1. A Too Young Review of (Weakly) Interacting Fermi Gas.

$$\hat{H} = \sum_{p,\sigma} \frac{p^2}{2m} c^{+}_{p\sigma} c_{p\sigma} + \frac{1}{2}\sum{}' u \, c^{+}_{p_1'\sigma_1'} c^{+}_{p_2'\sigma_2'} c_{p_2\sigma_2} c_{p_1\sigma_1}. \qquad (11.7.1)$$

Sort u by its spin indexes. $\begin{cases} \sigma_1 = \pm\frac{1}{2},\ \sigma_2 = \mp\frac{1}{2},\ \sigma_1' = \pm\frac{1}{2},\ \sigma_2' = \mp\frac{1}{2}. \\ \sigma_1 = \pm\frac{1}{2},\ \sigma_2 = \mp\frac{1}{2},\ \sigma_1' = \mp\frac{1}{2},\ \sigma_2' = \pm\frac{1}{2}. \end{cases}$

$$\hat{H} = \sum_{p,\sigma} \frac{p^2}{2m} c^{+}_{p\sigma} c_{p\sigma} + \frac{u_0}{V}\sum{}' c^{+}_{p_1'+} c^{+}_{p_2'-} c_{p_2-} c_{p_1+}, \quad \frac{u_0}{V} = \left(u^0_{+-+-} - u^0_{+--+}\right)$$

(11.7.3/4)

Perturbation.

Zeroth – diagonal terms (without interacting terms)

$$E^{(0)} = \sum_{p,\sigma} \frac{p^2}{2m} n_{p\sigma}, \quad \bar{n}_{p\sigma} = \frac{1}{z^{-1}\exp\left(p^2/2mkT\right)+1}, \quad E^{(0)} = V\cdot\frac{3kT}{\lambda^3} f_{5/2}(z_0)$$

$$\lambda = \frac{h}{\sqrt{2\pi mkT}}. \quad f_\nu(z_0) = \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}\,dx}{z_0^{-1}e^x+1}, \quad N = V\cdot\frac{2}{\lambda^3} f_{3/2}(z_0).$$

First order – diagonal elements of the interacting terms. ($p_1' = p_1$ & $p_2' = p_2$)

$$E^{(1)} = \frac{u_0}{V}\sum_{p_1,p_2} n_{p_1+} n_{p_2-} = \frac{u_0}{V} N^+ N^- = V\frac{u_0}{\lambda^6}\{f_{3/2}(z_0)\}^2$$

$$= V\cdot\frac{2kT}{\lambda^3}\left(\frac{a}{\lambda}\right)\{f_{3/2}(z_0)\}^2$$

Second Order. – two components contribute

$$E^{(2)}_n = \sum_{m\neq n}\frac{|V_{nm}|^2}{E_n - E_m} \qquad n\to(p_1,p_2) \quad m\to(p_1',p_2') \quad V_{nm} = \frac{u_0}{V} c^{+}_{p_1'+} c^{+}_{p_2'-} c_{p_2-} c_{p_1+}.$$

$$|V_{nm}|^2 = \frac{u_0^2}{V^2}\left(c^{+}_{p_1'+} c^{+}_{p_2'-} c_{p_2-} c_{p_1+}\right)\left(c^{+}_{p_1+} c^{+}_{p_2-} c_{p_2'-} c_{p_1'+}\right)$$

$$= n_{p_1+} n_{p_2-} (1-n_{p_1'+})(1-n_{p_2'-}). \qquad \to (11.7.16)$$

~~perturbation~~ renormalization of coupling constant.

$$\frac{4\pi a\hbar^2}{mV} \simeq \frac{u_0}{V} + 2\frac{u_0^2}{V^2}\sum_{p_1p_2p_1'}\frac{1}{(p_1^2+p_2^2-p_1'^2-p_2'^2)/2m}$$

$$u_0 \simeq \frac{4\pi a\hbar^2}{m}\left[1 - \frac{8\pi a\hbar^2}{mV}\sum_{p_1p_2p_1'}\frac{1}{(p_1^2+p_2^2-p_1'^2-p_2'^2)/2m}\right] \to (2)$$

$$E^{(1)}_2 = -2\left(\frac{4\pi a\hbar^2}{mV}\right)^2\sum_{p_1p_2p_1'}\frac{n_{p_1+} n_{p_2-}}{(\ - - - -\ )/2m}.$$

$$E_2^{(2)} = 2 \cdot \left(\frac{4\pi a \hbar^2}{mv}\right)^2 \sum_{p_1 p_2 p_1'} \frac{n_{p_1} + n_{p_2} - (1 - n_{p_1'+})(1 - n_{p_1'-})}{(- \quad - \quad)/m}$$

