The Algebranomicon

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Contents

Still need to find a quote that works for this chapter. In the meantime, we have this.

Author
Description of author

Chapter 1

Linear Systems

Every chapter should have a lead paragraph – even just a short one – that appears before the first heading. This is a placeholder paragraph which will at some point be replaces by actual content.

1.1 Mathematical Modeling

Mathematics has many applications to the sciences, medicine, business, and economics. Often, though, we have to "translate" some problem in the real world so that it can be interpreted mathematically. It is usually necessary to break down the situation into a set of variables and equations, then look at them all together and see how they interact with each other. Doing this is called **mathematical modeling**.

We can model the financials of a business, weather patterns, transportation infrastructure, airline schedules, missile guidance systems, factory efficiency — and many other phenomena — in this way.

Startup Exploration: I ♥ YeardleighCorp

Yeardleigh is a good business woman. She realizes that in order to grow her business, she must brand her corporation by selling things like t-shirts with her company logo on it. She spends \$270 on a design. Since she plans on printing a huge number of shirts, she finds a supplier who will sell her blank cotton shirts in a variety of sizes and colors for \$3.00 per shirt, and a printer who will print the logo in full color for \$1.50 per shirt. If Yeardleigh decides to sell the shirts for \$12.00 each, how many shirts must Yeardleigh sell before she can start making a profit on t-shirt sales?

1.1.1 Simultaneous Equations

One approach to mathematical modeling is to model problems using multiple functions, and then exploring how those functions relate to one another.

System of Equations

A set of two or more equations with the same variables is called a **system of equations**. They are also sometimes called *simultaneous equations*.

In the startup exploration, the amount of money Yeardleigh earns will depend on how many t-shirts she buys and sells. So, suppose we let x, the independent variable, represent the number of t-shirts. Then we let y, the dependent variable, represent the amount of money that those t-shirts cost (or that they generate for the company).

Yeadrleigh earns \$12.00 for every t-shirt she sells, so one equation involved with the scenario is

$$y = 12.00x \qquad \text{Yeardleigh's income equation.}$$

She has already spent \$270 on her design, and every t-shirt she prints costs an additional \$4.50 (that's \$3.00 for the shirt and \$1.50 for printing). So, another equation in the scenario is

$$y = 270 + 4.50x \qquad \text{Yeardleigh's cost equation.}$$

We now have to linear equations. Observe what happens when we graph these two equations on the same set of axes (fig. 1.1). What does this graph tell us about the problem?

On the graph in the figure, the red line is Yeardleigh's cost equation and the green line is Yeardleigh's income equation. The x-coordinate shows the independent variable (number of shirts), and the y-coordinate shows the dependent variable (amount of money spent or earned).

Notice that these two lines intersect (in other words: they cross over one another). Does this intersection point have any special significance?

The intersection shows that for a certain number of t-shirts, cost and income are the same. The points to the left of the intersection show where cost is greater than income, points to the right of the intersection show where income is greater than cost. So that intersection point is really important for the problem!

The coordinates of the intersection point are $(36, 432)$. So, if Yeardleigh sells 36 t-shirts, she will make 432 in income... and it will also cost her exactly that much to produce the t-shirts. If she sells more than 36 t-shirts, she will earn more than it costs to make the shirts.

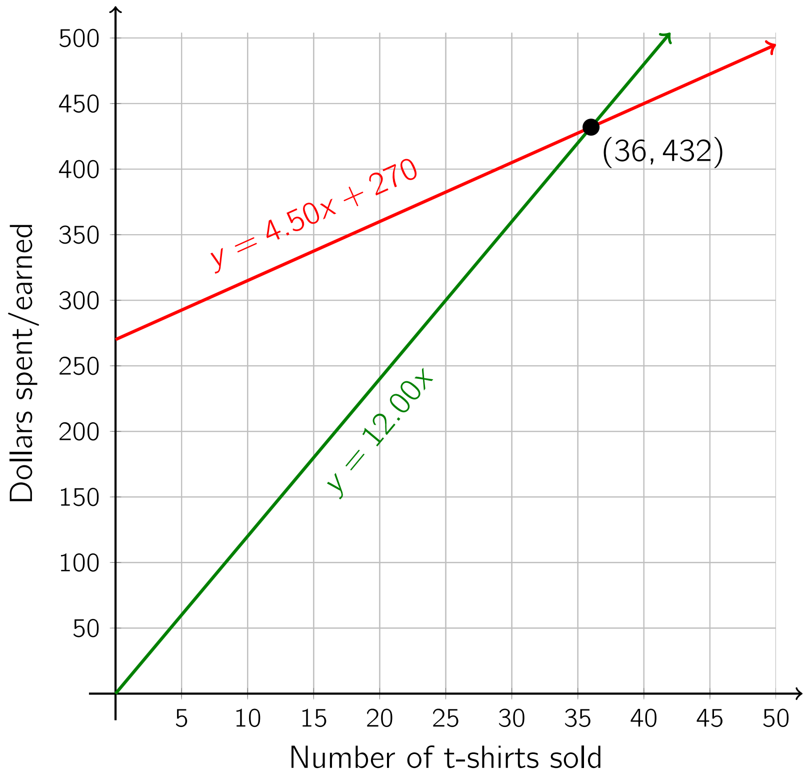


Figure 1.1: Graph showing both Yeardleigh's cost and income equations.

We have found Yeardleigh's *break-even point*, a very important idea in economics and business. It is the point where a company's costs and income are the same. Before this point, the company will be losing money. After this point, the company will start to make a profit. In the context of our problem, Yeardleigh will make a profit on t-shirt sales only if she sells more than 36 shirts.

1.1.2 Writing a System of Equations

When we have two (or more) equations that we want to treat as a system, we usually write them stacked vertically, and we very often add a big curly brace (usually on the left-hand side only) to show that they are meant to be considered a group. In the case of the startup exploration, we write:

$$\begin{cases} y = 12.00x \\ y = 4.50x + 270 \end{cases}$$

1.1.3 Solving Systems of Linear Equations

Solution to a System of Equations

The set of all points that are common to all equations in the system. Graphically, these are the points where all of the graphs of the functions intersect. Note: a system of equations may have no solution.

The solution to a system of linear equations is the intersection point of the graphs of those functions. Can a system of equations have more than one solution?

Imagine that we have a system of two linear equations. Try to visualize the types of solutions we might get. What are the different ways that two straight lines in the plane can intersect?

- The lines might intersect at a single point. Such a system has a unique solution.
- The lines might be parallel and not intersect at all. In this case, the system has no solution.
- The lines might overlap completely (that is, the "two lines" might actually be the same line). In this case, the system has infinitely many solutions: every point one line is also on the other line — they're the same line!

Since we have straight lines, these are the only possibilities. For example, a system of linear equations cannot have "exactly two soltuions". (Can you explain why not?) Later (in algebra 2, for instance), systems will get more complex. Then, we won't have to limit ourselves to linear equations! A system of two quadratic equations, for example, might have no solution, one solution, two solutions, or infinitely many solutions. (Can you picture how each of those might happen? Look back at some of the graphs we drew in **??** and **??**.)

1.1.4 Checking a Solution to a System

Suppose we have a point that we think might be a solution to a system of equations. How can we check?

Example 1.1

Determine whether or not the point $(-1, 5)$ is a solution to each of the given systems.

$$\text{System A: } \begin{cases} x + y = 4 \\ x = -1 \end{cases} \qquad \text{System B: } \begin{cases} y = -x + 4 \\ y = -\frac{1}{5}x \end{cases}$$

To check whether the point is a solution to System A, we'll sustitute x and y into both equations. If the point makes both equations true, then the point is a solution to the system.

Checking System A, we have:

$$\begin{aligned} x + y &= 4 & \text{and} \qquad x &= -1 \\ (-1) + 5 &\stackrel{?}{=} 4 & -1 &\stackrel{\checkmark}{=} -1 \\ 4 &\stackrel{\checkmark}{=} 4 \end{aligned}$$

The given point makes both of the equations true, so yes, the point $(-1, 5)$ is a solution to System A.

Checking System B, we have:

$$\begin{aligned} &y = -x + 4 & \text{and} \qquad &y = -\tfrac{1}{5}x \\ &5 \stackrel{?}{=} -(-1) + 4 & &5 \stackrel{?}{=} -\tfrac{1}{5}(-1) \\ &5 \stackrel{\checkmark}{=} 5 & &5 \neq \tfrac{1}{5} \end{aligned}$$

The given point works for the first equation, but not for the second. To be a solution, the point has to satisfy both equations, so no, the point $(-1, 5)$ is not a solution to System B.

1.1.5 Writing the Solution to a System

NEED SOME VISUALS HERE...?

When we were solving equations — meaning one equation at a time — we use set notation to record our solutions. Since the solution to a system involves two numbers, we have to be mindful about how we write our answers.

Two cases are pretty straightforward. If the system has a unique point as its solution, we can write that point as the solution set. For example, the solution to System A in the previous example is $\mathcal{S} = \{(-1, 5)\}$. Note that we've just enclosed the point $(-1, 5)$ inside the curly braces to show that the set includes that point.

The second easy case is when a system has no solution. Then, we can use the same approach for when a single equation has no solution: $\mathcal{S} = \{\ \}$ or $\mathcal{S} = \emptyset$.

The trickier case is when two lines overlap completely. We can't say that the solutions in the case are "all real numbers" (as we said for a single equation).

1.1.6 Techniques for solving linear systems

Over the next few sections, we will study three different techniques for solving systems of linear equations. A fourth technique, which we only mention here, will be an important topic in algebra 2.

1. **Graphing the system.** To solve a system by graphing, we graph the equations on the same set of coordinate axes and look for the intersection point. If we are lucky, we'll get "nice" points with integers for their coordinates. Graphing with technology works a little better, since trace functionality can give us good estimates for the coordinates if they are not integers.

2. **Collapsing the system by substitution.** To collapse a system is to do something algebraically that fuses two equations into one. If done correctly, we can turn two equations in two variables into one equation in one variable! One method uses the property of substitution to achieve this.

3. **Collapsing the system by elimination.** This is an alternative approach to collapsing a system, that is based on a mathematical structure called a matrix (although we won't go into detail about matrix operations.)

4. **Modeling the system as a matrix.** A *matrix* (plural: matrices) is a rectangular arrangment of numbers. Matrix operations are good for solving systems with more than two variables, which we won't see in algebra 1. We will learn much more about matrices (in general) in algebra 2, along with a number of ways of using them to solve systems.

