Physics
物理

Do not ask what it is. Ask what you can say about it.

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Chapter 1
The formulation of Classical Mechanics

1.1 Lagrangian Formulation

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt, \quad \delta q_i(t_1) = \delta q_i(t_2) = 0$$

$$\delta S = 0 \to \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

1. If we transform the coordinates q to the Q as $q = q(Q, t)$, the new Lagrangian will be

$$\bar{L}(Q, \dot{Q}, t) \equiv L(q(Q, t), \dot{q}(Q, \dot{Q}, t), t)$$

   We can verify that

$$\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{Q}} - \frac{\partial \bar{L}}{\partial Q} = 0$$

2. If $L_1 = L + \frac{d}{dt}f(q, t)$, then L and L_1 is equivalent and will generate the same dynamical equation.

Example:

1. The form of Lagrangian for an isolated system of particles in inertial frame:

$$L = \sum_a \frac{1}{2}m_a v_a^2 - U(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots,)$$

   The equation of motion is

$$m_i\ddot{\boldsymbol{r}}_i = -\nabla_{\boldsymbol{r}_i}U$$

   To get the form of Lagrangian for a system of interacting particles, we must assume:

   - Space and time are homogeneous and isotropic in inertial frame;
   - Galileo's relativity principle and Galilean transformation;
   - Spontaneous interaction between particles;

2. Consider a reference frame K. Suppose the K is moving with the velocity $\boldsymbol{V}(t)$ and rotating with angular velocity $\boldsymbol{\Omega}$ relative to the inertial reference frame. We use the coordinates of the mass point in K as general coordinates, i.e. $\boldsymbol{r} = (x_k, y_k, z_k)$. Then the Lagrangian of the mass point will be

$$L = \frac{1}{2}m\boldsymbol{v}^2 + m\boldsymbol{v} \cdot (\boldsymbol{\Omega} \times \boldsymbol{r}) + \frac{m}{2}(\boldsymbol{\Omega} \times \boldsymbol{r})^2 - m\dot{\boldsymbol{V}} \cdot \boldsymbol{r} - U$$

The equation of motion will be

$$m\frac{d\boldsymbol{v}}{dt} = -\frac{\partial U}{\partial \boldsymbol{r}} - m\dot{\boldsymbol{V}} + m(\boldsymbol{r} \times \dot{\boldsymbol{\Omega}}) + 2m(\boldsymbol{v} \times \boldsymbol{\Omega}) + m[\boldsymbol{\Omega} \times (\boldsymbol{r} \times \boldsymbol{\Omega})]$$

1.2 Symmetry and Conservation Laws(1)

Theorem 1.1 Nother's theorem

For $q_i \to q_i + \delta q_i$ and $L \to L + \delta L$, if $\delta L = \frac{df(q,\dot{q},t)}{dt}$,then we get

$$\frac{d}{dt}\left(\sum_i p^i \delta q_i - f\right) = 0 \quad \left(p^i = \frac{\partial L}{\partial \dot{q}_i}\right)$$

Example: For an isolated system of particles in inertial frame,
$\delta L = 0$ when $\delta \boldsymbol{r}_i \to \boldsymbol{r}_i + \delta \boldsymbol{a}$, so

$$\frac{d}{dt}\left(\sum_i \boldsymbol{p}_i\right) = 0$$

$\delta L = 0$ when $\delta \boldsymbol{r}_i \to \boldsymbol{r}_i + \boldsymbol{r}_i \times \delta \boldsymbol{\theta}$, so

$$\frac{d}{dt}\left(\sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i\right) = 0$$

Homogeneity of time   If $\frac{\partial L}{\partial t} = 0$,then we get

$$\frac{dE}{dt} = 0 \quad \left(E = \sum_i \dot{q}_i p^i - L\right)$$

1.3 Hamilton formulation

$$p^i = \frac{\partial L}{\partial \dot{q}_i}$$

$$H(q,p,t) = \sum_i p^i \dot{q}_i - L$$

$$\dot{p}^i = -\frac{\partial H}{\partial q_i} \quad \dot{q}_i = \frac{\partial H}{\partial p^i}$$

Example: For an isolated system of particles in inertial frame,

$$\boldsymbol{p}_i = m_i \boldsymbol{v}_i$$

$$H(q, p, t) = \sum_i \frac{p_i^2}{2m} + U(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots)$$

$$\dot{\boldsymbol{p}}_i = -\nabla_{\boldsymbol{r}_i} U \quad \dot{\boldsymbol{r}}_i = \frac{\boldsymbol{p}_i}{m_i}$$

1.3.1 Poisson Brackets

First, we assume the bracket operation has the following properties:

$$[f, g] = -[g, f]$$

$$[\alpha_1 f_1 + \alpha_2 f_2, \beta_1 g_1 + \beta_2 g_2] = \alpha_1 \beta_1 [f_1, g_1] + \alpha_1 \beta_2 [f_1, g_2] + \alpha_2 \beta_1 [f_2, g_1] + \alpha_2 \beta_2 [f_2, g_2]$$

$$[f_1 f_2, g_1 g_2] = f_1 [f_2, g_1] g_2 + f_1 g_1 [f_2, g_2] + g_1 [f_1, g_2] f_2 + [f_1, g_1] g_2 f_2$$

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$$

Here, f, g, h are functions of p^i, q_i, t. Then, we assume that

$$[q_i, p^k] = \delta_i^k$$

we can derive that

$$[f, g] = \sum_k \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p^k} - \frac{\partial f}{\partial p^k} \frac{\partial g}{\partial q_k} \right)$$

So the Hamilton equation can be written as

$$\dot{p}^i = [p^i, H] \quad \dot{q}_i = [q_i, H]$$

And we can also get

$$\frac{df}{dt} = [f, H] + \frac{\partial f}{\partial t} \quad \frac{d}{dt}[f, g] = [\frac{df}{dt}, g] + [f, \frac{dg}{dt}]$$

Example: For an isolated system of particles in inertial frame,

$$[r_{ia}, p_{jb}] = \delta_{ab} \delta_{ij}$$

we define $l_a = \epsilon_{abc} r_a p_b$, then

$$[l_a, r_b] = \epsilon_{abc} r_c \quad [l_a, p_b] = \epsilon_{abc} p_c \quad [l_a, l_b] = \epsilon_{abc} l_c$$

