Physics

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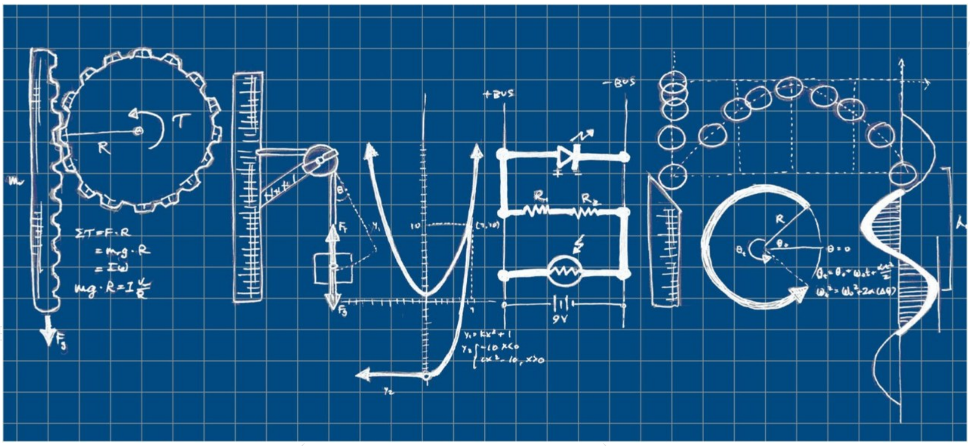

Summary is the best way to say "Good Bye"

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1.1.1 Nonabelian symmetries

Consider the theory of N real scalar fields ϕ_i

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{2}m^2\phi_i\phi_i - \frac{1}{16}\lambda(\phi_i\phi_i)^2$$

This lagrangian is clearly invariant under the $SO(N)$ transformation

$$\phi_i(x) \to R_{ij}\phi_j(x)$$

where R is an orthogonal matrix with a positive determinant: $R^T = R^{-1}$ and $\det R = +1$. Consider an infinitesimal $SO(N)$ transformation

$$R_{ij} = \delta_{ij} + \theta_{ij} + O(\theta^2)$$

Orthogonality of R_{ij} implies that θ_{ij} is real and antisymmetric. It is convenient to express θ_{ij} in terms of a basis set of hermitian matrices T^a_{ij}. The index a runs from 1 to $\frac{1}{2}N(N-1)$, the number of linearly independent, hermitian, antisymmetric, $N \times N$ matrices. Commonly, we demand these matrices obey the normalization condition

$$\mathrm{Tr}(T^aT^b) = 2\delta^{ab}$$

In terms of them, we can write

$$\theta_{ij} = -i\theta^a T^a_{ij}$$

The T^as are the generator matrices of $SO(N)$. The product of any two $SO(N)$ transformations is another $SO(N)$ transformation; this implies that the commutator of any two generator matrices must be a linear combination of generator matrices,

$$[T^a, T^b] = if^{abc}T^c$$

The numerical factors f^{abc} are the structure coefficients of the group. If $f^{abc} = 0$, the group is abelian. Otherwise, it is nonabelian. Under our normalization condition, we have

$$f^{abc} = -\frac{i}{2}\mathrm{Tr}\left([T^a, T^b]T^c\right)$$

Using the cyclic property of the trace, we find that f^{abc} must be completely antisymmetric. Taking the complex conjugate of equation above, we find that f^{abc} must be real.

Example: The simplest nonabelian group is $SO(3)$. In this case, we can choose $T^a_{ij} = \epsilon^{aij}$. The commutation relations become

$$[T^a, T^b] = i\epsilon^{abc}T^c$$

Consider now the theory of N complex scalar fields ϕ_i

$$\mathcal{L} = -\partial_\mu \phi_i^\dagger \partial^\mu \phi_i - m^2 \phi_i^\dagger \phi_i - \frac{1}{4}\lambda(\phi_i^\dagger \phi_i)^2$$

This lagrangian is clearly invariant under the $U(N)$ transformation

$$\phi_i(x) \to U_{ij}\phi_j(x)$$

where U is a unitary matrix, $R^\dagger = R^{-1}$. We can write $U_{ij} = e^{-i\theta}\widetilde{U}_{ij}$, where θ is a real parameter and $\det \widetilde{U} = 1$. $\widetilde{U}_{ij}$ is called a special unitary matrix. Clearly the product of two special unitary matrices is another special unitary matrix; the $N \times N$ special unitary matrices form the group $SU(N)$. The group $U(N)$ is the direct product of the group $U(1)$ and the group $SU(N)$.