The different approaches have pros and cons, and you may find that you prefer a particular approach. Generally speaking, you should solve problems using whatever techniques make the most sense to you, and some assignments will be open and allow you to choose your own method. Other assignments may ask you to use a specific technique, though, so its important to read directions carefully.

1.2 Solving Linear Systems By Graphing

We solved the startup exploration at the start of *section* 1.1 by graphing two equations on the same set of axes. That's the idea behind this solution technique.

Graphing is easy with the right technology.

Graphing on the calculator is actually a good way to check your answer, not the only one though.

However, you might not be able to get the exact answer if it is fractional. Don't become dependent on graphing. Some are actually solved easier and more accurately algebraically. Also, if you are prone to type things in wrong, you may have a problem. Scale can also be an issue. I'm not going to solve Yeardleigh's t-shirt problem via graphing.

Things to be careful of: a. Be careful if you have to convert the equation from one form into another, like standard into slope-intercept b. Don't rely on "trace" if it appears that the answer is fractional. Technique: If you are given a word problem, write it as a system of equations. Then, graph the equations to find any intersection points. You can either graph by hand or with a graphing calculator. Your work is the graph. The answer is the intersection point!

a. Graphing By Hand: Use graph paper. You can't use the short-cut way of graphing standard form either. The points will be "nice" if you are asked to graph by hand. Remember that the graph is not the answer. It is the work. This means that it doesn't need to be a high quality graph. Your answer is the intersection point.

b. Graphing By Calculator: They have to be in function form for the graphing calculators that we have in class. The "trace" function is pretty useless as I expect exact answers. You should look at the table to find exact answers. On the table you will notice that the solution will show up as 1 x-value with the same y-value for the two equations.

Example 2 Problem: Solve these systems by graphing. If a system has infinite solutions, give one point that will satisfy the system. y=1x 3+ (a) ïčť 3 (b) y=x y=5x (c) y=1 y=x (d) 2x+y=3 (e) 3xâĹŠy=7 xâĹŠ2y=4 y=3x 7âĹŠ ïčťy=1x 3 âĹŠ ïčť 3

Solution: If you graph with the calculator, the equation has to be in function notation (either slope-intercept or point-slope). IF you do use a calculator, you must sketch a graph to show your work. If you do not use a calculator, you have to make sure to graph very precisely, using graph paper. If I ask you to solve with graphing, I will try to make sure your solutions are integers. If not, approximate them. (a) S = Null Set (lines are parallel) (b) S = {(0,0)} (c) S = {(1,1)} (d) S = {(2, -1)} (e) Any point on the line y = 3x-7, examples (0, -7); (1, -4) in set notation, the solution looks like S=∀(x,y)∈ 3xâĹŠ y= 7 . {}

Please remember, I want you to think about the problem and map out your strategy before you jump right into the work. If you can determine, upon inspection, that lines are parallel, or if you transform some of the equations and notice that they are parallel or the same line, you don't have to do all of the work of graphing the lines (or in the future substitution or elimination). Just write the solution set and a little statement like "upon inspection, the lines are parallel" or "the same line".

1.2.1 Finding an intersection point using a calculator

When finding an intersection point using a calculator, we can use the `trace` function on the calculator. But, `trace` can only give us a decimal approximation, and this could be a problem for some solutions. (For example, a solution that is not whole numbers or a fraction with "convenient" denominator.)

Using `trace` is good for approximating a solution, but then we can use an alternative way: looking at the table. Here, we can search the table for a place where `y1` and `y2` are the same for a particular value of `x`.

For the t-shirt example, go to the `table` screen on your calculator, and scroll to where `x` is 36. You should see that both `y1` and `y2` are 432. This will be how the intersection point will show up in a table.

1.3 Solving Linear Systems By Substitution

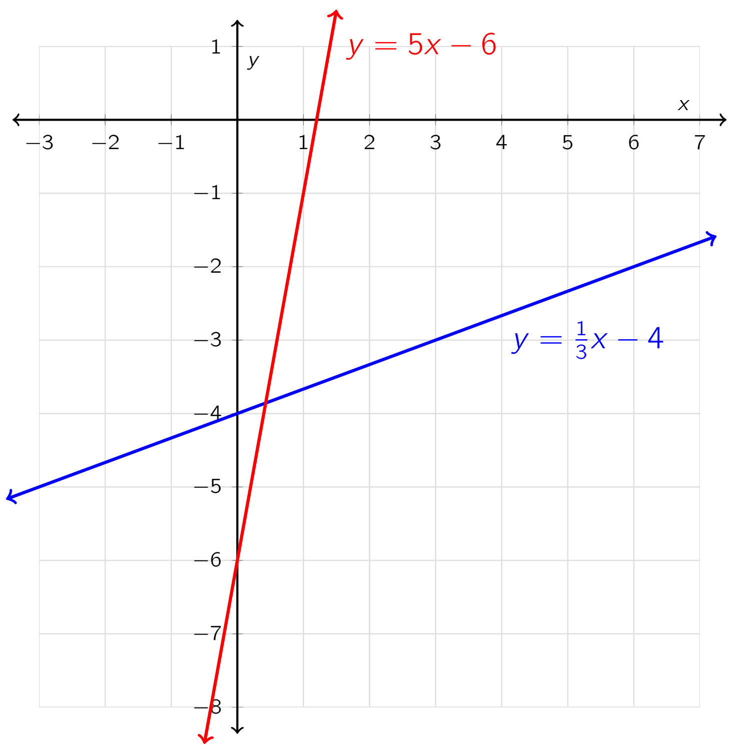
Startup Exploration: Ugly Graphing

Solve the following system by graphing.

$$\begin{cases} y = \frac{1}{3}x - 4 \\ y = 5x - 6 \end{cases}$$

What are the challenges that arise when solving this system by graphing?

The challenge we uncover in the startup exploration is that the intersection point does not have integer coordinates. The x-coordinate is between 0 and 1, and the y-coordinate is between -3 and -4, but beyond that the going is pretty rough even using graphing technology (since the decimal values of the intersection point are not all that pretty).



In this section we will explore the first of two algebraic methods for solving systems of equations. These algebraic approahes will avoid the challenges demonstrated by the startup exploration. The first algebraic technique is the *substitution method*.

We've been substituting numbers for variables for a long time. For example, given $f(x) = 3x + 1$, we know how to find $f(5)$ by substituting 5 for x in the equation:

$$f(x) = 3x + 1 \qquad \Longrightarrow \qquad f(5) = 3(5) + 1 = 15 + 1 = 16$$

The clever idea behind solving a system using substitution is that rather than just replacing variables with letters, we can substitute one algebraic expression for another.

Substitution Method

A method for solving a system of equations that involves solving one of the equations for one variable and substituting the resulting expression into a different equation.

Example 1.2

Solve the following system using substitution.

$$\begin{cases} y = 3x \\ y = 2x - 4 \end{cases}$$

Solution: We are looking for a point where the two equations have the same x value and the same y value. So, let's assume that the y's in these two equations are equal. If the two y's are equal, then it means "$3x$" is equal to "$2x - 4$" and that is a Level 4 linear equation (one with variables on both sides)!

$$\begin{aligned} 3x &= 2x - 4 && \text{substitution} \\ x &= -4 && \text{SPOE: subtract } 2x \text{ from both sides} \end{aligned}$$

So, if the two y values are equal, as we assumed they were, then it must be that $x = -4$. We just found the x-coordinate of the intersection point! How can we find the y-coordinate?

Notice that the first equation says $y = 3x$. So, if $x = -4$, it must be that $y = 3x = 3(-4) = -12$. So, the solution to the system is the point $(-4, -12)$. Or we can write $\mathcal{S} = \{(-4, -12)\}$.

To check out work, we can either graph the system, or plug the x and y values into the other equation to verify that equality holds.

$$\begin{aligned} y &= 2x - 4 && \text{the second equation we were given} \\ -12 &\stackrel{?}{=} 2(-4) - 4 && \text{substitute our proposed solution} \\ -12 &\stackrel{?}{=} -8 - 4 && \text{simplify right-hand side} \\ -12 &\stackrel{\checkmark}{=} -12 && \end{aligned}$$

When we assume that the two y's were equal — as they must be if we have found a solution to the system — then substitution means that we can replace one y with the other.

We can carry this out for the problem in the startup exploration. Since we have two equations in $y =$ format, and since any point that satisfies the system has a y-coordinate that satisfies both equations, we can set the equations equal to one another:

$$\begin{aligned} \frac{1}{3}x - 4 &= 5x - 6 && \text{substitution} \\ 2 &= \frac{14}{3}x && \text{SPOE: variable terms to the right, constants to the left} \\ \frac{6}{14} &= x && \text{MPOE} \\ \frac{3}{7} &= x && \text{simplify the fraction} \end{aligned}$$

Aha! Even though this x-value is inconvenient to graph by hand (since it's not an integer) or using technology (since it's not a very tidy decimal)... it emerges with no trouble using substitution.

To find the y-coordinate, we use our newly-discovered x-coordinate and one of the two original equations. Why not pick the second equation to minimize the fractions?

$$\begin{aligned} y &= 5x - 6 && \text{the second equation} \\ y &= 5\left(\frac{3}{7}\right) - 6 && \text{substitute in our new } x\text{-value} \\ y &= \frac{15}{7} - 6 && \text{multiply fractions} \\ y &= \frac{15}{7} - \frac{42}{7} && \text{write with a common denominator} \\ y &= \frac{-27}{7} && \text{simplify the fraction} \end{aligned}$$

To check our work, we can substitute our candidate values for x and y back into the first equation. Note that since we used the second equation already to figure out the answer, we use the first equation to check it.

$$\begin{aligned} y &= \frac{1}{3}x - 4 && \text{the first equation} \\ \frac{-27}{7} &\stackrel{?}{=} \frac{1}{3}\left(\frac{3}{7}\right) - 4 && \text{substitute in our candidate solution} \\ \frac{-27}{7} &\stackrel{?}{=} \frac{3}{21} - 4 && \text{multiply fractions} \\ \frac{-27}{7} &\stackrel{?}{=} \frac{3}{21} - \frac{84}{21} && \text{write with a common denominator} \\ \frac{-27}{7} &\stackrel{?}{=} \frac{-81}{21} && \text{subtract fractions} \\ \frac{-27}{7} &\stackrel{\checkmark}{=} \frac{-27}{7} && \text{simplify the right-hand side} \end{aligned}$$

So, we have found our solution $\mathcal{S} = \left\{\left(\frac{3}{7}, -\frac{27}{7}\right)\right\}$.

