HOMEWORK 1: **Representations of SU(2) and SU(3)**

COURSE:	Physics 262, *Group Theory for Physicists* (Fall 2019)
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DUE BY:	Be ready to discuss on Friday, Feb 8
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1 The spin-1 representation of SU(2)

The spin-1 representation of SU(2) is the three dimensional representation with highest weight $j = 1$ and states $|1\rangle$, $|0\rangle$, and $|-1\rangle$. A vector in this space is:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\,|1\rangle + b\,|0\rangle + c\,|-1\rangle \ . \tag{1.1}$$

1.1 Normalizations

We chose a normalization of our states so that:

$$T^- \,|m\rangle = N_m \,|m-1\rangle \tag{1.2}$$

$$T^+ \,|m-1\rangle = N_m \,|m\rangle \ , \tag{1.3}$$

where $|n\rangle$ is orthonormal. Write out N_m for each of the weights $m = \pm 1, 0$.

1.2 Raising, lowering, Cartan

Write out the explicit form of the generators in this representation: $d(T^+)$, $d(T^-)$, $d(T^3)$.

1.3 Generators in the 'usual' basis

Write out the explicit form of the generators in this representation: $d(T^1)$, $d(T^2)$, $d(T^3)$. Recall that:

$$d(T^\pm) = d(T^1) \pm i d(T^2) \ . \tag{1.4}$$

2 The adjoint representation of SU(2)

[**Flip**: Updated 1/24: typo on ε^{123}, $(-i)$ on $\text{ad}(T^a)^{bc} = -if^{abc}$. Version below is corrected.]

The **adjoint** representation is one where the generators themselves are states. The action of a generator (as a matrix) on a generator (as a state) is given by the **structure constant**, f^{abc}. Recall that the structure constant is defined by

$$[T^a, T^b] = if^{abc}T^c \ . \tag{2.1}$$

For SU(2) $f^{abc} = \varepsilon^{abc}$, the totally antisymmetric tensor with $\varepsilon^{123} = 1$. The matrices of the adjoint representation are:

$$\text{ad}(T^a)^{bc} = -if^{abc} \ . \tag{2.2}$$

In other words, the action of a generator in the adjoint representation $\text{ad}(T^a)$ on a state $\left|T^b\right\rangle$ is

$$\text{ad}(T^a)\left|T^b\right\rangle = -if^{abc}\left|T^c\right\rangle \ . \tag{2.3}$$

[**Flip**: Update: removed irrelevant information.]

2.1 Dimension of the adjoint representation

What is the dimension of the adjoint representation? (How many basis states are there?) Answer: three. (You may want to write one sentence explaining why the answer is three. If the answer is more than one sentence, then it's probably wrong.)

2.2 Dimension of the adjoint representation

[**Flip**: Update: clarified the basis. Why is the $\left|T^\pm\right\rangle, \left|T^3\right\rangle$ basis a useful one to use instead of $\left|T^{1,2}\right\rangle, \left|T^3\right\rangle$? (Think about the weight of the state.)].

Write out the explicit matrix form of $\text{ad}(T^3)$ acting on a basis of states $\left|T^\pm\right\rangle, \left|T^3\right\rangle$. Confirm that it matches $d(T^3)$ in the spin-1 representation acting on states $\left|m = \pm 1\right\rangle, \left|m = 0\right\rangle$. Write out the explicit matrix forms of $\text{ad}(T^1)$ and $\text{ad}(T^2)$ acting on the basis $\left|T^\pm\right\rangle, \left|T^3\right\rangle$. Observe that this is *not* quite the same as $d(T^1)$ and $d(T^2)$ of the spin-1 repesentation.

2.3 What gives?

Confirm that $\text{ad}(T^1)$, $\text{ad}(T^2)$, and $\text{ad}(T^3)$ satisfy the SU(2) commutation relations:

$$[\text{ad}(T^a), \text{ad}(T^b)] = i\varepsilon^{abc}\text{ad}(T^c) \ . \tag{2.4}$$

Confirm that if some set of matrices $d(T)$ are a representation—that is, they satisfy the algebra's commutation relations—then another set of matrices that differ by a unitary transformation, $\widetilde{d}(T) \equiv Ud(T)U^\dagger$, is also a representation. Write down the matrix U that transforms the spin-1 representation into the adjoint representation. This proves that the spin-1 and adjoint representations are, in fact, the same. The difference between them is purely "cosmetic." Hint: `https://physics.stackexchange.com/q/279880/166736`. The difference has to do with the spherical basis of rotations.

[**Flip**: Remark: the basis $T^\pm$, T^3 forms an algebra that has different structure constants than ε^{abc}. That's okay. This basis is useful for building the representations of the group and understanding how they transform, but $T^\pm$ is not a Hermitian matrix.]

3 Properties of Algebras

Show that:

- If a matrix Lie group is defined to be **special** (unit determinant), then the algebra is made up of traceless matrices.
- If a matrix Lie group is defined to be **unitary**, then the algebra is made up of Hermitian matrices.

4 Adjoint representation of a group, algebra

[**Flip**: Update: Clarification and discussion.] Let G be a matrix Lie group and call the generators T^a. We can write elements of G that are "sufficiently close to the origin" as $g = \exp(-i\theta^a T^a)$. The adjoint representation of the *group* is the action of G on elements of its algebra:

$$\text{Ad}(g)\,|X\rangle = gXg^{-1} \ , \tag{4.1}$$

where $X = x^a T^a$ is an element of the algebra. Show that if $g = \exp(-i\epsilon^a T^a)$ for small angles ϵ^a, the adjoint action of the *group* reduces to the adjoint action of the *algebra*:

$$Ad(g)\,|X\rangle = \left(1 + c^a \text{ad}(T^a) + \mathcal{O}(\epsilon^2)\right) X \ . \tag{4.2}$$

Determine what c^a is. Observe that the action of $\text{ad}(T^a)$ on X is a commutator:

$$\text{ad}(T^a)\left|T^b\right\rangle = -if^{abc}\left|T^c\right\rangle = -\left|[T^a, T^b]\right\rangle \ . \tag{4.3}$$

Hint: See Gutowski sections 2.14 - 2.26.

Extra Credit

These problems are for your own edification. You are encouraged to explore them according to your own personal and research interests. **Relevant**: https://youtu.be/0obMRztklqU.

BCH

Derive the Baker-Campbell-Hausdorff formula. Start by looking up what the Baker-Campbell-Hausdorff formula is. I don't have anything deep to say about this, but going through the derivation once gives you a feel for how to think of the group and algebra as a manifold and tangent space.