1.3.2 Canonical transformations

In Hamiltonian mechanics, a canonical transformation is a change of canonical coordinates that preserves the form of Hamilton's equations (that is, the new Hamilton's equations resulting from the transformed Hamiltonian may be simply obtained by substituting the new coordinates for the old coordinates), although it might not preserve the Hamiltonian itself.

$$Q_i = Q_i(p, q, t) \quad P_i = P_i(p, q, t)$$

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i} \quad \dot{P}_i = -\frac{\partial H'}{\partial Q_i}$$

> **Proposition 1.1 Canonical condition**
>
> If $(q_i, p^i, H) \to (Q_i, P^i, H)$ is a canonical transformation, then there exists a generating function $F(q_i, Q_i, t)$ satisfying that
>
> $$\sum_i p^i \dot{q}_i - H(p^i, q_i) = \sum_i P^i \dot{Q}_i - H'(Q_i, P^i) + \frac{dF}{dt}$$

Applying Legendre transformation, we can get four kinds of generating function.

1.

$$\frac{dF}{dt} = \sum_i p^i \dot{q}_i - \sum_i P^i \dot{Q}^i + (H' - H)$$

Assume $\Phi(q_i, Q_i, t) = F$, so

$$p^i = \frac{\partial \Phi}{\partial q_i} \quad P^i = -\frac{\partial \Phi}{\partial Q_i} \quad H' = H + \frac{\partial \Phi}{\partial t}$$

2.

$$\frac{d}{dt}\left(F + \sum_i P^i Q_i\right) = \sum_i p^i \dot{q}_i + \sum_i Q_i \dot{P}^i + (H' - H)$$

Assume $\Phi(q_i, P^i, t) = F + \sum_i P^i Q_i$, so

$$p^i = \frac{\partial \Phi}{\partial q_i} \quad Q_i = \frac{\partial \Phi}{\partial P^i} \quad H' = H + \frac{\partial \Phi}{\partial t}$$

3.

$$\frac{d}{dt}\left(F - \sum_i p^i q_i\right) = -\sum_i q_i \dot{p}^i - \sum_i P^i \dot{Q}_i + (H' - H)$$

Assume $\Phi(p^i, Q_i, t) = F - \sum_i p^i q_i$, so

$$q_i = -\frac{\partial \Phi}{\partial p^i} \quad P^i = -\frac{\partial \Phi}{\partial Q_i} \quad H' = H + \frac{\partial \Phi}{\partial t}$$

4.

$$\frac{d}{dt}\left(F + \sum_i P^i Q_i - \sum_i p^i q_i\right) = -\sum_i q_i \dot{p}^i + \sum_i Q_i \dot{P}^i + (H' - H)$$

Assume $\Phi(p^i, P^i, t) = F + \sum_i P^i Q_i - \sum_i p^i q_i$, so

$$q_i = -\frac{\partial \Phi}{\partial p^i} \quad Q_i = \frac{\partial \Phi}{\partial P^i} \quad H' = H + \frac{\partial \Phi}{\partial t}$$

Theorem 1.2 The invariance of Poisson Bracket

Suppose that $(q, p, H) \to (Q, P, H')$ is a canonical transformation and $f(q, p, t) = F(Q, P, t)$, $g(q, p, t) = G(Q, P, t)$, then

$$[f, g]_{q,p} = [F, G]_{Q,P}$$

♣

As a result, the condition for canonical transformation can also be stated as

$$[Q_i, Q_j]_{q,p} = 0 \quad [P^i, P^j]_{p,q} = 0 \quad [Q_i, P^j]_{q,p} = \delta_i^j$$

1.3.3 Evolution as canonical transformations

Let q_t, p_t be the values of the canonical variables at time t, and $q_{t+\tau}, p_{t+\tau}$ their values at another time $t + \tau$. The latter are some functions of the former:

$$q_{t+\tau} = q(q_t, p_t, t, \tau) \quad p_{t+\tau} = p(q_t, p_t, t, \tau)$$

If these formulae are regarded as a transformation from the variables q_t, p_t to $q_{t+\tau}$, $p_{t+\tau}$, then this transformation is canonical. This is evident from the expression

$$dS = p_t dq_t + p_{t+\tau} dq_{t+\tau} - (H_{t+\tau} - H_t)dt$$

for the differential of the action $S(q_t, q_{t+\tau}, t, \tau)$, taken along the true path, passing through the points q, and $q_{t+\tau}$ at times t and $t + \tau$ for a given τ. $-S$ is the generating function of the transformation. So we have the following communication relation

$$[q_{i\,t+\tau}, q_{j\,t+\tau}]_{q_t, p_t} = 0 \quad [p_{t+\tau}^i, p_{t+\tau}^j]_{q_t, p_t} = 0 \quad [q_{i\,t+\tau}, p_{t+\tau}^j]_{q_t, p_t} = \delta_i^j$$

1.3.4 Liouville's theorem

Lemma 1

Let D be the Jacobian of the canonical transformation

$$\frac{\partial(Q_1, \cdots, Q_s, P^1, \cdots, P^s)}{\partial(q_1, \cdots, q_s, p^1, \cdots, p^s)}$$

Then we have

$$D = 1$$

♣

> **Theorem 1.3 Liouville's theorem**
>
> The phase-space distribution function is constant along the trajectories of the system

☞ **Proof:** The phase volume is invariant under canonical transformation. The change in p and q during the motion can be regarded as a canonical transformation. Suppose that each point in the region of phase space moves in the course of time in accordance with the equations of motion of the mechanical system. The region as a whole therefore moves also, but its volume remains unchanged.   □

1.4 Symmetry and Conservation Laws(2)

Suppose g is a function of p and q. If the transformation of q and p can be described as

$$q \to q + \epsilon[q, g]$$

$$p \to p + \epsilon[p, g]$$

We can prove that

$$H \to H + \epsilon[H, g]$$

So if H is invariant under the transformation, then $[H, g] = 0$, that means $\frac{dg}{dt} = 0$, i.e. g is a conserved quantity of the motion.

