The Algebranomicon

Patty C. Hill
Jason L. Ermer

Draft: June 23, 2014

Contents

Introduction

> Science is operated according to the judicial system. A theory is assumed to be true if there is enough evidence to prove it "beyond all reasonable doubt". On the other hand, mathematics does not rely on evidence from fallible experimentation, but it is built on infallible logic.
>
> Simon Singh, *Fermat's Last Theorem*

Welcome to your first real mathematics course!

The mathematics courses you have taken before now have focused on arithmetic, which is a branch of mathematics that, for the most part, involves combining numbers by addition, subtraction, multiplication, and division. You have worked hard on developing number sense and perfecting your computational skills with different types of numbers.

In algebra, on the other hand, we will focus our study on mathematical relationships. We will develop the skills needed to describe these relationships abstractly using variables, manipulate them through the use of fundamental laws and properties, and use the patterns and features of these relationships to solve problems.

Learning mathematics is much like learning a language. The best way to learn is to be immersed in the subject and to practice the skills you pick up along the way. We have designed this course with that in mind.

Our approach is based on the idea of a "function". Early on, we will discuss what a mathematical function is, and learn different ways to represent functions. As we proceed through the year, we will study specific types, or "families", of functions in depth, and learn the various rules, properties, and skills that relate to that family of functions.

We split each of the main units into two components. One component we might call "mathematical grammar". Here, we will learn the vocabulary, notation, rules, and properties associated with the big idea of the unit. We will develop sets of tools that will allow us to manipulate algebraic expressions and solve equations. In the second component of each unit we will explore the big ideas of the unit and use the skills we developed earlier to solve problems.

0.0.1 Other ideas...?

The intro is how we describe how our book is different.

I say mention them in this section describing that we have a, this is probably not the right word but, holistic approach to creating a student of mathematics.

Our appendices can be useful. Most books have appendices that are related to test prep, but ours could be more like guides for students and teachers, inserts for notebooks, etc. Is there a way to put links in that portion of the intro to those specific appendices?

Color guide and icon guide for the text boxes and activities...

To include here... or maybe some of these go in an appendix?

1. How to be a student of mathematics (tips, responsibilities)?
2. Expectations (like collaborating in groups)?
3. How to use the resources found here most effectively?
4. How to take notes?
5. Organization?

A discussion of how problem solving plays a role. "My dear, all life is a series of problems which we must try and solve: first one, then the next, then the next... until at last we die. Why don't you get us an ice cream." – The Dowager Countess of Grantham

0.0.2 Link to some unsolved problems

Might be fun to mention some of the relevant unsolved problems in mathematics?

http://mathworld.wolfram.com/UnsolvedProblems.html

http://en.wikipedia.org/wiki/List_of_unsolved_problems_in_mathematics

Chapter 1

Numbers

> *Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.*
>
> God made the integers, all the rest is the work of man.
>
> – Leopold Kronecker, German mathematician

The number system that we use every day, both in mathematics class and in our daily lives, developed over many generations. Men and women from all over the world, both famous and anonymous, have helped to make mathematics what it is today. Yet, people argue over whether mathematical ideas are "invented" or "discovered".

For example: **imaginary numbers** (which we will discuss briefly in this course, and study in more detail in the next course), first appeared on the mathematical scene in the 1500s. Italian mathematician Gerolamo Cardano first wrote about these new numbers in his work trying to solve certain types of problems that otherwise would have been impossible to solve. Since he was the first person to describe this new kind of number, we might say that Cardano *invented* imaginary numbers.

But, we now know that imaginary numbers have practical applications in, for example, electrical engineering. The laws of electromagnetism haven't changed since the 1500s. (Well, our understanding of the laws has changed, but the physics has not.) So, maybe imaginary numbers have been there all along, lurking within the fabric of the universe. In this case, Cardano *discovered* imaginary numbers.

It's not clear which is the more accurate description. No matter what side of the debate you find more convincing, there is a certain beautiful interconnectedness to our system of numbers and mathematical laws. To illustrate this, we'd like to tell a story.

1.1 The Story of Numbers

Once upon a time, there was a simple farmer. Knut Krumbli lived in rural Sweden, raising goats and making goat cheese. He and his family led an uncomplicated life and they didn't have much need for mathematics. In fact, they really only needed numbers to count their goats: 1 goat, 2 goats, 3 goats, 4 goats...

But one day, after a terrible storm, Knut went to the field to count the goats and discovered, much to his dismay, that there were no goats to count. He hadn't needed a number to describe this situation before, but now people were asking him hard questions, like "How many of your goats made it through that crazy storm?" (But, you know, in Swedish.)

Knut and his family couldn't very well survive without any goats, so he went to his neighbor for help. The neighbor agreed to loan Knut some goats to restart his herd but, of course, Knut would have to repay his goat-debt later. The village hadn't needed to do much accounting before the storm, but now they needed a system of numbers that could keep track of debts and credits.

Over time, Knut's family got back on their feet and thrived. They paid back their debts and eventually grew to raise more goats (and to make more goat cheese) than they could eat. They began to trade with their neighbors for other foods or services. Of course, everything had relative value: two wheels of goat cheese were worth three bales of hay. So, the village developed a system of numbers for describing exchange rates of this kind.

As the village grew, Knut's family farm led the development of a booming goat cheese industry. They invested their profits into bank accounts that paid interest. In certain situations, everyone was surprised to discover, interest-bearing accounts led to a new system of numbers that no one had seen before.

Eventually, some of Knut's ancestors emigrated to America and, years later, a pair of twins – Knut's great-great-great-grandchildren – would grow up to change the world. But let's not get too far ahead of ourselves. More about the Krumbli twins later...

1.1.1 Dissecting the Story

Mathematically speaking, it's natural to begin our discussion of numbers exactly where Knut began: counting things. The numbers we use to count are called the natural numbers (also known as the counting numbers, for obvious reasons).

Natural Number

A **natural number** is a member of the list of numbers that starts $1, 2, 3, 4, \ldots$ and continues forever. The set of all natural numbers is denoted using the symbol $\mathbb{N}$, so we can write $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$.

The natural numbers have some interesting properties. If we add two natural numbers, their sum will always be a natural number. The same goes for multiplication: the product of two natural numbers is again a natural number.

Mathematically speaking, we call this **closure**. We say that the natural numbers are closed under the operation of addition. Also, the natural numbers are closed under the operation of multiplication.

Notice that we didn't include 0 among the natural numbers. Zero is a bit tricky because it seems like a counting number. For example, "zero" is (probably) the answer to the counting question, "How many live elephants are there in the room with you right now?" But if there are no elephants to count, can we really count them? That's a philosophical question.[1]

Practically speaking, we usually exclude 0 from the set of natural numbers. We will always be very clear when we come to situation where we want to consider 0 to be a natural number.

The natural numbers are closed under the operations of addition and multiplication, but they are *not* closed under the operation of subtraction. Sometimes, the difference of two natural numbers is a natural number: for example $8 - 6 = 2$, no problem. But some subtraction sentences don't work: for example $10 - 13 = {}^{-}3$, and ${}^{-}3$ is not a natural number.

Integer

An **integer** is a natural number, or the opposite of a natural number, or zero. The set of all integers is denoted $\mathbb{Z}$, so we sometimes write $\mathbb{Z} = \{\dots, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, \dots\}$. We could also write $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$.

The symbol $\mathbb{Z}$ comes from *Zahl*, the German word for number.

Note that every natural number is an integer. We say that the set of natural numbers is a **subset** of the set of integers.[2]

The integers are closed under the operations of addition and multiplication. Plus – bonus! – the integers are closed under the operation of subtraction. Whenever we subtract one integer from another, we always get another integer as the result.

But (as you may have anticipated) we have a problem with division. In certain cases, the quotient of two integers is itself an integer: $15 \div {}^{-}3 = {}^{-}5$, no problem. But other times we get a quotient that is not an integer:

[1] Philosophy and mathematics have historically gone hand-in-hand. Many important discoveries (inventions?) in mathematics are attributed to people who are considered both "philosopher" and "mathematician". For example: French mathematician René Descartes, for whom the Cartesian coordinate system is named, is also the philosopher who said "I think therefore I am".

[2] For those who are into mathematical symbols and notation, the sentence "The natural numbers are a subset of the integers" is denoted $\mathbb{N} \subseteq \mathbb{Z}$.

$^-3 \div 15 = {}^-0.2$ and $^-0.2$ is not an integer. In other words, the integers are not closed under the operation of division.

Rational Number

A **rational number** is any number that can be written as the ratio of two integers $\frac{a}{b}$, where b is not zero. This set includes all of your classic fractions, as well as all terminating decimals and all repeating decimals.

The set of all rational numbers is denoted by the symbol $\mathbb{Q}$, which comes from the word *quotient*.

Fractions have a lousy reputation among math students[3], but the rational numbers are great because they are closed under all four of the basic operations. When we add, subtract, multiply, or divide any two rational numbers, the result will always be another rational number. What more could we ask for?

Note that the rational numbers include all of the terminating decimals (like 0.5 and 1.678), and all of the repeating decimals (like $0.\overline{3}$ and $^-12.34\overline{56}$). But, consider the number

$$0.10\,110\,1110\,11110\,111110\ldots$$

This number does not terminate, but it does not repeat either (can you explain why not?). So, this number is *not* a rational number.

Irrational Number

An **irrational number** is a number that cannot be expressed as the ratio of two integers. In decimal form, an irrational number never terminates and never repeats.

Sadly, the set of all irrational numbers doesn't have a standard notation or fancy letter to denote it.

You have likely encountered irrational numbers before. A famous example is the number π, which shows up when we study circles. We usually approximate π to be about 3.14, but in fact, the decimal representation of π goes on forever without stopping or repeating:

$$\pi \approx 3.1415926535\ 8979323846\ 2643383279\ 502884197\ 6939937510\ 5820974944\ 5923078164\ldots$$

When we group together all of the rational numbers and all of the irrational numbers, we will have accounted for all possible decimal representations. This combined set of numbers is going to be of key importance to us in algebra 1.

[3] Fractions, the F-word of mathematics?

Real Number

A **real number** is any rational or irrational number. The set of all real numbers is denoted $\mathbb{R}$. The symbol $\mathbb{R}$, naturally enough, comes from the word *real*.

Like the set $\mathbb{Q}$, the set $\mathbb{R}$ is closed under the four fundamental operations. $\mathbb{R}$ is the number system we will use in algebra 1. Other types and sets of numbers exist (like $\mathbb{C}$, the set of so-called "complex numbers"), but we won't get into them very much until algebra 2 and beyond.

1.2 Integers

You may have been working with positive and negative numbers for a while now, so this section will review the most important terms and algorithms for working with signed numbers. To get the ball rolling, have a think about this:

Warm-up Problem (TODO: Better name for these?)

In each of the expressions below, x and y are natural numbers and $x < y$ (x is less than y). Will the result be greater than 0, less than 0, equal to 0, or is there not enough information to tell? Why?

a. $x + (-y)$ b. $x - (-y)$ c. $x \cdot (-y)$ d. $x \div (-y)$

1.2.1 Language of the signed numbers

As we saw in section 1.1, the integers include the natural numbers and their opposites (and also zero). Numbers now include two pieces of information: they have a "size" and a "direction", either positive or negative.[4] Sometimes, we care only about the magnitude of a number, in which case we refer to:

Absolute Value

The **absolute value** of a number x is its distance away from zero on the number line. To express this in mathematical symbols, we write $|x|$ to mean "the absolute value of x".

For those who like mnemonic devices and memory aids, it may be helpful to think of the absolute value bars as a little numerical shower stall or car wash. A number goes in and all its negativity gets washed away.

[4] Later in mathematics and the physical sciences, we'll encounter mathematical objects with both magnitude and direction again. They're called vectors.

Example 1

Compute each of the following.

