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# **Part I**

## **Classical Mechanics**

# Chapter 1

## The formulation of Classical Mechanics



### 1.1 Lagrangian Formulation

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt, \quad \delta q_i(t_1) = \delta q_i(t_2) = 0$$

$$\delta S = 0 \rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

1. If we transform the coordinates  $q$  to the  $Q$  as  $q = q(Q, t)$ , the new Lagrangian will be

$$\bar{L}(Q, \dot{Q}, t) \equiv L(q(Q, t), \dot{q}(Q, \dot{Q}, t), t)$$

We can verify that

$$\frac{d}{dt} \frac{\partial \bar{L}}{\partial \dot{Q}} - \frac{\partial \bar{L}}{\partial Q} = 0$$

2. If  $L_1 = L + \frac{d}{dt} f(q, t)$ , then  $L$  and  $L_1$  is equivalent and will generate the same dynamical equation.

**Example:**

1. The form of Lagrangian for an isolated system of particles in inertial frame:

$$L = \sum_a \frac{1}{2} m_a v_a^2 - U(\mathbf{r}_1, \mathbf{r}_2, \dots, )$$

The equation of motion is

$$m_i \ddot{\mathbf{r}}_i = -\nabla_{\mathbf{r}_i} U$$

To get the form of Lagrangian for a system of interacting particles, we must assume:

- Space and time are homogeneous and isotropic in inertial frame;
  - Galileo's relativity principle and Galilean transformation;
  - Spontaneous interaction between particles;
2. Consider a reference frame  $K$ . Suppose the  $K$  is moving with the velocity  $\mathbf{V}(t)$  and rotating with angular velocity  $\boldsymbol{\Omega}$  relative to the inertial reference frame. We use the coordinates of the mass point in  $K$  as general coordinates, i.e.  $\mathbf{r} = (x_k, y_k, z_k)$ . Then the Lagrangian of the mass point will be

$$L = \frac{1}{2} m \mathbf{v}^2 + m \mathbf{v} \cdot (\boldsymbol{\Omega} \times \mathbf{r}) + \frac{m}{2} (\boldsymbol{\Omega} \times \mathbf{r})^2 - m \dot{\mathbf{V}} \cdot \mathbf{r} - U$$

The equation of motion will be

$$m \frac{d\mathbf{v}}{dt} = -\frac{\partial U}{\partial \mathbf{r}} - m\dot{\mathbf{V}} + m(\mathbf{r} \times \dot{\boldsymbol{\Omega}}) + 2m(\mathbf{v} \times \boldsymbol{\Omega}) + m[\boldsymbol{\Omega} \times (\mathbf{r} \times \boldsymbol{\Omega})]$$

## 1.2 Symmetry and Conservation Laws(1)

### Theorem 1.1 Nother's theorem

For  $q_i \rightarrow q_i + \delta q_i$  and  $L \rightarrow L + \delta L$ , if  $\delta L = \frac{df(q, \dot{q}, t)}{dt}$ , then we get

$$\frac{d}{dt} \left( \sum_i p^i \delta q_i - f \right) = 0 \quad (p^i = \frac{\partial L}{\partial \dot{q}_i})$$

**Example:** For an isolated system of particles in inertial frame,  
 $\delta L = 0$  when  $\delta \mathbf{r}_i \rightarrow \mathbf{r}_i + \delta \mathbf{a}$ , so

$$\frac{d}{dt} \left( \sum_i \mathbf{p}_i \right) = 0$$

$\delta L = 0$  when  $\delta \mathbf{r}_i \rightarrow \mathbf{r}_i + \mathbf{r}_i \times \delta \boldsymbol{\theta}$ , so

$$\frac{d}{dt} \left( \sum_i \mathbf{r}_i \times \mathbf{p}_i \right) = 0$$

**Homogeneity of time** If  $\frac{\partial L}{\partial t} = 0$ , then we get

$$\frac{dE}{dt} = 0 \quad (E = \sum_i \dot{q}_i p^i - L)$$

## 1.3 Hamilton formulation

$$p^i = \frac{\partial L}{\partial \dot{q}_i}$$

$$H(q, p, t) = \sum_i p^i \dot{q}_i - L$$

$$\dot{p}^i = -\frac{\partial H}{\partial q_i} \quad \dot{q}_i = \frac{\partial H}{\partial p^i}$$

**Example:** For an isolated system of particles in inertial frame,

$$\mathbf{p}_i = m_i \mathbf{v}_i$$



$$H(q, p, t) = \sum_i \frac{p_i^2}{2m} + U(\mathbf{r}_1, \mathbf{r}_2, \dots)$$

$$\dot{\mathbf{p}}_i = -\nabla_{\mathbf{r}_i} U \quad \dot{\mathbf{r}}_i = \frac{\mathbf{p}_i}{m_i}$$

### 1.3.1 Poisson Brackets

First, we assume the bracket operation has the following properties:

$$[f, g] = -[g, f]$$

$$[\alpha_1 f_1 + \alpha_2 f_2, \beta_1 g_1 + \beta_2 g_2] = \alpha_1 \beta_1 [f_1, g_1] + \alpha_1 \beta_2 [f_1, g_2] + \alpha_2 \beta_1 [f_2, g_1] + \alpha_2 \beta_2 [f_2, g_2]$$

$$[f_1 f_2, g_1 g_2] = f_1 [f_2, g_1] g_2 + f_1 g_1 [f_2, g_2] + g_1 [f_1, g_2] f_2 + [f_1, g_1] g_2 f_2$$

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$$

Here,  $f, g, h$  are functions of  $p^i, q_i, t$ . Then, we assume that

$$[q_i, p^k] = \delta_i^k$$

we can derive that

$$[f, g] = \sum_k \left( \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p^k} - \frac{\partial f}{\partial p^k} \frac{\partial g}{\partial q_k} \right)$$

So the Hamilton equation can be written as

$$\dot{p}^i = [p^i, H] \quad \dot{q}_i = [q_i, H]$$

And we can also get

$$\frac{df}{dt} = [f, H] + \frac{\partial f}{\partial t} \quad \frac{d}{dt}[f, g] = \left[ \frac{df}{dt}, g \right] + \left[ f, \frac{dg}{dt} \right]$$

**Example:** For an isolated system of particles in inertial frame,

$$[r_{ia}, p_{jb}] = \delta_{ab} \delta_{ij}$$

we define  $l_a = \epsilon_{abc} r_a p_b$ , then

$$[l_a, r_b] = \epsilon_{abc} r_c \quad [l_a, p_b] = \epsilon_{abc} p_c \quad [l_a, l_b] = \epsilon_{abc} l_c$$



### 1.3.2 Canonical transformations

In Hamiltonian mechanics, a canonical transformation is a change of canonical coordinates that preserves the form of Hamilton's equations (that is, the new Hamilton's equations resulting from the transformed Hamiltonian may be simply obtained by substituting the new coordinates for the old coordinates), although it might not preserve the Hamiltonian itself.

$$Q_i = Q_i(p, q, t) \quad P_i = P_i(p, q, t)$$

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i} \quad \dot{P}_i = -\frac{\partial H'}{\partial Q_i}$$

#### Proposition 1.1 Canonical condition

If  $(q_i, p^i, H) \rightarrow (Q_i, P^i, H')$  is a canonical transformation, then there exists a generating function  $F(q_i, Q_i, t)$  satisfying that

$$\sum_i p^i \dot{q}_i - H(p^i, q_i) = \sum_i P^i \dot{Q}_i - H'(Q_i, P^i) + \frac{dF}{dt}$$

Applying Legendre transformation, we can get four kinds of generating function.

1.

$$\frac{dF}{dt} = \sum_i p^i \dot{q}_i - \sum_i P^i \dot{Q}_i + (H' - H)$$

Assume  $\Phi(q_i, Q_i, t) = F$ , so

$$p^i = \frac{\partial \Phi}{\partial q_i} \quad P^i = -\frac{\partial \Phi}{\partial Q_i} \quad H' = H + \frac{\partial \Phi}{\partial t}$$

2.

$$\frac{d}{dt}(F + \sum_i P^i Q_i) = \sum_i p^i \dot{q}_i + \sum_i Q_i \dot{P}^i + (H' - H)$$

Assume  $\Phi(q_i, P^i, t) = F + \sum_i P^i Q_i$ , so

$$p^i = \frac{\partial \Phi}{\partial q_i} \quad Q_i = \frac{\partial \Phi}{\partial P^i} \quad H' = H + \frac{\partial \Phi}{\partial t}$$

3.

$$\frac{d}{dt}(F - \sum_i p^i q_i) = -\sum_i q_i \dot{p}^i - \sum_i P^i \dot{Q}_i + (H' - H)$$

Assume  $\Phi(p^i, Q_i, t) = F - \sum_i p^i q_i$ , so

$$q_i = -\frac{\partial \Phi}{\partial p^i} \quad P^i = -\frac{\partial \Phi}{\partial Q_i} \quad H' = H + \frac{\partial \Phi}{\partial t}$$



4.

$$\frac{d}{dt}(F + \sum_i P^i Q_i - \sum_i p^i q_i) = - \sum_i q_i \dot{p}^i + \sum_i Q_i \dot{P}^i + (H' - H)$$

Assume  $\Phi(p^i, P^i, t) = F + \sum_i P^i Q_i - \sum_i p^i q_i$ , so

$$q_i = -\frac{\partial \Phi}{\partial p^i} \quad Q_i = \frac{\partial \Phi}{\partial P^i} \quad H' = H + \frac{\partial \Phi}{\partial t}$$

### Theorem 1.2 The invariance of Poisson Bracket

Suppose that  $(q, p, H) \rightarrow (Q, P, H')$  is a canonical transformation and  $f(q, p, t) = F(Q, P, t)$ ,  $g(q, p, t) = G(Q, P, t)$ , then

$$[f, g]_{q,p} = [F, G]_{Q,P}$$

As a result, the condition for canonical transformation can also be stated as

$$[Q_i, Q_j]_{q,p} = 0 \quad [P^i, P^j]_{p,q} = 0 \quad [Q_i, P^j]_{q,p} = \delta_i^j$$

### 1.3.3 Evolution as canonical transformations

Let  $q_t, p_t$  be the values of the canonical variables at time  $t$ , and  $q_{t+\tau}, p_{t+\tau}$  their values at another time  $t + \tau$ . The latter are some functions of the former:

$$q_{t+\tau} = q(q_t, p_t, t, \tau) \quad p_{t+\tau} = p(q_t, p_t, t, \tau)$$

If these formulae are regarded as a transformation from the variables  $q_t, p_t$  to  $q_{t+\tau}, p_{t+\tau}$ , then this transformation is canonical. This is evident from the expression

$$dS = p_t dq_t + p_{t+\tau} dq_{t+\tau} - (H_{t+\tau} - H_t)dt$$

for the differential of the action  $S(q_t, q_{t+\tau}, t, \tau)$ , taken along the true path, passing through the points  $q$ , and  $q_{t+\tau}$  at times  $t$  and  $t + \tau$  for a given  $\tau$ .  $-S$  is the generating function of the transformation. So we have the following communication relation

$$[q_{it+\tau}, q_{jt+\tau}]_{q_t, p_t} = 0 \quad [p_{it+\tau}^i, p_{jt+\tau}^j]_{q_t, p_t} = 0 \quad [q_{it+\tau}, p_{jt+\tau}^j]_{q_t, p_t} = \delta_i^j$$

### 1.3.4 Liouville's theorem

#### Lemma 1

Let  $D$  be the Jacobian of the canonical transformation


$$\frac{\partial(Q_1, \dots, Q_s, P^1, \dots, P^s)}{\partial(q_1, \dots, q_s, p^1, \dots, p^s)}$$


Then we have

$$D = 1$$



**Theorem 1.3 Liouville's theorem**

The phase-space distribution function is constant along the trajectories of the system 

 **Proof:** The phase volume is invariant under canonical transformation. The change in  $p$  and  $q$  during the motion can be regarded as a canonical transformation. Suppose that each point in the region of phase space moves in the course of time in accordance with the equations of motion of the mechanical system. The region as a whole therefore moves also, but its volume remains unchanged.  $\square$

## 1.4 Symmetry and Conservation Laws(2)

Suppose  $g$  is a function of  $p$  and  $q$ . If the transformation of  $q$  and  $p$  can be described as

$$q \rightarrow q + \epsilon[q, g]$$

$$p \rightarrow p + \epsilon[p, g]$$

We can prove that

$$H \rightarrow H + \epsilon[H, g]$$

So if  $H$  is invariant under the transformation, then  $[H, g] = 0$ , that means  $\frac{dg}{dt} = 0$ , i.e.  $g$  is a conserved quantity of the motion.

## 1.5 Hamilton-Jacobi equation

We define

$$S(q, t) = \left( \int_{q_0, t_0}^{q, t} L dt \right) |_{\text{extremum}}$$

We can prove that

$$p = \frac{\partial S}{\partial q}, \quad H = -\frac{\partial S}{\partial t}$$

So, we have

$$-\frac{\partial S}{\partial t} = H(q, \frac{\partial S}{\partial q})$$

This is called Hamiltonian-Jacobi equation.

Suppose the complete integral of the Hamilton-Jacobi equation is

$$S = f(t, q_1, \dots, q_s; \alpha^1, \dots, \alpha^s) + A$$

where  $\alpha^1, \dots, \alpha^s$  and  $A$  are arbitrary constants. We effect a canonical transformation from the variables  $q, p$  to new variables, taking the function  $f(t, q, \alpha)$  as the generating function, and the quantities  $\alpha^1, \dots, \alpha^s$  as the new momenta. Let the new co-ordinates be  $\beta_1, \dots, \beta_2$ .

$$p^i = \frac{\partial f}{\partial q_i} \quad \beta_s = \frac{\partial f}{\partial \alpha_s} \quad H' = H + \frac{\partial f}{\partial t} = 0$$



So,

$$\alpha^s = \text{constant}, \beta_s = \text{constant}$$

By means of the  $s$  equations  $\beta_s = \frac{\partial f}{\partial \alpha^s}$ , the  $s$  coordinates  $q$  can be expressed in terms of the time and the  $2s$  constants. This gives the general integral of the equations of motion.

## 1.6 Symmetry and Conservation Laws(3)

If  $S$  is invariant under transformation  $q_i \rightarrow q_i + \delta q_i$ , then

$$\delta S = \left( \sum_i p^i \delta q_i \right) \Big|_{q_0, t_0}^{q, t} = 0$$

So, we have

$$\frac{d}{dt}(p^i \delta q_i) = 0$$

Further more, if

$$\delta S = \left( \sum_i p^i \delta q_i \right) \Big|_{q_0, t_0}^{q, t} = f(q_i, \dot{q}_i, t) \Big|_{q_0, t_0}^{q, t}$$

we will have conserved quantity

$$\frac{d}{dt}(p^i \delta q_i - f) = 0$$





# Chapter 2

## Two body problem



### 2.1 Reduced mass and central field

The Lagrangian for a two-body system is

$$L = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 + U(|\mathbf{r}_1 - \mathbf{r}_2|)$$

Let  $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$  be the relative position vector and let the origin be at the centre of mass, i.e.  $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0$ . These two equations give

$$\mathbf{r}_1 = \frac{m_2}{m_1 + m_2}\mathbf{r} \quad \mathbf{r}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{r}$$

Then, we have

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 - U(r)$$

where

$$m = \frac{m_1m_2}{m_1 + m_2}$$

is called reduced mass. The Lagrangian is formally identical with the Lagrangian of a particle of mass  $m$  moving in an external field  $U(r)$  which is symmetrical about a fixed origin.

$L$  is isotropic, so angular momentum is conserved, i.e.  $\mathbf{M} = \mathbf{r} \times \mathbf{p} = \text{const}$ . Since  $\mathbf{r}$  is always perpendicular to  $\mathbf{M}$ , the path of the particle lies in one plane. Using polar coordinates, we have

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

And it is easy to see that

$$M = mr^2\dot{\phi} = \text{const} \quad E = \frac{1}{2}m\dot{r}^2 + \frac{M^2}{2mr^2} + U(r) = \text{const}$$

So,

$$\frac{dr}{dt} = \sqrt{\frac{2(E - U(r))}{m} - \frac{M^2}{m^2r^2}}$$
$$\frac{d\phi}{dr} = \frac{M}{r^2\sqrt{2m(E - U(r)) - M^2/r^2}}$$

The radial part of the motion can be regarded as taking place in one dimension in a field where the effective potential energy is

$$U_{eff} = U(r) + \frac{M^2}{2mr^2}$$

The values of  $r$  for which

$$U(r) + \frac{M^2}{2mr^2} = E$$

determine the limits of the motion as regards distance from the centre. When equation above is satisfied, the radial velocity  $\dot{r}$  is zero. This does not mean that the particle comes to rest as in true one-dimensional motion, since the angular velocity is not zero. The value  $\dot{r} = 0$  indicates a turning point of the path, where  $r(t)$  begins to decrease instead of increasing, or vice versa. If the range in which  $r$  may vary is limited only by the condition  $r \geq r_{min}$ , the motion is infinite: the particle comes from, and returns to, infinity. If the range of  $r$  has two limits  $r_{min}$  and  $r_{max}$ , the motion is finite and the path lies entirely within the annulus bounded by the circles  $r = r_{min}$  and  $r = r_{max}$ . This does not mean that the path must be a closed curve. During the time in which  $r$  varies from  $r_{min}$  to  $r_{max}$  and back, the radius vector turns through an angle

$$\Delta\phi = 2 \int_{r_{min}}^{r_{max}} \frac{M}{r^2 \sqrt{2m(E - U(r)) - M^2/r^2}} dr$$

The condition for the path to be closed is that this angle should be a rational fraction of  $2\pi$ . There are only two types of central field in which all finite motions take place in closed paths. They are those in which the potential energy of the particle varies as  $\frac{1}{r}$  or as  $r^2$ .

The presence of the centrifugal energy when  $M \neq 0$ , which becomes infinite as  $\frac{1}{r^2}$  when  $r \rightarrow 0$ , generally renders it impossible for the particle to reach the centre of the field, even if the field is an attractive one. A fall of the particle to the centre is possible only if the potential energy tends sufficiently rapidly to  $-\infty$  as  $r \rightarrow 0$ . From the inequality

$$\frac{1}{2}m\dot{r}^2 = E - U(r) - \frac{M^2}{2mr^2} > 0$$

it follows that  $r$  can take values tending to zero only if

$$[r^2 U(r)]_{r \rightarrow 0} < -\frac{M^2}{2m}$$

## 2.2 Kepler Problem

An important class of central fields is formed by those in which the potential energy is inversely proportional to  $r$ . They include the fields of N Newtonian gravitational attraction and of Coulomb electrostatic interaction; the latter may be either attractive or repulsive.

Let us first consider an attractive field, where

$$U = -\frac{\alpha}{r}$$

with  $\alpha$  a positive constant. The effective potential energy

$$U_{eff} = -\frac{\alpha}{r} + \frac{M^2}{2mr^2}$$

As  $r \rightarrow 0$ ,  $U_{eff}$  tends to  $+\infty$ , and as  $r \rightarrow \infty$  it tends to zero from negative values; for  $r = \frac{M^2}{m\alpha}$  it has a minimum value

$$U_{eff,min} = -\frac{m\alpha^2}{2M^2}$$



The motion is finite for  $-\frac{m\alpha^2}{2M^2} \leq E < 0$  and infinite for  $E \geq 0$ .

The shape of path is

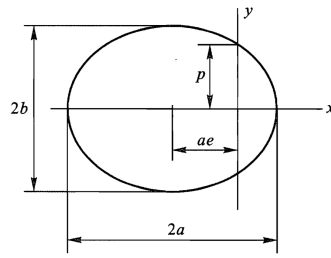
$$\frac{p}{r} = 1 + e \cos \phi$$

Here,

$$p = \frac{M^2}{m\alpha} \quad e = \sqrt{1 + \frac{2EM^2}{m\alpha^2}}$$

This is the equation of a conic section with one focus at the origin;  $2p$  is called the latus rectum of the orbit and  $e$  the eccentricity. Our choice of the origin is such that the point where  $\phi = 0$  is the point nearest to the origin (called the perihelion).

If  $E < 0$ , the orbit is an ellipse and the motion is finite.



**Figure 2.1:** Attractive Kepler orbit with  $e < 1$

The major and minor semi-axes of the ellipse are

$$a = \frac{p}{1 - e^2} = \frac{\alpha}{2|E|} \quad b = \frac{p}{\sqrt{1 - e^2}} = \frac{M}{\sqrt{2m|E|}}$$

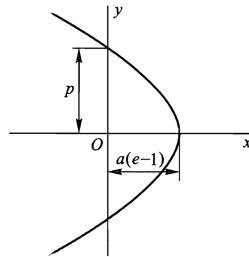
The least and greatest distances from the centre of the field (the focus of the ellipse) are

$$r_{min} = \frac{p}{1 + e} = a(1 - e) \quad r_{max} = \frac{p}{1 - e} = a(1 + e)$$

The period of revolution in an elliptical orbit is

$$T = \frac{\pi ab}{\frac{1}{2}r^2\dot{\phi}} = 2\pi a^{3/2} \sqrt{\frac{m}{\alpha}} = \pi\alpha \sqrt{\frac{m}{2|E|^3}}$$

If  $E > 0$ , the path is a hyperbola with the origin as internal focus.



**Figure 2.2:** Attractive Kepler orbit with  $e > 1$



The distance of the perihelion from the focus is

$$r_{min} = \frac{p}{1+e} = a(1-e)$$

where  $a = \frac{p}{(1-e^2)^2} = \frac{\alpha}{2E}$  is the semiaxis of the hyperbola.

If  $E = 0$ , the eccentricity  $e = 1$ , and the particle moves in a parabola with perihelion distance  $r_{min} = \frac{p}{2}$ . This case occurs if the particle starts from rest at infinity.

Let us now consider motion in a repulsive field, where

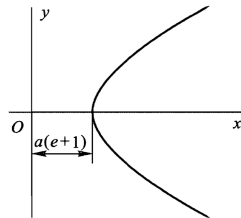
$$U = \frac{\alpha}{r} \quad (\alpha > 0)$$

Here the effective potential energy is

$$U_{eff} = \frac{\alpha}{r} + \frac{M^2}{2mr^2}$$

and decreases monotonically from  $+\infty$  to zero as  $r$  varies from zero to infinity. The energy of the particle must be positive, and the motion is always infinite. The calculations are exactly similar to those for the attractive field. The path is a hyperbola:

$$\frac{p}{r} = -1 + e \cos \phi$$



**Figure 2.3:** Repulsive Kepler orbit

The perihelion distance is

$$r_{min} = \frac{p}{-1+e} = a(1+e)$$

There is an integral of the motion which exists only in fields  $U = \frac{\alpha}{r}$ . It is easy to verify by direct calculation that the quantity

$$\mathbf{v} \times \mathbf{M} + \frac{\alpha \mathbf{r}}{r}$$

is constant. The direction of the conserved vector is along the major axis from the focus to the perihelion, and its magnitude is  $\alpha e$ . This is most simply seen by considering its value at perihelion.



## 2.3 Disintegration and collisions of particles

Let us consider a spontaneous disintegration of a particle into two constituent parts. This process is most simply described in a frame of reference in which the particle is at rest before the disintegration. The law of conservation of momentum shows that the sum of the momenta of the two particles formed in the disintegration is then zero; that is, the particles move apart with equal and opposite momenta. The magnitude  $p_0$  of either momentum is given by the law of conservation of energy:

$$E_i = E_{1i} + \frac{p_0^2}{2m_1} + E_{2i} + \frac{p_0^2}{2m_2}$$

here  $m_1$  and  $m_2$  are the masses of the particles,  $E_{1i}$  and  $E_{2i}$ , their internal energies, and  $E_i$  the internal energy of the original particle. If  $\epsilon$  is the disintegration energy, i.e. the difference

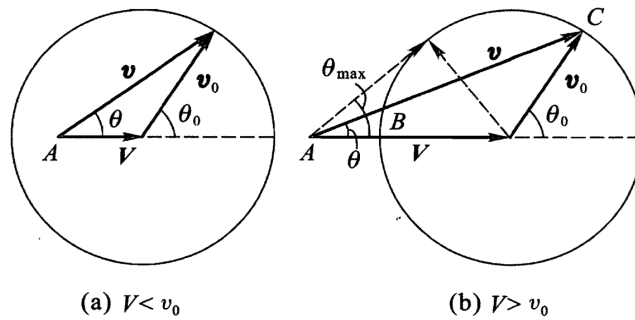
$$\epsilon = E_i - E_{1i} - E_{2i}$$

which must obviously be positive, then

$$\epsilon = \frac{p_0^2}{2m}$$

here  $m$  is the reduced mass of the two particles.

Let us now change to a frame of reference in which the primary particle moves with velocity  $V$  before the break-up. This frame is usually called the laboratory system, or L system, in contradistinction to the centre-of-mass system, or C system, in which the total momentum is zero. Let us consider one of the resulting particles, and let  $v$  and  $v_0$  be its velocities in the L and the C system-respectively. It can be represented by



**Figure 2.4:** Disintegration in L and C frame

The relation between the angles  $\theta$  and  $\theta_0$  in the L and C systems is evidently.

$$\tan \theta = \frac{v_0 \sin \theta_0}{V + v_0 \cos \theta_0}$$

In physical applications we are usually concerned with the disintegration of not one but many similar particles, and this raises the problem of the distribution of the resulting particles in direction, energy, etc. We shall assume that the primary particles are randomly oriented in



space, i.e. isotropically on average.

In the C system, every resulting particle has the same energy, and their directions of motion are isotropically distributed. The fraction of particles entering a solid angle element  $do$  is  $\frac{do}{4\pi}$ . So the distribution with respect to the angle  $\theta_0$  is

$$\frac{1}{2} \sin \theta_0 d\theta_0$$

The corresponding distributions in the L system are obtained by an appropriate transformation. For example, let us calculate the kinetic energy distribution in the L system. Since

$$v^2 = V^2 + v_0^2 + 2Vv_0 \cos \theta_0$$

we have  $d(v^2) = d \cos \theta_0$ . So the kinetic energy can be distributed uniformly over between  $T_{min} = \frac{1}{2}(v_0 - V)^2$  and  $T_{max} = \frac{1}{2}m(v_0 + V)^2$ .

A collision between two particles is said to be elastic if it involves no change in their internal state. The collision is most simply described in the C system. The velocities of the particles before the collision are related to their velocities  $\mathbf{v}_1$  and  $\mathbf{v}_2$  in the L system by  $\mathbf{v}_{10} = m_2 \mathbf{v} / (m_1 + m_2)$ ,  $\mathbf{v}_{20} = -m_1 \mathbf{v} / (m_1 + m_2)$ , where  $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$ .

Because of the law of conservation of momentum, the momenta of the two particles remain equal and opposite after the collision, and are also unchanged in magnitude, by the law of conservation of energy. Thus, in the C system the collision simply rotates the velocities, which remain opposite in direction and unchanged in magnitude. The velocities of the two particles after the collision are

$$\mathbf{v}'_{10} = \frac{m_2 v \mathbf{n}_0}{m_1 + m_2} \quad \mathbf{v}'_{20} = -\frac{m_1 v \mathbf{n}_0}{m_1 + m_2}$$

The velocities in the L system after the collision are therefore

$$\mathbf{v}'_1 = \frac{m_2 v \mathbf{n}_0}{m_1 + m_2} + \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2} \quad \mathbf{v}'_2 = -\frac{m_1 v \mathbf{n}_0}{m_1 + m_2} + \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2}$$

Multiplying equations by  $m_1$  and  $m_2$  respectively, we obtain

$$\mathbf{p}'_1 = m v \mathbf{n}_0 + \frac{m_1(\mathbf{p}_1 + \mathbf{p}_2)}{m_1 + m_2} \quad \mathbf{p}'_2 = -m v \mathbf{n}_0 + \frac{m_2(\mathbf{p}_1 + \mathbf{p}_2)}{m_1 + m_2}$$

It can be represented by

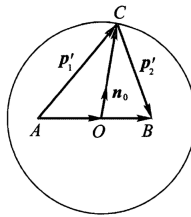
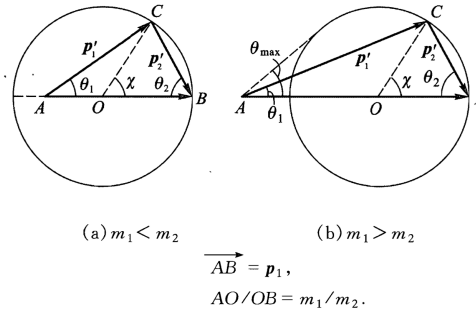


Figure 2.5: Collision in L and C frame

Let us consider in more detail the case where one of the particles ( $m_2$ , say) is at rest before the



**Figure 2.6:** Collision with 2 at rest

collision. In that case the distance  $OB = \frac{m_2 p_1}{m_1 + m_2} = mv$  is equal to the radius. The vector  $\overrightarrow{AB}$  is equal to the momentum  $\mathbf{p}_1$  of the particle  $m_1$  before the collision.  $\theta_1$  and  $\theta_2$  can be expressed in terms of  $\chi$  by

$$\tan \theta_1 = \frac{m_2 \sin \chi}{m_1 + m_2 \cos \chi} \quad \theta_2 = \frac{1}{2}(\pi - \chi)$$

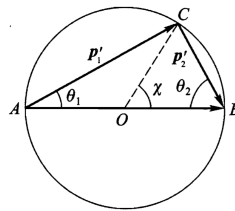
The magnitudes of the velocities of the two particles after the collision in terms of  $\chi$  are

$$v'_1 = \frac{\sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \cos \chi}}{m_1 + m_2} v \quad v'_2 = \frac{2m_1 v}{m_1 + m_2} \sin \frac{1}{2}\chi$$

If  $m_1 < m_2$ , the velocity of  $m_1$  after the collision can have any direction. If  $m_1 > m_2$ , this particle can be deflected only through an angle not exceeding  $\theta_{max}$  from its original direction. Evidently

$$\sin \theta_{max} = \frac{m_2}{m_1}$$

The collision of two particles of equal mass, of which one is initially at rest, is especially simple. In this case both B and A lie on the circle, so

**Figure 2.7:** Collision of equal mass

Then

$$\theta_1 = \frac{1}{2}\chi \quad \theta_2 = \frac{1}{2}(\pi - \chi)$$

$$v'_1 = v \cos \frac{1}{2}\chi \quad v'_2 = v \sin \frac{1}{2}\chi$$

After the collision the particles move at right angles to each other.



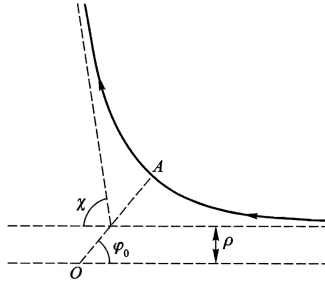
## 2.4 Scattering and cross section

Scattering is a general physical process where some forms of radiation, such as light, sound, or moving particles, are forced to deviate from a straight trajectory by one or more paths due to localized non-uniformities in the medium through which they pass. In classical mechanics, scattering generally refer to particle-particle collisions.

The definition of cross section of a scattering process is

$$\sigma \equiv \frac{\text{Number of Events per target}}{\text{Time} \times \text{Incident Flux}}$$

Here, the incident flux are measured in the frame of target particle. Recall that when we reduce two body problem in to one body problem,  $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$  is the coordinates of projectile in the frame of target. So the scattering process can be represented by the reduced mass moving in the central field.



**Figure 2.8:** Scattering in central field

And we have

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{M}{r^2 \sqrt{2m(E - U(r)) - \frac{M^2}{r^2}}} \quad \chi = |\pi - 2\phi_0|$$

Since

$$E = \frac{1}{2}mv_{\infty}^2 \quad M = mpv_{\infty},$$

we can get the relation between  $\chi$  and  $\rho$ . Suppose the number density of the particles is  $n$ , then the incident flux is  $nv_{\infty}$ , number of events that particles are scattered into the solid angle  $d\Omega = \sin \chi d\chi d\phi$  at  $(\chi, \phi)$  in time  $T$  is

$$d\rho \rho(\chi) d\phi n v_{\infty} T$$

So, we have

$$d\sigma = \rho(\chi) d\rho d\phi = \frac{\rho(\chi)}{\sin \chi} \left| \frac{d\rho}{d\chi} \right| d\chi d\phi$$

In C system, we have  $\mathbf{r}_1 = \frac{m_2}{m_1 + m_2} \mathbf{r}$ , so the scattering angle of particle 1 is the same as  $\chi$ . While in L system (particle 2 is at rest before scattering), we must making corresponding transformation to get the right expression for cross section.





**Rutherford's formula**

One of the most important applications of the formulae derived above is to the scattering of charged particles in a Coulomb field. As  $U = \frac{\alpha}{r}$ , we have

$$\phi_0 = \arccos \frac{\alpha/mv_\infty^2 \rho}{\sqrt{1 + (\alpha/mv_\infty^2 \rho)^2}}$$

Recall that  $\chi = \frac{1}{2}(\pi - \phi_0)$ , we can obtain that

$$\rho^2 = \frac{\alpha^2}{m^2 v_\infty^4} \cot^2 \frac{1}{2} \chi$$

and

$$d\sigma = \left( \frac{\alpha}{2mv_\infty^2} \right)^2 \frac{do}{\sin^4 \frac{1}{2} \chi}$$

This is Rutherford's formula. It may be noted that the effective cross-section is independent of the sign of  $\alpha$ , so that the result is equally valid for repulsive and attractive Coulomb fields.

Formula above gives the effective cross-section in the frame of reference in which the centre of mass of the colliding particles is at rest. The transformation to the laboratory system is effected by means of

$$\tan \theta_1 = \frac{m_2 \sin \chi}{m_1 + m_2 \cos \chi} \quad \theta_2 = \frac{1}{2}(\pi - \chi)$$

For particles initially at rest, we have

$$d\sigma_2 = \left( \frac{\alpha}{mv_\infty^2} \right)^2 \frac{do_2}{\cos^3 \theta_2}$$

The same transformation for the incident particles leads, in general, to a very complex formula, and we shall merely note two particular cases.

If the mass  $m_2$  of the scattering particle is large compared with the mass  $m_1$  of the scattered particle, then  $\chi = \theta_1$  and  $m = m_1$ , so that

$$d\sigma_1 = \left( \frac{\alpha}{4E_1} \right)^2 \frac{do_1}{\sin^2(\frac{1}{2}\theta_1)}$$

here  $E_1 = \frac{1}{2}m_1 v_\infty^2$  is the energy of the incident particle.

If the masses of the two particles are equal, then by  $\theta_1 = \frac{\chi}{2}$ , we have

$$d\sigma_1 = \left( \frac{\alpha}{E_1} \right)^2 \frac{\cos \theta_1}{\sin^4 \theta_1} do_1$$

If the particles are entirely identical, that which was initially at rest cannot be distinguished after the collision. The total effective cross-section for all particles is obtained by adding  $do_1$  and  $do_2$ , so

$$d\sigma = \left( \frac{\alpha}{E_1} \right)^2 \left( \frac{1}{\sin^4 \theta} + \frac{1}{\cos^4 \theta} \right) \cos \theta do$$

Let us return to the general formula and use it to determine the distribution of the scattered particles with respect to the energy lost in the collision. When the masses of the scattered



and scattering particles are arbitrary, the velocity acquired by the latter is given in terms of the angle of scattering in the C system by

$$v'_2 = \frac{2m_1}{m_1 + m_2} v_\infty \sin \frac{\chi}{2}$$

The energy acquired by 2 and lost by 1 is therefore

$$\epsilon = \frac{2m^2}{m_2} v_\infty^2 \sin^2 \frac{\chi}{2}$$

Expressing  $\sin \frac{\chi}{2}$  in terms of  $\epsilon$ , we obtain

$$d\sigma = 2\pi \frac{\alpha^2}{m_2 v_\infty^2} \frac{d\epsilon}{\epsilon^2}$$

This is the required formula: it gives the effective cross-section as a function, of the energy loss  $\epsilon$ , which takes values from zero to  $\epsilon_{max} = 2m^2 v_\infty^2 / m_2$ .



## Chapter 3

### Small Oscillation



#### 3.1 Small oscillation in one dimensional

Let us consider the motion in one dimension, the potential energy of the particle is  $V = V(q)$ . If we choose the coordinate of equilibrium point as  $q = 0$ , then  $\frac{\partial V}{\partial q}|_{q=0} = 0$ . Expand  $V(q)$  around  $q = 0$ , we have

$$V(q) = V(0) + \frac{1}{2} \frac{\partial^2 V}{\partial q^2} \bigg|_{q=0} q^2 + \dots$$

If the equilibrium point is stable, we have

$$\frac{\partial^2 V}{\partial q^2} \bigg|_{q=0} \equiv V''(0) > 0$$

For small oscillation, we can neglect the higher orders of  $q$  and the Lagrangian can be written as

$$L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} V''(0) q^2$$

The Euler-Lagrangian equation is

$$\ddot{q} + \omega_0^2 q = 0 \quad \omega_0^2 = \frac{V''(0)}{m}$$

The general solution is

$$q = A \cos(\omega_0 t + \phi)$$

$A$  and  $\phi$  depends on the initial condition.

If there is a damped force which is proportional to the velocity of the particle, then we have

$$\ddot{q} + \frac{1}{Q} \dot{q} + \omega_0^2 q = 0$$

If  $Q > \frac{1}{2\omega_0}$ , we have

$$q = A e^{-\frac{t}{2Q}} \cos(\omega t + \phi) \quad \omega = \sqrt{\omega_0^2 - \frac{1}{4Q^2}}$$

It is called under damped oscillation.

If  $Q < \frac{1}{2\omega_0}$ , we have

$$q = A e^{\lambda_+ t} + B A e^{\lambda_- t}$$

Here,

$$\lambda_{\pm} = -\frac{1}{2Q} \pm \sqrt{\frac{1}{4Q^2} - \omega_0^2}$$

It is called over damped oscillation.

If  $Q = \frac{1}{2\omega_0}$ , we have

$$q = C(1 + Dt)e^{-\omega_0 t}$$

It is called critical damped oscillation.

## 3.2 Forced oscillation

The equation of motion for forced oscillation is

$$\ddot{q} + \frac{1}{Q}\dot{q} + \omega_0^2 q = F(t)$$

The form of solution is

$$q = q_s + q_g$$

and  $q_g$  is the general solution of the homogeneous equation and  $q_s$  is an arbitrary special solution of the equation. In order to get  $q_s$ , we consider the following equation

$$\ddot{G} + \frac{1}{Q}\dot{G} + \omega_0^2 G = \delta(t - t')$$

and its solution is  $G(t, t')$ . Then we have

$$q_s = \int_{-\infty}^{\infty} F(t')G(t - t')dt'$$

For under damped oscillation, we have

$$G(t, t') = \begin{cases} \frac{e^{-\frac{t-t'}{2Q}}}{\omega} \sin \omega(t - t') & t > t' \\ 0 & t < t' \end{cases}$$

And we can verify that

$$q_s = \int_0^{\infty} F(t - t') \frac{e^{-\frac{t'}{2Q}}}{\omega} \sin \omega t' dt'$$

A special case is that

$$F(t) = F_0 \cos \Omega t$$

and we have

$$q(t) = \frac{F_0}{\sqrt{(\Omega^2 - \omega_0^2 + \frac{1}{2Q^2})^2 + \frac{\omega^2}{Q^2}}} \cos(\Omega t + \phi)$$

Here

$$\tan \phi = \frac{\frac{\Omega}{Q}}{\omega_0^2 - \Omega^2}$$



When

$$\Omega = \sqrt{\omega_0^2 - \frac{1}{2Q^2}}$$

we have

$$q_{max} = \frac{QF_0}{\omega}$$

It is called resonance.

### 3.3 Non-linear oscillation and perturbation theory

Consider the equation of motion

$$\ddot{q} + \omega_0^2 q + \epsilon q^3 = 0$$

Suppose  $\epsilon$  is very small, then we can expand

$$q = q_0 + \epsilon q_1 + \epsilon^2 q_2 + \dots$$

and

$$\omega = \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \dots$$

The equation of motion can be written as

$$\ddot{q} + \omega^2 q = (\omega^2 - \omega_0^2)q - \epsilon q^3$$

Let  $\tau \equiv \omega t$  and  $q' \equiv \frac{dq}{d\tau} = \frac{\dot{q}}{\omega}$ , we have

$$q'' + q = (1 - \frac{\omega_0^2}{\omega^2})q - \frac{\epsilon}{\omega^2} q^3 = 0$$

We can solve the equation above power by power and get

$$q_0'' + q_0 = 0$$

$$q_1'' + q_1 = -\frac{q_0^3}{\omega_0^2} + \frac{2\omega_1}{\omega_0} q_0$$

and so on. When doing the perturbation, we must adjust the  $\omega_i$  to avoid the resonance solution. The details will be neglect here.

Now let us consider the non-linear oscillation with drive force. The equation of motion is

$$\ddot{q} + \frac{1}{Q}\dot{q} + \omega_0^2 q + \epsilon q^3 = F_0 \cos \omega t$$

It can be rewritten as

$$\ddot{q} + \omega^2 q = -\frac{1}{Q}\dot{q} + (\omega^2 - \omega_0^2)q - \epsilon q^3 + F_0 \cos \omega t$$



We treat the right hand of the equation as perturbation, so we multiply it by a parameter  $\mu$  and let it be 1 later,

$$\ddot{q} + \omega^2 q = \mu \left( -\frac{1}{Q} \dot{q} + (\omega^2 - \omega_0^2) q - \epsilon q^3 + F_0 \cos \omega t \right)$$

Concerning on the phase lagging effect, we redefine the "time" as

$$\tau \equiv \omega t - \delta$$

so,

$$q'' + q = \mu \left[ -\frac{1}{Q\omega} q' + \left( 1 - \frac{\omega_0^2}{\omega^2} \right) q - \frac{\epsilon}{\omega^2} q^3 + \frac{F_0}{\omega^2} \cos(\tau + \delta) \right]$$

The expansion series of  $q$  and  $\delta$  are

$$q = q_0 + \mu q_1 + \mu^2 q_2 + \dots$$

$$\delta = \delta_0 + \mu \delta_1 + \mu^2 \delta_2 + \dots$$

We can solve the equation above power by power and get

$$q_0'' + q_0 = 0$$

$$q_1'' + q_1 = -\frac{1}{Q\omega} q_0' + \left( 1 - \frac{\omega_0^2}{\omega^2} \right) q_0 - \frac{\epsilon}{\omega^2} q_0^3 + \frac{F_0}{\omega^2} \cos(\tau + \delta_0)$$

and so on. The solution of the zeroth order perturbation is

$$q_0 = A_0 \cos \tau$$

Substitute it into the first order equation, we have

$$q_1'' + q_1 = \left( \frac{A_0}{Q\omega} - \frac{F_0}{\omega^2} \sin \delta_0 \right) \sin \tau + \left[ \left( 1 - \frac{\omega_0^2}{\omega^2} \right) A_0 - \frac{3\epsilon A_0^3}{4\omega^2} + \frac{F_0}{\omega^2} \cos \delta_0 \right] \cos \tau - \frac{\epsilon A_0^3}{4\omega^2} \cos 3\tau$$

To avoid non-physical solution, we have

$$\sin \delta_0 = \frac{A_0 \omega}{F_0 Q}$$

and

$$\left( 1 - \frac{\omega_0^2}{\omega^2} \right) A_0 - \frac{3\epsilon A_0^3}{4\omega^2} + \frac{F_0}{\omega^2} \cos \delta_0 = 0$$

At last, we can get

$$A_0 = \frac{F_0}{\sqrt{[(\omega^2 - \omega_0^2) - \frac{3}{4}\epsilon A_0^2]^2 + \frac{\omega^2}{Q^2}}}$$

If we define

$$x \equiv \frac{\omega}{\omega_0} \quad y \equiv \frac{A_0 \omega_0^2}{F_0}$$

the equation above can be written as

$$y^2 = \frac{1}{(x^2 - 1 - ay^2)^2 + bx^2}$$



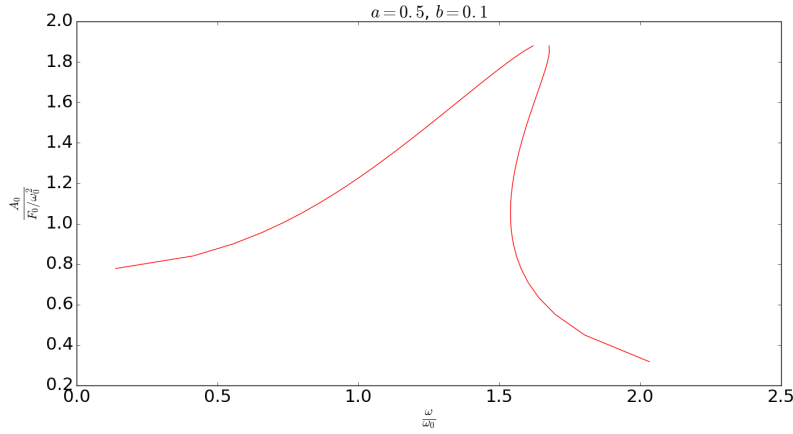
Here

$$a = \frac{3\epsilon F_0^2}{4\omega_0^6} \quad b = \frac{1}{\omega_0^2 Q^2}$$

We than can solve for  $x$  in terms of  $y$ .,

$$x^2 = \frac{2 + 2ay - b \pm \sqrt{(b - 2 - 2ay)^2 - 4(a^2y^2 + 2ay + 1 - \frac{1}{y})}}{2}$$

So the resonance curve has two branches, corresponding to the two roots of the equation above.



**Figure 3.1:** Resonance curve

When the frequency of the drive force increase from left, the amplitude of oscillation will become larger and larger. But when it comes to the point of inflection, the amplitude will drop to the low-right part of the curve. When the frequency of the drive force decrease from right, the amplitude of oscillation will also become larger and larger. When it comes to the point of inflection, the amplitude will jump to the hight-left part of the curve. This effect is called hysteresis.

### 3.4 Oscillations of systems with more than one degree of freedom

Let's look at a system with many degrees of freedom; we have

$$L = \frac{1}{2} \sum_{i,j} T_{ij} \dot{q}_i \dot{q}_j - V(q_1, \dots, q_n)$$

Let  $q_{0,i}$  be an equilibrium position and expand about this point  $q_i = q_{0,i} + \eta_i$ , so  $\dot{q}_i = \dot{\eta}_i$ . We can expand the potential energy to give

$$V(q_1, \dots, q_n) = V(q_{0,1}, \dots, q_{0,n}) + \sum_i \left( \frac{\partial V}{\partial q_i} \right)_{q_{0,i}} \eta_i + \frac{1}{2} \sum_{i,j} \left( \frac{\partial^2 V}{\partial q_i \partial q_j} \right)_{q_{0,i}} \eta_i \eta_j + \dots$$



The first term is a constant with respect to  $\eta_i$  and constant terms do not affect the motion. The second term is zero, because  $q_{0,i}$  is a point of equilibrium so we are left with

$$L = \frac{1}{2} \sum_{i,j} (T_{ij} \dot{\eta}_i \dot{\eta}_j - V_{ij} \eta_i \eta_j)$$

where

$$T_{ij} = T_{ij}(q_{0,1}, \dots, q_{0,n}) \quad V_{ij} = \left( \frac{\partial^2 V}{\partial q_i \partial q_j} \right)_{q_{0,i}}$$

yielding the equations of motion

$$\sum_j (T_{ij} \ddot{\eta}_j - V_{ij} \eta_j) = 0$$

This is a linear differential equation with constant coefficients. We can try the solution

$$\eta_i = C a_i e^{-i\omega t}$$

so we have

$$\sum_j (V_{ij} a_j - \omega^2 T_{ij} a_j) = 0$$

This is a matrix equation such that

$$\vec{A} \cdot \vec{a} = 0$$

with

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_j \end{bmatrix}$$

and

$$\vec{A} = \begin{bmatrix} V_{11} - \omega^2 T_{11} & V_{12} - \omega^2 T_{12} & \cdots \\ V_{21} - \omega^2 T_{21} & V_{22} - \omega^2 T_{22} & \cdots \\ \vdots & & \end{bmatrix}$$

This equation only has a solution is  $\det \vec{A} = 0$ . This gives a  $n$ th-degree polynomial to solve for  $\omega^2$ . We will get  $n$  solutions for  $\omega^2$  that we can substitute into the matrix equation and solve for  $a_j$ .





# Chapter 4

## Motion of a Rigid Body



### 4.1 Angular velocity

Suppose there are two coordinate frames. The frame 2 is rotating relatively to frame 1. If the coordinates of the particles in frame 2 is  $r_2 = (x_2, y_2, z_2)$ . The the coordinates of the particle in frame 1 is

$$r_1 = O(t)r_2$$

and we have

$$O^T O = I$$

So, we can derive that

$$\Omega + \Omega^T = 0 \quad \Omega \equiv O^T \frac{dO}{dt}$$

It is easy to verify that

$$\frac{dr_1}{dt} = \frac{dO}{dt}r_2 + O\frac{dr_2}{dt} = OO^T\frac{dO}{dt}r_2 + Ov_2 = O(\Omega r_2 + v_2)$$

Suppose

$$\Omega = \begin{bmatrix} 0 & -\omega_{2z} & \omega_{2y} \\ \omega_{2z} & 0 & -\omega_{2x} \\ -\omega_{2y} & \omega_{2x} & 0 \end{bmatrix}$$

So, we have

$$v_1 = O(\omega_2 \times r_2 + v_2)$$

Now, we define

$$\omega_1 \equiv O\omega_2$$

We can derive that

$$O\Omega O^T = \begin{bmatrix} 0 & -\omega_{1z} & \omega_{1y} \\ \omega_{1z} & 0 & -\omega_{1x} \\ -\omega_{1y} & \omega_{1x} & 0 \end{bmatrix}$$

So, we have

$$v_1 = \omega_1 \times r_1 + Ov_2$$

$\omega$  is the so-called angular velocity.



**Note:**  $\omega_1$  is independent of the base vector we choose for frame 2. If we choose frame 1 differently,  $\omega_1$  will transform like an vector.

## 4.2 Dynamics of rigid body

### Inertial tensor of rigid body

Suppose there is frame attached to the rigid body, then the coordinate of all the mass point of the rigid body is constant in this frame, i.e.

$$\mathbf{r}_1 = \mathbf{r}_0(t) + \mathbf{O}(t)\mathbf{r}_2 \quad \mathbf{r}_2 \text{ is a constant}$$

So we have

$$\mathbf{v}_1 = \mathbf{V} + \boldsymbol{\omega}_1 \times \mathbf{r}_1 = \mathbf{V} + \mathbf{O}(\boldsymbol{\omega}_2 \times \mathbf{r}_2)$$

The kinetic energy of the rigid body is

$$T = \sum \frac{m}{2} (\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r})^2 = \sum \frac{m}{2} V^2 + \sum m \mathbf{V} \cdot (\boldsymbol{\omega} \times \mathbf{r}) + \sum \frac{m}{2} (\boldsymbol{\omega} \times \mathbf{r})^2$$

If we choose the origin of the frame 2 to be the center of mass of the rigid body, we have

$$T = \frac{\mu V^2}{2} + \frac{1}{2} \sum m [\omega^2 r^2 - (\boldsymbol{\omega} \cdot \mathbf{r})^2]$$

If we define the inertial tensor as

$$I_{ik} = \sum m (x_i^2 \delta_{ik} - x_i x_k)$$

the kinetic energy of the rigid body can be rewritten as

$$T = \frac{\mu V^2}{2} + \frac{1}{2} I_{ik} \omega_i \omega_k$$

and the Lagrangian of the rigid body is

$$L = \frac{\mu V^2}{2} + \frac{1}{2} I_{ik} \omega_i \omega_k - U$$

If the body is regarded as continuous, the sum becomes an integral over the volume of the body:

$$I_{ik} = \int \rho (x_i^2 \delta_{ik} - x_i x_k) dV$$

Like any symmetrical tensor of rank two, the inertia tensor can be reduced to diagonal form by an appropriate choice of the directions of the axes  $x_1, x_2, x_3$ . These directions are called the principal axes of inertia, and the corresponding values of the diagonal components of the tensor are called the principal moments of inertia; we shall denote them by  $I_1, I_2, I_3$ . When the axes  $x_1, x_2, x_3$  are so chosen, the kinetic energy of rotation takes the very simple form

$$T_{rot} = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2)$$

A body whose three principal moments of inertia are all different is called an asymmetrical top.



If two are equal ( $I_1 = I_2 \neq I_3$ ), we have a symmetrical top. In this case the direction of one of the principal axes in the  $x_1x_2$ -plane may be chosen arbitrarily.

If all three principal moments of inertia are equal, the body is called a spherical top, and the three axes of inertia may be chosen arbitrarily as any three mutually perpendicular axes.

The determination of the principal axes of inertia is much simplified if the body is symmetrical, for it is clear that the position of the centre of mass and the directions of the principal axes must have the same symmetry as the body.

For example, if the body has a plane of symmetry, the centre of mass must lie in that plane, which also contains two of the principal axes of inertia, while the third is perpendicular to the plane.

If a body has an axis of symmetry of any order, the centre of mass must lie on that axis, which is also one of the principal axes of inertia, while the other two are perpendicular to it.

If the axis is of order higher than the second, the body is a symmetrical top. For any principal axis perpendicular to the axis of symmetry can be turned through an angle different from  $\pi$  about the latter, i.e. the choice of the perpendicular axes is not unique, and this can happen only if the body is a symmetrical top.

Finally, we may note one further result concerning the calculation of the inertia tensor. Although this tensor has been defined with respect to a system of co-ordinates whose origin is at the centre of mass, it may sometimes be more conveniently found by first calculating a similar tensor,

$$I'_{ik} = \sum m(x'_i{}^2\delta_{ik} - x'_ix'_k)$$

defined with respect to some other origin  $O'$ . If the distance  $OO'$  is represented by a vector  $\mathbf{a}$ , then  $\mathbf{r} = \mathbf{r}' + \mathbf{a}$ ; since, by the definition of  $O$ ,  $\sum m\mathbf{r} = 0$ , we have

$$I'_{ik} = I_{ik} + \mu(a^2\delta_{ik} - a_ia_k)$$

Using this formula, we can easily calculate  $I_{ik}$  if  $I'_{ik}$  is known.

### Angular momentum

The value of the angular momentum of systems depends on the point with respect to which it is defined. In the mechanics of a rigid body, the most appropriate point to choose for this purpose is the origin of the moving system of co-ordinates, i.e. the centre of mass of the body. Then we have

$$\mathbf{M} = \sum m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r} + \mathbf{V}) = \sum m [r^2\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{r})\mathbf{r}]$$

or, in tensor notation,

$$M_i = I_{ik}\omega_k$$

If the axes  $x_1, x_2, x_3$  are the same as the principal axes of inertia, we have

$$M_1 = I_1\omega_1 \quad M_2 = I_2\omega_2 \quad M_3 = I_3\omega_3$$



### Equation of motion

Since a rigid body has, in general, six degrees of freedom, the general equations of motion must be six in number. They can be put in a form which gives the time derivatives of two vectors, the momentum and the angular momentum of the body. The first equation is obtained by simply summing the equations  $\dot{\mathbf{p}} = \mathbf{f}$  for each particle in the body. In terms of the total momentum of the body

$$\mathbf{P} = \sum \mathbf{p} = \mu \mathbf{V}$$

and total force acting on it  $\mathbf{F} = \sum \mathbf{f}$ , we have

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}$$

Although  $\mathbf{F}$  has been defined as the sum of all the forces  $\mathbf{f}$  acting on the various particles, including the forces due to other particles,  $\mathbf{F}$  actually includes only external forces: the forces of interaction between the particles composing the body must cancel out, since if there are no external forces the momentum of the body, like that of any closed system, must be conserved, i.e. we must have  $\mathbf{F} = 0$ .

Let us now derive the second equation of motion, which gives the time derivative of the angular momentum  $\mathbf{M}$ . To simplify the derivation, it is convenient to choose the fixed (inertial) frame of reference in such a way that the centre of mass is at rest in that frame at the instant considered. We have

$$\dot{\mathbf{M}} = \frac{d}{dt} \sum \mathbf{r} \times \mathbf{p} = \sum \dot{\mathbf{r}} \times \mathbf{p} + \sum \mathbf{r} \times \dot{\mathbf{p}}$$

Our choice of the frame of reference (with  $\mathbf{V} = 0$ ) means that the vectors  $\dot{\mathbf{r}}$  and  $\mathbf{p} = m\mathbf{v}$  are parallel, so  $\dot{\mathbf{r}} \times \mathbf{p} = 0$ . We have finally

$$\frac{d\mathbf{M}}{dt} = \mathbf{K}$$

where

$$\mathbf{K} = \sum \mathbf{r} \times \mathbf{f}$$

Since  $\mathbf{M}$  has been defined as the angular momentum about the centre of mass, it is unchanged when we go from one inertial frame to another. We can therefore deduce that the equation of motion, though derived for a particular frame of reference, is valid in any other inertial frame, by Galileo's relativity principle.

The vector  $\mathbf{r} \times \mathbf{f}$  is called the moment of the force  $\mathbf{f}$ , and so  $\mathbf{K}$  is the total torque, i.e. the sum of the moments of all the forces acting on the body. Like the total force,  $\sum \mathbf{r} \times \mathbf{f}$  need include only the external forces: by the law of conservation of angular momentum, the sum of the moments of the internal forces in a closed system must be zero.

### Euler's equation

Let  $\frac{d\mathbf{A}}{dt}$  be the rate of change of any vector  $\mathbf{A}$  with respect to the fixed system of co-ordinates. We have

$$\frac{d\mathbf{A}}{dt} = \frac{d'\mathbf{A}}{dt} + \boldsymbol{\omega} \times \mathbf{A}$$



where  $\frac{d'A}{dt}$  is the rate of change of the  $A$ 's components in the body system of coordinates. So, we have

$$\frac{d'M}{dt} + \boldsymbol{\omega} \times \mathbf{M} = \mathbf{K}$$

Suppose the principal axes of inertia are  $x_1, x_2, x_3$ , so we have

$$\begin{aligned} I_1 \frac{d\omega_1}{dt} + (I_3 - I_2)\omega_2\omega_3 &= K_1 \\ I_2 \frac{d\omega_2}{dt} + (I_1 - I_3)\omega_1\omega_3 &= K_2 \\ I_3 \frac{d\omega_3}{dt} + (I_2 - I_1)\omega_1\omega_2 &= K_3 \end{aligned}$$

This is called Euler's equation.

### 4.3 Eulerian angle

The motion of a rigid body can be described by means of the three co-ordinates of its centre of mass and any three angles which determine the orientation of the axes  $x_1, x_2, x_3$  in the moving system of coordinates relative to the fixed system  $X, Y, Z$ . These angles may often be conveniently taken as what are called Eulerian angles. The moving  $x_1x_2$ -plane intersects the

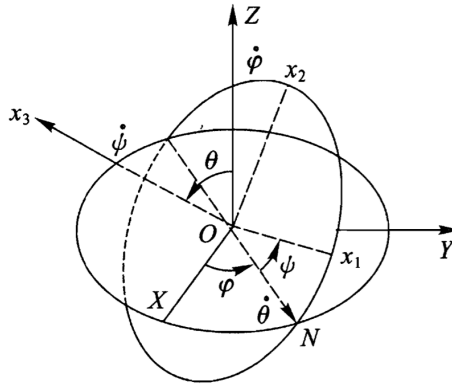


Figure 4.1: Eulerian angle

fixed  $XY$ -plane in some line  $ON$ , called the line of nodes. This line is evidently perpendicular to both the  $Z$ -axis and the  $x_3$ -axis; we take its positive direction as that of the vector product  $\mathbf{z} \times \mathbf{x}_3$ . We take, as the quantities defining the position of the axes  $x_1, x_2, x_3$  relative to the axes  $X, Y, Z$  the angle  $\theta$  between the  $Z$  and  $x_3$  axes, the angle  $\phi$  between the  $X$ -axis and  $ON$ , and the angle  $\psi$  between the  $x_1$  and  $ON$ .

Let us now express the components of the angular velocity vector  $\boldsymbol{\omega}$  along the moving axes  $x_1, x_2, x_3$  in terms of the Eulerian angles and their derivatives. To do this, we must find the components along those axes of the angular velocities  $\dot{\theta}, \dot{\phi}, \dot{\psi}$ . The angular velocity  $\dot{\theta}$  is along the line of nodes  $ON$ . The angular velocity  $\dot{\phi}$  is along the  $Z$ -axis. The angular velocity  $\dot{\psi}$  is



along the  $x_3$ -axis. Collecting the components along each axis, we have

$$\begin{aligned}\omega_1 &= \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi \\ \omega_2 &= \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi \\ \omega_3 &= \dot{\phi} \cos \theta + \dot{\psi}\end{aligned}$$

For a symmetrical top, by using the fact that the choice of directions of the principal axes  $x_1$ ,  $x_2$  is arbitrary for a symmetrical top. If the  $x_1$  axis is taken along the line of nodes  $ON$ , i.e.  $\psi = 0$ , the components of the angular velocity are simply

$$\omega_1 = \dot{\theta} \quad \omega_2 = \dot{\phi} \sin \theta \quad \omega_3 = \dot{\phi} \cos \theta + \dot{\psi}$$

For the free motion of a symmetrical top, we take the  $Z$ -axis of the fixed system of coordinates in the direction of the constant angular momentum  $M$  of the top. The  $x_3$ -axis of the moving system is along the axis of the top; let the  $x_1$ -axis coincide with the line of nodes at the instant considered. Then the components of the vector  $M$  are

$$M_1 = I_1 \dot{\theta} \quad M_2 = I_2 \dot{\phi} \sin \theta \quad M_3 = I_3 (\dot{\phi} \cos \theta + \dot{\psi})$$

Since the  $x_1$ -axis is perpendicular to the  $Z$ -axis, we have

$$M_1 = 0 \quad M_2 = M \sin \theta \quad M_3 = M \cos \theta$$

Comparison gives

$$\dot{\theta} = 0 \quad \dot{\phi} = \frac{M}{I_1} \quad \dot{\phi} \cos \theta + \dot{\psi} = \frac{M \cos \theta}{I_3}$$

The first of these equations gives  $\theta = \text{constant}$ , i.e. the angle between the axis of the top and the direction of  $M$  is constant. The second equation gives the angular velocity of precession  $\dot{\phi} = \frac{M}{I_1}$ . Finally, the third equation gives the angular velocity with which the top rotates about its own axis  $\omega_3 = \frac{M \cos \theta}{I_3}$ .



## **Part II**

# **Classical Field Theory**

# Chapter 5

## Special relativity



### 5.1 The principle of relativity

First, we assume there is an upper limit of velocity of propagation of interaction  $c$ . Second, we assume that inertial reference frame are all the same in describing the law of physics. Then, we can find the invariant intervals when transforming from one inertial reference frame to another,  $ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$ . (In the following, we assume that  $c = 1$ .) This transformation is called Lorentz transformation, which can be written as

$$\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu$$

The invariant symbol of the vector representation of Lorentz transformation is  $\eta^{\mu\nu}$

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu},$$

where,

$$\eta^{\mu\nu} \equiv \begin{bmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{bmatrix}$$

The inverse matrix of  $\eta^{\mu\nu}$  is

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{bmatrix}$$

We can use  $\eta^{\mu\nu}$  and its inverse  $\eta_{\mu\nu}$  to raise and lower vector indices,

$$x_\mu \equiv \eta_{\mu\nu} x^\nu$$

And we can verify the following equations,

$$\Lambda^\mu{}_\rho \Lambda_\nu{}^\rho = \delta_\mu^\nu$$

$$x^\mu = \eta^{\mu\nu} x_\nu$$

$$\bar{x}_\mu = \Lambda_\mu{}^\nu x_\nu$$

$$\Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \eta_{\rho\sigma} = \eta_{\mu\nu},$$



In a special case when the new reference frame move along  $\hat{1}$  direction with velocity  $\beta$ , we have

$$\bar{x}^0 = \gamma x^0 - \gamma \beta x^1$$

$$\bar{x}^1 = -\gamma \beta x^0 + \gamma x^1$$

Some physical quantity will behave like a tensor (vector, scalar) when transforming form one inertial frame to another. For example,

**scalar** proper time:  $d\tau$ , mass:  $m$ , electrical charge  $e$

**vector** four velocity:  $u^\mu = \frac{dx^\mu}{d\tau}$ , four momentum:  $p^\mu = m u^\mu$ , four acceleration:  $a^\mu = \frac{du^\mu}{d\tau}$ , four force:  $f^\mu = m a^\mu$ .

We can also define the corresponding three vector.

**three velocity** :  $\hat{v}^i = \frac{dx^i}{dt}$

$$u^0 = \gamma_v, u^i = \gamma \hat{v}^i$$

When the new reference frame move along  $\hat{1}$  direction with velocity  $\beta$ , we have

$$\bar{\hat{v}}^1 = \frac{\hat{v}^1 - \beta}{1 - \hat{v}^1 \beta}$$

$$\bar{\hat{v}}^2 = \frac{\hat{v}^2}{\gamma(1 - \hat{v}^2 \beta)}$$

$$\bar{\hat{v}}^3 = \frac{\hat{v}^3}{\gamma(1 - \hat{v}^3 \beta)}$$

**three momentum** :  $\hat{p}^i = p^i$

**three acceleration** :  $\hat{a}^i = \frac{dv^i}{dt}$

**three force** :  $\hat{f}^i = \frac{d\hat{p}^i}{dt}$

$$f^i = \gamma_v \hat{f}^i$$

## 5.2 Relativistic Mechanics

For a free particle, we have

$$\frac{dp^\mu}{d\tau} = 0$$

It can be formulated in several ways



### Lagrangian formulation

$$S = -m \int_a^b d\tau, \quad \delta x^\mu(a) = \delta x^\mu(b) = 0$$

$$\delta S = 0 \Rightarrow m \frac{du^\mu}{d\tau} = 0$$

### Hamiltonian formulation

$$S = -m \int_{t_1}^{t_2} \sqrt{1 - \dot{x}_i \dot{x}^i} dt$$

$$L = -m \sqrt{1 - \dot{x}_i \dot{x}^i}$$

$$\pi^i = \frac{\partial L}{\partial \dot{x}_i} = \gamma m \eta^{ij} \dot{x}_j$$

$$H = \pi^i \dot{x}_i - L = \gamma m = \sqrt{m^2 + \pi^i \pi_i}$$

So, the Hamiltonian equations are

$$\dot{\pi}^i = 0, \quad \dot{x}_i = \eta_{ij} \frac{\pi^j}{\sqrt{m^2 + \pi^k \pi_k}}$$

### Hamiltonian-Jacobi equation

$$H = -\frac{\partial S}{\partial t}, \quad \pi^i = \frac{\partial S}{\partial x_i}$$

If we define  $p^0 = H$ ,  $p^i = \pi^i$ , then we can verify that  $p^\mu = \frac{\partial S}{\partial x_\mu}$ . So,  $p^\mu$  is a vector under Lorentz transformation. The Hamiltonian-Jacobi equation can be written as

$$\left(\frac{\partial S}{\partial t}\right)^2 = m^2 + \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2$$

For a non-free particle, we have the revised newton's second law:

$$f^\mu = \frac{dp^\mu}{d\tau}$$

The formula is the definition of the four force. It can also be written in three vector as

$$\hat{f}^i = \gamma_v m \hat{a}^i + \gamma_v^3 (\hat{a}^j \hat{v}_j) m \hat{v}^i$$

If the system consists of more than one particles interacting with each other. We can derive the conservation laws from the symmetry.



### Translational symmetry and conservation of momentum

$$\bar{x}^\mu = x^\mu + \delta x^\mu$$

$$\delta S = \sum m u_\mu \delta x^\mu|_a^b = 0$$

$\sum p^\mu$  is conserved.

### Rotational symmetry and conservation of angular momentum

$$\bar{x}^\mu = x^\mu + x_\nu \delta \Omega^{\mu\nu}$$

$$\delta S = \sum m u^\mu x^\nu \delta \Omega_{\mu\nu}|_a^b = 0$$

$\sum M^{\mu\nu}$  is conserved, where  $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ .

## 5.3 Relativistic Scattering

### 5.3.1 Distribution function

The number of particles in the region  $\mathbf{r} + d\mathbf{r}$  and  $\mathbf{p} + d\mathbf{p}$  is  $f(\mathbf{p}, \mathbf{r}) dp_x dp_y dp_z dx dy dz$ . Then  $f(\mathbf{p}, \mathbf{r})$  is called distribution function.

We first determine the properties of the "volume element"  $dp_x dp_y dp_z$ , with respect to Lorentz transformations. If we introduce a four-dimensional coordinate system, on whose axes are marked the components of the four-momentum of a particle, then  $dp_x dp_y dp_z$ , can be considered as the zeroth component of an element of the hypersurface defined by the equation  $p^\mu p_\mu + m^2 = 0$ . The element of hypersurface is a four-vector directed along the normal to the hypersurface; in our case the direction of the normal obviously coincides with the direction of the four-vector  $p^\mu$ . From this it follows that the ratio

$$\frac{dp_x dp_y dp_z}{E}$$

is an invariant quantity, since it is the ratio of corresponding components of two parallel four-vectors.

Then, we notice that  $dV dt$  is invariant under Lorentz transformation and  $dt = \frac{E}{m} d\tau$ . So we can infer that

$$dx dv dz E$$

is an invariant quantity. Putting all together, we know the phase volume

$$dp_x dp_y dp_z dx dy dz$$

is an invariant volume. So, we have

$$f(\mathbf{r}, \mathbf{p}) = f'(\mathbf{r}', \mathbf{p}')$$

in coordinate transformation.



### 5.3.2 Invariant cross section

Recall the definition of cross section

$$\sigma \equiv \frac{\text{Number of Events per target}}{\text{Time} \times \text{Incident Flux}}$$

Here, the incident flux and time are measured in the frame of target particle.

Suppose that we have two colliding beams; we denote by  $n_1$  and  $n_2$  the particle densities in them and by  $\mathbf{v}_1$  and  $\mathbf{v}_2$  the velocities of the particles. In the reference system in which particle 2 is at rest, we are dealing with the collision of the beam of particles 1 with a stationary target. Then according to the usual definition of the cross-section  $\sigma$ , the number of collisions occurring in volume  $dV$  in time  $dt$  is

$$dN = \sigma v_{rel} n_1 n_2 dV dt$$

,where  $v_{rel}$  is the velocity of particle 1 in the rest system of particle 2 (which is just the definition of the relative velocity of two particles in relativistic mechanics).

The number  $dN$  is by its very nature an invariant quantity. We would like to express it in a form which is applicable in any reference system:

$$dN = A n_1 n_2 dV dt$$

where  $A$  is a number to be determined, for which we know that its value in the rest frame of one of the particles is  $v_{rel}\sigma$ . We shall always mean by  $\sigma$  precisely the cross-section in the rest frame of one of the particles, i.e. by definition, an invariant quantity. From its definition, the relative velocity  $v_{rel}$  is also invariant. The product  $dV dt$  is an invariant. Therefore the product  $A n_1 n_2$  must also be an invariant. The law of transformation of the particle density  $n$  is

$$n = \frac{n_0}{\sqrt{1 - v^2}} = n_0 E / m$$

so we can construct  $A$  in an arbitrary frame as

$$A = -\sigma v_{rel} \frac{p_1^\mu p_{2\mu}}{E_1 E_2}$$

Note that

$$-p_1^\mu p_{2\mu} = \frac{m_1}{\sqrt{1 - v_{rel}^2}} m_2 = m_1 m_2 \frac{1 - \mathbf{v}_1 \cdot \mathbf{v}_2}{\sqrt{(1 - \mathbf{v}_1^2) \cdot (1 - \mathbf{v}_2^2)}}$$

we can get the following expression for  $v_{rel}$ :

$$v_{rel} = \frac{\sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2}}{1 - \mathbf{v}_1 \cdot \mathbf{v}_2}$$

Finally, we have

$$dN = \sigma \sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2} n_1 n_2 dV dt$$

If the velocities  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are collinear, then we have

$$dN = \sigma |\mathbf{v}_1 - \mathbf{v}_2| n_1 n_2 dV dt$$

If we have only one target, then

$$dN = \sigma |\mathbf{v}_1 - \mathbf{v}_2| n_1 dt$$



### 5.3.3 Elastic scattering between two particles

$$(E_1, \mathbf{p}_1, E_2, \mathbf{p}_2) \rightarrow (E'_1, \mathbf{p}'_1, E'_2, \mathbf{p}'_2)$$

In lab frame (L frame),

$$E_2 = m_2 \quad \mathbf{p}_2 = 0$$

By the conservation of momentum, we can derive that

$$\cos \theta_1 = \frac{E'_1(E_1 + m_2) - E_1 m_2 - m_1^2}{p_1 p'_1}$$

$$\cos \theta_2 = \frac{(E_1 + m_2)(E'_2 - m_2)}{p_1 p'_2}$$

Here  $\theta_1(\theta_2)$  is the angle between  $\mathbf{p}'_1(\mathbf{p}'_2)$  with  $\mathbf{p}_1$ . In a special case where  $m_1 = 0$ , we have

$$E'_1 = \frac{m_2}{1 - \cos \theta_1 + \frac{m_2}{E_1}}$$

Suppose in the center of mass frame (C frame), the scattering angle is  $\chi$ , then we can derive that

$$E'_1 = E_1 - \Delta E \quad E'_2 = m_2 + \Delta E$$

Here,

$$\Delta E = \frac{m_2(E_1^2 - m_1^2)}{m_1^2 + m_2^2 + 2m_2 E_1} (1 - \cos \chi)$$

Next, suppose in L frame, let  $x = p'_1 \cos \theta_1, y = p'_1 \sin \theta_1$ , we can get that

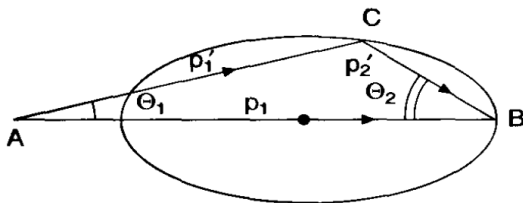
$$\frac{(x - c)^2}{a^2} + \frac{y^2}{b^2} = 1$$

Here,

$$a = \frac{p_1(E_1 m_2 + m_2^2)}{m_1^2 + m_2^2 + 2m_2 E_1} = \frac{m_2 V}{\sqrt{1 - V^2}} \quad b = \frac{m_2 p_1}{\sqrt{m_1^2 + m_2^2 + 2m_2 E_1}} = \frac{a}{\sqrt{1 - V^2}} \quad c = \frac{p_1(E_1 m_2 + m_1^2)}{m_1^2 + m_2^2 + 2m_2 E_1}$$

Here  $V = \frac{p_1}{E_1 + m_2}$  is the velocity of particle 2 before scattering in C frame. And it is easy to see that  $a + c = p_1$ . The result above can be represented by a picture.

(a)  $m_1 > m_2$



(b)  $m_1 < m_2$

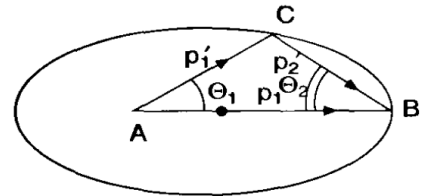


Figure 5.1: Relativistic scattering



# Chapter 6

## Classical field theory



### 6.1 Lagrangian formulation

$$S = \int \mathcal{L}(\phi_a, \dot{\phi}_a, \nabla \phi_a) d^4x, \quad \delta\phi_a|_{\Sigma} = 0$$

$$\delta S = 0 \Rightarrow \partial_{\mu} \left( \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} \phi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0$$

#### Locality of the theory

There are no terms in the Lagrangian coupling  $\phi(\mathbf{x}, t)$  directly to  $\phi(\mathbf{y}, t)$  with  $\mathbf{x} \neq \mathbf{y}$ . The closest we get for the  $\mathbf{x}$  label is coupling between  $\phi(\mathbf{x}, t)$  and  $\phi(\mathbf{x} + \delta\mathbf{x}, t)$  through the gradient term  $\nabla \phi$ .

