P262 LEC 5: REPS of SU(3) 23 JAN '19

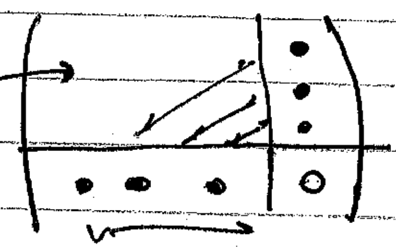
REVIEW: Generators of SU(3)

$\boxed{\text{RAISING \& LOWERING}}$ $\boxed{\text{CARTAN SUBALGEBRA}}$

Q: HOW MANY?
3. WE CALLED THEM $T^\pm, V^\pm, U^\pm$

Q. if $H_1, H_2 \in$ CARTAN what is $[H_1, H_2]$?
$\to 0$

Q: HOW MANY for SU(4)?
6.



this block has $T^\pm, U^\pm, V^\pm$

3 ADDITIONAL

Q. what is generic form of $[H_1, T^\pm]$?
$\to$ # $T^\pm$
can be negative
encodes "geometry" (really trigonometry)

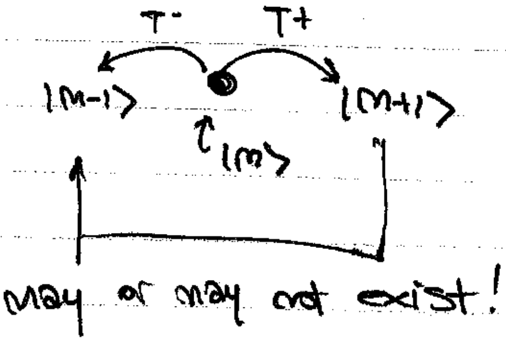
Q. HOW MANY ELEMENTS IN CARTAN OF SU(N)?
$\to (N-1)$

diagonal, traceless, Hermitian $N \times N$

- diag + Hermitian: $\mathbb{R}$ entries on diag
- traceless: one constraint on N diag elem.

<u>BACK TO su(3)</u>: 3 pairs of raising/lowering
→ 3 DIRECTIONS WHERE YOU CAN HOP FROM ONE STATE TO ANOTHER

<u>of su(2)</u>:



nb: "HOPPING BETWEEN STATES" is the "ROTATION"
this is how one element of a vector turns into another

RECALL: Spin-1 : $\begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$ ← $|1\rangle$ component, ← $|0\rangle$, ← $|-1\rangle$

s.t. $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a|1\rangle + b|0\rangle + c|-1\rangle$

what you can do: using $[\cdot,\cdot]$ relations, write $d(T^a)$ as a 3×3 matrix

finite ROTATION : $D(g(\theta)) = \exp(-i\theta d(T))$

the off diag elem of $d(T)$ ⟷ lin comb of RAISING/LOWERING (back to SU(3))

3 RAISING/LOWERING → "but" only 2 cartan elements

# of cartan elements tells you about the indices of states

do not confuse w/ indices of vector in rep.

$|p, q\rangle$

if you want, you can call these quantum #'s (nothing to do w/ QUANTUM MECHANICS!)

A REP. of SU(3):

some dimension.

$$\begin{pmatrix} \vdots \\ \vdots \end{pmatrix} \quad \begin{matrix} |p_1, q_1\rangle \\ |p_2, q_2\rangle \\ \vdots \end{matrix}$$

for A SET OF STATES indexed by $(p_1, q_1), (p_2, q_2), \ldots$

s.t. $\begin{pmatrix} a \\ b \\ \vdots \end{pmatrix} = a|p_1, q_1\rangle + b|p_2, q_2\rangle + \cdots$

so now the problem is: how do we find the set of (p_i, q_i) for a given representation?

using our conventions in su(2) pdf

in su(2): $T^- |m\rangle = N_m |m-1\rangle$
$T^+ |m-1\rangle = N_m |m\rangle$

s.t. $\langle m | m \rangle = 1$

$$\Rightarrow \boxed{N_m = \sqrt{(j-m+1)(j+m)}}$$

- we ~~picked~~ j to be highest weight
  $T^+ |j\rangle = 0$
  which we can see in N_m: $(j+1-(j+1)) = 0$ when $m-1=j$
  (of $T^+|m\rangle$ eqn above)

- WE ALSO SEE THAT $\boxed{m=-j}$ IS LOWEST WEIGHT.
  $(j+m)=0$ when $m=-j$

- T^- DECREASES m BY one unit.
  So the DISTANCE BTWN $m_{max}=j$ & $m_{min}=-j$ MUST BE A COUNTING #:

$$n = m_{max} - m_{min} = 2j \rightarrow \boxed{j = n/2}$$

APPLY THIS TO SU(3):
↳ no additional work!

