P262



# $\boxed{\text{LEC 7}}$ WEIGHT DIAGRAMS

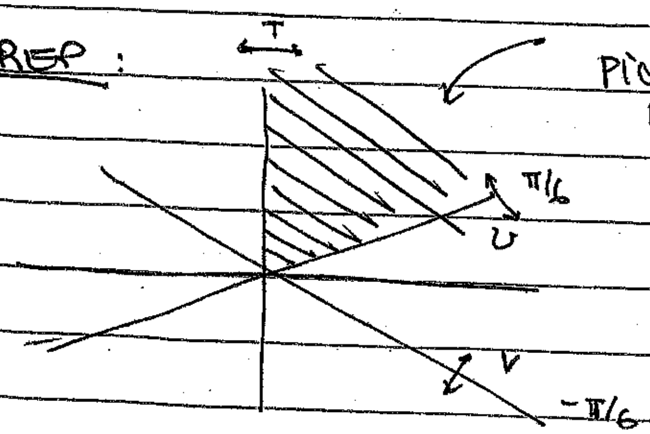
1/28/19

LAST TIME: use the cross raising/lowering [...]

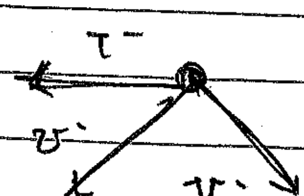
$$[T^\pm, V^\mp] = \mp U^\mp$$
$$[T^\pm, U^\pm] = \pm V^\pm$$
$$[V^\pm, U^\mp] = \pm T^\pm$$

& all others zero.

BUILD A REP:




→ then hit it with all combos of lowering operators




@ this point: you know all the states up to multiplicities
↳ you know each su(2) terminates
& you know symmetry axes: $\pm\pi/6$, $\pi/2$

# TRIANGULAR DIAGRAMS

ANSWER: EACH STATE HAS MULTIPLICITY 1
SO THE WEIGHT DIAGRAM IS EASY.

WHY? STATES ARE LABELLED BY

$$\underbrace{\prod (T^-, V^-, U^-)}_{\text{some product of lowering ops, in some order}} \; \underbrace{|\text{MAX}\rangle}_{\text{HIGHEST WEIGHT}}$$

WHY NOT RAISING OPS?
CAN ALWAYS COMMUTE THEM
TO THE RIGHT UNTIL THEY
ANNIHILATE $|\text{MAX}\rangle$

either: LEAVE BEHIND PROD. OF LOWERING, OR SOME NEW RAISING OPS W/ FEWER TERMS IN THE PRODUCT ... REPEAT UNTIL GONE

WE OBSERVED: for  weight diagram.

(I) $T^- |\text{MAX}\rangle = 0$

(II) $[V^-, U^-] = 0$

(III) $[T^-, U^-] = -V^-$

(IV) $[T^-, V^-] = 0$

(II) → ORDER OF $V^-$, $U^-$ DOESN'T MATTER.
A STATE WRITTEN AS $n$ POWERS OF $V^-$
$m$ POWERS OF $U^-$
IS <u>EQUIVALENT</u> TO

$$(V^-)^n (U^-)^m \, |MAX\rangle$$

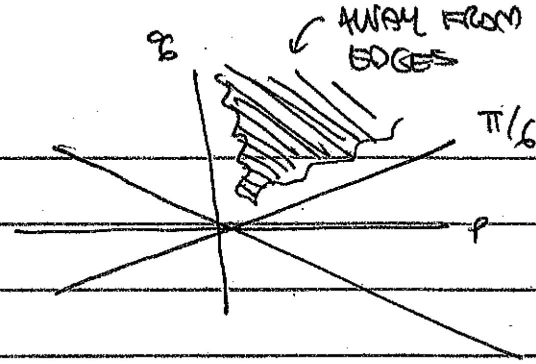
(I) → A STATE W/ $(T^-)$ SOMEWHERE IN $\Pi(T^-, U^-, V^-)$
CAN BE WRITTEN IN TERMS OF STATE(S)
WITH JUST $U^-$ AND $V^-$

(D) → $[T^-, V^-] = 0$ : COMMUTE $T^-$ TO RIGHT OF
ANY $(V^-)$'s. → IF YOU HIT $|MAX\rangle$,
ANNIHILATE

$[T^-, U^-] = -V^-$ : COMMUTE $T^-$ TO RIGHT OF
$(U^-)$, GET $(-V^-)$ ...
WHICH IS A TERM WHICH
FALLS INTO $(V^-)^n (U^-)^m |MAX\rangle$
FORMAT.

→ *That's it.* [There are many ways to argue this
page ~ use whatever makes sense
to you.]

