Summary on Classical Mechanics

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September 21, 2016

1 Lagrangian Formulation

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt, \quad \delta q_i(t_1) = \delta q_i(t_2) = 0$$

$$\delta S = 0 \rightarrow \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0$$

1.1 Example

The form of Lagrangian for a system of particles in inertial frame:

$$L = \sum_a \frac{1}{2} m_a v_a^2 - U(\vec{r_1}, \vec{r_2}, \cdots,)$$

To get the form of Lagrangian for a system of interacting particles, we must assume:
(1) Space and time are homogeneous and isotropic in inertial frame;
(2) Galileo's relativity principle and Galilean transformation;
(3) Spontaneous interaction between particles;

2 Symmetry and Conservation Laws(1)

2.1 Nother's theorem

For $q_i \rightarrow q_i + \delta q_i$ and $L \rightarrow L + \delta L$, if $\delta L = \frac{df(q,\dot{q},t)}{dt}$,then we get

$$\frac{d}{dt}(p^i \delta q_i - f) = 0 \quad (p^i = \frac{\partial L}{\partial \dot{q}_i})$$

2.2 Homogeneity of time

If $\frac{\partial L}{\partial t} = 0$,then we get

$$\frac{dH}{dt} = 0 \quad (H = \sum_i \dot{q}_i p^i - L)$$

3 Hamilton formulation

$$p^i = \frac{\partial L}{\partial \dot{q}_i}$$

$$H(q,p,t) = \sum_i p^i \dot{q}_i - L$$

$$\dot{p^i} = -\frac{\partial H}{\partial q_i} \qquad \dot{q}_i = \frac{\partial H}{\partial p^i}$$

3.1 Poisson Brackets

First, we assume the bracket operation has the following properties:

$$[f,g] = -[g,f]$$

$$[\alpha_1 f_1 + \alpha_2 f_2, \beta_1 g_1 + \beta_2 g_2] = \alpha_1\beta_1 [f_1,g_1] + \alpha_1\beta_2 [f_1,g_2] + \alpha_2\beta_1 [f_2,g_1] + \alpha_2\beta_2 [f_2,g_2]$$

$$[f_1 f_2, g_1 g_2] = f_1 [f_2,g_1] g_2 + f_1 g_1 [f_2,g_2] + g_1 [f_1,g_2] f_2 + [f_1,f_2] g_2 f_2$$

$$[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$$

Here, f,g,h are functions of p^i, q_i, t. Then, if we assume that

$$\left[q_i, p^k\right] = \delta_i^k$$

we can deduce that

$$[f,g] = \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p^k} - \frac{\partial f}{\partial p^k}\frac{\partial g}{\partial q_k}\right)$$

The Hamilton equation can be written as

$$\dot{p^i} = \left[p^i, H\right] \qquad \dot{q}_i = [q_i, H] \tag{1}$$

And we can also derive that $\frac{df}{dt} = [f,H]$.

4 Symmetry and Conservation Laws(2)

Suppose g is a function of p and q. If the transformation of q and p can be described as

$$q \to q + \epsilon[q,g]$$

$$p \to p + \epsilon[p,g]$$

We can prove that

$$H \to H + \epsilon[H,g]$$

So if H is invariant under the transformation, then $[H,g] = 0$, that means $\frac{dg}{dt} = 0$, i.e. g is a conserved quantity of the motion.

5 Hamilton-Jacobi equation

We define

$$S(q,t) = \left(\int_{q_0,t_0}^{q,t} L dt\right)|_{extremum}$$

We can prove that

$$p = \frac{\partial S}{\partial q}, \quad H = -\frac{\partial S}{\partial t}$$

So, we have

$$-\frac{\partial S}{\partial t} = H(q, \frac{\partial S}{\partial q})$$

This is called Hamiltonian-Jacobi equation.

6 Symmetry and Conservation Laws(3)

If S is invariant under transformation $q_i \to q_i + \delta q_i$, then

$$\delta S = (\sum_i p^i \delta q_i)|_{q_0,t_0}^{q,t} = 0$$

So, we have

$$\frac{d}{dt}(p^i \delta q_i) = 0$$

Further more, if

$$\delta S = (\sum_i p^i \delta q_i)|_{q_0,t_0}^{q,t} = f(q_i, \dot{q}_i, t)|_{q_0,t_0}^{q,t}$$

we will have conserved quantity

$$\frac{d}{dt}(p^i \delta q_i - f) = 0$$