1. $|4|$ — The absolute value of a positive number is just the original number, so $|4| = 4$.

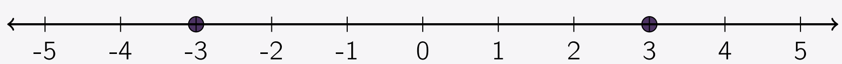
2. $|-8|$ — Absolute value ignores the sign and tells us how far a number is from zero. -8 is eight units away from zero, so $|-8| = 8$.

3. $-|-6|$ — The absolute value bars only apply to what's inside, but then this negative sign *outside* the absolute value bars will make the final answer negative again! So, $-|-6| = -6$.

Every nonzero number has a counterpart that is the same distance away from zero, but on the opposite side of the number line. This is a simple idea, but one that is important enough for us to give it a name.

Opposite

The **opposite** of a number x is the number $-x$. In other words, the opposite of a number is the number with the *same absolute value*, but the *opposite sign*. For example, 3 and -3 are opposites.



Note that zero is its own opposite.

Note also that the sum of opposites is always 0. For this reason, we sometimes use the term **additive inverse** to describe the opposite of a number.

The set of math skills that you have already acquired should include techniques for adding, subtracting, multiplying, and dividing positive and negative numbers quickly and accurately. In the next few sections we will review and summarize the algorithms.

1.2.2 Adding Signed Numbers

Like matter and antimatter, combining positive and negative numbers leads to annihilation. (Dramatic, no?)

For example when we bring together $+8$ and -6, we can picture 8 units of "matter" and 6 units of "antimatter". Particles and antiparticles annihilate one another (both disappearing in the process). Since we have more matter than antimatter in this case, all of the antimatter is consumed, leaving behind 2 units of matter.

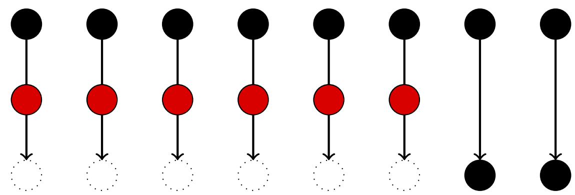


Figure 1.1: Eight units of matter versus six units of antimatter!

We might visualize annihilation with a drawing like in Figure 1.1 (black circles are units of matter, red circles are units of antimatter). Or, we could write a number sentence like $8 + {}^{-}6 = 2$. Of course, no annihilations occur when we scrape together a big pile of matter (or a big pile of antimatter, for that matter). It's only when they mix that anything interesting happens.

Adding Signed Numbers

If the two numbers being added have the same sign, then we add the absolute values of the numbers, and use the sign that they share.

Otherwise, if the two numbers being added have different signs, then we find the different between their absolute values, and use the sign of the number with larger absolute value.

Example 2

Compute each of the following.

1. $8 + 12$ — We add as usual, like we've been doing since elementary school: $8 + 12 = 20$
2. ${}^{-}8 + {}^{-}12$ — Both numbers are negative. That's a big pile o' antimatter: ${}^{-}8 + {}^{-}12 = {}^{-}20$
3. ${}^{-}8 + 12$ — The numbers have different signs, so prepare for annihilation! $|12|$ is larger than $|{}^{-}8|$, so we will have matter left over. Therefore, ${}^{-}8 + 12 = 4$
4. $8 + {}^{-}12$ — This time, we have more antimatter than matter: $8 + {}^{-}12 = {}^{-}4$

1.2.3 Subtracting Signed Numbers

Addition and subtraction are called opposite (or inverse) operations because they "undo" one another. The act of adding 5 to a number can be "undone" by subtracting 5. On the other hand, the act of adding 5 units of matter can be undone by *adding 5 units of antimatter*. In other words:

Subtracting Signed Numbers

Subtracting a number is the same as *adding the opposite of the number*.

When faced with a subtraction problem, change it to an "addition of the opposite" problem and then follow the rules for adding signed numbers.

The benefit of this approach is that we can avoid having to learn a whole new set of rules for subtracting signed numbers. All we need are the rules for addition, plus one new rule about how to change subtraction problems into addition problems!

Example 3

Compute each of the following:

1. $8 - 12$ "Subtracting 12" is the same as "adding negative 12", so $8 - 12 = 8 + {}^{-}12$. Then, we can apply the rules for adding signed numbers: $8 + {}^{-}12 = {}^{-}4$
2. ${}^{-}3 - 14$ This is the same as ${}^{-}3 + {}^{-}14$, and we follow the rules for adding numbers with the same sign: ${}^{-}3 + {}^{-}14 = {}^{-}17$
3. $4 - {}^{-}9$ This is the same as $4 + 9$, which is an easy addition: $4 + 9 = 13$
4. ${}^{-}6 - {}^{-}5$ This is the same as ${}^{-}6 + 5 = {}^{-}1$

When dealing with addition and subtraction of signed numbers in a problem, a good habit is to simplify the signs by rewriting subtraction as addition-of-the-opposite, before doing any computations.

Explaining the Warm-up Problem

In the **warm-up problem** that started this section, we have two natural numbers x and y where $x < y$. Natural numbers are all positive, so this means that $|x|$ is less than $|y|$.

Part a. asks us to consider $x + (-y)$. Since opposite numbers have the same absolute value, $|y|$ is the same as $|-y|$. Therefore when we add, the number with larger absolute value is negative and the sum will take on the negative sign. So, when x and y are natural numbers and $x < y$, we know that $x + (-y)$

is *always negative* so the result is always less than 0.

Part b. asks us to consider $x - (-y)$. We can change this subtraction expression into addition-of-the-opposite: $x - (-y) = x + y$. Since both x and y are natural numbers, their sum is a natural number. In other words, when x and y are natural numbers, $x - (-y)$ is always positive, in other words always greater than 0.

1.2.4 Multiplying and Dividing Signed Numbers

Like addition and subtraction, the operations of multiplication and division are inverse operations. We'll discuss multiplication below, though the rules about the signs of products also apply to the signs of quotients.

Recall, from your elementary school days, that one way to think about whole number multiplication is as *repeated addition*. We interpret $a \cdot b$ as "a groups with b items in each group". So, $3 \cdot 5$ means "3 groups of 5", which we can write as an addition sentence: $3 \cdot 5 = 5 + 5 + 5$.

Using this interpretation, we can easily explain the product of a positive number and a negative number. The expression $4 \cdot {}^-8$ means "four groups of negative eight": $4 \cdot {}^-8 = {}^-8 + {}^-8 + {}^-8 + {}^-8 = {}^-32$. No problem!

But what about ${}^-5 \cdot 6$? What does it mean to have "negative five groups of six"? Or even worse, what about ${}^-3 \cdot {}^-7$, "negative three groups of negative seven"? Rather than try to twist the metaphor to fit these new situations, let's just admit that multiplication can *not* always be represented by repeated addition.[5]

For the moment, we'll simply review the rules for multiplying (and dividing) signed numbers. We can explain why these rules work using the so-called "field axioms for the real numbers". More on all of that in a few chapters.

Multiplying (and Dividing) Signed Numbers

The absolute value of the product of two numbers is the product of their absolute values: $|\,a{\cdot}b\,| = |\,a\,|{\cdot}|\,b\,|$. If the two numbers have the same sign, then the product is positive. If the two numbers have opposite signs, then the product is negative.

There are several clever ways to remember this rule. Some people remember that every pair of negatives in a product cancel one another. Other people use a triangle with one positive sign and two negative signs drawn on the vertices. We present another way of looking at it in the next section. Choose whichever mnemonic[6] method is most helpful to you!

---

[5] The "multiplication as repeated addition" analogy breaks down when we have a negative number of groups, but also for rational and irrational numbers. What's the repeated addition problem for $\frac{2}{3} \cdot \frac{1}{2}$, or for $\sqrt{2} \cdot \sqrt{3}$?

[6] mnemonic ($na \cdot MON \cdot ic$, the first "m" is silent): A learning aid that helps to remember or retain information.

1.2.5 Karmic Multiplication

Karma, an underlying concept of many Eastern religions, is a belief that a person's actions and intentions shape their future. Performing good deeds will contribute to one's "good karma" and will lead to future happiness. Bad deeds contribute to one's "bad karma" and will lead to future suffering.

So, karma suggests that good things happen to good people, and that bad things happen to bad people. Of course we know that the universe does not always operate in accordance with karma.

Karmic Multiplication

When good things happen to good people, that's good!

When bad things happen to good people, that's bad!

When good things happen to bad people, that's bad!

When bad things happen to bad people, that's good!

For example: Mahatma Gandhi used nonviolent means to inspire civil rights movement around the world. Gandhi was a good person. On the other hand, Adolf Hitler was chancellor of Nazi Germany during World War II and orchestrated appalling crimes against humanity. Hitler was a bad person.[7]

In terms of life events, winning the lottery is a good thing. Getting hit by a truck is a bad thing.

If Gandhi had won the lottery, that would have been in accordance with all his good karma. That's good! If Gandhi had been hit by a truck, that would have been in opposition to all of his good karma. That's bad!

If Hitler had won the lottery, that would have been in opposition to his evil karma. That's bad! If Hitler had been hit by a truck, it would have served him right! Go karma! Good news!

Of course, in this metaphor good things and good people represent positive numbers. Bad things and bad people represent negative numbers. When karma is operating as it should, we get a positive result. When the laws of karma are broken, we get a negative result.

[7] understatement (*UN · der · state · ment*): The act of representing something in a weak or restrained way, to a lesser degree than is borne out by the facts.

Example 4

Compute each of the following:

1. $8 \cdot {-12}$ — We multiply the absolute values and, since the two factors have different signs, we know the answer is negative: $8 \cdot {-12} = {-96}$

2. ${-72} \div {-3}$ — We divide absolute values and, since the two factors have the same sign (both negative in this case, so that's "Hitler gets his by a truck"), the answer is positive: ${-72} \div {-3} = 24$

Note: When multiplying (or dividing) more than two numbers, we can approach things in two different ways. We might simplify the product two factors at a time, and keep track of the sign as we go. Or, we could treat the signs as a separate problem: first multiply all of the absolute values, then go back and count up the negative signs.

Example 5

Multiply: $(2)(-2)(1)(-2)(-2)(1)(1)(-2)(-1)(2)$

Solution: If we count up the negative signs, we find there are five. Pairs of negatives will have a positive product, so we'll have two pairs of negatives plus one left over. Our final answer, then, will be negative. All that remains is to multiply the 2s (of which there are six):

$$(2)(-2)(1)(-2)(-2)(1)(1)(-2)(-1)(2) = -64$$

An alternative solution would be to multiply from left to right and accumulate the product as we go along.

Explaining the Warm-up Problem

In the **warm-up problem** for this section, x and y are natural numbers and $x < y$.

Since x are y are natural numbers they are both positive, and then $-y$ is negative. Part c. asks us to consider $x \cdot (-y)$. This is have a positive number times a negative number, so the product is *always negative*, always less than 0.

Part d. asks us to consider $x \div (-y)$. Again, we have a positive number and a negative number, so the quotient is *always* less than 0.

1.3 Rational Numbers

As with integers, you've probably been working with fractions and decimals for a number of years. In this section, we review the key terms and algorithms for working with rational numbers. As we get going, think about this:

Warm-up Problem (TODO: Better name for these?)

In each of the expressions below, a and b are rational numbers where $0 < a < b < 1$. Will the result be greater than 1, less than 1, equal to 1, or is there not enough information to tell? Why?

a. $a + b$ b. $a - b$ c. $a \cdot b$ d. $a \div b$

1.3.1 The Language of Rational Numbers

Sometimes in life we discover, much to our surprise, that some ridiculous and insignificant thing as been given a name.[8] We find ourselves wondering, "Who decided to give *that* a name? Why bother?" Prepare for one of those moments:

Vinculum

A **vinculum** (plural: vincula) is a horizontal bar used in mathematics to show grouping. For example, the fraction bar in the middle of $\frac{5}{2}$ is a vinculum.