1.5 Hamilton-Jacobi equation

We define

$$S(q, t) = \left(\int_{q_0, t_0}^{q, t} L dt \right) |_{extremum}$$

We can prove that

$$p = \frac{\partial S}{\partial q}, \quad H = -\frac{\partial S}{\partial t}$$

So, we have

$$-\frac{\partial S}{\partial t} = H(q, \frac{\partial S}{\partial q})$$

This is called Hamiltonian-Jacobi equation.

Suppose the complete integral of the Hamilton-Jacobi equation is

$$S = f(t, q_1, \cdots, q_s; \alpha^1, \cdots, \alpha^s) + A$$

where $\alpha^1, \cdots, \alpha^s$ and A are arbitrary constants. We effect a canonical transformation from the variables q, p to new variables, taking the function $f(t, q, \alpha)$ as the generating function, and the quantities $\alpha^1, \cdots, \alpha^s$ as the new momenta. Let the new co-ordinates be $\beta_1, \cdots, \beta_2$.

$$p^i = \frac{\partial f}{\partial q_i} \quad \beta_s = \frac{\partial f}{\partial \alpha_s} \quad H' = H + \frac{\partial f}{\partial t} = 0$$

So,

$$\alpha^s = \text{ constant} , \beta_s = \text{ constant}$$

By means of the s equations $\beta_s = \frac{\partial f}{\partial \alpha^s}$, the s coordinates q can be expressed in terms of the time and the $2s$ constants. This gives the general integral of the equations of motion.

1.6 Symmetry and Conservation Laws(3)

If S is invariant under transformation $q_i \to q_i + \delta q_i$, then

$$\delta S = (\sum_i p^i \delta q_i)|_{q_0,t_0}^{q,t} = 0$$

So, we have

$$\frac{d}{dt}(p^i \delta q_i) = 0$$

Further more, if

$$\delta S = (\sum_i p^i \delta q_i)|_{q_0,t_0}^{q,t} = f(q_i, \dot{q}_i, t)|_{q_0,t_0}^{q,t}$$

we will have conserved quantity

$$\frac{d}{dt}(p^i \delta q_i - f) = 0$$

Chapter 2
Two body problem

2.1 Reduced mass and central field

The Lagrangian for a two-body system is

$$L = \frac{1}{2}m_1\dot{\boldsymbol{r}}_1^2 + \frac{1}{2}m_2\dot{\boldsymbol{r}}_2^2 + U(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$$

Let $\boldsymbol{r} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_2$ be the relative position vector and let the origin be at the centre of mass, i.e. $m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 = 0$. These two equations give

$$\boldsymbol{r}_1 = \frac{m_2}{m_1 + m_2}\boldsymbol{r} \quad \boldsymbol{r}_2 = \frac{m_1}{m_1 + m_2}\boldsymbol{r}$$

Then, we have

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 - U(r)$$

where

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

is called reduced mass. The Lagrangian is formally identical with the Lagrangian of a particle of mass m moving in an external field $U(r)$ which is symmetrical about a fixed origin.

L is isotropic, so angular momentum is conserved, i.e. $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{p} = \text{ const}$. Since $\boldsymbol{r}$ is always perpendicular to $\boldsymbol{M}$, the path of the particle lies in one plane. Using polar coordinates, we have

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

And it is easy to see that

$$M = mr^2\dot{\phi} = \text{ const} \quad E = \frac{1}{2}m\dot{r}^2 + \frac{M^2}{2mr^2} + U(r) = \text{ const}$$

So,

$$\frac{dr}{dt} = \sqrt{\frac{2(E - U(r))}{m} - \frac{M^2}{m^2 r^2}}$$

$$\frac{d\phi}{dr} = \frac{M}{r^2\sqrt{2m(E - U(r)) - M^2/r^2}}$$

The radial part of the motion can be regarded as taking place in one dimension in a field where the effective potential energy is

$$U_{eff} = U(r) + \frac{M^2}{2mr^2}$$

The values of r for which

$$U(r) + \frac{M^2}{2mr^2} = E$$

determine the limits of the motion as regards distance from the centre. When equation above is satisfied, the radial velocity $\dot{r}$ is zero. This does not mean that the particle comes to rest as in true one-dimensional motion, since the angular velocity is not zero. The value $\dot{r} = 0$ indicates a turning point of the path, where $r(t)$ begins to decrease instead of increasing, or vice versa. If the range in which r may vary is limited only by the condition $r \geq r_{min}$, the motion is infinite: the particle comes from, and returns to, infinity. If the range of r has two limits r_{min} and r_{max}, the motion is finite and the path lies entirely within the annulus bounded by the circles $r = r_{min}$ and $r = r_{max}$. This does not mean that the path must be a closed curve. During the time in which r varies from r_{min} to r_{max} and back, the radius vector turns through an angle

$$\Delta\phi = 2 \int_{r_{min}}^{r_{max}} \frac{M}{r^2\sqrt{2m(E - U(r)) - M^2/r^2}} dr$$

The condition for the path to be closed is that this angle should be a rational fraction of 2π. There are only two types of central field in which all finite motions take place in closed paths. They are those in which the potential energy of the particle varies as $\frac{1}{r}$ or as r^2.

The presence of the centrifugal energy when $M \neq 0$, which becomes infinite as $\frac{1}{r^2}$ when $r \to 0$, generally renders it impossible for the particle to reach the centre of the field, even if the field is an attractive one. A fall of the particle to the centre is possible only if the potential energy tends sufficiently rapidly to $-\infty$ as $r \to 0$. From the inequality

$$\frac{1}{2}m\dot{r}^2 = E - U(r) - \frac{M^2}{2mr^2} > 0$$

it follows that r can take values tending to zero only if

$$[r^2 U(r)]_{r\to 0} < -\frac{M^2}{2m}$$

2.2 Kepler Problem

An important class of central fields is formed by those in which the potential energy is inversely proportional to r. They include the fields of N Newtonian gravitational attraction and of Coulomb electrostatic interaction; the latter may be either attractive or repulsive.
Let us first consider an attractive field, where

$$U = -\frac{\alpha}{r}$$

with α a positive constant. The effective potential energy

$$U_{eff} = -\frac{\alpha}{r} + \frac{M^2}{2mr^2}$$

As $r \to 0$, U_{eff} tends to $+\infty$, and as $r \to \infty$ it tends to zero from negative values; for $r = \frac{M^2}{m\alpha}$ it has a minimum value

$$U_{eff,min} = -\frac{m\alpha^2}{2M^2}$$

The motion is finite for $-\frac{m\alpha^2}{2M^2} \leq E < 0$ and infinite for $E \geq 0$.