The raising & lowering pairs T, V, U all look like copies of su(2)

in general: let $X^\pm$ be a raising/lowering pair.

let $H_{1,2}$ be elements of the cartan

→ can be ANY elements (not nec the ones we chose explicitly)

→ subject to NORMALIZATION

in "defining rep" (really just from pdbc) → $\mathrm{Tr}(H^2) = \frac{1}{2}$

GUTOWSKI

[in fact: there is something like a metric in the ALGEBRA:

not a metric: not positive def. → $K(T^a, T^b) = \mathrm{Tr}\, \mathrm{ad}(T^a)\, \mathrm{ad}(T^b)$]

this is subtle. see eg. Cahn ch. III
Gutowski § 2.17

$$[X^+, X^-] = c_1 H_1 + c_2 H_2$$
$$[H_1, X^\pm] = \pm a_1 X^\pm$$
$$[H_2, X^\pm] = \pm a_2 X^\pm$$

~~the $\pm$ follows from H^T, diag, traceless, NEGATIVE ELEMENTS~~

the $\pm$ is easy: $X^+ = (X^-)^T$
and H is diagonal: $H^T = H$

$$[H, X^+] = C$$
$$HX^+ - X^+H = C$$

transpose

$$(X^+)^T H^T - H^T (X^+)^T = C^T$$
$$\shortparallel$$
$$-[H^T, X^-] = C^T$$
$$[H, X^-] = -C^T$$

$= X^-$ when $C = X^+$

<u>FACT</u>: $c_1 = 2a_1$, $c_2 = 2a_2$ } w/ our norm of $X^\pm$ (GOES BACK TO OUR SU(2) CONVENTIONS)

EXTRA CREDIT: PROVE THIS (JACOBI ID SHOULD DO IT) these commutation relations hold for **EACH** PAIR OF RAISE/LOWER OPS

in other words: for **EACH** RAISE/LOW:

$$[X^+, X^-] = 2X^3 \leftarrow X^3 = a_1 H_1 + a_2 H_2$$
$$[X^3, X^\pm] = \pm X^\pm$$

(IN CARTAN)

so for **EACH** raise/lower
there is some quantum number m_x
$\boxed{\text{AS AN } SU(2)}$

this is our trick!

SUCH THAT: $\exists$ heighest weight j_x
$X^\pm$ are raising/lowering
lowest weight is $-j_x$

AND $j_x = n_x/2$

for some COUNTING # n_x

THIS IS WHY $SU(2)$ IS THE "BASIC MODULE" OF LIE ALGEBRAS

SO NOW WE KNOW A LOT ABOUT <u>EACH</u> RAISING/LOWERING <u>SEPARATELY</u>.

PUTTING IT TOGETHER: only 2 indep. CARTAN elem.
→ really only 2 indices (p, q) for states

BUT 3 quantum numbers: m_T, m_V, m_U
these are related by the coefficients (a_1, a_2) ← in turn, fixed by NORMALIZATION OF H_1, H_2

LET'S GO BACK TO OUR EXPLICIT $T^\pm, V^\pm, U^\pm$ BASIS TO BE CLEAR:

$$m_T = \overset{a_1}{p}$$
$$m_V = \tfrac{1}{2}p + \tfrac{\sqrt{3}}{2}\overset{a_2}{q}$$
$$m_U = -\tfrac{1}{2}p + \tfrac{\sqrt{3}}{2}q \quad \leftarrow \text{not indep.}$$

<u>HEIGHEST WEIGHT</u>

$2j$ is a counting # (from SU(2))
$\Rightarrow 2p_{max} = n_T$
$\Rightarrow p_{max} + \sqrt{3}\,q_{max} = n_V$
$\Rightarrow -p_{max} + \sqrt{3}\,q_{max} = n_U$