In the two examples we have seen so far, both equations have been in $y =$ format. This is not a requirement for using substitution, as the following two examples will demonstrate.

Example 1.3

Solve the following system using substitution.

$$\begin{cases} y = 3x - 5 \\ x - y = 4 \end{cases}$$

Solution: Here, we have one equation in point-intercept form, and one equation in standard form. No problem! We simply substitute y from the first equation for y in the second equation. It's a good idea to use patentheses when doing the substitution: watch what happens with the negative signs in this example.

$$\begin{aligned} x - y &= 4 && \text{start with the second equation} \\ x - (3x - 5) &= 4 && \text{substitue } y \text{ from first equation, using parentheses} \\ x - 3x + 5 &= 4 && \text{distributive property} \\ -2x + 5 &= 4 && \text{combine like terms} \\ -2x &= -1 && \text{SPOE} \\ x &= \frac{1}{2} && \text{DPOE} \end{aligned}$$

This tells us that if the y's are equal, then it must be that $x = \frac{1}{2}$. To find the y-coordinate, we substitute the x value we just found back into one of the original equations. Let's choose the second equation, which seems a bit easier:

$$\begin{aligned} x - y &= 4 && \text{the second equation} \\ \frac{1}{2} - y &= 4 && \text{substitue } x = \tfrac{1}{2}\text{, which we just computed} \\ -y &= \frac{7}{2} && \text{SPOE} \\ y &= -\frac{7}{2} && \text{MPOE} \end{aligned}$$

So, $\mathcal{S} = \left\{\left(\frac{1}{2}, -\frac{7}{2}\right)\right\}$. To check we can plug these values back into the other equation (the first equation, in this case, since we used the second equation to find y).

$$\begin{aligned}
y &= 3x - 5 && \text{the first equation} \\
-\frac{7}{2} &\stackrel{?}{=} 3\left(\frac{1}{2}\right) - 5 && \text{substitue our proposed solution} \\
-\frac{7}{2} &\stackrel{?}{=} \frac{3}{2} - 5 && \text{simplify on the right-hand side} \\
-\frac{7}{2} &\stackrel{?}{=} \frac{3}{2} - \frac{10}{2} \\
-\frac{7}{2} &\stackrel{\checkmark}{=} \frac{-7}{2}
\end{aligned}$$

Notice that after our substitution step, we've been getting linear equations like the ones we saw in **??**.

In the first example in this section, we got an equation with the variable x on both sides. Recall that when equations have variables on both sides, we sometimes find that the equation had one unique solution, sometimes "no solution", and sometimes "infinitely many solutions". Do you see how these correspond to the different possible solutions to a linear system?

The key to the substitution method is that we need at least one equation to be in $x =$ or $y =$ form. What if neither equation is in this form? We can use our skills at transforming formulas to create what we want!

Example 1.4

Solve the system by substitution.

$$\begin{cases} 3x - 12y = 15 \\ x - 4y = 7 \end{cases}$$

Solution: We could transform either equation, but notice that there's less work to do in the second equation, since the x term has coefficient 1. So, let's transform the second equation to $x = 7 + 4y$ using APOE. Then, we can substitute that expression into the first equation.

$$\begin{aligned}
3x - 12y &= 15 && \text{the first equation} \\
3(7 + 4y) - 12y &= 15 && \text{substitute our expression for } x \\
21 + 12y - 12y &= 15 && \text{distributive property on the left-hand side} \\
21 &= 15 && \text{D'oh.}
\end{aligned}$$

We have arrived in one of those impossible situations that tells us the equation we created has no solution. In this case, it means that the system has no solution. In other words, that these lines were parallel all along. $\mathcal{S} = \emptyset$

That was the first "special case". The other special case is when the two lines completely overlap. What do you suppose will happen when we solve the system that lets us know that we're in this special case?

1.4 Solving Linear Systems By Elimination

Startup Exploration: Ugly Substitution

Study the following system of linear equations.

$$\begin{cases} 4x + 3y = 22 \\ 3x + 5y = 11 \end{cases}$$

What challenges would you face if you tried to solve this system by graphing? What challenges would you face if you tried to solve it by substitution?

The equations in the startup exploration would require quite a bit of work to graph them accurately by hand (even using our x- and y-intercept strategy for graphing lines in standard form), and we'd have to do some work so that one could be substituted into the other. The second algebraic approach to solving a system, called the **elimination method**, addresses these concerns.

This technique is particularly helpful when the given equations are both in standard form. The premise behind elimination is that we use the properties of equality to, say, add two equations together. Recall that the addition property of equality says that we can add the same quantity to both sides of an equation. If we know two things are equal, we can add one thing to one side of an equation and the other thing to the other side, and still maintian equality.

Example 1.5

Solve the following system using elimination.

$$\begin{cases} y = 3 \\ 5x - y = 12 \end{cases}$$

Solution: The first equation tells us that y is equal to 3. Of course, we could add y to both sides of an equation. . . or we could add 3 to both sides of an equation. . . but since we know y and 3 are equal we can add y to one side of an equation and 3 to the other side of that equation!

In other words, we can add y to the left-hand side of the second equation we are given while simultaneously add 3 to the right-hand side. Since we know $y = 3$, we know equality will be maintained! Watch what happens:

$$\begin{aligned} 5x - y &= 12 && \text{the second equation} \\ 5x - y + y &= 12 + 3 && \text{add on the first equation} \\ 5x &= 15 && \text{combine like terms: } y \text{ has been eliminated!} \\ x &= 3 && \text{DPOE} \end{aligned}$$

Having found x, and having y given in the original system, we have $\mathcal{S} = \{(3, 3)\}$.

Elimination Method

A method for solving a system of equations that involves adding two equations to eliminate a variable.

The method is called "elimination" because it eliminates a variable.[1]

Example 1.6

Solve the following system by elimination.

$$\begin{cases} 2x + 2y = 7 \\ 3x - 2y = 8 \end{cases}$$

Solution: Notice how the coefficients on the y terms are opposites. If we add these two equations together, the y's will be eliminated:

$$\begin{array}{rr} & 2x + 2y = 7 \\ + & 3x - 2y = 8 \\ \hline & 5x + 0y = 15 \\ & 5x = 15 \\ & x = 3 \end{array}$$

We then substitute $x = 3$ back into one of the equations to find $y = \frac{1}{2}$. So, $\mathcal{S} = \left\{\left(3, \frac{1}{2}\right)\right\}$.

The key to the elimination approach is to "add the equations together" so that one variable vanishes, just as the y disappeared on the third in the previous example. In order to faciliatate this elimination, one or both of the equations must sometimes be multiplied by a constant before we add them (an application of MPOE).

[1] This name is also related to a technique from the study of matrices called *Gaussian elimination*.

This approach is required to handle the startup exploration. Recall, that in that problem we were asked to solve the following system by elimination.

$$\begin{cases} 4x + 3y = 22 \\ 3x + 5y = 11 \end{cases}$$

Adding these two equations won't eliminate any variables, as we have encountered in the earlier examples. But, with a few clever applicatons on MPOE, we can put ourselves into exactly the situation that we desire.

Let's make it our goal to eliminate x. What could we do to these equations so that we have equivalent equations but ones in which the x terms have opposite coefficients?

One way (but not the only way) to accomplish this is to multiply the first equation by 3 and multiply the second equation by -4.

$$\begin{array}{lcl} 4x + 3y = 22 & \xrightarrow{\text{multiply through by 3}} & 12x + 9y = 66 \\ 3x + 5y = 11 & \xrightarrow{\text{multiply through by -4}} & -12x - 20y = -44 \end{array}$$

Then, we add equations:

$$\begin{array}{rr} & 12x + 9y = 66 \\ + & -12x - 20y = -44 \\ \hline & -11y = 22 \\ & y = -2 \end{array}$$

Substituting $y = -2$ into the second of the original equations, we get $3x + 5(-2) = 11$. This implies that $x = 7$. So, $\mathcal{S} = \{(7, -2)\}$.

1.5 Applications of Systems

[TODO] Systems: The word problems in this chapter should all connect into one story, IMHO... perhaps advancing the overall story.

"A train leaves New York traveling south at 180 miles per hour. . . " If you were to read that sentence out loud in a room full of adults, you might cause some folks to start sweating and mumbling nervously to themselves. People all over the world, for generations, have been scarred by algebra word problems.

But, word problems have a bad reputation.

Yes, word problems require us to ponder a bit, decipher the information we're given, and stumble around as we figure out how to proceed. But this is the process of problem solving! The algebra skills we have learned so far, along with a little determination and creativity, will serve us well as we tackle these algebra classics.

We discuss several kinds of problems below. Our general pattern is this: We'll think about the unknown quantities and relevant facts given in the problem. We'll choose variables to represent the unknowns and model the situation using equations. (Drawing pictures and making charts might help!) Then, we'll solve the equations and check to ensure your answer makes sense.

As we go along, and as you work on various assignments, remember to think back over these examples. If you have solved a similar problem before, that will give you a place to start when faced with something new.

Two unknowns, two facts

The first examples we did with substitution and elimination are of this type. We are given two unknowns and two different facts about them. The facts usually have different units, which hint at what two equations you need to write for your model.

One way to approach this type of problem is by creating a table to organize the data.

Evil Entertainment. . . Eviltainment?

Yeardleigh receives a shipment of entertainment centers, the basic model weighing 30 kilograms each and the deluxe weighing 50 kilograms each, has a total weight of 880 kilograms. If there are 20 centers altogether, how many weigh 50 kilograms?

The two unknowns are the number of 30kg and 50kg centers. One set of facts has to do with quantity, the other has to do with the weight. We make these into which are rows and columns in the table. I added "total" columns and rows. Let's use T to represent the number of 30kg centers, and F to represent the number of 50kg centers.

	Number of centers	Weight per unit	Total Weight
30kg centers	T	30	$30T$
50kg centers	F	50	$50F$
Combined	20	–	880

Once the table is set up, out two equations are there in the second and fourth columns.

$$\begin{cases} T + F = 20 \\ 30T + 50F = 880 \end{cases}$$

We can now solve this system using our favorite method. It looks like a good candidate for elimination! (Why?)

Mixture problems

The type of mixture problem we have previously solved were very straight forward. Every number we needed was given to us directly in the problem. This is a different type of problem. There is information in the problem that is not directly stated that we will have to infer. This is where the table is really helpful.