The shape of path is

$$\frac{p}{r} = 1 + e\cos\phi$$

Here,

$$p = \frac{M^2}{m\alpha} \quad e = \sqrt{1 + \frac{2EM^2}{m\alpha^2}}$$

This is the equation of a conic section with one focus at the origin; $2p$ is called the latus rectum of the orbit and e the eccentricity. Our choice of the origin is such that the point where $\phi = 0$ is the point nearest to the origin (called the perihelion).

If $E < 0$, the orbit is an ellipse and the motion is finite.
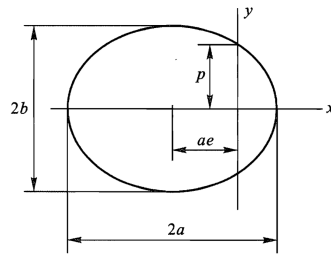
Figure 2.1: Attractive Kepler orbit with $e < 1$

The major and minor semi-axes of the ellipse are

$$a = \frac{p}{1 - e^2} = \frac{\alpha}{2|E|} \quad b = \frac{p}{\sqrt{1 - e^2}} = \frac{M}{\sqrt{2m|E|}}$$

The least and greatest distances from the centre of the field (the focus of the ellipse) are

$$r_{min} = \frac{p}{1 + e} = a(1 - e) \quad r_{max} = \frac{p}{1 - e} = a(1 + e)$$

The period of revolution in an elliptical orbit is

$$T = \frac{\pi ab}{\frac{1}{2}r^2\dot{\phi}} = 2\pi a^{3/2}\sqrt{\frac{m}{\alpha}} = \pi\alpha\sqrt{\frac{m}{2|E|^3}}$$

If $E > 0$, the path is a hyperbola with the origin as internal focus.

Figure 2.2: Attractive Kepler orbit with $e > 1$

The distance of the perihelion from the focus is

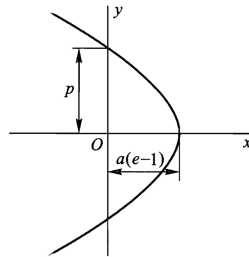$$r_{min} = \frac{p}{1+e} = a(1-e)$$

where $a = \frac{p}{(1-e^2)^2} = \frac{\alpha}{2E}$ is the semiaxis of the hyperbola.

If $E = 0$, the eccentricity $e = 1$, and the particle moves in a parabola with perihelion distance $r_{min} = \frac{p}{2}$. This case occurs if the particle starts from rest at infinity.

Let us now consider motion in a repulsive field, where

$$U = \frac{\alpha}{r} \quad (\alpha > 0)$$

Here the effective potential energy is

$$U_{eff} = \frac{\alpha}{r} + \frac{M^2}{2mr^2}$$

and decreases monotonically from $+\infty$ to zero as r varies from zero to infinity. The energy of the particle must be positive, and the motion is always infinite. The calculations are exactly similar to those for the attractive field. The path is a hyperbola:

$$\frac{p}{r} = -1 + e\cos\phi$$
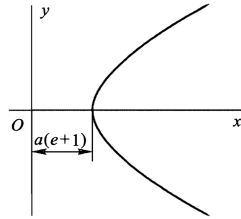
Figure 2.3: Repulsive Kepler orbit

The perihelion distance is

$$r_{min} = \frac{p}{-1+e} = a(1+e)$$

There is an integral of the motion which exists only in fields $U = \frac{\alpha}{r}$. It is easy to verify by direct calculation that the quantity

$$\boldsymbol{v} \times \boldsymbol{M} + \frac{\alpha\boldsymbol{r}}{r}$$

is constant. The direction of the conserved vector is along the major axis from the focus to the perihelion, and its magnitude is αe. This is most simply seen by considering its value at perihelion.

2.3 Disintegration and collisions of particles

Let us consider a spontaneous disintegration of a particle into two constituent parts. This process is most simply described in a frame of reference in which the particle is at rest before the disintegration. The law of conservation of momentum shows that the sum of the momenta of the two particles formed in the disintegration is then zero; that is, the particles move apart with equal and opposite momenta. The magnitude p_0 of either momentum is given by the law of conservation of energy:

$$E_i = E_{1i} + \frac{p_0^2}{2m_1} + E_{2i} + \frac{p_0^2}{2m_2}$$

here m_1 and m_2 are the masses of the particles, E_{1i} and E_{2i}, their internal energies, and E_i the internal energy of the original particle. If ϵ is the disintegration energy, i.e. the difference

$$\epsilon = E_i - E_{1i} - E_{2i}$$

which must obviously be positive, then

$$\epsilon = \frac{p_0^2}{2m}$$

here m is the reduced mass of the two particles.

Let us now change to a frame of reference in which the primary particle moves with velocity $\boldsymbol{V}$ before the break-up. This frame is usually called the laboratory system, or L system, in contradistinction to the centre-of-mass system, or C system, in which the total momentum is zero. Let us consider one of the resulting particles, and let $\boldsymbol{v}$ and $\boldsymbol{v_0}$ be its velocities in the L and the C system-respectively. It can be represented by

(a) $V < v_0$ **(b)** $V > v_0$
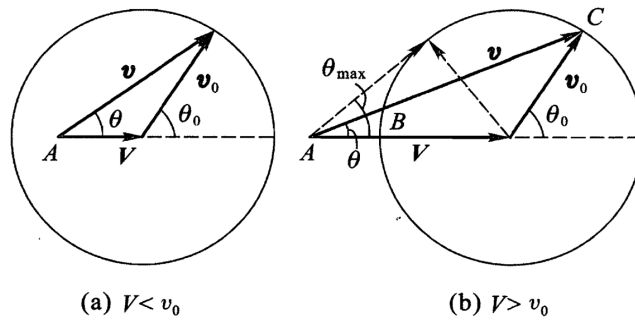
Figure 2.4: Disintegration in L and C frame

The relation between the angles θ and θ_0 in the L and C systems is evidently.

$$\tan \theta = \frac{v_0 \sin \theta_0}{V + v_0 \cos \theta_0}$$

In physical applications we are usually concerned with the disintegration of not one but many similar particles, and this raises the problem of the distribution of the resulting particles in direction, energy, etc. We shall assume that the primary particles are randomly oriented in

space, i.e. isotropically on average.