Note: Vincula are used in other contexts as well. For example, we use a vinculum to represent a repeating decimal such as $0.\overline{3}$.

With that definition in mind, we can continue with two of the most daunting words in elementary mathematics. You're definitely not alone if you have ever been confused about these.

[8] For instance, did you know that the little plastic sheath at the end of your shoelaces is called an "aglet"? Now you know.

Numerator and Denominator

In a fraction, the number above the vinculum is called the **numerator** of the fraction, and the number below the vinculum is called the **denominator** of the fraction.

For example in the number $\frac{3}{5}$, the numerator is 3 and the denominator is 5.

If we ask "*how many* fifths?", the numerator tells us "*three* fifths". The word numerator is related to the word "number", and the numerator counts the pieces.

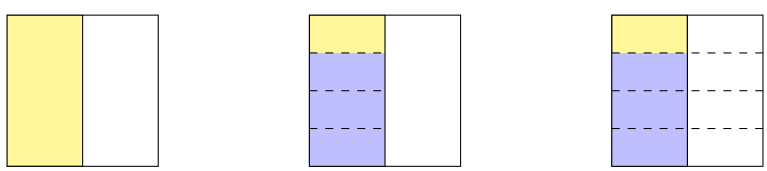
If we ask "three *whats*?", the denominator tells us "three *fifths*". The word denominator is related to the word "nominate" (as in "to nominate someone for president") which means "to name". The denominator *names* the fraction.

1.3.2 Multiplying Rational Numbers

Don't worry, your version of the Algebranomicon isn't missing any sections. Most textbooks would discuss adding and subtracting rational numbers, but we're going to start by studying the most helpful of the rational number operations: multiplication.

Suppose that at the cheese market, Knut Krumbli is selling chunks from a 10-bound block of cave-aged goat cheese.[9] Half of the original block of cheese is left, and a local weaver asks to buy three fourths of it. The original block had a value of 800 Swedish kronor. How much should Knut charge the weaver?

Let's draw a picture. In the images below, the square represents the original block of cheese. We divide the square in half vertically, and the region shaded yellow represents how much of the cheese remains. We can then divide the cheese into fourths horizontally and shade in three fourths (the amount that the weaver wants to buy) in blue.



What fraction of the whole block does this blue region represent? If we extend the lines, we can see that we've divided up the cheese into 8 equally-sized pieces. So the weaver is buying $\frac{3}{8}$ of the whole block and Knut should charge 300 kronor.

[9] When cheese is aged in the cool, humid air of underground caves, it can develop a denser texture and a more complex flavor, since small salt crystals form throughout its interior.

In the end, we solved a fraction multiplication problem: "How much is three fourths of one half of a whole block of cheese?"

$$\frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$$

Multiplying Rational Numbers

To multiply rational numbers, we multiply numerators to find the numerator of the product. We multiply denominators to find the denominator of the product.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Explaining the Warm-up Problem

In the **warm-up problem** for this section, $0 < a < b < 1$, and part (c) asks us to consider $a \cdot b$. Consider this from a block-of-cheese perspective. We start with less than a whole block of cheese (since b is less than 1) and we only want to buy a fraction of what's there (since a is less than 1).

So, we are certainly buying less than a whole block of cheese! Since both a and b are less than 1, we know that their product must be less than one.

[TODO] Extend box method to mult of mixed numbers.

Fancy Versions of One

Recall that any number times 1 is itself. This simple fact, along with the fraction multiplication procedure, gives us an extremely powerful tool that we'll use in various different ways throughout algebra. The key idea is that multiplying something by 1 doesn't change its value, even if we use a "fancy version of 1". Consider, for example:

$$\begin{aligned} \frac{5}{8} &= \frac{5}{8} \cdot 1 && \text{Multiplication by 1 doesn't change the number.} \\ &= \frac{5}{8} \cdot \frac{7}{7} && \text{We can rewrite 1 however we like, here } 1 = \frac{7}{7}. \\ &= \frac{35}{56} && \text{Multiply fractions. The representation has changed, but the value is the same!} \end{aligned}$$

The "fancy one" we chose here is $\frac{7}{7}$, but any other version of 1 would work the same way: $\frac{-140}{-140}$, $\frac{2\pi}{2\pi}$, $\frac{\sqrt{3}}{\sqrt{3}}$... The possibilities are endless.

The first application of the "fancy one" has to do with one of the themes we encounter throughout algebra: the idea of finding a "completely simplified" solution to a problem. Fractions introduce us to the first criteria for something being simplified.

Simplified Rational Numbers #1

Fractions should be simplified to **lowest terms**, meaning that the numerator and denominator of the fraction are **relatively prime** integers.

Two integers are said to be relatively prime (or coprime) if they have no common factors other than 1. So, the fraction $\frac{21}{34}$ is in lowest terms, since 21 the factors of 21 are $\{1, 3, 7, 21\}$ and the factors of 34 are $\{1, 2, 17, 34\}$. They have no factors in common, other than 1.

On the other hand, $\frac{18}{84}$ is not in lowest terms. Both 18 and 84 are even, and so both the numerator and denominator are divisible by at least 2. To write this fraction in lowest terms, we "undo" fraction multiplication and search for some fancy ones that we can eliminate:

$$\frac{18}{84} = \frac{2 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 3 \cdot 7} = \frac{2}{2} \cdot \frac{3}{3} \cdot \frac{3}{2 \cdot 7} = 1 \cdot 1 \cdot \frac{3}{14} = \frac{3}{14}$$

Once we know how and why this works, we can take a shortcut and "cancel" common factors from the numerator and denominator:

$$\frac{18}{84} = \frac{2 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 3 \cdot 7} = \frac{\cancel{2} \cdot \cancel{3} \cdot 3}{\cancel{2} \cdot 2 \cdot \cancel{3} \cdot 7} = \frac{3}{2 \cdot 7}$$

Simplify Before You Multiply

We can also use the "fancy one" to save ourselves some work! Consider having to multiply:

$$\frac{15}{8} \cdot \frac{4}{7} \cdot \frac{14}{3} \cdot \frac{1}{5}$$

The first helpful step is to factor the individual numerators and denominators to expose all of the factors that are going to be in the product.

$$\frac{3 \cdot 5}{2 \cdot 2 \cdot 2} \cdot \frac{2 \cdot 2}{7} \cdot \frac{2 \cdot 7}{3} \cdot \frac{1}{5}$$

Then, a common factor in the numerator and denominator is like multiplication by 1, so common factors can be crossed out. Look at what happens in this case!

$$\frac{\cancel{3} \cdot \cancel{5}}{\cancel{2} \cdot \cancel{2} \cdot \cancel{2}} \cdot \frac{\cancel{2} \cdot \cancel{2}}{\cancel{7}} \cdot \frac{\cancel{2} \cdot \cancel{7}}{\cancel{3}} \cdot \frac{1}{\cancel{5}} = 1$$

The alternative would have been to multiply all of those numbers together by hand and notice that they are the same[10], but only after bunch of work. It pays to be clever!

[10] Spoiler alert: it's $\frac{840}{840}$.

Example 6

Two worked examples of fraction multiplication.

1.3.3 Adding and Subtracting Rational Numbers

Knut Krumbli would tell you that brining together a herd of 8 goats and a herd of 11 goats results in a herd of 19 goats. It's goat herding, not rocket science.

But, combining a herd of 4 goats and a flock of 13 sheep doesn't really give us 17 of anything until we can find some shared characteristic that is common to both groups. For instance, we could say we have a group of "17 mammals", or "17 quadrupeds".

So it goes with fractions. When our fractions have a shared name (a common denominator) we can total up how many things we have with that name: 8 thirds and 11 thirds makes 19 thirds. In math symbols, we write

$$\frac{8}{3} + \frac{11}{3} = \frac{19}{3}$$

When our quantities *don't* share a common unit (a common denominator), we have to find one before we can add in a meaningful way.

Given a fraction, we can use multiplication by a "fancy one" to generate a new fraction that has the same value, but a different, perhaps more helpful, denominator.

Some people like to try and find the *least common denominator*, but that's not strictly necessary. Any common denominator will do. In fact, a guaranteed common denominator of any two fractions is the *product of their denominators*.

Adding and Subtracting Rational Numbers

To add rational numbers, we must have a common denominator, for example the product of the original denominators. Then we add the numerators, and keep the common denominator.

$$\frac{a}{b} + \frac{c}{d} = \left(\frac{a}{b} \cdot 1\right) + \left(1 \cdot \frac{c}{d}\right) = \left(\frac{a}{b} \cdot \frac{d}{d}\right) + \left(\frac{b}{b} \cdot \frac{c}{d}\right) = \frac{a \cdot d}{b \cdot d} + \frac{b \cdot c}{b \cdot d} = \frac{a \cdot d + b \cdot c}{b \cdot d}$$

To subtract rational numbers, change the subtraction problem to an "addition of the opposite" problem and then follow the algorithm for addition.

Negative Fractions

Where should we put the negative sign when we have a negative fraction? Does it matter? Consider the following three possibilities. Are they all equivalent?

$$-\frac{3}{4} = \frac{-3}{4} = \frac{3}{-4}$$

A fraction a way of writing a division problem. If four of the Krumbli kids share three bowls of lingonberries equally, then each kid will get $3 \div 4 = \frac{3}{4}$ of a bowl of berries.[11] The fraction $\frac{3}{4}$ is just another way of writing $3 \div 4$.

So, all three of the fractions above have the same value. In the first example, the whole fraction has been negated. In the second and third examples, the numbers have opposite signs and so the quotient will be negative. In other words it actually doesn't matter where we put the negative sign. We can put it where it is most convenient for the problem (very often, that's in the numerator of the fraction).

Mixed Numbers

Improper fractions have a numerator that is greater than or equal (in absolute value) to their denominator, like $\frac{5}{3}$ or $\frac{-84}{16}$. Improper fractions have been scorned by many elementary school mathematics teachers, who instead prefer mixed numbers: $1\frac{2}{3}$ or $-5\frac{1}{4}$. But, improper fractions are often much easier to work with than mixed numbers.[12] So:

Simplified Rational Numbers #2

Simplified **improper fractions** are preferred over **mixed numbers** and decimals. Only convert to a mixed number or decimal when the context (or the directions) require it.

We usually prefer exact fraction answers over decimal approximations. Writing the decimal $\frac{10}{7}$, for instance, is much preferred over 1.43, and better even than the exact answer $1.\overline{428571}$ (yep, that's a big chuck o' repeating decimal).

But, be sure to read questions carefully! There are exceptions to these rules. When working in a real-world context, a certain number format may make more sense. For example, when solving a problem about money, the answer \$3.50 makes a lot more sense than $\$\frac{7}{2}$. Likewise, the answer "$1\frac{1}{3}$ pounds of cheese" is better than "$\frac{4}{3}$ pounds of cheese". In ambiguous cases, we will make it clear what number format is preferred.

[11] Lingonberries are a popular fruit in Scandinavia and throughout northern, central, and eastern Europe. The berries are quite tart, and so they are usually mixed with sugar and preserved as jam or compote. In Sweden and Norway, reindeer is traditionally served with gravy and lingonberry sauce. Yes, Scandinavians eat reindeer.

[12] One situation where improper fractions are superior is when describing the slope of a line, as we will see in a few chapters.

When faced with mixed numbers in a problem, we have to be careful. When adding, we can convert all mixed numbers to improper fractions, or work with them "as is". Subtracting with mixed numbers is tricky, however, because we we may have to handle regrouping. Multiplication is even trickier.

