

$$V(k_0) = \sum_{t=0}^{\infty} [\beta^t \ln(1 - \alpha\beta) + \beta^t \alpha \ln k_t]$$

$$= \ln(1 - \alpha\beta) \sum_{t=0}^{\infty} \beta^t + \alpha \sum_{t=0}^{\infty} \beta^t \ln k_t$$

$$= \frac{\alpha}{1 - \alpha\beta} \ln k_0 + \frac{\alpha \ln(\alpha\beta)}{1 - \alpha\beta} + \alpha \ln(\alpha\beta) \sum_{t=0}^{\infty} \beta^t$$

$$= \frac{\alpha}{1 - \alpha\beta} \ln k_0 + \frac{\alpha \ln(1 - \alpha\beta)}{1 - \alpha\beta} + \frac{\alpha\beta}{(1 - \beta)(1 - \alpha\beta)} \ln(\alpha\beta)$$



$$\text{右边} = \max \{u(f(k) - y) + \beta V(y)\}$$

Do not ask what it is. Ask what you can say about it.

$$= \ln(k^\alpha - \alpha\beta k^\alpha) + \beta \left[\frac{\alpha}{1 - \alpha\beta} \ln \alpha\beta k^\alpha + A \right]$$

$$= \ln(k^\alpha - \alpha\beta k^\alpha) + \beta \left[\frac{\alpha}{1 - \alpha\beta} \ln \alpha\beta k^\alpha + A \right]$$

$$= \ln(1 - \alpha\beta) + \alpha \ln k + \beta \left[\frac{\alpha}{1 - \alpha\beta} [\ln \alpha\beta + \alpha \ln k] + k \right]$$

$$= \alpha \ln k + \frac{\alpha\beta}{1 - \alpha\beta} \alpha \ln k + \ln(1 - \alpha\beta) + \frac{\alpha\beta}{1 - \alpha\beta} \ln \alpha\beta + \beta A$$

$$= \frac{\alpha}{1 - \alpha\beta} \ln k + \ln(1 - \alpha\beta) + \frac{\alpha\beta}{1 - \alpha\beta} \ln \alpha\beta + \beta A$$

$$= \frac{\alpha}{1 - \alpha\beta} \ln k + (1 - \beta)A + \beta A$$

$$= \frac{\alpha}{1 - \alpha\beta} \ln k + A$$

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所以，左边 = 右边，证毕。

Contents



1	Approximation method	3
1.1	Time independent perturbation theory	3
1.1.1	Brillouin-Wigner perturbation theory	3
1.1.2	Nondegenerate perturbation theory	4
1.1.3	Degenerate perturbation theory	5
1.2	Application of time independent perturbation theory	6
1.2.1	Stark effect in hydrogen atom	6
1.2.2	Fine structure of hydrogen atom	8
1.3	Time dependent perturbation theory	9
1.4	Atomic Radiation	9
1.5	The classical limit	9
2	Many body problem	10
2.1	Identical particles	10
2.2	Non-relativistic quantum field theory	10
3	Scattering theory	11
3.1	Lippmann–Schwinger equation	11
3.2	Born approximation	13
3.3	Partial wave analysis	15

Chapter 1

Approximation method



1.1 Time independent perturbation theory

1.1.1 Brillouin-Wigner perturbation theory

We consider an unperturbed Hamiltonian H_0 with eigenvalues ϵ_k and eigenstates $|k\alpha\rangle$, where α is an index introduced to resolve degeneracies, so that

$$H_0|k\alpha\rangle = \epsilon_k|k\alpha\rangle$$

We pick one of these levels ϵ_n for study, so the index n will be fixed for the following discussion. We denote the eigenspace of the unperturbed system corresponding to eigenvalue ϵ_n by \mathcal{H} , so that the unperturbed eigenkets $\{|n\alpha\rangle, \alpha = 1, 2, \dots\}$ form a basis in this space.

