GROUP THY: AS PHYSICISTS, we are like CARPENTERS. group theory is a hammer. We should know how to use the hammer — BUT THE POINT IS NOT THE HAMMER. (~ T. TAIT)

P262 LEC 3

SU(2) normalizations on website

REVIEW

$\underbrace{\text{compact}}$ $\underbrace{\text{representations}}$ of $\underbrace{\text{Lie Algebras}}$

PERIODIC (VS. eg BOOSTS)

↑

UNITARY GROUP REP

↕

HERMITIAN ALG. REP.

matrices acting on vectors in a vector space

- DIMENSION
- $\mathbb{R}$ or $\mathbb{C}$

$\exp(-i\theta^A T^A)$

OBEY COMMUTATION RELATIONS

$[T^A, T^B] = i f^{ABC} T^C$

STRUCTURE CONSTANT.

sometimes : $D(g)$, $d(T)$

rep of group, rep of ALG.

$\boxed{SU(2)}$ → ALGEBRA ⟷ CARTAN SUBALGEBRA, H

$H = \{T^3\}$

DIAGONAL ⟷ COMMUTE w/ EACH OTHER

WEIGHTS → "quantum number"

→ LABEL : $T^3|m\rangle = m|m\rangle$

ROOT VECTORS — nothing to do w/ vector space ... ROOT SPACE!

$T^{\pm}$: moves the labels up & down

root of the symmetry structure

A ~~REP~~ REPRESENTATION OF SU(2):

- PICK AN ALLOWED HIGHEST WEIGHT of the REP.
  $\rightarrow$ eg $j = 1/2$
  j IS HALF INTEGER, POSITIVE

- $|m\rangle = |j\rangle$ is a state in the rep

- LOWER w/ LOWERING OPERATOR(S)
  EACH STATE IS in the rep

- EVENTUALLY, T^- WILL ANNIHILATE LOWEST STATE
  $\rightarrow$ in SU(2) notes: $T^-|-j\rangle = T^-|-1/2\rangle = 0$

THE COLLECTION OF STATES IS YOUR "<u>VECTOR</u>" TRANSFORMING IN THE "SPIN-j" REP OF SU(2)

$$\hookrightarrow \{ |\tfrac{1}{2}\rangle, |-\tfrac{1}{2}\rangle \} \longrightarrow \begin{pmatrix} a \\ b \end{pmatrix}$$

WHAT ARE THE ROTATION MATRICES?
$\hookrightarrow$ sufficient to write matrices for the <u>generators</u>

CARTAN: $T^3|m\rangle = m|m\rangle$

$$\begin{pmatrix} a \\ b \end{pmatrix} = a\underset{\substack{\uparrow \\ |\frac{1}{2}\rangle}}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} + b\underset{\substack{\uparrow \\ |-\frac{1}{2}\rangle}}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$\underbrace{d(T^3)}_{\text{2x2 matrix}} = \begin{pmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{pmatrix}$$

no surprise; this is exactly $T^3 = \frac{1}{2}\sigma^3$

ROOT VECTORS / "ROOTS" $T^{\pm}|m\rangle \propto |m\pm1\rangle$

w/ norm: $T^-|m\rangle = N_m|m-1\rangle$

$T^+|m-1\rangle = N_m|m\rangle$

$$N_m = \sqrt{\underbrace{(j-m+1)}_{\substack{\uparrow \\ \text{0 when you raise highest weight}}}\underbrace{(j+m)}_{\substack{\uparrow \\ \text{0 when you lower lowest weight}}}}$$

see SU(2) notes

for $j = 1/2$

$$N_{+1/2} = \sqrt{\left(\tfrac{1}{2} \underbrace{\cancel{}}_{-1/2} + 1\right)\left(\tfrac{1}{2} + 1/2\right)} = 1$$

$$d(T^+) = \begin{pmatrix} & 1 \\ 0 & \end{pmatrix}$$

$$d(T^-) = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}$$

NOW WE HAVE ~~REPS OF~~ MATRICES FOR THE ENTIRE ALGEBRA IN THE $j = 1/2$ REP. HOW DOES A VECTOR IN THIS VECTOR SPACE ROTATE?