#### Lorentz invariance

Scalar fields:

$$\bar{\phi}(x) = \phi(\Lambda^{-1}x)$$

Vector fields:

$$\bar{A}^{\mu}(x) = \Lambda^{\mu}_{\nu} A^{\nu}(\Lambda^{-1}x)$$

$$\bar{A}_{\mu}(x) = (\Lambda^{-1})^{\nu}_{\mu} A_{\nu}(\Lambda^{-1}x) = \Lambda_{\mu}^{\nu} A_{\nu}(\Lambda^{-1}x)$$

$$\overline{\partial_{\mu} \phi}(x) = (\Lambda^{-1})^{\nu}_{\mu} \partial_{\nu} \phi(\Lambda^{-1}x) = \Lambda_{\mu}^{\nu} \partial_{\nu} \phi(\Lambda^{-1}x)$$

Lagrangian is a scalar, or more loosely, action is invariant under Lorentz transformation.

## 6.2 Symmetry and conservation law

### Theorem 6.1 Noether's theorem

Every continuous symmetry of the Lagrangian gives rise to a conserved current  $j^\mu(x)$  such that the equation of motion imply  $\partial_\mu j^\mu = 0$ . Suppose that the infinitesimal transformation is

$$\phi_a \rightarrow \phi_a + \delta\phi_a$$

$$\mathcal{L} \rightarrow \mathcal{L} + \delta\mathcal{L}$$

and if  $\delta\mathcal{L} = \partial_\mu K^\mu$ , we can get

$$j^\mu = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a + K^\mu$$

### space-time translation

$$\bar{x} = x - a$$

$$j^\mu = a_\nu T^{\mu\nu}$$

$$T^{\mu\nu} \equiv -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^\nu\phi_a + \eta^{\mu\nu}\mathcal{L}$$

If we define  $P^\mu \equiv \int T^{0\mu} d^3x$ , then we have the law of momentum conservation:

$$\frac{dP^\mu}{dt} = 0$$

### Lorentz Transformation

$$\bar{x}^\mu = x^\mu + \delta\omega^\mu{}_\nu x^\nu$$

The infinitesimal Lorentz transformation can be written as  $I + \delta\omega^\mu{}_\nu$

$$\delta\omega^\mu{}_\nu = \begin{bmatrix} 0 & \beta_1 & \beta_2 & \beta_3 \\ \beta_1 & 0 & -\theta_3 & \theta_2 \\ \beta_2 & \theta_3 & 0 & -\theta_1 \\ \beta_3 & -\theta_2 & \theta_1 & 0 \end{bmatrix}$$

This time, we assume that

$$\bar{\phi}_a(x) = \mathcal{S}_a{}^b \phi_b(\Lambda^{-1}x)$$

In the limit of infinitesimal Lorentz transformation, we have

$$\mathcal{S}_a{}^b = \delta_a{}^b + \frac{1}{2}\delta\omega_{\alpha\beta}(\Sigma^{\alpha\beta})_a{}^b$$

$$j^\mu = \frac{1}{2}M^{\mu\nu\rho}\delta\omega_{\nu\rho}$$



$$M^{\mu\nu\rho} \equiv x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} (\Sigma^{\nu\rho})_a{}^b \phi_b$$

If we define  $M^{\nu\rho} \equiv \int M^{0\nu\rho} d^3x$ , then we have the law of angular momentum conservation:

$$\frac{dM^{\nu\rho}}{dt} = 0$$

## 6.3 Functional derivatives

### Definition 6.1 Functional derivatives

Given a manifold  $M$  representing (continuous/smooth) functions  $\rho$  (with certain boundary conditions etc.), and a functional  $F$  defined as

$$F: M \rightarrow \mathbb{R} \quad \text{or} \quad F: M \rightarrow \mathbb{C},$$

the functional derivative of  $F[\rho]$ , denoted  $\frac{\delta F}{\delta \rho}$ , is defined by

$$\begin{aligned} \int \frac{\delta F}{\delta \rho}(x) \phi(x) dx &= \lim_{\epsilon \rightarrow 0} \frac{F[\rho + \epsilon \phi] - F[\rho]}{\epsilon} \\ &= \left[ \frac{d}{d\epsilon} F[\rho + \epsilon \phi] \right]_{\epsilon=0}, \end{aligned}$$

where  $\phi$  is an arbitrary function. The quantity  $\epsilon \phi$  is called the variation of  $\rho$ .

Like the derivative of a function, the functional derivative satisfies the following properties, where  $F[\rho]$  and  $G[\rho]$  are functionals:

Linearity:

$$\frac{\delta(\lambda F + \mu G)[\rho]}{\delta \rho(x)} = \lambda \frac{\delta F[\rho]}{\delta \rho(x)} + \mu \frac{\delta G[\rho]}{\delta \rho(x)},$$

where  $\lambda, \mu$  are constants.

Product rule:

$$\frac{\delta(FG)[\rho]}{\delta \rho(x)} = \frac{\delta F[\rho]}{\delta \rho(x)} G[\rho] + F[\rho] \frac{\delta G[\rho]}{\delta \rho(x)},$$

Chain rules:

If  $F$  is a functional and  $G$  an operator, then

$$\frac{\delta F[G[\rho]]}{\delta \rho}(y) = \int dx \frac{\delta F[G[\rho]]}{\delta G[\rho]}(x) \cdot \frac{\delta G[\rho]}{\delta \rho}(x, y)$$

If  $G$  is an ordinary differentiable function  $g$ , then this reduces to

$$\frac{\delta F[g(\rho)]}{\delta \rho}(y) = \frac{\delta F[g(\rho)]}{\delta g[\rho]}(y) \frac{dg(\rho)}{d\rho}(y)$$





**Proposition 6.1 Properties of functional derivatives**

$$\frac{\delta F}{\delta \rho}(y) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \{F[\rho(x) + \epsilon \delta(x - y)] - F[\rho(x)]\}$$

$$\frac{\delta f(x)}{\delta f(y)} = \delta(x - y)$$

$$\frac{\delta}{\delta f(y)} \int g(f(x)) dx = g'(f(y))$$

$$\frac{\delta f'(x)}{\delta f(y)} = \frac{d}{dx} \delta(x - y)$$

$$\frac{\delta}{\delta f(y)} \int g(f'(x)) dx = -\frac{d}{dy} g'(f'(y))$$



## 6.4 Hamiltonian formulation

$$\pi^a(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}$$

$$\mathcal{H}(\phi_a, \nabla \phi_a, \pi^a) = \pi^a \dot{\phi}_a - \mathcal{L}$$

$$H = \int \mathcal{H} d^3x$$

Now, we can get the Hamilton equation form  $\delta S = 0$ ,

$$\dot{\phi}_a(\mathbf{x}, t) = \frac{\delta}{\delta \pi^a(\mathbf{x}, t)} H = \frac{\partial \mathcal{H}}{\partial \pi^a}$$

$$\dot{\pi}^a(\mathbf{x}, t) = -\frac{\delta}{\delta \phi_a(\mathbf{x}, t)} H = -\frac{\partial \mathcal{H}}{\partial \phi_a} + \left( \frac{\partial \mathcal{H}}{\partial \phi_{a,b}} \right)_{,b}$$

### 6.4.1 Poisson bracket

First, we define that

$$[\phi_a(\mathbf{x}), \phi_b(\mathbf{y})] \equiv [\pi^a(\mathbf{x}), \pi^b(\mathbf{y})] \equiv 0$$

$$[\phi_a(\mathbf{x}), \pi^b(\mathbf{y})] \equiv \delta_a^b \delta(\mathbf{x} - \mathbf{y})$$

then, we demand that the bracket operation has the same properties as the Poisson bracket in classical mechanics. And we also demand that

$$[\partial_x A(\mathbf{x}), B(\mathbf{y})] = \partial_x [A(\mathbf{x}), B(\mathbf{y})]$$

and

$$\left[ \int d^3x A(\mathbf{x}), B(\mathbf{y}) \right] = \int d^3x [A(\mathbf{x}), B(\mathbf{y})]$$



As a result, we can verify that

$$[W[\phi(\mathbf{x}), \pi(\mathbf{x})], Z[\phi(\mathbf{x}), \pi(\mathbf{x})]] = \int d^3x \left\{ \frac{\delta W}{\delta \phi(\mathbf{x})} \frac{\delta Z}{\delta \pi(\mathbf{x})} - \frac{\delta W}{\delta \pi(\mathbf{x})} \frac{\delta Z}{\delta \phi(\mathbf{x})} \right\}$$

Especially,

$$[\phi_a(\mathbf{x}), H] = \frac{\delta}{\delta \pi^a(\mathbf{x})} H, \quad [\pi^a(\mathbf{x}), H] = -\frac{\delta}{\delta \phi_a(\mathbf{x})} H$$

So, the Hamilton equation can be written as

$$\dot{\phi}_a = [\phi_a, H], \quad \dot{\pi}^a = [\pi^a, H]$$

Further more, we can prove that

$$\frac{dO(\phi, \pi, t)}{dt} = [O, H] + \frac{\partial O}{\partial t}$$

and

$$\frac{d[A, B]}{dt} = [A, \frac{dB}{dt}] + [\frac{dA}{dt}, B]$$

## 6.4.2 Momentum

It is easy to verify that

$$P^0 = H, \quad P^i = \int -\pi^a \partial^i \phi_a d^3x$$

And we can get the Poisson bracket

$$\begin{aligned} [\phi_a, P^\mu] &= -\partial^\mu \phi_a \\ [\pi^a, P^\mu] &= -\partial^\mu \pi^a \\ [P^\mu, P^\nu] &= 0 \end{aligned}$$

## 6.4.3 Angular momentum

It is easy to verify that

$$M^{\mu\nu} = \int (x^\mu T^{0\nu} - x^\nu T^{0\mu} - \pi^a (\Sigma^{\mu\nu})_a^b \phi_b) d^3x$$

We define that

$$\begin{aligned} M_L^{\mu\nu} &\equiv \int (x^\mu T^{0\nu} - x^\nu T^{0\mu}) d^3x & M_S^{\mu\nu} &\equiv \int (-\pi^a (\Sigma^{\mu\nu})_a^b \phi_b) d^3x \\ (L^{\mu\nu})_a^b &\equiv -(x^\mu \partial^\nu - x^\nu \partial^\mu) \delta_a^b & (S^{\mu\nu})_a^b &\equiv -(\Sigma^{\mu\nu})_a^b \end{aligned}$$

So, we can get the Poisson bracket

$$\begin{aligned} [\phi_a, M_L^{\mu\nu}] &= (L^{\mu\nu})_a^b \phi_b & [\phi_a, M_S^{\mu\nu}] &= (S^{\mu\nu})_a^b \phi_b \\ [\pi^a, M_L^{\mu\nu}] &= (L^{\mu\nu})_b^a \pi^b & [\pi^a, M_S^{\mu\nu}] &= -(S^{\mu\nu})_b^a \pi^b \end{aligned}$$



Because  $\frac{dM^{\mu\nu}}{dt} = 0$ ,  $M^{\mu\nu}$  can commute with  $\frac{d}{dt}$ , so

$$[[\phi(x), M^{\mu\nu}], M^{\rho\sigma}] = (L^{\mu\nu} + S^{\mu\nu})(L^{\rho\sigma} + S^{\rho\sigma})\phi(x)$$

At last, we can get the Poisson bracket

$$[\phi(x), [M^{\mu\nu}, M^{\rho\sigma}]] = (L^{\mu\nu}L^{\rho\sigma} - L^{\rho\sigma}L^{\mu\nu} + S^{\mu\nu}S^{\rho\sigma} - S^{\rho\sigma}S^{\mu\nu})\phi(x)$$

Since we can prove that

$$L^{\mu\nu}L^{\rho\sigma} - L^{\rho\sigma}L^{\mu\nu} = -\eta^{\nu\rho}L^{\mu\sigma} + \eta^{\sigma\mu}L^{\rho\nu} + \eta^{\mu\rho}L^{\nu\sigma} - \eta^{\sigma\nu}L^{\rho\mu}$$

if we demand that

$$S^{\mu\nu}S^{\rho\sigma} - S^{\rho\sigma}S^{\mu\nu} = -\eta^{\nu\rho}S^{\mu\sigma} + \eta^{\sigma\mu}S^{\rho\nu} + \eta^{\mu\rho}S^{\nu\sigma} - \eta^{\sigma\nu}S^{\rho\mu}$$

We will get the Poisson bracket of the  $M^{\mu\nu}$ ,

$$[M^{\mu\nu}, M^{\rho\sigma}] = -\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\sigma\mu}M^{\rho\nu} + \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\sigma\nu}M^{\rho\mu}$$

up to the possibility of a term on the right-hand side that commutes with  $\phi(x)$  and its derivatives.

We now define  $J_i \equiv \frac{1}{2}\epsilon_{ijk}M^{jk}$  and  $K_i \equiv M^{i0}$ , so we have

$$M^{\mu\nu} = \begin{bmatrix} 0 & -K_1 & -K_2 & -K_3 \\ K_1 & 0 & J_3 & -J_2 \\ K_2 & -J_3 & 0 & J_1 \\ K_3 & J_2 & -J_1 & 0 \end{bmatrix} \quad \left( \delta\omega_{\mu\nu} = \begin{bmatrix} 0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & 0 & -\theta_3 & \theta_2 \\ \beta_2 & \theta_3 & 0 & -\theta_1 \\ \beta_3 & -\theta_2 & \theta_1 & 0 \end{bmatrix} \right)$$

the Poisson bracket can be written as

$$\begin{aligned} [J_i, J_j] &= \epsilon_{ijk}J_k \\ [J_i, K_j] &= \epsilon_{ijk}K_k \\ [K_i, K_j] &= -\epsilon_{ijk}J_k \end{aligned}$$

We can use the similar method to derive that

$$[P^\mu, M^{\rho\sigma}] = \eta^{\mu\sigma}P^\rho - \eta^{\mu\rho}P^\sigma$$

It can also be written as

$$\begin{aligned} [J_i, H] &= 0 \\ [J_i, P_j] &= \epsilon_{ijk}P_k \\ [K_i, H] &= P_i \\ [K_i, P_j] &= \delta_{ij}H \end{aligned}$$

At last, we define  $L_i \equiv \frac{1}{2}\epsilon_{ijk}M_L^{jk}$  and  $S_i \equiv \frac{1}{2}\epsilon_{ijk}M_S^{jk}$  we can demonstrate that

$$\begin{aligned} [L_i, S_j] &= 0 \\ [S_i, P_j] &= 0 \\ [L_i, P_j] &= \epsilon_{ijk}P_k \end{aligned}$$



# Chapter 7

## Classical Electrodynamics



### 7.1 The formulation of classical electrodynamics

#### 7.1.1 Maxwell equations and Lorentz force

The Lagrangian of the EM field  $A^\mu$  when coupling with current is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + j^\mu A_\mu$$

Here

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad j^\mu = \rho_{e0}u^\mu$$

$\rho_{e0}$  is the charge density measured in the frame of charge. Then we can derive the Euler-Lagrange equation of EM field:

$$\partial_\nu F^{\mu\nu} = j^\mu$$

And we can derive the charge conservation equation

$$\partial_\mu j^\mu = 0$$

For a charged particle moving in EM field with orbit  $x^\mu(\tau)$ , we have

$$j^\mu = e_a \delta(\mathbf{r} - \mathbf{r}_a(t)) \sqrt{1 - \mathbf{v}_a^2} u_a^\mu$$

Then we can derive that

$$\int dV dt j^\mu A_\mu = \int dx^\mu e A_\mu$$

So, the action for a charged particle when coupling with EM field is

$$S = -m \int d\tau + e \int dx^\mu A_\mu$$

We can derive the Euler-Lagrangian equation of the particle:

$$ma_\mu = eF_{\mu\nu}u^\nu$$

The Hamiltonian formulation of electrodynamics will be discussed in detail in the Hamiltonian formulation of general relativity and canonical quantization formulation of quantum electrodynamics.

We now define the electric field and magnetic field as follows,

$$E^i \equiv F^{0i} = -\dot{A}^i - \nabla^i A^0 \quad B^i \equiv \epsilon_{ijk} \nabla^j A^k$$

We also define  $\rho_e \equiv j^0$  and  $J^i \equiv j^i$ , so the field equation can then be written

$$\begin{aligned}\nabla \times \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{E} &= \rho_e \\ \nabla \cdot \mathbf{B} &= 0\end{aligned}$$

That is the so-called Maxwell equation.

The equation of motion of the charged particle can be written as

$$\frac{d\mathbf{p}}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad \frac{d\mathcal{E}}{dt} = e\mathbf{E} \cdot \mathbf{v}$$

That is the so-called Lorentz force.

We note that the Field  $A$  cannot be completely determined by Maxwell and Lorentz equation. If we make the transformation  $A_\mu \rightarrow A_\mu + \partial_\mu \xi(x)$ ,  $\mathcal{L}$  and  $F^{\mu\nu}$  would be invariant, and the Maxwell and Lorentz equation is still valid. This arbitrariness of  $\xi$  is called gauge invariance. The topic will be discussed in detail in QED.

### 7.1.2 Lorentz transformation of fields

In the new coordinate system  $\bar{x}^\mu = \Lambda^\mu_\nu x^\nu$ , we have

$$\bar{A}^\mu(\bar{x}) = \Lambda^\mu_\nu A^\nu(\Lambda^{-1}\bar{x})$$

In a special case when the new reference frame move along  $\hat{1}$  direction with velocity  $\beta$ , we have

$$\bar{x}^0 = \gamma x^0 - \gamma\beta x^1 \quad \bar{x}^1 = -\gamma\beta x^0 + \gamma x^1$$

So, we have

$$\bar{A}^0 = \gamma A^0 - \gamma\beta A^1 \quad \bar{A}^1 = -\gamma\beta A^0 + \gamma A^1$$

We can further derive that

$$\begin{aligned}\bar{E}_1 &= E_1 & \bar{E}_2 &= \gamma E_2 - \gamma\beta B_3 & \bar{E}_3 &= \gamma E_3 + \gamma\beta B_2 \\ \bar{B}_1 &= B_1 & \bar{B}_2 &= \gamma B_2 + \gamma\beta E_3 & \bar{B}_3 &= \gamma B_3 - \gamma\beta E_2\end{aligned}$$

It can be written in the three vector form as

$$\begin{aligned}\bar{\mathbf{E}} &= \gamma(\mathbf{E}_\perp + \boldsymbol{\beta} \times \mathbf{B}) + \mathbf{E}_\parallel \\ \bar{\mathbf{B}} &= \gamma(\mathbf{B}_\perp - \boldsymbol{\beta} \times \mathbf{E}) + \mathbf{B}_\parallel\end{aligned}$$

If  $\beta \ll 1$ , neglect the higher order of  $\beta^2$ , we have

$$\begin{aligned}\bar{\mathbf{E}} &= \mathbf{E} + \boldsymbol{\beta} \times \mathbf{B} \\ \bar{\mathbf{B}} &= \mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}\end{aligned}$$

We also note that  $F_{\mu\nu}F^{\mu\nu}$  and  $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$  is invariant under Lorentz transformation. It can be represented by the electric and magnetic field as

$$E^2 - B^2 = \text{inv} \quad \mathbf{E} \cdot \mathbf{B} = \text{inv}$$



### 7.1.3 Energy-momentum tensor

For a free EM field, the energy-momentum tensor is

$$T_f^{\mu\nu} \equiv -\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \partial^\nu A_\rho + \eta^{\mu\nu} \mathcal{L} = \partial^\nu A^\rho F^\mu{}_\rho - \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$$

We note that the energy-momentum tensor defined above is not symmetric, so we define a modified energy-momentum tensor by adding a term  $-\partial^\rho A^\nu F^\mu{}_\rho$ , so

$$T_e^{\mu\nu} = F^{\nu\rho} F^\mu{}_\rho - \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$$

Note that for free EM field,  $\partial^\rho A^\nu F^\mu{}_\rho = \partial^\rho (A^\nu F^\mu{}_\rho)$ . So, we can get that

$$\partial_\mu T_f^{\mu\nu} = 0 \quad P_f'^\mu = P_f^\mu$$

So, from now on, we will use  $T_f'$  as the energy-momentum tensor of EM field and omit the prime of  $T_f'$  for simplicity. The momentum of the free EM field is

$$P_f^0 = \int dV \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) \equiv \int dV w \quad P_f^i = \int dV \mathbf{E} \times \mathbf{B} \equiv \int dV \mathbf{S}$$

If there also exists charged particles in the system, i.e. the source of EM field, we must also include the energy-momentum tensor of the particles to get the right conservation equation. The energy-momentum tensor of particles is defined as

$$T_p^{\mu\nu} \equiv \sum_a m_a \delta(\mathbf{r} - \mathbf{r}_a) \sqrt{1 - \mathbf{v}_a^2} u_a^\mu u_a^\nu$$

In this definition, we have

$$P_p^0 = \sum_a \frac{m_a}{\sqrt{1 - \mathbf{v}_a^2}} \quad \mathbf{P}_p = \sum_a \frac{m_a}{\sqrt{1 - \mathbf{v}_a^2}} \mathbf{v}_a$$

So, our definition is consistent with the definition of four momentum in previous chapter. Recall that

$$j_e^\mu = \sum_a e_a \delta(\mathbf{r} - \mathbf{r}_a) \sqrt{1 - \mathbf{v}_a^2} u_a^\mu$$

We can define the mass current as

$$j_m^\mu \equiv \sum_a m_a \delta(\mathbf{r} - \mathbf{r}_a) \sqrt{1 - \mathbf{v}_a^2} u_a^\mu$$

If there is no particle creation and annihilation, and the mass of the particle is constant during the motion, we would have

$$\partial_\mu j_{ma}^\mu = 0$$

Recall the Lorentz force equation  $ma_\mu = eF_{\mu\nu} u^\nu$ , it can be rewritten as

$$\rho_{m0a} \frac{du_\mu}{d\tau} = F_{\mu\nu} j_{ea}^\nu$$



Here,  $\rho_{m0a} \equiv m_a \delta(\mathbf{r} - \mathbf{r}_a) \sqrt{1 - \mathbf{v}_a^2}$  is the mass density measured in the rest frame of the particle. Then, on the one hand, we have

$$\partial_\mu T_p^{\mu\nu} = \sum_a \rho_{m0a} u^\mu \frac{\partial u_a^\nu}{\partial x^\mu} = \sum_a \rho_{m0a} \frac{du_a^\nu}{d\tau} = F^\nu{}_\mu j_e^\mu$$

On the other hand, we can derive that

$$\partial_\mu T_f^{\mu\nu} = -F^\nu{}_\mu j_e^\mu$$

by implementing Maxwell equation. Define

$$T^{\mu\nu} \equiv T_f^{\mu\nu} + T_p^{\mu\nu}$$

We have

$$\partial_\mu T^{\mu\nu} = 0$$

We define the Maxwell stress tensor as

$$f^{ij} \equiv -T_f^{ij} = E^i E^j + B^i B^j - w \delta^{ij}$$

Then, we can write down the conservation law of energy and momentum as

$$\begin{aligned} \frac{d}{dt} \left( P_p^0 + \int w dV \right) &= - \oint \mathbf{S} \cdot d\boldsymbol{\sigma} \\ \frac{d}{dt} \left( \mathbf{P}_p + \int \mathbf{S} dV \right) &= \oint \mathbf{f} \cdot d\boldsymbol{\sigma} \end{aligned}$$

We assume there is no particle cross the boundary.

At last, because  $T^{\mu\nu} = T^{\nu\mu}$ , we have

$$\partial_\mu (x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}) = 0$$

It is easy to write down the conservation law of angular momentum as

$$\frac{d}{dt} \left( \mathbf{L}_p + \int \mathbf{r} \times \mathbf{S} dV \right) = \oint \mathbf{r} \times \mathbf{f} \cdot d\boldsymbol{\sigma}$$

Here,

$$\mathbf{L}_p = \sum_a \frac{m_a}{\sqrt{1 - \mathbf{v}_a^2}} \mathbf{r}_a \times \mathbf{v}_a$$

### 7.1.4 Charged particles in a given EM field

Now we suppose the EM field is given, i.e. we will neglect the EM field generated by the test charged particles. Then we have

$$S = \int_{t_1}^{t_2} (-\sqrt{1 - v^2} m + e \mathbf{A} \cdot \mathbf{v} - e\phi) dt$$

So,

$$L = -\sqrt{1 - v^2} m + e \mathbf{A} \cdot \mathbf{v} - e\phi$$



The canonical momentum is

$$\boldsymbol{\pi} = \frac{\partial L}{\partial \mathbf{v}} = \gamma m \mathbf{v} + e \mathbf{A}$$

The Hamiltonian is therefore

$$H = \boldsymbol{\pi} \cdot \mathbf{v} - L = \gamma m + e\phi = \sqrt{m^2 + (\boldsymbol{\pi} - e\mathbf{A})^2} + e\phi$$

If  $v \ll 1$ , we would have

$$L = \frac{mv^2}{2} + e\mathbf{A} \cdot \mathbf{v} - e\phi \quad \boldsymbol{\pi} = m\mathbf{v} + e\mathbf{A} \quad H = \frac{(\boldsymbol{\pi} - e\mathbf{A})^2}{2m} + e\phi$$

If the EM field is constant in time, we have

$$\nabla \times \mathbf{E} = 0$$

we would choose the gauge that

$$\dot{\mathbf{A}} = 0 \quad \mathbf{E} = -\nabla\phi$$

Because  $\frac{\partial L}{\partial t} = 0$ , we can know that  $\gamma m + e\phi$  is a constant.

Now we would list some special case.

### Motion in a uniform and constant electric field

Suppose the the direction of electric field is  $\hat{x}$ , the orbit is in the  $x - y$  plane. So, the equation of motion is

$$\dot{p}_x = eE \quad \dot{p}_y = 0$$

The final solution is

$$x = \frac{1}{eE} \sqrt{\mathcal{E}_0^2 + (eEt)^2} \quad y = \frac{p_0}{eE} \operatorname{arcsinh} \frac{eEt}{\mathcal{E}_0}$$

Here, we assume when  $t = 0$ ,  $p_x = 0, p_y = p_0$ . The orbit function is

$$x = \frac{\mathcal{E}_0}{eE} \cosh \frac{eEy}{p_0}$$

### Motion in a uniform and constant magnetic field

Suppose the direction of magnetic field is  $\hat{z}$ . Note that particle's kinetic energy  $\mathcal{E} = \gamma m$  is constant if there are no electric field, we can derive the equation of motion

$$\dot{v}_x = \omega v_y \quad \dot{v}_y = -\omega v_x \quad \dot{v}_z = 0$$

Here,  $\omega = \frac{eB}{\gamma m}$ . The final solution is

$$x = x_0 + r \sin(\omega t + \alpha) \quad y = y_0 + r \cos(\omega t + \alpha) \quad z = z_0 + v_{0z}t$$

Here,  $x_0, y_0, z_0, r, \alpha, v_{0z}$  should be determined by initial condition.





### Motion in a uniform and constant EM field

We only focus on the case that the velocity of particle is much smaller the velocity of light. Suppose the direction of magnetic field is  $\hat{z}$ , the direction of electric field is within  $y-z$  plane. The equation of motion is

$$m\ddot{x} = eB\dot{y} \quad m\ddot{y} = eE_y - eB\dot{x} \quad m\ddot{z} = eE_z$$

The solution is

$$\dot{x} = a \cos \omega t + \frac{E_y}{B} \quad \dot{y} = -a \sin \omega t \quad \dot{z} = v_{0z} + \frac{eE_z}{m}t$$

Here,  $\omega = \frac{eB}{m}$ ,  $a, v_{0z}$  is determined by initial condition. As we suppose that  $v \ll 1$  is satisfied, we must have that

$$a \ll 1 \quad v_{0z} \ll 1 \quad \frac{eE_z t}{m} \ll 1 \quad E_y \ll B$$

## 7.2 Constant electromagnetic field

### 7.2.1 Coulomb' law

For constant electric field, the Maxwell equation take the form

$$\nabla \cdot \mathbf{E} = \rho_e \quad \nabla \times \mathbf{E} = 0$$

So, we have

$$\mathbf{E} = -\nabla \phi \quad \nabla^2 \phi = -\rho_e$$

The solution is

$$\phi(\mathbf{r}) = \int \frac{\rho_e(\mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|} dV'$$

If  $\rho_e(\mathbf{r}') = Q\delta(\mathbf{r}')$ , we have

$$\phi(\mathbf{r}) = \frac{Q}{4\pi|\mathbf{r}|} \quad \mathbf{E}(\mathbf{r}) = \frac{Q\mathbf{r}}{4\pi|\mathbf{r}|^3}$$

For a system of static charged particles, the total energy is

$$U = \frac{1}{2} \int E^2 dV = \frac{1}{2} \int \rho \phi dV = \frac{1}{2} \sum e_a \phi_a + \frac{1}{2} \sum e_a \Phi_a$$

Here,  $\phi_a$  is the electric potential at the point where  $e_a$  is located, produced by  $e_a$  itself, while  $\Phi_a$  is the potential produced by other charges. It is obvious that  $U_{self} = \frac{1}{2} e_a \phi_a$  is infinite, indicating that classical electrodynamics is no more valid in small distance. This problem will be solved in quantum electrodynamics: the mass of charged particle we measured is already renormalized to include the electromagnetic self energy. So, actually, we have

$$U = \frac{1}{2} \int E^2 dV - U_{self} = \frac{1}{2} \sum_{a \neq b} \frac{e_a e_b}{4\pi R_{ab}}$$



If the charged particle is moving with a constant velocity, we can derive the electric field it produced by Lorentz transformation, the final result is that

$$\mathbf{E} = \frac{e\mathbf{R}}{4\pi R^3} \frac{1 - V^2}{(1 - V^2 \sin^2 \theta)^{3/2}} \quad \mathbf{B} = \mathbf{V} \times \mathbf{E}$$

Here,  $\mathbf{R}$  is the vector point from the particle to the point we measure the electric field, and  $\theta$  is the angle between  $\mathbf{V}$  and  $\mathbf{R}$ . If  $V \sim 1$ , the electric field will be concentrated in the direction perpendicular to the  $\mathbf{V}$ . If  $V \ll 1$ , we have

$$\mathbf{E} = \frac{e\mathbf{R}}{4\pi R^3} \quad \mathbf{B} = \frac{e\mathbf{V} \times \mathbf{R}}{4\pi R^3}$$

## 7.2.2 Multipole moments

For a system of charged particles, the potential it produced at  $\mathbf{R}$  is

$$\phi = \sum_a \frac{e_a}{4\pi |\mathbf{R} - \mathbf{r}_a|}$$

If  $R \gg r_a$ , we can expand the equation around  $r_a = 0$ . Generally, we have

$$\frac{1}{|\mathbf{R} - \mathbf{r}|} = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{r^l}{R^{l+1}} \frac{4\pi}{2l+1} Y_{lm}^*(\Theta, \Phi) Y_{lm}(\theta, \phi)$$

So, we have

$$\phi = \sum \phi^{(l)}$$

Here,

$$\begin{aligned} \phi^{(l)} &= \frac{1}{4\pi R^{l+1}} \sum_{m=-l}^l \sqrt{\frac{4\pi}{2l+1}} Q_m^{(l)} Y_{lm}^*(\Theta, \Phi) \\ Q_m^{(l)} &= \sum_a e_a r_a^l \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta_a, \phi_a) \end{aligned}$$

Special case:

$$\phi^{(0)} = \frac{Q}{4\pi R} \quad E^{(0)} = \frac{Q\mathbf{n}}{4\pi R^2} \quad Q = \sum_a e_a$$

$$\phi^{(1)} = \frac{\mathbf{d} \cdot \mathbf{n}}{4\pi R^2} \quad E^{(1)} = \frac{3(\mathbf{d} \cdot \mathbf{n})\mathbf{n} - \mathbf{d}}{4\pi R^3} \quad \mathbf{d} = \sum_a e_a \mathbf{r}_a$$

$$\phi^{(2)} = \frac{\mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n}}{8\pi R^3} \quad E^{(2)} = \frac{5(\mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n})\mathbf{n} - (\mathbf{n} \cdot \mathbf{D} + \mathbf{D} \cdot \mathbf{n})}{8\pi R^4} \quad D_{ij} = \sum e(3x_i x_j - r^2 \delta_{ij})$$

For a system of charged particles in the electric field  $\phi(\mathbf{r})$ , if all the particles is near the  $r = 0$ , we can make the expansion

$$\phi(\mathbf{r}) = \sum_{l=0}^{\infty} r^l \sum_{m=-l}^{m=l} a_{lm} \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta, \phi)$$



So

$$U = \sum_{l=0}^{\infty} U^{(l)} \quad U^{(l)} = \sum_{m=-l}^l a_{lm} Q_m^{(l)}$$

Special case:

$$\begin{aligned} U^{(0)} &= Q\phi_0 & F^{(0)} &= Q\mathbf{E}_0 \\ U^{(1)} &= -\mathbf{d} \cdot \mathbf{E}_0 & F^{(1)} &= \mathbf{d} \cdot \nabla \mathbf{E}_0 & M^{(1)} &= \mathbf{d} \times \mathbf{E}_0 \\ U^{(2)} &= -\frac{1}{6} \mathbf{D} \cdot \nabla \mathbf{E}_0 & F^{(2)} &= \frac{1}{6} \mathbf{D} \cdot \nabla \nabla \mathbf{E}_0 & M^{(2)} &= \frac{1}{3} \nabla \cdot (\mathbf{D} \times \mathbf{E}_0) \end{aligned}$$

### 7.2.3 Biot-Savart law

Let us consider the magnetic field produced by charges which perform a finite motion, in which the particles are always within a finite region of space and the momenta also always remain finite. Consider the time average magnetic field  $\bar{\mathbf{B}}$ , produced by the charges; this field will now be a function only of the coordinates and not of the time. We take the time average of the Maxwell equations

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j}$$

Note that the average value of the derivative  $\partial \mathbf{E} / \partial t$ , like the derivative of any quantity which varies over a finite range, is zero. We can get

$$\nabla \cdot \bar{\mathbf{B}} = 0 \quad \nabla \times \bar{\mathbf{B}} = \bar{\mathbf{j}}$$

Recall that  $\mathbf{B} = \nabla \times \mathbf{A}$ . We impose the gauge condition  $\nabla \cdot \mathbf{A} = 0$ , we have

$$\nabla^2 \bar{\mathbf{A}} = -\bar{\mathbf{j}}$$

The solution is

$$\bar{\mathbf{A}}(\mathbf{r}) = \frac{1}{4\pi} \int \frac{\bar{\mathbf{j}}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' = \frac{1}{4\pi} \sum \frac{\overline{e_a \mathbf{v}_a}}{|\mathbf{r} - \mathbf{r}_a|}$$

And the magnetic field is

$$\bar{\mathbf{B}}(\mathbf{r}) = \frac{1}{4\pi} \int \frac{\bar{\mathbf{j}}(\mathbf{r}') \times \mathbf{R}}{R^3} dV' \quad \mathbf{R} = \mathbf{r} - \mathbf{r}'$$

### 7.2.4 Magnetic moment

For a system of charged particles, the potential it produced at  $\mathbf{R}$  is

$$\bar{\mathbf{A}} = \frac{1}{4\pi} \sum \frac{\overline{e_a \mathbf{v}_a}}{|\mathbf{R} - \mathbf{r}_a|}$$

If  $R \gg r_a$ , we can expand the equation around  $r_a = 0$  the first order,

$$4\pi \bar{\mathbf{A}} = \frac{1}{R} \sum e \bar{\mathbf{v}} - \sum e \mathbf{v} \left( \mathbf{r} \cdot \nabla \frac{1}{R} \right)$$



Firstly,

$$\sum e\bar{\mathbf{v}} = \overline{\frac{d}{dt} \sum e\mathbf{r}} = 0$$

Secondly,

$$-\sum e\mathbf{v} \left( \mathbf{r} \cdot \nabla \frac{1}{R} \right) = \frac{1}{R^3} \sum e\mathbf{v}(\mathbf{r} \cdot \mathbf{R})$$

Note that

$$\sum e\mathbf{v}(\mathbf{r} \cdot \mathbf{R}) = \frac{1}{2} \frac{d}{dt} e\mathbf{r}(\mathbf{r} \cdot \mathbf{R}) + \frac{1}{2} \left( \sum e\mathbf{r} \times \mathbf{v} \right) \times \mathbf{R}$$

Define the magnetic moment as

$$\mathbf{m} \equiv \frac{1}{2} \left( \sum e\mathbf{r} \times \mathbf{v} \right)$$

We can get

$$\bar{\mathbf{A}} = \frac{\bar{\mathbf{m}} \times \mathbf{R}}{4\pi R^3} \quad \bar{\mathbf{B}} = \frac{3\mathbf{n}(\bar{\mathbf{m}} \cdot \mathbf{n}) - \bar{\mathbf{m}}}{4\pi R^3}$$

If all the particles have the same charge-mass ratio, and the velocity of all the particles is much smaller than that of light, we have

$$\mathbf{m} = \frac{e}{2m} \sum m\mathbf{r} \times \mathbf{v} = \frac{e}{2m} \mathbf{M}$$

Let us consider a system of charges in an external constant uniform magnetic field. The time average of the force acting on the system,

$$\bar{\mathbf{F}} = \sum e\bar{\mathbf{v}} \times \bar{\mathbf{B}} = \overline{\frac{d}{dt} \sum e\mathbf{r} \times \mathbf{B}} = 0$$

The average value of the moment of the forces is

$$\bar{\mathbf{K}} = \sum e\mathbf{r} \times (\mathbf{v} \times \mathbf{B})$$

We can derive that

$$\bar{\mathbf{K}} = \bar{\mathbf{m}} \times \mathbf{B}$$

Let us consider the change in the average angular momentum  $\bar{\mathbf{M}}$  of the system. According to a well-known equation of mechanics, the derivative of  $\mathbf{M}$  is equal to the moment  $\mathbf{K}$  of the forces acting on the system. We therefore have

$$\frac{d\bar{\mathbf{M}}}{dt} = \bar{\mathbf{m}} \times \mathbf{B}$$

If the charge-mass ratio is the same for all particles of the system, the angular momentum and magnetic moment are proportional to one another, and we find:

$$\frac{d\bar{\mathbf{M}}}{dt} = -\boldsymbol{\Omega} \times \bar{\mathbf{M}} \quad \boldsymbol{\Omega} = \frac{e}{2m} \mathbf{B}$$

This equation states that the vector  $\bar{\mathbf{M}}$  rotates with angular velocity  $-\boldsymbol{\Omega}$  around the direction of the field, while its absolute magnitude and the angle which it makes with this direction remain fixed. This motion is called the Larmor precession. Let us consider the Lagrangian for a system of charges in an external constant uniform magnetic field, it contains the additional term

$$L_H = \sum e\mathbf{A} = \sum e(\mathbf{B} \times \mathbf{r}) \cdot \mathbf{v} = \sum e(\mathbf{r} \times \mathbf{v}) \cdot \mathbf{B} = \mathbf{m} \cdot \mathbf{B}$$



## 7.3 Electromagnetic waves

### 7.3.1 Electromagnetic waves

electromagnetic fields occurring in vacuum in the absence of charges are called electromagnetic waves. We choose the Coulomb's Gauge, i.e.

$$\phi = 0 \quad \nabla \cdot \mathbf{A} = 0$$

So,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Then, from Maxwell equation we can derive that

$$\nabla^2 \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0$$

This is the equation which determines the potentials of electromagnetic waves. It is called the d'Alembert equation, or the wave equation. We can verify that the electric and magnetic fields  $\mathbf{E}$  and  $\mathbf{H}$  satisfy the same wave equation.

We consider the special case of electromagnetic waves in which the fields depend only on one coordinate, say  $x$ . Such waves are said to be plane. In this case the equation for the field becomes

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} = 0$$

The solution is

$$f(t, x) = f_1(t - x) + f_2(t + x)$$

$f_1(t - x)$  represents a plane wave moving in the positive direction along the  $x$  axis.  $f_2(t + x)$  represents a plane wave moving in the negative direction along the  $x$  axis. The Coulomb's gauge would imply that  $A_x = 0$ , and we can obtain

$$\mathbf{E} = -\mathbf{A}' \quad \mathbf{B} = -\mathbf{n} \times \mathbf{A}' = \mathbf{n} \times \mathbf{E}$$

where the prime denotes differentiation with respect to  $t - x$  and  $\mathbf{n}$  is a unit vector along the direction of propagation of the wave. We see that the electric and magnetic fields  $\mathbf{E}$  and  $\mathbf{B}$  of a plane wave are directed perpendicular to the direction of propagation of the wave. For this reason, electromagnetic waves are said to be transverse.

The energy density and flux of the plane waves are

$$W = \mathbf{E}^2 \quad \mathbf{S} = W\mathbf{n}$$

### 7.3.2 Monochromatic wave

A very important special case of electromagnetic waves is a wave in which the field is a simply periodic function of the time. Such a wave is said to be monochromatic. All quantities (potentials, field components) in a monochromatic wave depend on the time through a factor



of the form  $\cos(\omega t + a)$ . The quantity  $\omega$  is called the cyclic frequency of the wave (we shall simply call it the frequency). For the monochromatic wave, the wave equation becomes

$$\frac{\partial^2 f}{\partial x^2} + \omega^2 f = 0$$

The vector potential of such a wave is most conveniently written as the real part of a complex expression

$$\mathbf{A} = \text{Re} \{ \mathbf{A}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \} \quad \mathbf{k} = \omega \mathbf{n}$$

The time average of the product of field intensity can be calculated as

$$\overline{\mathbf{A}\mathbf{B}} = \frac{1}{2} \text{Re} \{ \mathbf{A}_0 \mathbf{B}_0^* \}$$

The electric and magnetic field are

$$\mathbf{E} = ik\mathbf{A} \quad \mathbf{B} = i\mathbf{k} \times \mathbf{A}$$

And we can verify that  $(\omega, \mathbf{k})$  transform like a four-vector.

Generally, the electric field can be written as

$$E_y = A \cos(\phi) \quad E_z = B \cos(\phi + \delta) \quad \phi = kx - \omega t \quad -\pi < \delta \leq \pi$$

The end of the vector  $\mathbf{E}$  in  $y - z$  plane will form an ellipse. The magnitudes of the semiaxes of the polarized ellipse are

$$|\sqrt{A^2 + B^2 + 2AB \sin \delta} \pm \sqrt{A^2 + B^2 - 2AB \sin \delta}|$$

The angle  $\theta$  between the major axis and  $y$  axis satisfies the equation

$$\tan 2\theta = \frac{2AB \cos \delta}{A^2 - B^2}$$

If  $-\frac{\pi}{2} < \delta < \frac{\pi}{2}$ , major axis is in the second and forth quadrant. If  $\delta > \frac{\pi}{2}$  or  $\delta < -\frac{\pi}{2}$ , major axis is in the first and third quadrant. If  $\delta = \pm \frac{\pi}{2}$  and  $A > B$ , the major axis is  $y$  axis. If  $\delta = \pm \frac{\pi}{2}$  and  $A < B$ , the major axis is  $z$  axis. If  $\delta = \pm \frac{\pi}{2}$  and  $A = B$ , the ellipse becomes a circle.

If  $0 < \delta < \pi$ , the rotation is positive in the direction of  $x$  axis (left handed). If  $-\pi < \delta < 0$ , the rotation is negative in the direction of  $x$  axis (right handed). If  $\delta = 0, \pi$ , the ellipse becomes a line.

Any field is expandable in a Fourier integral containing a continuous or discrete distribution of different frequencies. Such an expansion has the form

$$f(t) = \int_{-\infty}^{\infty} f_{\omega} e^{-i\omega t} \frac{d\omega}{2\pi}$$

where the Fourier components are given in terms of the function  $f(t)$  by the integrals

$$f_{\omega} = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$



Because  $f(t)$  must be real, so

$$f_{-\omega} = f_{\omega}^*$$

The total intensity of the wave is

$$\int_{-\infty}^{\infty} f^2 dt = \int_{-\infty}^{\infty} |f_{\omega}|^2 \frac{d\omega}{2\pi} = 2 \int_0^{\infty} |f_{\omega}|^2 \frac{d\omega}{2\pi}$$

There is a special case that  $f(t)$  is a periodic function with angular frequency  $\omega_0$ , then  $f(t)$  can be expanded as

$$f(t) = \sum_{-\infty}^{\infty} f_n e^{-in\omega_0 t}$$

here,

$$f_n = \frac{1}{T} \int_0^T f(t) e^{in\omega_0 t} dt$$

The average intensity of the wave is

$$\frac{1}{T} \int_0^T f^2 dt = \sum_{-\infty}^{\infty} |f_n|^2$$

And we can verify that

$$f_{\omega} = \sum_{-\infty}^{\infty} 2\pi f_n \delta(\omega - n\omega_0)$$

### 7.3.3 Partially polarized light

Every monochromatic wave is necessarily polarized. However we usually have to deal with waves which are only approximately monochromatic, and which contain frequencies in a small interval  $\Delta\omega$ . We consider such a wave, and let  $\omega$  be some average frequency for it. Then its field at a fixed point in space can be written in the form

$$\mathbf{E}_0(t) e^{-i\omega t}$$

where the complex amplitude  $\mathbf{E}_0$  is some slowly varying function of the time. Since  $\mathbf{E}_0$  determines the polarization of the wave, this means that at each point of the wave, its polarization changes with time, such a wave is said to be partially polarized.

The polarization properties of electromagnetic waves are observed experimentally by passing the light to be investigated through various bodies and then observing the intensity of the transmitted light. From the mathematical point of view this means that we draw conclusions concerning the polarization properties of the light from the values of certain quadratic functions of its field. Here of course we are considering the time averages of such functions.

Quadratic functions of the field are made up of terms proportional to the products  $E_{\alpha} E_{\beta}$ ,  $E_{\alpha}^* E_{\beta}^*$  or  $E_{\alpha}^* E_{\beta}$ . Products of the form  $E_{\alpha} E_{\beta}$  and  $E_{\alpha}^* E_{\beta}^*$  contain the rapidly oscillating factors  $e^{-i2\omega t}$  and will give zero when the time average is taken. Thus we see that the polarization properties of the light are completely characterized by the tensor

$$J_{\alpha\beta} = E_{0\alpha} E_{0\beta}^*$$



The trace of the tensor

$$J \equiv J_{\alpha\alpha} = \mathbf{E}_0 \mathbf{E}_0^*$$

determines the intensity of the wave, as measured by the energy flux density. To eliminate this quantity which is not directly related to the polarization properties, we introduce the tensor

$$\rho_{\alpha\beta} = \frac{J_{\alpha\beta}}{J}$$

we call it the polarization tensor.

Generally, the polarization tensor is expressed as

$$\rho = \frac{1}{2} \begin{bmatrix} 1 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & 1 - p_3 \end{bmatrix}$$

If we introduce the Pauli matrix, i.e.

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The  $\rho$  can be expressed as

$$\frac{1}{2}(1 - P)I + \frac{1}{2}P(I + \mathbf{n} \cdot \boldsymbol{\sigma})$$

Here,

$$P = \sqrt{p_1^2 + p_2^2 + p_3^2} \quad \mathbf{n} = \left(\frac{p_1}{P}, \frac{p_2}{P}, \frac{p_3}{P}\right)$$

For a totally polarized light with polarization state  $|E\rangle = (\cos \frac{\theta}{2} e^{-i\frac{\phi}{2}}, \sin \frac{\theta}{2} e^{i\frac{\phi}{2}})$ , the polarization tensor  $\rho = |E\rangle\langle E|$ . We can verify that

$$P = 1 \quad \mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

So, for an arbitrary light,

$$\rho = (1 - P)\rho_n + P\rho_p \quad \rho_n = \frac{1}{2}I \quad \rho_p = \frac{1}{2}(I + \mathbf{n} \cdot \boldsymbol{\sigma})$$

Thus, we call  $P$  the Polarization degree.

Suppose there is a polarizing filter, which allow the light with polarization state

$$|D\rangle = (\cos \frac{\theta}{2} e^{-i\frac{\phi}{2}}, \sin \frac{\theta}{2} e^{i\frac{\phi}{2}})$$

to pass totally. If a light with polarization tensor  $\rho$  pass through the device, the relative intensity will become

$$\langle D|\rho|D\rangle = \frac{1}{2} + \frac{1}{2}\mathbf{p} \cdot \mathbf{m}$$

Here,  $\mathbf{p} = (p_1, p_2, p_3)$ ,  $\mathbf{m} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$





In optics, the Stokes vectors are defined as The Stokes parameters are defined by

$$\begin{aligned}
 I &\equiv \langle E_x^2 \rangle + \langle E_y^2 \rangle \\
 &= \langle E_a^2 \rangle + \langle E_b^2 \rangle \\
 &= \langle E_l^2 \rangle + \langle E_r^2 \rangle \\
 Q &\equiv \langle E_x^2 \rangle - \langle E_y^2 \rangle \\
 U &\equiv \langle E_a^2 \rangle - \langle E_b^2 \rangle \\
 V &\equiv \langle E_l^2 \rangle - \langle E_r^2 \rangle
 \end{aligned}$$

where the subscripts refer to three different bases of the space of Jones vectors: the standard Cartesian basis  $\hat{x}, \hat{y}$ , a Cartesian basis rotated by  $45^\circ$   $\hat{a}, \hat{b}$ , and a circular basis  $\hat{l}, \hat{r}$ . The symbols  $\langle \cdot \rangle$  represent expectation values. The light can be viewed as a random variable taking values in the space  $C^2$  of Jones vectors  $(E_1, E_2)$ . It is easy to verify that

$$Q = Ip_3 \quad U = Ip_2 \quad V = Ip_1$$

## 7.4 The field of moving charges

### 7.4.1 Retarded potential

This time we impose the Lorenz Gauge  $\partial_\mu A^\mu = 0$ . Now, Maxwell equations would imply that

$$\partial^2 A^\mu = j^\mu$$

We can rewrite it in three dimension form,

$$\nabla^2 \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mathbf{J} \quad \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = -\rho$$

To find the particular solution, we divide the whole space into infinitely small regions and determine the field produced by the charges located in one of these volume elements. Because of the linearity of the field equations, the actual field will be the sum of the fields produced by all such elements.

The charge  $de$  in a given volume element is a function of the time. If we choose the origin of coordinates in the volume element under consideration, then the charge density is  $de(t)\delta(\mathbf{R})$ , where  $R$  is the distance from the origin. Thus we must solve the equation

$$\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = -de(t)\delta(\mathbf{R})$$

The particular solution is

$$\phi = \frac{de(t - R)}{4\pi R}$$

For an arbitrary distribution of charges  $\rho(\mathbf{r}', t)$ , we have

$$\phi(\mathbf{r}, t) = \int \frac{\rho(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|)}{4\pi |\mathbf{r} - \mathbf{r}'|} dV'$$



Similarly we have for the vector potential

$$\mathbf{A}(\mathbf{r}, t) = \int \frac{\mathbf{J}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|)}{4\pi|\mathbf{r} - \mathbf{r}'|} dV'$$

The particular solution above is called retarded potential.

Now we consider an arbitrary charged particle with kinetic equation  $\mathbf{r} = \mathbf{r}_0(t)$ , so we have

$$\rho(\mathbf{r}, t) = e\delta(\mathbf{r} - \mathbf{r}_0(t))$$

$$\phi(\mathbf{r}, t) = \int \frac{e\delta(\mathbf{r}' - \mathbf{r}_0(t - |\mathbf{r} - \mathbf{r}'|))}{4\pi|\mathbf{r} - \mathbf{r}'|} dV' = \frac{e}{4\pi R^* (1 - \frac{\mathbf{R}^* \cdot \mathbf{v}^*}{R^*})}$$

Here,

$$\mathbf{R}^* = \mathbf{r} - \mathbf{r}_0(t^*) \quad \mathbf{v}^* = \mathbf{v}_0(t^*) \quad t^* = t - R^*$$

Similarly, we have

$$\mathbf{A}(\mathbf{r}, t) = \frac{e\mathbf{v}^*}{4\pi R^* (1 - \frac{\mathbf{R}^* \cdot \mathbf{v}^*}{R^*})}$$

The potential is called Lienard-Wiechert potentials.

We can derive that

$$\frac{\partial t^*}{\partial t} = \frac{1}{1 - \frac{\mathbf{R}^* \cdot \mathbf{v}^*}{R^*}} \quad \nabla t^* = \frac{\mathbf{R}^*}{R^* - \mathbf{R}^* \cdot \mathbf{v}^*}$$

With the help of the equation above, we can calculate the electric and magnetic field intensity.

$$\mathbf{E} = \frac{e}{4\pi(R^* - \mathbf{R}^* \cdot \mathbf{v}^*)^3} \left\{ (1 - v^{*2})(\mathbf{R}^* - R^*\mathbf{v}^*) + \mathbf{R}^* \times [(\mathbf{R}^* - R^*\mathbf{v}^*) \times \mathbf{a}^*] \right\}$$

$$\mathbf{B} = \frac{\mathbf{R}^*}{R^*} \times \mathbf{E}$$

The electric field consists of two parts of different type. The first term depends only on the velocity of the particle (and not on its acceleration) and varies at large distances like  $\frac{1}{R^2}$ . The second term depends on the acceleration, and for large  $R$  it varies like  $\frac{1}{R}$ . This latter term is related to the electromagnetic waves radiated by the particle.

### 7.4.2 Spectral resolution of the retarded potentials

Suppose

$$\rho(\mathbf{r}, t) = \int_{-\infty}^{\infty} \rho_{\omega}(\mathbf{r}) e^{-i\omega t} \frac{d\omega}{2\pi} \quad \phi(\mathbf{r}, t) = \int_{-\infty}^{\infty} \phi_{\omega}(\mathbf{r}) e^{-i\omega t} \frac{d\omega}{2\pi}$$

We can derive that

$$\phi_{\omega}(\mathbf{r}) = \int dV' \int_{-\infty}^{\infty} dt \frac{\rho(\mathbf{r}', t)}{4\pi R} e^{i\omega(R+t)}$$

If there is just one point charge, we set

$$\rho = e\delta(\mathbf{r} - \mathbf{r}_0(t))$$



we can get

$$\phi_{\omega}(\mathbf{r}) = \int_{-\infty}^{\infty} dt \frac{e}{4\pi R(t)} e^{i\omega[R(t)+t]} \quad \mathbf{R}(t) = \mathbf{r} - \mathbf{r}_0(t)$$

Similarly, for vector potential, we have

$$\mathbf{A}_{\omega}(\mathbf{r}) = \int_{-\infty}^{\infty} dt \frac{e\mathbf{v}_0(t)}{4\pi R(t)} e^{i\omega[R(t)+t]} \quad \mathbf{R}(t) = \mathbf{r} - \mathbf{r}_0(t)$$

For electric and magnetic field, we have

$$\mathbf{E}_{\omega} = -\nabla\phi_{\omega} + i\omega\mathbf{A}_{\omega}$$

$$\mathbf{B}_{\omega} = \nabla \times \mathbf{A}_{\omega}$$

## 7.5 Radiation

### 7.5.1 Far field approximation

We consider the field produced by a system of moving charges at distances large compared with the dimensions of the system. We choose the origin of coordinates  $O$  anywhere in the interior of the system of charges. The radius vector from  $O$  to the point  $P$ , where we determine the field, we denote by  $\mathbf{R}_0$ , and the unit vector in this direction by  $\mathbf{n}$ . Let the radius vector of the charge element be  $\mathbf{r}$ , and the radius vector from de to the point  $P$  be  $\mathbf{R}$ . At large distances from the system of charges,  $R_0 \gg r$ , and we have approximately,

$$R \approx R_0 - \mathbf{n} \cdot \mathbf{r}$$

We substitute this for the retarded potentials. In the denominator of the integrands we can neglect  $\mathbf{n} \cdot \mathbf{r}$  compared with  $R_0$ . In  $t - R$ , whether it is possible to neglect these terms is determined by how much the quantities  $e$  and  $\mathbf{j}$  change during the time  $\mathbf{n} \cdot \mathbf{r}$ . The potentials of the field at large distances from the system of charges are

$$\phi(\mathbf{R}_0, t) = \frac{1}{4\pi R_0} \int \rho(\mathbf{r}, t - R_0 + \mathbf{n} \cdot \mathbf{r}) dV$$

$$\mathbf{A}(\mathbf{R}_0, t) = \frac{1}{4\pi R_0} \int \mathbf{J}(\mathbf{r}, t - R_0 + \mathbf{n} \cdot \mathbf{r}) dV$$

At sufficiently large distances from the system of charges, the field over small regions of space can be considered to be a plane wave. For this it is necessary that the distance be large compared not only with the dimensions of the system, but also with the wavelength of the electromagnetic waves radiated by the system. We refer to this region of space as the wave zone of the radiation. In wave zone, we have

$$\mathbf{B} = \frac{\partial \mathbf{A}}{\partial t} \times \mathbf{n} \quad \mathbf{E} = \left( \frac{\partial \mathbf{A}}{\partial t} \times \mathbf{n} \right) \times \mathbf{n}$$

The energy flux is given by the Poynting vector which, for a plane wave, is

$$\mathbf{S} = B^2 \mathbf{n}$$



The intensity of radiation into the element of solid angle  $do$

$$dI = B^2 R_0^2 do$$

Since the field is inversely proportional to  $R_0$ , we see that the amount of energy radiated by the system in unit time into the element of solid angle  $do$  is the same for all distances. For the radiation produced by a single arbitrarily moving point charge, it turns out to be convenient to use the Lienard-Wiechert potentials. At large distances, we have

$$\mathbf{A}(\mathbf{R}_0, t) = \frac{e\mathbf{v}(t')}{4\pi R_0(1 - \mathbf{n} \cdot \mathbf{v}_0(t'))}$$

Here,

$$t' - \mathbf{n} \cdot \mathbf{r}_0(t') = t - R_0$$

For the spectral resolution of the field of the waves radiated by the system, we have

$$\mathbf{A}_\omega(\mathbf{R}_0) = \frac{e^{ikR_0}}{R_0} \int \mathbf{J}_\omega e^{-i\mathbf{k} \cdot \mathbf{r}} dV$$

In wave zone, we have

$$\mathbf{B}_\omega = i\mathbf{k} \times \mathbf{A}_\omega \quad \mathbf{E} = \frac{1}{\omega} (\mathbf{k} \times \mathbf{A}_\omega) \times \mathbf{k}$$

Suppose  $d\mathcal{E}_{\omega n}$  is the energy radiated into the element of solid angle  $do$  in the form of waves with frequencies in the interval  $d\omega$ . We have

$$d\mathcal{E}_{\omega n} = 2B_\omega^2 R_0^2 do \frac{d\omega}{2\pi}$$

For the radiation produced by a single arbitrarily moving point charge, we have

$$\begin{aligned} \mathbf{A}_\omega(\mathbf{R}_0) &= \frac{e e^{i\omega R_0}}{4\pi R_0} \int_{-\infty}^{\infty} e^{i\omega[t - \mathbf{n} \cdot \mathbf{v}_0]} \mathbf{v}_0 dt \\ \mathbf{B}_\omega(\mathbf{R}_0) &= \frac{e\omega e^{i\omega R_0}}{4\pi R_0} \int_{-\infty}^{\infty} e^{i\omega[t - \mathbf{n} \cdot \mathbf{v}_0]} \mathbf{n} \times d\mathbf{r}_0 \end{aligned}$$

### 7.5.2 Low velocity approximation

If the charge density (current density) distribution changes a little during time  $\mathbf{r} \cdot \mathbf{n}$ , then we can expand  $f(r, t - R_0 + \mathbf{r} \cdot \mathbf{n})$  in series of  $(\mathbf{r} \cdot \mathbf{n})^2$ . In this case, if the typical angular frequency of the motion is  $\omega$ , the typical scale of the motion is  $a$ , then we have

$$a \ll T \sim \frac{2\pi}{\omega} \sim \lambda$$

or

$$v \sim \frac{a}{T} \ll 1$$



which means that the velocity of the particles is much smaller than that of light, or the scale of the system is much smaller than the wavelength.

As for the zeroth order approximation, we just drop the  $\mathbf{r} \cdot \mathbf{n}$  in the equation, so

$$\mathbf{A}(\mathbf{R}_0, t) = \frac{1}{4\pi R_0} \int \mathbf{J}(\mathbf{r}, t - R_0) dt$$

For the radiation produced by arbitrarily moving point charges, we have

$$\mathbf{A}(\mathbf{R}_0, t) = \frac{1}{4\pi R_0} \sum e_a \mathbf{v}_a(\mathbf{r}, t - R_0) = \frac{1}{4\pi R_0} \dot{\mathbf{d}}$$

So,

$$\mathbf{B} = \frac{1}{4\pi R_0} \ddot{\mathbf{d}} \times \mathbf{n} \quad \mathbf{E} = \frac{1}{4\pi R_0} (\ddot{\mathbf{d}} \times \mathbf{n}) \times \mathbf{n}$$

Radiation of this kind is called dipole radiation. We note that a closed system of particles, for all of which the ratio of charge to mass is the same, cannot radiate (by dipole radiation). The intensity of the dipole radiation is

$$dI = \frac{\ddot{\mathbf{d}}^2}{16\pi^2} \sin^2 \theta d\Omega$$

where  $\theta$  is the angle between  $\ddot{\mathbf{d}}$  and  $\mathbf{n}$ . Integrate over all over the direction, we have

$$I = \frac{\ddot{\mathbf{d}}^2}{6\pi}$$

If we have just one charge moving in the external field, we have

$$I = \frac{e^2 w^2}{6\pi}$$

$w$  is the acceleration of the charge. For the spectral resolution of the intensity of dipole radiation, we have

$$d\mathcal{E}_\omega = \frac{|(\ddot{\mathbf{d}})_\omega|^2}{3\pi} = \frac{\omega^4}{3\pi} |\mathbf{d}_\omega|^2 \frac{d\omega}{2\pi}$$

More details on dipole radiation during collisions and Coulomb interaction can be found in section 68,69 and 70 of *The classical theory of fields* (L.D.Landau & E.M.Lifshitz).

If we keep the first order of  $\mathbf{n} \cdot \mathbf{r}$ , we have

$$\mathbf{A} = \frac{\dot{\mathbf{d}}}{4\pi R_0} + \frac{\ddot{\mathbf{D}}}{24\pi R_0} + \frac{\dot{\mathbf{m}} \times \mathbf{n}}{4\pi R_0}$$

Here,

$$\mathbf{D} = \sum e[3\mathbf{r}(\mathbf{n} \cdot \mathbf{r} - nr^2)] \quad \mathbf{m} = \frac{1}{2} \sum e\mathbf{r} \times \mathbf{v}$$

So

$$\begin{aligned} \mathbf{B} &= \frac{1}{4\pi R_0} \left[ \ddot{\mathbf{d}} \times \mathbf{n} + \frac{1}{6} \ddot{\mathbf{D}} \times \mathbf{n} + (\ddot{\mathbf{m}} \times \mathbf{n}) \times \mathbf{n} \right] \\ \mathbf{E} &= \frac{1}{4\pi R_0} \left[ (\ddot{\mathbf{d}} \times \mathbf{n}) \times \mathbf{n} + \frac{1}{6} (\ddot{\mathbf{D}} \times \mathbf{n}) \times \mathbf{n} + \mathbf{n} \times \ddot{\mathbf{m}} \right] \end{aligned}$$



$$I = \frac{1}{6\pi} \ddot{\mathbf{d}}^2 + \frac{1}{720\pi} \ddot{D}_{ij}^2 + \frac{1}{6\pi} \dot{\mathbf{m}}^2 \quad D_{ij} = \sum e(3r_i r_j - r^2 \delta_{ij})$$

The total radiation consists of three independent parts: dipole, quadrupole, and magnetic dipole radiation. The details of the derivation can be found in section 71 of *The classical theory of fields* (L.D.Landau & E.M.Lifshitz).

### 7.5.3 Radiation from a rapidly moving charge

Consider the particle in the system of reference in which the particle is at rest at a given moment; in this system of reference we can apply low velocity approximation. Here, we have

$$d\mathcal{E} = \frac{e^2 w^2}{6\pi} dt \quad d\mathbf{P} = 0$$

So, in an arbitrary reference, we have

$$dP^\mu = -\frac{e^2}{6\pi} \frac{du^\nu}{d\tau} \frac{du_\nu}{d\tau} u^\mu d\tau$$

Recall the Lorentz equation, the total four-momentum radiated during the time of passage of the particle through a given electromagnetic field is equal to

$$\Delta P^\mu = -\frac{e^4}{6\pi m^2} \int F_{\nu\rho} u^\rho F^{\nu\sigma} u_\sigma dx^\mu$$

Particularly, we have

$$\Delta\mathcal{E} = \frac{e^2}{6\pi} \int \frac{w^2 - (\mathbf{v} \times \mathbf{w})^2}{(1 - v^2)^3} dt = \frac{e^4}{6\pi m^2} \int \frac{(\mathbf{E} + \mathbf{v} \times \mathbf{B})^2 - (\mathbf{E} \cdot \mathbf{v})^2}{1 - v^2} dt$$

It is clear from formula above that for velocities close to the velocity of light, the total energy radiated per unit time is proportionally to the square of the energy of the moving particle. The only exception is motion in an electric field, along the direction of the field. In this case the factor  $(1 - v^2)$  standing in the denominator is cancelled by an identical factor in the numerator, and the radiation does not depend on the energy of the particle.

Now we discuss the angular distribution of the radiation from a rapidly moving charge. The radiation electric field is

$$\mathbf{E} = \frac{e}{4\pi R} \frac{\mathbf{n} \times [(\mathbf{n} - \mathbf{v}) \times \mathbf{w}]}{(1 - \mathbf{n} \cdot \mathbf{v})^3} \quad \mathbf{B} = \mathbf{n} \times \mathbf{E}$$

where all the quantities on the right sides of the equations refer to the retarded time  $t' = t - \frac{R}{c}$ . The intensity radiated into the solid angle  $do$  is

$$dI = \frac{e^2}{16\pi^2} \left\{ \frac{2(\mathbf{n} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{w})}{(1 - \mathbf{v} \cdot \mathbf{n})^5} + \frac{w^2}{(1 - \mathbf{v} \cdot \mathbf{n})^4} - \frac{(1 - v^2)(\mathbf{n} \cdot \mathbf{w})^2}{(1 - \mathbf{v} \cdot \mathbf{n})^6} \right\} do$$

If we want to determine the angular distribution of the total radiation throughout the whole motion of the particle, we must integrate the intensity over the time. In doing this, it is important to remember that the integrand is a function of  $t'$ ; therefore we must write

$$dt = (1 - \mathbf{n} \cdot \mathbf{v}) dt'$$



after which the integration over  $t'$  is immediately done.

In the ultrarelativistic case, the intensity is large within the narrow range of angles in which  $1 - \mathbf{v} \cdot \mathbf{n}$  is small. Thus an ultrarelativistic particle radiates mainly along the direction of its own motion, within the small range of angles around the direction of its velocity. We also point out that, for arbitrary velocity and acceleration of the particle, there are always two directions for which the radiated intensity is zero. These are the directions for which the vector  $\mathbf{n} - \mathbf{v}$  is parallel to the vector  $\mathbf{w}$ .

If the velocity and acceleration of the particle are parallel,

$$\mathbf{B} = \frac{e}{4\pi R} \frac{\mathbf{w} \times \mathbf{n}}{(1 - \mathbf{v} \cdot \mathbf{n})^3}$$

and the intensity is

$$dI = \frac{e^2}{16\pi^2} \frac{w^2 \sin^2 \theta}{(1 - v \cos \theta)^6}$$

It is naturally, symmetric around the common direction of  $\mathbf{v}$  and  $\mathbf{w}$ , and vanishes along ( $\theta = 0$ ) and opposite to ( $\theta = \pi$ ) the direction of the velocity. In the ultrarelativistic case, the intensity as a function of  $\theta$  has a sharp double maximum near  $\mathbf{v}$ , with a steep drop to zero for  $\theta = 0$ .

If the velocity and acceleration are perpendicular to one another,

$$dI = \frac{e^2 w^2}{16\pi^2} \left[ \frac{1}{(1 - v \cos \theta)^4} - \frac{(1 - v^2) \sin^2 \theta \cos^2 \phi}{(1 - v \cos \theta)^6} \right]$$

where  $\theta$  is again the angle between  $\mathbf{v}$  and  $\mathbf{n}$ , and  $\phi$  is the azimuthal angle of the vector  $\mathbf{n}$  relative to the plane passing through  $\mathbf{v}$  and  $\mathbf{w}$ .

The discussion of synchrotron radiation (magnetic bremsstrahlung) can be found in section 74 of *The classical theory of fields* (L.D.Landau & E.M.Lifshitz).

## 7.6 The interaction between charged particles and EM field

### 7.6.1 Radiation reaction

If a charged particle accelerates, it radiates away energy. This means that if an external force is applied to a charge, not all of the energy transferred to the charge by the force is converted to the kinetic energy of the charge; some of the energy is radiated away in the form of electromagnetic waves. From Newton's law  $\mathbf{F} = m\mathbf{a}$ , the net force on the charge must be less than the applied external force. In effect, the fields surrounding the charge exert a recoil or reaction force on the charge.

The fields of a moving point charge are

$$\mathbf{E} = \frac{e}{4\pi(\mathbf{R}^* \cdot \mathbf{u}^*)^3} \left\{ (1 - v^{*2}) \mathbf{R}^* \mathbf{u} + \mathbf{R}^* \times [\mathbf{R}^* \mathbf{u}^* \times \mathbf{a}^*] \right\} \quad \mathbf{B} = \frac{\mathbf{R}^*}{R^*} \times \mathbf{E}$$



Here,  $\mathbf{u}^* = \frac{\mathbf{R}^*}{R^*} - \mathbf{v}^*$ . It is only those terms that go as  $\frac{1}{R}$  that contribute to radiation energy. The other term falls off as  $\frac{1}{R^2}$  so contributes nothing to the integral of the Poynting vector over a large sphere. This term is called the velocity field and, although it doesn't contribute to the energy radiated away by the EM field, it does store energy, so some of the energy imparted by the force that gets the charge moving must be siphoned off to create these velocity fields. These velocity fields are curious beasts, however, for they contain energy that is never actually lost to the charge. If a charge is accelerated to some velocity, the velocity fields are constructed around the moving charge, but if the charge is then decelerated to rest again, the velocity fields disappear without having radiated away any energy. It would seem to be reabsorbed by the charge as it slows down.

If we look at a charged particle that starts off in some state, then goes through an acceleration followed by a deceleration, and finally ends up in the same state that it started from, what we can say is that the velocity fields are the same at the end as they were at the start, so over this period, the only energy that is truly lost from the particle is the energy that is radiated away. In non-relativistic case,

$$\int_{t_1}^{t_2} \mathbf{F}_{rad} \cdot \mathbf{v} dt = - \int_{t_1}^{t_2} P dt = - \frac{e^2}{6\pi} \int_{t_1}^{t_2} a^2 dt$$

Since

$$\int_{t_1}^{t_2} a^2 dt = \int_{t_1}^{t_2} \dot{\mathbf{v}} \cdot \dot{\mathbf{v}} dt = \mathbf{v} \cdot \dot{\mathbf{v}} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathbf{v} \cdot \ddot{\mathbf{v}} dt = - \int_{t_1}^{t_2} \mathbf{v} \cdot \ddot{\mathbf{v}} dt$$

So, we have

$$\mathbf{F}_{rad} = \frac{e^2}{6\pi} \cdot \mathbf{a}$$

This is known as the Abraham-Lorentz formula for radiation reaction. This equation can only be applied when the frequency and intensity of the EM field is not very big, i.e.

$$\lambda \gg \frac{e^2}{m} \quad H \ll \frac{m^2}{e^3}$$

The details can be found in section 75 of *The classical theory of fields* (L.D.Landau & E.M.Lifshitz).

We derive the relativistic expression for the radiation damping for a single charge, which is applicable also to motion with velocity comparable to that of light. This force is now a four-vector  $g^\mu$ , which must be included in the equation of motion of the charge, written in four-dimensional form:

$$m \frac{du^\mu}{d\tau} = e F^{\mu\nu} u_\nu + g^\mu$$

To determine  $g^\mu$  we note that for  $v \ll 1$ , its three space components must go over into the components of the vector  $\frac{e^2}{6\pi} \cdot \mathbf{a}$ . It is easy to see that the vector  $\frac{e^2}{6\pi} \frac{d^2 u^\mu}{d\tau^2}$  has this property. However, it does not satisfy the identity  $g^\mu u_\mu = 0$ , which is valid for any force four-vector. In order to satisfy this condition, we must add to the expression given a certain auxiliary four-vector, made up from the four-velocity  $u^\mu$  and its derivatives. The three space components of this vector must become zero in the limiting case  $v = 0$ . As a result we find

$$g^\mu = \frac{e^2}{6\pi} \left( \frac{d^2 u^\mu}{d\tau^2} - u^\mu u^\nu \frac{d^2 u^\nu}{d\tau^2} \right)$$





It is called Abraham–Lorentz–Dirac force.

The integral of the four-force  $g^\mu$  over the world line of the motion of a charge, passing through a given field, must coincide (except for opposite sign) with the total four-momentum  $\Delta P^\mu$  of the radiation from the charge. The first term in equation above goes to zero on performing the integration, since at infinity the particle has no acceleration. We integrate the second term by parts and get:

$$-\int g^\mu d\tau = -\frac{e^2}{6\pi} \int \frac{du^\nu}{d\tau} \frac{du_\nu}{d\tau} u^\mu d\tau = \Delta P^\mu$$

### 7.6.2 Scattering by free charges

If an electromagnetic wave falls on a system of charges, then under its action the charges are set in motion. This motion in turn produces radiation in all directions; there occurs, we say, a scattering of the original wave. The scattering is most conveniently characterized by the ratio of the amount of energy emitted by the scattering system in a given direction per unit time, to the energy flux density of the incident radiation. This ratio clearly has dimensions of area, and is called the reflective scattering cross section. Let  $dI$  be the energy radiated by the system into solid angle  $do$  per second for an incident wave with Poynting vector  $\mathbf{S}$ . Then the effective cross-section for scattering (into the solid angle  $do$ ) is

$$d\sigma = \frac{d\bar{I}}{\bar{S}}$$

(the dash over a symbol means a time average). The integral  $\sigma$  of  $d\sigma$  over all directions is the total scattering cross-section.

Let us consider the scattering produced by a free charge at rest. Suppose there is incident on this charge an approximately plane monochromatic wave (partially polarized light). We shall assume that the velocity acquired by the charge under the influence of the incident wave is small compared with the velocity of light. Then we can neglect the force exerted by magnetic field. We also assume the wavelength of the EM field is much larger than the displacement of the charge during its vibrations. So, we have

$$m\ddot{\mathbf{r}} = e\mathbf{E}_0(t)e^{-i\omega t}$$

and

$$\ddot{\mathbf{d}} = \frac{e^2}{m}\mathbf{E}_0(t)e^{-i\omega t}$$

Now we assume the incident direction of the EM wave is  $\hat{x}$ , the scattered direction of the EM wave is  $\mathbf{n}' = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ . The dipole radiation is

$$\bar{dI} = \frac{e^4}{16\pi^2 m^2} \overline{(\mathbf{E} \times \mathbf{n}')^2} do$$

Since

$$(\mathbf{E} \times \mathbf{n}')^2 = -2 E_y E_z \cos(\theta) \sin(\phi) \sin(\theta) - (E_y^2 \sin^2(\phi) - E_z^2) \sin^2(\theta) + E_y^2$$

we have

$$\bar{dI} = \frac{e^4 \bar{S}}{16\pi^2 m^2} [-(\rho_{12} + \rho_{21}) \cos(\theta) \sin(\phi) \sin(\theta) - (\rho_{11} \sin^2(\phi) - \rho_{22}) \sin^2(\theta) + \rho_{11}]$$



Here

$$\bar{S} = \overline{\text{Re}(\mathbf{E}) \cdot \text{Re}(\mathbf{E})} = \frac{1}{2} \text{Re} \overline{\mathbf{E}_0 \cdot \mathbf{E}_0^*}$$

The effective cross-section for scattering is

$$d\sigma = \frac{e^4}{16\pi^2 m^2} [-(\rho_{12} + \rho_{21}) \cos(\theta) \sin(\phi) \sin(\theta) - (\rho_{11} \sin(\phi)^2 - \rho_{22}) \sin(\theta)^2 + \rho_{11}] d\phi$$

The total cross section is

$$\sigma = \frac{8\pi}{3} \left( \frac{e^2}{4\pi m} \right)^2$$

If the incident light is totally linear polarized in  $\hat{z}$  direction, then we have

$$d\sigma = \frac{e^4}{16\pi^2 m^2} \sin^2 \theta d\phi$$

If the incident light is unpolarized, we have

$$d\sigma = \frac{e^4}{32\pi^2 m^2} (1 + \cos^2 \Theta) d\phi$$

Here,  $\cos \Theta = \cos \phi \sin \theta$ ,  $\Theta$  is the angle between the direction of incident light and scattered light.