We take the perturbed Hamiltonian to be $H = H_0 + \lambda H_1$, where λ is a formal expansion parameter that we allow to vary between 0 and 1 to interpolate between the unperturbed and perturbed system. When the perturbation is turned on, the unperturbed energy level ϵ_n may split and shift. We denote one of the exact energy levels that grows out of ϵ_n by E . We let $|\psi\rangle$ be an exact energy eigenket corresponding to energy E , so that

$$H|\psi\rangle = (H_0 + \lambda H_1)|\psi\rangle = E|\psi\rangle$$

Both E and $|\psi\rangle$ are understood to be functions of λ ; as $\lambda \rightarrow 0$, E approaches ϵ_n and $|\psi\rangle$ approaches some state lying in \mathcal{H}_n . We break the Hilbert space into the subspace \mathcal{H}_n and its orthogonal complement which we denote by \mathcal{H}_n^\perp . The components of $|\psi\rangle$ parallel and perpendicular to \mathcal{H}_n are conveniently expressed in terms of the projector P onto the subspace \mathcal{H}_n and the orthogonal projector Q , defined by

$$P = \sum_{\alpha} |n\alpha\rangle\langle n\alpha| \quad Q = \sum_{k \neq n, \alpha} |k\alpha\rangle\langle k\alpha|$$

These projectors satisfy

$$P^2 = P \quad Q^2 = Q \quad PQ = QP = 0 \quad P + Q = I \quad [P, H_0] = [Q, H_0] = 0$$

The component $P|\psi\rangle$ is a linear combination of the known unperturbed eigenstates $\{|n\alpha\rangle, \alpha = 1, 2, \dots\}$, and is easily characterized. The orthogonal component $Q|\psi\rangle$ is harder to find. It turns out it is possible to write a neat power series expansion for this solution. Firstly, we have

$$(E - H_0)|\psi\rangle = \lambda H_1|\psi\rangle$$

Now we define a new operator R

$$R \equiv \sum_{k \neq n, \alpha} \frac{|k\alpha\rangle\langle k\alpha|}{E - \epsilon_k}$$



Note: If there are other unperturbed energy levels ϵ_k lying close to ϵ_n , then the perturbation could push the exact energy E near to or past some of these other levels, and then other small denominators would make R ill defined. This will certainly happen if the perturbation is large enough. For the time being we will assume this does not happen, so that R is free of small denominators. When this is not the case we shall refer to "nearly degenerate perturbation theory", which is discussed later.

The operator R satisfies

$$PR = RP = 0 \quad QR = RQ = R \quad R(E - H_0) = (E - H_0)R = Q$$

Then we have

$$R(E - H_0)|\psi\rangle = Q|\psi\rangle = \lambda RH_1|\psi\rangle$$

and

$$|\psi\rangle = P|\psi\rangle + \lambda RH_1|\psi\rangle$$

$|\psi\rangle$ can be solved as a series of $P|\psi\rangle$:

$$|\psi\rangle = \frac{1}{1 - \lambda RH_1} P|\psi\rangle = P|\psi\rangle + \lambda RH_1 P|\psi\rangle + \lambda^2 RH_1 RH_1 P|\psi\rangle + \dots$$

1.1.2 Nondegenerate perturbation theory

In nondegenerate perturbation theory the level ϵ_n of H_0 is nondegenerate. Then the index α is not needed for the level ϵ_n , and we can write simply $|n\rangle$ for the corresponding eigenstate. We assume that $P|\psi\rangle$ is normalized rather than $\psi\rangle$ so that

$$P|\psi\rangle = |n\rangle$$

With this normalization convention, we have

$$\langle n|\psi\rangle = 1$$

Now the series becomes

$$|\psi\rangle = |n\rangle + \lambda \sum_{k \neq n, \alpha} |k\alpha\rangle \frac{\langle k\alpha|H_1|n\rangle}{E - \epsilon_k} + \lambda^2 \sum_{k \neq n, \alpha} \sum_{k' \neq n, \alpha'} |k\alpha\rangle \frac{\langle k\alpha|H_1|k'\alpha'\rangle \langle k'\alpha'|H_1|n\rangle}{(E - \epsilon_k)(E - \epsilon_{k'})}$$

To find an equation for E , we have

$$\langle n|E - H_0|\psi\rangle = E - \epsilon_n = \lambda \langle n|H_1|\psi\rangle$$

then we can get

$$\begin{aligned} E &= \epsilon_n + \lambda \langle n|H_1|n\rangle + \lambda^2 \langle n|H_1RH_1|n\rangle + \lambda^3 \langle n|H_1RH_1RH_1|n\rangle + \dots \\ &= \epsilon_n + \lambda \langle n|H_1|n\rangle + \lambda^2 \sum_{k \neq n, \alpha} \frac{\lambda \langle n|H_1|k\alpha\rangle \langle k\alpha|H_1|n\rangle}{E - \epsilon_k} \\ &\quad + \lambda^3 \sum_{k \neq n, \alpha} \sum_{k' \neq n, \alpha'} \frac{\langle n|H_1|k\alpha\rangle \langle k\alpha|H_1|k'\alpha'\rangle \langle k'\alpha'|H_1|n\rangle}{(E - \epsilon_k)(E - \epsilon_{k'})} + \dots \end{aligned}$$



