

Summary on Classical Field Theory

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1 Mechanics within special relativity

1.1 Basic Assumption

First, we assume there is an upper limit of velocity of propagation of interaction c . Second, we assume that inertial reference frame are all the same in describing the law of physics. Then, we can find the invariant intervals when transforming from one inertial reference frame to another, $ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$. (In the following, we assume that $c = 1$.) This transformation is called Lorentz transformation, which can be written as

$$\bar{x}^\mu = \Lambda^\mu_\nu x^\nu$$

and it is easy to verify that

$$\eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = \eta_{\rho\sigma},$$

where,

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{bmatrix}$$

and

$$(\Lambda^{-1})^\rho_\nu = \Lambda_\nu^\rho$$

In a special case when we boost along $\hat{1}$ direction, we have

$$\bar{x}^0 = \gamma x^0 - \gamma\beta x^1$$

$$\bar{x}^1 = -\gamma\beta x^0 + \gamma x^1$$

Some physical quantity will behave like a tensor (vector, scalar) when transforming from one inertial frame to another. For example,

scalar proper time: $d\tau$, mass: m , electrical charge e

vector four velocity: $v^\mu = \frac{dx^\mu}{d\tau}$, four momentum: $p^\mu = mv^\mu$, four acceleration: $a^\mu = \frac{du^\mu}{d\tau}$, four force: $f^\mu = ma^\mu$

1.2 "Three vector"

three velocity: $\hat{u}^i = \frac{dx^i}{dt}$

$$u^0 = \gamma_v, u^i = \gamma \hat{u}^i$$

Transformation of three velocity when we boost along $\hat{1}$ direction:

$$\bar{\hat{v}}^1 = \frac{\hat{v}^1 - \beta}{1 - \hat{v}^1 \beta}$$

$$\bar{\hat{v}}^2 = \frac{\hat{v}^2}{\gamma(1 - \hat{v}^1 \beta)}$$

$$\bar{\hat{v}}^3 = \frac{\hat{v}^3}{\gamma(1 - \hat{v}^1 \beta)}$$

three momentum: $\hat{p}^i = p^i$

$$\hat{p}^i \gamma_v \hat{v}^i$$

three acceleration: $\hat{a}^i = \frac{dv^i}{dt}$

three force: $\hat{f}^i = \frac{d\hat{p}^i}{dt}$

$$f^i = \gamma_v \hat{f}^i$$

Energy: $E = p^0 = mu^0 = \gamma_v m$

1.3 Mechanics

Revised newton's second law:

$$f^\mu = \frac{dp^\mu}{d\tau}$$

It can be written in three vector language as

$$\hat{f}^i = \gamma_v m \hat{a}^i + \gamma_v^3 (\hat{a}^j \hat{v}_j) m \hat{v}^i$$

1.4 Lagrangian formulation

$$S = -m \int_a^b d\tau, \quad \delta x^\mu(a) = \delta x^\mu(b) = 0$$

$$\delta S = 0 \Rightarrow m \frac{du^\mu}{d\tau} = 0$$

1.5 Hamiltonian formulation

$$S = -m \int_{t_1}^{t_2} \sqrt{1 - \dot{x}_i \dot{x}^i} dt$$

$$L = -m \sqrt{1 - \dot{x}_i \dot{x}^i}$$

$$\pi^i = \frac{\partial L}{\partial \dot{x}_i} = \gamma m \eta^{ij} \dot{x}_j$$

$$H = \pi^i \dot{x}_i - L = \gamma m = \sqrt{m^2 + \pi^i \pi_i}$$

1.5.1 Hamilton equation

$$\dot{\pi}^i = 0, \quad \dot{x}_i = \eta_{ij} \frac{\pi^j}{\sqrt{m^2 + \pi^k \pi_k}}$$

1.5.2 Hamiltonian-Jacobi equation

$$H = -\frac{\partial S}{\partial t}, \quad \pi^i = \frac{\partial S}{\partial x_i}$$

