

GROUP THEORY: AS PHYSICISTS, WE ARE LIKE CARPENTERS. GROUP THEORY IS A HAMMER. WE SHOULD KNOW HOW TO USE THE HAMMER. BUT THE POINT IS NOT THE HAMMER. (N.T. TAIT)

P262 LEC 3

1/14

SU(2) normalizations on website

REVIEW

Compact representations of Lie Algebras

PERIODIC
(vs. eg. BOOSTS)

↑
UNITARY GROUP
REP

↓
HERMITIAN ALG.
REP.

matrices acting
on vectors in
a vector space

└ DIMENSION
ℝ or ℂ

sometimes: $\underbrace{D(g)}_{\text{rep of group}}, \underbrace{d(T)}_{\text{rep of ALG.}}$

$\exp(-i\theta^a T^a)$

OPER COMMUTATION
RELATIONS

$$[T^a, T^b] = i f^{abc} T^c$$

↑
STRUCTURE
CONSTANT

SU(2)

ALGEBRA

→ CARTAN SUBALGEBRA, \mathcal{H}
 $\mathcal{H} = \{T^3\}$

DIAGONAL ↔ commute w/
EACH OTHER

WEIGHTS

"quantum number"

$$\text{LABEL: } T^3 |m\rangle = m |m\rangle$$

ROOT VECTORS

nothing to do w/ vector space
... ROOT SPACE!

↑ T^\pm : moves the labels up & down

└ root of the symmetry structure

A ~~REP~~ REPRESENTATION OF $SU(2)$:

- PICK AN ALLOWED HIGHEST WEIGHT of the REP.
→ eg $j = 1/2$ ↗ j IS HALF INTEGER, POSITIVE
- $|m\rangle = |j\rangle$ is a state in the rep
- LOWER w/ LOWERING OPERATOR(S)
EACH STATE IS in the rep
- EVENTUALLY, T^- WILL ANNIHILATE LOWEST STATE
→ in $SU(2)$ notes: $T^- | -j \rangle = T^- | -1/2 \rangle = 0$

THE COLLECTION OF STATES IS YOUR
"VECTOR" TRANSFORMING IN THE "SPIN- j " REP
OF $SU(2)$

$$\hookrightarrow \{ |1/2\rangle, |-1/2\rangle \} \longrightarrow \begin{pmatrix} a \\ b \end{pmatrix}$$

WHAT ARE THE ROTATION MATRICES?

↳ sufficient to write matrices for
the generators.

CARTAN: $T^3|m\rangle = m|m\rangle$

$$\begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

\uparrow \uparrow
 $|1/2\rangle$ $|1-1/2\rangle$

$$\underbrace{d(T^3)}_{2 \times 2 \text{ matrix}} = \begin{pmatrix} 1/2 & \\ & -1/2 \end{pmatrix}$$

2x2 matrix

no surprise; this is exactly $T^3 = \frac{1}{2}\sigma^3$

ROOT VECTORS / "ROOTS" $T^\pm|m\rangle \propto |m\pm 1\rangle$

w/ norm: $T^-|m\rangle = N_m|m-1\rangle$
 $T^+|m-1\rangle = N_m|m\rangle$

$$N_m = \sqrt{\underbrace{(j-m+1)}_{\uparrow} \underbrace{(j+m)}_{\downarrow}}$$

see SU(2) notes

o when you raise highest weight

o when you lower lowest weight

for $j = 1/2$

$$N_{+1/2} = \sqrt{\left(\frac{1}{2} \frac{1}{-1/2} + 1\right) \left(\frac{1}{2} + \frac{1}{2}\right)} = 1$$

$$d(T^+) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$d(T^-) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

NOW WE HAVE ~~REPS~~ ^{MATRICES} ~~OR~~ THE ENTIRE ALGEBRA
IN THE $j = 1/2$ REP. HOW DOES A VECTOR
IN THIS VECTOR SPACE ROTATE?

$$g = e^{-i(\theta^1 T^1 + \theta^2 T^2 + \theta^3 T^3)}$$

$$D(g) = \exp[-i\theta^1 d(T^1) - i\theta^2 d(T^2) - i\theta^3 d(T^3)]$$

↑
2x2 matrix

↑
2x2 matrix

↑
we wrote this!

what are these?

$$\underline{T^\pm \equiv T^1 \pm iT^2} \Rightarrow \underline{d(T^\pm) = d(T^1) \pm id(T^2)}$$