$$g = e^{-i(\theta^1 T^1 + \theta^2 T^2 + \theta^3 T^3)}$$

$$\downarrow$$

$$\underset{\substack{\uparrow \\ 2\times 2 \text{ MATRIX}}}{D(g)} = \exp\Big[-i\theta^1 \underset{\substack{\uparrow \\ 2\times 2 \text{ MATRIX}}}{d(T^1)} - i\theta^2 d(T^2) - i\theta^3 \underset{\substack{\uparrow \\ \text{we wrote this!}}}{d(T^3)}\Big]$$

what are these?

$$T^{\pm} \equiv T^1 \pm i T^2 \Rightarrow d(T^{\pm}) = d(T^1) \pm i\, d(T^2)$$

ABSTRACT (w/ COMMUTATION REL. of $\frac{1}{2}\underline{\sigma}$

explicit matrices (for $j=1/2$, these are exactly $\frac{1}{2}\underline{\sigma}$)

so the $d(T^1)$ & $d(T^2)$ matrices are

$$d(T^1) = \frac{1}{2}\left(d(T^+) + d(T^-)\right)$$
$$d(T^2) = \frac{1}{2i}\left(d(T^+) - d(T^-)\right)$$

$\hookrightarrow$ gives

$$d(T^1) = \frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$
$$d(T^2) = \frac{1}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix}$$

so there you go we have built up the usual spinor rep using only the commutation relations of the su(2) algebra.

eg. $\boxed{j = 1}$

Q: what is the dimension of rep?

3 $\rightarrow$ $\underbrace{|j\rangle}_{|1\rangle}$, $\underbrace{T^-|j\rangle}_{N_1|0\rangle}$, $\underbrace{(T^-)^2|j\rangle}_{N_2|-1\rangle}$

$\uparrow$ $|-j\rangle$ = BOTTOM

$$\begin{array}{l} \bullet\ |1\rangle \\ \downarrow \\ \bullet\ |0\rangle \\ \downarrow \\ \bullet\ |-1\rangle \end{array}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{|1\rangle} + b\underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{|0\rangle} + c\underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{|-1\rangle}$$

so the matrices $d(T^A)$ are 3×3.

EASY ONE:

$$d(T^3) = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

from $T^3|m\rangle = m|m\rangle$

what about $T^{\pm}$?

$$j = 1 \rightarrow \quad N_m = \sqrt{(2-m)(m+1)}$$

$$T^- |1\rangle = \sqrt{2}\, |0\rangle \qquad\qquad T^+ |0\rangle = \sqrt{2}\, |1\rangle$$
$$T^- |0\rangle = \sqrt{2}\, |-1\rangle \qquad\qquad T^+ |-1\rangle = \sqrt{2}\, |0\rangle$$

$$d(T^-) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\, d(T^-) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\, d(T^-) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \underbrace{c\, d(T^-) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{=0}$$

$$= \begin{pmatrix} 0 \\ \sqrt{2}\, a \\ \sqrt{2}\, b \end{pmatrix}$$

$$d(T^+) = \begin{pmatrix} \sqrt{2}\, b \\ \sqrt{2}\, c \\ 0 \end{pmatrix}$$

$$d(T^+) = \begin{pmatrix} & \sqrt{2} & \\ & 0 & \sqrt{2} \\ 0 & & \end{pmatrix} \qquad d(T^-) = \begin{pmatrix} & & 0 \\ \sqrt{2} & 0 & \\ & \sqrt{2} & \end{pmatrix}$$

$\boxed{8b}$: the GENERATORS in the more intuitive basis :

$$d(T^1) = \frac{1}{2}\left(d(T^+) + d(T^-)\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix}$$

$$d(T^2) = \frac{1}{2i}\left(d(T^+) - d(T^-)\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} & -i & \\ i & & -i \\ & i & \end{pmatrix}$$

$$d(T^3) \text{ from earlier } = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

AS BEFORE : a rotation of this object by $(\theta^1, \theta^2, \theta^3)$ is

$$D(g(\underline{\theta})) = \underbrace{\exp\left(-i \sum_A \theta^A d(T^A)\right)}$$

3x3 matrix acting on
3 component vectors
that transform as the $j = 1$
(OR 3D REP) of $SU(2)$.