If we define $p^0 = H$, $p^i = \pi^i$, then we can verify that $p^\mu = \frac{\partial S}{\partial x_\mu}$. So, p^μ is a vector under Lorentz transformation. The Hamiltonian-Jacobi equation can be written as

$$\left(\frac{\partial S}{\partial t}\right)^2 = m^2 + \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2$$

1.6 Symmetry and conservation law

1.6.1 Translational symmetry and conservation of momentum

$$\begin{aligned} \bar{x}^\mu &= x^\mu + \delta x^\mu \\ \delta S &= \sum m u_\mu \delta x^\mu|_a^b = 0 \end{aligned}$$

$\sum p^\mu$ is conserved.

1.6.2 Rotational symmetry and conservation of angular momentum

$$\begin{aligned} \bar{x}^\mu &= x^\mu + x_\nu \delta \Omega^{\mu\nu} \\ \delta S &= \sum m u^\mu x^\nu \delta \Omega_{\mu\nu}|_a^b = 0 \end{aligned}$$

$\sum M^{\mu\nu}$ is conserved, where $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$.

2 Classical field theory

2.1 Lagrangian formulation

$$\begin{aligned} S &= \int \mathcal{L}(\phi_a, \dot{\phi}_a, \nabla \phi_a) d^4x, \quad \delta \phi_a|_\Sigma = 0 \\ \delta S &= 0 \Rightarrow \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0 \end{aligned}$$

2.1.1 Locality of the theory

There are no terms in the Lagrangian coupling $\phi(\vec{x}, t)$ directly to $\phi(\vec{y}, t)$ with $\vec{x} \neq \vec{y}$. The closest we get for the \vec{x} label is coupling between $\phi(\vec{x}, t)$ and $\phi(\vec{x} + \delta \vec{x}, t)$ through the gradient term $\nabla \phi$.

2.1.2 Lorentz invariance

Scalar fields:

$$\bar{\phi}(x) = \phi(\Lambda^{-1}x)$$

Vector fields:

$$\begin{aligned} \bar{A}^\mu(x) &= \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) \\ \bar{A}_\mu(x) &= (\Lambda^{-1})^\nu{}_\mu A_\nu(\Lambda^{-1}x) = \Lambda_\mu{}^\nu A_\nu(\Lambda^{-1}x) \\ \bar{\partial}_\mu \bar{\phi}(x) &= (\Lambda^{-1})^\nu{}_\mu \partial_\nu \phi(\Lambda^{-1}x) = \Lambda_\mu{}^\nu \partial_\nu \phi(\Lambda^{-1}x) \end{aligned}$$

Lorentz invariance Lagrangian is a scalar, or more loosely, action is invariant under Lorentz transformation.

2.2 Symmetry and conservation law

2.2.1 Noether's theorem

Every continuous symmetry of the Lagrangian gives rise to a conserved current $j^\mu(x)$ such that the equation of motion imply $\partial_\mu j^\mu = 0$.

Suppose that the infinitesimal transformation is

$$\phi_a \rightarrow \phi_a + \delta\phi_a$$

$$\mathcal{L} \rightarrow \mathcal{L} + \delta\mathcal{L}$$

and if $\delta\mathcal{L} = \partial_\mu K^\mu = 0$, we can get

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a - K^\mu$$

space-time translation $\bar{x} = x - a$.

$$j^\mu = -a_\nu T^{\mu\nu}$$

$$T^{\mu\nu} \equiv -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^\nu\phi_a + \eta^{\mu\nu}\mathcal{L}$$

If we define $P^\mu = \int T^{0\mu} d^3x$, then we have

$$\frac{dP^\mu}{dt} = 0$$

Lorentz Transformation $\bar{x}^\mu = x^\mu + \delta\omega^\mu{}_\nu x^\nu$.

