The Algebranomicon

Patty C. Hill
Jason L. Ermer

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Acknowledgements

Shawn Tyson...

The online LaTeXcommunity...

God made the integers, all the rest is the work of man.

Leopold Kronecker
German mathematician

Chapter 1

Numbers

The number system that we use every day, both in mathematics class and in our regular lives, developed over many generations. Men and women from all over the world, both famous and anonymous, have helped to make mathematics what it is today. Yet, people argue over whether mathematical ideas are "invented" or "discovered".

For example: **imaginary numbers** (which we will discuss briefly in algebra 1, and study in more detail in algebra 2), first appeared on the mathematical scene in the 1500s. Italian mathematician Gerolamo Cardano first wrote about these new numbers in his work with certain types of problems that otherwise would have been impossible to solve. Since he was the first person to describe this new kind of number, we might say that Cardano *invented* imaginary numbers.

But, we now know that imaginary numbers have practical applications in, for example, electrical engineering. The laws of electromagnetism haven't changed since the 1500s. (Well, our understanding of the laws has changed, but the physics has not.) So, maybe imaginary numbers have been there all along, lurking within the fabric of the universe. In this case, Cardano *discovered* imaginary numbers.

It's not clear which is the more accurate description. No matter what side of the debate you find more convincing, there is a certain beautiful interconnectedness to our system of numbers and mathematical laws. To illustrate this, we'd like to tell a story.

1.1 Story of numbers

Once upon a time, there was a simple farmer. Knut Krumbli lived in rural Sweden, raising goats and making goat cheese. He and his family led an uncomplicated life and they didn't have much need for mathematics. In fact, they really only needed numbers to count their goats: 1 goat, 2 goats, 3 goats, 4 goats...

But one day, after a terrible storm, Knut went to the field to count the goats and discovered, much to his dismay, that there were no goats to count. He hadn't needed a number to describe this situation before, but *now people were asking him hard questions, like "How many of your goats made it through that crazy storm?" (But, you know, in Swedish.)*

Knut and his family couldn't very well survive without any goats, so he went to his neighbor for help. The neighbor agreed to loan Knut some goats to restart his herd but, of course, Knut would have to repay his goat-debt later. The village hadn't needed to do much accounting before the storm, but now they needed a system of numbers that could keep track of debts and credits.

Over time, Knut's family got back on their feet and thrived. They paid back their debts and eventually grew to raise more goats (and to make more goat cheese) than they could eat. They began to trade with their neighbors for other foods or services. Of course, everything had relative value: three wheels of goat cheese were worth two bales of hay. So, the village developed a system of numbers for describing exchange rates of this kind.

As the village grew, Knut's family farm led the development of a booming goat cheese industry. They invested their profits into bank accounts that paid interest. In certain situations, everyone was surprised to discover, interest-bearing accounts led to a new system of numbers that no one had seen before.

Eventually, some of Knut's ancestors emigrated to America and, years later, a pair of twins – Knut's great-great-great-grandchildren – would grow up to change the world. But let's not get too far ahead of ourselves. More about the Krumbli twins later...

1.1.1 Dissecting the story

Mathematically speaking, it's natural to begin our discussion of numbers exactly where Knut began: counting things. The numbers we use to count are called the natural numbers (also known as the counting numbers, for obvious reasons).

Natural number

A **natural number** is a member of the list of numbers that starts $1, 2, 3, 4, \ldots$ and continues forever. The set of all natural numbers is denoted using the symbol $\mathbb{N}$, so we can write $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$.

We use the {curly braces} here to indicate that we're collecting a group of numbers together as a **set**. A set written in this way is written in **set notation**.

The natural numbers have some interesting properties. If we add two natural numbers, their sum will always be a natural number. The same goes for multiplication: the product of two natural numbers is again a natural number.

Mathematically speaking, we call this **closure**. We say that the natural numbers are closed under the operation of addition. Also, the natural numbers are closed under the operation of multiplication.

Notice that we didn't include 0 among the natural numbers. Zero is a bit tricky because it seems like a counting number. For example, "zero" is (probably) the answer to the counting question, "How many live elephants are there in the room with you right now?" But if there are no elephants to count, can we really count them? That's a philosophical question.[1]

Practically speaking, we usually exclude 0 from the set of natural numbers. We will always be very clear when we come to situation where we want to consider 0 to be a natural number.

The natural numbers are closed under the operations of addition and multiplication, but they are *not* closed under the operation of subtraction. Sometimes, the difference of two natural numbers is a natural number: for example $8 - 6 = 2$, no problem. But some subtraction sentences don't work: for example $10 - 13 = {}^-3$, and $^-3$ is not a natural number.

Integer

An **integer** is a natural number, or the opposite of a natural number, or zero. The set of all integers is denoted $\mathbb{Z}$, so we sometimes write $\mathbb{Z} = \{\ldots, {}^-3, {}^-2, {}^-1, 0, 1, 2, 3, \ldots\}$. We could also write $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$.

The symbol $\mathbb{Z}$ comes from *Zahl*, the German word for number.

Note that every natural number is an integer. We say that the set of natural numbers is a **subset** of the set of integers.[2]

The integers are closed under the operations of addition and multiplication. Plus – bonus! – the integers are closed under the operation of subtraction. Whenever we subtract one integer from another, we always get another integer as the result.

But (as you may have anticipated) we have a problem with division. In certain cases, the quotient of two integers is itself an integer: $15 \div {}^-3 = {}^-5$, no problem. But other times we get a quotient that is not an integer: $^-3 \div 15 = {}^-0.2$ and $^-0.2$ is not an integer. In other words, the integers are not closed under the operation of division.

[1] Philosophy and mathematics have historically gone hand-in-hand. Many important discoveries (inventions?) in mathematics are attributed to people who are considered both "philosopher" and "mathematician". For example: French mathematician René Descartes, for whom the Cartesian coordinate system is named, is also the philosopher who said "I think therefore I am".

[2] For those who are into mathematical symbols and notation, the sentence "The natural numbers are a subset of the integers" is denoted $\mathbb{N} \subseteq \mathbb{Z}$.

Rational number

A **rational number** is any number that can be written as the ratio of two integers $\frac{a}{b}$, where b is not zero. This set includes all of your classic fractions, as well as all terminating decimals and all repeating decimals.

The set of all rational numbers is denoted by the symbol $\mathbb{Q}$, which comes from the word *quotient*.

Fractions have a lousy reputation among math students, but the rational numbers are great because they are closed under all four of the basic operations. When we add, subtract, multiply, or divide any two rational numbers, the result will always be another rational number. What more could we ask for? In some sense, $\mathbb{Q}$ is a complete number system, and the world was content with the rational numbers for a long time.

But, other numbers exist. Note that the rational numbers include all of the terminating decimals (like 0.5 and 1.678), and all of the repeating decimals (like $0.\overline{3}$ and $-12.34\overline{56}$). Now, consider the number

$$0.10\,110\,1110\,11110\,111110\ldots$$

This number does not terminate, but it does not repeat either.[3] So, this number is *not* a rational number.

Irrational number

An **irrational number** is a number that cannot be expressed as the ratio of two integers. In decimal form, an irrational number never terminates and never repeats.

You have likely encountered irrational numbers before. A famous example is the number π (pi), which shows up when we study circles. We usually approximate π to be about 3.14, but in fact, the decimal representation of π goes on forever without stopping or repeating:

$$\pi \approx 3.\,1415926535\ 8979323846\ 2643383279\ 502884197\ 6939937510\ 5820974944\ 5923078164\ldots$$

When we group together all of the rational numbers and all of the irrational numbers, we will have accounted for all possible decimal representations. This combined set of numbers is going to be of key importance to us in algebra 1.

[3] This number does have a pattern, but that is not the same as "repeating" in the sense of "repeating decimal".

Real number

A **real number** is any rational or irrational number. The set of all real numbers is denoted $\mathbb{R}$ which, naturally enough, comes from the word *real*.

Like the set $\mathbb{Q}$, the set $\mathbb{R}$ is closed under the four fundamental operations. $\mathbb{Q}$ and $\mathbb{R}$ are the number systems we will work with in algebra 1. Other types and sets of numbers exist (like $\mathbb{C}$, the set of so-called "complex numbers"), but we won't get into them very much until algebra 2 and beyond.

1.2 Integers

You have probably been working with positive and negative numbers for a while now, so this section will review the most important terms and algorithms for signed numbers. To get the ball rolling, think about this:

Startup exploration: Integer comparisons

In each of the expressions below, x and y are natural numbers and $x < y$ (x is less than y). Will the result be greater than 0, less than 0, equal to 0, or is there not enough information to tell? Why?

(a) $x + (^-y)$ (b) $x - (^-y)$ (c) $x \cdot (^-y)$ (d) $x \div (^-y)$

1.2.1 Language of signed numbers

As we saw in section 1.1, the integers include the natural numbers, their opposites, and zero. Numbers now include two pieces of information: they have a *size* (or *magnitude*) and a *direction*, either positive or negative.[4] Sometimes, we care only about the magnitude of a number, in which case we refer to a number's *absolute value*.

Absolute value

The **absolute value** of a number x is its distance away from zero on the number line. To express this in mathematical symbols, we write $|x|$ to mean "the absolute value of x".

For those who like mnemonic devices and memory aids, it may be helpful to think of the absolute value bars as a little numerical shower stall or car wash. A number goes in and all its negativity gets washed away.

Example 1.1

Compute each of the following.

1. $|4|$ The absolute value of a positive number is just the original number, so $|4| = 4$.

[4] Later in mathematics and the physical sciences, we'll encounter mathematical objects with both magnitude and direction again. They're called vectors.

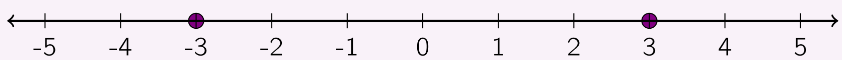
2. $|{-8}|$ Absolute value ignores the sign and tells us how far a number is from zero. $^-8$ is eight units away from zero, so $|{-8}| = 8$.

3. $-|{-6}|$ The absolute value bars only apply to what's inside, but then this negative sign *outside* the absolute value bars will make the final answer negative again! So, $-|{-6}| = {-6}$.

Every nonzero number has a counterpart that is the same distance away from zero, but on the opposite side of the number line. This is a simple idea, but one that is important enough for us to give it a name.

Opposite

The **opposite** of a number x is the number $-x$. In other words, the opposite of a number is the number with the *same absolute value*, but the *opposite sign*. For example, 3 and $^-3$ are opposites. The number 0 is its own opposite.



The sum of opposites is always 0. For this reason, we sometimes use the term **additive inverse** to describe the opposite of a number. (More on inverses in chapter 5.)

1.2.2 Adding signed numbers

Like matter and antimatter, combining positive and negative numbers leads to annihilation. (Dramatic, no?)

For example when we bring together $+8$ and $^-6$, we can picture 8 units of "matter" and 6 units of "antimatter". Particles and antiparticles annihilate one another, both disappearing in the process. Since we have more matter than antimatter in this case, all of the antimatter is consumed, leaving behind 2 units of matter.

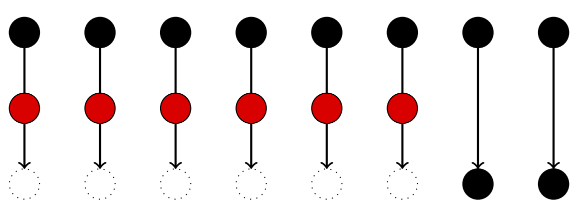


Figure 1.1: Eight units of matter versus six units of antimatter!

We might visualize annihilation with a drawing like in fig. 1.1 where black circles are units of matter, red circles are units of antimatter. Or, we could write a number sentence like $8 + {^-6} = 2$. Of course, no annihilations occur when we scrape together a big pile of matter (or a big pile of antimatter, for that matter). It's only when they mix that anything interesting happens.

Adding signed numbers

If the two numbers being added have the same sign, then we add the absolute values of the numbers, and use the sign that they share.

Otherwise, if the two numbers being added have different signs, then we find the difference between their absolute values, and use the sign of the number with larger absolute value.

Example 1.2

Compute each of the following.

1. $^{-}8 + {}^{-}12$ Both numbers are negative. That's a big pile o' antimatter: $^{-}8 + {}^{-}12 = {}^{-}20$

2. $^{-}8 + 12$ The numbers have different signs, so prepare for annihilation! $|\,12\,|$ is larger than $|\,{}^{-}8\,|$, so we will have matter left over. Therefore, $^{-}8 + 12 = 4$

1.2.3 Subtracting signed numbers

Addition and subtraction are called *opposite* (or *inverse*) *operations* because they "undo" one another. The act of adding 5 of matter can be "undone" by subtracting 5 units of matter. But on the other hand, the act of adding 5 units of matter can also be undone by *adding 5 units of antimatter*.

Subtracting signed numbers

Subtracting a number is the same as *adding the opposite of the number*.

When faced with a subtraction problem, change it to an "addition of the opposite" problem and then follow the rules for adding signed numbers.

The benefit of this approach is that we can avoid having to learn a whole new set of rules for subtracting signed numbers. All we need are the rules for addition, plus one new rule about how to change subtraction problems into addition problems!

Example 1.3

Compute each of the following:

1. $8 - 12$ — "Subtracting 12" is the same as "adding negative 12", so $8 - 12 = 8 + {}^{-}12$. Then, we can apply the rules for adding signed numbers: $8 + {}^{-}12 = {}^{-}4$

2. ${}^{-}3 - 14$ — This is the same as ${}^{-}3 + {}^{-}14$, and we follow the rules for adding numbers with the same sign: ${}^{-}3 + {}^{-}14 = {}^{-}17$

3. $4 - {}^{-}9$ — This is the same as $4 + 9$, which is an easy addition: $4 + 9 = 13$

4. ${}^{-}6 - {}^{-}5$ — This is the same as ${}^{-}6 + 5 = {}^{-}1$

When dealing with addition and subtraction of signed numbers in a problem, a good habit is to simplify the signs by rewriting subtraction as addition-of-the-opposite, before doing any computations.

In the startup exploration for this section, we have two natural numbers x and y where $x < y$. Natural numbers are all positive, so this means that $|x|$ is less than $|y|$.

Part (a) asks us to consider $x + (-y)$. Since opposite numbers have the same absolute value, $|y|$ is the same as $|-y|$. Therefore when we add, the number with larger absolute value is negative and the sum will take on the negative sign. So, when x and y are natural numbers and $x < y$, we know that $x + (-y)$ is *always negative* so the result is always less than 0.

Part (b) asks us to consider $x - (-y)$. We can change this subtraction expression into addition-of-the-opposite: $x - (-y) = x + y$. Since both x and y are natural numbers, their sum is a natural number. In other words, when x and y are natural numbers, $x - (-y)$ is always positive, in other words always greater than 0.

1.2.4 Multiplying and dividing signed numbers

Like addition and subtraction, the operations of multiplication and division are inverse operations. We'll discuss multiplication below, though the rules about the signs of products also apply to the signs of quotients.

Recall from your elementary school days that one way to think about whole number multiplication is as *repeated addition*. We interpret $a \cdot b$ as "a groups with b items in each group". So, $3 \cdot 5$ means "3 groups of 5", which we can write as an addition sentence: $3 \cdot 5 = 5 + 5 + 5$.

Using this interpretation, we can easily explain the product of a positive number and a negative number. The expression $4 \cdot {}^{-}8$ means "four groups of negative eight": $4 \cdot {}^{-}8 = {}^{-}8 + {}^{-}8 + {}^{-}8 + {}^{-}8 = {}^{-}32$. No problem!

But what about $^{-}5 \cdot 6$? What does it mean to have "negative five groups of six"? Or even worse, what about $^{-}3 \cdot {}^{-}7$, "negative three groups of negative seven"? Rather than try to twist the metaphor to fit these new situations, let's just admit that multiplication can not *always* be represented by repeated addition.[5]

For the moment, we'll simply review the rules for multiplying (and dividing) signed numbers. We can explain why these rules work using the so-called "field axioms for the real numbers". More on that later.

Multiplying and dividing signed numbers

The absolute value of the product of two numbers is the product of their absolute values: $|\, a \cdot b \,| = |\, a \,| \cdot |\, b \,|$. If the two numbers have the same sign, then the product is positive. If the two numbers have opposite signs, then the product is negative.

There are several clever ways to remember this rule. Some people remember that every pair of negatives in a product cancel one another. Other people use a triangle with one positive sign and two negative signs drawn on the vertices. We present another way of looking at it in the next section. Choose whichever mnemonic[6] method is most helpful to you!

Karmic multiplication

Karma, an underlying concept of many Eastern religions, is a belief that a person's actions and intentions shape their future. Performing good deeds will contribute to one's "good karma" and will lead to future happiness. Bad deeds contribute to one's "bad karma" and will lead to future suffering.

So, karma suggests that good things happen to good people, and that bad things happen to bad people. Of course we know that the universe does not always operate in accordance with karma.

Karmic multiplication

When good things happen to good people, that's good!

When bad things happen to good people, that's bad!

When good things happen to bad people, that's bad!

When bad things happen to bad people, that's good!

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[5] The "multiplication as repeated addition" analogy breaks down when we have a negative number of groups, but also for rational and irrational numbers. What's the repeated addition problem for $\frac{2}{3} \cdot \frac{1}{2}$, or for $\sqrt{2} \cdot \sqrt{3}$?

[6] mnemonic (*na · MON · ic*, the first "m" is silent): A learning aid that helps to remember or retain information.

For example: Mahatma Gandhi used nonviolent means to inspire civil rights movements around the world. Gandhi was a good person. On the other hand, Adolf Hitler was chancellor of Nazi Germany during World War II and orchestrated appalling crimes against humanity. Hitler was a bad person.[7]

In terms of life events, winning the lottery is a good thing. Getting hit by a truck is a bad thing.

If Gandhi had won the lottery, that would have been in accordance with all his good karma. That's good! If Gandhi had been hit by a truck, that would have been in opposition to all of his good karma. That's bad!

If Hitler had won the lottery, that would have been in opposition to his evil karma. That's bad! If Hitler had been hit by a truck, it would have served him right! That's good! Go karma!

Of course, in this metaphor good things and good people represent positive numbers. Bad things and bad people represent negative numbers. When karma is operating as it should, we get a positive result. When the laws of karma are broken, we get a negative result.

Example 1.4

Compute each of the following:

1. $8 \cdot {}^{-}12$ — We multiply the absolute values and, since the two factors have different signs, we know the answer is negative: $8 \cdot {}^{-}12 = {}^{-}96$

2. ${}^{-}72 \div {}^{-}3$ — We divide absolute values and, since the two factors have the same sign (both negative in this case, so that's "Hitler gets his by a truck"), the answer is positive: ${}^{-}72 \div {}^{-}3 = 24$

Note: When multiplying (or dividing) more than two numbers, we can approach things in two different ways. We might simplify the product two factors at a time, and keep track of the sign as we go. Or, we could treat the signs as a separate problem: first multiply all of the absolute values, then go back and count up the negative signs.

Example 1.5

Multiply: $(2)({}^{-}2)(1)({}^{-}2)({}^{-}2)(1)(1)({}^{-}2)({}^{-}1)(2)$

Solution: If we count up the negative signs, we find there are five. Pairs of negatives will have a positive

[7] understatement ($UN \cdot der \cdot state \cdot ment$): The act of representing something in a weak or restrained way, to a lesser degree than is borne out by the facts.

product, so we'll have two pairs of negatives plus one left over. Our final answer, then, will be negative. All that remains is to multiply the 2s (of which there are six):

$$(2)(-2)(1)(-2)(-2)(1)(1)(-2)(-1)(2) = -64$$

In the startup exploration for this section, x and y are natural numbers and $x < y$. Since x are y are natural numbers they are both positive, and then $-y$ is negative. Part (c) asks us to consider $x \cdot (-y)$. We have a positive number times a negative number, so the product is *always negative*, always less than 0.

Part (d) asks us to consider $x \div (-y)$. Again, we have a positive number and a negative number, so the quotient is *always* less than 0.

1.3 Rational numbers

As with integers, you've probably been working with fractions and decimals for a while now. In this section, we review the key terms and algorithms for working with rational numbers. As we get going, think about this:

Startup exploration: Rational comparisons

In each of the expressions below, a and b are rational numbers where $0 < a < b < 1$. Will the result be greater than 1, less than 1, equal to 1, or is there not enough information to tell? Why?

(a) $a + b$ (b) $a - b$ (c) $a \cdot b$ (d) $x \div (-y)$

1.3.1 Language of rational numbers

Sometimes in life we discover, much to our surprise, that some ridiculous and insignificant thing as been given a name.[8] We find ourselves wondering, "Who decided to give *that* a name? Why bother?" Prepare for one of those moments:

Vinculum

A **vinculum** (plural: vincula) is a horizontal bar used in mathematics to show grouping. For example, the fraction bar in the middle of $\frac{5}{2}$ is a vinculum.

Vincula are used in other contexts as well. For example, we use a vinculum to represent a repeating decimal such as $0.\overline{3}$.

With that definition in mind, we can continue with two of the most daunting words in elementary mathematics. You're definitely not alone if you have ever been confused about these.

Numerator and denominator

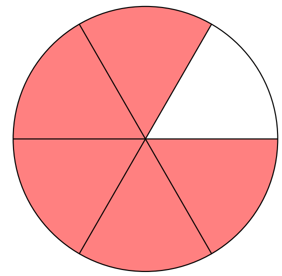
In a fraction, the number above the vinculum is called the **numerator** of the fraction, and the number below the vinculum is called the **denominator** of the fraction.

For example in the number $\frac{5}{6}$, the numerator is 5 and the denominator is 6.

[8] For instance, did you know that the little plastic sheath at the end of your shoelaces is called an "aglet"? Now you know.

If we ask "*how many* sixths?", the numerator tells us "*five* sixths". The word numerator is related to the word "number", and the numerator counts the pieces.

If we ask "five *whats*?", the denominator tells us "five *sixths*". The word denominator is related to the word "nominate" (as in "to nominate someone for president") which means "to name". The denominator *names* the fraction.



1.3.2 Multiplying rational numbers

Don't worry, your version of the 𝔄𝔩𝔤𝔢𝔟𝔯𝔞𝔫𝔬𝔪𝔦𝔠𝔬𝔫 isn't missing any sections. Most textbooks would discuss adding and subtracting rational numbers, but we're going to start by studying the most helpful of the rational number operations: multiplication.

Suppose that at the cheese market, Knut Krumbli is selling chunks from a 10-pound block of cave-aged goat cheese.[9] Half of the original block of cheese is left, and a local weaver asks to buy three-fourths of it. The original block had a value of 800 Swedish kronor. How much should Knut charge the weaver?

Let's draw a picture (fig. 1.2). In the images below, the square represents the original block of cheese. We divide the square in half vertically, and the region shaded yellow represents how much of the cheese remains. We can then divide the cheese into fourths horizontally and shade in three-fourths (the amount that the weaver wants to buy) in blue.

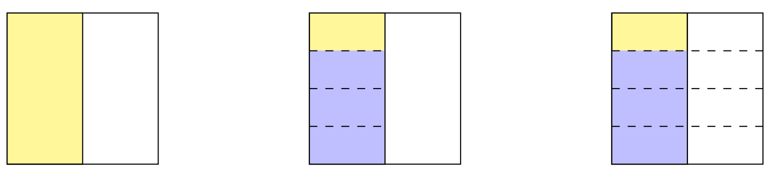


Figure 1.2: Selling three-fourths of one-half of a block of cheese.

What fraction of the whole block does this blue region represent? If we extend the lines, we can see that we've divided up the whole block of cheese into 8 equally-sized pieces. So the weaver is buying $\frac{3}{8}$ of the whole block and Knut should charge 300 kronor.

In the end, we solved a fraction multiplication problem: "How much is three-fourths of one-half of a whole block of cheese?"

$$\frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$$

[9] When cheese is aged in the cool, humid air of underground caves, it can develop a denser texture and a more complex flavor, since small salt crystals form throughout its interior.

Multiplying rational numbers

To multiply rational numbers, we multiply numerators to find the numerator of the product. We multiply denominators to find the denominator of the product.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

In the startup exploration for this section, $0 < a < b < 1$, and part (c) asks us to consider $a \cdot b$. Let's think about this from a block-of-cheese perspective. We start with less than a whole block of cheese (since b is less than 1) and we only want to buy a fraction of what's there (since a is less than 1).

So, we are certainly buying less than a whole block of cheese! Since both a and b are less than 1, we know that their product must be less than one.

Fancy versions of one

Recall that any number times 1 is itself. This simple fact, along with the fraction multiplication procedure, gives us an extremely powerful tool that we'll use in various different ways throughout algebra. The key idea is that multiplying something by 1 doesn't change its value, even if we use a "fancy version of 1".[10] Consider, for example:

$$\begin{aligned}
\frac{5}{8} &= \frac{5}{8} \cdot 1 && \text{multiplication by 1 doesn't change the number} \\
&= \frac{5}{8} \cdot \frac{7}{7} && \text{we can rewrite 1 however we like, here } 1 = \tfrac{7}{7} \\
&= \frac{35}{56} && \text{multiply fractions}
\end{aligned}$$

In the end, we have two equivalent fractions $\frac{5}{8} = \frac{35}{56}$. The representation has changed, but the value is the same! The "fancy one" we chose here is $\frac{7}{7}$, but any other version of 1 would work the same way: $\frac{-140}{-140}$, $\frac{2\pi}{2\pi}$, $\frac{\sqrt{3}}{\sqrt{3}}$... The possibilities are endless.

The first application of the "fancy one" has to do with one of the themes we encounter throughout algebra: the idea of finding a "completely simplified" solution to a problem. Fractions introduce us to the first criteria for something being simplified.

[10] Multiplication by a fancy version of 1 is an application of the identity property of multiplication, which we will study in more detail in chapter 5.

Simplified rational numbers #1

Fractions should be simplified to **lowest terms**, meaning that the numerator and denominator of the fraction are **relatively prime** integers.

Two integers are said to be relatively prime (or coprime) if they have no common factors other than 1. So, the fraction $\frac{21}{34}$ is in lowest terms, since the factors of 21 are $\{1, 3, 7, 21\}$ and the factors of 34 are $\{1, 2, 17, 34\}$. They have no factors in common, other than 1.

On the other hand, $\frac{18}{84}$ is not in lowest terms. Both 18 and 84 are even, and so both the numerator and denominator are divisible by at least 2. To write this fraction in lowest terms, we "undo" fraction multiplication and search for some fancy ones that we can eliminate:

$$\frac{18}{84} = \frac{2 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 3 \cdot 7} = \frac{2}{2} \cdot \frac{3}{3} \cdot \frac{3}{2 \cdot 7} = 1 \cdot 1 \cdot \frac{3}{14} = \frac{3}{14}$$

Once we know how and why this works, we can take a shortcut and "cancel" common factors from the numerator and denominator:

$$\frac{18}{84} = \frac{2 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 3 \cdot 7} = \frac{\cancel{2} \cdot \cancel{3} \cdot 3}{\cancel{2} \cdot 2 \cdot \cancel{3} \cdot 7} = \frac{3}{2 \cdot 7}$$

Simplify before you multiply

We can also use the "fancy one" to save ourselves some work! What if we have to multiply:

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6}$$

Let's write the product out "the long way" before we actually do any computations. The numerator of the product will be the product of the numerators, and likewise for the denominator:

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

Then, a common factor in the numerator and denominator is like multiplication by 1, so we can make some simplifications:

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = \frac{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5}}{\cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot 6} = \frac{1}{6}$$

The alternative would have been to multiply all of those numbers together by hand and then simplify the fraction to lowest terms[11], which is a bunch more work. It pays to be clever!

[11] That would have been $\frac{120}{720}$.

Example 1.6

Multiply: (a) $\frac{3}{4} \cdot \frac{7}{16}$ (b) $\frac{15}{8} \cdot \frac{4}{7} \cdot \frac{14}{3} \cdot \frac{1}{5}$

Problem (a) is nothing special. No common factors are shared between numerators and denominators, so we simply multiply as usual.

$$\frac{3}{4} \cdot \frac{7}{16} = \frac{3 \cdot 7}{4 \cdot 16} = \frac{21}{64}$$

Since there were no common factors to cancel before multiplying, we know the product is already in lowest terms.

To tackle (b), the first helpful step is to factor the individual numerators and denominators to expose all of the factors that are going to be in the product.

$$\frac{3 \cdot 5}{2 \cdot 2 \cdot 2} \cdot \frac{2 \cdot 2}{7} \cdot \frac{2 \cdot 7}{3} \cdot \frac{1}{5}$$

Then, a common factor in the numerator and denominator is like multiplication by 1, so common factors can be crossed out. Look at what happens in this case!

$$\frac{\cancel{3} \cdot \cancel{5}}{\cancel{2} \cdot \cancel{2} \cdot \cancel{2}} \cdot \frac{\cancel{2} \cdot \cancel{2}}{\cancel{7}} \cdot \frac{\cancel{2} \cdot \cancel{7}}{\cancel{3}} \cdot \frac{1}{\cancel{5}} = 1$$

Note that all of the factors cancel out in the denominator. A common mistake at this point is to replace the "empty" denominator with 0 – but remember, a common factor in the numerator and denominator is like a factor of 1, which is the fraction $\frac{1}{1}$, not $\frac{1}{0}$, so there's always a "phantom one" lurking there even if we don't write it.

1.3.3 Adding and subtracting rational numbers

Knut Krumbli would tell you that brining together a herd of 8 goats and a herd of 11 goats results in a herd of 19 goats. It's goat herding, not rocket science.

But, combining a herd of 4 goats and a flock of 13 sheep doesn't really give us 17 of anything until we can find some shared characteristic that is common to both groups. For instance, we could say we have a group of "17 mammals", or "17 quadrupeds".

So it goes with fractions. When our fractions have a shared name (a common denominator) we can total up how many things we have with that name: 8 thirds and 11 thirds makes 19 thirds. In math symbols, we write

$$\frac{8}{3} + \frac{11}{3} = \frac{19}{3}$$

When our quantities *don't* share a common unit (a common denominator), we have to find one before we can add in a meaningful way.

Given a fraction, we can use multiplication by a “fancy one” to generate a new fraction that has the same value but a different, perhaps more helpful, denominator.

Some people like to try and find the *least common denominator*, but that’s not strictly necessary. Any common denominator will do. In fact, a guaranteed common denominator of any two fractions is the *product of their denominators*.

Adding and subtracting rational numbers

To add rational numbers, we must have a common denominator, for example the product of the original denominators. Then we add the numerators, and keep the common denominator.

$$\frac{a}{b}+\frac{c}{d}=\frac{a\cdot d+b\cdot c}{b\cdot d}$$

To subtract rational numbers, change the subtraction problem to an “addition of the opposite” problem and then follow the algorithm for addition.

Here’s the derivation of fraction addition in a bit more detail. Note how we use multiplication by 1 to find a common denominator $b\cdot d$:

$$\frac{a}{b}+\frac{c}{d}=\left(\frac{a}{b}\cdot 1\right)+\left(1\cdot\frac{c}{d}\right)=\left(\frac{a}{b}\cdot\frac{d}{d}\right)+\left(\frac{b}{b}\cdot\frac{c}{d}\right)=\frac{a\cdot d}{b\cdot d}+\frac{b\cdot c}{b\cdot d}=\frac{a\cdot d+b\cdot c}{b\cdot d}$$

Negative fractions

Where should we put the negative sign when we have a negative fraction? Does it matter? Consider the following three possibilities. Are they all equivalent?

$$-\frac{3}{4}=\frac{-3}{4}=\frac{3}{-4}$$

A fraction is a way of writing a division problem. If Knut’s four children share three bowls of lingonberries equally, then each child will get $3\div 4=\frac{3}{4}$ of a bowl of berries.[12] The fraction $\frac{3}{4}$ is just another way of writing $3\div 4$. So, all three of the fractions above have the same value.

In the first example, the whole fraction has been negated. In the second and third examples, the numbers have opposite signs and so the quotient will be negative. In other words it actually doesn’t matter where we put the negative sign. We can put it where it is most convenient for the problem (very often, that’s in the numerator of the fraction).

[12] Lingonberries are a popular fruit in Scandinavia and throughout northern, central, and eastern Europe. The berries are quite tart, and so they are usually mixed with sugar and preserved as jam or compote. In Sweden and Norway, reindeer is traditionally served with gravy and lingonberry sauce. Yes, Scandinavians eat reindeer.

Mixed numbers

Improper fractions have a numerator that is greater than or equal (in absolute value) to their denominator, like $\frac{5}{3}$ or $\frac{-84}{16}$. Improper fractions have been scorned by many elementary school mathematics teachers, who instead prefer mixed numbers: $1\frac{2}{3}$ or $-5\frac{1}{4}$. But, improper fractions are often much easier to work with than mixed numbers.[13]

Simplified rational numbers #2

Simplified **improper fractions** are preferred over **mixed numbers** and decimals. Only convert to a mixed number or decimal when the context (or the directions) require it.

We usually prefer exact fraction answers over decimal approximations. Writing the decimal $\frac{10}{7}$, for instance, is much preferred over 1.43, and better even than the exact answer $1.\overline{428571}$ (yep, that's a big chuck o' repeating decimal).

But, be sure to read questions carefully! There are exceptions to these rules. When working in a real-world context, a certain number format may make more sense. For example, when solving a problem about money, the answer \$3.50 makes a lot more sense than $\$\frac{7}{2}$. Likewise, the answer "$1\frac{1}{3}$ pounds of cheese" is better than "$\frac{4}{3}$ pounds of cheese". In ambiguous cases, we will make it clear what number format is preferred.

When faced with mixed numbers in a problem, we have to be careful. When adding, we can convert all mixed numbers to improper fractions, or work with them "as is". Subtracting with mixed numbers is tricky, however, because we we may have to handle regrouping. Multiplication is even trickier.

Since we prefer improper fractions as final answers anyway, we recommend converting all mixed numbers to improper fractions before you start computations. To convert a mixed number to an improper fraction, all we have to do is think about the mixed number as an addition problem:

$$3\frac{5}{8} = 3 + \frac{5}{8} = \frac{3}{1} + \frac{5}{8} = \frac{3 \cdot 8 + 1 \cdot 5}{1 \cdot 8} = \frac{24 + 5}{8} = \frac{29}{8}$$

Here we used the fact that any integer has a "phantom one" in its denominator: $3 = \frac{3}{1}$. We don't usually write it, but it's there when we need it.

[13] One situation where improper fractions are superior is when describing the slope of a line, as we will see in chapter 7.

Example 1.7

Compute each of the following:

1. $\dfrac{3}{4}+\dfrac{5}{6}$

These fractions do not have a common denominator, so we'll have to find one. We could use their least common denominator (which is 12) or use the product of the denominators (which is 24). Let's use 24:

$$\frac{3}{4}+\frac{5}{6}=\frac{3\cdot 6+4\cdot 5}{4\cdot 6}=\frac{18+20}{24}=\frac{38}{24}=\frac{19}{12}$$

2. $1\dfrac{2}{5}-1\dfrac{7}{8}$

First, we'll convert to improper fractions and change the subtraction to addition-of-the-opposite, putting the negative sign in the numerator of the fraction. Then we add. In the end, we can adjust the negative sign again:

$$\frac{7}{5}-\frac{15}{8}=\frac{7}{5}+\frac{{}^-15}{8}=\frac{7\cdot 8+5\cdot {}^-15}{5\cdot 8}=\frac{56+{}^-75}{40}=\frac{{}^-19}{40}=-\frac{19}{40}$$

In the startup exploration for this section, $0<a<b<1$. Part (a) asks about $a+b$. Since both numbers are positive, their sum is positive, but we don't have enough information to tell whether the sum is greater than 1. If a and b are both less than $\frac{1}{2}$, for example, then their sum will be less than 1. On the other hand, if they are both greater than $\frac{1}{2}$, then their sum will be greater than 1.

Part (b) asks us to consider $a-b$. Since a is less than b, we're subtracting a larger number from a smaller number, and so the answer must be negative.

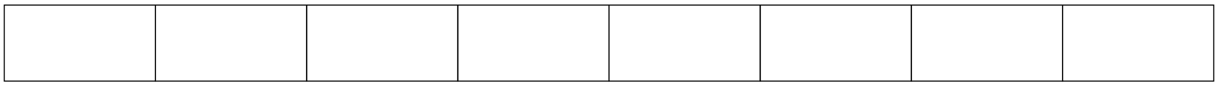
We can reason this out in another way: $a-b$ is the same as $a+({}^-b)$. The absolute value of b is the same as the absolute value of ${}^-b$. And so this sum will be negative because the negative number is the one with the greater absolute value. In any case, $a-b$ is negative and so we know for sure that it is less than 1.

1.3.4 Dividing rational numbers

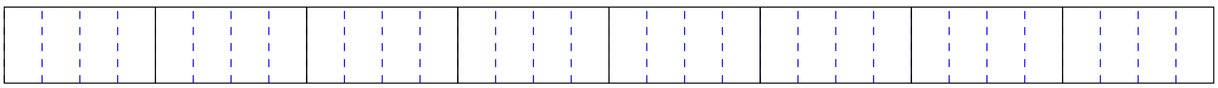
Fraction division may be the most poorly understood operation in all of arithmetic. The algorithm for dividing fractions seems arbitrary, and it's often difficult to judge whether our answers make sense. Let's pause for a moment to think about what fraction division means.

Suppose Jorunn Krumbli, Knut's wife, is making scarves for the goats (Scandinavian winters are chilly). She has 8 meters of burlap, and each scarf requires $\frac{3}{4}$ of a meter. How many scarves can she make?

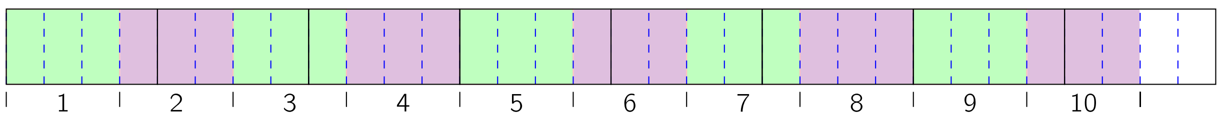
This question is asking us to compute $8 \div \frac{3}{4}$, but let's try to solve the problem by drawing a picture. Suppose this rectangle represents Jorunn's 8 meters of burlap.



To figure out how many pieces of $\frac{3}{4}$ meter are in there, let's first determine the number of pieces of size $\frac{1}{4}$ meter. To do that, we'll break each meter into four pieces. That gives us $8 \cdot 4 = 32$ pieces total.



Now, let's gather these pieces up into groups of three. We'll be able to make 10 whole groups, and we'll have 2 pieces left over. Since we have two pieces, but need three, we have $\frac{2}{3}$ of a group. So, Jorunn can make $10\frac{2}{3}$ scarves for the goats.



Let's retrace our steps. We set out to solve $8 \div \frac{3}{4}$, but in our picture we first multiplied to find the total number of fourths, and then we divided to find how many groups of three we could make. In other words: $8 \div \frac{3}{4}$ must have the same answer as $8 \cdot \frac{4}{3}$. Does that look familiar?

