today: still motivational

P262 LEC 2: the structure of SU(2)

LAST TIME : SO(3) ← rotations

Matrix Lie group: SPECIAL, ORTHOGONAL 3×3

determinant = 1

$M^T M = \mathbb{1}$

eg

$$g(0,0,\theta) = \begin{pmatrix} c_\theta & -s_\theta & \\ s_\theta & c_\theta & \\ & & 1 \end{pmatrix} = e^{-i\theta T_z}$$

ROT ABOUT $\hat{x}$ AXIS

ROT ABOUT $\hat{y}$ AXIS

ROT ABOUT $\hat{z}$ AXIS

generator

for example ... could also use Euler angles

FROM CAHN CH. 1

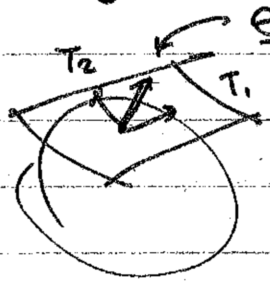
can work out products of group elements:

$$e^{-i\underline{\theta}\cdot\underline{T}}\, e^{-i\underline{\varphi}\cdot\underline{T}} = e^{-i(\underline{\theta}+\underline{\varphi})\cdot\underline{T} - \frac{1}{2}[\underline{\theta}\cdot\underline{T},\, \underline{\varphi}\cdot\underline{T}] + \cdots}$$

commutators

$\underline{\theta}\cdot\underline{T}$ is just some direction on the tangent space

... happens to be a matrix.



BCH FORMULA (you can derive if you want)

Remark: the moment we write out an explicit matrix for a group element, we are ~~[illegible]~~ working with a representation of the group.

cf. Magritte "Ceci n'est pas un pipe"

~~In the case above~~

A group can have many representations, but they all satisfy the same "groupiness" (SYMMETRY STRUCTURE)

e.g.
$$\left(\begin{array}{ccc|cc} c_\theta & -s_\theta & & & \\ s_\theta & c_\theta & & & \\ & & 1 & & \\ \hline & & & 1 & \\ & & & & 1 \end{array}\right)$$

$$g^{\text{WEIRD}} = g(\underline{\theta}) \oplus \mathbb{1}$$

The matrices will satisfy same commutation structure. They just act on different vector spaces

[matrix]
Representation [of a group]:

A vector space that transforms linearly (BY MATRICES) under the group.

SUBTLE: the groups we study are often defined w/rt matrices.

$SO(N) \leftarrow$ SPECIAL ORTHOGONAL $N \times N$
$\det M = 1 \quad M^T M = \mathbb{1}$

$SU(N) \leftarrow$ SPECIAL UNITARY $N \times N$
$\det M = 1 \quad M^\dagger M = 1$

these "defining" matrices are referred to as the fundamental rep.

↑ MORE PRECISELY: the vectors in this space are said to be in the fundamental rep.

if R is a rotation ($SO(N)$)

$$v^i \mapsto R^i{}_j v^j$$

IMPLICIT SUM OVER REPEATED INDICES; ~~all indices upper for now~~

THERE ARE OTHER REPRESENTATIONS

eg. Tensors

$$T^{ij} \mapsto R^i{}_k R^j{}_l T^{kl}$$

two indices
each one transforms

$$\text{vs} \quad M^i{}_j \mapsto R^i{}_k \boxed{(R^T)^l{}_j} M^k{}_l$$

$$R^i{}_k M^k{}_l (R^T)^l{}_j$$

eg Moment of inertia tensor
indices tell you how it transforms when you rotate your coords

~~even more exotic~~

commutators of generators

THIS CLASS: how structure of groups manifest themselves on representations

RECALL: DERIVATIVE of GROUP ELEMENT @ IDENTITY $\longrightarrow$ Generators

$$g_{\underline{\theta}} = e^{-i\,\underline{\theta}\cdot\underline{T}}$$

$= \theta^a T^a$ is some direction in the ALGEBRA; it generates g_θ

$$\nabla_{\underline{\theta}}\, g(\underline{\theta})\Big|_{\theta=0} = -i\,\underline{T}$$

@ the identity, where $g(0) = \mathbb{1}$

with our conventions, generators are [USUALLY] hermitian.

Generators obey commutation relations that encode everything about the group.

CLOSURE of GROUP $\leftrightarrow$ CLOSURE OF ALGEBRA UNDER COMMUTATOR

$[\cdot,\cdot]$ is the "MULTIPLY" of ALG.

$$[T^x, T^y] = iT^z \xrightarrow{\text{cyclic perm}} [T^a, T^b] = i\,\varepsilon^{abc}\ldots$$

as 3x3 matrices:

$$T^x = \begin{pmatrix} 0 & & \\ & & -i \\ & i & \end{pmatrix} \quad T^y = \begin{pmatrix} & & i \\ & 0 & \\ -i & & \end{pmatrix} \quad T^z = \begin{pmatrix} & -i & \\ i & & \\ & & 0 \end{pmatrix}$$

You've seen this commutation relation before:

$$\underline{J} = \left\{ \tfrac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \tfrac{1}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix}, \tfrac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right\}$$

$$= \tfrac{1}{2}\underline{\sigma} \leftarrow \text{PAULI MATRICES}$$

Are these related to SO(3)?

↳ subtle question!

ANSWER (if you can't wait until last week):
LOOK UP INDUCED REPRESENTATIONS
see, eg. Weinberg QFT vol 1 ch. 2

~~APPETIZER: UREAL~~

are these just a repackaging of so(3)?

→ so(3) generators in fundamental are pure imag.
$\exp(-i\underline{\theta}\cdot\underline{T})$ is pure real

→ PAULI MATRICES ARE MIXED $\mathbb{R}$ & $\mathbb{I}\mathrm{m}$ ...
generate $\mathbb{C}$ matrices.

note: $\underline{\sigma}$ are HERMITIAN
→ generate UNITARY matrices.

in fact: these $\underline{J} = \frac{1}{2}\underline{\sigma}$ are generators of a <u>COMPLETELY DIFFERENT</u> group:

$SU(2) \leftarrow$ <u>special</u> <u>unitary</u> 2×2

det $=1$ $\quad M^{+}M = \mathbb{1}$

but has the SAME ALGEBRA as $SO(3)$

LOCAL PROPERTIES
ARE IDENTICAL

so when we do small transformations, we can say the <u>fundamental</u> rep of $SU(2)$ is "sorta" like a rep of $SO(3)$...
<u>SPINOR REP</u>.

there is a more precise definition that we will postpone.

but from $SU(2)$ perspective: <u>SPINOR</u> is just the 2 component <u>vector</u> in $\mathbb{C}$ space that $\exp(-\frac{i}{2}\underline{\theta}\cdot\underline{\sigma})$ acts on

same θ's as $SO(3)$

OF COURSE... we know that there is a famous difference <u>GLOBALLY</u>

360° ROT in $SO(3)$ should give $\mathbb{1}$

<u>but</u> 360° ROT in $SU(2)$ gives $-\mathbb{1}$

↳ can trace this to the factor of $1/2$ in $\frac{1}{2}\underline{\sigma}$

could we just use $\underline{\sigma}$ instead?

<u>NO</u>: the $\frac{1}{2}$ is necessary for the commutation relations to work out!

((there is a sense of normalization))

Keep this in back of our mind.
WE WILL MOSTLY THINK ABOUT <u>LIE ALGEBRAS</u>, so saying "SPINOR REP" is ACCEPTABLE.