In the C system, every resulting particle has the same energy, and their directions of motion are isotropically distributed. The fraction of particles entering a solid angle element do is $\frac{do}{4\pi}$. So the distribution with respect to the angle θ_0 is

$$\frac{1}{2}\sin\theta_0 d\theta_0$$

The corresponding distributions in the L system are obtained by an appropriate transformation. For example, let us calculate the kinetic energy distribution in the L system. Since

$$v^2 = V^2 + v_0^2 + 2Vv_0\cos\theta_0$$

we have $d(v^2) = d\cos\theta_0$. So the kinetic energy can distributed uniformly over between $T_{min} = \frac{1}{2}(v_0 - V)^2$ and $T_{max} = \frac{1}{2}m(v_0 + V)^2$.

A collision between two particles is said to be elastic if it involves no change in their internal state. The collision is most simply described in the C system.The velocities of the particles before the collision are related to their velocities v_1 and v_2 in the L system by $v_{10} = m_2 v/(m_1 + m_2)$, $v_{20} = -m_1 v/(m_1 + m_2)$, where $v = v_1 - v_2$.

Because of the law of conservation of momentum, the momenta of the two particles remain equal and opposite after the collision, and are also unchanged in magnitude, by the law of conservation of energy. Thus, in the C system the collision simply rotates the velocities, which remain opposite in direction and unchanged in magnitude. The velocities of the two particles after the collision are

$$v'_{10} = \frac{m_2 v \boldsymbol{n_0}}{m_1 + m_2} \quad v'_{20} = -\frac{m_1 v \boldsymbol{n_0}}{m_1 + m_2}$$

The velocities in the L system after the collision are therefore

$$v'_1 = \frac{m_2 v \boldsymbol{n_0}}{m_1 + m_2} + \frac{m_1 \boldsymbol{v_1} + m_2 \boldsymbol{v_2}}{m_1 + m_2} \quad v'_2 = -\frac{m_1 v \boldsymbol{n_0}}{m_1 + m_2} + \frac{m_1 \boldsymbol{v_1} + m_2 \boldsymbol{v_2}}{m_1 + m_2}$$

Multiplying equations by m_1 and m_2 respectively, we obtain

$$\boldsymbol{p}'_1 = mv\boldsymbol{n_0} + \frac{m_1(\boldsymbol{p_1} + \boldsymbol{p_2})}{m_1 + m_2} \quad \boldsymbol{p}'_2 = -mv\boldsymbol{n_0} + \frac{m_2(\boldsymbol{p_1} + \boldsymbol{p_2})}{m_1 + m_2}$$
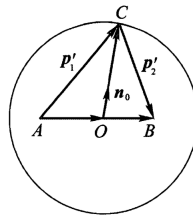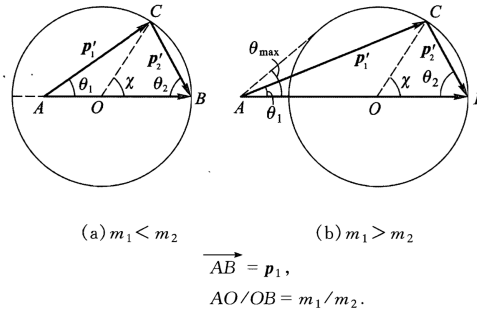
It can be represented by

Figure 2.5: Collision in L and C frame

Let us consider in more detail the case where one of the particles (m_2, say) is at rest before the

(a) $m_1 < m_2$ (b) $m_1 > m_2$

$$\overrightarrow{AB} = \boldsymbol{p}_1,$$
$$AO/OB = m_1/m_2.$$

Figure 2.6: Collision with 2 at rest

collision. In that case the distance $OB = \frac{m_2 p_1}{m_1 + m_2} = mv$ is equal to the radius. The vector $\overrightarrow{AB}$ is equal to the momentum $\boldsymbol{p}_1$ of the particle m_1 before the collision.

θ_1 and θ_2 can be expressed in terms of χ by

$$\tan \theta_1 = \frac{m_2 \sin \chi}{m_1 + m_2 \cos \chi} \quad \theta_2 = \frac{1}{2}(\pi - \chi)$$

The magnitudes of the velocities of the two particles after the collision in terms of χ are

$$v_1' = \frac{\sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \cos \chi}}{m_1 + m_2} v \quad v_2' = \frac{2m_1 v}{m_1 + m_2} \sin \frac{1}{2}\chi$$

If $m_1 < m_2$, the velocity of m_1 after the collision can have any direction. If $m_1 > m_2$, this particle can be deflected only through an angle not exceeding $\theta_m ax$ from its original direction. Evidently

$$\sin \theta_{max} = \frac{m_2}{m_1}$$

The collision of two particles of equal mass, of which one is initially at rest, is especially simple. In this case both B and A lie on the circle, so
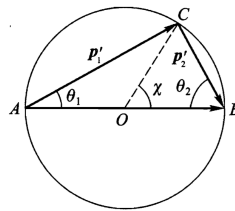
Figure 2.7: Collision of equal mass

Then

$$\theta_1 = \frac{1}{2}\chi \quad \theta_2 = \frac{1}{2}(\pi - \chi)$$

$$v_1' = v \cos \frac{1}{2}\chi \quad v_2' = v \sin \frac{1}{2}\chi$$

After the collision the particles move at right angles to each other.

2.4 Scattering and cross section

Scattering is a general physical process where some forms of radiation, such as light, sound, or moving particles, are forced to deviate from a straight trajectory by one or more paths due to localized non-uniformities in the medium through which they pass. In classical mechanics, scattering generally refer to particle-particle collisions.

