<u>Review</u> : HOW DOES SPIN 1/2 ⊗ SPIN 1/2 ⊗ SPIN 1/2

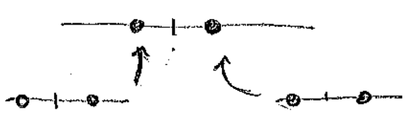
$\underline{2} \otimes \underline{2} \otimes \underline{2}$

DECOMPOSE INTO <u>IRREPS</u> ?

<u>principle</u> : • WEIGHTS ADD

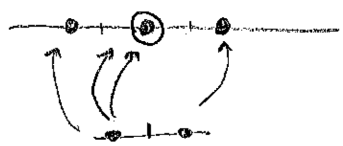
• ALGORITHM for IRREPS
- start @ highest avail weight
- lower w/ avail lowering ops

$\underline{2}$



COPIES OF SECOND $\underline{2}$ CENTERED @ EACH WEIGHT OF 1st $\underline{2}$

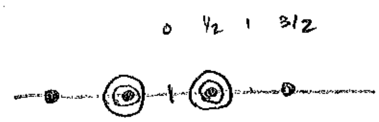
$\underline{2}\otimes\underline{2}$



you recognize from $\underline{2}\otimes\underline{2} = \underline{3}\oplus 1$

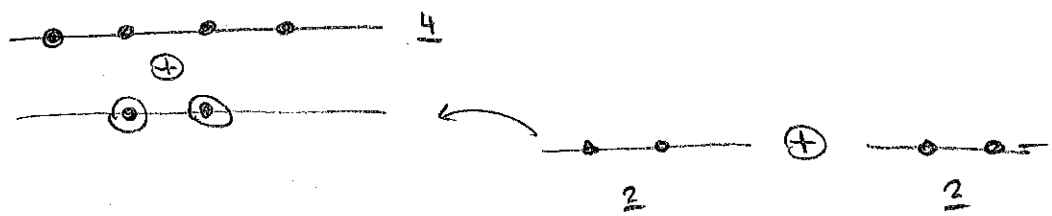
COPIES OF THIRD $\underline{2}$ @ EACH NODE INCLUDING MULTIPLICITY

$\underline{2}\otimes\underline{2}\otimes\underline{2}$



all the states
now LOWER FROM HIGHEST

=



$$\Rightarrow \underline{2}\otimes\underline{2}\otimes\underline{2} = \underline{4}\oplus\underline{2}\oplus\underline{2}$$

how are these $\underline{2}$'s Related?
the IRREPS are combinations of <u>all</u> the $\underline{2}$'s on the LHS

same principle for SU(3) diagrams
... just harder to draw → need to be careful. :-)

cheng & li
CH. 4.3

INDICES - *the FUNDAMENTAL*

$SU(N) \rightarrow N \times N$ unitary matrices, det = 1

these act on N-dim. $\mathbb{C}$ vec space

$$\psi^i \mapsto \boxed{\psi'^i = U^{ij} \psi^j}$$

reminder

$$\underline{\psi} = \sum_i \psi^i \underbrace{|e_{(i)}\rangle}$$

or $|\psi\rangle$

BASIS DOES NOT TRANSFORM

BASIS VECTOR
CARRIES VECTOR-NESS

set of numbers
[CARRIES INDEX THAT TRANSFORMS]

the $|e_{(i)}\rangle$ are the dots on our weight diagrams!

FOR SU(2): $|e_{(i)}\rangle \sim |m\rangle$

SU(3): $|e_{(i)}\rangle \sim |p, q\rangle$

SU(4): $|e_{(i)}\rangle \sim |p, q, r\rangle$

etc.

