METHOD of INDUCED REPS

A SKETCH of Weinberg QFT Vol. 1. §2.5

$\boxed{\text{what is a particle?}}$

state $\psi_\sigma(p)$ s.t. $\hat{P}^\mu \psi_\sigma(p) = p^\mu \psi_\sigma(p)$

$\psi_\sigma(p)$ ← QUANTUM #'s W/RT "SPIN" (WANT TO UNDERSTAND THIS)

$\hat{P}^\mu$ ← OPERATOR

p^μ ← EIG. VAL

(nb) NOW WE'RE DEALING W/ Poincaré GROUP

LORENTZ & TRANSLATIONS ← REP: $U(\ldots)$.

RECALL: $\boxed{U(\Lambda', a')\, U(\Lambda, a) = U(\Lambda'\Lambda, \Lambda' a + a')}$

momentum generates translations

$$U(\mathbb{1}, a)\psi(p) = e^{-ip\cdot a}\psi(p)$$

what about Lorentz?

THIS IS THE "WATCH CAREFULLY" MOMENT: $(U(\Lambda) = U(\Lambda, 0))$

$$\underbrace{\hat{P}^\mu U(\Lambda)\psi(p)}_{\text{what is momentum of } U(\Lambda)\psi\text{?}} = \underbrace{U(\Lambda)\left[U^{-1}(\Lambda)\right.}_{\mathbb{1}}\left. P^\mu U(\Lambda)\right]\psi$$

this is a useful (familiar) combination → like finite transformation of an Adjoint

(2.4.6) $$\underbrace{U(\Lambda, a)\, U(\mathbb{1}, \varepsilon)}_{\mathbb{1} - \varepsilon_\rho P^\rho}\, U^{-1}(\Lambda, 0) = \underbrace{U(\mathbb{1}, \Lambda\varepsilon)}_{\mathbb{1} - (\Lambda\varepsilon)_\mu P^\mu}$$

$$\eta_{\mu\nu}\Lambda^\nu{}_\rho\,\varepsilon^\rho = \Lambda_\mu{}^\rho\,\varepsilon_\rho$$

Lemma

$$\Rightarrow U(\Lambda, 0)\, P^\rho\, U^{-1}(\Lambda, 0) = \boxed{\Lambda_\mu{}^\rho P^\mu}$$

how to interpret funny position of indices?

REMARK: $$\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$$

$$\underbrace{\eta^{\sigma\alpha}\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma} = \delta^\alpha_\rho$$

$$\Lambda^\mu{}_\rho\Lambda_\mu{}^\alpha = \delta^\alpha{}_\rho$$

$$\Rightarrow \boxed{(\Lambda^{-1})^\alpha{}_\mu = \Lambda_\mu{}^\alpha}$$

$$\boxed{U(\Lambda)\, P^\rho\, U^{-1}(\Lambda) = (\Lambda^{-1})^\rho{}_\mu P^\mu}$$

ORDER OF INVERSES!

$$U^{-1}(\Lambda)\, P^\mu\, U(\Lambda)$$

continuing:

$$P^\mu U(\Lambda)\Psi(p) = U(\Lambda)\underbrace{(\Lambda^{+1})^\mu{}_\rho P^\rho}_{(\Lambda P)^\mu}\Psi$$

$$= (\Lambda p)^\mu$$ ← now just some #s!

$$= (\Lambda p)^\mu\, \boxed{U(\Lambda)\Psi(p)}$$

<u>So what?</u>

$$\hat{P}^{\mu} \boxed{U(\Lambda)\Psi(p)} = (\Lambda p)^{\mu} \boxed{U(\Lambda)\Psi(p)}$$

kind of obvious in retrospect!
the 4-MOMENTUM of $U(\Lambda)\Psi(p)$ is Λp!

<u>RESULT:</u> $U(\Lambda)\Psi(p)$ is some <u>combination</u> of states w/ 4-MOMENTUM Λp.

$$U(\Lambda)\Psi_{\sigma}(p) = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda, p)\, \Psi_{\sigma'}(\Lambda p)$$

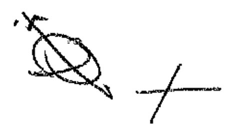
<u>not</u> LORENTZ 4-VEC INDICES!

SOME GENERAL INDEX for <u>some</u> kind OF LORENTZ REP.

<u>REM</u>: ASSUME $C_{\sigma\sigma'}(\Lambda, p)$ IS BLOCK-DIAGONAL

↳ EACH BLOCK IS AN IRREP of POINCARÉ

this is a good def for a <u>particle</u>.

( same as )

what can we know/learn re $C_{\sigma'\sigma}$?