It is easy to get E up to $O(\lambda^3)$,

$$E = \epsilon_n + \lambda \langle n | H_1 | n \rangle + \lambda^2 \sum_{k \neq n, \alpha} \frac{\lambda \langle n | H_1 | k \alpha \rangle \langle k \alpha | H_1 | n \rangle}{\epsilon_n - \epsilon_k} + O(\lambda^3)$$

and $|\psi\rangle$ up to $O(\lambda^2)$,

$$|\psi\rangle = |n\rangle + \lambda \sum_{k \neq n, \alpha} |k \alpha\rangle \frac{\langle k \alpha | H_1 | n \rangle}{\epsilon_n - \epsilon_k} + O(\lambda^2)$$

Higher corrections can be found on the internet.

1.1.3 Degenerate perturbation theory

In the case that the unperturbed energy level ϵ_n is degenerate, we have

$$P|\psi\rangle = \sum_{\alpha} |n \alpha\rangle c_{\alpha}$$

and

$$\langle n \alpha | P |\psi\rangle = \langle n \alpha | \psi\rangle = c_{\alpha}$$

Then we can obtain an equation for the c_{α} ,

$$\langle n \alpha | E - H_0 | \psi\rangle = c_{\alpha} (E - \epsilon_n) = \lambda \langle n \alpha | H_1 | \psi\rangle$$

then we can get

$$\begin{aligned} (E - \epsilon_n) c_{\alpha} &= \lambda \sum_{\beta} \langle n \alpha | H_1 | n \beta \rangle c_{\beta} + \lambda^2 \sum_{\beta} \langle n \alpha | H_1 R H_1 | n \beta \rangle c_{\beta} + \dots \\ &= \lambda \sum_{\beta} \langle n \alpha | H_1 | n \beta \rangle c_{\beta} + \lambda^2 \sum_{\beta} \sum_{k \neq n, \gamma} \frac{\lambda \langle n \alpha | H_1 | k \gamma \rangle \langle k \gamma | H_1 | n \beta \rangle}{E - \epsilon_k} c_{\beta} + \dots \end{aligned} \quad (1.1)$$

This equation must be solved simultaneously for the eigenvalues E and the unknown expansion coefficients c_{α} .

If we truncate the series at first order, we see that the corrections $E - \epsilon_n$ to the energies are determined as the eigenvalues of the matrix $\langle n \alpha | H_1 | n \beta \rangle$, and the coefficients c_{α} are the corresponding eigenvectors. This determines the energies to first order, but the coefficients c_{α} only to zeroth order. Then $P|\psi\rangle$ becomes known to zeroth order and $Q|\psi\rangle$ to first order.

The first order matrix may or may not have degeneracies itself. If it does not, then all degeneracies are lifted at first order; if it does, the remaining degeneracies may be lifted at a higher order, or may persist to all orders. Degeneracies that persist to all orders are almost always due to some symmetry of the system, which can usually be recognized at the outset.

The higher order corrections can be calculated step by step, which will not be listed here.

Now let us consider the case in which the unperturbed levels of H_0 , while not technically degenerate, are close to one another. Suppose to be specific that two levels, say, ϵ_n and ϵ_m , are close enough to one another that first order perturbations will push the exact level E close to



or onto the unperturbed level ϵ_m .

In this case we choose some energy, call it $\bar{\epsilon}$, which is close to ϵ_n and ϵ_m . Then let us take the original unperturbed Hamiltonian and perturbation and rearrange them in the form,

$$H = H_0 + H_1 = H'_0 + H'_1$$

where

$$\begin{aligned} H_0 &= \sum_{k\alpha} \epsilon_k |k\alpha\rangle \langle k\alpha| \\ H'_0 &= \sum_{k \neq m,n;\alpha} \epsilon_k |k\alpha\rangle \langle k\alpha| + \sum_{k=m,n;\alpha} \bar{\epsilon} |k\alpha\rangle \langle k\alpha| \\ H'_1 &= H_1 + \sum_{k=m,n;\alpha} (\epsilon_k - \bar{\epsilon}) |k\alpha\rangle \langle k\alpha| \end{aligned}$$

Then standard degenerate perturbation theory may be applied. We will call this procedure "nearly degenerate perturbation theory."

