LEC 4 : Adjoint REP, SU(3)

16 JAN 19

the ADJOINT : see notes from lecture 3. (we didn't get to the ADJOINT)

Reminder of nomenclature:

REPRESENTATION : vector space that "gets rotated"

group element ⟶ VERB
(ALGEBRA is infinitesimal)

vector (representation) ⟶ OBJECT

$$\begin{pmatrix} \cdot \\ \cdot \\ \vdots \\ \cdot \end{pmatrix} \Big\} \text{ some \# of components (DIMENSION of REP)}$$

each component is labelled by a weight

basis of vector space

$$\text{eg } \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a|1\rangle + b|0\rangle + c|-1\rangle$$

for SPIN-1 REP of SU(2)

the <u>weights</u> are eigenvalues of the diagonal generator(s)

<u>CARTAN SUBALGEBRA</u>, H

"maximal set of mutually commuting generators"

$$T^3 |m\rangle = m |m\rangle$$

$$\underline{d(T^3)} = \begin{pmatrix} j & & & & \\ & j-1 & & & \\ & & j-2 & & \\ & & & \ddots & \\ & & & & -j \end{pmatrix}$$

"REPRESENTATION of T^3"

MY MISTAKE — WRONG WORD! CALL THEM RAISING/LOWERING OPS

the off diagonal generators : $\boxed{\text{ROOTS}}$

(the rest of 'em)

OR: ROOT VECTORS

↳ can be paired into raising & lowering operators

CONSEQUENCE of HERMITICITY

$$\frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \; , \; \frac{1}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} & 0 \\ 1 & \end{pmatrix} \; , \; \begin{pmatrix} & 1 \\ 0 & \end{pmatrix}$$

RAISING / LOWERING: $\underbrace{\text{changes weight of state}}$

moves components of
the vector (rep) around

what is the weight of "$T^+|m\rangle$" ?

$$T^3\left[T^+|m\rangle\right] = (m+1)\left[T^+|m\rangle\right]$$

oh the weight is $(m+1)$

NOW GO TO SU(3)

↳ we wrote matrices in lec 3 notes

KEY POINTS :

CARTAN: $\{ T^3, T^8 \}$

BEST : 3 PAIRS of RAISING & LOWERING

obs: only 2 labels... but 3 directions!

How do we know that these are the generators?

SU(3) : 3×3 matrices M s.t.

$$\begin{aligned} &(1) \quad \det M = 1 \\ &(2) \quad M^\dagger M = \mathbb{1} \end{aligned}$$

<u>Semi pf</u> : WRITE $M = \exp(-iT\theta)$

STRATEGY: differentiate by θ @ $\theta = 0$

hard one: (1) ← use Jacobi's formula.

HAND WAVY:

M is unitary → diagonalizable

$M = U\hat{M}U^\dagger$ for $\hat{M}$ diagonal, U unitary

$$\begin{aligned} \Rightarrow \det(M) &= \det(U^\dagger \hat{M} U) \\ &= \det(\hat{M}) \\ &= \det(1 - i\theta\hat{T} + \cdots) \\ &= (1 - i\theta t_1)(1 - i\theta t_2)\cdots \end{aligned}$$

DIAGONAL TOO

$$\hat{T} = \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & \ddots \end{pmatrix}$$

$$\frac{d}{d\theta}\det(M)\Big|_{\theta=0} = -i\sum t_i$$

$$= -i\,\mathrm{Tr}\,\hat{T}$$

$$= -i\,\mathrm{Tr}\,U^{+}\hat{T}U$$

$$= 0 \qquad = -i\,\mathrm{Tr}\,T$$

REMARK.

if $UMU^{+} = \hat{M}$ $\cancel{M = U^{+}\hat{M}U}$

$$U\left(1 - i\theta T + \frac{(-i\theta)^2}{2}T^2 \cdots\right)U^{+}$$

$$UU^{+} - i\theta\, \underline{UTU^{+}} - \frac{(-i\theta)^2}{2}\, \underline{UTU^{+}}\,\underline{UTU^{+}} + \ldots$$

$$= e^{-i\theta\hat{T}} \longleftrightarrow \hat{T} = UTU^{+}$$

SIMILARLY:

