THE LORENTZ GROUP

so far: LORENTZ: $\underbrace{\Lambda^\mu{}_\rho \, \eta_{\mu\nu} \, \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}}$

$$so(\underline{3,1})$$

LIKE $\boxed{so(4)}$, BUT minus sign for one component

as if one component were pure imaginary

this minus sign has big implications
→ NON-COMPACTNESS of group "$e^{i\theta} \to e^{\eta}$"

~~FROM DYNKIN ANALYSIS, WE KNOW THAT THERE ARE TWO SU(2)'s IN so(4).~~
~~→ We explicitly constructed them for so(3,1)~~

GENERATORS of so(3,1)

$$(M_{\mu\nu})^{\rho\sigma} = i\left(\delta^\rho_\mu \delta^\sigma_\nu - \delta^\sigma_\mu \delta^\rho_\nu\right)$$

$$\text{s.t.} \quad \Lambda^\rho{}_\sigma = \exp\left(i\omega^{\mu\nu} (M_{\mu\nu})^\rho{}_\sigma\right)$$

index lowered w/ $\eta_{..}$

transformation parameter
ANTISYM, PURE Im → 6 INDEP.

eg $\omega^{12} = -\omega^{21} = \theta$ is a ROT ABOUT z
$\omega^{01} = -\omega^{10} = \eta$ is a BOOST ALONG x

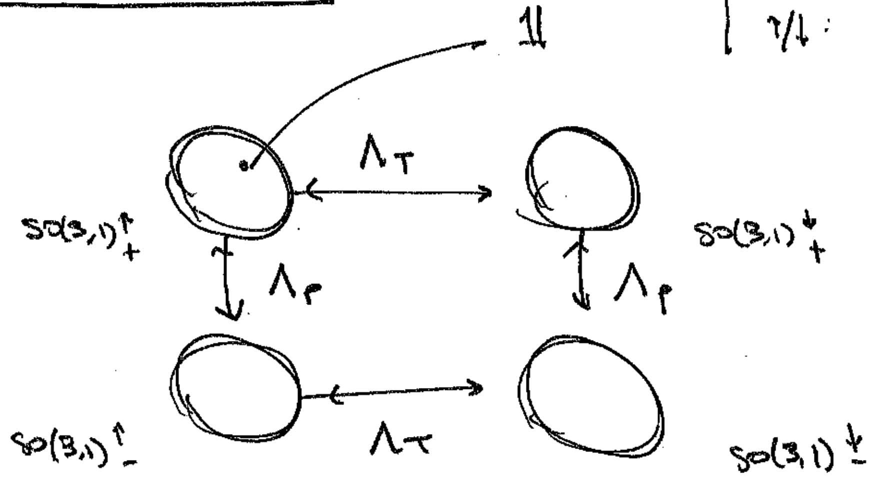
$$\boxed{J_i = \tfrac{1}{2}\varepsilon_{ijk} M_{jk}} \qquad \boxed{K_i = M_{0i}}$$

ROT. BOOST

DISCONNECTED-ness

$\pm$: $\det \Lambda = \pm$
$\uparrow/\downarrow$: $\Lambda^0{}_0 = \pm\sqrt{1+\Sigma(\Lambda^i{}_0)^2}$

$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$



$$\left.\begin{array}{l}\Lambda_T = (-,+,+,+)\\ \Lambda_P = (+,-,-,-)\end{array}\right\}\text{ discrete sym}$$

we live on $so(3,1)^\uparrow_+$: continuously connected to $\mathbb{1}$

makes sense to talk about infinitesimal transformations being exponentiated

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Dynkin: there are two $su(2)$'s living in $so(4)$
→ we explicitly constructed them

$$\boxed{A_i = \tfrac{1}{2}(J_i + iK_i)} \quad \boxed{B_i = \tfrac{1}{2}(J_i - iK_i)}$$

nb: nothing to do w/ roots! (NOT RAISING/LOWERING)

$$[A_i, A_j] = i\varepsilon_{ijk} A_k$$
$$[B_i, B_j] = i\varepsilon_{ijk} B_k$$
$$[A_i, B_j] = 0$$

$\underline{A}$, $\underline{B}$, not hermitian → certainly not $SU(2)\times SU(2)$
↳ but something LIKE $SU(2)\times SU(2)$

CLAIM : $\mathbb{C}$ lin comb of LORENTZ ALGEBRA are $\boxed{\text{ISOMORPHIC}}$ ← identical mathematically
to $\mathbb{C}$ lin comb of $SU(2)\times SU(2)$ ALGEBRA

CLAIM : this COMPLEXIFICATION of $SU(2)^2$ is the SPECIAL LINEAR GROUP
$SL(2,\mathbb{C})$ ← 2×2 MATRICES w/ $\mathbb{C}$ ELEM. & det 1
← -- SPECIAL : universal cover of LORENTZ

MORE IMPORTANT :

The Lorentz Group is isomorphic to $SL(2,\mathbb{C})/\mathbb{Z}_2$

DOUBLE COVER

4-vector in Minkowski ⟷ $\mathbb{C}$ HERMITIAN 2×2 MATRIX

$$V^\mu \longrightarrow V_\mu \sigma^\mu$$

$$\sigma^\mu = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} & -i \\ i & \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right\}$$

$$X = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \longleftrightarrow \underbrace{\begin{pmatrix} x_0 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix}}_{\text{n.b. lowered indices.}}$$

call this $\underset{\sim}{X}$

HOW DO WE GO THE OTHER WAY?

$$\mathrm{Tr}\,(\sigma^i \sigma^j) = 2\delta^{ij}$$
$$\mathrm{Tr}\,(\sigma^i \underbrace{\mathbb{1}}_{\sigma^0}) = 0$$

$$\boxed{x_0 = \tfrac{1}{2}\mathrm{Tr}\,\underset{\sim}{x}} \qquad \boxed{x_i = \tfrac{1}{2}\mathrm{Tr}\,(\underset{\sim}{x}\,\sigma^i)}$$

So far: 4 dof $\longleftrightarrow$ 4 dof. no big deal.