### Scattering by bound charges

The dynamic equation of the bound charges are

$$\ddot{\xi} = \frac{e}{m} \mathbf{E}_0 e^{-i\omega t} - \omega_0^2 \xi + \frac{e^2}{6\pi m} \ddot{\xi}$$

Suppose  $\xi = \xi_0 e^{-i\omega t}$ , we can get

$$\xi = \frac{e \mathbf{E}_0}{m(\omega_0^2 - \omega^2 - i\omega\gamma)} e^{-i\omega t}$$

Here,

$$\gamma = \frac{e^2 \omega^2}{6\pi m}$$

We can show that

$$\sigma = \sigma_0 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}$$

Here,  $\sigma_0$  is the total cross section when EM wave is scattered by free charges. When  $\omega \gg \omega_0$ ,  $\sigma$  is independent of  $\omega$ , i.e.  $\sigma \approx \sigma_0$ . When  $\omega \ll \omega_0$ ,  $\sigma_0$  is proportional to  $\omega^4$ , i.e.  $\sigma = \sigma_0 \left( \frac{\omega}{\omega_0} \right)^4$ , and it is called Rayleigh scattering.



## **Part III**

### **General relativity**

# Chapter 8

## Elementary Differential Geometry



### 8.1 Fundamental conception on differential manifolds

#### Definition 8.1 Manifold

**Manifold** Formally, a topological manifold is a second countable Hausdorff space that is locally homeomorphic to Euclidean space.

**Differentiable manifold** In formal terms, a differentiable manifold is a topological manifold with a globally defined differential structure.

**Tangent space** In mathematics, the tangent space of a manifold facilitates the generalization of vectors from affine spaces to general manifolds, since in the latter case one cannot simply subtract two points to obtain a vector pointing from one to the other.

**Cotangent space** Typically, the cotangent space is defined as the dual space of the tangent space at  $x$ .

#### Definition 8.2 Submanifold

##### Submanifold

**Immersed submanifolds** An immersed submanifold of a manifold  $M$  is the image  $S$  of an immersion map  $f : N \rightarrow M$ ; in general this image will not be a submanifold as a subset, and an immersion map need not even be injective (one-to-one) – it can have self-intersections.

**Injective immersion submanifolds** More narrowly, one can require that the map  $f : N \rightarrow M$  be an inclusion (one-to-one), in which we call it an injective immersion, and define an immersed submanifold to be the image subset  $S$  together with a topology and differential structure such that  $S$  is a manifold and the inclusion  $f$  is a diffeomorphism: this is just the topology on  $N$ , which in general will not agree with the subset topology: in general the subset  $S$  is not a submanifold of  $M$ , in the subset topology.

**Open submanifolds**

**Closed submanifolds**

**Definition 8.3 Embedded Submanifold**

An embedded submanifold (also called a regular submanifold), is an immersed submanifold for which the inclusion map is a topological embedding. That is, the submanifold topology on  $S$  is the same as the subspace topology. Given any embedding  $f : N \rightarrow M$  of a manifold  $N$  in  $M$  the image  $f(N)$  naturally has the structure of an embedded submanifold. That is, embedded submanifolds are precisely the images of embeddings.

**Proposition 8.1**

If an  $n$  dimensional injective immersed submanifold  $N$  of a  $m$  dimensional manifold  $M$  is a closed submanifold of an open submanifold of  $M$ , then for every point  $p \in f(N)$  there exists a chart  $(U \subset M, \phi : U \rightarrow \mathbb{R}^m)$  containing  $p$  such that  $\phi(f(N) \cap U)$  is the intersection of a  $n$ -dimensional plane with  $\phi(U)$ .

Closed submanifolds of an open submanifold are equal to embedded submanifolds.

## 8.2 Multi linear algebra

**Definition 8.4 Tensor**

**Vector space**

**Dual space**

In mathematics, any vector space  $V$  has a corresponding dual vector space (or just dual space for short) consisting of all linear functionals on  $V$  together with a naturally induced linear structure.

**Tensor product**

$$V \otimes W = \text{Span}\{v \otimes w\} = \mathcal{L}(V^*, W^*; F)$$

$$V^* \otimes W^* = \text{Span}\{v^* \otimes w^*\} = \mathcal{L}(V, W; F)$$

$$\mathcal{L}(V, W; Z) = \mathcal{L}(V \otimes W; Z)$$

$$(\phi \otimes \psi) \otimes \xi = \phi \otimes (\psi \otimes \xi)$$

**Tensor**

$$V_s^r = V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$$

$$x = x^{i_1 \cdots i_r}_{k_1 \cdots k_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{*k_1} \otimes \cdots \otimes e^{*k_s}$$

$$(x \otimes y)^{i_1 \cdots i_{r_1+r_2}}_{k_1 \cdots k_{s_1+s_2}} = x^{i_1 \cdots i_{r_1}}_{k_1 \cdots k_{s_1}} \cdot y^{i_{r_1+1} \cdots i_{r_1+r_2}}_{k_{s_1+1} \cdots k_{s_1+s_2}}$$



**Definition 8.5 Symmetric and antisymmetric tensor****Permutation** ( $\sigma \in \mathcal{P}(r)$ )

$$\sigma x(v^{*1}, \dots, v^{*r}) = x(v^{*\sigma(1)}, \dots, v^{*\sigma(r)})$$

**Symmetric contra-variant tensor**

$$\sigma x = x$$

**Antisymmetric contra-variant tensor**

$$\sigma x = \text{sgn } \sigma \cdot x$$

**Symmetrization operator**

$$S_r(x) = \frac{1}{r!} \sum_{\sigma \in \mathcal{P}(r)} \sigma x$$

**Antisymmetrization operator**

$$A_r(x) = \frac{1}{r!} \sum_{\sigma \in \mathcal{P}(r)} \text{sgn } \sigma \cdot \sigma x$$

**Definition 8.6 Exterior vector space****Exterior vector space**

$$\Lambda^r(V) = A_r(T^r(V))$$

$$\Lambda^0(V) = F \quad \Lambda^1(V) = V$$

**Wedge product**

$$\xi \wedge \eta \equiv \frac{(k+l)!}{k!l!} A_{k+l}(\xi \otimes \eta)$$

**Pull-back mapping**  $f : V \rightarrow W$  is a linear mapping, we define  $f^* : \Lambda^r(W^*) \rightarrow \Lambda^r(V^*)$  as

$$f^* \phi(v_1, \dots, v_r) = \phi(f(v_1), \dots, f(v_r)).$$

**Proposition 8.2 Properties of Wedge product**

$$(\xi_1 + \xi_2) \wedge \eta = \xi_1 \wedge \eta + \xi_2 \wedge \eta$$

$$\xi \wedge (\eta_1 + \eta_2) = \xi \wedge \eta_1 + \xi \wedge \eta_2$$

$$\xi \wedge \eta = (-1)^{kl} \eta \wedge \xi$$

$$(\xi \wedge \eta) \wedge \zeta = \xi \wedge (\eta \wedge \zeta) = \frac{(k+l+h)!}{k!l!h!} A_{k+l+h}(\xi \otimes \eta \otimes \zeta)$$

$$f^*(\phi \wedge \psi) = f^* \phi \wedge f^* \psi$$



**Proposition 8.3 Properties of exterior space**

$$\begin{aligned}
e_{i_1} \wedge \cdots \wedge e_{i_r}(v^{*1}, \dots, v^{*r}) &= \det\langle e_{i_\alpha}, v^{*\beta} \rangle \\
e_{i_1} \wedge \cdots \wedge e_{i_r}(e^{*j_1}, \dots, e^{*j_r}) &= \det\langle e_{i_\alpha}, e^{*j_\beta} \rangle = \delta_{i_1 \dots i_r}^{j_1 \dots j_r} \\
\Lambda^r(V) &= \text{Span} \{e_{i_1} \wedge \cdots \wedge e_{i_r}, 1 \leq i_1 < \cdots < i_r \leq n\} \\
(\Lambda^r(V))^* &= \Lambda^r(V^*)
\end{aligned}$$



## 8.3 Vector Bundle

**Definition 8.7 Fiber bundle**

**Fiber bundle** In mathematics, and particularly topology, a fiber bundle is a space that is locally a product space, but globally may have a different topological structure. Specifically, the similarity between a space  $E$  and a product space  $B \times F$  is defined using a continuous surjective map  $\pi : E \rightarrow B$  that in small regions of  $E$  behaves just like a projection from corresponding regions of  $B \times F$  to  $B$ . The map  $\pi$ , called the projection or submersion of the bundle, is regarded as part of the structure of the bundle. The space  $E$  is known as the total space of the fiber bundle,  $B$  as the base space, and  $F$  the fiber.

**Vector Bundle** In mathematics, a vector bundle is a topological construction that makes precise the idea of a family of vector spaces parameterized by another space  $X$  (for example  $X$  could be a topological space, a manifold, or an algebraic variety): to every point  $x$  of the space  $X$  we associate (or "attach") a vector space  $V(x)$  in such a way that these vector spaces fit together to form another space of the same kind as  $X$  (e.g. a topological space, manifold, or algebraic variety), which is then called a vector bundle over  $X$ .

**Tangent bundle** In differential geometry, the tangent bundle of a differentiable manifold  $M$  is a manifold  $TM$ , which assembles all the tangent vectors in  $M$ . As a set, it is given by the disjoint union of the tangent spaces of  $M$ . That is,

$$TM = \bigsqcup_{x \in M} T_x M = \bigcup_{x \in M} \{x\} \times T_x M = \bigcup_{x \in M} \{(x, y) | y \in T_x M\}$$

where  $T_x M$  denotes the tangent space to  $M$  at the point  $x$ . So, an element of  $TM$  can be thought of as a pair  $(x, v)$ , where  $x$  is a point in  $M$  and  $v$  is a tangent vector to  $M$  at  $x$ . There is a natural projection  $\pi : TM \rightarrow M$  defined by  $\pi(x, v) = x$ . This projection maps each tangent space  $T_x M$  to the single point  $x$ . A section of  $TM$  is a vector field on  $M$ , and the dual bundle to  $TM$  is the cotangent bundle, which is the disjoint union of the cotangent spaces of  $M$ .


**Cotangent bundle**  $T^*M = \bigcup_{x \in M} T_x^* M$

**Tensor bundle**  $T_s^r M = \bigcup_{x \in M} T_{sx}^r M$



## 8.4 Tangent vector field


### Theorem 8.1

Let  $M$  be a smooth manifold, and let  $Y : M \rightarrow TM$  be a vector field. If  $(U, (X_i))$  is any smooth coordinate chart on  $M$ , then  $Y$  is smooth on  $U$  if and only if its component functions with respect to this chart are smooth. 


### Theorem 8.2

Let  $M$  be a  $m$  dimensional smooth manifold and  $v$  a smooth tangent vector field on  $M$ .  $v : C^\infty(M) \rightarrow C^\infty$  satisfy that


- (1)  $\forall f, g \in C^\infty(M), v(f + g) = v(f) + v(g)$ ;
- (2)  $\forall f \in C^\infty(M), \alpha \in \mathbf{R}, v(\alpha f) = \alpha \cdot v(f)$ ;
- (3)  $\forall f, g \in C^\infty(M), v(fg) = f \cdot v(g) + g \cdot v(f)$ .

If  $\alpha : C^\infty(M) \rightarrow C^\infty(M)$  satisfy the three conditions above, there exists a unique smooth vector field  $v$  on  $M$  that  $\forall f \in C^\infty(M), v(f) = \alpha(f)$ . 

### Theorem 8.3

$\forall X, Y \in \mathcal{H}(M), [X, Y] = X \circ Y - Y \circ X \in \mathcal{H}(M)$ . 


### Proposition 8.4

- (1)  $[aX + bY, Z] = a[X, Z] + b[Y, Z]; [Z, aX + bY] = a[Z, X] + b[Z, Y]$ ;
  - (2)  $[X, Y] = -[Y, X]$ ;
  - (3)  $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ ;
  - (4)  $[X, Y]|_U = [X|_U, Y|_U] = (X_i \frac{\partial Y^j}{\partial u^i} - Y^i \frac{\partial X^j}{\partial u^i}) \frac{\partial}{\partial u^j}$ ;
  - (5)  $f_*[X, Y] = [f_*X, f_*Y]$ ;
- 

### Definition 8.8 One parameter differentiable transformation group

Let  $M$  be a smooth manifold and  $\phi : \mathbf{R} \times M \rightarrow M$  a smooth mapping, and  $\forall (t, p) \in \mathbf{R} \times M$ , denote  $\phi_t(p) = \phi(t, p)$ . If  $\phi$  satisfy that

- (1)  $\phi_0 = \text{id} : M \rightarrow M$ ;
- (2)  $\forall s, t \in \mathbf{R}, \phi_s \circ \phi_t = \phi_{s+t}$ ;

then  $\phi$  is called a one parameter differentiable transformation group acting on  $M$ . 


Trajectory of  $\phi$  through  $p$  on  $M$ :  $\gamma_p(t) = \phi(t, p)$ .

Vector field induced by  $\phi$ :  $X_p(f) = \langle \gamma_p, f \rangle$ .






**Proposition 8.5**


- (1)  $\gamma_q(t) = \phi(t, \phi(s, p)) = \phi(t + s, p) = \gamma_p(t + s)$ ;  
 (2)  $(\phi_s)_* X_p = X_{\phi_s(p)}$ ;  
 (3)  $\psi_* X_p = \tilde{X}_{\psi(p)}$  if  $X$  is induced by  $\phi$  and  $\tilde{X}$  is induced by  $\psi \circ \phi \circ \psi^{-1}$ .  $\psi$  is a smooth homeomorphism.   
 (4)  $[X, Y] = \lim_{t \rightarrow 0} \frac{Y_p - (\phi_t)_* Y_{\phi_t(p)}}{t} = \lim_{t \rightarrow 0} \frac{(\phi_{-t})_* Y_{\phi_t(p)} - Y}{t}$  if  $X$  is induced by  $\phi$ .

**Definition 8.9 Lie derivative**

$$\mathcal{L}_X Y \equiv \lim_{t \rightarrow 0} \frac{(\phi_{-t})_* Y_{\phi_t(p)} - Y}{t} = [X, Y]$$


$$\mathcal{L}_X f \equiv X(f)$$



**Proposition 8.6**

$$\begin{aligned}\mathcal{L}_X(Y + \lambda Z) &= \mathcal{L}_X Y + \lambda \mathcal{L}_X Z \\ \mathcal{L}_X(f \cdot Y) &= \mathcal{L}_X(f) \cdot Y + f \mathcal{L}_X Y \\ \mathcal{L}_X([Y, Z]) &= [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z]\end{aligned}$$


**Theorem 8.4**

Let  $M$  be a  $n$ -dimensional smooth manifold and  $X \in \mathcal{H}(M)$ . If  $p \in M$  and  $X_p \neq 0$ ,  $\exists(V, x^i)$  and  $p \in V$  that  $X|_V = \frac{\partial}{\partial y^1}$


**Definition 8.10 Distribution**

**Distribution** Let  $M$  be a  $C^\infty$  manifold of dimension  $m$ , and let  $n \leq m$ . Suppose that for each  $x \in M$ , we assign an  $n$ -dimensional subspace  $\Delta_x \subset T_x(M)$  of the tangent space in such a way that for a neighbourhood  $N_x \subset M$  of  $x$  there exist  $n$  linearly independent smooth vector fields  $X_1, \dots, X_n$  such that for any point  $y \in N_x$ ,  $X_1(y), \dots, X_n(y)$  span  $\Delta_y$ . We let  $\Delta$  refer to the collection of all the  $\Delta_x$  for all  $x \in M$  and we then call  $\Delta$  a distribution of dimension  $n$  on  $M$ , or sometimes a  $C^\infty$   $n$ -plane distribution on  $M$ . The set of smooth vector fields  $\{X_1, \dots, X_n\}$  is called a local basis of  $\Delta$ . 



### Definition 8.11 Involutive distributions

We say that a distribution  $\Delta$  on  $M$  is involutive if for every point  $x \in M$  there exists a local basis  $\{X_1, \dots, X_n\}$  of the distribution in a neighbourhood of  $x$  such that for all  $1 \leq i, j \leq n$ ,  $[X_i, X_j]$  is in the span of  $\{X_1, \dots, X_n\}$ . That is, if  $[X_i, X_j]$  is a linear combination of  $\{X_1, \dots, X_n\}$ . Normally this is written as  $[\Delta, \Delta] \subset \Delta$ .

Involutive distributions are the tangent spaces to foliations. Involutive distributions are important in that they satisfy the conditions of the Frobenius theorem, and thus lead to integrable systems. A related idea occurs in Hamiltonian mechanics: two functions  $f$  and  $g$  on a symplectic manifold are said to be in mutual involution if their Poisson bracket vanishes.

### Theorem 8.5 Frobenius Theorem

If distribution  $\Delta$  on  $M$  is involutive, then  $\forall p \in M, \exists (V, x^i)$  and  $p \in V$  that  $\Delta|_V = \text{Span}\left\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^h}\right\}$ .

### Definition 8.12 Integrable manifold

Let  $L^h$  be a smooth distribution on  $M$ . If  $\phi : N \rightarrow M$  is an injective immersion manifold, and  $\forall p \in N, \phi_*(T_p N) \subset L^h(\phi(p))$ , then  $(\phi, N)$  is called an integrable manifold of  $L^h$ . If  $\forall q \in M$ , there is an integrable manifold of  $L^h$  through it, we say that  $L^h$  is completely integrable.

### Theorem 8.6

Let

$$\tau : \underbrace{A^1(M) \times \dots \times A^1(M)}_p \times \underbrace{\mathcal{H}(M) \times \dots \times \mathcal{H}(M)}_q \rightarrow c^\infty(M)$$

be a  $p + q$  multi-linear mapping, if  $\forall 1 \leq a \leq p, 1 \leq b \leq q$  and  $\mu \in C^\infty(M)$ ,

$$\begin{aligned} & \tau(\alpha^1, \dots, \mu\alpha^a, \dots, \alpha^p, v_1, \dots, v_q) \\ &= \tau(\alpha^1, \dots, \alpha^p, v_1, \dots, \mu v_b, \dots, v_q) \\ &= \mu \cdot \tau(\alpha^1, \dots, \alpha^p, v_1, \dots, v_q) \end{aligned}$$

then the mapping  $\tau$  define a  $(p, q)$  tensor for all  $x \in M$  smoothly.



**Definition 8.13 Lie derivatives**

Let  $X$  be a smooth tangent vector field on  $M$  and  $\phi_t$  the one parameter differentiable transformation group inducing it. Denote the trajectory of  $\phi_t$  through  $x$  by  $\gamma_x(t)$ . So we have linear isomorphism

$$(\phi_t^{-1})_* = (\phi_{-t})_* : T_{\gamma_x(t)}M \rightarrow T_xM$$

$$(\phi_t)^* : T_{\gamma_x(t)}^* \rightarrow T_x^*M$$

So we can induce the linear isomorphism

$$\Phi_t : T_q^p(\gamma_x(t)) \rightarrow T_q^p(x)$$

If  $S$  and  $T$  are smooth tensor fields on  $M$ ,

(1) for all  $t$  which is small enough,  $\Phi_t S$  is a smooth tensor field on  $M$  which has the same type as  $S$ , and  $\lim_{t \rightarrow 0} \Phi_t(S(\gamma_p(t))) = S(p)$ ,  $\forall p \in M$ .

(2)  $\Phi_t(S \otimes T) = \Phi_t S \otimes \Phi_t T$ .

(3)  $\Phi_t(C_b^a(S)) = C_b^a(\Phi_t(S))$ ,  $C_b^a$  is a tag for contraction.

So, we can define the Lie derivative for smooth tensor field  $\tau$  on  $M$  as

$$\mathcal{L}_X(\tau) = \lim_{t \rightarrow 0} \frac{\Phi_t(\tau) - \tau}{t}$$

**Proposition 8.7**

$$\mathcal{L}_X(\tau_1 + \lambda\tau_2) = \mathcal{L}_X\tau_1 + \lambda\mathcal{L}_X\tau_2$$

$$\mathcal{L}_X(\tau_1 \otimes \tau_2) = \mathcal{L}_X\tau_1 \otimes \tau_2 + \tau_1 \otimes \mathcal{L}_X\tau_2$$

$$C_s^r(\mathcal{L}_X\tau) = \mathcal{L}_X(C_s^r(\tau))$$

$$(\mathcal{L}_X\omega)(Y) = X(\omega(Y)) - \omega([X, Y])$$

$$\mathcal{L}_{[X, Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X$$

$$\mathcal{L}_{X+Y} = \mathcal{L}_X + \mathcal{L}_Y$$

**Proposition 8.8**

$$((\mathcal{L}_X\tau)|_U)^{\mu_1, \dots, \mu_p}_{v_1, \dots, v_q} = X^\alpha \partial_\alpha \tau_{v_1, \dots, v_q}^{\mu_1, \dots, \mu_p} - \sum_{i=1}^p \tau_{v_1, \dots, v_q}^{\mu_1, \dots, \alpha, \dots, \mu_p} \partial_\alpha X^{\mu_i} + \sum_{j=1}^q \tau_{v_1, \dots, \alpha, \dots, v_q}^{\mu_1, \dots, \mu_p} \partial_{v_j} X^\alpha$$



## 8.5 Exterior differential

### Definition 8.14 Exterior form space

$$A(M) = \sum_{r=0}^m A^r(M)$$

For  $\tau \in A^r(M)$ ,

$$\tau|_U = \frac{1}{r!} \tau_{i_1 \dots i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r} = \tau_{|i_1 \dots i_r|} dx^{i_1} \wedge \dots \wedge dx^{i_r}$$

$$\tau_{i_1 \dots i_r} = \tau\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}}\right)$$

$$\begin{aligned} \tau(v_1, \dots, v_r)|_U &= \tau_{|i_1 \dots i_r|} dx^{i_1} \wedge \dots \wedge dx^{i_r}(v_1, \dots, v_r) \\ &= \tau_{|i_1 \dots i_r|} \begin{vmatrix} v_1^{i_1} & \dots & v_r^{i_1} \\ \vdots & & \vdots \\ v_1^{i_r} & \dots & v_r^{i_r} \end{vmatrix} \end{aligned}$$

It is a  $r$  multi-linear mapping, and for every variable, it is  $C^\infty(M)$  linear.

### Proposition 8.9 Pullback mapping

$$f : M \rightarrow N \Rightarrow f_* : T_p M \rightarrow T_{f(p)} N \Rightarrow f^* : \wedge^r(T_{f(p)}^* N) \rightarrow \wedge^r(T_p^* M)$$

$$f^* \phi(v_1, \dots, v_r) = \phi(f_* v_1, \dots, f_* v_r)$$

$$f^* \phi|_U = \frac{1}{r!} (\phi_{\alpha_1 \dots \alpha_r} \circ f) \cdot \frac{\partial f^{\alpha_1}}{\partial x^{i_1}} \dots \frac{\partial f^{\alpha_r}}{\partial x^{i_r}} dx^{i_1} \wedge \dots \wedge dx^{i_r}$$

$$f^*(\phi \wedge \psi) = f^* \phi \wedge f^* \psi$$

### Definition 8.15 Exterior differential

Let  $M$  be a  $m$ -dimensional smooth manifold. Then  $\exists$  a unique mapping  $d : A(M) \rightarrow A(M)$  satisfy that

$$(1) d(A^r(M)) \subset A^{r+1}(M)$$

$$(2) \forall \omega_1, \omega_2 \in A(M), d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$$

$$(3) \text{ if } \omega_1 \in A^r(M), \text{ then } d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge d\omega_2$$

$$(4) f \in A^0(M), df \text{ is just the differential of } f$$

$$(5) \forall f \in A^0(M), d(df) = 0$$

$d$  is called exterior differential.



**Theorem 8.7**

$\forall \omega \in A^1(M), X, Y \in \mathcal{H}(M),$

$$d\omega(X, Y) = X\langle Y, \omega \rangle - Y\langle X, \omega \rangle - \langle [X, Y], \omega \rangle$$

$\forall \omega \in A^r(M), X_1, \dots, X_{r+1} \in \mathcal{H}(M),$

$$\begin{aligned} d\omega(X_1, \dots, X_{r+1}) &= \sum_{i=1}^{r+1} (-1)^{i+1} X_i(\langle X_1 \wedge \dots \wedge \hat{X}_i \wedge \dots \wedge X_{r+1}, \omega \rangle) \\ &+ \sum_{1 \leq i < j \leq r+1} (-1)^{i+j} \langle [X_i, X_j] \wedge \dots \wedge \hat{X}_i \wedge \dots \wedge \hat{X}_j \wedge \dots \wedge X_{r+1}, \omega \rangle \end{aligned}$$

**Theorem 8.8**

$$f^*(d\omega) = d(f^*\omega)$$

**Lemma 1 Poincare Lemma**

1.  $d^2 = 0$
2. Let  $U = B_0(r)$  be a spherical neighbourhood with center origin  $O$  and radius  $r$  in  $R^n$ .  $\forall \omega \in A^r(U)$  and  $d\omega = 0$ ,  $\exists \tau \in A^{r-1}(U)$ , satisfy that  $\omega = d\tau$ .

**Definition 8.16 Pfaff equations**

Let  $\omega^\alpha (1 \leq \alpha \leq r) \in A^1(U)$  and  $U$  is an open set of  $m$ -dimensional smooth manifold  $M$ . Differential equation set  $\omega^\alpha = 0$  is called Pfaff equations.

**Definition 8.17 Integral manifold of Pfaff equations**

If there is an injective immersion submanifold  $\phi : N \rightarrow U$  satisfying that  $\phi^*\omega^\alpha = 0$ ,  $(\phi, N)$  is called an integral manifold of Pfaff equation set.



**Proposition 8.10 Partial differential equations and Pfaff equations**

There is a set of first order partial differential equations

$$\frac{\partial y^\alpha}{\partial x^i} = f_i^\alpha(x^1, \dots, x^m, y^1, \dots, y^n) \quad (1 \leq i \leq m, 1 \leq \alpha \leq n)$$

$f_i^\alpha(x, y)$  is a smooth function on the open set  $U \times V \subset \mathbb{R}^m \times \mathbb{R}^n$ . The equations sets can be written as Pfaff equations on  $U \times V$

$$\omega^\alpha \equiv dy^\alpha - f_i^\alpha(x, y)dx^i = 0$$

If the partial differential equations have solution

$$y^\alpha = g^\alpha(x^1, \dots, x^m)$$

then the submanifold  $\phi : U \rightarrow U \times V$ ,

$$\phi(x^1, \dots, x^m) = (x^1, \dots, x^m, g^1(x), \dots, g^n(x))$$

is an integral manifold of the Pfaff equations, i.e.  $\phi^*\omega^\alpha = 0$

**Proposition 8.11 Distribution and Pfaff equations**

Pfaff equations  $\omega^\alpha = 0$  on open set  $V \in M$  with rank  $r$  is equivalent to a  $h = m - r$  dimensional smooth distribution locally.

$$\Delta^h(p) = \{v \in T_p M : \omega^\alpha(v) = 0, 1 \leq \alpha \leq r\}$$

If  $\phi : N \rightarrow V$  is an integral manifold of  $\omega^\alpha, \forall X \in T_p N, \omega^\alpha(\phi_* X) = \phi^*\omega_\alpha(X) = 0$ . So  $\phi_* X \in \Delta^h(p)$ , and so  $\phi : N \rightarrow V$  is an integral manifold of  $\Delta^h$ .

**Definition 8.18 Completely integrable**

Suppose  $\omega^\alpha$  is a set of  $r$  linearly independent 1 forms defined on an open set  $U \subset M$ . If  $\forall p \in U$ , Pfaff equations

$$\omega^\alpha = 0 \quad (1 \leq \alpha \leq r)$$

has an  $h = \dim M - r$  dimensional integral manifold  $\phi : N \rightarrow V$  such that  $p \in V$ , Pfaff equations are called completely integrable.



**Definition 8.19 Frobenius condition**

Frobenius condition for Pfaff equations  $\omega^\alpha = 0 (1 \leq \alpha \leq r)$  is that

$$d\omega^\alpha \equiv 0 \pmod{(\omega^1, \dots, \omega^r)}$$

**Theorem 8.9 Frobenius theorem**

Pfaff equations satisfying Frobenius condition is completely integrable.

**Definition 8.20 Orientation of manifold**

Let  $\alpha : [0, 1] \rightarrow M$  be a path on  $M$ .  $\forall t \in [0, 1]$ , assign an orientation for  $T_{\alpha(t)}M$ , denoted by  $\mu_t$ . If for  $t_0 \in [0, 1]$ , there is a local coordinate  $(U; x_i)$  of  $\alpha(t_0)$  and a neighbourhood  $[t_0 - \delta_1, t_0 + \delta_2]$  of  $t_0$  that

$$\alpha([t_0 - \delta_1, t_0 + \delta_2]) \subset U$$

and

$$\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m} \right\} |_{\alpha(t)} \in \mu_t, \forall t \in [t_0 - \delta_1, t_0 + \delta_2],$$

$\mu$  is called a continuous topological orientation of  $\alpha$

**Definition 8.21 The propagation of orientation**

Let  $p, q \in M$  and  $\alpha : [0, 1] \rightarrow M$  a path connecting  $p, q$ . Assign an orientation  $\lambda$  of  $T_p M$ . If there is a continuous topological orientation of  $\alpha$   $\mu$  satisfying that  $\mu_0 = \lambda$ , then orientation  $\mu_1$  of  $T_q M$  is called the propagation of orientation  $\lambda$  along  $\alpha$ . The orientation of  $\mu_1$  is unique.

**Definition 8.22 Orientable manifold**

Let  $M$  be a  $m$  dimensional smooth manifold. If there is an atlases  $(\mathcal{A}, = \{(U_\alpha, \phi_\alpha)\})$ , making that if  $U_\alpha \cap U_\beta \neq \emptyset$ , the Jacobian of

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$$

is positive. Then  $M$  is called orientable manifold.



### Theorem 8.10

Let  $M$  be a orientable connected manifold.  $\forall p \in M$ , assign an orientation  $\lambda$  for  $T_p M$ , then for all point  $q \in M$ , the propagation of  $\lambda$  along an arbitrary path define a unique orientation  $\mu$  for  $T_q M$ .



### Definition 8.23 Manifold with boundary

A topological manifold with boundary is a Hausdorff space in which every point has a neighbourhood homeomorphic to an open subset of Euclidean half-space (for a fixed  $n$ ):

$$\mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$



### Definition 8.24 Boundary and interior

Let  $M$  be a manifold with boundary. The interior of  $M$ , denoted  $\text{Int } M$ , is the set of points in  $M$  which have neighbourhoods homeomorphic to an open subset of  $\mathbb{R}^n$ . The boundary of  $M$ , denoted  $\partial M$ , is the complement of  $\text{Int } M$  in  $M$ . The boundary points can be characterized as those points which land on the boundary hyperplane ( $x^n = 0$ ) of  $\mathbb{R}_+^n$  under some coordinate chart.

If  $M$  is a manifold with boundary of dimension  $n$ , then  $\text{Int } M$  is a manifold (without boundary) of dimension  $n$  and  $\partial M$  is a manifold (without boundary) of dimension  $n - 1$ .



### Theorem 8.11

Let  $M$  be a smooth manifold with boundary and  $\partial M \neq \emptyset$ . The differential structure of  $\partial M$  can be deduced from the  $M$ , making  $\partial M$  a  $m - 1$  dimensional smooth manifold and the inclusion map  $i : \partial M \rightarrow M$  is embedding map. If  $M$  is orientable, then  $\partial M$  is also orientable.



### Definition 8.25 Induced orientation

Let  $M$  be an orientable  $m$  dimensional smooth manifold with boundary and  $\partial M \neq \emptyset$ .  $\mathcal{A}$  is the orientation of  $M$ . For local coordinates  $(U; x^i) \in \mathcal{A}$ , when

$$\tilde{U} = U \cap \partial M = \{(x^1, \dots, x^m) \in U : x^m = 0\} \neq \emptyset$$

assign a local coordinate system  $((-1)^m \cdot x^1, x^2, \dots, x^{m-1})$  on  $\tilde{U}$ . The orientation defined by this local coordinate system is called induced orientation of  $\partial M$ .





**Definition 8.26 Support set**

Let  $M$  be a  $m$  dimensional orientable smooth manifold.  $\omega \in A^r(M)$ , the support set of  $\omega$  can be defined as

$$\text{Supp } \omega = \overline{\{p \in M : \omega(p) \neq 0\}}$$



All the  $r$ -form with compact support set is denoted as  $A_0^r(M)$ .

**Definition 8.27 Partition of unity**

Let  $\Sigma$  be an open cover of  $M$ . Then there is a family of smooth function  $g_\alpha$  on  $M$  that

1.  $\forall \alpha, 0 \leq g_\alpha \leq 1, \text{supp } g_\alpha$  is compact and there is an open set  $W_i \in \Sigma$  that  $\text{supp } g_\alpha \subset W_i$ .
2.  $\forall p \in M$ , it has a neighbourhood  $U$  which intersect finite  $\text{supp } g_\alpha$ .
3.  $\sum_\alpha g_\alpha = 1$

**Definition 8.28 Integral of differential form with compact support**

$$\phi = \left( \sum_\alpha g_\alpha \right) \cdot \phi = \sum_\alpha (g_\alpha \cdot \phi)$$

$$\int_M g_\alpha \cdot \phi = \int_{W_i} g_\alpha \cdot \phi = \int_{W_i} f(u^1, \dots, u^m) du^1 \wedge \dots \wedge du^m = \int_{W_i} f(u^1, \dots, u^m) du^1 \dots$$



$$\int_M \phi = \sum_\alpha \int_M g_\alpha \cdot \phi$$

**Theorem 8.12 Stokes Theorem**

Let  $M$  be an orientable  $m$  dimensional smooth manifold with boundary and  $\omega \in A_0^{m-1}(M)$ , then

$$\int_M d\omega = \int_{\partial M} i^* \omega$$



Here,  $\partial M$  has an orientation induced by  $M$  and  $i$  is embedding mapping.



## 8.6 Connection

### Definition 8.29 Connection

Let  $M$  be a smooth manifold and  $E$  a  $q$  dimensional real vector bundle on  $M$ .  $\Gamma(E)$  is the set of all smooth sections of  $E$  on  $M$ . The connection on  $E$  is a mapping:

$$D : \Gamma(E) \rightarrow \Gamma(T^*(M) \otimes E)$$

it satisfies that

1.  $\forall s_1, s_2 \in \Gamma(E), D(s_1 + s_2) = Ds_1 + Ds_2$
2.  $\forall s \in \Gamma(E)$  and  $\alpha \in C^\infty(M), D(\alpha s) = d\alpha \otimes s + \alpha Ds$

If  $X$  is a smooth tangent vector field on  $M$ ,  $s \in \Gamma(E)$ , then  $D_X s = \langle X, Ds \rangle$ , called absolute derivative of  $s$  along  $X$ .

### Proposition 8.12

Local representation of connection is:

$$Ds_\alpha = \sum_{1 \leq i \leq m, 1 \leq \beta \leq q} \Gamma_{\alpha i}^\beta du^i \otimes s_\beta$$

$$\omega_\alpha^\beta = \sum_{1 \leq i \leq m} \Gamma_{\alpha i}^\beta du^i$$

$$Ds_\alpha = \sum_{\beta=1}^q \omega_\alpha^\beta \otimes s_\beta$$

$$DS = \omega \otimes S$$

Transformation law of connection is:

$$S' = A \cdot S$$

$$\begin{aligned} DS' &= dA \otimes S + A \cdot DS \\ &= (dA + A \cdot \omega) \otimes S \\ &= (dA \cdot A^{-1} + A \cdot \omega \cdot A^{-1}) \otimes S' \\ \omega' &= dA \cdot A^{-1} + A \cdot \omega \cdot A^{-1} \end{aligned}$$



**Theorem 8.13**

For an arbitrary vector bundle, connection always exists.

**Theorem 8.14**

Let  $D$  be a connection of vector bundle  $E$ .  $\forall p \in M$ , there exists a local frame field  $S$  on the neighbourhood of  $p$  that  $\omega(p) = 0$ .

**Definition 8.30 Curvature matrix**

$$\Omega \equiv d\omega - \omega \wedge \omega$$

**Proposition 8.13 Transformation law of curvature matrix**

$$\Omega' = A \cdot \Omega \cdot A^{-1}$$

**Definition 8.31 Curvature operator**

$$s = \sum_{\alpha=1}^q \lambda^\alpha s_{\alpha|p}$$

$$R(X, Y)s = \sum_{\alpha, \beta=1}^q \lambda^\alpha \Omega_\alpha^\beta(X, Y) s_{\beta|p}$$

$$(R(X, Y)s)(p) = R(X_p, Y_p)s_p$$

**Proposition 8.14**

$$R(X, Y) = D_X D_Y - D_Y D_X - D_{[X, Y]}$$

**Theorem 8.15 Bianchi equation**

$$d\Omega = \omega \wedge \Omega - \Omega \wedge \omega$$



**Definition 8.32 Induced connection**

$$\begin{aligned} d\langle s, s^* \rangle &= \langle Ds, s^* \rangle + \langle s, Ds^* \rangle \\ \langle s_\alpha, s^{*\beta} \rangle &= \delta_\alpha^\beta \Rightarrow Ds^{*\beta} = - \sum_{\alpha=1}^q \omega_\alpha^\beta \otimes s^{*\alpha} \\ D(s_1 \oplus s_2) &\equiv Ds_1 \oplus Ds_2 \\ D(s_1 \otimes s_2) &\equiv Ds_1 \otimes s_2 + s_1 \otimes Ds_2 \end{aligned}$$



**Definition 8.33 Affine connection**

$$D \frac{\partial}{\partial u^i} \equiv \omega_i^j \otimes \frac{\partial}{\partial u^j} \equiv \Gamma_{ik}^j du^k \otimes \frac{\partial}{\partial u^j}$$



**Proposition 8.15**

$$\begin{aligned} \Gamma_{ik}^{'j} &= \Gamma_{pr}^q \frac{\partial w^j}{\partial u^q} \frac{\partial u^p}{\partial u^i} \frac{\partial u^r}{\partial u^k} + \frac{\partial^2 u^p}{\partial w^i \partial w^k} \frac{\partial w^j}{\partial u^p} \\ DX &= (dx^i + x^j \omega_j^i) \otimes \frac{\partial}{\partial u^i} = (x_{;j}^i + x^k \Gamma_{kj}^i) du^j \otimes \frac{\partial}{\partial u^i} = x_{;j}^i du^j \otimes \frac{\partial}{\partial u^i} \\ D\alpha &= (d\alpha_i - \alpha_j \omega_i^j) \otimes du^i = (\alpha_{i;j} - \alpha_k \Gamma_{ij}^k) du^j \otimes du^i = \alpha_{i;j} du^j \otimes du^i \end{aligned}$$



**Definition 8.34 Geodesic equation**

$$\begin{aligned} D\left(\frac{du^i(t)}{dt} \frac{\partial}{\partial u^i}\right) &= 0 \\ \frac{d^2 u^i}{dt^2} + \Gamma_{jk}^i \frac{du^j}{dt} \frac{du^k}{dt} &= 0 \end{aligned}$$



**Definition 8.35 Curvature tensor**

$$\begin{aligned} \Omega_i^j &= \frac{1}{2} R_{ikl}^j du^k \wedge du^l \\ R &\equiv R_{ikl}^j du^i \otimes \frac{\partial}{\partial u^j} \otimes du^k \otimes du^l \end{aligned}$$



**Proposition 8.16**

$$R_{ikl}^j = \frac{\partial \Gamma_{il}^j}{\partial u^k} - \frac{\partial \Gamma_{ik}^j}{\partial u^l} + \Gamma_{il}^h \Gamma_{hk}^j - \Gamma_{ik}^h \Gamma_{hl}^j$$

$$R_{ikl}^j = R_{prs}^q \frac{\partial w^j}{\partial u^q} \frac{\partial u^p}{\partial w^i} \frac{\partial u^r}{\partial w^k} \frac{\partial u^s}{\partial w^l}$$

$$R(X, \alpha_Y, Z, W) = \langle \alpha_Y, R(Z, W)X \rangle$$

$$R_{ikl}^j = \langle R(\frac{\partial}{\partial u^k} \frac{\partial}{\partial u^l}) \frac{\partial}{\partial u^i}, du^j \rangle$$

**Definition 8.36 Torsion tensor**

$$T_{ik}^j = \Gamma_{ki}^j - \Gamma_{ik}^j$$

$$T = T_{ik}^j \frac{\partial}{\partial u^j} \otimes du^i \otimes du^k$$

**Proposition 8.17**

$$T(X, Y) = T_{ij}^k X^i Y^j \frac{\partial}{\partial u^k}$$

$$T(X, Y) = D_X Y - D_Y X - [X, Y]$$

**Theorem 8.16**

Let  $D$  be an affine connection without torsion on  $M$ .  $\forall p \in M$ , there is a local coordinate system that  $\Gamma_{ik}^j(p)$  vanishes.

**Theorem 8.17**

Let  $D$  be an affine connection without torsion on  $M$ . Then we have Bianchi equation

$$R_{ikl;h}^j + R_{ihk;l}^j + R_{ilh;k}^j = 0$$



## 8.7 Riemann manifold

### Definition 8.37 Riemann manifold

Let  $M$  be a smooth manifold equipped with a smooth non-degenerate symmetric second order covariant tensor field  $G$ , then  $M$  is called general Riemann manifold and  $G$  is called the metric tensor of  $M$ .

If  $G$  is positive definite, then  $M$  is called Riemann manifold.

### Theorem 8.18

There must be a Riemann metric on  $m$  dimensional manifold  $M$ .

### Definition 8.38 Index lifting

$$f : T_p(M) \rightarrow T_p^*(M) \quad \alpha_X(Y) \equiv G(X, Y)$$

### Definition 8.39 Adapted connection

Let  $(M, G)$  be a general Riemann manifold and  $D$  a connection on  $M$ . If  $DG = 0$ , then  $D$  is called adapted connection on  $M$ .

### Proposition 8.18 Christoffel-Levi-Civita connection

Let  $M$  be a general Riemann manifold, then there is a unique adapted connection without torsion on  $M$ , called Christoffel-Levi-Civita connection.

As  $\omega_i^j = \Gamma_{ik}^j du^k$ ,  $dg_{ij} = g_{ik}\omega_j^k + g_{kj}\omega_i^k$ . If we denote that  $\omega_{ij} = \omega_i^j g_{jk}$  and  $\Gamma_{ijk} = \Gamma_{ik}^l g_{lj}$ , we have  $\omega_{ij} = \Gamma_{ijk} du^k$  and  $dg_{ij} = \omega_{ji} + \omega_{ij}$ . At last, we have

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left( \frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} \right)$$



**Proposition 8.19 Curvature tensor**

If we denote that  $\Omega_{ij} = \Omega_i^k g_{kj}$  and  $R_{ijkl} = R_{ikl}^h g_{hj}$ , we will have that

$$\Omega_{ij} + \Omega_{ji} = 0, \quad \Omega_{ij} = d\omega_{ij} + \omega_i^l \wedge \omega_{jl}, \quad \Omega_{ij} = \frac{1}{2} R_{ijkl} du^k \wedge du^l$$

The properties of curvature tensor:

$$R_{ijkl} = -R_{jikl} = -R_{ijlk}$$

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0$$

$$R_{ijkl} = R_{klij}$$

**Definition 8.40 Normal coordinates**

In differential geometry, normal coordinates at a point  $p$  in a differentiable manifold equipped with a symmetric affine connection are a local coordinate system in a neighbourhood of  $p$  obtained by applying the exponential map to the tangent space at  $p$ . In a normal coordinate system, the Christoffel symbols of the connection vanish at the point  $p$ , thus often simplifying local calculations. In normal coordinates associated to the Levi-Civita connection of a Riemann manifold, one can additionally arrange that the metric tensor is the Kronecker delta at the point  $p$ , and that the first partial derivatives of the metric at  $p$  vanish.

The properties of normal coordinates often simplify computations. In the following, assume that  $U$  is a normal neighbourhood centred at  $p$  in  $M$  and  $(x_i)$  are normal coordinates on  $U$ .

Let  $V$  be some vector from  $T_p M$  with components  $V^i$  in local coordinates, and  $\gamma_V$  be the geodesic with starting point  $p$  and velocity vector  $V$ , then  $\gamma_V$  is represented in normal coordinates by  $\gamma_V(t) = (tV^1, \dots, tV^n)$  as long as it is in  $U$ .

The coordinates of  $p$  are  $(0, \dots, 0)$

In Riemann normal coordinates at  $p$  the components of the Riemann metric  $g$  simplify to  $\delta_{ij}$ .

The Christoffel symbols vanish at  $p$ . In the Riemann case, so do the first partial derivatives of  $g_{ij}$ .

**Theorem 8.19**

Let  $M$  be a differentiable manifold equipped with a symmetric affine connection.  $\forall x_0 \in M$ , there is a neighbourhood  $W$  that for every point in  $W$ , there is a neighbourhood equipped with a normal coordinate system which contains  $W$ .



### Theorem 8.20

Let  $M$  be a Riemann manifold.  $\forall O \in M$ , there is a neighbourhood with normal coordinates  $W$  that:

- (1) For every point in  $W$ , there is a neighbourhood equipped with a normal coordinates which contains  $W$ .
- (2) The geodesic connecting  $O$  and  $p \in W$  is the only shortest path connecting these two points in  $W$ .



### Theorem 8.21

Let  $U$  be the neighbourhood with normal coordinates of  $O$ .  $\exists \epsilon > 0, \forall \delta \in (0, \epsilon)$ , the surface

$$\Sigma_\delta = \{p \in U \mid \sum_{i=1}^m (u^i(p))^2 = \delta^2\}$$

has following properties:

- (1)  $\forall p \in \Sigma_\delta$ , there is a unique shortest geodesic connecting  $p$  and  $O$  in  $U$ .
- (2) For all geodesics tangent to  $\Sigma_\delta$ , there is a neighbourhood of the cut point in which the geodesics lies outside of  $\Sigma_\delta$



### Theorem 8.22

Let  $M$  be a Riemann manifold and  $\forall p \in M$ , there is a  $\eta$ -spherical neighbourhood  $W$  that for arbitrary two points in  $W$ , there is a unique geodesic connecting these two points.



### Definition 8.41 Cross section curvature

$$R(X, Y, Z, W) \equiv R_{ijkl} X^i Y^j Z^k W^l$$

$$R(X, Y, Z, W) = (R(Z, W)X) \cdot Y$$

$$G(X, Y, Z, W) \equiv G(X, Z)G(Y, W) - G(X, W)G(Y, Z)$$

Let  $E$  be a two dimensional subspace of  $T_p(M)$  and  $X, Y$  two linearly independent tangent vector of  $E$ , then

$$K(E) = -\frac{R(X, Y, X, Y)}{G(X, Y, X, Y)}$$

is a function of  $E$ , which is independent of the choice of  $X, Y$ , called cross section curvature.





**Theorem 8.23**

Let  $M$  be a Riemann space. The curvature tensor of  $p \in M$  is uniquely determined by the cross section curvature of all the two dimensional subspace of  $T_p(M)$ .

**Definition 8.42 Constant curvature Riemann manifold**

Let  $M$  be a Riemann manifold. If all of  $K(E)$  on  $p$  is constant, then  $M$  is called isotropic on  $p$ .

If  $M$  is isotropic every where and  $K(p)$  is constant over  $M$ , then  $M$  is called constant curvature Riemann manifold.

**Theorem 8.24 F.Schur theorem**

Let  $M$  be a  $m$ -dimensional connected Riemann manifold that is isotropic every where. If  $m \geq 3$ , then  $M$  is constant curvature Riemann manifold.



# Chapter 9

## A Geometrical Description of Newton Theory



### 9.1 Introduction

We choose Euclidean coordinates for our absolute space and an absolute time  $t$ , than the equation of motion can be written as

$$\frac{d^2 t}{d\lambda^2} = 0$$

$$\frac{d^2 x^i}{d\lambda^2} + \frac{\partial \Phi}{\partial x^i} \left( \frac{d\lambda}{dt} \right)^2 = 0$$

It is convenient to define that  $\Gamma_{00}^i = \frac{\partial \Phi}{\partial x^i}$ , and all other  $\Gamma_{\beta\gamma}^\alpha$  vanish. Then we can write the equation of motion as

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0$$

Next, we can get the Riemann tensor given the connection above

$$R_{0j0}^i = -R_{00j}^i = \frac{\partial \Phi}{\partial x^i \partial x^j},$$

and all other terms vanish. It is straight forward to derive the expression of Ricci tensor,

$$R_{00} = \Phi_{ii} = \nabla^2 \Phi,$$

and all other terms vanish. So, newton gravity law can be written as

$$R_{00} = 4\pi\rho$$

### 9.2 Geometry structure of Newtonian Space-time

#### Stratification of space-time

Regard absolute time  $t$  as a scalar field defined once and for all in Newtonian space-time  $t = t(\mathcal{P})$ . The layers of space-time are the slices of constant  $t$ -the "space slices"-each of which has an identical geometric structure: the old "absolute space."

#### Flat Euclidean space

A given space slice is endowed with basis vectors  $e_i = \frac{\partial}{\partial x^i}$ ; and this basis has vanishing connection coefficients,  $\Gamma_{jk}^i = 0$ . Consequently, the geometry of each space slice is completely flat. Absolute space is Euclidean in its geometry. Each space slice is endowed with a three-dimensional metric, and its Galilean coordinate basis is orthonormal,  $e_i \cdot e_j = \delta_{ij}$ .

### Curvature of space-time

Parallel transport a vector around a closed curve lying entirely in a space slice; it will return to its starting point unchanged. But transport it forward in time by  $\Delta t$ , northerly in space by  $\Delta x_k$ , back in time by  $-\Delta t$ , and southerly by  $-\Delta x_k$  to its starting point; it will return changed by

$$\delta \mathbf{A} = -\mathcal{R}(\Delta t \frac{\partial}{\partial t}, \Delta x_k \frac{\partial}{\partial x_k}) \mathbf{A}$$

Geodesics of a space slice (Euclidean straight lines) that are initially parallel remain always parallel. But geodesics of space-time (trajectories of freely falling particles) initially parallel get pried apart or pushed together by space-time curvature,

$$\nabla_u \nabla_u \mathbf{n} + \mathcal{R}(\mathbf{n}, \mathbf{u}) \mathbf{u} = 0$$

## 9.3 Geometry formulation of Newtonian gravity

1. There exists a function  $t$  called "universal time", and a symmetric covariant derivative  $\nabla$ .
2. The 1-form  $dt$  is covariant constant, i.e.,

$$\nabla_u dt = 0 \text{ for all } \mathbf{u}.$$



**Note:** if  $\mathbf{w}$  is a spatial vector field, then  $\nabla_u \mathbf{w}$  is also spatial for every  $\mathbf{u}$ .

3. Spatial vectors are unchanged by parallel transport around infinitesimal closed curves; i.e.,

$$\mathcal{R}(\mathbf{n}, \mathbf{u}) \mathbf{w} = 0 \text{ if } \mathbf{w} \text{ is spatial, for every } \mathbf{u} \text{ and } \mathbf{n}.$$

4. All vectors are unchanged by parallel transport around infinitesimal, spatial, closed curves; i.e.,

$$\mathcal{R}(\mathbf{v}, \mathbf{w}) = 0 \text{ for every spatial } \mathbf{v} \text{ and } \mathbf{w}.$$

5. The Ricci curvature tensor has the form

$$\mathbf{Ricci} = 4\pi\rho d\mathbf{t} \otimes d\mathbf{t}$$

where  $\rho$  is the density of mass.

6. There exists a metric  $\cdot$  defined on spatial vectors only, which is compatible with the covariant derivative in this sense: for any spatial  $\mathbf{w}$  and  $\mathbf{v}$ , and for any  $\mathbf{u}$  whatsoever,

$$\nabla_u (\mathbf{w} \cdot \mathbf{v}) = (\nabla_u \mathbf{w}) \cdot \mathbf{v} + \mathbf{w} \cdot (\nabla_u \mathbf{v}).$$



**Note:** Axioms (1), (2), and (3) guarantee that such a spatial metric can exist.



7. The Jacobi curvature operator  $\mathcal{J}(\mathbf{u}, \mathbf{v})$ , defined for any vectors  $\mathbf{u}, \mathbf{n}, \mathbf{p}$  by

$$\mathcal{J}(\mathbf{u}, \mathbf{n})\mathbf{p} = \frac{1}{2}[\mathcal{R}(\mathbf{p}, \mathbf{n})\mathbf{u} + \mathcal{R}(\mathbf{p}, \mathbf{u})\mathbf{n}]$$

is "self-ad-joint" when operating on spatial vectors,i.e.,

$$\mathbf{v} \cdot [\mathcal{R}(\mathbf{u}, \mathbf{n})\mathbf{w}] = \mathbf{w} \cdot [\mathcal{R}(\mathbf{u}, \mathbf{n})\mathbf{v}] \text{ for all spactial } \mathbf{v}, \mathbf{w}; \text{ and for any } \mathbf{u}, \mathbf{n}.$$

8. "Ideal rods" measure the lengths that are calculated with the spatial metric; "ideal clocks" measure universal time  $t$  ( or some multiple thereof); and "freely falling particles" move along geodesics of  $\nabla$ .

## 9.4 Standard formulation of Newtonian gravity

1. There exist a universal time  $t$ , a set of Cartesian space coordinates  $x_i$  (called "Galilean coordinates"), and a Newtonian gravitational potential  $\Phi$ .
2. The density of mass  $\rho$  generates the Newtonian potential by Poisson's equation,

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^i \partial x^i} = 4\pi\rho.$$

3. The equation of motion for a freely falling particle is

$$\frac{d^2 x^i}{dt^2} + \frac{\partial \Phi}{\partial x^i} = 0.$$

4. "Ideal rods" measure the Galilean coordinate lengths; "ideal clocks" measure universal time.

## 9.5 Galilean coordinate system

The features of Galilean coordinate systems are

$$x^0(\mathcal{P}) = t(\mathcal{P})$$

$$\frac{\partial}{\partial x^i} \cdot \frac{\partial}{\partial x^j} = \delta_{ij}$$

$$\Gamma_{00}^j = \Phi_{,j} \text{ for some scalar field } \Phi, \text{ and all other } \Gamma_{\beta\gamma}^\alpha \text{ vanish.}$$

Consider following coordinate transformation:

- (1)  $x^{0'} = x^0 = t$ , both time coordinates must be universal time.
- (2) at fixed  $t$ , both sets of space coordinates must be Euclidean, so they must be related by a rotation and a translation:

$$\bar{x}^{i'}(t) = A_{i'j}(t)x^j + \bar{a}^{i'}(t)$$



We can get

$$\bar{\Gamma}_{0j'}^{i'} = \bar{\Gamma}_{j'0}^{i'} = A_{i'l} \dot{A}_{j'l}$$

$$\bar{\Gamma}_{00}^{i'} = \Phi_{,i'} + A_{i'j} (\ddot{A}_{j'k} \bar{x}^{k'} - \ddot{a}^j), \text{ here, } a^j = \bar{a}^{j'} A_{j'j}$$

and all other terms vanish. So, new coordinates have the standard Galilean form if and only if

$$\dot{A}_{i'j} = 0, \quad \Phi' = \Phi - \ddot{a}^{i'} x^{i'} + C$$

Were all the matter in the universe concentrated in a finite region of space and surrounded by emptiness ("island universe"), then one could impose the global boundary condition  $\Phi \rightarrow 0$  as  $r \equiv (x^i x^i)^{\frac{1}{2}} \rightarrow \infty$ . This would single out a subclass of Galilean coordinates ("absolute" Galilean coordinates), with a unique, common Newtonian potential. The transformation from one absolute Galilean coordinate system to any other is called Galilean transformation.

## 9.6 Coordinate transformation in space

We now consider a coordinate transformation of Galilean coordinate system purely in space without any terms related with time. That means that  $\bar{x}^{i'} = y^{i'}(x^i)$  and  $t' = t$ . We can calculate the connection term in the new coordinate system.

$$\bar{\Gamma}_{00}^{i'} = \Gamma_{00}^i \frac{\partial y^{i'}}{\partial x^i}$$

$$\bar{\Gamma}_{j'k'}^{i'} = \frac{\partial^2 x^m}{\partial y^{i'} \partial y^{k'}} \frac{\partial y^{i'}}{\partial x^m}$$

The equation of motion of free fall body is that

$$\frac{d^2 t'}{d\lambda^2} = 0$$

$$\frac{d^2 \bar{x}^{i'}}{d\lambda^2} + \bar{\Gamma}_{j'k'}^{i'} \frac{d\bar{x}^{j'}}{d\lambda} \frac{d\bar{x}^{k'}}{d\lambda} + \bar{\Gamma}_{00}^{i'} \frac{dt'}{d\lambda} \frac{dt'}{d\lambda} = 0$$

We can write it compactly as

$$\frac{d^2 \bar{x}^{i'}}{dt^2} + \bar{\Gamma}_{j'k'}^{i'} \frac{d\bar{x}^{j'}}{dt} \frac{d\bar{x}^{k'}}{dt} + \bar{\Gamma}_{00}^{i'} = 0$$

We can demonstrate that

$$\bar{\Gamma}_{j'k'}^{i'} = \frac{1}{2} \bar{g}^{i'p'} (\partial_{k'} \bar{g}_{j'p'} + \partial_{j'} \bar{g}_{k'p'} - \partial_{p'} \bar{g}_{j'k'})$$

and

$$\bar{\Gamma}_{00}^{i'} = \bar{g}^{i'j'} \partial_{j'} \Phi$$

Here,  $\bar{g}$  is the metric of the space in new coordinate system.



# Chapter 10

## Geometry of Space-time



### 10.1 More on the manifold of space-time

#### 10.1.1 Hodge dual

##### Definition 10.1 Hodge dual operator

The Hodge star operator on a vector space  $V$  with a non-degenerate symmetric bilinear form (herein referred to as the inner product) is a linear operator on the exterior algebra of  $V$ , mapping  $k$ -vectors to  $(n - k)$ -vectors where  $n = \dim V$ , for  $0 \leq k \leq n$ . It has the following property, which defines it completely: given two  $k$ -vectors  $\alpha, \beta$ ,

$$\alpha \wedge (\star\beta) = \langle \alpha, \beta \rangle \omega$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product on  $k$ -vectors and  $\omega$  is the preferred unit  $n$ -vector. The inner product  $\langle \cdot, \cdot \rangle$  on  $k$ -vectors is extended from that on  $V$  by requiring that

$$\langle \alpha, \beta \rangle = \det [\langle \alpha_i, \beta_j \rangle]$$

for any decomposable  $k$ -vectors  $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k$  and  $\beta = \beta_1 \wedge \cdots \wedge \beta_k$ . The unit  $n$ -vector  $\omega$  is unique up to a sign. The preferred choice of  $\omega$  defines an orientation on  $V$ .

Given an orthonormal basis  $(e_1, \dots, e_n)$  ordered such that  $\omega = e_1 \wedge \cdots \wedge e_n$ , we see that

$$\star(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) = e_{i_{k+1}} \wedge e_{i_{k+2}} \wedge \cdots \wedge e_{i_n}$$

where  $(i_1, i_2, \dots, i_n)$  is an even permutation of  $\{1, 2, \dots, n\}$ . Of these  $\frac{n!}{2}$ , only  $\binom{n}{k}$  are independent. The first one in the usual lexicographical order reads

$$\star(e_1 \wedge e_2 \wedge \cdots \wedge e_k) = e_{k+1} \wedge e_{k+2} \wedge \cdots \wedge e_n$$

### 10.1.2 Levi-Civita tensor

#### Definition 10.2 Levi-Civita tensor

$$\epsilon_{i_1, \dots, i_n} \equiv |g|^{\frac{1}{2}} \tilde{\epsilon}_{i_1, \dots, i_n},$$

where  $\epsilon$  is Levi-Civita symbol.

#### Proposition 10.1

$$\epsilon^{i_1, \dots, i_n} = g^{i_1 j_1} \dots g^{i_n j_n} \epsilon_{j_1, \dots, j_n} = \frac{|g|^{\frac{1}{2}}}{g} \tilde{\epsilon}^{i_1, \dots, i_n} = \text{sgn}(g) \frac{1}{|g|^{\frac{1}{2}}} \tilde{\epsilon}^{i_1, \dots, i_n}$$

Using tensor index notation, the Hodge dual is obtained by contracting the indices of a  $k$ -form with the  $n$ -dimensional completely antisymmetric Levi-Civita tensor.

#### Proposition 10.2

$$(\star \eta)_{i_1, i_2, \dots, i_{n-k}} = \frac{1}{(n-k)!} \eta^{j_1, \dots, j_k} \epsilon_{j_1, \dots, j_k, i_1, \dots, i_{n-k}},$$

where  $\eta$  is an arbitrary antisymmetric tensor in  $k$  indices.

### 10.1.3 Metric-induced properties of Riemann curvature tensor

1. In a  $n$  dimensional manifold with torsion-free affine connection, the number of independent components of Riemann tensor is

$$\frac{n^3(n-1)}{2} - \frac{n^2(n-1)(n-2)}{6} = \frac{(n^2-1)n^2}{3}$$

In a  $n$  dimensional Riemann manifold, the number of independent components of Riemann tensor is

$$\left(\frac{n(n-1)}{2}\right)^2 - \frac{n^2(n-1)(n-2)}{6} = \frac{(n^2-1)n^2}{12}$$

2. The double dual of Riemann tensor

$$\bar{G}^{\alpha\beta}_{\gamma\delta} = \frac{1}{2} \tilde{\epsilon}^{\alpha\beta\mu\nu} R_{\mu\nu}{}^{\rho\sigma} \frac{1}{2} \tilde{\epsilon}_{\rho\sigma\gamma\delta} = -\frac{1}{4} \delta^{\alpha\beta\mu\nu}_{\rho\sigma\gamma\delta} R_{\mu\nu}{}^{\rho\sigma}$$

contains precisely the same amount of information as Riemann tensor, and satisfies precisely the same set of symmetries.

3. The Einstein curvature tensor, which is symmetric

$$G^\beta_\delta = \bar{G}^{\mu\beta}_{\mu\delta}; \quad G_{\beta\delta} = G_{\delta\beta}$$



4. The Bianchi identity takes a particularly simple form when rewritten in Bianchi identities terms of the double dual  $\bar{G}$ :

$$\bar{G}^{\alpha\beta}{}_{\gamma\delta}{}^{;\delta} = 0$$

and it has the obvious consequence

$$G_{\beta\delta}{}^{;\delta} = 0$$

5. The Ricci curvature tensor, which is symmetric, and the curvature scalar

$$R^\beta{}_\delta = R^{\mu\beta}{}_{\mu\delta}; \quad R_{\beta\delta} = R_{\delta\beta}; \quad R = R^\beta{}_\beta$$

which are related to the Einstein tensor by

$$G^\beta{}_\delta = R^\beta{}_\delta - \frac{1}{2}\delta^\beta_\delta R$$

6. The Weyl conformal tensor

$$C^{\alpha\beta}{}_{\gamma\delta} = R^{\alpha\beta}{}_{\gamma\delta} - 2\delta^{[\alpha}{}_{[\gamma} R^{\beta]}{}_{\delta]} + \frac{1}{3}\delta^{[\alpha}{}_{[\gamma} \delta^{\beta]}{}_{\delta]} R$$

possesses the same symmetries as the Riemann tensor. Weyl tensor is completely "trace-free"; i.e., that contraction of  $C_{\alpha\beta\gamma\delta}$  on any pair of slots vanishes. Thus,  $C_{\alpha\beta\gamma\delta}$  can be regarded as the trace-free part of Riemann, and  $R_{\alpha\beta}$  can be regarded as the trace of Riemann. Riemann is determined entirely by its trace-free part  $C_{\alpha\beta\gamma\delta}$  and its trace  $R_{\alpha\beta}$

## 10.2 The coordinates of observer

### 10.2.1 Riemann normal coordinates

$$g_{\alpha\beta}(\mathcal{P}_0) = \eta_{\alpha\beta}$$

$$g_{\alpha\beta,\gamma}(\mathcal{P}_0) = 0$$

$$\Gamma^\alpha{}_{\beta\gamma}(\mathcal{P}_0) = 0$$

$$\Gamma^\alpha{}_{\beta\gamma,\mu}(\mathcal{P}_0) = -\frac{1}{3}(R^\alpha{}_{\beta\gamma\mu} + R^\alpha{}_{\gamma\beta\mu})$$

$$g_{\alpha\beta,\mu\nu}(\mathcal{P}_0) = -\frac{1}{3}(R_{\alpha\mu\beta\nu} + R_{\alpha\nu\beta\mu})$$

$$R_{\alpha\beta\gamma\delta}(\mathcal{P}_0) = g_{\alpha\delta,\beta\gamma} - g_{\alpha\gamma,\beta\delta}$$





### 10.2.2 The proper reference frame of an accelerated observer

1. Let  $\tau$  be proper time as measured by the accelerated observer's clock. Let  $\mathcal{P} = \mathcal{P}_0(\tau)$  be the observer's world line.
2. The observer carries with himself an orthonormal tetrad  $\{e_{\hat{\alpha}}\}$  with

$$e_{\hat{0}} = \mathbf{u} = \frac{d\mathcal{P}_0}{d\tau}$$

and with

$$e_{\hat{\alpha}} \cdot e_{\hat{\beta}} = \eta_{\alpha\beta}$$

3. The tetrad changes from point to point along the observer's world line, relative to parallel transport:

$$\nabla_{\mathbf{u}} e_{\hat{\alpha}} = -\boldsymbol{\Omega} \cdot e_{\hat{\alpha}}$$

$$\Omega^{\mu\nu} = a^{\mu}u^{\nu} - u^{\mu}a^{\nu} + u_{\alpha}\omega_{\beta}\epsilon^{\alpha\beta\mu\nu}$$

This transport law has the same form in curved space-time as in flat because curvature can only be felt over finite distances, not over the infinitesimal distance involved in the "first time-rate of change of a vector" (equivalence principle).

$$\mathbf{a} = \nabla_{\mathbf{u}} \mathbf{u}$$

$$\mathbf{u} \cdot \mathbf{a} = \mathbf{u} \cdot \boldsymbol{\omega} = 0$$

If  $\boldsymbol{\omega}$  were zero, the observer would be Fermi-Walker-transporting his tetrad (gyroscope-type transport). If both  $\mathbf{a}$  and  $\boldsymbol{\omega}$  were zero, he would be freely falling (geodesic motion) and would be parallel-transporting his tetrad.

4. The observer constructs his proper reference frame (local coordinate system) in a manner analogous to the Riemann-normal construction. From each event  $\mathcal{P}_0(\tau)$  on his world line, he sends out purely spatial geodesics (geodesics orthogonal to  $\mathbf{u}$ ), with affine parameter equal to proper length. The tangent vector has unit length, because the chosen affine parameter is proper length.
5. Each event near the observer's world line is intersected by precisely one of the geodesics  $\mathcal{G}[\tau, \mathbf{n}, s]$ . [Far away, this is not true; the geodesics may cross, either because of the observer's acceleration].
6. Pick an event  $\mathcal{P}$  near the observer's world line. The geodesic through it originated on the observer's world line at a specific time  $\tau$ , had original direction  $\mathbf{n} = n^{\hat{j}}e_{\hat{j}}$ ; and needed to extend a distance  $s$  before reaching  $\mathcal{P}$ . Hence, the four numbers

$$(x^{\hat{0}}, x^{\hat{1}}, x^{\hat{2}}, x^{\hat{3}}) \equiv (\tau, sn^{\hat{1}}, sn^{\hat{2}}, sn^{\hat{3}})$$

are a natural way of identifying the event  $\mathcal{P}$ . These are the coordinates of  $\mathcal{P}$  in the observer's proper reference frame.



Along the world line of observer:

$$\frac{\partial}{\partial x^{\hat{\alpha}}} = \mathbf{e}_{\hat{\alpha}}, \quad g_{\hat{\alpha}\hat{\beta}} = \mathbf{e}_{\hat{\alpha}} \cdot \mathbf{e}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}$$

$$\Gamma_{\hat{\alpha}\hat{0}}^{\hat{\beta}} = -\Omega_{\hat{\alpha}}^{\hat{\beta}}, \quad \Gamma_{\hat{0}\hat{0}}^{\hat{0}} = 0$$

$$\Gamma_{\hat{j}\hat{0}}^{\hat{0}} = \Gamma_{\hat{0}\hat{0}}^{\hat{j}} = a^{\hat{j}}, \quad \Gamma_{\hat{k}\hat{0}}^{\hat{j}} = -\omega^{\hat{i}} \epsilon_{0\hat{i}\hat{j}\hat{k}}$$

$$\Gamma_{\hat{j}\hat{k}}^{\hat{\alpha}} = 0$$

$$g_{\hat{\alpha}\hat{\beta},\hat{0}} = 0, \quad g_{\hat{j}\hat{k},\hat{l}} = 0$$

$$g_{\hat{0}\hat{0},\hat{j}} = -2a^{\hat{j}}, \quad g_{\hat{0}\hat{j},\hat{k}} = -\epsilon_{0\hat{j}\hat{k}\hat{l}}\omega^{\hat{l}}$$

## 10.3 Hypersurfaces

### 10.3.1 Description of hypersurfaces



**Note:** We only discuss timelike and spacelike hypersurfaces in this section.

#### Normal vector

$$n^{\alpha}n_{\alpha} = \epsilon \equiv \begin{cases} -1 & \text{if } \Sigma \text{ is spacelike} \\ +1 & \text{if } \Sigma \text{ is timelike} \end{cases}$$

#### Induced metric

Suppose that the hypersurface is parametrized with equation  $x^{\alpha} = x^{\alpha}(y^a)$ , Then

$$e_a^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^a}.$$

For displacements within  $\Sigma$ , we have

$$\begin{aligned} ds_{\Sigma}^2 &= g_{\alpha\beta} dx^{\alpha} dx^{\beta} \\ &= g_{\alpha\beta} \left( \frac{\partial x^{\alpha}}{\partial y^a} dy^a \right) \left( \frac{\partial x^{\beta}}{\partial y^b} dy^b \right) \\ &= h_{ab} dy^a dy^b \end{aligned}$$

where  $h_{ab} = g_{\alpha\beta} e_a^{\alpha} e_b^{\beta}$ . The completeness relation can be written as

$$g^{\alpha\beta} = \epsilon n^{\alpha} n^{\beta} + h^{ab} e_a^{\alpha} e_b^{\beta}$$



### 10.3.2 Integration on hypersurfaces

The positive volume element of the whole space time is  $dx^0 \wedge \cdots \wedge dx^{m-1}$ , the positive volume element of the hypersurfaces is  $dy^1 \wedge \cdots \wedge dy^{m-1}$ . Suppose that the coordinate in hypersurfaces is compatible with the coordinate of the whole space-time, which means that  $-dy^m \wedge dy^1 \wedge \cdots \wedge dy^{m-1}$  has the same orientation as  $dx^0 \wedge \cdots \wedge dx^{m-1}$ . Then we have that

$$\tilde{\epsilon}_{\alpha_m \alpha_1 \cdots \alpha_{m-1}} \frac{\partial x^{\alpha_m}}{\partial y^m} e_1^{\alpha_1} \cdots e_{m-1}^{\alpha_{m-1}} < 0$$

If we demand that the direction of  $n^\alpha$  is the opposite of  $\frac{\partial x^\alpha}{\partial y^m}$ , then we have

$$\tilde{\epsilon}_{\alpha_m \alpha_1 \cdots \alpha_{m-1}} n^{\alpha_m} e_1^{\alpha_1} \cdots e_{m-1}^{\alpha_{m-1}} > 0.$$

#### Surface element

We define the surface element of a hypersurface as

$$d\Sigma_\mu = \epsilon_{\mu\alpha\beta\gamma} e_1^\alpha e_2^\beta e_3^\gamma dy^1 \wedge dy^2 \wedge dy^3$$

It is easy to verify that

$$f^*(\sqrt{-g} dx^1 \wedge dx^2 \wedge dx^3) = d\Sigma_0$$

$$f^*(-\sqrt{-g} dx^0 \wedge dx^2 \wedge dx^3) = d\Sigma_1$$

and so on. We can demonstrate that

$$d\Sigma_\mu = \epsilon n_\mu |h|^{\frac{1}{2}} dy^1 \wedge dy^2 \wedge dy^3$$

#### Element of two-surface

Within the hypersurface  $\Sigma$ , we can define a two-surface  $S$ , which is parametrized with  $y^a = y^a(\theta_A)$ , then

$$e_A^a = \frac{\partial y^a}{\partial \theta^A}, \quad e_A^\alpha = \frac{\partial x^\alpha}{\partial \theta^A} = e_a^\alpha e_A^a$$

$$\sigma_{AB} = h_{AB} e_A^a e_B^b = g_{\alpha\beta} e_A^\alpha e_B^\beta$$

$$h^{ab} = \epsilon_r r^a r^b + \sigma^{AB} e_A^a e_B^b$$

$$g^{\alpha\beta} = \epsilon_n n^\alpha n^\beta + \epsilon_r r^\alpha r^\beta + \sigma^{AB} e_A^\alpha e_B^\beta$$

If we demand that the direction  $r^a$  is the opposite of that of  $\frac{\partial y^a}{\partial \theta^1}$ , then the condition of compatibility can be written as

$$\epsilon_{\mu\nu\beta\gamma} n^\mu r^\nu e_2^\beta e_3^\gamma > 0$$

We define the surface element of a two-surface as

$$dS_{\mu\nu} = \epsilon_{\mu\nu\beta\gamma} e_2^\beta e_3^\gamma d\theta^2 \wedge d\theta^3$$

It is easy to verify that

$$f^*(\sqrt{-g} dx^2 \wedge dx^3) = dS_{01}$$



and so on. We can demonstrate that

$$dS_{\alpha\beta} = \epsilon_n \epsilon_r (n_\alpha r_\beta - n_\beta r_\alpha) \sqrt{\sigma} d\theta^2 \wedge d\theta^3$$

### Theorem 10.1 Gauss-Stokes theorem

1.

$$\int_M d\omega = \int_{\partial M} i^* \omega$$

2.

$$\int_V A^\alpha_{;\alpha} \sqrt{-g} dx^4 = \oint_{\partial V} A^\alpha d\Sigma_\alpha$$

3.

$$\int_\Sigma B^{\alpha\beta}_{;\beta} d\Sigma_\alpha = \frac{1}{2} \oint_{\partial\Sigma} B^{\alpha\beta} d\Sigma_{\alpha\beta},$$

where  $B_{\alpha\beta}$  is an antisymmetric tensor

### 10.3.3 Differentiation of tangent vector fields

#### Tangent tensor field

$$A^{\alpha\beta\cdots} = A^{ab\cdots} e_a^\alpha e_b^\beta \cdots$$

$$A_{\alpha\beta\cdots} e_a^\alpha e_b^\beta \cdots = A_{ab\cdots} = h_{am} h_{bn} \cdots A^{mn\cdots}$$

#### Projection tensor

$$h^{\alpha\beta} \equiv h^{ab} e_a^\alpha e_b^\beta = g^{\alpha\beta} - \epsilon n^\alpha n^\beta$$

#### Intrinsic covariant derivative

$$A_{a|b} \equiv A_{\alpha;\beta} e_a^\alpha e_b^\beta = A_{a,b} - \Gamma_{ab}^c A_c$$

Here, the connection  $\Gamma_{ab}^c$  is compatible with  $h_{ab}$ .