Since we prefer improper fractions as final answers anyway, we recommend converting all mixed numbers to improper fractions before you start computations. To convert a mixed number to an improper fraction, all we have to do is think about the mixed number as an addition problem, and then use our trusty "fancy one" trick:

$$3\frac{5}{8} = 3 + \frac{5}{8} = \frac{3}{1} + \frac{5}{8} = \frac{3 \cdot 8 + 1 \cdot 5}{1 \cdot 8} = \frac{24 + 5}{8} = \frac{29}{8}$$

In addition to using the "fancy one" trick, we also used the fact that a whole number has a "phantom one" in its denominator: $3 = \frac{3}{1}$. We don't usually write it, but it's there when we need it.

Example 7

Compute each of the following:

1. $\frac{3}{4} + \frac{5}{6}$

These fractions do not have a common denominator, so we'll have to find one. We could use their least common denominator (which is 12) or use the product of the denominators (which is 24). Let's use 24:

$$\frac{3}{4} + \frac{5}{6} = \frac{3 \cdot 6 + 4 \cdot 5}{4 \cdot 6} = \frac{18 + 20}{24} = \frac{38}{24} = \frac{19}{12}$$

2. $\frac{2}{5} - \frac{7}{8}$

First, we'll change the subtraction to addition-of-the-opposite. We'll put the negative sign in the numerator of the fraction, and then add. At the end, we can adjust the negative sign again:

$$\frac{2}{5} - \frac{7}{8} = \frac{2}{5} + \frac{-7}{8} = \frac{2 \cdot 8 + 5 \cdot -7}{5 \cdot 8} = \frac{16 + -35}{40} = \frac{-19}{40} = -\frac{19}{40}$$

Explaining the Warm-up Problem

In the **warm-up problem** for this section, $0 < a < b < 1$.

Part (a) asks about $a + b$. Since both numbers are positive, their sum is positive, but we don't have enough information to tell whether the sum is greater than 1. If a and b are both less than $\frac{1}{2}$, for example, then their sum will be less than 1. On the other hand, if they are both greater than $\frac{1}{2}$, then their sum will be greater than 1.

Part (b) asks us to consider $a - b$. Since a is less than b, we're subtracting a larger number from a smaller number, and so the answer must be negative.

We can reason this out in another way: $a - b$ is the same as $a + (^-b)$. The absolute value of b is the same as the absolute value of ^-b. And so this sum will be negative because the negative number is the one with the greater absolute value.

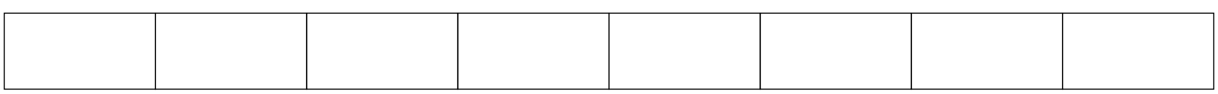
In any case, $a - b$ is negative and so we know for sure that it is less than 1.

1.3.4 Dividing Rational Numbers

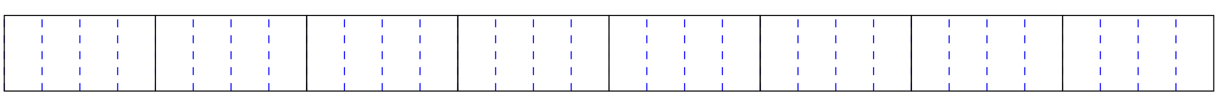
Fraction division may be the most poorly understood operation in all of arithmetic. The algorithm for dividing fractions seems arbitrary, and it's often difficult to judge whether our answers make sense. Let's pause for a moment to think about what fraction division means.

Suppose Jorunn Krumbli, Knut's wife, is making scarves for the goats (Scandinavian winters are chilly). She has 8 meters of burlap, and each scarf requires $\frac{3}{4}$ of a meter. How many scarves can she make?

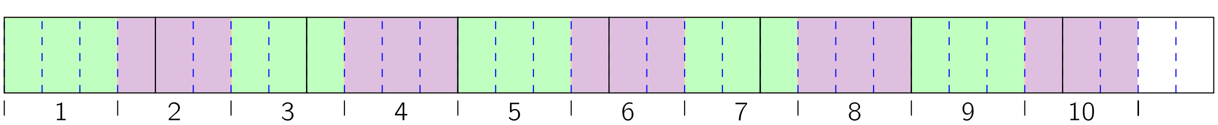
This question is asking us to compute $8 \div \frac{3}{4}$, but let's try and solve the problem by drawing a picture. Suppose this rectangle represents Jorunn's 8 meters of burlap.



To figure out how many pieces of $\frac{3}{4}$ meter are in there, let's first determine how many pieces of size $\frac{1}{4}$ meter. To do that, we'll break each meter into four pieces. That gives us $8 \cdot 4 = 32$ pieces total.



Now, let's gather these pieces up into groups of three. We'll be able to make 10 whole groups, and we'll have 2 pieces left over. Since we have two pieces, but need three, we have $\frac{2}{3}$ of a group. So, Jorunn can make $10\frac{2}{3}$ scarves for the goats.



Let's retrace our steps. We set out to solve $8 \div \frac{3}{4}$, but in our picture we first multiplied to find the total number of fourths, and then we divided to find how many groups of three we could make. In other words: $8 \div \frac{3}{4}$ must have the same answer as $8 \cdot \frac{4}{3}$. Does that look familiar?

Dividing Rational Numbers

Dividing by a number is the same as multiplying by the reciprocal of the number. So, we can change a given fraction division problem into an equivalent fraction multiplication problem

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$$

and then use the rules for fraction multiplication.

Recall that the **reciprocal** of a fraction is the fraction that interchanges the numerator and the denominator of the original fraction. The reciprocal of an integer (which is sitting on a phantom 1) is "one over the original integer".

Example 8

Compute each of the following:

1. $\frac{5}{6} \div -\frac{3}{4}$

We don't need a common denominator or anything, so we can just jump right in with fraction division. We don't even need to move the negative sign, since we know the answer will be negative.

$$\frac{5}{6} \div -\frac{3}{4} = \frac{5}{6} \cdot -\frac{4}{3} = -\frac{20}{18} = -\frac{10}{9}$$

2. $3\frac{3}{4} \div 5$

First, we'll convert to improper fractions, then we'll implement fraction division At the end, we can simplify before we multiply!

$$3\frac{3}{4} \div 5 = \frac{15}{4} \div \frac{5}{1} = \frac{15}{4} \cdot \frac{1}{5} = \frac{3 \cdot 5}{4} \cdot \frac{1}{5} = \frac{3 \cdot \cancel{5}}{4} \cdot \frac{1}{\cancel{5}} = \frac{3}{4}$$

Fractions Inside Fractions

As if fractions on their own weren't enough, you may soon face the turducken of arithmetic, the dreaded "fraction in a fraction".[13] Note that the first criteria for simplified rational numbers states that the numerator and denominator must be *integers*.

These creatures can look hard to handle, but don't be intimidated. Recall that a fraction is just a division problem. So, we can rewrite things using an **obelus** (which is the fancy mathematical name for the ÷ symbol. . . and another thing that you probably didn't know had a name), and divide as usual.

Example 9

Simplify each of the following:

1. $\dfrac{4}{\left(\frac{1}{3}\right)}$

All we have to do is rewrite the fraction-in-a-fraction as "numerator ÷ denominator", and then divide as usual.

$$\frac{4}{\left(\frac{1}{3}\right)} = 4 \div \frac{1}{3} = 4 \cdot \frac{3}{1} = 4 \cdot 3 = 12$$

2. $\dfrac{\left(\frac{2}{5}\right)}{\left(1\frac{3}{8}\right)}$

Here we must rewrite using the obelus, then convert to improper fractions, then divide!

$$\frac{\left(\frac{2}{5}\right)}{\left(1\frac{3}{8}\right)} = \frac{2}{5} \div 1\frac{3}{8} = \frac{2}{5} \div \frac{11}{8} = \frac{2}{5} \cdot \frac{8}{11} = \frac{16}{55}$$

Explaining the Warm-up Problem

In the **warm-up problem** for this section, we have rational numbers a and b where $0 < a < b < 1$. Part (d) asks us to consider $a \div b$. This division problem is another way of writing $\frac{a}{b}$. Since a is less than

[13] A "turducken" is a food product where a deboned chicken is stuffed inside a deboned duck, which is then stuffed inside a deboned turkey. In culinary terminology, this is an example of "engastration", a cooking method in which one animal is stuffed inside the gastric cavity of another. . . which is probably yet another phenomenon that you didn't know had a name.

b, though, we know that this fraction is a proper fraction (as opposed to an improper fraction), which means it is less than 1. So, $a \div b$ is less than 1.

1.3.5 Evil and Wrong

Everyone makes mistakes, but not all mistakes are created equal. Some mistakes are not just wrong, the are **Evil and Wrong**. Wrong because they are mistakes, evil because they are subtle, and sneaky, and tempting.

What we mean here is that sometimes we feel drawn to perform certain arithmetic or algebraic maneuvers... some things seem so logical, so easy, so natural, so tempting... but in reality, they are a total trap. For example:

WARNING!

Armed with the idea of "simplify before you multiply", we might want to try and pull this stunt in other places:

$$\frac{2+3}{2+7} = \frac{\cancel{2}+3}{\cancel{2}+7} = \frac{3}{7}$$

Seems logical, right? But so wrong! There is no "simplify before you add" maneuver, tempting though it may be. Such a thing is **Evil and Wrong**.

You may be thinking, "Nah, I'd never to that," and with numerical expressions like these, you may be right. But later, when faced with variable expressions like

$$\frac{x+3}{x+7}$$

this temptation may come back, in disguise. We'll draw attention to these **Evil and Wrong** mistakes as we go along because they are so tempting. Beware!

1.4 Order of Operations

We turn now to something else that it probably familiar, the **order of operations**. Consider the following problem as we get started:

Warm-up Problem (TODO: Better name for these?)

Three of the Krumbli kids performed the following computation:

$$14 + 6 \cdot 7 - 10$$

Sini got 130, Siri got 46, and Sisi got $^{-}60$. How did each of the girls arrive at their answer? Which of them performed the computation correctly?

The order of operations gives us a standard procedure for simplifying numeric expressions. **Numeric expression** is the algebra term for something you may have called just a "math problem" in elementary school. For example $12-5$ is a numeric expression. It is not in its simplest form because we can evaluate $12-5$ and write 7 instead.

Simplification Rule #1

A numeric expression is completely simplified if all grouping symbols have been eliminated and operations have been evaluated. The resulting quantity is called the *value* of the expression.

As we go along in algebra, we will learn many rules that maintain "mathematical equivalence." Expressions are mathematically equivalent if they represent the same quantity. For example, $\frac{4}{8}$ is equivalent to $\frac{1}{2}$, and $12 - 5$ is equivalent to 7. Our job, when we simplify, is to maintain the equivalence from one step to the next. The order of operations is a set of rules for how to do that.

Order of Operations

The **order of operations** is an agreed-upon order for simplifying numeric expressions. The big idea is that "more powerful" operations take priority over "less powerful" operations. When we want to alter the usual rules of precedence, we introduce grouping symbols to make our intentions clear.

When simplifying an expression, we work from left to right and evaluate things in the following order:[a]

First: Grouping symbols like (parentheses), [square brackets], {curly braces}, as well as other subtle grouping symbols like absolute value and the vinculum. Here we work from the innermost set of groupers[b] to the outermost.

Then: Exponents (and, later, roots and logarithms). In the case of a "stack" of exponents, work from the top down.

Then: Multiplication and division. Recall that dividing is the same as "multiplying by the reciprocal". So, these two operations have the same priority.

Finally: Addition and subtraction. Recall that "subtracting" is the same as "adding the opposite". So, these two operations have the same priority.