1.2 Application of time independent perturbation theory

1.2.1 Stark effect in hydrogen atom

The Stark effect concerns the behaviour of atoms in external electric fields. We choose hydrogen atom because it is single-electron atoms. The hydrogen atom will be modelled with the central force Hamiltonian

$$H_0 = \frac{p^2}{2m} - \frac{e^2}{4\pi r}$$

In this Hamiltonian we ignore spin and other small effects such as relativistic corrections, hyperfine effects and the Lamb shift. These effects cause a splitting and shifting of the energy levels of our simplified model, as well as the introduction of new quantum numbers and new degrees of freedom. But these effects are all small, and if the applied electric field is strong enough, it will overwhelm them and the physical consequences will be much as we shall describe them with our simplified model.

The unperturbed energy levels in hydrogen are given by

$$E_n = -\frac{1}{2n^2} \frac{e^2}{4\pi a_0}$$

where a_0 is the Bohr radius. These levels are n^2 degenerate.

As for the perturbation, let us write \mathbf{F} for the external electric field, and let us take it to lie in the z -direction. Thus, the perturbing potential has the form

$$V_1 = -(-e)\mathbf{F} \cdot \mathbf{x} = eFz$$

For small z , the attractive Coulomb field dominates the total potential and we have the usual Coulomb well that supports atomic bound states. However, for large negative z , the unperturbed potential goes to zero, while the perturbing potential becomes large and negative. At



intermediate values of negative z , the competition between the two potentials gives a maximum in the total potential. The electric force on the electron is zero at the maximum of the potential. Given the relative weakness of the applied field, the maximum must occur at a distance from the nucleus that is large in comparison to the Bohr radius a_0 . Atomic states with small principal quantum numbers n lie well inside this radius. The perturbation analysis we shall perform applies to these states.

The bound states of the unperturbed system are able to tunnel through the potential barrier. When an external electric field is turned on, the bound states of the atom cease to be bound in the strict sense, and become resonances. Electrons that tunnel through the barrier and emerge into the classically allowed region at large negative z will accelerate in the external field, leaving behind an ion. This is the phenomenon of field ionization. This effect can be neglected if the external field is weak enough and the lifetime of the "bound state" is long enough.

In the case of hydrogen, the ground state is $|100\rangle$. The first order shift in the ground state energy level is given by

$$\Delta E_{\text{gnd}}^{(1)} = \langle 100 | eFz | 100 \rangle = 0$$

which vanishes because the parity of z is odd, but $\langle 100 |$ and $|100\rangle$ have the same parity.

For the excited states of hydrogen, according to first order degenerate perturbation theory, the shifts in the energy levels E_n are given by the eigenvalues of the $n^2 \times n^2$ matrix,

$$\langle nlm | eFz | n'l'm' \rangle$$

According to the Wigner-Eckart theorem and parity, the matrix elements vanish unless $l - l' = \pm 1$ and $m = m'$. Consider, for example, the case $n = 2$. The four degenerate states are $|2, 0, 0\rangle$, $|2, 1, -1\rangle$, $|2, 1, 0\rangle$ and $|2, 1, 1\rangle$. Only the states $|2, 0, 0\rangle$ and $|2, 1, 0\rangle$ are connected by the perturbation. Therefore of the 16 matrix elements, the only nonvanishing ones are

$$\langle 2, 0, 0 | eFz | 2, 1, 0 \rangle = -W = -3eFa_0$$

and its complex conjugate. The matrix connecting the two states $|2, 0, 0\rangle$ and $|2, 1, 0\rangle$ is

$$\begin{pmatrix} 0 & -W \\ -W & 0 \end{pmatrix}$$

and its eigenvalues are the first order energy shifts in the $n = 2$ level,

$$\Delta E_2^{(1)} = \pm W$$

In addition, the two states $|2, 1, -1\rangle$ and $|2, 1, 1\rangle$ do not shift their energies at first order. The perturbed eigenfunctions are

$$|+W\rangle = \frac{|2, 0, 0\rangle - |2, 1, 0\rangle}{\sqrt{2}} \quad |-W\rangle = \frac{|2, 0, 0\rangle + |2, 1, 0\rangle}{\sqrt{2}}$$

This is zeroth order part of the exact eigenstates.

Now let us look at the exact symmetries of the full, perturbed Hamiltonian $H = H_0 + H_1$, without doing perturbation theory at all. Since $[H, L_z]$ the exact eigenstates of H can be chosen



to be eigenstates of L_z as well. Denote these by $|\gamma m\rangle$, where γ is an additional index needed to specify an energy eigenstate. Thus, we have

$$L_z|\gamma m\rangle = m|\gamma m\rangle \quad H|\gamma m\rangle = E_{\gamma m}|\gamma m\rangle$$

where $E_{\gamma m}$ is allowed to depend on m since the full rotational symmetry is broken.

