P262 : LEC 14: the Lorentz Group

2/20/19

WHY? ultimately → the spin representations
↳ hopefully broad interest, eg cond-mat

the EASY PART: You already know "the answers"
↳ SPECIAL RELATIVITY

So let's rediscover SPECIAL RELATIVITY from A GROUP THEORETICAL POINT OF VIEW.
↳ new for us: non-compact groups

LORENTZ GROUP (or "homogeneous Lorentz")

you have a $\boxed{\text{METRIC}}$, $\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

μ

DEFINES AN INNER PRODUCT ON VECTORS → RAISES/LOWERS INDICES.

$\mu, \nu = \underset{t}{0}, \underbrace{1,2,3}_{x,y,z}$

$$x \cdot y = \underbrace{\eta_{\mu\nu} x^\mu}_{= x_\nu} y^\nu = x^0 y^0 \underbrace{- x^1 y^1 - x^2 y^2 - x^3 y^3}_{= -\underline{x} \cdot \underline{y}}$$

$= x_\nu$ a "ROW VEC" or "DUAL VEC" (covar vs. contrav.)

$\underline{x} \cdot \underline{y}$: 3-vectors in SPACE

LORENTZ TRANSFORMATIONS PRESERVE THE METRIC:

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}$$

or: $\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \underbrace{(\Lambda^T)_\rho{}^\mu \eta_{\mu\nu} \Lambda^\nu{}_\sigma}_{} = \eta_{\rho\sigma}$

$$\underline{\underline{\Lambda}}^T \underline{\underline{\eta}} \underline{\underline{\Lambda}} = \underline{\underline{\eta}}$$

ANALOG to <u>ROTATIONS</u>

metric is just $g_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

$$\underline{x} \cdot \underline{y} = \underbrace{g_{ij} x^i}_{x_i} y^j$$

Rotations preserve the inner product

$$\underline{x} \mapsto R^i{}_j x^j \qquad \underline{x} \cdot \underline{y} \to g_{ij} R^i{}_k x^k R^j{}_l y^l$$

$$= \underbrace{x^k R^i{}_k}_{\text{ORDERING OF INDICES IS FUNNY}} g_{ij} R^j{}_l y^l$$

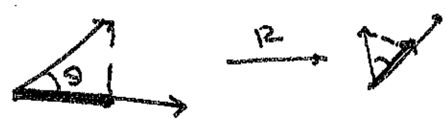
$$= x^k (R^T)_k{}^i \underbrace{g_{ij} R^j{}_l}_{\text{WANT THIS} = g_{kl}} y^l$$

$$= x^k g_{kl} y^l = \underline{x} \cdot \underline{y}$$

$$\Rightarrow \underline{\underline{R}}^T \underline{\underline{g}}\, \underline{\underline{R}} = \underline{\underline{g}}$$

BUT $\underline{\underline{g}} = \mathbb{1} \longrightarrow \underline{\underline{R}}^T \underline{\underline{R}} = \mathbb{1}$

in 2D: $\underline{x} \cdot \underline{y} = x^1 y^1 + x^2 y^2$



$x^2 = (x^1)^2 + (x^2)^2$

ah: can guess $R = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$

in 2D <u>MINKOWSKI</u>: $x^2 = (x^0)^2 - (x^1)^2$

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\eta & +\sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \qquad \cosh\eta = \frac{e^\eta + e^{-\eta}}{2} \qquad \sinh\eta = \frac{e^\eta - e^{-\eta}}{2}$$

HW: relate this to β & γ in RELATIVITY

~~GENERATO~~ OBSERVE : unbounded

eh... this probably requires some mathematical def...

AS $\eta \to$ HUGE : $\cosh \to \frac{1}{2}e^{\eta}$

$\sinh \to \frac{1}{2}e^{\eta}$

$$\Lambda x = \frac{1}{2} e^{\eta} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \frac{1}{2} e^{\eta} \begin{pmatrix} x^0 + x^1 \\ x^0 + x^1 \end{pmatrix}$$

$\eta \gg 1$ → "ARB far in the future & arb. far away.."

actually, this is not quite the right thing to act on.

USUALLY LORENTZ / MINKOWSKI b/c we're in a local, inertial frame ⇝ in tangent space

WHERE MOMENTA live

[will talk about translations soon]
BOOSTS ARE NOT TRANSLATIONS.

