Homework 6 – Quantum Mechanics

Ch 3: 1, 2, 4, 16 Blake Griffith

Exercise (3.1). Find the eigenvalues and eigenvectors of $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Suppose an electron is in the spin state $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. If S_y is measured, what is the probability of the result $\hbar/2$?

Proof. (a) We begin by solving the charachteristic equation to find the eigenvalues of σ_y:

$$\begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \implies \boxed{\lambda = \pm 1}$$

Now we find the corresponding eigenvectors, for $\lambda = -1$:

$$\sigma_y \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix} \implies -ib = -a \qquad ai = -b \implies \boxed{\mathbf{X}(\lambda = -1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}}$$

for $\lambda = 1$:

$$\sigma_y \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \implies -ib = a \qquad ai = b \implies \boxed{\mathbf{X}(\lambda = 1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}}$$

(b) The probability of measuring $S_y = \hbar/2$ (spin up in the y basis) is $|\langle \psi_{y+}| S_y |\psi\rangle|^2$. Where is $\langle \psi_{y+}|$ is the eigenstate corresponding to the S_y "up" eigenvalue ($\lambda = 1$). Which we found above since $S_y = \frac{\hbar}{2}\sigma_y$. So we have

$$|\langle \psi_{y+}| S_y |\psi\rangle|^2 \implies \frac{\hbar^2}{8} \left| \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right|^2 \implies \boxed{\frac{\hbar^2}{8}(\alpha^2 + \beta^2)}$$

□

Exercise (3.2). Find by explicit construction using Pauli matrices, the eigenvalues for the Hamiltonian

$$H = -\frac{2\mu}{\hbar} \mathbf{S} \cdot \mathbf{B}$$

for a spin $\frac{1}{2}$ particle in the presence of a magnetic field $\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}$.

Proof. We rotate our coordinate system so that $\mathbf{B}$ points in the $\hat{\mathbf{z}}$ direction at the point of the particle. Then our Hamiltonian becomes

$$H = -\frac{2\mu B}{\hbar} S_z = \omega S_z$$

Recall the eigenvalue for S_z is $\pm\frac{\hbar}{2}$, so at $t = 0$ we have:

$$H |\psi\rangle = \mp \frac{2\mu B}{2} |\psi\rangle$$

To calculate the time evolution note that the time evolution operator is:

$$U(t) = \exp \frac{-iHt}{\hbar} = \exp \frac{-i\omega t S_z}{\hbar} \implies U(t) |\psi\rangle = \exp \frac{\mp i\omega t}{2}$$

Combining these:

$$H |\psi(t)\rangle = HU(t) |\psi\rangle = \boxed{\mp \frac{2\mu B}{2} \exp\left(\frac{\mp i\omega t}{2}\right) |\psi\rangle}$$

Where do Pauli matrices come into this?

□

Exercise (3.4). The spin-dependent Hamiltonian of an electron-positron system in the presence of a uniform magnetic field in the z-direction can be written as

$$H = A\mathbf{S}^{e^-} \cdot \mathbf{S}^{\mathbf{e}^+} + \frac{eB}{mc}\left(S_z^{e^-} - S_z^{e^+}\right)$$

Suppose the spin function of the system is given by $\chi_+^{e^-}\chi_-^{e^+}$.

(a) Is this an eigenfunction of H in the limit $A \to 0$, $eB/mc \neq 0$? If it is, what is the energy eigenvalue? If it is not, what is the expectation value of H?

(b) Solve the same problem when $eB/mc \to 0$, $A \neq 0$.

Proof. (a) With $A \to 0$ the hamiltonian is just the term from the electron and positron independent of interactions:

$$H = \frac{eB}{mc}\left(S_z^{e^-} - S_z^{e^+}\right)$$

(b) With $eB/mc \to 0$, $A \neq 0$ we have:

$$H = A\mathbf{S}^{e^-} \cdot \mathbf{S}^{\mathbf{e}^+}$$

□

Exercise (3.16). Show that the orbital angular-momentum operator $\mathbf{L}$ commutes with both the operators $\mathbf{p}^2$ and $\mathbf{x}^2$; that is, prove (3.7.2).

Proof. (a) Looking at $[\mathbf{L}, \mathbf{p}^2] = \mathbf{L}$ we see that each p_i much commute with each $\mathbf{L}$ component so we have

$$[L_x, p_x^2 + p_y^2 + p_z^2] = [(\mathbf{x} \times \mathbf{p})_x, +p_y^2 + p_z^2] = [x_y p_z - x_z p_y, p_x^2 + p_y^2 + p_z^2]$$

So we expressions of the form $[x_i p_j, p_k^2] = \delta_{ij} i\hbar(2p_k p_j)$, we prove this:

$$[x_i p_j, p_k^2] = [x_i, p_k^2]p_j + x_i[p_k, p_j^2]$$

the second term is zero so we have:

$$[x_i p_j, p_k^2] = [x_i, p_k^2]p_j = [x_i, p_k p_k]p_j = [x_i, p_k]p_k p_j + p_k[x_i, p_k]p_j = 2i\hbar p_k p_j \delta_{ik}$$

So for each angular momentum component we have:

$$[L_x, p_x^2 + p_y^2 + p_z^2] = [x_y p_z - x_z p_y, p_x^2 + p_y^2 + p_z^2] = [x_y p_z, p_y^2] - [x_z p_y, p_z^2] = i2\hbar(p_y p_z - p_z p_y) = i2\hbar[p_y, p_z] = 0$$

Which implies $[\mathbf{L}, \mathbf{p}^2] = 0$.

(b) $[\mathbf{L}, \mathbf{x}^2]$ reduces to a similar relation:

$$[L_z, \mathbf{x}^2] = [xp_y - yp_x, x^2 + y^2 + z^2]$$

gives us expressions like $[x_i p_j, x_k^2] = x_i x_k(-i2\hbar\delta_{jk})$, proof:

$$[x_i p_j, x_k^2] = [x_i, x_k^2]p_j + x_i[p_j, x_k^2]$$

The first term is zero so:

$$x_i[p_j, x_k^2] = x_i([p_j, x_k]x_k + x_k[p_j, x_k]) = x_i x_k(-i2\hbar\delta_{jk})$$

Applying this identity we get

$$[L_z, \mathbf{x}^2] = [xp_y, y^2] - [yp_x, x^2] = -2i\hbar(xy - yx) = -2i\hbar[x, y] = 0$$

Which implies $[\mathbf{L}, \mathbf{x}^2]$.

□