Summary on QFT

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1 From classical field to quantum field

1.1 Heisenberg picture of fields

The state of the field is described by an element $|\psi\rangle$ in Hilbert space. The measurement of the field is described by an operator field $\phi_a(\vec{x}, t)$. In Heisenberg picture, the dynamic of the field satisfy the equation

$$\frac{d\phi_a(x)}{dt} = -i[\phi_a(x), H]$$

So, the mean value of the measurement of the field is described by Erenfest theorem

$$\frac{d\langle\psi|\phi_a|\psi\rangle}{dt} = -i\langle\psi|[\phi_a, H]|\psi\rangle$$

If $[\phi_a, H]_Q = i[\phi_a, H]_C$, we can reproduce the classical field equation. We also note that the bracket operation here $[A, B] = AB - BA$ has the same properties as the poission bracket in classical mechanics. So, what we need here is the canonical quantization

$$[\phi_a(\vec{x}, t), \phi_b(\vec{y}, t)] = 0 \quad [\pi^a(\vec{x}, t), \pi^b(\vec{y}, t)] = 0 \quad [\phi_a(\vec{x}, t), \pi^b(\vec{y}, t)] = i\delta_a^b\delta(\vec{x} - \vec{y})$$

and the definition of $\mathcal{L}$,π^a and H is the same as those in corresponding classical theory. Then we can recover the classical field theory.

1.2 Lorentz invariance in quantum field theory

$$|\bar{\psi}\rangle = U(\Lambda)|\psi\rangle$$

Scalar fields:

$$\langle\bar{\psi}|\phi(x)|\bar{\psi}\rangle = \langle\psi|\phi(\Lambda^{-1}x)|\psi\rangle$$

$$U^{-1}(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$$

Vector fields:

$$\langle\bar{\psi}|A^\mu(x)|\bar{\psi}\rangle = \langle\psi|\Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)|\psi\rangle$$

$$U^{-1}(\Lambda)A^\mu(x)U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$$

Lorentz invariance Lagrangian is a scalar, or more loosely, action is invariant under Lorentz transformation.

1.3 Momentum

The definition of momentum is the same as that in classical theory.

$$T^{\mu\nu} \equiv -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\partial^\nu \phi_a + \eta^{\mu\nu}\mathcal{L} \quad \partial_\mu T^{\mu\nu} = 0$$

and

$$P^\mu = \int T^{0\mu} d^3x \quad \frac{dP^\mu}{dt} = 0$$

$$P^0 = H, \quad P^i = \int -\pi^a \partial^i \phi_a d^3x$$

And we can get the commutation relationship that

$$\begin{aligned} [\phi_a, P^\mu] &= -i\partial^\mu \phi_a \\ [\pi^a, P^\mu] &= -i\partial^\mu \pi^a \\ [P^\mu, P^\nu] &= 0 \end{aligned}$$

We denote the translation operator as $T(s)$, so

$$T^{-1}(s)\phi_a(x)T(s) = \phi_a(x-s)$$

we can deduce that

$$T(\epsilon) = 1 - i\epsilon_\mu P^\mu \quad T(s) = e^{-iP^\mu s_\mu}$$

1.4 Angular Momentum

The definition of Angular momentum is the same as that in classical theory.

$$M^{\mu\nu\rho} \equiv x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}(\Sigma^{\nu\rho})_a{}^b \phi_b$$

and

$$M^{\nu\rho} = \int M^{0\nu\rho} d^3x \quad \frac{dM^{\nu\rho}}{dt} = 0$$

$$M^{\mu\nu} = \int (x^\mu T^{0\nu} - x^\nu T^{0\mu} - \pi^a (\Sigma^{\mu\nu})_a{}^b \phi_b) d^3x$$

We denote that

$$M_L^{\mu\nu} = \int (x^\mu T^{0\nu} - x^\nu T^{0\mu}) d^3x \quad M_S^{\mu\nu} = \int (-\pi^a (\Sigma^{\mu\nu})_a{}^b \phi_b) d^3x$$

$$(L^{\mu\nu})_a{}^b = -i(x^\mu \partial^\nu - x^\nu \partial^\mu)\delta_a{}^b \quad (S^{\mu\nu})_a{}^b = -i(\Sigma^{\mu\nu})_a{}^b$$

And we have the commutation relationship that

$$M^{\mu\nu} = M_L^{\mu\nu} + M_S^{\mu\nu}$$

$$[\phi_a, M_L^{\mu\nu}] = (L^{\mu\nu})_a{}^b \phi_b \quad [\phi_a, M_S^{\mu\nu}] = (S^{\mu\nu})_a{}^b \phi_b$$

$$[\pi^a, M_L^{\mu\nu}] = (L^{\mu\nu})_b{}^a \pi^b \quad [\pi^a, M_S^{\mu\nu}] = -(S^{\mu\nu})_b{}^a \pi^b$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(-g^{\nu\rho}M^{\mu\sigma} + g^{\sigma\mu}M^{\rho\nu} + g^{\mu\rho}M^{\nu\sigma} - g^{\sigma\nu}M^{\rho\mu})$$

We again define $J_i \equiv \frac{1}{2}\epsilon_{ijk}M^{jk}$ and $K_i \equiv M^{i0}$, the communication relationship can be written as

$$\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk}J_k \\
[J_i, K_j] &= i\epsilon_{ijk}K_k \\
[K_i, K_j] &= -i\epsilon_{ijk}J_k
\end{aligned}$$

Further more,

$$[P^\mu, M^{\rho\sigma}] = i(g^{\mu\sigma}P^\mu - g^{\mu\rho}P^\sigma)$$

$$\begin{aligned}
[J_i, H] &= 0 \\
[J_i, P_j] &= i\epsilon_{ijk}P_k \\
[K_i, H] &= iP_i \\
[K_i, P_j] &= i\delta_{ij}H
\end{aligned}$$

At last, we define $L_i \equiv \frac{1}{2}\epsilon_{ijk}M_L^{jk}$ and $S_i \equiv \frac{1}{2}\epsilon_{ijk}M_S^{jk}$. So

$$\begin{aligned}
[L_i, S_j] &= 0 \\
[S_i, P_j] &= 0 \\
[L_i, P_j] &= i\epsilon_{ijk}P_k
\end{aligned}$$

We denote the rotation operator as $U(\Lambda)$, so

$$U^{-1}(\Lambda)\phi_a(x)U(\Lambda) = S_a{}^b\phi_b(\Lambda^{-1}x)$$

and

$$S_a{}^b = \delta_a{}^b + \frac{i}{2}\delta\omega_{\alpha\beta}(S^{\alpha\beta})_a{}^b$$

we can deduce that

$$U(1+\delta\omega) = 1 + \frac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu} \quad U(\Lambda) = e^{\frac{i}{2}\theta_{\mu\nu}M^{\mu\nu}}$$

$$U^{-1}(\Lambda)P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu$$

$$U^{-1}(\Lambda)M^{\mu\nu}U(\Lambda) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma M^{\rho\sigma}$$