#### Extrinsic curvature

$$K_{ab} \equiv n_{\alpha;\beta} e_a^\alpha e_b^\beta$$

$$A^\alpha_{;\beta} e_b^\beta = A^\alpha_{|b} e_a^\alpha - \epsilon A^a K_{ab} n^\alpha$$

$$e^\alpha_{a;\beta} e_b^\beta = \Gamma_{ab}^c e_c^\alpha - \epsilon K_{ab} n^\alpha$$

$$K_{ab} = \frac{1}{2} (\mathcal{L}_n g_{\alpha\beta}) e_a^\alpha e_b^\beta$$

$$K \equiv h^{ab} K_{ab} = n^\alpha_{;\alpha}$$



**Theorem 10.2 Gauss-Codazzi theorem**

1.

$$R_{\alpha\beta\gamma}^{\mu} e_a^{\alpha} e_b^{\beta} e_c^{\gamma} = R_{abc}^m e_m^{\mu} + \epsilon(K_{ab|c} - K_{ac|b})n^{\mu} + \epsilon K_{ab} n_{;\gamma}^{\mu} e_c^{\gamma} - \epsilon K_{ac} n_{;\beta}^{\mu} e_b^{\beta}$$

2.

$$-2\epsilon G_{\alpha\beta} n^{\alpha} n^{\beta} = {}^3R + \epsilon(K^{ab} K_{ab} - K^2)$$

$$G_{\alpha\beta} e_a^{\alpha} n^{\beta} = K_{a|b}^b - K_{,a}$$

3.

$$R = {}^3R + \epsilon(K^2 - K^{ab} K_{ab}) + 2\epsilon(n_{;\beta}^{\alpha} n^{\beta} - n^{\alpha} n_{;\beta}^{\beta})_{;\alpha}$$



# Chapter 11

## Formulation of General Relativity



Give the fields that generate mass-energy, and their time-rates of change, and give 3-geometry of space and its time-rate of change, all at one time, and solve for the 4-geometry of spacetime at that one time. Four of the ten components of Einstein's law connect the curvature of space here and now with the distribution of mass-energy here and now, and the other six equations tell how the geometry as thus determined then proceeds to evolve.

### 11.1 Basic assumptions of general relativity

1. Space-time is a four dimensional pseudo-Riemann manifold.
2. The metric of the manifold is governed by the Einstein field equation

$$G = 8\pi T.$$

3. All special relativistic laws of physics are valid in local Lorentz frames of metric.

### 11.2 Lagrangian formulation

#### 11.2.1 Mechanics

$$S[q] = \int_{\tau_1}^{\tau_2} L(x^\alpha, \frac{dx^\alpha}{d\tau}) d\tau$$

$$\delta x^\alpha(\tau_1) = 0, \delta x^\alpha(\tau_2) = 0$$

$$\delta S = 0 \Rightarrow \frac{d}{d\tau} \frac{\partial L}{\partial u^\alpha} - \frac{\partial L}{\partial x^\alpha} = 0$$

**Example:**

$$L = -m(-g_{\mu\nu}u^\mu u^\nu)^{1/2} + eA_\mu u^\mu$$

$\Downarrow$

$$m(\frac{du_\alpha}{d\tau} - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} u^\mu u^\nu) = e(A_{\mu,\alpha} - A_{\alpha,\mu})u^\mu \Rightarrow ma_\alpha = eF_{\alpha\mu}u^\mu$$

### 11.2.2 Field Theory

$$\begin{aligned}
 S[q] &= \int_{\mathcal{V}} \mathcal{L}(q, q_\alpha) \sqrt{-g} d^4x \\
 \delta q|_{\partial\mathcal{V}} &= 0 \\
 \delta S = 0 &\Rightarrow \nabla_\alpha \left( \frac{\partial \mathcal{L}}{\partial q_{,\alpha}} \right) - \frac{\partial \mathcal{L}}{\partial q} = 0
 \end{aligned}$$

**Example:**

$$\begin{aligned}
 \mathcal{L} &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A_\mu j^\mu \\
 &\Downarrow \\
 F^{\mu\nu}{}_{;\nu} &= j^\mu
 \end{aligned}$$

### 11.2.3 General relativity

$$\begin{aligned}
 S_H[g] &= \frac{1}{16\pi} \int_{\mathcal{V}} R \sqrt{-g} d^4x \\
 S_B[g] &= \frac{1}{8\pi} \oint_{\partial\mathcal{V}} \epsilon K |h|^{\frac{1}{2}} d^3y \\
 S_0 &= \frac{1}{8\pi} \oint_{\partial\mathcal{V}} \epsilon K_0 |h|^{\frac{1}{2}} d^3y \\
 S_M[\phi; g] &= \int_{\partial\mathcal{V}} \mathcal{L}(\phi, \phi_{,\alpha}; g_{\alpha\beta}) \sqrt{-g} d^4x
 \end{aligned}$$

**Variation of Hilbert term**

$$(16\pi) \delta S_H = \int_{\mathcal{V}} G_{\alpha\beta} \delta g^{\alpha\beta} \sqrt{-g} d^4x - \oint_{\partial\mathcal{V}} \epsilon h^{\alpha\beta} \delta g_{\alpha\beta, \mu} n^\mu |h|^{\frac{1}{2}} d^3y$$

**Variation of boundary term**

$$16\pi \delta S_B = \oint_{\partial\mathcal{V}} \epsilon h^{\alpha\beta} \delta g_{\alpha\beta, \mu} n^\mu |h|^{\frac{1}{2}} d^3y$$

**Variation of matter action**

$$\delta S_M = \int_{\mathcal{V}} \left( \frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} - \frac{1}{2} \mathcal{L} g_{\alpha\beta} \right) \delta g^{\alpha\beta} \sqrt{-g} d^4x$$

Define

$$T_{\alpha\beta} \equiv -2 \frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} + \mathcal{L} g_{\alpha\beta}$$

**Example:**

$$\begin{aligned}
 \mathcal{L} &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \\
 &\Downarrow
 \end{aligned}$$



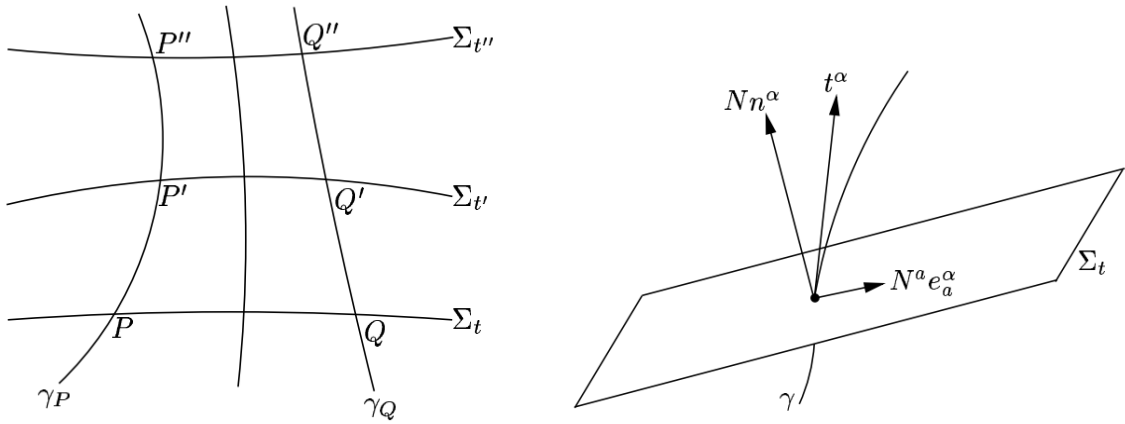
$$T_{\alpha\beta} = F_{\mu\alpha} F^\mu{}_\beta - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} g_{\alpha\beta}$$

Nondynamical term

$K_0$  = extrinsic curvature of  $\partial\mathcal{L}$  embedded in flat space-time.

## 11.3 Hamiltonian formulation

### 11.3.1 3+1 decomposition



**Figure 11.1:** Foliation of space-time by space-like hypersurfaces

**Figure 11.2:** Decomposition of  $t^\alpha$  into lapse and shift

The space-time is foliated by spacelike hypersurfaces  $\Sigma_t$  that is described by scalar function  $t(x^\alpha)$ .  $t$  is a single valued function and the unit normal to the hypersurfaces  $n_\alpha \propto \partial_\alpha t$  is a future directed timelike vector field.

Consider a congruence of curves  $\gamma$  intersecting  $\Sigma_t$ . We use  $t$  as a parameter on the curves and the vector  $t^\alpha$  is tangent to the congruence (so,  $t^\alpha \partial_\alpha t = 1$ ). Install coordinates  $y^a$  on  $\Sigma_t$  and impose  $y^a(P'') = y^a(P') = y^a(P)$ , so  $y^a$  is held constant on each member of the congruence. This construction defines a coordinate system  $(t, y^a)$  in  $\mathcal{V}$ .

base vector

$$t^\alpha = \left( \frac{\partial x^\alpha}{\partial t} \right)_{y^a}, \quad e_a^\alpha = \left( \frac{\partial x^\alpha}{\partial y^a} \right)_t, \quad \mathcal{L}_t e_a^\alpha = 0$$

Normal vector

$$n_\alpha = -N \partial_\alpha t, \quad n_\alpha e_a^\alpha = 0$$





Decomposition of  $t^\alpha$ 

$$t^\alpha = Nn^\alpha + N^a e_a^\alpha$$

## Metric

$$\begin{aligned} ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= g_{\alpha\beta} (t^\alpha dt + e_a^\alpha dy^a) (t^\beta dt + e_b^\beta dy^b) \\ &= -N^2 dt^2 + h_{ab} (dy^a + N^a dt) (dy^b + N^b dt) \\ \sqrt{-g} &= N\sqrt{h} \end{aligned}$$

## 11.3.2 Field theory

$$\begin{aligned} \dot{q} &= \frac{\partial q}{\partial t}, \quad p = \frac{\partial}{\partial \dot{q}} (\sqrt{-g} \mathcal{L}) \\ \mathcal{H}(p, q, q_a) &= p\dot{q} - \sqrt{-g} \mathcal{L} \\ H &= \int_{\Sigma_t} \mathcal{H}(p, q, q_a) d^3y \\ S &= \int_{t_1}^{t_2} dt \int_{\Sigma_t} (p\dot{q} - \mathcal{H}) d^3y \\ \delta S = 0 &\Rightarrow \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} + \left( \frac{\partial \mathcal{H}}{\partial q_{,a}} \right)_{,a}, \quad \dot{q} = \frac{\partial \mathcal{H}}{\partial p} \end{aligned}$$

**Example:** For electromagnetic field in 3+1 decomposition form, we define the electrical field as  $E_a = F_{\alpha\beta} n^\beta e_a^\alpha$ , the magnetic field as  $\epsilon_{abc} B^c = F_{\alpha\beta} e_a^\alpha e_b^\beta$ . In this definition, the equation of motion of particles in electromagnetic field can be written as

$$mA_a = \gamma e (E_a + \epsilon_{abc} v^b B^c)$$

Here,  $A_a = u_{\alpha;\beta} u^\beta e_a^\alpha$ ,  $\gamma = \frac{1}{\sqrt{1-v^2}}$ . So, the three force

$$\mathbf{f} = \frac{d\mathbf{p}}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

If we adopt the coordinates  $(t, y^a)$ , it is easy to verify that

$$E^a = NF^{0a}, \quad B^a = \frac{1}{2} \epsilon^{abc} F_{bc}$$

We further define

$$\mathcal{E}^a = \sqrt{h} E^a, \quad \mathcal{B}^a = \sqrt{h} B^a, \quad \phi = -A_0, \quad \rho_e = -j^\alpha n_\alpha = Nj^0$$

If we notice that

$$F_{0a} = -h_{ab} N^2 F^{0b} - F_{ab} N^b, \quad \tilde{\epsilon}_{abc} \tilde{\epsilon}_{ijk} h^{ai} h^{bj} = \frac{2h_{ck}}{h}$$



It is easy to verify that

$$\sqrt{-g}\mathcal{L} = -\mathcal{E}^a \dot{A}_a + \phi \mathcal{E}_{,a}^a - \frac{1}{2} N h^{-\frac{1}{2}} h_{ab} (\mathcal{E}^a \mathcal{E}^b + \mathcal{B}^a \mathcal{B}^b) + \tilde{\epsilon}_{abc} N^a \mathcal{E}^b \mathcal{B}^c - \sqrt{h} \phi \rho_e + N \sqrt{h} A_a j^a$$

So,  $\pi^a = -\mathcal{E}^a$ , and we can get the Hamilton density

$$\mathcal{H} = \phi \pi_{,a}^a + \frac{1}{2} N h^{-\frac{1}{2}} h_{ab} (\pi^a \pi^b + \mathcal{B}^a \mathcal{B}^b) + \tilde{\epsilon}_{abc} N^a \pi^b \mathcal{B}^c + \sqrt{h} \phi \rho_e - N \sqrt{h} A_a j^a$$

Then, the Hamilton equation can be written as

$$\begin{aligned} \dot{A}_a &= -\phi_a + N h^{-\frac{1}{2}} h_{ab} \pi^b + \tilde{\epsilon}_{abc} N^a \mathcal{B}^c \\ \dot{\pi}^a &= -\tilde{\epsilon}^{jab} (N h^{-\frac{1}{2}} h_{ij} \mathcal{B}^j)_{,b} - \tilde{\epsilon}^{cab} (\tilde{\epsilon}_{ijc} N^i \pi^j)_{,b} + N \sqrt{h} j^a \end{aligned}$$

and also the constraint equation  $\pi_{,a}^a + \sqrt{h} \rho_e = 0$ . After simplification, the Maxwell equations are

$$\begin{aligned} \frac{1}{\sqrt{h}} \frac{\partial}{\partial t} (\sqrt{h} \mathbf{E}) &= \nabla \times (N \mathbf{B} - \mathbf{N} \times \mathbf{E}) - N \mathbf{J} \\ \frac{1}{\sqrt{h}} \frac{\partial}{\partial t} (\sqrt{h} \mathbf{B}) &= -\nabla \times (N \mathbf{E} + \mathbf{N} \times \mathbf{B}) \\ \nabla \cdot \mathbf{E} &= \rho_e \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

### 11.3.3 General relativity

$$S_G = \frac{1}{16\pi} \int_{t_1}^{t_2} dt \left\{ \int_{\Sigma_t} ({}^3R + K^{ab} K_{ab} - K^2) N \sqrt{h} d^3y + 2 \oint_{\Sigma_t} (k - k_0) N \sqrt{\sigma} d^2\theta \right\}$$

$k_0$  = extrinsic curvature of  $S_t$  embedded in flat space.

#### Gravitational Hamiltonian

$$\dot{h}_{ab} \equiv \mathcal{L}_t h_{ab} = \mathcal{L}_t (g_{\alpha\beta} e_a^\alpha e_b^\beta) = \mathcal{L}_t (g_{\alpha\beta} e_a^\alpha e_b^\beta) = 2N K_{ab} + N_{a|b} + N_{b|a}$$

$$K_{ab} = \frac{1}{2N} (\dot{h}_{ab} - N_{a|b} - N_{b|a})$$

$$p^{ab} = \frac{\partial}{\partial \dot{h}_{ab}} (\sqrt{-g} \mathcal{L}_G) = \frac{\sqrt{h}}{16\pi} (K^{ab} - K h^{ab})$$

$$\sqrt{h} K^{ab} = 16\pi (p^{ab} - \frac{1}{2} p h^{ab})$$

$$\mathcal{H}_G = p^{ab} \dot{h}_{ab} - \sqrt{-g} \mathcal{L}_G$$

$$\begin{aligned} 16\pi H_G &= \int_{\Sigma_t} [N (K^{ab} K_{ab} - K^2 - {}^3R) - 2N_a (K^{ab} - K h^{ab})_{|b}] \sqrt{h} d^3y \\ &\quad - 2 \oint_{S_t} [N (k - k_0) - N_a (K^{ab} - K h^{ab}) r_b] \sqrt{\sigma} d^2\theta \end{aligned}$$



## Variation of gravitational Hamiltonian

$$\begin{aligned}
\delta N &= \delta N^a = \delta h_{ab} = 0 \text{ on } S_t \\
\delta H_G &= \int_{\Sigma_t} (\mathcal{P}^{ab} \delta h_{ab} + \mathcal{H}_{ab} \delta p^{ab} - \mathcal{C} \delta N - 2\mathcal{C}_a \delta N^a) d^3y \\
(16\pi)\mathcal{P}^{ab} &= N\sqrt{h}G^{ab} - \sqrt{h}(N^{[ab} - h^{ab}N^{c]}_{\phantom{ab}c}) \\
&\quad + (16\pi)[2p^{c(a}N^{b)}_{\phantom{ab}c} - \sqrt{h}(\frac{1}{\sqrt{h}}p^{ab}N^c)_{|c}] \\
&\quad + (16\pi)^2[\frac{2N}{\sqrt{h}}(p^a_c p^{bc} - \frac{1}{2}pp^{ab}) - \frac{N}{2\sqrt{h}}(p^{cd}p_{cd} - \frac{1}{2}p^2)h^{ab}] \\
\mathcal{H}_{ab} &= (16\pi)\frac{2N}{\sqrt{h}}(p_{ab} - \frac{1}{2}ph_{ab}) + 2N_{(a|b)} \\
\mathcal{C} &= \frac{\sqrt{h}}{16\pi}({}^3R + K^2 - K^{ab}K_{ab}) \\
\mathcal{C}^a &= \frac{\sqrt{h}}{16\pi}(K_a{}^b - K\delta_a{}^b)_{|b}
\end{aligned}$$

## Variation of electromagnetic Hamiltonian

$$\begin{aligned}
\delta H_E &= \int_{\Sigma_t} (-\frac{1}{2}N\sqrt{h}\mathcal{I}^{ab}\delta h_{ab} + \sqrt{h}\rho\delta N - \sqrt{h}s_a\delta N^a) \\
\mathcal{I}^{ab} &= \frac{1}{2}(E^c E_c + B^c B_c)h^{ab} - E^a E^b - B^a B^b \\
\rho &= \frac{1}{2}(E^c E_c + B^c B_c) \\
s_a &= \epsilon_{abc}E^b B^c
\end{aligned}$$

## Hamilton's equations

$$\begin{aligned}
\dot{h}_{ab} &= \mathcal{H}_{ab}, \quad \dot{p}^{ab} = -\mathcal{P}^{ab} + \frac{1}{2}N\sqrt{h}\mathcal{I}^{ab} \\
{}^3R + K^2 - K^{ab}K_{ab} &= 16\pi\rho \\
(K_a{}^b - K\delta_a{}^b)_{|b} &= -8\pi s_a
\end{aligned}$$



## **Part IV**

# **Quantum Mechanics**

# Chapter 12

## Linear Algebra



### 12.1 Linear Vector Space

#### 12.1.1 Definition

##### Definition 12.1 Linear vector space

A linear vector space is a set of elements, called vectors, which is closed under addition and multiplication by scalars. That is to say, if  $\phi$  and  $\psi$  are vectors then so is  $a\phi + b\psi$ , where  $a$  and  $b$  are arbitrary scalars. If the scalars belong to the field of complex (real) numbers, we speak of a complex (real) linear vector space. Henceforth the scalars will be complex numbers unless otherwise stated.

**Example:**

1. Discrete vectors, which may be represented as columns of complex numbers.
2. Spaces of functions of some type, for example the space of all differentiable functions

#### 12.1.2 Linear independence

##### Definition 12.2 Linear independence

A set of vectors  $\{\phi_n\}$  is said to be linearly independent if no non-trivial linear combination of them sums to zero; that is to say, if the equation  $\sum_n c_n \phi_n$  can hold only when  $c_n = 0$  for all  $n$ . If this condition does not hold, the set of vectors is said to be linearly dependent, in which case it is possible to express a member of the set as a linear combination of the others.

##### Definition 12.3 Dimension

The maximum number of linearly independent vectors in a space is called the dimension of the space.

### Definition 12.4 Base

A maximal set of linearly independent vectors is called a basis for the space. Any vector in the space can be expressed as a linear combination of the basis vectors.

## 12.1.3 Inner product

### Definition 12.5 Inner product

An inner product (or scalar product) for a linear vector space associates a scalar  $(\phi, \psi)$  with every ordered pair of vectors. It must satisfy the following properties:

1.  $(\phi, \psi) = a$  complex number
2.  $(\phi, \psi) = (\psi, \phi)^*$
3.  $(\phi, c_1\psi_1 + c_2\psi_2) = c_1(\phi, \psi_1) + c_2(\phi, \psi_2)$
4.  $(\phi, \phi) \geq 0$ , with equality holding if and only if  $\phi = 0$

### Example:

1. If  $\psi$  is the column vector with elements  $a_1, a_2, \dots$ , and  $\phi$  is the column vector with elements  $b_1, b_2, \dots$ , then

$$(\psi, \phi) = a_1^*b_1 + a_2^*b_2 + \dots$$

2. If  $\psi$  and  $\phi$  are functions of  $x$ , then

$$(\phi, \psi) = \int \phi^*(x)\psi(x)w(x)dx$$

where  $w(x)$  is some non-negative weight function.

### Definition 12.6 Norm

$$||\phi|| = (\phi, \phi)^{\frac{1}{2}}$$

### Theorem 12.1 Schwarz's inequality

$$|(\psi, \phi)|^2 \leq (\psi, \psi)(\phi, \phi)$$



**Theorem 12.2 triangle inequality**

$$||(\psi + \phi)|| \leq ||\phi|| + ||\psi||$$

**Definition 12.7 Orthonormal**

A set of vectors  $\{\phi_n\}$  is said to be orthonormal if the vectors are pairwise orthogonal and of unit norm; that is to say, their inner products satisfy  $(\psi_m, \phi_n) = \delta_{mn}$ .

**12.1.4 Dual space****Definition 12.8 Dual vector**

Corresponding to any linear vector space  $V$  there exists the dual space of linear functionals on  $V$ . A linear functional  $F$  assigns a scalar  $F(\phi)$  to each vector  $\phi$ , such that

$$F(a\phi + b\psi) = aF(\phi) + bF(\psi)$$

for any vectors for  $\phi$  and  $\psi$ , and any scalars  $a$  and  $b$ . The set of linear functionals may itself be regarded as forming a linear space  $V'$  if we define the sum of two functionals as

$$(F_1 + F_2)(\phi) = F_1(\phi) + F_2(\phi)$$

**Theorem 12.3 Riesz theorem**

There is a one-to-one correspondence between linear functionals  $F$  in  $V'$  and vectors  $f$  in  $V$ , such that all linear functionals have the form

$$F(\phi) = (f, \phi)$$

$f$  being a fixed vector, and  $\phi$  being an arbitrary vector. Thus the spaces  $V$  and  $V'$  are essentially isomorphic.

**12.1.5 Dirac's bra and ket notation**

In Dirac's notation, which is very popular in quantum mechanics, the vectors in  $V$  are called ket vectors, and are denoted as  $|\phi\rangle$ . The linear functionals in the dual space  $V'$  are called bra vectors, and are denoted as  $\langle F|$ . The numerical value of the functional is denoted as

$$F(\phi) = \langle F|\phi\rangle$$



According to the Riesz theorem, there is a one-to-one correspondence between bras and kets. Therefore we can use the same alphabetic character for the functional (a member of  $V'$ ) and the vector (in  $V$ ) to which it corresponds, relying on the bra,  $\langle F|$ , or ket,  $|F\rangle$ , notation to determine which space is referred to. So

$$\langle F|\phi\rangle = (F, \phi)$$

Note that the Riesz theorem establishes, by construction, an antilinear correspondence between bras and kets. If  $\langle F| \leftrightarrow |F\rangle$ , then

$$c_1^* \langle F_1| + c_2^* \langle F_2| \leftrightarrow c_1 |F_1\rangle + c_2 |F_2\rangle$$

## 12.2 Linear Operators

### Definition 12.9 Linear operators

An operator on a vector space maps vectors onto vectors. A linear operator satisfies

$$A(c_1\psi_1 + c_2\psi_2) = c_1A(\psi_1) + c_2A(\psi_2)$$

Define the sum and product of operators,

$$\begin{aligned}(A + B)\psi &= A\psi + B\psi \\ AB\psi &= A(B\psi)\end{aligned}$$

Define their action to the left on bra vectors as

$$(\langle\phi|A)\psi = \langle\phi|(A\psi)$$

So we may define the operation of  $A$  on the bra space of functionals as

$$AF_\phi(\psi) = F_\phi(A\psi)$$

According to the Riesz theorem there must exist a ket vector  $\chi$  such that

$$AF_\phi(\psi) = (\chi, \psi) = F_\chi(\psi)$$

Define operator  $A^\dagger$  as

$$AF_\phi = F_{A^\dagger\chi}$$

Therefore,

$$\begin{aligned}(A^\dagger\phi, \psi) &= (\phi, A\psi) \\ \langle\psi|A^\dagger|\phi\rangle^* &= \langle\phi|A|\psi\rangle\end{aligned}$$





**Definition 12.10 Outer product**

$$(|\psi\rangle\langle\phi|)|\lambda\rangle \equiv |\psi\rangle\langle\phi|\lambda\rangle$$

**Definition 12.11 Trace**

$$\text{Tr} A \equiv \sum \langle u_j | A | u_j \rangle$$

where  $\{u_j\}$  may be any orthonormal basis. It can be shown that the value of  $\text{Tr} A$  is independent of the particular orthonormal basis that is chosen for its evaluation.

**Proposition 12.1**

$$\begin{aligned} (cA)^\dagger &= c^* A^\dagger \\ (A+B)^\dagger &= A^\dagger + B^\dagger \\ (AB)^\dagger &= B^\dagger A^\dagger \\ (|\psi\rangle\langle\phi|)^\dagger &= |\phi\rangle\langle\psi| \end{aligned}$$



## 12.3 Self-Adjoint operators

**Definition 12.12 Self-Adjoint operators**

An operator  $A$  that is equal to its adjoint  $A^\dagger$  is called self-adjoint. This means that it satisfies

$$\langle\phi|A|\psi\rangle = \langle\psi|A|\phi\rangle^*$$



and that the domain of  $A$  coincides with the domain of  $A^\dagger$ . An operator that only satisfies above equation is called Hermitian.



**Theorem 12.4**

If  $\langle \psi | A | \psi \rangle = \langle \psi | A | \psi \rangle^*$  for all  $|\psi\rangle$ , then it follows that  $\langle \phi_1 | A | \phi_2 \rangle = \langle \phi_2 | A | \phi_1 \rangle^*$  for all  $|\phi_1\rangle$  and  $|\phi_2\rangle$ , and hence that  $A = A^\dagger$ .

If an operator acting on a certain vector produces a scalar multiple of that same vector,

$$A|\phi\rangle = a|\phi\rangle$$

we call the vector  $|\phi\rangle$  an eigenvector and the scalar  $a$  an eigenvalue of the operator  $A$ . The antilinear correspondence between bras and kets, and the definition of the adjoint operator  $A^\dagger$ , imply that the left-handed eigenvalue equation

$$\langle \phi | A^\dagger = a^* \langle \phi |$$

**Theorem 12.5**

If  $A$  is a Hermitian operator then all of its eigenvalues are real.

**Theorem 12.6**

Eigenvectors corresponding to distinct eigenvalues of a Hermitian operator must be orthogonal.

If the orthonormal set of vectors  $\{\phi_i\}$  is complete, then we can expand an arbitrary vector  $|v\rangle$  in terms of it:

$$|v\rangle = \sum |\phi_i\rangle (\langle \phi_i | v \rangle) = \left( \sum |\phi_i\rangle \langle \phi_i| \right) |v\rangle$$

So,

$$\sum |\phi_i\rangle \langle \phi_i| = I$$

If  $A|\phi_i\rangle = a_i|\phi_i\rangle$  and the eigenvectors form a complete orthonormal set, then the operator can be reconstructed in a useful diagonal form in terms of its eigenvalues and eigenvectors:

$$A = \sum a_i |\phi_i\rangle \langle \phi_i|$$

We can define a function of an operator

$$f(A) = \sum f(a_i) |\phi_i\rangle \langle \phi_i|$$

The Hermitian operators in a finite N-dimensional vector space have complete sets of eigenvectors. But This statement does not carry over to infinite-dimensional spaces. A Hermitian operator in an infinite dimensional vector space may or may not possess a complete set of eigenvectors, depending upon the precise nature of the operator and the vector space. Instead, we have spectral theorem.



**Theorem 12.7**

To each self-adjoint operator  $A$  there corresponds a unique family of projection operators,  $E(\lambda)$ , for real  $\lambda$ , with the properties:

1. If  $\lambda_1 < \lambda_2$  then  $E(\lambda_1)E(\lambda_2) = E(\lambda_2)E(\lambda_1) = E(\lambda_1)$
2. If  $\epsilon > 0$ , then  $E(\lambda + \epsilon)|\psi\rangle \rightarrow E(\lambda)|\psi\rangle$  as  $\epsilon \rightarrow 0$
3.  $E(\lambda)|\psi\rangle \rightarrow 0$  as  $\lambda \rightarrow -\infty$
4.  $E(\lambda)|\psi\rangle \rightarrow |\psi\rangle$  as  $\lambda \rightarrow \infty$
5.  $\int_{-\infty}^{\infty} \lambda dE(\lambda) = A$

We can define a function of an operator

$$f(A) = \int_{-\infty}^{\infty} f(\lambda) dE(\lambda)$$

Following Dirac's pioneering formulation, it has become customary in quantum mechanics to write a formal eigenvalue equation for an operator such as  $Q$  that has a continuous spectrum,

$$Q|q\rangle = q|q\rangle$$

The orthonormality condition for the continuous case takes the form

$$\langle q'|q''\rangle = \delta(q - q')$$

Evidently the norm of these formal eigenvectors is infinite, since  $\langle q|q\rangle \rightarrow \infty$ . Instead of the spectral theorem for  $Q$ , Dirac would write

$$Q = \int_{-\infty}^{\infty} q|q\rangle\langle q|dq$$

Dirac's formulation does not fit into the mathematical theory of Hilbert space, which admits only vectors of finite norm. The projection operator formally given by

$$E(\lambda) = \int_{-\infty}^{\lambda} |q\rangle\langle q|dq$$

is well defined in Hilbert space, but its derivative does not exist within the Hilbert space framework.

**Theorem 12.8**

If  $A$  and  $B$  are self-adjoint operators, each of which possesses a complete set of eigenvectors, and if  $AB = BA$ , then there exists a complete set of vectors which are eigenvectors of both  $A$  and  $B$ .



Let  $(A, B, \dots)$  be a set of mutually commutative operators that possess a complete set of common eigenvectors. Corresponding to a particular eigenvalue for each operator, there may be more than one eigenvector. If, however, there is no more than one eigenvector (apart from the arbitrary phase and normalization) for each set of eigenvalues  $(a_n, b_m, \dots)$ , then the operators  $(A, B, \dots)$  are said to be a complete commuting set of operators.

**Theorem 12.9**

Any operator that commutes with all members of a complete commuting set must be a function of the operators in that set.



## 12.4 Rigged Hilbert space

**Definition 12.13 Rigged Hilbert space**

Formally, a rigged Hilbert space consists of a Hilbert space  $\mathcal{H}$ , together with a subspace  $\Phi$  which carries a finer topology, that is one for which the natural inclusion  $\Phi \subseteq \mathcal{H}$  is continuous. It is no loss to assume that  $\Phi$  is dense in  $\mathcal{H}$  for the Hilbert norm. We consider the inclusion of conjugate space  $\mathcal{H}^X$  in  $\Phi^X$ .  $\Phi^X$  is the space of  $\tau_\Phi$  continuous antilinear functional on  $\Phi$ .

For any  $\phi \in \Phi$ ,  $F \in \Phi^X$ , we define

$$\langle \phi | F \rangle \equiv F(\phi)$$

$$\langle F | \phi \rangle \equiv [F(\phi)]^*$$

Now by applying the Riesz representation theorem we can identify  $\mathcal{H}^X$  with  $\mathcal{H}$ . Therefore, the definition of rigged Hilbert space is in terms of a sandwich:

$$\Phi \subseteq \mathcal{H} \subseteq \Phi^X$$



There may or may not exist any solutions to the eigenvalue equation  $A|a_n\rangle = a_n|a_n\rangle$  for a self-adjoint operator  $A$  on an infinite-dimensional vector space. However, the generalized spectral theorem asserts that if  $A$  is self-adjoint in  $\mathcal{H}$  then a complete set of eigenvectors exists in the extended space  $\Phi^X$ . The precise conditions for the proof of this theorem are rather technical, so the interested reader is referred to *Gel'fand and Vilenkin (1964)* for further details.

There are many examples of rigged-Hilbert-space triplets. A Hilbert space  $\mathcal{H}$  is formed by those functions that are square-integrable. That is,  $\mathcal{H}$  consists of those functions  $\psi(x)$  for which

$$\langle \psi | \psi \rangle = \int_{-\infty}^{\infty} |\psi(x)|^2 dx \text{ is finite}$$

A nuclear space  $\Phi$  is made up of functions  $\psi(x)$  which satisfy the infinite set of conditions,

$$\int_{-\infty}^{\infty} |\psi(x)|^2 (1 + |x|)^m dx \text{ is finite for } m = 0, 1, 2, \dots$$




The functions  $\psi(x)$  which make up  $\Phi$  must vanish more rapidly than any inverse power of  $x$  in the limit  $|x| \rightarrow \infty$ . The extended space  $\Phi^X$ , which is conjugate to  $\Phi$ , consists of those functions  $\chi(x)$  for which

$$\langle \chi | \psi \rangle = \int_{-\infty}^{\infty} \chi^*(x) \psi(x) dx \text{ is finite for any } \psi \text{ in } \Phi$$

In addition to the functions of finite norm, which also lie in  $\mathcal{H}$ ,  $\Phi^X$  will contain functions that are unbounded at infinity provided the divergence is no worse than a power of  $x$ . Hence  $\Phi^X$  contains  $e^{ikx}$ , which is an eigenfunction of the operator  $D = i \frac{d}{dx}$ . It also contains the Dirac delta function,  $\delta(x - \lambda)$ , which is an eigenfunction of the operator  $X$ , defined by  $X\psi(x) = x\psi(x)$ . These two examples suffice to show that rigged Hilbert space seems to be a more natural mathematical setting for quantum mechanics than is Hilbert space.

## 12.5 Unitary operators

### Definition 12.14 Unitary operator

A unitary operator is a bounded linear operator  $U : H \rightarrow H$  on a Hilbert space  $H$  that satisfies  $UU^\dagger = U^\dagger U = I$ , where  $U^\dagger$  is the adjoint of  $U$ , and  $I : H \rightarrow H$  is the identity operator. 

Consider a family of unitary operators,  $U(s)$ , that depend on a single continuous parameter  $s$ . Let  $U(0) = I$  be the identity operator, and let  $U(s_1 + s_2) = U(s_1)U(s_2)$ . We can demonstrate that

$$\left. \frac{dU}{ds} \right|_{s=0} = iK \text{ with } K = K^\dagger$$

The Hermitian operator  $K$  is called the generator of the family of unitary operators because it determines  $U(s)$ , not only for infinitesimal  $s$ , but for all  $s$ . This can be shown by differentiating

$$U(s_1 + s_2) = U(s_1)U(s_2)$$

with respect to  $s_2$  and we can get

$$\left. \frac{dU}{ds} \right|_{s=s_1} = U(s_1)iK$$

This first order differential equation with initial condition  $U(0) = I$  has the unique solution

$$U(s) = e^{iKs}$$



## 12.6 Antiunitary operators

### Definition 12.15 Antiunitary operator

In mathematics, an antiunitary transformation, is a bijective antilinear map

$$U : H_1 \rightarrow H_2$$

between two complex Hilbert spaces such that

$$\langle Ux, Uy \rangle = \overline{\langle x, y \rangle}$$

for all  $x$  and  $y$  in  $H_1$ , where the horizontal bar represents the complex conjugate. If additionally one has  $H_1 = H_2$  then  $U$  is called an antiunitary operator.

### Proposition 12.2

1.  $\langle Ux, Uy \rangle = \overline{\langle x, y \rangle} = \langle y, x \rangle$  holds for all elements  $x, y$  of the Hilbert space and an antiunitary  $U$ .
2. When  $U$  is antiunitary then  $U^2$  is unitary. This follows from

$$\langle U^2x, U^2y \rangle = \overline{\langle Ux, Uy \rangle} = \langle x, y \rangle$$

3. For unitary operator  $V$  the operator  $VK$ , where  $K$  is complex conjugate operator, is antiunitary. The reverse is also true, for antiunitary  $U$  the operator  $UK$  is unitary.
4. For antiunitary  $U$  the definition of the adjoint operator  $U^*$  is changed into

$$\langle U^*x, y \rangle = \overline{\langle x, Uy \rangle}$$

5. The adjoint of an antiunitary  $U$  is also antiunitary and  $UU^* = U^*U = 1$ .



# Chapter 13

## Formulation of quantum mechanics



### 13.1 Axioms of quantum mechanics

1. The properties of a quantum system are completely defined by specification of its state vector  $|\psi\rangle$ . The state vector is an element of a complex Hilbert space  $\mathcal{H}$  called the space of states.
2. With every physical property  $A$  (energy, position, momentum, angular momentum, ...) there exists an associated linear, Hermitian operator  $A$  (usually called observable), which acts in the space of states. The eigenvalues of the operator are the possible values of the physical properties.
3.
  - If  $|\psi\rangle$  is the vector representing the state of a system and if  $|\phi\rangle$  represents another physical state, there exists a probability  $P(|\psi\rangle, |\phi\rangle)$  of finding  $|\psi\rangle$  in state  $|\phi\rangle$ , which is given by the squared modulus of the scalar product on  $\mathcal{H}$  :  $P(|\psi\rangle, |\phi\rangle) = |\langle\psi|\phi\rangle|^2$  (Born Rule)
  - If  $A$  is an observable with eigenvalues  $a_k$  and eigenvectors  $|k\rangle$ , given a system in the state  $|\psi\rangle$ , the probability of obtaining  $a_k$  as the outcome of the measurement of  $A$  is  $|\langle k|\psi\rangle|^2$ . After the measurement the system is left in the state projected on the subspace of the eigenvalue  $a_k$  (Wave function collapse).
4. The evolution of a closed system is unitary. The state vector  $|\psi(t)\rangle$  at time  $t$  is derived from the state vector  $|\psi(t_0)\rangle$  at time  $t_0$  by applying a unitary operator  $U(t, t_0)$ , called the evolution operator:  $|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle$ .

### 13.2 Transformations of States

A transformation of states can be described by  $|\psi\rangle \rightarrow U(\tau)|\psi\rangle \equiv |\psi'\rangle$ . And we demand that

$$|\langle\phi|\psi\rangle| = |\langle\phi'|\psi'\rangle|$$

#### Theorem 13.1 Wigner Theorem

Any mapping of the vector space onto itself that preserves the value of  $|\langle\phi|\psi\rangle|$  may be implemented by an operator  $U$  with  $U$  being either unitary (linear) or antiunitary (antilinear).



### Continuous transformation

Only linear operators can describe continuous transformations because every continuous transformation has a square root. Suppose, for example, that  $U(l)$  describes a displacement through the distance  $l$ . This can be done by two displacements of  $U(l/2)$ , and hence  $U(l) = U(l/2)U(l/2)$ . The product of two antilinear operators is linear, since the second complex conjugation nullifies the effect of the first. Thus, regardless of the linear or antilinear character of  $U(l/2)$ , it must be the case that  $U(l)$  is linear. A continuous operator cannot change discontinuously from linear to antilinear as a function of  $l$ , so the operator must be linear for all  $l$ .

### Transformations of observables

For an observable  $Q$ ,

$$\langle \phi' | Q | \phi' \rangle = \langle \phi | U^{-1} Q U | \phi \rangle$$

If  $U(\tau)^{-1} Q U(\tau) = \tau Q$ , we can prove that

$$U|q\rangle = |\tau q\rangle$$

Here,  $|q\rangle$  is the eigenvector of  $Q$  with eigenvalue  $q$ .

## 13.3 Schrödinger equation

$U(t, t_0)$  is unitary and  $U(t_2, t_0) = U(t_2, t_1)U(t_1, t_0)$ . We can define  $H(t_0)$  as

$$\left. \frac{d}{dt} U(t, t_0) \right|_{t=t_0} = -iH(t_0) \text{ with } H(t_0) = H(t_0)^\dagger$$

We can demonstrate that

$$\left. \frac{dU(t, t_0)}{dt} \right|_{t=t_1} = -iH(t_1)U(t_1, t_0)$$

The formal solution of the differential equation is

$$U(t, t_0) = I + (-i)^n \sum_{n=1}^{\infty} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n H(t_1)H(t_2) \cdots H(t_n)$$

Suppose that  $T$  stands for time ordering, placing all operators evaluated at later times to the left, the above equation can be written as

$$U(t, t_0) = I + \frac{(-i)^n}{n!} \sum_{n=1}^{\infty} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n T \{ H(t_1)H(t_2) \cdots H(t_n) \} \equiv \exp \left[ -iT \left\{ \int_{t_0}^t H(t') dt' \right\} \right]$$

If the Hamiltonian operator  $H$  is time-dependent but the  $H$ 's at different times commute. The equation above can be simplified to

$$U(t, t_0) = \exp \left[ -i \int_{t_0}^t H(t') dt' \right]$$





If the  $H$  is time-independent, then

$$U(t, t_0) = \exp[-iH(t - t_0)]$$

Since  $|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle$ , we can derive the Schrödinger equation

$$\frac{d|\psi(t)\rangle}{dt} = -iH(t)|\psi(t)\rangle$$

The expectation value of an observable  $Q$  is  $\langle\psi|Q|\psi\rangle$ , denoted by  $\langle Q\rangle$ . We can then derive that

$$\frac{d\langle Q\rangle}{dt} = -i \left\{ \langle [Q, H] \rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle \right\}$$

This is called Ehrenfest's theorem.

## 13.4 Position operators

In three dimensional space, for a particle, we have three operators corresponding to the observations of its position in space,  $\mathbf{X} = (X_1, X_2, X_3)$ . If the particle has some other internal degrees of freedom, then  $\mathbf{X}$  plus some other observables  $S$ 's will form a complete commuting set of operators. The eigenstate state will be denoted by  $|\mathbf{x}, s\rangle$ , satisfying that

$$X_i|\mathbf{x}, s\rangle = x_i|\mathbf{x}, s\rangle$$

It describes a particle posited in  $\mathbf{x}$  with internal state  $s$ . And we will normalize  $|\mathbf{x}, s\rangle$  by

$$\langle \mathbf{x}, s' | \mathbf{x}, s \rangle = \delta_{ss'} \delta(\mathbf{x} - \mathbf{x}')$$

## 13.5 Momentum operators and canonical quantization

Since  $\mathbf{X}$  plus some other observables  $S$ 's form a complete commuting set of operators. So, the momentum operators can not be independent of them. Numerous experiments shows that the position and momentum of particles can not be measured simultaneously. So, we expect  $[X, P] \neq 0$ .

**Guess** For a system which has a classical correspondence, the classical equation of motion of a particle is

$$\begin{aligned} \dot{x} &= [x, H_C(x, p, t)]_C \\ \dot{p} &= [p, H_C(x, p, t)]_C \end{aligned}$$

$[\ ]_C$  is the Poisson bracket in classical mechanics. In quantum mechanics,

$$\begin{aligned} \frac{d\langle X \rangle}{dt} &= -i\langle [X, H] \rangle \\ \frac{d\langle P \rangle}{dt} &= -i\langle [P, H] \rangle \end{aligned}$$



If we assume that the classical equation of motion of a particle is an approximation of quantum mechanics, we may expect

$$[ \quad ] = i[ \quad ]_C$$

Since the Poisson bracket in classical mechanics and commutation bracket in quantum mechanics have the same algebra structure. To get the right classical equation of motion of the particle, we demand that

$$[X_i, X_j] = 0 \quad [X_i, X_j] = 0 \quad [X_i, P_j] = i\delta_{ij}$$

and

$$H = H_C(X, P, t)$$

For a general system, we formally define momentum operator  $\mathbf{P}$  by

$$[X_i, P_j] = i\delta_{ij}$$

The form of  $H$  can not be given as a priori, which can be specified only by the hints from classical theory and experiments.

## 13.6 Momentum operators and translation of states

### Theorem 13.2

$$\exp(iG\lambda)A\exp(-iG\lambda) = A + i\lambda[G, A] + \cdots + \frac{i^n \lambda^n}{n!} [G, [G, [G, \cdots [G, A]]] \cdots] + \cdots$$

Define  $T(\mathbf{a}) \equiv e^{-i\mathbf{P} \cdot \mathbf{a}}$  We can get

$$T(\mathbf{a})^{-1} \mathbf{X} T(\mathbf{a}) = \mathbf{X} + \mathbf{a}$$

$$T(\mathbf{a})|x\rangle = |x + \mathbf{a}\rangle$$

So,  $T(\mathbf{a})$  is the space translation operator. Now, we can also define the momentum operator as the generator of space translation.

## 13.7 Angular momentum operators and rotation of states

We define the angular momentum operators  $\mathbf{J}$  as the generator of rotation.

$$R(\boldsymbol{\theta}) \equiv e^{-i\mathbf{J} \cdot \mathbf{n}\theta}$$

If the operator  $\mathbf{M} = (M_1, M_2, M_3)$  is a vector in configuration space and can be rotated by  $R$ , then we can demonstrate that

$$[J_i, M_j] = i\epsilon_{ijk}M_k$$

Especially,

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$



### Orbital angular momentum

Orbital angular momentum of a particle is defined as  $\mathbf{L} = \mathbf{X} \times \mathbf{P}$ . It is the generator of rotation of the position of the particle, since

$$[L_i, X_j] = i\epsilon_{ijk}X_k \quad [L_i, P_j] = i\epsilon_{ijk}P_k \quad [L_i, L_j] = i\epsilon_{ijk}L_k$$

### Spin angular momentum

Experiments show that some microscopic particles possess a property called spin. The state of the spin is denoted by  $|s\rangle$ . The corresponding operators are  $\mathbf{S} = [S_1, S_2, S_3]$ , which measure the spin along the 1, 2, 3 direction. Spin operator is the generator of rotation of the spin of the particle, so we have

$$[S_i, S_j] = i\epsilon_{ijk}S_k$$

And the rotation of position and spin is independent, so

$$[S_i, L_j] = 0$$

### Total angular momentum

The total angular momentum of the particle is

$$\mathbf{J} = \mathbf{L} + \mathbf{S}$$

It is the generator of the rotation of the entire system, which is equivalent to the rotation of the coordinates in opposite direction.

## 13.8 Heisenberg picture

Define

$$Q_H = U^\dagger(t, t_0)QU(t, t_0)$$

We can derive that

$$\frac{dQ_H(t)}{dt} = -i[Q_H(t), H_H(t)] + \left(\frac{\partial Q}{\partial t}\right)_H$$

Here,  $H_H(t) = U^\dagger(t, t_0)H(t)U(t, t_0)$  If the state of the system at  $t_0$  is  $|\phi_0\rangle$ , then

$$\langle Q \rangle_t = \langle \phi(t) | Q | \phi(t) \rangle = \langle \phi_0 | Q_H(t) | \phi_0 \rangle$$

If the state  $|q\rangle$  is the eigenstate of the  $Q$  with the eigenvalue  $q$ , then  $U^\dagger(t, t_0)|q\rangle$  is the eigenstate of the  $Q_H$  with eigenvalue  $q$ , which can be denoted by  $|q_H(t)\rangle$ , so we have

$$\langle q | \phi(t) \rangle = \langle q_H(t) | \phi_0 \rangle$$



## 13.9 Symmetries and conservation laws

Let  $U = e^{iKs}$  be a continuous unitary transformation with generator  $K = K^\dagger$ . To say that the Hamiltonian operator  $H$  is invariant under this transformation means that

$$U(s)^{-1}H(t)U(s) = H(t)$$

Then we can deduce that

$$[K, H(t)] = 0$$

Usually,  $K$  does not depend on time explicitly. If the above equation hold for all  $t$ , then in Heisenberg picture,

$$K_H(t) = K \quad |k_H(t)\rangle = |k\rangle$$

So,

$$\langle K \rangle_t = \langle K \rangle_{t_0} \quad \langle k|\phi(t)\rangle = \langle k|\phi_0\rangle$$

The probability distribution of the measurement of the observable  $K$  will not change with time for an arbitrary initial state. We can assume that the  $K$  is a constant of motion.



**Note:** The concept of a constant of motion should not be confused with the concept of a stationary state. Suppose that the Hamiltonian operator  $H$  is independent of  $t$ , and that the initial state vector is an eigenvector of  $H$ ,  $|\phi_0\rangle = |E_n\rangle$  with  $H|E_n\rangle = E_n|E_n\rangle$ . This describes a state having a unique value of energy  $E_n$ . So

$$|\phi(t)\rangle = e^{-iE_nt}|\phi_0\rangle$$

From this result it follows that the average of any dynamical variable  $R$ ,

$$\langle \phi(t)|R|\phi(t)\rangle = \langle E_n|R|E_n\rangle$$

is independent of  $t$  for such a state. By considering functions of  $R$  we can further show that the probability distribution is independent of time. In a stationary state the averages and probabilities of all dynamical variables are independent of time, whereas a constant of motion has its average and probabilities independent of time for all states.



# Chapter 14

## Coordinate and Momentum Representation



### 14.1 Coordinate representation

To form a representation of an abstract linear vector space, one chooses a complete orthonormal set of basis vectors  $\{|u_i\rangle\}$  and represents an arbitrary vector  $|\psi\rangle$  by its expansion coefficients  $\{c_i\}$ , where  $|\psi\rangle = \sum c_i |u_i\rangle$ . The array of coefficients  $\langle u_i|\psi\rangle$  can be regarded as a column vector (possibly of infinite dimension), provided the basis set is discrete.

Coordinate representation is obtained by choosing as the basis set the eigenvectors  $\{|\mathbf{x}\rangle\}$  of the position operator. Since this is a continuous set, the expansion coefficients define a function of a continuous variable,

$$\psi(\mathbf{x}) \equiv \langle \mathbf{x}|\psi\rangle$$

We can show that the inner product of the state vector in coordinate representation is

$$\langle \phi|\psi\rangle = \int \phi^*(\mathbf{x})\psi(\mathbf{x})d^3\mathbf{x}$$

It is a matter of taste whether one says that the set of functions forms a representation of the vector space, or that the vector space consists of the functions  $\psi(\mathbf{x})$ .

The action of an operator  $A$  on the function space is related to its action on the abstract vector space by the rule

$$A\psi(\mathbf{x}) \equiv \langle \mathbf{x}|A|\psi\rangle$$

The action of an position operator in coordinate representation is

$$\mathbf{X}\psi(\mathbf{x}) = \mathbf{x}\psi(\mathbf{x})$$

The action of an momentum operator in coordinate representation is

$$\mathbf{P}\psi(\mathbf{x}) = -i\nabla\psi(\mathbf{x})$$

For a spin-less particle in the scalar potential  $W(\mathbf{x})$ ,  $H = \frac{\mathbf{P}^2}{2M} + W(\mathbf{X})$ . The equation of motion in the coordinate representation is

$$\left[ -\frac{1}{2M}\nabla^2 + W(\mathbf{x}) \right] \psi(\mathbf{x}, t) = i\frac{\partial}{\partial t}\psi(\mathbf{x}, t)$$

## 14.2 Galilei transformation of Schrödinger equation

For simplicity we shall treat only one spatial dimension. Let us consider two frames of reference:  $F$  with coordinates  $x$  and  $t$ , and  $F'$  with coordinates  $x'$  and  $t'$ .  $F'$  is moving uniformly with velocity  $v$  relative to  $F$ , so that

$$x = x' + vt' \quad t = t'$$

The potential energy is given by  $W(x, t)$  in  $F$ , and by  $W'(x', t')$  in  $F'$ , with

$$W(x, t) = W'(x', t')$$

Because the requirement of invariance under Galilei transformation, we expect in  $F'$  the Schrödinger equation has the form

$$\left[ -\frac{1}{2M} \frac{\partial^2}{\partial x'^2} + W'(x') \right] \psi'(x', t') = i \frac{\partial}{\partial t'} \psi'(x', t')$$

where  $\psi'(x', t')$  is the wave function in  $F'$ . The probability density at a point in space-time must be the same in the two frames of reference

$$|\psi(x, t)|^2 = |\psi'(x', t')|^2$$

and hence we must have

$$\psi(x, t) = e^{if} \psi'(x', t')$$

where  $f$  is a real function of the coordinates. Put all the conditions above together, we can derive

$$f(x, t) = Mvx - \frac{1}{2}Mv^2t$$

apart from an irrelevant constant term.

## 14.3 Probability flux and conditions on wave functions

Define the probability flux vector

$$\mathbf{J}(\mathbf{x}, t) = \frac{1}{M} \text{Im}(\psi^* \nabla \psi)$$

We can get a continuity equation

$$\frac{d}{dt} |\psi(\mathbf{x}, t)|^2 + \nabla \cdot \mathbf{J}(\mathbf{x}, t) = 0$$

Applying the divergence theorem, we obtain

$$\frac{\partial}{\partial t} \int_{\Omega} |\psi(\mathbf{x}, t)|^2 d^3x = - \oint_{\sigma} \mathbf{J} \cdot d\mathbf{s}$$

The equations of continuity require that the probability flux  $\mathbf{J}(\mathbf{x}, t)$  be continuous across any surface, since otherwise the surface would contain sources or sinks. Although this condition applies to all surfaces, implying that  $\mathbf{J}(\mathbf{x}, t)$  must be everywhere continuous, its practical applications are mainly to surfaces separating regions in which the potential has different analytic forms. Usually, we have the following conditions,



1.

$$\psi(x)|_{x+0} = \psi(x)|_{x-0} \quad \frac{d\psi}{dx}|_{x+0} = \frac{d\psi}{dx}|_{x-0}$$

2.

$$\psi(x)|_{x+0} = \psi(x)|_{x-0} = 0 \quad \frac{d\psi}{dx}|_{x+0} - \frac{d\psi}{dx}|_{x-0} \text{ is finite}$$

Consider next the behavior at a singular point, assumed for convenience to be the origin of coordinates. Let  $S$  be a small sphere of radius  $r$  surrounding the singularity. The probability that the particle is inside  $S$  must be finite. Suppose that  $\psi = \frac{u}{r^\alpha}$ , where  $u$  is a smooth function that does not vanish at  $r = 0$ . Then we must have  $|\psi|^2 r^3$  convergent at the origin, which implies that  $\alpha < \frac{3}{2}$ .

The net outward flow through the surface  $S$  is  $F = \oint_S J \cdot dS$ . It must vanish in the limit  $r \rightarrow 0$ , since otherwise the origin would be a point source or sink. One has  $\frac{\partial \psi}{\partial r} = r^{-\alpha} \frac{\partial u}{\partial r} - \alpha u r^{-\alpha-1}$ . The second term does not contribute to the flux, so we obtain

$$F \propto r^{2-2\alpha}$$

where the integration is over solid angle. If the integral does not vanish, then we must have  $\alpha < 1$  in order for  $F$  to vanish in the limit  $r \rightarrow 0$ . This is a stronger condition than that derived from the probability density.

Since  $|\psi|^2$  is a probability density, it must vanish sufficiently rapidly at infinity so that its integral over all configuration space is convergent and equal to 1.

The conditions that we have discussed apply to wave functions  $\psi(x)$  which represent physically realizable states, but they need not apply to the eigenfunctions of operators that represent observables. Those eigenfunctions,  $\chi(x)$ , which play the role of filter functions in computing probabilities, are only required to lie in the extended space,  $\Phi^X$ , of the rigged-Hilbert-space triplet. It has been suggested that  $\psi(x)$  be restricted to the nuclear space  $\Phi$ , rather than merely to the Hilbert space  $\mathcal{H}$ . In many cases this would amount to requiring that  $\psi(x)$  should vanish at infinity more rapidly than any inverse power of the distance.

## 14.4 Path integrals

### Theorem 14.1 Gaussian integration

$$\int dx e^{-\frac{1}{2}ax^2 + Jx} = \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} e^{\frac{J^2}{2a}}$$



The time evolution of a quantum state vector,  $|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle$ , can be regarded as the propagation of an amplitude in configuration space,

$$\psi(x, t) = \int G(x, t; x', t_0) \psi(x', t_0) dx'$$

where

$$G(x, t; x', t_0) = \langle x, t | U(t, t_0) | x', t_0 \rangle$$



is often called the propagator.

Making use of the multiplicative property of the time development operator, it follows that the propagator can be written as

$$G(x, t; x_0, t_0) = \int \cdots \int G(x, t; x_N, t_N) \cdots G(x_1, t_1; x_0, t_0) dx_N \cdots dx_1$$

The N-fold integration is equivalent to a sum over zigzag paths that connect the initial point  $(x_0, t_0)$  to the final point  $(x, t)$ . If we now pass to the limit of  $N \rightarrow \infty$  and  $\Delta t = t_i - t_{i-1} \rightarrow 0$ , we will have the propagator expressed as a sum (or, rather, as an integral) over all paths that connect the initial point to the final point. We can show that

$$\langle x | e^{-iH\Delta t} | x' \rangle = \sqrt{\frac{M}{2i\pi\Delta t}} \exp \left\{ i \left[ \frac{M(x - x')^2}{2\Delta t^2} - V(x') \right] \Delta t \right\} \quad \Delta t \rightarrow 0$$

So,

$$G(x, t; x_0, t_0) = \lim_{N \rightarrow \infty} \int \cdots \int \left( \frac{M}{2i\pi\Delta t} \right)^{\frac{N+1}{2}} \exp \left\{ i \sum_{j=0}^N \left[ \frac{M(x_{j+1} - x_j)^2}{2\Delta t^2} - V(x_{j+1}) \right] \Delta t \right\} dx_1 \cdots dx_N$$

The final result can be expressed as

$$G(x, t; x_0, t_0) = \int \mathcal{D}[x(\tau)] e^{iS[x(\tau)]}$$

Here,  $S[x(\tau)]$  is the action associated with the path

$$S[x(\tau)] = \int_{x(\tau)} L(x, \dot{x}) d\tau$$

The integral is a functional integration over all paths  $x(\tau)$  which connect the initial point  $(x_0, t_0)$  to the final point  $(x, t)$ .

To conclude this section, let us generalize our path-integral formula to a more complicated systems. Consider a very general quantum system, described by arbitrary set of coordinates  $q_i$ , conjugate momentum  $p^i$ , and Hamiltonian  $H(q, p)$ . We can show that

$$\langle q_{k+1} | e^{-i\epsilon H} | q_k \rangle = \left( \prod_i \int \frac{dp_k^i}{2\pi} \right) \exp \left[ -i\epsilon H \left( \frac{q_{k+1} + q_k}{2}, p_k \right) \right] \exp \left[ i \sum_i p_k^i (q_{i,k+1} - q_{i,k}) \right]$$

so,

$$\langle q_N | U(t, t_0) | q_0 \rangle = \left( \prod_{i,k} \int \frac{dp_k^i dq_{i,k}}{2\pi} \right) \exp \left[ i \sum_k \left( \sum_i p_k^i (q_{i,k+1} - q_{i,k}) - \epsilon H \left( \frac{q_{k+1} + q_k}{2}, p_k \right) \right) \right]$$

There is one momentum integral for each  $k$  from 0 to  $N$ , and on coordinate integral for each  $k$  from 1 to  $N$ . The final result can be expressed as

$$\langle q_N | U(t, t_0) | q_0 \rangle = \left( \prod_i \int \mathcal{D}q(t) \mathcal{D}p(t) \right) \exp \left[ i \int_0^T dt \left( \sum_i p^i \dot{q}_i - H(q, p) \right) \right]$$

where the functions  $q(t)$  are constrained at the endpoints, but  $p(t)$  are not. The details of this generalization can be found in chapter 9.1 of *An introduction to quantum field theory* (M.E.Peskin & D.V.Schroeder)





## 14.5 Momentum representation

Momentum representation is obtained by choosing as the basis set the eigenvectors  $\{|p\rangle\}$  of the momentum operator. The orthonormality condition takes the form

$$\langle p|p'\rangle = \delta(p - p')$$

Then we can derive that

$$\langle x|p\rangle = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{p}\cdot\mathbf{x}}$$

and

$$\phi(\mathbf{p}) \equiv \langle p|\psi\rangle = \frac{1}{(2\pi)^{3/2}} \int e^{-i\mathbf{p}\cdot\mathbf{x}} \langle x|\psi\rangle d^3x$$

The effect of position operator in momentum representation is

$$\mathbf{X}\phi(\mathbf{p}) = i\nabla\phi(\mathbf{p})$$

### Bloch's Theorem

A crystal is unchanged by translation through a vector displacement of the form

$$\mathbf{R}_n = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3$$

where  $n_1, n_2$  and  $n_3$  are integers, and  $\mathbf{a}_1, \mathbf{a}_2$  and  $\mathbf{a}_3$  form the edges of a unit cell of the crystal. Corresponding to such a translation, there is a unitary operator,  $U(\mathbf{R}_n) = \exp(-i\mathbf{P} \cdot \mathbf{R}_n)$ , which leaves the Hamiltonian of the crystal invariant:

$$U^{-1}(\mathbf{R}_n) H U(\mathbf{R}_n) = H$$

These unitary operators for translations commute with each other, as well as with  $H$ , so there must exist a complete set of common eigenvectors for all of these operators,

$$H|\psi\rangle = E|\psi\rangle \quad U(\mathbf{R}_n)|\psi\rangle = c(\mathbf{R}_n)|\psi\rangle$$

By the composition relation of the translation operators, we can deduce that

$$c(\mathbf{R}_n) = \exp(-i\mathbf{p} \cdot \mathbf{R}_n)$$

So, in coordinate representation, we have

$$\psi(\mathbf{x} - \mathbf{R}_n) = U(\mathbf{R}_n)\psi(\mathbf{x}) = \exp(-i\mathbf{p} \cdot \mathbf{R}_n)\psi(\mathbf{x})$$

The vector  $\mathbf{p}$  is called the Bloch wave vector of the state. If we expand a function of the Bloch form in a series of plane waves,

$$\psi(\mathbf{x}) = \sum_{\mathbf{k}'} a(\mathbf{k}') e^{i\mathbf{k}'\cdot\mathbf{x}}$$

we can show that for all  $\mathbf{R}_n$

$$\exp(i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{R}_n) = 1$$

So  $\mathbf{G}_n \equiv \mathbf{k}' - \mathbf{k}$  is a vector of the reciprocal lattice. So, the expansion can be written as

$$\psi(\mathbf{x}) = \sum_{\mathbf{G}_m} a(\mathbf{k} + \mathbf{G}_m) e^{i(\mathbf{k} + \mathbf{G}_m)\cdot\mathbf{x}}$$



## 14.6 Harmonic oscillator

A harmonic oscillator is an object that is subject to a quadratic potential energy, which produces a restoring force against any displacement from equilibrium that is proportional to the displacement. The Hamiltonian for such an object whose motion is confined to one dimension is

$$H = \frac{1}{2M}P^2 + \frac{M\omega^2}{2}Q^2$$

where  $P$  is the momentum,  $Q$  is the position, and  $M$  is the mass.

### 14.6.1 Algebraic solution

We have the commutation relation

$$[Q, P] = i$$

and the self-adjointness of the operators  $P$  and  $Q$ ,

$$P = P^\dagger \quad Q = Q^\dagger$$

Define

$$p \equiv (M\omega)^{-1/2}P \quad q \equiv (M\omega)^{1/2}Q$$

So

$$[q, p] = i \quad H = \frac{1}{2}\omega(p^2 + q^2)$$

We further define

$$a \equiv \frac{q + ip}{\sqrt{2}}$$

We can verify that

$$[a, a^\dagger] = 1 \quad H = \frac{1}{2}\omega(aa^\dagger + a^\dagger a) = \omega(aa^\dagger - \frac{1}{2}) = \omega(a^\dagger a + \frac{1}{2})$$

Introduce  $N \equiv a^\dagger a$ , we have

$$[N, a] = -a \quad [N, a^\dagger] = a^\dagger$$

Let  $N|\nu\rangle = \nu|\nu\rangle$ , with  $\langle\nu|\nu\rangle \neq 0$ . We have

$$Na|\nu\rangle = a(N-1)|\nu\rangle = (\nu-1)a|\nu\rangle$$

If  $\nu \neq 0$ , then  $a|\nu\rangle$  cannot be 0,  $a|\nu\rangle$  must be an eigenvector of  $N$  with eigenvalue  $\nu - 1$ . On the other hand, we have

$$(\langle\nu|a^\dagger)(a|\nu\rangle) = \langle\nu|N|\nu\rangle = \nu\langle\nu|\nu\rangle$$

Since the norm must be nonnegative, it follows that  $\nu \geq 0$ , and thus an eigenvalue cannot be negative. By applying the operator  $a$  repeatedly, it would appear that one could construct an indefinitely long sequence of eigenvectors having the eigenvalues  $\nu - 1, \nu - 2, \nu - 3$  and so on. The contradiction can be avoided only if the sequence terminates with the value  $\nu = 0$



and  $a|0\rangle = 0$ .

We also have that

$$Na^\dagger|\nu\rangle = a^\dagger(N+1)|\nu\rangle = (\nu+1)a^\dagger|\nu\rangle$$

The squared norm of the vector  $a^\dagger|\nu\rangle$  is

$$\langle\nu|a\rangle\langle a^\dagger|\nu\rangle = \langle\nu|N+1|\nu\rangle = (\nu+1)\langle\nu|\nu\rangle$$

which never vanishes because  $\nu \geq 0$ . Thus  $a^\dagger|\nu\rangle$  is an eigenvector of  $N$  with eigenvalue  $\nu+1$ . By repeatedly applying the operator  $a^\dagger$ , one can construct an unlimited sequence of eigenvectors, each having an eigenvalue one unit greater than that of its predecessor. The sequence begins with the eigenvalue  $\nu = 0$ . Therefore the spectrum of  $N$  consists of the nonnegative integers,  $\nu = n$ .

The orthonormal eigenvectors of  $N$  will be denoted as  $|n\rangle$ , and we can verify that

$$|n\rangle = n^{-1/2}a^\dagger|n-1\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle$$

The matrix elements of  $a^\dagger$  and  $a$  are

$$\langle n'|a^\dagger|n\rangle = (n+1)^{1/2}\delta_{n',n+1} \quad \langle n'|a|n\rangle = (n)^{1/2}\delta_{n',n-1}$$

Finally we note that the eigenvalues and eigenvectors of the harmonic oscillator Hamiltonian are

$$H|n\rangle = E_n|n\rangle \quad E_n = (n + \frac{1}{2})\omega$$

### 14.6.2 Solution in coordinate representation

In the coordinate representation, we have

$$-\frac{1}{2M}\frac{d^2}{dx^2}\psi(x) + \frac{M\omega^2}{2}x^2\psi(x) = E\psi(x)$$

Define

$$q \equiv (M\omega)^{1/2}x \quad \lambda \equiv \frac{2E}{\omega}$$

The differential equation becomes

$$\frac{d^2u}{dq^2} + (\lambda - q^2)u = 0$$

When  $q \rightarrow \pm\infty$ , we have  $u(q) \sim e^{q^2/2}$  or  $e^{-q^2/2}$ . The first of these is unacceptable, because it diverges so severely as to be outside of both Hilbert space and rigged Hilbert space. We would seek solutions of the form

$$u(q) = H(q)e^{-\frac{1}{2}q^2}$$

We then have

$$H'' - 2qH' + (\lambda - 1)H = 0$$

It is the so-called **Hermite differential equation**. When  $\lambda - 1 = 2n$ , we have regular solutions. Solutions are **Hermite polynomials**, and will be denoted as  $H_n(q)$ . After appropriate normalization, we have

$$\psi_n(x) = \left[ \frac{\alpha}{\pi^{1/2}2^n n!} \right]^{1/2} H_n(\alpha x) e^{-\frac{1}{2}\alpha^2 x^2} \quad E_n = (n + \frac{1}{2})\omega$$

where  $\alpha \equiv (M\omega)^{1/2}$ .



### 14.6.3 Path integral solution

The propagator of harmonic oscillator in terms of path integral are

$$\begin{aligned} G(x_b, t_b; x_a, t_a) &= \int \mathcal{D}[x(t)] e^{i \int_{t_a}^{t_b} (\frac{1}{2} M \dot{x}^2 - \frac{1}{2} M \omega^2 x^2) dt} \\ &= \lim_{N \rightarrow \infty} \int \left( \frac{M}{2i\pi\Delta t} \right)^{\frac{N+1}{2}} \exp \left\{ i \sum_{j=0}^N \left[ \frac{M(x_{j+1} - x_j)^2}{2\Delta t^2} - \frac{1}{2} M \omega^2 x_{j+1}^2 \right] \Delta t \right\} dx_1 \cdots dx_N \end{aligned}$$

where

$$x_0 = x_a \quad x_{N+1} = x_b \quad \Delta t = \frac{t_b - t_a}{N+1}$$

Suppose  $x_c(t)$  is the classical path of harmonic oscillator and we define

$$\delta x(t) \equiv x(t) - x_c(t)$$

Substitute it into the equation above, the terms which is linear in  $\delta x$  can be dropped, because

$$\frac{\delta S}{\delta x} \Big|_{x(t)=x_c(t)} = 0$$

And since  $x_c(t_a) = x_a$ ,  $x_c(t_b) = x_b$ , we have

$$\delta x_0 = \delta x_{N+1} = 0$$

At last, we have

$$\begin{aligned} G(x_b, t_b; x_a, t_a) &= e^{iS_c} \lim_{N \rightarrow \infty} \int \left( \frac{M}{2i\pi\Delta t} \right)^{\frac{N+1}{2}} \exp \left\{ i \sum_{j=1, k=1}^N \delta x_j S_{jk} \delta x_k \right\} d\delta x_1 \cdots d\delta x_N \\ &= e^{iS_c} \lim_{N \rightarrow \infty} \left( \frac{M}{2i\pi\Delta t} \right)^{\frac{N+1}{2}} \sqrt{\frac{\pi^N}{\det(-iS)}} \end{aligned} \quad (14.1)$$

where

$$-iS = \frac{M}{2i\Delta t} \begin{pmatrix} 2 - \omega^2 \Delta t^2 & -1 & 0 & \cdots \\ -1 & 2 - \omega^2 \Delta t^2 & -1 & \cdots \\ 0 & -1 & 2 - \omega^2 \Delta t^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The details of the following calculation will be eliminated here, and some discussion can be found in section 2.1.4 of *Quantum Field Theory of Many-body Systems (Xiao-Gang Wen)*. The final result is

$$G(x_b, t; x_a, 0) = \left( \frac{M\omega}{2\pi i \sin \omega t} \right)^{1/2} \exp \left[ \frac{iM\omega}{2 \sin \omega t} ((x_a^2 + x_b^2) \cos \omega t - 2x_a x_b) \right]$$

In the limit  $\omega \rightarrow 0$ , we have

$$G(x_b, t; x_a, 0) = \left( \frac{M}{2\pi i t} \right)^{1/2} \exp \left[ \frac{iM}{2t} (x_b - x_a)^2 \right]$$

This is just the propagator of free particle.



# Chapter 15

## Angular Momentum



### 15.1 Eigenvalues of angular momentum operator

The commutation relations among the angular momentum operators are

$$[J_i, J_j] = \epsilon_{ijk} J_k$$

And these three operators are self-adjoint. We first introduce the operator  $J^2 = J_x^2 + J_y^2 + J_z^2$ . We can verify that  $[J^2, \mathbf{J}] = 0$ . Thus there exists a complete set of common eigenvectors of  $J^2$  and any one component of  $\mathbf{J}$ . Particularly, we have the pair of eigenvalue equations

$$J^2|\beta, m\rangle = \beta|\beta, m\rangle \quad J_z|\beta, m\rangle = m|\beta, m\rangle$$

Since

$$\langle\beta, m|J^2|\beta, m\rangle = \langle\beta, m|J_x^2|\beta, m\rangle + \langle\beta, m|J_y^2|\beta, m\rangle + \langle\beta, m|J_z^2|\beta, m\rangle$$

we have  $m^2 \leq \beta$ . Thus for a fixed value of  $\beta$  there must be maximum and minimum values for  $m$ .

Define

$$J_+ \equiv J_x + iJ_y \quad J_- \equiv J_x - iJ_y$$

we have the commutation relations

$$[J_z, J_+] = J_+ \quad [J_z, J_-] = -J_- \quad [J_+, J_-] = 2J_z$$

So

$$J_z J_+ |\beta, m\rangle = J_+ (J_z + 1) |\beta, m\rangle = (m + 1) J_+ |\beta, m\rangle$$

Therefore, either  $J_+ |\beta, m\rangle$  is an eigenvector of  $J_z$  with the raised eigenvalue  $m+1$ , or  $J_+ |\beta, m\rangle = 0$ . Now for fixed  $\beta$  there is a maximum value of  $m$ , which we shall denote as  $j$ . It must be the case that

$$J_+ |\beta, j\rangle = 0$$

Since

$$J_+ J_- = J^2 - J_z^2 - J_z$$

it is obvious that  $\beta = j(j + 1)$ . By similar method, we can show the minimum eigenvalue of  $J_z$  for fixed  $\beta$  satisfy that  $\beta = k(k - 1)$ . So, we have  $k = -j$ .

We have thus shown the existence of a set of eigenvectors corresponding to integer spaced  $m$  values in the range  $-j \leq m \leq j$ . Since the difference between the maximum value  $j$  and the minimum value  $-j$  must be an integer, it follows that  $j = \text{integer}/2$ . Henceforth we shall adopt the common and more convenient notation of labeling the eigenvectors by  $j$  instead of by  $\beta$ . Thus the vector that was previously denoted as  $|\beta, m\rangle$  will now be denoted as  $|j, m\rangle$ .

## 15.2 Orbital Angular Momentum and Spin

Let  $\psi(\mathbf{x})$  be a one-component state function in coordinate representation. When it is subjected to a rotation it is transformed into

$$\mathbf{R}\psi(\mathbf{x}) = \psi(R^{-1}\mathbf{x})$$

where  $\mathbf{R}$  is the rotation operator generated by  $\mathbf{R} = \exp(-i\mathbf{J} \cdot \mathbf{n}\theta)$ . For a rotation through infinitesimal angle  $\epsilon$  about the  $z$  axis, we have

$$\mathbf{R}_z(\epsilon)\psi(x, y, z) = \psi(x + \epsilon y, y - \epsilon x, z) = \psi(x, y, z) + \epsilon(y \frac{\partial \psi}{\partial x} - x \frac{\partial \psi}{\partial y})$$

On the other hand,

$$\mathbf{R}_z(\epsilon) = I - i\epsilon J_z$$

so, we have  $J_z = -i(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x})$ . This is just the  $z$  component of the orbital angular momentum operator  $L = \mathbf{X} \times \mathbf{P}$ .

For a multicomponent state function, we have

$$\mathbf{R} \begin{pmatrix} \psi_1(\mathbf{x}) \\ \psi_2(\mathbf{x}) \\ \vdots \end{pmatrix} = D \begin{pmatrix} \psi_1(R^{-1}\mathbf{x}) \\ \psi_2(R^{-1}\mathbf{x}) \\ \vdots \end{pmatrix}$$

Thus the general form of the rotation operator will be

$$\mathbf{R}_n(\theta) = e^{-i\mathbf{L} \cdot \mathbf{n}\theta} D_n(\theta)$$

The two factors commute because the first acts only on the coordinate and the second acts only on the components of the column vector. The matrix  $D$  must be unitary, and so it can be written as

$$D_n(\theta) = e^{-i\mathbf{S} \cdot \mathbf{n}\theta}$$

The angular momentum operator  $\mathbf{J}$  has the form

$$\mathbf{J} = \mathbf{L} + \mathbf{S}$$

with  $\mathbf{L} = \mathbf{X} \times \mathbf{P}$  and  $[L_\alpha, S_\beta] = 0$ . In the particular representation used in this section, we have  $\mathbf{L} = -i\mathbf{x} \times \nabla$ , and the components of  $\mathbf{S}$  are discrete matrices. The operators  $\mathbf{L}$  and  $\mathbf{S}$  are called the orbital and spin parts of the angular momentum.