Dividing rational numbers

Dividing by a number is the same as multiplying by the reciprocal of the number. So, we can change a given fraction division problem into an equivalent fraction multiplication problem and then use the rules for fraction multiplication.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$$

Recall that the **reciprocal** of a fraction is the fraction that interchanges the numerator and the denominator of the original fraction. The reciprocal of an integer (which is sitting on a phantom 1) is "one over the original integer".

Example 1.8

Compute each of the following:

1. $\frac{5}{6} \div -\frac{3}{4}$

We don't need a common denominator or anything, so we can just jump right in with fraction division. We don't even need to move the negative sign, since we know the answer will be negative.

$$\frac{5}{6} \div -\frac{3}{4} = \frac{5}{6} \cdot -\frac{4}{3} = -\frac{20}{18} = -\frac{10}{9}$$

2. $3\frac{3}{4} \div 5$

First, we'll convert to improper fractions, then we'll implement fraction division At the end, we can simplify before we multiply!

$$3\frac{3}{4} \div 5 = \frac{15}{4} \div \frac{5}{1} = \frac{15}{4} \cdot \frac{1}{5} = \frac{3 \cdot 5}{4} \cdot \frac{1}{5} = \frac{3 \cdot \cancel{5}}{4} \cdot \frac{1}{\cancel{5}} = \frac{3}{4}$$

Let's think about this last answer for a second: does it makes sense? Look again at the problem we were given. If we interpret this as the question"how many groups of 5 are in $3\frac{3}{4}$?", then we can see that we can't even make one whole group: $3\frac{3}{4}$ is less than 5! So, it makes sense for our answer to be less than 1.

Fractions inside fractions

As if fractions on their own weren't enough, you may soon face the turducken[14] of arithmetic, the dreaded "fraction in a fraction". Note that the first criteria for simplified rational numbers states that the numerator and denominator must be *integers*.

These creatures can look hard to handle, but don't be intimidated. Recall that a fraction is just a division problem. So, we can rewrite things using a division sign and divide as usual.[15]

Example 1.9

Simplify each of the following:

[14] A "turducken" is a food product where a deboned chicken is stuffed inside a deboned duck, which is then stuffed inside a deboned turkey. In culinary terminology, this is an example of "engastration", a cooking method in which one animal is stuffed inside the gastric cavity of another... which is probably yet another phenomenon that you didn't know had a name.

[15] The fancy mathematical name for the ÷ symbol is *obelus* (plural: obeli)... another thing that you probably didn't know had a name).

1. $\dfrac{4}{\left(\frac{1}{3}\right)}$

All we have to do is rewrite the fraction-in-a-fraction as "numerator $\div$ denominator", and then divide as usual.

$$\frac{4}{\left(\frac{1}{3}\right)} = 4 \div \frac{1}{3} = 4 \cdot \frac{3}{1} = 4 \cdot 3 = 12$$

2. $\dfrac{\left(\frac{2}{5}\right)}{\left(1\frac{3}{8}\right)}$

Here we must rewrite using the obelus, then convert to improper fractions, then divide!

$$\frac{\left(\frac{2}{5}\right)}{\left(1\frac{3}{8}\right)} = \frac{2}{5} \div 1\frac{3}{8} = \frac{2}{5} \div \frac{11}{8} = \frac{2}{5} \cdot \frac{8}{11} = \frac{16}{55}$$

In the startup exploration for this section, we have rational numbers a and b where $0 < a < b < 1$. Part (d) asks us to consider $a \div b$. This division problem is another way of writing $\frac{a}{b}$. Since a is less than b, though, we know that this fraction is a proper fraction (as opposed to an improper fraction), which means it is less than 1. So, $a \div b$ is less than 1.

1.3.5 Evil and Wrong

Everyone makes mistakes, but not all mistakes are created equal. Some mistakes are not just wrong, the are **Evil and Wrong**. "Wrong" because they are mistakes, "evil" because they are subtle, and sneaky, and tempting.

What we mean here is that sometimes we feel drawn to perform certain arithmetic or algebraic maneuvers – some things seem so logical, so easy, so natural, so tempting – when in reality, they are a total trap. For example:

Warning!

Armed with the idea of "simplify before you multiply", we might want to try and pull this stunt in other places:

$$\frac{2+3}{2+7} = \frac{\cancel{2}+3}{\cancel{2}+7} = \frac{3}{7}$$

Seems logical, right? But so wrong! There is no "simplify before you add" maneuver, tempting though it may be. Such a thing is **Evil and Wrong**.

You may be thinking, "Nah, I'd never to that," and with numerical expressions like these, you may be right. But later, when faced with variable expressions like

$$\frac{x+3}{x+7}$$

this temptation may come back, in disguise. We'll draw attention to these **Evil and Wrong** mistakes as we go along because they are so tempting. Beware!

1.4 Order of operations

We turn now to something else that it probably familiar, the **order of operations**. Consider the following problem as we get started:

Startup exploration: Who's right?

The four Krumbli kids each performed the following computation:

$$14 - 6 \cdot 7 + 10$$

Sini got $^-38$, Siri got $^-18$, Sten got 66, and Stig got 136. Which of them performed the computation correctly? What mistakes did the others make?

The order of operations gives us a standard procedure for simplifying numeric expressions. A **numeric expression** is the algebra way of describing something you may have called just a "math problem" in elementary school. For example $12 - 5$ is a numeric expression. It is not in its simplest form because we can evaluate $12 - 5$ and write 7 instead.

Simplification rule #1

A numeric expression is completely simplified if all operations have been evaluated and all grouping symbols have been eliminated. The resulting quantity is called the *value* of the expression.

As we go along in algebra, we will learn many rules that maintain "mathematical equivalence." Expressions are mathematically equivalent if they represent the same quantity. For example, $\frac{4}{8}$ is equivalent to $\frac{1}{2}$, and $12 - 5$ is equivalent to 7. Our job when we simplify is to maintain the equivalence from one step to the next. The order of operations is a set of rules for how to do that.

Order of operations

The **order of operations** is an agreed-upon order for simplifying numeric expressions. The big idea is that "more powerful" operations take priority over "less powerful" operations. When we want to alter the usual rules of precedence, we introduce grouping symbols to make our intentions clear.

When simplifying an expression, we evaluate things in the following order:[a]

First: Grouping symbols like (parentheses), [square brackets], {curly braces}, as well as other subtle grouping symbols like absolute value and the vinculum. Here we work from the innermost set of groupers[b] to the outermost.

Then: Exponents (and, later, roots and logarithms). In the case of a "stack" of exponents, work from the top down.

Then: Multiplication and division. Recall that dividing is the same as "multiplying by the reciprocal". So, these two operations have the same priority and we work from left to right.

Finally: Addition and subtraction. Recall that "subtracting" is the same as "adding the opposite". So, these two operations have the same priority and we work from left to right.

[a] PEMDAS is an mnemonic acronym for remembering the order of operations: Parentheses, Exponents, Multiplication, Division, Addition, Subtraction. In Canada it's BEDMAS (B for Brackets). In the UK and Australia it's BIDMAS or BODMAS (Indices or Orders, which are other words for exponent). A better acronym might be GEMS or PEMA, which group together the pairs of operations that have the same priority (the G is for Grouping symbols).

[b] As in grouping symbols, not the fish.

Example 1.10

Simplify: $14 - 6 \cdot 7 + 10$

Solution: The correct answer follows the order of operations, like so:

$$\begin{aligned} 14 - 6 \cdot 7 + 10 &= 14 - 42 + 10 && \text{work mult'n before addition or subtraction} \\ &= {}^{-}28 + 10 && \text{work add'n and sub'n left to right} \\ &= {}^{-}18 \end{aligned}$$

This is the problem from the startup exploration, so it is Siri who had the right answer.

Students who just memorize a clever mnemonic device might be tempted to do the Addition before the Subtraction, but don't be fooled! Those two operations have the same priority. (Sini made this mistake and got ${}^{-}38$.)

Sten worked the operations straight through from left to right, all at the same priority and got 66. Stig did something very creative, computing $(14 - 6) \cdot (7 + 10) = 8 \cdot 17 = 136$.

Notice how we showed the work going down the page, simplifying the problem one step at a time. Some people prefer to work across, and that's OK too. The point is that it's important to show work in an organized way when we solve a problem so that our reasoning and thought process are clear.

The work you show is the road map to your solution. Whoever reads over your work must be able to follow and understand your steps, without having to make any assumptions about what you actually did to reach your answer. It's a bad habit to skip steps, and it's no help to people reading your work to say you did a step in your head. Every step you take needs to be written down clearly and neatly.

Abuse of the equal sign

Consider the following work, written out by a student to simplify $8 \cdot 4 + 10$

$$8 \cdot 4 = 32 + 10 = 42 \qquad \text{OK or not OK?}$$

The student has reached the correct value, and it may even be clear what the student is thinking: "8 times 4 is 32, plus 10 more makes 42". This, however, is a heinous abuse of the equal sign! Look at what the first part of the work says:

$$8 \cdot 4 = 32 + 10 \qquad \text{These are not equal!}$$

One way to avoid this misuse of the equal sign is to write your work going down the page (as shown above). If you prefer to write across the page, be sure to write out the whole problem as you perform each simplification:

$$8 \cdot 4 + 10 = 32 + 10 = 42$$

In the next few sections, we'll look more closely at some trickier aspects of the order of operations.

1.4.1 About grouping symbols

As expressions get complicated, we may have grouping symbols inside of other grouping symbols. If there are different symbols, we can more easily see where different groups begin and end. But, we may just find a bunch of parentheses, like in the example below. In that case, we have to be a bit careful about what's being grouped together.

Example 1.11

Simplify: $12 + (3 - (4 - 2) + 5)$

Solution: When faced with multiple grouping symbols, we must start with the innermost set of grouping symbols and evaluate our way to the outermost. Once we have simplified the expression inside a set of grouping symbols down to a single quantity, we can write that quantity without the groupers.

$$\begin{aligned} 12+(3-(4-2)+5) &= 12+(3-2+5) \\ &= 12+(1+5) \\ &= 12+6 \\ &= 18 \end{aligned}$$

Vinculum is a grouping symbol

The **vinculum** (as in the fraction bar) is a grouping symbol. For example, if the task is to simplify a fraction such as:

$$\frac{20+2^2\cdot(14-9)}{(2-4)^3}$$

then we must think of the expression in the numerator as a group, and likewise for the denominator. In other words, like so:

$$\Big(20+2^2\cdot(14-9)\Big)\div\Big((2-4)^3\Big)$$

We have two options for simplifying this. We might keep it in a fraction the whole time, or we could simplify the numerator and denominator separately and then squish them back into a fraction at the end.

Example 1.12

Simplify: $\dfrac{20+2^2\cdot(14-9)}{(2-4)^3}$

Solution: Let's dismantle this thing and handle it in two pieces. The numerator works like this:

$$\begin{aligned} 20+2^2\cdot(14-9) &= 20+2^2\cdot 5 \\ &= 20+4\cdot 5 \\ &= 20+20 \\ &= 40 \end{aligned}$$

The denominator works like this: $(2-4)^3 = (^-2)^3 = {}^-8$. Then, we can put the pieces back into their original fraction configuration:

$$\frac{20+2^2\cdot(14-9)}{(2-4)^3} = \frac{40}{^-8} = {}^-5$$

1.4.2 About exponents

An expression of the form a^b is read "a to the power of b" or "a to the b^{th} power". In such an expression, a is called the **base** and b is called the **exponent**. We call the whole thing a **power** of a, since a is the base.

We will get into more detail about exponents later on, but we'll pause here to mention two key ideas. First, recall that we can think about an exponent as shorthand for a repeated multiplication.[16]

$$a^b = \underbrace{a \cdot a \cdot a \cdots a}_{b \text{ times}}$$

That part you probably knew already. This next fact may be new.

Raising to the power zero

For any nonzero number a, the expression $a^0 = 1$. In other words, any nonzero number raised to the power 0 equals 1.

When we say that a is a "nonzero number" this means, naturally enough, that a cannot be 0. The expression $0^0 \neq 1$. What *does* it equal? That's a tricky question that will have to wait for later. 0^0 is an unusual mathematical creature![a]

[a] It's not the only one, either. In 1872, Karl Weierstrass (or Weierstraß, if you prefer the German double-S) shook the foundations of calculus with his mathematical monster, now called the "Weierstraß function". It's a bit complicated to get into the details, but he described a mathematical rule which behaves like the geometric "fractals" that we'll study in chapter 2... and it made some other mathematicians very upset.

We'll get into the "hows and whys" of exponents in chapter 11, and we'll return to the idea of the zero exponent. In the meantime, here's an example of how this fact might come in handy.

Example 1.13

Simplify: $\left(\dfrac{120 - (24 - 5^2)}{7^2 \cdot 400 \div 6^3}\right)^0$

Solution: If we go on "auto-pilot" we might follow all of the simplification rules, work from the inside to the outside, simplify the numerator, simplify the denominator...

But, the expression is raised to the power 0. So the answer is probably 1! We must check that we don't

[16] As with "multiplication is repeated addition", this interpretation breaks down eventually. Expressions like 5^{-3} and $16^{\frac{1}{2}}$ don't really translate well into "repeated multiplication". Don't panic about the idea of a negative number or a fraction up there in the exponent! All will be revealed as the course goes on.

have 0^0, but we can use a little number sense to do a quick check of the numerator in the fraction, and see that it will not equal zero. (Can you see why, without having to work it all out?)

We must also check that the denominator is not zero. A quick check there shows that it is not zero either. (Can you see why?)

So, this one's easy:

$$\left(\frac{120-(24-5^2)}{7^2\cdot 400\div 6^3}\right)^0=1$$

The lesson here is to look at the entire problem and plan an efficient solution strategy *before* just jumping in and crunching numbers. In this case, we can save ourselves a lot of work with a bit of careful observation.

Fractions versus exponents

If we have a fraction raised to a power, we must be very mindful of the notation and to what, exactly, the exponent applies.

Example 1.14

Simplify each of the following:

1. $\left(\frac{2}{3}\right)^4$ The parentheses indicate that we are multiplying together 4 copies of the fraction:

$$\left(\frac{2}{3}\right)^4=\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)=\frac{16}{81}$$

2. $\frac{2^4}{3}$ Remember that the numerator is a group, and so the exponent applies only to the 2, not the whole fraction:

$$\frac{2^4}{3}=\frac{2\cdot 2\cdot 2\cdot 2}{3}=\frac{16}{3}$$

Keep this last example in mind when writing your own expressions, as well. Our notation must match our intentions. What exactly so we intend an exponent to apply to? Do we need to add parentheses to make our intentions clear?

One of the trickiest concepts in algebra 1

What is the difference between the following three expressions?

$$(-3)^2 \qquad -(3^2) \qquad -3^2$$

Tricky, right? This is an important difference that will come back over and over again, and will look a little different every time.

In the first case, the parentheses make it clear: $(-3)^2$ means "raise negative three to the second power".

$$(-3)^2 = -3 \cdot -3 = 9$$

As usual, the product of two negatives is positive (Hitler gets hit by a truck). The second case is clear as well. The parentheses indicate that we should simplify the exponent first and then take the opposite of the result:

$$-(3^2) = -(3 \cdot 3) = -(9) = -9$$

The third case, -3^2, is the tricky one. It looks kind of ambiguous, since there are no parentheses. But, suppose we wrote the problem like this:

$$0 - 3^2$$

Now, it's clear what to do even without the parentheses, and that is the key.

Opposites of numbers to a power

The expression $-a^n$, written without parentheses, is equivalent to the parenthesized expression $-(a^n)$. For example: $-3^2 = -(3^2) = -9$.

You might be saying to yourself, "No sweat. I get it." But (if history and human nature are any indication) you may find yourself tripping over this concept at some point. Confusion around the notation can pop up, for instance, when we're typing expressions into a graphing calculator.[17]

1.4.3 About multiplication

In algebra we frequently use x (the letter) to stand for a unknown or variable quantity. So, we never use $\times$ to show multiplication. It would be too confusing to have a mix of letter-x's and multiplication-$\times$'s in the same

[17] A note to old-school parents who are used to working with calculators that use reverse Polish notation (RPN) or a stack, and might want to argue that $-3^2 = 9$. When using an RPN calculator we punch in -3 and press enter to push that quantity onto the stack. The number and the negative are both in the stack together, so squaring the entry on the top of the stack means squaring negative three, which is equivalent to $(-3)^2$ and not the same as -3^2.

expression. Can you imagine trying to decode something like:

$$x \times 3 + 4 \times x = x + 4x \times x \times x \qquad \text{Yikes!}$$

No more x for multiplication!

To show the operation of multiplication, use an asterisk, a dot, or parentheses. Instead of writing 3×4 to represent "3 times 4", we write: $3 * 4$, or $3 \cdot 4$, or $3(4)$.

Implied operations

A key aspect of algebra will be learning to read the notation and use it correctly when writing expressions and equations. Algebra is very much like a language, and all languages have special rules. For example, in English we sometimes smash up two words as a contraction: instead of writing "do not" we can write "don't".

We used contractions (of a kind) in mathematics as well. We don't usually write a positive sign in front of positive numbers. We don't usually write the phantom 1 that is in the denominator of an integer.

Another kind of mathematical contraction is the use of an **implied operation**. This arises most often when dealing with multiplication. Here's an example of how implied operations can sneak into a problem.

Example 1.15

Simplify: $12 - 5(2 + 8)$

Solution: Be on the lookout for implied operations!

$$\begin{aligned} 12 - 5(2+8) &= 12 - 5(10) \\ &= 12 - 50 && \text{That 5(10) means mult'n, and must happen first!} \\ &= {}^{-}38 \end{aligned}$$

A very common mistake is to do the $12 - 5$ first, instead of the $5(10)$, but that would violate the order of operations!

In this chapter we have thought deeply about fundamental ideas regarding numbers and operations, and perhaps you have seen some familiar ideas in a new light. Armed with this knowledge, we now venture deeper into the algebraic wilderness.

A mathematician, like a painter or a poet, is a maker of patterns.

G. H. Hardy
British mathematician

Chapter 2

Sequences

In chapter 1 we reviewed some key ideas from arithmetic. Arithmetic involves the manipulation of numbers via certain operations (like addition and multiplication). In algebra, on the other hand, we often use symbols to replace numbers. This can be disorienting at first, but it is useful because it allows us to speak about *relationships and patterns* involving numbers, rather than specific numbers.

Given a few numbers that form a pattern, for example, we can use algebraic symbols to describe all of the numbers that fit the given pattern – even though there may be *infinitely many* numbers that fit the pattern! Since patterns lie at the heart of algebra, and they are the focus of this chapter.

2.1 Sequences and recursion

Startup exploration: Communicating a pattern

Predict the next few numbers in the number pattern shown below.

$$2, 5, 8, 11, 14, 17, \ldots$$

How would you describe this pattern to a partner who could not see it? Could you communicate the pattern *without actually listing all the numbers*? What's the minimum amount of information your partner would need to recreate the pattern?

Informally, we call this a number pattern. Mathematically speaking, an ordered list of numbers like this is called a **sequence**. Each of the numbers in the list is called a **term** of the sequence.

Sequences often have patterns within them. Perhaps, when thinking about how you'd describe this sequence to a partner, you thought about a rule like "add 3" or "+3". (Do you see how this applies to the given pattern?)

But “add 3” is not enough to recreate the sequence. Consider the sequence: $1, 4, 7, 10, 13, \ldots$. And what about $-10, -7, -4, -1, 2, \ldots$? The phrase “add 3” also applies to these sequences, even though they are different from the sequence in the startup exploration.

To distinguish these different sequences, we must include the starting value in our description. We can describe the original sequence clearly and unambiguously by saying something like: “Start with 2, then add 3 to the previous value.”

So, when describing the pattern of a sequence we are really describing how to generate the sequence from scratch. To do that, we have to answer these two questions: First, how does the sequence begin? Second, what must we do to the *current* term to find the *next* term of the sequence? This is called the **recursive** description of the sequence.

Recursive

Describes a procedure that is applied over and over again, starting with a number or a geometric figure, to produce a sequence of numbers or figures.

As the definition says, we can start a recursive procedure with a number or a geometric figure. We'll start our exploration of sequences by studying geometric figures called **fractals**.

Fractal

A geometric figure that has undergone infinitely many applications of a recursive procedure and exhibits the property of self-similarity.

Fractal geometry is often called “the geometry of nature”. If we look around the natural world, it is not like we see a lot of perfectly straight lines, rigid rectangles, and regular pentagons. But, the growth of a tree can be described using a recursive procedure: grow towards the sun for a bit, branch off at an angle, repeat. Trees exhibit self-similarity. If we break off a branch of a tree and stick it in the ground, looks just like a little tree!

Clouds, coastlines, mountains, trees, Romanesco broccoli, the folds of your brain, your vascular system, your bronchial tubes, the lining of your small intestines... all of these are a kind of fractal. [1] Technically speaking, natural fractals only have their recursive procedure applied a handful of times (we say the procedure has a handful of *iterations*) so they aren't true mathematical fractals. A mathematical fractal undergoes an infinite number of iterations. [2]

[1] In 1968, Hungarian biologist Aristid Lindenmayer developed a method for writing recursive rules that can be used to model the growth of algae. Called “Lindenmayer systems” or “L-systems” today, his methods have been used to model more complex organisms, as well as purely mathematical structures.

[2] Fractals may play an interesting role later on in your study of mathematics, for example the Mandelbrot set is a fractal that involves the complex numbers. Do an internet search for “Mandelbrot set” and check out the pictures!

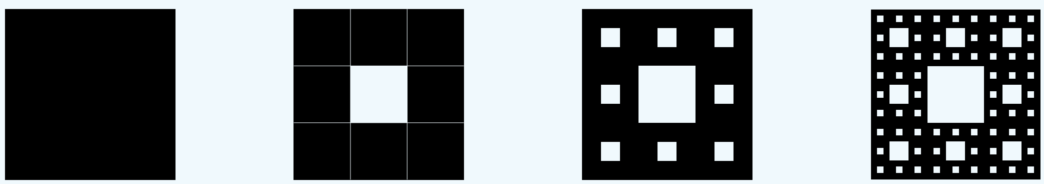
We will begin our study of sequences by looking at some famous mathematical fractals that were first studied by Polish mathematician Wacław Sierpiński.

Extended exploration: Sierpiński's triangle

[TODO] Click here to visit the extended exploration: Sierpiński's triangle

Startup exploration: Sierpiński's carpet

To draw Sierpiński's carpet, we begin with a square called "stage 0". We subdivide this square into nine congruent sub-squares and remove the one in the center. We repeat the process with the remaining eight sub-squares. Stages 0 through 3 are shown below.



Stage 0 Stage 1 Stage 2 Stage 3

There is one solid square in stage 0, and there are eight (smaller) solid squares in stage 1. How many of the smallest solid squares are there in each stage 2? What about stage 3?

2.1.1 Algebra of Sierpiński's carpet

Sierpiński's carpet generates some interesting sequences of numbers. For example, if we consider the number of (smallest) squares at each stage of the fractal. We have one square in stage 0, and eight squares in stage 1.

To create stage 2, we divide each of the eight stage-one squares into 9 pieces, and then remove the center square. So each of the eight squares from stage 1 turns into eight new, tiny squares in stage 2. So there are $8 \cdot 8 = 64$ tiny squares in stage 2. To create stage 3, each of these sixty-four tiny squares becomes 8 super-tiny squares, so there are $64 \cdot 8 = 512$ super-tiny squares in stage 3. Together, we have a sequence that begins:

$$1, 8, 64, 512, \ldots$$

What is a recursive rule for this sequence? The sequence starts with 1, and then to go from one number to the next, we multiply by 8. So, that's our rule: "start with 1, multiply the previous value by 8". In other words, to find the next term in the sequence, take the previous term (the last term in the sequence that we know) and multiply by 8.

We can use this rule to generate the next few terms of our sequence, but watch out! We quickly end up with a lot of squares and you don't want to get any on your shoes.

$$1,\ 8,\ 64,\ 512,\ 4096,\ 32\,768,\ 262\,144,\ 2\,097\,152,\ 16\,777\,216,\ \ldots$$

We can always write a recursive rule as a sentence, as we did above. Another way to capture a recursive procedure is using a "now-next" rule, sometimes called a "start-now-next" rule. If we have the sentence: "Start with 1, multiply the previous value by 8," we could write the now-next rule as follows:

$$\text{START} = 1$$
$$\text{NEXT} = \text{NOW} \cdot 8$$

It's pretty obvious that the first part of the rule says where to start. The second part of the rule says: "To find the *next* number in the pattern, we take the number we have *now* and multiply by 8."

2.1.2 Recursive rules and formulas

Recursive rules are easy to write in sentence form, and now-next equations are nice and succinct, but there is a more mathematical way. We are going to write what we call a recursive formula.

We use the variable a with a subscript to represent a specific term of the sequence. So, a_1 represents the first term of the sequence, a_2 represents the second term of the sequence, and a_{98} would represent the 98th term of the sequence.

For example: given the sequence $4, 12, 36, 108, \ldots$, we have:

$$a_1 = 4$$
$$a_2 = 12$$
$$a_3 = 36$$
$$a_4 = 108$$

We use a_n to represent any old term of the sequence. Then, a_{n+1} represents the *next* term in the sequence. (Can you explain why?)

We can write the recursive rule either as "start with 4, multiply the previous term by 3", or "START = 4, NEXT = NOW · 3". Here's how we can translate this into a recursive formula.

"Start with 4" means that the first term of the sequence is 4. We write $a_1 = 4$, since a_1 represents the first term of the sequence. This is just like "START = 4" in the now-next rule. To translate "NEXT = NOW · 3", we write: $a_{n+1} = a_n \cdot 3$.

A note about notation: When multiplying a number and a letter, we usually write the number first and we don't usually write a multiplication symbol in between.[3] So, we have created the recursive formula:

$$a_1 = 4$$
$$a_{n+1} = 3a_n$$

Example 2.1

Write the recursive formula for the sequence $1, 5, 25, 125, 625, \ldots$.

Solution: With a little exploration, we see that the sentence version of this rule is "Start with 1, multiply previous by 5", and the now-next version is "START = 1, NEXT = NOW · 5". So, we have the recursive formula: $a_1 = 1, \quad a_{n+1} = 5a_n$.

Example 2.2

Write out the first five terms of the sequence generated by each rule.

1. "Start with 128, multiply previous by $\frac{1}{2}$"

   Solution: The rule states clearly that the first term is 128, no trouble. Then, to find the second term, we multiply the first term by $\frac{1}{2}$, that means $128 \cdot \frac{1}{2} = 64$. To find the third term, we multiply the second term by one-half: $64 \cdot \frac{1}{2} = 32$. We repeat for the next few terms, which gives:

   $$128, 64, 32, 16, 8, \ldots$$

2. $a_1 = 12, \quad a_{n+1} = {}^{-}2 \cdot a_n$

   Solution: The first term is a_1, and the formula says that's 12. Then, to find a_2, the second term, we have

   $$a_2 = {}^{-}2 \cdot a_1 = {}^{-}2 \cdot 12 = {}^{-}24.$$

   We continue to multiply by $^{-}2$ each step of the way and get:

   $$12, {}^{-}24, 48, {}^{-}96, 192, \ldots$$

[3] More on working with letters, or variables, in chapter 3.

Recursive rules and formulas are handy for describing a sequence, but suppose we want to skip around and find random terms of the sequence. In this situation, the recursive rule is the worst possible rule to have!

For example, how could we use the rule $a_1 = 4, \quad a_{n+1} = 3 \cdot a_n$ to find the value of the 1000th term in the sequence, a_{1000}? The rule tells us that $a_{1000} = 3 \cdot a_{999}$. But, what's a_{999}?

Well, $a_{999} = 3 \cdot a_{998}$ But, what's a_{998}?

Hmm. $a_{998} = 3 \cdot a_{997}$ But... oh boy. Can you see the problem here?

If we want to skip around and find random terms in a sequence, it's much easier to use a different kind of formula, called an "apparent" or "explicit formula". More on those in the next section!

2.2 Geometric sequences

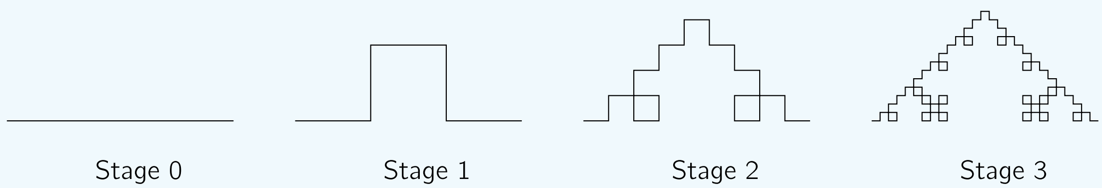
We are going to continue our study of sequences by looking at another fractal. In 1904, Swedish mathematician Helge von Koch first described a kind of fractal that has since come to be known as a Koch curve. There are several variants of the Koch curve. We'll look at two different forms in the following explorations.

Extended exploration: Koch curve, triangular version

[TODO] Click here to visit the extended exploration: Koch curve

Startup exploration: Koch curve, square version

We begin in stage 0 with a line segment of length 1. To create stage 1, we alter the segment as follows: cut it into three pieces, and replace the center piece with three sides of a square. We repeat the process for each line segment in the previous figure to create stages 2 and 3.



Write a recursive formula describing the number of segments in each stage of the fractal.

2.2.1 Algebra of the Koch curve

The Koch curves are beautiful things, at once incredibly simple and incredibly complex. As the square-based version above grows, each line segment is replaced by five shorter segments. The recursive rule is "start with 1, multiply the previous value by 5".

To compute the number of segments in each stage, we might organize our work in a list like this:

$$\begin{array}{rl}
1 & \text{segment in stage 0} \\
(1)\cdot 5 & \text{segments in stage 1} \\
(1\cdot 5)\cdot 5 & \text{segments in stage 2} \\
(1\cdot 5\cdot 5)\cdot 5 & \text{segments in stage 3} \\
(1\cdot 5\cdot 5\cdot 5)\cdot 5 & \text{segments in stage 4} \\
(1\cdot 5\cdot 5\cdot 5\cdot 5)\cdot 5 & \text{segments in stage 5}
\end{array}$$

We can use a bit of shorthand, and write this repeated multiplication using an exponent.

$$\begin{aligned} 1 &= 5^0 && \text{segment in stage 0} \\ 1 \cdot 5 &= 5^1 && \text{segments in stage 1} \\ 1 \cdot 5 \cdot 5 &= 5^2 && \text{segments in stage 2} \\ 1 \cdot 5 \cdot 5 \cdot 5 &= 5^3 && \text{segments in stage 3} \\ 1 \cdot 5 \cdot 5 \cdot 5 \cdot 5 &= 5^4 && \text{segments in stage 4} \\ 1 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 &= 5^5 && \text{segments in stage 5} \end{aligned}$$

Notice that the exponent is equal to the stage number. This "5 to a power" notation works even for stage 0, since since $5^0 = 1$.

So, if we want to know how many segments are in stage 8 of the fractal, we can use this pattern to predict that there will be $1 \cdot 5^8$ segments. If we let x represent the stage number, then stage x of the fractal will have 5^x segments.

We have discovered a way of calculating the number of segments that is *not* recursive, because it doesn't rely on our knowing any of the previous terms. Instead, to produce the value of a certain term, all we need is the *number of the term*. We can compute the number of line segments in stage x without having to know anything about the stages that came before it.

2.2.2 Explicit formulas for sequences

In our discussion of fractals, we have always described the first image as "stage 0" of the fractal. But, when we write out a sequence, the first term is, well, the *first* term (not the *zero*th term).[4]

In other words, the same pattern of values has a slightly different numbering, depending on whether we're describing stages of a fractal or terms in a sequence.

Value	1	5	25	125
Fractal Stage Number	0	1	2	3
Sequence Term Number	1	2	3	4

So, if we want to write a recursive formula for the terms of a sequence, we have to make a little adjustment:

$$\begin{aligned} a_1 &= 1 && = 5^0 \\ a_2 &= 5 && = 5^1 \\ a_3 &= 25 && = 5^2 \\ a_4 &= 125 && = 5^3 \end{aligned}$$

[4] In some scientific disciplines, it is customary to start counting with zero: for example, in computer science. Jason, one of the authors of the 𝔄𝔩𝔤𝔢𝔟𝔯𝔞𝔫𝔬𝔪𝔦𝔠𝔬𝔫, is a computer scientist by training and thinks this way. Jason also prefers to include 0 as one of the natural numbers. Patty, the other author of the 𝔄𝔩𝔤𝔢𝔟𝔯𝔞𝔫𝔬𝔪𝔦𝔠𝔬𝔫, is a mathematician by training and prefers to start counting at 1.

Can you see the relationship between the subscript and the exponent? If we let a_n represents any term of the sequence, then our rule is:

$$a_n = 5^{n-1}$$

Rules of this kind are called apparent formulas or explicit formulas. One benefit of rules like this is that if we want to know, say, the number of segments in the curve at stage 1904, we can compute simply:

$$a_{1904} = 5^{1903}$$

By the way, this number is enormous. It's more than 1300 digits long!

Example 2.3

Write explicit formulas for the other features of the Koch curve.

Solution: *Length of one segment.* Each segment in a certain stage is one-third the length of the segment in the stage before. So, the sequence generated by the length of one segment in each stage is

$$\left(\frac{1}{3}\right)^0, \left(\frac{1}{3}\right)^1, \left(\frac{1}{3}\right)^2, \ldots, \quad \text{and so} \quad a_n = \left(\frac{1}{3}\right)^{n-1}.$$

Total length of the curve. Since we know the number of segments and the length of each segment, we can multiply to find the total length of the curve. We have

$$\left(\frac{5}{3}\right)^0, \left(\frac{5}{3}\right)^1, \left(\frac{5}{3}\right)^2, \ldots, \quad \text{and so} \quad a_n = \left(\frac{5}{3}\right)^{n-1}.$$

Note that we're putting these fractions in parentheses! Our notation has to match our intentions and in this case we want to show that the *whole fraction* is being raised to a given power.

Example 2.4

What if the stage 0 figure had been a segment of length 7, rather than length 1? How would that change our formula?

Solution: The number of segments would not change, but the length of each segment (and the total length of the curve) would! The new sequence for the length of one segment would be generated as follows:

$$
\begin{aligned}
7 &= 7 * \left(\frac{1}{3}\right)^0 && \text{length of one segment in stage 0} \\
7 * \left(\frac{1}{3}\right) &= 7 * \left(\frac{1}{3}\right)^1 && \text{length of one segment in stage 1} \\
7 * \left(\frac{1}{3}\right) * \left(\frac{1}{3}\right) &= 7 * \left(\frac{1}{3}\right)^2 && \text{length of one segment in stage 2} \\
7 * \left(\frac{1}{3}\right) * \left(\frac{1}{3}\right) * \left(\frac{1}{3}\right) &= 7 * \left(\frac{1}{3}\right)^3 && \text{length of one segment in stage 3}
\end{aligned}
$$

Again we can use an exponent to simplify the repeated multiplication of $\frac{1}{3}$. This generates the sequence

$$7 * \left(\frac{1}{3}\right)^0, \quad 7 * \left(\frac{1}{3}\right)^1, \quad 7 * \left(\frac{1}{3}\right)^2, \; \ldots .$$

If we let n represent the term number, then the recursive formula for this sequence is

$$a_n = 7 * \left(\frac{1}{3}\right)^{n-1}$$

2.2.3 Geometric sequences

So far, all of our sequences have had recursive rules like "start with A, *multiply* the previous term by B". Sequences with recursive rules of this type are called **geometric sequences**. Geometric sequences belong to the family of *exponential relationships*, because the $(n-1)$ expression appears as an exponent.

To generate the next term of a geometric sequence, we multiply the previous term by a fixed value. This fixed value is sometimes called, naturally enough, the *constant multiplier*. More often, it is called the *common ratio*.

Geometric sequence

A sequence where the ratio between each pair of successive terms is constant. The constant ratio is called the *common ratio*, usually denoted r. Geometric sequences are exponential relationships.

Example 2.5

Determine whether or not the sequence $4, 12, 36, 108, \ldots$ is a geometric sequence.

Solution: If this is a geometric sequence, then it must have a rule of the form "start with A, multiply the previous term by B"? Let's check.

To go from 4 to 12, we multiply by $\frac{12}{4} = 3$.

To go from 12 to 36, we multiply by $\frac{36}{12} = 3$. Looking good so far!

To go from 36 to 108, we multiply by $\frac{108}{36} = 3$. Nice! Based on the four terms given, the sequence is geometric.

Now, look at what we did to determine this: we created ratios of successive terms, and found that they were all the same.

$$\frac{12}{4} = \frac{36}{12} = \frac{108}{36} = 3$$

So, the *common ratio* for this sequence is 3.

Example 2.6

Write recursive and explicit formulas for the geometric sequence $32, 24, 18, 13\frac{1}{2}, \ldots$.

Solution: To get from 32 to 24, our first instinct might be to subtract: $32 - 8 = 24$. But, we're told in the problem that this is a *geometric* sequence, and that means that the recursive rule involves multiplication, not subtraction.

How can get from 32 to 24 using multiplication? The constant multiplier must be less than one (can you explain why?), and we can divide to find what it is:

$$\frac{24}{32} = \frac{3}{4}$$

So, $\frac{3}{4}$ is a good candidate for the constant ratio of the sequence. Let's check the other terms to see if we're right. We multiply 24 by $\frac{3}{4}$ to see if that gives us the next term in the sequence:

$$24 \cdot \frac{3}{4} = \frac{24}{1} \cdot \frac{3}{4} = \frac{\cancel{4} \cdot 6}{1} \cdot \frac{3}{\cancel{4}} = \frac{6}{1} \cdot \frac{3}{1} = 18 \qquad \text{Check!}$$

Now see if 18 times $\frac{3}{4}$ gives the next term:

$$18 \cdot \frac{3}{4} = \frac{18}{1} \cdot \frac{3}{4} = \frac{\cancel{2} \cdot 9}{1} \cdot \frac{3}{\cancel{2} \cdot 2} = \frac{9}{1} \cdot \frac{3}{2} = \frac{27}{2} = 13\tfrac{1}{2} \qquad \text{Check!}$$

So, we have found the correct constant multiplier based on the information we were given. The recursive formula is

$$a_1 = 32, \quad a_{n+1} = \frac{3}{4} \cdot a_n,$$

and the explicit formula is

$$a_n = 32 \cdot \left(\frac{3}{4}\right)^{n-1}.$$

If we look back over the explicit rules for the sequences in this section, we might notice that the formulas have a formula! In other words, the apparent rule for a geometric sequence always has a certain structure, which we summarize here.

Apparent formula for a geometric sequence

Given a geometric sequence with first term a_1 and common ratio r, in other words, a sequence of the form

$$a_1 \ , \quad a_1 * r \ , \quad a_1 * r^2 \ , \quad a_1 * r^3 \ , \quad \ldots$$

The apparent or explicit formula for the sequence is

$$a_n = a_1 * r^{n-1}$$

2.3 Arithmetic sequences

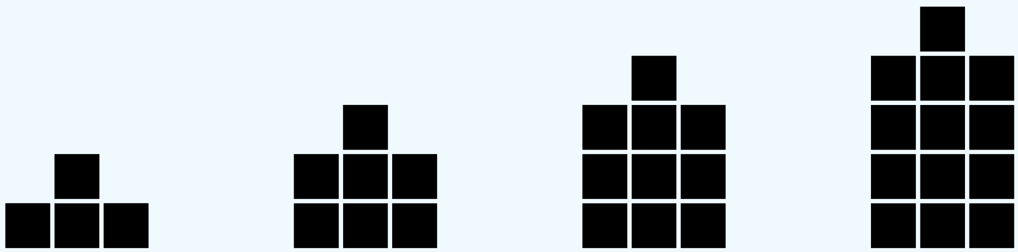
Not all sequences are geometric sequences, of course. Let's explore some other types of sequences.