2 Spin 0 Fields

2.1 Klein-Gordon fields

Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 + \Omega_0$$

Field equation

$$(\partial^\mu\partial_\mu - m^2)\phi = 0$$

Hamiltonian

$$\pi = \dot{\phi}$$

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 - \Omega_0$$

$$H = \int \mathcal{H} d^3x$$

Momentum and angular momentum

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}(\frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 - \Omega_0)$$

$$P^0 = H \quad P^i = \int -\pi\nabla^i\phi d^3x$$

$$J_k = \int -\pi\epsilon_{ijk}x^j\nabla^k\phi d^3x$$

2.2 Canonical quantization Formulation

Canonical quantization

$$\begin{aligned} [\phi(\vec{x},t),\phi(\vec{y},t)] &= 0 \\ [\pi(\vec{x},t),\pi(\vec{y},t)] &= 0 \\ [\phi(\vec{x},t),\pi(\vec{y},t)] &= i\delta(\vec{x}-\vec{y}) \end{aligned}$$

Fourier expansion

$$\phi(\vec{x},t) = \int \widetilde{dk}\left[a(\vec{k})e^{ikx} + a^\dagger(\vec{k})e^{-ikx}\right]$$

$$\pi(\vec{x},t) = -i\int \widetilde{dk}\omega\left[a(\vec{k})e^{ikx} - a^\dagger(\vec{k})e^{-ikx}\right]$$

Here, $k^2 = \mathbf{k}^2 - \omega^2 = -m^2$, $kx = \mathbf{k}\cdot\mathbf{x} - \omega t$, $\widetilde{dk} = \frac{d^3}{(2\pi)^2 2\omega}$

$$a(\vec{k}) = \int d^3x e^{-ikx}(i\pi + \omega\phi)$$

$$a^\dagger(\vec{k}) = \int d^3x e^{ikx}(-i\pi + \omega\phi)$$

$$\begin{aligned} [a(\vec{p}),a(\vec{q})] &= 0 \\ \left[a^\dagger(\vec{p}),a^\dagger(\vec{q})\right] &= 0 \\ \left[a(\vec{p}),a^\dagger(\vec{q})\right] &= (2\pi)^3 2\omega\delta(\vec{p}-\vec{q}) \end{aligned}$$

Operator represented by a and $a^\dagger$

$$H = \int \widetilde{dk}\,\omega\,a^\dagger(\vec{k})a(\vec{k}) + (\mathcal{E}_0 - \Omega_0)V \quad \mathcal{E}_0 = \frac{1}{2}(2\pi)^{-3}\int d^3k\,\omega$$

$$P^i = \int \widetilde{dk}\,k^i\,a^\dagger(\vec{k})a(\vec{k})$$

Particles

$$[H,a(\vec{k})] = -\omega a(\vec{k}) \quad [H,a^\dagger(\vec{k})] = \omega a^\dagger(\vec{k})$$

$$[P^i,a(\vec{k})] = -k^i a(\vec{k}) \quad [P^i,a^\dagger(\vec{k})] = k^i a^\dagger(\vec{k})$$

Let $|p\rangle = a^\dagger(\vec{p})|0\rangle$,so

$$H|p\rangle = \omega_p|p\rangle \quad P^i|p\rangle = p^i|p\rangle$$

So, we interpret the state $|\vec{p}\rangle$ as the momentum eigenstate of a single particle of mass m. We can also show that $J_i|\vec{p}=0\rangle = 0$, so the particle carries no internal angular momentum.

Causality The amplitude for a particle to propagate from y to x is $\langle 0|\phi(x)\phi(y)|0\rangle$, denoted by $D(x-y)$.

$$D(x-y) = \int \widetilde{dk} e^{ik(x-y)}$$

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x)$$

If $x-y$ is space-like, a continuous Lorentz transformation can take $(x-y)$ to $-(x-y)$. So $[\phi(x), \phi(y)] = 0$ for space-like $x-y$. A measurement performed at one point can not affect a measurement at another point whose separation is space-like.

The Klein-Gordon propagator

$$D_R(x-y) \equiv \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{-i}{p^2+m^2}e^{ip(x-y)}$$

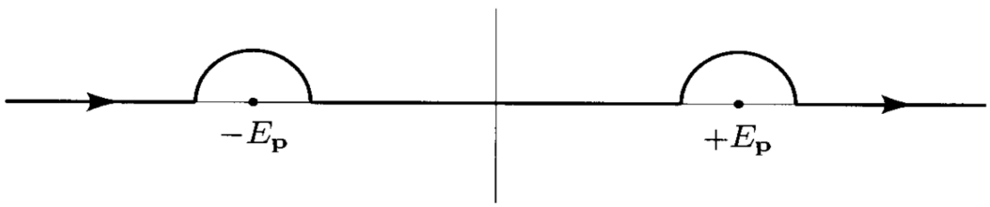


Figure 1: Retarded Green Function

$$(\partial^2 - m^2)D_R(x-y) = i\delta(x-y)$$

$$D_F(x-y) \equiv \langle 0|T\phi(x)\phi(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{-i}{p^2+m^2-i\epsilon}e^{ip(x-y)}$$

Here, T stands for time ordering, placing all operators evaluated at later times to the left.

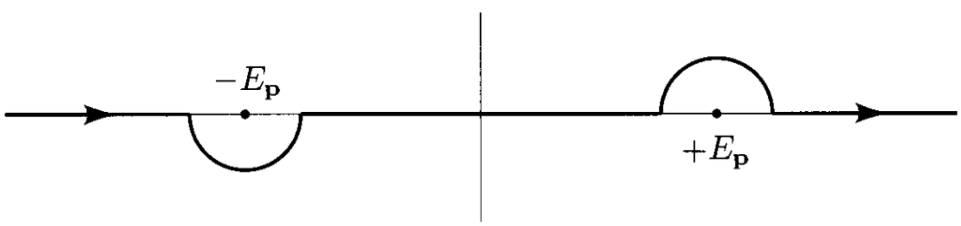


Figure 2: Feynman Green Function

2.3 Perturbation theory for canonical quantization

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4$$

$$H = H_0 + H_{int} \quad H_{int} = \int d^3x \frac{\lambda_0^4}{4!}\phi^4(\vec{x})$$

2.3.1 Perturbation expansion of correlation functions

Note The ground state of the interaction field theory is denoted by $|\Omega\rangle$, the ground state of the free field theory is denoted by $|0\rangle$. The zero of energy is defined by $H_0|0\rangle = 0$ and $E_0 = \langle\Omega|H|\Omega\rangle$.

$$\phi(t_0, \vec{x}) = \int \frac{d^3p}{(2\pi)^3}(a(\vec{p})e^{i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}})$$
$$\phi(t, \vec{x}) = e^{iH(t-t_0)}\phi(t_0, \vec{x})e^{-iH(t-t_0)}$$
$$\phi_I(t, \vec{x}) \equiv e^{iH_0(t-t_0)}\phi(t_0, \vec{x})e^{-iH_0(t-t_0)}$$
$$H_I(x) = \int d^3x \frac{\lambda_0^4}{4!}\phi_I^4$$

The perturbation expansion of correlation functions is

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\left\{\phi_I(x)\phi_I(y)\text{exp}\left[-i\int_{-T}^{T} dtH_I\right]\right\}|0\rangle}{\langle 0|T\left\{\text{exp}\left[-i\int_{-T}^{T} dtH_I\right]\right\}|0\rangle}$$

The proof can be found in chapter 4.2 of *An introduction to quantum field theory (M.E.Peskin & D.V.Schroeder)*

2.3.2 Wick's theorem

$$T\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\} = N\{\phi(x_1)\phi(x_2)\cdots\phi(x_n) + \text{ all possible contractions }\}$$

Normal order : all the a's are to the right of all the $a^\dagger$.