(nnnn? symmetric reason)

$$E^{(2)*} = -2 \left(\frac{4\pi a \hbar^2}{mv}\right)^2 \sum_{p_1 p_2 p_1'} \frac{n_{p_1} + n_{p_2} - (n_{p_1'+} + n_{p_1'-})}{(- - - \cdots)/2m}$$

$$= -4 \left(\frac{4\pi a \hbar^2}{mV}\right)^2 \sum_{\substack{p_1 p_2,\\ p_1' p_2'}} \frac{n_{p_1} + n_{p_2} - n_{p_1'} + \delta_{p_1+p_2,\, p_1'+p_2'}}{(- \quad - \quad)/2m}$$

$$\stackrel{?}{=} V \frac{\gamma kT}{\lambda^3} \left(\frac{a^2}{\lambda^2}\right) F(z_0) \qquad \Big\downarrow \; T \to 0. \; (z_0 \to \infty)$$

2. A Too Simple Introduction to Fermi Liquid Theory.

Preliminaries. (T. G. many-body)

- Interacting Hamiltonian on lattice.

$$H = \sum_{k\sigma} \varepsilon(k)\, c_{k\sigma}^+ c_{k\sigma} + \frac{1}{2\Omega} \sum_{\substack{k_1 k_2 q\\ \sigma_1 \sigma_2}} V(q)\, c_{k_1+q\sigma_1}^+ c_{k_2-q\sigma_2}^+ c_{k_2\sigma_2} c_{k_1\sigma_1}.$$

$\rightarrow$ in BZ.

- Green Function (single particle) and Spectral Function.

Create a particle at $(\vec{r}_1, t_1)$ and recycle one from at $(\vec{r}_2, t_2)$.

$$|\psi(t)\rangle = e^{-iHt_0} c_{r_1}^+ |G\rangle. \qquad |\psi'(t)\rangle = c_{r_2}^+ e^{-iHt_0} |G\rangle.$$

$$\langle \psi'(t) | \psi(t) \rangle = \langle G | \underbrace{e^{iHt_0} c_{r_2} e^{-iHt_0}}_{c_{r_2}(t)} c_{r_1}^+ |G\rangle.$$

e.g. linear response of perturbation $H_{pert} = \int dr \left[\lambda(r,t) c_r^+ + \lambda^*(r,t) c_r\right]$

$$\langle c_{r_2} \rangle_t = \int dr_1\, dt_1\; G(r_2 - r_1, t_2 - t_1)\, \lambda(r_1, t_1).$$

$$\downarrow$$

$$-i\theta(t_2 - t_1) \langle [c_{r_2 t_2}, c_{r_1 t_1}^+] \rangle.$$

$G(r,t) \to G(k,\omega)$. Fourier Transformation.

$$G^0(k,t) = -i\theta(t) \langle [c_{kt}, c_{k0}^+] \rangle = -i\theta(t) e^{-i\xi(k)t}.$$

$$G^0(k,\omega) = \int dt\, e^{i(\omega + i\delta)t} G^0(k,t) = \frac{1}{\omega - \xi(k) + i\delta}.$$

$$A(k,\omega) = -\frac{1}{\pi} \mathrm{Im}\, G(k,\omega). \longrightarrow \text{Spectral Function}.$$

$A^0(k,\omega) = \delta(\omega - \xi(k))$.

Expand in the basis of H $\langle\cdots\rangle \to \frac{1}{Z}\mathrm{Tr}[e^{-\beta H}\cdots]$.

$$G(k,\omega) = \frac{1}{Z}\sum_{n,m} \langle n|c_k|m\rangle\langle m|c_k^\dagger|n\rangle \frac{e^{-\beta E_n}+e^{-\beta E_m}}{\omega+E_n-E_m+i\delta} \quad ($$

$$A(k,\omega) = \frac{1}{Z}\sum_{n,m}\langle n|c_k|m\rangle\langle m|c_k^\dagger|n\rangle(e^{-\beta E_n}+e^{-\beta E_m})\delta(\omega+E_n-E_m).$$

$$= \frac{1+e^{-\beta\omega}}{Z}\sum_{n,m}|\langle m|c_k^\dagger|n\rangle|^2 e^{-\beta E_n}\delta(\omega+E_n-E_m)$$

~~probability to find in~~ $= \frac{1}{f_F(\omega)}\frac{1}{Z}\sum_{n,m}|\langle m|c_k^\dagger|n\rangle|^2 e^{-\beta E_n}\delta(\omega+E_n-E_m)$

Spectral function measures the probability of finding an excitation of energy ω and momentum K.