Extended exploration: Squares, triangles, segments

[TODO] Click here to visit the extended exploration: Squares, triangles, segments

Startup exploration: Tile pattern #1

The pictures below represent stages 1, 2, 3, and 4 for a pattern of square tiles.



Stage 1 Stage 2 Stage 3 Stage 4

Draw pictures representing stages 5 and 6 in the pattern. Write a sentence or two to describe the pattern in the pictures. What would the stage 0 figure look like?

Write out the sequence for the number of tiles at each stage (starting with stage 1). Write a recursive rule to describe your sequence. How is this rule different from the rules in section 2.2?

The number of tiles in each stage of the pattern creates the sequence

$$4, 7, 10, 13, 16, 19, \ldots$$

We might write recursive rules that go something like "start with 4, add 3 to the previous term", or "START = 4, NEXT = NOW + 3". The fact that we're adding in the rule is a clear difference from the rules we saw when studying geometric sequences.

Sequences like these are called **arithmetic sequences**.[5] Instead of having a common ratio, these sequences have a *common difference*. Arithmetic sequences belong to the family of *linear relationships*.

[5] A word about pronunciation. The branch of mathematics that deals with calculations and operations on numbers is called "arithmetic". When used as a noun in this way, the word is pronounced with the emphasis on the second syllable: $a{\cdot}RITH{\cdot}me{\cdot}tic$. The sequences we're talking about in this section are "arithmetic sequences". When the word is used as an adjective, the emphasis is on the third syllable: $a \cdot rith \cdot ME \cdot tic$.

Arithmetic sequence

A sequence where the difference between each pair of successive terms is constant. The constant difference is called the *common difference*, usually denoted d. Arithmetic sequences are linear relationships.

Example 2.7

Verify that the given sequence is arithmetic and write a recursive formula for it: $12, 17, 22, 27, \ldots$

Solution: In order for a sequence to be arithmetic, we must add the same quantity as we go from term to term. We can check this by subtracting (which is why the thing we add is called a "common difference"). So, let's check:

$$17 - 12 = 5$$
$$22 - 17 = 5$$
$$27 - 22 = 5$$

Check! This is an arithmetic sequence with a common difference of 5.

To write the recursive formula we know that the common difference is added to the current term in order to find the next term. We also know the first term. So:

$$a_1 = 12, \quad a_{n+1} = a_n + 5$$

is the recursive formula for the sequence.

2.3.1 Explicit formulas for arithmetic sequences

Of course, we can write an apparent or explicit formula (that is, a non-recursive formula) for an arithmetic sequence. Consider the sequence from the startup exploration: $4, 7, 10, 13, \ldots$ We know where each of the terms come from:

$$a_1 = 4$$
$$a_2 = (4) + 3$$
$$a_3 = (4 + 3) + 3$$
$$a_4 = (4 + 3 + 3) + 3$$
$$a_5 = (4 + 3 + 3 + 3) + 3$$

Notice the repeated addition of 3. This is a case where we can reinterpret repeated addition as multiplication:

$$\begin{aligned}
a_1 &= 4 &&= 4 + 3 \cdot 0 \\
a_2 &= 4 + 3 &&= 4 + 3 \cdot 1 \\
a_3 &= 4 + 3 + 3 &&= 4 + 3 \cdot 2 \\
a_4 &= 4 + 3 + 3 + 3 &&= 4 + 3 \cdot 3 \\
a_5 &= 4 + 3 + 3 + 3 + 3 &&= 4 + 3 \cdot 4
\end{aligned}$$

Notice now that these multiplications are 3 times "one less than the stage number"! Therefore, we can write

$$a_n = 4 + 3(n - 1)$$

As with geometric sequences, there is a formula for these formulas, too:

Apparent formula for an arithmetic sequence

Given an arithmetic sequence with first term a_1 and common difference d, in other words, a sequence of the form

$$a_1 \ , \quad a_1 + d \ , \quad a_1 + 2d \ , \quad a_1 + 3d \ , \quad \ldots$$

The apparent or explicit formula for the sequence is

$$a_n = a_1 + (n - 1) * d$$

2.3.2 Using stage zero

There is another way to write the apparent rule for an arithmetic sequence. We can use this approach when we know (or can find) the "zeroth" term. Then, we interpret the stage 1 figure not as the *start*, but rather as though we are joining a sequence that is "already in progress".

For example, in the tile sequence from the startup exploration, to find the stage 0 figure we have to "back up a step". Since the pattern goes forward by adding 3, to back up one step we must subtract 3. So, the stage 0 figure is just 1 square tile.

Stage	Value	Start with stage 1?		Start with stage 0?	
1	4	4	$= 4 + 3(0)$	$1 + 3$	$= 1 + 3(1)$
2	7	$4 + 3$	$= 4 + 3(1)$	$1 + 3 + 3$	$= 1 + 3(2)$
3	10	$4 + 3 + 3$	$= 4 + 3(2)$	$1 + 3 + 3 + 3$	$= 1 + 3(3)$
4	13	$4 + 3 + 3 + 3$	$= 4 + 3(3)$	$1 + 3 + 3 + 3 + 3$	$= 1 + 3(4)$

One benefit of this new rule is that we find ourselves multiplying the constant difference by the term number itself (before we multiplied by one less than the term number). In other words, we can write the apparent rule as follows:

Apparent formula for an arithmetic sequence (zero version)

Given an arithmetic sequence with first term a_1 and common difference d, we can write the apparent or explicit formula for the sequence is

$$a_n = a_0 + n * d$$

Where a_0 represents the "zeroth" term of the sequence (the term that comes before the first term).

In later chapters, we will explore in more detail the connections between the "stage 1 version" and the "stage 0 version" of the rule for arithmetic sequences, and we will learn techniques for writing "stage 0 versions" of the rules for geometric sequences.

Example 2.8

Write a stage zero version of the explicit rule for the arithmetic sequence: $43, 35, 27, 19, \ldots$

Solution: This sequence is decreasing, so we must be adding a negative number in the rule. In other words, the common difference must be negative. Subtracting neighboring terms, we can find that the common difference is $^-8$.

To write a zero-based rule, we have to know the zeroth term, and to find that we have to back up from the first term. So, we have $a_0 = 43 - {}^-8 = 43 + 8 = 51$. This value makes sense: Since the sequence is decreasing, the zero term should be larger than the first term.

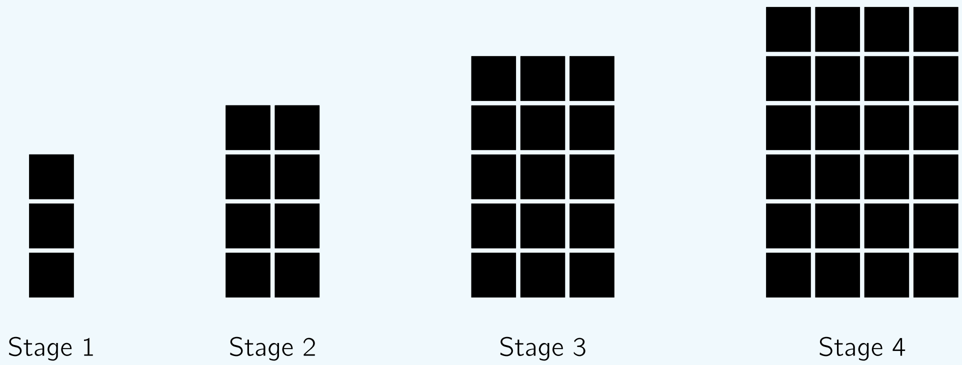
Knowing the common difference and the zero term, we can write a zero-based explicit rule:

$$a_n = a_0 + n * d = 51 + n * {}^-8 = 51 + {}^-8n$$

2.4 Other types of sequences

Startup exploration: Tile pattern #2

The pictures below represent stages 1, 2, 3, and 4 for a new pattern of square tiles.



Draw pictures representing stages 5 and 6 in the pattern. Write a sentence or two to describe the pattern in the pictures. What would the Stage 0 figure look like?

Write out the sequence for the number of tiles at each stage (starting with stage 1). Write a recursive rule to describe your sequence. How is this rule different from the rules in the last few sections?

You may have noticed that these sequences are a bit harder to work with. They belong to the family of *quadratic relationships*, and we'll study lots more about quadratic relationships later.

The figures in the startup exploration generate the sequence:

$$3, 8, 15, 24, 35, 48, \ldots$$

Is this sequence geometric? Let's check for a common ratio: the ratio between the first two terms is $\frac{8}{3}$, and the ratio between the next two terms is $\frac{15}{8}$. Those are different ratios, since if we write them with a common denominator, we have $\frac{8}{3} = \frac{64}{24}$ and $\frac{15}{8} = \frac{45}{24}$. So, the sequence is *not geometric*.

Is the sequence arithmetic? Let's check for a common difference:

$$\begin{aligned} 8 - 3 &= 5 \\ 15 - 8 &= 7 \\ 24 - 15 &= 9 \\ 35 - 24 &= 11 \\ 48 - 35 &= 13 \end{aligned}$$

The sequence does not have a common difference, so it is *not arithmetic*. But take a look at those differences! The differences have a pattern of their own: They go up by 2 every time. In other words, the differences form an arithmetic sequence! It's a sequence in a sequence! The turducken of sequences![6]

To describe this sequence with a recursive rule, we'll need to give the starting value, as usual: "start with 3". Then, we must describe the pattern in the differences: in this case, we're adding consecutive odd numbers (starting with 5). So, one way to express this recursive rule is "start with 3, add consecutive odd numbers (starting with 5) to the previous term". Note that we kind of sneak in two starting places: one for the start of the sequence (3, in this case) and one for the start of the sequence of numbers that are being added on (5, in this case). Tricky!

Example 2.9

Verify that the given sequence is quadratic, and write a recursive rule: $1, 4, 10, 19, 31, 46, \ldots$.

Solution: In order for a sequence to be quadratic the differences between successive terms must form an arithmetic sequence. Let's check:

$$\begin{aligned} 4 - 1 &= 3 \\ 10 - 4 &= 6 \\ 19 - 10 &= 9 \\ 31 - 19 &= 12 \\ 46 - 31 &= 15 \end{aligned}$$

The differences are: $3, 6, 9, 12,\ 15, \ldots,$ and that's an arithmetic sequence with common difference 3. So, yes, the original sequence is quadratic.

Now let's try to write a recursive rule (in sentences). Clearly, we start with 1. Then, we add consecutive multiples of three, starting with 3. So, our rule is "start with 1, add consecutive multiples of three (starting with 3) to the previous term".

At this point, our goal is just to recognize that these sequences are neither arithmetic nor geometric, but follow a different kind of pattern. Writing the formulas for them can be quite challenging – but our brains grow when we stretch them around new ideas! Let's give it a shot.

[6] This sequence-in-a-sequence stuff can get pretty involved. Here, we found an arithmetic sequence in the *differences* between the terms in our quadratic sequence. But why not build the sequence $1, 4, 12, 27, 51, \ldots,$ in which the sequence of differences is our quadratic sequence! Of course we could keep building sequences like this for as long as we wanted. This isn't just a turducken, it's a *rôti sans pareil*! That's French for "roast without equal", a dish that which calls for 17 different birds, each one stuffed into the body cavity of the next. Since the dish was first proposed in 1807 by the French gastronomist Grimod de La Reyniére, several of the birds called for in the recipe have become endangered species.

2.4.1 (;,,;) Recursive formulas for quadratic sequences

Extension sections

Sections marked with the Cthulhu (;,,;) emoticon, like this one, are extension sections that might be a bit more intense than the norm. We encourage you to explore the concepts, but don't feel discouraged if you find the material challenging.

Your math brain grows when you think deeply about mathematics, so hard work is valuable, even if things aren't completely clear right away. Many of the ideas in the optional sections will appear again in later chapters, so you'll probably find that your confidence grows as time goes on.

In the last section, we looked at the sequence, which came from a rectangular pattern of tiles:

$$3, 8, 15, 24, 35, \ldots$$

We wrote the recursive rule in sentences: "start with 3, add consecutive odd numbers (starting with 5) to the previous term". Can we translate this into a recursive formula?

The first step is easy: $a_1 = 3$. Hooray for small victories!

In order to describe the recursive step, we need to describe the sequence of differences: $5, 7, 9, 11, \ldots$. Since this is an arithmetic sequence, we know how to write its explicit rule. Let's use the symbol b, so we don't get our sequences confused. Then this sequence is $b_n = 5+(n-1)\cdot 2$ or, if we use a zero-based rule, $b_n = 3+n\cdot 2$.

Let's try and put these together:

$$\begin{aligned}
a_1 &= 3 \\
a_2 &= 8 && = a_1 + 5 && = a_1 + b_1 \\
a_3 &= 15 && = a_2 + 7 && = a_2 + b_2 \\
a_4 &= 24 && = a_3 + 9 && = a_3 + b_3 \\
a_5 &= 35 && = a_4 + 11 && = a_4 + b_4
\end{aligned}$$

So, our recursive step is that $a_{n+1} = a_n + b_n$. Since we have an explicit formula for the b_n's, we can replace that part with their explicit rule! Altogether we have:

$$a_1 = 3, \quad a_{n+1} = a_n + 3 + n \cdot 2$$

How can we check to see if we're right? One way is to use the rule to try and re-generate the sequence. Our rule states that $a_1 = 3$. To find a_2, we can use the rule with $n = 1$ and $n + 1 = 2$:

$$a_2 = a_1 + 3 + 1 \cdot 2 = 3 + 3 + 1 \cdot 2 = 3 + 3 + 2 = 8.$$

Then, we can take one step forward and apply the rule again. Now, $n = 2$ and $n+1 = 3$:

$$a_3 = a_2 + 3 + 2 \cdot 2 = 8 + 3 + 2 \cdot 2 = 8 + 3 + 4 = 15.$$

Let's go one more step and try $n = 3$ and $n + 1 = 4$:

$$a_4 = a_3 + 3 + 3 \cdot 2 = 15 + 3 + 3 \cdot 2 = 15 + 3 + 6 = 24.$$

Phew! It pays to be patient when working out a convoluted rule like this, but in the end, we can see that our rule is working as intended!

Example 2.10

Write a recursive rule for the quadratic sequence: $1, 4, 9, 16, 25, \ldots$.

Solution: A bit of tinkering leads us to the rule "start with 1, add consecutive odd numbers (starting with 3) to the previous term". So $a_1 = 1$.

How do we write the apparent formula for the odd number pattern? The common difference is 2, and the pattern starts at 3, so $b_n = 3 + (n-1) \cdot 2$ is the apparent formula for the differences.

Putting the pieces together:

$$\begin{aligned}
a_1 &= 1 & & \\
a_2 &= 4 &= a_1 + 3 &= a_1 + b_1 \\
a_3 &= 9 &= a_2 + 5 &= a_2 + b_2 \\
a_4 &= 16 &= a_3 + 7 &= a_3 + b_3 \\
a_5 &= 25 &= a_4 + 9 &= a_4 + b_4
\end{aligned}$$

So, again, we have $a_{n+1} = a_n + b_n$. Then, we can replace the b_n with the apparent formula we created for the sequence of differences:

$$a_1 = 1, \quad a_{n+1} = a_n + 3 + (n-1) \cdot 2.$$

If we would rather use a zero-based rule for the pattern in the differences, we could write:

$$a_1 = 1, \quad a_{n+1} = a_n + 1 + n \cdot 2.$$

Note that even though these two rules look quite different, they are equivalent ways of describing the sequence. In later chapters, we will learn techniques that will help us to explain why these two different-looking rules give us the same result.

2.4.2 (;,;) Explicit formulas for quadratic sequences

It seems only proper to discuss a method for writing a non-recursive formula for a quadratic sequence.

There are, in fact, multiple methods for writing rules like this. There is a way that requires knowledge of calculus, there is a method that uses *systems of equations* (more on those in chapter 8), there is the not-so-efficient method of guess and check, and so on. Most of these require knowledge of the structure of a quadratic relationship which (seeing as how we're only here in chapter 2) we haven't discussed yet.

But, there is a clever approach that requires a bit of pattern-hunting and detective work. That's the approach we'll explore here.

Let us once again consider the sequence $1, 4, 9, 16, 25, \ldots$. You might have recognized these numbers are the **perfect squares**. That name comes from the idea that we can view these numbers as the areas of squares, as shown in fig. 2.1. The first number is the area of a 1-by-1 square, the second is the area of a 2-by-2 square, then a 3-by-3 square, and so on. Knowing this, we can write any term of the sequence: $a_n = n \cdot n$.

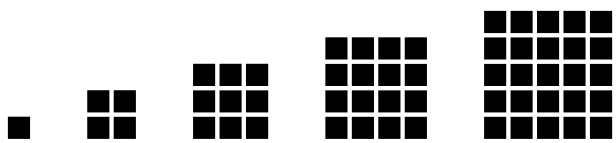


Figure 2.1: Perfect squares

With other quadratic sequences, we can sometimes crack the code if we think about areas of rectangles and triangles. We'll look for pairs of integers that could give us the areas we're after, and then look for arithmetic sequences among those integers. The next thing you know, we'll have a non-recursive formula for the quadratic!

Let's go back to the other example we've been studying, the sequence $3, 8, 15, 24, 35, \ldots$. This sequence came from the tile pattern at the start of section 2.4. Go back and take another look at those pictures. What do you notice?

Stage	Number of Squares	Dimensions of Rectangle
1	3	1×3
2	8	2×4
3	15	3×5
4	24	4×6

In stage n, the size of the rectangle is n units tall and $(n + 2)$ units wide! So, we can use those two values to write an explicit formula:

$$a_n = n \cdot (n + 2)$$

How cool is that?

Example 2.11

Write a non-recursive formula to generate the sequence $6, 12, 20, 30, 42, \ldots$.

Solution: It's not obvious how rectangles are related to these numbers, but if we assume we're looking for rectangles with integer side lengths, then there are a limited number of options.

For example, if the first number represents the area of a rectangle with integer side lengths, then it could be either a 1×6 rectangle or a 2×3 rectangle. Let's organize the different options in a table:

Stage	Value	Possible Rectangles							
1	6	1×6	or	2×3					
2	12	1×12	or	2×6	or	3×4			
3	20	1×20	or	2×10	or	4×5			
4	30	1×30	or	2×15	or	3×10	or	5×6
5	42	1×42	or	2×21	or	3×14	or	6×7

Now comes the detective work. We are looking for patterns in the factors as they progress through the terms. We've highlighted the key patterns below.

Stage	Value	Possible Rectangles						
1	6	1×6	or	2×3				
2	12	1×12	or	2×6	or	3×4		
3	20	1×20	or	2×10	or	4×5		
4	30	1×30	or	2×15	or	3×10	or	5×6
5	42	1×42	or	2×21	or	3×14	or	6×7

Notice that the first set of factors (in yellow) form the arithmetic sequence $2, 3, 4, 5, \ldots$, and the second set (in green) form the arithmetic sequence $3, 4, 5, 6, \ldots$.

The yellow sequence is always one more than the term number. The green sequence is always two more than the term number. So, we have our explicit formula!

$$a_n = (n+1) \cdot (n+2)$$

If these last two sections felt a bit overwhelming, don't worry. After we have some more algebraic tools in our toolbox, we'll return to quadratic relationships and describe them in more detail.

There is a magic in graphs. The profile of a curve reveals in a flash a whole situation – the life history of an epidemic, a panic, or an era of prosperity. The curve informs the mind, awakens the imagination, convinces.

Henry D. Hubbard
US National Bureau of Standards

Chapter 3

Graphs and data

In chapter 2 we investigated pictures patterns (fractals by Koch and Sierpiński, patterns of made of square tiles) and used those pictures to generate sequences of numbers. We begin this chapter with a discussion of another way to represent our sequences of numbers visually: by making a coordinate graph. Then, we will extend these ideas to create visual representations of other mathematical objects, and of scientific data.

3.1 Coordinate graphing

Figure 3.1 summarizes the familiar landmarks of the **coordinate plane**. We see the horizontal **x-axis** and the vertical **y-axis**. Using the two axes as number lines, we can locate **ordered pairs** of numbers using the convention (x, y). The points $(7, -2)$ and $(-6, 3)$ have been plotted as examples. The point $(0, 0)$ where the the two axes meet, is a special point called the **origin**.

The axes chop the plane into four regions called **quadrants**, which are numbered starting in the upper right and moving counter-clockwise (as shown in the figure). The signs in parentheses indicate the signs of x- and y-coordinates in each quadrant. The x-coordinates of the points in Quadrants I and IV are positive, while points in Quadrants II and III have x-coordinates that are negative. Points with positive y-coordinates lie in Quadrants I and II, while points with negative y-coordinates fall in Quadrants III and IV.[1]

[1] A point that lies on an axis doesn't actually lie in any of the four quadrants. So, the point $(4, 0)$ lives on the positive x-axis, but not in Quadrant I nor in Quadrant IV.

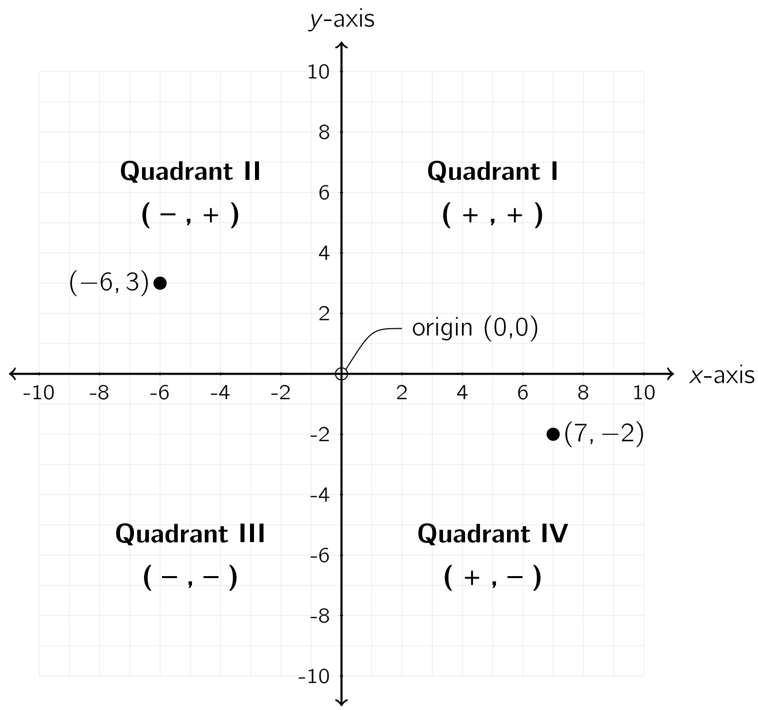


Figure 3.1: The coordinate plane and its important landmarks

Startup exploration: Pizza intersections

Bob's friend Melvin "Hambone" Jones once worked delivering pizzas in his hometown of Euclid, Ohio. The town has streets running north-south and east-west.[a]

Hambone is currently parked at the intersection we will call $(0, 0)$. If he drives one block east, he will arrive at the intersection $(1, 0)$. If he then turns right and drives one block south, he will arrive at the intersection $(1, -1)$.

Starting from $(0, 0)$, describe are all the intersections that Hambone can reach by driving a total distance of exactly 10 blocks.

[a] Euclid, Ohio is also the home of the Polka Hall of Fame, though that has nothing to do with this problem. Also, Euclid isn't laid out in a grid as this problem implies, though it should be, given its name.

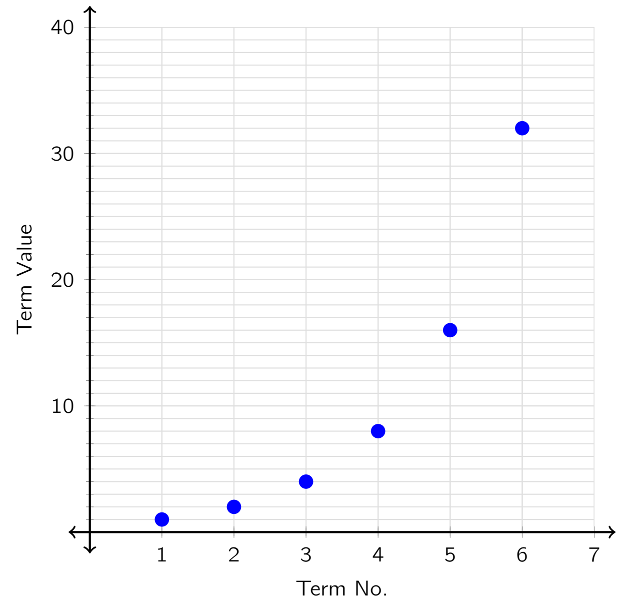
3.1.1 Graphing a sequence

Our goal is to make a visual representation of a sequence on this coordinate plane.

"But," you may be asking yourself, "a sequence is just a list of numbers. How do we make a coordinate graph, which needs coordinate pairs of numbers? It takes *two numbers* to make a *pair*!"

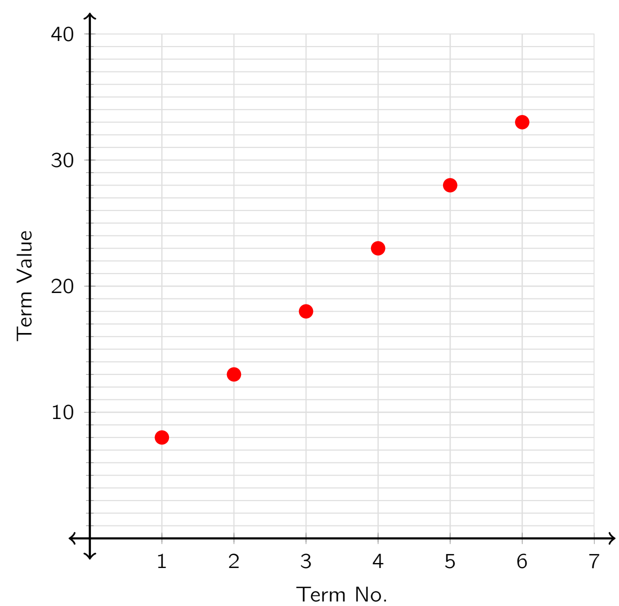
To graph a sequence, we use the term number as the x-coordinate and the term's value as the y-coordinate. Consider the sequence $1, 2, 4, 8, 16, 32, \ldots$. The first term is 1, the second term is 2, and so on. We can organize this in a table, and write out the ordered pairs. Then, we can plot those ordered pairs and see a visual representation of our sequence!

Term No. x	Term Value y	Coord. Pair (x, y)
1	1	$(1, 1)$
2	2	$(2, 2)$
3	4	$(3, 4)$
4	8	$(4, 8)$
5	16	$(5, 16)$
6	32	$(6, 32)$



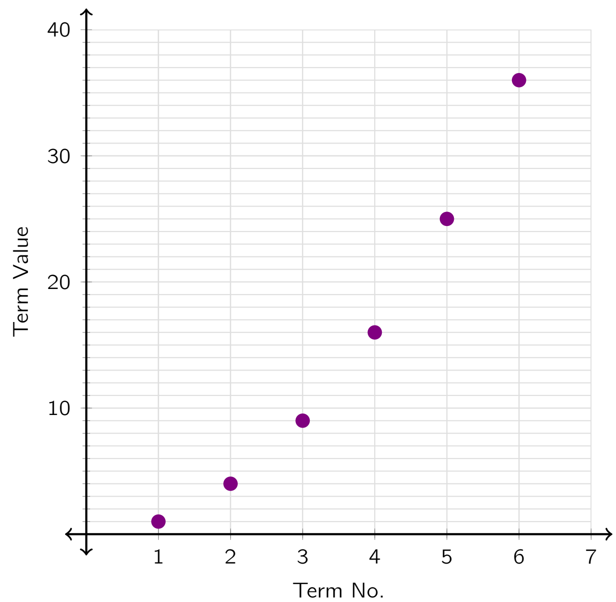
Recall that that sequence above, where the terms have a constant ratio, is called a geometric sequence. Let's look at an example of an arithmetic sequence, in which the terms have a constant difference. For example, consider the sequence $8, 13, 18, 23, 28, 33, \ldots$. This sequence produces the following table and graph:

Term No. x	Term Value y	Coord. Pair (x, y)
1	8	$(1, 8)$
2	13	$(2, 13)$
3	18	$(3, 18)$
4	23	$(4, 23)$
5	28	$(5, 28)$
6	33	$(6, 33)$



Finally, let's see what a quadratic pattern looks like. For example, the perfect squares form a quadratic sequence $1, 4, 9, 16, 25, 36, \ldots$. Their table and graph go like this:

Term No. x	Term Value y	Coord. Pair (x, y)
1	1	$(1, 1)$
2	4	$(2, 4)$
3	9	$(3, 9)$
4	16	$(4, 16)$
5	25	$(5, 25)$
6	36	$(6, 36)$



Take a moment to compare the graphs of these three sequences. How are they alike? How are they different?

3.1.2 Features of the graph of a sequence

Note that since the term number is always greater than 0, our graphs only show the positive part of the x-axis. We'll soon see that this is an artificial limitation: the negative part of the x-axis is just as important as the positive part.

Note also that we haven't connected the dots: a sequence has a first term and a second term, but no one-and-a-halfth term. We'll soon see that this is an artificial limitation, too. The fractional values in between the points we've plotted are also important.

In fact, when we generate a sequence we use only the natural numbers as the input values to the formula. However, if we change our point of view and allow the input to be any real number, we will have turned our sequence into a mathematical relationship called a **function**.[2] We'll discuss the details of what functions are (and what they are not) in chapter 4.

[2] Technically, a sequence is *already* a function. The distinction we make here has to do with what kinds of input values are allowed. Later, once we have some additional concepts under our belts, we'll talk about a sequence as "a function whose domain is the natural numbers".

3.2 Algebraic expressions

Startup exploration: Don't take all year

Find three natural numbers x, y, and z which satisfy the equation $28x + 30y + 31z = 365$. Can you find more than one set of numbers x, y, z that satisfy the equation?

In chapter 1 we used the order of operations to simplify numeric expressions, which are made up of numbers and arithmetic operators. For example,

$$3 \cdot 4 - 8(4^2 - 1)$$

is a numeric expression. It contains only numbers and operators and, in the end, it simplifies down to a single number.[3] An **algebraic expression** on the other hand can contain letters in addition to numbers and operators, for example

$$3x - 5y + 18.$$

These letters, called *variables*, stand in for numbers that we don't know or which may change.

Variable

A **variable** is representation of a value that can change. In algebra, variables are often represented by letters. We usually use letters from the Latin alphabet $(a, b, c, d, \ldots)$, but sometimes also use other symbols, such as letters from the Greek alphabet $(\alpha, \beta, \gamma, \delta, \ldots)$.

Algebraic expression

A symbolic representation of mathematical operations that can involve both numbers and variables.

3.2.1 Numbers and variables

In your mathematical career so far, you have probably worked with letters that stand in for numbers. Recall the formula for the circumference of a circle:

$$C = \pi d.$$

[3] Spoiler alert: It's -108.

This formula explains the relationship between d, which stands in for the diameter of some circle, and C, which stands for the circumference of that circle. The letters d and C are variables. They stand in for numbers that can change, depending on which circle we're talking about.[4]

Once we start to introduce letters into our expressions, we have to discuss some standard notation and terminology. As we have seen, we don't usually write any multiplication symbol when multiplying a number times a variable. Rather than writing $3 \cdot x$ or $3(x)$, we can write $3x$ without anything in between.

An algebraic expression that is built using only multiplication (or division) is called a **term**. For example, $3x$ and $\frac{1}{2}m$ are terms. On the other hand, the expression $3x + 2y$ is not a term because it includes addition. In fact, this expression is the sum of two terms.

Term

An algebraic expression that represents only multiplication and division between variables and constants.

When we have the product of a number and a variable – like $3x$ or ^-11g – the number part is called the **coefficient** of the term. So, the coefficient of $3x$ is 3, and $^-11$ is the coefficient of ^-11x. If we have a variable all alone without a number attached – like y or w – then we picture a "phantom 1" lurking there as the coefficient: y is the same as $1y$ and $1w$ is the same as w.

Coefficient

The numerical factor in a term with a variable. If no number is explicitly written, the coefficient is understood to be 1.

3.2.2 Evaluating algebraic expressions

A variable is a "placeholder" that stands in for a number. We can only determine the value of an algebraic expression if we know what numbers the different variables represent.

Consider the expression $3x$. If we know that x represents 15, then we can **evaluate** the expression $3x$ in the case that $x = 15$. In that case, it must be that $3x$ represents $3(15) = 45$.

When we evaluate an algebraic expression, we substitute in values for its variables, and then simplify the resulting numeric expression using the order of operations. It is a really good habit always to use parentheses when substituting numeric values for variables. This can avoid confusion about negative numbers!

[4] Note that π is *not* a variable. We use a (Greek) letter in this case not because the value of π might change, but because it's an irrational number that is impossible to write out in full. We often use letters to stand in for mathematical objects that are inconvenient, sometimes impossible, to write down in another way: e, i, ϕ, and $\aleph_0$ each has special mathematical meaning.

Example 3.1

Evaluate the expressions $6x + 4$ and $x^2 - 5$ for the x values 3, $^-1$, and $\frac{1}{2}$.

Solution:

(a) To evaluate the expression $6x + 4$ for the given values of x, we simply substitute and follow the order of operations.

When $x = 3$:
$$\begin{aligned} 6x + 4 &= 6(3) + 4 \\ &= 18 + 4 \\ &= 22 \end{aligned}$$

When $x = {^-1}$:
$$\begin{aligned} 6x + 4 &= 6({^-1}) + 4 \\ &= {^-6} + 4 \\ &= {^-2} \end{aligned}$$

When $x = \frac{1}{2}$:
$$\begin{aligned} 6x + 4 &= 6\left(\tfrac{1}{2}\right) + 4 \\ &= 3 + 4 \\ &= 7 \end{aligned}$$

(b) We do the same in order to evaluate $x^2 - 5$ for the given x values.

When $x = 3$:
$$\begin{aligned} x^2 - 5 &= (3)^2 - 5 \\ &= 9 - 5 \\ &= 4 \end{aligned}$$

When $x = {^-1}$:
$$\begin{aligned} x^2 - 5 &= ({^-1})^2 - 5 \\ &= 1 - 5 \\ &= {^-4} \end{aligned}$$

When $x = \frac{1}{2}$:
$$\begin{aligned} x^2 - 5 &= \left(\tfrac{1}{2}\right)^2 - 5 \\ &= \tfrac{1}{4} - 5 \\ &= -\tfrac{19}{4} \end{aligned}$$

Note how the parentheses help out when $x = {^-1}$! Without those parentheses, we would have run the risk of making the most common mistake in algebra 1: remember the difference between $({^-1})^2$ and ${^-1}^2$.

3.3 Graphing a function

The graphs of sequences that we created earlier were quite limited. Since sequences use only the natural numbers as input values, the only points we had available to plot were the points where $x = 1, 2, 3, 4, \ldots$. But now, knowing how to evaluate algebraic expressions, we can create more complete graphs by choosing a wider range of x values.

Startup exploration: Extending our sequences

Write a zero-based explicit rule for the arithmetic sequence shown below (this is the second example from section 3.1). First write the rule in terms of n and a_n, then translate your rules into a graphable format in terms of x and y.

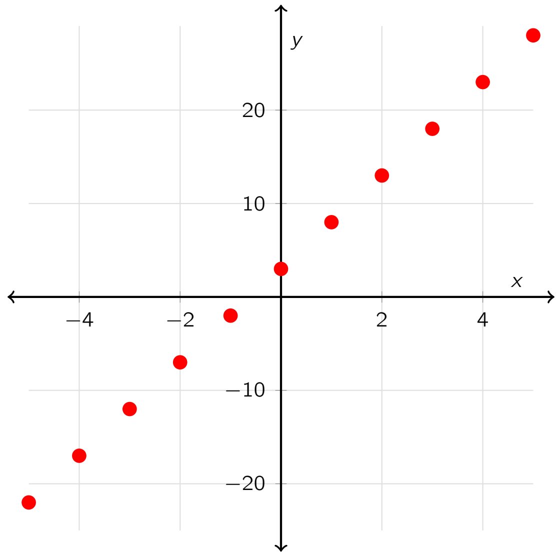
$$8, 13, 18, 23, 28, \ldots$$

The given sequence represents the y-values of the rule for the x-values 1, 2, 3, 4, and 5. Evaluate your rule for the x-values 0, -1, -2, -3, -4, and -5. Then, plot these 11 points on a coordinate grid.

The rule for the sequence in the startup exploration is $a_n = 5n + 3$, or in terms of x and y, we have the rule $y = 5x + 3$. To create the coordinate graph, we can substitute the different x-values into the rule and compute the y-values. The middle column in the table below is our "process column" in which we substitute an x-value and compute the corresponding y-value.

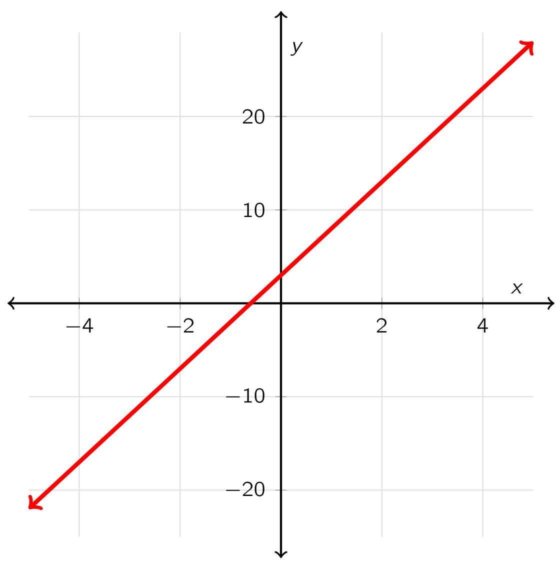
x	$y = 5x + 3$	(x, y)
0	$y = 5(0) + 3 = 0 + 3 = 3$	$(0, 3)$
-1	$y = 5(-1) + 3 = -5 + 3 = -2$	$(-1, -2)$
-2	$y = 5(-2) + 3 = -10 + 3 = -7$	$(-2, -7)$
-3	$y = 5(-3) + 3 = -15 + 3 = -12$	$(-3, -12)$
-4	$y = 5(-4) + 3 = -20 + 3 = -17$	$(-4, -17)$
-5	$y = 5(-5) + 3 = -25 + 3 = -22$	$(-5, -22)$

In the end, we generate 6 new coordinate pairs, which we can graph alongside the five points that we were given.



Can you anticipate the location of the point that we plot when $x = \frac{1}{3}$? What about when $x = -\frac{5}{2}$? What would our graph look like if we used all of the points in $\mathbb{R}$ as the x-values?

