Summary on QFT

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1 From classical field to quantum field

1.1 Heisenberg picture of fields

The state of the field is described by an element $|\psi\rangle$ in Hilbert space. The measurement of the field is described by an operator field $\phi_a(\vec{x},t)$. In Heisenberg picture, the dynamic of the field satisfy the equation

$$\frac{d\phi_a(x)}{dt} = -i[\phi_a(x), H]$$

So, the mean value of the measurement of the field is described by Erenfest theorem

$$\frac{d\langle\psi|\phi_a|\psi\rangle}{dt} = -i\langle\psi|[\phi_a, H]|\psi\rangle$$

If $[\phi_a, H]_Q = i[\phi_a, H]_C$, we can reproduce the classical field equation. We also note that the bracket operation here $[A, B] = AB - BA$ has the same properties as the poission bracket in classical mechanics. So, what we need here is the canonical quantization

$$[\phi_a(\vec{x},t), \phi_b(\vec{y},t)] = 0 \quad [\pi^a(\vec{x},t), \pi^b(\vec{y},t)] = 0 \quad [\phi_a(\vec{x},t), \pi^b(\vec{y},t)] = i\delta_a^b\delta(\vec{x}-\vec{y})$$

and the definition of $\mathcal{L}$,π^a and H is the same as those in corresponding classical theory. Then we can recover the classical field theory.

1.2 Lorentz invariance in quantum field theory

$$|\bar{\psi}\rangle = U(\Lambda)|\psi\rangle$$

Scalar fields:

$$\langle\bar{\psi}|\phi(x)|\bar{\psi}\rangle = \langle\psi|\phi(\Lambda^{-1}x)|\psi\rangle$$

$$U^{-1}(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$$

Vector fields:

$$\langle\bar{\psi}|A^\mu(x)|\bar{\psi}\rangle = \langle\psi|\Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)|\psi\rangle$$

$$U^{-1}(\Lambda)A^\mu(x)U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$$

Lorentz invariance Lagrangian is a scalar, or more loosely, action is invariant under Lorentz transformation.

1.3 Momentum

The definition of momentum is the same as that in classical theory.

$$T^{\mu\nu} \equiv -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\partial^\nu \phi_a + \eta^{\mu\nu}\mathcal{L} \quad \partial_\mu T^{\mu\nu} = 0$$

and

$$P^\mu = \int T^{0\mu} d^3x \quad \frac{dP^\mu}{dt} = 0$$

$$P^0 = H, \quad P^i = \int -\pi^a \partial^i \phi_a d^3x$$

And we can get the commutation relationship that

$$\begin{aligned} [\phi_a, P^\mu] &= -i\partial^\mu \phi_a \\ [\pi^a, P^\mu] &= -i\partial^\mu \pi^a \\ [P^\mu, P^\nu] &= 0 \end{aligned}$$

We denote the translation operator as $T(s)$, so

$$T^{-1}(s)\phi_a(x)T(s) = \phi_a(x-s)$$

we can deduce that

$$T(\epsilon) = 1 - i\epsilon_\mu P^\mu \quad T(s) = e^{-iP^\mu s_\mu}$$

1.4 Angular Momentum

The definition of Angular momentum is the same as that in classical theory.

$$M^{\mu\nu\rho} \equiv x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}(\Sigma^{\nu\rho})_a{}^b \phi_b$$

and

$$M^{\nu\rho} = \int M^{0\nu\rho} d^3x \quad \frac{dM^{\nu\rho}}{dt} = 0$$

$$M^{\mu\nu} = \int (x^\mu T^{0\nu} - x^\nu T^{0\mu} - \pi^a (\Sigma^{\mu\nu})_a{}^b \phi_b) d^3x$$

We denote that

$$M_L^{\mu\nu} = \int (x^\mu T^{0\nu} - x^\nu T^{0\mu}) d^3x \quad M_S^{\mu\nu} = \int (-\pi^a (\Sigma^{\mu\nu})_a{}^b \phi_b) d^3x$$

$$(L^{\mu\nu})_a{}^b = -i(x^\mu \partial^\nu - x^\nu \partial^\mu)\delta_a{}^b \quad (S^{\mu\nu})_a{}^b = -i(\Sigma^{\mu\nu})_a{}^b$$

And we have the commutation relationship that

$$M^{\mu\nu} = M_L^{\mu\nu} + M_S^{\mu\nu}$$

$$[\phi_a, M_L^{\mu\nu}] = (L^{\mu\nu})_a{}^b \phi_b \quad [\phi_a, M_S^{\mu\nu}] = (S^{\mu\nu})_a{}^b \phi_b$$

$$[\pi^a, M_L^{\mu\nu}] = (L^{\mu\nu})_b{}^a \pi^b \quad [\pi^a, M_S^{\mu\nu}] = -(S^{\mu\nu})_b{}^a \pi^b$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(-g^{\nu\rho}M^{\mu\sigma} + g^{\sigma\mu}M^{\rho\nu} + g^{\mu\rho}M^{\nu\sigma} - g^{\sigma\nu}M^{\rho\mu})$$