$$\frac{d}{dt}\left(M^\dagger M\right) = 0$$

$$\dot{M}^\dagger M + M^\dagger \dot{M}$$

$$= +i\theta T^\dagger - i\theta T$$

$$\Rightarrow \boxed{T^\dagger = T}$$

SU(3)

$$T^+ = \frac{1}{2}\left(\begin{array}{cc|c} 0 & 1 & \\ \hline & & \\ \end{array}\right) \quad ; \quad T^-$$

$$V^+ = \frac{1}{2}\begin{pmatrix} & & 1 \\ & 0 & \\ 0 & & \end{pmatrix} \quad ; \quad V^-$$

$$U^+ = \frac{1}{2}\left(\begin{array}{c|cc} & & \\ \hline & & 1 \\ & 0 & \end{array}\right) \quad ; \quad U^-$$

$$T^3 = \frac{1}{2}\left(\begin{array}{cc|c} 1 & & \\ & -1 & \\ \hline & & \end{array}\right) \qquad T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

COMMUTATION RELATIONS

$$[T^+, T^-] = 2T^3$$
$$[T^3, T^\pm] = \pm T^\pm$$
$$[T^8, T^\pm] = 0$$

$$[V^+, V^-] = T^3 + \sqrt{3}T^8$$
$$\left.\begin{aligned}[T^3, V^\pm] &= \pm \tfrac{1}{2}V^\pm \\ [T^8, V^\pm] &= \pm \tfrac{\sqrt{3}}{2}V^\pm\end{aligned}\right\} \quad [\tfrac{1}{2}T^3 + \tfrac{\sqrt{3}}{2}T^8, V^\pm] = \pm V^\pm$$

$$[U^+, U^-] = -T^3 + \sqrt{3}T^8$$
$$\left.\begin{aligned}[T^3, U^\pm] &= \mp \tfrac{1}{2}U^\pm \\ [T^8, U^\pm] &= \pm \tfrac{\sqrt{3}}{2}U^\pm\end{aligned}\right\} \quad [-\tfrac{1}{2}T^3 + \tfrac{\sqrt{3}}{2}T^8, U^\pm] = \pm U^\pm$$

$$[T^\pm, V^\pm] = \mp U^\mp$$
$$[T^\mp, U^\pm] = \pm V^\pm$$
$$[V^\pm, U^\pm] = \pm T^\pm$$

"3 copies of SU(2)" ; state: $|p, q\rangle$ — (p,q) WEIGHT VECTOR

$$T^3 |p, q\rangle = p |p, q\rangle$$
$$T^8 |p, q\rangle = q |p, q\rangle$$

SUPPOSE WE RAISE/LOWER $|p,q\rangle$.
EITHER $\longrightarrow$ annihilate (highest p, lowest p
highest q, lowest q)

OR:

$T^{\pm}|p,q\rangle$ HAS WEIGHT $(p,q) \pm (1,0)$

$V^{\pm}|p,q\rangle$ —— " —— $(p,q) \pm (\frac{1}{2}, \frac{\sqrt{3}}{2})$

$U^{\pm}|p,q\rangle$ —— " —— $(p,q) \pm (-\frac{1}{2}, \frac{\sqrt{3}}{2})$

$\boxed{SU(2)}$: HIGHEST WEIGHT IS $j = \overset{1,2,3,\dots}{n}/2$
LOWEST —————— $-j$

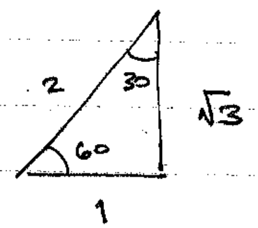
$\boxed{SO}$: $p_{max} = n_1/2$

$$\frac{1}{2} p_{max} + \frac{\sqrt{3}}{2} q_{max} = n_2/2$$

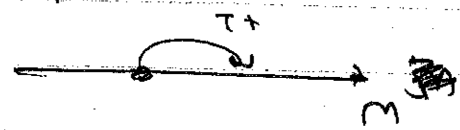
$$-\frac{1}{2} p_{max} + \frac{\sqrt{3}}{2} q_{max} = n_3/2$$