WE KNOW that A GIVEN 4-MOMENTUM p^μ HAS TWO INVARIANTS:

$$p^2 \quad \& \quad \text{sign}(p^0)$$

So: for EACH CHOICE OF THESE, DEFINE a STANDARD 4-MOMENTUM

$$K^\mu \qquad \text{s.t.} \quad p^\mu = \underbrace{L^\mu{}_\nu(p)}\, K^\nu$$

eg $(M, \underline{0})$ for MASSIVE
$(K,0,0,K)$ for MASSLESS

standard

SOME LORENTZ TO TURN STANDARD TO THE SPECIFIC 4-MOMENTUM YOU WANT.

implicit DEP on K.

this is just saying: LET'S LOOK @ EACH PARTICLE "in the same way"

an electron moving @ 100 GeV is still an electron $\Rightarrow$ might as well talk about it in the rest frame.

thus we can relate $\Psi_\sigma(p)$ to $\Psi_\sigma(K)$

$$\Psi_\sigma(p) \equiv \underbrace{N(p)}_{\text{NORM.}}\ \underbrace{U(L(p))\Psi_\sigma(K)}_{\text{state w/ 4-momentum } p = LK}$$

nb: σ labels unaffected.

now ask: How does Λ affect $\Psi(p)$?

$$U(\Lambda)\Psi_\sigma(p) = N(p)\, U(\Lambda L(p))\, \Psi_\sigma(k)$$
$$= N(p)\, \underline{U(L(\Lambda p))}\, \underline{U(L^{-1}(\Lambda p))}\, U(\Lambda L(p))\, \Psi_\sigma(k)$$
$$= N(p)\, U(L(\Lambda p))\, U[\underbrace{L^{-1}(\Lambda p)\Lambda L(p)}]\, \Psi_\sigma(k)$$

this is a lorentz transf. that takes $k^\mu \to k^\mu$

$$W^\mu{}_\nu = L^{-1}(\Lambda p)\, \Lambda\, L(p) \quad : k \to k$$

this is part of a SUBGROUP called the Little Group. → reshuffles σ indices

$$U(W)\Psi_\sigma(k) = \sum_{\sigma'} \underbrace{D_{\sigma\sigma'}(W)}\Psi_{\sigma'}(k)$$

representation of the LITTLE GROUP

$$D(WW') = D(W)D(W')$$

$W = L^{-1}(\Lambda p)\,\Lambda\, L(p)$

$$U(\Lambda)\Psi_\sigma(p) = N(p)\, U(L(\Lambda p)) \sum_{\sigma'} \underbrace{D_{\sigma\sigma'}(W)}_{\text{"just \#'s"}} \Psi_{\sigma'}(k)$$

$$= N(p) \sum_{\sigma'} D_{\sigma\sigma'}(W)\, \boxed{U(L(\Lambda p))\Psi_{\sigma'}(k)}$$

$$= \frac{1}{N(\Lambda p)}\Psi_{\sigma'}(\Lambda p) \quad \text{from p.4}$$

$$U(\Lambda)\Psi_\sigma(p) = \frac{N(p)}{N(\Lambda p)} \sum_{\sigma'} D_{\sigma\sigma'}(W)\ \Psi_{\sigma'}(\Lambda p)$$

(SO): $C_{\sigma\sigma'}$ boils down to understanding <u>little group</u> *method of induced ~~✱~~ reps.*

			LITTLE GROUP
(A)	$p^2 = m^2$	$k^\mu = (M, \underline{0})$	$SO(3)$
(B)	$p^2 = 0$	$k^\mu = (k, 0, 0, k)$	$ISO(2)$ ← 2D transl + rot.

<u>MASSIVE PARTICLE</u>: same transformation as in NR QM

} all that 1/2 integer stuff ⟶ { spherical harmonics, clebsch-Gordon, etc carry over

⟶ this is actually somewhat surprising!

<u>MASSLESS</u>: more subtle.

$$U(W(\theta, \underbrace{\alpha, \beta}_{\text{trans.}})) = 1 + i\alpha A + i\beta B + i\theta J_3$$

rot.

$A = J_2 + K_1$
$B = -J_1 + K_2$

give continuous spin quantum #

⟶ must be zero radius.

EIGENVALUE

$J_3 \Psi_{k,\sigma} = \sigma \Psi_{k,\sigma}$

<u>HELICITY</u>

why 1/2 integer.

$$U(W) = e^{i\theta\sigma}\Psi_\sigma(k) \rightarrow D_{\sigma'\sigma}(W) = e^{i\theta\sigma}\delta_{\sigma\sigma'}$$