This time, we assume that

$$\bar{\phi}_a(x) = S_a{}^b \phi_b(\Lambda^{-1}x)$$

In the limit of infinitesimal Lorentz transformation, we have

$$S_a{}^b = \delta_a{}^b + \frac{1}{2}\delta\omega_{\alpha\beta}(\Sigma^{\alpha\beta})_a{}^b$$

$$j^\mu = -\frac{1}{2}M^{\mu\nu\rho}\delta\omega_{\nu\rho}$$

$$M^{\mu\nu\rho} \equiv x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(\Sigma^{\nu\rho})_a{}^b \phi_b$$

If we define $M^{\nu\rho} = \int M^{0\nu\rho} d^3x$, then we have

$$\frac{dM^{\nu\rho}}{dt} = 0$$

2.3 Hamiltonian formulation

$$\pi^a(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_a}$$

$$\mathcal{H}(\phi_a, \nabla\phi_a, \pi^a) = \pi^a \dot{\phi}_a - \mathcal{L}$$

$$H = \int \mathcal{H} d^3x$$

Now, we define

$$\frac{\delta}{\delta f(q_0)} F = \frac{\delta}{\delta f}(F\delta(q - q_0)) = \frac{\delta\mathcal{F}}{\delta f}(q_0),$$

where $F = \int \mathcal{F}(f(q))dq$. Now, we can get the Hamilton equation form $\delta S = 0$,

$$\begin{aligned}\dot{\phi}_a(\vec{x}, t) &= \frac{\delta}{\delta \pi^a(\vec{x}, t)} H = \frac{\partial \mathcal{H}}{\partial \pi^a} \\ \dot{\pi}^a(\vec{x}, t) &= \frac{\delta}{\delta \phi_a(\vec{x}, t)} H = -\frac{\partial \mathcal{H}}{\partial \phi_a} + \left(\frac{\partial \mathcal{H}}{\partial \phi_{a,b}} \right)_{,b}\end{aligned}$$

2.3.1 Poission bracket

First, we define that

$$[\phi_a(\vec{x}), \pi^b(\vec{y})] = \delta_a^b \delta(\vec{x} - \vec{y})$$

then, we assume the bracket operation has the same properties as the Poission bracket in classical mechanics. And we also assume that

$$[\partial_x A(\vec{x}, t), B(\vec{y}, t)] = \partial_x [A(\vec{x}, t), B(\vec{y}, t)]$$

and

$$[\int d^3x A(\vec{x}, t), B(\vec{y}, t)] = \int d^3x [A(\vec{x}, t), B(\vec{y}, t)]$$

It is easy to verify that

$$[\phi_a(\vec{x}), H] = \frac{\delta}{\delta \pi^a(\vec{x})} H, \quad [\pi^a(\vec{x}), H] = \frac{\delta}{\delta \phi_a(\vec{x})} H$$

So, the Hamilton equation can be written as

$$\dot{\phi}_a = [\phi_a, H], \quad \dot{\pi}^a = [\pi^a, H]$$

Further more, we can prove

$$\frac{dO(\phi, \pi, t)}{dt} = [O, H] + \frac{\partial O}{\partial t}$$

and

$$\frac{d[A, B]}{dt} = [A, \frac{dB}{dt}] + [\frac{dA}{dt}, B]$$

2.3.2 Momentum

It is easy to verify that

$$P^0 = H, \quad P^i = \int -\pi^a \partial^i \phi_a d^3x$$

And we can get the commutation relationship that

$$\begin{aligned}[\phi_a, P^\mu] &= -\partial^\mu \phi_a \\ [\pi^a, P^\mu] &= -\partial^\mu \pi^a \\ [P^\mu, P^\nu] &= 0\end{aligned}$$

2.3.3 Angular momentum

It is easy to verify that

$$M^{\mu\nu} = \int (x^\mu T^{0\nu} - x^\nu T^{0\mu} - \pi^a (\Sigma^{\mu\nu})_a^b \phi_b) d^3x$$

We denote that

$$M_L^{\mu\nu} = \int (x^\mu T^{0\nu} - x^\nu T^{0\mu}) d^3x \quad M_S^{\mu\nu} = \int (-\pi^a (\Sigma^{\mu\nu})_a^b \phi_b) d^3x$$

$$(L^{\mu\nu})_a{}^b = -(x^\mu \partial^\nu - x^\nu \partial^\mu) \delta_a{}^b \quad (S^{\mu\nu})_a{}^b = -(\Sigma^{\mu\nu})_a{}^b$$