Example

$$\begin{aligned}\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\}|0\rangle &= D_F(x_1-x_2)D_F(x_3-x_4)\\ &= D_F(x_1-x_3)D_F(x_2-x_4)\\ &= D_F(x_1-x_4)D_F(x_2-x_3)\end{aligned}$$

2.3.3 Feynman diagram

Expand $\langle 0|T\left\{\phi_I(x)\phi_I(y)\text{exp}\left[-i\int_{-T}^{T} dtH_I\right]\right\}|0\rangle$ to the first order of λ_0

$$\begin{aligned}&\langle 0|T\left\{\phi_I(x)\phi_I(y)\frac{-i\lambda_0}{4!}\int dz^4\phi_I(z)\phi_I(z)\phi_I(z)\phi_I(z)\right\}|0\rangle\\ =\;& 3\cdot(\frac{-i\lambda_0}{4!})D_F(x-y)\int d^4zD_F(z-z)D_F(z-z)\\ +\;& 12\cdot(\frac{-i\lambda_0}{4!})\int d^4zD_F(x-z)D_F(y-z)D_F(z-z)\end{aligned}$$

It can be represented by the so called Feynman diagram.

The symmetry factor of the first diagram is $S = \frac{4!}{3} = 8$. The symmetry factor of the second diagram is $S = \frac{4!}{12} = 2$. The Feynman rules for ϕ^4 theory are:
(1) For each propagator, $P = D_F(x - y)$;
(2) For each vertex, $V = (-i\lambda_0)\int d^4z$;
(3) For each external point, $E = 1$;
(4) Divided by the symmetry factor.
At last, we can prove that

$$\langle\Omega|T\{\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\}|\Omega\rangle = \text{ sum of all E-connected diagrams with n external points}$$

Here, the "E-disconnected" means disconnected from all external points", being called "vacuum bubbles". They vacuum bubbles in $\langle 0|T\left\{\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\exp\left[-i\int_{-T}^{T} dtH_I\right]\right\}|0\rangle$ are all cancelled by the $\langle 0|T\left\{\exp\left[-i\int_{-T}^{T} dtH_I\right]\right\}|0\rangle$.

2.4 Path integral formulation

2.4.1 Basic equation

Recall the path integrals in quantum mechanics, we abandon the Hamiltonian formalism and define the Hamiltonian dynamics as

$$\langle\phi_b(\vec{x})|e^{-iHT}|\phi_a(\vec{x})\rangle = \int \mathcal{D}\phi\exp\left[i\int_0^T d^4x\mathcal{L}\right]$$

Here, $\langle\phi_b(\vec{x})|$ is the eigenstate of $\phi_S(\vec{x}) = \phi_H(\vec{x}, 0)$ with eigenvalue $\phi_b(\vec{x})$ at time $t = T$,$|\phi_a(\vec{x})\rangle$ is the eigenstate of $\phi_S(\vec{x})$ with eigenvalue $\phi_a(\vec{x})$ at time $t = 0$.
We emphasize that in this subsection, ϕ_H denotes the Heisenberg picture of field whose value is operators,ϕ_S denotes the Schrödinger picture of field, $\phi(x)$ is classical field whose value is ordinary number.

2.4.2 Correlation function

$$\langle\Omega|T\phi_H(x_1)\phi_H(x_2)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\phi(x_1)\phi(x_2)\exp\left[i\int_T^T d^4x\mathcal{L}\right]}{\int \mathcal{D}\phi\exp\left[i\int_T^T d^4x\mathcal{L}\right]}$$

The proof can be found in chapter 9.2 of *An introduction to quantum field theory (M.E.Peskin & D.V.Schroeder).*

2.4.3 Functional derivatives and the generating functional

We define the generating functional as

$$Z[J] \equiv \int \mathcal{D}\phi\exp\left[i\int d^4x\mathcal{L} + J(x)\phi(x)\right]$$

We can prove that

$$\langle\Omega|T\phi_H(x_1)\cdots\phi_H(x_n)|\Omega\rangle = \frac{1}{Z_0}\left(-i\frac{\delta}{\delta J(x_1)}\right)\cdots\left(-i\frac{\delta}{\delta J(x_n)}\right)Z[J]|_{J=0}$$

Here, $Z_0 \equiv Z[J = 0]$.

2.4.4 Free field theory

In Klein-Gordon field theory,

$$\int d^4x[\mathcal{L}_0(\phi) + J\phi] = \int d^4x[\frac{1}{2}\phi(\partial^2 - m^2 + i\epsilon)\phi + J\phi]$$

Define

$$\phi'(x) \equiv \phi(x) + \int d^4y(-iD_F(x-y))J(y)$$

Recall that $(\partial^2 - m^2)D_F(x-y) = i\delta(x-y)$, we can derive that

$$\int d^4x[\mathcal{L}_0 + J\phi] = \int d^4x[\frac{1}{2}\phi'(\partial^2 - m^2 + i\epsilon)\phi'] - \int d^4xd^4y\frac{1}{2}J(x)[-iD_F(x-y)]J(y)$$

After integration, we can know that

$$Z[J] = Z_0\text{exp}[-\frac{1}{2}\int d^4xd^4yJ(x)D_F(x-y)J(y)]$$

So,

$$\langle 0|T\phi_H(x_1)\phi_H(x_2)|0\rangle = -\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\text{exp}[-\frac{1}{2}\int d^4xd^4yJ(x)D_F(x-y)J(y)]|_{J=0} = D_F(x_1 - x_2)$$

2.5 Perturbation theory for path integral quantization

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4 \quad \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \quad \mathcal{L}_1 = \frac{\lambda_0^4}{4!}\phi^4(\vec{x}) \\
Z[J] &= \int \mathcal{D}\phi e^{i\int d^4x[\mathcal{L}_0 + \mathcal{L}_1 + J\phi]} \\
&= e^{i\int d^4y\mathcal{L}_1(\frac{1}{i}\frac{\delta}{\delta J(y)})}\int \mathcal{D}\phi e^{i\int d^4x[\mathcal{L}_0 + J\phi]} \\
&\propto e^{i\int d^4x\mathcal{L}_1(\frac{1}{i}\frac{\delta}{\delta J(x)})}\text{exp}[-\frac{1}{2}\int d^4yd^4zJ(y)D_F(y-z)J(z)] \\
&= \sum_{V=0}^{\infty}\frac{1}{V!}[\frac{-i\lambda_0}{4!}\int d^4x(\frac{1}{i}\frac{\delta}{\delta J(y)})^4]^V \times \sum_{P=0}^{\infty}\frac{1}{P!}[-\frac{1}{2}\int d^4yd^4zJ(y)D_F(y-z)J(z)]^P
\end{aligned} \tag{1}$$

If we focus on a term with particular values of V and P, then the number of surviving sources (after we take all the functional derivatives) is $E = 2P - 4V$. The $4V$ functional derivatives can act on the 2P sources in $\frac{(2P)!}{(2P-4V)!}$ different combinations. However, many of the resulting expressions are algebraically identical.

