Physics 262: SU(2) Conventions

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Abstract

Some conventions for group theory. Corrections and additions welcome. We roughly follow chapter 3.2 of Georgi with slightly different conventions.

Contents

1 Basis of Generators

The SU(2) algebra is

$$[T^a, T^b] = i\varepsilon^{abc}T^c \ . \tag{1.1}$$

Let us decompose these generators into a pair of raising/lowering operators $T^\pm$ and the diagonal generator, T^3. We choose to define the raising/lowering operators as:

$$T^\pm = T^1 \pm iT^2 \tag{1.2}$$

note that Georgi's definition differs by a factor of $1/\sqrt{2}$. These satisfy the commutation relations:

$$[T^3, T^\pm] = \pm T^\pm \tag{1.3}$$
$$[T^+, T-] = 2T^3 \ . \tag{1.4}$$

2 Weights

The **weight**, m, of a state is the T^3 eigenvalue:

$$T^3 |m\rangle = m |m\rangle \ . \tag{2.1}$$

The weight is an index for the different components of the representation.

3 Raising/Lowering Weights

The $T^\pm$ operators increment the weight of a state by ± 1. We can see this by applying T^3 to the state $T^\pm |m\rangle$:

$$T^3 T^\pm |m\rangle = \left([T^3, T^\pm] + T^\pm T^3\right)|m\rangle = (m \pm 1)\, T^\pm |m\rangle \ . \tag{3.1}$$

The result is that acting on a state with $T^\pm$ moves you to a new

4 Finite Dimensional Representations

Let us examine representations of SU(2) that have a finite number of weights—that is, a finite number of components. The $T^\pm$ operators mean that given a state with some state $|m\rangle$, you can "rotate" to a different state $|m \pm 1\rangle$. We are interested in finite dimensional representations—objects with only a finite number of components (weights) that rotate into each other. Thus this ladder must truncate for some maximum and minimum weights. Let j be the **maximum weight** of the representation:

$$T^+ |j\rangle = 0 \ . \tag{4.1}$$

Let us sequentially derive the states of the representation by applying the lowering operator. What we want to find is the value of the *minimum weight* so that we will know that we have found all of the states.

We can assume that $|j\rangle$ is normalized. However, we have no guarantee that $T^- |j\rangle$ is normalized. Thus let us define a normalization

$$T^- |j\rangle = N_j |j-1\rangle \ , \tag{4.2}$$

where $|j-1\rangle$ is normalized. Let's look at this first state below the maximum weight state.

4.1 Normalization of $|j-1\rangle$

We determine the normalization N_j by calculating the norm $||T^- |j\rangle ||$:

$$\begin{aligned} \langle j| \left(T^-\right)^\dagger T^- |j\rangle &= \langle j| T^+ T^- |j\rangle & (4.3)\\ &= \langle j| \left([T^+, T^-] + T^- T^+\right) |j\rangle & (4.4)\\ &= \langle j| 2T^3 |j\rangle & (4.5)\\ &= 2j \ . & (4.6) \end{aligned}$$

We have used $T^+ |j\rangle = 0$. Using the definition of the normalized state $|j-1\rangle$, this must equal:

$$2j = |N_j|^2 \langle j-1|j+1\rangle \ , \tag{4.7}$$

so that we have found the normalization

$$N_j = \sqrt{2j} \ . \tag{4.8}$$

Next we should check the normalization of the $T^+ |j-1\rangle$. To do this, we use the one strategy available: *use the commutation relations to simplify the expression.* Use the facts that

1. T^+ annihilates $|j\rangle$
2. $T^3 |m\rangle = m$.

Thus the raising of $|j-1\rangle$ gives:

$$T^+ |j-1\rangle = T^+ \left(\frac{1}{N_j} T^- |j\rangle \right) \tag{4.9}$$
$$= \frac{1}{N_j} \left([T^+, T^-] + T^- T^+ \right) |j\rangle \tag{4.10}$$
$$= \frac{2j}{N_j} |j\rangle \tag{4.11}$$
$$= N_j |j\rangle \ . \tag{4.12}$$

