Solutions to Pathria's Statistical Mechanics
Chapter 2

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Problem 2.1

Problem 2.2

Problem 2.3

The Hamiltonian of the rotator is a function of the angular momentum L.

$$H = f(L)$$

Now we divide the phase space into cells with volume h by lines with constant energies:

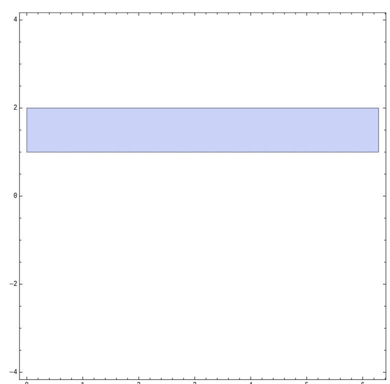

Figure 1: A quantum state in phase space

Since the angle φ varies between 0 and 2π, angular momentum should be quantized as shown:

$$h = 2\pi\Delta L \quad \Rightarrow \quad \Delta L = \hbar$$

Since we starting from the zero line, the eigenvalues of energy should be

$$E_m = f(m\hbar) \tag{1}$$

Notice that we get this result by cutting the phase space into slices without solving the Schrödinger Equation. Because the Hamiltonian commutes with the angular momentum, the eigenenergy is given by eigenvalues of the angular momentum:

$$-i\hbar\frac{\partial\psi}{\partial\varphi} = L\psi \quad \psi(\varphi) = \psi(\varphi + 2\pi)$$

solve the differential equation and the result is:

$$L = m\hbar \quad m \in \mathbb{Z}\,.$$

Now we find that the result we get from the eigenfunction of angular momentum operator is the same as we get from cutting the phase space into cells.

Problem 2.4

If we just consider about the orbital angular momentum, it can be written as a function of p_θ and p_φ which are the canonical momentum conjugate to the spherical coordinate variables θ and φ:

$$L^2 = p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta} \tag{2}$$

thus the phase volume of the region which satisfies $L^2 \leq M^2$ is

$$\begin{aligned} \boldsymbol{\Omega} &= \int_0^\pi d\theta \int_0^{2\pi} d\varphi \int_{L^2 \leq M^2} dp_\theta dp_\varphi \\ &= \int_0^\pi d\theta \int_0^{2\pi} d\varphi\, \pi M^2 \sin\theta \\ &= 4\pi^2 M^2 \end{aligned} \tag{3}$$

Thus the number of microstates is $\Omega = \boldsymbol{\Omega}/h^2 = M^2/\hbar^2$. Then let us calculate the number by quantized angular momentum. By summing up all the eigenstates of the angular momentum, we get:

$$\Omega = \sum_{j=0}^{j_{\max}} (2j+1) = (j_{\max}+1)^2 \tag{4}$$

Now we have to determine the number $j_{\max}$. Since we want the absolute value of the angular momentum $\sqrt{j_{\max}(j_{\max}+1)}\hbar < M$, we can find that $j_{\max}$ is determined by the following equation:

$$j_{\max} = \left\lfloor \frac{\sqrt{1+\frac{4M^2}{\hbar^2}}-1}{2} \right\rfloor \tag{5}$$

It is obvious that $j_{\max} = \lfloor M/\hbar \rfloor - 1$ or $j_{\max} = \lfloor M/\hbar \rfloor$. So the total number of microstates will be:

$$\Omega = \left\lfloor \frac{M}{\hbar} \right\rfloor^2 \text{ or } \left(\left\lfloor \frac{M}{\hbar} \right\rfloor + 1 \right)^2 \tag{6}$$

If we take the classical limit that $M \gg \hbar$, the result will be:

$$\Omega \simeq \frac{M^2}{\hbar^2} \tag{7}$$

Problem 2.5

In this problem we need to use the WKB approximation in Quantum Mechanics. In D. Griffiths' book we find that the WKB wave function between two classical turning point x_1 and x_2 is:

$$\psi(x) = \frac{2D}{\sqrt{p(x)}} \sin\left[\frac{1}{\hbar} \int_x^{x_2} p(x')dx' + \frac{\pi}{4} \right] \qquad x < x_2$$

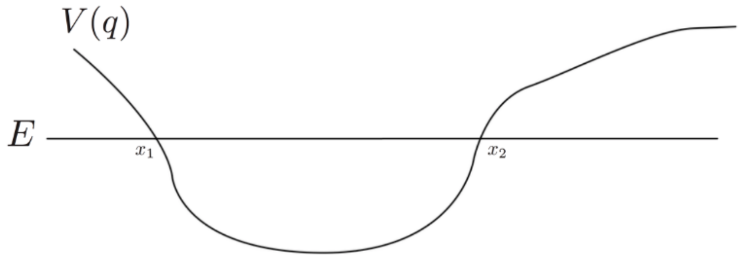

Figure 2: WKB Approximation

or

$$\psi(x) = -\frac{2D'}{\sqrt{p(x)}}\sin\left[-\frac{1}{\hbar}\int_{x_1}^{x} p(x')dx' - \frac{\pi}{4}\right] \qquad x > x_1$$

in which $p(x) = \sqrt{2m[E - V(x')]}$. We can define:

$$\begin{aligned}
\theta_1(x) &= \frac{1}{\hbar}\int_{x_1}^{x} p(x')dx' + \frac{\pi}{4} \\
\theta_2(x) &= \frac{1}{\hbar}\int_{x}^{x_2} p(x')dx' + \frac{\pi}{4}
\end{aligned}$$

Since the two solutions should be the same, the difference between the two θ functions should be $n\pi, n \in \mathbb{Z}$:

$$n\pi - \frac{\pi}{2} = \frac{1}{\hbar}\int_{x_1}^{x_2} p(x')dx' \tag{8}$$

The integral in classical phase space is

$$\oint p\, dq = 2\int_{x_1}^{x_2} p(x')dx'$$

so finally we can find that

$$\oint p\, dq = h\left(n - \frac{1}{2}\right) \quad n \in \mathbb{Z}\,. \tag{9}$$

Problem 2.6

The equation of phase space orbit is:

$$\frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}mgl\theta^2 = E$$

This is a eclipse whose area is:

$$S = \pi\sqrt{2Eml^2}\sqrt{\frac{2E}{mgl}} = 2\pi E\sqrt{\frac{l}{g}} = E\tau$$

Problem 2.7

(i) Assume that these N SHOs are distinguishable.To distribute total energy E into such N SHOs, there are

$$C_{E/\hbar\omega+N/2-1}^{N-1}$$

ways. Let $E/\hbar\omega >> N$,we get the approximate result:

$$\frac{1}{(N-1)!}(\frac{E}{\hbar\omega})^{N-1}$$

(ii) The total energy of N classical SHOs is:

$$\sum_{i=1}^{N}(\frac{p_i^2}{2m}+\frac{kx_i^2}{2})=E$$

The phase space volume is:

$$(\frac{2}{\omega})^N E^{N-1}\pi^N\frac{1}{(N-1)!}dE$$

while dE=$\hbar\omega$, we get $\omega_0=h^N$

Problem 2.8

Problem 2.9