$\Big\}$ $2\sqrt{3}\,q_{max}$ is COUNTING #
↑
(1 error in LEC 4))

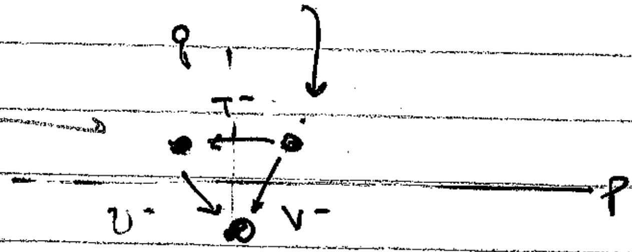
SO: 2 CHOICES : $p_{max} = n_p/2$

$\sqrt{3}\, q_{max} = n_q/2$

FUNDAMENTAL : "$\underline{3}$ of SU(3)"

pick $(p, q)_{max} = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$

$\left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$



$\left(0, \frac{1}{2\sqrt{3}} - \frac{\sqrt{3}}{2}\right) = \left(0, -\frac{1}{\sqrt{3}}\right)$

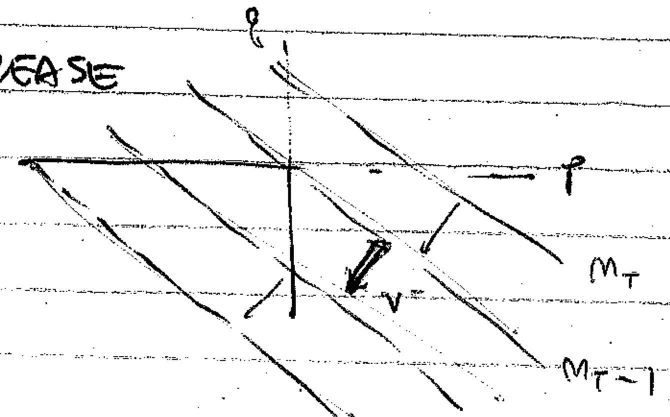
T^- DECREASES $M_T = p$ BY 1

V^- DECREASES $M_T = \frac{1}{2}p + \frac{\sqrt{3}}{2}q$ BY 1

DECREASE

OR: $[T^3, V^-] = -\frac{1}{2}V^-$

$[T^8, V^-] = -\frac{\sqrt{3}}{2}V^-$



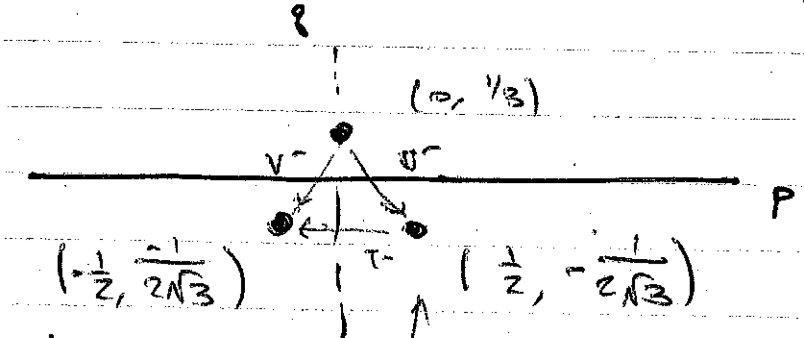
LINES OF CONST M_T

$q = \frac{2}{\sqrt{3}} M_T - \frac{1}{\sqrt{3}} p$

SIMILARLY FOR U^-

Another rep : $\overline{3}$, antifundamental ← or $\overline{\mathbf{3}}$, "three-bar"

$$(p, q)_{max} = \left(0, \frac{1}{\sqrt{3}}\right) \leftarrow n_p = 0, \; n_q = 2$$



$$\begin{cases} -p_{max} + \sqrt{3}\, q_{max} = 1 \\ p_{max} + \sqrt{3}\, q_{max} = 1 \\ 2p_{max} = 0 \end{cases}$$

on p.8

BUT → nb: ∃ states w/ HIGHER p than "p_{max}"

it's not p_{max} ... it's $(p,q)_{max}$!

ALSO: this is just $(p, q) \longmapsto (-p, -q)$ from $\mathbf{3}$!