P282 - WIN 2019

LEC13 : LIE ALG IN SOME GENERALITY 13 FEB 2019

FROM LAST TIME

HAMED'S Q: $J_{12} = \begin{pmatrix} & i \\ -i & \end{pmatrix}$ FOR $SO(2)$

DIAGONALIZING IT:

$$U \begin{pmatrix} & i \\ -i & \end{pmatrix} U^{-1} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

UNITARY

CLEARLY $\exp(-i\theta J_{12})$ IS REAL
(& ACTS ON $\mathbb{R}$ VECTORS)

HOW IS $\exp\left(-i\theta \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\right)$ A REAL REP?
IT IS EQUIVALENT

$$\begin{pmatrix} c & s \\ -s & c \end{pmatrix} = e^{-i\theta J_{12}} = e^{-i\theta U^{-1} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} U}$$

$$= U^{-1} \boxed{e^{-i\theta \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}} U$$

COMPLEX

EQUIVALENT TO REAL
(SIMILARITY TRANSFORM)

WHAT'S THE MEANING OF

$$e^{\left(-i\theta \begin{pmatrix}1 & \\ & -1\end{pmatrix}\right)} \;?$$

$$\begin{pmatrix} x^1+ix^2 \\ x^1-ix^2\end{pmatrix} \to \begin{pmatrix} e^{-i\theta} & x^1+ix^2 \\ e^{i\theta} & x^1-ix^2\end{pmatrix} \approx \begin{pmatrix}(x^1+\theta x^2)+i(x^2-\theta x^1)\\ (x^1+\theta x^2)-i(x^2-\theta x^1)\end{pmatrix}$$

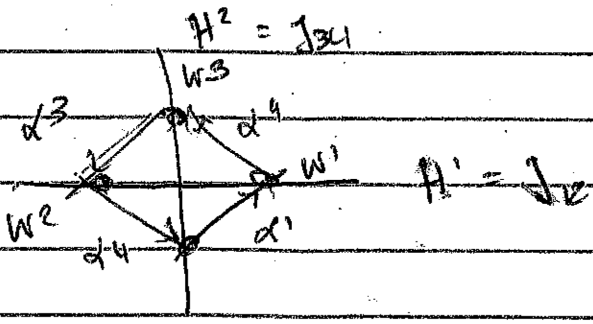
CIRCULAR/POLAR

vs $\begin{pmatrix}x^1\\ x^2\end{pmatrix} \to \begin{pmatrix}x^1+\theta x^2\\ x^2-\theta x^1\end{pmatrix}$

CARTESIAN.

Zee VI.2 SIMILAR THING GOING ON BTWN spin-1 & ad of SU(2).

LEXI'S QUESTION: so(4)



why no roots in vert/horiz directions?

1) we've exhausted the 6 gens of so(4)
2) WE'RE IN CIRCULAR BASIS
$((x^1+ix^2), (x^1-ix^2), (x^3+ix^4), (x^3-ix^4))$

no way to transform these into each other.

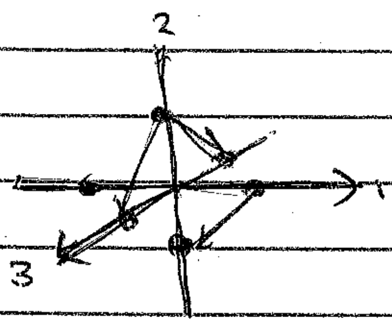
SO(6) : Bee

$H^1 = \mathrm{diag}(1, -1, \cdot \cdot \cdot \cdot)$
$H^2 = \mathrm{diag}(\cdot, \cdot, 1, -1, \cdot, \cdot)$
$H^3 = \mathrm{diag}(\cdot, \cdot, \cdot, \cdot, 1, -1)$
} circular basis

WEIGHTS of fundamental ← 6 BASIS VEC.