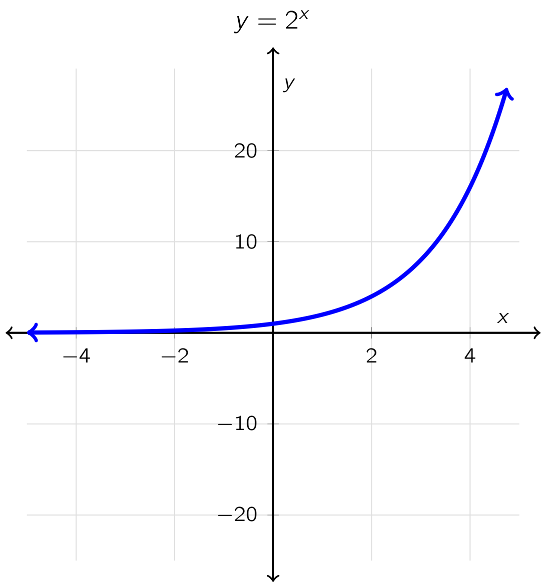
When we use all of the real numbers as input to the rule $y = 5x + 3$, the resulting graph is a straight line. Of course, it would be impossible to actually plot *all* of the points (there are infinitely many of them), but the pattern holds true, and so we can replace the dots with a continuous line.

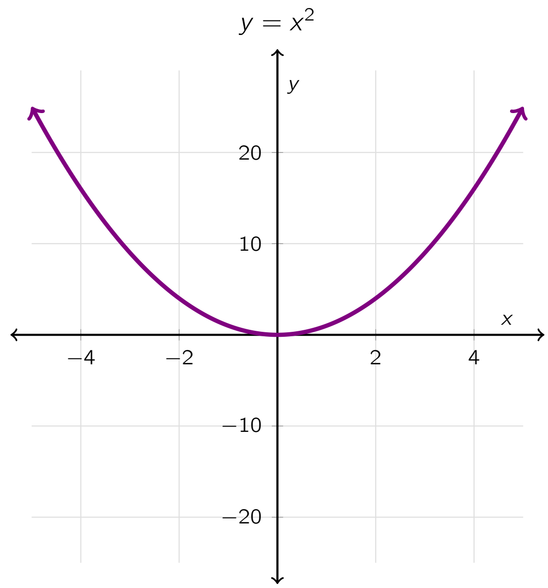


The rules below correspond to the other two sequences we studied in section 3.1. Under each rule is the graph that is created when we plot the rule over $\mathbb{R}$.

Are you surprised by these two graphs? Back in section 3.1, the graphs of these two sequence looked similar. But, we were only looking at the first quadrant! Their graphs are very different for negative values of x.

The moral of the story is that when we are asked to graph an equation by hand, we need to use a variety of different x-values, including negative numbers and fractions. Often, a problem will clearly indicate exactly what values to use.





3.3.1 Criteria for high quality graphs

In algebra we make a distinction between “sketching” and “graphing”. A sketch is just a quick drawing and it doesn't need to be super accurate. A sketch can be scribbled on a napkin or a piece of notebook paper.

On the other hand, a proper graph is meant to communicate something to the viewer. A graph must be accurately made on graph paper. The most important features of whatever you're graphing – a sequence, a function, a plot of a data set – must be clear, accurate, and neatly represented. Here are some guidelines for creating high-quality graphs.

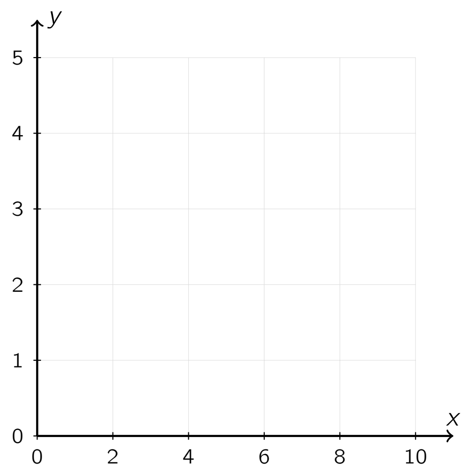
High quality graphs should be drawn on graph paper. An individual graph does not have to take up an entire piece of graph paper, although it should be drawn large enough to be easily understood.

Use a ruler or straightedge to make straight lines, in particular the coordinate axes or any linear data. Plus, we should draw arrowheads on anything that goes forever. This includes axes, lines, and curves.

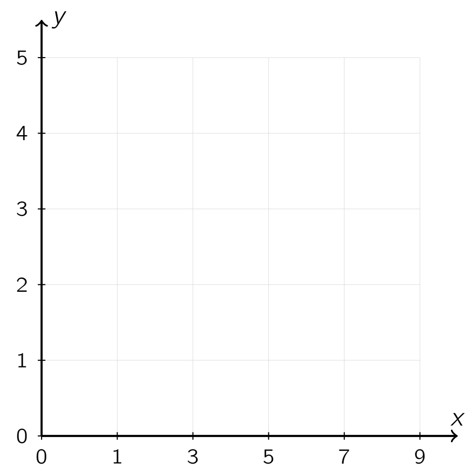
Be mindful when choosing a scale for the axes. Choose a scale that fits the data and ensures that your graph shows all important features of the curve. The origin does not necessarily need to be in the center of the grid. For example, if we are plotting data that includes only positive values, then we only need the first quadrant. The origin can be in the lower left-hand corner of the graph.

Scales for the x- and y-axes can be different, and in some cases must be different, so this can take a little bit of planning. The scale must be the same for the whole length of an axis, and cannot have any “jumps” or “breaks”. Changes in the scale will distort the shape of the graph, which defeats the purpose! Compare the two graphs below. The graph on the left is fine, but the graph on the right includes a common mistake. Can you identify the problem?

OK!



Not OK!



High quality graphs are clearly labeled. The scale should be indicated on the axes, and the axes should be labeled. For graphs of equations, this might mean simply labeling the axes x and y. When plotting data, include more informative names like "time" or "distance".

A final question about data points: to connect or not to connect? In the next section, we will discuss this question in some detail. But we've already seen an important piece of this puzzle: When graphing an equation by plotting data points, we should connect data points with a smooth curve and *not* individual line segments. If we use a straight line to connect points, then we are telling our audience that the data in between the points is linear, which may not be the case! Go back and have a look at the graph of $y = x^2$. It doesn't come to a point at the bottom. Rather it's a smooth curve that passes through the origin.

Extended exploration: Big graphs

[TODO] Click here to visit the extended exploration: Big Graphs

3.4 Patterns in data

The graphs that we have been working with so far have been very orderly. Technical and scientific data, however, are not always so tidy. Data can be noisy, messy, and incomplete. We will need some tools that can help us to see and describe patterns that may (or may not) exist in experimental data.

Extended exploration: Who is the best age guesser?

[TODO] Click here to visit the extended exploration: Who Is the Best Age Guesser?

Startup exploration: Water consumption

The graph shown in fig. 3.2 depicts water consumption in Edmonton, capital of the Canadian province of Alberta, during the gold medal men's ice hockey game at the 2010 Winter Olympics in Vancouver. The game was played between Canada and United States.

Water consumption during the game is shown in blue, while data from the same time period on the previous day is shown in green.

Write down anything you notice or wonder about the data presented in this graph.

A few notes: Ice hockey games are played in three 20-minute periods with breaks in between. In this particular game, the score was tied at the end of regular play. Canada scored the game winning goal in overtime and was awarded the gold medal.

In the graph of Edmonton water consumption, the amount of water being used varies depending on the time of day (not the other way around). We say that "time of day" is the *independent variable* and "water demand" is the *dependent variable*.

Independent variable

A variable whose values affect the values of another variable. In a graph of the relationship between two variables, the quantity represented on the horizontal axis (the x-axis) usually represents the independent variable.

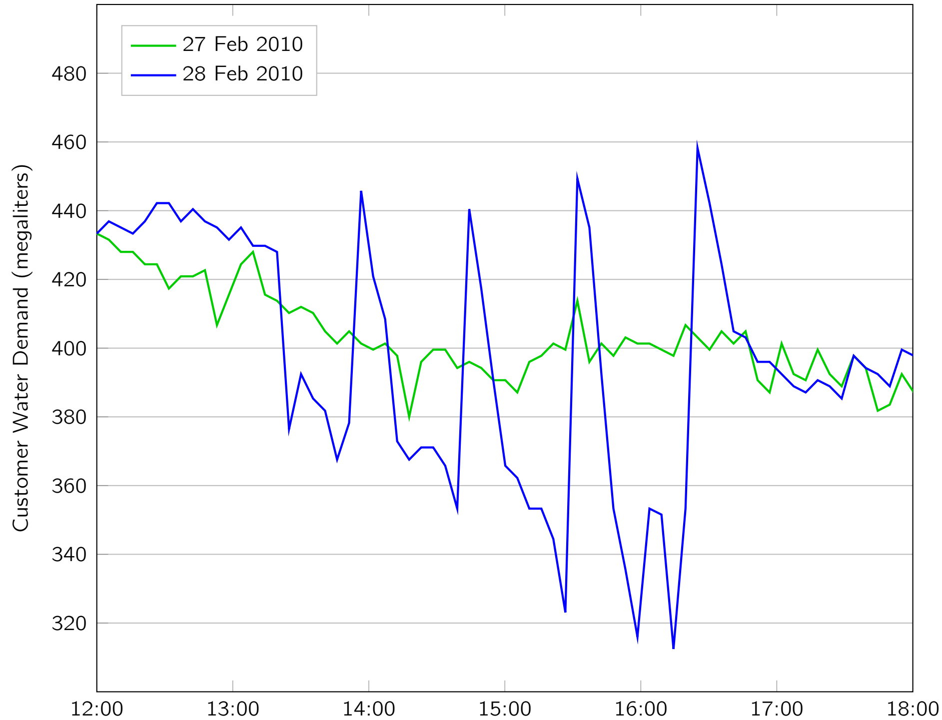


Figure 3.2: Water consumption in Edmonton on 27 and 28 February 2010 (source: EPCOR)

Dependent variable

A variable whose values depend on the values of another variable. In a graph of the relationship between two variables, the quantity represented on the vertical axis (the y-axis) usually represents the dependent variable.

3.4.1 Correlation

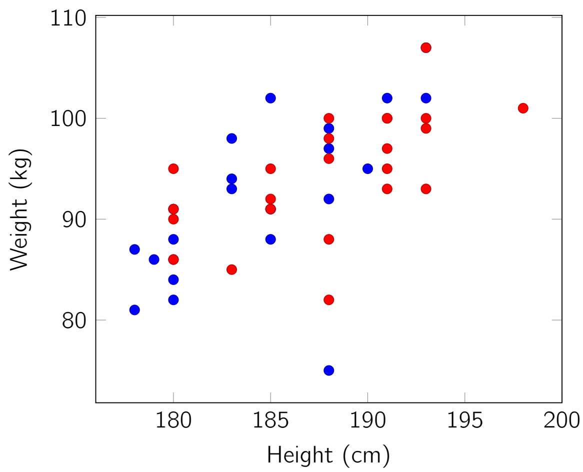
We can compare just about any two quantities. One way to do this is with a graph called a **scatter plot**.

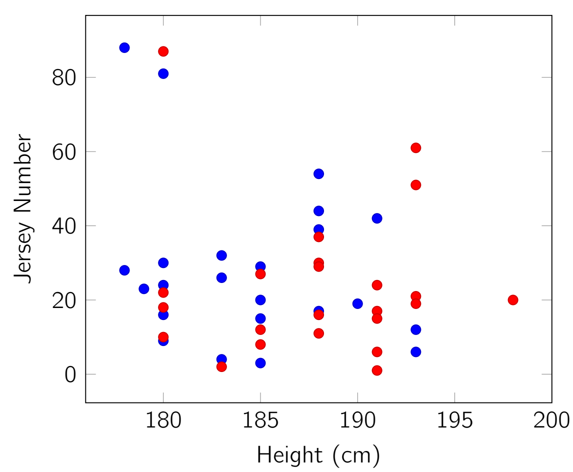
Scatter plot

A graph that relates data of two different sets. The two sets of data are displayed as ordered pairs.

Suppose we wish to compare, say, the height and weight for all of the players on the top two 2010 Olympic men's ice hockey teams. Let's make a graph comparing every player's height and weight as ordered pairs: (height, weight). This graph is shown on the left below. Blue dots represent players on the United States team, red dots represent players on team Canada.[5]

While we're at it, let's make another comparison. The graph on the right plots every player's height and jersey number as ordered pairs (height, jersey number). What do you notice about these two graphs? How are they the same? How are they different?





Notice that the graph of weight and height has a clear upward slant. This seems reasonable: the taller someone is, the more we might expect that person to weigh. There are some data points that don't fit the trend, but generally speaking the data points are increasing as we look toward the right on the graph. We say that this data shows a positive **correlation**.

Correlation

A trend between two sets of data, as seen in a scatter plot. A trend can show positive, negative, or no correlation. Positive correlation shows an **increasing** trend in data. Negative correlation shows a **decreasing** trend in data.

The graph of jersey number versus height is more of a blob. There's no clear trend in this data, and so we say that is shows *no correlation*. This makes sense, too: there's no logical connection between a player's height and the number they wear on their shirt.

A third possibility would be data that shows a negative correlation, meaning that the data are decreasing as we look towards the right on the graph. Can you imagine two variables that might show a negative correlation when compared on a scatter plot?

Graphing experimental data on a scatter plot helps us to see if there is a relationship between variables. If there is, a pattern will emerge in the graph. The points will fall (approximately) in a line or a curve and will have a correlation.

[5] Data from the International Olympic Committee, as reported in Wikipedia.

If the scatter plot shows a positive correction, it means that as the independent variable increases, the dependent variable increases. If the scatter plot shows a negative correlation, it means that as the independent variable increases, the dependent variable decreases. If a scatter plot shows no correlation, it indicates that there is no relationship between the two variables.

A key thing to remember when it comes to looking at data is that "correlation does not imply causation". In other words: If we see that two variables are correlated, we might be tempted to assume that the change in one variable *causes* the change in the other. This is sometimes true, but not always.

For example, it seems reasonable to believe that a change in height will cause a change in weight. But, there is data that shows a positive correlation between "consumption of mozzarella cheese per person" and "number of civil engineering doctorates awarded". This has to be a coincidence! There's no (good) reason to think that changing one of these variables would cause a change in the other one.[6]

3.4.2 Continuous and discrete data

When we drew the graph of a sequence, we didn't connect the dots. A sequence has a first term and a second term, but no one-and-a-halfth term. The x-values have no "in-betweens".

Similarly, imagine a graph showing "time" as the independent variable and "number of hockey players on the ice" as the dependent variable. In this case, the y-values would have no "in-betweens". There could be 11 players or 12 players on the ice, but never 11.5 players. This is called *discrete data*.

Discrete data

Data for which it doesn't make sense for measurements to exist between given data points. Discrete data often involves *counting items*, such at the number of cars in a parking lot over time.

On the other hand, when we started to picture the graph of a rule that could accept any real number as input, we drew a continuous line on the graph. The graph of water consumption in Edmonton, is jagged, spiky, and irregular – but it's a continuous line. We can measure how much water has been used at any point in time, and we can measure the amount of water in fractions of a unit.

[6] This fact is courtesy of the website Spurious Correlations, which has many graphs of interesting and ridiculous data that show correlation but not causation. Sometimes, it's fun to try to "explain" the correlation. Perhaps in this case the increase in mozzarella cheese consumption is due to an increase in takeout pizza demand, which leads to more pizza delivery shops, which in turn requires more roads, which means we need more engineers to design them.

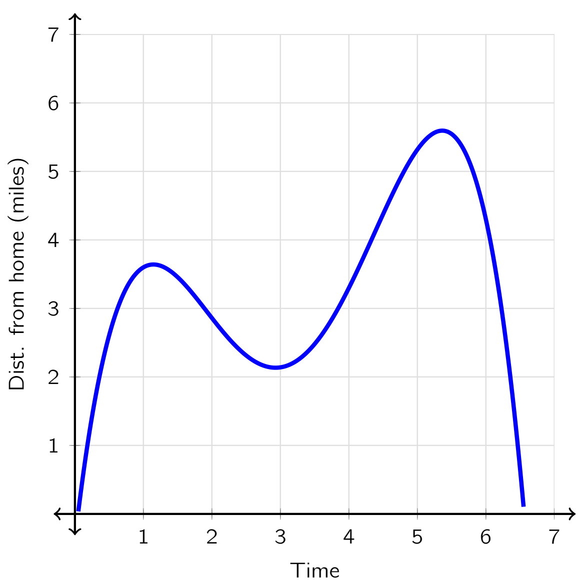
Continuous data

Data that has no holes, gaps, or breaks. Continuous data often involves *measuring some value* where measurements exist (and may change) between data points. For example, a person's height over time.

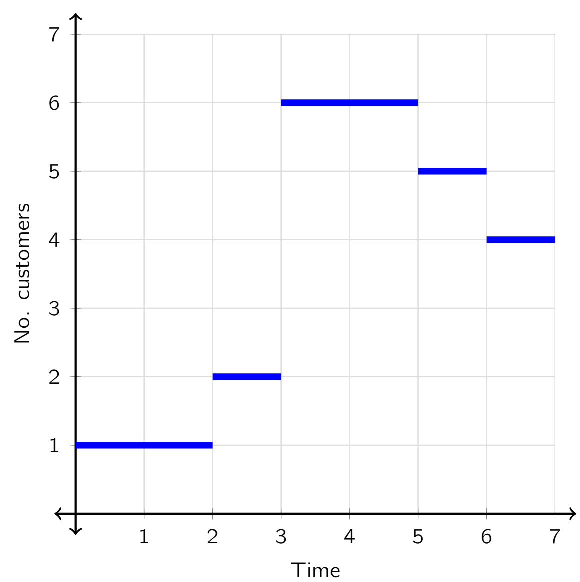
Consider a data collection scenario in which we want to graph "Bob's distance from home at any given time (in kilometers)". Bob is always a certain distance away from home (perhaps 0 km, if he is at home), and he could be any distance (even fractions of a kilometer). So, this is continuous data and our graph should be an unbroken line.

Now consider the scenario in which we graph "number of customers in line at the cheese counter at Middle Market." Although there is always a certain number of people in line (maybe zero), there can never be 5.7 people in line. We must count people, and so this data is discrete. Our graph would have to "jump" from 5 people to 6 people without going through the in-between values.

Continuous data



Discrete data



3.5 Interpreting graphs

In this section, our goal is to hone our skills at understanding what a graph is *communicating*.

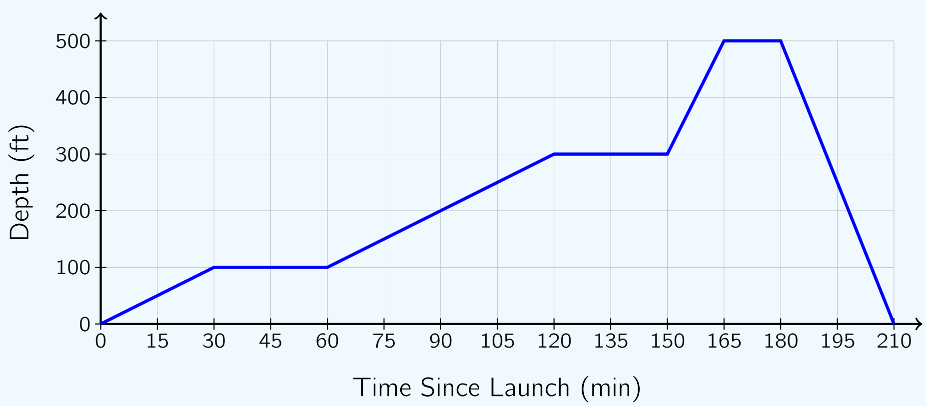
Extended exploration: Interpreting graphs with distance match

[TODO] Click here to visit the extended exploration: Interpreting Graphs with Distance Match

Startup exploration: Yeardleigh's submersible

Always questing after the most delicious ingredients, Yeardleigh buys an underwater submersible vehicle so that she can hunt the ocean floor for interesting sea plants. The graph below shows the depth of Yeardleigh's submersible over time.

Study the graph. What can you tell about what's happening? Write a short paragraph telling, in words, the same story that the graph is telling visually.



At the beginning of the trip, Yeardleigh's submersible dives a total of 100 feet in the first 30 minutes. At the end of the trip, it returns to the surface, rising 500 feet in the last 30 minutes. This tells us that the depth of the submersible was *changing much faster* at the end of the journey compared to the beginning.

Note that the graph reflects this: the line is quite steep at the end of the trip and not so steep at the beginning. The steeper the line, the faster the dependent variable is changing with respect to the independent variable.[7]

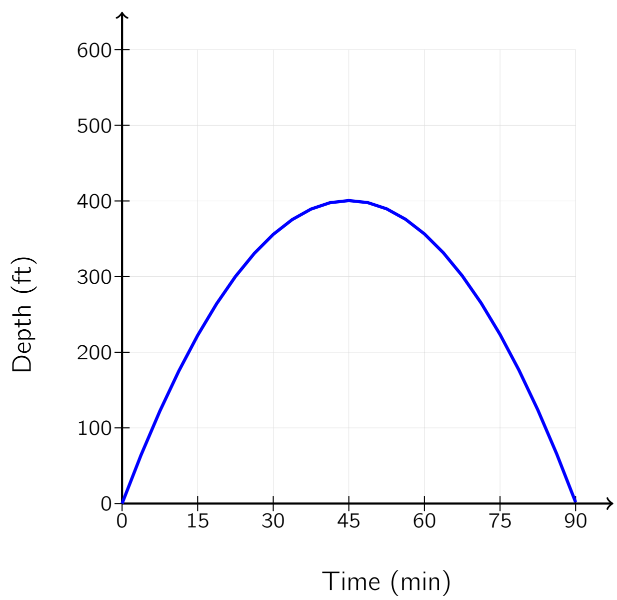
Plus, we can tell from the graph when the submersible is getting deeper (the depth is increasing; the line shows a positive trend) and when it is getting shallower (the depth is decreasing over time; the line shows a negative trend).

[7] We call this the "rate of change", and it will become an important focus of our work in chapter 7.

Notice that other parts of the graph are horizontal, for example between 30 and 60 minutes. This tells us that the depth of the submersible stayed the same during that time. Of course Yeardleigh could still be moving around below the surface, but she remains at a constant depth, neither diving nor surfacing.

3.5.1 Interpreting curves

Straight lines are fine, but what if the graph shows curved lines? Consider this graph showing the submersible's depth over time. What story would we tell about this graph?



One way to get a feel for what's happening is to imagine leaning a ruler or pencil against the rounded edge of a soda can. Then picture the ruler rolling along the side of the can, and how the angle of the ruler will change as it rolls.

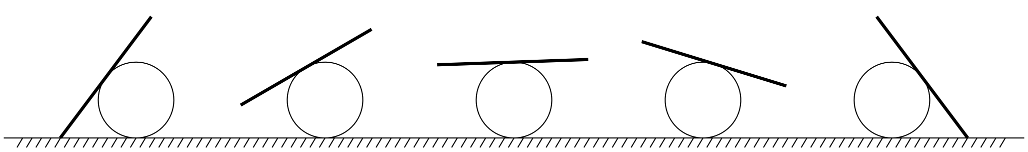
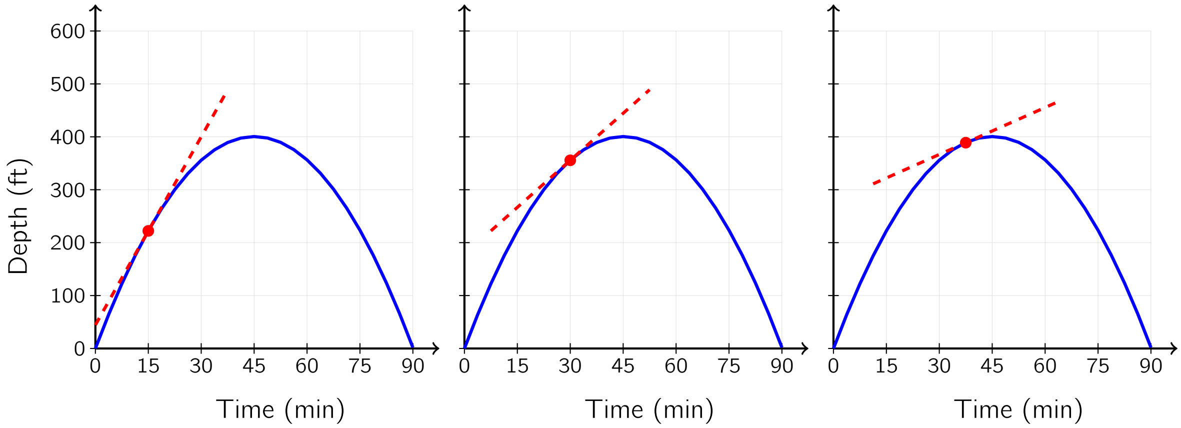


Figure 3.3: Pencil rolling along the side of a can

Now picture a straight line rolling along the surface of the curved line in the graph above. The straight line approximates the curved line at the point where they touch.

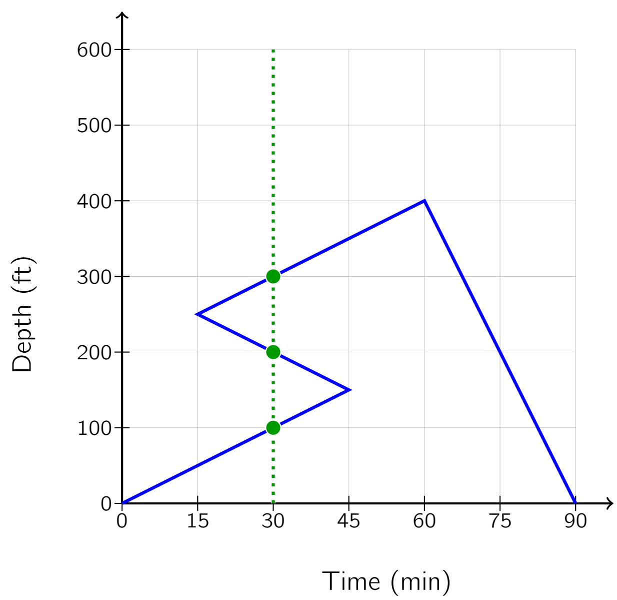
Using the straight line, we can see that the submersible starts out diving at a fairly high rate. It gradually slows its rate of descent until it eventually stops diving. Then it gradually accelerates as it returns to the surface.[8]

[8] Believe it or not, this idea – approximating a curved line with a straight line – is one of the fundamental motivating ideas in calculus. As you work to interpret these curved graphs, you're growing your calculus brain, right here in algebra 1. How cool is that?



3.5.2 Impossible situations

Consider this graph showing the submersible's depth over time. What's going on here?



This graph is a problem, given the context of "depth of the submersible over time". Consider this question: How deep it the submersible 30 minutes after launch?

According to the graph, the submersible is 100 feet deep... and 200 feet deep... *and* 300 feet deep... all at the same time! That's impossible!

This kind of problem could pop up in other places. For example, in a distance-time graph, points that line up vertically mean that something is in more than one place at one instant in time. Though we might wish reality were different, nothing can be in two (or more) places at the same time.

Graphs like this – where several y-values stack up vertically over the same x-value – violate a certain requirement that we will learn about in the next chapter, as we delve into the important mathematical idea of a *function*.

On two occasions I have been asked, "Pray, Mr. Babbage, if you put into the machine wrong figures, will the right answers come out?" I am not able rightly to apprehend the kind of confusion of ideas that could provoke such a question.

Charles Babbage, English inventor of the first mechanical computer

Chapter 4

Fundamentals of functions

We have seen a number of relationships so far: the relationship between distance and time of a moving object, for example, or the relationship between a number's position in a sequence and that number's value. A *function* is a certain kind of relationship that is of key importance to us in algebra, and indeed throughout mathematics. In this chapter we'll get an overview of functions, and then in the rest of the course we will look closely at specific kinds of functions.

4.1 Mathematical relationships

Name of Extended Exploration

[TODO] Click here to visit the extended exploration: NAME

Startup exploration: Number machines

A number machine accepts numbers as input and produces numbers as output. When machine A receives the number 8 as input, it produces the output 28. When this machine receives $^{-}12$ as input, the output is $^{-}42$.

Machine B has a different mechanism for producing output values. When machine B receives the number 8 as input, it outputs 15. When it receives $^{-}12$ as input, the output is 23.

What do you suppose each machine will output when given the number $^{-}7$ as input? Write a sentence explaining how you believe each machine works.

The number machines described in the startup exploration relate certain input values to certain output values. Mathematically speaking, a number machine defines a *relation*.

Relation

A **relation** defines how certain numbers (or other objects) are connected to other numbers (or objects). We can think of a relation as a collection of ordered pairs (x, y), meaning "x is related to y".

We might think of a number machine as a set of pairs of numbers: (input, output). Machine A in the startup exploration is defined by the pairs $(8, 28)$ and $(-12, -42)$. We sometimes say that "machine A maps 8 to 28" and that "-12 is mapped to -42".

One possible explanation is that the machine multiplies the input value by 3.5 to produce the output value. In this case, we'd expect the input -7 to produce the output -24.5. In other words, $(-7, -24.5)$ is a third point associated with machine A.

For machine B we are given the points $(8, 15)$ and $(-12, 23)$. One, somewhat complicated, explanation is that the machine doubles the input, takes the absolute value of the result, and then subtracts 1. In this case, we'd have $(-7, 13)$ as another input-output pair for machine B.[1]

We can define relations that compare things other than numbers. For example, we might define the "name-the-mother" relation that accepts a person as input and which gives that person's biological mother as output. Or, we might define the "name-the-pet" relation, which accepts a human as input and outputs other animals.

Note that there is a difference between these two relations we have just defined. Not all people own pets, so some inputs to the name-the-pet macine have no output. Plus, some people own multiple pets, so some inputs to the name-the-pet machine have multiple outputs. On the other hand, the name-the-mother machine is guaranteed to always produce a unique output: every person has a biological mother, and every person has exactly one biological mother. It is impossible for someone to have more than one biological mother![2]

We saw something similar in section 3.5, where certain graphs could be drawn in a way that would suggest an impossible situation. For example, we drew a graph (shown in fig. 4.1) that suggested Yeardleigh's submersible could at three different depths simultaneously, which – like have more than one biological mother – is nonsense.

Relationships that follow this very natural rule – any given input produces a unique output – have a special status in algebra. They're called **functions**.

[1] There are other rules that we could write based on the two data points we were given, and perhaps you thought of different rules than the ones we describe here. Can you find alternative explanations for how each machine works? Be creative!

[2] We don't deny that a family with two mothers is still a family! It is true that a person can have multiple mothers in a legal or emotional sense. . . but a person has exactly one *biological* mother.

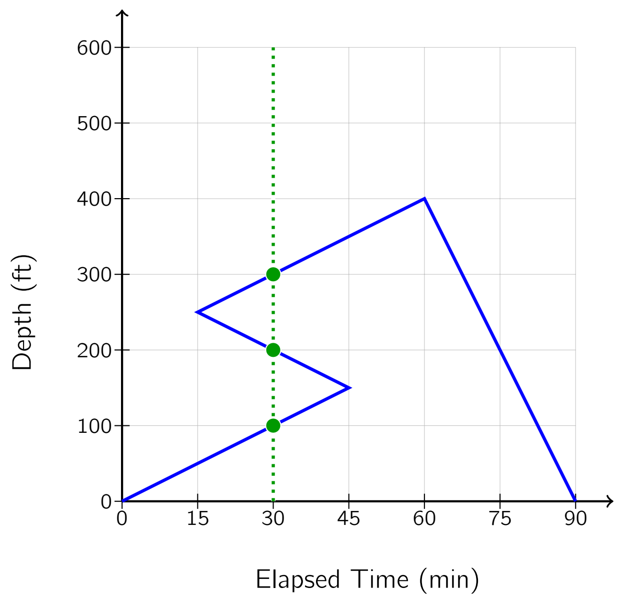


Figure 4.1: A graph of depth over time? Impossible!

Function

A **function** is a special type of relation in which the ordered pairs have the following property: each x-value is paired with one, and only one, y-value.

The "one and only one" requirement is what makes a function special. The graph in fig. 4.1 violates this requirement: certain x-values have more than one associated y-value (look at x values between 15 and 45 minutes). So, this graph does not depict a function. Any given sequence is a function, since each x value (position in the sequence) is occupied by exactly one y value.

A sequence is a function: it would be silly to suggest that a sequence had more than one value as its third term. Depending on how we think about number machines, we might expect that a machine would act predictably and always produce the same output for a given input. A number machine that behaves in this way defines a function.

Representing relations

Ther are several ways to represent a mathematical relation. We might represent a relation using:

- A graph. This representation gives is a visual of the relationship between input values and output values.
- A table of values. This representation gives us an organized list of input values and their corresponding output values. (Related forms include a *mapping diagram*, or just a collection of ordered pairs.)

- An equation. This representation gives us a rule for turning input values into output values, the way a number machine might do.

Since functions hold a special position in the universe of all relations, our first task is to determine whether a given mathematical relationship is, in fact, a function. To do this we have to check it against the definition of what it means to be a function. In other words, we must make sure each x-value corresponds to one and only one y-value.

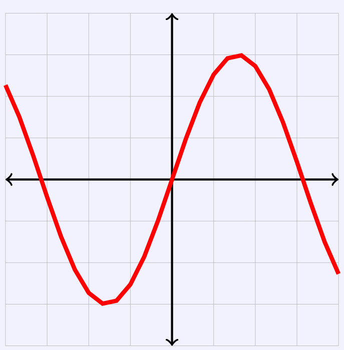
4.1.1 Graphs of functions

In order for a graph to be a function it must avoid the situation in which multiple y-values stack up above a particular x-value. A handy way of remembering this requirement is that the graph of a function passes the **vertical line test**. The vertical line test states that for the graph of a function, any vertical line drawn on the graph will intersect the graph *exactly once*.

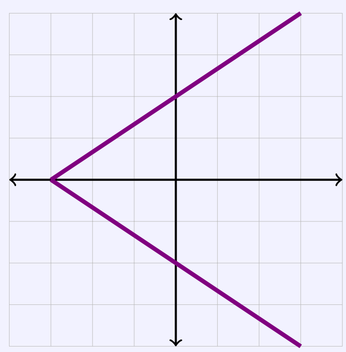
Example 4.1

Which of the graphs below, if any, represents a function? How do you know?

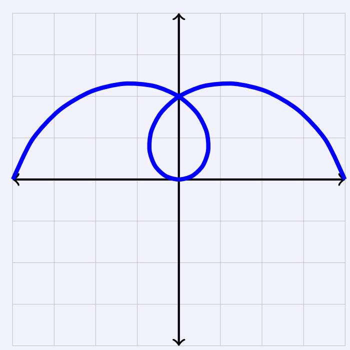
Graph A



Graph B



Graph C

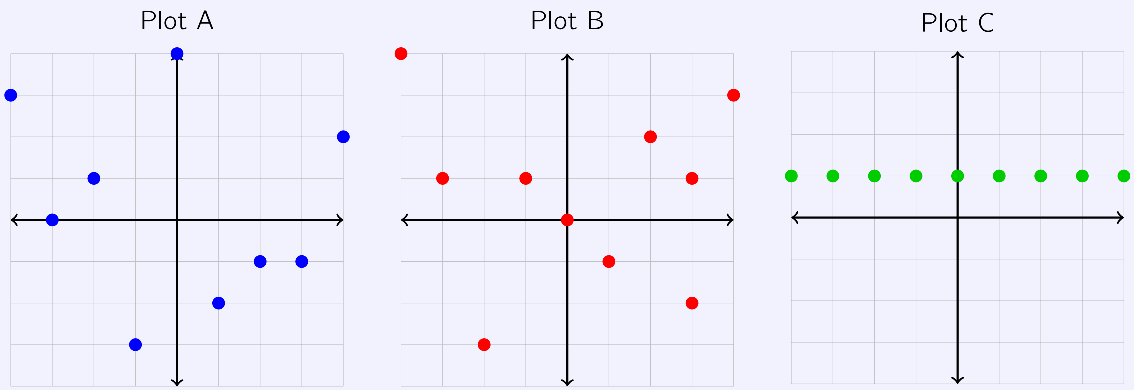


Solution: Only graph A represents a function. We can draw vertical lines on graphs B and C which intersect the graph at more than one point. This means that those x-values have more than one y-value, violating the definition of function.

Note that in graph C, certain sections of the graph pass the vertical line test. There is no partial credit, though. A graph fails the vertical line test if any vertical line (even just one) crosses the graph in more than one point. This happens also with the graph in fig. 4.1. That graph fails the vertical line test for some vertical lines. Therefore the relationship depicted is not a function.

Example 4.2

Which of the scatter plots below, if any, represents a function? How do you know? You may have to look closely!



Solution: Plots A and C both pass the vertical line test, and therefore represent functions. Plot B fails the vertical line test because it fails for the vertical line at $x = 3$.

Note that we don't care if different x's map to the same y. For example, in plot C, all of the x-values map to the same y-value. That's OK. In other words, a function *can* fail the "horizontal line test" without penalty.[3]

If we think about the name-the-mother function, it is not uncommon for two different people to have the same mother: they're called siblings!

4.1.2 Functions from points

A scatter plot gives a visual representation of a collection of ordered pairs. If the ordered pairs are presented in an alternative format – say, a table of values – we can use the same reasoning to determine whether the given relation is a function.

Example 4.3

Which of the data sets below, if any, represents a function? How do you know?

[3] The horizontal line test *is* a thing, and passing or failing the horizontal line test does have a mathematical meaning. But, a discussion of this will have to wait until later.

Set A

x	y
-3	4
-2	6
-1	4
0	2
1	0
2	-2
3	0

Set B

x	y
2	1
1	2
-1	7
4	3
3	4
1	0
0	6

Set C

$(0,4)$ $(-1,6)$

$(2,3)$ $(0,4)$

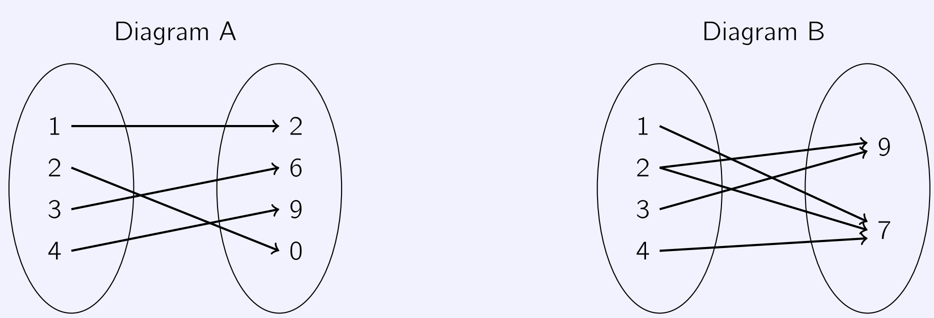
$(1,5)$ $(-2,5)$

Solution: Set A is a function. There are no repeated x-values! Set B is not a function. The x-value 1 appears twice and with two different related y-values (0 and 2). This violates the definition of function. Set C is a function. The x value 0 appears twice, but it is mapped to the same value in both cases.

A **mapping diagram** is a similar way of representing a relationship between specific input and output values. Here we draw two regions (represented by a box or an oval). In one region, we list the input values, in the other region we list the output values. Then we draw arrows to show which input maps to which output.

