$

CLOSURE: $e^{i\pi/4} \cdot e^{i\pi/2} = \boxed{e^{i3\pi/4}}$

must be in group.

similarly, $\boxed{e^{i2\pi} = 1}$ must be in!

$e^{i5\pi/4} \equiv e^{i\pi/4}$ "by definition"

SO THAT'S ALL WE NEED FROM CLOSURE

$G = \{ e^{i\pi/4}, e^{i\pi/2}, e^{i3\pi/4}, 1 \}$

has a name: $\mathbb{Z}_4$

observe: this is a subgroup of the much "bigger" group of all rotations "do nothing"

- IDENTITY: the trivial transformation is part of the Group.

- INVERSE: if you can transform one way, you can transform in the opposite way.

eg. if $g(\theta) = e^{i\theta}$

then $g(\theta)^{-1} = g(-\theta)$

such that $g(\theta)^{-1} g(\theta) = e^{i\theta} e^{-i\theta} = 1$

the identity

OBSERVE: the group of all phase transformations $g(\theta) = e^{i\theta}$ has an ∞ number of elements.

the group of $90°$ phase transforms $g(\theta) = e^{i\pi/4 \cdot n}$ $n = 0, 1, 2, 3$ HAS FOUR elements.

In this class: we will only look at the infinite dimensional groups.

These are <u>continuous</u> transformations

↳ called: $\boxed{\text{Lie Groups}}$

PRONOUNCED "Lee"

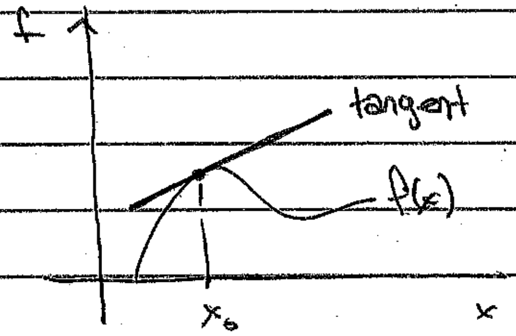
- finite dim groups also important... different machinery & application

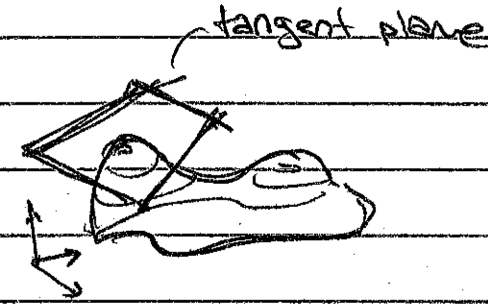
<u>continuous</u> ⟷ group is a <u>manifold</u>

"YOU CAN DO CALCULUS ON IT"

more precisely: DIFFERENTIAL GEOMETRY. (so this is an interesting course to take alongside GR!)

can take derivatives w/rt <u>parameter</u>
(the GROUP TRANSFORMATION PARAMETER, θ)





tangent space (FORMALLY: tangent bundle)

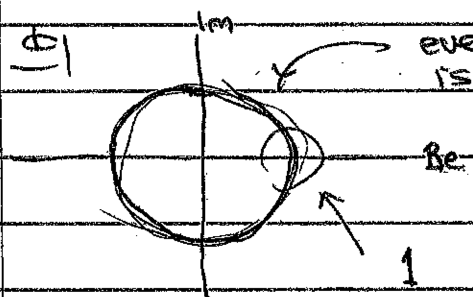
→ space of "velocities"
infinitesimal transformations

you may recall from QUANTUM MECHANICS that finite transformations come from exponentiation of $\boxed{\text{generators}}$

↑

infinitesimal transformations

eg. U(1): group of phase transforms



every point on the unit circle is an element of U(1) → $g(\theta) = e^{i\theta}$

infinitesimal transf @ ORIGIN:

$1 \mapsto 1 + i\varepsilon \xrightarrow{\exp} e^{i\varepsilon}$

cont'd: for U(1), "the generator is 1"

IN GENERAL (physicist convention):
a (finite) transformation in the group G is ~~antihermitian trans~~ the exponentiation of a Hermitian (infinitesimal) transformation.

MATHEMATICIANS SOMETIMES LIKE ANTIHERMITIAN GENERATORS.

We call these GENERATORS, and will label them
T^a ← index, if there are many.

$$\boxed{g(\theta^1, \theta^2, \dots) = e^{-i\underline{\theta}\cdot\underline{T}}}$$

I WILL OFTEN GET THIS SIGN CONVENTION WRONG.

PARAMETERS OF FINITE TRANSFORM

"vector" of GENERATORS

nb: finite number of generators, even if group ~~is~~ has ∞ elements.

nb: there is a subtlety re: local vs. global that we are ignoring for now

is every element the exp. of a generator @ the origin?? [ans: no.]