$\begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}$

ROOTS: CONNECT DIFFERENT 2x2 BLOCKS
RECALL IN CIRCULAR BASIS: no way to
ROTATE $x^1 + i x^2 \rightarrow x^1 - i x^2$



ROOTS CONNECT THESE POINTS ACROSS "DIAGONALS"
(along axes: same 2x2 block)

(W1.2) Bee notation : $\pm e^1 \pm e^2, \pm e^2 \pm e^3, \pm e^1 \pm e^3$
uncorrelated signs

pos:
$e^1 \pm e^2$
$e^2 \pm e^3$
$e^1 \pm e^3$

simple :
$e^1 - e^2$
$e^2 - e^3$
$e^2 + e^3$

$\hookrightarrow 3 \times (2 \times 2) = 12$ ROOTS
$+ \; 3$ CARTAN
15 GENERATORS ✓

$\left(\frac{6 \times (6-1)}{2}\right) \longrightarrow$

# of elem
in upper rt. block

GENERALIZE: $SO(2N)$, $SO(2N+1)$

$$H^1 = \mathrm{diag}(1,-1,0,0\ldots)$$
$$H^2 = \mathrm{diag}(0,0,1,-1,\ldots)$$
$$\vdots$$
$$H^N = \mathrm{diag}(0,0,\ldots,1,-1)$$

WEIGHTS OF BASIS VECTORS: ± 1 on ea axis

N COMP. (not N+1)

$$\left\{ \begin{pmatrix} \pm 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 1 \\ 0 \\ \vdots \end{pmatrix}, \ldots, \begin{pmatrix} \vdots \\ 0 \\ \pm 1 \end{pmatrix} \right. \left[+ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ for } so(2N+1) \right]$$

$\left(4 \times \frac{N(N-1)}{2}\right)$ + N CARTAN $= \frac{2N(2N-1)}{2}$ GENS ✓ $so(2N)$

ROOTS: connect points on diff axes

$so(2N)$ $\boxed{\pm e^i \pm e^j \quad i<j}$ signs uncorrelated (DET. WHICH POINTS ON THE i & j AXES CONNECT)

$\boxed{so(2N+1) \quad \pm e^i}$

$+2N$ GENS

tot! $\frac{2N(2N+1)}{2}$ ✓

POSITIVE ROOTS: $\boxed{e^i \pm e^j \quad i<j}$ | $\boxed{\begin{array}{c} so(2N+1) \\ + e^i \end{array}}$

hierarchy of axes

SIMPLE ROOTS: $\boxed{\begin{array}{ll} e^{i-1} - e^i & (i=2,\ldots,N) \\ so(2N) \quad e^{N-1} + e^N & \end{array}}$

$$\boxed{\begin{array}{ll} so(2N+1): \; e^{i-1} - e^i & (i=2,\ldots,N) \\ \quad e^N & \end{array}}$$

SU(N): more annoying b/c of $\sqrt{\ }$ factors

$H^1 = \mathrm{diag}\,(1, -1, 0, \dots)/\sqrt{2}$ ← $\sqrt{2}$ (OUR BASIS)

$H^2 = \mathrm{diag}\,(1, 1, -2, \dots)/\sqrt{2\cdot 3}$ s.t. $\mathrm{Tr}(H^i H^j) = \delta^{ij}$ not $\delta^{ij}/2$

$\vdots$

$H^i = \mathrm{diag}\,(\underbrace{1, 1, \dots, 1}_{i \text{ ones}}, -i, 0, \dots 0)/\sqrt{i(i+1)}$

$\vdots$

$H^{N-2} = \mathrm{diag}\,(1, \dots, 1, -(N-2), 0)/\sqrt{(N-2)(N-1)}$

$H^{N-1} = \mathrm{diag}\,(\underbrace{1, \dots, 1, -(N-1)}_{N \text{ components}})/\sqrt{(N-1)N}$

N components (N×N) matrices

CHECK NORM: $\mathrm{Tr}(H^i H^i) = (i + i^2)/(i+1)i = 1$

READ OFF <u>WEIGHTS</u> of FUNDAMENTAL

$$\begin{pmatrix}1\\0\\ \vdots\\ \vdots\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\\ \vdots\\ 1\end{pmatrix} \quad \cdots \quad \begin{pmatrix}0\\0\\ \vdots\\ \vdots\\ 1\end{pmatrix} \Bigg\} \text{ N states}$$

$$W^1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \dots, \underbrace{\frac{1}{\sqrt{m(m+1)}}}_{m\text{th}}, \dots, \frac{1}{\sqrt{(N-1)N}}\right)$$

$$W^2 = \left(-\frac{1}{\sqrt{2}}, \text{———}\ {''}\ \text{———}\right)$$

$$W^3 = \left(0, -\frac{2}{\sqrt{6}}, \text{———}\ {''}\ \text{———}\right)$$

$-\frac{3}{\sqrt{12}}$

$$W^{m+1} = \left(0 \text{ ——— } 0, \frac{-m}{\sqrt{m(m+1)}}, \text{———}\ {''}\ \text{———}\right)$$

$$W^{N-1} = \left(0 \text{ ——— } 0, \frac{-N+1}{\sqrt{(N-1)N}}\right)$$

just READ this off!