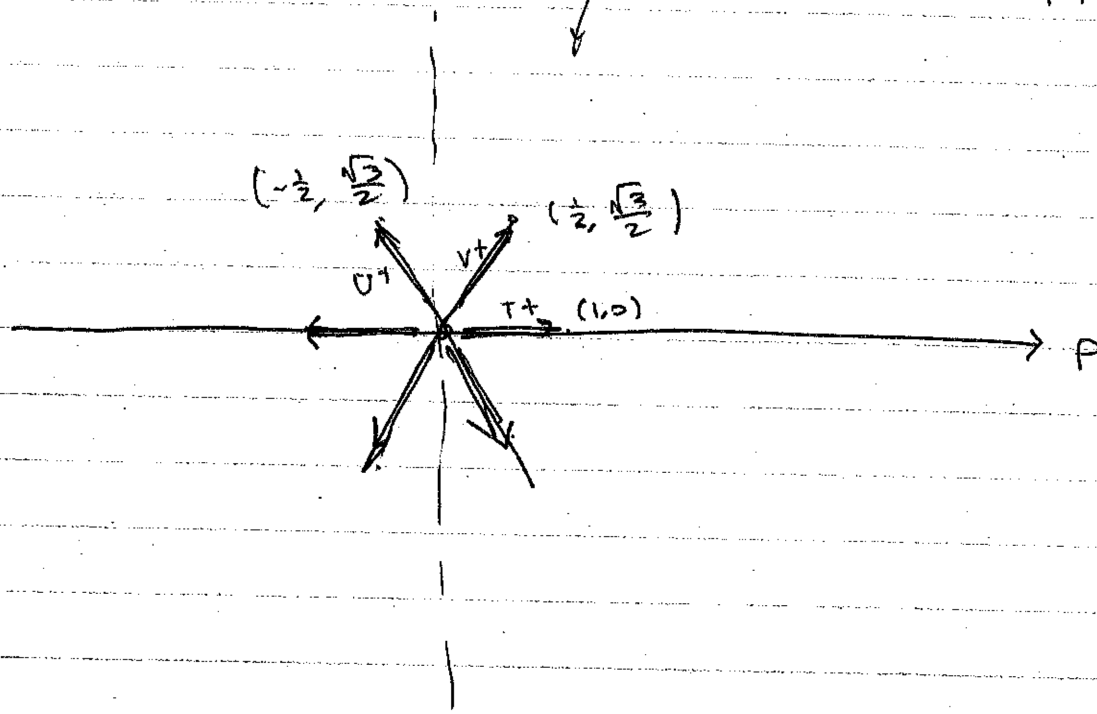
$\Rightarrow$ $\boxed{\frac{2}{\sqrt{3}} q_{max} \text{ is a counting number}}$

FACT:

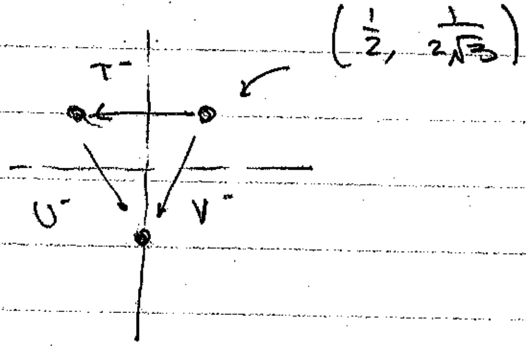


compare to





eg: SUPPOSE HEIGHEST WEIGHT IS $\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$



why this? analog of $j = \frac{1}{2}$

$p_{max} = \frac{1}{2}$

$\frac{1}{2} p_{max} + \frac{\sqrt{3}}{2} q_{max} = \frac{1}{2}$

↓

$q_{max} = \frac{1}{2\sqrt{3}}$

$$\underbrace{-\frac{1}{2} p_{max} + \frac{\sqrt{3}}{2} q_{max} = 0}_{\text{"trivial w/rt U"}}$$

THE ADJOINT (REP)

the matrices (OF THE FUNDAMENTAL $d(T)$) are themselves a basis of a representation.

USE: $[T^A, T^B] = i f^{ABC} T^C$

So now the $T \to$ matrix $ad(T)$
$\to$ basis $|T\rangle$

the F^{ABC} GIVE THE MATRIX ELEMENTS

$$\boxed{ad(T^A)_{BC} = -i f^{ABC}}$$

$$ad(T^A)|T^B\rangle = |-i[T^A, T^B]\rangle$$

$$= f^{ABC}|T^C\rangle$$

$\to$ see eq @ end of LEC 3.