---

[a] PEMDAS is an mnemonic acronym for remembering the order of operations: Parentheses, Exponents, Multiplication, Division, Addition, Subtraction. In Canada it's BEDMAS (B for Brackets). In the UK and Australia it's BIDMAS or BODMAS (Indices or Orders, which are other words for exponent). A better acronym might be GEMS or PEMA, which group together the pairs of operations that have the same priority (the G is for Grouping symbols).

[b] As in grouping symbols, not the fish.

Example 10

Simplify: $12 + 10 \div 5 \cdot 3$

Solution: Students who just memorize a clever mnemonic device might be tempted to do the M before the D, but don't be fooled! Multiplication and division have the same priority.

$$\begin{aligned}
& 12 + 10 \div 5 \cdot 3 & \\
&= 12 + 2 \cdot 3 && \text{work multiplication and division from left to right} \\
&= 12 + 6 && \text{work multiplication before addition} \\
&= 12 + 6 & \\
&= 18 &
\end{aligned}$$

Notice how we showed the work going down the page, simplifying the problem one step at a time. Some people prefer to work across, and that's OK too. The point is that it's important to show work in an organized way when we solve a problem so that our reasoning and thought process is clear.

[TODO] Criteria for showing work.

The work you show is the roadmap to your solution. Whoever reads over your work must be able to follow and understand your steps, without having to make any assumptions about what you actually did to reach your answer.

It's a bad habit to skip steps, and it's no help to people reading your work to say you did a step in your head. Every step you take needs to be written down clearly and neatly.

Abuse of the Equal Sign

Consider the following work, written out by a student to simplify $8 \cdot 4 + 10$

$$8 \cdot 4 = 32 + 10 = 42 \qquad \text{OK or not OK?}$$

The student has reached the correct value, and it may even be clear what the student is thinking: "8 times 4 is 32, plus 10 more makes 42". This, however, is a heinous abuse of the equal sign! Look at what the first part of the work says:

$$8 \cdot 4 = 32 + 10 \qquad \text{These are not equal!}$$

One way to avoid this misuse of the equal sign is to write your work going down the page (as shown above). If you prefer to write across the page, be sure to write out the whole problem as you perform each simplification:

$$8 \cdot 4 + 10 = 32 + 10 = 42$$

In the next few sections, we'll look more closely at some trickier aspects of the order of operations.

1.4.1 About Grouping Symbols

As expressions get complicated, we may have grouping symbols inside of other grouping symbols. If there are different symbols, we can more easily see where different groups begin and end. But, we may just find a bunch of parentheses, like in the example below. In that case, we have to be a bit careful about what's being grouped together.

Example 11

Simplify: $12 + (3 - (4 - 2) + 5)$

Solution: When faced with multiple grouping symbols, we must start with the innermost set of grouping symbols and evaluate our way to the outermost. Once we have simplified the expression inside a set of grouping symbols down to a single quantity, we can write that quantity without the groupers.

$$\begin{aligned} &12 + (3 - (4 - 2) + 5) \\ =\ &12 + (3 - 2 + 5) \\ =\ &12 + (1 + 5) \\ =\ &12 + 6 \\ =\ &18 \end{aligned}$$

The Vinculum Is a Grouping Symbol

The **vinculum** (as in the fraction bar) is a grouping symbol. For example, if the task is to simplify a fraction such as:

$$\frac{20 + 2^2 \cdot (14 - 9)}{(2 - 4)^3}$$

then we must think of the expression in the numerator as a group, and likewise for the denominator. In other words, like so:

$$\Big(20 + 2^2 \cdot (14 - 9)\Big) \div \Big((2 - 4)^3\Big)$$

We have two options for simplifying this. We might keep it in a fraction the whole time, or we could simplify the numerator and denominator separately and then squish them back into a fraction at the end. Let's try!

Example 12

Simplify: $\dfrac{20 + 2^2 \cdot (14 - 9)}{(2 - 4)^3}$

Solution: Let's dismantle this thing and handle it in two pieces. The numerator works like this:

$$\begin{aligned} &20 + 2^2 \cdot (14 - 9) \\ &= 20 + 2^2 \cdot 5 \\ &= 20 + 4 \cdot 5 \\ &= 20 + 20 \\ &= 40 \end{aligned}$$

The denominator works like this: $(2-4)^3 = (-2)^3 = -8$. Then, we can put the pieces back into their original fraction configuration:

$$\frac{20 + 2^2 \cdot (14 - 9)}{(2-4)^3} = \frac{40}{-8} = -5$$

1.4.2 About Exponents

An expression like this a^b is read "a to the power of b" or "a to the b^{th} power". In such an expression, a is called the **base** and b is called the **exponent**. Since a is the base, we call the whole thing a **power** of a.

We will get into more detail about exponents later on, but we'll pause here to mention two key ideas. First, recall that we can think about an exponent as shorthand for a repeated multiplication.[14]

$$a^b = \underbrace{a \cdot a \cdot a \cdots a}_{b \text{ times}}$$

That part you probably knew already. This next fact may be new.

Raising to the Power Zero

For any nonzero number a, the expression $a^0 = 1$. In other words, any nonzero number raised to the power 0 equals 1.

Note that a cannot be 0. The expression $0^0 \neq 1$. What *does* it equal? That's a tricky question that will have to wait for later. 0^0 is an unusual mathematical creature![a]

[a] It's not the only one, either. In 1872, Karl Weierstrass (or Weierstraß, if you prefer the German double-s) shook the foundations of calculus with his mathematical monster, now called the "Weierstraß function". It's a bit complicated to get into the details, but he described a function that behaves like the fractals we'll see in ??.

[14] As with "multiplication is repeated addition", this interpretation breaks down eventually. Expressions like 5^{-3} and $16^{\frac{1}{2}}$ don't really translate well into "repeated multiplication". Don't panic about the idea of a negative number or a fraction up there in the exponent! All will be revealed as the course goes on.

We'll get into the "hows and whys" of exponents in **??**, and we'll return to the idea of the zero exponent. In the meantime, here's an example of how this fact might come in handy.

Example 13

Simplify: $\left(\dfrac{120-(24-5^2)}{7^2\cdot 400\div 6^3}\right)^0$

Solution: If we go on "auto-pilot" we might follow all of the simplification rules, work from the inside to the outside, simplify the numerator, simplify the denominator...

But, the expression is raised to the power 0. So the answer is probably 1! We must check that we don't have 0^0, but we can use a little number sense to do a quick check of the numerator in the fraction, and see that it will not equal zero. (Can you see why, without having to work it all out?)

We must also check that the denominator is not zero. A quick check there shows that it is not zero either. (Can you see why?)

So, this one's easy:

$$\left(\frac{120-(24-5^2)}{7^2\cdot 400\div 6^3}\right)^0=1$$

So, what is the lesson here? Look at the entire problem and see if there is an alternate solution strategy. In this case, we save ourselves a lot of work by reading carefully.

Fractions Vs. Exponents

If we have a fraction raised to a power, we must be very mindful of the notation and where the exponent applies.

Example 14

Simplify each of the following:

1. $\left(\dfrac{2}{3}\right)^4$ The parentheses indicate that we are multiplying together 4 copies of the fraction:

$$\left(\frac{2}{3}\right)^4=\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)=\left(\frac{16}{81}\right)$$

2. $\dfrac{2^4}{3}$

Remember that the numerator is a group, and so the exponent applies only to the 2, not the whole fraction:

$$\frac{2^4}{3} = \frac{2 \cdot 2 \cdot 2 \cdot 2}{3} = \frac{16}{3}$$

One of the Trickiest Concepts in Algebra 1

What is the difference between the following three expressions?

$$(-3)^2 \qquad -(3^2) \qquad -3^2$$

Tricky, right? This is an important difference that will come back over and over again, and will look a little different every time.

In the first case, the parentheses make it clear because of the order of operations: $(-3)^2$ means "raise negative three to the second power".

$$(-3)^2 = -3 \cdot -3 = 9$$

As usual, the product of two negatives is positive (Voldemort gets the flu).

[TODO] Michael suggested that making a pop-culture reference like Voldemort would make things dated (as opposed to Hitler, who will forever be known as evil). I kind of like the idea of sprinkling in various villains, as needed. Related: should we choose fictional characters throughout (even in the original Gandhi vs Hitler)? I wonder whether invoking Hitler may run the risk of being considered insensitive. Thoughts?

The second case is clear as well. The parentheses indicate that we should simplify the exponent first and then take the opposite of the result:

$$-(3^2) = -(3 \cdot 3) = -(9) = -9$$

The third case, -3^2, is the tricky one. It looks kind of ambiguous, since there are no parentheses. But, suppose we wrote the problem like this:

$$0 - 3^2$$

Now, it's clear what to do even without the parentheses, and that is the key.

Opposites of Numbers to a Power

The expression $-a^n$, written without parentheses, is equivalent to the parenthesized expression $-(a^n)$. For example: $-3^2 = -(3^2) = -9$.

You might be saying to yourself, "No sweat. I get it." But (if history and human nature are any indication) you may find yourself tripping over this concept at some point. Confusion around the notation can pop up, for instance, when we're typing expressions into a graphing calculator.[15]

1.4.3 About Multiplication

In algebra we use x (the letter) to stand for a unknown or variable quantity. So, we never use $\times$ to show multiplication. It would be too confusing to have a mix of letter-x's and multiplication-$\times$'s in the same expression. Can you imagine trying to decode something like:

$$x \times 3 + 4 \times x = x + 4x \times x \times x \qquad \text{Yikes!}$$

No more x for multiplication!

To show the operation of multiplication, use an asterisk, a dot, or parentheses. Instead of writing 3×4 to represent "3 times 4", we write: $3 * 4$, or $3 \cdot 4$, or $3(4)$.

Implied Operations

A key aspect of algebra will be learning to read the notation and use it correctly when writing expressions and equations. Algebra is very much like a language, and all languages have special rules. For example, in English we sometimes smash up two words as a contraction: instead of writing "do not" we can write "don't".

We used contractions (of a kind) in mathematics as well. We don't usually write a positive sign in front of positive numbers. We don't usually write the phantom 1 that is in the denominator of an integer.

Another kind of mathematical contraction is the use of an **implied operation**. This arises most often when dealing with multiplication. Here's an example of how implied operations can sneak into a problem.

[15] A note to old-school parents who are trained as engineers and are used to working with calculators that use reverse Polish notation (RPN) or a stack, and might want to argue that $-3^2 = 9$: when using an RPN calculator we punch in -3 and press enter to push that quantity onto the stack. The number and the negative are both in the stack together, so squaring the entry on the top of the stack means squaring negative three, which is equivalent to $(-3)^2$.

Example 15

Simplify: $12 - 5(2 + 8)$

Solution:

$$\begin{aligned} & 12 - 5(2+8) \\ &= 12 - 5(10) && \text{Aha! That 5(10) means multiplication!} \\ &= 12 - 50 \\ &= -38 \end{aligned}$$

A very common mistake is to do the $12 - 5$ first, instead of the $5(10)$, but that would totally violate the order of operations!

[TODO] Each chapter should probably have some sort of a closing remark that sets the stage for what comes next. These remarks could be derived from our thoughts on flow.

Glossary

Symbols · **A** · **B** · **C** · **D** · **E** · **F** · **G** · **H** · **I** · **L** · **M** · **N** · **O** · **P** · **Q** · **R** · **S** · **T** · **U** · **V** · **X** · **Y** · **Z**

Symbols

Δ Delta, the fourth letter of the Greek alphabet. Used to represent change. They symbol Δx is read "delta x" or "the change in x".