The definition of cross section of a scattering process is

$$\sigma \equiv \frac{\text{Number of Events per target}}{\text{Time} \times \text{Incident Flux}}$$

Here, the incident flux are measured in the frame of target particle. Recall that when we reduce two body problem in to one body problem, $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ is the coordinates of projectile in the frame of target. So the scattering process can be represented by the reduced mass moving in the central field.
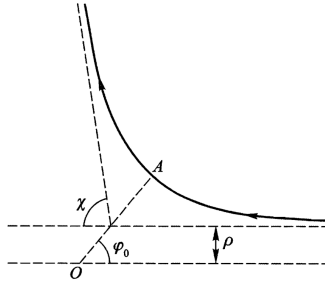

Figure 2.8: Scattering in central field

And we have

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{M}{r^2 \sqrt{2m(E - U(r)) - \frac{M^2}{r^2}}} \quad \chi = |\pi - 2\phi_0|$$

Since

$$E = \frac{1}{2}mv_\infty^2 \quad M = m\rho v_\infty,$$

we can get the relation between χ and ρ. Suppose the number density of the particles is n, then the incident flux is nv_∞, number of events that particles are scattered into the solid angle $do = \sin\chi d\chi d\phi$ at (χ, ϕ) in time T is

$$d\rho\rho(\chi)d\phi nv_\infty T$$

So, we have

$$d\sigma = \rho(\chi)d\rho d\phi = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| do$$

In C system, we have $\boldsymbol{r}_1 = \frac{m_2}{m_1 + m_2}\boldsymbol{r}$, so the scattering angle of particle 1 is the same as χ. While in L system (particle 2 is at rest before scattering), we must making corresponding transformation to get the right expression for cross section.

Rutherford's formula

One of the most important applications of the formulae derived above is to the scattering of charged particles in a Coulomb field. As $U = \frac{\alpha}{r}$, we have

$$\phi_0 = \arccos \frac{\alpha/mv_\infty^2 \rho}{\sqrt{1 + (\alpha/mv_\infty^2 \rho)^2}}$$

Recall that $\chi = \frac{1}{2}(\pi - \phi_0)$, we can obtain that

$$\rho^2 = \frac{\alpha^2}{m^2 v_\infty^4} \cot^2 \frac{1}{2}\chi$$

and

$$d\sigma = \left(\frac{\alpha}{2mv_\infty^2}\right)^2 \frac{do}{\sin^4 \frac{1}{2}\chi}$$

This is Rutherford's formula. It may be noted that the effective cross-section is independent of the sign of α, so that the result is equally valid for repulsive and attractive Coulomb fields. Formula above gives the effective cross-section in the frame of reference in which the centre of mass of the colliding particles is at rest. The transformation to the laboratory system is effected by means of

$$\tan \theta_1 = \frac{m_2 \sin \chi}{m_1 + m_2 \cos \chi} \quad \theta_2 = \frac{1}{2}(\pi - \chi)$$

For particles initially at rest, we have

$$d\sigma_2 = \left(\frac{\alpha}{mv_\infty^2}\right)^2 \frac{do_2}{\cos^3 \theta_2}$$

The same transformation for the incident particles leads, in general, to a very complex formula, and we shall merely note two particular cases.

If the mass m_2 of the scattering particle is large compared with the mass m_1 of the scattered particle, then $\chi = \theta_1$ and $m = m_1$, so that

$$d\sigma_1 = \left(\frac{\alpha}{4E_1}\right)^2 \frac{do_1}{\sin^2(\frac{1}{2}\theta_1)}$$

here $E_1 = \frac{1}{2}m_1 v_\infty^2$ is the energy of the incident particle.

If the masses of the two particles are equal, then by $\theta_1 = \frac{\chi}{2}$, we have

$$d\sigma_1 = \left(\frac{\alpha}{E_1}\right)^2 \frac{\cos \theta_1}{\sin^4 \theta_1} do_1$$

If the particles are entirely identical, that which was initially at rest cannot be distinguished after the collision. The total effective cross-section for all particles is obtained by adding do_1 and do_2, so

$$d\sigma = \left(\frac{\alpha}{E_1}\right)^2 \left(\frac{1}{\sin^4 \theta} + \frac{1}{\cos^4 \theta}\right) \cos \theta do$$

Let us return to the general formula and use it to determine the distribution of the scattered particles with respect to the energy lost in the collision. When the masses of the scattered

and scattering particles are arbitrary, the velocity acquired by the latter is given in terms of the angle of scattering in the C system by

$$v_2' = \frac{2m_1}{m_1 + m_2} v_\infty \sin \frac{\chi}{2}$$

The energy acquired by 2 and lost by 1 is therefore

$$\epsilon = \frac{2m^2}{m_2} v_\infty^2 \sin^2 \frac{\chi}{2}$$

Expressing $\sin \frac{\chi}{2}$ in terms of ϵ, we obtain

$$d\sigma = 2\pi \frac{\alpha^2}{m_2 v_\infty^2} \frac{d\epsilon}{\epsilon^2}$$

This is the required formula: it gives the effective cross-section as a function, of the energy loss ϵ, which takes values from zero to $\epsilon_{max} = 2m^2 v_\infty^2 / m_2$.

Chapter 3
Small Oscillation

3.1 Small oscillation in one dimensional

Let us consider the motion in one dimension, the potential energy of the particle is $V = V(q)$. If we choose the coordinate of equilibrium point as $q = 0$, then $\frac{\partial V}{\partial q}|_{q=0} = 0$. Expand $V(q)$ around $q = 0$, we have

$$V(q) = V(0) + \frac{1}{2}\frac{\partial^2 V}{\partial q^2}|_{q=0} \, q^2 + \cdots$$

If the equilibrium point is stable,we have

$$\frac{\partial^2 V}{\partial q^2}|_{q=0} \equiv V''(0) > 0$$

For small oscillation, we can neglect the higher orders of q and the Lagrangian can be written as

$$L = \frac{1}{2}m\dot{q} - \frac{1}{2}V''(0)q^2$$

The Euler-Lagrangian equation is

$$\ddot{q} + \omega_0^2 q = 0 \quad \omega_0^2 = \frac{V''(0)}{m}$$

The general solution is

$$q = A\cos(\omega_0 t + \phi)$$

A and ϕ depends on the initial condition.

If there is a damped force which is proportional to the velocity of the particle, then we have

$$\ddot{q} + \frac{1}{Q}\dot{q} + \omega_0^2 q = 0$$

If $Q > \frac{1}{2\omega_0}$, we have

$$q = Ae^{-\frac{t}{2Q}}\cos(\omega t + \phi) \quad \omega = \sqrt{\omega_0^2 - \frac{1}{4Q^2}}$$

It is called under damped oscillation.
If $Q < \frac{1}{2\omega_0}$, we have

$$q = Ae^{\lambda_+ t} + BAe^{\lambda_- t}$$

Here,

$$\lambda_{\pm} = -\frac{1}{2Q} \pm \sqrt{\frac{1}{4Q^2} - \omega_0^2}$$

It is called over damped oscillation.