Evil Trail Mix

YeardleighCorp opens health food stores, which sell an Evil Trail Mix of raisins and roasted nuts. Raisins sell for \$3.50 per kg, and roasted nuts sell for \$4.75 per kg. How many kg of each should be mixed to make 20 kg of Evil Trail Mix worth \$4.00 per kg?

Our unknowns are number of kg of raisins (let's call that R) and number of kg of nuts (let's call that N). We also have the prices of each ingredient per kg, but nothing about a total cost.

	Amount (kg)	Unit price	Total price
Raisins	R	3.50	$3.50R$
Nuts	N	4.75	$4.75N$
Mixture	20	4.00	??

How can we figure out total cost? Well, look at that row of the table: 20 kg of Evil Trail Mix at \$4.00 per kg = \$80 worth of trail mix! We can calculate this value, even though that information is not given directly in the problem. Sneaky!

$$\begin{cases} R + N = 20 \\ 3.50R + 4.75N = 80 \end{cases}$$

And now, we can solve the system. Once again, this looks like a job for elimination. (Why?)

RTD: Same direction

Rate-time-distance problems (RTD for short) come in a variety of flavors. In "same direction" RTD problems, the travelers are – surprise! – going in the same direction. The often involve a "race" of some kind where someone gets a head start, and our task is to determine when someone catches up or passes the other.

It will be helpful to draw a simple picture at the start to understand how the problem is set up. Many times, there is something not directly stated that is important to the solution, but which you can figure out from the situation. The picture can help you with that.

Same Direction

At 5:00 pm Bob leaves the Chtulhu-Chip factory driving 25 miles per hour. At 5:30, Yeardleigh leaves the Cthulhu-Chip factory and follows the same route, driving 40 miles per hour. At what time does Yeardleigh pass Bob?

The problem is asks when Yeardleigh will pass Bob. In order to know when she will pass, we need to know when they have gone exactly the same amount of distance from the factory (this is the moment she passes him). So we'll write two distance equations, one for Bob and one for Yeardleigh, and compare them to try and figure out at what time their distances from the factories are equal.

[TODO] Systems: Some pictures of the word problems would be helpful.

Our table columns are – naturally enough – rate, time, and distance. The rows deal with the people driving. The problem asks a "when" question, so we must be solving for time. Let's use t to represent the amount of time Bob has been driving, in hours.

Note that we didn't make the variable represent actual time "on the clock". (Why not? What challenges would that introduce into the mathematics?)

Yeardleigh started 30 minutes later, which means she has 30 minutes less travel time than Bob. Note that we have to be careful about units: 30 minutes is 1/2 of an hour. Since the speeds are in miles *per hour*, it makes sense to work in hours: $t - \frac{1}{2}$. If we want to use $t30$ instead, we can... but we'll have to also convert the given speeds into miles *per minute*.

	Rate (mph)	Time (hours)	Distance (miles)
Bob	25	t	$25t$
Yeardleigh	40	$t - \frac{1}{2}$	$40\left(t - \frac{1}{2}\right)$

$$\begin{cases} d = 25t \\ d = 40\left(t - \dfrac{1}{2}\right) \end{cases}$$

Solving this system (substitution looks like a good choice!) will give us a value for t. What does that value represent? How can we use it to determine when (on the clock) Yeardleigh will pas Bob?

RTD: Opposite directions

Here we might have travelers starting from the same location and traveling away from one another (in which case you usually are asked to determine when they will be a certain distance apart). Or, we might have travelers who are coming from two distinct locations and moving towards each other (in which we are usually asked to figure out when or where they will meet).

A common mistake with this second type is assuming that the travelers will meet at exactly the half-way point. They don't necessarily. It depends on when they leave and speed at which they traveling!

Opposite Directions

Yeardleigh hamster-naps Feta and holds her for ransom. The twins agree to meet in a neutral location. Bob and Yeardleigh each set out at noon from points 60 km apart and drive toward each other, meeting at 1:30pm. Bob's speed was 4 km/h greater than Yeardleigh's. How fast was each of them driving?

We could choose either Bob's speed or Yeardleigh's speed as the reference point. Let's use Yeardleigh's speed and call it v (for velocity). Then, Bob's speed is 4 km/h greater than that.

	Rate (mph)	Time (hours)	Distance (miles)
Yeardleigh	v	1.5	$1.5v$
Bob	$v+4$	1.5	$1.5(v+4)$

Now, to figure out how to make that table into some helpful equations. Since we have one variable, we only need one equation. The missing key component is that the distance Bob and Yeardleigh traveled together is 60 km, which gives us the extra piece of the puzzle: The two distances add up to 60 km:

$$1.5v + 1.5(v+4) = 60$$

RTD: Roundtrip

"Round trip" means that someone travels from a given location to some other location and back again, as in the following example.

Round Trip

Yeardleigh uses the ransom money from hamster-napping Feta to go on an evil ski trip. An evil ski lift carries Yeardleigh up the slope at the rate of 6 km/h, and then she skies back down the same slope at 34 km/h. The round trip takes 30 minutes. How long is the ski slope, and much time does it take Yeardlight to ski down it?

Let's let t represent the amount of time that Yeardleigh spends skiing. The whole trip takes 30 minutes (that's $\frac{1}{2}$ hour, remember to watch out for the units!) so her time on the chair lift is $\frac{1}{2} - t$. Here we chop the total time into two pieces using one variable, just like we did with distance in the last problem.

	Rate (km/h)	Time (hours)	Distance (km)
On the ski lift	6	$\frac{1}{2} - t$	$6\left(\frac{1}{2} - t\right)$
Skiing	34	t	$34t$

In a round trip problem, the traveler goes over the same path twice. So, the distance that Yeardleigh travels on the ski lift, is the same as when she is skiing. We don't know what that distance is (yet!) but we know the two distances in our table above are the same. So, our equation is:

$$6\left(\frac{1}{2} - t\right) = 34t$$

Solve this, and we'll get the amount of time she was skiing. Then, we can use this time, together with her skiing speed, to figure out the distance traveled.

Uniform motion in a current

Don't let that complicated-sounding name fool you, problems of this type are not that complicated to solve.

Have you ever tried to walk *up* the *down* escalator at the mall? (Be honest!) When you walk down the down elevator – that is, when you walk in the same direction that the escalator is moving – the motion of the escalator helps you to go faster than if you were just walking down regular stairs. On the other hand, if you try walk in the wrong direction on an escalator, the motion of the escalator will make you go slower than you would walk on regular stairs.

This is the idea of motion in a current. When you go "with the current" (or "downstream", if you're floating on a river) the current adds to your usual rate of motion. Going "against the current" (or "upstream"), the current reduces your usual rate of motion.

Motion in a Current

Bob decides to go on vacation. He takes a day trip on a river boat. The boat travels 60 km upstream (against the current) in 5 hours. The boat travels the same distance downstream in 3 hours. What is the rate of the boat in still water? What is the rate of the river's current?

Notive that Bob makes a round trip journey. So, the distance upstream is the same as the distance downstream. We don't know the speed of the water or the speed of the boat. So, let's let b represent the speed of the boat, and c represent the speed of the current.

The boat goes $b - c$ km/hour upstream, because the current is taking away from how fast the boat can go. The boat will go $b + c$ km/hour downstream, because the current will help the boat go faster!

	Rate (km/h)	Time (hours)	Distance (km)
Upstream	$b - c$	5	60
Downstream	$b + c$	3	60

Our system is:

$$\begin{cases} 5(b-c) = 60 \\ 3(b+c) = 60 \end{cases}$$

To solve this, we'll need the distributive property and the elimination method!

Work problems

In a work problem, a group of workers split up a job. The key is knowing that everyone does a fraction of the job, and that all together they complete one whole job. We will use an equation very similar to DRT:

$$\text{work rate } \times time = \text{ work done.}$$

The "work rate" we will think of "number of jobs per unit of time". If it takes me 3 days to paint an apartment, my work rate for painting is "one third of an apartment per day".

Take a whack at this example:

Work

Ivan can chop a cord of wood* in 4 days, and his father, François, can chop a cord of wood in 2 days. How long will it take them to split a cord of wood if they work together?

(One cord, which is a unit of measure of volume for firewood in the US and Canada, is about 128 cubic feet. . . but that's not really important to this problem.)

If we think of the "job" as "chopping one cord of wood", then Ivan's work rate is "1 job in 4 days", or "$\frac{1}{4}$ job per day". Similarly, François's work rate is "$\frac{1}{2}$ job per day". When the two guys work together, they will finish the job in some amount of time. Let's call that t. The rows in our table are for the individual workers.

	Work rate	Work time	Amount done
Ivan	$\frac{1}{4}$	t	$\frac{t}{4}$
François	$\frac{1}{2}$	t	$\frac{t}{2}$

Together, they complete 1 whole job (they chop one whole cord of wood). This gives us the equation:

$$\frac{t}{4} + \frac{t}{2} = 1$$

[TODO] Systems: Some conclusion might be good?

Still need to find a quote that works for this chapter. In the meantime, we have this.

Author
Description of author

Chapter 2

Inequalities

Every chapter should have a lead paragraph – even just a short one – that appears before the first heading. This is a placeholder paragraph which will at some point be replaces by actual content.

2.1 Equations Versus Inequalities

Startup Exploration: Honest Senators

Imagine that the following statements are released from a government watchdog group: "(1) There exists at least one honest senator. (2) Given any two senators, at least one of them is dishonest."

If both of these statements are true, what can we say about our 100 senators? Determine the number of honest senators or, explain why we don't have enough information to determine exactly how many are honest.

An equation states that two expressions are equal and thus includes an equal sign "=". When we solve an equation (by using one of the properties to simplify it), we create a series of equivalent equations until we can determine the solution set. Recall that the solution set is the collection of all the numbers that make the original equation "true".

Inequalities express an unequal relationship and thus include a mathematical symbol of inequality. Rather than "x is equal to 5" we might have "x is less than 5". To solve an inequality is to find all of the values that make it true. But whereas an equation may have only one solution, there could be many values that make an inequality true. Perhaps infinitely many! All of these are part of the solution set.

2.1.1 Inequality Symbols

Equations all include the equal sign, but we have more options for expressing inequality:

$\neq$	not equal to
$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to

Using these symbols we can write true and false statements. The statement $4 < 5$ is true statement, and the statement $3 > 10$ is false.[1] Note that the statement $12 < 12$ (12 is less than 12) is false. In contrast, the statement $12 \leq 12$ (12 is less than or equal to 12) is true, because the "or equal to".