To organize them, we introduce Feynman diagrams similar to that in perturbation theory of canonical quantization. In these diagrams, a line segment stands for a propagator $D_F(x-y)$, a filled circle at one end of a line segment for a source $i\int d^4xJ(x)$, and a vertex joining four line segments for $-i\lambda_0\int d^4z$.

For each diagram, we can assign a symmetry factor S_P similar to that in perturbation theory for canonical quantization. Due to the fact that some external sources are identical here, usually $S_P \neq S_C$. But when calculating the correlation function, the exchange of the order of functional derivatives to identical sources can eliminate the difference.

We can demonstrate that

$$Z[J] \propto \text{exp}(\sum_I C_I)$$

Here, C_I stands for a particular connected diagram, including its symmetry factor. We define $E[J]$ as

$$Z[J] \equiv Z_0\text{exp}(-iE[J])$$

As, $E[0] = 0$, we know

$$-iE[J] = \sum_{I\neq\{0\}} C_I$$

The notation $I \neq \{0\}$ means that the vacuum diagrams are omitted from the sum.

The detailed discussion can be found in chapter 9 of *Quantum field theory (M. Srednicki)*.

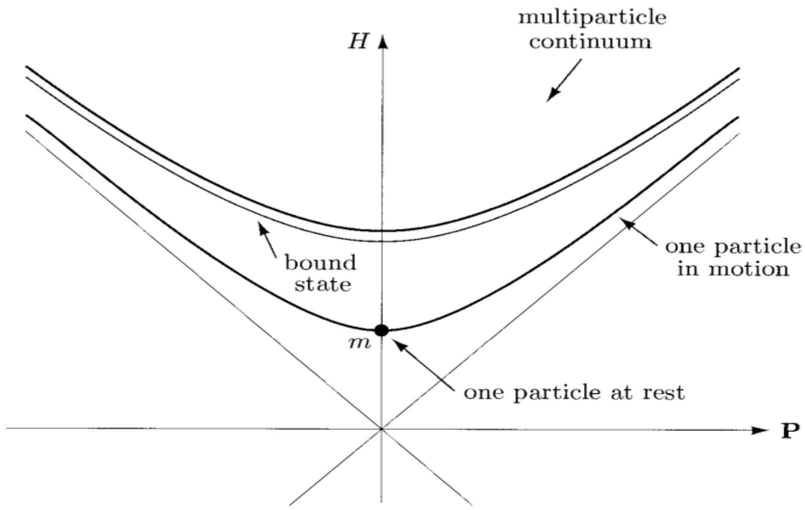
2.6 LSZ reduction formula

2.6.1 Field strength renormalization

The completeness relation:

$$\mathbf{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|\lambda_{\mathbf{p}}\rangle\langle\lambda_{\mathbf{p}}|$$

Here, $E_{\mathbf{p}} = \sqrt{m_\lambda^2 + \mathbf{p}^2}$



Assume for now $x^0 > y^0$ and define $\langle\Omega|\phi(x)\phi(y)|\Omega\rangle_C = \langle\Omega|\phi(x)\phi(y)|\Omega\rangle - \langle\Omega|\phi(x)|\Omega\rangle\langle|\Omega\phi(y)|\Omega\rangle$ as connected two point function. (The term $\langle\Omega|\phi(x)|\Omega\rangle\langle|\Omega\phi(y)|\Omega\rangle$ is usually zero by symmetry; for higher spin fields, it is zero by Lorentz invariance.) The connected two point function is

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle_C = \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\langle\Omega|\phi(x)|\lambda_{\mathbf{p}}\rangle\langle\lambda_{\mathbf{p}}|\phi(y)|\Omega\rangle$$

It can be verified that

$$\langle\Omega|\phi(x)|\lambda_{\mathbf{p}}\rangle = \langle\langle\Omega|\phi(0)|\lambda_0\rangle e^{ipx}|_{p^0=E_{\mathbf{p}}}$$

So,

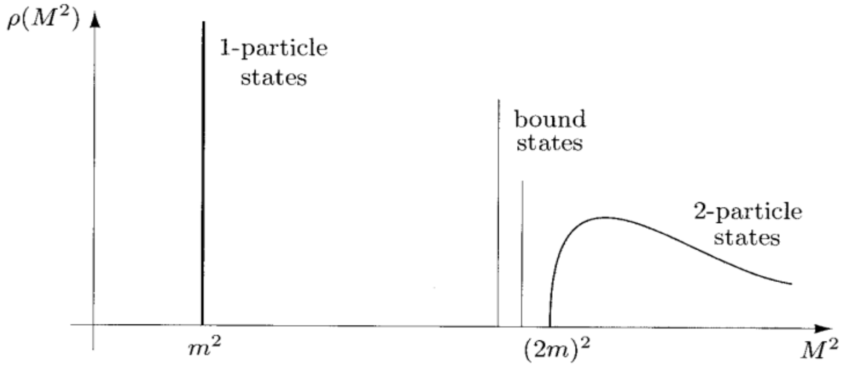
$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle_C = \sum_\lambda \int \frac{d^4p}{(2\pi)^4}\frac{-i}{p^2+m_\lambda^2-i\epsilon}e^{ip(x-y)}|\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

Analogous expressions hold for the case $y^0 > x^0$, and both cases can be summarized as

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle_C = \int_0^\infty \frac{dM^2}{2\pi}\rho(M^2)D_F(x-y;M^2)$$

and

$$\rho(M^2) = \sum_\lambda (2\pi)\delta(M^2-m^2)|\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$



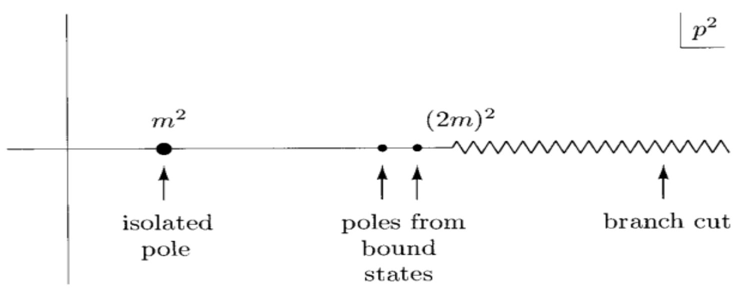
The structure of the spectral density function $\rho(M^2)$

The one-particle state contribute an isolated delta function to the spectral density function, so

$$\rho(M^2) = 2\pi\delta(M^2 - m^2)\cdot Z + \text{ (nothing else until } M^2 >\sim (2m)^2)$$

$Z = |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$ is called field-strength renormalization. m is the physical mass of a single particle of the ϕ boson. The Fourier transformation of the two point function is

$$\int d^4x e^{-ipx}\langle\Omega|T\phi(x)\phi(0)|\Omega\rangle_C = \int_0^\infty \frac{dM^2}{2\pi}\rho(M^2)\frac{-i}{p^2 + M^2 - i\epsilon} = \frac{-iZ}{p^2 + m^2 - i\epsilon} + \int_{\sim 4m^2}^\infty \frac{dM^2}{2\pi}\rho(M^2)\frac{-i}{p^2 + M^2 - i\epsilon}$$