Example 4.4

Which of the mapping diagrams below, if any, represents a function? How do you know?



Solution: The first mapping is a function. Each x-value maps to exactly one y-value. The second mapping is not a function: the value 2 maps to both 7 and 9.

Note again that the problem in diagram 2 is *not* that multiple arrows point to each of the output numbers. It's OK for multiple arrows to go *toward* a certain output value. The problem is that the input value 2 has multiple arrows *leaving* and going to different targets.

4.1.3 Equations of functions

The "number machine" metaphor is often helpful when thinking about a function. A number (an x-value) goes into the machine, the machine's rule works on the number, and another number comes out of the machine (the y-value). Most of the time, when we have a particular rule and a particular input, we get a single, predictable output.

For example, the equation $y = 3x$ defines a function. We can substitute in 5 for x. The function machine multiplies 5 by 3 and outputs 15. This generates the ordered pair $(5, 15)$ and we say that "5 maps to 15". If we choose other values of x, we can generate more ordered pairs.

Not every equation defines a function, though. In order to be a function, every input value must produce one and only one output value. In algebra 2 we will study many interesting and important mathematical relationships that are not functions. We will learn, for instance, how to write the equation for a circle – and a circle is not a function. (Can you explain why not?)

Our focus in algebra 1, though, will be on three main "families" of functions, and our rules will all be of the form "y equals some expression in terms of x".

4.1.4 Function notation

We can thank Swiss mathematician Leonhard Euler for the concept of a function.[4] He was the first to coin the term "function" and was the first to use a handy way of writing a function, called *function notation*.

Up until now, we have used "$y =$ something in terms of x" to define all of our rules. Euler's function notation looks a bit different, but it is quite helpful in certain situations. The generic form of function notation is "$f(x) =$ something in terms of x". In this case, we simply replace y with the notation $f(x)$, which is read aloud as "f of x" or "f as a function of x".

In its generic form, the letter f is used to name the function. We use the letter f, of course, because it stands for "function". And as usual, we use x to represent the independent variable.

What's nice about this notation is that it allows us great flexibility to use other letters or symbols that can be more descriptive. For example, if we wanted to describe how the height of a bouncing ball changes over time, we might want to let the variable t represent time, and then name the function h for height.

So, we could write a function "$h(t) =$ some expression in terms of t". That's read "h of t", which does a pretty good job of capturing the idea that we're describing *height* in terms *of time*.

We might write a function like $d(t) = 60t$ which described distance traveled d as a function of time t. This is exactly the same equation as $y = 60x$, but we've changed the names of things to better match the context.

[4] The surname Euler is German, and so it is pronounced $OIL \cdot er$ (and *not* $YOO \cdot ler$).

Evaluation using function notation

Function notation has another convenience. We can use the notation to indicate a specific choice of value for the independent variable. Earlier, we said: "Evaluate the function $y = x^2 - 4$ when $x = 3$." This was fine, but function notation gives us a way to shorten these instructions.

For example, if we have $f(x) = x^2 - 4$ then we might want to evaluate $f(3)$, which is said aloud "f of three". The x in the function notation was replaced by 3, which is exactly what is means to evaluate the function at that value! The notation is telling us to input 3 for x and find the output value. So if $f(x) = x^2 - 4$, then

$$f(3) = (3)^2 - 4 = 9 - 4 = 5.$$

We will use function notation and $y =$ notation through this course. The flexibility of function notation means that it is the preferred way of writing functions in higher level mathematics courses.

4.2 Domain and range

Machines (meaning real, mechanical devices) are physical things that have physical limitations. Number machines (meaning functions) often have similar operational restrictions.

Startup exploration: The ins and outs of functions

Consider the function machines below.

$$f(x) = |\,x\,| \qquad\qquad g(x) = \frac{1}{x}$$

What sorts of values are allowed as input to each of these machines? What sorts of values will each one produce as output?

A blender is a machine that has limited operating parameters. Anything we put into the blender comes out "blended" in some way. But, of course, there are some things we can't put in a blender: a piano, for example.[5] Similarly, we wouldn't put bread into a blender, hit the button, and expect toast to pop out. That kind of output is associated with a different machine.

4.2.1 Regarding input values

An important component of the definition of any function is a specification about what sorts of numbers it can accept as input. In the language of mathematics, we call the set of all "legal" input values the *domain* of the function.

Domain

The set of all allowed input values to a function. This is the set of all allowed x-values, or all possible values of the independent variable.

We have two functions in the startup exploration. The first function, $f(x) = |\,x\,|$, is very welcoming: any real number can be used as unput to this number machine: whole numbers, fractions, positive numbers, negative numbers, it doesn't matter. The other function is a little more finicky.

In chapter 1, we discussed the illegality of division by zero. So, if we try to put 0 into the function $g(x) = \frac{1}{x}$, then we're going to have a problem: $g(0)$ – that's the function g evaluated at $x = 0$ – is undefined. So, our

[5] We'll pause here to note that "blended" is defined rather loosely: smoothies come out "blended", ice comes out "crushed", chickpeas come out "puréed", magazines come out "shredded"… We consider all of these to be a kind of "blended". Speaking of magazines, we'll also mention that there are things which one *can* put into a blender, but *shouldn't*.

user manual for the function g has to include a note that only numbers other than 0 are allowed as input. We say that the function $g(x)$ has a *limited* or *restricted domain*.

There are other circumstances where we will want or need to limit the domain. For example, if we write an equation that shows distance as a function of time, it usually makes sense to limit the domain to only include positive time-values. So, a domain can be limited by context.

A sequence is a function whose domain is limited to the set of natural numbers (the positive integers). We saw this in chapters 2 and 3: a sequence can have a first term and a second term, but no one-and-a-halfth term. We exclude from the domain of a sequence all numbers except $1, 2, 3, 4, \ldots$.

We can also define a specific domain for convenience. On an assignment, for example, we might be asked to evaluate the function $y = 4x + 5$ for the x-vales -1, 0, and 1. We did something like this in example 3.1. This is another example of limiting the domain.

4.2.2 Regarding output values

In addition to knowing about input values, it is often important for us to know what sorts of numbers we can expect as output from a certain function. We call the set of all output values the *range* of the function.

Range

The set of all output values from a function. This is the set of all y-values, or all possible values of the dependent variable.

The function $f(x) = |\,x\,|$ can accept any number as input, but only non-negative numbers ever come out. We say that the range of f is non-negative real numbers.

In the case of $g(x) = \frac{1}{x}$, notice that the only way for a fraction to equal 0 is if the numerator is equal to zero. No matter what nonzero input value we put into the function g, we will never get 0 as the output! The range of g is nonzero real numbers.

Piecewise definition

An interesting way to control the output of a function is to give it different definitions depending on the input. For example, suppose we define a function like so:

$$f(x) = \begin{cases} x & \text{if } x \text{ is positive} \\ -x & \text{if } x \text{ is negative} \\ 0 & \text{if } x \text{ is equal to } 0 \end{cases}$$

This unusual notation gives us several definitions at once, depending on which of the "if" statements applies to a given input value x.

If the input x is positive, then the output is simply x itself. If the input is negative, then the output is the opposite of x (so again, we get a positive value). Finally, in the third case, if the input is 0, then we get 0 as the output. Do you recognize this function? This case-by-case definition is really just a complicated way of saying $f(x) = |x|$.

Here's a more interesting example. Suppose we define a function whose domain is the non-negative integers as follows:

$$z(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ -\dfrac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

This function produces different output depending on whether the input value is even or odd. Let's try the first few input values to see what happens. What happens when we input 0? Zero is an even number, so the first case applies and:

$$z(0) = \frac{0}{2} = 0.$$

One is an odd number, so the second case applies:

$$z(1) = -\frac{1+1}{2} = -\frac{2}{2} = -1$$

Two is even, and three is odd. So we have:

$$z(2) = \frac{2}{2} = 1 \qquad \text{and} \qquad z(3) = -\frac{3+1}{2} = -\frac{4}{2} = -2.$$

This function produces the sequence:

$$0, -1,\ 1, -2,\ 2, -3,\ 3, -4,\ 4, \cdots$$

which is a list of all the integers. In other words: this machine turns the natural numbers $\mathbb{N}$ (plus zero) into the set of integers $\mathbb{Z}$.[6]

[6] This function suggests a shocking idea. Most people would say that there are more integers than there are natural numbers. After all, since the integers include all of the natural numbers *and* all of the opposites of the natural numbers, it must be that the set of integers has *twice as many members*... right? Nope! The set of natural numbers and the set of integers are *exactly the same size*. Think of it this way: the function z defined above pairs up natural numbers and integers. If I give you a natural number (input), can you tell me what integer (output) it partners up with? If I give you an integer (output), can you tell me what natural number (input) it partners up with? If so, then the two sets match up exactly and neither set has any leftovers. They must therefore be the same size! Mind blown!

4.3 Three families of algebra 1

The majority of algebra 1 will focus on the in-depth study of three key families of functions: the linear family, the exponential family, and the quadratic family. We've encountered each of these families before, but now it's time for a more formal introduction.

Extended exploration: Exploring the three families

[TODO] Click here to visit the extended exploration: Exploring the Three Families

4.3.1 Parent functions

For a given function to be considered a member of a certain family, it must share the characteristics of other members of that family as seen in its graph, its equation, and the patterns it exhibits in a table of values. These features are shown in their most basic form by the *parent function* of each family.

The linear parent is $y = x$ and the quadratic parent is $y = x^2$. In this course, we will consider $y = 2^x$ to be the parent function for the exponential family, though other choices are possible.[7]

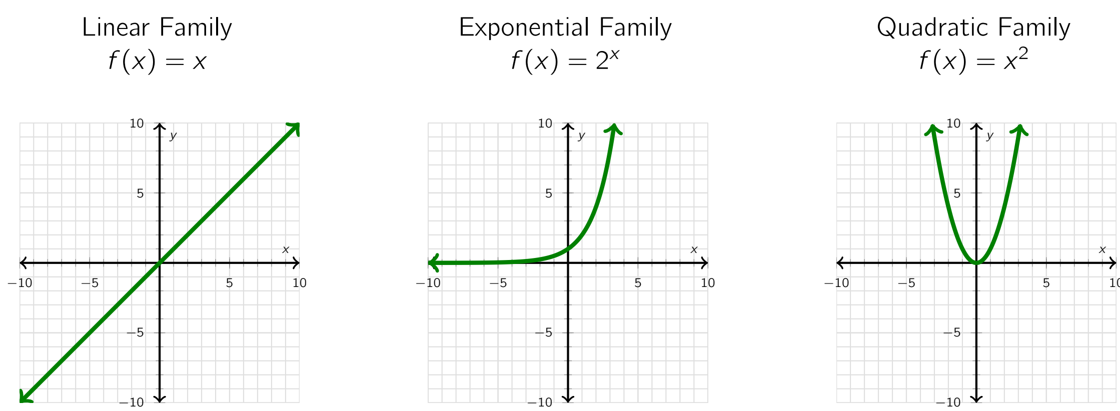


Figure 4.2: The parent functions of the three families

Compare the equations for each of the parent functions: How are they the same? How are they different? Compare the graphs: How are they the same? How are they different? When presented with a new equation of graph, what features might we look for to decide which family it belongs to?

[7] A more natural choice for the exponential parent is the function $y = e^x$, where e is Euler's number (yes, that's the same Euler as the function notation guy). Euler's number has many important connections to the family of exponential functions, but those will have to wait until later. By the way e, like π, is an irrational number: $e \approx 2.71828\,18284\,59045\,23536\,02875\ldots$

These are not the only three families of functions out there. Invent a rule or draw a graph which *does not fit* into any of these three families. What makes this rule or graph different?

In section 4.1 we learned techniques for identifying whether a given relation is a function. One of our tasks going forward will be to learn techniques for distinguishing the function families from one another.

4.3.2 Transforming the parents

As we have seen many times already, a graph can be particularly helpful way to see, and therefore understand, a mathematical relationship. One of the skills we'll develop as we go forward in this course is the ability to connect features of the equation of a certain function to features of its graph.

Over time and with practice, we'll get better at picturing the graph of a function in our mind's eye without having to draw the graph on paper. Drawing graphs will still be informative, of course, but some features of the graph will start to "jump out at us" simply from looking at the equation of the function.

Startup exploration: Not far from the tree

Study the three sets of animations below. Each set of images shows a way in which we can change the parent equation and produce a change in its graph.

Figure 4.3 shows what happens when we multiply the parent function by a constant value. Figure 4.4 shows what happens when we add a constant value to the parent. Figure 4.5 shows what happens when we add a constant value to the agrument (the x value) before that value is used as input to the parent function.

Study the animations, and then write a sentence or two summarizing what is happening in each case.

Note about the animations: The images will only appear animated when this PDF document is viewed using Adobe Reader. Otherwise, please visit OUR WEBSITE for an interactive versions of these animations.

[TODO] Create something on the website for this. Maybe links to desmos pages w/ sliders.

4.3.3 Fundamentals of linear functions

The first family we will study in depth is the family of linear functions. Here we summarize the key features of this family, all of which will return in future chapters.

We will learn several different forms for the equation of a linear function. The feature they all share is that the variable x has no exponent, or rather the highest power of the variable x is 1. For example, the parent of this

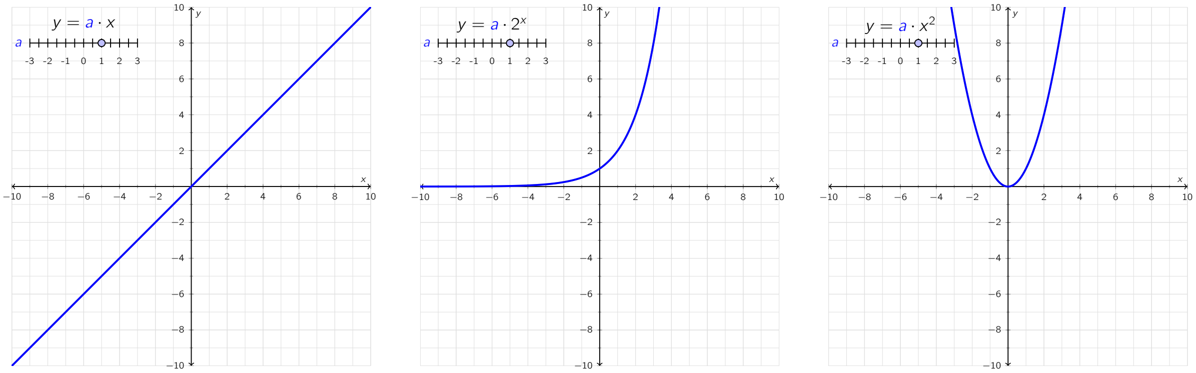


Figure 4.3: Comparing the parent function $f(x)$ with offspring $a \cdot f(x)$.

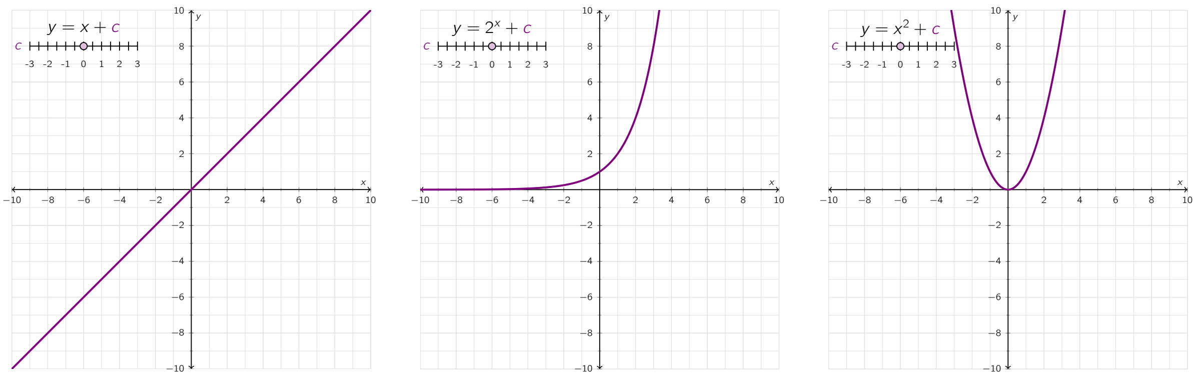


Figure 4.4: Comparing the parent function $f(x)$ with offspring $f(x) + c$.

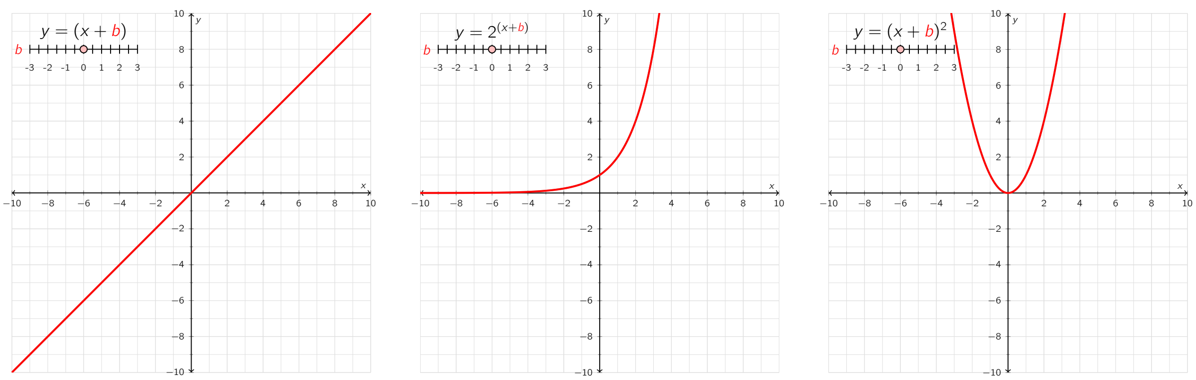


Figure 4.5: Comparing the parent function $f(x)$ with offspring $f(x + b)$.

family, $y = x$, is the same as $y = x^1$. That's a "phantom 1" there in the exponent, and we don't usually write it.

The graph of a linear function is a straight, non-vertical line. (Can you explain why we have to exclude vertical lines from this family of functions?) We sometimes also exclude horizontal lines from this family: a function of the form "$y =$ a number" is a horizontal line, for example $y = 6$.

Recall that arithmetic sequences are in the family of linear functions. Arithmetic sequences have a rule that involves repeatedly adding on a constant difference at each step. So, this is the feature we see in the table of a linear function: when the x-values increase by a fixed amount, the y-values increase by a (possibly different) fixed amount.

There are two key transformations of the parent function. Multiplying by a constant value, as in $y = a{\cdot}x$ changes the "steepness" of the graph. Certain a values (negative ones) cause the graph to show a decreasing trend. Adding a value as in $y = x + c$ shifts the graph upwards or downwards. Viewed from a different perspective, we could also say that this shifts the graph left or right. Up-down shifts and left-right shifts look the same for a straight line. The distinction is more clear in the other families.

4.3.4 Fundamentals of exponential functions

The second family of functions we will explore in this course is the exponential family. In fact, we will begin our study of this family in algebra 1, and then return to it again in algebra 2.

An exponential equation is one in which the variable x appears as the exponent (hence the name). The graph of an exponential function is a smooth, curving shape that we sometimes describe as "J-like". This is only an approximation, though, since an exponential graph may sometimes look more like a backwards J or an upside-down J.

Exponentials have the interesting property that while they get almost vertical on one side, and almost horizontal on the other side. In other words, one side gets "infinitely large" while on the other side we have tinier and tinier fractions that get closer and closed to zero... but never actually disappear! Mathematically speaking, we say that the function approaches an **asymptote** at the flat horizontal portion.

Recall that geometric sequences are in the family of exponential functions. Geometric sequences have a rule that involves repeatedly multiplying by a constant ratio. This is the feature we look for in a table.

Multiplying the parent by a value, as in $y = a \cdot 2^x$, changes the steepness of the graph and – if a is negative – causes it to be reflected over the x-axis.

Adding a value to the parent function – $y = 2^x + c$ – causes a shift up or down. Adding to the argument – $y = 2^{(x+b)}$ – gives us a shift left or right.

4.3.5 Fundamentals of quadratic functions

We will study the quadratic family last in this course. In fact, it will be our gateway into the study of techniques that will become helpful as we learn about other families and types of functions in algebra 2.

The key feature of a quadratic equation is that the highest exponent on the variable x is 2. The equation may also contain an x term, but that's the same as x^1, so the highest power of x is still 2!

The graph of a quadratic function is a smooth U-like graph called a **parabola**. Of course, under certain circumstances, the U opens downwards.

The transformations of the parent function are perhaps more clear with this family. Multiplying by a constant, $y = a \cdot x^2$, changes the "steepness" of the graph. And, in the case that a is negative, controls whether the graph opens upwards or downwards.

Adding to the parent function $y = x^2 + c$ produces a clear vertical shift, and adding to the argument $y = (x+b)^2$ produces a clear horizontal shift. Interestingly, the horizontal shifts appear to be "backwards" compared to what we might expect. Adding to the argument shifts the graph to the left, whereas subtracting from the argument (or adding a negative value) shifts the graph to the right. More on this unusual behavior later!

Recall from our study of sequences that quadratic patterns show a constant second difference. In other words, they have a recursive rule that states: "Start with A, add to each previous term the members of some arithmetic sequence." This constant second difference will be the feature that we look for in a data table.

4.3.6 Domain and range of the three families

It's quite easy to identify the domains of the three main families of algebra 1. Linear, exponential, and quadratic functions all have "all real numbers" as their domain. We can always use any number as input. This is convenient!

The range of a linear function is also pretty simple: it is all real numbers. As we saw as we transformed linear functions, the a (non-horizontal) line will eventually stretch up (or down) as high (or as low) as you want to go on the y-axis.

The ranges for exponential and quadratic functions are a little more complicated, but we can see what is happening by examining their graphs. Notice, for example, that the graphs of the parents $y = 2^x$ and $y = x^2$ never dip below the x-axis.

This feature of the exponential and quadratic families will always be retained, in some form, when we transform these equations. The range will always be limited by some minimum or maximum value.

For example, the U-shaped quadratic function always has a lowest point (if the parabola opens upwards) or a highest point (if the parabola opens downwards). We will learn more about identifying and describing the maxima and minima when we get into each family in depth.

[TODO] Closing Remarks of some kind, since we're venturing off into the linear unit now...

Mathematics is the art of giving the same name to different things.

Henri Poincaré
French mathematician

Chapter 5

Solving Equations

Every chapter should have a lead paragraph – even just a short one – that appears before the first heading. This is a placeholder paragraph which will at some point be replaces by actual content.

5.1 Equivalence

Equivalence is an important concept in algebra, and indeed throughout mathematics. The kernel of the idea is that we can manipulate mathematical objects in ways that change their appearance without changing their value.

Startup Exploration: Never the Tween Shall Meet?

Bob thinks: "Since $\frac{3}{7}$ and $\frac{4}{7}$ are 'right next to each other' on the number line, there must be no fractions in between them." Use what you know about equivalent fractions to find a fraction in between $\frac{3}{7}$ and $\frac{4}{7}$.

Given any two fractions $\frac{a}{b}$ and $\frac{c}{d}$, describe a procedure for finding a fraction that lies between them on the number line. Does your procedure work for the fractions $\frac{3}{5}$ and $\frac{5}{8}$.

Name of Extended Exploration

[TODO] Click here to visit the extended exploration: NAME

We've studied equivalence before and know, for instance, that multiplication by 1 is an operation which maintains equivalence. When we multiply some quantity by 1 – even a very fancy version of 1 – the quantity remains unchanged. This is the idea behind equivalent fractions:

$$\frac{3}{7} = \frac{3}{7} \cdot 1 = \frac{3}{7} \cdot \frac{18}{18} = \frac{54}{126}$$

We also know the order of operations, which gives us a process for simplifying numerical expressions. The order of operations guarantees we always have exactly the same quantity we started with, even though it may look different.

As we progress through this chapter (and beyond), we will learn more tools that we can use to simplify algebraic expressions and solve equations. We will have to be careful to use the tools correctly, though, so that the maneuvers we perform are sure to maintain equivalence. The devil is in the details, so it is a good habit to pay attention and proceed carefully.

The first key piece of the equivalence puzzle is something that probably seems obvious:

Substitution

We may replace one quantity with another quantity that we know has the same value.

This is a property about numbers that we use all the time, for example when we replace 4+3 with 7. We do a series of substitutions when we solve a problem using the order of operations. Given $2+3 \cdot 6$ we first substitute 18 for $3 \cdot 6$, giving $2+18$. Then we substitute 20 for $2+18$.

5.1.1 The Field Axioms

An **axiom** is a statement that is accepted as true without proof. The real numbers $\mathbb{R}$ are built upon several axioms, called the **field axioms**, which are the properties and laws that make arithmetic and algebra work.[1]

Axiom: Closure Properties

When we add two real numbers, their sum is a real number. Mathematically speaking, we say that *the real numbers are closed under the operation of addition.*

Similarly, when we multiply two real numbers, their product is a real number. We say that *the real numbers are closed under the operation of multiplication.*

[1] A *field* is a mathematical object that pairs up a set of numbers with some operations on those numbers. The number system we know so well – the real numbers $\mathbb{R}$, along with the operations of addition and multiplication – are a field. But many other fields exist! If you continue to study algebra in college, the focus gradually shifts from studying familiar fields like the real numbers, to studying the properties of fields in general.

This might seem obvious, but we saw in Chapter 1 that closure is not guaranteed for all sets and all operations. For example, the integers are not closed under the operation of division.

Axiom: Identity Properties

The **identity property of addition** states that for any real number a,

$$0 + a = a + 0 = a$$

The number 0 is called the **additive identity**.

The **identity property of multiplication** states that for any real number a,

$$1 \cdot a = a \cdot 1 = a$$

The number 1 is called the **multiplicative identity**.

As we saw above, the identity property of multiplication is the axiom that allows us to create equivalent fractions.

Axiom: Inverse Properties

The **inverse property of addition** states that for any real number a, there exists a real number ^-a such that

$$a + {}^-a = {}^-a + a = 0$$

The number ^-a is called the **additive inverse** of a. Very often we will call it the **opposite** of a.

The **inverse property of multiplication** states that for any nonzero real number a, there exists a real number $\frac{1}{a}$ such that

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$$

The number $\frac{1}{a}$ is called the **multiplicative inverse** of a. Very often we will call it the **reciprocal** of a.

Note that when discussing the multiplicative inverse of a, this axiom states that a has to be nonzero. In ordinary arithmetic, division by zero is undefined, and expressions such as $5 \div 0$ and $\frac{5}{0}$ have no meaning.

Tangent: Explaining Division By Zero

Consider the following two equations. How are they related?

$$72 \div 8 = \square \qquad \text{and} \qquad \square \cdot 8 = 72$$

Now consider these two equations. How are they related?

$$5 \div 0 = \square \qquad \text{and} \qquad \square \cdot 0 = 5$$

The first pair of equations demonstrates the relationship between multiplication and division, which tells us that the same number goes in both boxes. Whatever number makes the first equation true, must also make the second equation true.[a]

This second pair of equations is related in the same way. But, there's a problem with the multiplication sentence $\boxed{} \cdot 0 = 5$. *Anything times zero is zero.* So, no number exists that could make that multiplication sentence true! Therefore, there can be no answer in the related division sentence.

At this point, troublemakers always like to ask about $0 \div 0 = \boxed{}$ because that sentence, they say, is related to the multiplication sentence $\boxed{} \cdot 0 = 0$, and *every number in the universe* makes that sentence true. They're right, of course. We say that $a \div 0$ is "undefined" when $a \neq 0$, whereas $0 \div 0$ is called an "indeterminate form". It doesn't really matter as far as we're concerned, though. Dividing *anything* by zero means that we don't get a clear answer. It's mathematically off-limits.

[a] Spoiler alert: it's 9.

Axiom: Commutative Properties

The **commutative property of addition** states that for real numbers a and b:

$$a + b = b + a$$

The **commutative property of multiplication** states that for real numbers a and b:

$$a \cdot b = b \cdot a$$

The commutative properties allow us to rearrange the order of things when we add or multiply. For instance, when adding up a string of numbers, it's sometimes helpful to "make a ten":

$$\begin{aligned} & 12 + 6 + 8 && \text{grouping up the 12 and the 8 would be easier than adding left-to-right} \\ = {} & 12 + 8 + 6 && \text{commutative property of addition says } 6 + 8 = 8 + 6\text{, so we can swap them} \\ = {} & 20 + 6 \\ = {} & 26 \end{aligned}$$

Note that this property is for *addition and multiplication*, but not subtraction or division! To move things around in, say, a subtraction problem, we must use the definition of subtraction as "addition of the opposite" to create an equivalent addition problem.

$$6 - 5 \neq 5 - 6 \qquad \text{but} \qquad 6 + (^-5) = (^-5) + 6$$

Axiom: Associative Properties

The **associative property of addition** states that for real numbers a, b, and c:

$$(a+b)+c = a+(b+c)$$

The **associative property of multiplication** states that for real numbers a, b, and c:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

The associative properties allow us to move around certain parentheses and regroup addition and multiplication. Here, the numbers don't move; the parentheses do.

$$\begin{aligned} & 5 \cdot (8 \cdot 7) && \text{order of operations would force us to do what's inside the parentheses first} \\ &= (5 \cdot 8) \cdot 7 && \text{associative property of multiplication says we can move the parentheses} \\ &= 40 \cdot 7 \\ &= 280 \end{aligned}$$

A few words of warning: We can only use this property around addition or multiplication. Subtraction, for example, would have to be transformed into addition of the opposite.

$$(8-4)+3 \neq 8-(4+3) \qquad \text{but} \qquad (8+{}^{-}4)+3 = 8+({}^{-}4+3)$$

Also, we can't apply the associative property when there are different operations involved:

$$6 \cdot (5+4) \neq (6 \cdot 5)+4$$

Here's a sneaky one, can you explain why these two equations are not equivalent?

$$5+6(3+4) \neq (5+6)3+4$$

To handle situations like these last two, we need the final field axiom:

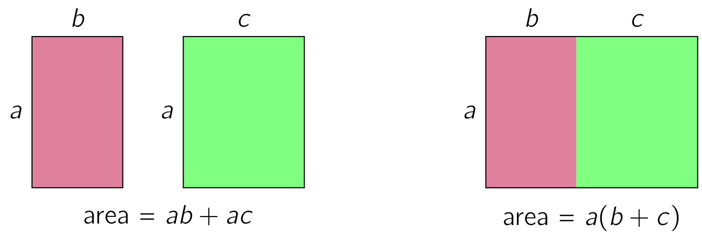
Axiom: Distributive Property

The **distributive property** states that for real numbers a, b, and c:

$$a \cdot (b+c) = a \cdot b + a \cdot c$$

This property explains how multiplication and addition are related to each other. To get a feel for this, suppose we have two rectangles: one that is a units tall and b units wide, and a second rectangle that is a units tall and c units wide. Together, their combined area is $a \cdot b + a \cdot c$.

But, since the two rectangles have the same height, we could glue them together perfectly and create one conjoined rectangle with height a and width $(b+c)$. The area of this rectangle is $a \cdot (b+c)$. But of course, the area hasn't changed, so these two different ways of expressing the area are equal.



Another way to think about the distributive property is to think of taking the a (in this case) and sprinkling it over the parentheses so that it becomes a part of each of the terms inside.

$$a(b + c) = ab + ac$$

Both the rectangle-area metaphor and the sprinkling metaphor can be extended to include expressions like

$$a(b + c + d) = ab + ac + ad,$$

and to expressions with any number of terms inside the parentheses.

5.2 Simplified Algebraic Expressions

The axioms are a set of tools that we can use to manipulate algebraic objects such that the result is equivalent to the original. That's great... but why would we ever need to do that?

Clarity and simplicity are important when it comes to communicating mathematical ideas. The fraction $\frac{16}{24}$ is fine, but the simplified fraction $\frac{2}{3}$ communicates the same information more clearly. For example, it's much easier to visualize a circle with $\frac{2}{3}$ shaded than it is to visualize a circle with $\frac{16}{24}$ shaded.

If a movie starts at 8 o'clock, we would all prefer to see that simple number rather than have to compute $(4+5)\cdot 2-10$ o'clock (yes, even algebra teachers feel this way). A simpler form will almost always be easier to understand.

So, as we did with rational numbers and arithmetic expressions, we will define what it means for an algebraic expression to be "simplified". Recall from our work in section 3.2 that an **algebraic expression** could be a single term, or the sum of difference of terms. A **term** could be a number, or a variable, or the product/quotient of numbers and variables.

Criteria for Simplified Algebraic Expressions

An algebraic expression is considered completely simplified if...

1. Numerical expressions have been evaluated
2. Redundant negative signs have been rewritten
3. Terms have been arranged in order of decreasing degree, with coefficients written first
4. It contains no explicit grouping symbols
5. Different variable terms appear at most once

Several of these criteria are old news. Criteria #1 just means that we have to simplify numerical expressions using the order of operations: the expression $2\cdot 3\cdot x$ is not simplified until we evaluate "$2\cdot 3$" and write "6" instead. The equivalent expression $6\cdot x$ or $6x$ is simplified.

Criteria #2 is also something we know how to handle. In an expression like $x+{}^{-}6$, where there are two signs in a row, we can simplify using the definition of subtraction: $x+{}^{-}6=x-6$.

Criteria #3 is just a bit of cosmetics. We say that the **degree of a term** is the power to which the variable is raised in a variable term. So, if we have an expression that includes a variable raised to different powers, it's often helpful to see the terms arranged in order from greatest degree to smallest degree. For example, instead of:

$$2x + 3x^4 + 7 - 6x^3 - 18x^2$$

it's often nicer to write:

$$3x^4 - 6x^3 - 18x^2 + 2x + 7$$

Note that when there's a variable with no exponent written, as in the $2x$ here, we think of it has having a "phantom 1" in the exponent: $2x = 2x^1$. Also note that the term 7 has no variable part at all. In this case, we picture a different kind of "phantom 1": $7 = 7 \cdot 1 = 7x^0$. If we make these substitutions, we can really see how the terms are arranged by degree:

$$3x^4 - 6x^3 - 18x^2 + 2x^1 + 7x^0$$

Also note that this criteria asks us to write the coefficients first. So, $2x$ is preferred over $x2$. This is, primarily, because it might be tricky to tell the difference between $x2$ and x^2. It would be especially confusing in a term has both a coefficient and an exponent: x^43 looks a bit too much like x^{43}, whereas $3x^4$ is much clearer.

The only criteria left are criteria #4 and criteria #5. These are more interesting, and so each gets its own section.

5.2.1 The Distributive Property

To eliminate grouping symbols, as expressed by criteria #4, we can rely on the field axioms for help.[2] The expression $(x + 2) + 2$ is not simplified, but this can be fixed easily enough using the associative property of addition.

$$(x + 2) + 2 = x + (2 + 2) = x + 4$$

The expression $3(x + 4)$ is not simplified either, and in this case the distributive property comes to the rescue. Get sprinkling!

$$3(x + 4) = 3 \cdot x + 3 \cdot 4 = 3x + 12$$

There are subtleties to using the distributive property. Distribution and negative numbers can lead to some easy-to-miss mistakes. We have to be on the lookout, so study the following examples carefully.

[2] Some special grouping symbols, like the vinculum, can remain in the final expression. Generally, however, we must get rid of grouping symbols that do not double as another operation.

Example 5.1

Simplify: (a) $^-3(2x-4)$ and (b) $8-5(6x-9)$

Solution: In problem (a) we distribute a negative number. Notice what happens with the signs of the terms in the result.

$$\begin{aligned}
& {}^-3(2x-4) & \\
&= {}^-3(2x+{}^-4) && \text{change subtraction to addition of the opposite}\\
&= {}^-3\cdot 2x+{}^-3\cdot{}^-4 && \text{distributive property}\\
&= ({}^-3\cdot 2)x+{}^-3\cdot{}^-4 && \text{commutative property of multiplication}\\
&= {}^-6x+12 && \text{substitution (in both terms)}
\end{aligned}$$

The first thing we did here was to change the subtraction to addition of the opposite. This was helpful because – look! – we end up with *positive 12* in the final expression. That might otherwise have been an easy thing to miss.

In problem (b) notice that we have an implied operation that will require us to apply the distributive property. In other words, the first step is *not* to do $8-5$! As with the last problem, we'll start by changing to all-addition.

$$\begin{aligned}
& 8-5(6x-9) & \\
&= 8+{}^-5(6x+{}^-9) && \text{change subtraction to addition of the opposite}\\
&= 8+{}^-5\cdot 6x+{}^-5\cdot{}^-9 && \text{distributive property}\\
&= 8+{}^-30x+45 && \text{substitution}\\
&= {}^-30x+8+45 && \text{commutative property of addition}\\
&= {}^-30x+53 && \text{substitution}
\end{aligned}$$

Again here, we get a term ${}^-5\cdot{}^-9=45$. Changing to all-addition before distributing is a helpful technique for getting these signs right.

Example 5.2

Simplify: $2x-(x-4)$

Solution: This one is subtle and sneaky! The negative sign is stuck on the parentheses, which means we'll have to "distribute the negative sign" or – more accurately – we again have an implied operation, hiding in there as a "phantom one". This expression is the same as $2x-1(x-4)$.

Observe:

$$
\begin{aligned}
& 2x - (x - 4) & \\
&= 2x - 1(x - 4) && \text{rewrite to expose the "phantom one"} \\
&= 2x + {}^{-}1(x + {}^{-}4) && \text{change subtraction to addition of the opposite} \\
&= 2x + {}^{-}1x + {}^{-}1 \cdot {}^{-}4 && \text{distributive property} \\
&= 1x + 4 && \text{substitution} \\
&= x + 4 && \text{rewrite to hide the "phantom one"}
\end{aligned}
$$

Did you try this problem on your own before reading the solution? Did you get $x - 4$? If so, don't be too upset: you're in good company. This is one of the most common mistakes in algebra 1. Always be on the looking when you see the distributive property mixed up with subtraction and negative signs.

5.2.2 Combining Like Terms

Startup Exploration: Expression Sort

Take a moment to sort the following mathematical objects into different groups, creating whatever categories make the most sense to you. After sorting, describe how you made your decisions. What features define each of the groups you have created?

$17x$	$2xy$	$-4x^2$	8	$0.5xy^2$	$-y$	$-15z$
-3	$8x^2y$	$4y$	xy^2	xyz	$11z$	x
$4xy^2$	$-2x^2y$	$3xyz$	x^2	$-6y$	$4x$	$3x^2$

One way to sort the terms above is to group **like terms** together. Like terms have the same variable factors raised to the same powers. For example, $17x$ and $4x$ are like terms. Also, x^2 and $-4x^2$ are like terms. But, $17x$ and $11z$ are *not* like terms since they have different variable parts. Though it may seem confusing at first, x and x^2 are *not* like terms. They both have x's, but those x's are raised to different powers.

Explaining the Startup Exploration

The terms in the startup exploration can be grouped using many different schemes. The categorization below is based on concept of grouping "like terms" together.

