

HOMEWORK 2: Tensor Stuff

COURSE: Physics 262, *Group Theory for Physicists* (Fall 2019)
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DUE BY: Be ready to discuss on Mon, Feb 25

1 Weight Diagram Example

Draw the $SU(3)$ weight diagram for the $SU(3)$ representation with highest weight $(3/2, \sqrt{3}/2)$. What is the dimension of this representation? It may be helpful to use the triangular graph paper available on the course website.

2 Tensor Representation

What is the decomposition of $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ into $SU(3)$ irreducible representations? Is there a combination that is invariant?

3 Interpretation: $SU(3)$ Color Symmetry

3.1 The Δ^{++}

The Δ^{++} is a composite fermion that is a cousin of the proton. One puzzle from the early days of modern particle physics was why the Δ^{++} existed. We suspected that the Δ^{++} was composed of three identical fermions, each with charge $2/3$ (we now know this as the up quark). However, identical fermion wavefunctions must be antisymmetrized. Thus the Δ^{++} could not be made up of identical fermions. Explain why this was indirect motivation for $SU(3)$ color symmetry for which the up quark is in the fundamental representation. HINT: your explanation should make use of a $SU(3)$ invariant tensor.

3.2 Not $SU(2)$

Argue that in order to account for the Δ^{++} , the new symmetry group could not have been $SU(2)$. There are many ways to argue this, but for this problem, argue based on tensor representations of $SU(2)$.

3.3 Not $SU(4)$

Argue that in order to account for the Δ^{++} , the new symmetry group could not have been $SU(4)$. There are many ways to argue this, but for this problem, argue based on invariant tensors.

3.4 “Technigluons”

Suppose I have a new theory based on $SU(N)$ symmetry. I have the following particles:

1. A techni-quark in the fundamental representation.
2. A techni-tensor with $(N - 1)$ fundamental (upper) indices.
3. A techni-gluon in the adjoint representation.

Particles are allowed to interact if I can write down an invariant combination of their tensor product. Argue that the three particles above may interact with one another.

4 Adjoint vs. Fundamental–Anti-fundamental

We saw that the tensor product of a fundamental and an anti-fundamental of $SU(2)$ decomposes as

$$\mathbf{2} \otimes \bar{\mathbf{2}} = \mathbf{3} \oplus \mathbf{1}.$$

One way to understand the $\mathbf{3}$ on the right-hand side is to see how it transforms as a spin-1 representation versus as a tensor product.

4.1 Tensor Product Transformation

Recall that $SU(N)$ indices transform as follows:

$$\psi^i \rightarrow U^i_j \psi^j \qquad \chi_i \rightarrow \chi_j (U^\dagger)^j_i , \qquad (4.1)$$

where U is a finite $SU(N)$ transformation.

1. Write the infinitesimal transformation of Ψ^i_j under an arbitrary, small $SU(2)$ transformation.
2. Let $\mathbf{v} = (v^+, v^0, v^-)$ be a state in the $\mathbf{3}$ of the aforementioned tensor product. With respect to the fundamental and the antifundamental indices, \mathbf{v} may be written

$$\mathbf{v} = v^+ (T^+)^i_j + v^0 (T^3)^i_j + v^- (T^-)^i_j . \qquad (4.2)$$

. Write the components of \mathbf{v} after performing an infinitesimal $SU(2)$ transformation in the T^1 direction according to the transformation rule you wrote above.

4.2 Triplet Transformation

Alternatively, we may think of \mathbf{v} as an object with only a triplet index:

$$\mathbf{v} = v^+ |m = 1\rangle + v^0 |m = 0\rangle + v^- |m = -1\rangle . \qquad (4.3)$$

1. Write the infinitesimal transformation of v^a under an arbitrary, small $SU(2)$ transformation, where a is a triplet (spin-1) index.
2. Write the components of \mathbf{v} after performing an infinitesimal $SU(2)$ transformation in the T^1 direction according to the transformation rule you wrote above.

DISCUSSION: In this way, the tensor T is used to convert between different kinds of indices. Observe that the transformation on the components in \mathbf{v} are the same whichever picture you use.

5 Invariance of $\varepsilon^{i_1 \cdots i_N}$

Show that $\varepsilon^{i_1 \cdots i_N}$ is invariant under an $SU(N)$ transformation. HINT: you can check for $SU(2)$ explicitly. Use a handy expression for the determinant of an $N \times N$ matrix.

Extra Credit

These problems are for your own edification. You are encouraged to explore them according to your own personal and research interests.

Normalizations once and for all

In case you'd like to keep track of normalization, write a set of notes (for yourself) that go from first principles to the normalizations that we use in our general study of roots and weights. Start with Georgi section 2.4 which starts with commutation relations and leads up to (2.37):

$$\text{Tr}(T^a T^b) = \lambda \delta^{ab} . \quad (5.1)$$

Then jump to section 6.3 and the inner product defined on the adjoint representation:

$$\langle E_\alpha | E_\beta \rangle = \frac{1}{\lambda} \text{Tr} (E_\alpha^\dagger E_\beta) = \delta_{\alpha\beta} . \quad (5.2)$$

Confirm that this also gives $\langle H_i | H_j \rangle = \delta_{ij}$ on the Cartan subalgebra. Confirm that with this convention,

$$[E_\alpha, E_{-\alpha}] = \alpha \cdot H . \quad (5.3)$$

HINT: We know that $[E_\alpha, E_{-\alpha}]$ is some linear combination of the Cartan elements because $E_\alpha | E_{-\alpha} \rangle$ has weight zero.

Finally, show that the $SU(2)$ subalgebra along the α direction is generated by

$$E^3 = \frac{\alpha \cdot H}{\alpha^2} \qquad E^\pm = \frac{E_{\pm\alpha}}{|\alpha|} . \quad (5.4)$$

Note that $[E^+, E^-] = E^3$. This probably indicate that the normalization that we've been using in this course has been the 'wrong' one.