The math sentence x < 5 is an inequality that has infinitely many solutions. For example, we can replace x with 4, π, 3, 2, 1.889, 1, $\frac{1}{2}$, 0, $-6\ldots$ in fact, every number x that is less than the "endpoint" 5 is included in the solution set! But again, 5 itself is not included in the solution set for this inequality.

Because of their behavior around the endpoints, the symbols $<$ and $>$ are said to be "exclusive inequalities", meaning they do not include the endpoint as a solution: $5 < 5$ is a false statement. These are often described as "strictly less than" or "strictly greater than". In contrast, the symbols $\leq$ and $\geq$ are said to be "inclusive" because they do include the end point as a solution: $5 \leq 5$ is a true statement.[2]

[1] Remember your elementary school days: the symbols are like hungry alligators who open their mouths to eat the larger number. Nom, nom, nom.

[2] Another way to describe these "or equal to" inequalities inequalities is using the terms "at most" or describe "less than or equal to", and "at least" to describe "greater than or equal to". For example, if I ask you to give me "at least 5 dollars", then I'm asking you for "greater than or equal to 5 dollars".

2.2 One-variable Inequalities

Startup Exploration: Wire Triangle

A nine-inch piece of wire is bent at two points such that its ends come together to form a triangle. If the bending points must be on the inch marks, how many possible choices of bending points are there?

Consider the equation $3x = 12$. This equation is true when x has the value 4 and we say the solution set $\mathcal{S} = \{4\}$. Now consider the inequality $3x \leq 12$. What values of x make this inequality true? The solution set will be the set of all of those values.

This means that the solution set contains "all real numbers less than or equal to 4" (notice that 4 is included in the solution set). To write this in set notation, we intorduce a few new symbols:

$$\mathcal{S} = \{x \in \mathbb{R} \mid x \leq 4\}$$

The symbols in this set notation can all be translated into regular English. The symbol $\in$, which looks like a little stylized 'e', means "is an element of". So, the phrase $x \in \mathbb{R}$ means "x is an element of $\mathbb{R}$", or in other words: "x is a real number".

The little vertical line has a meaning, too! We can think of the bar as saying "such that" of "for which". So what we have is a math sentence that reads "$\mathcal{S}$ is the set of real numbers x such that x is less than or equal to 4", or a bit more succinctly "all real numbers less than or equal to 4".

2.2.1 Graphing One-variable Inequalities

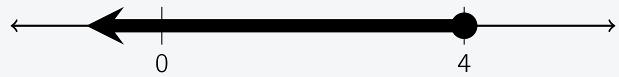
Instead of writing the set notation, another way to show the solution to a one-variable inequality is to draw a graph of the solution. These graphs aren't like the two-dimensional graphs of linear functions. Since we have just one variable, we'll have a one-dimensional graph! Rather than graphing on the coordinate plane, we draw these graphs on a number line.

These graphs may seem pretty basic, but stay tuned! In algebra 2 we will solve "combined" and "polynomial" inequalities and we won't be able to figure out the answers without graphs like these!

Example 2.1

Draw a graph of $x \leq 4$.

Solution: First, we'll show the graph, and then we'll explain what decisions we made in drawing it.



We first found the boundary (which is 4) and plotted it on the number line using a filled in dot. Then we drew an arrow (a ray) pointing to the left because we are interested in values that are less than 4, in addition to 4 itself. We also placed the zero on the number line, for reference.

Contrast the previous example with the next example, and you will probably have a good idea about how to graph any one-variable inequality!

Example 2.2

Draw a graph of $x > -2$.

Solution: Here's the graph. What features are different compared to the previous example?

First, we found the boundary (it's -2) and plotted it on the number line, along with 0. The boundary is excluded from the solution set, so we used an hollow circle to show that -2 itself is not part of the solution. Then we used an arrow (a ray) pointing to the right because we are interested in values that are greater than -2.

The following comments may not be necessary, given the previous to examples, but we'll summarize them here for clarity. There are two main things to consider when graphing a one-variable inequality.

Consideration #1: Boundary. What is the greatest or least possible value of the solution, and is that point included in the solution set?

If we have an inclusive inequality ($\leq$, $\geq$, and $=$), then the boundary point is included in the solution set. On the graph, we use a filled in circle or "closed" endpoint to show that the boundary is part of the solution. We call the boundary point an *inclusive boundary*.

If, on the other hand, we have an exclusive inequality ($<$, $>$, and $\neq$), then the boundary point is not included in the solution set. On the graph we draw a hollow or "open" endpoint to show that the boundary is not part of the solution. We call the boundary an **exclusive boundary**.

Consideration #2: Direction. Are the solutions greater than the boundary (stretching to the right, towards positive infinity) or are they less than the bounday (stretching to the left towards negative infinity)? We draw a heavy arrow in that direction.

We always label the boundary point, and also place the zero on the number line for reference.

2.2.2 (;,;) Interval Notation

In addition to set notation and a number line graph, there is a third way to describe an inequality. In many higher level math classes (like algebra 2 and, later, calculus), we use a technique called *interval notation*. With interval notation, we describe the endpoints of a region on the number line.

Inclusive boundaries are denoted with square brackets: [and]. Exclusive boundaries are denoted using round brackets: (or). Lines that extend forever have no endpoint, so we use positive infinity ∞ or negative infinity $-\infty$ to indicate that the set extends forever in the positive (or negative) direction.

In the first example above, we have $x \leq 4$. To write this in interval notation, we write $x \in (-\infty, 4]$. To write the solution to the second example, $x > -2$, in interval notation, we write $x \in (-2, \infty)$.

Try drawing the graph corresponding to $x \in [8, \infty)$? What about $x \in (-\infty, -10)$? What do you suppose the graph of $x \in [-3, 7)$ might look like?

2.3 Solving Inequalities

Startup Exploration: TODO

TODO

When we solve an equation using the POEs, we create a series of equivalent equations until we can determine the solution set. The big question of this section is: Can we do something similar for inequalities? What does it mean to have *equivalent inequalities*?

Equivalent inequalities is not an oxymoron.[3] It just means that two inequality statements have the same solution set! For example, $3x < 12$ and $x < 4$ are equivalent because the exact same values of x will work in both statements. They have the same graph!

But, how can we transform an inequality into an equivalent inequality? Can we use the same properties as we used for solving equations? In essence: are there Properties of Inequalities like there are Properties of Equality (POEs)?

Strictly seaking, an inequality doesn't only state that two numbers are not-equal.[4] An inequality really shows that two numbers appear in a certain *order on the number line*. If a certain inequality states a "less than" relationship, we want to find properties that maintain that.

Since we are looking for rules which "maintain order", we will call these rules the "Properties of Order", or POOs.[5]

2.3.1 Properties of Order (POOs)

So what are the POOs, and do they work like POEs? Yes. . . almost.

Suppose we start with an inequality that we know is true, say, $5 < 6$. What operations can we perform to this inequality that maintain its ordered relationship?

[3] An oxymoron is a figure of speech that brings together two things that appear to be contradictory. For example, the titles of the movie "Night of the Living Dead" and the song "Along Together" contain oxymorons.

[4] Okay, yes, this is exactly what the $\neq$ symbol does. But, we're not going to do much with the $\neq$ symbol. Solving problems with $\neq$ isn't usually very interesting, as we'll see later in this chapter.

[5] We considered using "POI" to stand for "Property of Inequality", but POO is both more mathematically accurate, and a lot funnier.

Could we add (or subtract) the same things on both sides and maintain order? To take just one example, adding 3 to both sides gives $5+3<6+3$, which maintains order since $8<9$. If we now add -8 to both sides we have $0<1$, and order is again maintained.

In fact, adding the same thing to both sides does nothing to change their relative order on the number line. Adding simply translates the original inequality to the right or left on the number line. (Note that translations to the left happen when adding a negative number, which is the same as subtracting!)

Properties of Order: Addition and Subtraction

The **addition property of order** (APOO) states that for all real numbers a, b, and c

$$\text{if } a<b, \text{ then } a+c<b+c$$

The **subtraction property of order** (SPOO) states that for all real numbers a, b, and c

$$\text{if } a<b, \text{ then } a-c<b-c$$

Note! Rather than write out the same sentence with all the different inequality symbols, we simply note here that APOO and SPOO hold for $a\leq b$, $a>b$, and $a\geq b$.

What about multiplication and division? If we take our sample inequality $5<6$ and multiply both sides by 4, we have $5\cdot 4<6\cdot 4$, which maintains order, since $20<24$. If we now divide both sides by 2 we have $10<12$, and order is maintained.

This is encouraging! But... we have to be a bit cautious. Notice that we have not explored multiplication by a negative number. Observe that if we start with $5<6$ and multiply both sides by -4, we get -20 and -24 and $-20>-24$ — not the other way around. The order of the inequality has changed!

There's a geometric interpretation to this: Multiplication (and division) perform a scaling of the points away from (or towards) zero. Multiplying by a positive number performs that scaling on the same side of zero as the original numbers. But, multiplying by a negative number first reflects our inequality about zero. That reflection changes the order of the numbers!

Properties of Order: Multiplication and Division

The **multiplication property of order** (MPOO) states that for all real numbers a, b, and c:

$$\text{if } a<b \text{ and } c>0, \text{ then } ac<bc$$

$$\text{if } a<b \text{ and } c<0, \text{ then } ac>bc$$

The **division property of order** (DPOO) states that for all real numbers a, b, and c:

$$\text{if } a<b \text{ and } c>0, \text{ then } \frac{a}{c}<\frac{b}{c}$$

$$\text{if } a<b \text{ and } c<0, \text{ then } \frac{a}{c}>\frac{b}{c}$$

Note that, as before, MPOO and DPOO behave the same for all of the inequality symbols, not just $<$.

For the most part, we can solve an inequality just like we solve an equation. The addition and subtraction POOs are no different than the POEs. However, we have to be a bit more careful with multiplication and division. When multiplying or dividing both sides of an inequality by a negative number, you must change the direction of the inequality.

Example 2.3

Solve and graph: $-3x < 21$.

Solution: We can divide both sides by -3 and change the direction of the inequality: $x > -7$. To check, we might try plugging in values that are on both sides of the boundary (like -8 and -6), into the original inequality:

$-3 \cdot -8 = 24$ and $24 > 21$, so -8 *does not* make the original inequality true. The solution set cannot include this value.

$-3 \cdot -6 = 24$ and $18 < 21$, so -6 *does* make the original inequality true. The solution set must include this value, so the solutions must extend from the boundary (-7) toward to the right (hitting -6 and all the points toward positive infinity).