YOU CAN GO ON & ON. $j = 3/2$

$$\left.\begin{array}{ll} \circ & m = 3/2 \\ T^- \downarrow & \\ \circ & m = 1/2 \\ T^- \downarrow & \\ \circ & m = -1/2 \\ T^- \downarrow & \\ \circ & m = -3/2 \end{array}\right\} \begin{array}{l} \text{4 component vector} \\ \{ |3/2\rangle, |1/2\rangle, |-1/2\rangle, |-3/2\rangle \} \end{array}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{etc.}$$

Use:
$$T^3 |m\rangle = m |m\rangle$$
$$T^- |m\rangle = N_m |m-1\rangle$$
$$T^+ |m-1\rangle = N_m |m\rangle$$

$$N_m = \sqrt{(j-m+1)(j+m)}$$

construct matrix reps $d(T^3), d(T^\pm)$

$$\updownarrow$$

$$d(T^1), d(T^2)$$

vs infinitesimal

then finite ROT is

$$D(g(\theta)) = \exp\left(-i \sum_A \theta^A d(T^A)\right)$$

THERE ARE TWO SPECIAL REPRESENTATIONS

① FUNDAMENTAL "defining rep"

$\boxed{j = \frac{1}{2}}$

We defined the su(2) algebra as "the one the $\frac{1}{2}\sigma$ matrices satisfy"

↳ BUT $T^{1,2,3}$ were abstract objects, not matrices that we chose/defined to satisfy $[T^A, T^B] = i\varepsilon^{ABC}T^C$

fundamental: the rep where

$$d(T^A) \equiv \frac{1}{2}\sigma^A$$

$SU(2)$ ← 2x2 unitary, det=1

naturally acts on 2d $\mathbb{C}$ VEC SPACE

FUNDAMENTAL

↳ $U\begin{pmatrix} a \\ b \end{pmatrix}$ where $U = D(g)$

REMARK: ALSO ANTIFUNDAMENTAL REP

transforms as $U^\dagger$ ...

what are generators?

② ADJOINT REP

the generators themselves form a representation of the algebra.

(this is really weird the first time you hear it!)

REP of A GENERATOR T^A: matrix in vec space
LINEAR TRANSFORMATION that takes a VECTOR into another VECTOR.

$$\text{eg: } \underbrace{\theta^A d(T^A)}_{\sum_A \frac{\theta^A}{2}\sigma^A} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix}$$

ADJOINT: $d(T^A)$ OR $ad(T^A)$

FOR ANY ALGEBRA:

$$ad(T^A)\,T^B \equiv \frac{1}{i}[T^A, T^B] = \frac{i}{i}\underbrace{f^{ABC}}_{\varepsilon^{ABC} \text{ in } su(2)} T^C$$

CHECK: is it linear? yes. DO NOT CONFUSE T^A the "matrix" ((really: $[T^A, \circ]$)) with T^B the BASIS of VECTOR SPACE!!

eg. vector space of SU(2) ADJOINT:

$$\left\{ \left|\tfrac{1}{2}\sigma^1\right\rangle, \left|\tfrac{1}{2}\sigma^2\right\rangle, \left|\tfrac{1}{2}\sigma^3\right\rangle \right\}$$

$$\begin{aligned}
\mathrm{ad}(T^1)\left|\tfrac{1}{2}\sigma^2\right\rangle &\equiv \tfrac{1}{i}\left[T^1, \tfrac{1}{2}\sigma^2\right] \\
&= \tfrac{1}{i}\left[\tfrac{1}{2}\sigma^1, \tfrac{1}{2}\sigma^2\right] \\
&= \cancel{i}\,\tfrac{1}{2}\sigma^3 \\
&\equiv \cancel{i}\left|\tfrac{1}{2}\sigma^3\right\rangle
\end{aligned}$$