Consider an infinitesimal $SU(N)$ transformation

$$\widetilde{U}_{ij} = \delta_{ij} - i\theta^a T^a_{ij} + O(\theta^2)$$

where θ^a is a set of real, infinitesimal parameters. Unitarity of $\widetilde{U}$ implies that the generator matrices T are hermitian, and $\det \widetilde{U} = 1$ implies that each T is traceless. The index a runs from 1 to $N^2 - 1$, the number of linearly independent, hermitian, traceless, $N \times N$ matrices. We can choose these matrices to obey the normalization condition

$$\mathrm{Tr}(T^a T^b) = 2\delta^{ab}$$

Example: For $SU(2)$, we can choose $T^a_{ij} = \frac{1}{2}\sigma^a_{ij}$. The commutation relations become

$$[T^a, T^b] = i\epsilon^{abc}T^c$$

1.1.2 Nonabelian gauge theory

Consider a lagrangian with N scalar or spinor fields $\phi^i(x)$ that is invariant under a continuous $SU(N)$ symmetry,

$$\phi_i(x) = U_{ij}\phi_j(x)$$

It is called a global symmetry transformation, because the matrix U does not depend on the space-time label x.

If we want to generalize the symmetry of lagrangian to local transformation

$$\phi_i(x) = U_{ij}(x)\phi_j(x)$$

terms with derivatives, such as $\partial^\mu \psi^\dagger \partial_\mu \phi_i$, will not remain invariant under local transformation. So we must include a traceless hermitian $N \times N$ gauge field $A_\mu(x)$, and promote ordinary derivatives ∂_μ to covariant derivatives $D_\mu = \partial_\mu - igA_\mu$ to ensure that

$$D_\mu \phi \to U D_\mu \phi$$

As a result, the gauge field must transform as

$$A_\mu(x) \to U(x) A_\mu(x) U^\dagger(x) + \frac{i}{g} U(x) \partial_\mu U^\dagger(x)$$

Replacing all ordinary derivatives in $\mathcal{L}$ with covariant derivatives renders $\mathcal{L}$ gauge invariant (assuming, of course, that $\mathcal{L}$ originally had a global $SU(N)$ symmetry).
We can write $U(x)$ in terms of the generator matrices as $\exp[-ig\Gamma(x)T^a]$. If the structure constant $f^{abc} \neq 0$, we have a nonabelian gauge theory.

We still need a kinetic term for $A_\mu(x)$. Let us define the field strength

$$F_{\mu\nu}(x) \equiv \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

We can verify that the field strength transform as

$$F_{\mu\nu}(x) \to U(x) F_{\mu\nu}(x) U^\dagger(x)$$

Therefore, a reasonable kinetic term is

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2} \mathrm{Tr}(F^{\mu\nu} F_{\mu\nu})$$

Since we have taken A_μ to be hermitian and traceless, we can expand it in terms of the generator matrices:

$$A_\mu(x) = A_\mu^a(x) T^a$$

Similarly, we have

$$F_{\mu\nu}(x) = F_{\mu\nu}^a(x) T^a$$

We can get

$$F_{\mu\nu}^c = \partial_\mu A_\nu^c - \partial_\nu A_\mu^c + g f^{abc} A_\mu^a A_\nu^b$$
$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} F^{c\mu\nu} F_{c\mu\nu}$$

Everything we have just said about $SU(N)$ also goes through for $SO(N)$, with unitary replaced by orthogonal, and traceless replaced by antisymmetric. There is also another class of compact nonabelian groups called $Sp(2N)$, and five exceptional compact groups: $G(2)$, $F(4)$, $E(6)$, $E(7)$ and $E(8)$. Compact means that $\mathrm{Tr}(T^a T^b)$ is a positive definite matrix. Nonabelian gauge theory must be based on a compact group, because otherwise some of the terms in $\mathcal{L}_{\text{kin}}$ would have the wrong sign, leading to a Hamiltonian that is unbounded below.

As a specific example, let us consider quantum chromodynamics, or QCD, which is based on the gauge group $SU(3)$. There are several Dirac fields corresponding to quarks. Each quark comes in three colors; these are the values of the $SU(3)$ index. There are also six flavours: up, down, strange, charm, bottom, and top. Thus we consider the Dirac field $\Psi_{iI}(x)$, where i is the color index and I is the flavour index. The Lagrangian is

$$\mathcal{L} = i\overline{\Psi}_{iI}\not{D}_{ij}\Psi_{jI} - m_I\overline{\Psi}_I\Psi_I - \frac{1}{2}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu})$$

The different quark flavours have different masses, ranging from a few MeV for the up and down quarks to 174 GeV for the top quark. The covariant derivative is

$$D_{\mu ij} = \delta_{ij}\partial_\mu - igA_\mu^a T_{ij}^a$$

The index a on A_μ^a runs from 1 to 8, and the corresponding massless spin-one particles are the eight gluons.

In a nonabelian gauge theory in general, we can consider scalar or spinor fields in different representations of the group. A representation of a compact nonabelian group is a set of finite-dimensional hermitian matrices T_R^a that obey the same commutation relations as the original generator matrices T^a. Given such a set of $D(R) \times D(R)$ matrices, and a field $\phi(x)$ with $D(R)$ components, we can write its covariant derivative as $D_\mu = \partial_\mu - igA_\mu^a T_R^a$. Under a gauge transformation, $\phi(x) \to U_R(x)\phi(x)$. The theory will be gauge invariant provided that

$$A_\mu^c \to A_\mu^c + g\theta^a A_\mu^b f^{abc} - \partial_\mu\theta^c$$

under infinitesimal transformation, which is independent of representation.