If $Q = \frac{1}{2\omega_0}$, we have

$$q = C(1 + Dt)e^{-\omega_0 t}$$

It is called critical damped oscillation.

3.2 Forced oscillation

The equation of motion for forced oscillation is

$$\ddot{q} + \frac{1}{Q}\dot{q} + \omega_0^2 q = F(t)$$

The form of solution is

$$q = q_s + q_g$$

and q_g is the general solution of the homogeneous equation and q_s is an arbitrary special solution of the equation. In order to get q_s, we consider the following equation

$$\ddot{G} + \frac{1}{Q}\dot{G} + \omega_0^2 G = \delta(t - t')$$

and its solution is $G(t, t')$. Then we have

$$q_s = \int_{-\infty}^{\infty} F(t')G(t - t')dt'$$

For under damped oscillation, we have

$$G(t, t') = \begin{cases} \frac{e^{-\frac{t-t'}{2Q}}}{\omega} \sin\omega(t - t') & t > t' \\ 0 & t < t' \end{cases}$$

And we can verify that

$$q_s = \int_0^{\infty} F(t - t')\frac{e^{-\frac{t'}{2Q}}}{\omega} \sin\omega t' dt'$$

A special case is that

$$F(t) = F_0 \cos\Omega t$$

and we have

$$q(t) = \frac{F_0}{\sqrt{(\Omega^2 - \omega_0^2 + \frac{1}{2Q^2})^2 + \frac{\omega^2}{Q^2}}} \cos(\Omega t + \phi)$$

Here

$$\tan\phi = \frac{\frac{\Omega}{Q}}{\omega_0^2 - \Omega^2}$$

When

$$\Omega = \sqrt{\omega_0^2 - \frac{1}{2Q^2}}$$

we have

$$q_{max} = \frac{QF_0}{\omega}$$

It is called resonance.

3.3 Non-linear oscillation and perturbation theory

Consider the equation of motion

$$\ddot{q} + \omega_0^2 q + \epsilon q^3 = 0$$

Suppose ϵ is very small, then we can expand

$$q = q_0 + \epsilon q_1 + \epsilon^2 q_2 + \cdots$$

and

$$\omega = \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \cdots$$

The equation of motion can be written as

$$\ddot{q} + \omega^2 q = (\omega^2 - \omega_0^2)q - \epsilon q^3$$

Let $\tau \equiv \omega t$ and $q' \equiv \frac{dq}{d\tau} = \frac{\dot{q}}{\omega}$, we have

$$q'' + q = (1 - \frac{\omega_0^2}{\omega^2})q - \frac{\epsilon}{\omega^2}q^3 = 0$$

We can solve the equation above power by power and get

$$q_0'' + q_0 = 0$$

$$q_1'' + q_1 = -\frac{q_0^3}{\omega_0^2} + \frac{2\omega_1}{\omega_0}q_0$$

and so on. When doing the perturbation, we must adjust the ω_i to avoid the resonance solution. The details will be neglect here.

Now let us consider the non-linear oscillation with drive force. The equation of motion is

$$\ddot{q} + \frac{1}{Q}\dot{q} + \omega_0^2 q + \epsilon q^3 = F_0 \cos \omega t$$

It can be rewritten as

$$\ddot{q} + \omega^2 q = -\frac{1}{Q}\dot{q} + (\omega^2 - \omega_0^2)q - \epsilon q^3 + F_0 \cos \omega t$$

We treat the right hand of the equation as perturbation, so we multiply it by a parameter μ and let it be 1 later,

$$\ddot{q} + \omega^2 q = \mu \left(-\frac{1}{Q}\dot{q} + (\omega^2 - \omega_0^2)q - \epsilon q^3 + F_0 \cos \omega t \right)$$

Concerning on the phase lagging effect, we redefine the "time" as

$$\tau \equiv \omega t - \delta$$

so,

$$q'' + q = \mu \left[-\frac{1}{Q\omega}q' + \left(1 - \frac{\omega_0^2}{\omega^2} \right) q - \frac{\epsilon}{\omega^2}q^3 + \frac{F_0}{\omega^2} \cos(\tau + \delta) \right]$$

The expansion series of q and δ are

$$q = q_0 + \mu q_1 + \mu^2 q_2 + \cdots$$

$$\delta = \delta 0 + \mu \delta 1 + \mu^2 \delta 2 + \cdots$$

We can solve the equation above power by power and get

$$q_0'' + q_0 = 0$$

$$q_1'' + q_1 = -\frac{1}{Q\omega}q_0' + \left(1 - \frac{\omega_0^2}{\omega^2} \right) q_0 - \frac{\epsilon}{\omega^2}q_0^3 + \frac{F_0}{\omega^2} \cos(\tau + \delta_0)$$

and so on. The solution of the zeroth order perturbation is

$$q_0 = A_0 \cos \tau$$

Substitute it into the first order equation, we have

$$q_1'' + q_1 = \left(\frac{A_0}{Q\omega} - \frac{F_0}{\omega^2} \sin \delta_0 \right) \sin \tau + \left[\left(1 - \frac{\omega_0^2}{\omega^2} \right) A_0 - \frac{3\epsilon A_0^3}{4\omega^2} + \frac{F_0}{\omega^2} \cos \delta_0 \right] \cos \tau - \frac{\epsilon A_0^3}{4\omega^2} \cos 3\tau$$

To avoid non-physical solution, we have

$$\sin \delta_0 = \frac{A_0 \omega}{F_0 Q}$$

and

$$\left(1 - \frac{\omega_0^2}{\omega^2} \right) A_0 - \frac{3\epsilon A_0^3}{4\omega^2} + \frac{F_0}{\omega^2} \cos \delta_0 = 0$$

At last, we can get

$$A_0 = \frac{F_0}{\sqrt{\left[(\omega^2 - \omega_0^2) - \frac{3}{4}\epsilon A_0^2 \right]^2 + \frac{\omega^2}{Q^2}}}$$