- Terms with no variable: -3 and 8

- Terms with x as the variable: x, $4x$, and $17x$
- Terms with x^2 as the variable: x^2, $3x^2$, and $-4x^2$
- Terms with y as the variable: $-y$, $4y$, and $-6y$
- Terms with z as the variable: $-15z$, and $11z$
- Terms with xy as the variable: $2xy$
- Terms with x^2y as the variable: $-2x^2y$, and $8x^2y$
- Terms with xy^2 as the variable: $0.5xy^2$, xy^2, and $4xy^2$
- Terms with xyz as the variable: xyz and $3xyz$

Just as 3 goats and 2 goats combine to make 5 goats, so too do $3x$ and $2x$ combine to make $5x$. Similarly, $8x^2 + x^2 = 9x^2$ (in this case, x^2 really means $1x^2$: there's a phantom 1 lurking there as the coefficient).

The expression $3xy^2 + 3x^2y + 3x^2y^2$ is simplified, since the variables and their corresponding powers do not match (look closely!).

The process of uniting like terms under a single coefficient is called **combining like terms**.[3] This is not technically a property in itself, but a "shortcut" for a somewhat longer chain of events:

$$\begin{aligned} & 3x + 2x & \\ &= x(3+2) && \text{the distributive property, in reverse} \\ &= x(5) && \text{substitution} \\ &= 5x && \text{commutative property of multiplcation} \end{aligned}$$

At first, it's best to commute the terms and group them up. Then we can add up the coefficients of the like terms. We'll see this approach, and a shortcut, in the examples below.

Example 5.4

Simplify: $3x^2 + 2x - 2 + x^3 + 4x^2 - 3x - 2x^3 + 8$

Solution: To make sure we keep track of the signs, we'll convert to all-addition and then move the

[3] Also sometimes called "collecting" like terms, or "gathering" like terms, and sometimes abbreviated "CLT".

terms around to put like terms next to each other.

$$
\begin{aligned}
& 3x^2 + 2x - 2 + x^3 + 4x^2 - 3x - 2x^3 + 8 & \\
= {} & 3x^2 + 2x + {}^-2 + x^3 + 4x^2 + {}^-3x + {}^-2x^3 + 8 & \text{definition of subtraction} \\
= {} & x^3 + {}^-2x^3 + 3x^2 + 4x^2 + 2x + {}^-3x + {}^-2 + 8 & \text{commutative property of addition} \\
= {} & -1x^3 + 7x^2 + {}^-x + 6 & \text{combine like terms} \\
= {} & x^3 + 7x^2 - x6 & \text{simplify signs}
\end{aligned}
$$

Example 5.5

Simplify $3x - 4y + 2 - 5x + 7y - 6 + 8y$

Solution: There is no commutative property of subtraction, remember, so we can rewrite all subtraction into adding the opposite, as in the previous example. A shortcut is to remember that *the sign goes with the term* when you rearrange. Really, this is why we rewrite subtraction in the first place! If we're careful, though, we can avoid the rewriting step.

If we look at the given expression, we see that there are three "types" of terms that need combining: terms with x, terms with y and constant terms (numbers).

Let's pick a type of term (let's choose ones with x) and group them up – signs and all! In the given expression, we have $3x - 5x$, to we have ${}^-2x$ in all.

Pick another type of term and repeat as necessary! If we look at the y terms, we have $-4y + 7y + 8y$ so that's $11y$ in all. Looking at the constant terms, we have $2 - 6 = {}^-4$.

Putting it together, we have ${}^-2x + 11y + {}^-4$ or (after simplifying signs) ${}^-2x + 11y - 4$.

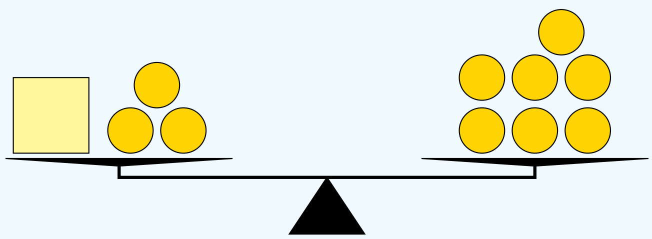
5.3 The Properties of Equality (POEs)

Extended Exploration: Mystery Numbers

[TODO] Click here to visit the extended exploration: Mystery Numbers

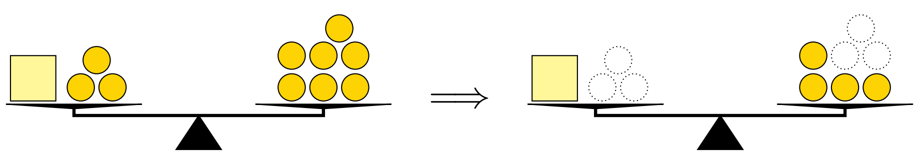
Startup Exploration: Cheese Scale

Bob put a block of edam cheese and three wheels of gouda on one side of a two-pan balance. On the other pan he put seven wheels of gouda. He discovered that the scale was in perfect balance.



If all of the wheels of gouda are identical in weight, how many wheels will balance the block of edam?

If Bob removes a wheel of gouda from the right-hand pan, it will tip the scales... unless he also removes an equivalent amount of gouda from the left-hand pan. In particular, the scale will still be in balance after Bob removes three wheels from each pan of the balance.[4]



Taking three wheels from each side of the balance is helpful because it means that we will have gotten the block of edam alone on one side. From there it's easy to see that the block weighs the same as four wheels.

The idea of maintaining balance is the heart of the algebra of solving equations.

[4] Edam and gouda are traditional Dutch cheeses. Every summer the city of Alkmaar, in the Netherlands, hosts an open-air cheese market. Cheese merchants demonstrate how cheese was bought and sold in medieval times. It operates on Waagplein, which means "weighing square" in Dutch, because once the buyer and seller agreed on a price, the huge wheels of cheese were taken to the scales by official "cheese carriers", and then weighed very carefully to determine their value. The cheese carriers' guild was established in 1593.

Equation

An **equation** is a number sentence which states that two algebraic expressions are equal.

To create an algebraic version of the balance problem, we need a way to represent the weight of the block of edam. In other words, we need a way to write down the weight of the block *before we know what the weight actually is*.

Unknown

An **unknown** is a quantity whose value is not known. We usually represent an unknown using a letter.

Both variables and unknowns are represented by letters though, technically speaking, an unknown is different from a variable. A variable's value can vary or change, whereas an unknown has a fixed value (or set of values) that must be determined.

Let's use B to stand for the unknown weight of the block, and let's assume that each wheel weighs 1 unit. Then, we can translate our balance problem into algebraic symbols:

$$B + 3 = 7.$$

If we take three (wheels) from each side of the balance, we have

$$B = 4$$

which tells us that B, the weight of the block, is four units. Our key task here was transforming the first equation into the second equation.

In section 5.1 we used the field axioms to simplify individual algebraic expressions. An equation is a math sentence with an expression on each side of the equal sign. So, we need some rules that will allow us to manipulate two expressions at the same time, but in such a way that their equality remains intact.

5.3.1 The Properties of Equality

An equation states that the two pans of a balance (the expressions on either side of the equal sign) are in alignment. If we were to, say, add five to the expression on one side of the equation, then we would tip the scales out of balance... unless we also add five to the expression on other side of the equation.

This idea, and others like it, are captured mathematically is a set of rules called the Properties of Equality, or POEs for short.

Properties of Equality: Addition and Subtraction

The **addition property of equality** (APOE) states that for all real numbers a, b, and c

$$\text{if } a = b, \text{ then } a + c = b + c$$

The **subtraction property of equality** (SPOE) states that for all real numbers a, b, and c

$$\text{if } a = b, \text{ then } a - c = b - c$$

Note that since "subtraction" is the same as "addition of the opposite", SPOE is really just APOE in disguise.

Properties of Equality: Multiplication and Division

The **multiplication property of equality** (MPOE) states that for all real numbers a, b, and c, where $c \neq 0$

$$\text{if } a = b, \text{ then } a \cdot c = b \cdot c$$

The **division property of equality** (DPOE) states that for all real numbers a, b, and c, where $c \neq 0$

$$\text{if } a = b, \text{ then } \frac{a}{c} = \frac{b}{c}$$

Here again, note that since "division" is the same as "multiplication by the reciprocal", DPOE is really just MPOE in disguise. Note too that c must not equal 0 when using these rules. Using 0 with DPOE must certainly be avoided, lest we be flummoxed upon division by zero.[5]

Let's use the properties of equality to solve the problem of Bob's cheese balance.

Example 5.6

Bob put a block of edam cheese and three wheels of gouda on one side of a two-pan balance. On the other pan he put seven wheels of gouda. He discovered that the scale was in perfect balance. If all of the wheels of gouda are identical in weight, how many wheels will balance the block of edam?

Solution: Let B represent the weight of the block of edam. Then, the given information suggests the equation $B + 3 = 7$. We want to isolate B.

$$\begin{aligned} B + 3 &= 7 \\ B + 3 - 3 &= 7 - 3 && \text{SPOE: subtract 3 from both sides of the equation} \\ B &= 4 \end{aligned}$$

[5] Why must we be cautious about multiplying both sides of an equation by zero? In the end we'll get $0 = 0$, which is a true statement. What's the problem? Note that multiplication by 0 is *irreversible*. If we multiply by 2 (or any nonzero number) we can reverse or "undo" this operation by dividing by 2. But, since division by 0 is not allowed, we cannot undo multiplication by 0. Later, we will see that irreversible operations can lead us to "fake" solutions to our equations, so-called *extraneous solutions*.

Note that in our work above, we started out by defining what variable we would use to represent our unknown. Then, we wrote an equation based on the information given in the problem. Our goal was to get B by itself on one side of the equation, or to *isolate* B. Since 3 had been added to B, we had to "undo" this. So, we subtracted 3 from both sides using SPOE.

Inverse Operations (a.k.a. Opposite Operations)

Operations that will "undo" one another. For example, addition and subtraction are inverse operations. Multiplication and division are inverse operations.

When we solve an equation, we use inverse operations to undo the order of operations. Once we get a hang of the rules, solving equations can become a game. Plus, like the best video games, equation-solving has different levels of challenge so that the game stays interesting even as you get better and better at it.

5.4 Solving Level 1 and Level 2 Linear Equations

5.4.1 Level 1 Equations

Level 1 linear equations are the simplest kind to solve because it takes only one step to isolate the variable. The example we saw in the last section was a Level 1 linear equation.

Example 5.7

Consider each of the equations below. What step should we perform to isolate the variable?

$$x \div 4 = 21 \qquad w - 9 = 50 \qquad y + 13 = {}^{-}24 \qquad 8x = 72$$

Partial solution: In the first equation we must multiply both sides of the equation by 4 (MPOE).

$$\begin{aligned} x \div 4 &= 21 \\ \frac{x}{4} &= 21 && \text{definition of division} \\ \frac{x}{4} \cdot 4 &= 21 \cdot 4 && \text{MPOE: multiply both sides of the equation by 4} \\ x &= 84 \end{aligned}$$

In the second equation we should add 9 to both sides (APOE). In the third equation, we should subtract 13 (SPOE). Alternatively, we can think of this as APOE in which we add ${}^{-}13$ to both sides.

In the last example, we should divide both sides of the equation by 8 (DPOE). Or, we can think of this as MPOE in which we multiply both sides by $\frac{1}{8}$.

5.4.2 Level 2 Equations

Here's an example of a level 2 equation. How is it different from Level 1?

$$3x + 16 = 43$$

Notice that in this equation, two things were done to the unknown x. First it was multiplied by 3, and then 16 was added to the result. To isolate the variable, the rule of thumb is to undo whatever was done to the unknown. Usually this means that we will have to undo the order of operations in reverse. Most of the time, we can rely on the "underpants analogy".

The Underpants Analogy

When you get dressed for school in the morning, you put your underpants on *before* you put on your jeans. When you get undressed to go to bed, you do the *opposite* of each step, and you do the steps in the *opposite order*. In other words: you have to take your jeans off first, before you can take your underpants off.[a]

[a] Try doing it the other way around, we dare you. Later.

To undo the order of operations, we have to work upside-down and backwards:

$$\begin{aligned}
3x + 16 &= 43 & \\
3x + 16 - 16 &= 43 - 16 & &\text{SPOE: subtract 16 from both sides of the equation}\\
3x &= 27 & &\text{Simplify and substitute}\\
\frac{3x}{3} &= \frac{27}{3} & &\text{DPOE: divide both sides of the equation by 3}\\
x &= 9 & &\text{Simplify and substitute}
\end{aligned}$$

In the end we have isolated x, and so this final equation tells us that 9 is a solution to the original equation.

Solution & Solution Set

A **solution** to an equation is any value that will make the equation true.

Sometimes an equation will have more than one solution. The set of all solutions to an equation are called a **solution set**. To write a solution set, we use set notation and write $\mathcal{S} = \{list\ of\ solutions\}$.

As we go through the process of solving an equation, each application of a Property of Equality produces an equation that is equivalent to the original.

Equivalent Equations

Equations that have the same solution set.

5.4.3 Showing and Checking Our Work

As with problems that involved the order of operations, it is often best to show your work going down the page as we have done in the earlier sections. It is also very helpful to anyone reading your work if you write beside each step the property that you used to justify turning one equation into an equivalent equation.

A note on vocabulary: we *simplify and expression* and we *solve an equation*. Even though the work we show might be very similar, the goal of our steps is quite different.

Having found a solution, it is a great habit to check the solution by substituting it back into the original equation. In the example above, we had the equation $3x + 16 = 43$ and found that this was equivalent to the equation $x = 9$. To check our solution, we plug 9 back in for x in the original equation:

$$\begin{aligned} 3x + 16 &= 43 && \text{original equation} \\ 3(9) + 16 &\stackrel{?}{=} 43 && \text{substitute in 9 for } x \\ 27 + 16 &\stackrel{?}{=} 43 && \text{carry out the order of operations} \\ 43 &\stackrel{\checkmark}{=} 43 && \text{Check!} \end{aligned}$$

When we substitute in 9, we put little question marks over out equal sign, since we are checking the solution – we're not sure whether the two expressions are equal yet! In the end, when we discover that the left-hand side is equal to the right-hand side, we replace our question with a check mark of confirmation!

The key here is to realize that, ultimately, the process we use to solve an equation yields *candidates* for the value of the unknown. We are not guaranteed that all of these candidates actually satisfy the original equation, until we check them to know for sure.[6]

Once we have verified that our solution satisfies the original equation, we should write our answer in set notation. The equation $x = 5$ isn't our solution, but an equivalent equation – in fact, it's the simplest equation that has the same solution as the original!

In other words, the equation solving process creates simpler and simpler equations that all have the same solution. Eventually, we get to the an equation that shows the solution clearly. We use set notation to record our solution because it is a notation that suggests the solution works for all of the intermediate equations.

Plus, solution set notation works nicely with all types of equations. Soon we will have equations with multiple solutions (even infinitely many!) and the $x =$ notation starts to be become quite awkward.

So, in our ongoing example, we write $\mathcal{S} = \{5\}$. Yes, a mathematical "set" may contain just a single element![7] The alternative to using solution set notation is to write the sentence "The solution to the equation is 5."

[6] There will come a time when we will do everything correctly, and still some of our solution-candidates will have to be rejected! Get in the habit of checking solutions now!

[7] This is another situation in which we find a conflict between everyday language and the language of mathematics. Lots of words have a special meaning when used in a mathematical context: similar, odd, mean, product... What other examples can you think of?

5.5 Solving Level 3 Linear Equations

Level 1 equations are called, naturally enough, *one-step equations*, and Level 2 equations are called – you guessed it! – two-step equations. In Level 3, we increase complexity by combining the tasks of expression-simplifying and equation-solving.

Extended Exploration: Evil Mystery Numbers

[TODO] Click here to visit the extended exploration: Evil Mystery Numbers

Startup Exploration: Linear Level 3

Determine the value x, given the equation

$$2x + 7 - 5x + 12 = 15.$$

Here we find an equation that must be simplified using the field axioms and related tools. Let's start out by trying to simplify the left-hand side.

$$\begin{aligned}
2x + 7 - 5x + 12 &= 15 && \\
2x + 7 + {}^-5x + 12 &= 15 && \text{rewrite as addition, by def'n of subtraction} \\
2x + {}^-5x + 7 + 12 &= 15 && \text{commutative property of addition} \\
{}^-3x + 19 &= 15 && \text{combine like terms – this is now a Level 2 equation!} \\
{}^-3x + 19 - 19 &= 15 - 19 && \text{SPOE} \\
{}^-3x &= {}^-4 && \\
\frac{{}^-3x}{{}^-3} &= \frac{{}^-4}{{}^-3} && \text{DPOE} \\
x &= \frac{4}{3} && \text{Simplify}
\end{aligned}$$

Notice that after a few lines, we had turned the Level 3 equation into a Level 2 equation. We know how to solve those! So, our goal will be to try to turn Level 3 equations into a Level 1 or Level 2 equations. But, since there are more steps in the process, there's more chance we can make a mistake. So, it's a good habit to remember to check our answer at the end.

We can check our answer for the last problem by substituting our solution back into the original equation and simplifying using the order of operations.

$$\begin{aligned}
2x + 7 - 5x + 12 &= 15 && \text{original equation}\\
2\left(\frac{4}{3}\right) + 7 - 5\left(\frac{4}{3}\right) + 12 &\stackrel{?}{=} 15 && \text{substitute in our solution } x = \tfrac{4}{3}\\
2\left(\frac{4}{3}\right) + 7 + {}^{-}5\left(\frac{4}{3}\right) + 12 &\stackrel{?}{=} 15 && \text{rewrite as addition, by def'n of subtraction}\\
\frac{8}{3} + 7 + \frac{-20}{3} + 12 &\stackrel{?}{=} 15 && \text{multiply fractions}\\
\frac{8}{3} + \frac{21}{3} + \frac{-20}{3} + \frac{36}{3} &\stackrel{?}{=} 15 && \text{rewrite left-hand side with common denominator}\\
\frac{45}{3} &\stackrel{?}{=} 15 && \text{add fractions}\\
15 &\stackrel{\checkmark}{=} 15 && \text{boom!}
\end{aligned}$$

Writing our solution as a solution set, we have $\mathcal{S} = \{15\}$.

Example 5.8

Determine the value of w, given the equation

$$7w + 2({}^{-}3w + 1) = 12.$$

Solution: Our goal is to try and simplify the left-hand side so that it looks like a Level 1 or Level 2 equation.

$$\begin{aligned}
7w + 2({}^{-}3w + 1) &= 12 && \\
7w + {}^{-}6w + 2 &= 12 && \text{distributive property}\\
1w + 2 &= 12 && \text{combine like terms -- that's a Level 1 equation!}\\
w + 2 - 2 &= 12 - 2 && \text{SPOE}\\
w &= 10 &&
\end{aligned}$$

Let's check our solution:

$$\begin{aligned}
7w + 2({}^{-}3w + 1) &= 12 && \text{original equation}\\
7(10) + 2({}^{-}3(10) + 1) &\stackrel{?}{=} 12 && \text{substitute in our solution } w = 10\\
70 + 2({}^{-}30 + 1) &\stackrel{?}{=} 12 && \text{carry out the order of operations on the left-hand side}\\
70 + 2({}^{-}29) &\stackrel{?}{=} 12 &&\\
70 + {}^{-}58 &\stackrel{?}{=} 12 &&\\
12 &\stackrel{\checkmark}{=} 12 &&
\end{aligned}$$

We're going to look at one more example, and discuss two alternative ways to approach it.

Example 5.9

Determine the value of g, given the equation

$$-4(2g-7)=36.$$

Solution: Approach #1. Those parentheses are just asking to be simplified using the distributive property. Be mindful of the signs, though!

$$\begin{aligned}
-4(2g-7) &= 36 \\
-8g+28 &= 36 && \text{distributive property – that's a Level 2 equation!} \\
-8g+28-28 &= 36-28 && \text{SPOE} \\
-8g &= 8 \\
\frac{-8g}{-8} &= \frac{8}{-8} && \text{DPOE} \\
g &= -1
\end{aligned}$$

Approach #2. Let's solve the same equation again. This time, notice that we can divide both sides by -4 as the first step, which eliminates the need for the distributive property.

$$\begin{aligned}
-4(2g-7) &= 36 \\
\frac{-4(2g-7)}{-4} &= \frac{36}{-4} && \text{DPOE} \\
2g-7 &= -9 && \text{simplify – that's a Level 2 equation!} \\
2g-7+7 &= -9+7 && \text{APOE} \\
2g &= -2 \\
\frac{2g}{2} &= \frac{-2}{2} && \text{DPOE} \\
g &= -1
\end{aligned}$$

The two different approaches taken in the previous example are not always available. Compare the solutions to examples 5.8 and 5.9. Can we avoid the distributive property in example 5.8? Why or why not?

5.5.1 Checking Work

Every step we take in solving an equation generates an equivalent equation that has the same solution set. If we make a mistake, we accidentally create an equation with a different solution set. Knowing this can help us check our answers on the more complex equations (ones that have more steps).

Example 5.10

Solve for x, given the equation $2x - 3(4x - 1) = -53$.

Let's say we mess up the signs when carrying out the distributive property – a very common mistake.

$$\begin{aligned} 2x - 3(4x - 1) &= {}^-53 && \text{Line 1.} \\ 2x - 12x - 3 &= {}^-53 && \text{Line 2. That's a mistake!} \\ {}^-10x - 3 &= {}^-53 && \text{Line 3.} \\ {}^-10x &= {}^-50 && \text{Line 4.} \\ x &= 5 && \text{Line 5.} \end{aligned}$$

Now, we go back to check our work by substituting the solution into the original equation. The answer doesn't work in line 1, so we know it is not the correct solution.

$$2(5) - 3\big(4(5) - 4\big) = {}^-53 \implies {}^-38 \neq {}^-53 \quad \text{(something went wrong!)}$$

But, our answer *does* work in line 2 (and also lines 3,4, and 5).

$$2(5) - 12(5) - 3 = {}^-53 \implies {}^-53 = {}^-53 \quad \text{(that works!)}$$

This is the clue helps us pinpoint the location of the mistake. It tells us the mistake must have happened when transforming line 1 into line 2.

If, on a different problem, we find that our solution doesn't work for steps 1, 2, or 3, but *does work* in step 4, then it means that our mistake must have happened when transforming step 3 into step 4!

5.6 Solving Level 4 Linear Equations

In Level 4, we add a wrinkle, which can lead to some very unusual results.

Extended Exploration: Super Evil Mystery Numbers

[TODO] Click here to visit the extended exploration: Super Evil Mystery Numbers

Startup Exploration: Linear Level 4

Determine the value of x given the equation

$$13x - 5x - 5 = x + 7 + x.$$

The goal is the same: to isolate the variable on one side of the equal sign. We begin by simplifying each side.

$$\begin{aligned} 13x - 5x - 5 &= x + 7 + x && \\ 8x - 5 &= 2x + 7 && \text{combine like terms on both sides} \\ 8x - 5 - 2x &= 2x + 7 - 2x && \text{SPOE: subtract } 2x \text{ from both sides – clever!} \\ 6x - 5 &= 7 && \text{combine like terms again – that's a Level 2 equation!} \\ 6x &= 12 && \text{APOE: add 5 to both sides} \\ x &= 2 && \text{DPOE: divide both sides by 6} \end{aligned}$$

In this example, we subtracted $2x$ from both sides of the equation. This might seem like a tricky move, but it's a completely legal application of SPOE. We are allowed to do the same thing to both sides of the balanced equation, even if that means subtracting an unknown amount from both sides.[8]

Let's see what happens if we isolate the unknown on *the other side of the equal sign*. (It had better give us the

[8] Of course, you have to subtract the same unknown amount from both sides. It's no fair to subtract x from one side of an equation and y from the other side. This will put the equation out of balance unless we know that these two unknown amounts are equal... that is, unless we know that $x = y$. Caution: can you think of a situation in which it might be dangerous to apply a POE using an unknown? Think about what might happen if we use DPOE to divide both sides by an unknown value x.

same answer!)

$$
\begin{aligned}
13x - 5x - 5 &= x + 7 + x && \text{same equation as before} \\
8x - 5 &= 2x + 7 && \text{combine like terms on both sides, as before} \\
8x - 5 - 8x &= 2x + 7 - 8x && \text{SPOE: subtract } 8x \text{ from both sides} \\
{}^{-}5 &= {}^{-}6x + 7 && \text{combine like terms} \\
{}^{-}12 &= {}^{-}6x && \text{SPOE: subtract 7 from both sides} \\
2 &= x && \text{DPOE: divide both sides by } {}^{-}6
\end{aligned}
$$

So, it doesn't matter which side we choose to eliminate the unknown. As long as we apply APOE or SPOE correctly, the solution will be the same. One strategy is to choose the side that will result in a *positive coefficient* for the variable. This isn't required, but it helps avoid the chances of losing a negative sign along the way (maybe that's happened to you).[9]

Plus, if you don't like the side of the equation that the unknown is on, you can always rewrite the equation. For example if you have the equation $3 = 4 + x$ and you really want the x on the left-hand side, you can just rewrite the equation as $4 + x = 3$. Those are equivalent equations![10]

Example 5.11

Determine the value of x given the equation

$$3(x - 2) + 3x = 4x - 6.$$

Solution: Let's jump right in and start simplifying the left-hand side.

$$
\begin{aligned}
3(x - 2) + 3x &= 4x - 6 && \\
3x - 6 + 3x &= 4x - 6 && \text{distributive property} \\
6x - 6 &= 4x - 6 && \text{combine like terms} \\
2x - 6 &= -6 && \text{SPOE: subtract } 4x \text{ from both sides} \\
2x &= 0 && \text{APOE: add 6 to both sides} \\
x &= 0 && \text{DPOE: divide both sides} by 2
\end{aligned}
$$

[9] Napoleon ordered the creation of the first modern lost-and-found office in Paris in 1805, although lost-property systems have existed in Japan since the early 700s.

[10] Even these obvious properties have been given names. For example, this is a fact about equations: if $a = b$, then $b = a$. To describe this, we say that equality is *symmetric*. An even more obvious statement is that $a = a$ (we say that equality is *reflexive*). A third property is a little more interesting: if $a = b$ and $b = c$, then $a = c$ (we say that equality is *transitive*).

Yes, we can get 0 as the solution to an equation. We can always check to be sure:

$$
\begin{aligned}
3(x-2)+3x &= 4x-6 && \text{original equation} \\
3(0-2)+3(0) &\stackrel{?}{=} 4(0)-6 && \text{substitute in our solution } x=0 \\
3(^-2)+0 &\stackrel{?}{=} {}^-6 && \text{simplify} \\
{}^-6 &\stackrel{\checkmark}{=} {}^-6
\end{aligned}
$$

5.6.1 Special Cases

There are some things to be careful about with when it comes to solving Level 4 linear equations. Here is an example of something strange that can happen when we have unknowns on both sides of the equal sign.

Example 5.12

Determine the value of x given the equation

$$12x+36 = 2(6x-10).$$

Solution: We proceed as we have done in earlier problems:

$$
\begin{aligned}
12x+36 &= 2(6x-10) \\
12x+36 &= 12x-20 && \text{distributive property} \\
12x+36-12x &= 12x-20-12x && \text{SPOE: subtract } 12x \text{ from both sides} \\
36 &= {}^-20 && \text{Huh?}
\end{aligned}
$$

In this case the variable term disappears completely from both sides of the equation. And then, what is left is obviously *not equal*. In a sense, the original "equation" is not an equation at all. The two given expressions are not equal, and never will be.

This means that there is no value of x what will ever work to make the equation true. We say that this equation *has no solution*.

As unsatisfying as it might sound, it is possible to have an equation that does not have a solution.[11]

[11] Equations without solutions are very important in the history of mathematics. The French mathematician Pierre de Fermat made a conjecture in 1637 stating that a certain equation had no integer solutions. His claim, which came to be known as Fermat's Last Theorem, wasn't officially proven to be correct for more than 350 years. British mathematician Andrew Wiles published the first complete proof of Fermat's Last Theorem in 1995. The theorem states that the equation $a^n + b^n = c^n$ has no integer solutions for a, b, and c when the exponent n is a natural number greater than 2.

How do you write down the solution to a problem that has no solution? This is not quite as philosophical as it sounds. We can, of course, write the words “no solution”. Or, we could write a solution set that contains no numbers, $\mathcal{S} = \{\ \}$.

Or, there is a third option. You may or may not be surprised to learn that mathematicians have invented a symbol for just such an occasion. We use the mathematical symbol $\varnothing$, called the “empty set” or “null set”, as a way to show that that our solution set is empty.[12] We write $\mathcal{S} = \varnothing$ to mean “the solution set is empty”.

Notice that we don’t write the curly braces around the empty set. In other words, we write $\mathcal{S} = \varnothing$ and not $\mathcal{S} = \{\varnothing\}$. The former says “$S$ is the empty set”, which is what we mean when we have an equation with no solutions. The latter says “S is the set which contains the empty set” which – believe it or not – is not the same thing.[13]

For the record, the empty set is not the same as zero. So, an equation can have the solution set $\mathcal{S} = \{0\}$, like example 5.6. This is different from an equation having the solution set $\mathcal{S} = \{\ \}$, like example 5.7.

If you look back at the reason why the last example has no solution, you may be wondering about whether there is another possibility. Study the following example.

Example 5.13

Determine the value of x given the equation

$$5x - 3(x + 2) = 2x - 6.$$

Solution: Let’s go for it:

$$
\begin{aligned}
5x - 3(x+2) &= 2x - 6 && \\
5x - 3x - 6 &= 2x - 6 && \text{distributive property, watch the signs} \\
2x - 6 &= 2x - 6 && \text{combine like terms – something is fishy already.} \\
2x - 6 - 2x &= 2x - 6 - 2x && \text{SPOE: subtract } 2x \text{ from both sides} \\
-6 &= -6 && \text{But, of course.}
\end{aligned}
$$

In this case the variable term disappears again, and the leftovers are *obviously equal*. This is the opposite of what happened earlier. If that example had no solutions, this one has all of them. We can replace

[12] The symbol $\varnothing$, which was introduced by French mathematician André Weil in 1939, is a letter in the Norwegian alphabet.

[13] The empty set is not the same as zero, and it is not the same as nothing. It is a set with nothing inside it, like an empty bag. So, writing $\mathcal{S} = \varnothing$ is stating that the solution set is like an empty bag. Extending this metaphor, $\{\varnothing\}$ is a bag with an empty bag in it and, therefore, a set which contains the empty set is not empty. Om.

x with literally any real number and the equation will be true. That means that we have a solution set that contains every real number there is.

We say that this equation's solutions are *all real numbers*.

In this example, our solution set is the set of real numbers. We can write this out in words, "all real numbers", or use the shorthand symbol that stands for the real numbers and write, $\mathcal{S} = \mathbb{R}$. Notice no curly braces are used here either.

Checking Our Answers in Special Cases

In the case that we find "all real numbers" as as the solution to an equation, we know that every number is going to work. So to check, we might pick a couple of different numbers (easy ones, like 0 and 1) and substitute those into the original equation. Any number that you choose should satisfy the equation.

If we find that an equation has "no solutions", then any number we use to check will fail to satisfy the equation. Not very informative. In this case, it may be best just to check back over our work to make sure that we did each of the steps correctly.

5.7 Solving Level 5 Linear Equations

Over the last few sections we have added complexity to the equation-solving picture. We have learned properties that can be used to "undo" whatever has been "done" to the variable. Sometimes this has led to situations where the equation has no solutions, or infinitely many solutions. We turn now to equations that involve the absolute value of an unknown.[14]

Extended Exploration: Dreadful Mystery Numbers

[TODO] Click here to visit the extended exploration: Dreadful Mystery Numbers

Startup Exploration: Linear Level 5

Recall the definition of absolute value. Determine the value of x given the equation

$$|x| = 6.$$

Recall that the absolute value of a number is that number's distance away from zero on the number line. Distances are always positive: -6 is 6 units away from 0, and so $|-6| = 6$. Of course, $|6| = 6$ as well. So, the equation above has two solutions $x = 6$ <u>or</u> -6.[15] In solution set notation, we write $\mathcal{S} = \{6, -6\}$.

The presence of the absolute value gives us the possibility of having two solutions, since whatever is inside the absolute value bars could be either the positive or the negative version of the value.

Here's an extension to the idea. Determine the value of x given the equation

$$|2x| = 16$$

This is telling us that the number "$2x$" is 16 units away from zero on the number line. In other words

$$2x = 16 \ \underline{\text{or}} \ -16.$$

This is really two equations written at once: $2x = 16$ and $2x = -16$. We can apply DPOE and divide everything by 2, which will isolate x. That is, we have

$$x = 8 \ \underline{\text{or}} \ -8.$$

[14] We're discussing absolute value equations here in the chapter on linear equations, though we put "linear" in quotes. Absolute value equations of the kind that we're discussing here are linear-like enough that they fit here. Other types of absolute value equations are possible, though we won't get into the full range of variations in this course.

[15] Note, we mean *or* and not *and*. Both 6 and -6 are solutions to the equation, but we must write $x = 6$ <u>or</u> -6, since x can have only take on one value at a time.

The following two examples carry this idea a bit further.

Example 5.14

Determine the value of x given the equation

$$|{-3x}+9| = 12.$$

Solution: The stuff inside the absolute value bars can equal 12 or −12. So, this means, we have two equations:

$$\begin{array}{rcrl} -3x+9=12 & \underline{\text{or}} & -3x+9=-12 & \\ -3x=3 & & -3x=-21 & \text{SPOE: subtract 9 throughout} \\ x=-1 & & x=7 & \text{DPOE: divide by } -3 \text{ throughout} \end{array}$$

This equation has solutions $\mathcal{S} = \{-1, 7\}$.

Note that the steps we took for solving were the same for both equations (in this case, first we used SPOE, and then DPOE). So, we can streamline our work a bit, as shown in the next example.

The next example also has things going on outside of the absolute value symbols. For these types of problems, we want to isolate the absolute value expression first (using the properties of equality), then separate into two equations (if necessary).[16]

Example 5.15

Determine the value of x given the equation: $3|{-4x}+4| = 48$.

Solution:

$$\begin{array}{rl} 3|{-4x}+4| = 48 & \\ |{-4x}+4| = 16 & \text{DPOE, to isolate the absolute value expression} \\ -4x+4 = 16 \ \underline{\text{or}} \ -16 & \text{definition of absolute value} \\ -4x = 12 \ \underline{\text{or}} \ -20 & \text{SPOE: subtract 4 throughout} \\ x = -3 \ \underline{\text{or}} \ 5 & \text{DPOE: divide by } -4 \text{ throughout} \end{array}$$

[16] In case you are wondering, we won't get into equations that include multiple abvolute value expressions. . . although you might enjoy exploring an equation of this kind, just for fun. For example, what values of x satisfy this equation? $|x+2| = 2|x| - 1$

So, this equation has solutions $\mathcal{S} = \{-3, 5\}$.

5.7.1 Special Considerations

A few sneaky things can pop up when it comes to absolute value equations. Recall that the only number that has absolute value zero is zero itself: $|\,0\,| = 0$. So, not all absolute value equations have *two* solutions. Consider

$$|\,3x + 51\,| = 0.$$

We can't split this into two different equations. All we have is

$$3x + 51 = 0,$$

and that's a Level 2 linear equation with just a single solution, $\mathcal{S} = \{-17\}$. To demonstrate another tricky aspect, consider the equation

$$|\,4x\,| + 16 = 5.$$

No problem! We begin by subtracting 16 from both sides:

$$|\,4x\,| = -11$$

But, the absolute value of a number can never be negative. So, this equation has no solution. In other words, we have $\mathcal{S} = \varnothing$.

Don't be too quick with the "no solutions" talk, though. If we encounter the equation

$$-3|\,x + 2\,| = -12,$$

we may see the -12 and assume there is no solution. But recall that we must isolate the absolute value expression first. To do that we will divide both sides by -3 and then we will have

$$|\,x + 2\,| = 4,$$

an equivalent equation that surely has a solution (two solutions, in fact). So, don't jump to conclusions!

Warning!

Some folks like to use the notation "plus/minus" in situations that involve absolute value. For example, writing "± 6" with the little stacked-up plus and minus signs to stand for "6 <u>or</u> -6". We recommend avoiding $\pm$ notation when solving equations.

Here's why. See if you can spot the error in the following work.

$$\begin{aligned} |\,x - 5\,| &= 12 \\ x - 5 &= \pm 12 \\ x &= \pm 17 \qquad \text{APOE: add 5 to both sides} \end{aligned}$$

And so, $\mathcal{S} = \{17, {}^-17\}$. This work looks reasonable, but let's check the two solutions.

$$|\,17 - 5\,| \stackrel{?}{=} 12 \implies |\,12\,| \stackrel{?}{=} 12 \implies 12 \stackrel{\checkmark}{=} 12$$

$$|\,{}^-17 - 5\,| \stackrel{?}{=} 12 \implies |\,{}^-22\,| \stackrel{?}{=} 12 \implies 22 \neq 12$$

So, using the $\pm$ notation has led us to get one correct solution and one incorrect solution for the equation! This is why we recommend writing out the word "or" explicitly, or breaking into two separate equations.

(By the way, the correct solution is $\mathcal{S} = \{17, {}^-7\}$. Try working out the problem again using a more reliable technique. Can you see exactly where the work using $\pm$ went astray?)

5.8 Applications of Equation Solving

Now that we know how to handle different types of equations, let's explore a few applications. First, we'll discuss writing (and then solving) equations based on problem situations. Then, we'll see how to apply the properties of equality to manipulate formulas from science and other disciplines. We'll close this chapter with an extension section about explaining some fundamental ideas using the field axioms.

5.8.1 Writing Equations to Solve Word Problems

There are lots of ways to approach problems like this: educated guessing-and-checking, wishful thinking, making an organized list, and so on. One way is to model the situation with (and then solve) an equation. Converting a word problem into an equation is an important skill for your mathematical toolbox.

Extended Exploration: Mixed Bag 'O Problems

[TODO] Click here to visit the extended exploration: Mixed Bag 'O Problems

Startup Exploration: Trick or Treat

On Halloween night, Hildegaard and Ingvar (Bob and Yeardleigh's parents) had 200 candies in their trick-or-treat bowl. They gave 5 candies to each kid who came to their door, and at the end of the night they had 10 candies left. How many kids came trick-or-treating?

The key in word problems like this is first to identify the unknown in the problem. We choose a variable to stand in for the unknown, and write an equation based on the other information given in the problem.

In our example, the unknown is "how many kids came trick-or-treating", so let's use k to represent the number of kids. Hildegaard and Ingvar gave 5 candies to every kid, so they gave away $5k$ candies. They started with 200 and gave away $5k$. Giving away suggests subtraction, so they were left with $200 - 5k$ candies at the end of the night. The problem tells us that they had 10 candies let over, so:

$$200 - 5k = 10.$$

That's a Level 2 equation! We'll leave it for you to solve and put the solution in the footnote.[17]

[17] Solving this equation using the POEs leads to the result $k = 38$. So, the answer to the question is that 38 kids came trick-or-treating at the Krumbli house on Halloween night.

Here's a second example. "Consecutive integer problems" are a clever type of mystery number problem that might at first appear impossible to solve. There doesn't seem to be enough information!

Name of Startup Exploration

The sum of three consecutive integers is 39. What are the numbers?

What does it mean to have three consecutive integers? Consecutive means "in an unbroken sequence", so three consecutive integers are three integers in a row, such as $4, 5, 6$.