As for time reversal, the state $T|\gamma m\rangle$ must be an eigenstate of energy with eigenvalue $E_{\gamma m}$ since $TH = HT$. But because $T^{-1}L_zT = -L_z$, it also follows that $T|\gamma m\rangle$ is an eigenstate of L_z with eigenvalue $-m$. If $m \neq 0$, we must have a degeneracy of at least two. The only energy levels that can be nondegenerate are those with $m = 0$. In the example above, even higher order corrections cannot separate $|2, 1, -1\rangle$ and $|2, 1, 1\rangle$.

1.2.2 Fine structure of hydrogen atom

Fine structure of atoms concerns the effects of relativity and spin on the dynamics of the electron. Both these effects are of the same order of magnitude, and must be treated together in any realistic treatment of the atomic structure.

The fine structure terms account for relativistic effects through order v^2 , and have the effect of enlarging the Hilbert space by the inclusion of the spin degrees of freedom, introducing new quantum numbers, and shifting and splitting the energy levels of the electrostatic model. The splitting in particular means that spectral lines that appear as singlets under low resolution become closely spaced multiplets under higher resolution.

Derivation of the exact form of relativistic corrections of Hamiltonian in quantum mechanics can be very rigorous and needs some reasonable guess. The details of derivation can be found in [lecture notes on fine structure](#) by Robert G. Littlejohn. Here we just list the result.

$$H_{FS} = H_{RKE} + H_D + H_{SO}$$

The term H_{RKE} is due to the second order term of the expansion series of $E = \sqrt{p^2 + m^2}$. (The first order term is just the kinetic energy in non relativistic quantum mechanics). We have

$$H_{RKE} = -\frac{p^4}{8m^3}$$

The term H_D comes out as a result of virtual process $e^- \rightarrow e^- + e^- + e^+$ in the region whose scale is smaller than the Compton length $\lambda_C = \frac{1}{m} = \alpha a_0$ of electrons. Such virtual states appear in perturbation theory when one sums over intermediate states, which derive ultimately from a resolution of the identity. The effect is to smear out the position of the atomic electron over a distance of order λ_C . We have

$$H_D = \frac{1}{8m^2} \nabla^2 V$$

The term H_{SO} arises because the electric field of nuclei generates a magnetic field in the rest frame of electron. We have

$$H_{SO} = \frac{1}{2m^2} \frac{1}{r} \frac{dV}{dr} \mathbf{L} \cdot \mathbf{S}$$

The unperturbed energy levels in hydrogen are given by

$$E_n = -\frac{1}{2n^2} \frac{e^2}{4\pi a_0}$$



When spin of electron is taken into account, these levels are $2n^2$ degenerate. One choice of base is $|nlm_l s\rangle$. It is the eigenvector of operator L^2 , L_z and S_z . However, L_z and S_z do not commute with H_{SO} . A better choice of base is $|nljm_j\rangle$. It is the eigenvector of operator L^2 , J^2 and J_z . H_{SO} , H_{RKE} and H_{SO} are all commute with L^2 , J^2 and J_z . So

$$\langle nl'j'm'_j | H | nljm_j \rangle$$

vanishes unless $l' = l$, $j = j'$ and $m'_j = m_j$. The final results are

$$\langle nljm_j | H_{RKE} | nljm_j \rangle = -\alpha^2 E_n \frac{1}{n^2} \left(\frac{3}{4} - \frac{n}{l + \frac{1}{2}} \right)$$

$$\langle nljm_j | H_D | nljm_j \rangle = -\alpha^2 E_n \frac{1}{n} \delta_{l0}$$

$$\langle nljm_j | H_{SO} | nljm_j \rangle = -\alpha^2 E_n \frac{1}{2n} \frac{j(j+1) - l(l+1) - \frac{3}{4}}{l(l + \frac{1}{2})(l+1)}$$

When we add them up to get the total energy shift due to the fine structure we find

$$\Delta E_{FS} = -\alpha^2 E_n \frac{1}{n^2} \left(\frac{3}{4} - \frac{n}{j + \frac{1}{2}} \right)$$

It is independent of the orbital angular momentum quantum number l , although each of the individual terms does depend on l . However, the total energy shift does depend on j in addition to the principal quantum number n , so when we take into account the fine structure corrections, the energy levels of hydrogen atom have the form E_{nj} .