ABSTRACT
(w/ commutation rel.
of $\frac{1}{2}\sigma$)

explicit matrices
(for $j = \frac{1}{2}$, these are
exactly $\frac{1}{2}\sigma$)

so the $d(T^1)$ & $d(T^2)$ matrices are

$$d(T^1) = \frac{1}{2} (d(T^+) + d(T^-))$$

$$d(T^2) = \frac{1}{2i} (d(T^+) - d(T^-))$$

$$\hookrightarrow \text{gives } d(T^1) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

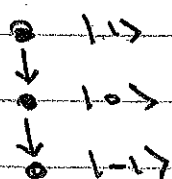
$$d(T^2) = \frac{1}{2} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$$

so there you go we have built up the
usual spinor rep using only the
commutation relations of the $su(2)$
algebra.

eg. $j=1$

Q: what is the dimension of rep?

③ \rightarrow $\underbrace{|j\rangle}_{|1\rangle}, \underbrace{T|j\rangle}_{N_1|0\rangle}, \underbrace{(T^-)^2|j\rangle}_{N_0|-1\rangle}$



\uparrow
 $|-j\rangle = \text{bottom}$

$\rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{|1\rangle} + b \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{|0\rangle} + c \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{|-1\rangle}$

so the matrices $d(T^{\pm})$ are 3×3 .

EASY ONE:

$$d(T^3) = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

from $T^3|m\rangle = m|m\rangle$

What about T^\pm ?

$$j=1 \rightarrow N_m = \sqrt{(2-m)(m+1)}$$

$$T^- |1\rangle = \sqrt{2} |0\rangle$$

$$T^- |0\rangle = \sqrt{2} |-1\rangle$$

$$T^+ |0\rangle = \sqrt{2} |1\rangle$$

$$T^+ |-1\rangle = \sqrt{2} |0\rangle$$

$$\begin{aligned} d(T^-) \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= a d(T^-) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b d(T^-) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c d(T^-) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \sqrt{2} a \\ \sqrt{2} b \end{pmatrix} \end{aligned}$$

$$d(T^+) = \begin{pmatrix} \sqrt{2} b \\ \sqrt{2} c \\ 0 \end{pmatrix}$$

$$d(T^+) = \left(\begin{array}{c|c} \sqrt{2} & \\ \hline 0 & \sqrt{2} \\ \hline 0 & \end{array} \right)$$

$$d(T^-) = \left(\begin{array}{c|c} & 0 \\ \hline \sqrt{2} & 0 \\ \hline 0 & \sqrt{2} \end{array} \right)$$

80 : the GENERATORS in the more intuitive basis :

$$d(T^1) = \frac{1}{2} (d(T^+) + d(T^-)) = \frac{1}{\sqrt{2}} \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$$

$$d(T^2) = \frac{1}{2i} (d(T^+) - d(T^-)) = \frac{1}{\sqrt{2}} \begin{pmatrix} & -i & \\ i & & \\ & & -i \end{pmatrix}$$

$$d(T^3) \text{ from earlier} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

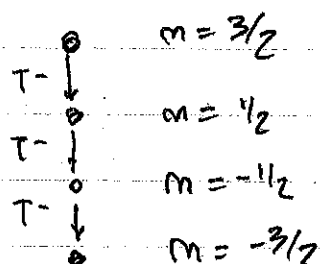
As before : a rotation of this object by $(\theta^1, \theta^2, \theta^3)$ is

$$D(g(\underline{\theta})) = \exp \left(-i \sum_A \theta^A d(T^A) \right)$$

where

3x3 matrix acting on
3 component vectors
that transform as the $j=1$
(or 3D rep) of $SU(2)$.