ROOT VECTORS : $\underbrace{W^i - W^j}$ for $i,j \in (1,\ldots,N)$

$N(N-1)$ choices

connect any weight of fund. to any other weight

PLUS: $N-1$ CARTAN $= (N+1)(N-1) = N^2-1$ GENS. ✓

POSITIVE ROOTS : $\alpha^{ij} = W^i - W^j \quad i<j$

$\subset \boxed{N(N-1)/2 \text{ CHOICES}}$

LIVE IN $(N-1)$ DIM WEIGHT SPACE

$\subset$ RANK $=$ dim (CARTAN)

$(N-1)$ SIMPLE ROOTS : $\left(W^i - W^{i+1}\right) \quad i = 1,\ldots,(N-1)$

remark: we're going to skip $Sp(2N)$

SYMPLECTIC GROUP. → in CLASSICAL MECHANICS

$$\hookrightarrow R^T \underbrace{\left(\begin{array}{c|c} & 1\!\!1 \\ \hline -1\!\!1 & \end{array}\right)} R = \left(\begin{array}{c|c} & 1\!\!1 \\ \hline -1\!\!1 & \end{array}\right)$$

symplectic

recall: $\begin{pmatrix} \dot{p} = -\partial H/\partial q \\ \dot{q} = \partial H/\partial p \end{pmatrix}$

HW ↑ → See IV.13 on E(2) ↓

MASTER FORMULA (Georgi 6.36)

$$\boxed{\frac{\alpha\cdot\mu}{\alpha^2} = \frac{q-p}{2}}$$

DISTANCE TO BOTTOM / TOP OF α-LADDER

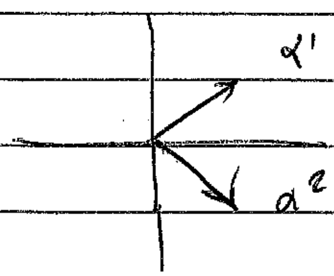
weight of a state

some SU(2) axis (ROOT)

GIVEN SIMPLE ROOTS α, β; $(\alpha-\beta)$ not a root

$\hookrightarrow$ $$\frac{\alpha\cdot\beta}{\alpha^2} = \frac{-p}{2} \qquad \frac{\beta\cdot\alpha}{\beta^2} = \frac{-p'}{2}$$

eg SO(4): α^1 & α^2 ARE SIMPLE



$$\boxed{\cos^2\theta = 0}$$

so: $$\frac{\alpha\cdot\beta}{\alpha^2} = 0 \leftrightarrow E_\alpha|E_\beta\rangle = 0$$

$$\frac{\beta\cdot\alpha}{\beta^2} = 0 \rightarrow E_\beta|E_\alpha\rangle = 0$$

no $(\alpha^1+\alpha^2)$ root.

WEYL REFLECTION : generaliz. of "HEX SYM" of SU(3)

if α, β are ROOTS (ROOT VECTORS)

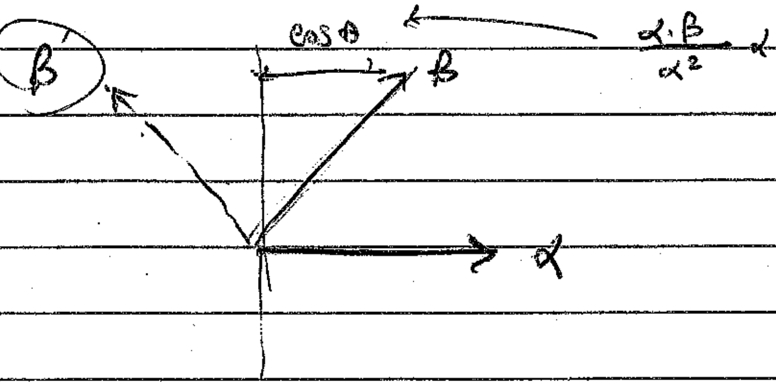
claim : $\beta' = \beta - 2\frac{\alpha \cdot \beta}{|\alpha|^2}\alpha = \beta - (q-p)\alpha$
is also a ROOT.