1.3 Time dependent perturbation theory

1.4 Atomic Radiation

1.5 The classical limit



Chapter 2

Many body problem



2.1 Identical particles

2.2 Non-relativistic quantum field theory

Chapter 3

Scattering theory



3.1 Lippmann–Schwinger equation

Imagine a particle coming in and getting scattered by a short-ranged potential $V(x)$ located around the origin $x \sim 0$. The time-independent Schrödinger equation is simply

$$(H_0 + V)|\psi\rangle = E|\psi\rangle$$

Here, $H_0 = \frac{p^2}{2m}$ is the free-particle Hamiltonian operator. We can write the solution as

$$|\psi^{(\pm)}\rangle = \frac{1}{E - H_0 \pm i\epsilon} V |\psi^{(\pm)}\rangle + |\phi\rangle$$

Here, $H_0|\phi\rangle = E|\phi\rangle$. In coordinate representation,

$$\psi^{(\pm)}(\mathbf{x}) = \phi(\mathbf{x}) + \int d^3x' \langle \mathbf{x} | \frac{1}{E - H_0 \pm i\epsilon} | \mathbf{x}' \rangle V(\mathbf{x}') \psi^{(\pm)}(\mathbf{x}')$$

Here, $\phi(\mathbf{x}) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{\frac{3}{2}}}$. Define the Green function as

$$G_{\pm}(\mathbf{x}, \mathbf{x}') \equiv \frac{1}{2m} \langle \mathbf{x} | \frac{1}{E - H_0 \pm i\epsilon} | \mathbf{x}' \rangle$$

We can derive that

$$G_{\pm}(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi} \frac{e^{\pm ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|}$$

where $k = \sqrt{2mE}$. And it is easy to show that

$$(\nabla^2 + k^2)G_{\pm}(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$$

So, we have

$$\psi^{(\pm)}(\mathbf{x}) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{\frac{3}{2}}} - 2m \int d^3x' \frac{1}{4\pi} \frac{e^{\pm ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} V(\mathbf{x}') \psi^{(\pm)}(\mathbf{x}')$$

We now can interpret $\psi^+(\mathbf{x})$ as a superposition of incident plane wave and scattered wave which propagate from scatterer to outside region. From now on, we will denote it as $\psi(\mathbf{x})$.

The experiment is done typically by placing the detector far away from the scatterer $|\mathbf{x}| \ll a$ where a is the "size" of the scatterer. The integration over \mathbf{x}' , on the other hand, is limited within the "size" of the scatterer because of the $V(\mathbf{x}')$ factor. Therefore, we are in the situation $|\mathbf{x}| \ll |\mathbf{x}'|$, and hence can use the approximation

$$|\mathbf{x} - \mathbf{x}'| \approx |\mathbf{x}| - \frac{\mathbf{x}' \cdot \mathbf{x}}{|\mathbf{x}|}$$

Under this limit,

$$\psi(\mathbf{x}) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{\frac{3}{2}}} - 2m \frac{e^{ikr}}{4\pi r} \int d^3x' e^{-\mathbf{k}'\cdot\mathbf{x}'} V(\mathbf{x}') \psi(\mathbf{x}')$$

Here, $r = |\mathbf{x}|$ and $\mathbf{k}' = k \frac{\mathbf{x}}{r}$. It is customary to write this equation in the form

$$\psi(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \left(e^{i\mathbf{k}\cdot\mathbf{x}} + f(\mathbf{k}, \mathbf{k}') \frac{e^{ikr}}{r} \right)$$

Here,

$$f(\mathbf{k}, \mathbf{k}') \equiv -\frac{m}{2\pi} (2\pi)^3 \langle \mathbf{k}' | V | \psi \rangle$$

Recall the definition of cross section

$$\sigma \equiv \frac{\text{Number of Events}}{\text{Time} \times \text{Incident Flux}}$$

So, the differential cross section for particles being scattered into the solid angle is

$$d\sigma = \frac{|\mathbf{j}_{\text{scatt}}| r^2 d\Omega}{|\mathbf{j}_{\text{inc}}|} = |f(\mathbf{k}, \mathbf{k}')|^2 d\Omega$$

In a more realistic situation, we should use wave packets to describe the scattering process. The basic picture is a free wave packet approaches the scattering center. After a long time, we have both the original wave packet moving in the original direction plus a spherical wave front that moves outward. The details can be found in the section 3 of the lecture notes *Scattering Theory I (Hitoshi Murayama)*.