### Orbital angular momentum

The form of the gradient operator in spherical coordinates is

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$$

The orbital angular momentum operator then has the form

$$\mathbf{L} = r\mathbf{e}_r \times (-i\nabla) = (-i) \left[ \mathbf{e}_\phi \frac{\partial}{\partial \theta} - \mathbf{e}_\theta \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right]$$



So, we have

$$L_z = \mathbf{L} \cdot \mathbf{e}_z = -i \frac{\partial}{\partial \phi}$$

$$L^2 = \mathbf{L} \cdot \mathbf{L} = - \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]$$

We must now solve the two coupled differential equations,

$$L_z Y(\theta, \phi) = m Y(\theta, \phi) \quad L^2 Y(\theta, \phi) = l(l+1) Y(\theta, \phi)$$

Apart from normalization, we have  $Y(\theta, \phi) = e^{im\phi} P_l^m(\cos \theta)$ . Here,  $P_l^m$  is the **associated Legendre polynomials**. If we assume that the solution must be single-valued under rotation, then it will follow that  $m$  must be an integer. If we further assume that it must be nonsingular at  $\theta = 0$  and  $\theta = \pi$ , then from the standard theory of the Legendre equation it will follow that  $l$  must be a nonnegative integer in the range  $l \geq |m|$ . The normalized solutions that result from these assumptions are the well-known **spherical harmonics**

$$Y_l^m(\theta, \phi) = (-1)^{(m+|m|)/2} \left[ \frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!} \right]^{1/2} e^{im\phi} P_l^{|m|}(\cos \theta)$$

## Spin

A particular species of particle is characterized by a set of quantum numbers that includes the value of its spin  $s$ , it is often sufficient to treat the spin operators  $\mathbf{S}$  as acting on the space of dimension  $2s+1$  that is spanned by the eigenvectors of for a fixed value of  $s$ .

If  $s = \frac{1}{2}$ , we have

$$S_x = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad S_y = \frac{1}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad S_z = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The spin operator in direction  $\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$  is

$$\mathbf{S}_n = \frac{1}{2} \begin{bmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{bmatrix}$$

The eigenvectors are

$$\begin{bmatrix} e^{-i\phi/2} \cos \frac{\theta}{2} \\ e^{i\phi/2} \sin \frac{\theta}{2} \end{bmatrix} \quad \begin{bmatrix} -e^{-i\phi/2} \sin \frac{\theta}{2} \\ e^{i\phi/2} \cos \frac{\theta}{2} \end{bmatrix}$$

corresponding to eigenvalues  $\frac{1}{2}$  and  $-\frac{1}{2}$ .

If  $s = 1$ , we have

$$S_x = \sqrt{\frac{1}{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad S_y = \sqrt{\frac{1}{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad S_z = \sqrt{\frac{1}{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The spin operator in direction  $\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$  is

$$\mathbf{S}_n = \begin{bmatrix} \cos \theta & \sin \theta e^{-i\phi} \sqrt{\frac{1}{2}} & 0 \\ \sin \theta e^{i\phi} \sqrt{\frac{1}{2}} & 0 & \sin \theta e^{-i\phi} \sqrt{\frac{1}{2}} \\ 0 & \sin \theta e^{i\phi} \sqrt{\frac{1}{2}} & -\cos \theta \end{bmatrix}$$



The eigenvectors are

$$\begin{bmatrix} \frac{1}{2}(1 + \cos \theta)e^{-i\phi} \\ \sqrt{\frac{1}{2}} \sin \theta \\ \frac{1}{2}(1 - \cos \theta)e^{i\phi} \end{bmatrix} \quad \begin{bmatrix} -\sqrt{\frac{1}{2}} \sin \theta e^{-i\phi} \\ \cos \theta \\ \sqrt{\frac{1}{2}} \sin \theta e^{i\phi} \end{bmatrix} \quad \begin{bmatrix} \frac{1}{2}(1 - \cos \theta)e^{-i\phi} \\ -\sqrt{\frac{1}{2}} \sin \theta \\ \frac{1}{2}(1 + \cos \theta)e^{i\phi} \end{bmatrix}$$

corresponding to eigenvalues 1, 0 and  $-1$ .

### 15.3 Rotation operator

Three parameters are required to describe an arbitrary rotation. A common parameterization is by the Euler angles. From the fixed system of axes  $Oxyz$ , a new rotated set of axes  $Ox'y'z'$  is produced in three steps:

- Rotate through angle  $\alpha$  about  $Oz$ , carrying  $Oy$  into  $Ou$
- Rotate through angle  $\beta$  about  $Ou$ , carrying  $Oz$  into  $Oz'$
- Rotate through angle  $\gamma$  about  $Oz'$ , carrying  $Ou$  into  $Oy'$

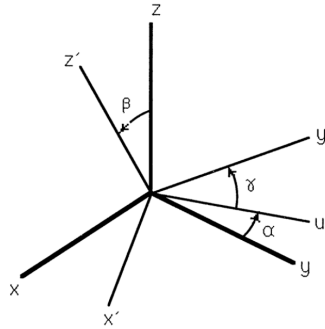


Figure 15.1: Euler angles

The net rotation is

$$\mathbf{R}(\alpha, \beta, \gamma) = \mathbf{R}_{z'}(\gamma) \mathbf{R}_u(\beta) \mathbf{R}_z(\alpha) = e^{-i\gamma J_{z'}} e^{-i\beta J_u} e^{-i\alpha J_z}$$

Since  $J_u = \mathbf{R}_z(\alpha) J_y \mathbf{R}_z(-\alpha)$ , we have  $\mathbf{R}_u(\beta) = \mathbf{R}_z(\alpha) \mathbf{R}_y(\beta) \mathbf{R}_z(-\alpha)$ . Similarly, we can obtain  $\mathbf{R}_{z'}(\gamma) = \mathbf{R}_u(\beta) \mathbf{R}_z(\gamma) \mathbf{R}_u(-\beta)$ . So, the rotation operator is

$$\mathbf{R}(\alpha, \beta, \gamma) = \mathbf{R}_z(\alpha) \mathbf{R}_y(\beta) \mathbf{R}_z(\gamma) = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}$$

The matrix representation of the rotation operator in the basis  $|j, m\rangle$

$$\langle j', m' | \mathbf{R}(\alpha, \beta, \gamma) | j, m \rangle = \delta_{jj'} D_{m'm}^{(j)}(\alpha, \beta, \gamma)$$

gives rise to the rotation matrices,

$$D_{m'm}^{(j)}(\alpha, \beta, \gamma) = \langle j', m' | e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} | j, m \rangle = e^{-i(\alpha m' + \gamma m)} d_{mm'}^{(j)}(\beta)$$





where

$$d_{mm'}^{(j)}(\beta) = \langle j', m' | e^{-i\beta J_y} | j, m \rangle$$

For the case of  $j = \frac{1}{2}$ , we have  $J_y = \frac{1}{2}\sigma_y$  and  $\sigma_y^2 = I$ . We can obtain

$$d^{(1/2)}(\beta) = \begin{bmatrix} \cos \frac{\beta}{2} & -\sin \frac{\beta}{2} \\ \sin \frac{\beta}{2} & \cos \frac{\beta}{2} \end{bmatrix}$$

Notice that this matrix is periodic in  $\beta$  with period  $4\pi$ , but it changes sign when  $2\pi$  is added to  $\beta$ . This double-valuedness under rotation by  $2\pi$  is a characteristic of the full rotation matrix whenever  $j$  is a half odd-integer. The matrix is single-valued under rotation by  $2\pi$  whenever  $j$  is an integer.

The operator for a rotation through  $2\pi$  about an axis along the unit vector  $\mathbf{n}$  is  $\mathbf{R}_n(2\pi) = e^{-2\pi i \mathbf{n} \cdot \mathbf{J}}$ . Its effect on the standard angular momentum eigenvectors is

$$\mathbf{R}_n(2\pi) = (-1)^{2j} |j, m\rangle$$

We assume a rotation through  $2\pi$  as a trivial operation that leaves everything unchanged, i.e. all dynamical variables are invariant under  $2\pi$  rotation:

$$\mathbf{R}(2\pi) A \mathbf{R}^{-1}(2\pi) = A$$

where  $A$  may represent any physical observable.

the operator  $\mathbf{R}_{2\pi}$  divides the vector space into two subspaces. A typical vector in the first subspace, denoted as  $|+\rangle$ , has the property  $\mathbf{R}(2\pi)|+\rangle = |+\rangle$ , whereas a typical vector in the second subspace, denoted as  $|-\rangle$ , has the property  $\mathbf{R}(2\pi)|-\rangle = -|-\rangle$ . Now, if  $A$  represents any physical observable, we have  $\langle + | \mathbf{R}(2\pi) A | - \rangle = \langle + | A \mathbf{R}(2\pi) | - \rangle$ , leading to

$$\langle + | A | - \rangle = 0$$

No physical observable can have nonvanishing matrix elements between states with integer angular momentum and states with half odd-integer angular momentum. This fact forms the basis of a superselection rule: There is no observable distinction among the state vectors of the form

$$|\Psi_\omega\rangle = |+\rangle + e^{i\omega} |-\rangle$$

for different values of the phase  $\omega$ .

## 15.4 Addition of angular momentum

Let us consider a two-component system, each component of which has angular momentum degrees of freedom. Basis vectors for the composite system can be formed from the basis vectors of the components by taking all binary products of a vector from each set

$$|j_1, j_2, m_1, m_2\rangle = |j_1, m_1\rangle^{(1)} |j_2, m_2\rangle^{(2)}$$



These vectors are common eigenvectors of the four commutative operators  $\mathbf{J}^{(1)} \cdot \mathbf{J}^{(1)}$ ,  $\mathbf{J}^{(2)} \cdot \mathbf{J}^{(2)}$ ,  $\mathbf{J}_z^{(1)}$ , and  $\mathbf{J}_z^{(2)}$ . It is often desirable to form eigenvectors of the total angular momentum operators,  $\mathbf{J} \cdot \mathbf{J}$  and  $\mathbf{J}_z$ , where the total angular momentum vector operator is

$$\mathbf{J} = \mathbf{J}^{(1)} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{J}^{(2)}$$

This is useful when the system is invariant under rotation as a whole, but not under rotation of the two components separately. The eigenvectors of  $\mathbf{J} \cdot \mathbf{J}$  and  $\mathbf{J}_z$  may be denoted as  $|\alpha, J, M\rangle$ . It is easy to verify that the four operators  $\mathbf{J}^{(1)} \cdot \mathbf{J}^{(1)}$ ,  $\mathbf{J}^{(2)} \cdot \mathbf{J}^{(2)}$ ,  $\mathbf{J} \cdot \mathbf{J}$  and  $\mathbf{J}_z$  are mutually commutative, and hence they possess a complete set of common eigenvectors. Since the set of product vectors and the new set of total angular momentum eigenvectors are both eigenvectors of  $\mathbf{J}^{(1)} \cdot \mathbf{J}^{(1)}$  and  $\mathbf{J}^{(2)} \cdot \mathbf{J}^{(2)}$ , the eigenvalues  $j_1$  and  $j_2$  will be constant in both sets. Therefore we may confine our attention to the vector space of dimension  $(2j_1 + 1)(2j_2 + 1)$  that is spanned by product vectors with fixed values of  $j_1$  and  $j_2$ .

Now the  $2J + 1$  vectors  $|\alpha, J, M\rangle$ , with  $M$  in the range  $-J \leq M \leq J$ , span an irreducible subspace. Therefore if the vector  $|\alpha, J, M\rangle$ , for a particular value of  $M$ , can be constructed in the space under consideration, then so can the entire set of  $2J + 1$  such vectors with  $M$  in the range  $-J \leq M \leq J$ . For a particular value of  $J$ , it might be possible to construct one such set of vectors, two or more linearly independent sets, or none at all.

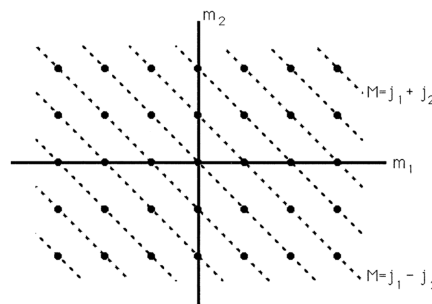
Let  $N(J)$  denote the number of independent sets that can be constructed. Let  $n(M)$  be the degree of degeneracy, in this space, of the eigenvalue  $M$ . The relation between these two quantities is

$$n(M) = \sum_{J \geq |M|} N(J)$$

and hence

$$N(J) = n(J) - n(J + 1)$$

The product vectors  $|j_1, m_1\rangle |j_2, m_2\rangle$  are eigenvectors of the operator  $\mathbf{J}_z$ , with eigenvalue  $m_1 + m_2$ , and the degree of degeneracy  $n(M)$  is equal to the number of pairs  $(m_1, m_2)$  such that  $M = m_1 + m_2$ . Therefore,



**Figure 15.2:** Possible values of  $M = m_1 + m_2$ , illustrated for  $j_1 = 3$ ,  $j_2 = 2$

$$n(M) = \begin{cases} 0 & |M| > j_1 + j_2 \\ j_1 + j_2 + 1 - |M| & |j_1 - j_2| \leq M \leq j_1 + j_2 \\ 2j_{\min} + 1 & 0 \leq |M| \leq |j_1 - j_2| \end{cases}$$



It then follows that

$$N(J) = \begin{cases} 1 & |j_1 - j_2| \leq J \leq |j_1 + j_2| \\ 0 & \text{otherwise} \end{cases}$$

It has turned out that  $N(J)$  is never greater than 1, and so the vectors  $|\alpha, J, M\rangle$  can be uniquely labeled by the eigenvalues of the four operators  $\mathbf{J}^{(1)} \cdot \mathbf{J}^{(1)}$ ,  $\mathbf{J}^{(2)} \cdot \mathbf{J}^{(2)}$ ,  $\mathbf{J} \cdot \mathbf{J}$  and  $J_z$ . Henceforth these total angular momentum eigenvectors will be denoted as  $|j_1, j_2, J, M\rangle$ . And we have the unitarity transformation

$$|j_1, j_2, J, M\rangle = \sum_{m_1, m_2} |j_1, j_2, m_1, m_2\rangle \langle j_1, j_2, m_1, m_2 | j_1, j_2, J, M\rangle$$

The coefficients of this transformation are called the Clebsch–Gordan coefficients, denoted as  $(j_1, j_2, m_1, m_2 | J, M)$ . We fix the phase of  $|j_1, j_2, J, M\rangle$  by requiring that  $(j_1, j_2, m_1, m_2 | J, M)$  be real and positive. The CG coefficient vanishes unless the following conditions are satisfied:

- $m_1 + m_2 = M$
- $|j_1 - j_2| \leq J \leq |j_1 + j_2|$
- $j_1 + j_2 + J = \text{an integer}$

It is possible to calculate the values of the CG coefficients by successive application of the raising or lowering operator to

$$|j_1, j_2, J, M\rangle = \sum_{m_1, m_2} |j_1, j_2, m_1, m_2\rangle \langle j_1, j_2, m_1, m_2 | j_1, j_2, J, M\rangle$$

The details of the calculation can be found in section 7.7 of *Quantum mechanics - a modern development* (Leslie E. Ballentine). And we have [Table of CG coefficients](#) and [Calculator of CG coefficients](#) on the internet. A special case of angular momentum addition is spin–orbit coupling of spin  $\frac{1}{2}$  particles, and we list the corresponding CG coefficients  $(l, \frac{1}{2}, M - m_s, m_s | J, M)$  here:

	$J = l + \frac{1}{2}$	$J = l - \frac{1}{2}$
$m_s = \frac{1}{2}$	$\left[ \frac{l+M+\frac{1}{2}}{2l+1} \right]^{\frac{1}{2}}$	$-\left[ \frac{l-M+\frac{1}{2}}{2l+1} \right]^{\frac{1}{2}}$
$m_s = -\frac{1}{2}$	$\left[ \frac{l-M+\frac{1}{2}}{2l+1} \right]^{\frac{1}{2}}$	$\left[ \frac{l+M+\frac{1}{2}}{2l+1} \right]^{\frac{1}{2}}$

Table 15.1: Spin-Orbit coupling

### Magnetic moment

The magnetic moment operator for an atom has the form

$$\boldsymbol{\mu} = \frac{-e}{2m_e c} (g_L \mathbf{L} + g_S \mathbf{S})$$

The parameters  $g_L$  and  $g_S$  have approximately the values  $g_L = 1$  and  $g_S = 2$ . The former is an generalization of the magnetic moment we calculated in classical electrodynamics for a



system of charged particles. The latter will be discussed in quantum field theory. We define the effective Lande factor as

$$\langle \tau, J, M' | \boldsymbol{\mu} | \tau, J, M \rangle = \frac{-e}{2m_e c} g_{eff} \langle J, M' | \mathbf{J} | J, M \rangle$$

So, we have

$$g_{eff} = \frac{\langle \tau, J, M | g_L \mathbf{L} \cdot \mathbf{J} + g_s \mathbf{S} \cdot \mathbf{J} | \tau, J, M \rangle}{J(J+1)} = 1 + \frac{J(J+1) - L(L+1) + S(S+1)}{2J(J+1)}$$

A formal discussion can be found in section 7.8 of *Quantum mechanics - a modern development* (Leslie E. Ballentine).

## 15.5 Spherical potential well

The stationary states of a particle in the spherical potential well are determined by

$$-\frac{1}{2M} \nabla^2 \Psi + W(r) \Psi = E \Psi$$

In spherical coordinates,

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial}{\partial r} \right] + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left[ \sin \theta \frac{\partial}{\partial \theta} \right] + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

So, the eigenvalue equation becomes

$$-\frac{1}{2M} \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial \Psi}{\partial r} \right] + \frac{L^2}{2Mr^2} \Psi + W(r) \Psi = E \Psi$$

Suppose the eigenfunctions have the factored form

$$\Psi(r, \theta, \phi) = Y_l^m(\theta, \phi) \frac{u(r)}{r}$$

The radial function then satisfies the equation

$$-\frac{1}{2M} \frac{d^2 u(r)}{dr^2} + \left[ \frac{l(l+1)}{2Mr^2} + W(r) \right] u(r) = E u(r)$$

The radial function must satisfy the boundary condition  $u(0) = 0$  since  $\Psi(r, \theta, \phi)$  would otherwise have an  $r^{-1}$  singularity at the origin. The normalization  $\langle \Psi | \Psi \rangle = 1$  implies that

$$\int_0^\infty |u(r)|^2 dr = 1$$

### The hydrogen atom

The hydrogen atom is a two-particle system consisting of an electron and a proton. The Hamiltonian is

$$H = \frac{P_e^2}{2M_e} + \frac{P_p^2}{2M_p} - \frac{e^2}{4\pi |\mathbf{Q}_e - \mathbf{Q}_p|}$$



We take as independent variables the center of mass and relative coordinates of the particles

$$\mathbf{Q}_c = \frac{M_e \mathbf{Q}_e + M_p \mathbf{Q}_p}{M_e + M_p} \quad \mathbf{Q}_r = \mathbf{Q}_e - \mathbf{Q}_p$$

The corresponding momentum operators are

$$\mathbf{P}_c = \mathbf{P}_e + \mathbf{P}_p \quad \mathbf{P}_r = \frac{M_p \mathbf{P}_e - M_e \mathbf{P}_p}{M_e + M_p}$$

We can verify that

$$[Q_{c\alpha}, P_{c\beta}] = [Q_{r\alpha}, P_{r\beta}] = i\delta_{\alpha\beta} \quad [Q_{c\alpha}, P_{r\beta}] = [Q_{r\alpha}, P_{c\beta}] = 0$$

The Hamiltonian becomes

$$H = \frac{P_c^2}{2(M_e + M_p)} + \frac{p_r^2}{2\mu} - \frac{e^2}{4\pi|\mathbf{Q}_r|}$$

where  $\mu$  is called the reduced mass, and is defined by  $\mu \equiv \frac{M_e M_p}{M_e + M_p}$ . The center of mass behaves as a free particle, and its motion is not coupled to the relative coordinate. We shall confine our attention to the internal degrees of freedom described by the relative coordinate  $\mathbf{Q}_r$ . The energy eigenvalue equation in coordinate representation is

$$-\frac{1}{2\mu} \nabla^2 \Psi(\mathbf{r}) - \frac{e^2}{4\pi r} \Psi(\mathbf{r}) = E \Psi(\mathbf{r})$$

Suppose  $\Psi(r, \theta, \phi) = Y_l^m(\theta, \phi) \frac{u(r)}{r}$ , we have

$$-\frac{1}{2\mu} \frac{d^2 u(r)}{dr^2} + \left[ \frac{l(l+1)}{2\mu r^2} - \frac{e^2}{4\pi r} \right] u(r) = E u(r)$$

Define

$$\rho \equiv \alpha r \quad \alpha \equiv \sqrt{8\mu|E|} \quad \lambda \equiv \frac{e^2}{4\pi} \sqrt{\frac{\mu}{2|E|}}$$

we have

$$\frac{d^2 u}{d\rho^2} + \left[ -\frac{1}{4} + \frac{\lambda}{\rho} - \frac{l(l+1)}{\rho^2} \right] u = 0$$

As  $\rho \rightarrow \infty$ , we have  $u \sim e^{-\rho/2}$ . And as  $\rho \rightarrow 0$ , we have  $u \sim \rho^{l+1}$ . So, we can suppose

$$u(\rho) = \rho^{l+1} e^{-\rho/2} v(\rho)$$

And we can get

$$\rho \frac{d^2 v}{d\rho^2} + (2l+2-\rho) \frac{dv}{d\rho} + (\lambda-l-1)v = 0$$

It is the so-called **Confluent Hypergeometric Differential Equation**. When  $\lambda - 1 - l = n_r$ , we have regular solutions. Solutions are **Associated Laguerre Polynomial**, and will be denoted as  $L_{n-l-1}^{2l+1}(\rho)$  ( $n = n_r + l + 1$ ). The energy levels are

$$E_n = -\frac{\mu e^4}{32\pi^2 n^2}$$



The degeneracy of an eigenvalue  $E_n$  is

$$\sum_{l=0}^{n-1} (2l+1) = n^2$$



**Note:** The degeneracy of an energy level of a hydrogen atom is greater than this by a factor of 4, which arises from the two-fold orientational degeneracies of the electron and proton spin states. This four-fold degeneracy is modified by the hyperfine interaction between the magnetic moments of the electron and the proton.

The orthonormal energy eigenfunctions for the hydrogen atom are

$$\Psi_{nlm}(r, \theta, \phi) = \left[ \frac{4(n-l-1)!}{(na_0)^3 n [(n+l)!]^3} \right]^{\frac{1}{2}} \rho^l L_{n-l-1}^{2l+1}(\rho) e^{-\rho/2} Y_l^m(\theta, \phi)$$

where  $\rho = \alpha r = \frac{2r}{na_0}$ , and  $a_0 \equiv \frac{4\pi}{\mu e^2}$  is a characteristic length for the atom, known as the Bohr radius. The ground state wave function is

$$\Psi_{000} = (\pi a_0^3)^{-\frac{1}{2}} e^{-\frac{r}{a_0}}$$

A measure of the spatial extent of the bound states of hydrogen is given by the averages of various powers of the distance  $r$ .

$$\begin{aligned} \langle r \rangle &= n^2 a_0 \left\{ 1 + \frac{1}{2} \left[ 1 - \frac{l(l+1)}{n^2} \right] \right\} \\ \langle r^2 \rangle &= n^4 a_0^2 \left\{ 1 + \frac{3}{2} \left[ 1 - \frac{l(l+1) - 1/3}{n^2} \right] \right\} \\ \left\langle \frac{1}{r} \right\rangle &= \frac{1}{n^2 a_0} \end{aligned}$$



# Chapter 16

## Quantum-mechanical Particles and Classical Electromagnetic Field



### 16.1 Charged Particle in EM field

In classical electrodynamics, if the velocity of the charged particle is much smaller than that of light, then the Hamiltonian of charged particles in a given EM field is

$$H = \frac{(\boldsymbol{\pi} - e\mathbf{A})^2}{2m} + e\phi$$

In corresponding quantum theory, we will suppose the Hamiltonian operator to be

$$H = \frac{[\mathbf{P} - e\mathbf{A}(\mathbf{X})]^2}{2M} + e\phi(\mathbf{X})$$

The corresponding field operators are

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

In Heisenberg picture, we have

$$\frac{d\mathbf{X}}{dt} = -i[\mathbf{X}, H] = \frac{1}{M}(\mathbf{P} - e\mathbf{A})$$

Define kinetic momentum  $\mathbf{K}$  by

$$\mathbf{K} = \mathbf{P} - e\mathbf{A}$$

We have

$$[K_i, K_j] = ie(\partial_i A_j - \partial_j A_i) = ie\epsilon_{ijk} B_k$$

So

$$M\frac{d^2\mathbf{X}}{dt^2} = -i[\mathbf{K}, H] + \frac{\partial\mathbf{K}}{\partial t} = e \left[ \mathbf{E} + \frac{1}{2} \left( \frac{d\mathbf{X}}{dt} \times \mathbf{B} - \mathbf{B} \times \frac{d\mathbf{X}}{dt} \right) \right]$$

In coordinate representation of Schrödinger picture, we have

$$\frac{1}{2M} [-i\nabla - e\mathbf{A}] \cdot [-i\nabla - e\mathbf{A}] \psi(\mathbf{x}, t) + e\phi(\mathbf{x})\psi(\mathbf{x}, t) = i\frac{\partial\psi(\mathbf{x}, t)}{\partial t}$$

Define probability current  $\mathbf{j}$  as

$$\mathbf{j} \equiv \frac{1}{M} \text{Im}(\psi^* \nabla \psi) - \frac{e}{M} \mathbf{A} |\psi|^2$$

we can verify that

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0$$

Transformation

$$\phi \rightarrow \phi - \frac{\partial \Lambda}{\partial t} \quad \mathbf{A} \rightarrow \mathbf{A} + \nabla \Lambda$$

will leave  $\mathbf{E}$  and  $\mathbf{B}$  unchanged. The transformation is called gauged transformation. In classical electrodynamics, gauge transformation will not change the trajectory of particles, (which is the only thing we can observed in experiment). In quantum theory, suppose the state vector  $|\psi\rangle$  will transform as

$$|\psi(t)\rangle \rightarrow O(t)|\psi(t)\rangle$$

where  $O(t)$  is an unitary operator. If the Schrödinger equation is always satisfied, we can derive that

$$H'O - OH = i \frac{\partial O}{\partial t}$$

where  $H'$  is the Hamiltonian operator after gauge transformation. Generally, we have

$$O(t) = \exp [ie\Lambda(\mathbf{X}, t)]$$

And so

$$O^{-1} \mathbf{X} O = \mathbf{X} \quad O^{-1} \mathbf{P} O = \mathbf{P} + e \nabla \Lambda \quad O^{-1} (\mathbf{P} - e \mathbf{A}') O = \mathbf{P} - e \mathbf{A}$$

So, the expectation value of  $\mathbf{X}$  and  $\mathbf{K}$  is invariant under gauge transformation. We can also verify that  $\mathbf{j}$  is also invariant under gauge transformation. A special case is that

$$\phi \rightarrow \phi + \phi_0(t) \quad \mathbf{A} \rightarrow \mathbf{A}$$

In this case, we have

$$O(t) = \exp \left[ -i \int_{t_0}^t dt' e \phi_0(t') \right]$$

If  $\phi_0$  is a constant, then

$$O(t) = \exp [-ie\phi_0(t - t_0)]$$

## 16.2 Charged particle in a magnetic field

### 16.2.1 Motion in a uniform static magnetic field

Suppose that the magnetic field be of magnitude  $B$  in the  $z$  direction. The Hamiltonian  $H = H_{xy} + H_z$  with  $H_{xy} = \frac{K_x^2 + K_y^2}{2M}$  and  $H_z = \frac{P_z^2}{2M}$ . Since  $B_x = B_y = 0$ , then  $P_z$  commutes with  $P_x$  and  $P_y$ . Hence the operators  $H_{xy}$  and  $H_z$  are commutative, and every eigenvalue of  $H$  is just the sum of an eigenvalue of  $H_{xy}$  and an eigenvalue of  $H_z$ .

Define

$$Q' \equiv \frac{K_x}{\gamma} \quad P' \equiv \frac{K_y}{\gamma} \quad \gamma \equiv \sqrt{|eB|}$$





Then we have

$$H_{xy} = \frac{1}{2} \frac{|eB|}{M} (Q'^2 + P'^2) \text{ with } [Q', P'] = i \text{ or } -i$$

therefore the eigenvalues of  $H_{xy}$  must be equal to  $(n + \frac{1}{2}) \frac{|eB|}{M}$ , where  $n$  is any non-negative integer.

The spectrum of  $K_z$  can be shown to be gauge-invariant. Because the magnetic field is uniform and in the  $z$  direction, it is possible to choose the vector potential such that  $A_z = 0$ . Therefore the spectrum of  $K_z$  is continuous from  $-\infty$  to  $\infty$ , like that of  $P_z$ .

Thus the energy eigenvalues for a charged particle in a uniform static magnetic field  $B$  are

$$E_n(p_z) = \frac{(n + \frac{1}{2})|eB|}{M} + \frac{p_z^2}{2M}$$

The motion parallel to the magnetic field is not coupled to the transverse motion, and is unaffected by the field. The classical motion in the plane perpendicular to the field is in a circular orbit with angular frequency  $\omega_c = \frac{eB}{M}$ , and it is well known that periodic motions correspond to discrete energy levels whose separation is  $\omega_c$ .

Now let us choose the vector potential to be  $A_x = -yB, A_y = A_z = 0$ . The Hamiltonian now becomes

$$H = \frac{(P_x + yeB)^2 + P_y^2 + P_z^2}{2M}$$

$P_x$  and  $P_z$  commute with  $H$ , so it is possible to construct a complete set of common eigenvectors of  $H, P_x$  and  $P_z$ . In coordinate representation, the eigenvalue equation now takes the form

$$-\frac{1}{2M} \nabla^2 \psi - \frac{ieB}{M} y \frac{\partial}{\partial x} \psi + \frac{e^2 B^2}{2M} y^2 \psi = E \psi$$

Substitute

$$\psi(x, y, z) = \exp(ik_x x + ik_z z) \phi(y)$$

The equation then takes the form

$$-\frac{1}{2M} \frac{d^2 \phi(y)}{dy^2} + \left[ \frac{M\omega_c^2}{2} (y - y_0)^2 - E' \right] \phi(y) = 0$$

where  $\omega_c = \frac{eB}{M}$  is the classical cyclotron frequency, and  $E' = E - \frac{k_z^2}{2M}$  is the energy associated with motion in the  $xy$  plane. This is just the energy eigenvalue equation for a simple harmonic oscillator with angular frequency  $\omega = |\omega_c|$ , whose eigenvalues are  $E' = (n + 1/2)\omega$ . Thus the energies for the charged particle in the magnetic field must be  $E = (n + 1/2)|\omega_c| + \frac{k_z^2}{2M}$ . Apart from a normalization constant, the eigenfunction will be

$$\psi = \exp(ik_x x + ik_z z) H_n[\alpha(y - y_0)] \exp[-\frac{1}{2}\alpha^2(y - y_0)^2]$$

with  $\alpha = \sqrt{M\omega} = \sqrt{|eB|}$ , and  $y_0 = -\frac{k_x}{eB}$ .

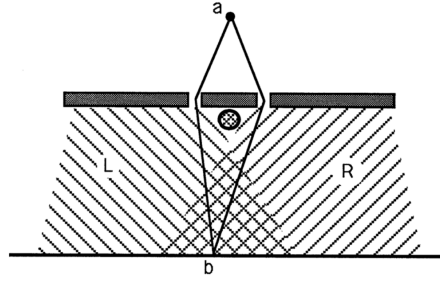
For fixed  $n$  and  $k_z$ , the energy eigenvalue is highly degenerate. For convenience, we assume that the system is confined to a rectangle of dimension  $D_x \times D_y$  and subject to periodic boundary conditions. The allowed values of  $k_x$  are  $k_x = \frac{2\pi n_x}{D_x}$ , with  $n_x = 0, \pm 1, \dots$ . The orbit center



coordinate  $y_0 = -\frac{2\pi n_x}{D_x e B}$  must lie in the range  $[0, D_y]$ . In the limit as  $D_x$  and  $D_y$  become large, we may ignore problems associated with orbits lying near the boundary, since they will be a negligible fraction of the total. In this limit the number of degenerate states corresponding to fixed  $n$  and  $k_z$  will be  $\frac{D_x D_y |eB|}{2\pi}$ .

### 16.2.2 The Aharonov-Bohm effect

A long solenoid is placed perpendicular to the plane of the figure, so that a magnetic field can be created inside the solenoid while the region external to the solenoid remains field-free. The solenoid is located in the unilluminated shadow region so that no particles will reach it, and moreover it may be surrounded by a cylindrical shield that is impenetrable to the charged particles. Nevertheless it can be shown that the interference pattern depends upon the magnetic flux through the cylinder.



**Figure 16.1:** The Aharonov–Bohm experiment

Let  $\Psi^{(0)}(\mathbf{x}, t)$  be the solution of the Schrödinger equation and boundary conditions of this problem for the case in which the vector potential is everywhere zero. Now let us consider the case in which the magnetic field is non-zero inside the cylinder but zero outside of it. The vector potential  $\mathbf{A}$  will not vanish everywhere in the exterior region, even though  $\mathbf{B} = \nabla \times \mathbf{A} = 0$  outside of the cylinder. This follows by applying Stokes's theorem to any path surrounding the cylinder

$$\oint \mathbf{A} \cdot d\mathbf{x} = \int \int (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int \int \mathbf{B} \cdot d\mathbf{S} = \Phi$$

If the flux through the cylinder is not zero, then the vector potential must be nonzero on every path that encloses the cylinder. However in any simply connected region outside of the cylinder, it is possible to express the vector potential as the gradient of a scalar, from the zero potential solution by means of a gauge transformation,  $\Psi = \Psi^{(0)} \exp(i e \Lambda)$ .

In region  $L$ , which contains the slit on the left, the wave function can be written as  $\Psi_L = \Psi_L \exp(i e \Lambda_1)$ , where  $\Psi_L$  is the zero potential solution in region  $L$ , and  $\Lambda_1(\mathbf{x}, t) = \int \mathbf{A} \cdot d\mathbf{x}$ , with the integral taken along a path within region  $L$ . A similar form can be written for the wave function in the region  $R$ , which contains the slit on the right.

At the point  $b$ , in the overlap of regions  $L$  and  $R$ , the wave function is a superposition of contributions from both slits. Hence we have

$$\Psi(\mathbf{b}) = \Psi_L \exp(i e \Lambda_1) + \Psi_R \exp(i e \Lambda_2)$$



The interference pattern depends  $\exp(ie(\Lambda_1 - \Lambda_2)) = \exp(ie\Phi)$ . Therefore the interference pattern is sensitive to the magnetic flux inside of the cylinder, even though the particles never pass through the region in which the magnetic field is nonzero. The AB effect is a topological effect, in that the effect depends on the flux encircled by the paths available to the particle, even though the paths may never approach the region of the flux.

At last, we conclude that in quantum theory, the potentials themselves are physically significant; however, they are subject to the requirement that all observable effects be invariant under gauge transformations.

### 16.2.3 The Zeeman effect

The Zeeman effect refers to the effect of a uniform magnetic field on atomic states and energy levels. The Hamiltonian for an electron in the atom is

$$H = \frac{(\mathbf{P} + e\mathbf{A})^2}{2M} + W(r) = \frac{P^2}{2M} + \frac{e}{M}\mathbf{A} \cdot \mathbf{P} + \frac{e^2 A^2}{2M} + W(r)$$

where the mass of the electron is  $M$ , and its charge is  $-e$ . We have used the simplification  $\mathbf{P} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{P}$ , which is valid if  $\nabla \cdot \mathbf{A} = 0$ . We shall take vector potential to be  $\mathbf{A}(\mathbf{X}) = \frac{1}{2}\mathbf{B} \times \mathbf{X}$ . It then follows that

$$\mathbf{A} \cdot \mathbf{P} = \frac{1}{2}(\mathbf{B} \times \mathbf{X}) \cdot \mathbf{P} = \frac{1}{2}(\mathbf{X} \times \mathbf{P}) \cdot \mathbf{B} = \frac{1}{2}\mathbf{B} \cdot \mathbf{L}$$

For weak magnetic fields it is convenient to write this Hamiltonian in the form

$$H = H_a + \frac{e}{2M}\mathbf{B} \cdot \mathbf{L} + \frac{e^2}{8M}(\mathbf{B} \times \mathbf{X})^2$$

where  $H_a = \frac{P^2}{2M} + W(r)$  is the Hamiltonian of the free atom. If we neglect the last term of  $H$ , which is of the second order in the magnetic field, and choose the magnetic field to lie along the  $z$  axis, then eigenfunctions  $\Psi_{nlm}$  of  $H_a$  will also be an eigenfunction of  $H$ . To the first order in the magnetic field, the atomic energy levels will be displaced by an amount

$$E^{(1)} = \frac{eB}{2M}m$$

The degeneracy of the  $2l + 1$  fold multiplet of fixed  $n$  and  $l$ , due to the spherical symmetry of the atom, is broken by the magnetic field.



**Note:** There is also a magnetic dipole moment associated with the spin,  $\boldsymbol{\mu}_S = -\frac{e}{M}\mathbf{S}$ , and so in practice one must also add the spin term  $-\boldsymbol{\mu}_S \cdot \mathbf{B}$  to the Hamiltonian.

For strong magnetic fields the term in the Hamiltonian that is proportional to  $B^2$  becomes important. The details of the discussion can be found in section 11.5 of *Quantum mechanics - a modern development*(Leslie E. Ballentine).



# Chapter 17

## Discrete Symmetries



### 17.1 Space inversion

The space inversion transformation is  $\mathbf{x} \rightarrow -\mathbf{x}$ . The corresponding operator on state vector space is usually called the parity operator. It will be denoted by  $P$ . By definition, the parity operator reverses the signs of the position operator and the momentum operator

$$P^{-1} \mathbf{X} P = -\mathbf{X} \quad P^{-1} \mathbf{P} P = -\mathbf{P}$$

It follows that the orbital angular momentum,  $\mathbf{L} = \mathbf{X} \times \mathbf{P}$ , is unchanged by the parity transformation. This property is extended, by definition, to any angular momentum operator,

$$P^{-1} \mathbf{J} P = \mathbf{J}$$

We can verify that  $P$  must be linear by applying space inversion to the commutation relation  $[X_i, P_i] = i$ . Therefore the parity operator is a unitary operator rather than an anti-unitary operator. Since two consecutive space inversions produce no change at all, it follows that the states described by  $|\psi\rangle$  and by  $P^2|\psi\rangle$  must be the same. Thus the operator  $P^2$  can differ from the identity operator by at most a phase factor. This phase factor is left arbitrary. It is most convenient to choose that phase factor to be unity, and hence we have

$$P = P^{-1} = P^\dagger$$

Further more, we can derive that

$$P|\mathbf{x}\rangle = |-\mathbf{x}\rangle$$

So the effect of  $P$  on a wave function is

$$P\psi(\mathbf{x}) \equiv \langle \mathbf{x} | P | \psi \rangle = \langle -\mathbf{x} | \psi \rangle = \psi(-\mathbf{x})$$

From the fact the  $P^2 = 1$ , it follows that  $P$  has eigenvalues  $\pm 1$ . Any even function,  $\psi_e(\mathbf{x}) = \psi_e(-\mathbf{x})$ , is an eigenfunction on  $P$  with eigenvalue 1, and any odd function,  $\psi_o(\mathbf{x}) = -\psi_o(-\mathbf{x})$ , is an eigenfunction of  $P$  with eigenvalue  $-1$ . A function corresponding to parity  $+1$  is also said to be of even parity, and a function corresponding to parity  $-1$  is said to be of odd parity.

#### Example

Under space inversion,  $\mathbf{x} \rightarrow -\mathbf{x}$ , the spherical harmonic undergoes the transformation

$$Y_l^m(\theta, \phi) \rightarrow Y_l^m(\pi - \theta, \phi + \pi) = (-1)^l Y_l^m(\theta, \phi)$$

Hence the single particle orbital angular momentum eigenvector  $|l, m\rangle$  is also an eigenvector of parity, with parity  $(-1)^l$ .

A total orbital angular momentum eigenvector for a two- electron atom is of the form

$$|l_1, l_2, L, M\rangle = \sum_{m_1, m_2} \langle l_1, l_2, m_1, m_2 | l_1, l_2, L, M \rangle |l_1, m_1\rangle \otimes |l_2, m_2\rangle$$

It is apparent that

$$P|l_1, l_2, L, M\rangle = (-1)^{l_1+l_2} |l_1, l_2, L, M\rangle$$

and that  $(-1)^{l_1+l_2} \neq (-1)^L$ . Thus we see that, in general, the parity of an angular momentum state is not determined by its total angular momentum.

If the parity operator  $P$  commutes with the Hamiltonian  $H$ , then parity eigenvalue  $\pm 1$  is a conserved quantity. In that case an even parity state can never acquire an odd parity component, and an odd parity state can never acquire an even parity component.

If  $|\psi(t)\rangle$  is a physical process of the system with Hamiltonian  $H$ , then we have Schrödinger equation

$$H|\psi\rangle = i\frac{\partial|\psi\rangle}{\partial t}$$

If  $PH = HP$ , we can verify that

$$HP|\psi\rangle = i\frac{\partial P|\psi\rangle}{\partial t}$$

So, the space inversion of  $|\psi(t)\rangle$ ,  $P|\psi(t)\rangle$ , can also be a possible physical process of the system. Experiments have shown that parity in  $\beta$  decay is not conserved.

## 17.2 Time reversal

The effect of the time reversal operator  $T$  is to reverse the linear and angular momentum while leaving the position unchanged. Thus we require, by definition,

$$T^{-1}\mathbf{X}T = \mathbf{X} \quad T^{-1}\mathbf{P}T = -\mathbf{P} \quad T^{-1}\mathbf{J}T = -\mathbf{J}$$

We can verify that  $T$  must be anti-linear by applying space inversion to the commutation relation  $[X_i, P_i] = i$ . Therefore the parity operator is a anti-unitary operator.

The time evolution of a system satisfied Schrödinger equation

$$H|\psi(t)\rangle = i\frac{\partial|\psi(t)\rangle}{\partial t}$$

Suppose that  $TH = HT$ , we can derive that

$$HT|\psi(t)\rangle = -i\frac{\partial T|\psi(t)\rangle}{\partial t}$$

i.e.

$$HT|\psi(-t)\rangle = i\frac{\partial T|\psi(-t)\rangle}{\partial t}$$



$T|\psi(-t)\rangle$  is also a possible physical process of the system.

In coordinate representation the Schrödinger equation takes the form

$$\left[-\frac{1}{2M}\nabla^2 + W(\mathbf{x})\right]\psi(\mathbf{x}, t) = i\frac{\partial\psi(\mathbf{x}, t)}{\partial t}$$

Its complex conjugate is

$$\left[-\frac{1}{2M}\nabla^2 + W^*(\mathbf{x})\right]\psi^*(\mathbf{x}, t) = -i\frac{\partial\psi^*(\mathbf{x}, t)}{\partial t}$$

The condition for the Hamiltonian to be invariant under complex conjugation is that the potential be real:  $W = W^*$ . In that case it is apparent that if  $\psi(\mathbf{x}, t)$  is a solution then so is  $\psi^*(\mathbf{x}, t)$ . This suggests that we may identify the time reversal operator with the complex conjugation operator in this representation,

$$T = K_0$$

where, by definition,  $K_0\psi(\mathbf{x}, t) = \psi^*(\mathbf{x}, t)$ . In this case  $T$  is its own inverse. The formal expression for an arbitrary vector in coordinate representation is  $|\psi\rangle = \int \psi(\mathbf{x})|\mathbf{x}\rangle d^3\mathbf{x}$ , where the basis vector  $|\mathbf{x}\rangle$  is an eigenvector of the position operator. Since  $T$  is equal to the complex conjugation operator, its effect is simply  $T|\psi\rangle = \int \psi^*(\mathbf{x})|\mathbf{x}\rangle d^3\mathbf{x}$ , with  $T|\mathbf{x}\rangle = |\mathbf{x}\rangle$ .

In momentum representation, an arbitrary vector can be written as

$$|\psi\rangle = \int \psi(\mathbf{p})|\mathbf{p}\rangle d^3\mathbf{p}$$

Since

$$T|\mathbf{p}\rangle = \int \langle\mathbf{x}|\mathbf{p}\rangle^*|\mathbf{x}\rangle d^3\mathbf{x} = |-\mathbf{p}\rangle$$

so we have

$$T|\psi\rangle = \int \psi^*(\mathbf{p})|-\mathbf{p}\rangle d^3\mathbf{p} = \int \psi^*(-\mathbf{p})|\mathbf{p}\rangle d^3\mathbf{p}$$

The time reversal operator must reverse the angular momentum. For spin operator, we have

$$T^{-1}\mathbf{S}T = -\mathbf{S}$$

In the standard representation of the spin operators,  $S_x$  and  $S_z$  are real, while  $S_y$  is imaginary. The time reversal operator  $T$  cannot be equal to the complex conjugation operator  $K_0$  in this representation, since the effect of the latter is

$$K_0S_xK_0 = S_x \quad K_0S_yK_0 = -S_y \quad K_0S_zK_0 = S_z$$

Let us write the time reversal operator as  $T = YK_0$ , where  $Y$  is a linear operator.  $Y$  must have the following properties:

$$Y^{-1}S_xY = -S_x \quad Y^{-1}S_yY = S_y \quad Y^{-1}S_zY = -S_z$$



And  $Y$  must operate only on the spin degrees of freedom. A reasonable choice is that  $Y = e^{-i\pi S_y}$ , whose effect is to rotate spin (and only spin) through the angle  $\pi$  about the  $y$  axis. Therefore the explicit form of the time reversal in this representation is

$$T = e^{-i\pi S_y} K_0$$

Two successive applications of the time reversal transformation, must leave the physical situation unchanged. Therefore

$$T^2|\Psi\rangle = c|\Psi\rangle$$

where  $|c| = 1$ . And we have

$$T^2(T|\Psi\rangle) = T(T^2|\Psi\rangle) = T(c|\Psi\rangle) = c^*T|\Psi\rangle$$

so

$$T^2(|\Psi\rangle + T|\Psi\rangle) = c|\Psi\rangle + c^*T|\Psi\rangle = c'(|\Psi\rangle + T|\Psi\rangle)$$

And we can determine that  $c' = c^* = c$ . Thus we must have  $c = \pm 1$ , i.e.

$$T^2|\Psi\rangle = \pm|\Psi\rangle$$

In the particularly representation, we have

$$T^2 = e^{-i\pi S_y} K_0 e^{-i\pi S_y} K_0 = e^{-i2\pi S_y}$$

This may equivalently be written as

$$T^2 = e^{-i2\pi J_y}$$

since  $e^{-i2\pi L_y} = I$ . So we have an identity

$$T^2 = R(2\pi)$$

### Kramer's theorem

Let us consider the energy eigenvalue equation,  $H|\Psi\rangle = E|\Psi\rangle$ , for a time-reversal-invariant Hamiltonian,  $TH = HT$ . Then  $HT|\Psi\rangle = TH|\Psi\rangle = ET|\Psi\rangle$ , and so both  $|\Psi\rangle$  and  $T|\Psi\rangle$  are eigenvectors with energy eigenvalue  $E$ . There are two possibilities: (a)  $|\Psi\rangle$  and  $T|\Psi\rangle$  are linearly dependent, and so describe the same state. (b)  $|\Psi\rangle$  and  $T|\Psi\rangle$  are linearly independent, and so describe two degenerate states.

Suppose that (a) is true, in which case we must have  $T|\Psi\rangle = a|\Psi\rangle$  with  $|a| = 1$ . A second application of  $T$  yields  $T^2|\Psi\rangle = |\Psi\rangle$ . So for those states that satisfy  $T^2|\Psi\rangle = -|\Psi\rangle$  it is necessarily true that  $|\Psi\rangle$  and  $T|\Psi\rangle$  are linearly independent, degenerate states. This result is known as Kramer's theorem: any system for which  $T^2|\Psi\rangle = -|\Psi\rangle$  has only degenerate energy levels.



# Chapter 18

## Approximation method



**18.1 Time independent perturbation theory**

**18.2 Time dependent perturbation theory**

**18.3 Atomic Radiation**

**18.4 The classical limit**



# Chapter 19

## Many body problem



### 19.1 Identical particles

### 19.2 Non-relativistic quantum field theory

# Chapter 20

## Scattering theory



### 20.1 Lippmann–Schwinger equation

Imagine a particle coming in and getting scattered by a short-ranged potential  $V(x)$  located around the origin  $x \sim 0$ . The time-independent Schrödinger equation is simply

$$(H_0 + V)|\psi\rangle = E|\psi\rangle$$

Here,  $H_0 = \frac{p^2}{2m}$  is the free-particle Hamiltonian operator. We can write the solution as

$$|\psi^{(\pm)}\rangle = \frac{1}{E - H_0 \pm i\epsilon} V |\psi^{(\pm)}\rangle + |\phi\rangle$$

Here,  $H_0|\phi\rangle = E|\phi\rangle$ . In coordinate representation,

$$\psi^{(\pm)}(\mathbf{x}) = \phi(\mathbf{x}) + \int d^3x' \langle \mathbf{x} | \frac{1}{E - H_0 \pm i\epsilon} | \mathbf{x}' \rangle V(\mathbf{x}') \psi^{(\pm)}(\mathbf{x}')$$

Here,  $\phi(\mathbf{x}) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{\frac{3}{2}}}$ . Define the Green function as

$$G_{\pm}(\mathbf{x}, \mathbf{x}') \equiv \frac{1}{2m} \langle \mathbf{x} | \frac{1}{E - H_0 \pm i\epsilon} | \mathbf{x}' \rangle$$

We can derive that

$$G_{\pm}(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi} \frac{e^{\pm ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|}$$

where  $k = \sqrt{2mE}$ . And it is easy to show that

$$(\nabla^2 + k^2)G_{\pm}(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$$

So, we have

$$\psi^{(\pm)}(\mathbf{x}) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{\frac{3}{2}}} - 2m \int d^3x' \frac{1}{4\pi} \frac{e^{\pm ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} V(\mathbf{x}') \psi^{(\pm)}(\mathbf{x}')$$

We now can interpret  $\psi^+(\mathbf{x})$  as a superposition of incident plane wave and scattered wave which propagate from scatterer to outside region. From now on, we will denote it as  $\psi(\mathbf{x})$ .

The experiment is done typically by placing the detector far away from the scatterer  $|\mathbf{x}| \ll a$  where  $a$  is the "size" of the scatterer. The integration over  $\mathbf{x}'$ , on the other hand, is limited within the "size" of the scatterer because of the  $V(\mathbf{x}')$  factor. Therefore, we are in the situation  $|\mathbf{x}| \ll |\mathbf{x}'|$ , and hence can use the approximation

$$|\mathbf{x} - \mathbf{x}'| \approx |\mathbf{x}| - \frac{\mathbf{x}' \cdot \mathbf{x}}{|\mathbf{x}|}$$

Under this limit,

$$\psi(\mathbf{x}) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{\frac{3}{2}}} - 2m \frac{e^{ikr}}{4\pi r} \int d^3x' e^{-\mathbf{k}'\cdot\mathbf{x}'} V(\mathbf{x}') \psi(\mathbf{x}')$$

Here,  $r = |\mathbf{x}|$  and  $\mathbf{k}' = k \frac{\mathbf{x}}{r}$ . It is customary to write this equation in the form

$$\psi(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \left( e^{i\mathbf{k}\cdot\mathbf{x}} + f(\mathbf{k}, \mathbf{k}') \frac{e^{ikr}}{r} \right)$$

Here,

$$f(\mathbf{k}, \mathbf{k}') \equiv -\frac{m}{2\pi} (2\pi)^3 \langle \mathbf{k}' | V | \psi \rangle$$

Recall the definition of cross section

$$\sigma \equiv \frac{\text{Number of Events}}{\text{Time} \times \text{Incident Flux}}$$

So, the differential cross section for particles being scattered into the solid angle is

$$d\sigma = \frac{|\mathbf{j}_{\text{scatt}}| r^2 d\Omega}{|\mathbf{j}_{\text{inc}}|} = |f(\mathbf{k}, \mathbf{k}')|^2 d\Omega$$

In a more realistic situation, we should use wave packets to describe the scattering process. The basic picture is a free wave packet approaches the scattering center. After a long time, we have both the original wave packet moving in the original direction plus a spherical wave front that moves outward. The details can be found in the section 3 of the lecture notes *Scattering Theory I (Hitoshi Murayama)*.

Furthermore, if we require that the normalization of the wave function should always satisfy  $\int dx^3 |\psi(\mathbf{x})|^2$  for any  $t$ , as guaranteed by the unitarity of time evolution operator. This requirement leads to a special requirement on the scattered wave, and hence  $f(\mathbf{k}, \mathbf{k}')$ , from which we can derive the optical theorem.

#### Theorem 20.1 Optical theorem

$$\text{Im} f(\theta = 0) = \frac{k\sigma_{\text{tot}}}{4\pi}$$

where

$$f(\theta = 0) \equiv f(\mathbf{k}, \mathbf{k}),$$

the setting of  $\mathbf{k} \equiv \mathbf{k}'$  imposes scattering in the forward direction, and

$$\sigma_{\text{tot}} = \int \frac{d\sigma}{d\Omega} d\Omega$$

The meaning of this theorem is clear. Because the scattered wave takes the probability away to different directions, the total probability for the particle to go to the forward direction (unscattered) should decrease. This decrease is caused by the interference between the unscattered and scattered waves and hence is proportional to  $f(0)$ . On the other hand, the amount of decrease in the forward direction should equal the total probability at other directions, which is proportional to the total cross section. The proof can be found in the section 4 of the lecture notes *Scattering Theory I (Hitoshi Murayama)*.



## 20.2 Born approximation

If  $|\psi\rangle = |\phi\rangle + O(V)$  is close to  $|\phi\rangle$ , we can solve the Lippman-Schwinger equation by perturbation theory. The lowest order approximation in  $V$  is

$$|\psi\rangle = \frac{1}{E - H_0 + i\epsilon} V |\phi\rangle + |\phi\rangle$$

This is called Born approximation. In coordinate representation,

$$f^{(1)}(\mathbf{k}, \mathbf{k}') = -\frac{m}{2\pi} \int d^3x V(\mathbf{x}) e^{i\mathbf{q}\cdot\mathbf{x}}$$

Here,  $\mathbf{q} = |\mathbf{k} - \mathbf{k}'|$ . If the potential is central, we can derive that

$$f^{(1)}(\mathbf{k}, \mathbf{k}') = -\frac{2m}{q} \int_0^\infty dr r V(r) \sin(qr)$$

### Yukawa potential

$$V = \frac{\alpha}{r} e^{-\mu r}$$

So, we can derive

$$f(\theta) = -\frac{2m\alpha}{q^2 + \mu^2}$$

Differential cross section is therefore given by

$$\frac{d\sigma}{d\Omega} = (2m\alpha)^2 \frac{1}{[2k^2(1 - \cos\theta) + \mu^2]^2}$$

The total cross section is obtained by integrating over  $d\Omega$ ,

$$\sigma = (2m\alpha)^2 \frac{4\pi}{4k^2\mu^2 + \mu^4}$$

### Coulomb potential

$$V = \frac{\alpha}{r}$$

Take the limit  $\mu \rightarrow 0$ , we can get

$$f(\theta) = -\frac{2m\alpha}{q^2}$$

Differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{4E}\right)^2 \frac{1}{\sin^4 \frac{\theta}{2}}$$

The total cross section diverges. The divergence is in the  $\cos\theta$  integral when  $\theta \rightarrow 0$ . In other words, the divergence occurs for the small momentum transfer  $q \rightarrow 0$ , which corresponds to large distances. The reason why the total cross section diverges is because the Coulomb potential is actually a long-range force. No matter how far the incident particles are from the charge, there is always an effect on the motion of the particles and they get scattered.



### Form factor

If the source of Coulomb potential has an distribution  $\rho_N(\mathbf{x})$ , then

$$V(\mathbf{x}) = \int d^3x' \frac{\alpha}{|\mathbf{x} - \mathbf{x}'|} \rho(\mathbf{x}')$$

Note that the potential is mathematically a convolution of the Coulomb potential and the probability density. Since the first Born amplitude is nothing but the Fourier transform of the potential, the convolution becomes a product of Fourier transforms, one for the Coulomb potential and the other for the probability density. So

$$f(\theta) = f(\theta)_{\text{pointlike}} F(q)$$

Here,

$$F(q) \equiv \int d^3x \rho_N(\mathbf{x}) e^{i\mathbf{q} \cdot \mathbf{x}},$$

being called form factor.

### Born expansion

Define T-matrix by

$$V|\psi\rangle = T|\phi\rangle$$

Using the definition of the T-matrix, we find

$$f(\mathbf{k}, \mathbf{k}') = -\frac{m}{2\pi} (2\pi)^3 \langle \mathbf{k}' | T | \mathbf{k} \rangle$$

Using the Lippmann–Schwinger equation and multiplying the both sides by  $V$  from left, we find

$$T|\phi\rangle = V \frac{1}{E - H_0 + i\epsilon} T|\phi\rangle + V|\phi\rangle$$

A formal solution to the T-matrix is

$$T = \frac{1}{1 - V \frac{1}{E - H_0 + i\epsilon}} V$$

By Taylor expanding this operator in geometric series, we find

$$T = V + V \frac{1}{E - H_0 + i\epsilon} V + V \frac{1}{E - H_0 + i\epsilon} V \frac{1}{E - H_0 + i\epsilon} V + \dots$$

So,

$$|\psi\rangle = \left( 1 + \frac{1}{E - H_0 + i\epsilon} V + \frac{1}{E - H_0 + i\epsilon} V \frac{1}{E - H_0 + i\epsilon} V + \dots \right) |\phi\rangle$$

The first term is the wave which did not get scattered. The second term is the wave that gets scattered at a point in the potential and then propagates outwards by the propagator. In the third term, the wave gets scattered at a point in the potential, propagates for a while, and gets scattered again at another point in the potential, and propagates outwards. In the  $n + 1$ -th term, there are  $n$  times scattering of the wave before it propagates outwards.



## 20.3 Partial wave analysis

### Partial wave expansion

When the potential is **central**, angular momentum is conserved due to Noether's theorem. Therefore, we can expand the wave function in the eigenstates of the angular momentum. Obtained waves with definite angular momenta are called partial waves. We can solve the scattering problem for each partial wave separately, and then in the end put them together to obtain the full scattering amplitude. The plane wave can be expanded as follows.

$$e^{ikz} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos \theta)$$

Here,  $j_l(kr)$  is spherical Bessel functions of first kind. The asymptotic behaviour of  $j_l(kr)$  at large  $r$  can be written as

$$j_l(kr) \sim \frac{\sin(kr - \frac{l\pi}{2})}{kr}$$

so,

$$e^{ikz} \sim \frac{1}{2ikr} \sum_{l=0}^{\infty} (2l+1) (e^{ikr} - (-1)^l e^{-ikr}) P_l(\cos \theta)$$

Meanwhile, the  $f$  factor can be expanded as

$$f(\theta) = \sum_{l=0}^{\infty} f_l (2l+1) P_l(\cos \theta)$$

### Optical theorem constraint

The cross section can be represented by expansion coefficient of  $f$  factor as

$$\sigma = 4\pi \sum_l (2l+1) |f_l|^2$$

On the other hand,

$$\text{Im} f(0) = \sum_l (2l+1) \text{Im} f_l$$

From optical theorem we can derive that

$$|f_l|^2 = \frac{1}{k} \text{Im} f_l$$

This constraint can be rewritten as

$$|1 + 2ikf_l|^2 = 1$$

So we can define a phase  $\delta_l$  as

$$1 + 2ikf_l = e^{i\delta_l}$$

or equivalently,

$$f_l = \frac{1}{k} e^{i\delta_l} \sin(\delta_l)$$



### Phase shifts

We can derive the asymptotic behaviour of the wave function as

$$\psi(\mathbf{x}) \sim \frac{1}{2ikr} \sum_l (2l+1) P_l(\cos \theta) [e^{ikr} e^{2i\delta_l} - (-1)^l e^{-ikr}]$$

Compare it to the case of the plane wave without scattering. What this equation says is that the wave converging on the scatterer has the well-defined phase factor  $-(-1)^l$ , the same as in the case without scattering. On the other hand, the wave that emerges from the scatterer has an additional phase factor  $e^{2i\delta_l}$ . All what scattering did is to shift the phase of the emerging wave by  $2\delta_l$ . The reason why this is merely a phase factor is the conservation of probability. What converged to the origin must come out with the same strength. But this shift in the phase causes the interference among all partial waves different from the case without the phase shifts, and the result is not a plane wave but contains the scattered wave.

In terms of the phase shifts, the cross section is given by

$$\sigma = \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2 \delta_l$$

Actual calculation of phase shifts is basically to solve the Schrödinger equation for each partial waves,

$$\left[ -\frac{1}{r} \frac{d^2}{dr^2} r + \frac{l(l+1)}{r^2} + 2mV(r) \right] R_l(r) = k^2 R_l(r)$$

After solving the equation, we take the asymptotic limit  $r \rightarrow \infty$ , and write  $R_l(r)$  as a linear combination of  $j_l(kr) \cos \delta_l + n_l(kr) \sin \delta_l$ . The relative coefficients of  $j_l$  and  $n_l$  determines the phase shift  $\delta_l$ , and hence the cross section.



## **Part V**

# **Quantum Field Theory**



# Chapter 21

## From classical field to quantum field



### 21.1 Motivation

The state of the field is described by an element  $|\psi\rangle$  in Hilbert space. The measurement of the field is described by an operator field  $\phi_a(\mathbf{x})$ . The mean value of the measurement of the field is described by Erenfest's theorem

$$\frac{d\langle\psi|\phi_a(\mathbf{x})|\psi\rangle}{dt} = -i\langle\psi|[\phi_a(\mathbf{x}), H_S]|\psi\rangle$$

If  $[\phi_a(\mathbf{x}), H_S]_Q = i[\phi_a(\mathbf{x}), H_S]_C$ , we can reproduce the classical field theory as an average effect of quantum field theory. We also note that the commutation relation here  $[A, B] = AB - BA$  obeys the same operation laws as the Poisson bracket in classical field theory. So, what we need here is

1. the canonical quantization

$$[\phi_a(\mathbf{x}), \phi_b(\mathbf{y})] = 0 \quad [\pi^a(\mathbf{x}), \pi^b(\mathbf{y})] = 0 \quad [\phi_a(\mathbf{x}), \pi^b(\mathbf{y})] = i\delta_a^b \delta(\mathbf{x} - \mathbf{y})$$

2. the same definition of  $\mathcal{L}, \pi^a$  and  $H$  as those in corresponding classical theory.

In Heisenberg picture, we define

$$A_H \equiv U^{-1}(t, t_0) A_S U(t, t_0)$$

Especially,

$$\phi_a(x) \equiv U^{-1}(t, t_0) \phi_a(\mathbf{x}) U(t, t_0)$$

$$\pi^a(x) \equiv U^{-1}(t, t_0) \pi^a(\mathbf{x}) U(t, t_0)$$

If  $A = f(\phi_a(\mathbf{x}), \pi^a(\mathbf{x}), t)$ , then we have  $A_H = f(\phi_a(x), \pi^a(x), t)$ .

The canonical quantization can be generalized to the field operator in any time

$$[\phi_a(\mathbf{x}, t), \phi_b(\mathbf{y}, t)] = 0 \quad [\pi^a(\mathbf{x}, t), \pi^b(\mathbf{y}, t)] = 0 \quad [\phi_a(\mathbf{x}, t), \pi^b(\mathbf{y}, t)] = i\delta_a^b \delta(\mathbf{x} - \mathbf{y})$$

The dynamics of the field can be describe by Heisenberg equation

$$\frac{d\phi_a(x)}{dt} = -i[\phi_a(x), H_H]$$

$$\frac{d\pi^a(x)}{dt} = -i[\pi^a(x), H_H]$$

whose form will be equivalent to the classical field equation.

## 21.2 Lorentz invariance in quantum field theory

$$|\bar{\psi}\rangle = U(\Lambda)|\psi\rangle$$

Scalar fields:

$$\langle\bar{\psi}|\phi(x)|\bar{\psi}\rangle = \langle\psi|\phi(\Lambda^{-1}x)|\psi\rangle$$

$$U^{-1}(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$$

Vector fields:

$$\langle\bar{\psi}|A^\mu(x)|\bar{\psi}\rangle = \langle\psi|\Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)|\psi\rangle$$

$$U^{-1}(\Lambda)A^\mu(x)U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$$

Lorentz invariance means Lagrangian must be a scalar, or more loosely, action must be invariant under Lorentz transformation.

## 21.3 Momentum

The definition of momentum is the same as that in classical theory:

$$T^{\mu\nu} \equiv -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^\nu\phi_a + \eta^{\mu\nu}\mathcal{L} \quad \partial_\mu T^{\mu\nu} = 0$$

and

$$P^\mu \equiv \int T^{0\mu}d^3x \quad \frac{dP^\mu}{dt} = 0$$

$$P^0 = H, \quad P^i = \int -\pi^a \partial^i \phi_a d^3x$$

We can get the commutation relation

$$[\phi_a, P^\mu] = -i\partial^\mu\phi_a$$

$$[\pi^a, P^\mu] = -i\partial^\mu\pi^a$$

$$[P^\mu, P^\nu] = 0$$

We define the translation operator  $T(s)$  by

$$T^{-1}(s)\phi_a(x)T(s) = \phi_a(x-s)$$

then we can derive that

$$T(\epsilon) = 1 - i\epsilon_\mu P^\mu \quad T(s) = e^{-iP^\mu s_\mu}$$



## 21.4 Angular Momentum

The definition of angular momentum is the same as that in classical theory:

$$M^{\mu\nu\rho} \equiv x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} (\Sigma^{\nu\rho})_a^b \phi_b$$

and

$$M^{\nu\rho} \equiv \int M^{0\nu\rho} d^3x \quad \frac{dM^{\nu\rho}}{dt} = 0$$

$$M^{\mu\nu} = \int (x^\mu T^{0\nu} - x^\nu T^{0\mu} - \pi^a (\Sigma^{\mu\nu})_a^b \phi_b) d^3x$$

We also define that

$$M_L^{\mu\nu} \equiv \int (x^\mu T^{0\nu} - x^\nu T^{0\mu}) d^3x \quad M_S^{\mu\nu} \equiv \int (-\pi^a (\Sigma^{\mu\nu})_a^b \phi_b) d^3x$$

$$(L^{\mu\nu})_a^b \equiv -i(x^\mu \partial^\nu - x^\nu \partial^\mu) \delta_a^b \quad (S^{\mu\nu})_a^b \equiv -i(\Sigma^{\mu\nu})_a^b$$

We can get the commutation relation

$$M^{\mu\nu} = M_L^{\mu\nu} + M_S^{\mu\nu}$$

$$[\phi_a, M_L^{\mu\nu}] = (L^{\mu\nu})_a^b \phi_b \quad [\phi_a, M_S^{\mu\nu}] = (S^{\mu\nu})_a^b \phi_b$$

$$[\pi^a, M_L^{\mu\nu}] = (L^{\mu\nu})_b^a \pi^b \quad [\pi^a, M_S^{\mu\nu}] = -(S^{\mu\nu})_b^a \pi^b$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(-\eta^{\nu\rho} M^{\mu\sigma} + \eta^{\sigma\mu} M^{\rho\nu} + \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\sigma\nu} M^{\rho\mu})$$

We again define  $J_i \equiv \frac{1}{2} \epsilon_{ijk} M^{jk}$  and  $K_i \equiv M^{i0}$ , the commutation relation can be written as

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k \\ [J_i, K_j] &= i\epsilon_{ijk} K_k \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k \end{aligned}$$

Further more,

$$[P^\mu, M^{\rho\sigma}] = i(\eta^{\mu\sigma} P^\rho - \eta^{\mu\rho} P^\sigma)$$

$$\begin{aligned} [J_i, H] &= 0 \\ [J_i, P_j] &= i\epsilon_{ijk} P_k \\ [K_i, H] &= iP_i \\ [K_i, P_j] &= i\delta_{ij} H \end{aligned}$$

At last, we also define  $L_i \equiv \frac{1}{2} \epsilon_{ijk} M_L^{jk}$  and  $S_i \equiv \frac{1}{2} \epsilon_{ijk} M_S^{jk}$ . So

$$\begin{aligned} [L_i, S_j] &= 0 \\ [S_i, P_j] &= 0 \\ [L_i, P_j] &= i\epsilon_{ijk} P_k \end{aligned}$$



We define the rotation operator  $U(\Lambda)$  by

$$U^{-1}(\Lambda)\phi_a(x)U(\Lambda) = \mathcal{S}_a^b \phi_b(\Lambda^{-1}x)$$

where

$$\mathcal{S}_a^b = \delta_a^b + \frac{i}{2}\delta\omega_{\alpha\beta}(S^{\alpha\beta})_a^b$$

We can derive that

$$U(1 + \delta\omega) = 1 + \frac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu} \quad U(\Lambda) = e^{\frac{i}{2}\theta_{\mu\nu}M^{\mu\nu}}$$

$$U^{-1}(\Lambda)P^\mu U(\Lambda) = \Lambda^\mu_\nu P^\nu$$

$$U^{-1}(\Lambda)M^{\mu\nu}U(\Lambda) = \Lambda^\mu_\rho \Lambda^\nu_\sigma M^{\rho\sigma}$$

## 21.5 Anticommutation relation

We define the anticommutation relation of operators as  $\{A, B\} \equiv AB + BA$ . Suppose that the field operator and its canonical momentum operator has the following anticommutation relation

$$\{\phi_a(\mathbf{x}, t), \phi_b(\mathbf{y}, t)\} = 0 \quad \{\pi^a(\mathbf{x}, t), \pi^b(\mathbf{y}, t)\} = 0 \quad \{\phi_a(\mathbf{x}, t), \pi^b(\mathbf{y}, t)\} = i\delta_a^b \delta(\mathbf{x} - \mathbf{y})$$

If the operator  $A$  composes of the terms like  $\pi^a \mathcal{E}_a^b \phi_b$ , we can show that the value of  $[\phi_a, A]$  and  $[\pi^a, A]$  is the same as those in the theory quantized with commutation relation. It is easy to verify that the operator  $P^i$  and  $M_S^{\mu\nu}$  has the required form. The form of  $H$  is determined by the specific theory. As we can see later, the Hamiltonian of Dirac field has the required form. When it is quantized with anticommutation relation, the commutation relation between field operator and momentum, angular momentum discussed in previous section will be hold automatically.



# Chapter 22

## Spin 0 Fields



### 22.1 Klein-Gordon fields

Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 + \Omega_0$$

Field equation

$$(\partial^\mu\partial_\mu - m^2)\phi = 0$$

Hamiltonian

$$\begin{aligned}\pi &= \dot{\phi} \\ \mathcal{H} &= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 - \Omega_0 \\ H &= \int \mathcal{H}d^3x\end{aligned}$$

Momentum and angular momentum

$$\begin{aligned}T^{\mu\nu} &= \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\left(\frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 - \Omega_0\right) \\ P^0 &= H \quad P^i = \int -\pi\nabla^i\phi d^3x \\ J_k &= \int -\pi\epsilon_{ijk}x^j\nabla^k\phi d^3x\end{aligned}$$

### 22.2 Canonical quantization Formulation

Canonical quantization

$$\begin{aligned}[\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] &= 0 \\ [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] &= 0 \\ [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] &= i\delta(\mathbf{x} - \mathbf{y})\end{aligned}$$

## Fourier expansion

$$\phi(\mathbf{x}, t) = \int \widetilde{dk} [a(\mathbf{k})e^{ikx} + a^\dagger(\mathbf{k})e^{-ikx}]$$

$$\pi(\mathbf{x}, t) = -i \int \widetilde{dk} \omega [a(\mathbf{k})e^{ikx} - a^\dagger(\mathbf{k})e^{-ikx}]$$

Here,  $k^2 = \mathbf{k}^2 - \omega^2 = -m^2$ ,  $kx = \mathbf{k} \cdot \mathbf{x} - \omega t$ ,  $\widetilde{dk} = \frac{d^3}{(2\pi)^2 2\omega}$

$$a(\mathbf{k}) = \int d^3x e^{-ikx} (i\pi + \omega\phi)$$

$$a^\dagger(\mathbf{k}) = \int d^3x e^{ikx} (-i\pi + \omega\phi)$$

$$\begin{aligned} [a(\mathbf{p}), a(\mathbf{q})] &= 0 \\ [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{q})] &= 0 \\ [a(\mathbf{p}), a^\dagger(\mathbf{q})] &= (2\pi)^3 2\omega \delta(\mathbf{p} - \mathbf{q}) \end{aligned}$$

Operator represented by  $a$  and  $a^\dagger$ 

$$H = \int \widetilde{dk} \omega a^\dagger(\mathbf{k}) a(\mathbf{k}) + (\mathcal{E}_0 - \Omega_0) V \quad \mathcal{E}_0 = \frac{1}{2} (2\pi)^{-3} \int d^3k \omega$$

$$P^i = \int \widetilde{dk} k^i a^\dagger(\mathbf{k}) a(\mathbf{k})$$

## Particles

$$\begin{aligned} [H, a(\mathbf{k})] &= -\omega a(\mathbf{k}) & [H, a^\dagger(\mathbf{k})] &= \omega a^\dagger(\mathbf{k}) \\ [P^i, a(\mathbf{k})] &= -k^i a(\mathbf{k}) & [P^i, a^\dagger(\mathbf{k})] &= k^i a^\dagger(\mathbf{k}) \end{aligned}$$

Let  $|p\rangle = a^\dagger(\mathbf{p})|0\rangle$ , so

$$H|p\rangle = \omega_p |p\rangle \quad P^i |p\rangle = p^i |p\rangle$$

So, we interpret the state  $|p\rangle$  as the momentum eigenstate of a single particle of mass  $m$ . We can also show that  $J_i |p\rangle = 0$ , so the particle carries no internal angular momentum.

## Causality

The amplitude for a particle to propagate from  $y$  to  $x$  is  $\langle 0 | \phi(x) \phi(y) | 0 \rangle$ , denoted by  $D(x - y)$ .

$$D(x - y) = \int \widetilde{dk} e^{ik(x-y)}$$

$$[\phi(x), \phi(y)] = D(x - y) - D(y - x)$$

If  $x - y$  is space-like, a continuous Lorentz transformation can take  $(x - y)$  to  $-(x - y)$ . So  $[\phi(x), \phi(y)] = 0$  for space-like  $x - y$ . A measurement performed at one point can not affect a measurement at another point whose separation is space-like.



## The Klein-Gordon propagator

$$D_R(x - y) \equiv \theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{-i}{p^2 + m^2} e^{ip(x-y)}$$



Figure 22.1: Retarded Green Function

$$(\partial^2 - m^2)D_R(x - y) = i\delta(x - y)$$

$$D_F(x - y) \equiv \langle 0 | T \phi(x) \phi(y) | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{-i}{p^2 + m^2 - i\epsilon} e^{ip(x-y)}$$

Here,  $T$  stands for time ordering, placing all operators evaluated at later times to the left.