A

abscissa The x-coordinate of a point in the coordinate plane.

absolute value For real numbers, it is the distance a number is away from zero on a number line. It is a scalar quantity, meaning it just has a magnitude and no direction (sign). The absolute value of a number is always non-negative. In the order of operations, it works like a grouping symbol. The "absolute value of x" is denoted $|x|$.

addition property of equality For all real numbers a, b, and c: If $a = b$, then $a + c = b + c$. This axiom is used when solving equations.

addition property of order For all real numbers a, b, and c: If $a > b$, then $a + c > b + c$. This axiom is used when solving inequalities and also applies to inclusive symbols of order.

additive identity The number which, when added to a given number x, leaves x unchanged. In the real number system, 0 is the additive identity. The existence of the additive identity is a **field axiom**.

additive inverse The number which, when added to a given number x, gives a sum of 0, the additive identity. The opposite of a number is its additive inverse. The existence of the additive inverse is a **field axiom**.

algebraic expression A symbolic representation of mathematical operations that can involve both numbers and variables. There is no equal sign in an expression.

algebraic number A number that is the root on a nonzero polynomial equation in one variable with rational coefficients. The set of algebraic numbers is a subset of the real numbers.

arithmetic sequence A sequence where the difference between each pair of successive terms is constant. The constant difference is called the "common difference", usually denoted d.

associative property of addition For all real numbers a, b, and c: $a+(b+c)=(a+b)+c$. This field axiom allows for the regrouping of longer strings of addition.

associative property of multiplication For all real numbers a, b, and c: $a(bc)=(ab)c$. This field axiom allows for the regrouping of longer strings of multiplication.

asymptote A line that a curve approaches as they both tend towards infinity. There are three types of asymptotes: vertical, horizontal, and oblique (slant). Exponential functions have a horizontal asymptote.

axiom A property or statement that is accepted without proof.

axis One of two perpendicular number lines used to locate points in the coordinate plane. The plural form is "axes".

axis of symmetry The line about which one can reflect an image onto itself. For example, a parabola has an axis of symmetry. Given the graph of a quadratic function, the axis of symmetry is a vertical line through the vertex. When written in standard form, the equation for the line of symmetry is given by $x=-\frac{b}{2a}$.

B

base (1) For triangles: A side of the triangle. (2) For expressions: A term or expression that is raised to a power.

binomial A polynomial with exactly 2 terms.

boundary For a one-variable **inequality**, the boundary is a point on the number line. Inclusive boundaries are drawn as closed or filled in points, and exclusive boundaries are draw as open circles. For a two-variable inequality, the boundary is a line or curve. Inclusive boundaries are drawn as solid lines/curves, and exclusive boundaries are lines/curves drawn with a dashed or dotted line. For a linear inequalities, the boundary line separates the plane into two **half-planes**, one of which will contain the solutions to the inequality.

C

Cartesian plane See **coordinate plane**.

closure A set is said to be to "have closure" (or to "be closed") under an operation performing the operation on members from the set always yields a result that is also a member of the set. The **natural numbers**, for example, are closed under the operation of addition, since the sum of any two natural numbers is itself a natural number. The natural numbers are not closed under the operation of subtraction.

coefficient The numerical factor in a term with a variable. If the number is not explicitly written, the coefficient is understood to be 1.

colinear To be on the same line.

combining like terms A short-cut used to add terms that have exactly the same variables raised to the same exponents.

common difference In an arithmetic sequence, it is the constant difference between successive terms.

common monomial factor A monomial that is a factor of every term in a polynomial expression.

common ratio In an geometric sequence, it is the constant ratio between successive terms.

commutative property of addition For all real numbers a and b: $a + b = b + a$. This field axiom allows for the reordering longer strings of addition.

commutative property of multiplication For all real numbers a and b: $ab = ba$. This field axiom allows for the reordering longer strings of multiplication.

completing the square Using the properties of equality on a quadratic equation to convert one side into a perfect square trinomial. Completing the square can be used as a technique to solve quadratic equations.

complex number A member of the set of numbers that consists of real and imaginary numbers. The set is denoted $\mathbb{C}$.

compound interest A way to calculate interest based on both the principal amount and any interest already accrued. This type of interest is an exponential relationship. The formula is $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where P is the principal amount, r is the rate of interest, t is the amount of time over which interest is to be computed, and n is the number of compounding periods per unit of time.

constant A value that does not change.

constant function A function whose graph is a horizontal line. It is of the form $f(x) = c$, where c is a constant. Constant functions are polynomial functions of degree zero.

constant multiplier In a sequence that grows or decays exponentially, the number each term is multiplied by to get the next term. Also known as the "common multiplier", or "common ratio".

constant of variation The constant ratio in a direct variation or the constant product in an inverse variation. It is designated with the variable k.

constant term A term that includes no variable.

constraint The limitations on the values of the variables in a problem. Equations, inequalities and systems are used model the constraints in real-world situations.

continuous data Data that has no breaks and has measurements that can change between data points. Graphically, the measured data points are connected with lines or curves.

continuous function A function that has no breaks in the domain or range. The graph of a continuous function is a line or curve with no holes, gaps, or vertical asymptotes.

converse of the Pythagorean theorem If a triangle has sides a, b, and c, such that $a^2 + b^2 = c^2$, then the triangle is a right triangle with a hypotenuse of length c.

conversion factor A ratio used to convert measurement from one unit to another.

coordinate plane A plane with a pair of scaled, perpendicular axes allowing one to locate points with ordered pairs and to represent lines and curves by equations. Also known as the Cartesian plane, named for its creator, French philosopher René Descartes.

coprime See **relatively prime**.

correlation Used in describing data graphed in a scatter plot. It is a trend between two variables. A trend can show positive, negative, or no correlation. Positive correlation shows an **increasing** trend in data. Negative correlation shows a **decreasing** trend in data.

cubic A function, number, or expression raised to the third power. Called cubic as it relates to the volume of a cube.

D

decay factor In exponential decay, the constant multiplier used to calculate the amount of decay after each unit of time. In the formula $y = (1 - r)^x$, it is the quantity $(1 - r)$. It represents the quantity remaining and is the common multiplier in the exponential relationship.

decay rate In exponential decay, the fraction or percentage by which a population decreases for each unit of time. In the formula $y = (1 - r)^x$, it is the quantity r.

decreasing A function is said to be decreasing if as x increases, y decreases. Lines with negative slopes are decreasing.

degree of a polynomial The degree of the term in a polynomial with the highest degree.

degree of a term The power (exponent) to which the variable is raised in a variable term. If there is no exponent explicitly written on a variable in a term, the term is understood to be of degree 1. The degree of a constant term is zero.

denominator The number or expression below the **vinculum** in a **rational number** or **rational expression**. For example, in the number $\frac{5}{2}$, the denominator is 2.

dependent variable A variable whose values depend on the values of another variable. In a graph of the relationship between the two variables, the values on the vertical axis represent the values of the dependent variable. The generic variable used is y.

difference of squares A binomial of the form $a^2 - b^2$.

dimensional analysis A strategy for converting a measurement from one unit to another using multiplication by a string of conversion factors. The key is to include the units with the numbers. It is used often in science.

direct variation In algebra 1, a relationship in which the ratio of two variables is constant. A direct variation has an equation of the form $y = kx$. The quantities represented by x and y are said to be **directly proportional**. The value k is called the **constant of variation**.

directly proportional Used to describe two variables whose values have a constant ratio.

discrete data Data that can only take on certain values. Discrete data usually involves a count of items.

discrete function A function whose domain and range have breaks or are made up of distinct values rather than intervals of real numbers. The graph of a discrete function will have breaks or will be made up of distinct points.

discriminant The expression under the square root in the quadratic formula, used to determine the number and nature of the roots of a quadratic. If a quadratic equation is written in standard form, then the discriminant is $b^2 - 4ac$. If the value of the discriminant is greater than 0, there are two real solutions to the quadratic equation. If it is equal to zero, there is one real solution. If it is less than zero, there are no real solutions to the quadratic equation.

distance formula A formula based on the Pythagorean theorem that uses the coordinates of two points to calculate the distance between the two points. The formula for the distance d between any two points (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

distributive property For all real numbers a, b, and c: $a(b + c) = ab + ac$. This field axiom allows one to simplify an expression without having the evaluate the sum inside the grouping symbol first.

division property of equality For all real numbers a, b, and c, where $c \neq 0$: if $a = b$, then $\frac{a}{c} = \frac{b}{c}$. This property is a version of the multiplication property of equality. It is used when solving equations.

division property of order For all real numbers a, b, and c where $c > 0$: if $a < b$, then $\frac{a}{c} < \frac{b}{c}$. If, on the other hand, $c < 0$, then $a < b$ implies $\frac{a}{c} > \frac{b}{c}$. This property is a version of the multiplication property of order. It is used when solving inequalities.

domain The set of all input values of a function, or the x-values. In a problem context it is represented by the independent variable.

domain restriction Values that cannot be used in the domain of a function. Radical and rational functions have domain restrictions.

doubling time In exponential growth, the amount of time it takes for a population, or amount, to double in size. It is constant for an exponential relationship.

E

elimination method A method for solving a system of equations that involves adding or subtracting multiples of the equations in order to eliminate a variable. It is based on Gaussian Elimination, a method to solve systems of equations that have been converted into matrices.

equation A statement that says the value of one expression is the same as the value of another expression.

equivalent equations Equations that have the same solution set.

equivalent inequalities Inequalities that have the same solution set.

evaluate To find the value of an expression. If an expression contains variables, values must be substituted for the variable before the expression can be evaluated.

exclusive boundary See **boundary**.

exclusive inequality See **inequality**.

exponent A number or variable written as a small superscript to a number or a variable, called the **base**, that indicates how many times the base is being used as a factor.

exponential decay A decreasing pattern in which amounts decrease by a constant percent.

exponential equation An equation in which a variable appears in the exponent.

exponential form The form of an expression in which repeated multiplication is written using exponents.

exponential function A function that repeatedly multiplies an initial amount by the same positive number. They can all be modeled using $y = ab^x$ where a is the initial amount and b is the constant multiplier.

exponential growth An increasing pattern in which amounts increase by a constant percent.

extraneous solution An apparent solution of an equation that does not satisfy the original equation. They occur when the transformation of an equation changes the solution set of the original equation, for example squaring both sides of an equation or multiplying by a quantity that can be zero.

F

factor One of the numbers, variables, or expressions multiplied to obtain a product.

factored form The form of an expression when it is written as the product of factors. The factors can be numbers, variables, or expressions. Factored form is not simplified.

factoring The process of rewriting an expression as a product of factors.

family of functions Similar functions that are all transformations of the same parent function.

field axiom One of a set of axioms including closure, identity, inverse, associative, commutative, and distributive properties. Along with a few definitions and properties of equality, they create the foundation upon which algebra is built.

FOIL A mnemonic for remembering the procedure to multiply two binomials. F stands for multiplying the first term in each binomial. O stands for multiplying the outer terms of the binomials. I stands for multiplying the inner terms of the binomials. L stands for multiplying that last term in each binomial.

fractal A geometric figure that has undergone infinite applications of a recursive procedure and which exhibits the property of self-similarity.

function A relation in which there is exactly one output value for each input value. The graph of a function must pass the vertical line test.

function notation A notation in which a function is named by a letter and the input is shown in parenthesis after the function name, generically, $f(x)$, read "f of x". The variables used may be changed to better represent quantities in a problem, for example $d(t)$ may represent distance d as a function of time t. When graphing in the **coordinate plane**, $f(x)$ is another way to write y. When x is replaced by a number, it indicates that one should evaluate the function at that value. The notation was first used by Swiss mathematician Leonhard Euler.

function rule An expression that represents the relationship between the variables of a function.

G

GCF Greatest Common Factor

geometric sequence A sequence where the ratio between each pair of successive terms is constant. The constant ratio is called the "common ratio", usually denoted r. Geometric sequences are exponential.

growth factor In exponential growth, the constant multiplier used to calculate the amount of growth after each unit of time. In the formula $y = (1 + r)^x$, it is the quantity $(1 + r)$. It is the common multiplier in the exponential relationship.

growth rate In exponential growth, the fraction or percentage by which a population increases for each unit of time. In the formula $y = (1 + r)^x$, it is the quantity r.