Furthermore, if we require that the normalization of the wave function should always satisfy $\int dx^3 |\psi(\mathbf{x})|^2$ for any t , as guaranteed by the unitarity of time evolution operator. This requirement leads to a special requirement on the scattered wave, and hence $f(\mathbf{k}, \mathbf{k}')$, from which we can derive the optical theorem.

Theorem 3.1 Optical theorem

$$\text{Im} f(\theta = 0) = \frac{k\sigma_{\text{tot}}}{4\pi}$$

where

$$f(\theta = 0) \equiv f(\mathbf{k}, \mathbf{k}),$$

the setting of $\mathbf{k} \equiv \mathbf{k}'$ imposes scattering in the forward direction, and

$$\sigma_{\text{tot}} = \int \frac{d\sigma}{d\Omega} d\Omega$$

The meaning of this theorem is clear. Because the scattered wave takes the probability away to different directions, the total probability for the particle to go to the forward direction (unscattered) should decrease. This decrease is caused by the interference between the unscattered and scattered waves and hence is proportional to $f(0)$. On the other hand, the amount of decrease in the forward direction should equal the total probability at other directions, which is proportional to the total cross section. The proof can be found in the section 4 of the lecture notes *Scattering Theory I (Hitoshi Murayama)*.



3.2 Born approximation

If $|\psi\rangle = |\phi\rangle + O(V)$ is close to $|\phi\rangle$, we can solve the Lippmanmn-Schwinger equation by perturbation theory. The lowest order approximation in V is

$$|\psi\rangle = \frac{1}{E - H_0 + i\epsilon} V |\phi\rangle + |\phi\rangle$$

This is called Born approximation. In coordinate representation,

$$f^{(1)}(\mathbf{k}, \mathbf{k}') = -\frac{m}{2\pi} \int d^3x V(\mathbf{x}) e^{i\mathbf{q}\cdot\mathbf{x}}$$

Here, $\mathbf{q} = |\mathbf{k} - \mathbf{k}'|$. If the potential is central, we can derive that

$$f^{(1)}(\mathbf{k}, \mathbf{k}') = -\frac{2m}{q} \int_0^\infty dr r V(r) \sin(qr)$$

Yukawa potential

$$V = \frac{\alpha}{r} e^{-\mu r}$$

So, we can derive

$$f(\theta) = -\frac{2m\alpha}{q^2 + \mu^2}$$

Different cross section is therefore given by

$$\frac{d\sigma}{d\Omega} = (2m\alpha)^2 \frac{1}{[2k^2(1 - \cos\theta) + \mu^2]^2}$$

The total cross section is obtained by integrating over $d\Omega$,

$$\sigma = (2m\alpha)^2 \frac{4\pi}{4k^2\mu^2 + \mu^4}$$

Coulomb potential

$$V = \frac{\alpha}{r}$$

Take the limit $\mu \rightarrow 0$, we can get

$$f(\theta) = -\frac{2m\alpha}{q^2}$$

Different cross section is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{4E}\right)^2 \frac{1}{\sin^4 \frac{\theta}{2}}$$

The total cross section diverges. The divergence is in the $\cos\theta$ integral when $\theta \rightarrow 0$. In other words, the divergence occurs for the small momentum transfer $q \rightarrow 0$, which corresponds to large distances. The reason why the total cross section diverges is because the Coulomb potential is actually a long-range force. No matter how far the incident particles are from the charge, there is always an effect on the motion of the particles and they get scattered.



Form factor

If the source of Coulomb potential has an distribution $\rho_N(\mathbf{x})$, then

$$V(\mathbf{x}) = \int d^3x' \frac{\alpha}{|\mathbf{x} - \mathbf{x}'|} \rho(\mathbf{x}')$$

Note that the potential is mathematically a convolution of the Coulomb potential and the probability density. Since the first Born amplitude is nothing but the Fourier transform of the potential, the convolution becomes a product of Fourier transforms, one for the Coulomb potential and the other for the probability density. So

$$f(\theta) = f(\theta)_{\text{pointlike}} F(q)$$

Here,

$$F(q) \equiv \int d^3x \rho_N(\mathbf{x}) e^{i\mathbf{q} \cdot \mathbf{x}},$$

being called form factor.