2.3.2 What About the Field Axioms?

Recall that the field axioms were our other tools for simplifying equations, but they were used on one side of an equation only. Since they are not performed on both sides, there's no risk of them messing with order.

So, we can distribute and combine like terms without it having any effect on the truth of the inequality. We can use them to simplify, just like we did when we were solving equations

Example 2.4

Solve and state the solution set: $4x + 3 - 2(3x + 1) > 13$.

Solution: We simplify on the left-hand side first, and then apply the POOs.

$$
\begin{aligned}
4x + 3 - 2(3x + 1) &> 13 \\
4x + 3 - 6x - 2 &> 13 && \text{distributive propoerty (check your signs!)} \\
-2x + 1 &> 13 && \text{combine like terms} \\
-2x &> 12 && \text{APOO} \\
x &< -6 && \text{MPOO with a negative number}
\end{aligned}
$$

In set notation, we write $\mathcal{S} = \{x \in \mathbb{R} \mid x < -6\}$.

2.3.3 Special Case Solutions

As with equations, inequalities with variables on both sides can have no solution or all real numbers as its solution.[6] The variables may vanish, as they sometimes do, but now we have to look at the inequality that is left over to determine whether it is always true or always false. If the remaining inequality is a true statement, then our solution set is *all real numbers*. If the remaining inequality is false, then our solution set is empty.

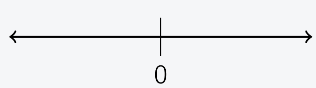
Example 2.5

Solve and state the solution set: $6x - 2 > 2(3x + 1)$.

$$
\begin{aligned}
6x - 2 &> 2(3x + 1) \\
6x - 2 &> 6x + 2 && \text{distributive propoerty} \\
-2 &> 2 && \text{subtract } 6x \text{ from both sides (APOO)}
\end{aligned}
$$

When we examine the remaining inequality, we see that it's false: -2 is not greater than 2. Therefore, the original inequality has no solution. To write the answer in set notation: $\mathcal{S} = \varnothing$.

The graph of "no solution" is just an empty number line. In that case, we can either write "no graph" or "empty graph" or just draw a blank number line. Boring, perhaps, but mathematically accurate.



The mirror-image situation, as you might expect, is demonstrated by the following example.

[6] We can't simply say "infinitely many solutions", since an inequality like $x < 0$ has infinitely many solutions.

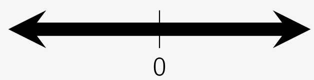
Example 2.6

Solve and state the solution set: $5x + 4 \geq 2x + 3x - 1$.

$$\begin{aligned} 5x + 4 &\geq 2x + 3x - 1 && \\ 5x + 4 &\geq 5x - 1 && \text{combine like terms} \\ 4 &\geq -1 && \text{subtract } 5x \text{ from both sides (APOO)} \end{aligned}$$

This time when we interpret the remaining inequality, we see that it's true: 4 is greater than or equal to -1. Therefore, every real number is a solution to the original inequality. To write the answer in set notation: $\mathcal{S} = \mathbb{R}$.

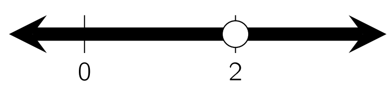
The graph of all real numbers is a graph with the entire number line colored in.



2.3.4 Not-Equal

We haven't talked much about inequalities that use $\neq$, but that's because they're pretty uninteresting. For example: Solve $8x + 3 \neq 19$. We can subtract 3 from both sides, then divide both sides by 8, yielding: $x \neq 2$. So, the only number that *doesn't* work in this inequality is when $x = 2$.

In set notation, we write: $\mathcal{S} = \{x \in \mathbb{R} \mid x \neq 2\}$. The graph is the whole number line, except that single point:



2.4 Two-Variable Inequalities

Startup Exploration: TODO

TODO

Having spent all of that time talking about one-variable inequalities, we can now get on with something more interesting: turning a linear equation like $y = -2x + 6$ into a two-variable inequality like $y \leq -2x + 6$.

What does this inequality mean? What counts as solution? To explore this a bit more, suppose we want to check whether the random points $(4, 5)$ and $(-3, 2)$ satisfy the inequality $y \leq -2x + 6$.

To accomplish this, we substitute the values $x = 4$ and $y = 5$ into the given inequality, and then see whether the resulting inequality is true or false:

$$\begin{aligned}
y &\leq -2x + 6 && \text{original inequality} \\
5 &\overset{?}{\leq} -2(4) + 6 && \text{substitute candidate point} \\
5 &\overset{?}{\leq} -8 + 6 \\
5 &\not\leq -2
\end{aligned}$$

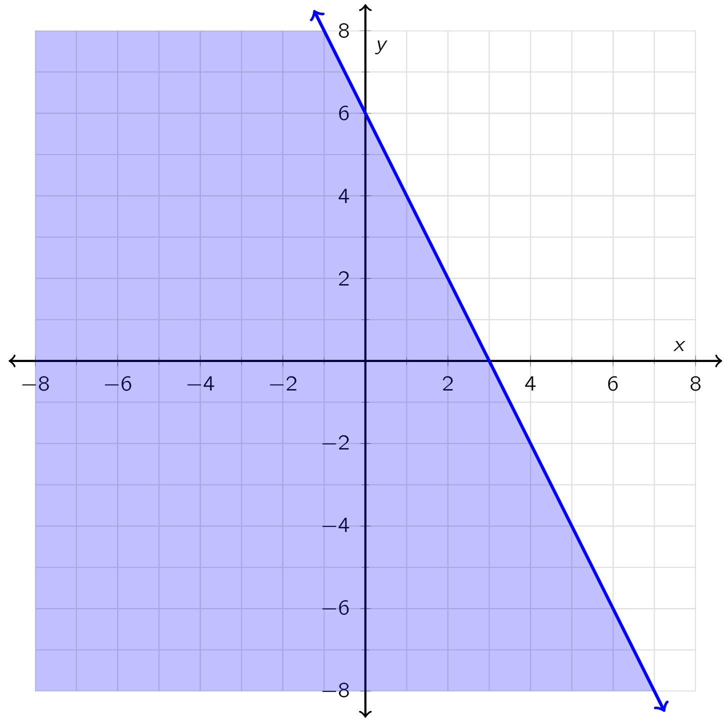
The final inequality is false, so the answer is no, the point $(4, 5)$ is not part of the solution set to the inequality $y \leq -2x + 6$. If we test the other point, $(-3, 2)$, we have:

$$\begin{aligned}
y &\leq -2x + 6 && \text{original inequality} \\
2 &\overset{?}{\leq} -2(-3) + 6 && \text{substitute candidate point} \\
2 &\overset{?}{\leq} 6 + 6 \\
2 &\leq 12
\end{aligned}$$

This statement is true, and so the points $(-3, 2)$ is part of the solution set. But our work with one-variable inequalities suggests that this probably isn't the *only* point in the solution set. What other points might be included?

Note that the inequality is inclusive, so any point on the line $y = -2x + 6$ must be part of the solution set (that's the line y *equals* $-2x + 6$).

In addition, the solution set must include any point that is "less than" that line. In other words, any point on the same side of the line as $(-3, 2)$, the point that we checked above, and which we know is in the solution set. So, our solution must look like the graph below.



So, graphs of two-variable inequalities share some of the features of one-variable inequalities. We will have a boundary, but rather than a dot or enpoint, it will be a line (or a curve). We will shade a section of the graph to show all the solutions, but this will be a whole region of the plane, rather than a ray on the number line.

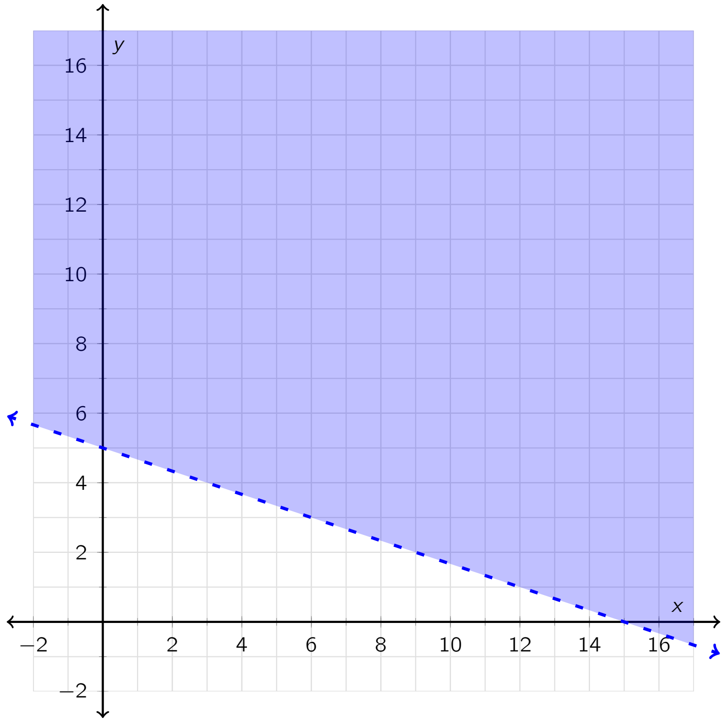
Let's work through another example in detail: graphing the inequality

$$x + 3y > 15.$$

Here we have a line in standard form. An easy way to graph standard form is to find the x- and y-intercepts. When we plug in $x = 0$, we have $y = 5$. When we plug in $y = 0$, we have $x = 15$. So, we can draw our graph by connecting the points $(0, 5)$ and $(15, 0)$.

Before we draw the line, though, note that we have an exclusive inequality: "greater than". For one-variable inequalities, we drew a hollow dots which allowed us to show where the boundary was without actually including the boundary itself. In the graph of a two-variable inequality, we will draw a dotted or dashed line to show that the line itself is not included in the solution set.

Since our inequality is "greater than" it seems reasonable to shade the region above the line. The resulting graph is shown below.