pf/ this is just using the α LADDER that passes through $|\beta\rangle$

the ladder spans $(\beta + p\alpha), \ldots (\beta - q\alpha)$ by definition of p & q RELATIVE TO $|\beta\rangle$

so this is true by $p \geq p-q \geq q$. ✓

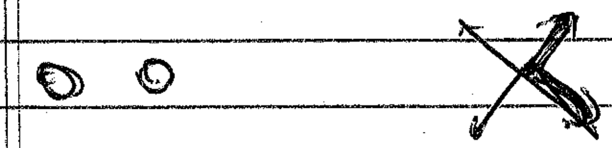
Result : EACH ROOT IS PERPENDICULAR TO AN AXIS OF REFLECTION SYM.



helps us draw root diagram from simple roots

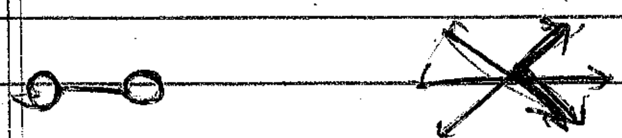
ALL RANK-2 LIE ALGEBRAS (weight diagrams can draw)

math notation

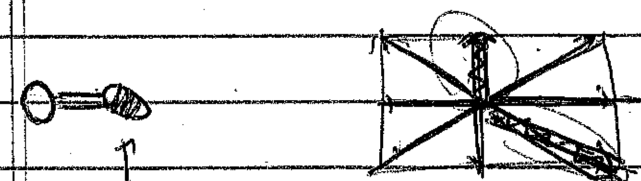


$so(4) = D_2$

2 indep dir : $so(4) \cong su(2) \times su(2)$

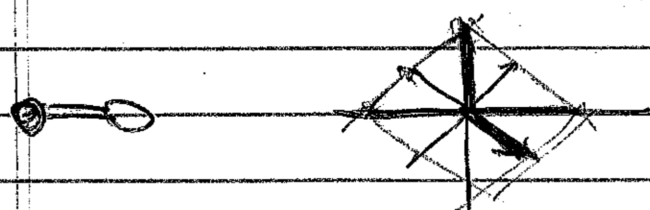


$su(3) = A_2$

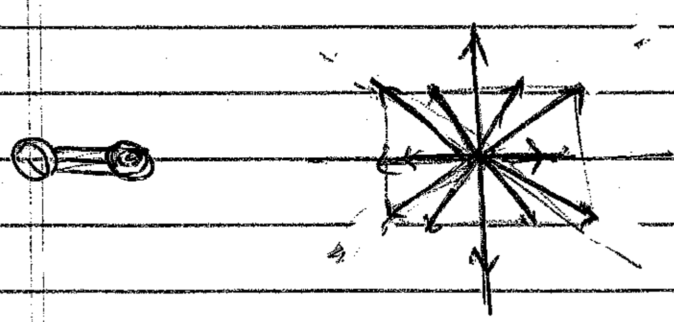


$so(5) = B_2$

DIFF LENGTH



$sp(4) = C_2 \cong so(5)$



G_2 ← one of the "exceptional" groups (only for certain N's)

Dynkin Diagrams

$$\boxed{\frac{\alpha\cdot\beta}{\alpha^2} = \frac{p}{2}} \qquad \boxed{\frac{\beta\cdot\alpha}{\alpha^2} = \frac{p'}{2}} \qquad \boxed{\frac{\alpha^2}{\beta^2} = \frac{p'}{p}}$$

for simple roots $\alpha, \beta \longrightarrow$ circles on Dynkin Diag.

lines connecting the circles

$$= \boxed{4\frac{(\alpha\cdot\beta)^2}{\alpha^2\beta^2} = pp'}$$

$\hookrightarrow$ or: $\cos\theta = \frac{-\sqrt{pp'}}{2} = \boxed{\frac{-1}{2}, \frac{-1}{\sqrt{2}}, \frac{-\sqrt{3}}{2}}$