We again define $J_i \equiv \frac{1}{2}\epsilon_{ijk}M^{jk}$ and $K_i \equiv M^{i0}$, the communication relationship can be written as

$$\begin{aligned}[J_i, J_j] &= i\epsilon_{ijk}J_k \\ [J_i, K_j] &= i\epsilon_{ijk}K_k \\ [K_i, K_j] &= -i\epsilon_{ijk}J_k\end{aligned}$$

Further more,

$$[P^\mu, M^{\rho\sigma}] = i(g^{\mu\sigma}P^\mu - g^{\mu\rho}P^\sigma)$$

$$\begin{aligned}[J_i, H] &= 0 \\ [J_i, P_j] &= i\epsilon_{ijk}P_k \\ [K_i, H] &= iP_i \\ [K_i, P_j] &= i\delta_{ij}H\end{aligned}$$

At last, we define $L_i \equiv \frac{1}{2}\epsilon_{ijk}M_L^{jk}$ and $S_i \equiv \frac{1}{2}\epsilon_{ijk}M_S^{jk}$. So

$$\begin{aligned}[L_i, S_j] &= 0 \\ [S_i, P_j] &= 0 \\ [L_i, P_j] &= i\epsilon_{ijk}P_k\end{aligned}$$

We denote the rotation operator as $U(\Lambda)$, so

$$U^{-1}(\Lambda)\phi_a(x)U(\Lambda) = S_a{}^b\phi_b(\Lambda^{-1}x)$$

and

$$S_a{}^b = \delta_a{}^b + \frac{i}{2}\delta\omega_{\alpha\beta}(S^{\alpha\beta})_a{}^b$$

we can deduce that

$$U(1+\delta\omega) = 1 + \frac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu} \quad U(\Lambda) = e^{\frac{i}{2}\theta_{\mu\nu}M^{\mu\nu}}$$
$$U^{-1}(\Lambda)P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu$$
$$U^{-1}(\Lambda)M^{\mu\nu}U(\Lambda) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma M^{\rho\sigma}$$

2 Spin 0 Fields

2.1 Canonical quantization of Klein-Gordon fields

Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 + \Omega_0$$

Field equation

$$(\partial^\mu\partial_\mu - m^2)\phi = 0$$

Hamiltonian

$$\pi = \dot{\phi}$$
$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 - \Omega_0$$
$$H = \int \mathcal{H}d^3x$$

Momentum and angular momentum

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}(\frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 - \Omega_0)$$

$$P^0 = H \quad P^i = \int -\pi\nabla^i\phi d^3x$$

$$J_k = \int -\pi\epsilon_{ijk}x^j\nabla^k\phi d^3x$$

Canonical quantization

$$\begin{aligned}
[\phi(\vec{x},t),\phi(\vec{y},t)] &= 0 \\
[\pi(\vec{x},t),\pi(\vec{y},t)] &= 0 \\
[\phi(\vec{x},t),\pi(\vec{y},t)] &= i\delta(\vec{x}-\vec{y})
\end{aligned}$$

Fourier expansion

$$\phi(\vec{x},t) = \int \widetilde{dk}\left[a(\vec{k})e^{ikx} + a^\dagger(\vec{k})e^{-ikx}\right]$$

$$\pi(\vec{x},t) = -i\int \widetilde{dk}\omega\left[a(\vec{k})e^{ikx} - a^\dagger(\vec{k})e^{-ikx}\right]$$

Here, $k^2 = \mathbf{k}^2 - \omega^2 = -m^2$, $kx = \mathbf{k}\cdot\mathbf{x} - \omega t$, $\widetilde{dk} = \frac{d^3}{(2\pi)^2 2\omega}$

$$a(\vec{k}) = \int d^3x e^{-ikx}(i\pi + \omega\phi)$$

$$a^\dagger(\vec{k}) = \int d^3x e^{ikx}(-i\pi + \omega\phi)$$

$$\begin{aligned}
[a(\vec{p}),a(\vec{q})] &= 0 \\
\left[a^\dagger(\vec{p}),a^\dagger(\vec{q})\right] &= 0 \\
\left[a(\vec{p}),a^\dagger(\vec{q})\right] &= (2\pi)^3 2\omega\delta(\vec{p}-\vec{q})
\end{aligned}$$

Operator represented by a and $a^\dagger$

$$H = \int \widetilde{dk}\,\omega\,a^\dagger(\vec{k})a(\vec{k}) + (\mathcal{E}_0 - \Omega_0)V \quad \mathcal{E}_0 = \frac{1}{2}(2\pi)^{-3}\int d^3k\,\omega$$

$$P^i = \int \widetilde{dk}\,k^i\,a^\dagger(\vec{k})a(\vec{k})$$

particles

$$[H,a(\vec{k})] = -\omega a(\vec{k}) \quad [H,a^\dagger(\vec{k})] = \omega a^\dagger(\vec{k})$$

$$[P^i,a(\vec{k})] = -k^i a(\vec{k}) \quad [P^i,a^\dagger(\vec{k})] = k^i a^\dagger(\vec{k})$$

Let $|p\rangle = a^\dagger(\vec{p})|0\rangle$,so

$$H|p\rangle = \omega_p|p\rangle \quad P^i|p\rangle = p^i|p\rangle$$

So, we interpret the state $|\vec{p}\rangle$ as the momentum eigenstate of a single particle of mass m. We can also show that $J_i|\vec{p}=0\rangle = 0$, so the particle carries no internal angular momentum.