<u>WANT</u>: INVARIANCE of $x \cdot x$, LORENTZ TRANSF ...

$$x \cdot x = \underbrace{x_0^2 - x_1^2 - x_2^2 - x_3^2} \longleftarrow |x|^2$$

for $\underset{\sim}{x}$, this combination shows up in the <u>DET</u>

$$\det\begin{pmatrix} (x_0 + x_3) & (x_1 + ix_2) \\ (x_1 - ix_2) & (x_0 - x_3) \end{pmatrix} = |x|^2$$

further, consider $\underset{\sim}{N} \in SL(2, \mathbb{C})$

then $\underset{\sim}{N}^\dagger \underset{\sim}{x} \underset{\sim}{N}$ is also in space of HERMITIAN 2x2 MATRICES

SAME SPACE AS $\underset{\sim}{x}$

BUT THAT MEANS THAT THERE IS SOME y^μ s.t.

$$\underset{\sim}{N}^\dagger \underset{\sim}{x} \underset{\sim}{N} = \underset{\sim}{y}$$
$$= (\Lambda x)_\mu \sigma^\mu$$

this is some Lorentz transf of x^μ

$$\underset{\sim}{N}^{+}(x^{\mu}\sigma_{\mu})\underset{\sim}{N} = (\Lambda x)^{\mu}\sigma_{\mu}$$

two diff. representations of a Lorentz transform! one is a "matrix rotation", other is the 'usual' $x \to \Lambda x$.

<u>OBSERVE</u>: there is an apparent REDUNDANCY. $\underset{\sim}{N}$ & $-\underset{\sim}{N}$ yield the same transform.

$$\boxed{\underset{\sim}{N}^{+}\underset{\sim}{x}\underset{\sim}{N} = (-\underset{\sim}{N})^{+}\underset{\sim}{x}(-\underset{\sim}{N})} = (\Lambda_N x)^{\mu}\sigma_{\mu}$$

<u>some</u> Λ corresp. to $\pm\underset{\sim}{N}$

$\Rightarrow$ $SO(3,1)$ isn't isomorphic to $SL(2,\mathbb{C})$

... but to $SL(2,\mathbb{C})/\mathbb{Z}_2$

<u>So WHAT?</u> you may <u>miss something</u> if you look only at REPS OF $SO(3,1)$ & not $SL(2,\mathbb{C})$.

$\hookrightarrow$ *the* SPINOR.

<u>eh?!</u> what's so great about $SL(2,\mathbb{C})$ over $SO(3,1)$??

$\rightarrow$ $\boxed{SL(2,\mathbb{C}) \text{ is SIMPLY CONNECTED}}$

as a group manifold.

SIMPLE CONNECTIVITY of $SL(2,\mathbb{C})$

(sketch)

POLAR DECOMP: $g \in SL(2,\mathbb{C})$

$$g = (\text{UNITARY})\, e^{(\text{TRACELESS HERMITIAN})}$$

$$\underbrace{\underbrace{\begin{pmatrix} d+ie & f-ig \\ f+ig & d-ie \end{pmatrix}}_{d^2+e^2+f^2+g^2=1}}_{S^3} \qquad \underbrace{\underbrace{\begin{pmatrix} c & a-ib \\ a+ib & -c \end{pmatrix}}_{\text{unconstr.}}}_{\mathbb{R}^3}$$

so TOPOLOGICALLY, $SL(2,\mathbb{C}) = \underbrace{\mathbb{R}^3 \times S_3}$

EACH IS SIMPLY CONNECTED, SO PRODUCT IS, TOO.

in contrast, $SL(2,\mathbb{C})/\mathbb{Z}_2 = SO(3,1)^{\uparrow}$ is not simply connected.

so what: SIMPLY CONNECTED: CAN REACH ANY ELEMENT FROM EXPONENTIATING A GENERATOR @ $\mathbb{1}$

or: the Algebra is what is important!

... the elements of $SO(3,1)$ connected to the $\mathbb{1}$ miss the SPINOR

FACT. for ANY LIE GROUP, $\exists$ UNIQUE MINIMAL SIMPLY CONNECTED GROUP THAT IS IDENTICAL (HOMEOMORPHIC) TO IT

└ universal covering group.

$\forall\ G \quad \exists$ universal cover $\tilde{G}$

s.t. $\exists$ homomorphism

$\Pi : \tilde{G} \to G$ where $G \cong \tilde{G} / \underbrace{\ker \Pi}$

DISCRETE SUBGROUP of CENTER OF $\tilde{G}$

the key take-away: THE LORENTZ GROUP IS COVERED BY $SL(2,\mathbb{C})$

sometimes called $Spin(3,1)$

ANOTHER PERSPECTIVE: why the SPINOR is more "FUNDAMENTAL" than the VECTOR.

GROUP THY: $\underbrace{U(g_1)}_{\text{REP OF GROUP}} U(g_2) = \underbrace{U(g_1 g_2)}_{\text{REP. INHERITS GROUP STRUCTURE}}$

QUANTUM MECH: physical states are invariant under rephasing: ψ & $e^{i\alpha}\psi$ describe same state.

so: $U(g_1)U(g_2) = U(g_1 g_2) \underbrace{e^{i\phi(g_1,g_2)}}_{\text{REP "UP TO A PHASE"}}$

These are called PROJECTIVE REPRESENTATIONS.