$120^\circ \quad 135^\circ \quad 150^\circ$

for our favorite groups:

simple roots

$SU(N) = A_{N-1} \qquad w^i - w^{i+1} \qquad i = 1, \ldots, N-1$

$SO(2N+1) = B_N \qquad (e^{i-1} - e^i), (e^N) \qquad i = 2, \ldots, N$

$SO(2N) = D_N \qquad (e^{i-1} - e^i), (e^{N-1} + e^N) \qquad i = 2, \ldots, N$

SU(N) has (N-1) simple Roots

$$\underbrace{(w^{i} - w^{i+1})^2}_{(\alpha^{i(i+1)})^2} = \left(\begin{array}{l} \left(0,\dots,0, \overset{(i-1)^{th}}{\frac{-(i-1)}{\sqrt{(i-1)(i)}}}, \overset{i}{\frac{1}{\sqrt{i(i+1)}}}, \dots\right) \\ - \left(0,\dots,0, 0, \frac{-i}{\sqrt{i(i+1)}}, \dots\right) \end{array} \right)^2$$

$$= \frac{+(i-1)^2}{(i-1)i} + \frac{(i+1)^2}{i(i+1)}$$

$$= \boxed{2}$$

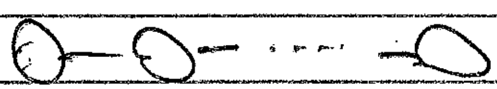
i ~~clearly~~: $(w^{i} - w^{i+1}) \cdot (w^{j} - w^{j+1}) = 0$ for $j \neq i \pm 1$

since the $\alpha^{i(i+1)}$ root only has nonzero components in i & $i+1$ components

$= -1$ if $j = i \pm 1$

$$\Rightarrow \cos^2\theta = \frac{(-1)^2}{2 \cdot 2} = \frac{1}{4} \rightarrow \boxed{\theta = 120^\circ}$$

so neighboring roots connected by a line
non-neighboring roots are $90^\circ \rightarrow$ no line



N-1 of them.

@ most 2 lengths

SO(2N+1): DIFFERENT LENGTHS

↳ fill circle of short root.

SO(2N+1): $(N-1)$ LONG ROOTS $(e^{i-1} - e^{i})$

1 SHORT ROOT e^{N}

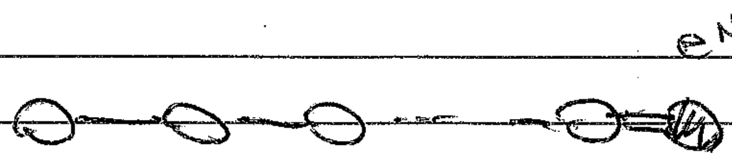
CAN SEE AGAIN: LONG ROOTS CONNECTED BY A SINGLE LINE:

$$\underbrace{(e^{i-1} - e^{i})}_{\text{length } \sqrt{2} \text{ as before}} \cdot (e^{j-1} - e^{j}) = -\delta^{j(i-1)} - \delta^{i(j-1)}$$

$\cos^2\theta = \frac{(-1)^2}{2 \cdot 2}$ ✓

BUT THE SHORT ROOT CONNECTS TO THE LAST LONG ROOT

$$\cos^2\theta = \frac{-1}{2 \cdot \underset{\text{SHORT}}{1}} = \frac{1}{2} \rightarrow 135^\circ$$

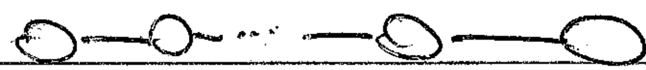


obs: so(3): only one circle - $\simeq$ su(2) ?!

$i=2,\dots,N$

SO(2N): $(e^{i-1}-e^{i})$, $\underbrace{(e^{N-1}+e^{N})}_{\text{same length } = \sqrt{2}}$

gives chain



$(e^{N-2}-e^{N-1})$

$(e^{N-1}-e^{N})$

OBS: $(e^{N-1}+e^{N})\cdot(e^{N-1}-e^{N})=0$!

BUT $\underbrace{(e^{N-1}\pm e^{N})}_{\cancel{\text{either}}\text{ of last two}}\cdot(e^{N-2}-e^{N-1}) = -1$

same for $(e^{N-1}\pm e^{N})$!