Let N represent the smallest of the three consecutive integers. Then, the next integer is $N+1$, and the integer after that is $N+2$. The problem says their sum is 39. So we have

$$\begin{aligned} N + (N+1) + (N+2) &= 39 \\ 3N + 3 &= 39 && \text{combine like terms} \\ 3N &= 36 && \text{SPOE} \\ N &= 12 && \text{DPOE} \end{aligned}$$

At this point, many students would be tempted to draw a box around their answer and call it a day. After all, we solved the equation, right? But, look back at the question: it asks us to find the *numbers*, plural. Our answer to the question should list all three of the numbers!

What did we actually compute when we solved the equation? We chose N to represent the smallest of the three consecutive integers, and that's 12. So, the full answer is that the three integers are 12, 13, and 14. (A quick check shows that these three do, in fact, add up to 39.)

The moral of the story here is always to check that we are answering the question that has been asked. This step is a helpful way to avoid giving an irrelevant or incomplete solution.

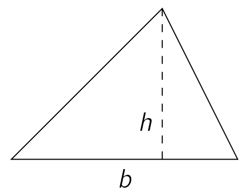
Finally, when it comes to stating the solution to a word problem, we don't usually use set notation. A handy rule of thumb is to think about how we'd say the answer out loud to someone. When we solve an equation with no context, we'd probably say "The solution is 4." Set notation is appropriate, since the answer is just a number, $\mathcal{S} = \{4\}$.

On the other hand, if we were answering a word problem that asked how many pounds of cream cheese Bob ate, we'd say "Bob ate 4 pounds of cream cheese." When the context of the problem implies a unit on the answer (pounds, dollars, goats, miles per hour), we write the answer and the unit rather than use solution set notation: 4 pounds of cream cheese.

5.8.2 Transforming Formulas

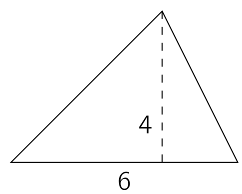
In all of the examples in this chapter so far, we have solved equations and found the value of an unknown. In this section, we'll discuss a different use the field axioms and the properties of equality.

In the past, you may have learned a formula for the area of a triangle. If we have a triangle with base of length b and height h, then its area A has a tidy little formula.



$A = \frac{1}{2}bh$

If we know the base and height of a specific triangle, we can find its area.



$A = \frac{1}{2}(6)(4) = 12$

But what if we are given the area and the base length: Could we work backwards to find the height? Could we "transform" the area formula so that it computes h instead of A?

Transforming a formula means to isolate a specific variable, even though the other side of the equal sign might not simplify down to a single number. The only difference between this and what we have been doing before now is that the equations may have multiple variables (and not unknowns).

Example 5.16

Transform the triangle area formula, $A = \frac{1}{2}bh$, to isolate h.

Solution: We'll start with the given formula, and then apply the properties of equality to get h by itself on one side of the equal sign.

$$\begin{aligned}
A &= \frac{1}{2}bh \\
2 \cdot A &= 2 \cdot \frac{1}{2}bh && \text{MPOE, to eliminate the fraction} \\
2A &= bh && \text{simplify on the righthand side} \\
\frac{2A}{b} &= \frac{bh}{b} && \text{DPOE, to isolate } h \\
\frac{2A}{b} &= h && \text{simplify on the righthand side: mission accomplished!}
\end{aligned}$$

So, our transformed formula expresses the height of a triangle in terms of its area and the length of a base:

$$h = \frac{2A}{b}$$

This is a skill used in science (especially physics) and higher levels of mathematics. Generally, we have to study the equation we're given and find the variable we are trying to isolate. Then, think about what has been done to that variable which must be undone. We undo these using the POEs.

[TODO] Remember: Since we moved transforming formulas before the linear equations formulae, we should add a bit about transforming standard from later.

Even though our answers won't be just a number, they should be simplified as much as possible, for instance we must avoid fractions-in-fractions. Parentheses, too, can lead to tricky situations. It often pays to formulate a plan before rushing to work.

Example 5.17

$P = 2(w + d)$ for w

Solution: Sometimes we have to distribute, other times it is easier to avoid it all together. Compare the two options below.

Option 1: Avoiding the distributive property

$$\begin{aligned} P &= 2(w+d) && \\ \tfrac{P}{2} &= w - d && \text{DPOE} \\ \tfrac{P}{2} - d &= w && \text{SPOE} \end{aligned}$$

Option 2: Distributive property first

$$\begin{aligned} P &= 2(w+d) && \\ P &= 2w + 2d && \text{distributive property} \\ P - 2d &= 2w && \text{SPOE} \\ \tfrac{P-2d}{2} &= w && \text{DPOE} \\ \tfrac{P}{2} - \tfrac{2d}{2} &= w && \text{undo fraction subtraction} \\ \tfrac{P}{2} - d &= w && \text{write second fraction in lowest terms} \end{aligned}$$

In the previous example, the second approach is a bit more work. The distributive property can't always be avoided though, as we saw back in example 5.8. So, it pays to be observant and do a bit of thinking before we start crunching the numbers.

[TODO] fix example numbering here

5.8.3 (;,;) Proving Foundational Results Using the Field Axioms

As we mentioned in section 5.1, an axiom is a statement that is accepted without proof. In some sense, we "take it for granted" that adding 0 to any number doesn't change the number: $a+0=a$ for any real number a.

Similarly, there's a field axiom about multiplication by 1. But, that there is no axiom about multiplication by 0. Why not? When thinking about the field axioms, an interesting thing to consider is why some rules are omitted.[18]

Surprisingly, we can use the axioms given in section 5.1 to explain other basic results of arithmetic. In other words: our axioms are the fundamental building blocks. We can use them to build complex structures (like Level 5 equations) or simple structures. Sometimes, we discover that we can build really simple – even obvious-looking – mathematical statements from *even simpler statements*!

Multiplication by Zero

We know that $a \cdot 0 = 0$, but there's no axiom stating this. That's because we can explain multiplication by zero using the axioms that we already have. Consider this argument, where we state the property that we used to transform each line:

$$\begin{aligned} 0+0 &= 0 && \text{identity property of addition} \\ a\cdot(0+0) &= a\cdot 0 && \text{MPOE: multiply both sides by } a \\ a\cdot 0 + a\cdot 0 &= a\cdot 0 && \text{distribuive property on left-hand side} \\ a\cdot 0 &= 0 && \text{SPOE: subtract } a\cdot 0 \text{ from both sides} \end{aligned}$$

In the end, we used only the given axioms and the properties of equality to demonstrate that any number times 0 is zero. We don't need an axiom for this rule, because we can *build it* from the given set of axioms!

[18] Remember that the Cthulhu icon (;,;) indicates that this is an extension section. Don't feel bad if some of the ideas feel like swimming in the deep end! Stick with it!

Multiplication and Negative Numbers: Property #1

We are given an axiom stating that $1 \cdot a = a$ (that's the identity property of multiplication). But, we have no axiom stating that $^-1 \cdot a = {^-a}$. It turns out that we don't need one:

$$
\begin{aligned}
1 + {^-1} &= 0 && \text{inverse property of addition} \\
a \cdot (1 + {^-1}) &= a \cdot 0 && \text{MPOE: multiply both sides by } a \\
a \cdot 1 + a \cdot {^-1} &= a \cdot 0 && \text{distribuive property on left-hand side} \\
a + a \cdot {^-1} &= 0 && \text{simplifications: } a \cdot 1 = a \text{ and } a \cdot 0 = 0 \\
a \cdot {^-1} &= {^-a} && \text{SPOE: subtract } a \text{ from both sides}
\end{aligned}
$$

We could apply the commutative property of multiplication on the left-hand side, if we want, to have the new rule $^-1 \cdot a = {^-a}$.

Multiplication and Negative Numbers: Property #2

Another fundamental result that seems obvious is that if we take the "double negative" of a number, we get back to the original number. In other words, the opposite of the opposite of a is a itself: $^-({^-a}) = a$.

Consider this: we know that $a + {^-a} = 0$ (that's the inverse property of addition). We know that ^-a also has an additive inverse: $^-a + {^-({^-a})} = 0$. Since these are both equal to 0, they are equal to each other. So:

$$
\begin{aligned}
a + {^-a} &= {^-a} + {^-({^-a})} && \text{both equal 0, so they equal each other} \\
a &= {^-({^-a})} && \text{SPOE: subtract } {^-a} \text{ from both sides!}
\end{aligned}
$$

Multiplication and Negative Numbers: Property #3

We can use the axioms to explain why the product of a negative number and a positive number is negative. Let a and b be positive numbers:

$$
\begin{aligned}
a \cdot 0 &= 0 && \text{multiplication by zero} \\
a \cdot (b + {^-b}) &= 0 && \text{rewrite 0 using the additive inverse property} \\
a \cdot b + a \cdot {^-b} &= 0 && \text{distributive property} \\
a \cdot {^-b} &= {^-(a \cdot b)} && \text{SPOE: subtract } a \cdot b \text{ from both sides}
\end{aligned}
$$

So, if we have "the opposite of a" times b, the result is the opposite of the product of a and b.

Multiplication and Negative Numbers: Property #4

We can use property #3 to prove that the product of two negative numbers is positive. Let a and b be positive numbers:

$$\begin{aligned} {}^{-}a \cdot 0 &= 0 && \text{multiplication by zero} \\ {}^{-}a \cdot (b + {}^{-}b) &= 0 && \text{rewrite 0 using the additive inverse property} \\ {}^{-}a \cdot b + {}^{-}a \cdot {}^{-}b &= 0 && \text{distributive property} \\ {}^{-}(a \cdot b) + {}^{-}a \cdot {}^{-}b &= 0 && \text{property \#3} \\ {}^{-}a \cdot {}^{-}b &= a \cdot b && \text{APOE: add } a \cdot b \text{ to both sides!} \end{aligned}$$

So, the two products are the same! The product of two negative numbers is the same as the product of their opposites (that is, their positive partners).

Mind Blown!

We've been taking the properties of equality for granted, but *even the properties of equality* can be derived from the field axioms. The POEs are the properties that allow us to make a cancellation, for example taking

$$x + 4 = 10$$

and thinking about it as

$$x + 4 = 6 + 4.$$

We can cancel the 4 from both sides and get $x = 6$ (we call this SPOE, the subtraction property of equality). Suppose we have real numbers a, b, and c, and we know that $a + c = b + c$. Then, consider this chain of reasoning:

$$\begin{aligned} a &= a + 0 && \text{additive identity property} \\ &= a + (c + {}^{-}c) && \text{rewrite 0 using the additive inverse property} \\ &= (a + c) + {}^{-}c && \text{associative property of addition} \\ &= (b + c) + {}^{-}c && \text{substitution: we assumed that } a + c = b + c \\ &= b + (c + {}^{-}c) && \text{associative property of addition} \\ &= b + 0 && \text{additive inverse property} \\ &= b && \text{additive identity property} \end{aligned}$$

What does this tell us? Notice that we started our equation with a and ended up with b. This means $a = b$. Altogether, we have shown that if we know $a + c = b + c$, then it must be that $a = b$. That's one of our POEs!

In the end, we have been able to used the field axioms to prove that other fundamental properties hold true. Don't worry if it feels overwhelming at first. This section has gotten into some pretty abstract territory! Let these ideas sink in and read through this section again later. It takes time for our brains to incorporate challenging ideas.

Give me the place to stand, and I shall move the earth.

Archimedes
Ancient Greek mathematician and philosopher

Chapter 6

Proportional Reasoning

Every chapter should have a lead paragraph – even just a short one – that appears before the first heading. This is a placeholder paragraph which will at some point be replaces by actual content.

6.1 Proportions as Equations

Mathematicians, like Yeardleigh's team of gourmet chefs, won't use tools, techniques, or ingredients unless they know exactly where they come from. This is the attitude we adopt, for the most part, in algebra. This point of view, however, may upset some students' mathematical status quo.

Case in point: solving proportions. Many of us have been taught a mysterious method called "cross multiplication" as a means of solving a proportion. Unless we can explain it's inner workings – Where does it come from? Why does it give us the correct answer? – then the technique must be considered off-limits.

We'll explain so-called cross-multiplication in this chapter, and explore alternative (easier) methods of handling proportions.

Extended Exploration: Multiply and Conquer

[TODO] Click here to visit the extended exploration: Multiply and Conquer

Startup Exploration: Hamster and Superhamster

Genetic engineers at YeardleighCorp have genetically engineered superhamsters that weigh 15 ounces. Tests indicate that one superhamster can carry a 40-ounce packet of food on its back. If a human could carry the same amount as a superhamster, relative to body mass, how many pounds could a 120-pound teenager carry?

We have mentioned that the study of mathematical relationships is a central idea of algebra. One helpful skill for studying relationships is the ability to make sound mathematical comparisons.

Ratio

A comparison of two quantities, often expressed as a fraction.

There are two main types of ratios: part-to-part ratios and part-to-whole ratios. For example, "number of girls in class to number of boys in class" is a part-to-part ratio, whereas "number of girls in class to number of students in class" is a part-to-whole ratio.

Proportion

An equation stating that two ratios are equal.

Proportions are just equations, and so we don't need any special techniques like "cross-multiplication" to solve them. We can use the trusty properties of equality, just as we do with any other equation.

Example 6.1

Solve for x: $\dfrac{x}{15} = \dfrac{44}{60}$

Solution: Our suggestion is not to think of the left side as a fraction at all. Think of it as x divided by 15. That's just a Level 1 linear equation! We need to get rid of the "divide by 15" and so we multiply both sides by 15.

$$15 \cdot \frac{x}{15} = 15 \cdot \frac{44}{60}$$

$$\cancel{15} \cdot \frac{x}{\cancel{15}} = \cancel{15} \cdot \frac{44}{\cancel{15} \cdot 4}$$

$$x = 11$$

After a bit of simplifying, we have isolated x with just one application of MPOE.

We call this approach "clear the denominator". Of course, it's not the only way to solve a proportion, and not always the easiest way.[1]

6.1.1 Adding a Bit More Complexity

In their most basic form, proportions have the unknown in the numerator of one of the ratios. These are just Level 1 (one-step) equations. All we need to do to solve them is multiply both sides of the equation by the denominator of the unknown (that's MPOE in action).

But what if the unknown is in the denominator of a fraction?

Example 6.2

The startup exploration problem says that a 15-ounce superhamster can carry a 40-ounce packet of food on its back. At that rate of strength, how many pounds could a 120-pound teenager carry?

Solution: Let's write a proportion comparing the ratio of weight to carrying capacity, and let us use P to represent the amount that the teenage can carry, in pounds. Then we have:

$$\frac{15 \text{ ounces}}{40 \text{ ounces}} = \frac{120 \text{ pounds}}{P \text{ pounds}}$$

We wrote these ratios as "weight/carrying capacity", but of course we could also have written the ratios as "carrying capacity/weight". So, one solution approach is to take the reciprocal of both sides. That will put the unknown conveniently in the numerator!

$$\frac{40 \text{ ounces}}{15 \text{ ounces}} = \frac{P \text{ pounds}}{120 \text{ pounds}}$$

$$120 \cdot \frac{40}{15} = 120 \cdot \frac{P}{120}$$

$$320 = P$$

So, a 120-pound teenage of superhamster strength could carry 320 pounds of food. For comparison, according to the US census, the average American ate approximately 257 pounds of fruit in 2009

[1] You may have learned other approaches for solving proportions, and that's a good thing! One very handy approach is to scale up (or down) one (or both) of the ratios so that they have a common numerator or denominator: another application of the identity property of multiplication!

(including fresh fruit, dried fruit, and fruit juice). So this teenager could carry 25% more fruit than they would typically eat in a year.

Very often in mathematical comparisons, it is helpful to create a comparison "per unit" of some quantity: miles *per hour*, dollars *per pound*, and so on. These "per unit" comparisons mean "per *one* unit". For example, if Bob buys 3 pounds of Swiss cheese for $19.50, then we can find an equivalent ratio

$$\frac{19.50}{3} = \frac{6.50}{1}$$

and see that the cheese costs "6.50 dollars per (one) pound".

Unit Rate

A **unit rate** is a ratio in which the value of one of the quantities is 1. For example "miles per hour", meaning "miles traveled in *one* hour".

Before we move on, the maneuver we made in the last example — taking the reciprocal of both sides — bears another moment of reflection.

Tangent: Explaining Reciprocal of Both Sides

Is "take the reciprocal of both sides" a property of equality? In other words, if we know that $a = b$, do we know for sure that $\frac{1}{a} = \frac{1}{b}$ is always true?

The fact that a and b end up in the denominator of a fraction means that we must require that a and b are non-zero at the start. If a and b are both non-zero, then we can use DPOE with both numbers.

$a = b$	We assume that this is true, and that a and b are non-zero
$\frac{a}{a} = \frac{b}{a}$	DPOE, using a
$1 = \frac{b}{a}$	Simplify
$\frac{1}{b} = \frac{b}{a \cdot b}$	DPOE, using b
$\frac{1}{b} = \frac{\cancel{b}}{a \cdot \cancel{b}}$	Cancel the common factor
$\frac{1}{b} = \frac{1}{a}$	Voilà

So, in the end, if two numbers (both nonzero) are equal, then we know that their reciprocals are equal as well. "Take the reciprocal of both sides" is a property of equality.

Of course, we might be faced with numerators and denominators that are more complicated. Here's an example with variables on both sides of the equation. Our approach is to use MPOE to clear *both of the denominators*.

Example 6.3

Solve for x: $\frac{x+1}{2} = \frac{x+2}{5}$

Solution:

$$\begin{aligned} \frac{x+1}{2} &= \frac{x+2}{5} \\ \cancel{2} \cdot 5 \cdot \left(\frac{x+1}{\cancel{2}}\right) &= 2 \cdot \cancel{5} \cdot \left(\frac{x+2}{\cancel{5}}\right) && \text{MPOE two times!} \\ 5(x+1) &= 2(x+2) \\ 5x+5 &= 2x+4 && \text{distributive property} \\ 3x &= 1 && \text{SPOE two times!} \\ x &= \frac{1}{3} && \text{DPOE} \end{aligned}$$

To write our answer in set notation, we have $\mathcal{S} = \left\{\frac{1}{3}\right\}$.

The previous example may give you some insight into how cross-multiplication works, and might come in handy when working on the problems and exercises.

Warning!

Solve: $\frac{x+2}{5} = \frac{7}{6}$

There may be a temptation to subtract 2 from both sides in the first step.

$$\frac{x+2-2}{5} = \frac{7-2}{6} \qquad \Longrightarrow \qquad \frac{x}{5} = \frac{5}{6}$$

But, to do so would be Evil and Wrong! The x+2 is grouped together by the fraction bar (vinculum) and that little group is divided by 5. Before we can manipulate the group, we have to undo the division by 5. Then, the 2 is free to be subtracted.

Note that this is the same kind of structure as

$$4(x + 2) = 16$$

Here we have to divide both sides by 4, before we can use SPOE on the $x + 2$. For some reason the temptation to subtract first is stronger when in fraction form (maybe because the grouping isn't obvious without the parentheses). In any case: beware!

6.2 Applications of Proportional Reasoning

Proportional reasoning encompasses a key set of skills that are important to have in your problem solving toolbox. In this section we'll discuss a few of the classic ways that ratio and proportion show up in problem solving contexts.

6.2.1 Percent

Startup Exploration: Spam "Less Sodium"

Bob is counting cans of Spam in his pantry. For every 3 cans of "low-sodium" Spam, he has 5 cans of Spam with the usual (higher) amount of sodium. What percent of the Spam in his pantry is low-sodium?

Break the word "percent" into its component parts, "per cent", and think about what each part means. The word "cent" indicates 100 (there are 100 cents in a dollar, and 100 years in a *cent*ury), while the word "per" indicates that we are making a comparison (earning $10 *per* hour at a job means you earn $10 for every hour that you work).

Putting the words back together, percent means "per 100" or "for every 100". When we write a percent as 75%, or say out loud "seventy-five percent", we're really mean "75 per 100" or "75 out of 100". We could write the decimal 0.75, or the fraction $\frac{75}{100}$.

The fraction $\frac{75}{100}$ isn't in lowest terms, so we could write instead:

$$\frac{3}{4} = \frac{75}{100}$$

Proportion is the heart of percent, and percent problems are all really about the percent proportion:

$$\frac{\text{part}}{\text{whole}} = \frac{\text{percent}}{100}$$

So given a percent question, we just have to ask ourselves which pieces of the percent proportion we know, and which we are trying to find.

Example 6.4

Write proportions that could be used to solve each of the following:

1. What number is 10% of 20?

2. What percent of 75 is 50?

3. 15 is 25% of what number?

Partial solution:

1. $\frac{x}{20} = \frac{10}{100}$ We're given the percent (10%) and the whole (20). The word "of" is one hint that 20 is the whole. We're looking for the percent, so that's where we'll write the unknown (x, or whatever variable you like). Solve the proportion for the unknown, and we'll have our answer.

2. $\frac{50}{75} = \frac{x}{100}$ Here we are asked for the percent. The other information has to be untangled a bit, but we can see that 75 is the whole and 50 is the part.

3. $\frac{15}{x} = \frac{15}{100}$ We're given the percent again, only this time the question asks "of what number", indicating that we're seeking the whole. The unknown ends up in the denominator, but that's no problem if we remember to reciprocalize!

In Bob's case, he has 3 cans of "low-sodium" Spam for every 5 cans of "regular sodium". That's a part-to-part comparison. To write that as a part-to-whole ratio, note that 3 cans are "low-sodium" for every 8 cans that he counts. We setup the percent proportion to find the percent of cans that were "low-sodium":

$$\frac{3 \text{ less sodium cans}}{8 \text{ cans}} = \frac{p \text{ ``low-sodium'' cans}}{100 \text{ cans}}$$

Which means 37.5% of the cans in Bob's pantry are "low-sodium".

6.2.2 Probability

Questions about chance and probability are very often just proportional reasoning questions in fancy clothes. We won't get into much detail about probability in this course, except to hit some of the highlights.

Startup Exploration: 1d12

Suppose we were to roll a standard 12-sided die (with faces showing the numbers 1–12) 1000 times. How many times would we expect to roll a prime number?

There are 12 numbers on the die, and these numbers comprise the **sample space** of the experiment:

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

Five of the numbers in the sample space are prime numbers $\{2, 3, 5, 7, 11\}$. So, the probability of rolling a prime number on a single toss of the die is $\frac{5}{12}$.

Sample Space

The sample space of a probability experiment is the set of all possible outcomes of the experiment.

Generally speaking, the probability of a certain desired outcome occurring in an experiment is the ratio

$$\frac{\text{number of ways the desired outcome can occur}}{\text{number of outcomes in the sample space}}$$

This probability $\frac{5}{12}$ tells us that if we threw the die 12 times and wrote down the numbers, we'd expect to see approximately 5 prime numbers among our data. To predict how many prime numbers we might see when throwing the die 1000 times, we set up a proportion. Let p be the number of primes we expect in 1000 trials.

$$\frac{5 \text{ primes}}{12 \text{ trials}} = \frac{p \text{ primes}}{1000 \text{ trials}}$$

Solving for p, we see that $p \approx 417$. So, we predict 417-ish primes in 1000 trials.[2]

Startup Exploration: 2d6

Suppose we were to roll *two* standard 6-sided dice and add the two numbers. If we perform this experiment 1000 times, how many times would we expect to roll a prime number?

Determining the sample space for this experiment requires a bit of thinking. It might be tempting to simply list all of the possible sums — the lowest is 2, the highest is 24 — but these sums are not all equally likely. There's only one way to roll a 2, but there are multiple ways to roll a 9. A clever strategy here is to organize the outcomes in a table like the one shown in table 6.1.

From the table we can see that there are, for instance, four ways to roll a 9. With a bit of counting in the table, we can see that there are 15 cells that contain a prime number (out of 36 cells total). So, the probability of rolling a prime number in this experiment is $\frac{15}{36}$. We set up and solve our proportion:

$$\frac{15}{36} = \frac{p}{1000} \qquad \Longrightarrow \qquad p \approx 417.$$

[2] Of course, nothing is guaranteed when it comes to probability experiments. It could be that we get *1000* prime numbers! It's not very likely that this could happen, in fact it's practically impossible. In smaller experiments — throwing the die 50 times, say — it's quite likely that we'd get many more (or many less) than the 20 or some primes we would expect in that case.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Table 6.1: Sample space for rolling two 6-sided dice.

In 1000 trials, then, we would expect to see approximately 417 prime numbers when rolling two 6-sided dice and finding their sum.[3]

The examples above discuss *theoretical probability*. We were working in a "perfect world" and basing our predictions and conclusions on properties of the dice. The alternative would have been to conduct an experiment and then use the data from that experiment to predict the outcome of future experiments. We call that *experimental probability*, and an example from that department will close out this section.

6.2.3 Experimental Data

In an ongoing project about goat safety, cryptozoologists at YeardleighCorp Labs have proposed a study of the hunting behavior of the chupacabra.[4] The "scientists" plan to release a chupacabra into the hills around YeardleighCorp Labs, and then study the impact on the local wild goat population.

Before the study can begin, the scientists need to estimate the existing population of goats. It's clearly impossible to actually count all of the wild goats, so to make an estimate the biologists propose to use the *capture-recapture* (or *capture-mark-recapture*) process.

First, they will capture a sample of goats and mark them with ear tags. Then these tagged goats will be released and allowed to mingle with all the untagged goats. The biologists will wait a few weeks for the goats to mix thoroughly, then capture second sample of goats.

The assumption is that the proportion of "tagged goats to total goats *in the second sample*" is the same as the proportion of "tagged goats to total goats *in the study area*". (A second assumption is that the population

[3] Isn't it interesting that the probability of rolling a prime number on a single 12-sided die is the same as rolling a prime number on a pair of 6-sided dice? Is this a coincidence, or is there a mathematical connection between these two probabilities that suggests why they are the same?

[4] Cryptozoology (which comes from the Greek "crypto", meaning "hidden", and "zoology", the study of animals) is a pseudoscience (the prefix "pseudo" means "false") based on the study of animals that have not been proven to exist. The "animals" in question are called "cryptids". Famous cryptids include the Loch Ness monster, bigfoot, the yeti, and the chupacabra. The chupacabra, Spanish for "goat sucker" is, allegedly, a creature that attacks and drinks the blood of livestock, especially goats.

remains constant during this process.) Solving the resulting proportion will give an estimate for the number of goats in the study area.

Example 6.5

Suppose this team of cryptozoologists initially captures and tags 25 goats, then releases them. Several weeks later, they capture a second sample of 100 goats, 6 of which have tags. What is a reasonable estimate for the size of the goat population?

Solution: Our assumption is that the ratio of *tagged goats* to *all goats* represented by the second sample is equivalent to the ratio for the whole population. Originally, 25 goats were tagged. Then the second sample of 100 contained 6 tagged goats. So, we can estimate the total population as follows:

$$\frac{\text{6 tagged in second sample}}{\text{100 total in second sample}} = \frac{\text{25 tagged in population}}{g \text{ total in population}}$$

Solving the proportion:

$$\begin{aligned} \frac{6}{100} &= \frac{25}{g} \\ \frac{100}{6} &= \frac{g}{25} && \text{take reciprocal of both sides} \\ \frac{100}{6} \cdot 25 &= \frac{g}{25} \cdot 25 && \text{MPOE} \\ 417 &\approx g \end{aligned}$$

Based on the data, a reasonable estimate is that there are approximately 417 goats in the study area.

There is a great deal more that could be said about probability, some of which you have probably seen before. There are other ways to represent a sample space (like a tree diagram), there are other kinds of experiments in which certain events influence other events (you may recall that it sometimes matters whether or not you "put the marble back into the bag")... but for now, we just wish to mention the aspects of probability include a taste of proportional reasoning.

We now turn to another application of proportional reasoning, and our first in-depth look at a certain kind of linear function.

6.3 Direct Variation

Startup Exploration: Rupee Exchange

In her search for delicious new flavors, Yeardleigh travels to India to learn about traditional cooking techniques and flavor combinations. In preparing for her visit, she looks up looks up the currency exchange rate and learns that 1 US dollar (USD) is equal to approximately 60 Indian rupees (INR).

Make a table and a graph showing how many rupees Yeardleigh will get in exchange for 100, 200, 300, 400, 500 USD.

How many rupees will Yeardleigh get for exchanging 250 USD? How much is 22 500 INR worth in USD?

6.3.1 Variation and Direct Variation

When scientists begin to understand a phenomenon, like a law of physics, it's often quite helpful to understand it in terms of a relationship: how one quantity changes with respect to another.

For example: The greater the mass of an object, the greater the gravitational force acting on that object.[5] We say that the force varies directly with mass. There is an equation that describes this relationship, but the key idea is that the force acting on an object due to gravity is directly proportional to the object's mass.

Direct Variation

The variables x and y are said to be **directly proportional** if their values have a constant ratio. In other words, $\frac{y}{x} = k$, where k is a constant called the **constant of variation**.

An equation of the form $y = kx$ is called a **direct variation**. The quantities x and y are directly proportional (can you see how this equation is related to the equation in the previous paragraph?).

Since we have a proportional relationship, we can solve direct variation problems using our knowledge of proportions. We will also see that the graphs, tables, and equations for direct variations share special characteristics.

[5] When the masses of the two objects are very different (for example, if we are comparing your mass to the mass of the Earth), then the larger object's mass dominates the interaction. The equation describing this relationship is part of Isaac Newton's "Law of Universal Gravitation". It states that any two bodies in the universe attract each other with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

This also means that all of the proportion problems we solved earlier can be solved by looking at them as linear functions. If we want to estimate how many goats live in the chupacabra study area, we can model the situation using a linear equation and find the answer as a point on the line.

Let's look back at Yeardleigh's currency exchange. The rate "60 rupees per dollar" is constant. The number of Indian rupees Yeardleigh receives is directly proportional to the number of US dollars she exchanges. The two quantities are in direct variation.

If we let y represent the number of rupees, and x represent the number of dollars, then we have the equation

$$y = 60x,$$

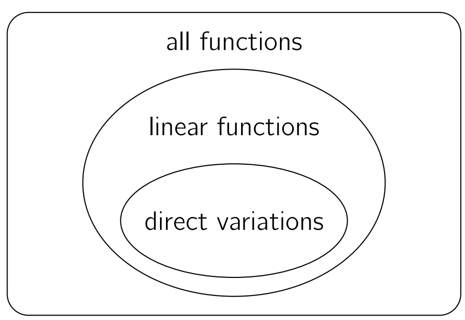
or we can write this equation to highlight the ratio

$$\frac{y}{x} = 60$$

and see that the constant of variation in this scenario is the exchange rate of 60 rupees per dollar.

6.3.2 Direct Variation: a Type of Linear Function

The family of direct variations is a subset of the family of linear functions, in the same way that the integers $\mathbb{Z}$ are a subset of the rational numbers $\mathbb{Q}$.



Every direct variation is a linear function, but not every linear function is a direct variation. What makes direct variation "special"?

Graph of a Direct Variation

This is the easiest way to spot a direct variation. It must be a straight line that goes through the origin. That's it, really! Since every direct variation is of the form $y = kx$, it's quite easy to see that when $x = 0$, we have $y = k \cdot 0 = 0$. So the point $(0, 0)$ is always on the graph of a direct variation. This makes sense in the context of the startup exploration: if Yeardleigh exchanges 0 dollars, she gets back 0 rupees.

Equation of a Direct Variation

The equation of a direct variation will be of the form $y = kx$. It seems simple enough, but trickiest thing that might come up here is if we encounter an equation which, at first glance, doesn't appear to be a direct variation but which, with a little manipulation (using the POEs), can be transformed into a direct variation.

Example 6.6

Which of the following rules show that x and y are directly proportional? If a rule is a direct variation, state the constant of variation.

(a) $\dfrac{y}{x} = \dfrac{1}{2}$ (b) $2y = 8x$

(c) $y + 7 = 2x + 4$ (d) $3y + 1 = x + 1$

Solution: Equations (a), (b), and (d) represent direct variation; equation (c) does not. Equation (a) is clearly a direct variation with $k = \frac{1}{2}$. With equation (b) we can use DPOE to divide both sides by 2. The result is an equation of the form $y = 4x$, and that's another direct variation with $k = 4$.

With equation (c) we can use SPOE to get y by itself, but the result is $y = 2x - 3$, and this is not a direct variation. When we isolate y with equation (d) — first using SPOE and then DPOE — we have $y = \frac{1}{3}x$. That's a direct variation with $k = \frac{1}{3}$.

The lesson here is: if we can transform and equation using the POEs into and equation of the form $y = kx$, then we have a direct variation.

Data Table of a Direct Variation

Up until this point, we have studied techniques for identifying a table of data (or a sequence) as linear, exponential, or quadratic. We've always been given tables that count sequentially through x-values $(0, 1, 2, 3, \dots)$. As we study the different types of functions in more depth, we will be able to identify functions from tables which aren't sequential, or which jump around.

For directly proportional relationships, we should be able to recognize not just that they are "linear", but that they are specifically "direct variation". The test for a direct variation goes back to the defining characteristic of a direct variation: the fact that $\frac{y}{x}$ is constant.

Example 6.7

Do the following data tables show direct variation? If so, write the equation for the direct variation.

(a)

x	y
-7	21
9	-27
13	-39

(b)

x	y
40	10
16	64
-20	-5

Solution: Table (a) shows a direct variation, since for each line of the table, we have the same ratio:

$$\frac{y}{x} = \frac{21}{-7} = \frac{-27}{9} = \frac{-39}{13} = -3$$

The equation for this direct variation is $\frac{y}{x} = -3$ or $y = -3x$.

Table (b) does not represent a direct variation. The first and last rows have the same ratio, but the ratio of the middle row is different:

$$\frac{10}{40} = \frac{-5}{-20} = \frac{1}{4} \quad \text{but this is different from} \quad \frac{64}{16} = 4$$

The ratio $\frac{y}{x}$ has to be the same for all points, otherwise we do not have a direct variation. This is one of the things that makes a direct variation special. A table can still be linear and not be a direct variation: there are lots of straight lines that *do not* go through the origin! In chapter 7 we will learn a different way to determine whether a table of data represents a (generic) linear function.

Here's one more example of the kind of question we might encounter about direct variation.

Example 6.8

Find the missing value in each case: (a) A direct variation includes the points $(16, -2)$ and $(x, 4)$. Determine the value of x. (b) A different direct variation includes the points $(6, 30)$ and $(-10, y)$. Determine the value of y.

Solution: We are told that we have a direct variation, and so we know that the two points must share the same ratio. We can therefore set up a proportion to solve part (a):

$$\frac{-2}{16} = \frac{4}{x}$$

and solve to find the value of x. We find that $x = -32$.

Let's use a different approach to solve part (b). Knowing one point is enough to write the equation for the direct variation. We know that the point $(6, 30)$ is on the direct variation, and so

$$k = \frac{y}{x} = \frac{30}{6} = 5.$$

Therefore, we have the equation $y = 5x$. We can then substitute -10 for x and solve to discover that $y = -50$.

A final note: Technically, we can write all of these ratio with x over y and find the answers we are looking for. The reason we stick so closely to "y over x" is the relationship that this quantity has to the unit rate in the problem. Plus, we want the notation that we use with direct variations to be consistent with the notation we use with linear functions generally.[6]

[TODO] Graphs and visuals for DV

[6] We will meet a very important ratio called *rate of change* in the next chapter. In this ratio we set the change in the independent variable over the change in the dependent variable.

6.4 Inverse Variation

Extended Exploration: Teeter Totter Nickels

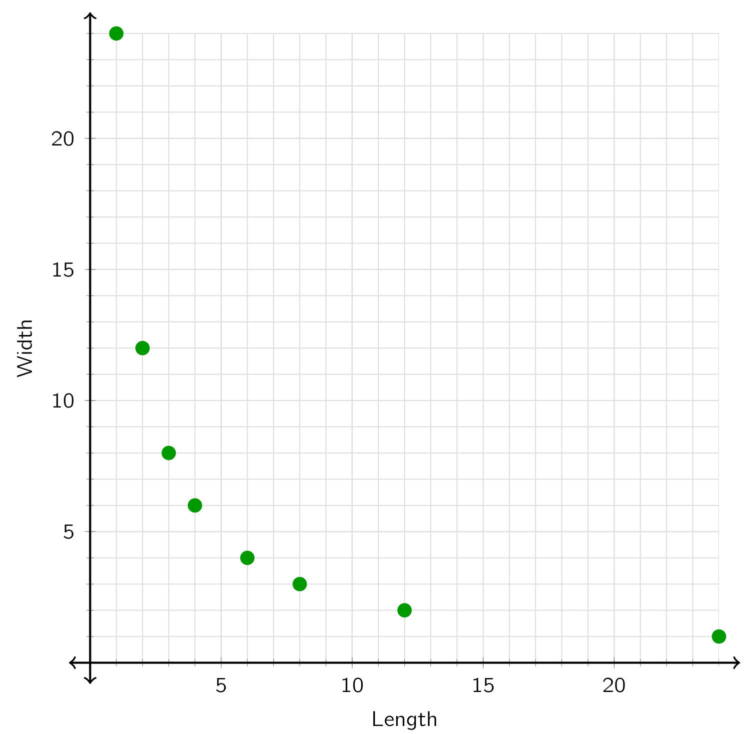
[TODO] Click here to visit the extended exploration: Teeter Totter Nickels

Startup Exploration: Herb Garden

François has enough mulch to plant an herb garden covering 24 square meters. Plus, he has whole barn full of snap-together fencing units that are each 1-meter long. How many different herb gardens can François create, if he wants to use all the mulch and surround the garden with a rectangular fence?

To solve François's problem, we need to find a rectangle that has natural number side lengths and whose area is 24 square meters. One candidate is the long, skinny rectangle that is 1 meter by 24 meters. Of course, that's not the only one! We list the possible rectangles below and plot a graph of the data.

Length (x)	Width (y)
1	24
2	12
3	8
4	6
6	4
8	3
12	2
24	1



The first thing that should jump out about this graph is that it is definitely not linear! The J-like shape suggests that it could be an exponential relationship... but is it? What could we do to determine whether or not this is an exponential relationship?

Full Disclosure

Even though we're beginning our discussion of linear functions, inverse variation is *not* a linear relationship. We discuss it here just as a way to compare and contrast inverse and direct variation.

Inverse Variation

The variables x and y are said to be *inversely proportional* if their values have a constant product. In other words, $xy = k$, where k is a constant called the *constant of variation*.

An equation of the form $y = \frac{k}{x}$ is called an *inverse variation*. The quantities x and y are inversely proportional.

In an inverse variation (where the variables are inversely proportional), the most important thing to remember is the fact that $x \cdot y$ is constant. For every point (x, y) the product of x and y is always the same number.

When we force the product of two numbers to be the same, certain patterns arise. Most notably, if the value of one of the quantities goes up, the other one has to go down.

Data Table of an Inverse Variation

If we step away from context and think purely about data, the test for whether some data shows an inverse variation is whether $xy = k$, some constant value.

Example 6.9

Do the tables below show inverse variation? If so, write the equation for the direct variation.

(a)

x	y
2.5	20
-10	-5
8	5.5

(b)

x	y
3.5	4
1	14
-7	-2

Solution: Table (a) does not represent an inverse variation. The first and second rows of the table have the same product (50), but the last row has a different product (44) Table (b) does represent an inverse variation: all three rows have the same product (14). So, table (b) has equation $xy = 14$.