The structure of the two point function in Fourier space

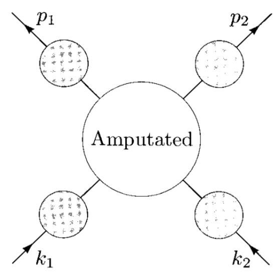
2.6.2 LSZ reduction formula

$$\prod_1^n \int d^4x_i e^{-ip_ix_i}\prod_1^m d^4y_i e^{ik_jy_j}\langle\omega|T\{\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)\}|\Omega\rangle$$
$$\underset{p_i^0\to E_{\mathbf{p}_i}\, k_i^0\to E_{\mathbf{k}_i}}{\sim} \left(\prod_1^n \frac{-\sqrt{Z}i}{p_i^2 + m^2 - i\epsilon}\right)\left(\prod_1^m \frac{-\sqrt{Z}i}{k_i^2 + m^2 - i\epsilon}\right)\langle\mathbf{p}_1\cdots\mathbf{p}_n|S|\mathbf{k}_1\cdots\mathbf{k}_m\rangle$$

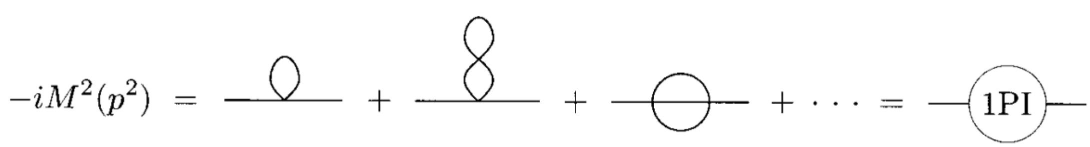
The $\sim$ means the two sides of the expression share the same singular structure around $p_i^0 \to E_{\mathbf{p}_i}$, $k_i^0 \to E_{\mathbf{k}_i}$. The proof can be found in chapter 7.2 of *An introduction to quantum field theory (M.E.Peskin & D.V.Schroeder)*. To express the formula above in the language of Feynman diagrams, we consider the S-matrix element for 2-particle $\to$ 2-particle for example. Note the disconnected diagram should be disregarded because they do not have the singularity structure with a product of four poles indicated by on the right side of the LSZ reduction formula. So, the exact four point function

$$\prod_1^2 \int d^4x_i e^{-ip_ix_i} \prod_1^2 d^4y_i e^{ik_jy_j} \langle\omega|T\{\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)\}|\Omega\rangle$$

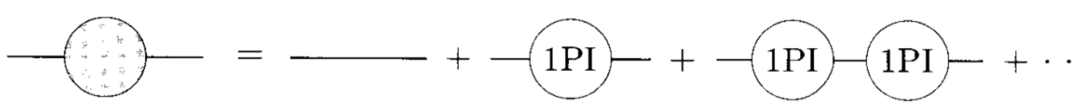
has the general form showed as below.



Note One-particle-irreducible, or 1PI for short, refers to diagrams that is still connected after one line is cut.
We can sum up the corrections to each external leg. Let $-iM^2(p^2)$ denote the sum of all 1PI insertions into the scalar propagator:



Then the exact propagator can be written as a geometric series as



The result is $\frac{-i}{p^2+m_0^2+M^2}$. If we expand each resummed propagator about the physical particle pole, we see that each external leg of the four-point amplitude contributes

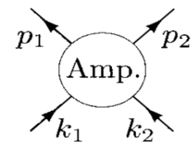
$$\frac{-i}{p^2+m_0^2+M^2} \underset{p^0 \to E_{\mathbf{p}}}{\sim} \frac{-iZ}{p^2+m^2} + \text{ (regular)}$$

Thus, the sum of diagrams contains a product of four point poles:

$$\frac{-iZ}{p_1^2+m^2}\frac{-iZ}{p_2^2+m^2}\frac{-iZ}{k_1^2+m^2}\frac{-iZ}{k_2^2+m^2}$$

So, the S matrix element can be represented by

$$\langle \mathbf{p}_1\mathbf{p}_2 | \, S \, |\mathbf{k}_1\mathbf{k}_2\rangle = \left(\sqrt{Z}\right)^4$$



It is easy to be generalized to the more complicated scattering cases. After Fourier transforming the n-point function to momentum space and cutting off the external legs, the Feynman rules for S-matrix element can be stated as follows:
(1) For each propagator, $P = \frac{-i}{p^2+m_0^2-i\epsilon}$;
(2) For each vertex, $V = -i\lambda_0$;
(3) For each external point, $E = 1$;
(4) Impose momentum conservation at each vertex;
(5) Integrate over each undetermined loop momentum: $\int \frac{d^4p}{(2\pi)^4}$;
(6) Divided by the symmetry factor;
(7) Multiply the total momentum conservation factor $(2\pi)^4\delta(\sum p_f - \sum p_i)$ We can write $\langle f|S|i\rangle = i\mathcal{M}(2\pi)^4\delta(\sum p_f - \sum p_i)$ for convenience.

2.7 Renormalization

Renormalization, the procedure in quantum field theory by which divergent parts of a calculation, leading to nonsensical infinite results, are absorbed by redefinition into a few measurable quantities, so yielding finite answers.

2.7.1 Counting of ultraviolet divergence

Consider a pure scalar theory in d dimensions with a ϕ^n interaction term

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{n!}\phi^n$$

Let N be the number of external lines in the diagram, P the number of propagators, V the number of vertices. The number of the loops in the diagram is $L = P - V + 1$. There are n lines meeting at each vertex, so $nV = 2P + N$. Loosely speaking, each loop has an integral d^dp, each propagator has a factor p^{-2}, so the superficial degrees of divergence is

$$D = dL - 2P = d + [n(\frac{d-2}{2}) - d)]V - (\frac{d-2}{2})N$$

According the superficial degrees of divergence of the diagram. These three possible types of ultraviolet behaviour of quantum field theories. We will refer to them as follows:
(1) Super-Renormalizable theory: Only a finite number of Feynman diagrams superficially diverge.
(2) Renormalizable theory: Only a finite number of amplitudes superficially diverge; however, divergences occur at all orders in perturbation theory.
(3) Non-Renormalizable theory: All amplitudes are divergent at a sufficiently high order in perturbation theory.
So, for ϕ^4 theory in four dimension, $D = 4 - N$. It is a renormalizable theory. For ϕ^3 theory in four dimension, $D = 4 - V - N$. It is a super-renormalizable theory. For ϕ^6 theory in four dimension, $D = 4 + 2V - N$. It is a Non-renormalizable theory.
The superficial degrees of freedom can also be derived from dimensional analysis. The dimension of λ is $d - \frac{n(d-2)}{2}$. Now consider an arbitrary diagram with N external lines. One way that such a diagram could arise is from an interaction term

$\eta\phi^N$ in the Lagrangian. The dimension of η would then be $d - \frac{N(d-2)}{2}$, and therefore we conclude that any (amputated) diagram with N external lines has dimension $d - \frac{N(d-2)}{2}$. In our theory with only the $\lambda\phi^n$ vertex, if the diagram has V vertices, its divergent part is proportional to $\lambda^V \Lambda^D$, where Λ is a high momentum cut-off and D is the superficial degree of divergence. Applying dimensional analysis, we find

$$d - \frac{N(d-2)}{2} = V[d - \frac{n(d-2)}{2}] + D$$

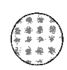
Note that the quantity that multiplies V in this expression is just the dimension of the coupling constant λ. Thus we can characterize the three degrees of renormalizability in a second way:
(1) Super-Renormalizable: Coupling constant has positive mass dimension.
(2) Renormalizable: Coupling constant is dimensionless.
(3) Non-Renormalizable: Coupling constant has negative mass dimension.