YOU CAN GO ON \uparrow ON. $j = 3/2$



~~4 component vector~~
 4 component vector
 $\{ |3/2\rangle, |1/2\rangle, |-1/2\rangle, |-3/2\rangle \}$

~~etc.~~

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{etc.}$$

Use:

$$T^3 |m\rangle = m |m\rangle$$

$$T^- |m\rangle = N_m |m-1\rangle$$

$$T^+ |m-1\rangle = N_m |m\rangle$$

$$N_m = \sqrt{(j-m+1)(j+m)}$$

construct matrix reps $d(T^3), d(T^\pm)$

$$\downarrow$$

$$d(T^1), d(T^2)$$

\swarrow vs infinitesimal

then finite ROT is

$$D(g(\theta)) = \exp(-i \sum_{\alpha} \theta^{\alpha} d(T^{\alpha}))$$

THERE ARE TWO SPECIAL REPRESENTATIONS

① FUNDAMENTAL "defining rep"

↑
$$j = \frac{1}{2}$$

We defined the $SU(2)$ algebra
as "the one the $\frac{1}{2}$ matrices
satisfy"

↳ BUT $T^{1,2,3}$ were abstract
objects, not matrices
that we chose/defined
to satisfy $[T^A, T^B] = i\epsilon^{ABC} T^C$

Fundamental: the rep where

$$d(T^A) \equiv \frac{1}{2} \sigma^A$$

$SU(2) \leftarrow 2 \times 2$ unitary, $\det = 1$

naturally acts on 2d C VEC SPACE

FUNDAMENTAL

↳ $U \begin{pmatrix} a \\ b \end{pmatrix}$ where $U = D(g)$

REMARK: ALSO ANTIFUNDAMENTAL REP
transforms as U^\dagger
what are generators?

(2) ADJOINT REP

the generators themselves
form a representation of
the algebra.

(this is really weird the
first time you hear it!)

REP of A GENERATOR T^A : matrix in vec space
LINEAR TRANSFORMATION that takes
a VECTOR into another VECTOR.

$$\text{eg: } \mathcal{O}^A d(T^A) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix}$$

$$\sum_A \frac{\theta^A}{2} \sigma^A$$

ADJOINT: $d(T^A)$ or $\text{ad}(T^A)$

σ^{ABC} in $SU(2)$

for ANY ALGEBRA:

$$\text{ad}(T^A) T^B \equiv \frac{1}{i} [T^A, T^B] \equiv \frac{1}{i} f^{ABC} T^C$$

CHECK: is it linear? yes. DO NOT CONFUSE
 T^A the "matrix" ((really: $[T^A, \cdot]$))
with T^B the BASIS of VECTOR SPACE!!

eg. vector space of $\mathfrak{su}(2)$ ADJOINT:

$$\left\{ \left| \frac{1}{2} \sigma^1 \right\rangle, \left| \frac{1}{2} \sigma^2 \right\rangle, \left| \frac{1}{2} \sigma^3 \right\rangle \right\}$$

$$\begin{aligned} \text{ad}(T^1) \left| \frac{1}{2} \sigma^2 \right\rangle &= \frac{1}{i} [T^1, \frac{1}{2} \sigma^2] \\ &= \frac{1}{i} \left[\frac{1}{2} \sigma^1, \frac{1}{2} \sigma^2 \right] \\ &= \cancel{i} \frac{1}{2} \sigma^3 \\ &= \cancel{i} \left| \frac{1}{2} \sigma^3 \right\rangle \end{aligned}$$

$$\text{ad}(T^1) \left| \frac{1}{2} \sigma^3 \right\rangle = -\cancel{i} \left| \frac{1}{2} \sigma^2 \right\rangle$$

$$\text{ad}(T^1) \left| \frac{1}{2} \sigma^1 \right\rangle = 0$$

$$\text{ad}(T^1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\text{acting on } \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a \left| \frac{1}{2} \sigma^1 \right\rangle + b \left| \frac{1}{2} \sigma^2 \right\rangle + c \left| \frac{1}{2} \sigma^3 \right\rangle$$

what is the dimension of this REP? (3)

↳ hey! we've seen that...

$$\boxed{\text{ad}(T^A) |T^B\rangle = \frac{1}{i} [T^A, T^B]}$$

CHOOSE \otimes
 normaliz.

PICK A BASIS: $\{ |T^3\rangle, \frac{1}{\sqrt{2}} T^\pm \}$

$$ad(T^3) \left| \frac{1}{\sqrt{2}} T^\pm \right\rangle = \pm \left| \frac{1}{\sqrt{2}} T^\pm \right\rangle$$

$$ad(T^\pm) |T^3\rangle = \mp \sqrt{2} \left| \frac{1}{\sqrt{2}} T^\pm \right\rangle$$

$$ad(T^\pm) \left| \frac{1}{\sqrt{2}} T^\mp \right\rangle = \pm \sqrt{2} |T^3\rangle$$

$$ad(T^3) = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \begin{matrix} \left| \frac{1}{\sqrt{2}} T^+ \right\rangle \\ |T^3\rangle \\ \left| \frac{1}{\sqrt{2}} T^- \right\rangle \end{matrix}$$

$$ad(T^+) = \begin{pmatrix} & -\sqrt{2} & \\ & & \\ & & \sqrt{2} \end{pmatrix}$$

$$ad(T^-) = \begin{pmatrix} & \sqrt{2} & \\ -\sqrt{2} & & \\ & & -\sqrt{2} \end{pmatrix}$$

signs are a bit different.