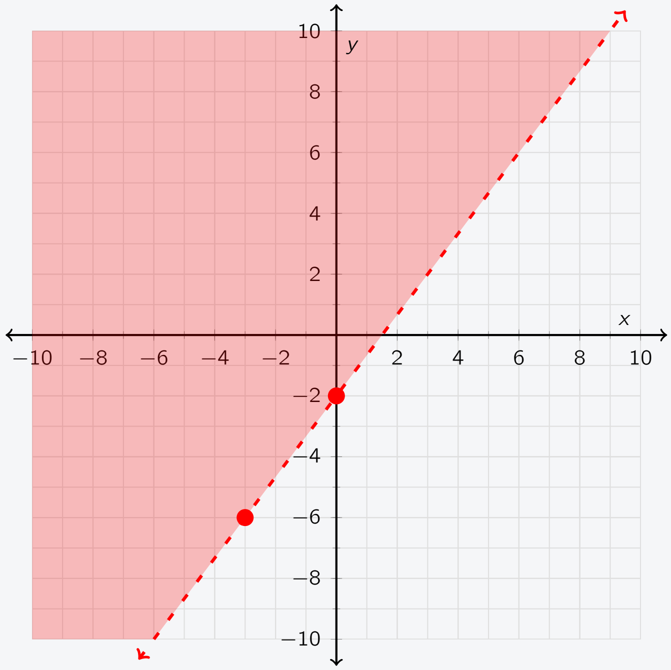
We can check that we've shaded the correct side by testing a point. A convenient point to test is the origin — all of that multiplication by zero makes this easy! If we plug in $x = 0$ and $y = 0$ to our equation, we have

$$\begin{aligned} x + 3y &> 15 && \text{original inequality} \\ 0 + 3(0) &\overset{?}{>} 15 && \text{substitute candidate point} \\ 0 &\not> 15 \end{aligned}$$

We have a false statement in the end, so the origin is not included in the solution set. This means the solution set must lie on the other side of the line. Our graph agrees with this.

Example 2.7

Write the inequality pictured in the graph below.



Solution: We can use the two given points to find that the slope of the line is $\frac{4}{3}$ (perhaps using a slope triangle). We're also given the y-intercept, so we can write the equation for the boundary line: $y = \frac{4}{3}x - 2$.

Since the line is dotted, we know that we're dealing with an exclusive inequality, and the shading lines above the line, so this graph depicts the inequality

$$y > \frac{4}{3}x - 2.$$

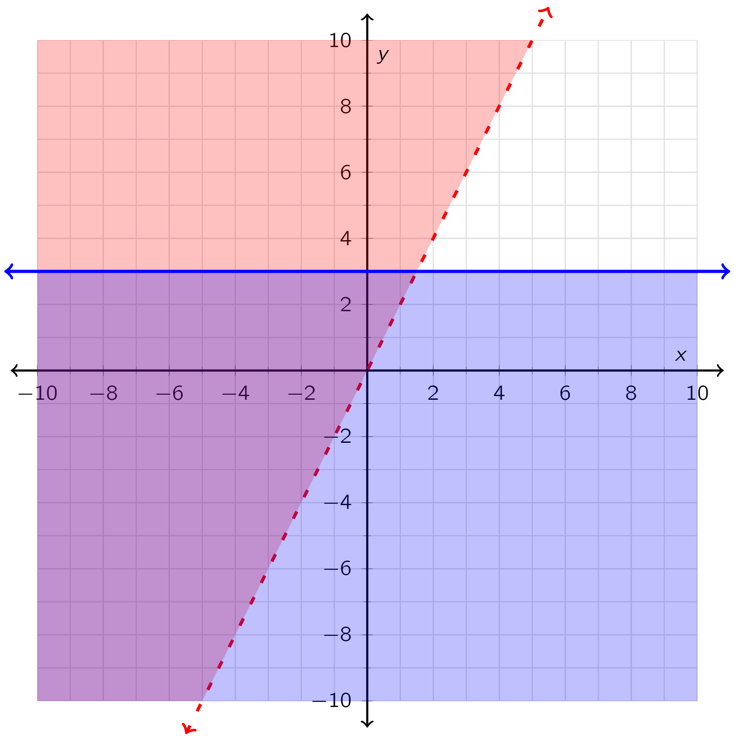
Finally: Remember to keep in mind the criteria for high-quality graphs (graphs should be done on graph paper, we should be mindful about how we place and scale the axes, and so on.)

2.5 Systems of Inequalities

Startup Exploration: TODO

TODO

By now you can perhaps anticipate where this is heading! Here you see a graph of a system of inequalities. The solution to the system is the set of points that the graphs share. In other words, the region where they "overlap".



System of Inequalities

A set of two or more inequalities with the same variables.

Solution to a System of Inequalities

The set of all points common to each inequality in a system. Graphically, the region where the graphs overlap.

The only way to find the solution to a system of inequalities is to graph the system! That is, to graph both inequalities on the same set of axes. There is really no meaningful set notation for the solution; the solution is the graph. In that visual we can see the region of the plane that contains the solution set, boundaries and all.

Graphing Tips

Systems of inequalities can be quite fun to graph. We can use different colors for each inequality, for example yellow and blue colored pencils. Where the graphs overlap, we'll get green!

If we're using just a regular pencil and have only one color choice, we might shade one inequality with horizontal lines and shade the other inequality with vertical lines. The region where we see the checkerboard shading is the solution set.

In any case, it will be important to make sure the solution region is clear. If the colors or shading are muddled and hard to interpret, consider using an arrow or label to identify the solution set.

Example 2.8

Write the system pictured in the graph at the beginning of this section.

Solution: The horizontal line goes through the point $(0, 3)$. It is a solid line and shaded below, so that means we have the inequality "less than or equal to". So, the blue graph depicts $y \leq 3$.

The red graph is a direct variation (a straight line through the origin) through the point $(1, 2)$. So the equation for the line is $y = 2x$. This line is dashed and shaded above, so we have the inequality $y > 2x$.

So, together we have the system of inequalities

$$\begin{cases} y \leq 3 \\ y > 2x \end{cases}.$$

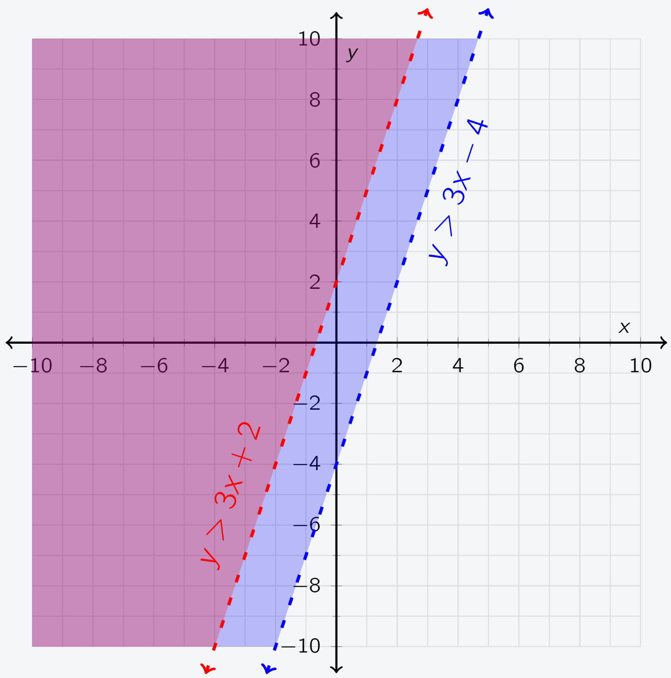
2.5.1 Special Case Systems

We have been graphing lines for a while now, and shading on one side of a line doesn't usually add that much of a challenge. Sometimes, however, we encounter some unexpected images.

Example 2.9

Example. Graph the system $\begin{cases} y > 3x - 4 \\ y > 3x + 2 \end{cases}$

Solution: Before we start graphing, note that the lines are parallel! Our first instinct might be to say that this system has "no solution". That would be correct, if we were dealing with a system of *equations*, but here we have inequalities! Let's take a look at the graph.



Note that the red region is entirely covered over by the blue region! So the overlapping region is, in fact, the red inequality $y > 3x + 2$.

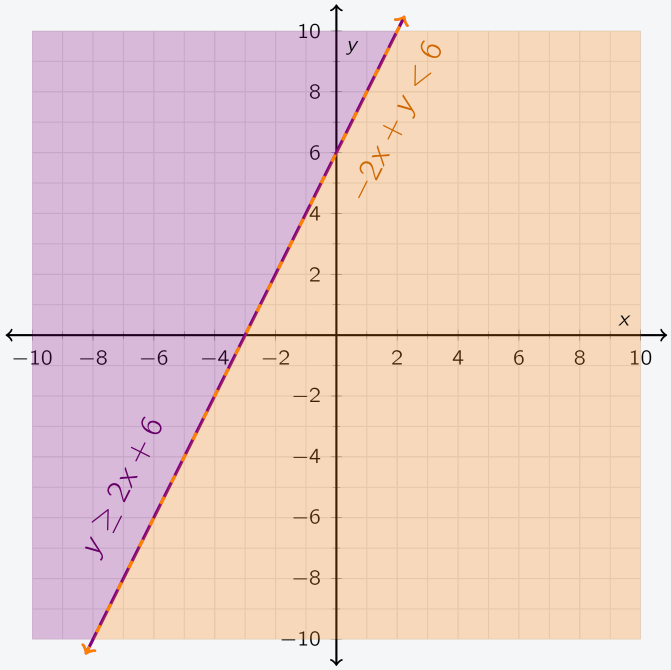
As the previous example shows, a pair of parallel lines can create a pair of inequalities which overlap. But, could the shading have gone differently? There are four different scenarios that can arise when turning two parallel lines into inequalities (one of this is shown in the example). Can you draw pictures of the other three scenarios? How does the solution set look in each case?

Example 2.10

Graph the system:

$$\begin{cases} y \geq 2x + 6 \\ -2x + y < 6 \end{cases}.$$

Solution: Let's fast-forward to the graph and see what's happening here.



It might not have looked like it at first, but we have two different ways of expressing the same line! Both inequalities have the same boundary, but they are shaded in opposite directions, so those regions do not overlap anywhere except the boundary itself.

Since the boundary is exclusive for one of the inequalities, the boundary cannot be part of the solution set. So in this case, there is no part of the graph where the two shaded regions over lap. This system of inequalities has no solution!

What are the other cases that might arise when we have two inequalities that share the same boundary line? Can you draw graphs that express these different possibilities?

2.5.2 Checking a Solution

We jumped right in to graphing a system, but suppose we want simply to check to see if a given point is a solution to a given system of inequalities?