Nomenclature: the infinitesimal transforms (THE GENERATORS / TANGENT SPACE) are called the **ALGEBRA** of the group.

We will spend most of our time on this. no canonical convention (closest one uses gothic...)

sometimes we may say $\mathcal{L}(so(3))$ to distinguish from the group $SO(3)$

SO FAR: [GROUP DEF
↳ LIE GROUPS: continuous
↳ LIE ALGEBRAS: infinitesimal

turn symmetries into "math"

Next: what do these transformations act on?

→ main idea: $\boxed{\text{Representation}}$

Group elements: "platonic ideals" of a symmetry transformation.

Representation of a group: explicit matrices that act on a VECTOR SPACE to enact those transformations.

there are many ways to represent a group. the structure of the group will always be the same

NB: it's convenient to define groups w/rt a matrix. eg:

eg. $SO(N)$ ORTHOG, det = 1, $N \times N$ ($\mathbb{R}$) matrices

$M^T M = 1$

eg. $SU(N)$ HERMITIAN, det = 1, $N \times N$ $\mathbb{C}$ matrices

$M^\dagger M = 1$

this is subtly different from representation
(OR: this is the FUNDAMENTAL REP)

The notation can get clunky here, too.

$\underbrace{\quad}_{}$ TRANSF PARAMS

$g(\theta^1, \theta^2, \ldots)$ is a group element

$D(g(\underline{\theta}))^i{}_j$ is a matrix representation of the group.

indices!!
WE CAN MAKE A BIG DEAL ABOUT THESE LATER

$\underline{\underline{D}}(g(\theta))$ acts on vectors $\underline{v}$ as [GENERALIZED] rotations.

some transformation of $\underline{v}$ that is the action of the symmetry we care about example : $SO(3) \simeq SU(2)$

orthogonal, 3x3 matrices, det = 1

3D (spatial) ROTATIONS
the most intuitive non trivial symmetry
PARAMETERS: EULER ANGLES for eg.

FUNDAMENTAL (defining)
Representation of $SO(3)$:

from CAHN CH.I

Start w/ Rotation about $\hat{z}$

$$\underbrace{g(0,0,\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{pmatrix}}$$

or $\mathcal{D}(g(0,0,\theta))$

INFINITESIMAL: $g(0,0,\varepsilon) = \underbrace{g(0,0,0)}_{} + \underbrace{\varepsilon \frac{d}{d\theta} g(0,0,\theta)|_{\theta=0}}_{}$

$= \mathbb{1} \underbrace{-}_{\text{convention}} i \varepsilon \underbrace{T_z}_{\text{GENERATOR}}$

OBS: HERMITIAN

$$T_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

generator of rotations about $\hat{z}$

SIMILARLY : $T_x = \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix}$

$$T_y = \begin{pmatrix} 0 & & i \\ & 0 & \\ -i & & 0 \end{pmatrix}$$

to go from infinitesimal to finite, we exponentiate

$$\begin{pmatrix} c_\theta & -s_\theta & \\ s_\theta & c_\theta & \\ & & 1 \end{pmatrix} = e^{-i\theta T_z}$$

def by series exp.

These are just rotations of vectors in the freshman physics classes.

So: lets get back to book rotations!

$$g(\alpha,0,0)\, g(0,\beta,0) = e^{-i\alpha T_x} e^{-i\beta T_y}$$

$$= \left(1 - i\alpha T_x - \tfrac{1}{2}\alpha^2 T_x^2\right)\left(1 - i\beta T_y - \tfrac{1}{2}\beta^2 T_y^2\right)$$

$$= 1 - i(\alpha T_x + \beta T_y) - i\alpha\beta T_x T_y - \tfrac{1}{2}\alpha^2 T_x^2 - \tfrac{1}{2}\beta^2 T_y^2$$

$$g(0,\beta,0)\, g(\alpha,0,0)$$

$$= 1 - i(\beta T_y + \alpha T_x) - i\alpha\beta T_y T_x - \tfrac{1}{2}\alpha^2 T_x^2 - \tfrac{1}{2}\beta^2 T_y^2$$

$$g(\alpha,0,0)\, g(0,\beta,0) - g(0,\beta,0)\, g(\alpha,0,0)$$

$$= -i\alpha\beta\, [T_x, T_y]$$

commutators will define the algebra.

Remarks :

- this is a commutator
- this is not quantum in any sense !!
  - ↳ still freshmen vector analysis !

Next mtg: WED 4:15

↳ EITHER CONF. ROOM OR MSIE 117

Reminder : talk to derek