H

half-life The time needed for an amount of a substance to exponentially decay to half the original amount. Half-life is constant for an exponential relationship.

half-plane The set of points on a plane that fall on one side of a boundary line. Part of the solution of a linear inequality in two variables is a half-plane.

hypotenuse The side of a right triangle opposite the right angle. It is the longest side of the triangle.

I

identity When solving equations with variables on both sides, identities occur when the equation is true for every value of the variable. The solution set S is written as $S = \mathbb{R}$.

identity property of addition The sum of any number and 0 is that number. For every real number a, $a+0 = a$ and $0 + a = a$. The existence of the **additive identity** is a **field axiom**.

identity property of multiplication The product of any number and 1 is that number. For every real number a, $a \cdot 1 = a$ and $1 \cdot a = a$. The existence of the multiplicative identity is a **field axiom**.

imaginary number A member of the set of numbers that is created by taking the square root of a negative number. In the set of imaginary numbers, the square root of -1 is represented by the letter i. The set of imaginary numbers is a subset of the complex number system. The sets of real and imaginary numbers are disjoint, meaning they have no common members.

implied operation An operation that is not explicitly written. For example, in $3(x + 4)$ the multiplication between 3 and $(x + 4)$ is an implied operation, since no multiplication symbol is explicitly written in between.

improper fraction A fraction whose **numerator** is greater than its **denominator**. For example, $\frac{5}{2}$ is an improper fraction. A fraction that is not an improper fraction is called a **proper fraction**. See also **mixed number**.

inclusive boundary See **boundary**.

inclusive inequality See **inequality**.

increasing A function is said to be decreasing if as x increases, y increases. Lines with positive slopes are decreasing.

independent variable A variable whose values affect the values of another variable. In a graph of the relationship between the two variables, the values on the horizontal axis represent the values of the dependent variable. The generic variable used is x.

inequality A statement that one quantity is less than or greater than another. An inequality may exclusive or inclusive. The exclusive inequalities are $<$ and $>$, read "less than" and "greater than". The inclusive inequalities are $\leq$ and $\geq$, read "less than or equal to" and "greater than or equal to".

initial value The starting value of a sequence or exponential function.

integer A member of the set of natural numbers, their opposites, and zero. The set is denoted $\mathbb{Z}$, and we may write $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$. The integers are a subset of the rational numbers.

intercept The point which a graph intersects one of the axes.

interest A percentage of the balance added to an account at regular time intervals.

interest rate The percentage used to calculate interest.

inverse property of addition For any real number a, there exists a real number ^{-}a such that $a + {}^{-}a = 0$. The number ^{-}a is called the **additive inverse** of a. Very often we will call it the **opposite** of a.

inverse property of multiplication For any nonzero real number a, there exists a real number $\frac{1}{a}$ such that $a \cdot \frac{1}{a} = 1$. The number $\frac{1}{a}$ is called the **multiplicative inverse** of a. Very often we will call it the **reciprocal** of a.

inverse variation In algebra 1, a relationship in which the product of two variables is constant. An inverse variation has an equation in the form $xy = k$, or $y = \frac{k}{x}$. The quantities represented by x and y are said to be **inversely proportional**. The value k is called the **constant of variation**.

inversely proportional Used to describe two variables whose values have a constant product.

irrational number A number that cannot be expressed as the ratio of two integers. In decimal form, an irrational number has an infinite number of digits and does not repeat. The set of irrational numbers consist of algebraic and transcendental numbers. The set of irrational numbers is a subset of the real numbers.

irreversible operation An operation performed when solving an equation that changes the solution set of the equation. Multiplying or dividing both sides of an equation by an expression that might equal zero are considered irreversible operations.

L

leg One of the perpendicular sides of a right triangle.

like terms Terms with exactly the same variable factors in a variable expression. The variables and the powers to which the variables are raised must be identical for the terms to be considered like terms.

limited domain The restricted domain of a function. Domains are usually limited in real world contexts. For example, we rarely allow negative values for a variable that represents "time". For this reason it is often referred to as a reasonable domain.

line of best fit A line used to model a set of data. A line of best fit shows general direction of the data. When hand-drawn, one should have about the same number of data points above and below the line. When using the linear regression tool on the calculator, the correlation coefficient will show how well the line fits the data.

line of symmetry See **axis of symmetry**.

linear In the shape of a line or represented by a line. In mathematics, a linear equation or expression has variables raised only to the power of 1.

linear function A function characterized by a constant rate of change. The graph of a linear function is a non-vertical line. It is a polynomial of degree one.

linear inequality An inequality of two variables whose boundary is formed by a linear function. It describes a region of the coordinate plane that consists of a boundary line and a half-plane.

linear programming A method to optimize a quantity that uses an objective function to represent the quantity and a system of linear inequalities to represent the constraints on the variables involved. The system of inequalities are graphed to represent a set of feasible solutions and the vertices of the region will describe the optimal amount of the quantity.

linear relationship A relationship that can be represented by a linear function. A linear relationship is characterized by a constant rate of change.

linear term A term of degree 1.

lowest terms The form of a fraction in which the numerator and denominator are **relatively prime**. A fraction in lowest terms is also called a reduced fraction.

M

mapping diagram A diagram used to determine if a relation is a function. The values of the domain and range are written in circles. Arrows are drawn from the elements of the domain to the corresponding elements of the range. It is a visual that shows how the members of the domain map to the members of the range.

mathematical equivalence The idea that numbers, expressions, equations, functions, or other mathematical objects can be algebraically manipulated, using specific rules, such that their representations and appearance are changed while other fundamental properties remain unchanged.

mathematical modeling Translating a real-world scenario with a given set of constraints into an abstract representation that can be manipulated and studied mathematically. For example, creating a set of variables and equations to solve a **linear programming** problem.

maximum The greatest value. In a quadratic function, the vertex will be a maximum if the coefficient of the quadratic term is negative.

midpoint The point on a line segment halfway between the endpoints. The coordinates of the midpoint are found by averaging the abscissas and ordinates of the endpoints.

midpoint formula The formula that can be used to compute the midpoint of a line segment. Given a line segment with endpoints (x_1, y_1) and (x_2, y_2), the midpoint of the segment has coordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

minimum The smallest value. In a quadratic function, the vertex will be a minimum if the coefficient of the quadratic term is positive.

mixed number The sum of a nonzero **integer** and a **proper fraction**. For example $2\frac{3}{5}$ is a mixed number. See also **improper fraction**.

monomial A polynomial with only one term.

multiplication property of equality For all real numbers a, b, and c: if $a = b$ then $ac = bc$. This property is used to solve equations.

multiplication property of order For all real numbers a, b, and c and $c > 0$: if $a < b$ then $ac < bc$. If, on the other hand, $c < 0$, then $a < b$ implies $ac > bc$. This property is used to solve equations.

multiplicative identity The number which, when multiplied by a given number x, leaves x unchanged. In the real number system, 1 is the multiplicative identity. The existence of the multiplicative identity is a **field axiom**.

multiplicative inverse The number which, when multiplied by a given nonzero number x, gives a product of 1, the multiplicative identity. The reciprocal of a number is its multiplicative inverse. The existence of the multiplicative inverse is a **field axiom**.

N

natural number A member of the set $\{1, 2, 3, 4, \ldots\}$, denoted $\mathbb{N}$. Also called the counting numbers. The number 0 is sometimes included as a natural number.

negative correlation See **correlation**.

null set A set that contains no elements. Also called the empty set. Used to show that there is no solution to an equation. Denoted $\varnothing$ or $\{\}$.

numerator The number or expression above the **vinculum** in a **rational number** or **rational expression**. For example, in the number $\frac{5}{2}$, the numerator is 5.

numeric expression An expression containing only numbers and mathematical operations.

O

obelus The division symbol $\div$.

one-variable data Data that measures only one trait or quantity. A one-variable data set consists of single values (as opposed to ordered pairs) and is graphed on a number line. Compare with: **two-variable data**.

opposite See **additive inverse**.

optimization To maximize or minimize a quantity given constraints. For example a company will want to optimize (maximize) their profits while faced with constraints such as the cost and availability of labor and materials.

order of magnitude A way of expressing the size of an very large or very small number by giving the power of 10 associated with the number.

order of operations The agreed-upon order in which operations are carried out when evaluating an expression.

ordered pair A pair of numbers named in an order that matters. The coordinates of a point are given as an ordered pair in which the first number is the x-coordinate (abscissa) and the second number is the y-coordinate (ordinate).

ordinate The y-coordinate of a point in the coordinate plane.

origin The point where the coordinate axes intersect. In a coordinate plane it has the coordinates $(0, 0)$.

P

parabola The set of all points whose distance from a fixed point (called the focus) is equal to the distance from a fixed line (called the directrix). Also known as the smooth "U" shaped curve of a quadratic function.

parallel lines Lines in the same plane that never intersect. They are always the same distance apart in Euclidean geometry. The slopes of parallel lines are the same.

parent function The most basic form of a function. A parent function can be transformed to create a family of functions.

percent change The percent by which an amount differs from its original amount. It is calculated by taking the amount of the change and dividing it by the original amount.

perfect cube A number that is equal to the cube of an integer, or a polynomial that is equal to the cube of another polynomial.

perfect square A number that is equal to the square of an integer, or a polynomial that is equal to the square of another polynomial.

perfect square trinomial A trinomial generated by squaring a binomial. For example, squaring the binomial $(a + b)$ yields $(a + b)^2 = a^2 + 2ab + b^2$. Thus, $a^2 + 2ab + b^2$ is a perfect square trinomial.

period of compounding The number of times interest is calculated during a year for compound interest. It is represented by n in the **compound interest** formula.

perpendicular lines Lines that intersect at a right angle. The slopes of perpendicular lines are opposites and reciprocals. The slopes of perpendicular lines multiply to -1.

point-slope form The form of a linear equation that uses the slope and any point on the line. It is written either $y - y_1 = m(x - x_1)$ or $y = m(x - x_1) + y_1$, where m is the slope of the line and (x_1, y_1) is a point on the line. It can be derived from the slope formula and represents the transformation of the line $y = mx$ where a vertical shift of y_1 and a horizontal shift of x_1 has occurred.

polynomial A sum of terms that have positive integer exponents. In algebra 1, all polynomials are in one variable.

positive correlation See **correlation**.

power An expression of the form a^n is called a power of a.

principal amount The original amount invested in a situation that involves accumulating interest. It is represented by P in the **compound interest** and simple interest formulas.

principal square root The positive square root of a number.

proper fraction A fraction whose **numerator** is less than its **denominator**. For example, the fraction $\frac{7}{9}$ is a proper fraction. A fraction that is not a proper fraction is called an **improper fraction**.

proportion An equation stating that two ratios are equal.

Pythagorean theorem A formula that expresses the relationship between the sides of a right triangle. It states that the sum of the squares of the legs of a right triangle is equal to the square of the **hypotenuse**.

Q

quadrant One of the four regions that a coordinate plane is divided into by the two axes. The quadrants are numbered I, II, III, and IV, starting in the upper right and moving counterclockwise.

quadratic formula The formula used to find the exact solution to any quadratic equation. Given that $ax^2 + bx + c = 0$, the formula states

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is derived by completing the square on the standard form quadratic equation.

quadratic function A function with an equation of the form $y = ax^2 + bx + c$ where $a \neq 0$. The graph of a quadratic function is a **parabola**.

quadratic term A term of degree 2.