EUCLIDEAN INNER PRODUCT: $\boxed{\langle \psi, \chi \rangle = \sum_i (\psi^i)^* \chi^i}$

this is invariant under rotations.

how do we see it? there are no indices

INDICES w/rt GROUP (ALG.) REP

↳ transformation properties

CARRIES info about the $\underline{\text{IRREP}}$.

irreducible rep (things that transf. into each other)

why is $\langle \psi, \phi \rangle$ invariant?

$$\psi^{*i} \to \underbrace{U^{*ij}}_{\uparrow} \psi^j$$

conjugate: $\bar{d}(T^a) = -d(T^a)^*$

$$e^{-i\theta \bar{d}(T)} = e^{+i\theta d(T)} \quad \text{by Hermiticity}$$

then $(\psi^*)^i \chi^i \mapsto (U^{*ij}\psi^{*j})(U^{ik}\chi^k)$

$$\psi^{*j} \underbrace{(U^\dagger)^{ji}}_{\uparrow} U^{ik}\chi^k$$

$$(M^T)^{ij} = M^{ji}$$

<u>USEFUL NOTATION</u>: only contract upper & lower

$\psi_i = \psi^{*i}$ LOWER INDEX: conjugate rep

$U_i{}^j = (U^*)^{ij}$ — U^{ij} before

so: $\psi^i \mapsto \psi'^i = U^i{}_j \psi^j$

$\psi_i \mapsto \psi'_i = \underline{U_i{}^j} \psi_j \qquad \psi_i \to \psi_j (U^{-1})^j{}_i$

$= \psi_j U_i{}^j \qquad (U^{-1})^j{}_i$

$(U^*)^{ij} = (U^\dagger)^{ji}$

$$\langle \psi, \chi \rangle = \psi_i \chi^i \to \psi_j \underbrace{\boxed{U_i{}^j U^i{}_k}}_{=\delta^j{}_k} \chi^k$$

INVARIANT TENSORS of SU(N)

- Kronecker - δ

$$U_k{}^i U^k{}_j = \delta^i{}_j$$

this is kind of odd

$$U_k{}^i = (U^{-1})^i{}_k \qquad \text{from bottom of p. 3}$$

$$U_k{}^i = (U^\dagger)^{ki} = (U^\dagger)^{ki}$$

in "all upper" notation

$$\text{so: } \underbrace{U_k{}^i U^k{}_j}_{(U^{-1})^i{}_k U^k{}_j} = U_k{}^i \underbrace{\delta^k_\ell}_{\substack{\text{trivial } \delta \\ \text{"insert } \mathbb{1}\text{"}}} U^\ell{}_j = \delta^i{}_j$$

$$\Rightarrow \delta^i{}_j \longmapsto U_k{}^i U^\ell{}_j \delta^k_\ell = \delta^i{}_j \quad \text{INVT !}$$

- N-COMPONENT LEVI-CIVITA SYMBOL.

$$\varepsilon^{i_1 \cdots i_N} = \varepsilon_{i_1 \cdots i_N} = \begin{cases} 1 & \text{if even perm. of } (1, \ldots, N) \\ -1 & \text{if odd} \\ 0 & \text{otherwise} \end{cases}$$

$$\varepsilon^{i_1 \cdots i_N} \longmapsto U^{i_1}{}_{j_1} U^{i_2}{}_{j_2} \cdots U^{i_N}{}_{j_N} \varepsilon^{j_1 \cdots j_N}$$

HW

$$= \underbrace{\det(U)}_{=1} \varepsilon^{i_1 \cdots i_N}$$

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

FIX: $i=1$, $k=2$

CHECK: for SU(2): $U^i{}_j U^k{}_\ell \varepsilon^{j\ell} = U^1{}_1 U^2{}_2 - U^1{}_2 U^2{}_1$

$$= \det U \; \varepsilon^{12}$$

INVARIANT TENSORS GIVE US THINGS TO THROW AT TENSOR REPS TO GET IRREPS.

often we don't care about full decomp into IRREPS ⟶ we just want to get singlets

eg $SU(3)$: $\underline{3} \otimes \underline{3}$

$$\psi^{ij} = S^{ij} + \boxed{A^{ij}}$$

$\underline{6}$

what is this? 3 COMPONENTS. $\underline{3}$ or $\bar{\underline{3}}$?

$$A^{ij} \underline{\varepsilon_{ijk}} = A_k$$

antisym

SU(3) inv

can only contract UPPER + LOWER

one lower index : $\bar{3}$!