$$\mathrm{ad}(T^1)\left|\tfrac{1}{2}\sigma^3\right\rangle = -\cancel{i}\left|\tfrac{1}{2}\sigma^2\right\rangle$$

$$\mathrm{ad}(T^1)\left|\tfrac{1}{2}\sigma^1\right\rangle = 0$$

$$\mathrm{ad}(T^1) = \left(\begin{array}{c|cc} 0 & & \\ \hline & & -1 \\ & 1 & \end{array}\right)$$

acting on $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\left|\tfrac{1}{2}\sigma^1\right\rangle + b\left|\tfrac{1}{2}\sigma^2\right\rangle + c\left|\tfrac{1}{2}\sigma^3\right\rangle$

what is the dimension of this REP? 3

↳ hey! we've seen that...

$$\boxed{\mathrm{ad}(T^A)\left|T^B\right\rangle = \left|\tfrac{1}{i}[T^A, T^B]\right\rangle}$$

CHOOSE normaliz.

PICK A BASIS : $\left\{ |T^3\rangle, \ |\tfrac{1}{\sqrt{2}}T^\pm\rangle \right\}$

$$\mathrm{ad}(T^3)\,|\tfrac{1}{\sqrt{2}}T^\pm\rangle = \pm\,|\tfrac{1}{\sqrt{2}}T^\pm\rangle$$
$$\mathrm{ad}(T^\pm)\,|T^3\rangle = \mp\sqrt{2}\,|\tfrac{1}{\sqrt{2}}T^\pm\rangle$$
$$\mathrm{ad}(T^\pm)\,|\tfrac{1}{\sqrt{2}}T^\mp\rangle = \pm\sqrt{2}\,|T^3\rangle$$

$$\begin{array}{cc} \mathrm{ad}(T^3) = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} & \begin{array}{l} |\tfrac{1}{\sqrt{2}}T^+\rangle \\ |T^3\rangle \\ |\tfrac{1}{\sqrt{2}}T^-\rangle \end{array} \end{array}$$

$$\mathrm{ad}(T^+) = \begin{pmatrix} & -\sqrt{2} & \\ & & \sqrt{2} \\ & & \end{pmatrix}$$

$$\mathrm{ad}(T^-) = \begin{pmatrix} & & \\ -\sqrt{2} & & \\ & \sqrt{2} & \end{pmatrix}$$

signs are a bit different.

HM. DUNNO WHY...

see: physics.stackexchange.com

→ QUESTION 279880

HAS TO DO W/ SPHERICAL BASIS

<u>homework</u>: $\exists\ U$ unitary s.t.

$$d^{(3)}(T^A) = U\,\mathrm{ad}(T^A)\,U^{\dagger}$$

<u>Next time</u>: $SU(3)$ (see Cahn ch. 11)

PAULI MATRICES $\longleftrightarrow$ GellMann MATRICES

$$\lambda^1 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & \end{pmatrix} \qquad \lambda^2 = \begin{pmatrix} & -i & \\ i & & \\ & & \end{pmatrix} \qquad \lambda^3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix} \qquad \lambda^5 = \begin{pmatrix} & & -i \\ & 0 & \\ i & & \end{pmatrix}$$

$$\lambda^6 = \begin{pmatrix} & & \\ & & 1 \\ & 1 & \end{pmatrix} \qquad \lambda^7 = \begin{pmatrix} & & \\ & & -i \\ & i & \end{pmatrix}$$

"missing"?

$$\lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \longleftarrow ?$$

T^x, T^y, T^z
V^x, V^y
U^x, U^y
Y

$\left.\begin{matrix} T^\pm \\ V^\pm \\ U^\pm \end{matrix}\right\}$ ROOTS

DIAGONAL GENERATORS
(CARTAN)

2 WEIGHTS

Weights: $(h_1, h_2) \rightarrow |h_1, h_2\rangle$

LOOKS LIKE COPIES of SU(2)
... how many copies? 2–3 ish...

note: definitely <u>NOT</u> $SU(2) \times SU(2)$
because commutators don't close
into subalgebras.

eg $[\tfrac{1}{2}T^+, U^+] = i V^+$