If we define

$$x \equiv \frac{\omega}{\omega_0} \qquad y \equiv \frac{A_0 \omega_0^2}{F_0}$$

the equation above can be written as

$$y^2 = \frac{1}{(x^2 - 1 - ay^2)^2 + bx^2}$$

Here

$$a = \frac{3\epsilon F_0^2}{4\omega_0^6} \quad b = \frac{1}{\omega_0^2 Q^2}$$

We than can solve for x in terms of y.,

$$x^2 = \frac{2 + 2ay - b \pm \sqrt{(b - 2 - 2ay)^2 - 4(a^2 y^2 + 2ay + 1 - \frac{1}{y})}}{2}$$

So the resonance curve has two branches, corresponding to the two roots of the equation above.
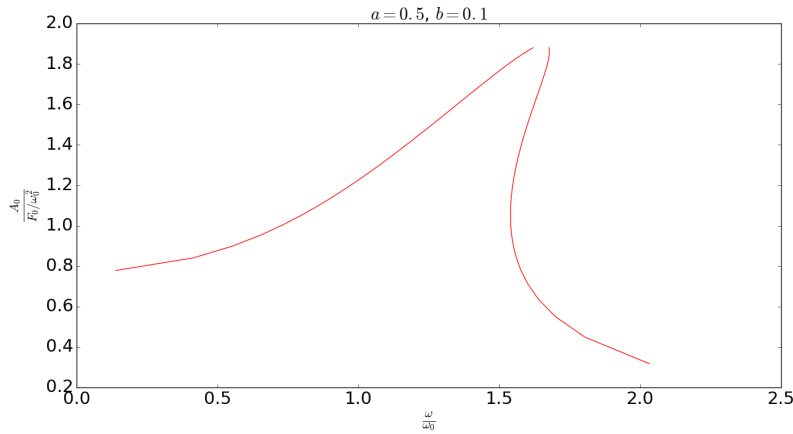

Figure 3.1: Resonance curve

When the frequency of the drive force increase from left, the amplitude of oscillation will become larger and larger. But when it comes to the point of inflection, the amplitude will drop to the low-right part of the curve. When the frequency of the drive force decrease from right, the amplitude of oscillation will also become larger and larger. When it comes to the point of inflection, the amplitude will jump to the hight-left part of the curve. This effect is called hysteresis.

3.4 Oscillations of systems with more than one degree of freedom

Let's look at a system with many degrees of freedom; we have

$$L = \frac{1}{2} \sum_{i,j} T_{ij} \dot{q}_i \dot{q}_j - V(q_1, \dots q_n)$$

Let $q_{0,i}$ be an equilibrium position and expand about this point $q_i = q_{0,i} + \eta_i$, so $\dot{q}_i = \dot{\eta}_i$. We can expand the potential energy to give

$$V(q_1, \dots q_n) = V(q_{0,1}, \dots q_{0,n}) + \sum_i \left(\frac{\partial V}{\partial q_i}\right)_{q_{0,i}} \eta_i + \frac{1}{2} \sum_{i,j} \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_{q_{0,i}} \eta_i \eta_j + \cdots$$

The first term is a constant with respect to η_i and constant terms do not affect the motion. The second term is zero, because $q_{0,i}$ is a point of equilibrium so we are left with

$$L = \frac{1}{2} \sum_{i,j} \left(T_{ij} \dot{\eta}_i \dot{\eta}_j - V_{ij} \eta_i \eta_j \right)$$

where

$$T_{ij} = T_{ij} \left(q_{0,1}, \ldots q_{0,n} \right) \quad V_{ij} = \left(\frac{\partial^2 V}{\partial q_i \partial q_j} \right)_{q_{0,i}}$$

yielding the equations of motion

$$\sum_j \left(T_{ij} \ddot{\eta}_j - V_{ij} \eta_j \right) = 0$$

This is a linear differential equation with constant coefficients. We can try the solution

$$\eta_i = C a_i e^{-i\omega t}$$

so we have

$$\sum_j \left(V_{ij} a_j - \omega^2 T_{ij} a_j \right) = 0$$

This is a matrix equation such that

$$\vec{\vec{A}} \cdot \vec{a} = 0$$

with

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_j \end{bmatrix}$$

and

$$\vec{\vec{A}} = \begin{bmatrix} V_{11} - \omega^2 T_{11} & V_{12} - \omega^2 T_{12} & \cdots \\ V_{21} - \omega^2 T_{21} & V_{22} - \omega^2 T_{22} & \cdots \\ \vdots & & \end{bmatrix}$$

This equation only has a solution is $\det \vec{\vec{A}} = 0$. This gives a nth-degree polynomial to solve for ω^2. We will get n solutions for ω^2 that we can substitute into the matrix equation and solve for a_j.

Chapter 4
Motion of a Rigid Body

4.1 Angular velocity

Suppose there are two coordinate frames. The frame 2 is rotating relatively to frame 1. If the coordinates of the particles in frame 2 is $r_2 = (x_2, y_2, z_2)$. The the coordinates of the particle in frame 1 is

$$r_1 = O(t)r_2$$

and we have

$$O^T O = I$$

So, we can derive that

$$\Omega + \Omega^T = 0 \quad \Omega \equiv O^T \frac{dO}{dt}$$

It is easy to verify that

$$\frac{dr_1}{dt} = \frac{dO}{dt}r_2 + O\frac{dr_2}{dt} = OO^T\frac{dO}{dt}r_2 + Ov_2 = O(\Omega r_2 + v_2)$$

Suppose

$$\Omega = \begin{bmatrix} 0 & -\omega_{2z} & \omega_{2y} \\ \omega_{2z} & 0 & -\omega_{2x} \\ -\omega_{2y} & \omega_{2x} & 0 \end{bmatrix}$$

So, we have

$$v_1 = O(\omega_2 \times r_2 + v_2)$$

Now, we define

$$\omega_1 \equiv O\omega_2$$

We can derive that

$$O\Omega O^T = \begin{bmatrix} 0 & -\omega_{1z} & \omega_{1y} \\ \omega_{1z} & 0 & -\omega_{1x} \\ -\omega_{1y} & \omega_{1x} & 0 \end{bmatrix}$$

So, we have

$$v_1 = \omega_1 \times r_1 + Ov_2$$

ω is the so-called angular velocity.

Note: ω_1 *is independent of the base vector we choose for frame 2.*
If we choose frame 1 differently, ω_1 will transform like an vector.