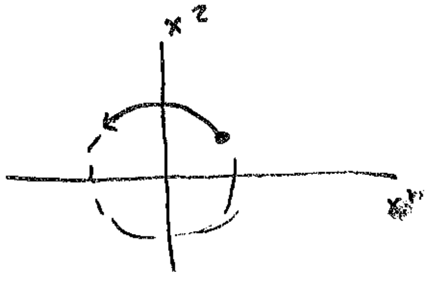
$\eta \gg 1$

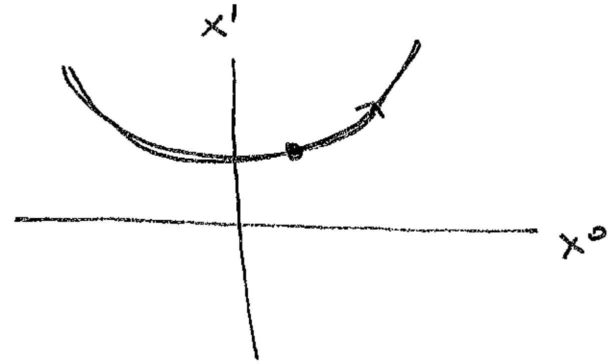
$$\Lambda p = \frac{1}{2} e^{\eta} \begin{pmatrix} E \\ p \end{pmatrix}$$ ← ARBITRARILY Hi ENERGY & ARBITRARILY Hi MOMENTUM

go hand-in-hand

$t \begin{pmatrix} E \\ p \end{pmatrix}$

VS:





ROTATIONS: NEVER LEAVE SOME FINITE REGION

never comes back

$\sim e^{i\theta}$

$\sim e^{\eta}$

Generators of LORENTZ

$$\underline{\underline{\Lambda}} = e^{it\underline{\underline{W}}} \longleftarrow \text{plug into } \underline{\underline{\Lambda}}^T \eta \underline{\underline{\Lambda}} = \eta$$

$$= 1 + it\underline{\underline{W}} \qquad \text{\& take leading order}$$

$$(1 + it\underline{\underline{W}}^T)\,\eta\,(1 + it\underline{\underline{W}}) = \eta$$

$$= \cancel{\eta} + it\underbrace{(\underline{\underline{W}}^T\eta + \eta\underline{\underline{W}}^T)}_{=0} = \cancel{\eta}$$

$$(W^T)_\mu{}^\alpha \eta_{\alpha\nu} + \eta_{\mu\alpha} W^\alpha{}_\nu = 0$$

$$= (W^T)_{\mu\nu} + W_{\mu\nu} = \boxed{W_{\mu\nu} + W_{\nu\mu} = 0}$$

well: pure IM

in 4D: antisymmetric, REAL, 4x4

sounds a lot like so(4)

... BUT so(4) didn't have those η's floating around.

how many generators? 6

$\boxed{\text{so 6 PARAMETERS for THE TRANSFORMATION.}}$

LET'S PACKAGE THE PARAMETERS AS antisym, $\mathbb{R}$, 4x4's.

$$itW^{\lambda\sigma} = i\,t_{(A)} W_{(A)}^{\lambda\sigma} = i\,\boxed{\omega^{\mu\nu}}\,(M_{\mu\nu})^{\lambda\sigma}$$

$A = 1, \cdots, 16$

" $t_{(A)}$ " WHERE $\mu,\nu \in 0,\dots,3$ $\omega^{\mu\nu}$ $\mathbb{R}$. antisym.

" $W_{(A)}{}^{\lambda\sigma}$ " BUT USING ANTISYM μ,ν AS INDEX.

A USEFUL BASIS:

$$(M_{\mu\nu})^{\rho\sigma} \equiv i\left(\delta_\mu^\rho \delta_\nu^\sigma - \delta_\mu^\sigma \delta_\nu^\rho\right) = i\,\delta_\mu^{[\rho}\delta_\nu^{\sigma]}$$

SO WE CAN "ROTATE" BY θ about 1-2 axis:

$$\omega^{12} = -\omega^{21} = \theta, \text{ all others zero} \leftarrow \text{PARAM}$$

PICKS OUT $M_{12} = M_{21}$ generator

$$M_{12} = i \begin{array}{cc} & \sigma \\ \rho & \begin{pmatrix} 0 & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & 0 \end{pmatrix} \end{array} \qquad i\begin{pmatrix} 0 & & & \\ & 0 & -1 & \\ & 1 & 0 & \\ & & & 0 \end{pmatrix} = -M_{12}$$

$$i\,\omega^{\mu\nu}(M_{\mu\nu})^{\rho\sigma} = i\,\overset{\theta}{\omega^{12}}(M_{12})^{\rho\sigma} + i\,\overset{-\theta}{\omega^{21}}(M_{21})^{\rho\sigma}$$

sum over μ & ν

... most are zero

$$= 2i\begin{pmatrix} 0 & & & \\ & 0 & \theta & \\ & -\theta & 0 & \\ & & & 0 \end{pmatrix}$$