R

radical The root symbol $\sqrt{\ }$, used to denote square roots, cube roots, and so on. The symbol $\sqrt[n]{x}$ is read "nth root of x." If n is not stated, as in $\sqrt{x}$, it is understood to be 2 and the radical indicates the square root.

radical expression An expression containing a radical (square root, cube root, or any nth root).

radical function A function where the independent variable is under a radical (square root, cube root, or any nth root).

radioactive decay The process by which an unstable element loses mass with a release of energy, transforming it into a different element or isotope.

range (1) In statistics it is the difference between the greatest value in a data set and the smallest value in a data set. (2) In the study of functions it is the set of all output values of a function. It is represented by the dependent variable.

rate A **ratio** that measures two quantities with different units.

rate of change A measurement of how quickly one quantity changes relative to another quantity. Given values (x_1, y_1) and (x_2, y_2), the rate of change of y with respect to x is $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$. The patterns of the rate of change of a set of data can be used to determine what type of data is represented by the pattern. For example, the rate of change of linear data is constant.

ratio A comparison between two quantities, often written in fraction form.

rational expression An expression that can be written as a ratio of two polynomials. The value of the variable cannot make the denominator 0.

rational function A function that is expressed as the ratio of two polynomial expressions. The values of the independent variables that make the denominator zero are restricted from the domain.

rational number A number that can be written as a ratio of two integers $\frac{a}{b}$ where $b \neq 0$. Their decimal forms are either terminating or repeating. The set of rational numbers is denoted $\mathbb{Q}$. The rational numbers are a subset of the real numbers.

rationalizing the denominator The process of making the denominator of a fraction a rational number without changing the value of the expression. It is used to eliminate a radical from the denominator of a fraction.

real number Denoted $\mathbb{R}$, the set of real numbers include the integers, rational numbers, and irrational numbers, but not imaginary numbers. This is the number set used in algebra 1. The set is closed under the operations of addition and multiplication. Members can be graphed on the standard number line. The real numbers is a subset of the complex numbers.

reciprocal The multiplicative inverse. The reciprocal of a given number is the number it must be multiplied by to get 1 (the multiplicative identity). To find the reciprocal of a number, we can write the number as a fraction and then invert the fraction. The reciprocal of n is $\frac{1}{n}$.

recursive Describes a procedure that is applied over and over again, starting with a number or a geometric figure, to produce a sequence of numbers or figures. The procedure requires previous entries in the pattern to find subsequent entries.

recursive rule Instructions for producing each stage of a sequence from the previous stage. It must contain a description of "stage 0", or the starting value.

recursive sequence An ordered list of numbers defined by a starting value and a recursive rule. We generate a recursive sequence by applying the rule to the starting value, then applying the rule to the resulting value, and so on.

relation Any set of ordered pairs.

relatively prime Two numbers are said to be relatively prime (or coprime) if they have no common factors other than 1. For example, 16 and 21 are relatively prime. In contrast, 21 and 24 are not relatively prime, since both numbers are divisible by 3.

repeating decimal A decimal representation of a rational number with a digit or group of digits after the decimal point that repeat infinitely.

root A zero or an x-intercept of a function.

S

scatter plot A two-variable data display in which values on a horizontal axis represent values of one variable and values on the vertical axis represent values of the other variable. The coordinates of each point represent a pair of data values.

scientific notation A notation in which a number is written as the product of a number greater than or equal to 1 but less than 10, multiplied by an integer power of 10.

sequence A function whose domain is the set of positive integers. A sequence is an ordered list of objects, like numbers. The individual objects are called terms. Unlike a set, order matters, and terms may be repeated.

set An unordered collection of items. Often denoted by listing the elements inside a set of braces.

set notation Using curly braces { and } to designate quantities that belong to a set. Certain sets do not require the use of braces, as they have symbols used to denote them, like the **null set**, the set of **integers**, and the set of **real numbers**.

simple interest Interest calculated using the formula $I = Prt$. The interest is only ever calculated using the initial investment (called the **principal amount**) and show linear growth.

simplified radical form A radical written so that (1) no perfect square factors exist under the radical (2) no fractions are under the radical and (3) there are no radicals in the denominator of the fraction.

simplify Using algebraic laws and properties which maintain equivalence in order to write an answer so that it fits a set of criteria. The criteria depend on what is being simplified.

slope The measurement of the steepness of a line, or the rate of change of a linear relationship. Often denoted m, and referred to as "rise over run." Given points (x_1, y_1) and (x_2, y_2), the slope of the line between the points is calculated as $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.

slope-intercept form The form $y = mx + b$ of a linear equation. The value of m is the slope and the value of b is the y-intercept. It is the simplified version of **point-slope form**.

solution A solution to an equation (or inequality) is any value of the variable (or variables) in the equation (or inequality) that make the equation (or inequality) true. The solution to a system of equations (or inequalities) is the set of all of the points common to all equations in the system. If there is no solution, the system is said to be inconsistent. If there are infinitely many solutions to a system, the system is said to be dependent. If there is a single solution, the system is said to be independent. In a system of two equations in two variables, the solution is the intersection point of the two lines.

solution set The set of values that make an equation, inequality, or system true.

solution set notation One way to denote the solution set to an equation, written as $S = \{ \ solutions \ \}$.

solve To find the solution set of an equation.

square root The square root of a number a, denoted $\sqrt{a}$, is the number b such that that $b \cdot b = a$. Every positive number has two square roots, a **principal square root** and a negative square root. The set of real numbers is not closed under the operation of square root.

standard form (1) For linear equations, it is an equation of the form $Ax + By = C$, in which A and B are not both 0. (2) For a polynomial, it is an expression written such that it is simplified and the terms are written in decreasing order of degree (highest degree term appears first). (3) For quadratic equations, it is an equation of the form $ax^2 + bx + c$, where $a \neq 0$.

subset A subset is a set that consists entirely of members from another set. If a set A is a subset of a set B, then every item in A is in B.

substitution To replace a quantity with another one that is equivalent.

substitution method A method for solving a system of equations that involves solving one of the equations for one variable and substituting the resulting expression into the other equation. See also: **elimination method**.

subtraction property of equality For all real numbers a, b, and c: if $a = b$ then $a - c = b - c$. This property is a restatement of the **addition property of equality** and is used to solve equations.

subtraction property of order For all real numbers a, b, and c: If $a > b$, then $a - c > b - c$. This property is a restatement of the **addition property of order** and is used to solve inequalities.

system of equations A set of two or more equations with the same variables. The equations act as constraints on the variables.

system of inequalities A set of two or more inequalities with the same variables. The inequalities act as constraints on the variables.

T

term An algebraic expression that represents only multiplication and division between variables and constants.

terminating decimal A decimal number with a finite number of nonzero digits after the decimal point.

transcendental number An irrational number that is not algebraic. The number π is transcendental because it is not the root of a polynomial equation in one variable with rational coefficients.

trinomial A polynomial with exactly three terms.

two-variable data A collection of data that measure two traits or quantities. A two-variable data set consists of pairs of values. Compare with: **one-variable data**.

U

unit rate A ratio in which one of the quantities has the value of 1.

unknown A quantity in an equation whose value is not known. In algebra, letters are often used to represent unknowns.

V

variable A trait or quantity whose value can change, or vary. In algebra, letters are often used to represent variables.

vector A quantity that has both a size (or magnitude) and a direction Vectors play an important role in physics and engineering, since many physical quantities (such as velocity, acceleration, and force) are best represented using vectors.

vertex Of a parabola, the point where the graph changes direction from increasing to decreasing or from decreasing to increasing.

vertex form A form of a quadratic equation. Given that (h, k) is the vertex, this form is written either as $y - k = a(x - h)^2$ or $y = a(x - h)^2 + k$. It can be derived by completing the square on standard form and represents the transformation of $y = ax^2$ by translation h units horizontally and k units vertically.

vertical line test A method for determining whether a graph on the coordinate plane represents a function. If all possible vertical lines cross the graph only once or not at all, the graph represents a function. If even one vertical line crosses the graph in more than one point, the graph does not represent a function.

vertical motion formula When an object is dropped or launched vertically, its height can be expressed using $h(t) = at^2 + vt + h_0$, where $h(t)$ is the object's height at time t, v is its initial vertical velocity, h_0 is its starting height, an a is the acceleration of gravity. For dropped objects, v is zero. This formula is used in the study of projectile motion.

vinculum A bar used in mathematics to show grouping. Examples of vincula include: the fraction bar (as in $\frac{1}{x+2}$), the bar used to show repeating digits (as in $0.\overline{3}$), and the horizontal bar of a radical (as in $\sqrt{2+5}$).

X

x-axis The horizontal number line on a coordinate graph. The independent variable is drawn on the x-axis.

x-intercept Any point at which a graph intersects the x-axis.

Y

y-axis The vertical number line on a coordinate graph. The dependent variable is drawn on the y-axis.

y-intercept Any point at which a graph intersects the y-axis.

Z

zero Of a function, the values of the independent variable that make the corresponding values of the function equal to zero, also known as the **roots** or x-intercepts of the function.

zero product property Property of real numbers stating that if the product of two or more factors equals zero, then at least one of the factors must equal zero. This property is used along with factoring as a method for solving a polynomial equation.

Bibliography

[1] Richard G. Brown, Mary P. Dolciani, Robert H. Sorgenfrey, and William L. Cole. *Algebra: Structure and Method, Book 1*. McDougal Littell, 2000.

[2] Richard G. Brown, Mary P. Dolciani, Robert H. Sorgenfrey, and Robert B. Kane. *Algebra and Trigonometry: Structure and Method, Book 2*. McDougal Littell, 1999.

[3] Edward B. Burger. *Extending the frontiers of mathematics: Inquiries into proof and argumentation*. Key College Publishing, 2007.

[4] Edward B. Burger and Michael Starbird. *The heart of mathematics: An invitation to effective thinking*. Key Curriculum Press, 2000.

[5] Sergey Dorichenko. *A Moscow math circle: Week-by-week problem sets*. American Mathematical Society, 2011.

[6] Paul A. Foerster. *Algebra and Trigonometry: Functions and Applications*. Addison-Wesley, 2005.

[7] Timothy Gowers. *Mathematics: A very short introduction*. Oxford University Press, 2002.

[8] Ken Johnson and Ted Herr. *Problem Solving Strategies: Crossing the River with Dogs and Other Mathematical Adventures*. Key Curriculum Press, 2001.

[9] Ron Larson and Robert P. Hostetler. *Algebra and Trigonometry*. Cengage Learning, 2003.

[10] Sandor Lehoczky and Richard Rusczyk. *The Art of Problem Solving, Vol. 1: The Basics*. AoPS Incorporated, 2006.

[11] Carol Meyer and Tom Sallee. *Make it simpler*. Dale Seymour Publications, 1983.

[12] Frederick Mosteller. *Fifty challenging problems in probability with solutions*. Dover Publications, 1965.

[13] Jerald Murdock, Ellen Kamischke, and Eric Kamischke. *Discovering advanced algebra: An investigative approach*. Key Curriculum Press, 2004.

[14] Jerald Murdock, Ellen Kamischke, and Eric Kamischke. *Discovering algebra: An investigative approach*. Key Curriculum Press, 2007.

[15] Alfred S. Posamentier and Charles T. Salkind. *Challenging problems in algebra*. Dover Publications, 1996.

[16] Paula Rozell. *Plotting Pictures: Coordinate Graphing and Number Skills Practice, Grades 5-10*. Pearson Education, 1995.

[17] Richard Rusczyk and Sandor Lehoczky. *The Art of Problem Solving, Vol. 2: And Beyond*. AoPS Incorporated, 2006.

[18] Reuben Schadler. *Algebra Problems: One Step Beyond*. Pearson Learning, 1990.

[19] Harold L. Schoen, editor. *Teaching mathematics through problem solving: Grades 6-12*. National Council of Teachers of Mathematics, 2006.

[20] Michael Serra. *Discovering geometry: An investigative approach*. Key Curriculum Press, 2003.

[21] Peter Winkler. *Mathematical puzzles: A Connoisseur's Collection*. A K Peters/CRC Press, 2003.

[22] Paul Zeitz. *The art and craft of problem solving*. John Wiley and Sons, 2007.

Index