Born expansion

Define T-matrix by

$$V|\psi\rangle = T|\phi\rangle$$

Using the definition of the T-matrix, we find

$$f(\mathbf{k}, \mathbf{k}') = -\frac{m}{2\pi} (2\pi)^3 \langle \mathbf{k}' | T | \mathbf{k} \rangle$$

Using the Lippmann–Schwinger equation and multiplying the both sides by V from left, we find

$$T|\phi\rangle = V \frac{1}{E - H_0 + i\epsilon} T|\phi\rangle + V|\phi\rangle$$

A formal solution to the T-matrix is

$$T = \frac{1}{1 - V \frac{1}{E - H_0 + i\epsilon}} V$$

By Taylor expanding this operator in geometric series, we find

$$T = V + V \frac{1}{E - H_0 + i\epsilon} V + V \frac{1}{E - H_0 + i\epsilon} V \frac{1}{E - H_0 + i\epsilon} V + \dots$$

So,

$$|\psi\rangle = \left(1 + \frac{1}{E - H_0 + i\epsilon} V + \frac{1}{E - H_0 + i\epsilon} V \frac{1}{E - H_0 + i\epsilon} V + \dots \right) |\phi\rangle$$

The first term is the wave which did not get scattered. The second term is the wave that gets scattered at a point in the potential and then propagates outwards by the propagator. In the third term, the wave gets scattered at a point in the potential, propagates for a while, and gets scattered again at another point in the potential, and propagates outwards. In the $n + 1$ -th term, there are n times scattering of the wave before it propagates outwards.



3.3 Partial wave analysis

Partial wave expansion

When the potential is **central**, angular momentum is conserved due to Noether's theorem. Therefore, we can expand the wave function in the eigenstates of the angular momentum. Obtained waves with definite angular momenta are called partial waves. We can solve the scattering problem for each partial wave separately, and then in the end put them together to obtain the full scattering amplitude. The plane wave can be expanded as follows.

$$e^{ikz} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos \theta)$$

Here, $j_l(kr)$ is spherical Bessel functions of first kind. The asymptotic behaviour of $j_l(kr)$ at large r can be written as

$$j_l(kr) \sim \frac{\sin(kr - \frac{l\pi}{2})}{kr}$$

so,

$$e^{ikz} \sim \frac{1}{2ikr} \sum_{l=0}^{\infty} (2l+1) (e^{ikr} - (-1)^l e^{-ikr}) P_l(\cos \theta)$$

Meanwhile, the f factor can be expanded as

$$f(\theta) = \sum_{l=0}^{\infty} f_l (2l+1) P_l(\cos \theta)$$

Optical theorem constraint

The cross section can be represented by expansion coefficient of f factor as

$$\sigma = 4\pi \sum_l (2l+1) |f_l|^2$$

On the other hand,

$$\text{Im} f(0) = \sum_l (2l+1) \text{Im} f_l$$

From optical theorem we can derive that

$$|f_l|^2 = \frac{1}{k} \text{Im} f_l$$

This constraint can be rewritten as

$$|1 + 2ikf_l|^2 = 1$$

So we can define a phase δ_l as

$$1 + 2ikf_l = e^{i\delta_l}$$

or equivalently,

$$f_l = \frac{1}{k} e^{i\delta_l} \sin(\delta_l)$$



Phase shifts

We can derive the asymptotic behaviour of the wave function as

$$\psi(\mathbf{x}) \sim \frac{1}{2ikr} \sum_l (2l+1) P_l(\cos \theta) [e^{ikr} e^{2i\delta_l} - (-1)^l e^{-ikr}]$$

Compare it to the case of the plane wave without scattering. What this equation says is that the wave converging on the scatterer has the well-defined phase factor $-(-1)^l$, the same as in the case without scattering. On the other hand, the wave that emerges from the scatterer has an additional phase factor $e^{2i\delta_l}$. All what scattering did is to shift the phase of the emerging wave by $2\delta_l$. The reason why this is merely a phase factor is the conservation of probability. What converged to the origin must come out with the same strength. But this shift in the phase causes the interference among all partial waves different from the case without the phase shifts, and the result is not a plane wave but contains the scattered wave.

In terms of the phase shifts, the cross section is given by

$$\sigma = \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2 \delta_l$$

Actual calculation of phase shifts is basically to solve the Schrödinger equation for each partial waves,

$$\left[-\frac{1}{r} \frac{d^2}{dr^2} r + \frac{l(l+1)}{r^2} + 2mV(r) \right] R_l(r) = k^2 R_l(r)$$

After solving the equation, we take the asymptotic limit $r \rightarrow \infty$, and write $R_l(r)$ as a linear combination of $j_l(kr) \cos \delta_l + n_l(kr) \sin \delta_l$. The relative coefficients of j_l and n_l determines the phase shift δ_l , and hence the cross section.