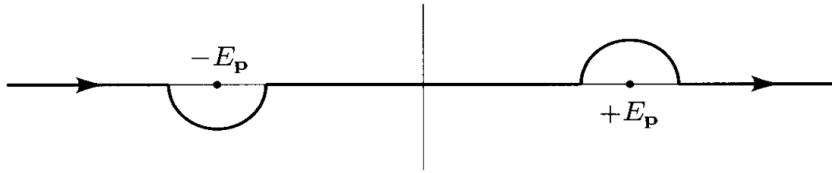


Figure 22.2: Feynman Green Function

## 22.3 Perturbation theory for canonical quantization

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m_0^2 \phi^2 - \frac{\lambda_0}{4!} \phi^4$$

$$H = H_0 + H_{int} \quad H_{int} = \int d^3 x \frac{\lambda_0}{4!} \phi^4(\mathbf{x})$$

## 22.3.1 Perturbation expansion of correlation functions



**Note:** The ground state of the interaction field theory is denoted by  $|\Omega\rangle$ , the ground state of the free field theory is denoted by  $|0\rangle$ . The zero of energy is defined by  $H_0|0\rangle = 0$  and  $E_0 = \langle \Omega | H | \Omega \rangle$ .

$$\phi(t_0, \mathbf{x}) = \int \frac{d^3 p}{(2\pi)^3} (a(\mathbf{p}) e^{i\mathbf{p} \cdot \mathbf{x}} + a^\dagger(\mathbf{p}) e^{-i\mathbf{p} \cdot \mathbf{x}})$$

$$\phi(t, \mathbf{x}) = e^{iH(t-t_0)} \phi(t_0, \mathbf{x}) e^{-iH(t-t_0)}$$

$$\phi_I(t, \mathbf{x}) \equiv e^{iH_0(t-t_0)} \phi(t_0, \mathbf{x}) e^{-iH_0(t-t_0)}$$



$$H_I(x) = \int d^3x \frac{\lambda_0^4}{4!} \phi_I^4$$

The perturbation expansion of correlation functions is

$$\langle \Omega | T \{ \phi(x) \phi(y) \} | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | T \left\{ \phi_I(x) \phi_I(y) \exp \left[ -i \int_{-T}^T dt H_I \right] \right\} | 0 \rangle}{\langle 0 | T \left\{ \exp \left[ -i \int_{-T}^T dt H_I \right] \right\} | 0 \rangle}$$

The proof can be found in chapter 4.2 of *An introduction to quantum field theory* (M.E.Peskin & D.V.Schroeder)

### Theorem 22.1 Wick's theorem

$$T \{ \phi(x_1) \phi(x_2) \cdots \phi(x_n) \} = N \{ \phi(x_1) \phi(x_2) \cdots \phi(x_n) + \text{all possible contractions} \}$$

Normal order : all the  $a$ 's are to the right of all the  $a^\dagger$ .

The proof can be found in chapter 4.3 of *An introduction to quantum field theory* (M.E.Peskin & D.V.Schroeder)

**Example:**

$$\begin{aligned} \langle 0 | T \{ \phi_I(x_1) \phi_I(x_2) \phi_I(x_3) \phi_I(x_4) \} | 0 \rangle &= D_F(x_1 - x_2) D_F(x_3 - x_4) \\ &+ D_F(x_1 - x_3) D_F(x_2 - x_4) \\ &+ D_F(x_1 - x_4) D_F(x_2 - x_3) \end{aligned}$$

### 22.3.2 Feynman diagram

Expand  $\langle 0 | T \left\{ \phi_I(x) \phi_I(y) \exp \left[ -i \int_{-T}^T dt H_I \right] \right\} | 0 \rangle$  to the first order of  $\lambda_0$

$$\begin{aligned} &\langle 0 | T \left\{ \phi_I(x) \phi_I(y) \frac{-i\lambda_0}{4!} \int d^4z \phi_I(z) \phi_I(z) \phi_I(z) \phi_I(z) \right\} | 0 \rangle \\ &= 3 \cdot \left( \frac{-i\lambda_0}{4!} \right) D_F(x - y) \int d^4z D_F(z - z) D_F(z - z) \\ &+ 12 \cdot \left( \frac{-i\lambda_0}{4!} \right) \int d^4z D_F(x - z) D_F(y - z) D_F(z - z) \end{aligned}$$

It can be represented by the so called Feynman diagram.

$$\left( \begin{array}{c} \text{---} \cdot \text{---} \cdot \text{---} \\ x \quad y \end{array} \right) \bigcirc_z + \left( \begin{array}{c} \text{---} \cdot \text{---} \cdot \text{---} \\ x \quad z \quad y \end{array} \right)$$

**Figure 22.3:** Feynman diagram representation of perturbation expansion

The symmetry factor of the first diagram is  $S = \frac{4!}{3} = 8$ . The symmetry factor of the second diagram is  $S = \frac{4!}{12} = 2$ . The Feynman rules for  $\phi^4$  theory are:





1. For each propagator,  $P = D_F(x - y)$
2. For each vertex,  $V = (-i\lambda_0) \int d^4z$
3. For each external point,  $E = 1$
4. Divided by the symmetry factor

At last, we can prove that

$$\langle \Omega | T \{ \phi_I(x_1) \phi_I(x_2) \cdots \phi_I(x_n) \} | \Omega \rangle = \text{sum of all E-connected diagrams with n external points}$$

Here, the "E-disconnected" means disconnected from all external points", being called "vacuum bubbles". They vacuum bubbles in  $\langle 0 | T \{ \phi_I(x_1) \phi_I(x_2) \cdots \phi_I(x_n) \exp \left[ -i \int_{-T}^T dt H_I \right] \} | 0 \rangle$  are all cancelled by the  $\langle 0 | T \{ \exp \left[ -i \int_{-T}^T dt H_I \right] \} | 0 \rangle$ .

## 22.4 Path integral formulation

### 22.4.1 Basic formulation

Recall the path integrals formulation in quantum mechanics, we have

$$\langle \phi_b(\mathbf{x}) | e^{-iHT} | \phi_a(\mathbf{x}) \rangle = \int \mathcal{D}\phi \mathcal{D}\pi \exp \left[ i \int_0^T d^4x \left( \pi \dot{\phi} - \frac{1}{2} \pi^2 - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right) \right]$$

Here,  $\langle \phi_b(\mathbf{x}) |$  is the eigenstate of  $\phi_S(\mathbf{x}) = \phi_H(\mathbf{x}, 0)$  with eigenvalue  $\phi_b(\mathbf{x})$  at time  $t = T$ ,  $| \phi_a(\mathbf{x}) \rangle$  is the eigenstate of  $\phi_S(\mathbf{x})$  with eigenvalue  $\phi_a(\mathbf{x})$  at time  $t = 0$ .

Since the exponential is quadratic in  $\pi$ , we can complete the square and evaluate the  $\mathcal{D}(\pi)$  integral to obtain

$$\langle \phi_b(\mathbf{x}) | e^{-iHT} | \phi_a(\mathbf{x}) \rangle = \int \mathcal{D}\phi \exp \left[ i \int_0^T d^4x \mathcal{L} \right]$$

Now we can abandon the Hamiltonian formalism and take the the equation above to define the Hamiltonian dynamics.



**Note:** We emphasize that in this subsection,  $\phi_H$  denotes the Heisenberg picture of field whose value is operators,  $\phi_S$  denotes the Schrödinger picture of field,  $\phi(x)$  is classical field whose value is ordinary number.

### Correlation function

$$\langle \Omega | T \phi_H(x_1) \phi_H(x_2) | \Omega \rangle = \lim_{T \rightarrow \infty (1-i\epsilon)} \frac{\int \mathcal{D}\phi \phi(x_1) \phi(x_2) \exp \left[ i \int_T^T d^4x \mathcal{L} \right]}{\int \mathcal{D}\phi \exp \left[ i \int_T^T d^4x \mathcal{L} \right]}$$

The proof can be found in chapter 9.2 of *An introduction to quantum field theory* (M.E.Peskin & D.V.Schroeder).



### Functional derivatives and the generating functional

We define the generating functional as

$$Z[J] \equiv \int \mathcal{D}\phi \exp \left[ i \int d^4x \mathcal{L} + J(x)\phi(x) \right]$$

We can prove that

$$\langle \Omega | T \phi_H(x_1) \cdots \phi_H(x_n) | \Omega \rangle = \frac{1}{Z_0} \left( -i \frac{\delta}{\delta J(x_1)} \right) \cdots \left( -i \frac{\delta}{\delta J(x_n)} \right) Z[J] |_{J=0}$$

Here,  $Z_0 \equiv Z[J = 0]$ .

### 22.4.2 Free field theory

In Klein-Gordon field theory,

$$\int d^4x [\mathcal{L}_0(\phi) + J\phi] = \int d^4x \left[ \frac{1}{2} \phi (\partial^2 - m^2 + i\epsilon) \phi + J\phi \right]$$

Define

$$\phi'(x) \equiv \phi(x) + \int d^4y (-iD_F(x-y))J(y)$$

Recall that  $(\partial^2 - m^2)D_F(x-y) = i\delta(x-y)$ , we can derive that

$$\int d^4x [\mathcal{L}_0 + J\phi] = \int d^4x \left[ \frac{1}{2} \phi' (\partial^2 - m^2 + i\epsilon) \phi' \right] - \int d^4x d^4y \frac{1}{2} J(x) [-iD_F(x-y)] J(y)$$

After integration, we can know that

$$Z[J] = Z_0 \exp \left[ -\frac{1}{2} \int d^4x d^4y J(x) D_F(x-y) J(y) \right]$$

So,

$$\langle 0 | T \phi_H(x_1) \phi_H(x_2) | 0 \rangle = \frac{1}{Z_0} - \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \exp \left[ -\frac{1}{2} \int d^4x d^4y J(x) D_F(x-y) J(y) \right] |_{J=0} = D_F(x_1 - x_2)$$

## 22.5 Perturbation theory for path integral quantization

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m_0^2 \phi^2 - \frac{\lambda_0}{4!} \phi^4 \\ \mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_1 \quad \mathcal{L}_1 = -\frac{\lambda_0}{4!} \phi^4(x) \\ Z[J] &= \int \mathcal{D}\phi e^{i \int d^4x [\mathcal{L}_0 + \mathcal{L}_1 + J\phi]} \\ &= e^{i \int d^4x \mathcal{L}_1 \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right)} \int \mathcal{D}\phi e^{i \int d^4x [\mathcal{L}_0 + J\phi]} \\ &\propto e^{i \int d^4x \mathcal{L}_1 \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right)} \exp \left[ -\frac{1}{2} \int d^4y d^4z J(y) D_F(y-z) J(z) \right] \\ &= \sum_{V=0}^{\infty} \frac{1}{V!} \left[ \frac{-i\lambda_0}{4!} \int d^4x \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right)^4 \right]^V \times \sum_{P=0}^{\infty} \frac{1}{P!} \left[ -\frac{1}{2} \int d^4y d^4z J(y) D_F(y-z) J(z) \right]^P \end{aligned}$$



If we focus on a term with particular values of  $V$  and  $P$ , then the number of surviving sources (after we take all the functional derivatives) is  $E = 2P - 4V$ . The  $4V$  functional derivatives can act on the  $2P$  sources in  $\frac{(2P)!}{(2P-4V)!}$  different combinations. However, many of the resulting expressions are algebraically identical.

To organize them, we introduce Feynman diagrams similar to that in perturbation theory of canonical quantization. In these diagrams, a line segment stands for a propagator  $D_F(x - y)$ , a filled circle at one end of a line segment for a source  $i \int d^4x J(x)$ , and a vertex joining four line segments for  $-i\lambda_0 \int d^4z$ .

For each diagram, we can assign a symmetry factor  $S_P$  similar to that in perturbation theory for canonical quantization. Due to the fact that some external sources are identical here, usually  $S_P \neq S_C$ . But when calculating the correlation function, the exchange of the order of functional derivatives to identical sources can eliminate the difference.

We can demonstrate that

$$Z[J] \propto \exp\left(\sum_I C_I\right)$$

Here,  $C_I$  stands for a particular connected diagram, including its symmetry factor. We define  $W[J]$  as

$$Z[J] \equiv Z_0 \exp(-iW[J])$$

As,  $W[0] = 0$ , we know

$$-iW[J] = \sum_{I \neq \{0\}} C_I$$

The notation  $I \neq \{0\}$  means that the vacuum diagrams are omitted from the sum.

The detailed discussion can be found in chapter 9 of *Quantum field theory* (M. Srednicki).

## 22.6 Symmetries in the functional formalism

### Equations of motion

The correlation function of the field theory is given by

$$\langle \Omega | T \phi_H(x_1) \cdots \phi_H(x_n) | \Omega \rangle = Z_0^{-1} \int \mathcal{D}\phi e^{iS} \phi(x_1) \cdots \phi(x_n)$$

The equation of motion of classical field theory will be give by

$$\frac{\delta S}{\delta \phi(x)} = 0$$

In quantum field theory, we derive the equation of motion by claim that the path integral will be invariant under the infinitesimal change of field, i.e.  $\phi(x) \rightarrow \phi(x) + \epsilon(x)$ . Define

$$Z[\phi(x_1), \cdots, \phi(x_n)] \equiv \int \mathcal{D}\phi e^{iS} \phi(x_1) \cdots \phi(x_n)$$



We know

$$\delta Z = \int \mathcal{D}\phi e^{iS} \left\{ \int d^4x \epsilon(x) \left[ i \frac{\delta S}{\delta \phi(x)} \phi(x_1) \cdots \phi(x_n) + \delta(x - x_1) \cdots \phi(x_n) + \cdots + \phi(x_1) \cdots \delta(x - x_n) \right] \right\}$$

so

$$\left\langle \frac{\delta S}{\delta \phi(x)} \phi(x_1) \cdots \phi(x_n) \right\rangle = \sum_{i=1}^n \langle \phi(x_1) \cdots (i\delta(x - x_i)) \cdots \phi(x_n) \rangle$$

**Example:** For Klein-Gordon field,

$$\mathcal{L} = -\frac{1}{2} \phi^\mu \phi \partial_\mu \phi - \frac{1}{2} m \phi^2$$

The variation of  $S$  gives

$$\frac{\delta S}{\delta \phi(x)} = (\partial^2 - m^2) \phi(x)$$

So, we can get

$$(\partial^2 - m^2) \langle T \phi(x) \phi(x_1) \rangle = i \delta(x - x_1)$$

### Conservation law

Consider a local field theory of a set of fields  $\phi_a(x)$ , governed by a Lagrangian  $\mathcal{L}(\phi)$ . An infinitesimal symmetry transformation on the fields  $\phi_a$  will be of the general form

$$\phi_a(x) \rightarrow \phi_a(x) + \epsilon \Delta \phi_a(x)$$

We assume that when  $\epsilon$  is a constant, the action is invariant under this transformation. Then the Lagrangian must be invariant up to a total divergence:

$$\mathcal{L}[\phi] \rightarrow \mathcal{L}[\phi] + \epsilon \partial_\mu K^\mu$$

If the symmetry parameter  $\epsilon$  depend on  $x$ , the variation of Lagrangian will be

$$\mathcal{L}[\phi] \rightarrow \mathcal{L}[\phi] + (\partial_\mu \epsilon) \Delta \phi_a \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} + \epsilon \partial_\mu K^\mu$$

So,

$$\frac{\delta S}{\delta \epsilon(x)} = \partial_\mu j^\mu \quad j^\mu = -\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \Delta \phi_a + K^\mu$$

If the measure  $\mathcal{D}\phi$  is invariant under the transformation, we can derive similarly that

$$\langle \partial_\mu j^\mu(x) \phi(x_1) \cdots \phi(x_n) \rangle = \sum_{i=1}^n \langle \phi(x_1) \cdots (i \Delta \phi(x_i) \delta(x - x_i)) \cdots \phi(x_n) \rangle$$



## 22.7 Cross section and the S-matrix

For a relativistic collinear scattering process, we have

$$dN = \sigma |\mathbf{v}_1 - \mathbf{v}_2| n_1 dt$$

Consider a  $2 \rightarrow n$  process:

$$p_1 + p_2 \rightarrow \{p_j\}$$

Suppose the volume of the space in which the scattering process takes place is  $V$ , the duration of the scattering process is  $T$ . So, the density of incident particle is

$$n_1 = \frac{1}{V}$$

and the number of events is

$$N = \frac{|\langle i|S|f\rangle|^2}{|\langle i|i\rangle||\langle f|f\rangle|} d\Pi.$$

Here,  $d\Pi$  is the region of final state momenta at which we are looking

$$d\Pi = \prod_j \frac{V}{(2\pi)^3} d^3 p_j$$

Recall that

$$\delta^{(3)}(0) = \frac{V}{(2\pi)^3} \quad \delta^{(4)}(0) = \frac{VT}{(2\pi)^4} \quad \langle p|p\rangle = (2\pi)^3 2\omega \delta^{(3)}(0)$$

We can calculate that

$$|\langle i|S|f\rangle|^2 = |\mathcal{M}|^2 VT (2\pi)^4 \delta(\sum p_j - p_1 - p_2) \quad \langle i|i\rangle = (2E_1 V)(2E_2 V) \quad \langle f|f\rangle = \prod_j (2E_j V)$$

At last, putting everything together, we have

$$d\sigma = \frac{1}{(2E_1)(2E_2)|\mathbf{v}_1 - \mathbf{v}_2|} |\mathcal{M}|^2 d\Pi_{\text{LIPS}}$$

where,

$$d\Pi_{\text{LIPS}} = \prod_j \frac{d^3 p_j}{(2\pi)^3} \frac{1}{2E_j} (2\pi)^4 \delta(\sum p_j - p_1 - p_2)$$

is called the Lorentz-invariant phase space (LIPS).

All the factors of  $V$  and  $T$  have dropped out, so now it is trivial to take  $V \rightarrow \infty$  and  $T \rightarrow \infty$ .

### Decay rates

The definition of decay rates is

$$\Gamma \equiv \frac{\text{Number of Events}}{\text{Time}}$$

Consider a  $1 \rightarrow n$  process:

$$p_1 \rightarrow \{p_j\}$$

We can derive the decay rates by the similar method,

$$d\Gamma = \frac{1}{2E_1} |\mathcal{M}|^2 d\Pi_{\text{LIPS}}$$



### Special cases

For  $2 \rightarrow 2$  scattering in the center-of-mass frame

$$p_1 + p_2 \rightarrow p_3 + p_4$$

Then

$$d\Pi_{\text{LIPS}} = \frac{d^3 p_3}{(2\pi)^3} \frac{1}{2E_3} \frac{d^3 p_4}{(2\pi)^3} \frac{1}{2E_4} (2\pi)^4 \delta(p_3 + p_4 - p_1 - p_2)$$

We can now integrate over  $\mathbf{p}_4$  to give

$$d\Pi_{\text{LIPS}} = \frac{1}{16\pi^2} d\Omega \int dp_f \frac{p_f^2}{E_3 E_4} \delta(E_3 + E_4 - E_{CM})$$

where  $p_f = |\mathbf{p}_3| = |\mathbf{p}_4|$ ,  $E_3 = \sqrt{p_f^2 + m_3^2}$ ,  $E_4 = \sqrt{p_f^2 + m_4^2}$  and  $E_{CM} = E_1 + E_2$ . Define  $x(p_f) \equiv E_3 + E_4 - E_{CM}$ , we can get

$$\frac{dx}{dp_f} = \frac{E_3 + E_4}{E_3 E_4} p_f$$

So,

$$d\Pi_{\text{LIPS}} = \frac{1}{16\pi^2} d\Omega \frac{p_f}{E_{CM}} \theta(E_{CM} - m_3 - m_4)$$

Plugging this into the general equation for cross section, we can get

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{CM}^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|} |\mathcal{M}|^2 \theta(E_{CM} - m_3 - m_4)$$

If all the masses are equal then this formula simplifies further

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{CM}^2} |\mathcal{M}|^2$$



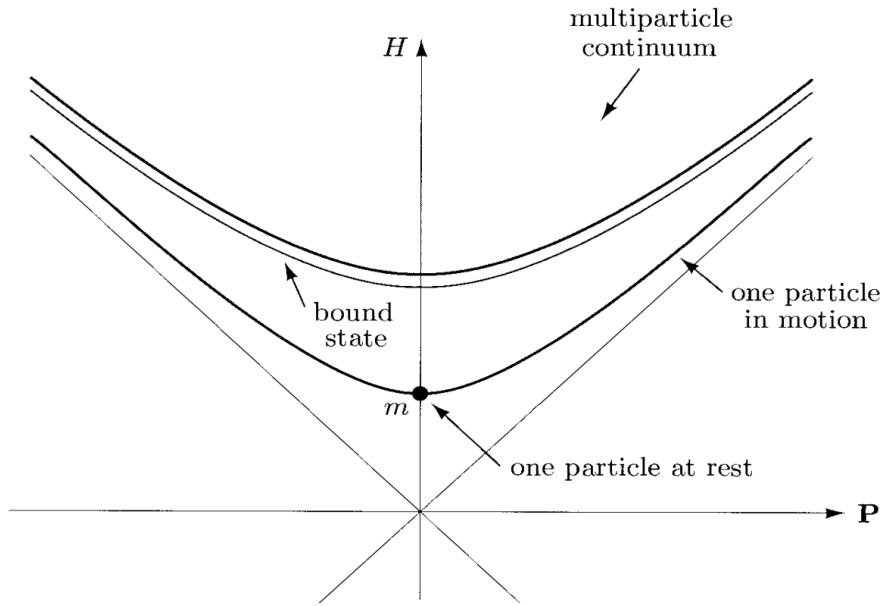
## 22.8 LSZ reduction formula

### 22.8.1 Field strength renormalization

The completeness relation:

$$1 = |\Omega\rangle\langle\Omega| + \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\lambda_p\rangle\langle\lambda_p|$$

Here,  $E_p = \sqrt{m_\lambda^2 + \mathbf{p}^2}$



**Figure 22.4:** Particle's energy-momentum relation

Assume for now  $x^0 > y^0$  and define  $\langle\Omega|\phi(x)\phi(y)|\Omega\rangle_C = \langle\Omega|\phi(x)\phi(y)|\Omega\rangle - \langle\Omega|\phi(x)|\Omega\rangle\langle\Omega|\phi(y)|\Omega\rangle$  as connected two point function. (The term  $\langle\Omega|\phi(x)|\Omega\rangle\langle\Omega|\phi(y)|\Omega\rangle$  is usually zero by symmetry; for higher spin fields, it is zero by Lorentz invariance.) The connected two point function is

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle_C = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \langle\Omega|\phi(x)|\lambda_p\rangle\langle\lambda_p|\phi(y)|\Omega\rangle$$

It can be verified that

$$\langle\Omega|\phi(x)|\lambda_p\rangle = \langle\Omega|\phi(0)|\lambda_0\rangle e^{ipx}|_{p^0=E_p}$$

So,

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle_C = \sum_{\lambda} \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2 + m_\lambda^2 - i\epsilon} e^{ip(x-y)} |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

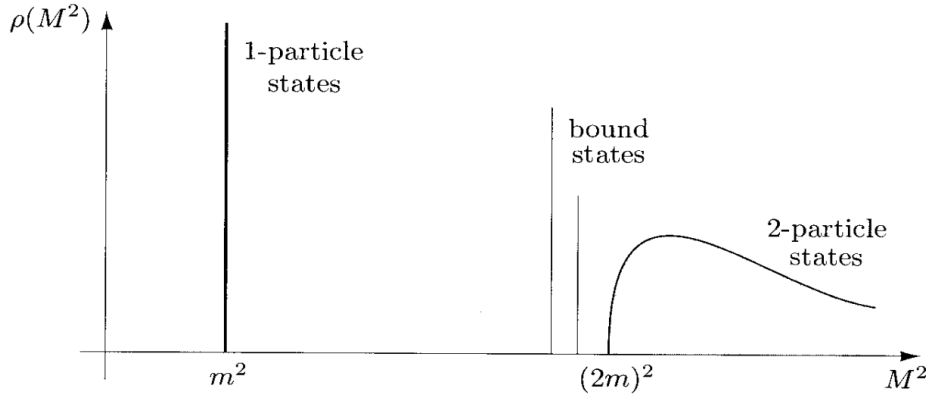
Analogous expressions hold for the case  $y^0 > x^0$ , and both cases can be summarized as

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle_C = \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) D_F(x-y; M^2)$$



and

$$\rho(M^2) = \sum_{\lambda} (2\pi) \delta(M^2 - m^2) |\langle \Omega | \phi(0) | \lambda_0 \rangle|^2$$



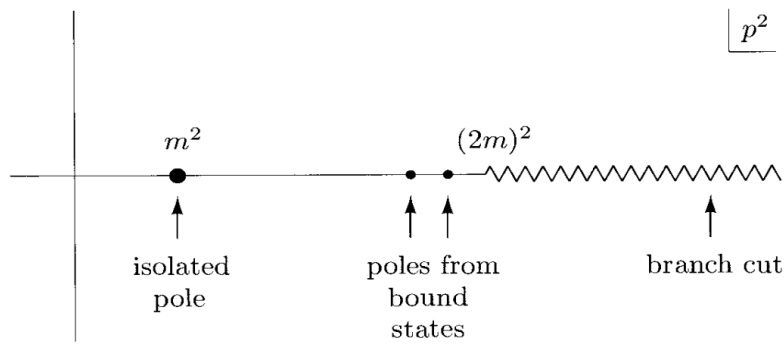
**Figure 22.5:** The structure of the spectral density function  $\rho(M^2)$

The one-particle state contribute an isolated delta function to the spectral density function, so

$$\rho(M^2) = 2\pi \delta(M^2 - m^2) \cdot Z + (\text{nothing else until } M^2 \gtrsim (2m)^2)$$

$Z = |\langle \Omega | \phi(0) | \lambda_0 \rangle|^2$  is called field-strength renormalization.  $m$  is the physical mass of a single particle of the  $\phi$  boson. The Fourier transformation of the two point function is

$$\begin{aligned} & \int d^4x e^{-ipx} \langle \Omega | T \phi(x) \phi(0) | \Omega \rangle_C \\ &= \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) \frac{-i}{p^2 + M^2 - i\epsilon} = \frac{-iZ}{p^2 + m^2 - i\epsilon} + \int_{\sim 4m^2}^\infty \frac{dM^2}{2\pi} \rho(M^2) \frac{-i}{p^2 + M^2 - i\epsilon} \end{aligned}$$



**Figure 22.6:** The structure of the two point function in Fourier space









**Note:** One-particle-irreducible, or 1PI for short, refers to diagrams that is still connected after one line is cut

Then the exact propagator can be written as a geometric series in Figure 2.9.



**Figure 22.9:** Diagram representation of exact propagator

The result is  $\frac{-i}{p^2 + m_0^2 + M^2}$ . If we expand each resummed propagator about the physical particle pole, we see that each external leg of the four-point amplitude contributes

$$\frac{-i}{p^2 + m_0^2 + M^2} \underset{p^0 \rightarrow E_p}{\sim} \frac{-iZ}{p^2 + m^2} + (\text{regular})$$

Thus, the sum of diagrams contains a product of four point poles:

$$\frac{-iZ}{p_1^2 + m^2} \frac{-iZ}{p_2^2 + m^2} \frac{-iZ}{k_1^2 + m^2} \frac{-iZ}{k_2^2 + m^2}$$

So, the S matrix element can be represented by

$$\langle \mathbf{p}_1 \mathbf{p}_2 | S | \mathbf{k}_1 \mathbf{k}_2 \rangle = (\sqrt{Z})^4 \text{Amp.},$$

**Figure 22.10:** Feynman diagram representation of LSZ reduction formula

It is easy to be generalized to the more complicated scattering cases. After Fourier transforming the n-point function to momentum space and cutting off the external legs, the Feynman rules for S-matrix element can be stated as follows:

1. For each propagator,  $P = \frac{-i}{p^2 + m_0^2 - i\epsilon}$ ;
2. For each vertex,  $V = -i\lambda_0$ ;
3. For each external point,  $E = 1$ ;
4. Impose momentum conservation at each vertex;
5. Integrate over each undetermined loop momentum:  $\int \frac{d^4 p}{(2\pi)^4}$ ;
6. Divided by the symmetry factor;
7. Multiply the total momentum conservation factor  $(2\pi)^4 \delta(\sum p_f - \sum p_i)$

We can write  $\langle f | S | i \rangle = (Z_1)^{\frac{n_s}{2}} i \mathcal{M} (2\pi)^4 \delta(\sum p_f - \sum p_i)$  for convenience.



## 22.9 Renormalization

Renormalization, the procedure in quantum field theory by which divergent parts of a calculation, leading to nonsensical infinite results, are absorbed by redefinition into a few measurable quantities, so yielding finite answers.

### 22.9.1 Counting of ultraviolet divergence

Consider a pure scalar theory in  $d$  dimensions with a  $\phi^n$  interaction term

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{n!}\phi^n$$

Let  $N$  be the number of external lines in the diagram,  $P$  the number of propagators,  $V$  the number of vertices. The number of the loops in the diagram is  $L = P - V + 1$ . There are  $n$  lines meeting at each vertex, so  $nV = 2P + N$ . Loosely speaking, each loop has an integral  $d^d p$ , each propagator has a factor  $p^{-2}$ , so the superficial degrees of divergence is

$$D = dL - 2P = d + [n(\frac{d-2}{2}) - d]V - (\frac{d-2}{2})N$$

According the superficial degrees of divergence of the diagram. These three possible types of ultraviolet behaviour of quantum field theories. We will refer to them as follows

1. Super-renormalizable theory: Only a finite number of Feynman diagrams superficially diverge.
2. renormalizable theory: Only a finite number of amplitudes superficially diverge; however, divergences occur at all orders in perturbation theory.
3. Non-renormalizable theory: All amplitudes are divergent at a sufficiently high order in perturbation theory.

So, for  $\phi^4$  theory in four dimension,  $D = 4 - N$ . It is a renormalizable theory. For  $\phi^3$  theory in four dimension,  $D = 4 - V - N$ . It is a super-renormalizable theory. For  $\phi^6$  theory in four dimension,  $D = 4 + 2V - N$ . It is a Non-renormalizable theory.

The superficial degrees of freedom can also be derived from dimensional analysis. The dimension of  $\lambda$  is  $d - \frac{n(d-2)}{2}$ . Now consider an arbitrary diagram with  $N$  external lines. One way that such a diagram could arise is from an interaction term  $\eta\phi^N$  in the Lagrangian. The dimension of  $\eta$  would then be  $d - \frac{N(d-2)}{2}$ , and therefore we conclude that any (amputated) diagram with  $N$  external lines has dimension  $d - \frac{N(d-2)}{2}$ . In our theory with only the  $\lambda\phi^n$  vertex, if the diagram has  $V$  vertices, its divergent part is proportional to  $\lambda^V \Lambda^D$ , where  $\Lambda$  is a high momentum cut-off and  $D$  is the superficial degree of divergence. Applying dimensional analysis, we find

$$d - \frac{N(d-2)}{2} = V[d - \frac{n(d-2)}{2}] + D$$

Note that the quantity that multiplies  $V$  in this expression is just the dimension of the coupling constant  $\lambda$ . Thus we can characterize the three degrees of renormalizability in a second way:




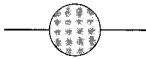
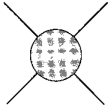
1. Super-renormalizable: Coupling constant has positive mass dimension.
2. renormalizable: Coupling constant is dimensionless.
3. Non-renormalizable: Coupling constant has negative mass dimension.

### 22.9.2 Renormalized perturbation theory

The Lagrangian of  $\phi^4$  theory is

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4$$

We write  $m_0$  and  $\lambda_0$ , to emphasize that these are the bare values of the mass and coupling constant, not the values measured in experiments. Since the theory is invariant under  $\phi \rightarrow -\phi$ , all amplitudes with an odd number of external legs vanish. The only divergent amplitudes are therefore

	(unobservable vacuum energy shift);
	$\sim \Lambda^2 + p^2 \log \Lambda + (\text{finite terms});$
	$\sim \log \Lambda + (\text{finite terms}).$

**Figure 22.11:** Divergence of  $\phi^4$  theory

Ignoring the vacuum diagram, these amplitudes contain three infinite constants. Our goal is to absorb these constants into the three unobservable parameters of the theory: the bare mass, the bare coupling constant, and the field strength. To accomplish this goal, it is convenient to reformulate the perturbation expansion so that these unobservable quantities do not appear explicitly in the Feynman rules. Recall that the exact two-point function has the form

$$\int d^4x \langle \Omega | \phi(x) \phi(0) | \Omega \rangle e^{-ipx} = \frac{-iZ}{p^2 + m^2} + \text{terms regular at } p^2 = m^2$$

We can eliminate the  $Z$  from this equation by rescaling the field:  $\phi = Z^{\frac{1}{2}}\phi_r$ . We also define

$$\delta_Z = Z - 1 \quad \delta_m = Zm_0^2 - m^2 \quad \delta_\lambda = \lambda_0 Z^2 - \lambda$$

Then the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi_r\partial_\mu\phi_r - \frac{1}{2}m^2\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 - \frac{1}{2}\delta_Z\partial^\mu\phi_r\partial_\mu\phi_r - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta\lambda}{4!}\phi_r^4$$

The last three terms, known as counter-terms, have absorbed the infinite but unobservable shifts between the bare parameters and the physical parameters. We give precise definitions of



$$\begin{aligned}
 \text{---} \bigcirc \text{---} &= \frac{-i}{p^2 + m^2} + (\text{terms regular at } p^2 = m^2); \\
 \left( \text{---} \bigcirc \text{---} \right)_{\text{amputated}} &= -i\lambda \quad \text{at } s = 4m^2, t = u = 0.
 \end{aligned}$$

Figure 22.12: Renormalization condition

the physical mass and coupling constant as Figure 2.12.

The renormalization scheme here is called on-shell (OS) scheme. Other renormalization scheme would be introduced later. These equations are called renormalization conditions. Our new Lagrangian gives a new set of Feynman rules as Figure 2.13.

$$\begin{aligned}
 \text{---} \overleftarrow{p} \text{---} &= \frac{-i}{p^2 + m^2 + i\epsilon} \\
 \text{---} \bullet \text{---} &= -i\lambda \\
 \text{---} \otimes \text{---} &= -i(p^2 \delta_Z + \delta_m) \\
 \text{---} \otimes \text{---} &= -i\delta_\lambda
 \end{aligned}$$

Figure 22.13: Feynman rules for renormalized perturbation theory

We can use these new Feynman rules to compute any amplitude in  $\phi^4$  theory. The procedure is as follows. Compute the desired amplitude as the sum of all possible diagrams created from the propagator and vertices shown above. The loop integrals in the diagrams will often diverge, so one must introduce a regulator. The result of this computation will be a function of the three unknown parameters  $\delta_Z$ ,  $\delta_m$ , and  $\delta_\lambda$ . Adjust ( or "renormalise") these three parameters as necessary to maintain the renormalization conditions. After this adjustment, the expression for the amplitude should be finite and independent of the regulator.

This procedure, using Feynman rules with counter-terms, is known as renormalized perturbation theory.

### Mandelstam variable

In theoretical physics, the **Mandelstam variable** are numerical quantities that encode the energy, momentum, and angles of particles in a scattering process in a Lorentz-invariant fashion. They are used for scattering processes of two particles to two particles. The Mandelstam vari-



ables  $s, t, u$  are then defined by

$$\begin{aligned} s &= -(p_1 + p_2)^2 = -(p_3 + p_4)^2 \\ t &= -(p_1 - p_3)^2 = -(p_2 - p_4)^2 \\ u &= -(p_1 - p_4)^2 = -(p_2 - p_3)^2 \end{aligned}$$

Where  $p_1$  and  $p_2$  are the four-momenta of the incoming particles and  $p_3$  and  $p_4$  are the four-momenta of the outgoing particles.  $s$  is also known as the square of the center-of-mass energy (invariant mass) and  $t$  is also known as the square of the four-momentum transfer.

We can verify that

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

### 22.9.3 Techniques for renormalization

#### Feynman's formula

##### Theorem 22.3 Feynman's formula

$$\frac{1}{A_1 \cdots A_n} = \int dF_n (x_1 A_1 + \cdots + x_n A_n)^{-n}$$

where the integration measure over the Feynman parameters  $x_i$  is

$$\int dF_n = (n-1)! \int_0^1 dx_1 \cdots dx_n \delta(x_1 + \cdots + x_n - 1)$$

This measure is normalized so that

$$\int dF_n = 1$$

A generalization of Feynman's formula is

$$\frac{1}{A_1^{\alpha_1} \cdots A_n^{\alpha_n}} = \frac{\Gamma(\sum_i \alpha_i)}{\prod_i \Gamma(\alpha_i)} \frac{1}{(n-1)!} \int dF_n \frac{\prod_i x_i^{\alpha_i-1}}{(\sum_i x_i A_i)^{\sum_i \alpha_i}}$$

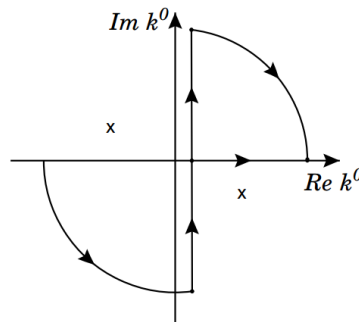


Figure 22.14: Wick rotation



### Wick rotation

For an integral  $\int d^d q f(q^2 - i\epsilon)$ , if the integrand vanishes fast enough as  $|q_0| \rightarrow \infty$ , we can rotate this contour clockwise by  $\frac{\pi}{2}$ , so that it runs from  $-i\infty$  to  $i\infty$ . In making this Wick rotation, the contour does not pass over any poles. (The  $i\epsilon$  are needed to make this statement unambiguous.) Thus the value of the integral is unchanged. It is now convenient to define a Euclidean d-dimensional vector  $\bar{q}$  via  $q^0 = i\bar{q}_d$  and  $q_j = \bar{q}_j$ ; then  $q^2 = \bar{q}^2$ , where

$$\bar{q}^2 = \bar{q}_1^2 + \cdots + \bar{q}_d^2$$

Also,  $d^d q = i d^d \bar{q}$ . Therefore, in general,

$$\int d^d q f(q^2 - i\epsilon) = i \int d^d \bar{q} f(\bar{q}^2)$$

### Dimensional regularization

Dimensional regularization is a method for regularizing integrals in the evaluation of Feynman diagrams. For example, if one wishes to evaluate a loop integral which is logarithmically divergent in four dimensions, like

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 + m^2)^2}$$

One first rewrites the integral in some way so that the number of variables integrated over does not depend on  $d$ , and then we formally vary the parameter  $d$ , to include non-integral values like  $d = 4 - \epsilon$ .

$$\int_0^\infty \frac{dp}{(2\pi)^{4-\epsilon}} \frac{2\pi^{(4-\epsilon)/2}}{\Gamma(\frac{4-\epsilon}{2})} \frac{p^{3-\epsilon}}{(p^2 + m^2)^2} = \frac{2^{\epsilon-4} \pi^{\frac{\epsilon}{2}-1}}{\sin(\frac{\pi\epsilon}{2}) \Gamma(1 - \frac{\epsilon}{2})} m^{-\epsilon} = \frac{1}{8\pi^2 \epsilon} - \frac{1}{16\pi^2} \left( \ln \frac{m^2}{4\pi} + \gamma \right) + \mathcal{O}(\epsilon)$$

There is a useful formula for calculating the integral

$$\int \frac{d^d \bar{q}}{(2\pi)^d} \frac{(\bar{q}^2)^a}{(\bar{q}^2 + D)^b} = \frac{\Gamma(b - a - \frac{1}{2}d) \Gamma(a + \frac{1}{2}d)}{(4\pi)^{d/2} \Gamma(b) \Gamma(\frac{1}{2}d)} D^{-(b-a-d/2)}$$

If  $a = 0$ , then the formula will be

$$\int \frac{d^d \bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2 + D)^b} = \frac{\Gamma(b - \frac{1}{2}d)}{(4\pi)^{d/2} \Gamma(b)} D^{-(b-d/2)}$$

#### 22.9.4 One loop structure of $\phi^4$ theory

First consider the basic two-particle scattering amplitude, If we define  $p = p_1 + p_2$ , then the second diagram of Figure 2.15 is

$$\frac{(-i\lambda)^2}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{-i}{k^2 + m^2} \frac{-i}{(k + m)^2 + m^2} \equiv (-i\lambda)^2 iV(-p^2)$$

So the entire amplitude is therefore

$$i\mathcal{M} = -i\lambda + (-i\lambda)^2 [iV(s) + iV(t) + iV(u)] - i\delta_\lambda + \mathcal{O}(\lambda^3)$$



$$\begin{aligned}
 i\mathcal{M}(p_1 p_2 \rightarrow p_3 p_4) &= \\
 &= \text{[Feynman diagram: a circle with four external lines labeled } p_1, p_2, p_3, p_4 \text{]} \\
 &= \text{[Feynman diagram: a cross]} + \left( \text{[Feynman diagram: a circle with two internal lines]} + \text{[Feynman diagram: a circle with two internal lines]} + \text{[Feynman diagram: a circle with two internal lines]} \right) + \text{[Feynman diagram: a cross]} + \dots
 \end{aligned}$$

**Figure 22.15:** Feynman diagram representation of two-particle scattering to one loop

To keep  $\lambda$  dimensionless in dimensional regularization, we can make the transformation  $\lambda \rightarrow \lambda \tilde{\mu}^\epsilon$ . Here,  $\mu$  is an arbitrary number with mass dimension 1 and  $\epsilon \equiv 4 - d$ .

We can calculate that

$$V(-p^2) = -\frac{1}{32\pi^2} \int_0^1 \left( \frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{D(-p^2)}\right) \right)$$

where  $\mu \equiv \sqrt{4\pi} e^{-\gamma/2} \tilde{\mu}$ ,  $D(-p^2) = x(1-x)p^2 + m^2$

The renormalization condition implies that

$$\delta_\lambda = -\lambda^2 [V(4m^2) + 2V(0)] + \mathcal{O}(\lambda^3)$$

So,

$$i\mathcal{M} = -i\lambda - \frac{i\lambda^2}{32\pi^2} \int_0^1 dx \left[ \ln\left(\frac{D(s)}{D(4m^2)}\right) + \ln\left(\frac{D(t)}{D(0)}\right) + \ln\left(\frac{D(u)}{D(0)}\right) \right] + \mathcal{O}(\lambda^3)$$

To determine  $\delta_Z$  and  $\delta_m$  we must compute the two-point function. Define  $-iM(p^2)$  as the sum of all one-particle-irreducible insertions into the propagator. The full two-point function is given by

$$\frac{-i}{p^2 + m^2 + M^2}$$

The renormalization conditions require that the pole in this full propagator occur at  $p^2 = -m^2$  and have residue 1. These two conditions are equivalent, respectively, to

$$M^2(p^2)|_{p^2=-m^2} = 0 \quad \frac{d}{dp^2} M^2(p^2)|_{p^2=-m^2} = 0$$

We can calculate that

$$-iM^2(p^2) = \frac{i\lambda}{32\pi^2} \left( \frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{m^2}\right) + 1 \right) m^2 - i(p^2 \delta_Z + \delta_m)$$

So, to the order of  $\lambda$ ,

$$\delta_Z = \mathcal{O}(\lambda^2) \quad \delta_m = \frac{\lambda}{32\pi^2} \left( \frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{m^2}\right) + 1 \right) m^2 + \mathcal{O}(\lambda^2) \quad M^2(p^2) = \mathcal{O}(\lambda^2)$$

The detailed calculation can be found in chapter 10.2 of *An introduction to quantum field theory* (M.E.Peskin & D.V.Schroeder) and will be eliminated here.





### Perturbation theory to all orders

We begin by summing all one-particle irreducible diagrams with two external lines; this gives us the self-energy. We next sum all 1PI diagrams with four external lines; this gives us the four-point vertex function  $V_4(k_1, k_2, k_3, k_4)$ . Order by order in  $\lambda$ , we must adjust the value of the lagrangian coefficients  $\delta_Z$ ,  $\delta_m$ , and  $\delta_\lambda$  to maintain the conditions  $M^2(-m^2) = 0$ ,  $\frac{dM^2}{dp^2}(-m^2) = 0$ , and  $V_4(s = 4m^2) = 0$ .

Next we will construct the  $n$ -point vertex functions  $V_n$  with  $4 < n \leq E$ , where  $E$  is the number of external lines in the process of interest. We compute these using a skeleton expansion. This means that we draw all the contributing 1PI diagrams, but omit diagrams that include either propagator or three-point vertex corrections. That is, we include only diagrams that are not only 1PI, but also 2PI and 3PI: they remain connected when any one, two, or three lines are cut. (Cutting three lines may isolate a single treelevel vertex, but nothing more complicated.) Then we take the propagators and vertices in these diagrams to be given by the exact propagator  $\frac{-i}{p^2 + m^2 + M^2(p^2)}$  and vertex  $V_4(k_1, k_2, k_3, k_4)$ , rather than by the tree-level propagator and vertex. We then sum these skeleton diagrams to get  $V_n$  for  $4 < n \leq E$ . Order by order in  $\lambda$ , this procedure is equivalent to computing  $V_n$  by summing the usual set of contributing 1PI diagrams.

Next we draw all tree-level diagrams that contribute to the process of interest (which has  $E$  external lines), including not only four-point vertices, but also  $n$ -point vertices. Then we evaluate these diagrams using the exact propagator for internal lines, and the exact 1PI vertices  $V_n$ ; external lines are assigned a factor of one. We sum these tree diagrams to get the scattering amplitude. Order by order in  $\lambda$ , this procedure is equivalent to computing the scattering amplitude by summing the usual set of contributing diagrams.

Thus we now know how to compute an arbitrary scattering amplitude to arbitrarily high order. The procedure is the same in any quantum field theory; only the form of the propagators and vertices change, depending on the spins of the fields.

### 22.9.5 General renormalization theory

Recall some of the major results and methods of renormalization theory:

1. In perturbation theory, bare and physical quantities are related by ultraviolet-divergent expressions

$$m_{phys} = m_0 + \Delta_m$$

where  $m_{phys}$  is finite,  $\Delta_m$  is ultraviolet-divergent, and so  $m_0$  is necessarily ultraviolet-divergent.

2. We express the Lagrangian in terms of physical quantities, and separate it into

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I + \mathcal{L}_{CT}$$

where  $\mathcal{L}_0$  is the canonically normalized free Lagrangian for physical fields and masses,  $\mathcal{L}_I$  contains the interaction, again in terms of physical parameters, and  $\mathcal{L}_{CT}$  contains the counterterms with ultraviolet divergent coefficients. From  $\mathcal{L}_0$ , we obtain the propagators of the physical fields.  $\mathcal{L}_I$  and  $\mathcal{L}_{CT}$  give interaction vertices.



3. At the one-loop level, the self-energy is given by the effective two-point vertices: the 1PI two-point vertex of the interaction and the counter-term two-point vertex. The counterterms absorb ultraviolet divergences, and the finite parts of the counterterms are determined by renormalization conditions, which ensure the quantities in  $\mathcal{L}_0 + \mathcal{L}_I$  are physical. The conditions constrain the self-energy and the effective vertices, and give a finite, uniquely-determined value for the counterterms.

### Degrees of divergences

Now, for a general theory in  $d$  spacetime dimensions, the field content is given by  $\phi_f$ ,  $f = 1, 2, \dots$ , where  $f$  labels the field type.  $[\phi_f] = \Delta f$  and  $\Delta f$  in all physical theories. We have interaction vertices of type  $i$ ,  $i = 1, 2, \dots$  contributing a term of the form

$$\lambda_i \partial^{n_i} \prod_f \phi_f^{n_{if}}$$

Here,  $\lambda_i$  is the coupling constant, with dimension

$$[\lambda_i] = \kappa_i = d - n_i - \sum_f n_{if} \Delta f$$

Now consider a 1PI diagram in such a theory:

$$E_f \equiv \text{number of external lines of } \phi_f$$

$$V_i \equiv \text{number of vertices of type } i.$$

Then  $M \sim \Lambda^D \prod_i \lambda_i^{V_i}$  and so  $D = [M] - \sum_i V_i \kappa_i$ . Again

$$[M] - d = \sum_f E_f \Delta f$$

and so the general expression for the superficial degree of divergence is given by

$$D = d - \sum_f E_f \Delta f - \sum_i V_i \kappa_i$$

Diagrams which are ultraviolet divergent satisfy

$$\sum_f E_f \Delta f + \sum_i V_i \kappa_i < d$$

We can now divide all theories into

1. All  $\kappa_i > 0$ : superrenormalizable theories.
2. All  $\kappa_i \geq 0$ : renormalizable theories.
3. There exists at least one  $\delta_i < 0$ : non-renormalizable theories.

These terms also apply to individual interactions for a vertex of type  $i$ :

1.  $\kappa_i > 0$ : super-renormalizable, relevant interaction.
2.  $\kappa_i \geq 0$ : renormalizable, marginal interaction.
3.  $\kappa_i < 0$ : non-renormalizable, irrelevant interaction.



### Cancellation of divergences

Consider a generic divergent diagram  $M$  of degree  $D$ ; that is,

$$M = \int^\Lambda ds s^{D-1}$$

if all loop momenta are taken proportional to  $s$ . Generically, internal propagators have the form

$$\frac{1}{(as + p)^\alpha \dots} \sim \frac{1}{s^\alpha}$$

for large  $s$ , where  $a$  is a numerical constant and  $p$  is a combination of the external momenta. Differentiating  $M$ , with respect to  $p$ ,  $n$  times gives a term proportional to

$$\frac{1}{(as + p)^{\alpha+n}} \sim \frac{1}{p^{\alpha+n}}$$

and so  $D + 1$  derivatives with respect to the external momenta will make  $M$  finite. This means that

$$M(p) = M_0 + M_1 p + \dots + M_D p^D + \text{finite terms}$$

where the argument  $p$  of the function represents the collection of external momenta, we have suppressed the index structure, and  $M_0, M_1, \dots, M_D$  are potentially divergent constants. Suppose that  $M$  has  $E_f$  external lines of the field  $\phi_f$ . Then divergences of  $M(p)$  can be cancelled by counterterms of the form

$$\sum_{j=0}^D A_j (\partial)^j \prod_f \phi_f^{E_f}$$

where the  $A_j$  are divergent coefficients in order to cancel divergences in  $M_j$ . The index structure in  $A_j \partial^j$  should match the suppressed index structure of  $M(p)$ .

## 22.10 Renormalization group

### 22.10.1 Modified minimal-subtraction scheme

The Lagrangian of  $\phi^4$  theory is

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 - \frac{1}{2} \delta_Z \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} \delta_m \phi^2 - \frac{\delta \lambda}{4!} \phi^4$$

For minimal-subtraction scheme, we do not demand that  $m$  be the physics mass of the field and  $\phi$  create a normalized one-particle state. The physical meaning of  $\lambda$  is not expressed directly as well. Instead we choose  $\delta_Z$ ,  $\delta_m$  and  $\delta_\lambda$  to cancel the infinities, and nothing more; we say that  $\delta_Z$ ,  $\delta_m$  and  $\delta_\lambda$  have no finite parts. It is called the modified minimal-subtraction or  $\overline{\text{MS}}$  scheme. ("modified" because we introduced  $\mu$  via  $\lambda \rightarrow \lambda \tilde{\mu}^\epsilon$ , with  $\mu \equiv \sqrt{4\pi} e^{-\gamma/2} \tilde{\mu}$ ; had we set  $\mu = \tilde{\mu}$  instead, the scheme would be just plain minimal subtraction or MS.)

For loop corrections to propagator,

$$\delta_Z = \mathcal{O}(\lambda^2) \quad \delta_m = \left[ \frac{\lambda}{16\pi^2} + \mathcal{O}(\lambda^2) \right] \frac{1}{\epsilon} m^2 \quad M^2(p^2) = \frac{\lambda}{32\pi^2} (\ln(\frac{m^2}{\mu^2}) - 1) m^2 + \mathcal{O}(\lambda^2)$$



Firstly, in the  $\overline{\text{MS}}$  scheme, the propagator will no longer have a pole at  $k^2 = -m^2$ . The pole will be somewhere else. However, by definition, the actual physical mass  $m_{ph}$  of the particle is determined by the location of this pole:  $k^2 = -m_{ph}^2$ . Thus, the Lagrangian parameter  $m$  is no longer the same as  $m_{ph}$ . The relation of  $m$  and  $m_{ph}$  is

$$m_{ph}^2 = M^2(-m_{ph}^2) + m^2$$

To the lowest order,

$$m_{ph}^2 = \left[ 1 + \frac{\lambda}{32\pi^2} (\ln(\frac{m^2}{\mu^2}) - 1) \right] m^2$$

Because  $m_{ph}$  is independent of  $\mu$ , according to  $\frac{d}{d\mu} m_{ph} = 0$ , it can be derived that

$$\frac{dm}{d \ln \mu} = \left[ \frac{\lambda}{32\pi^2} + \mathcal{O}(\lambda^2) \right] m$$

Furthermore, the residue of this pole is no longer one. Let us call the residue  $R$ . So, in the LSZ formula, we get a net factor of  $\sqrt{R}$  for each external line when using the  $\overline{\text{MS}}$  scheme. And in  $\phi^4$  theory,

$$R = 1 + \mathcal{O}(\lambda^2)$$

For loop corrections to vertex,

$$\delta_\lambda = \left[ \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3) \right] \frac{1}{\epsilon}$$

$$i\mathcal{M} = -i\lambda - \frac{i\lambda^2}{32\pi^2} \int_0^1 dx \left[ \ln\left(\frac{D(s)}{\mu^2}\right) + \ln\left(\frac{D(t)}{\mu^2}\right) + \ln\left(\frac{D(u)}{\mu^2}\right) \right] + \mathcal{O}(\lambda^3)$$

For a process with  $p^2 \gg m^2$ , we have

$$D \approx x(1-x)p^2$$

In OS renormalization scheme, the one-loop correction to propagator or vertex generally includes a factor

$$\ln\left(\frac{D}{D_0}\right) \sim \ln\frac{p^2}{m^2}$$

so perturbation theory is no longer a good approximation when  $p^2 \gg m^2$ . In  $\overline{\text{MS}}$  renormalization scheme, introducing  $\mu$  allows us to address this problem: if we choose  $\mu \sim p$ , no such logarithm arises. If we choose  $\mu$  appropriately, that is, to be comparable to the momentum scale of the physical process, we can improve our perturbation expansion. So  $\lambda(\mu)$  and  $m(\mu)$  can be considered as the scale-dependent coupling constants. And the reason we get large logarithmic terms in the on-shell scheme is that we are trying to use coupling defined at one scale to describe physics at very different scales.

### 22.10.2 Beta and gamma function

The Lagrangian of  $\phi^4$  theory is

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \phi_0 \partial_\mu \phi_0 - \frac{1}{2} m_0^2 \phi_0^2 - \frac{\lambda_0}{4!} \phi_0^4$$



It can be written as

$$\mathcal{L} = -\frac{1}{2}Z_\phi\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}Z_m m^2\phi^2 - Z_\lambda\mu^\epsilon\frac{\lambda}{4!}\phi^4$$

So,

$$\phi_0 = Z_\phi^{1/2}\phi \quad m_0 = Z_\phi^{-1/2}Z_m^{1/2}m \quad \lambda = Z_\phi^{-2}Z_\lambda\lambda\tilde{\mu}^\epsilon$$

After using dimensional regularization, the infinities coming from loop integrals take the form of inverse powers of  $\epsilon$ . In the  $\overline{\text{MS}}$  renormalization scheme, we choose the  $Z$ s to cancel off these powers of  $1/\epsilon$ , and nothing more. Therefore the  $Z$ s can be written as

$$\begin{aligned} Z_\phi &= 1 + \sum_{n=1}^{\infty} \frac{a_n(\lambda)}{\epsilon^n} \\ Z_m &= 1 + \sum_{n=1}^{\infty} \frac{b_n(\lambda)}{\epsilon^n} \\ Z_\lambda &= 1 + \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{\epsilon^n} \end{aligned}$$

In  $\phi^4$  theory,  $a_1 = \mathcal{O}(\lambda^2)$ ,  $b_1 = \frac{\lambda}{16\pi^2} + \mathcal{O}(\lambda^2)$ ,  $c_1 = \frac{3\lambda}{16\pi^2} + \mathcal{O}(\lambda^2)$

Remember that bare fields and parameters must be independent of  $\mu$ . Define

$$G(\lambda, \epsilon) \equiv \ln(Z_\phi^{-2}Z_\lambda) = \sum_{n=1}^{\infty} \frac{G_n(\lambda)}{\epsilon^n}$$

We can calculate  $G_1 = c_1 - 2a_1 = \frac{3\lambda}{16\pi^2} + \mathcal{O}(\lambda^2)$ . As  $\ln \lambda_0 = G + \ln \lambda + \epsilon \ln \tilde{\mu}$ . From the independence of  $\lambda_0$ , we can derive

$$\left(1 + \frac{\lambda G'_1}{\epsilon} + \dots\right) \frac{d\lambda}{d \ln \mu} + \epsilon \lambda = 0$$

$\frac{d\lambda}{d \ln \mu}$  is the rate at which  $\lambda$  must change to compensate for a small change in  $\ln \mu$ . If compensation is possible at all, this rate should be finite in the  $\epsilon \rightarrow 0$  limit. In a renormalizable theory, we should have

$$\frac{d\lambda}{d \ln \mu} = -\epsilon \lambda + \beta(\lambda)$$

So

$$\beta(\lambda) = \lambda^2 G'_1(\lambda)$$

In  $\phi^4$  theory, we have

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3)$$

Define

$$M(\lambda, \epsilon) \equiv \ln(Z_m^{1/2}Z_\phi^{-1/2}) = \sum_{n=1}^{\infty} \frac{M_n(\lambda)}{\epsilon^n}$$

We can calculate  $M_1 = \frac{1}{2}b_1 - \frac{1}{2}a_1 = \frac{\lambda}{32\pi^2} + \mathcal{O}(\lambda^2)$ . As  $\ln m_0 = M + \ln m$ , define the anomalous dimension of the mass

$$\gamma_m(\lambda) \equiv \frac{1}{m} \frac{dm}{d \ln \mu}$$



From the independence of  $m_0$ , we can derive

$$\gamma_m(\lambda) = \lambda M'_1$$

In  $\phi^4$  theory, we have

$$\gamma_m(\lambda) = \frac{\lambda}{32\pi^2} + \mathcal{O}(\lambda^2)$$

We can expand  $\ln Z_\phi$  as

$$\ln Z_\phi = \frac{a_1}{\epsilon} + \dots$$

Define the anomalous dimension of the field

$$\gamma_\phi(\lambda) \equiv \frac{1}{2} \frac{d \ln Z_\phi}{d \ln \mu}$$

We can derive

$$\gamma_\phi(\lambda) = -\frac{1}{2} \lambda a'_1$$

In  $\phi^4$  theory, we have

$$\gamma_\phi(\lambda) = \mathcal{O}(\lambda^2)$$

### 22.10.3 Callen-Symanzik equation

Consider the  $n$  point green function,

$$G^{(n)}(x_1, \dots, x_n) \equiv \langle \Omega | T \phi(x_1) \dots \phi(x_n) | \Omega \rangle_C$$

As  $G_0^{(n)} = Z_\phi^{n/2} G^{(n)}$ , from the independence of bare Green's function, we have

$$\left( \frac{\partial}{\partial \ln \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + \gamma_m(\lambda) m \frac{\partial}{\partial m} + n \gamma_\phi(\lambda) \right) G^n(x_1, \dots, x_n; \lambda, m, \mu) = 0$$

From now on, we will focus on the  $\phi^4$  theory in massless limit. Firstly, consider the two point Green's function in momentum space  $G^{(2)}(p)$ , we can express its dependence on  $p$  and  $\mu$  as

$$G^{(2)} = \frac{-i}{p^2} g(-p^2/\mu^2, \lambda(\mu))$$

Then the C-S equation can be written as

$$\left[ p \frac{\partial}{\partial p} - \beta(\lambda) \frac{\partial}{\partial \lambda} + 2 - 2\gamma_\phi \right] G^{(2)}(p, \lambda(\mu), \mu) = 0$$

Here,  $p \equiv (-p^2)^{1/2}$  In the free theory,  $\beta$  and  $\gamma$  vanish and we recover the trivial result

$$G^{(2)}(p) = \frac{-i}{p^2}$$

In an interacting theory, the solution to the C-S equation can be expressed as

$$G^{(2)}(p, \lambda_0, \mu_0) = G^{(2)}(\mu_0, \lambda_p, \mu_0) \exp \left( - \int_{p'=\mu_0}^{p'=p} d \ln(p'/\mu_0) \cdot 2[1 - \gamma_\phi(\lambda_{p'})] \right)$$



Here,  $\lambda_0 = \lambda(\mu_0)$ ,  $\lambda_p$  satisfy the following equation

$$\frac{\partial}{\partial \ln p} \lambda_p(p, \lambda_0) = \beta(\lambda_p) \quad \lambda_p(\mu_0, \lambda_0) = \lambda_0$$

The solution can be checked directly by noticing that

$$\left( p \frac{\partial}{\partial p} - \beta(\lambda_0) \frac{\partial}{\partial \lambda_0} \right) \lambda_p(p, \lambda_0) = 0$$

A convenient way of writing the solution is

$$G^{(2)}(p, \lambda_0, \mu_0) = \frac{-i}{p^2} g^{(2)}(\mu_0, \lambda_p, \mu_0) \exp \left( 2 \int_{\mu_0}^p d \ln(p'/\mu_0) \gamma_\phi(\lambda_{p'}) \right)$$

And  $g(\lambda_p) = 1 + O(\lambda_p^2)$

Now consider the connected four-point function of  $\phi^4$  theory evaluated at spacelike momenta  $p_i$  such that  $p_i^2 = P^2$ ,  $p_i \cdot p_j = 0$ , so that  $s, t$ , and  $u$  are of order  $-P^2$ . To leading order in perturbation theory, this function is given by

$$G^{(4)}(P) = \left( \frac{i}{P^2} \right)^4 (-i\lambda)$$

The solution of C-S equation is

$$G^{(2)}(p, \lambda_0, \mu_0) = \frac{1}{P^8} g^{(4)}(\mu_0, \lambda_p, \mu_0) \exp \left( 4 \int_{\mu_0}^p d \ln(p'/\mu_0) \gamma_\phi(\lambda_{p'}) \right)$$

And  $g(\lambda_p) = -i\lambda_p + O(\lambda_p^2)$ .

The ordinary Feynman perturbation series for a Green's function depends both on the coupling constant  $\lambda$  and on the dimensionless parameter  $\ln(p^2/\mu_0^2)$ . The perturbation theory can be badly behaved even when  $\lambda$  is small if the ratio  $p^2/\mu_0^2$  is large. The solutions of C-S equation reorganize this dependence into a function of the running coupling constant and an exponential scale factor.

The first factor of the solution is a function of the running coupling constant, evaluated at the momentum scale  $p$ . If  $p$  were of order  $\mu_0$ , this function would essentially be the ordinary perturbation evaluation of the Green's function. We can make use of this same expression at the scale  $p$ , but to replace  $\lambda_0$  with a new coupling constant  $\lambda_p$  appropriate to that scale. The exponential factor is the accumulated field strength resealing of the correlation function from the reference point  $\mu_0$  to the actual momentum  $p$  at which the Green's function is evaluated. This factor receives a multiplicative contribution from each intermediate scale between  $\mu$  and  $p$ .

In  $\phi^4$  theory, we have

$$\frac{d}{d \ln p} \lambda_p = \frac{3\lambda_p^2}{16\pi^2}$$

The solution is

$$\lambda_p = \frac{\lambda_0}{1 - \frac{3\lambda_0}{16\pi^2} \ln(p/\mu_0)}$$



This expression for the running coupling constant goes to zero at a logarithmic rate as  $p \rightarrow 0$ . If we expand the running coupling constant  $\lambda_p$  in powers of  $\lambda_0$ , we find that the successive powers of the coupling constant are multiplied by powers of logarithms,

$$\lambda_0^{n+1} (\ln p/\mu_0)^n$$

which become large and invalidate a simple perturbation expansion for  $p$  much greater or much less than  $\mu_0$ . If the running coupling constant becomes large, as happens in  $\phi^4$  theory for  $p \rightarrow \infty$ , the perturbation expansion will break down anyway, and we will need more advanced methods. However, if the running coupling constant becomes small, as for  $\phi^4$  theory as  $p \rightarrow 0$ , we will have successfully organized the powers of logarithms into a meaningful and controlled expression.

### 22.10.4 Running of coupling constants

In the limit of  $\epsilon \rightarrow 0$ , the coupling constant satisfies the differential equation

$$\frac{\partial \lambda}{\partial \ln \mu} = \beta(\lambda)$$

Three behaviours are possible in the region of small  $\lambda$ :

1.  $\beta(\lambda) > 0$
2.  $\beta(\lambda) = 0$
3.  $\beta(\lambda) < 0$

In theories of the first class, the running coupling constant goes to zero in the infra-red, leading to definite predictions about the small-momentum behaviour of the theory. However, the running coupling constant becomes large in the region of high momenta. Thus the short-distance behaviour of the theory cannot be computed using Feynman diagram perturbation theory. A Feynman diagram analysis is useful in such theories if one is mainly interested in large-distance or macroscopic behaviour.

In theories of the second class, the coupling constant does not flow. In these theories, the running coupling constant is independent of the momentum scale, and thus equal to the bare coupling. This means that there can be no ultraviolet divergences in the relation of coupling constants. The only possible ultraviolet divergences in such theories are those associated with field resealing, which automatically cancel in the computation of S-matrix elements. Such theories are called finite quantum field theories. Before the emergence of our modern understanding of renormalization, these theories would have been embraced as the solution to the problem of ultraviolet infinities. But in fact the known finite field theories in four dimensions are very special constructions the so-called gauge theories with extended supersymmetry with no known physical application.

In theories of the third class, the running coupling constant becomes large in the large-distance regime and becomes small at large momenta or short distances. Such theories are called asymptotically free. In theories of this class, the short-distance behaviour is completely





solvable by Feynman diagram methods. Though ultraviolet divergences appear in every order of perturbation theory, the renormalization group tells us that the sum of these divergences is completely harmless.

In the region of strong coupling, the approximation we have made, ignoring the higher-order terms in the  $\beta$  function is no longer valid. It is a logical possibility that the leading order term is positive while the higher terms of the  $\beta$  function are negative, so that the  $\beta$  function has a zero at a non-zero value  $\lambda_*$ . When  $\lambda_*$  approaches this value, the renormalization group flow slows to a halt; thus  $\lambda = \lambda_*$  would be a non-trivial fixed point of the renormalization group.

If the  $\beta$  function behaves in the vicinity of the fixed point as

$$\beta \approx -B(\lambda - \lambda_*)$$

where  $B$  is a positive constant. For  $\lambda$  near  $\lambda_*$

$$\frac{d}{d \ln \mu} \lambda \approx -B(\lambda - \lambda_*)$$

The solution of this equation is

$$\lambda(\mu) = \lambda_* + C \left( \frac{\mu_0}{\mu} \right)^B$$

Thus,  $\lambda$  indeed tends to  $\lambda_*$  as  $\mu \rightarrow \infty$ , and the rate of approach is governed by the slope of the  $\beta$  function at the fixed point. The fixed point here is called ultraviolet-stable fixed point.

If  $p$  is sufficiently large,  $\lambda(p)$  is close to  $\lambda_*$ . In the massless limit, the solution of C-S equation for two point Green function in momentum space becomes

$$G^{(2)}(p, \lambda_0, \mu_0) \approx G^{(2)}(\mu_0, \lambda_*, \mu_0) \exp(-\ln(p/\mu_0) \cdot 2[1 - \gamma_\phi(\lambda_*)])$$

Thus the naive scaling law  $G(p) \sim k^{-2}$  is changed to  $G(p) \sim k^{-2[1-\gamma_\phi(\lambda_*)]}$ . This has applications in the theory of critical phenomena.

A similar behaviour is possible in an asymptotically free theory. If the  $\beta$  function behaves in the vicinity of the fixed point as

$$\beta \approx B(\lambda - \lambda_*)$$

where  $B$  is a positive constant. the running coupling constant will tend to a fixed point as  $\mu \rightarrow 0$ . The fixed point is called infrared-stable fixed points.

## 22.11 Spontaneous symmetry breaking

### 22.11.1 Effective action

$$Z[J] = e^{-iE[J]} = \int \mathcal{D}\phi \exp \left[ i \int d^4x (\mathcal{L}[\phi] + J\phi) \right]$$

Define

$$\phi_{\text{cl}}(x) \equiv \langle \Omega | \phi(x) | \Omega \rangle_J$$



So, we can derive

$$\frac{\delta}{\delta J(x)} E[J] = -\phi_{\text{cl}}(x)$$

Define

$$\Gamma[\phi_{\text{cl}}] \equiv -E[J] - \int d^4y J(y) \phi_{\text{cl}}(y)$$

We can verify that

$$\frac{\delta}{\delta \phi_{\text{cl}}(x)} \Gamma[\phi_{\text{cl}}] = -J(x)$$

If the external source is set to zero, the effective action satisfy the equation

$$\frac{\delta}{\delta \phi_{\text{cl}}(x)} \Gamma[\phi_{\text{cl}}] = 0$$

The solution to this equation are the values of  $\langle \phi(x) \rangle$  in the stable quantum states of the theory. For a translational-invariant vacuum state, we will find a solution in which  $\phi_{\text{cl}}$  is independent of  $x$ . For the field theory that the possible vacuum states are invariant under translations and Lorentz transformations, for each possible vacuum states, the corresponding solution  $\phi_{\text{cl}}$  will be a constant, independent of  $x$ . If  $T$  is the time extent of the region and  $V$  is its three dimensional volume, we can write

$$\Gamma[\phi_{\text{cl}}] = -(VT) \cdot V_{\text{eff}}(\phi_{\text{cl}})$$

The coefficient  $V_{\text{eff}}$  is called effective potential. The condition that  $\Gamma[\phi_{\text{cl}}]$  has an extreme then reduces to the simple equation

$$\frac{\partial}{\partial \phi_{\text{cl}}} V_{\text{eff}}(\phi_{\text{cl}}) = 0$$

A system with spontaneously broken symmetry will have several minimum of  $V_{\text{eff}}$ , all with the same energy by virtue of the symmetry. The choice of one among these vacuum is the spontaneous symmetry breaking.

### 22.11.2 Computation of the effective action

Decompose the Lagrangian into a piece depending on renormalized parameters and one containing the counter-terms

$$\mathcal{L} = \mathcal{L}_1 + \delta \mathcal{L}$$

Define  $J_1$  by

$$\frac{\delta \mathcal{L}_1}{\delta \phi} |_{\phi=\phi_{\text{cl}}} + J_1(x) = 0$$

Define  $\delta J$  by

$$J(x) = J_1(x) + \delta J(x)$$

So, we have

$$e^{-iE[J]} = \int \mathcal{D}\phi e^{i \int d^4x (\mathcal{L}_1 + J_1 \phi)} e^{i \int d^4x (\delta \mathcal{L} + \delta J \phi)}$$



Replace  $\phi$  by  $\phi_{\text{cl}} + \eta$ ,

$$\begin{aligned} \int d^4x (\mathcal{L}_1 + J_1\phi) &= \int d^4x (\mathcal{L}_1[\phi_{\text{cl}}] + J_1\phi_{\text{cl}}) + \int d^4x \eta(x) \left( \frac{\delta \mathcal{L}_1}{\delta \phi} + J_1 \right) \\ &+ \frac{1}{2} \int d^4x d^4y \eta(x) \eta(y) \frac{\delta^2 \mathcal{L}_1}{\delta \phi(x) \delta \phi(y)} \\ &+ \frac{1}{3!} \int d^4x d^4y d^4z \eta(x) \eta(y) \eta(z) \frac{\delta^3 \mathcal{L}_1}{\delta \phi(x) \delta \phi(y) \delta \phi(z)} + \dots \end{aligned}$$

The term linear in  $\eta$  vanishes by definition of  $J_1$ . Keeping only the term up to quadratic order in  $\eta$  and still neglecting the counter-terms, we have a pure Gaussian integral, which can be evaluated in terms of a functional determinant:

$$\begin{aligned} &\int \mathcal{D}\eta \exp \left[ i \left( \int (\mathcal{L}_1[\phi_{\text{cl}}] + J_1\phi_{\text{cl}}) + \frac{1}{2} \int \eta \frac{\delta^2 \mathcal{L}_1}{\delta \phi \delta \phi} \eta \right) \right] \\ &= \exp \left[ i \int (\mathcal{L}_1[\phi_{\text{cl}}] + J_1\phi_{\text{cl}}) \right] \left( \det \left[ \frac{\delta^2 \mathcal{L}_1}{\delta \phi \delta \phi} \right] \right)^{-\frac{1}{2}} \end{aligned}$$

Finally, put back the effects of the counter-term Lagrangian, writing it as

$$(\delta \mathcal{L}[\phi_{\text{cl}}] + \delta J \phi_{\text{cl}}) + (\delta \mathcal{L}[\phi_{\text{cl}} + \eta] - \delta \mathcal{L}[\phi_{\text{cl}}] + \delta J \eta)$$

Define

$$\mathcal{L}_2 = \left( \frac{1}{3!} \int d^4x d^4y d^4z \eta(x) \eta(y) \eta(z) \frac{\delta^3 \mathcal{L}_1}{\delta \phi(x) \delta \phi(y) \delta \phi(z)} + \dots \right) + (\delta \mathcal{L}[\phi_{\text{cl}} + \eta] - \delta \mathcal{L}[\phi_{\text{cl}}] + \delta J \eta)$$

So

$$e^{-iE[J]} = \exp \left[ i \int (\mathcal{L}_1[\phi_{\text{cl}}] + J_1\phi_{\text{cl}} + \delta \mathcal{L}[\phi_{\text{cl}}] + \delta J \phi_{\text{cl}}) \right] e^{i \int \mathcal{L}_2(\frac{1}{i} \frac{\delta}{\delta I})} \int \mathcal{D}\eta e^{i \int \left( \frac{1}{2} \eta \frac{\delta^2 \mathcal{L}_1}{\delta \phi \delta \phi} \eta + I \eta \right)}$$

Therefore, define propagator as

$$D_F \equiv i \left( \frac{\delta^2 \mathcal{L}_1}{\delta \phi \delta \phi} \right)^{-1}$$

We have

$$e^{-iE[J]} = \exp \left[ i \int (\mathcal{L}_1[\phi_{\text{cl}}] + J_1\phi_{\text{cl}} + \delta \mathcal{L}[\phi_{\text{cl}}] + \delta J \phi_{\text{cl}}) \right] \left( \det \left[ \frac{\delta^2 \mathcal{L}_1}{\delta \phi \delta \phi} \right] \right)^{-\frac{1}{2}} e^{i \int \mathcal{L}_2(\frac{1}{i} \frac{\delta}{\delta I})} \int \mathcal{D}\eta e^{i \int (-\frac{1}{2} I D_F I)} \Big|_{I=0}$$

Similar to the procedure in the perturbation theory for path integral quantization, we can get a perturbation expansion for  $iE[J]$  using connected Feynman diagram,

$$-iE[J] = i \int (\mathcal{L}_1[\phi_{\text{cl}}] + J_1\phi_{\text{cl}} + \delta \mathcal{L}[\phi_{\text{cl}}] + \delta J \phi_{\text{cl}}) - \frac{1}{2} \log \det \left[ \frac{\delta^2 \mathcal{L}_1}{\delta \phi \delta \phi} \right] + \text{connected diagrams}$$

From this equation,  $\Gamma$  follows directly:

$$\Gamma[\phi_{\text{cl}}] = \int d^4x \mathcal{L}_1[\phi_{\text{cl}}] + \frac{i}{2} \log \det \left[ \frac{\delta^2 \mathcal{L}_1}{\delta \phi \delta \phi} \right] - i \text{connected diagrams} + \int d^4x \delta \mathcal{L}[\phi_{\text{cl}}]$$



Notice that there are no terms remaining that depend explicitly on  $J$ ; thus,  $\Gamma$  is expressed as a function of  $\phi_{\text{cl}}$ , as it should be. The Feynman diagrams contributing to  $\Gamma[\phi_{\text{cl}}]$  have no external lines, and the simplest ones turn out to have two loops. The lowest-order quantum correction to  $\Gamma$  is given by the functional determinant.

The last term provides a set of counter-terms that can be used to satisfy the renormalization conditions on  $\Gamma$  and, in the process, to cancel divergences that appear in the evaluation of the functional determinant and the diagrams. The renormalization conditions will determine all of the counter-terms in  $\delta\mathcal{L}$ . However, the formalism we have constructed contains a new counter-term  $\delta J$ . That coefficient is determined by  $\langle\eta\rangle = 0$ . In practice, we will satisfy this condition by simply ignoring any one-particle-irreducible one-point diagram, since any such diagram will be cancelled by adjustment of  $\delta J$ .