Some other properties: $\int_{-\infty}^{+\infty} d\omega\, A(k,\omega) = 1$. $\quad G(\omega) = \int d\omega' \frac{A(\omega')}{\omega-\omega'}$

$$\int_{-\infty}^{+\infty} d\omega\, f_F(\omega) A(k,\omega) = \langle c_k^\dagger c_k\rangle.$$

• Self Energy: life time $(+i\delta)$ and effective mass.

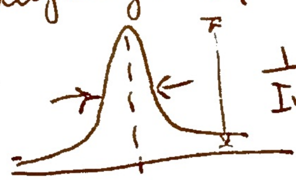
can be obtained by Dyson Equation

Kramers-Kronig Relation. (between real & imaginary time)

~~Imaginary~~ $G(k,\omega) = \frac{1}{\omega-\xi(k)-\Sigma(k,\omega)+i\delta}$.

$$A(k,\omega) = -\frac{1}{\pi}\frac{\mathrm{Im}\Sigma(k,\omega)}{(\omega-\xi(k)-\mathrm{Re}\Sigma)^2+(\mathrm{Im}\Sigma)^2}.$$

Imaginary: lifetime $\to$ FT.



$\frac{1}{\mathrm{Im}\Sigma} = \tau$. $\quad (\mathrm{Re}\Sigma \to 0)$

$\frac{1}{\tau} = \mathrm{Im}\Sigma$

Real part: Effective mass & quasiparticle weight.

$(\mathrm{Im}\Sigma \to 0)$

$A(k,\omega) = \delta(\omega-\xi(k)-\mathrm{Re}\Sigma)$

$$\boxed{E(k)-\xi(k)-\mathrm{Re}\Sigma(k,\omega=E(k)) = 0}.$$

Expand ~~the~~ the vicinity of fermi surface

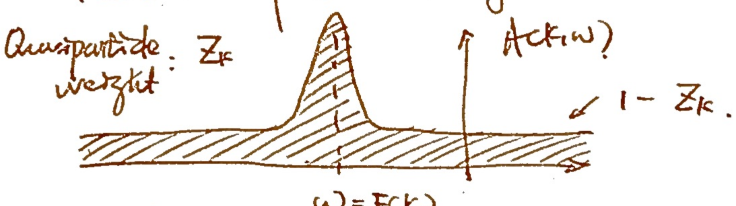
$$\xi(k) \simeq \frac{k_F}{m}(k-k_F) \qquad E(k) \simeq \frac{k_F}{m^*}(k-k_F)$$

$$\rightarrow \frac{k_F}{m^*} = \frac{\frac{k_F}{m} + \frac{\partial \mathrm{Re}\Sigma}{\partial k}\Big|_{\omega=E(k)}}{1 - \frac{\partial \mathrm{Re}\Sigma}{\partial \omega}\Big|_{\omega=E(k)}} \rightarrow \frac{m}{m^*} = \frac{1 + \frac{m}{k_F}\frac{\partial \mathrm{Re}\Sigma}{\partial k}\Big|_{\omega=E(k)}}{1 - \frac{\partial \mathrm{Re}\Sigma}{\partial \omega}\Big|_{\omega=E(k)}}.$$

$$A(k,\omega) = Z_k\, \delta(\omega - E(k))$$

$$Z_k = \frac{1}{1 - \frac{\partial \mathrm{Re}\Sigma}{\partial \omega}\Big|_{\omega=E(k)}}.$$

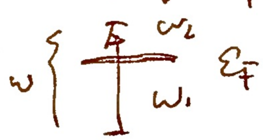
Fermi Liquid Theory.



lorentzian shape.

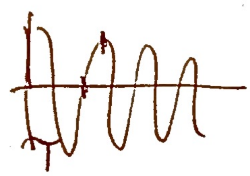
Why is the lifetime of quasiparticle so long?

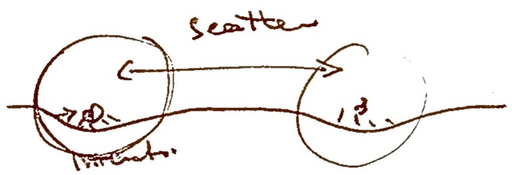
$\omega = \omega_1 + \omega_2$



$$P \propto \int_{-\omega}^{0} d\omega_1 \int_{0}^{\omega+\omega_1} d\omega_2 = \frac{1}{2}\omega^2.$$

$$\tau \sim \frac{1}{\omega^2} \qquad T \sim \frac{1}{\omega} \qquad \frac{\tau}{T} = \frac{1}{\omega} \rightarrow \infty.$$





3. An Sometimes Naive Explanation of BEC-BCS Crossover

Under construction.