So, we have the commutation relationship that

$$\begin{aligned} M^{\mu\nu} &= M_L^{\mu\nu} + M_S^{\mu\nu} \\ [\phi_a, M_L^{\mu\nu}] &= (L^{\mu\nu})_a{}^b \phi_b \quad [\phi_a, M_S^{\mu\nu}] = (S^{\mu\nu})_a{}^b \phi_b \\ [\pi^a, M_L^{\mu\nu}] &= (L^{\mu\nu})^a{}_b \pi^b \quad [\pi^a, M_S^{\mu\nu}] = -(S^{\mu\nu})^a{}_b \pi^b \end{aligned}$$

Because $\frac{dM^{\mu\nu}}{dt} = 0$, we can prove that

$$[[\phi(x), M^{\mu\nu}], M^{\rho\sigma}] = (L^{\mu\nu} + S^{\mu\nu})(L^{\rho\sigma} + S^{\rho\sigma})\phi(x)$$

and then we can get the communication relationship from the Jacobi identity,

$$\begin{aligned} [\phi(x), [M^{\mu\nu}, M^{\rho\sigma}]] &= (L^{\mu\nu} L^{\rho\sigma} - L^{\rho\sigma} L^{\mu\nu} + S^{\mu\nu} S^{\rho\sigma} - S^{\rho\sigma} S^{\mu\nu})\phi(x) \\ L^{\mu\nu} L^{\rho\sigma} - L^{\rho\sigma} L^{\mu\nu} &= -g^{\nu\rho} L^{\mu\sigma} + g^{\sigma\mu} L^{\rho\nu} + g^{\mu\rho} L^{\nu\sigma} - g^{\sigma\nu} L^{\rho\mu} \end{aligned}$$

If we demand that

$$S^{\mu\nu} S^{\rho\sigma} - S^{\rho\sigma} S^{\mu\nu} = -g^{\nu\rho} S^{\mu\sigma} + g^{\sigma\mu} S^{\rho\nu} + g^{\mu\rho} S^{\nu\sigma} - g^{\sigma\nu} S^{\rho\mu}$$

We can get the communication relationship of the $M^{\mu\nu}$,

$$[M^{\mu\nu}, M^{\rho\sigma}] = -g^{\nu\rho} M^{\mu\sigma} + g^{\sigma\mu} M^{\rho\nu} + g^{\mu\rho} M^{\nu\sigma} - g^{\sigma\nu} M^{\rho\mu}$$

up to the possibility of a term on the right-hand side that commutes with $\phi(x)$ and its derivatives.

We now define $J_i \equiv \frac{1}{2} \epsilon_{ijk} M^{jk}$ and $K_i \equiv M^{i0}$, the communication relationship can be written as

$$\begin{aligned} [J_i, J_j] &= \epsilon_{ijk} J_k \\ [J_i, K_j] &= \epsilon_{ijk} K_k \\ [K_i, K_j] &= -\epsilon_{ijk} J_k \end{aligned}$$

We can use the similar method to derive that

$$[P^\mu, M^{\rho\sigma}] = g^{\mu\sigma} P^\rho - g^{\mu\rho} P^\sigma$$

It can also be written as

$$\begin{aligned} [J_i, H] &= 0 \\ [J_i, P_j] &= \epsilon_{ijk} P_k \\ [K_i, H] &= P_i \\ [K_i, P_j] &= \delta_{ij} H \end{aligned}$$

At last, we define $L_i \equiv \frac{1}{2} \epsilon_{ijk} M_L^{jk}$ and $S_i \equiv \frac{1}{2} \epsilon_{ijk} M_S^{jk}$ we can demonstrate that

$$\begin{aligned} [L_i, S_j] &= 0 \\ [S_i, P_j] &= 0 \\ [L_i, P_j] &= \epsilon_{ijk} P_k \end{aligned}$$