### 22.11.3 The effective action as a generating functional

$E[J]$  is called the generating of connected correlation functions,

$$\frac{\delta^n E[J]}{\delta J(x_1) \cdots \delta J(x_n)} = i^{n+1} \langle \phi(x_1) \cdots \phi(x_n) \rangle_{\text{conn}}$$

The effective action  $\Gamma[\phi_{\text{cl}}]$  is the generating functional of one-particle-irreducible correlation functional,

$$\frac{\delta \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(x)} = 0$$

$$\frac{\delta^2 \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(x) \delta \phi_{\text{cl}}(y)} = iD^{-1}(x, y)$$

Here,  $D(x, y) = \langle \phi(x) \phi(y) \rangle_{\text{conn}}$ . When  $n \geq 3$ ,

$$\frac{\delta^n \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(x_1) \cdots \delta \phi_{\text{cl}}(x_n)} = -i \langle \phi(x_1) \cdots \phi(x_n) \rangle_{\text{1PI}}$$

The proof of statements above can be found in chapter 10.2 of *An introduction to quantum field theory* (M.E.Peskin & D.V.Schroeder)

The chapter 21 of *Quantum field theory* (M. Srednicki) gives an constructive way to define the effective action.

$$\Gamma[\phi] \equiv \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{\phi}(-k) (-k^2 - m^2 - M^2(k^2)) \tilde{\phi}(k) + \frac{1}{n!} \int \frac{d^d k_1}{(2\pi)^d} \cdots \frac{d^d k_n}{(2\pi)^d} (2\pi)^d \delta(k_1 + \cdots + k_n) V_n(k_1, \cdots, k_n)$$

Here  $\tilde{\phi}(k) = \int d^d x e^{-ikx} \phi(x)$ , and  $iV_n(k_1, \cdots, k_n)$  equals the value of 1PI Feynman diagram in momentum space. The effective action has the property that the tree-level Feynman diagrams it generates give the complete scattering amplitude of the original theory. The author also proved that this definition is equivalent to the definition from *An introduction to quantum field theory* (M.E.Peskin & D.V.Schroeder).



### 22.11.4 Renormalization and symmetry

Consider first the computation of the effective potential for constant classical fields, in a field theory with an arbitrary number of fields  $\phi^i$ . The effective potential has mass dimension 4, so we expect that  $V_{\text{eff}}(\phi_{\text{cl}})$  will have divergent terms up to  $\Lambda^4$ . To understand these divergences, expand  $V_{\text{eff}}(\phi_{\text{cl}})$  in a Taylor series:

$$V_{\text{eff}}(\phi_{\text{cl}}) = A_0 + A_2^{ij} \phi_{\text{cl}}^i \phi_{\text{cl}}^j + A_4^{ijkl} \phi_{\text{cl}}^i \phi_{\text{cl}}^j \phi_{\text{cl}}^k \phi_{\text{cl}}^l + \dots$$

In theories without a symmetry of  $\phi \rightarrow \phi$ , there might also be terms linear and cubic in  $\phi^i$ ; we omit these for simplicity. The coefficients  $A_0$ ,  $A_2$ ,  $A_4$  have mass dimension, respectively, 4, 2, and 0; thus we expect them to contain  $\Lambda^4$ ,  $\Lambda^2$ , and  $\log \Lambda$  divergences, respectively. The power-counting analysis predicts that all higher terms in the Taylor series expansion should be finite.

The constant term  $A_0$  is independent of  $\phi_{\text{cl}}$ ; it has no physical significance. However, the divergences in  $A_2$  and  $A_4$  appear in physical quantities, since these coefficients enter the inverse propagator and the irreducible four-point function and therefore appear in the computation of S-matrix elements. There is one further coefficient in the effective action that has non-negative mass dimension by power counting; this is the coefficient of the term quadratic in  $\partial_\mu \phi_{\text{cl}}$ , which appears when the effective action is evaluated for a non-constant background field:

$$\Delta\Gamma[\phi_{\text{cl}}] = \int d^4x B_2^{ij} \partial_\mu \phi_{\text{cl}}^i \partial^\mu \phi_{\text{cl}}^j$$

All other coefficients in the Taylor expansion of the effective action in powers of  $\phi_{\text{cl}}$  are finite by power counting.

We can now argue that the counter-terms of the original Lagrangian suffice to remove the divergences that might appear in the computation of  $\Gamma[\phi_{\text{cl}}]$ . The argument proceeds in two steps. We first use the BPHZ theorem to argue that the divergences of Green's functions can be removed by adjusting a set of counter-terms corresponding to the possible operators that can be added to the Lagrangian with coefficients of mass dimension greater than or equal to zero. The coefficients of these counter-terms are in 1-to-1 correspondence with the coefficients  $A_2$ ,  $A_4$ , and  $B_2$  of the effective action. Next, we use the fact that the effective action is manifestly invariant to the original Symmetry group of the model. This is true even if the vacuum state of the model has spontaneous symmetry breaking, since the method we presented for computing the effective action is manifestly invariant to the original symmetry of the Lagrangian. Combining these two results, we conclude that the effective action can always be made finite by adjusting the set of counter-terms that are invariant to the original symmetry of the theory, even if this symmetry is spontaneously broken.



### 22.11.5 Goldstone's theorem

#### Theorem 22.4 Goldstone's theorem

Goldstone's theorem examines a generic continuous symmetry which is spontaneously broken; i.e., its currents are conserved, but the ground state is not invariant under the action of the corresponding charges. Then, necessarily, new massless (or light, if the symmetry is not exact) scalar particles appear in the spectrum of possible excitations. There is one scalar particle - called a Nambu-Goldstone boson — for each generator of the symmetry that is broken, i.e., that does not preserve the ground state.

A general continuous symmetry transformation has the form

$$\phi^a \rightarrow \phi^a + \alpha \Delta^a(\phi)$$

where  $\alpha$  is an infinitesimal parameter and  $\Delta^a$  is some function of all the  $\phi$ 's. Specialize to constant fields; then the derivative terms in  $\mathcal{L}$  vanish and the potential alone must be invariant. This condition can be written

$$V(\phi^a) = V(\phi^a + \alpha \Delta^a(\phi)) \quad \text{or} \quad \Delta^a(\phi) \frac{\partial}{\partial \phi^a} V(\phi) = 0$$

The effective potential  $V_{\text{eff}}$  encapsulates the full solution to the theory, including all orders of quantum corrections. At the same time, it satisfies the general properties of the classical potential: It is invariant to the symmetries of the theory, and its minimum gives the vacuum expectation value of  $\phi_{\text{cl}}$ . So

$$\Delta^a(\phi) \frac{\partial}{\partial \phi^a} V_{\text{eff}}(\phi) = 0$$

Now differentiate with respect to  $\phi^b$ , and set  $\phi = \phi_{\text{cl}}$

$$0 = \left( \frac{\partial \Delta^a}{\partial \phi^b} \right)_{\phi_{\text{cl}}} \left( \frac{\partial V_{\text{eff}}}{\partial \phi^a} \right)_{\phi_{\text{cl}}} + \Delta^a(\phi_{\text{cl}}) \left( \frac{\partial^2}{\partial \phi^a \partial \phi^b} V_{\text{eff}} \right)_{\phi_{\text{cl}}}$$

The first term vanishes since  $\phi_{\text{cl}}$  is a minimum of  $V_{\text{eff}}$ , so the second term must also vanish. If the transformation leaves  $\phi_{\text{cl}}$  unchanged (i.e., if the symmetry is respected by the ground state), then  $\Delta^a(\phi_{\text{cl}}) = 0$  and this relation is trivial. A spontaneously broken symmetry is precisely one for which  $\Delta^a(\phi_{\text{cl}}) \neq 0$ ; in this case  $\Delta^a(\phi_{\text{cl}})$  is the vector with eigenvalue zero.

We now argue that the presence of such a zero eigenvalue implies the existence of a massless scalar particle. Effective action's second functional derivative is the inverse propagator

$$i\tilde{D}_{ij}^{-1}(p^2) = \int d^4x e^{-ip(x-y)} \frac{\delta \Gamma}{\delta \phi^i \delta \phi^j}(x, y)$$

A particle of mass  $m$  corresponds to a zero eigenvalue of this matrix equation at  $p^2 = -m^2$ . Now set  $p = 0$ . This implies that we differentiate  $\Gamma[\phi_{\text{cl}}]$  with respect to constant fields. Thus, we



can replace  $\Gamma[\phi_{\text{cl}}]$  by its value with constant classical fields, which is just the effective potential. We find that the quantum field theory contains a scalar particle of zero mass when the matrix of second derivatives,

$$\frac{\partial^2}{\partial \phi_{\text{cl}}^i \partial \phi_{\text{cl}}^j} V_{\text{eff}}$$

has a zero eigenvalue. This completes the proof of Goldstone's theorem.

## 22.12 Linear sigma model

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{2} \mu^2 (\phi^i)^2 - \frac{\lambda}{4} [(\phi^i)^2]^2$$

The Lagrangian is invariant under the symmetry

$$\phi^i \rightarrow R^{ij} \phi^j$$

for any  $N \times N$  orthogonal group in  $N$  dimensions, also called the  $N$ -dimensional orthogonal group or simply  $O(N)$ .



# Chapter 23

## Spin 1/2 Field



### 23.1 Representation of the Lorentz group

Recall the rotation of the field

$$U^{-1}(\Lambda)\phi_a(x)U(\Lambda) = \mathcal{S}_a{}^b\phi_b(\Lambda^{-1}x)$$

For infinitesimal rotation,

$$\mathcal{S}_a{}^b = \delta_a{}^b + \frac{i}{2}\delta\omega_{\alpha\beta}(S^{\alpha\beta})_a{}^b$$

The matrix  $S^{\alpha\beta}$  satisfy the following commutation relation

$$[S^{\mu\nu}, S^{\rho\sigma}] = -\eta^{\nu\rho}S^{\mu\sigma} + \eta^{\sigma\mu}S^{\rho\nu} + \eta^{\mu\rho}S^{\nu\sigma} - \eta^{\sigma\nu}S^{\rho\mu}$$

Define  $C_i \equiv \frac{1}{2}\epsilon_{ijk}S^{jk}$ ,  $D_i \equiv S^{i0}$ , so we can get

$$[C_i, C_j] = i\epsilon_{ijk}C_k \quad [C_i, D_j] = i\epsilon_{ijk}D_k \quad [D_i, D_j] = -i\epsilon_{ijk}C_k$$

We further define  $N_i \equiv \frac{1}{2}(C_i - iD_i)$  and  $N_i^\dagger \equiv \frac{1}{2}(C_i + iD_i)$ , then the commutation relation will be

$$[N_i, N_j] = i\epsilon_{ijk}N_k \quad [N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}N_k^\dagger \quad [N_i, N_j^\dagger] = 0$$

We see that we have two different  $SU(2)$  Lie algebras that are exchanged by hermitian conjugation. As we just discussed, a representation of the  $SU(2)$  Lie algebra is specified by an integer or half integer; we therefore conclude that a representation of the Lie algebra of the Lorentz group in four spacetime dimensions is specified by two integers or half-integers  $n$  and  $n'$ .

We will label these representations as  $(2n + 1, 2n' + 1)$ ; the number of components of a representation is then  $(2n + 1)(2n' + 1)$ . Different components within a representation can also be labeled by their angular momentum representations. We have  $C_i = N_i + N_i^\dagger$ . Thus, deducing the allowed values of  $j$  given  $n$  and  $n'$  becomes a standard problem in the addition of angular momenta. The general result is that the allowed values of  $j$  are  $|n - n'|, |n - n'| + 1, \dots, n + n'$ , and each of these values appears exactly once.



## 23.2 Spin-statistics theorem

### Theorem 23.1 Spin-statistics theorem

States with identical particles of integer spin are symmetric under the interchange of the particles, while states with identical particles of half-integer spin are antisymmetric under the interchange of the particles.

This is equivalent to the statement that the creation and annihilation operators for integer spin particles satisfy canonical commutation relations, while creation and annihilation operators for halfinteger spin particles satisfy canonical anticommutation relations.

Particles quantized with canonical commutation relations are called bosons, and satisfy Bose Einstein statistics, and particles quantized with canonical anticommutation relations are called fermions, and satisfy Fermi Dirac statistics.

Roughly speaking, one way to interchange two particles is to rotate them around their midpoint by  $\pi$ . For a particle of spin  $s$ , this rotation will introduce a phase factor of  $e^{i\pi s}$ . Thus, a two-particle state with identical particles both of spin  $s$  will pick up a factor of  $e^{i2\pi s}$ . For  $s$  a half-integer, this will give a factor of  $-1$ ; for  $s$  an integer, it will give a factor of  $+1$ . So, the creation and annihilation operators for integer spin particles satisfy canonical commutation relations, while creation and annihilation operators for halfinteger spin particles satisfy canonical anticommutation relations. The detailed proof can be found in chapter 12.1 and 12.2 from *Quantum Field Theory and the Standard Model* (Matthew D. Schwartz).

## 23.3 Spinor field

Consider a left-handed spinor field  $\psi_a(x)$ , also known as a left-handed Weyl field, which is in the  $(2, 1)$  representation of the Lie algebra of the Lorentz group. Here the index  $a$  is a left-handed spinor index that takes on two possible values. Under a Lorentz transformation, we have

$$U(\Lambda)^{-1}\psi_a(x)U(\Lambda) = L_a{}^b(\Lambda)\psi_b(\Lambda^{-1}x)$$

For an infinitesimal transformation, we can write

$$L_a{}^b(1 + \delta\omega) = \delta_a{}^b + \frac{i}{2}\delta\omega_{\mu\nu}(S_L^{\mu\nu})_a{}^b$$

$n = 1, \quad n' = 0$  implies that

$$(S_L^{ij}) = \frac{1}{2}\epsilon^{ijk}\sigma_k \quad (S_L^{k0}) = \frac{1}{2}i\sigma_k$$

where  $\sigma_k$  is Pauli matrix. Then the infinitesimal transformation can be written as

$$L(1 + \delta\omega) = I - \frac{i}{2}\theta_i\sigma_i - \beta_i\sigma_i$$



Recall that hermitian conjugation swaps the two  $SU(2)$  Lie algebras that comprise the Lie algebra of the Lorentz group. Therefore, the hermitian conjugate of a field in the  $(2, 1)$  representation should be a field in the  $(1, 2)$  representation; such a field is called a right-handed spinor field or a right-handed Weyl field. We will distinguish the indices of the  $(1, 2)$  representation from those of the  $(2, 1)$  representation by putting dots over them. Thus, we write

$$[\psi_a(x)]^\dagger = \psi_{\dot{a}}^\dagger(x)$$

Under a Lorentz transformation, we have

$$U(\Lambda)^{-1} \psi_{\dot{a}}^\dagger(x) U(\Lambda) = R_a^{\dot{b}}(\Lambda) \psi_{\dot{b}}^\dagger(x) (\Lambda^{-1}x)$$

For an infinitesimal transformation, we can write

$$R_{\dot{a}}^{\dot{b}}(1 + \delta\omega) = \delta_{\dot{a}}^{\dot{b}} + \frac{i}{2} \delta\omega_{\mu\nu} (S_R^{\mu\nu})_{\dot{a}}^{\dot{b}}$$

We can prove that

$$(S_R^{\mu\nu})_{\dot{a}}^{\dot{b}} = -[(S_L^{\mu\nu})_a^b]^*$$

Define

$$\epsilon_{ab} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Group theory can show that

$$L_a^c(\Lambda) L_b^d(\Lambda) \epsilon_{cd} = \epsilon_{ab}$$

which means that  $\epsilon_{ab}$  is an invariant symbol of the Lorentz group: it does not change under a Lorentz transformation that acts on all of its indices. The inverse matrix of  $\epsilon_{ab}$  is

$$\epsilon^{ab} \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

So,

$$\epsilon_{ab} \epsilon^{bc} = \delta_a^c \quad \epsilon^{ab} \epsilon_{bc} = \delta_c^a$$

We can use  $\epsilon_{ab}$  and its inverse  $\epsilon^{ab}$  to raise and lower left-handed spinor indices,

$$\psi^a(x) \equiv \epsilon^{ab} \psi_b(x)$$

We also notice the minus sign when we contract indices,

$$\psi^a \chi_a = -\psi_a \chi^a$$

And we can verify the following equations,

$$-L_b^a L_a^c = \delta_b^c$$

$$\psi_a(x) = \epsilon_{ab} \psi^b(x)$$

$$L_c^a(\Lambda) L_d^b(\Lambda) \epsilon^{cd} = \epsilon^{ab}$$

$$U(\Lambda)^{-1} \psi^a(x) U(\Lambda) = -L_b^a(\Lambda) \psi^b(\Lambda^{-1}x)$$



In right handed representation; we can also deduce the existence of an invariant symbol  $\epsilon_{\dot{a}\dot{b}}$ . The value of  $\epsilon_{\dot{a}\dot{b}}$  is the same as that of  $\epsilon_{ab}$ .

Define

$$\sigma_{a\dot{a}}^\mu \equiv (I, \vec{\sigma})$$

It is an invariant symbol of the group  $(2, 1) \otimes (1, 2) \otimes (2, 2)$

The properties of invariance symbol can be used to derive the following equations. The detailed derivation can be found in chapter 35 from *Quantum Field Theory* (Mark Srednicki).

### Proposition 23.1

1.

$$\begin{aligned}\sigma_{a\dot{a}}^\mu \sigma_{\mu b\dot{b}} &= -2\epsilon_{ab}\epsilon_{\dot{a}\dot{b}} \\ \epsilon^{ab}\epsilon^{\dot{a}\dot{b}} \sigma_{a\dot{a}}^\mu \sigma_{b\dot{b}}^\nu &= -2\eta^{\mu\nu}\end{aligned}$$

2.

$$\begin{aligned}(S_L^{\mu\nu})_{ab} &= (S_L^{\mu\nu})_{ba} \\ (S_R^{\mu\nu})_{\dot{a}\dot{b}} &= (S_R^{\mu\nu})_{\dot{b}\dot{a}}\end{aligned}$$

3. Define

$$\bar{\sigma}^{\mu\dot{a}a} \equiv \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\sigma_{b\dot{b}}^\mu$$

Numerically,

$$\bar{\sigma}^{\mu\dot{a}a} = (I, -\vec{\sigma})$$

So,

$$\begin{aligned}(S_L^{\mu\nu})_a^b &= +\frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_a^b \\ (S_R^{\mu\nu})_{\dot{a}}^{\dot{b}} &= -\frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)_{\dot{a}}^{\dot{b}}\end{aligned}$$

We adopt the following convention: a missing pair of contracted, undotted indices is understood to be written as  $^c_c$ , and a missing pair of contracted, dotted indices is understood to be written as  $^{\dot{c}}_{\dot{c}}$ . Thus, if  $\chi$  and  $\psi$  are two left-handed Weyl fields, we have

$$\chi\psi = \chi^a\psi_a \quad \chi^\dagger\psi^\dagger = \chi_a^\dagger\psi^{\dagger\dot{a}}$$

We expect Weyl fields to describe spin-one-half particles, and (by the spinstatistics theorem) these particles must be fermions. Therefore the corresponding fields must anticommute, rather than commute. That is, we should have

$$\chi_a(x)\psi_b(y) = -\psi_b(x)\chi_a(x)$$

Thus, we can get

$$\chi\psi = \psi\chi$$



Using the above convention, we can derive

**Proposition 23.2**

1.

$$(\chi\psi)^\dagger = \psi^\dagger\chi^\dagger$$

2.

$$[\psi^\dagger\bar{\sigma}^\mu\chi]^\dagger = \chi^\dagger\bar{\sigma}^\mu\psi$$



## 23.4 Lagrangians for spinor fields

Weyl field

$$\mathcal{L} = i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi - \frac{1}{2}m\psi\psi - \frac{1}{2}m\psi^\dagger\psi^\dagger$$



**Note:**

$$(i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi)^\dagger = i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi - i\partial_\mu(i\psi^\dagger\bar{\sigma}^\mu\psi)$$

The second term is a total divergence, and vanishes (with suitable boundary conditions on the fields at infinity) when we integrate it over  $d^4x$  to get the action  $S$ . Thus  $i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi$  has the hermiticity properties necessary for a term in  $\mathcal{L}$ .

We can derive the equation of motion from Hamilton principle,

$$-i\bar{\sigma}^\mu\partial_\mu\psi + m\psi^\dagger = 0$$

$$-i\sigma^\mu\partial_\mu\psi^\dagger + m\psi = 0$$

Define gamma matrix as

$$\gamma^\mu \equiv \begin{bmatrix} 0 & \sigma_{a\dot{c}}^\mu \\ \bar{\sigma}^{\mu\dot{a}c} & 0 \end{bmatrix}$$

We can prove that

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$$

Define a four-component Majorana field as

$$\Psi \equiv \begin{bmatrix} \psi_c \\ \psi^{\dagger\dot{c}} \end{bmatrix}$$

The equation of motion can be written as

$$(-i\gamma^\mu\partial_\mu + m)\Psi = 0$$



**Dirac field**

$$\mathcal{L} = i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi - \frac{1}{2}m\chi\xi - \frac{1}{2}m\xi^\dagger \chi^\dagger$$

It is invariant under

$$\chi \rightarrow e^{-i\alpha} \chi \quad \xi \rightarrow e^{i\alpha} \xi$$

Define a four-component Dirac field

$$\Psi \equiv \begin{bmatrix} \chi_a \\ \xi^{\dagger \dot{a}} \end{bmatrix}$$

So we take the hermitian conjugate of  $\Psi$  to get

$$\Psi^\dagger = (\chi_a^\dagger, \xi^a)$$

Introduce the matrix

$$\beta \equiv \begin{bmatrix} 0 & \delta^{\dot{a}}_c \\ \delta_a^c & 0 \end{bmatrix}$$

Given  $\beta$ , we define

$$\bar{\Psi} \equiv (\xi^a, \chi_a^\dagger)$$

Detailed calculation shows that

$$\bar{\Psi} \gamma^\mu \partial_\mu \Psi = \chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + \xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi + \partial_\mu (\xi^\dagger \sigma^\mu \xi)$$

So, the Lagrangian can be written as

$$\mathcal{L} = i\bar{\Psi} \gamma^\mu \partial_\mu \Psi - m\bar{\Psi} \Psi$$

It is invariant under the  $U(1)$  transformation

$$\Psi \rightarrow e^{-i\alpha} \Psi \quad \bar{\Psi} \rightarrow e^{i\alpha} \bar{\Psi}$$

The corresponding Noether current is

$$j^\mu = \bar{\Psi} \gamma^\mu \Psi = \chi^\dagger \bar{\sigma}^\mu \chi - \xi^\dagger \bar{\sigma}^\mu \xi$$

The equation of motion is

$$(-i\gamma^\mu \partial_\mu + m)\Psi = 0$$

**Charge conjugation**

Charge conjugation simply exchanges  $\xi$  and  $\chi$ . We can define a unitary charge conjugation operator  $C$  that implements this,

$$C^{-1} \xi_a(x) C = \chi_a(x)$$

$$C^{-1} \chi_a(x) C = \xi_a(x)$$



We then have  $C^{-1}\mathcal{L}(x)C = \mathcal{L}(x)$ .

Introduce the charge conjugation matrix

$$C \equiv \begin{bmatrix} \varepsilon_{ab} & \\ & \varepsilon^{\dot{a}\dot{b}} \end{bmatrix}$$

Take the transpose of  $\Psi$

$$\bar{\Psi}^T = \begin{bmatrix} \xi^a \\ \chi_{\dot{a}}^\dagger \end{bmatrix}$$

Define the charge conjugate of  $\Psi$ ,

$$\Psi^C \equiv C\bar{\Psi}^T = \begin{bmatrix} \xi_a \\ \chi^{\dagger\dot{a}} \end{bmatrix}$$

We therefore have

$$C^{-1}\Psi(x)C = \Psi^C(x)$$

for a Dirac field.

The charge conjugation matrix has a number of useful properties.

$$C^T = C^\dagger = C^{-1} = -C$$

$$C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$$

Now let us return to the Majorana field. It is obvious that a Majorana field is its own charge conjugate, that is,  $\Psi^C = \Psi$ . This condition is analogous to the condition  $\phi^\dagger = \phi$  that is satisfied by a real scalar field. A Dirac field, with its  $U(1)$  symmetry, is analogous to a complex scalar field, while a Majorana field, which has no  $U(1)$  symmetry, is analogous to a real scalar field. For a Majorana field, we have  $\bar{\Psi} = \Psi^T C$ . Then the Lagrangian can be written as

$$\mathcal{L} = \frac{i}{2}\Psi^T C\gamma^\mu \partial_\mu \Psi - \frac{1}{2}m\Psi^T C\Psi$$

### Projection matrix

We can also recover the Weyl components of a Dirac or Majorana field by means of a suitable projection matrix. Define

$$\gamma^5 \equiv \begin{bmatrix} -\delta_a^c & 0 \\ 0 & \delta^{\dot{a}}_{\dot{c}} \end{bmatrix}$$

Then we can define left and right projection matrices

$$P_L \equiv \frac{1}{2}(1 - \gamma^5) = \begin{bmatrix} \delta_a^c & 0 \\ 0 & 0 \end{bmatrix}$$

$$P_R \equiv \frac{1}{2}(1 + \gamma^5) = \begin{bmatrix} 0 & 0 \\ 0 & \delta^{\dot{a}}_{\dot{c}} \end{bmatrix}$$

The matrix  $\gamma^5$  can also be expressed as

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{24}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$$



The  $\gamma^5$  has the following properties,

$$\begin{aligned}(\gamma^5)^\dagger &= \gamma^5 \\ (\gamma^5)^2 &= 1 \\ \{\gamma^5, \gamma^\mu\} &= 0\end{aligned}$$

### The behavior of a Dirac field under a Lorentz transformation

Define

$$S^{\mu\nu} \equiv \begin{bmatrix} (S_L^{\mu\nu})_a{}^b & 0 \\ 0 & -(S_R^{\mu\nu})^{\dot{a}}{}_{\dot{b}} \end{bmatrix} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

Numerically, we have

$$S^{i0} = \frac{i}{2} \begin{bmatrix} \sigma^i & \\ & -\sigma^i \end{bmatrix} \quad S^{ij} = \frac{1}{2} \epsilon_{ijk} \begin{bmatrix} \sigma^k & \\ & \sigma^k \end{bmatrix}$$

Then, for either a Dirac or Majorana field  $\Psi$ , we can write

$$U(\Lambda)^{-1} \Psi(x) U(\Lambda) = D(\Lambda) \Psi(\Lambda^{-1}x)$$

where, for an infinitesimal transformation,

$$D(1 + \delta\omega) = I + \frac{i}{2} \delta\omega_{\mu\nu} S^{\mu\nu}$$

So the  $D(\Lambda)$  can be written as

$$D = \begin{bmatrix} L_a{}^b & 0 \\ 0 & -R^{\dot{a}}{}_{\dot{b}} \end{bmatrix}$$

We can verify that

$$U(\Lambda)^{-1} \bar{\Psi}(x) U(\Lambda) = \bar{\Psi}(\Lambda^{-1}x) [D(\Lambda)]^{-1}$$

From the identity

$$\sigma_{a\dot{a}}^\mu = L_a{}^b R_{\dot{a}}{}^{\dot{b}} \Lambda^\mu{}_\nu \sigma_{b\dot{b}}^\nu \quad \bar{\sigma}^{\mu\dot{a}a} = L^a{}_b R^{\dot{a}}{}_{\dot{b}} \Lambda^\mu{}_\nu \bar{\sigma}^{\nu\dot{b}b}$$

We have

$$L^a{}_b R_{\dot{a}}{}^{\dot{b}} \sigma_{a\dot{a}}^\mu = \Lambda^\mu{}_\nu \sigma_{b\dot{b}}^\nu \quad L_a{}^b R_{\dot{a}}{}^{\dot{b}} \bar{\sigma}^{\mu\dot{a}a} = \Lambda^\mu{}_\nu \bar{\sigma}^{\nu\dot{b}b}$$

So we can verify that

$$D^{-1} \gamma^\mu D = \Lambda^\mu{}_\nu \gamma^\nu$$

Recall the commutation relation of  $M^{\mu\nu}$  and  $P^\mu$ , we have

$$[\gamma^\mu, S^{\rho\sigma}] = i(\eta^{\mu\sigma} \gamma^\rho - \eta^{\mu\rho} \gamma^\sigma)$$

We also have

$$[\gamma^5, S^{\mu\nu}] = 0$$

which means that

$$D^{-1} \gamma^5 D = \gamma^5$$



## 23.5 Canonical quantization of Dirac field

### Canonical momentum and Hamiltonian

$$\Pi \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi)} = i\bar{\Psi}\gamma^0 = i\Psi^\dagger \quad (\bar{\Psi} = -i\Pi\gamma^0 \quad \Psi^\dagger = -i\Pi)$$

$$\mathcal{H} = -\Pi(\vec{\alpha} \cdot \vec{\nabla} + i\beta m)\Psi \quad (\alpha_i = \gamma^0\gamma^i \quad \beta = \gamma_0)$$

$$H = \int \mathcal{H} d^3x$$

### Momentum and angular momentum

$$T^{\mu\nu} = i\bar{\Psi}[\eta^{\mu\nu}\gamma^\rho\partial_\rho - \gamma^\mu\partial^\nu]\Psi - m\eta^{\mu\nu}\bar{\Psi}\Psi$$

$$P^0 = H \quad P^i = - \int \Pi \nabla^i \Psi d^3x$$

$$J_i = -\epsilon_{ijk} \int \Pi(x^j \nabla^k + \frac{i}{2} S^{jk}) \Psi d^3x$$

Define  $\Sigma_i \equiv \frac{1}{2}\epsilon_{ijk}S^{jk}$ , so

$$\Sigma_i = \frac{1}{2} \begin{bmatrix} \sigma^i & \\ & \sigma^i \end{bmatrix}$$

The above equation can be written as

$$\vec{J} = - \int \Pi(\vec{x} \times \vec{\nabla} + i\vec{\Sigma})\Psi d^3x$$

### Canonical quantization

$$\{\Psi_a(\mathbf{x}, t), \Psi_b(\mathbf{x}, t)\} = 0$$

$$\{\Psi_a(\mathbf{x}, t), \Pi^b(\mathbf{y}, t)\} = i\delta_a^b \delta(\mathbf{x} - \mathbf{y})$$

$$\{\Psi_a(\mathbf{x}, t), \Psi^{\dagger b}(\mathbf{y}, t)\} = \delta_a^b \delta(\mathbf{x} - \mathbf{y})$$

### Solution of Dirac equation

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp} [b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}]$$

Here, we introduce the Feynman slash: given any four-vector  $a^\mu$ , we define

$$\not{a} \equiv a_\mu \gamma^\mu$$

The Dirac equation implies that

$$(\not{p} + m)u(\mathbf{p}) = 0$$

$$(-\not{p} + m)v(\mathbf{p}) = 0$$





Each of these equations has two solutions, which we label via  $s = +$  and  $s = -$ . For  $m \neq 0$ , we can go to the rest frame,  $\mathbf{p} = 0$ . We will then distinguish the two solutions by the eigenvalue of the spin matrix  $\Sigma_3$ . Specifically, we will require

$$\Sigma_3 u_{\pm}(\mathbf{0}) = \pm \frac{1}{2} u_{\pm}(\mathbf{0})$$

$$\Sigma_3 v_{\pm}(\mathbf{0}) = \mp \frac{1}{2} v_{\pm}(\mathbf{0})$$

The solutions are

$$\begin{aligned} u_+(\mathbf{0}) &= \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} & u_-(\mathbf{0}) &= \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\ v_+(\mathbf{0}) &= \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} & v_-(\mathbf{0}) &= \sqrt{m} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

For later use we also compute the barred spinors

$$\bar{u}_s(\mathbf{p}) \equiv u_s^\dagger(\mathbf{p})\beta \quad \bar{v}_s(\mathbf{p}) \equiv v_s^\dagger(\mathbf{p})\beta \quad (\beta = \gamma^0)$$

We get

$$\bar{u}_+(\mathbf{0}) = \sqrt{m}(1, 0, 1, 0)$$

$$\bar{u}_-(\mathbf{0}) = \sqrt{m}(0, 1, 0, 1)$$

$$\bar{v}_+(\mathbf{0}) = \sqrt{m}(0, -1, 0, 1)$$

$$\bar{v}_-(\mathbf{0}) = \sqrt{m}(1, 0, -1, 0)$$

We can now find the spinors corresponding to an arbitrary three-momentum  $\mathbf{p}$  by applying to  $u_s(\mathbf{0})$  and  $v_s(\mathbf{0})$  the matrix  $D(\Lambda)$  that corresponds to an appropriate boost. This is given by

$$D(\Lambda) = \exp(i\eta \hat{\mathbf{p}} \cdot \mathbf{D})$$

where  $\hat{\mathbf{p}}$  is a unit vector in the  $\mathbf{p}$  direction,  $D^i = \frac{i}{4}[\gamma^i, \gamma^0] = \frac{i}{2}\gamma^i\gamma^0$  is the boost matrix, and  $\eta$  is the rapidity. Thus we have

$$u_s(\mathbf{p}) = \exp(i\eta \hat{\mathbf{p}} \cdot \mathbf{D}) u_s(\mathbf{0})$$

$$v_s(\mathbf{p}) = \exp(i\eta \hat{\mathbf{p}} \cdot \mathbf{D}) v_s(\mathbf{0})$$

We also have

$$\bar{u}_s(\mathbf{p}) = \bar{u}_s(\mathbf{0}) \exp(-i\eta \hat{\mathbf{p}} \cdot \mathbf{D})$$

$$\bar{v}_s(\mathbf{p}) = \bar{v}_s(\mathbf{0}) \exp(-i\eta \hat{\mathbf{p}} \cdot \mathbf{D})$$

This follows from  $\bar{K}^j = K^j$ , where, for any general combination of gamma matrices,

$$\bar{A} \equiv \beta A^\dagger \beta$$



In particular, it turns out that

$$\gamma^\mu S^{\mu\nu} - i\gamma^5 \gamma^\mu \gamma^5 = i\gamma^5 S^{\mu\nu}$$

all satisfy  $\bar{A} = A$  The barred spinors satisfy the equations

$$\bar{u}_s(\mathbf{p})(\not{p} + m) = 0 \quad \bar{v}_s(\mathbf{p})(-\not{p} + m) = 0$$

### Proposition 23.3

1.

$$\bar{u}_{s'}(\mathbf{p})u_s(\mathbf{p}) = 2m\delta_{ss'}$$

$$\bar{v}_{s'}(\mathbf{p})v_s(\mathbf{p}) = -2m\delta_{ss'}$$

$$\bar{u}_{s'}(\mathbf{p})v_s(\mathbf{p}) = 0$$

$$\bar{v}_{s'}(\mathbf{p})u_s(\mathbf{p}) = 0$$

2.

$$2m\bar{u}_{s'}(\mathbf{p}')\gamma^\mu u_s(\mathbf{p}) = \bar{u}_{s'}[(p' + p)^\mu - 2iS^{\mu\nu}(p' - p)_\nu]u_s(\mathbf{p})$$

$$-2m\bar{v}_{s'}(\mathbf{p}')\gamma^\mu v_s(\mathbf{p}) = \bar{v}_{s'}[(p' + p)^\mu - 2iS^{\mu\nu}(p' - p)_\nu]v_s(\mathbf{p})$$

3.

$$\bar{u}_{s'}(\mathbf{p})\gamma^\mu u_s(\mathbf{p}) = 2p^\mu \delta_{ss'}$$

$$\bar{v}_{s'}(\mathbf{p})\gamma^\mu v_s(\mathbf{p}) = 2p^\mu \delta_{ss'}$$

$$\bar{u}_{s'}(\mathbf{p})\gamma^0 v_s(-\mathbf{p}) = 0$$

$$\bar{v}_{s'}(\mathbf{p})\gamma^0 u_s(-\mathbf{p}) = 0$$

4.

$$\sum_{s=\pm} u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) = -\not{p} + m$$

$$\sum_{s=\pm} v_s(\mathbf{p})\bar{v}_s(\mathbf{p}) = -\not{p} - m$$

5.

$$u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) = \frac{1}{2}(1 - s\gamma^5 \not{z})(-\not{p} + m)$$

$$v_s(\mathbf{p})\bar{v}_s(\mathbf{p}) = \frac{1}{2}(1 - s\gamma^5 \not{z})(-\not{p} - m)$$

Here,  $z^\mu$  is the boost of the vector  $(0, 0, 0, 1)$  from the frame with  $\mathbf{p}' = 0$  to the frame with  $\mathbf{p}' = \mathbf{p}$ .

The proof can be found in chapter 38 from *Quantum field theory* (Mark Srednicki)

It is interesting to consider the extreme relativistic limit of this formula. Let us take the three-momentum to be in the z direction, so that it is parallel to the spin-quantization axis.



The component of the spin in the direction of the three-momentum is called the helicity. A fermion with helicity  $+1/2$  is said to be right-handed, and a fermion with helicity  $-1/2$  is said to be left-handed. For rapidity  $\eta$ , we have

$$\frac{p^\mu}{m} = (\cosh \eta, 0, 0, \sinh \eta) \quad z^\mu = (\sinh \eta, 0, 0, \cosh \eta)$$

In the limit of large  $\eta$ ,

$$z^\mu = \frac{p^\mu}{m} + \mathcal{O}(e^{-\eta})$$

Then the equations 3.3.5 can be written as

$$u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) \rightarrow \frac{1}{2}(1 + s\gamma^5)(-\not{p})$$

$$v_s(\mathbf{p})\bar{v}_s(\mathbf{p}) \rightarrow \frac{1}{2}(1 - s\gamma^5)(-\not{p})$$

So, we can see the spinor corresponding to a right-handed fermion is  $u_+(\mathbf{p})$  for a b-type particle and  $v_-(\mathbf{p})$  for a d-type particle. The spinor corresponding to a left-handed fermion is  $u_-(\mathbf{p})$  for a b-type particle and  $v_+(\mathbf{p})$  for a d-type particle.

Note that  $\beta u_s(\mathbf{0}) = +u_s(\mathbf{0})$  and  $\beta v_s(\mathbf{0}) = -v_s(\mathbf{0})$ . Also,  $\beta D^j = -D^j \beta$ . We then have

$$u_s(-\mathbf{p}) = \beta u_s(\mathbf{p}) \quad v_s(-\mathbf{p}) = -\beta v_s(\mathbf{p})$$

For charge conjugation matrix, note that that  $\mathcal{C}\bar{u}_s^T(\mathbf{0}) = v_s(\mathbf{0})$ ,  $\mathcal{C}\bar{v}_s^T(\mathbf{0}) = u_s(\mathbf{0})$ , and  $\mathcal{C}^{-1}D^j\mathcal{C} = -(D^j)^T$ , we have

$$\mathcal{C}\bar{u}_s^T(\mathbf{p}) = v_s(\mathbf{p}) \quad \mathcal{C}\bar{v}_s^T(\mathbf{p}) = u_s(\mathbf{p})$$

Taking the complex conjugate of the above equation, we have

$$u_s^*(\mathbf{p}) = \mathcal{C}\beta v_s(\mathbf{p}) \quad v_s^*(\mathbf{p}) = \mathcal{C}\beta u_s(\mathbf{p})$$

Note that  $\gamma^5 u_s(\mathbf{0}) = +s v_{-s}(\mathbf{0})$  and  $\gamma^5 v_s(\mathbf{0}) = -s u_{-s}(\mathbf{0})$ , and that  $\gamma^5 D^j = D^j \gamma^5$ , we have

$$\gamma^5 u_s(\mathbf{p}) = +s v_{-s}(\mathbf{p}) \quad \gamma^5 v_s(\mathbf{p}) = -s u_{-s}(\mathbf{p})$$

Combine the above equations, we can derive

$$u_{-s}^*(\mathbf{p}) = -s\mathcal{C}\gamma^5 u_s(\mathbf{p}) \quad v_{-s}^*(\mathbf{p}) = -s\mathcal{C}\gamma^5 v_s(\mathbf{p})$$

### Fourier expansion

$$\begin{aligned} \Psi(x) &= \sum_{s=\pm} \int \widetilde{dp} [b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}] \\ \Pi(x) &= i \sum_{s=\pm} \int \widetilde{dp} [b_s^\dagger(\mathbf{p})u_s^\dagger(\mathbf{p})e^{-ipx} + d_s(\mathbf{p})v_s^\dagger(\mathbf{p})e^{+ipx}] \\ b_s(\mathbf{p}) &= \int d^3x e^{-ipx} \bar{u}_s(\mathbf{p})\gamma^0 \Psi(x) \end{aligned}$$



$$b_s^\dagger(\mathbf{p}) = \int d^3x e^{ipx} \bar{\Psi}(x) \gamma^0 u_s(\mathbf{p})$$

$$d_s(\mathbf{p}) = \int d^3x e^{-ipx} \bar{\Psi}(x) \gamma^0 v_s(\mathbf{p})$$

$$d_s^\dagger(\mathbf{p}) = \int d^3x e^{ipx} \bar{v}_s(\mathbf{p}) \gamma^0 \Psi(x)$$

We can get the anticommutation relation in terms of  $b, b^\dagger, d, d^\dagger$ . The only non-vanishing terms are

$$\{b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\} = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') 2\omega \delta_{ss'}$$

$$\{d_s(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}')\} = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') 2\omega \delta_{ss'}$$

Operator represented by  $b, b^\dagger, d, d^\dagger$

Define

$$N^+(\mathbf{p}, s) = b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) \quad N^-(\mathbf{p}, s) = d_s^\dagger(\mathbf{p}) d_s(\mathbf{p})$$

So we can derive

$$H = \sum_{s=\pm} \int \widetilde{dp} \omega [N^+(\mathbf{p}, s) + N^-(\mathbf{p}, s)] - 4\mathcal{E}_0 V$$

$$P^i = \sum_{s=\pm} \int \widetilde{dp} p^i [N^+(\mathbf{p}, s) + N^-(\mathbf{p}, s)]$$

$$S_3 = \sum_{s=\pm} \int \widetilde{dp} \frac{s}{2} [N^+(\mathbf{p}, s) + N^-(\mathbf{p}, s)]$$

$$Q = \sum_{s=\pm} \int \widetilde{dp} [N^+(\mathbf{p}, s) - N^-(\mathbf{p}, s)]$$

Causality

Firstly, we derive the anticommutation relation for field operators at any space-time.

$$\{\bar{\Psi}_a(x), \Psi_b(y)\} = (i\partial_x + m)_{ab} i\Delta(x - y)$$

Here,

$$i\Delta(x - y) = \int \widetilde{dp} [e^{ip(x-y)} - e^{-ip(x-y)}]$$

For  $(x - y)^2 > 0$  the anti-commutators vanish, because  $\Delta(x - y)$  also vanishes. We then can verify that

$$[\bar{\Psi}_a(x) \Psi_b(x), \bar{\Psi}_c(y) \Psi_d(y)] = 0$$

for  $(x - y)^2 > 0$ .

In this way the microscopic causality is satisfied for the physical observables, such as the charge density or the momentum density.



### The Dirac propagator

$$\langle 0 | \Psi_a(x) \bar{\Psi}_b(y) | 0 \rangle = (i\cancel{\partial}_x + m)_{ab} \int \widetilde{dp} e^{ip(x-y)}$$

$$\langle 0 | \bar{\Psi}_b(y) \Psi_a(x) | 0 \rangle = -(i\cancel{\partial}_x + m)_{ab} \int \widetilde{dp} e^{ip(y-x)}$$

Define retarded green function as

$$S_R(x-y)_{ab} \equiv \theta(x^0 - y^0) \langle 0 | \{ \Psi_a(x) \bar{\Psi}_b(y) \} | 0 \rangle$$

It is easy to verify that

$$S_R(x-y) = (i\cancel{\partial}_x + m) D_R(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\cancel{p} - m)}{p^2 + m^2} e^{ip(x-y)}$$

and

$$(i\cancel{\partial}_x - m) S_R(x-y) = i\delta(x-y) \cdot \mathbf{1}_{4 \times 4}$$

Now, we define the time ordered product for fermion fields

$$T\eta(x)\eta(y) \equiv \theta(x^0 - y^0)\eta(x)\eta(y) - \theta(y^0 - x^0)\eta(y)\eta(x)$$

So,

$$S_F(x-y) \equiv \langle 0 | T \Psi(x) \bar{\Psi}(y) | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i(\cancel{p} - m)}{p^2 + m^2 - i\epsilon} e^{ip(x-y)}$$

It is easy to verify that

$$\langle 0 | T \bar{\Psi}_a(x) \Psi_b(y) | 0 \rangle = -\langle 0 | T \Psi_b(y) \bar{\Psi}_a(x) | 0 \rangle = -S_F(y-x)_{ba}$$

## 23.6 Parity,time reversal and charge conjugation

### Parity

We assume that

$$P^{-1} b_s(\mathbf{p}) P = \eta_b b_s(-\mathbf{p})$$

$$P^{-1} d_s(\mathbf{p}) P = \eta_d d_s(-\mathbf{p})$$

Here,  $\eta_b$  and  $\eta_d$  is the phase factor. Then, we can verify that

$$P^{-1} \mathbf{P} P = -\mathbf{P} \quad P^{-1} \mathbf{S} P = \mathbf{S}$$

So a parity transformation reverse the three momentum while leaving the spin direction unchanged. Further more, we have

$$P^{-1} \Psi(x) P = \sum_{s=\pm} \int \widetilde{dp} [\eta_b b_s(\mathbf{p}) \beta u_s(\mathbf{p}) e^{ip\mathcal{P}x} - \eta_d^* d_s^\dagger(\mathbf{p}) \beta v_s(\mathbf{p}) e^{-ip\mathcal{P}x}]$$

where,  $\mathcal{P}^\mu_\nu = \text{diag}(1, -1, -1, -1)$ . So, we should demand that  $\eta_b = -\eta_d^*$ . And we can get

$$P^{-1} \Psi(x) P = \eta_b \beta \Psi(\mathcal{P}x) \quad P^{-1} \bar{\Psi}(x) P = \eta_b^* \bar{\Psi}(\mathcal{P}x) \beta$$

Generally, we have

$$P^{-1} (\bar{\Psi} A \Psi) P = \bar{\Psi} (\beta A \beta) \Psi$$



### Time reversal

Note in quantum mechanics, we have shown that time reversal operator is antiunitary. Firstly, we assume that

$$T^{-1}b_s(\mathbf{p})T = \zeta_{b,s}b_{-s}(-\mathbf{p})$$

$$T^{-1}d_s(\mathbf{p})T = \zeta_{d,s}d_{-s}(-\mathbf{p})$$

Then we can verify that

$$T^{-1}\mathbf{P}T = -\mathbf{P} \quad T^{-1}\mathbf{S}T = -\mathbf{S}$$

So a parity transformation reverse the three momentum and the spin direction. Further more, we have

$$T^{-1}\Psi(x)T = \sum_{s=\pm} \int \widetilde{d\mathbf{p}} -s\mathcal{C}\gamma^5 [\zeta_{b,-s}b_s(\mathbf{p})u_s(\mathbf{p})e^{ip\mathcal{T}x} + \zeta_{d,-s}^*d_s^\dagger(\mathbf{p})\beta v_s(\mathbf{p})e^{-ip\mathcal{T}x}]$$

where,  $\mathcal{T}^\mu_\nu = \text{diag}(-1, 1, 1, 1)$ . So, we should demand that  $\zeta_{s,b} = s\zeta$  and  $\zeta_{s,d} = s\zeta^*$ . And we can get

$$T^{-1}\Psi(x)T = \zeta\mathcal{C}\gamma^5\Psi(\mathcal{T}x) \quad T^{-1}\bar{\Psi}(x)T = \zeta^*\bar{\Psi}(\mathcal{T}x)\gamma^5\mathcal{C}^{-1}$$

Generally, we have

$$T^{-1}(\bar{\Psi}A\Psi)T = \bar{\Psi}(\gamma^5\mathcal{C}^{-1}A\mathcal{C}\gamma^5)\Psi$$

### Charge conjugation

We have already shown that

$$C^{-1}\Psi(x)C = \mathcal{C}\bar{\Psi}^T(x) \quad C^{-1}\bar{\Psi}(x)C = \Psi^T(x)\mathcal{C}$$

Generally, we have

$$C^{-1}(\bar{\Psi}A\Psi)C = \bar{\Psi}(\mathcal{C}^{-1}A^T\mathcal{C})\Psi$$

### Summary

The transformation properties of the various fermion bilinears under  $C, P$  and  $T$  are summarized in the table below. Here we use the shorthand  $(-1)^\mu \equiv 1$  for  $\mu = 0$  and  $(-1)^\mu \equiv -1$  for  $\mu = 1, 2, 3$ .

	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$	$\partial_\mu$
$P$	+1	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
$T$	+1	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
$C$	+1	+1	-1	+1	-1	+1
$CPT$	+1	+1	-1	-1	+1	-1

**Figure 23.1:** Transformation of fermion bilinears under CPT



**Theorem 23.2 CPT theorem**

Any hermitian combination of any set of fields (scalar, vector, Dirac, Majorana) and their derivatives that is a Lorentz scalar (and so carries no indices) is even under  $CPT$ . Since the lagrangian must be formed out of such combinations, we have  $\mathcal{L}(x) \rightarrow \mathcal{L}(-x)$  under  $CPT$ , and so the action  $S = \int d^4x \mathcal{L}$  is invariant.



## 23.7 Perturbation theory for canonical quantization

We use Yukawa theory as an example.

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M_0^2\phi^2 + i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m_0\bar{\Psi}\Psi - g_0\bar{\Psi}\Psi\phi$$

$$H = H_0 + H_{int} \quad H_{int} = \int d^3x g_0\bar{\Psi}\Psi\phi$$

Similar to  $\phi^4$  theory, the perturbation expansion of correlation functions is

$$\langle\Omega|T\{\Psi(x)\bar{\Psi}(y)\phi(z)\}|\Omega\rangle = \lim_{T\rightarrow\infty(1-i\epsilon)} \frac{\langle 0|T\left\{\Psi_I(x)\bar{\Psi}_I(y)\phi_I(z)\exp\left[-i\int_{-T}^T dt H_I\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int_{-T}^T dt H_I\right]\right\}|0\rangle}$$

Before we state the Wick's theorem, we must note the following conventions:

1. The time-ordered product picks up one minus sign for each interchange of operators that is necessary to put the fields in time order.
2. The normal-ordered product picks up one minus sign for each interchange of operators that is necessary to put the fields in normal order.
3. Define contractions under the normal-ordering symbol to include minus signs for operator interchanges.

With these conventions, Wick's theorem takes the same form as before:

$$T\{\Psi_I(x_1)\bar{\Psi}_I(x_2)\Psi_I(x_3)\cdots\} = N\{\Psi_I(x_1)\bar{\Psi}_I(x_2)\Psi_I(x_3)\cdots + \text{all possible contractions}\}$$

**Example:**

$$\begin{aligned} \langle 0|T\{\Psi_{Ia}(x_1)\bar{\Psi}_{Ib}(x_2)\Psi_{Ic}(x_3)\bar{\Psi}_{Id}(x_4)\}|0\rangle &= S_F(x_1 - x_2)_{ab}S_F(x_3 - x_4)_{cd} \\ &- S_F(x_1 - x_4)_{ad}S_F(x_3 - x_2)_{cb} \end{aligned}$$

Then we can derive the Feynman rule for Yukawa theory. Expand

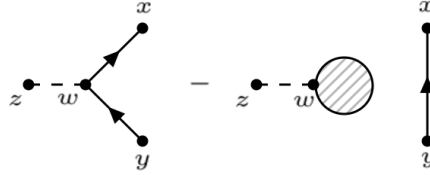
$$\langle 0|T\left\{\Psi_{Ia}(x)\bar{\Psi}_{Ib}(y)\phi_I(z)\exp\left[-i\int_{-T}^T dt H_I\right]\right\}|0\rangle$$



to the first order of  $g_0$

$$\begin{aligned}
 & \langle 0|T \left\{ \Psi_{Ia}(x) \bar{\Psi}_{Ib}(y) \phi_I(z) (-ig_0) \int d^4w \bar{\Psi}_I(w) \Psi_I(w) \phi_I(w) \right\} |0\rangle \\
 &= -(-ig_0) S_F(x-y)_{ab} \int d^4w D_F(z-w) \text{Tr}[S_F(w-z)] \\
 &+ (-ig_0) \int d^4w [S_F(x-w) S_F(w-y)]_{ab} D_F(w-z)
 \end{aligned}$$

It can be represented by the so called Feynman diagram.



**Figure 23.2:** Feynman diagram representation of perturbation expansion

The Feynman rules for Yukawa theory are:

1. For each Fermion propagator from  $y$  to  $x$ ,  $P = S_F(x-y)$
2. For each scalar propagator,  $P = D_F(x-y)$
3. For each vertex,  $V = (-ig_0) \int d^4w$
4. For each external point,  $E = 1$
5. Divided by the symmetry factor

## 23.8 Path integral quantization

### 23.8.1 Grassmann numbers

#### Formal definition

Grassmann numbers are individual elements or points of the exterior algebra generated by a set of  $n$  Grassmann variables or Grassmann directions or supercharges  $\{\theta_i\}$  with  $n$  possibly being infinite.

The Grassman variables are the basis vectors of a vector space (of dimension  $n$ ) They form an algebra over a field, with the field usually being taken to be the complex numbers, although one could contemplate other fields, such as the reals. The algebra is a unital algebra, and the generators are anti-commuting:

$$\theta_i \theta_j = -\theta_j \theta_i$$

Since the  $\theta_i$  form a vector space over the complex numbers, it is trivial that they commute with the complex numbers; this is by definition. That is, for complex  $x$ , one has

$$\theta_i x = x \theta_i$$





The squares of the generators vanish:

$$\theta_i \theta_i = -\theta_i \theta_i$$

In other words, a Grassmann variable is a non-zero square-root of zero.

Let  $V$  denote this  $n$ -dimensional vector space of Grassman variables. Note that it is independent of the choice of basis. The corresponding exterior algebra is defined as

$$\Lambda = \mathbb{C} \oplus V \oplus (V \wedge V) \oplus (V \wedge V \wedge V) \oplus \dots$$

where  $\wedge$  is the exterior product and  $\oplus$  is the direct sum. The individual elements of this algebra are then called Grassman numbers. It is standard to completely omit the wedge symbol  $\wedge$  when writing a Grassman number; it is used here only to clearly illustrate how the exterior algebra is built up out of the Grassman variables. Thus, a completely general Grassman number can be written as

$$z = \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_k} c_{i_1 i_2 \dots i_k} \theta_{i_1} \theta_{i_2} \dots \theta_{i_k}$$

where the  $c$ 's are complex numbers, or, equivalently,  $c_{i_1 i_2 \dots i_k}$  is a complex-valued, completely antisymmetric tensor of rank  $k$ . Again, the  $\theta_i$  can be clearly seen here to be playing the role of a basis vector of a vector space.

Observe that the Grassmann algebra generated by  $n$  linearly independent Grassmann variables has dimension  $2^n$ ; this follows from the binomial theorem applied to the above sum, and the fact that the  $n + 1$ -fold product of variables must vanish, by the anti-commutation relations, above. In other words, for  $n$  variables, the sum terminates

$$\Lambda = \mathbb{C} \oplus \Lambda^1 V \oplus \Lambda^2 V \oplus \dots \oplus \Lambda^n V$$

where  $\Lambda^k V$  is the  $k$ -fold alternating product. The dimension of  $\Lambda^k V$  is given by  $\binom{n}{k}$ , the binomial coefficient. The special case of  $n = 1$  is called a dual number, and was introduced by William Clifford in 1873.

### Integral over Grassmann number

Single Variable:

$$\int d\theta (A + B\theta) \equiv B$$

Multi-variable:

$$\int d\theta d\eta \eta \theta \equiv 1$$

Complex Grassmann number:

$$(\theta \eta)^* \equiv \eta^* \theta^* = -\theta^* \eta^*$$

$$\int d\theta^* d\theta \theta \theta^* \equiv 1$$

Gaussian integral over a complex Grassmann number:

$$\int d\theta^* d\theta e^{-\theta^* b \theta} = b$$



$$\int d\theta^* d\theta \theta \theta^* e^{-\theta^* b \theta} = 1$$

Unitary transformation:

If  $\theta'_i = U_{ij} \theta_j$  and  $U$  is unitary matrix, then we can derive

$$\prod_i \theta'_i = (\det U) \left( \prod_i \theta_i \right)$$

In a general integral

$$\left( \prod_i \int d\theta_i^* d\theta_i \right) f(\theta)$$

the only term of  $f(\theta)$  that survives has exactly one factor of each  $\theta_i$  and  $\theta_i^*$ ; it is proportional to  $(\prod_i \theta_i)(\prod_i \theta_i^*)$ . If we replace  $\theta$  by  $U\theta$ , this term acquires a factor of  $\det U \det U^* = 1$ , so the integral is unchanged under the unitary transformation.

Gaussian integral over multiple complex Grassmann numbers:

$$\left( \prod_i \int d\theta_i^* d\theta_i \right) e^{-\theta^* B_{ij} \theta_j} = \det B$$

$$\left( \prod_i \int d\theta_i^* d\theta_i \right) \theta_k \theta_l^* e^{-\theta^* B_{ij} \theta_j} = B_{kl}^{-1} \det B$$

### 23.8.2 Path integral formulation for free Dirac field

A Grassmann field is a function of space-time whose values are Grassmann numbers. The classical Dirac field being used to evaluate path integral is a Grassmann field. The correlation function is given by

$$\langle \Omega | T \Psi_H(x_1) \bar{\Psi}_H(x_2) | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp \left[ i \int_{-T}^T d^4x \bar{\Psi} (i\not{\partial} - m) \Psi \right] \Psi(x_1) \bar{\Psi}(x_2)}{\int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp \left[ i \int_{-T}^T d^4x \bar{\Psi} (i\not{\partial} - m) \Psi \right]}$$

The generating function is

$$Z[\bar{\eta}, \eta] = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp \left[ i \int d^4x \bar{\Psi} (i\not{\partial} - m) \Psi + \bar{\eta} \Psi + \bar{\Psi} \eta \right]$$

Here,  $\eta(x)$  is a Grassmann-valued source field. Define

$$\Psi'(x) \equiv \Psi(x) - i \int d^4y S_F(x-y) \eta(y)$$

Then we can derive that

$$\bar{\Psi}'(x) \equiv \bar{\Psi}(x) - i \int d^4y \bar{\eta}(y) S_F(y-x)$$

Recall that

$$(i\not{\partial}_x - m) S_F(x-y) = i\delta(x-y)$$



and

$$S_F(y - x)(i\cancel{\partial}_x + m) = -i\delta(x - y)$$

we can derive that

$$\int d^4x \bar{\Psi}(i\cancel{\partial} - m)\Psi + \bar{\eta}\Psi + \bar{\Psi}\eta = \int d^4x \bar{\Psi}'(i\cancel{\partial} - m)\Psi' + i \int d^4x d^4y \bar{\eta}(x) S_F(x - y) \eta(y)$$

After integration, we have

$$Z[\bar{\eta}, \eta] = Z_0 \exp \left[ - \int d^4x d^4y \bar{\eta}(x) S_F(x - y) \eta(y) \right]$$

If we adopt the convention that

$$\frac{d}{d\eta} \theta \eta = -\frac{d}{d\eta} \eta \theta = -\theta$$

the two point correlation functions are

$$\langle 0 | T \Psi_H(x_1) \bar{\Psi}_H(x_2) | 0 \rangle = Z_0^{-1} \left( -i \frac{\delta}{\delta \bar{\eta}(x_1)} \right) \left( i \frac{\delta}{\delta \eta(x_2)} \right) Z[\bar{\eta}, \eta] |_{\bar{\eta}, \eta=0} = S_F(x_1 - x_2)$$

### 23.8.3 Perturbation theory for path integral quantization

We use Yukawa theory as an example.

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} M_0^2 \phi^2 + i \bar{\Psi} \gamma^\mu \partial_\mu \Psi - m_0 \bar{\Psi} \Psi - g_0 \bar{\Psi} \Psi \phi \\ \mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_1 \quad \mathcal{L}_1 = -g_0 \bar{\Psi} \Psi \phi \\ Z[J] &= \int \mathcal{D}\phi \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{i \int d^4x [\mathcal{L}_0 + \mathcal{L}_1 + J\phi + \bar{\eta}\Psi + \bar{\Psi}\eta]} \\ &= e^{i \int d^4x \mathcal{L}_1(\frac{1}{i} \frac{\delta}{\delta J(x)}, \frac{1}{i} \frac{\delta}{\delta \bar{\eta}(x)}, -\frac{1}{i} \frac{\delta}{\delta \eta(x)})} \int \mathcal{D}\phi \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{i \int d^4y [\mathcal{L}_0 + J\phi + \bar{\eta}\Psi + \bar{\Psi}\eta]} \\ &\propto e^{i \int d^4x \mathcal{L}_1(\frac{1}{i} \frac{\delta}{\delta J(x)}, \frac{1}{i} \frac{\delta}{\delta \bar{\eta}(x)}, -\frac{1}{i} \frac{\delta}{\delta \eta(x)})} \exp \left[ - \int d^4y d^4z \frac{1}{2} J(y) D_F(y - z) J(z) + \bar{\eta}(y) S_F(y - z) \eta(z) \right] \\ &= \sum_{V=0}^{\infty} \frac{1}{V!} \left[ -i g_0 \int d^4x \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \cdot -\frac{1}{i} \frac{\delta}{\delta \eta(x)} \cdot \frac{1}{i} \frac{\delta}{\delta \bar{\eta}(x)} \right) \right]^V \\ &\times \sum_{P_1=0}^{\infty} \frac{1}{P_1!} \left[ -\frac{1}{2} \int d^4y_1 d^4z_1 J(y_1) D_F(y_1 - z_1) J(z_1) \right]^{P_1} \\ &\times \sum_{P_2=0}^{\infty} \frac{1}{P_2!} \left[ - \int d^4y_2 d^4z_2 \bar{\eta}(y_2) S_F(y_2 - z_2) \eta(z_2) \right]^{P_2} \end{aligned}$$

If we focus on a term with particular values of  $V$ ,  $P_1$  and  $P_2$ , then the number of surviving scalar sources is  $E_1 = 2P_1 - V$ , the number of surviving fermion sources is  $E_2 = 2P_2 - 2V$ . We can introduce Feynman diagrams as in the  $\phi^4$  theory. In these diagrams, a dashed line segment stands for a scalar propagator  $D_F(x - y)$ , a line with an arrow pointing from  $y$  to  $x$  for a fermion propagator  $S_F(x - y)$ , a filled circle at one end of a dashed line segment for a scalar source  $i \int d^4x J(x)$ , a filled circle at the start of a line with an arrow for a fermion source  $i \int d^4x \eta(x)$ , a filled circle at the end of a line with an arrow for an anti-fermion source  $i \int d^4x \bar{\eta}(x)$ , a vertex joining three line segments for  $-i g_0 \int d^4x$ .



## 23.9 LSZ reduction formula

Similarly to the scalar field theory, we can derive the structure of the exact propagator of Dirac fermions in interaction theory.

$$\int d^4x e^{-ipx} \langle \Omega | T \Psi(x) \bar{\Psi}(0) | \Omega \rangle_C = \frac{iZ_2(\not{p} - m)}{p^2 + m^2 - i\epsilon} + \dots$$

We eliminate the terms contributing none isolate poles for  $p^2$ . Here,  $m$  is the exact mass of the fermions. The constant  $Z_2$  is the probability for the quantum field to create or annihilate an exact one-particle eigenstate of  $H$ :

$$\langle \Omega | \Psi(0) | p, s \rangle = \sqrt{Z_2} u^s(p)$$

The LSZ reduction formula for fermions would take the form as

### Theorem 23.3 LSZ reduction formula

$$\begin{aligned} & \langle \mathbf{p}_1 \cdots \mathbf{p}_n \bar{\mathbf{p}}_1 \cdots \bar{\mathbf{p}}_{\bar{n}} | S | \mathbf{k}_1 \cdots \mathbf{k}_m \bar{\mathbf{k}}_1 \cdots \bar{\mathbf{k}}_{\bar{m}} \rangle \\ &= \prod_1^n \int d^4x_i e^{-ip_i x_i} \prod_1^{\bar{n}} \int d^4\bar{x}_i e^{-i\bar{p}_i \bar{x}_i} \prod_1^m \int d^4y_j e^{ik_j y_j} \prod_1^{\bar{m}} \int d^4\bar{y}_i e^{i\bar{k}_j \bar{y}_j} \\ &\times [\bar{u}_{s_1}(\mathbf{p}_1)(\not{p}_1 + m)] \cdots [\bar{u}_{s_n}(\mathbf{p}_n)(\not{p}_n + m)] \\ &\times [\bar{v}_{r_1}(\bar{\mathbf{k}}_1)(\not{\bar{k}}_1 - m)] \cdots [\bar{v}_{r_{\bar{m}}}(\bar{\mathbf{k}}_{\bar{m}})(\not{\bar{k}}_{\bar{m}} - m)] \\ &\times \langle \Omega | T \{ \Psi(x_1) \cdots \Psi(x_n) \bar{\Psi}(\bar{x}_1) \cdots \bar{\Psi}(\bar{x}_{\bar{n}}) \bar{\Psi}(y_1) \cdots \bar{\Psi}(y_m) \Psi(\bar{y}_1) \cdots \Psi(\bar{y}_{\bar{m}}) \} | \Omega \rangle \\ &\times [(\not{k}_1 + m)u_{r_1}(\mathbf{k}_1)] \cdots [(\not{k}_m + m)u_{r_m}(\mathbf{k}_m)] \\ &\times [(\not{p}_1 - m)v_{s_1}(\bar{\mathbf{p}}_1)] \cdots [(\not{p}_{\bar{n}} - m)v_{s_{\bar{n}}}(\bar{\mathbf{p}}_{\bar{n}})] \\ &\times \left( \frac{i}{\sqrt{Z_2}} \right)^{m+\bar{m}+n+\bar{n}} \end{aligned}$$

From this equation, we can see that the scattering amplitude would vanish unless  $n + \bar{n} = \bar{n} + m$ , which implies the conservation of  $Q$ . The term  $e^{ipx}$  will impose the condition of momentum conservation, and the term  $\not{p} \pm m$  will remove the external legs.

Finally, we list the Feymann rules of Yukawa theory in momentum space as follows:

1. For each incoming electron, draw a solid line with an arrow pointed towards the vertex, and label it with the electron's four-momentum,  $k_i$ .
2. For each outgoing electron, draw a solid line with an arrow pointed away from the vertex, and label it with the electron's four-momentum,  $p_i$ .
3. For each incoming positron, draw a solid line with an arrow pointed away from the vertex, and label it with minus the positron's four-momentum,  $-\bar{k}_i$ .
4. For each outgoing positron, draw a solid line with an arrow pointed towards the vertex, and label it with minus the positron's four-momentum,  $-\bar{p}_i$ .



5. For each incoming scalar, draw a dashed line with an arrow pointed towards the vertex, and label it with the scalar's four-momentum,  $q_i$ .
6. For each outgoing scalar, draw a dashed line with an arrow pointed away from the vertex, and label it with the scalar's four-momentum,  $q'_i$ .
7. The only allowed vertex joins two solid lines, one with an arrow pointing towards it and one with an arrow pointing away from it, and one dashed line (whose arrow can point in either direction). Using this vertex, join up all the external lines, including extra internal lines as needed. In this way, draw all possible diagrams that are topologically inequivalent.
8. Assign each internal line its own four-momentum. Think of the four-momenta as flowing along the arrows, and conserve four-momentum at each vertex. For a tree diagram, this fixes the momenta on all the internal lines.
9. The value of a diagram consists of the following factors:
  - for each incoming or outgoing scalar, 1;
  - for each incoming electron,  $u_r(\mathbf{k})$ ;
  - for each outgoing electron,  $\bar{u}_s(\mathbf{p})$ ;
  - for each incoming positron,  $\bar{v}_{\bar{r}}(\bar{\mathbf{k}})$ ;
  - for each outgoing positron,  $v_{\bar{s}}(\bar{\mathbf{p}})$ ;
  - for each vertex,  $-ig_0$ ;
  - for each internal scalar,  $\frac{-i}{p^2 + M^2 - i\epsilon}$ ;
  - for each internal fermion,  $\frac{i(\not{p} - m)}{p^2 + m^2 - i\epsilon}$
10. Spinor indices are contracted by starting at one end of a fermion line: specifically, the end that has the arrow pointing away from the vertex. The factor associated with the external line is either  $\bar{u}$  or  $\bar{v}$ . Go along the complete fermion line, following the arrows backwards, and write down (in order from left to right) the factors associated with the vertices and propagators that you encounter. The last factor is either a  $u$  or  $v$ . Repeat this procedure for the other fermion lines, if any.
11. The overall sign of a tree diagram is determined by drawing all contributing diagrams in a standard form: all fermion lines horizontal, with their arrows pointing from left to right, and with the left endpoints labeled in the same fixed order (from top to bottom); if the ordering of the labels on the right endpoints of the fermion lines in a given diagram is an even (odd) permutation of an arbitrarily chosen fixed ordering, then the sign of that diagram is positive (negative).
12. Each closed fermion loop contributes an extra minus sign.
13. Value of  $i\mathcal{M}$  is given by a sum over the values of the contributing diagrams.
14.  $\langle f|S|i\rangle = (Z_1)^{\frac{n_s}{2}} (Z_2)^{\frac{n_f}{2}} i\mathcal{M} \delta(\sum p_f - \sum p_i)$



# Chapter 24

## Vector Field



### 24.1 Vector field

Consider a vector field  $A^\mu(x)$ . Here the index  $\mu$  is a vector index that takes on four possible values. Under a Lorentz transformation, we have

$$U(\Lambda)^{-1} A^\mu(x) U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$$

For an infinitesimal transformation, we can write

$$\delta^\mu{}_\nu + \delta\omega^\mu{}_\nu = \delta^\mu{}_\nu + \frac{i}{2} \delta\omega_{\rho\sigma} (S_V^{\rho\sigma})^\mu{}_\nu$$

Here

$$(S_V^{\rho\sigma})^\mu{}_\nu = -i(\eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu)$$

It is obvious that  $A^{\dagger\mu}$  is also a vector field. We know that  $\eta^{\mu\nu}$  is invariant under Lorentz transformation, i.e.

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu}$$

We can use  $\eta^{\mu\nu}$  and its inverse  $\eta_{\mu\nu}$  to raise and lower vector indices of the vector field,

$$A_\mu \equiv \eta_{\mu\nu} A^\nu$$

And we can verify the following equations

$$\Lambda^\mu{}_\nu \Lambda_\mu{}^\rho = \delta_\nu^\rho$$

$$A^\mu(x) = \eta^{\mu\nu} A_\nu(x)$$

$$\Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \eta_{\rho\sigma} = \eta_{\mu\nu}$$

$$U(\Lambda)^{-1} A_\mu(x) U(\Lambda) = \Lambda_\mu{}^\nu A_\nu(\Lambda^{-1}x)$$

Define  $C_i \equiv \frac{1}{2} \epsilon_{ijk} S_V^{jk}, D_i \equiv S_V^{i0}$ . For example, we have

$$(C_3)_\mu{}^\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The eigenvectors of  $C_3$  are

$$[(-1, [(0, 1, -i, 0)], 1), (1, [(0, 1, i, 0)], 1), (0, [(1, 0, 0, 0), (0, 0, 0, 1)], 2)]$$

We further define  $N_i \equiv \frac{1}{2}(C_i - iD_i)$  and  $N_i^\dagger \equiv \frac{1}{2}(C_i + iD_i)$ . For example, we have

$$(N_1)_\mu{}^\nu = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}i \\ 0 & 0 & \frac{1}{2}i & 0 \end{pmatrix}$$

The eigenvectors of  $N_1$  are

$$\left[ \left( -\frac{1}{2}, [(1, 1, 0, 0), (0, 0, 1, -i)], 2 \right), \left( \frac{1}{2}, [(1, -1, 0, 0), (0, 0, 1, i)], 2 \right) \right]$$

And we can conclude that vector is in the  $(2, 2)$  representation of the Lie algebra of the Lorentz group.

## 24.2 Electromagnetic field and gauge invariance

The Lagrangian of EM field is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Here,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{and} \quad A^\mu = (\phi, \mathbf{A})$$

So,

$$F_{0i} = \dot{A}^i + \nabla_i \phi \equiv -E^i \quad \text{and} \quad F_{ij} = \nabla_i A^j - \nabla_j A^i \equiv \epsilon_{ijk} B^k$$

We can derive the equation of motion of the EM field by variation method,

$$\partial_\mu F^{\mu\nu} = 0$$

It can be rewritten in terms of  $\mathbf{E}$  and  $\mathbf{B}$ , i.e. Maxwell equations:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 & \frac{\partial \mathbf{E}}{\partial t} &= \nabla \times \mathbf{B} \\ \nabla \cdot \mathbf{B} &= 0 & \frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \end{aligned}$$

The massless vector field  $A_\mu$  has 4 components, which would naively seem to tell us that the gauge field has 4 degrees of freedom. But there are two related comments which will ensure that quantizing the gauge field  $A_\mu$  gives rise to 2 degrees of freedom, rather than 4.

- The field  $A_0$  has no kinetic term  $\dot{A}_0$  in the Lagrangian: it is not dynamical. This means that if we are given some initial data  $A_i$  and  $\dot{A}_i$  at a time  $t_0$ , then the field  $A_0$  is fully determined by the equation of motion  $\nabla \cdot \mathbf{E} = 0$ , which, expanding out, reads

$$\nabla^2 A_0 = \nabla \cdot \frac{\partial \mathbf{A}}{\partial t}$$

So  $A_0$  is not independent: we don't get to specify  $A_0$  on the initial time slice.



- If we transform the EM field as

$$A_\mu \rightarrow A_\mu + \partial_\mu \lambda(x)$$

we can derive that

$$F_{\mu\nu} \rightarrow F_{\mu\nu} \quad \mathcal{L} \rightarrow \mathcal{L}$$

The seemed infinite number of symmetries, one for each function  $\lambda(x)$ , is to be viewed as a redundancy in our description. That is, two states related by a gauge symmetry are to be identified: they are the same physical state. One way to see that this interpretation is necessary is to notice that Maxwell's equations are not sufficient to specify the evolution of  $A_\mu$ . The equations read,

$$(\eta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) A^\nu = 0$$

But the operator  $(\eta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu)$  is not invertible: it annihilates any function of the form  $\partial_\mu \lambda$ . This means that given any initial data, we have no way to uniquely determine  $A_\mu$  at a later time since we can't distinguish between  $A_\mu$  and  $A_\mu + \partial_\mu \lambda$ . This would be problematic if we thought that  $A_\mu$  is a physical object. However, if we're happy to identify  $A_\mu$  and  $A_\mu + \partial_\mu \lambda$  as corresponding to the same physical state, then our problems disappear.

The picture that emerges for the theory of electromagnetism is of an enlarged phase space, foliated by gauge orbits. All states that lie along a given gauge orbit can be reached by a gauge transformation and are identified. To make progress, we pick a representative from each gauge orbit. It doesn't matter which representative we pick after all, they're all physically equivalent. But we should make sure that we pick a "good" gauge, in which we cut the orbits. Here we'll look at two different gauges:

- Coulomb Gauge:  $\nabla \cdot \mathbf{A} = 0$

We can make use of the residual gauge transformations in Lorentz gauge to pick  $\nabla \cdot \dot{\mathbf{A}} = 0$ . We have as a consequence  $A_0 = 0$ . Coulomb gauge is sometimes called radiation gauge.

- Lorentz Gauge:  $\partial^\mu A_\mu = 0$

In fact this condition doesn't pick a unique representative from the gauge orbit. We're always free to make further gauge transformations with  $\partial^\mu \partial_\mu \lambda = 0$ , which also has non-trivial solutions. As the name suggests, the Lorentz gauge has the advantage that it is Lorentz invariant.