2.7.2 Renormalized Perturbation Theory

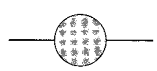
The Lagrangian of ϕ^4 theory is

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4$$

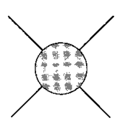
We write m_0 and λ_0, to emphasize that these are the bare values of the mass and coupling constant, not the values measured in experiments. Since the theory is invariant under $\phi \to -\phi$, all amplitudes with an odd number of external legs vanish. The only divergent amplitudes are therefore



(unobservable vacuum energy shift);



$\sim\Lambda^2 + p^2 \log\Lambda +$ (finite terms);



$\sim \log\Lambda +$ (finite terms).

Ignoring the vacuum diagram, these amplitudes contain three infinite constants. Our goal is to absorb these constants into the three unobservable parameters of the theory: the bare mass, the bare coupling constant, and the field strength. To accomplish this goal, it is convenient to reformulate the perturbation expansion so that these unobservable quantities do not appear explicitly in the Feynman rules. Recall that the exact two-point function has the form

$$\int d^4x \langle\Omega|\phi(x)\phi(0)|\Omega\rangle e^{-ipx} = \frac{-iZ}{p^2+m^2} + \text{ terms regular at } p^2 = m^2$$

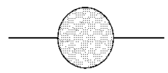
We can eliminate the Z from this equation by rescaling the field: $\phi = Z^{\frac{1}{2}}\phi_r$ We also define

$$\delta_Z = Z - 1 \quad \delta_m = Zm_0^2 - m^2 \quad \delta_\lambda = \lambda_0 Z^2 - \lambda$$

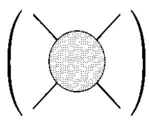
Then the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi_r\partial_\mu\phi_r - \frac{1}{2}m^2\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 - \frac{1}{2}\delta_Z\partial^\mu\phi_r\partial_\mu\phi_r - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta\lambda}{4!}\phi_r^4$$

The last three terms, known as counterterms, have absorbed the infinite but unobservable shifts between the bare parameters and the physical parameters.
We give precise definitions of the physical mass and coupling constant as follows. The renormalization scheme here is called on-shell (OS) scheme. Other renormalization scheme would be introduced later.



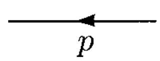
$$= \frac{-i}{p^2 + m^2} + (\text{terms regular at } p^2 = m^2);$$



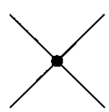
amputated

$$= -i\lambda \qquad \text{at } s = 4m^2,\ t = u = 0.$$

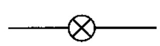
These equations are called renormalization conditions. Our new Lagrangian gives a new set of Feynman rules,



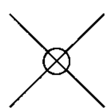
$$= \frac{-i}{p^2 + m^2 + i\epsilon}$$



$$= -i\lambda$$



$$= -i(p^2\delta_Z + \delta_m)$$



$$= -i\delta_\lambda$$

We can use these new Feynman rules to compute any amplitude in ϕ^4 theory. The procedure is as follows. Compute the desired amplitude as the sum of all possible diagrams created from the propagator and vertices shown above. The loop integrals in the diagrams will often diverge, so one must introduce a regulator. The result of this computation will be a function of the three unknown parameters δ_Z, δ_m, and δ_λ. Adjust (or "renormalize") these three parameters as necessary to maintain the renormalization conditions. After this adjustment, the expression for the amplitude should be finite and independent of the regulator.
This procedure, using Feynman rules with counterterms, is known as renormalized perturbation theory.

2.7.3 Mandelstam variable

In theoretical physics, the **Mandelstam variable** are numerical quantities that encode the energy, momentum, and angles of particles in a scattering process in a Lorentz-invariant fashion. They are used for scattering processes of two particles to two particles. The Mandelstam variables s, t, u are then defined by

$$
\begin{aligned}
s &= -(p_1+p_2)^2 = -(p_3+p_4)^2 \\
t &= -(p_1-p_3)^2 = -(p_2-p_4)^2 \\
u &= -(p_1-p_4)^2 = -(p_2-p_3)^2
\end{aligned}
$$

Where p_1 and p_2 are the four-momenta of the incoming particles and p_3 and p_4 are the four-momenta of the outgoing particles. s is also known as the square of the center-of-mass energy (invariant mass) and t is also known as the square of the four-momentum transfer.
We can verify that

$$s+t+u = m_1^2+m_3^2+m_3^2+m_4^2$$

2.7.4 Feynman's formula

$$\frac{1}{A_1\cdots A_n} = \int dF_n(x_1A_1+\cdots+x_nA_n)^{-n}$$

where the integration measure over the Feynman parameters x_i is

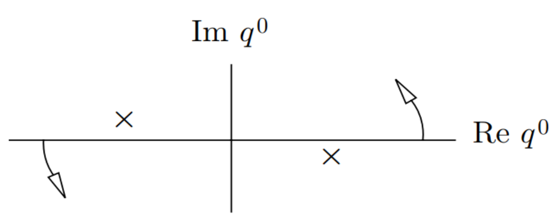
$$\int dF_n = (n-1)!\int_0^1 dx_1\cdots dx_n\delta(x_1+\cdots+x_n-1)$$

This measure is normalized so that

$$\int dF_n = 1$$

A generalization of Feynman's formula is

$$\frac{1}{A_1^{\alpha_1}\cdots A_n^{\alpha_n}} = \frac{\Gamma(\sum_i\alpha_i)}{\prod_i\Gamma(\alpha_i)}\frac{1}{(n-1)!}\int dF_n\frac{\prod_i x_i^{\alpha_i-1}}{(\sum_i x_iA_i)^{\sum_i\alpha_i}}$$



Wick rotation

2.7.5 Wick rotation

For an integral $\int d^dqf(q^2-i\epsilon)$, if the integrand vanishes fast enough as $|q_0|\to\infty$, we can rotate this contour clockwise by $\frac{\pi}{2}$, so that it runs from $-i\infty$ to $i\infty$. In making this Wick rotation, the contour does not pass over any poles. (The $i\epsilon$ are needed to make this statement unambiguous.) Thus the value of the integral is unchanged. It is now convenient to define a Euclidean d-dimensional vector $\bar{q}$ via $q^0 = i\bar{q}_d$ and $q_j = \bar{q}_j$; then $q^2 = \bar{q}^2$, where

$$\bar{q}^2 = \bar{q}_1^2+\cdots+\bar{q}_d^2$$

Also, $d^dq = id^d\bar{q}$.Therefore, in general,

$$\int d^dqf(q^2-i\epsilon) = i\int d^d\bar{q}f(\bar{q}^2)$$

2.7.6 Dimensional regularization

Dimensional regularization is a method for regularizing integrals in the evaluation of Feynman diagrams. For example, if one wishes to evaluate a loop integral which is logarithmically divergent in four dimensions, like