Hm. dunno why..



see: physics.stackexchange.com

→ QUESTION 279880

HAS TO DO W/ SPHERICAL BASIS

homework: \exists U unitary s.t.

$$d^{(3)}(T^A) = U d(T^A) U^\dagger$$

Next time: $SU(3)$ (see cahn at 11)

PAULI MATRICES \longleftrightarrow Gellmann MATRICES

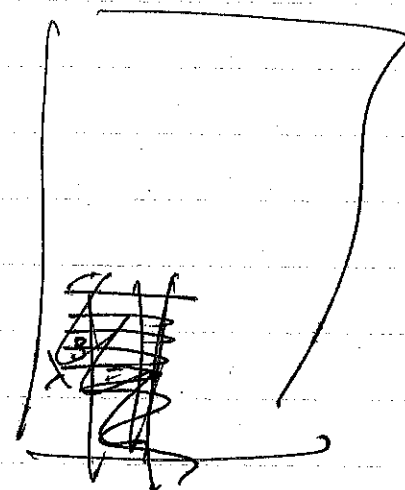
$$\lambda^1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} & -i & \\ i & & \\ & & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} & & -i \\ & 0 & \\ i & & \end{pmatrix}$$

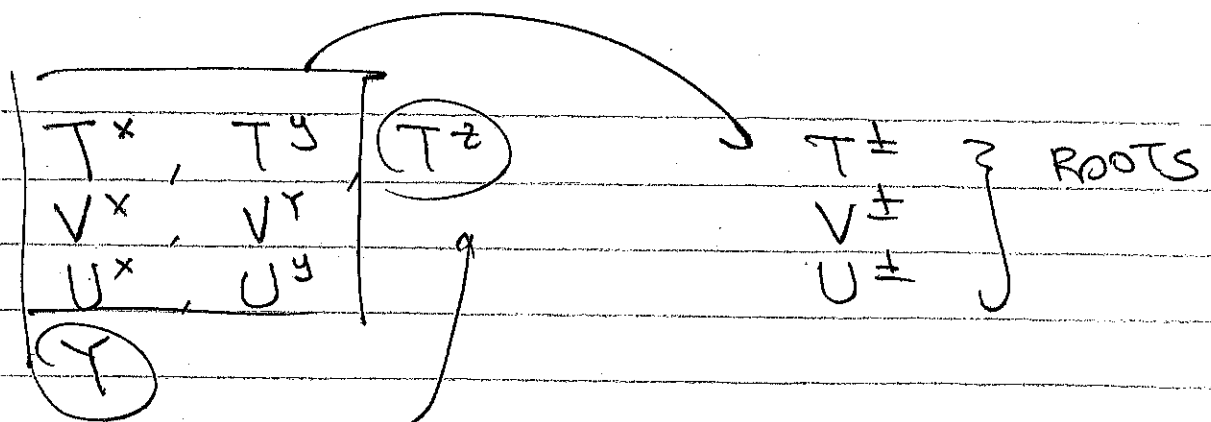
$$\lambda^6 = \begin{pmatrix} & & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{pmatrix} \quad \lambda^7 = \begin{pmatrix} & & & \\ & & & -i \\ & & i & \\ 1 & & & \end{pmatrix}$$

$$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -2 \end{pmatrix}$$

$\leftarrow ?$



\uparrow "missing"?



DIAGONAL GENERATORS
(CARTAN)

2 WEIGHTS

Weights: $(h_1, h_2) \rightarrow |h_1, h_2\rangle$

LOOKS LIKE COPIES OF $SU(2)$

... how many copies? 2-3 ish...

note: definitely NOT $SU(2) \times SU(2)$
because commutators don't close
into subalgebras.

$$\text{eg } [\frac{1}{2}T^+, U^+] = iV^+$$