$\boxed{\text{HW}}$ : ▽ DIAGRAMS ARE ALSO MULTIPLICITY = 1



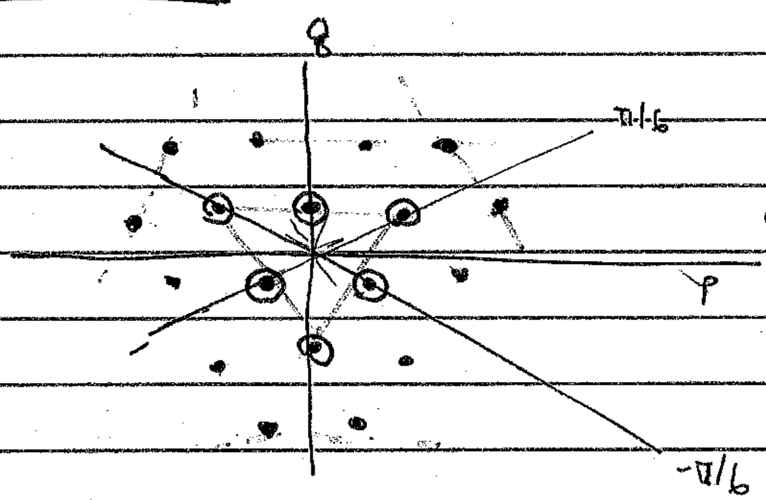

## HEX DIAGRAMS

what is different?

$T^-|max\rangle \neq 0$
$V^-|max\rangle \neq 0$
$U^-|max\rangle \neq 0$

BTW: something may feel asymm: we had $|max\rangle$ w/ $T^-|max\rangle = 0$, $V^-|max\rangle = 0$ ... why not $U^-|max\rangle = 0$? (positivity)

DRAW AN EXAMPLE (helps to start from inside)



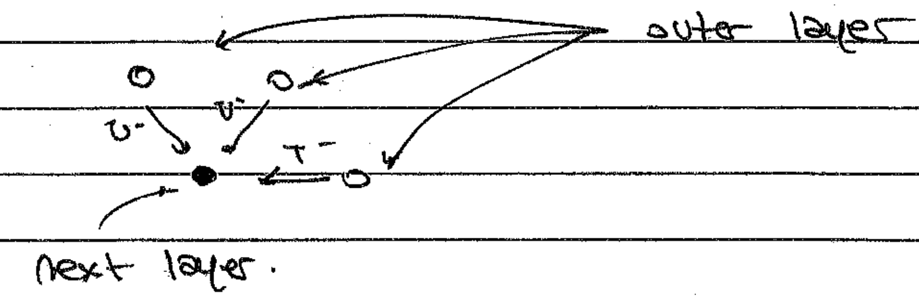

GEN FEAT: OUTER HEX
INNER TRI

① OUTERMOST (hex) LAYER : MULTIPLICITY = 1
reminder: $U^-|M\rangle$ vs. $T^+V^-|M\rangle = -U^-|M\rangle$ ($|MAX\rangle$)
then use symmetry of diagram w/rt $\pi/2$, $\pm\pi/6$
(symmetry of ALGEBRA)

(II) <u>next layer</u> : multiplicity increases by 1

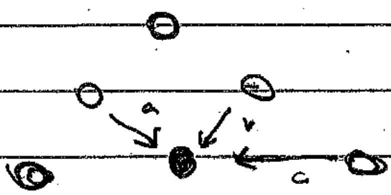
[ more formal pf. in Gutowski (4.1.3) - but I don't find it illustrative. ]

<u>why?</u>




THERE ARE 3 WAYS TO GET TO A $2^{nd}$ layer STATE.
↳ only 2 are independent.

COMPARE TO TRIANGULAR




a) $U^- \, (V^- |m\rangle)$
b) $V^- \, (U^- |m\rangle)$ } IDENTICAL
c) $\boxed{T^- \, (U^-)^2 |m\rangle}$

$$= [T^-, U^-] U^- |m\rangle + U^- T^- U^- |m\rangle$$

$$= -V^- U^- |m\rangle + U^- [T^-, U^-] |m\rangle + (U^-)^2 \underbrace{T^- |m\rangle}_{=0}$$

$$= \underbrace{-2 U^- V^- |m\rangle}_{\text{same as } U^+ 2} + \underbrace{\boxed{(U^-)^2 T^- |m\rangle}}_{\text{not zero for hex!}}$$

so (& you should play with this to check)

the second layer of a hex weight diagram has multiplicity 2

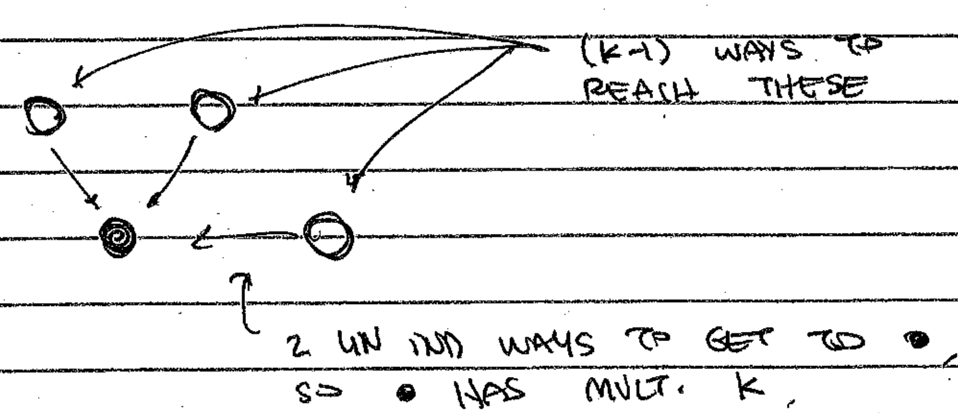
why? because there are two distinct ways to write a series of lowering ops on |MAX⟩

↳ it was critical that $T^-$, $U^-$, $V^-$ all did not annihilate |MAX⟩

HW → for some $n$, $(T^-)^n$, $(U^-)^n$, $(V^-)^n$ (EACH) will annih. |MAX⟩ (min such $n$)
↳ RELATE THIS $n$ to the LENGTH OF THE SHORTEST EDGE OF THE HEX.