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2+m^2)^2}$$

one first rewrites the integral in some way so that the number of variables integrated over does not depend on d, and then we formally vary the parameter d, to include non-integral values like $d = 4-\epsilon$.

$$\int_0^\infty \frac{dp}{(2\pi)^{4-\varepsilon}} \frac{2\pi^{(4-\varepsilon)/2}}{\Gamma\left(\frac{4-\varepsilon}{2}\right)} \frac{p^{3-\varepsilon}}{(p^2+m^2)^2} = \frac{2^{\varepsilon-4}\pi^{\frac{\varepsilon}{2}-1}}{\sin(\frac{\pi\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})} m^{-\varepsilon} = \frac{1}{8\pi^2\varepsilon} - \frac{1}{16\pi^2}\left(\ln\frac{m^2}{4\pi} + \gamma\right) + \mathcal{O}(\varepsilon)$$

There is a useful formula for calculating the integral

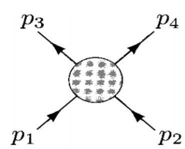
$$\int \frac{d^d\bar{q}}{(2\pi)^d} \frac{(\bar{q}^2)^a}{(\bar{q}^2+D)^b} = \frac{\Gamma(b-a-\frac{1}{2}d)\Gamma(a+\frac{1}{2}d)}{(4\pi)^{d/2}\Gamma(b)\Gamma(\frac{1}{2}d)} D^{-(b-a-d/2)}$$

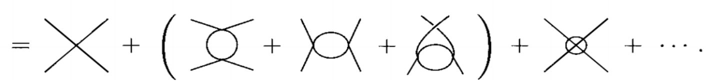
If $a = 0$, then the formula will be

$$\int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2+D)^b} = \frac{\Gamma(b-\frac{1}{2}d)}{(4\pi)^{d/2}\Gamma(b)} D^{-(b-d/2)}$$

2.7.7 One loop structure of ϕ^4 theory

First consider the basic two-particle scattering amplitude,

$i\mathcal{M}(p_1p_2 \to p_3p_4) =$ 



If we define $p = p_1 + p_2$, then the second diagram is

$$\frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2+m^2} \frac{-i}{(k+m)^2+m^2} \equiv (-i\lambda)^2 iV(-p^2)$$

So the entire amplitude is therefore

$$i\mathcal{M} = -i\lambda + (-i\lambda)^2[iV(s) + iV(t)iV(u)] - i\delta_\lambda + \mathcal{O}(\lambda^3)$$

To keep λ dimensionless in dimensional regularization, we can make the transformation $\lambda \to \lambda\tilde{\mu}^\epsilon$. Here, μ is an arbitrary number with mass dimension 1 and $\epsilon \equiv 4-d$. We can calculate that

$$V(-p^2) = -\frac{1}{32\pi^2}\int_0^1 (\frac{2}{\epsilon} + \ln(\frac{\mu^2}{D(-p^2)}))$$

where $\mu \equiv \sqrt{4\pi}e^{-\gamma/2}\tilde{\mu}$, $D(-p^2) = x(1-x)p^2 + m^2$
The renormalization condition implies that

$$\delta_\lambda = -\lambda^2[V(4m^2) + 2V(0)] + \mathcal{O}(\lambda^3)$$

So,

$$i\mathcal{M} = -i\lambda - \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left[\ln(\frac{D(s)}{D(4m^2)}) + \ln(\frac{D(t)}{D(0)}) + \ln(\frac{D(u)}{D(0)})\right] + \mathcal{O}(\lambda^3)$$

To determine δ_Z and δ_m we must compute the two-point function. Define $-iM(p^2)$ as the sum of all one-particle-irreducible insertions into the propagator.The full two-point function is given by

$$\frac{-i}{p^2 + m^2 + M^2}$$

The renormalization conditions require that the pole in this full propagator occur at $p^2 = -m^2$ and have residue 1. These two conditions are equivalent, respectively, to

$$M^2(p^2)|_{p^2=-m^2} = 0 \quad \frac{d}{dp^2}M^2(p^2)|_{p^2=-m^2} = 0$$

We can calculate that

$$-iM^2(p^2) = \frac{i\lambda}{32\pi^2}(\frac{2}{\epsilon} + \ln(\frac{\mu^2}{m^2}) + 1)m^2 - i(p^2\delta_Z + \delta_m)$$

So, to the order of λ,

$$\delta_Z = \mathcal{O}(\lambda^2) \quad \delta_m = \frac{\lambda}{32\pi^2}(\frac{2}{\epsilon} + \ln(\frac{\mu^2}{m^2}) + 1)m^2 + \mathcal{O}(\lambda^2) \quad M^2(p^2) = \mathcal{O}(\lambda^2)$$

The detailed calculation can be found in chapter 10.2 of *An introduction to quantum field theory (M.E.Peskin & D.V.Schroeder)* and will be eliminated here.

2.7.8 Perturbation theory to all orders

In order to get perturbation theory to all orders, We begin by summing all one-particle irreducible diagrams with two external lines; this gives us the self-energy $M^2(k^2)$. Since the theory is invariant under $\phi \to -\phi$, all amplitudes with an odd number of external legs vanish. So we next sum all 1PI diagrams with four external lines; this gives us the four-point vertex function $V_4(k_1, k_2, k_3, k_4)$. Order by order in λ, we must adjust the value of the lagrangian coefficients δ_Z, δ_m, and δ_λ to maintain the conditions $M^2(-m^2) = 0$, $(\frac{dM^2}{dp^2})(-m^2) = 0$, and $V_4(s = 4m^2) = \lambda$.

Next we will construct the n-point vertex functions $V_n(k_1, \cdots, k_n)$ with $4 < n < E + 1$, where E is the number of external lines in the process of interest. We compute these using a skeleton expansion. This means that we draw all the contributing 1PI diagrams, but omit diagrams that include either propagator or four-point vertex corrections. Then we take the propagators and vertices in these diagrams to be given by the exact propagator $\frac{-i}{p^2+m^2+M^2(p^2)}$ and vertex $V_4(k_1, k_2, k_3, k_4)$, rather than by the tree-level propagator $\frac{-i}{p^2+m^2}$ and vertex λ. We then sum these skeleton diagrams to get V_n for $4 < n < E+1$. Order by order in λ, this procedure is equivalent to computing V_n by summing the usual set of contributing 1PI diagrams.

Next we draw all tree-level diagrams that contribute to the process of interest (which has E external lines), including not only three-point vertices, but also n-point vertices for $n = 4, \cdots, E$. Then we evaluate these diagrams using the exact propagator for internal lines, and the exact 1PI vertices V_n; external lines are assigned a factor of one. We sum these tree diagrams to get the scattering amplitude. Order by order in λ, this procedure is equivalent to computing the scattering amplitude by summing the usual set of contributing diagrams.

Thus we now know how to compute an arbitrary scattering amplitude to arbitrarily high order. The procedure is the same in any quantum field theory; only the form of the propagators and vertices change, depending on the spins of the fields.