In the last line we explicitly use $N_j = \sqrt{2j}$. We observe that raising $|j-1\rangle$ and lowering $|j\rangle$ produce the same normalization:

$$T^- |j\rangle = N_j |j-1\rangle \tag{4.13}$$
$$T^+ |j-1\rangle = N_j |j\rangle \ . \tag{4.14}$$

4.2 Recursion relation

Now suppose that for some general weight $|m\rangle$ in our representation, we know that:

$$T^- |m+1\rangle = N_{m+1} |m\rangle \tag{4.15}$$
$$T^+ |m\rangle = N_{m+1} |m+1\rangle \ , \tag{4.16}$$

where N_{m+1} is known and the states $|m\rangle$, $|m+1\rangle$ are normalized. This is the case with the 'top of the ladder,' $(m+1) = 1$. We now derive N_m and show that the same relation holds for $m \to (m-1)$. N_m is defined by

$$T^- |m\rangle = N_m |m-1\rangle \ . \tag{4.17}$$

N_m is fixed by requiring that $|m-1\rangle$ is normalized, $\langle m-1|m-1\rangle = 1$. Then we have:

$$N_m^2 = \langle m| T^+ T^- |m\rangle \tag{4.18}$$
$$= \langle m| \left([T^+, T^-] + T^- T^+ \right) |m\rangle \tag{4.19}$$
$$= 2m + N_{m+1}^2 \ . \tag{4.20}$$

We thus have a recursion relation for N_m in terms of N_{m+1}.

To complete the recursion, we must prove that

$$T^+ |m-1\rangle = N_m |m\rangle \ . \tag{4.21}$$

The left-hand side is

$$T^+ |m-1\rangle = T^+ \left(\frac{1}{N_m} T^- |m\rangle \right) \tag{4.22}$$

$$= \frac{1}{N_m} \left([T+, T-] + T^- T^+ \right) |m\rangle \tag{4.23}$$

$$= \frac{1}{N_m} \left(2m |m\rangle + N_{m+1} T^- |m+1\rangle \right) \tag{4.24}$$

$$= \frac{1}{N_m} \left(2m |m\rangle + N_{m+1}^2 |m\rangle \right) \tag{4.25}$$

$$= N_m |m\rangle \ , \tag{4.26}$$

where we have used the explicit result, (4.20).

4.3 Closed expression for the normalization

We can imagine writing down the relations for the squared normalizations:

$$\begin{array}{ccccc} N_j^2 & & & = & 2j \\ N_{j-1}^2 & - & N_j^2 & = & 2(j-1) \\ N_{j-2}^2 & - & N_{j-1}^2 & = & 2(j-2) \\ \vdots & & & & \vdots \\ N_{j-k}^2 & - & N_{j-k-1}^2 & = & 2(j-k) \end{array} \tag{4.27}$$

Sum both sides of the equations and recognizing that there are $(k+1)$ terms,

$$N_{j-k}^2 = 2(k-1)j - 2\frac{k(k+1)}{2} \tag{4.28}$$

$$= (k+1)(2j-k) \ . \tag{4.29}$$

Rather than using "k steps below the highest weight, " it is more tidy to use the normalization of the m^{th} rung of the ladder via $k = j - m$:

$$N_m = \sqrt{(j-m+1)(j+m)} \ . \tag{4.30}$$

5 Bottom of the ladder

In order for the representation to be finite, we need the ladder to truncate at the bottom. There is thus some a limit to the number of times you can hit the highest weight state $|j\rangle$ with T^- before it annihilates. Let this number be ℓ. Then we want to use

$$T^- |j - \ell\rangle = 0 \tag{5.1}$$

with the normalization $N_{j-\ell}$ to determine what this ℓ is.

$$N_{j-\ell} = \sqrt{(\ell+1)(2j-\ell)} \ . \tag{5.2}$$

Observe that $(\ell+1) > 0$ since ℓ is a positive natural number. Then the requirement that the ladder terminates, $N_{j-\ell} = 0$, requires

$$\ell = 2j \ . \tag{5.3}$$

From this we observe:

1. The highest weight $m_{\text{max}} = j$ is half integer. It can be written as $\ell/2$ for some positive integer ℓ.
2. The lowest weight is simply $m_{\text{min}} = -j$.