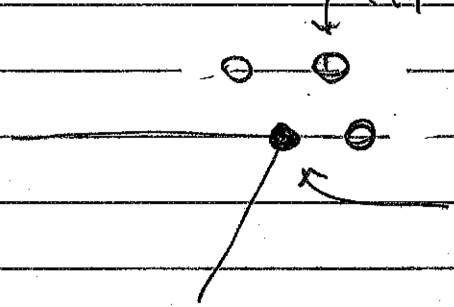
③ RECURSIVE STEP

in general: the $k^{th}$ layer, if hexagonal will have multiplicity $K$



(4) THIS CONTINUES UNTIL THE FIRST TRIANGULAR LAYER.

$(K_T - 1)$ LAYER



$K_T$ LAYER: 1st LAYER THAT IS TRIANGULAR

this is heighest weight of a tri ... happens to have mult. $K_T$

NOW: ALL STATES IN TRIANGLE HAVE MULTIPLICITY $K_T$ (no more incrementing)

HW → how many states in the hex diagram that we drew? (? DIMENSIONALITY of REP)

CORNER OF TRI

Why: $\bullet = \begin{cases} U^{-}\ \boxed{u}\ |max\rangle \\ V^{-}\ \boxed{v}\ |max\rangle \\ T^{-}\ \boxed{t}\ |max\rangle \end{cases}$

may contain some $U^{-}$'s.

cont'd

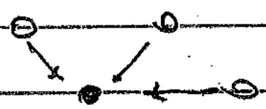
uniting ladder

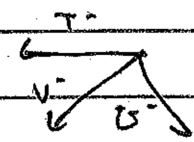
in our diagram (p.4):

$U^-|MAX\rangle \neq 0$, but $\boxed{(U^-)^2|MAX\rangle = 0}$

makes us hex.

So consider 1st to 2nd layer:





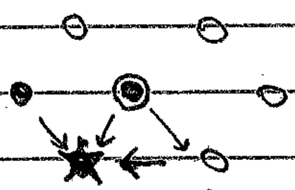

$U^-(T^-|m\rangle)$

$V^-|m\rangle$

$[T^-,U^-] = -V^-$ $\quad T^-(\underbrace{U^-|m\rangle}_{\neq 0})$

$$U^-T^-|m\rangle = -V^-|m\rangle + T^-(\underbrace{U^-|m\rangle}_{\neq 0})$$

Next one down (still 2ND layer)




⊙ $= V^-|m\rangle,\ T^-U^-|m\rangle$

$$\bigstar \begin{cases} = U^-|\bullet\rangle \\ = (V^-)^2|m\rangle \\ = V^-T^-U^-|m\rangle \\ = T^-V^-U^-|m\rangle \end{cases}$$

(I) $U^-|\circ\rangle$ is the same as

$$\{ (U^-)^2|m\rangle, \; V^-T^-U^-|m\rangle \}$$

eg: if $|\circ\rangle = V^-T^-|MAX\rangle$

$$\begin{aligned} U^-|\circ\rangle &= U^-V^-T^-|MAX\rangle \\ &= V^-U^-T^-|MAX\rangle \\ &= V^-T^-U^-|MAX\rangle \quad \checkmark \end{aligned}$$

( ? we know $|\circ\rangle$ is a state of multiplicity 2 )

So $U^-|\circ\rangle$ is not a new state

(II) $T^-V^-U^-|MAX\rangle = -\underbrace{(U^-)^2|MAX\rangle}_{=0} + \underbrace{V^-T^-U^-|MAX\rangle}_{\text{REDUNDANT}}$

so not a new state

Rest follows as a degenerate triangle.

## conjugate of rep

$$[d(T^a), d(T^b)] = i f^{abc} d(T^c)$$

$$\downarrow$$

$$[-d(T^a)^*, -d(T^b)^*] = i f^{abc}(-d(T^c)^*)$$

So $\boxed{\bar{d}(T^a) = -d(T^a)^*}$ is a rep.

$\hookrightarrow$ if $\bar{d}(T) = d(T) \longrightarrow \mathbb{R}$ representation

OTHERWISE: $\mathbb{C}$

$\boxed{\text{CARTAN GENERATORS}} \rightarrow -d(H_i)^*$

CARTAN IS HERMITIAN
SO CONJUGATE REP HAS <u>NEGATIVE</u> WEIGHTS COMPARED TO $d$.