2.7.9 Modified minimal-subtraction scheme

The Lagrangian of ϕ^4 theory is

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{1}{2}\delta_Z\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}\delta_m\phi_r - \frac{\delta\lambda}{4!}\phi^4$$

For minimal-subtraction scheme, we do not demand that m be the physics mass of the field and ϕ create a normalized one-particle state. The physical meaning of λ is not expressed directly as well. Instead we choose δ_Z, δ_m and δ_λ to cancel the infinities, and nothing more; we say that δ_Z, δ_m and δ_λ have no finite parts. It is called the modified minimal-subtraction or $\overline{\text{MS}}$ scheme. (Modified because we introduced μ via $\lambda \to \lambda\tilde{\mu}^\epsilon$, with $\mu \equiv \sqrt{4\pi}e^{-\gamma/2}\tilde{\mu}$; had we set $\mu = \tilde{\mu}$ instead, the scheme would be just plain minimal subtraction or MS.)
For loop corrections to propagator,

$$\delta_Z = \mathcal{O}(\lambda^2) \quad \delta_m = \left[\frac{\lambda}{16\pi^2} + \mathcal{O}(\lambda^2)\right]\frac{1}{\epsilon}m^2 \quad M^2(p^2) = \frac{\lambda}{32\pi^2}(\ln(\frac{m^2}{\mu^2}) - 1)m^2 + \mathcal{O}(\lambda^2)$$

Firstly, in the $\overline{\text{MS}}$ scheme, the propagator will no longer have a pole at $k^2 = -m^2$. The pole will be somewhere else. However, by definition, the actual physical mass m_{ph} of the particle is determined by the location of this pole: $k^2 = -m_{ph}^2$. Thus, the lagrangian parameter m is no longer the same as m_{ph}. The relation of m and m_{ph} is

$$m_{ph}^2 = M^2(-m_{ph}^2) + m^2$$

To the lowest order,

$$m_{ph}^2 = \left[1 + \frac{\lambda}{32\pi^2}(\ln(\frac{m^2}{\mu^2}) - 1)\right]m^2$$

Because m_{ph} is a independent of μ, according to $\frac{d}{d\mu}m_{ph} = 0$, it can be derived that

$$\frac{dm}{d\ln\mu} = \left[\frac{\lambda}{32\pi^2} + \mathcal{O}(\lambda^2)\right]m$$

Furthermore, the residue of this pole is no longer one. Let us call the residue R. So, in the LSZ formula, we get a net factor of $\sqrt{R}$ for each external line when using the $\overline{\text{MS}}$ scheme. And in ϕ^4 theory,

$$R = 1 + \mathcal{O}(\lambda^2)$$

For loop corrections to vertex,

$$\delta_\lambda = \left[\frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3)\right]\frac{1}{\epsilon}$$

$$i\mathcal{M} = -i\lambda - \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left[\ln(\frac{D(s)}{\mu^2}) + \ln(\frac{D(t)}{\mu^2}) + \ln(\frac{D(u)}{\mu^2})\right] + \mathcal{O}(\lambda^3)$$

2.7.10 The renormalization group

The Lagrangian of ϕ^4 theory is

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi_0\partial_\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$$

It can be written as

$$\mathcal{L} = -\frac{1}{2}Z_\phi\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}Z_m m^2\phi^2 - Z_\lambda\mu^\epsilon\frac{\lambda}{4!}\phi^4$$

So,

$$\phi_0 = Z_\phi^{1/2}\phi \quad m_0 = Z_\phi^{-1/2}Z_m^{1/2}m \quad \lambda = Z_\phi^{-2}Z_\lambda\lambda\tilde{\mu}^\epsilon$$

After using dimensional regularization, the infinities coming from loop integrals take the form of inverse powers of ϵ. In the $\overline{\mathrm{MS}}$ renormalization scheme, we choose the Zs to cancel off these powers of $1/\epsilon$, and nothing more. Therefore the Zs can be written as

$$Z_\phi = 1 + \sum_{n=1}^{\infty} \frac{a_n(\lambda)}{\epsilon^n}$$

$$Z_m = 1 + \sum_{n=1}^{\infty} \frac{b_n(\lambda)}{\epsilon^n}$$

$$Z_\lambda = 1 + \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{\epsilon^n}$$

In ϕ^4 theory, $a_1 = \mathcal{O}(\lambda^2)$, $b_1 = \frac{\lambda}{16\pi^2} + \mathcal{O}(\lambda^2)$,$c_1 = \frac{3\lambda}{16\pi^2} + \mathcal{O}(\lambda^2)$
Remember that bare fields and parameters must be independent of μ.Define

$$G(\lambda, \epsilon) \equiv \ln(Z_\phi^{-2} Z_\lambda) = \sum_{n=1}^{\infty} \frac{G_n(\lambda)}{\epsilon^n}$$

We can calculate $G_1 = c_1 - 2a_1 = \frac{3\lambda}{16\pi^2} + \mathcal{O}(\lambda^2)$. As $\ln \lambda_0 = G + \ln \lambda + \epsilon \ln \tilde{\mu}$. From the independence of λ_0, we can derive

$$\left(1 + \frac{\lambda G_1'}{\epsilon} + \cdots\right) \frac{d\lambda}{d \ln \mu} + \epsilon \lambda = 0$$

In a renormalizable theory, we should have

$$\frac{d\lambda}{d \ln \mu} = -\epsilon \lambda + \beta(\lambda)$$

So

$$\beta(\lambda) = \lambda^2 G_1'(\lambda)$$

In ϕ^4 theory, we have $\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3)$.Define

$$M(\lambda, \epsilon) \equiv \ln(Z_m^{1/2} Z_\phi^{-1/2}) = \sum_{n=1}^{\infty} \frac{M_n(\lambda)}{\epsilon^n}$$

We can calculate $M_1 = \frac{1}{2}b_1 - \frac{1}{2}a_1 = \frac{\lambda}{32\pi^2} + \mathcal{O}(\lambda^2)$. As $\ln m_0 = M + \ln m$, define the anomalous dimension of the mass

$$\gamma_m(\lambda) \equiv \frac{1}{m} \frac{dm}{d \ln \mu}$$

From the independence of m_0, we can derive

$$\gamma_m(\lambda) = \lambda M_1'$$

In ϕ^4 theory, we have $\gamma_m(\lambda) = \frac{\lambda}{32\pi^2} + \mathcal{O}(\lambda^2)$. We can expand $\ln Z_\phi$ as

$$\ln Z_\phi = \frac{a_1}{\epsilon} + \frac{a_2 - \frac{1}{2}a_1^2}{\epsilon^2}$$

Define the anomalous dimension of the field

$$\gamma_\phi(\lambda) = \frac{1}{2} \frac{d \ln Z_\phi}{d \ln \mu}$$

We can derive

$$\gamma_\phi(\lambda) = -\frac{1}{2} \lambda a_1'$$

In ϕ^4 theory, we have $\gamma_m(\lambda) = \mathcal{O}(\lambda^2)$.

Callen-Symanzik equation

$$G^{(n)}(x_1, \cdots, x_n) \equiv \langle\Omega|T\phi(x_1)\cdots\phi(x_n)|\Omega\rangle_C$$

As $G_0^{(n)} = Z_\phi^{n/2} G^{(n)}$, from the independence of bare Green's function, we have

$$\left(\frac{\partial}{\partial \ln \mu} + \beta(\lambda)\frac{\partial}{\partial \lambda} + \gamma_m(\lambda) m \frac{\partial}{\partial m} + n\gamma_\phi(\lambda)\right) G^n(x_1, \cdots, x_n; \lambda, m, \mu) = 0$$