Example 2.11

Determine whether the point $(-2, 1)$ is a solution to the system:

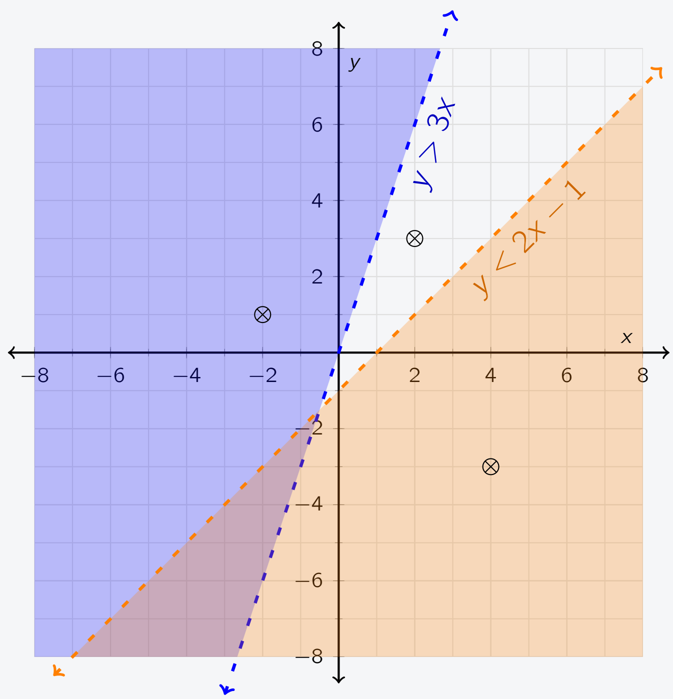
$$\begin{cases} y > 3x \\ y < 2x - 1 \end{cases}.$$

Solution: To answer this question, we don't have to graph. We can just substitute the point into each inequality and check to see whether it makes both true. Let's check $(-2, 1)$:

$$\begin{array}{lll} y > 3x & \text{and} & y < 2x - 1 \\ 1 \stackrel{?}{>} 3(-2) & & 1 \stackrel{?}{<} 2(-2) - 1 \\ 1 \stackrel{\checkmark}{>} -6 & & 1 \not< -5 \end{array}$$

The point satisfies the first inequality, but not the second. So, $(-2, 1)$ is not a solution to the system.

Graphing is not required to answer this question, but it may be helpful to see a visual. Note where the given point lies on the graph. Predict what would happen if we tested the non-solution $(2, 3)$. Do the same for the non-solution $(4, -3)$.



2.6 Optimization Using Linear Programming

Startup Exploration: Evil Vegan Appliances

A subsidiary of YeardleighCorp, Evil Vegan Appliances, manufactures solar-powered soymilk makers. They manufacture 2 types: a large capacity model for commercial use, and a smaller one for home use. Since these soymilk makers delicate technology, the factory can only hand-make a total of 16 machines per day.

In order to keep demand up — and because she enjoys toying with the emotions of her customers — Yeardleigh decides to restrict production in another way: She decides that they should build no more than 10 commercial models and no more than 12 family models per day, just to keep everyone wanting more.

If Evil Vegan Appliances makes \$75 dollar profit on each family size soy milk maker and \$100 profit on each commercial model, how many of each type should they build each day to maximize profit? What will their maximum daily profit be in this case?

Businesses want to maximize their profits and minimize their costs. At the same time, businesses have constraints on the availablity and cost of resources. They have to hire workers, buy materials, build production facilities, and so on. A key challenge for any business is to navigate the various constraints so that they can configure their operations in the optimal way.

Optimization

Maximizing or minimizing a quantity, given a set constraints.

There are multiple techniques for optimization. The approach we will learn here is called **linear programming**. Real-world linear programming problems have many variables, sometimes numbering in the millions. Since we're only in algebra 1, we'll stick with just a few variables.

2.6.1 The Process of Linear Programming

To begin the process of modeling the startup exploration, our first goal is to identify what quantity we are trying to optimize, and what variables are at play in the scenario.

In our case, the quantity that we wish to optimize is *profit*, and the variables are the number of commercial machines and number of family machines that can be made per day.

We use the variables and information from the problem to write what is called the *objective function*. This is the equation we use to calculate the quantity we want to optimize.

Let P represent the total profit, let C represent the number of commercial machines that can be made per day, and let F represent the number of family machines that can be made per day. The company makes \$100 per commercial machine and \$75 per family machine, so our objective function (profit function) is

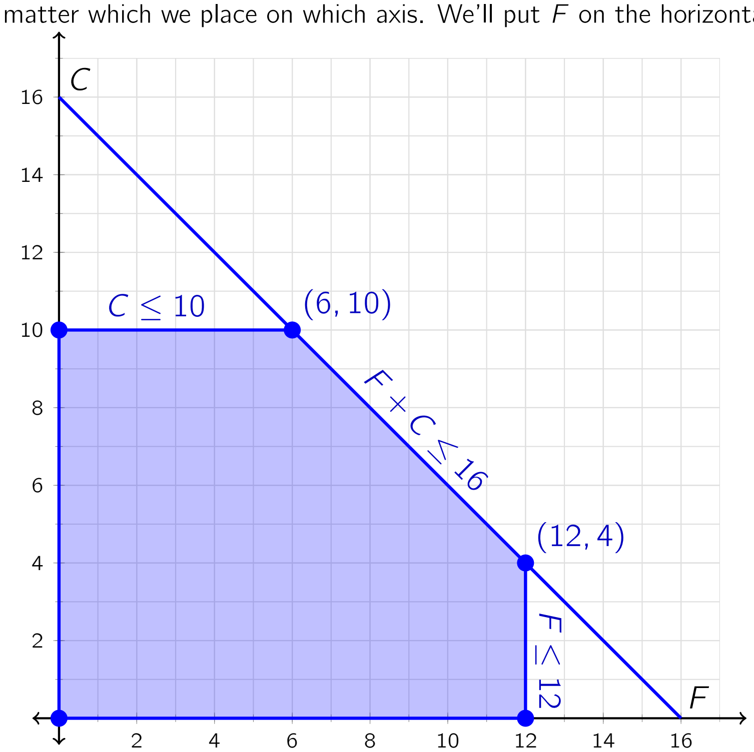
$$P = 100C + 75F.$$

We then model the constraints as a system of inequalities. The constraints on our system include: Per day they can make at most 16 machines (of any kind). In particular, no more than 10 commercial machines, and no more than 12 family machines per day. There are also natural "minimum" constraints at zero, since the factory can't make a negative number of machines.

Now, we combine our variables with our constraints to write a system of inequalities:

$$\begin{cases} F + C \leq 16 & \text{limit on the total number of machines} \\ C \leq 10 & \text{limit on the number of commerical machines} \\ F \leq 12 & \text{limit on the number of family machines} \\ C \geq 0 & \text{floor on the number of commercial machines} \\ F \geq 0 & \text{floor on the number of family machines} \end{cases}$$

We graph all five of these inequalities on the same set of axes. F and C aren't really dependent or independent variables, so it doesn't matter which we place on which axis. We'll put F on the horizontal axis.



The graph is a polygon, and every point inside this region represents a number of family size machines and a number of commercial size machines that the factory could possibly make, given the limits on production. Now, the task is to figure out which point earns the most profit.

The key insight here is that we should find the intersection points on the system. These intersection points show us where we utilize multiple resources to their fullest extent. It turns out that one of these intersection points will be the point where we optimize the objective.[7]

The intersection points are $(0, 0)$; $(0, 10)$; $(12, 0)$; $(6, 10)$; and $(12, 4)$. Those coordinates are of the form (F, C). It's important to keep track of what the numbers represent!

Now, we plug these coordinates into our objective function for total profit, $P = 100C + 75F$. One of these points will give the maximum profit and one will give the minimum profit.

- The point $(0, 0)$ means 0 Family, 0 Commercial: $P = 75(0) + 100(0) = \$0$
- The point $(0, 10)$ means 0 Family, 10 Commercial: $P = 75(0) + 100(10) = \$1000$
- The point $(12, 0)$ means 12 Family, 0 Commercial: $P = 75(12) + 100(0) = \$900$
- The point $(6, 10)$ means 0 Family, 0 Commercial: $P = 75(6) + 100(10) = \$1450$
- The point $(12, 4)$ means 0 Family, 0 Commercial: $P = 75(12) + 100(4) = \$1300$

So, now we have our solution: To maximize daily profit, the factory needs to make 6 family and 10 commercial machines per day, for a maximum daily profit of \$1450.

Discussion

The first vertex $(0, 0)$ is kind of a silly point to test. You might not be surprised to see that it's the scenario that creates the minimum profit.

The points $(0, 10)$ and $(12, 0)$ have the company maximizing only one constraint — the number of a certain type of machine they can make — but completely ignoring the fact that they can make a total of 16 machines per day.

The other two vertices show the company making all 16 machines they can make per day, and maximizing one of the other constraints. We can't maximize all three constraints at the same time. (Can you explain why not?)

Finally, our solution makes sense because it seems reasonable that we would want to find the solution that maximizes overall production, but also maximizes production of the machines that make more money for the company!

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[7] See "History of Linear Programming" below to learn a little about the people who created and proved that this process works.

In a real linear programming problem, there would likely be many more constraints. For example, the maximum number of machines per day might depend on the number of employees we have, the amount of time each employee can work, how productive those workers are during that time, the their salaries, how much money there is in the payroll account, how many parts we need, how many parts we have in stock, how much it costs to make new parts, how long it will take to make them,... well, this could go on for a while. At any rate: business work quite hard to figure out challenges like this!

2.6.2 History of Linear Programming

A Soviet mathematician named Leonid Kantorovich was the first to use linear programming in 1939. During World War II he was helping the Soviet army minimize their costs while at the same time maximize losses for their enemies, Nazi Germany. His technique was effective and the Soviets kept it a secret. Because of his work, Kantorovich won a Nobel Prize in Economics, the only Soviet economist to ever win in that field.

In 1947, George Dantzig, and American mathematician, created and published his own algorithm for linear programming. The example he used was a problem that would otherwise have taken vast amounts of computing time to figure out because the number of possibilities to account for was more than the number of particles in the universe! In 1947, they did not have computers that could handle that much data. Rather than test each of the vast number of possibilities, Dantzig's "simplex" method for linear programming took just a few moments and a few calculations.

A funny side-note about Dantzig: one time he was late to a college statistics class at Berkeley. He saw two problems written on the board. He just thought they were really hard homework problems, so he solved them and turned them into his professor.

It turns out that the two problems were actually "open problems" in mathematics. An open problem is a problem in mathematics that no one has been able to solve yet. His professor spent the beginning portion of the class discussing these problems, but since Dantzig was late, he missed all of that.

Also in 1947, another American mathematician, John van Neumann, who contributed quite a bit to mathematics and physics, developed another way to approach linear programming called the "theory of duality". There are other mathematicians who advanced the process over the years. Now, every business and government agency uses linear programming to determine the best way to use their resources.

Back Matter

Glossary, bibliography, and index removed during editing process.