Global PARTITIONING of the LORENTZ GROUP

$$\underline{\underline{\Lambda}}^T \eta \underline{\underline{\Lambda}} = \eta$$

$$\boxed{\begin{aligned} (\det \underline{\Lambda})^2 &= 1 \\ (\Lambda^0{}_0)^2 - \sum_i (\Lambda^i{}_0)^2 &= 1 \end{aligned}}$$

$$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$$

$\rho, \sigma = 0$

TWO BINARY CHOICES. ← (4) LORENTZ GROUPS

$$\boxed{\det \Lambda = \pm 1}$$

$\det \Lambda = +1$ contains $\mathbb{1}$
& preserves PARITY

$$\boxed{\Lambda^0{}_0 = \pm\sqrt{1 + \sum_i (\Lambda^i{}_0)^2}}$$

$\Lambda^0{}_0 > 0$ preserves arrow of time.

no way to continuously go to $\det \Lambda = -1$
... <u>disconnected</u> GROUPS!

the Lorentz group that we care about most is the ORTHOCHRONUS LORENTZ

$$\boxed{SO(3,1)^{\uparrow}_{+}}$$

$\uparrow$ ← time
$+$ ← $\det \Lambda$

ROT. BOOSTS
(rel minus in η)

DISCRETE TRANSFORMATIONS THAT JUMP BETWEEN DISCONNECTED ISLANDS OF THE LORENTZ GROUP.

$\Lambda_P = \mathrm{diag}(+,-,-,-)$

$\Lambda_T = \mathrm{diag}(-,+,+,+)$

<u>LOCALLY</u>: contains $\mathbb{1}$

THE LORENTZ GROUP is RELATED to $SU(2)\times SU(2)$

$$SO(3,1) \approx SU(2)\times SU(2)$$

I DON'T KNOW the appropriate relation

6 GENERATORS, $(M_{\mu\nu})$.

3 ROTATIONS: $J_i \equiv \frac{1}{2}\varepsilon_{ijk} M_{jk}$

3 BOOSTS: $K_i \equiv M_{0i}$

ALGEBRA:

$$[J_i, J_j] = i\varepsilon_{ijk} J_k$$
$$[K_i, K_j] = -i\varepsilon_{ijk} J_k$$
$$[J_i, K_j] = i\varepsilon_{ijk} K_k$$

clever step:

$$A_i \equiv \frac{1}{2}(J_i + iK_i)$$
$$B_i \equiv \frac{1}{2}(J_i - iK_i)$$

$\longrightarrow$

$$\left.\begin{aligned} [A_i, A_j] &= i\varepsilon_{ijk} A_k \\ [B_i, B_j] &= i\varepsilon_{ijk} B_k \\ [A_i, B_j] &= 0 \end{aligned}\right\} \text{ they separate!}$$

$\boxed{\text{BUT}} \to$ A_i & B_i ARE NOT HERMITIAN (not ROOTS, either!)

LIE GROUP $\to$ HERMITIAN GENS, no?

@ LEAST $SU(2)\times SU(2)$ IS HERMITIAN! (COMPACT)

so $SO(3,1) \neq SU(2)\times SU(2)$

Physics: A & B $\Longleftrightarrow$ LH & RH SPINORS.

technical :

isomorphic

$\mathbb{C}$ lin comb of $\underset{\text{ALG}}{SO(3,1)} \cong \mathbb{C}$ lin comb of $\underset{\text{ALG}}{SU(2)\times SU(2)}$.

$$\mathcal{L}_{\mathbb{C}}[SO(3,1)] \cong \mathcal{L}_{\mathbb{C}}[SU(2)\times SU(2)]$$

complexification

turns out that
the COMPLEXIFICATION
OF $SU(2)\times SU(2)$ IS
$SL(2, \mathbb{C})$
SPECIAL LINEAR GROUP.

Poincaré :

translations

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu$$

$(\Lambda, a) \in$ POINCARÉ IDENTITY: $(\mathbb{1}, 0)$

$$(\Lambda_2, a_2)\cdot(\Lambda_1, a_1) = (\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2)$$

$$\left.\begin{aligned} (\Lambda, 0)\cdot(\mathbb{1}, a) &= (\Lambda, \Lambda a) \\ (\mathbb{1}, a)\cdot(\Lambda, a) &= (\Lambda, a) \end{aligned}\right\} \text{ don't commute!}$$

called a <u>SEMI-DIRECT PRODUCT</u>

<u>ALGEBRA</u> :

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(M^{\mu\sigma}\eta^{\nu\rho} + \cdots)$$

$$[P^\mu, P^\nu] = 0$$

$$[M^{\mu\nu}, P^\sigma] = i(P^\mu \eta^{\nu\sigma} - P^\nu \eta^{\mu\sigma})$$