## 24.3 Canonical quantization of EM field

### 24.3.1 Canonical quantization in Coulomb gauge

Canonical momentum and Hamiltonian

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \quad \pi^i = \frac{\partial \mathcal{L}}{\partial (\partial_0 A_i)} = \dot{A}^i + \nabla_i \phi = -E^i$$





$$\mathcal{H} = \frac{1}{2}(\boldsymbol{\pi}^2 + \mathbf{B}^2) + (\boldsymbol{\pi} \cdot \nabla)A_0$$

Integration by parts can give

$$H = \int d^3x \frac{1}{2}(\boldsymbol{\pi}^2 + \mathbf{B}^2)$$

### Momentum and angular momentum

$$P^0 = H \quad \vec{P} = \int -\boldsymbol{\pi} \vec{\nabla} \mathbf{A} d^3x$$

$$\vec{J} = - \int \boldsymbol{\pi} (\vec{x} \times \vec{\nabla} + i\vec{C}) \mathbf{A} d^3x \quad \vec{S} = -i \int \boldsymbol{\pi} \vec{C} \mathbf{A} d^3x$$

### Canonical quantization

In Coulomb gauge, we have

$$A_0 = \pi^0 = 0 \quad \pi^i = \dot{A}^i$$

Three pairs of  $A_i$  and  $\pi^i$  are not independent from each other. They must satisfy the constraint equations

$$\nabla \cdot \mathbf{A} = 0 \quad \nabla \cdot \boldsymbol{\pi} = 0$$

A reasonable quantization condition can be written as

$$[A_i(\mathbf{x}, t), A_j(\mathbf{x}', t)] = 0 \quad [\pi^i(\mathbf{x}, t), \pi^j(\mathbf{x}', t)] = 0$$

$$[A_i(\mathbf{x}, t), \pi^j(\mathbf{x}', t)] = i \left( \delta_i^j - \frac{\partial_i \partial^j}{\nabla^2} \right) \delta(\mathbf{x} - \mathbf{x}') \equiv i \int \frac{d^3k}{(2\pi)^3} \left( \delta_i^j - \frac{k_i k^j}{\mathbf{k}^2} \right) e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')}$$

In this case, we can verify that

$$\dot{A}_i = -i[A_i(\mathbf{x}, t), H] = \pi_i(\mathbf{x}, t)$$

$$\dot{\pi}^i = -i[\pi^i(\mathbf{x}, t), H] = \nabla^2 A^i(\mathbf{x}, t)$$

It is constant with the field equation we derive from Euler-Lagrange equation.

### Fourier expansion

$$\mathbf{A}(x) = \sum_{r=\pm} \int \widetilde{dp} [a_r(\mathbf{p}) \boldsymbol{\epsilon}_r(\mathbf{p}) e^{ipx} + a_r^\dagger(\mathbf{p}) \boldsymbol{\epsilon}_r^*(\mathbf{p}) e^{-ipx}]$$

And we can derive from constraint condition that

$$\boldsymbol{\epsilon} \cdot \mathbf{p} = 0$$

We will choose  $\boldsymbol{\epsilon}$  to satisfy that

$$\boldsymbol{\epsilon}_r \cdot \boldsymbol{\epsilon}_s^* = \delta_{rs}$$

So, the completeness relation for the polarization vectors is

$$\sum_{r=\pm} \epsilon_r^i(\mathbf{p}) \epsilon_r^{*j}(\mathbf{p}) = \delta^{ij} - \frac{p^i p^j}{|\mathbf{p}|^2}$$



**Example:** If  $\mathbf{p} = (0, 0, p)$ , we usually choose

$$\epsilon_+ = \frac{1}{\sqrt{2}}(1, i, 0) \quad \epsilon_- = \frac{1}{\sqrt{2}}(1, -i, 0)$$

$\epsilon_+$  corresponds to anticlockwise rotation and it is the eigenvectors of the space-part of  $C_3$  with eigenvalue  $+1$ .  $\epsilon_-$  corresponds to clockwise rotation and it is eigenvector of the space-part of  $C_3$  with eigenvalue  $-1$ .

We can further derive from above discussion that

$$\pi(x) = -i \sum_{r=\pm} \int \widetilde{dp} \omega [a_r(\mathbf{p}) \epsilon_r(\mathbf{p}) e^{ipx} - a_r^\dagger(\mathbf{p}) \epsilon_r^*(\mathbf{p}) e^{-ipx}]$$

$$a_r(\mathbf{p}) = \epsilon_r^* \int d^3x e^{-ikx} (i\boldsymbol{\pi} + \omega \mathbf{A})$$

$$a_r^\dagger(\mathbf{p}) = \epsilon_r \int d^3x e^{ikx} (-i\boldsymbol{\pi} + \omega \mathbf{A})$$

$$[a_r(\mathbf{p}), a_{r'}(\mathbf{p}')] = 0 \quad [a_r^\dagger(\mathbf{p}), a_{r'}^\dagger(\mathbf{p}')] = 0 \quad [a_r(\mathbf{p}), a_{r'}^\dagger(\mathbf{p}')] = (2\pi)^3 2\omega \delta_{rr'} \delta(\mathbf{p} - \mathbf{p}')$$

### Operator represented by $a$ and $a^\dagger$

Define that

$$N(\mathbf{p}, r) \equiv a_r^\dagger(\mathbf{p}) a_r(\mathbf{p})$$

So, we can derive

$$H = \sum_{r=\pm} \int \widetilde{dp} \omega N(\mathbf{p}, r) + 2\mathcal{E}_0 V$$

$$\vec{P} = \sum_{r=\pm} \int \widetilde{dp} \vec{p} N(\mathbf{p}, r)$$

$$\vec{S} = \sum_{r,s=\pm} \int \widetilde{dp} \frac{1}{2} (\epsilon_s^* \vec{C} \epsilon_r - \epsilon_r \vec{C} \epsilon_s^*) a_s^\dagger(\mathbf{p}) a_r(\mathbf{p})$$

From above equation, we can say that  $a_r^\dagger(\mathbf{p})$  create an photon with energy  $\omega$ , momentum  $\mathbf{p}$  and spin angular momentum along the direction of momentum  $r$ .

### Propagator

$$G_F(x-y)_{ij} \equiv \langle 0 | T A_i(x) A_j(y) | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2 - i\epsilon} \left( \delta_{ij} - \frac{p_i p_j}{|\mathbf{p}|^2} \right) e^{ip(x-y)}$$

## 24.3.2 Canonical quantization in Lorentz gauge

### Undefined metric formalism

Modify the Maxwell Lagrangian introducing a new term

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2$$



The equations of motion are now

$$\partial^2 A_\mu - (1 - \frac{1}{\xi}) \partial^\mu (\partial \cdot A) = 0$$

Canonical momentums are

$$\pi^0 = \frac{1}{\xi} \partial \cdot A = \frac{1}{\xi} (-\dot{A}_0 + \partial_i A^i) \quad \pi^i = \dot{A}^i + \nabla^i A^0 = -E^i$$

Hamiltonian is

$$\mathcal{H} = \frac{1}{2} (\boldsymbol{\pi}^2 + \mathbf{B}^2 - \xi \pi^0 \pi^0) + (\boldsymbol{\pi} \cdot \boldsymbol{\nabla}) A_0 + \pi^0 (\boldsymbol{\nabla} \cdot \mathbf{A})$$

$$H = \int \left[ \frac{1}{2} (\boldsymbol{\pi}^2 + \mathbf{B}^2 - \xi \pi^0 \pi^0) - A_0 (\boldsymbol{\nabla} \cdot \boldsymbol{\pi}) + \pi^0 (\boldsymbol{\nabla} \cdot \mathbf{A}) \right] d^3x$$

We remark that the above Lagrangian and the equations of motion, reduce to Maxwell theory in the gauge  $\partial \cdot A = 0$ . This why we say that our choice corresponds to a class of Lorenz gauges with parameter  $\xi$ . With this abuse of language (in fact we are not setting  $\partial \cdot A = 0$ , otherwise the problems would come back) the value of  $\xi = 1$  is known as the Feynman gauge and  $\xi = 0$  as the Landau gauge. From now on we will take the case of the so-called Feynman gauge, where  $\xi = 1$ . Then the equation of motion coincide with the Maxwell theory in the Lorenz gauge. In Feynmann gauge, the canonical quantization conditions can be written as

$$[A_\mu(\mathbf{x}, t), A_\nu(\mathbf{x}', t)] = 0 \quad [\pi^\mu(\mathbf{x}, t), \pi^\nu(\mathbf{x}', t)] = 0 \quad [A_\mu(\mathbf{x}, t), \pi^\nu(\mathbf{x}', t)] = i\delta_\mu^\nu \delta(\mathbf{x} - \mathbf{x}')$$

we can also derive that

$$[\dot{A}_\mu(\mathbf{x}, t), \dot{A}_\nu(\mathbf{x}', t)] = 0 \quad [A_\mu(\mathbf{x}, t), \dot{A}_\nu(\mathbf{x}', t)] = i\eta_{\mu\nu} \delta(\mathbf{x} - \mathbf{x}')$$

### Fourier expansion

$$A(x) = \sum_{\lambda=0}^3 \int \widetilde{dp} [a_\lambda(\mathbf{p}) \epsilon_\lambda(\mathbf{p}) e^{ipx} + a_\lambda^\dagger(\mathbf{p}) \epsilon_\lambda^*(\mathbf{p}) e^{-ipx}]$$

where  $\epsilon_{\lambda\mu}$  are a set of four independent 4-vectors. We will now make a choice for these 4-vectors. We choose  $\epsilon_{1\mu}$  and  $\epsilon_{2\mu}$  orthogonal to  $k^\mu$  and  $n^\mu$ , such that

$$\epsilon_{\lambda\mu} \epsilon_\lambda^{*\mu} = \delta_{\lambda\lambda'} \quad \lambda, \lambda' = 1, 2$$

After, we choose  $\epsilon_{3\mu}$  in the plane  $(k^\mu, n^\mu)$  and perpendicular to  $n^\mu$  such that

$$\epsilon_{3\mu} n^\mu = 0 \quad \epsilon_{3\mu} \epsilon_3^{*\mu} = 1$$

Finally we choose  $\epsilon_{0\mu} = n_\mu$ . The vectors  $\epsilon_{1\mu}$  and  $\epsilon_{2\mu}$  are called transverse polarizations, while  $\epsilon_{3\mu}$  and  $\epsilon_{0\mu}$  longitudinal and scalar polarizations, respectively.

In general we can show that

$$\epsilon_\lambda \cdot \epsilon_{\lambda'}^* = \eta_{\lambda\lambda'} \quad \eta^{\lambda\lambda'} \epsilon_{\lambda\mu} \epsilon_{\lambda'\nu}^* = \eta_{\mu\nu}$$



We can further derive from above discussion that

$$\begin{aligned}\dot{A}(x) &= -i \sum_{\lambda=0}^3 \int \widetilde{dp} \omega [a_{\lambda}(\mathbf{p}) \epsilon_{\lambda}(\mathbf{p}) e^{ipx} - a_{\lambda}^{\dagger}(\mathbf{p}) \epsilon_{\lambda}^*(\mathbf{p}) e^{-ipx}] \\ a_{\lambda}(\mathbf{p}) &= \eta_{\lambda\lambda'} \epsilon_{\lambda'}^* \cdot \int d^3x e^{-ipx} (i\dot{A} + \omega A) \\ a_{\lambda}^{\dagger}(\mathbf{p}) &= \eta_{\lambda\lambda'} \epsilon_{\lambda'} \cdot \int d^3x e^{ipx} (-i\dot{A} + \omega A) \\ [a_{\lambda}(\mathbf{p}), a_{\lambda'}(\mathbf{p}')] &= 0 \quad [a_{\lambda}^{\dagger}(\mathbf{p}), a_{\lambda'}^{\dagger}(\mathbf{p}')] = 0 \quad [a_{\lambda}(\mathbf{p}), a_{\lambda'}^{\dagger}(\mathbf{p}')] = (2\pi)^3 2\omega \eta_{\lambda\lambda'} \delta(\mathbf{p} - \mathbf{p}')\end{aligned}$$

### Indefinite metric problem

We Introduce the vacuum state defined by

$$a_{\lambda}(\mathbf{p})|0\rangle = 0$$

To see the problem with the sign we construct the one-particle state with scalar polarization, that is

$$|1\rangle = \int \widetilde{dp} f(p) a_0^{\dagger}(\mathbf{p}) |0\rangle$$

and calculate its norm

$$\langle 1|1\rangle = -\langle 0|0\rangle \int \widetilde{dp} |f(p)|^2$$

The state  $|1\rangle$  has a negative norm.

To solve this problem we note that we are not working anymore with the classical Maxwell theory because we modified the Lagrangian. What we would like to do is to impose the condition  $\partial \cdot A = 0$ , but that is impossible as an equation for operators. We can, however, require that condition on a weaker form, as a condition only to be verified by the physical states.

More specifically, we require that the part of  $\partial \cdot A$  that contains the annihilation operator (positive frequencies) annihilates the physical states,

$$\partial^{\mu} A_{\mu}^{+} |\psi\rangle = 0$$

The states  $|\psi\rangle$  can be written in the form

$$|\psi\rangle = |\psi_T\rangle |\phi\rangle$$

where  $|\psi_T\rangle$  is obtained from the vacuum with creation operators with transverse polarization and  $|\phi\rangle$  with scalar and longitudinal polarization.

$\partial^{\mu} A_{\mu}^{+}$  contains only scalar and longitudinal polarizations

$$\partial^{\mu} A_{\mu}^{+} = i \sum_{\lambda=0,3} \int \widetilde{dp} a_{\lambda}(\mathbf{p}) (p \cdot \epsilon_{\lambda}(\mathbf{p})) e^{ipx}$$

Therefore the previous condition becomes

$$i \sum_{\lambda=0,3} (p \cdot \epsilon_{\lambda}(\mathbf{p})) a_{\lambda}(\mathbf{p}) |\phi\rangle = 0$$



The condition is equivalent to,

$$(a_0(\mathbf{p}) - a_3(\mathbf{p}))|\phi\rangle = 0$$

We can construct  $|\phi\rangle$  as a linear combination of states  $|\phi\rangle$  with  $n$  scalar or longitudinal photons:

$$|\phi\rangle = C_0|\phi_0\rangle + C_1|\phi\rangle + \dots \quad \text{Here, } |\phi_0\rangle \equiv |0\rangle$$

The states  $|\phi_n\rangle$  are eigenstates of the operator number for scalar or longitudinal photons

$$N'|\phi_n\rangle = n|\phi_n\rangle$$

where,

$$N' = \int \widetilde{dp} [a_3^\dagger(\mathbf{p})a_3(\mathbf{p}) - a_0^\dagger(\mathbf{p})a_0(\mathbf{p})]$$

Then

$$n\langle\phi_n|\phi_n\rangle = \langle\phi_n|N'|\phi_n\rangle = 0$$

This means that

$$\langle\phi_n|\phi_n\rangle = \delta_{n0}$$

that is, for  $n \neq 0$ , the state  $|\phi_n\rangle$  has zero norm. We have then for the general state  $|\phi\rangle$ ,

$$\langle\phi|\phi\rangle = |C_0|^2 \geq 0$$

and the coefficients  $C_i (i = 1, 2, \dots)$  are arbitrary.

### Operator represented by $a$ and $a^\dagger$

Define that

$$N'(\mathbf{p}) \equiv a_3^\dagger(\mathbf{p})a_3(\mathbf{p}) - a_0^\dagger(\mathbf{p})a_0(\mathbf{p})$$

$$N(\mathbf{p}, 1) \equiv a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) \quad N(\mathbf{p}, 2) \equiv a_2^\dagger(\mathbf{p})a_2(\mathbf{p}) \quad N_T(\mathbf{p}) \equiv N(\mathbf{p}, 1) + N(\mathbf{p}, 2)$$

We have that

$$\langle\psi|N'(\mathbf{p})|\psi\rangle = 0 \quad \langle\psi|N_T(\mathbf{p})|\psi\rangle = \langle\psi_T|N_T(\mathbf{p})|\psi_T\rangle$$

We can derive

$$H = \int \widetilde{dp} \omega [N'(\mathbf{p}) + N_T(\mathbf{p})] + 2\mathcal{E}_0 V$$

$$\vec{P} = \int \widetilde{dp} \vec{p} [N'(\mathbf{p}) + N_T(\mathbf{p})]$$

So, the arbitrariness of  $C_i (i = 1, 2, \dots)$  does not affect the physical observables. Only the physical transverse polarizations contribute to the result. Two states that differ only in their timelike and longitudinal photon content,  $|\phi_n\rangle$  with  $n \geq 1$  are said to be physically equivalent. We can think of the gauge symmetry of the classical theory as descending to the Hilbert space of the quantum theory.

It is important to note that although for the average values of the physical observables only the transverse polarizations contribute, the scalar and longitudinal polarizations are necessary for the consistency of the theory. In particular they show up when we consider complete sums over the intermediate states.



### Propagator

$$G_F(x-y)_{\mu\nu} \equiv \langle 0|T A_\mu(x) A_\nu(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{p^2 - i\epsilon} e^{ip(x-y)}$$

It is easy to verify that  $G_F(x-y)_{\mu\nu}$  is the Green's function of the equation of motion, that for  $\xi = 1$  is the wave equation, that is

$$\partial^2 G_F(x-y)_{\mu\nu} = i\eta_{\mu\nu} \delta(x-y)$$

For the general case,  $\xi \neq 0$ , the equal times commutation relations are more complicated. And the propagator will be

$$G_F(x-y)_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \left[ \frac{-i\eta_{\mu\nu}}{p^2 - i\epsilon} + i(1-\xi) \frac{p_\mu p_\nu}{(p^2 - i\epsilon)^2} \right] e^{ip(x-y)}$$

## 24.4 Perturbation theory for canonical quantization

### 24.4.1 Lagrangian of QED

The Lagrangian of QED is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\not{D} - m_0)\Psi + e_0 j^\mu A_\mu,$$

where  $j^\mu \equiv \bar{\Psi}\gamma^\mu\Psi$ . Usually, we also define a covariant derivative,

$$D_\mu \Psi \equiv \partial_\mu \Psi - ie_0 A_\mu \Psi$$

So, the Lagrangian can also be written as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\not{D} - m_0)\Psi$$

The Lagrangian is invariant under the gauge transformation

$$A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e_0} \partial_\mu \alpha(x) \quad \Psi(x) \rightarrow e^{i\alpha(x)} \Psi(x)$$

### 24.4.2 Coulomb gauge

The constraint equations are

$$\nabla \cdot \mathbf{A} = 0 \quad \nabla^2 A_0 = e_0 j^0$$

The solution for  $A_0$  is

$$A_0(\mathbf{x}, t) = -e_0 \int d^3x' \frac{j^0(\mathbf{x}', t)}{4\pi|\mathbf{x} - \mathbf{x}'|}$$

Finally, we can derive that

$$H = H_D + H_M + H_{\text{int}}$$



where,

$$H_D = \int d^3x \left[ -\Pi(\vec{\alpha} \cdot \vec{\nabla} + i\beta m)\Psi \right] \quad H_M = \int d^3x \frac{1}{2}(\pi^2 + \mathbf{B}^2)$$

$$H_{\text{int}} = \int d^3x \left[ -e_0 \mathbf{j} \cdot \mathbf{A} + \frac{e_0^2}{2} \int d^3x' \frac{j^0(\mathbf{x})j^0(\mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|} \right]$$

The perturbation expansion of correlation functions is

$$\langle \Omega | T \{ \Psi(x) \bar{\Psi}(y) A(z) \} | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | T \left\{ \Psi_I(x) \bar{\Psi}_I(y) A_I(z) \exp \left[ -i \int_{-T}^T dt H_I \right] \right\} | 0 \rangle}{\langle 0 | T \left\{ \exp \left[ -i \int_{-T}^T dt H_I \right] \right\} | 0 \rangle}$$

Here,  $\int_{-T}^T dt H_I$  can be written as

$$\left[ - \int d^4x e_0 \bar{\Psi}_I \gamma \Psi_I \cdot \mathbf{A}_I \right] + \left[ \int d^4x \int d^4x' \frac{e_0^2 \delta(t-t')}{4\pi|\mathbf{x} - \mathbf{x}'|} \frac{1}{2} \bar{\Psi}_I(\mathbf{x}, t) \gamma^0 \Psi_I(\mathbf{x}, t) \bar{\Psi}_I(\mathbf{x}', t') \gamma^0 \Psi_I(\mathbf{x}', t') \right]$$

Wick's theorem for photons takes is similar to that in  $\phi^4$  theory:

$$T \{ A_I(x_1) A_I(x_2) A_I(x_3) \cdots \} = N \{ A_I(x_1) A_I(x_2) A_I(x_3) \cdots + \text{all possible contractions} \}$$

**Example:**

$$\begin{aligned} & \langle 0 | T \{ A_{Ii}(x_1) A_{Ij}(x_2) A_{Ik}(x_3) A_{Il}(x_4) \} | 0 \rangle \\ &= G_F(x_1 - x_2)_{ij} G_F(x_3 - x_4)_{kl} + G_F(x_1 - x_3)_{ik} G_F(x_2 - x_4)_{jl} + G_F(x_1 - x_4)_{il} G_F(x_2 - x_3)_{jk} \end{aligned}$$

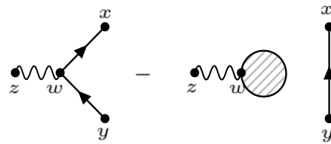
Now we can derive the Feynman rule for QED theory. Firstly, we evaluate this term,

$$\langle 0 | T \left\{ \Psi_{Ia}(x) \bar{\Psi}_{Ib}(y) A_{Ii}(z) (ie_0) \int d^4w \bar{\Psi}_I(w) \gamma \Psi_I(w) \cdot A_I(w) \right\} | 0 \rangle$$

After contraction, it can be written as

$$\begin{aligned} & - (ie_0) S_F(x - y)_{ab} \int d^4w G_F(z - w)_{ik} \text{Tr}[\gamma^k S_F(w - w)] \\ & + (ie_0) \int d^4w G_F(w - z)_{ik} [S_F(x - w) \gamma^k S_F(w - y)]_{ab} \end{aligned}$$

It can be represented by the following Feynman diagram.



**Figure 24.1:** Feynman diagram representation of perturbation expansion

Secondly, we evaluate this term,

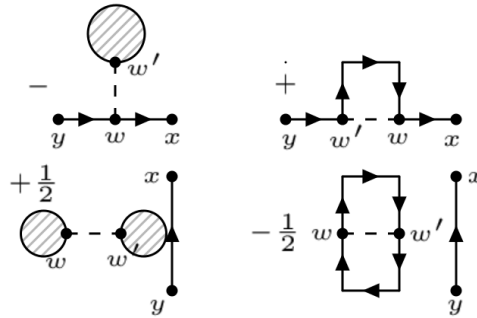
$$\langle 0 | T \left\{ \Psi_{Ia}(x) \bar{\Psi}_{Ib}(y) \int d^4w \int d^4w' \frac{-ie_0^2 \delta(w^0 - w'^0)}{4\pi|\mathbf{w} - \mathbf{w}'|} \frac{1}{2} \bar{\Psi}_I(w) \gamma^0 \Psi_I(w) \bar{\Psi}_I(w') \gamma^0 \Psi_I(w') \right\} | 0 \rangle$$



After contraction, it can be written as

$$\begin{aligned}
& - (ie_0)^2 \int d^4w d^4w' \frac{i\delta(w^0 - w'^0)}{4\pi|\mathbf{w} - \mathbf{w}'|} [S_F(x - w)\gamma^0 S_F(w - y)]_{ab} \text{Tr}[\gamma^0 S_F(w' - w')] \\
& + (ie_0)^2 \int d^4w d^4w' \frac{i\delta(w^0 - w'^0)}{4\pi|\mathbf{w} - \mathbf{w}'|} [S_F(x - w)\gamma^0 S_F(w - w')\gamma^0 S_F(w' - y)]_{ab} \\
& + \frac{1}{2}(ie_0)^2 S_F(x - y)_{ab} \int d^4w d^4w' \frac{i\delta(w^0 - w'^0)}{4\pi|\mathbf{w} - \mathbf{w}'|} \text{Tr}[\gamma^0 S_F(w - w)] \text{Tr}[\gamma^0 S_F(w' - w')] \\
& - \frac{1}{2}(ie_0)^2 S_F(x - y)_{ab} \int d^4w d^4w' \frac{i\delta(w^0 - w'^0)}{4\pi|\mathbf{w} - \mathbf{w}'|} \text{Tr}[\gamma^0 S_F(w - w')\gamma^0 S_F(w' - w)]
\end{aligned}$$

It can be represented by the following Feynman diagram.



**Figure 24.2:** Feynman diagram representation of perturbation expansion

Before we write down Feynmann rules, we notice that the offending non-local interaction comes from the  $A_0$  component of the gauge field, we could try to redefine the propagator to include a  $G_F(x - y)_{00}$  piece which will capture this term. We can verify that

$$\frac{i\delta(w^0 - w'^0)}{4\pi|\mathbf{w} - \mathbf{w}'|} = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{ip(w-w')}}{|\mathbf{p}|^2}$$

So we can combine the non-local interaction with the transverse photon propagator by defining a new photon propagator

$$G_F(p)_{\mu\nu} \equiv \begin{cases} \frac{i}{|\mathbf{p}|^2} & \mu, \nu = 0 \\ \frac{-i}{p^2 - i\epsilon} \left( \delta_{ij} - \frac{p_i p_j}{|\mathbf{p}|^2} \right) & \mu = i \neq 0, \nu = j \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

With this propagator, the wavy photon line now carries a  $\mu\nu = 0, 1, 2, 3$  index, with the extra  $\mu = 0$  component taking care of the instantaneous interaction.

The Feynman rules for QED are:

1. For each Fermion propagator from  $y$  to  $x$ ,  $P = S_F(x - y)$
2. For each vector propagator,  $P = G_F(x - y)$
3. For each vertex,  $V = (ie_0\gamma^\mu) \int d^4w$
4. For each external point,  $E = 1$
5. Divided by the symmetry factor





### 24.4.3 Lorentz Gauge

In Lorentz gauge,

$$\mathcal{H}_{int} = -e_0 \bar{\Psi} \gamma^\mu \Psi A_\mu$$

The Feynman rules for QED in Lorentz gauge will be the same as that in Coulomb gauge expect for that the vector propagator will be

$$G_F(p)_{\mu\nu} = \frac{-i\eta_{\mu\nu}}{p^2 - i\epsilon} + i(1 - \xi) \frac{p_\mu p_\nu}{(p^2 - i\epsilon)^2}$$

Especially, for Feynman gauge  $\xi = 1$ , we have

$$G_F(p)_{\mu\nu} = \frac{-i\eta_{\mu\nu}}{p^2 - i\epsilon}$$

## 24.5 Path integral quantization

### 24.5.1 Path integral formulation for free EM field

The correlation function is given by

$$\langle \Omega | T A_H(x_1) A_H(x_2) | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\int \mathcal{D}A \exp \left[ i \int_{-T}^T d^4x \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) \right] A(x_1) A(x_2)}{\int \mathcal{D}A \exp \left[ i \int_{-T}^T d^4x \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) \right]}$$

The generating function is

$$Z[J] = \int \mathcal{D}A \exp \left[ i \int d^4x \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) + J^\mu A_\mu \right]$$

We can verify that

$$S = \int d^4x \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) = \frac{1}{2} \int d^4x A_\mu(x) (\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu(x)$$

Notice that  $(\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu)$  is singular, since for any  $\alpha(x)$ ,

$$(\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu) \partial_\mu \alpha(x) = 0$$

This difficulty is due to gauge invariance:  $\alpha(x)$  is gauge equivalent to 0. The functional is badly defined because we are redundantly integrating over a continuous infinity of physically equivalent field configurations. To fix the problem, we would like to isolate the interesting part of the functional integral, which counts each physical configuration only once.

Let  $G(A)$  be some function that we wish to set equal zero as a gauge-fixing condition. We could constrain the functional integral to cover only the configurations with  $G(A) = 0$  by inserting a functional delta function,  $\delta(G(A))$ . To do so, we insert 1 in the path integral:

$$1 = \int \mathcal{D}\alpha(x) \delta(G(A(\alpha))) \det \left( \frac{\delta G}{\delta \alpha} \right)$$



where,

$$A_\mu(\alpha(x)) = A_\mu(x) + \frac{1}{e_0} \partial_\mu \alpha(x)$$

We set the gauge fixing function as  $G(A) = \partial^\mu A_\mu - \omega(x)$ , so  $G(A(\alpha)) = \partial^\mu A_\mu + \frac{1}{e_0} \partial^2 \alpha - \omega(x)$ . It is obvious that  $\det\left(\frac{\delta G}{\delta \alpha}\right)$  is equivalent to  $\det(\partial^2)/e_0$ , which is independent of  $A$ . So,

$$Z_0[A] = \det\left(\frac{\delta G}{\delta \alpha}\right) \int \mathcal{D}\alpha \int \mathcal{D}A e^{iS[A]} \delta(G(A(\alpha)))$$

Now change variables from  $A$  to  $A(\alpha)$ . This is a simple shift, so  $\mathcal{D}A = \mathcal{D}A(\alpha)$ . Also, by gauge invariance,  $S[A] = S[A(\alpha)]$ . Since  $A(\alpha)$  is now just a dummy integration variable, we can rename it back to  $A$ , so

$$Z_0[A] = \det\left(\frac{\delta G}{\delta \alpha}\right) \int \mathcal{D}\alpha \int \mathcal{D}A e^{iS[A]} \delta(\partial^\mu A_\mu - \omega(x))$$

Since the above equation is hold for any  $\omega(x)$ , so we have

$$\begin{aligned} Z_0[A] &= N(\xi) \int \mathcal{D}\omega \exp\left[-i \int d^4x \frac{\omega^2}{2\xi}\right] \det(\partial^2) \int \mathcal{D}\alpha \int \mathcal{D}A e^{iS[A]} \delta(\partial^\mu A_\mu - \omega(x)) \\ &= N(\xi) \frac{\det(\partial^2)}{e_0} \int \mathcal{D}\alpha \int \mathcal{D}A e^{iS[A]} \exp\left[-i \int d^4x \frac{1}{2\xi} (\partial^\mu A_\mu)^2\right] \\ &= W(\xi) \int \mathcal{D}A \exp\left[\frac{i}{2} \int d^4x A_\mu (\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu + \frac{1}{\xi} \partial^\mu \partial^\nu) A_\nu\right] \end{aligned}$$

And we rewrite the generating function as

$$Z[J] = W(\xi) \int \mathcal{D}A \exp\left[\frac{i}{2} \int d^4x A_\mu (\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu + \frac{1}{\xi} \partial^\mu \partial^\nu) A_\nu + J^\mu A_\mu\right]$$

Define

$$A'(x) = A(x) - i \int d^4y G_F(x-y) J(y)$$

Recall that

$$(\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu + \frac{1}{\xi} \partial^\mu \partial^\nu) G_F(x-y)_{\nu\rho} = i \delta_\rho^\mu \delta(x-y)$$

We can derive that

$$Z[J] = Z_0[A] \exp\left[-\frac{1}{2} \int d^4x d^4y J^\mu(x) G_F(x-y)_{\mu\nu} J^\nu(y)\right]$$

The two point correlation functions are

$$\langle 0|T A_H(x_1) A_H(x_2)|0\rangle = Z_0^{-1} \left(-i \frac{\delta}{\delta J(x_1)}\right) \left(-i \frac{\delta}{\delta J(x_2)}\right) Z[J] \Big|_{J=0} = G_F(x_1 - x_2)$$



### 24.5.2 Perturbation theory for path integral quantization

We use QED theory as an example.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\cancel{\partial} - m_0)\Psi + e_0\bar{\Psi}\gamma^\mu\Psi A_\mu$$

As we stated in section 4.1, the Lagrangian of QED is also invariant under a general gauge transformation. And we also notice that the measure  $\mathcal{D}\Psi\mathcal{D}\bar{\Psi}$  is invariant under gauge transformation. By the similar method, we can show that

$$Z_0 \equiv \int \mathcal{D}A\mathcal{D}\Psi\mathcal{D}\bar{\Psi} e^{iS[A,\Psi,\bar{\Psi}]} = W(\xi) \int \mathcal{D}A\mathcal{D}\Psi\mathcal{D}\bar{\Psi} e^{iS[A,\Psi,\bar{\Psi}]} \exp\left[-i \int d^4x \frac{1}{2\xi} (\partial^\mu A_\mu)^2\right]$$

So, define

$$\mathcal{L}_0 \equiv -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + \bar{\Psi}(i\cancel{\partial} - m_0)\Psi \quad \mathcal{L}_1 \equiv e_0\bar{\Psi}\gamma^\mu\Psi A_\mu$$

$$\begin{aligned} Z[J] &= W(\xi) \int \mathcal{D}A\mathcal{D}\bar{\Psi}\mathcal{D}\Psi e^{i \int d^4x [\mathcal{L}_0 + \mathcal{L}_1 + JA + \bar{\eta}\Psi + \bar{\Psi}\eta]} \\ &= W(\xi) e^{i \int d^4x \mathcal{L}_1(\frac{1}{i} \frac{\delta}{\delta J(x)}, \frac{1}{i} \frac{\delta}{\delta \bar{\eta}(x)}, -\frac{1}{i} \frac{\delta}{\delta \eta(x)})} \int \mathcal{D}\phi \mathcal{D}\bar{\Psi} \mathcal{D}A e^{i \int d^4y [\mathcal{L}_0 + JA + \bar{\eta}\Psi + \bar{\Psi}\eta]} \\ &\propto e^{i \int d^4x \mathcal{L}_1(\frac{1}{i} \frac{\delta}{\delta J(x)}, \frac{1}{i} \frac{\delta}{\delta \bar{\eta}(x)}, -\frac{1}{i} \frac{\delta}{\delta \eta(x)})} \exp\left[- \int d^4y d^4z \frac{1}{2} J(y) G_F(y-z) J(z) + \bar{\eta}(y) S_F(y-z) \eta(z)\right] \\ &= \sum_{V=0}^{\infty} \frac{1}{V!} [ie_0 \int d^4x (\frac{1}{i} \frac{\delta}{\delta J^\mu(x)} \cdot -\frac{1}{i} \frac{\delta}{\delta \eta(x)} \cdot \gamma^\mu \cdot \frac{1}{i} \frac{\delta}{\delta \bar{\eta}(x)})]^V \\ &\times \sum_{P_1=0}^{\infty} \frac{1}{P_1!} [-\frac{1}{2} \int d^4y_1 d^4z_1 J(y_1) G_F(y_1 - z_1) J(z_1)]^{P_1} \\ &\times \sum_{P_2=0}^{\infty} \frac{1}{P_2!} [- \int d^4y_2 d^4z_2 \bar{\eta}(y_2) S_F(y_2 - z_2) \eta(z_2)]^{P_2} \end{aligned}$$

If we focus on a term with particular values of  $V$ ,  $P_1$  and  $P_2$ , then the number of surviving scalar sources is  $E_1 = 2P_1 - V$ , the number of surviving fermion sources is  $E_2 = 2P_2 - 2V$ . We can introduce Feynman diagrams as in the  $\phi^4$  theory. In these diagrams, a wavy line segment stands for a vector propagator  $G_F(x - y)$ , a line with an arrow pointing from  $y$  to  $x$  for a fermion propagator  $S_F(x - y)$ , a filled circle at one end of a wavy line segment for a vector source  $i \int d^4x J(x)$ , a filled circle at the start of a line with an arrow for a fermion source  $i \int d^4x \eta(x)$ , a filled circle at the end of a line with an arrow for an anti-fermion source  $i \int d^4x \bar{\eta}(x)$ , a vertex joining three line segments for  $ie_0\gamma^\mu \int d^4x$ .

### 24.5.3 Ward-Takahashi identity (1)

The Noether current of the symmetry  $\Psi \rightarrow e^{i\alpha}\Psi$  is  $j^\mu = \bar{\Psi}\gamma^\mu\Psi$ . Recall the conservation law in functional formalism

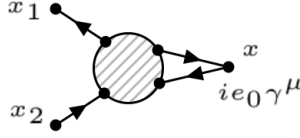
$$\langle \partial_\mu j^\mu(x) \phi(x_1) \cdots \phi(x_n) \rangle = \sum_{i=1}^n \langle \phi(x_1) \cdots (i\Delta\phi(x_i)\delta(x - x_i)) \cdots \phi(x_n) \rangle$$



So, we can write the charge conservation law as

$$ie_0 \partial_\mu \langle \Omega | T j^\mu \Psi(x_1) \bar{\Psi}(x_2) | \Omega \rangle = -ie_0 \delta(x-x_1) \langle \Omega | T \Psi(x_1) \bar{\Psi}(x_2) | \Omega \rangle + ie_0 \delta(x-x_2) \langle \Omega | T \Psi(x_1) \bar{\Psi}(x_2) | \Omega \rangle$$

Notice that  $i \langle \Omega | T j^\mu \psi(x_1) \bar{\Psi}(x_2) | \Omega \rangle$  can be represented by



**Figure 24.3:** Feynman diagram representation of correlation function

From the diagram, we have

$$\langle A_\nu(y) \rangle = \int d^4x G_F(x-y)_{\mu\nu} ie_0 \langle j^\mu(x) \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ipy} G_F(p)_{\mu\nu} \int d^4x e^{ipx} ie_0 \langle j^\mu(x) \rangle$$

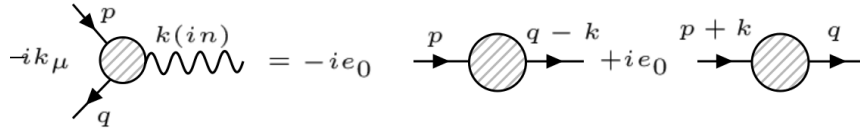
So,

$$\int d^4x \langle A_\nu(x) \rangle e^{ikx} = G_F(p)_{\mu\nu} \int d^4x e^{ikx} ie_0 \langle j^\mu(x) \rangle$$

Compute the Fourier transformation of the equation of charge conservation by integrating

$$\int d^4x e^{ikx} \int d^4x_1 e^{-iqx_1} \int d^4x_2 e^{ipx_2}$$

We can get



**Figure 24.4:** Feynman diagram representation of Ward identity

Note that in the diagram above, the external leg of photon will be cut-off, but external leg of fermion will remain. The above equation can be generated to the diagram with  $n$  external fermions. Another proof of Ward-Takahashi identity by calculating the Feynman diagram directly can be found in chapter 7.4 of *An introduction to quantum field theory* (M.E.Peskin & D.V.Schroeder)

## 24.6 Exact propagator of photon

### 24.6.1 Photon self-energy

The exact propagator of photon is

$$\mathcal{G}(x)_{\mu\nu} = \langle \Omega | T \{ A_\mu(x) A_\nu(0) | \Omega \rangle_C$$

Its Fourier transformation can be represented by the following diagram.





**Figure 24.5:** Feynman diagram representation of exact propagator of photon

Let us define  $i\Pi^{\mu\nu}$  to be the sum of all 1-particle-irreducible insertions into the photon propagator. So, we have

$$\mathcal{G}(k) = G_F(k) + G_F(k)(i\Pi(k))G_F(k) + \cdots = G_F(k) \frac{1}{1 - i\Pi(k)G_F(k)}$$

Hence,

$$(i\mathcal{G})^{-1} = (iG_F)^{-1} - \Pi$$

Recall that

$$iG_F(p)_{\mu\nu} = \frac{\eta_{\mu\nu}}{k^2 - i\epsilon} - (1 - \xi) \frac{k_\mu k_\nu}{(k^2 - i\epsilon)^2} = \frac{1}{k^2 - i\epsilon} (P_{\mu\nu}^T + \xi P_{\mu\nu}^L)$$

Here,

$$P_{\mu\nu}^T \equiv \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \quad P_{\mu\nu}^L \equiv \frac{k_\mu k_\nu}{k^2}$$

We can derive that

$$(iG_F)^{-1}(k)_{\mu\nu} = k^2 (P_{\mu\nu}^T + \frac{1}{\xi} P_{\mu\nu}^L)$$

We may also expand  $i\Pi^{\mu\nu}$  as

$$\Pi^{\mu\nu} = P_T^{\mu\nu} f_T(k^2) + P_L^{\mu\nu} f_L(k^2) = \eta^{\mu\nu} f_T + \frac{k^\mu k^\nu}{k^2} (f_L - f_T)$$

Therefore,

$$(i\mathcal{G})^{-1}(k)_{\mu\nu} = (k^2 - f_T(k^2)) P_{\mu\nu}^T + (\frac{k^2}{\xi} - f_L(k^2)) P_{\mu\nu}^L$$

$$\mathcal{G}(k)_{\mu\nu} = \frac{-i}{k^2 - f_T(k^2)} P_{\mu\nu}^T + \frac{-i}{\frac{k^2}{\xi} - f_L(k^2)} P_{\mu\nu}^L$$

We observe that if  $f_{T,L}(k^2 = 0) \neq 0$ , a mass will be generated for the photon. Because  $\Pi(k)$  comes from 1PI diagrams, it should not be singular at  $k^2 = 0$ , and so  $f_L - f_T = O(k^2)$ , as  $k \rightarrow 0$ . We will show that gauge invariance ensures that no mass is generated from the loop corrections

### 24.6.2 Ward identities(2)

We define the generating functional for connected diagrams

$$Z[J, \eta, \bar{\eta}] = e^{-iE[J, \eta, \bar{\eta}]}$$

For example,

$$\mathcal{G}(x - y)_{\mu\nu} = -i \frac{\delta^2 E[J, \eta, \bar{\eta}]}{\delta J^\mu(x) \delta J^\nu(y)} \Big|_{J, \eta, \bar{\eta} = 0}$$



Recall, for infinitesimal gauge transformations,  $\delta A_\mu = \partial_\mu \lambda$ ,  $\delta \Psi = ie_0 \lambda \Psi$  and  $\delta \bar{\Psi} = -ie_0 \lambda \bar{\Psi}$ . For a change of variables in the path integral,  $Z[J, \eta, \bar{\eta}]$  will remain the same. Recall that

$$Z[J, \eta, \bar{\eta}] = W(\xi) \int \mathcal{D}A \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{i \int d^4x [\mathcal{L}_0 + \mathcal{L}_1 + JA + \bar{\eta}\Psi + \bar{\Psi}\eta]}$$

The change of action is

$$\delta S = -\frac{1}{\xi} \int d^4x \partial_\mu A^\mu \partial^2 \lambda + \int d^4x J^\mu \partial_\mu \lambda + ie_0 \bar{\eta} \Psi \lambda - ie_0 \bar{\Psi} \eta \lambda$$

Hence, we must have

$$\int d^4x \lambda(x) W(\xi) \int \mathcal{D}A \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{iS} \left[ -\frac{1}{\xi} \partial^2 \partial_\mu A^\mu - \partial_\mu J^\mu + ie_0 (\bar{\eta} \Psi - \bar{\Psi} \eta) \right]$$

Since

$$\langle A_\mu(x) \rangle_{J, \eta, \bar{\eta}} = -\frac{\delta E}{\delta J^\mu} \quad \langle \Psi(x) \rangle_{J, \eta, \bar{\eta}} = -\frac{\delta E}{\delta \bar{\eta}} \quad \langle \bar{\Psi}(x) \rangle_{J, \eta, \bar{\eta}} = \frac{\delta E}{\delta \eta}$$

The above equation can be written as

$$\frac{1}{\xi} \partial^2 \partial^\mu \frac{\delta E}{\delta J^\mu} - \partial_\mu J^\mu - ie_0 \left[ \bar{\eta} \frac{\delta E}{\delta \bar{\eta}} + \frac{\delta E}{\delta \eta} \eta \right] = 0$$

We can derive that

$$\frac{1}{\xi} \partial^2 \partial^\mu \frac{\delta^2 E[J, \eta, \bar{\eta}]}{\delta J^\mu(x) \delta J^\nu(y)} \Big|_{J, \eta, \bar{\eta}=0} - \partial_\nu \delta(x-y) = 0$$

that is,

$$\frac{i}{\xi} \partial^2 \partial^\mu \mathcal{G}(x-y)_{\mu\nu} + \partial_\nu \delta(x-y) = 0$$

or, written in momentum-space,

$$-\frac{i}{\xi} k^2 k^\mu \mathcal{G}(k)_{\mu\nu} + k_\nu = 0$$

So

$$-\frac{k^2}{k^2 - \xi f_L(k^2)} k_\nu + k_\nu = 0$$

Which means  $f_L(k^2) = 0$  and so, we have  $f_T(k^2) \rightarrow O(k^2)$  as  $k^2 \rightarrow 0$ . The exact propagator of photon is

$$\mathcal{G}(k)_{\mu\nu} = \frac{-i}{k^2(1 - \pi(k^2))} P_{\mu\nu}^T + \frac{-i\xi}{k^2} P_{\mu\nu}^L$$

where  $\pi(k^2) \equiv \frac{f_T(k^2)}{k^2}$



## 24.7 LSZ reduction formula

### 24.7.1 LSZ reduction formula and Feynman rules

Suppose that the probability for the quantum field to create or annihilate an exact one-particle eigenstate of  $H$  is  $Z_3$ , i.e.

$$\langle \Omega | A(0) | p, \lambda \rangle = \sqrt{Z_3} \epsilon_\lambda(p)$$

In Feynman gauge, because the norm of  $|p, 0\rangle$  is negative, the expansion of orthogonal complete set will be written as

$$\frac{d^3q}{(2\pi)^3} \frac{1}{2E_q} \eta^{\lambda\lambda'} |p, \lambda\rangle \langle p, \lambda'|$$

We have demonstrated that photon will remain massless when interacting with charged fermions. And recall that  $\sum_\lambda \xi_\lambda(p) \xi_\lambda^*(p) = \eta_{\mu\nu}$ . So, we can derive by the similar method in  $\phi^4$  theory that

$$\int d^4x e^{-ipx} \langle \Omega | T A_\mu(x) A_\nu(0) | \Omega \rangle_C = \frac{-iZ_3 \eta_{\mu\nu}}{p^2 - i\epsilon} + \dots$$

The LSZ reduction formula for fermions would take the form as

#### Theorem 24.1 LSZ reduction formula

$$\begin{aligned} & \langle p_1 \cdots p_n | S | k_1 \cdots k_m \rangle \\ &= \prod_1^n \int d^4x_i e^{-ip_i x_i} \prod_1^m \int d^4y_j e^{ik_j y_j} \\ &\times \left( \frac{i}{\sqrt{Z_3}} \right)^{m+n} [p_1^2 \epsilon_{\lambda_1}^{*\mu_1}(p_1)] \cdots [p_n^2 \epsilon_{\lambda_n}^{*\mu_n}(p_n)] [k_1^2 \epsilon_{\lambda'_1}^{\nu_1}(k_1)] \cdots [k_m^2 \epsilon_{\lambda'_m}^{\nu_m}(p_m)] \\ &\times \langle \Omega | T \{ A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n) A_{\nu_1}(y_1) \cdots A_{\nu_m}(y_m) \} | \Omega \rangle \end{aligned}$$

The LSZ reduction formula in other gauge would give similar procedure for calculating scattering amplitude: Fourier transform the Green function in position space to momentum space, cut-off the external legs and multiply the polarization vector of asymptotic states. (Note there is still an extra factor  $\sqrt{Z_3}^{m+n}$  to multiply, similar to the  $\phi^4$  theory).

Finally, we list the Feynman rules of QED in momentum space as follows:

1. For each incoming electron, draw a solid line with an arrow pointed towards the vertex, and label it with the electron's four-momentum,  $p_i$ .
2. For each outgoing electron, draw a solid line with an arrow pointed away from the vertex, and label it with the electron's four-momentum,  $p'_i$ .
3. For each incoming positron, draw a solid line with an arrow pointed away from the vertex, and label it with minus the positron's four-momentum,  $-p_i$ .
4. For each outgoing positron, draw a solid line with an arrow pointed towards the vertex, and label it with minus the positron's four-momentum,  $-p'_i$ .



5. For each incoming photon, draw a wavy line with an arrow pointed towards the vertex, and label it with the photon's four-momentum,  $k_i$ .
6. For each outgoing photon, draw a wavy line with an arrow pointed away from the vertex, and label it with the photon's four-momentum,  $k'_i$ .
7. The only allowed vertex joins two solid lines, one with an arrow pointing towards it and one with an arrow pointing away from it, and one wavy line. Using this vertex, join up all the external lines, including extra internal lines as needed. In this way, draw all possible diagrams that are topologically inequivalent.
8. Assign each internal line its own four-momentum. Think of the four-momenta as flowing along the arrows, and conserve four-momentum at each vertex.
9. The value of a diagram consists of the following factors:
  - for each incoming photon,  $\epsilon^\mu_\lambda(k)$ ; for each outgoing photon,  $\epsilon^{*\mu}_\lambda(k)$ ;
  - for each incoming electron,  $u_r(\mathbf{k})$ ; for each outgoing electron,  $\bar{u}_s(\mathbf{p})$ ;
  - for each incoming positron,  $\bar{v}_r(\mathbf{k})$ ; for each outgoing positron,  $v_s(\mathbf{p})$ ;
  - for each vertex,  $ie_0\gamma^\mu$ ; for each internal photon,  $G_F(p)$ ; for each internal fermion,  $S_F(p)$ .
10. Spinor indices are contracted by starting at one end of a fermion line: specifically, the end that has the arrow pointing away from the vertex. The factor associated with the external line is either  $\bar{u}$  or  $\bar{v}$ . Go along the complete fermion line, following the arrows backwards, and write down (in order from left to right) the factors associated with the vertices and propagators that you encounter. The last factor is either a  $u$  or  $v$ . Repeat this procedure for the other fermion lines, if any. The vector index on each vertex is contracted with the vector index on either the photon propagator (if the attached photon line is internal) or the photon polarization vector (if the attached photon line is external).
11. The overall sign of a tree diagram is determined by drawing all contributing diagrams in a standard form: all fermion lines horizontal, with their arrows pointing from left to right, and with the left endpoints labeled in the same fixed order (from top to bottom); if the ordering of the labels on the right endpoints of the fermion lines in a given diagram is an even (odd) permutation of an arbitrarily chosen fixed ordering, then the sign of that diagram is positive (negative)
12. Each closed fermion loop contributes an extra minus sign.
13. Value of  $i\mathcal{M}$  is given by a sum over the values of the contributing diagrams.
14.  $\langle f|S|i\rangle = (Z_2)^{\frac{n_f}{2}}(Z_3)^{\frac{n_p}{2}}i\mathcal{M}\delta(\sum p_f - \sum p_i)$





### 24.7.2 Ward Takahashi identity (3)

Suppose the invariant matrix element for a process is  $\mathcal{M}$ , if we replace the polarization state vector  $\epsilon_\lambda^\mu$  (or  $\epsilon_\lambda^{*\mu}$ ) of one incoming (or outgoing) photon with its momentum vector  $k^\mu$ , we have

$$k^\mu \mathcal{M}_\mu = 0$$

**Proof:** Without losing generality, we can consider a physical process with a single incoming and outgoing fermion lines respectively. So, the ward identities states that

$$-ik_\mu F^\mu(k; p, q) = ie_0 [F_0(p + k, q) - F_0(p, q - k)]$$

Here,  $F$  keeps the external fermion legs but cuts external photon lines. According to the LSZ reduction formula, from each diagram a contribution to an S matrix element by taking the coefficient of the product of poles

$$\left( \frac{-i}{\not{p} + m} \right) \left( \frac{-i}{\not{q} + m} \right)$$

But the terms on the right hand side contain one of these poles, but neither contains both poles. So they contribute nothing to S-matrix. So, we can have

$$k^\mu \mathcal{M}_\mu = 0$$

□

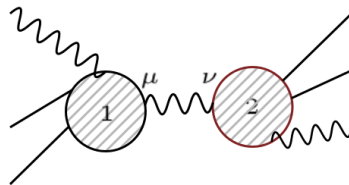
We also note that when calculating invariant matrix element, the main difference between different gauge are photon propagator. In Coulomb gauge, we have

$$G_F(p)_{\mu\nu} \equiv \begin{cases} \frac{i}{|\mathbf{p}|^2} & \mu, \nu = 0 \\ \frac{-i}{p^2 - i\epsilon} \left( \delta_{ij} - \frac{p_i p_j}{|\mathbf{p}|^2} \right) & \mu = i \neq 0, \nu = j \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

In Lorentz gauge, we have

$$G_F(p)_{\mu\nu} = \frac{-i\eta_{\mu\nu}}{p^2 - i\epsilon} + i(1 - \xi) \frac{p_\mu p_\nu}{(p^2 - i\epsilon)^2}$$

We now argue that the lead to the same  $\mathcal{M}$  element. For a general process, it can be represented as follows.



**Figure 24.6:** Feynman diagram representation of a QED process

The value of the diagram is

$$\mathcal{M}_1^\mu G_F(k)_{\mu\nu} \mathcal{M}_2^\nu$$



and  $k_\mu \mathcal{M}_1^\mu = 0$ ,  $k_\nu \mathcal{M}_2^\nu = 0$ . So the factor  $\xi$  in Lorentz gauge is irrelevant to the value of  $\mathcal{M}$ . As for Coulomb gauge, denote  $\mathcal{M}_1^\mu$  as  $\alpha^\mu$ ,  $\mathcal{M}_2^\mu$  as  $\beta^\mu$ , so

$$\alpha^\mu G_F(k)_{\mu\nu} \beta^\nu = i \left( -\frac{\alpha \cdot \beta}{k^2} + \frac{(\alpha \cdot \mathbf{k})(\beta \cdot \mathbf{k})}{k^2 \mathbf{k}^2} + \frac{\alpha^0 \beta^0}{\mathbf{k}^2} \right)$$

Using the fact that  $\alpha \cdot \mathbf{k} + \alpha^0 k_0 = 0$ , we can verify that

$$\alpha^\mu G_F(k)_{\mu\nu} \beta^\nu = \alpha^\mu \left( -\frac{i\eta_{\mu\nu}}{k^2} \right) \beta^\nu$$

So, the invariant matrix element is gauge invariant.

## 24.8 Renormalization

### 24.8.1 Renormalized quantum electrodynamics

The Lagrangian of QED is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\not{\partial} - m_0)\Psi + e_0 j^\mu A_\mu,$$

Suppose the number of external photons is  $N_\gamma$ , the number of external fermions is  $N_e$ , the number of vertex is  $V$ . The superficial divergence is

$$D = 4 - N_\gamma - \frac{3}{2} N_e$$

Recall that  $C$  is a symmetry of QED, so  $C|\Omega\rangle = |\Omega\rangle$ . But  $j^\mu$  changes sign under charge conjugation,  $C j^\mu(x) C^\dagger = -j^\mu(x)$ , so its vacuum expectation value must vanish:

$$\langle \Omega | T j^\mu(x) | \Omega \rangle = \langle \Omega | T C^\dagger C j^\mu(x) C^\dagger C | \Omega \rangle = -\langle \Omega | T j^\mu(x) | \Omega \rangle = 0$$

So, the amplitude with  $N_\gamma = 1$ ,  $N_e = 0$  will vanish. Similarly, we can verify that the diagram with  $N_\gamma = 3$ ,  $N_e = 0$  will also vanish.

As for the amplitude with  $N_\gamma = 4$ , the superficial divergence  $D = 0$ . So the only divergence must be of the form  $\log \Lambda$ . But the ward identity implies that

$$K^\mu \mathcal{M}_{\mu\nu\sigma\rho} = 0$$

So, the divergent terms must vanish.

If we neglect the vacuum term with  $N_\gamma = 0$ ,  $N_e = 0$ , there are only three divergent amplitude terms left.

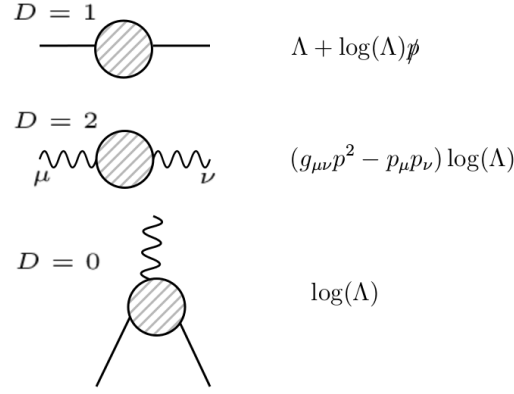
We need four counter terms to eliminate all the divergence. The Lagrangian can be written as

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_{ct}$$

Here,

$$\mathcal{L}_1 = -\frac{1}{4} F_r^{\mu\nu} F_{r\mu\nu} + i \bar{\Psi}_r \gamma^\mu \partial_\mu \Psi_r - m \bar{\Psi}_r \Psi_r + e \bar{\Psi}_r \gamma^\mu \Psi_r A_{r\mu}$$





**Figure 24.7:** Feynman diagram representation of divergent amplitude in QED

$$\mathcal{L}_{ct} = -\frac{1}{4}\delta_3 F_r^{\mu\nu} F_{r\mu\nu} + i\delta_2 \bar{\Psi}_r \gamma^\mu \partial_\mu \Psi_r - \delta_m \bar{\Psi}_r \Psi_r + e\delta_1 \bar{\Psi}_r \gamma^\mu \Psi_r A_{r\mu}$$

$$A = \sqrt{Z_3} A_r \quad \Phi = \sqrt{Z_2} \Phi_r \quad \delta_3 = Z_3 - 1 \quad \delta_2 = Z_2 - 1$$

$$\delta_m = Z_2 m_0 - m \quad \delta_1 = Z_1 - 1 = (e_0/e) Z_2 Z_3^{\frac{1}{2}} - 1$$

The Feynman rules of counter terms are:

$$-\frac{1}{4}\delta_3 F_r^{\mu\nu} F_{r\mu\nu} \quad -i(\eta^{\mu\nu} q^2 - q^\mu q^\nu) \delta_3$$

$$i\bar{\Psi}_r (\delta_2 \not{\partial} - \delta_m) \Psi_r \quad -i(\delta_2 \not{p} + \delta_m)$$

$$e\delta_1 \bar{\Psi}_r \gamma^\mu \Psi_r A_{r\mu} \quad ie\gamma^\mu \delta_1$$

We also denote the renormalized 1PI component of exact propagator of photon as  $i(\eta^{\mu\nu} q^2 - q^\mu q^\nu) \Pi_r(q^2)$ , the renormalized 1PI component of exact propagator of fermion as  $-i\Sigma_r(\not{p})$ , the renormalized exact amputated photon-fermion-antifermion vertex as  $ie\Gamma_r^\mu(p, p')$ .

So, renormalized exact propagator of photon is

$$\mathcal{G}_r(q)_{\mu\nu} = \frac{-i}{q^2(1 - \Pi_r(q^2))} P_{\mu\nu}^T$$

renormalized exact propagator of fermion is

$$\mathcal{S}_r(p) = \frac{-i}{\not{p} + m + \Sigma_r(\not{p})}$$

The on-shell renormalization conditions are

$$\begin{aligned} \Sigma_r(\not{p} = -m) &= 0 \\ \left. \frac{d}{d\not{p}} \Sigma_r(\not{p}) \right|_{\not{p} = -m} &= 0 \\ \Pi_r(q^2 = 0) &= 0 \\ ie\Gamma_r^\mu(p = p', p^2 = -m^2) &= ie\gamma^\mu \end{aligned}$$



Recall the ward identity, we have

$$ie\sqrt{Z_2}\partial_\mu\langle\Omega|Tj_r^\mu\Psi_r(x_1)\bar{\Psi}_r(x_2)|\Omega\rangle = -ie\delta(x-x_1)\langle\Omega|T\Psi_r(x_1)\bar{\Psi}_r(x_2)|\Omega\rangle + ie\delta(x-x_2)\langle\Omega|T\Psi_r(x_1)\bar{\Psi}_r(x_2)|\Omega\rangle$$

In momentum space, we have

$$-k_\mu Z_2 Z_1^{-1} \mathcal{S}_r(p+k) [ie\Gamma_r^\mu(p+k, p)] \mathcal{S}_r(p) \frac{1}{1 - \Pi_r(k^2)} = e(\mathcal{S}_r(p+k) - \mathcal{S}_r(p))$$

So,

$$Z_2 Z_1^{-1} k_\mu [\Gamma_r^\mu(p+k, p)] \frac{1}{1 - \Pi_r(k^2)} = \not{k} + \Sigma_r(\not{k} + \not{p}) - \Sigma_r(\not{p})$$

Since  $\Gamma_r$ ,  $\Pi_r$  and  $\Sigma_r$  are all finite by renormalization, so  $Z_1/Z_2$  must be finite. In  $\overline{MS}$  renormalization scheme, we immediately get

$$Z_1 = Z_2$$

In on-shell renormalization scheme, taking the limit of  $k \rightarrow 0$ , we can also get that

$$Z_1 = Z_2$$

So, we know

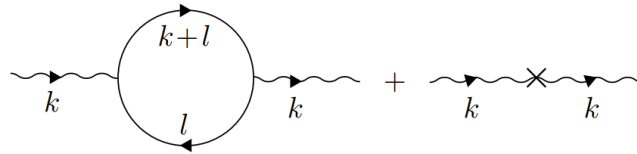
$$e = \sqrt{Z_3} e_0$$

This means that the relation between the bare and renormalized electric charge depends only on the photon field strength renormalization, not on quantities particular to the fermions, leading to a universal electric charge which has the same value for all species.

In the following subsection, we would omit the subscript  $r$  unless it is necessary to emphasis the difference of bare field and renormalized fields.

## 24.8.2 One loop structure of QED

### Photon propagator



**Figure 24.8:** The one-loop and counterterm corrections to the photon propagator

$$\Pi(k^2) = -\frac{e^2}{\pi^2} \int_0^1 dx x(1-x) \left[ \frac{1}{\epsilon} - \frac{1}{2} \ln\left(\frac{D}{\mu^2}\right) \right] - \delta_3 + O(e^4)$$

where  $D = x(1-x)k^2 + m^2 - i\epsilon$  and  $\mu^2 = 4\pi e^{-\gamma} \tilde{\mu}^2$ . Impose the OS renormalization condition  $\Pi(0) = 0$ , we have

$$\delta_3 = -\frac{e^2}{6\pi^2} \left[ \frac{1}{\epsilon} - \ln\left(\frac{m}{\mu}\right) \right] + O(e^4)$$

$$\Pi(k^2) = \frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \ln\left(\frac{D}{m^2}\right) + O(e^4)$$



### Fermion propagator

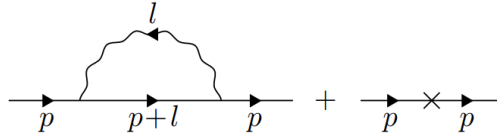
The exact renormalized fermion propagator in OS renormalization can be written in Lehmann-Kallen form as

$$i\mathcal{S}(\not{p}) = \frac{1}{\not{p} + m - i\epsilon} + \int_{m_{th}}^{\infty} ds \frac{\rho_{\Psi}(s)}{\not{p} + \sqrt{s} - i\epsilon}$$

We see that the first term has a pole at  $\not{p} = -m$  with residue one. This residue corresponds to the field normalization that is needed for the validity of the LSZ formula.

There is a problem, however: in quantum electrodynamics, the threshold mass  $m_{th}$  is  $m$ , corresponding to the contribution of a fermion and a zero energy photon. Thus the second term has a branch point at  $\not{p} = -m$ . The pole in the first term is therefore not isolated, and its residue is ill defined.

This is a reflection of an underlying infrared divergence, associated with the massless photon. To deal with it, we must impose an infrared cutoff that moves the branch point away from the pole. The most direct method is to change the denominator of the photon propagator from  $k^2$  to  $k^2 + m_{\gamma}^2$ , where  $m_{\gamma}$  is a fictitious photon mass. Ultimately, we must deal with this issue by computing cross sections that take into account detector inefficiencies. In quantum electrodynamics, we must specify the lowest photon energy  $\omega_{min}$  that can be detected. Only after computing cross sections with extra undetectable photons, and then summing over them, is it safe to take the limit  $m_{\gamma} \rightarrow 0$ .



**Figure 24.9:** The one-loop and counterterm corrections to the fermion propagator

$$\Sigma(\not{p}) = \frac{e^2}{8\pi^2} \int_0^1 dx \left( (2 - \epsilon)(1 - x)\not{p} + (4 - \epsilon)m \right) \left[ \frac{1}{\epsilon} - \frac{1}{2} \ln\left(\frac{D}{\mu^2}\right) \right] + \delta_2 \not{p} + \delta_m + O(e^4)$$

where  $D = x(1 - x)p^2 + xm^2 + (1 - x)m_{\gamma}^2$ . The fitness of  $\Sigma(\not{p})$  requires that

$$\delta_2 = -\frac{e^2}{8\pi^2} \left( \frac{1}{\epsilon} + \text{finite} \right) + O(e^4)$$

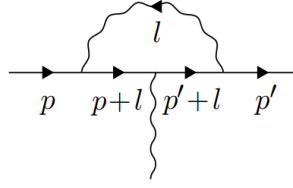
$$\delta_m/m = -\frac{e^2}{2\pi^2} \left( \frac{1}{\epsilon} + \text{finite} \right) + O(e^4)$$

Impose the OS renormalization condition  $\Sigma(-m) = 0$  and  $\Sigma'(-m) = 0$ , we have

$$\Sigma(\not{p}) = -\frac{e^2}{8\pi^2} \int_0^1 dx \left( (1 - x)\not{p} + 2m \right) \ln\left(\frac{D}{D_0}\right) + \kappa_2(\not{p} + m) + O(e^4)$$

where  $D_0 = x^2m^2 + (1 - x)m_{\gamma}^2$  and  $\kappa_2 = -2 \ln(m/m_{\gamma}) + 1$ .





**Figure 24.10:** The one-loop correction to the photon-fermion-fermion vertex

### Vertex

$$\Gamma^\mu(p, p') = (1 + \delta_1)\gamma^\mu + \frac{e^2}{8\pi^2} \left[ \left( \frac{1}{\epsilon} - 1 - \frac{1}{2} \int dF_3 \ln(D/\mu^2) \right) \gamma^\mu + \frac{1}{4} \int dF_3 \frac{\tilde{N}^\mu}{D} \right] + O(e^4)$$

where

$$\int dF_3 = 2 \int_0^1 dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1)$$

$$D = x_1(1 - x_1)p^2 + x_2(1 - x_2)p'^2 - 2x_1x_2p \cdot p' + (x_1 + x_2)m^2 + x_3m_\gamma^2$$

$$\tilde{N}^\mu = \gamma_\nu [x_1 \not{p} - (1 - x_2) \not{p}' + m] \gamma^\mu [-(1 - x_1) \not{p} + x_2 \not{p}' + m] \gamma^\nu$$

Fitness of  $\Gamma^\mu$  requires that

$$\delta_1 = -\frac{e^2}{8\pi^2} \left( \frac{1}{\epsilon} + \text{finite} \right) + O(e^4)$$

Impose the OS renormalization condition  $\Gamma_r^\mu(p = p', p^2 = -m^2) = \gamma^\mu$ , we have

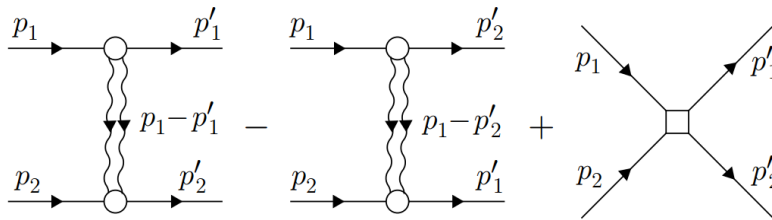
$$\Gamma^\mu(p, p') = \gamma^\mu - \frac{e^2}{16\pi^2} \int dF_3 \left[ (\ln(D/D_0) + 2\kappa_1) \gamma^\mu - \frac{\tilde{N}^\mu}{2D} \right] + O(e^4)$$

where

$$D_0 = (1 - x_3)^2 m^2 + x_3 m_\gamma^2 \quad \kappa_1 = -2 \ln(m/m_\gamma) + \frac{5}{2}$$

### Vertex function

Consider the process of electron-electron scattering In order to compute the contribution of



**Figure 24.11:** Diagrams for the exact electron-electron scattering amplitude. The vertices and photon propagator are exact; external lines stand for the usual  $u$  and  $\bar{u}$  spinor factors, times the unit residue of the pole at  $p^2 = -m^2$ .



these diagrams, we must evaluate  $\bar{u}_{s'}(\mathbf{p}')\Gamma^\mu(p', p)u_s(\mathbf{p})$  with  $p^2 = p'^2 = -m^2$ , but with  $q^2 = (p - p')^2$  arbitrary. Finally, we have

$$\bar{u}_{s'}(\mathbf{p}')\Gamma^\mu(p', p)u_s(\mathbf{p}) = \bar{u}' \left[ F_1(q^2)\gamma^\mu - \frac{i}{m}F_2(q^2)S^{\mu\nu}q_\nu \right] u_s(\mathbf{p})$$

where we have defined the form factors

$$\begin{aligned} F_1(q^2) &= 1 - \frac{e^2}{16\pi^2} \int dF_3 \ln \left( 1 + \frac{x_1 x_2 q^2 / m^2}{(1 - x_3)^2} \right) \\ &\quad - \frac{e^2}{16\pi^2} \int dF_3 \frac{1 - 4x_3 + x_3^2}{(1 - x_3)^2 + x_3 m_\gamma^2 / m^2} + \frac{(x_3 + x_1 x_2) q^2 / m^2 - (1 - 4x_3 + x_3^2)}{x_1 x_2 q^2 / m^2 + (1 - x_3)^2 + x_3 m_\gamma^2 / m^2} + O(e^4) \\ F_2(q^2) &= \frac{e^2}{8\pi^2} \int dF_3 \frac{x_3 - x_3^2}{x_1 x_2 q^2 / m^2 + (1 - x_3)^2} + O(e^4) \end{aligned}$$

The loop corrections of QED are related to anomalous magnetic moment of electrons and lamb shift of energy level in hydrogen atom. We will not discuss these interesting phenomenon in this notes. Discussion on anomalous magnetic moment can be found in chapter 6.2 of *An introduction to quantum field theory* (M.E.Peskin & D.V.Schroeder). You can also find some comments on lamb shift due to vacuum polarization in chapter 7.5. And a detailed treatment of infrared divergence is available in chapter 6.1, 6.4 and 6.5.

### 24.8.3 Renormalization group

#### $\overline{MS}$ renormalization scheme

$$\Pi(k^2) = \frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \ln \left( \frac{x(1-x)k^2 + m^2}{\mu^2} \right) + O(e^4)$$

$$Z_3 = 1 + \delta_3 = 1 - \frac{e^2}{6\pi^2} \frac{1}{\epsilon} + O(e^4)$$

$$\Sigma(p) = -\frac{e^2}{8\pi^2} \int_0^1 dx \ln \left( \frac{x(1-x)p^2 + xm^2 + (1-x)m_\gamma^2}{\mu^2} \right) ((1-x)\not{p} + 2m) + \frac{1}{2}\not{p} + m + O(e^4)$$

$$Z_2 = 1 + \delta_2 = 1 - \frac{e^2}{8\pi^2} \frac{1}{\epsilon} + O(e^4)$$

$$Z_m = 1 + \delta_m/m = 1 - \frac{e^2}{2\pi^2} \frac{1}{\epsilon} + O(e^4)$$

$$\Gamma^\mu(p, p') = \gamma^\mu + \frac{e^2}{8\pi^2} \left[ - \left( 1 + \frac{1}{2} \int dF_3 \ln(D/\mu^2) \right) \gamma^\mu + \frac{1}{4} \int dF_3 \frac{\tilde{N}^\mu}{D} \right] + O(e^4)$$

$$D = x_1(1-x_1)p^2 + x_2(1-x_2)p'^2 - 2x_1x_2p \cdot p' + (x_1+x_2)m^2 + x_3m_\gamma^2$$

$$Z_1 = 1 + \delta_1 = 1 - \frac{e^2}{8\pi^2} \frac{1}{\epsilon} + O(e^4)$$



### Renormalization group

The Lagrangian of QED is

$$\mathcal{L} = -\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} + \bar{\Psi}_0(i\cancel{\partial} - m_0)\Psi_0 + e_0\bar{\Psi}_0\gamma^\mu\Psi_0 A_{0\mu},$$

It can be written as

$$\mathcal{L} = -\frac{1}{4}Z_3F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(iZ_2\cancel{\partial} - Z_m m)\Psi + Z_1e\bar{\Psi}\gamma^\mu\Psi A_\mu,$$

So,

$$\Psi_0 = Z_2^{1/2}\phi \quad m_0 = Z_2^{-1}Z_m m \quad A_0 = Z_3^{1/2}A \quad e_0 = Z_2^{-1}Z_1Z_3^{-1/2}e\tilde{\mu}^{\epsilon/2} = Z_3^{-1/2}e\tilde{\mu}^{\epsilon/2}$$

After using dimensional regularization, the infinities coming from loop integrals take the form of inverse powers of  $\epsilon$ . In the  $\overline{\text{MS}}$  renormalization scheme, we choose the  $Z$ s to cancel off these powers of  $1/\epsilon$ , and nothing more. Therefore the  $Z$ s can be written as

$$\begin{aligned} Z_1 &= 1 + \sum_{n=1}^{\infty} \frac{a_n(\lambda)}{\epsilon^n} & Z_2 &= 1 + \sum_{n=1}^{\infty} \frac{b_n(\lambda)}{\epsilon^n} \\ Z_3 &= 1 + \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{\epsilon^n} & Z_m &= 1 + \sum_{n=1}^{\infty} \frac{d_n(\lambda)}{\epsilon^n} \end{aligned}$$

In QED,  $a_1 = -\frac{e^2}{8\pi^2} + O(e^4)$ ,  $b_1 = -\frac{e^2}{8\pi^2} + O(e^4)$ ,  $c_1 = -\frac{e^2}{6\pi^2} + O(e^4)$ ,  $d_1 = -\frac{e^2}{2\pi^2} + O(e^4)$ . Remember that bare fields and parameters must be independent of  $\mu$ . Define

$$E(e, \epsilon) \equiv \ln(Z_3^{-1/2}) = \sum_{n=1}^{\infty} \frac{E_n(e)}{\epsilon^n}$$

We can calculate  $E_1 = -\frac{1}{2}c_1 = \frac{e^2}{12\pi^2} + O(e^4)$ . As  $\ln e_0 = E + \ln e + \frac{\epsilon}{2} \ln \tilde{\mu}$ . From the independence of  $e_0$ , we can derive

$$\left(1 + \frac{eE'_1}{\epsilon} + \dots\right) \frac{de}{d\ln \mu} + \frac{\epsilon}{2}e = 0$$

In a renormalizable theory, we should have

$$\frac{de}{d\ln \mu} = -\frac{\epsilon}{2}e + \beta(e)$$

So

$$\beta(e) = \frac{e^2}{2}E'_1(\lambda)$$

In QED, we have

$$\beta(e) = \frac{e^3}{12\pi^2} + O(e^5)$$

Define

$$M(e, \epsilon) \equiv \ln(Z_m Z_2^{-1}) = \sum_{n=1}^{\infty} \frac{M_n(\lambda)}{\epsilon^n}$$





We can calculate  $M_1 = d_1 - b_1 = -\frac{3e^2}{8\pi^2} + O(e^4)$ . As  $\ln m_0 = M + \ln m$ , define the anomalous dimension of the mass

$$\gamma_m(e) \equiv \frac{1}{m} \frac{dm}{d \ln \mu}$$

From the independence of  $m_0$ , we can derive

$$\gamma_m(e) = \frac{e}{2} M'_1$$

In QED theory, we have

$$\gamma_m(e) = -\frac{3e^2}{8\pi^2} + O(e^4)$$

Expand  $\ln Z_2$  as

$$\ln Z_2 = \frac{b_1}{\epsilon} + \dots$$

Define the anomalous dimension of the fermion field

$$\gamma_2(e) \equiv \frac{1}{2} \frac{d \ln Z_2}{d \ln \mu}$$

We can derive

$$\gamma_2(e) = -\frac{1}{4} e b'_1$$

In QED theory, we have

$$\gamma_2(e) = \frac{e^2}{16\pi^2} + O(e^4)$$

Expand  $\ln Z_3$  as

$$\ln Z_3 = \frac{c_1}{\epsilon} + \dots$$

Define the anomalous dimension of the EM field

$$\gamma_3(e) \equiv \frac{1}{2} \frac{d \ln Z_3}{d \ln \mu}$$

We can derive

$$\gamma_3(e) = -\frac{1}{4} e c'_1$$

In QED theory, we have

$$\gamma_3(e) = \frac{e^2}{12\pi^2} + O(e^4)$$



## **Part VI**

### **Statistical mechanics**

# Chapter 25

## Thermodynamics



# Chapter 26

## Principles of Statistical Mechanics and Ensembles



**26.1 Density matrix**

**26.2 Micro-canonical ensemble**

**26.3 Canonical ensemble**

**26.4 Grand-canonical ensemble**

**26.5 Quantum statistics and Classical statistics**

# Chapter 27

## Ideal Gas



## Chapter 28

### Interaction



#### **28.1 Cluster expansion**

#### **28.2 Quantum field theory formulation**

#### **28.3 Path integrals**

## **Chapter 29**

# **Phase Transitions and the Renormalization Group**