HW: PV.

$$\underline{3} \otimes \bar{\underline{3}} : \underbrace{\psi^i{}_j}_{\text{cannot sym/asym mixed indices}} = \psi^i{}_i \, \delta^{ij}_{i} + \underline{\psi^i{}_{j \neq 1}}$$

take the trace

what's this? WE KNOW IT'S AN $\underline{8}$

there are matrices that convert indices

↳ the most useful one to know: T^a themselves
(THE ONLY ONE I KNOW)

$$(T^a)^i{}_j$$

in general: $d(T^a)$ for the reps

j: antifundamental: contracts w/ fundamental (DUAL VECTOR)

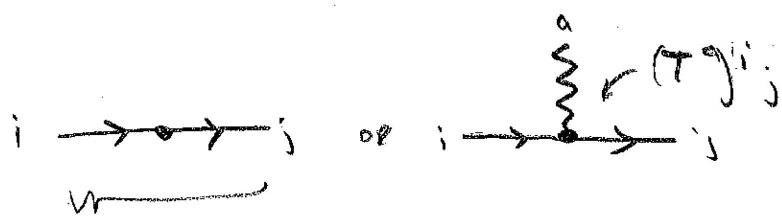
i: fundamental: spits out fundamental

a: ADJOINT (labels generators)

<u>claim</u>: $\psi_i \chi^j = \underbrace{\psi\cdot\chi}_{\text{singlet}} \oplus \underbrace{\psi_i (T^a)^i{}_j \chi^j}_{\text{transforms like ADJOINT}}$

$\psi_i \chi^j$: fundamental ⊗ antifundamental

DIAGRAMATICALLY:



in GAUGE THY: MASS TERM

GAUGE VERTEX

HIGHER RANK TENSORS:

$$\psi^{i_1 \cdots i_p}_{j_1 \cdots j_q} \mapsto \left(U^{i_1}{}_{k_1} \cdots U^{i_p}{}_{k_p}\right)\left(U_{j_1}{}^{l_1} \cdots U_{j_q}{}^{l_q}\right) \psi^{k_1 \cdots k_p}_{l_1 \cdots l_q}$$

SUM OVER REPEATED UPPER/LOWER INDICES

this is not typically an irrep.
BUT WE CAN NOTICE SOME SHORTCUTS

$$\psi^{ij} \mapsto \underline{U^i{}_k U^j{}_l} \, \psi^{kl}$$

by relabelling:

just relabel dummy indices

$$\psi^{ji} \mapsto U^j{}_k U^i{}_l \, \psi^{kl} = U^j{}_l U^i{}_k \psi^{lk}$$

$$= \underline{U^i{}_k U^j{}_l} \, \psi^{lk}$$

same

so ψ^{ij} & ψ^{ji} transform the same.
↳ LINEAR COMB. TRANSFORM THE SAME

in general: $\psi^{\cdots}_{\cdots}$ & $\psi^{\cdots}_{\cdots}$
PERMUTATIONS

transform the same.

SO: CONSIDER: $S^{ij} = \frac{1}{2}\psi^{(ij)} \equiv \frac{1}{2}(\psi^{ij} + \psi^{ji})$

$A^{ij} = \frac{1}{2}\psi^{[ij]} \equiv \frac{1}{2}(\psi^{ij} - \psi^{ji})$

CLEARLY: $S^{ij} \to U^i{}_k U^j{}_l \, S^{kl}$

$A^{ij} \to U^i{}_k U^j{}_l \, A^{kl}$

FURTHER: S'^{ij} & A'^{ij} RETAIN THEIR SYMM.

$$\text{eg } A'^{ji} = U^j{}_k U^i{}_l A^{kl}$$

$$A^{kl} \to -A^{lk}$$

$$= -(U^i{}_l)(U^j{}_k)A^{lk}$$
$$= -U^i{}_k U^j{}_l A^{kl}$$
$$= -A'^{ij}$$

RELABEL DUMMY INDICES

So SU(N) transforms leave the sym & antisym pieces sym/antisym.

↳ these form INVARIANT SUBSPACES

for SU(2):

$$\psi^{ij} = S^{ij} + A^{ij}$$

eg $s^i_a \otimes s^i_b$

sym. 2x2 → 3 comp.

antisym 2x2 → 1 comp.

$|\uparrow\rangle|\uparrow\rangle$
$\sim |\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle$
$|\downarrow\rangle|\downarrow\rangle$

$\sim |\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle$

$$\Rightarrow \underline{2} \otimes \underline{2} = \underline{3} \oplus \underline{1}$$

for mixed upper/lower INDICES: sym > trace (contract upper & lower)

↳ which is a <u>singlet</u>