Example 6.10

Find the missing values in the table, given that the data represent an inverse variation.

(a)

x	y
3	a
9	4
2	b
c	8

Solution: The second row of the table gives us the clue we need: $9 \cdot 4 = 36$, so that must be the constant of variation for this inverse variation. Then, it is simply a matter of finding the partner of each value that multiplies to get 36.

In the first row, $3a = 36$ implies that $a = 12$. In the third row, $2b = 36$ implies that $b = 18$. In the fourth row, $8c = 36$ implies that $c = 4.5$.

Equation of an Inverse Variation

We have seen that the equation for an inverse variation is either $yx = k$ or $y = \frac{k}{x}$. Writing the equation for an inverse variation brings out two important features that we must be aware of. Consider the equation

$$y = \frac{k}{x}$$

First, what happens when $x = 0$? Yikes! Remember, division by zero is undefined, and so 0 is an illegal value for x. We say that 0 must be *restricted from the domain* of the function.

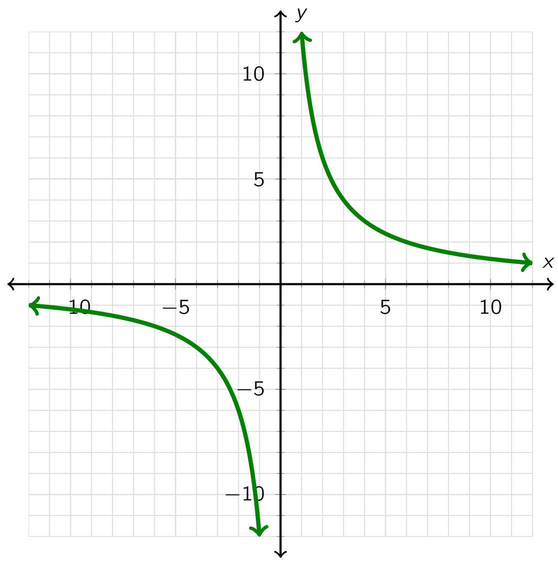
Second, what happens when $y = 0\ldots$ or is that even possible? Note that as x gets larger and larger, the value of the fraction $1/x$ gets smaller and smaller (closer and closer to zero). But, no matter how big x gets, $1/x$ will never be *equal to* zero. It will get close to zero. Super close! Impossibly close! But it will never equal zero.

We can see this unusual behavior reflected in the graph.

[TODO] asymptote in I.V. section

Graph of an Inverse Variation

Not being able to have an x or y value of zero does very interesting things to the graphs. Consider the inverse variation $xy = 12$ or $y = \frac{12}{x}$:



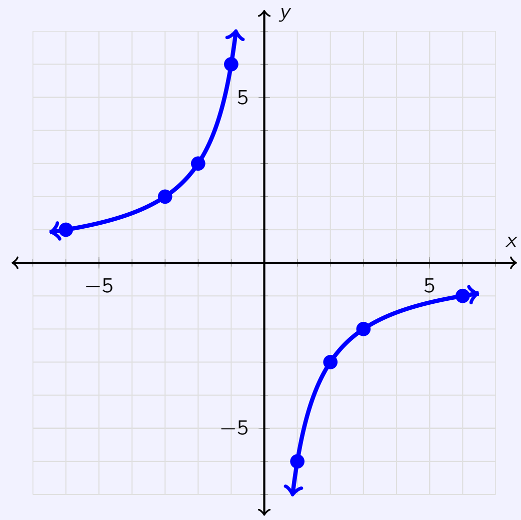
Notice a few things: First, we can see from the graph that this is not an exponential relationship. Although the portion of the graph in the first quadrant might look exponential, when we plug in negative input values we get different behavior that we'd expect from an exponential.

Secondly, the graph is in two disconnected pieces, and since it never crosses the x- or y-axis, it appears entirely in the first and third quadrants. It appears in those particular quadrants because of the defining feature of inverse variation: x and y have a constant product.

This graph is the picture of all points whose coordinates have a product of (positive) 12. The only way to get a positive product is to either multiply two positive numbers, or two negative numbers — exactly the description of the coordinates in the first and third quadrants! What do you suppose the graph of $y = \frac{-12}{x}$ might look like?

Example 6.11

Write the equation for the inverse variation pictured in the graph below.



Solution: Since we know that this is an inverse variation, all we need is to find the coordinates of one point. Some are plotted for us, so that's good news. If we use the point $(-2, 3)$, we can see that $k = xy = -2 \cdot 3 = -6$. Once we have found the constant of variation, we can easily write the equation: $xy = -6$ or $y = \frac{-6}{x}$.

About the "Family" of Inverse Variation

You may have noticed that inverse variation doesn't seem to fit into any of the function families we have looked at so far. This is true. Inverse variation doesn't belong to linear, quadratic, exponential families of functions, and so we won't do much more with inverse variation for quite a while.

Inverse variation is actually a member of the super-family of functions called *rational functions*, which is more of a focus in algebra 2. This rational super-family has a number of interesting and surprising features, and they can take on a whole variety of bizarre shapes.

Inverse variation graphs do have something in common with exponential functions: asymptotes. As we saw, inverse variations get impossibly close to both the x- and y-axis, but never reach them. Generally, these graphs have two asymptotes, one horizontal, one vertical.

A curve does not exist in its full power until contrasted with a straight line.

Robert Henri
American painter

Chapter 7

Linear Functions

So far we have learned quite a bit about the family of linear functions. From our work with sequences we learned how to tell if a sequence was arithmetic. We can write recursive rules for arithmetic sequences, and we can change those recursive rules into into explicit formulas.

We turned arithmetic sequences into tables of data, and when we graphed some linear functions, we saw that all were straight lines. We saw a preview of how different transformations might change the way its graph looks.

Most recently we learned about a specific type of linear function, the direct variation. In this chapter we will bring together all of the pieces we have learned previously into a single coherent, and very important, idea.

7.1 Rate of Change and Slope

Extended Exploration: Calculating Speed

[TODO] Click here to visit the extended exploration: Calculating Speed

Startup Exploration: CCC

Bob enrolls at Cheeseville Community College, and the admissions counselor there gives him the following table estimating the cost to attend. The total cost of a semester at CCC is made up of two expenses: (a) the fixed fees, which are the same for every student every semester (technology fee, printing allowance, and so on), and (b) the tuition rate, which varies per credit hour.

Most courses at CCC are either 3 or 4 credit hours per semester. The table summarizes several combinations of 3- and 4-hour courses.

No. Credit Hours	3	4	6	7	10
Cost (dollars)	501	654	960	1113	1572

What can you learn from the table? How much is tuition per credit hour? How much does CCC estimate Bob will pay in fees?

7.1.1 Unit Rates in Data

Since we're fresh from a discussion of proportional reasoning, perhaps our first guess might be that this is a direct variation. Unfortunately, a quick check of the ratios $\frac{y}{x}$ shoots that idea down:

$$\frac{501}{3} = 167 \qquad \text{but} \qquad \frac{654}{4} = 163.5$$

Maybe we have an arithmetic sequence? If so, then the terms of the "cost" sequence will have a constant difference. But, if we take the differences between neighboring terms, we have different values:

$$654 - 501 = 153 \qquad \text{but} \qquad 906 - 654 = 306$$

But, hang on. Look at the corresponding "number of credit hours". These numbers don't increase by the same amount at every step, like they did for our arithmetic sequences. There's a jump from 4 to 6. We're going to have to be a bit more clever to account for this!

Notice that when "credit hours" increases by 1, "cost" increases by \$153. And then, when "credit hours" increases by 2, "cost" increases by \$306 – the increase in both values has doubled! This even works at the end of the table: When the number of hours jumps by 3 and the cost jumps by \$459, the rate "\$dollars per 1 credit hour" says the same:

$$\frac{153}{1} = \frac{306}{2} = \frac{459}{3} = \frac{\text{change in the total cost}}{\text{change in the number of hours}}$$

So, even though the table skips over some numbers, the *change* in the cost still corresponds to the *change* in the number of hours. So, the tuition rate at CCC must be \$153 per credit hour.

Example 7.1

Does the data in the table below represent someone moving at a constant speed? If so, what is the speed?

Time (sec)	Distance (m)
5	17.5
12	42.0
31	108.5

Solution: Comparing the first two data points, we see that the change in distance is $42.0 - 17.5 = 24.5$ meters, and that this occurs over $12-5 = 7$ seconds. This ratio "24.5 meters per 7 seconds" corresponds to the unit rate "3.5 meters per second".

Comparing the second pair of data points, we see that a change in distance of $108.5 - 42.0 = 66.5$ meters happens over $31 - 12 = 19$ seconds. The ratio "66.5 meters per 19 seconds" corresponds to the unit rate "3.5 meters per second".

So, yes, this table does represent movement at a constant speed of 3.5 meters per second.

7.1.2 Rate of Change

We want to be able to tell if *any* data set is linear. The examples of speed and hourly tuition begin to suggest a general approach. We are seeking a constant **rate of change**.

Rate of Change

Rate of change is a measurement of how quickly the dependent variable changes relative to a one-unit change in the independent variable.

To compute the rate of change of y with respect to x, we want to "unitize" a change in y (the dependent variable) with respect to the corresponding change in x (the independent variable).

To write this out using mathematical notation, we use an abbreviation for the idea of "change": the Greek letter delta, Δ. This makes the definition of rate of change look like the following:

$$R.O.C. = \frac{\Delta \text{ dependent variable}}{\Delta \text{ independent variable}} = \frac{\Delta y}{\Delta x}$$

The symbol "Δy" is pronounced "delta y" or "change in y". We sometimes say that rate of change is "delta y over delta x".

Speed is an example of a rate of change. It is a measurement of how distance changes relative to a one-unit change in time: meters per (one) second. Bob's tuition at Cheeseville Community College is another example that measures how his total expense changes relative to a unit-change in the number of credit hours he takes: dollars per (one) hour.

7.1.3 Slope

Rate of change is a very important concept in mathematics and it is a general term that doesn't just apply to straight lines. Different types of functions have different patterns emerge in their rates of change. The study of rate of change led to the development of calculus!

But, the rate of change of a straight line is special: it's always the same. When it comes to linear relationships, the key is knowing that the *rate of change for linear data is constant*. We have a special term for the rate of change in a linear relationship.

Slope

Slope is a measurement of the steepness of a line.

For a line, slope and rate of change are describing the same thing. We don't really talk about the "slope" of a curve (like the graph of an exponential and quadratic functions), since the steepness changes, as we saw in section 3.5.

In mathematical notation, the letter most often used to abbreviate slope is m.[1] So, we can update our notation from earlier:

$$m = \frac{\Delta y}{\Delta x}$$

Slope from Data

As we have seen in the earlier examples, finding slope from a data table is rather easy. We find the change between two y-values and find the change between the corresponding x-values. When we unitize this ratio – in other words, when we divide Δy by Δx – we've got slope. Woot!

This works even if the data is not given in table form. You may have heard the old saying, "The shortest distance between two points is a straight line." Well... what's the slope of that line?

Example 7.2

Find the slope of the line connecting the points $(11, 4)$ and $(16, 29)$.

[1] "Why m?" we bet you're wondering. Many people have looked into this question, digging back through mathematical writing to try and learn where this abbreviation comes from. It seems that the earliest recorded use of m for slope was in 1757 by Italian mathematician Vincenzo Riccati. But his work includes no explanation of *why* the letter m was used.

Solution: The change in the y-values is $\Delta y = 29 - 4 = 25$, and the change in the x-values is $\Delta x = 16 - 11 = 5$. So, the slope of the line between these two points is

$$m = \frac{\Delta y}{\Delta x} = \frac{25}{5} = 5.$$

So, the line connecting the points $(11, 4)$ and $(16, 29)$ has a slope of 5.

Slope From a Graph: The Slope Triangle

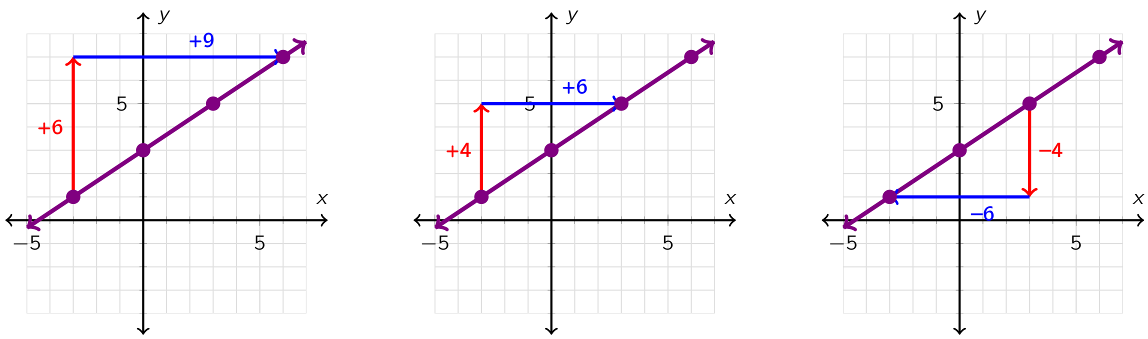
Suppose we're given the graph of a line and asked to find the slope. We could create a data table using points on the line and find the slope from that data. Or, we could inspect the graph directly using a *slope triangle*. A slope triangle is a right triangle that connects two points with a single vertical move and a single horizontal move.

Here is the graph of a line with some points marked. To find the slope of the line, we draw a slope triangle.

the red arrow shows the vertical change. It starts at 5 and goes to 7, a vertical distance of $7 - 5 = 2$ units. The blue arrow shows the horizontal change. It starts at 3 and goes to 6, a horizontal change of $6 - 3 = 3$ units. Now we can calculate slope

$$m = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{2}{3}.$$

Since the rate of change is constant for a straight line, it doesn't matter which two points we choose, or which point we choose as the starting place. Each of the slope triangles below can be used to compute slope of this line!



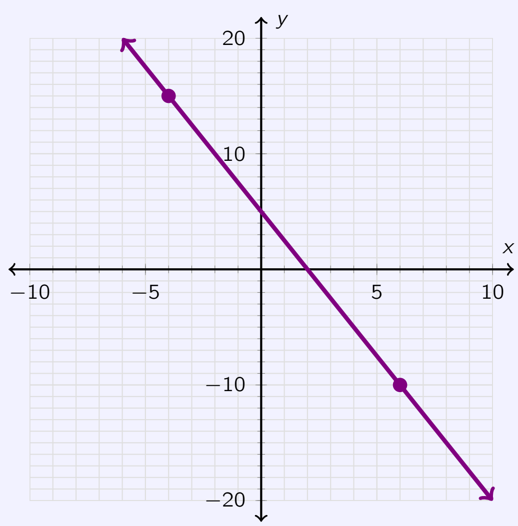
Again, we don't have to use a slope triangle. We can just convert at least two of the points on the graph into a table and then use the techniques for finding slope from a data set.

There are two commonly mixed up things about slope. First: reciprocalizing the ratio. Remember it's change in y over change in x, not the other way around.

Second, folks sometimes have problems with sign. A rule of thumb: if the data has a positive correlation, the slope is positive. If the data has a negative correlation, the slope is negative. It's a good habit always to double-check the sign of a slope to verify that it makes sense in context.

Example 7.3

Find the slope of the line depicted in the following graph.



Solution: This problem exhibits two key features: the slope of the line will be negative (since the line decreases as we move to the right), and we're going to have to be careful about the scale on the axes! We might see a vertical change of one "tick mark" on the axis, but note that one tick mark is a change of 5 units. Similarly, one tick on the x-axis is a change of two units.

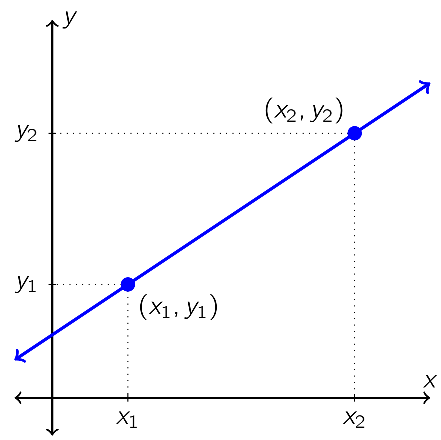
If we start at the point on the left and move to the point on the right, we have a horizontal change of

+10 units in the horizontal direction and −25 units in the vertical direction (a negative vertical change means moving downwards). So, the slope is

$$m = \frac{-25}{10} = \frac{-5}{2} = -\frac{5}{2}$$

7.1.4 The Slope Formula

The process of computing slope is always basically the same, so we can derive a formula for slope. Study the picture below, showing a line through two "generic points" called (x_1, y_2) and (x_2, y_2).



The vertical change between the two points is $y_2 - y_1$, and the horizontal change is $x_2 - x_1$. So we can compute the slope between these two points using our familiar formula

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

The trickiest part about using this formula is being consistent about which point we call (x_1, y_1) and which point we call (x_2, y_2). It doesn't matter which point is which – but we have to be consistent.

Tangent: The Order of the Points

That last sentence says "it doesn't matter which point is which." Why is that? Explain why it doesn't matter which point we call "point 1" and which we call "point 2". In other words, if we have two points called (x_1, y_1) and (x_2, y_2), explain why

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

(Hint: Experiment with some numbers first, then try to extend your observations to the two "generic" points given.)

7.1.5 Horizontal and Vertical Lines

If we think about how slope is calculated – vertical change over horizontal change – a horizontal line has *no vertical change*, or a vertical change of zero. If we pick two points on a horizontal line and calculate slope, we would get 0 divided by some number, which always equals zero! So the slope of a horizontal line is always 0. This makes sense if we remember the slope is a measure of the steepness: a horizontal line is flat, and that means no "steepness" at all!

If we pick two points on a vertical line and try to calculate slope, we will get a horizontal change of 0. So we'll have some number divided by 0... and that's off limits. We say that the slope of a vertical line is *undefined*, since division by 0 is undefined.

7.1.6 Using Slope

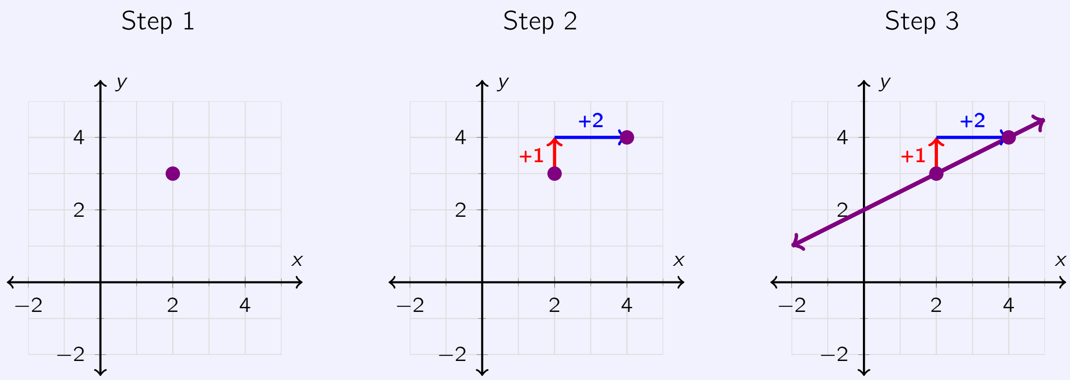
Before, when we were graphing a line we had to take some domain values, like the integers between 3 and 3, plug them in to the equation, and plot every single point, one at a time. Now we have an alternative!

All we really need to graph a line is two points. If we have one point on the line and the *slope*, we can find another point on the line.

Example 7.4

Graph the line that goes through $(2, 3)$ and has a slope of $\frac{1}{2}$.

Solution: Step 1: Plot the given point. Step 2: The slope $\frac{1}{2}$ tells us how to find another point on the line: the numerator is the change in y, so we move up 1 unit. The denominator is the change in x, so we move 2 units to the right. Now we have a second point: $(4, 4)$. Step 3: Connect the dots and you have your graph!



By the way, perhaps now you see why we've been such fans of improper fractions. Seriously, which form is easier to use when creating a graph: a line with slope $\frac{4}{3}$ or a line with slope 1.33333...?

Colinearity

Points that lie on the same line are said to be *colinear*. If we are asked to determine whether two points are colinear... well, that's easy! Any two points will define a line, so any pair of points will be colinear.

But what about three points? How could we determine whether three points are colinear – in other words, whether they all lie on the same line?

Example 7.5

Are the points $(4, 1)$; $(-1, 5)$; and $(1, 2)$ colinear? Why or why not?

Solution: The slope between the first pair of points is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1-5}{4 - -1} = \frac{-4}{5} = -\frac{4}{5}.$$

The slope between the second pair of points

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{5-2}{-1-1} = \frac{3}{-2} = -\frac{3}{2}.$$

Those two slopes are not equal, so: nope! These points won't line up in a single line! They are not colinear.

7.2 Point-Slope Form

We are going to learn three different ways to write equations for lines: the point-slope form, the slope-intercept form, and the standard form. Each form tells you something a little different about the line, and each one has its own set of pros and cons. Over the next few sections we will learn about these different forms, what they tell us about lines, what they are good for, and how to convert from one form to another.

Extended Exploration

[TODO] Click here to visit the extended exploration: Deriving the Point Slope Form

Startup Exploration: One Point, One Slope

A line with slope 2 passes through the point $(4, 3)$. Name three other points that also lie on this line. Can you find a point on the line that lies in the second quadrant? The third quadrant? The fourth quadrant?

The first form that we will study is called *point-slope form*. As the name suggests, all we need to write an equation in this form is the slope of the line and a point on the line.

In the startup exploration, we are asked to find different points on the line that has slope 2 and passes through the point $(4, 3)$. Finding other points isn't that hard. We saw how to do this in the last example in section 7.1.[2] It is more interesting to consider a generic point (x, y) on this line.

We now have two points on the line $(4, 3)$ and (x, y), and we know the slope is 2. We can arrange all of this information using the slope formula:

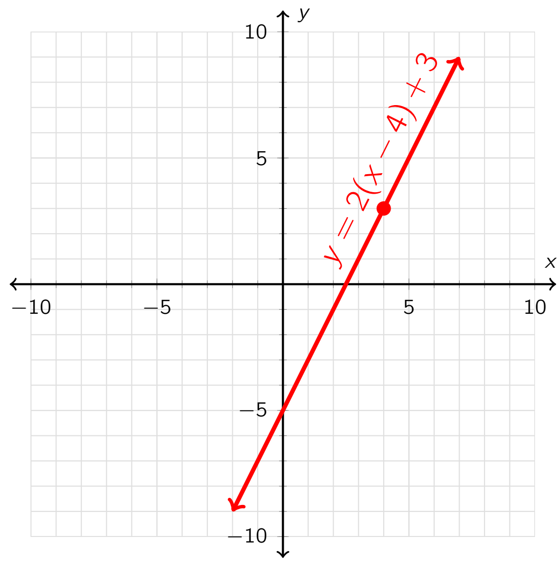
$$2 = \frac{y-3}{x-4}.$$

There should be something familiar about this: we've got an equation with a y and an x in it... so this is starting to look like a graphable equation. All we have to do it transform this equation to isolate y. Let's do that!

$$\begin{aligned} 2 &= \frac{y-3}{x-4} \\ 2(x-4) &= y-3 && \text{MPOE: multiply both sides by } (x-4) \\ 2(x-4)+3 &= y && \text{APOE: add 3 to both sides} \end{aligned}$$

[2] Well, the line doesn't pass through the second quadrant, so finding a point that meets that requirement is impossible. Something to ponder: are the following statements true or false? (a) A linear function can pass through at most 3 of the four quadrants. (b) A linear function will always pass through exactly 3 of the 4 quadrants.

If we flip this around to write it in a more familiar format, we have $y = 2(x - 4) + 3$. That's an equation we can graph! If we plot some points (or pull out our favorite piece of graphing technology) we can graph this equation:



What do you know! Graphing this equation gives us exactly the line we were looking for: a line with slope 2 that goes through the point $(4, 3)$. So the equation $y = 2(x - 4) + 3$ describes the line. Can you see the slope and the given point hiding there in the equation?

7.2.1 Deriving the Point-Slope Form

Let's make things even more generic. Suppose we have a line with slope m that goes through the specific point (x_1, y_1). Then let (x, y) be any other random point on the line. We can summarize this using the slope formula, and then transform our equation into a graphable, $y = something$ format:

$$m = \frac{y - y_1}{x - x_1}$$

$$m(x - x_1) = y - y_1 \qquad \text{MPOE: multiply both sides by } (x - x_1)$$

$$m(x - x_1) + y_1 = y \qquad \text{APOE: add } y_1 \text{ to both sides}$$

This last line is what we call the **point-slope form** of a line.

Point-Slope Form

The form of a linear equation that uses the slope and any point on the line. It is written either $y - y_1 = m(x - x_1)$ or $y = m(x - x_1) + y_1$, where m is the slope of the line and (x_1, y_1) is a point on the line.

7.2.2 Using Point-Slope Form

The trickiest part about the point-slope form – as you might anticipate – is handling the plusses and minuses. It can be tricky to remember the $-x_1$ and the $+y_1$, so be on the lookout and always check your signs.

Example 7.6

State the slope of each of the following lines, and name a point on each line.

(a) $y = -3(x-1) - 4$ (b) $y = 5 - 8(x+4)$ (c) $y - 8 = 4(x-2)$

Solution: Let us compare the given equation (a) to the generic point-slope formula

$$y = -3(x-1) - 4 \qquad \Longleftrightarrow \qquad y = m(x - x_1) + y_1,$$

We can see that the slope is -3. The equation has "-4" whereas the formula wants "$+y_1$", so we change the rule to reflect addition outside the parentheses: $y = -3(x-1) - 4 = -3(x-1) + {}^-4$. Now we can see that a point on the line is $(3, {}^-4)$.

With (b), note that things are a bit out of order, but we can commute the terms (taking the sign along, remember!) and have $y = {}^-8(x+4) + 5$. Now, we have addition in the parentheses, whereas the rule wants subtraction in there. We can make that adjustment easily enough: $y = {}^-8(x - {}^-4) + 5$. From this version, we can see that the slope is -8 and a point is $({}^-4, 5)$.

The definition of point-slope forms gives us this form as an alternative (check the box above)! So, the slope of this line is 4 and a point on the line is $(2, 8)$.

Back in chapter 3, we had to pick some values for the domain, substitute them into the equation, find the values of the range, and plot each point. In section 7.1, we learned that we really just need a point and a slope. This information is ready to be extracted from the point-slope form!

Example 7.7

Graph (by hand) the line $y = 3(x-4) - 5$.

Partial solution: We can see that the slope is 3 and that a point on the line is $(4, {}^-5)$. With this information we find a second point, and then graph the line!

How would we go about writing an equation in point-slope form for data given to us in a table, or as a collection of points?

Example 7.8

Write the equation of the line that goes through the points $(6, 10)$ and $(-1, -4)$.

Solution: First, we find the slope of this line:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10 - {}^-4}{6 - {}^-1} = \frac{14}{7} = 2.$$

Then, we can use either of the two points as the "point" in point-slope form:

$$y = 2(x - 6) + 10 \quad \underline{\text{or}} \quad y = 2(x + 1) - 4$$

7.2.3 Relating Point-Slope to Earlier Concepts

There is a way to look at the point-slope form that relates to transformations in a plane. In section 6.3, we studied direct variations of the form $y = kx$. This is a straight line through the origin, and k is the slope. So let's rewrite this as $y = mx$.

If we want to move the graph of the direct variation up or down, we add something to the equation. For example, if we want to move the graph 12 units up, the equation would become $y = mx + 12$. If we want to move down 7 units, the equation would become $y = mx + {}^-7$. In general, if I wanted to move the graph vertically y_1 units the equation would become $y = mx + y_1$.

If we want to move the graph of the direct variation to the right 5 units, we would replace x with $(x - 5)$ and our new equation would be $y = m(x - 5)$. If we want to move the graph to the left 8 units, we have the equation $y = m(x + 8)$. In general, if we wanted to move the graph horizontally x_1 units, the equation would become $y = m(x - x_1)$.

If we want to cause *both* a horizontal and vertical shift, we would do both transformations to get $y = m(x - x_1) + y_1$. We can think of this transformation as "moving the origin" of the direct variation from $(0, 0)$ to (x_1, y_1).

Yet another way to look at point-slope form is to think back to the apparent rules we wrote for linear sequences back in section 2.3. Given a sequence like $4, 7, 10, 13, \ldots$, we wrote the rule

$$a_n = 4 + 3(n - 1).$$

Can you see the connection to point slope form? In this rule the slope is 3 and the point is $(1, 4)$. The slope is telling us the constant difference: the amount that we add on for each step forward in the sequence. The point $(1, 4)$ is telling us that the first term in the sequence is 4.

Back in section 2.3 we also wrote zero-based rules for arithmetic sequences. What would be the zero-based rule for the sequence given above? These zero-based rules are at the core of the form that we will study in the next section.

7.3 Slope-Intercept Form

Point-slope form is handy and quite easy to write in many cases. But, it has those parentheses in it which means that we could simplify it even further. We explore that possibility in this section.

Name of Startup Exploration

Graph the line that goes through the points $(-1, 7)$ and $(2, -2)$. Find the slope through these two points and write two equations in point-slope form for the line.

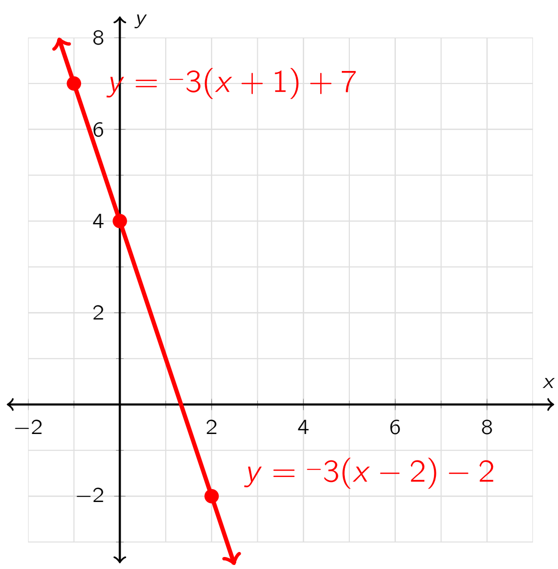
Use your toolbox of equivalence rules to eliminate the parentheses and rewrite each of your point-slope equations in simplest form. What happens? How is your new equation reflected in the graph?

Point-slope form is the easiest form for algebra students to use when writing the equation of a line. Once we know the slope of the line we only need one point on the line – any point at all – write the equation.

However, point-slope form is a pain for algebra *teachers* to grade. Why? Well, how many points are there on a line? Each one of those points can be used to write an equation for that line. So, there are infinitely many different point-slope equations for a single line! If a student gets creative about what point to use, the teacher may have to do some work to determine whether the student's answer is correct or not.[3]

One way to simplify things would be to designate a kind of "standard point" that we always use at *the point* when writing a point-slope form equation. Luckily, there's a natural point to set as the standard.

The startup exploration asks us to graph, and write two different equations for, the given line.



[3] Don't get any ideas.

We can eliminate the parentheses using the distributive property, and then we can combine like terms as needed. If we use the point $(-1, 7)$, then we have:

$$\begin{aligned} y &= {}^{-}3(x+1)+7 \\ &= {}^{-}3x + {}^{-}3(1) + 7 && \text{distributive property} \\ &= {}^{-}3x + {}^{-}3 + 7 && \text{substitution} \\ &= {}^{-}3x + 4 && \text{substitution/combining like terms} \end{aligned}$$

If we use the point $(2, -2)$, then we have:

$$\begin{aligned} y &= {}^{-}3(x-2)-2 \\ &= {}^{-}3(x + {}^{-}2) + {}^{-}2 && \text{adjust negative signs, for safety} \\ &= {}^{-}3x + {}^{-}3({}^{-}2) + {}^{-}2 && \text{distributive property} \\ &= {}^{-}3x + 6 + {}^{-}2 && \text{substitution} \\ &= {}^{-}3x + 4 && \text{substitution/combining like terms} \end{aligned}$$

After this simplification process, the two different point-slope form equations have turned into the *same equation*! It turns out that this always happens: if we simplify two different point-slope equations for the same line (by getting rid of the parentheses and combining like terms), we will always end up with the same equation. This simplified version is called the **slope-intercept form**.

Slope-Intercept Form

The **slope-intercept form** of a linear equation is an equation of the form

$$y = mx + b.$$

The value m is the slope of the line, and b is the value at which the line crosses the y-axis. The point $(0, b)$ is called the y-intercept of the line.

Note from the definition that the equation $y = {}^{-}3x + 4$ contains two pieces of information. It tells us the slope of the line is ${}^{-}3$, and it tells us that the line passes through the point $(0, 4)$. We can verify this in the graph above.

Since a linear function has a constant slope and and can only have one y-intercept (two straight lines can't cross more than once!), a linear function has only one slope-intercept equation.

Graphing a line in slope-intercept form is exactly the same as graph a line in point-slope form. In fact, it's easier! Figuring out the "point" in point-slope form can lead to some challenges with positive and negative numbers. With slope-intercept form, the "point" is much easier to identify. Then, we can use the slope to find a second point and graph the line.

7.3.1 Writing Equations in Slope-Intercept Form

Suppose we are given the slope of a line and a point on the line that is *not* the y-intercept. There are two primary ways to get to an equation for a line in slope-intercept form.

Method 1: Simplify point-slope form. This method works just as we saw earlier. We use the slope and the point to write an equation in point-slope form, which we then simplify using the distributive property and combining like terms. The result is slope-intercept form.

Method 2: The "old school" method for folks who don't like (or never learned) point-slope form. In this approach, we use the slope and the x and y coordinates of the point to write an equation that we can use to calculate b.

Example 7.9

A line as a slope of 2 and passes through the point $(3, 7)$. Write the slope-intercept equation of the line.

Solution: We're given $m = 2$, so we can start writing an equation in slope-intercept form: $y = 2x + b$. We don't know b yet, but we do know a pair (x, y) that is on the line! So, we plug in the x and y values from the point: $x = 3$ and $y = 7$. In other words:

$$7 = 2(3) + b$$

We can now solve for b by subtracting 6 from both sides (SPOE). We find that $b = 1$. So, the slope-intercept equation is $y = 2x + 1$.

These two methods are equally valid mathematically, so pick the approach that you understand the best and feel most confident about.

Example 7.10

Write an equation in slope-intercept form for the line containing the following data points.

x	y
5	12
6	8
7	4
8	0

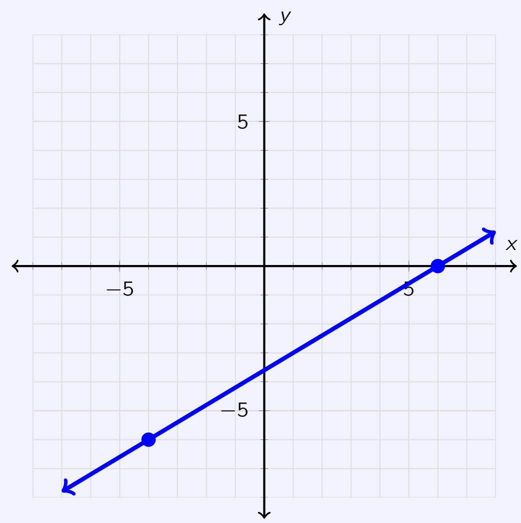
Solution: If a table of data ever includes the y-intercept – the point with x-coordinate 0 – then our job is quite easy. In this case, though, we don't have that. To write the equation, we could use one of the two methods above, or we could do a bit of detective work and find the y-intercept using the pattern in the table.

Notice that the y-values decrease by 4 as x increases by 1. That means the slope of the line is $^-4$. It also means we can work our way backwards in the table to the point where $x = 0$ and $y = 32$. So, our equation is $y = {^-4}x + 32$.

These equations should look familiar. We saw equations just like these when we were studying arithmetic sequences! The value of the sequence at stage 0 was just the y-intercept. The common difference (the value we added at each step) was the slope!

Example 7.11

Write an equation in slope-intercept form for the line pictured in the graph below.



Solution: If a graph has a clear y-intercept then our job is quite easy. In this case, though, we don't have that. We might try to guess about the point – maybe it's $-3.5\ldots$ish? Would you bet on that? – but we need another approach to be sure.

So, we'll use a slope triangle to find the slope between the two given points $\frac{6}{10} = \frac{3}{5}$. Then, we'll write an equation in point-slope form using the point $(6, 0)$: $y = \frac{3}{5}(x - 6) + 0$. Simplifying using the distributive property and combining like terms gives:

$$y = \frac{3}{5}x - \frac{18}{5},$$

and $-\frac{18}{5} = -3.6$, which is close to -3.5, but not the same thing!

7.3.2 (;,;) Equivalence of the Point-Slope and Slope-Intercept Forms

In this extension section, we'll look in more detail at a claim we made earlier in this chapter. If the content here feels like too much, take a break and come back to it later. Your brain keeps working on things even when you're not thinking about them consciously!

In section 7.3 we made the claim that *always get the same answer* when we start with an equation in point-slope form and we convert to an equation in slope-intercept form (using the distributive property and combining like terms).

Have you started to become suspicious of claims like this? Have you started to wonder, "How do we know that we *always* get the same answer?" This is a good question to ask. After all there are infinitely many different point-slope equations we could write. Surely we're not going to try and convert them all and check that we get the same answer every time.

Here's one way to think about this comparison. We have a line. Since we are going to assume we have a linear function, we know that this line is not vertical. So, that means it must cross over the y-axis at some point $(0, b)$. Suppose (x_1, y_1) is another point on that line.

This line has a slope that we can represent using the slope formula. We'll use the y-intercept as "point 2":

$$m = \frac{b - y_1}{0 - x_1} = \frac{b - y_1}{{}^{-}x_1}$$

Trust us on this next bit: it will be helpful. We're going to use the POEs to isolate b so that we can make a comparison.

$$\begin{aligned} m &= \frac{b - y_1}{{}^{-}x_1} & \\ {}^{-}mx_1 &= b - y_1 & \text{MPOE: multiply both sides by } {}^{-}x_1 \\ {}^{-}mx_1 + y_1 &= b & \text{APOE: add } y_1 \text{ to both sides} \end{aligned}$$

We're going to come back to this last line in a moment.

Our goal is to show that point-slope form and slope-intercept form are equivalent. So, let's write the point-slope form of the equation using the generic point (x_1, y_1) and try to convert it to slope-intercept form.

$$\begin{aligned} y &= m(x - x_1) + y_1 & \text{point-slope form} \\ y &= m(x + {}^{-}x_1) + y_1 & \text{rewrite subtraction} \\ y &= mx + {}^{-}mx_1 + y_1 & \text{distributive property} \\ y &= mx + {}^{-}mx_1 + y_1 & \text{the expression in blue is equal to } b \\ y &= mx + b & \text{slope-intercept form} \end{aligned}$$

So, in the end, we have shown that if we have a line with y-intercept $(0, b)$, then the point-slope equation for the line – using any point on the line! – can be converted into the slope-intercept equation of the line.

7.4 Standard Form

So far, we know a lot about writing equations for lines. We can find the slope of a line, we can write equations in point-slope form, and we can convert point-slope form into slope-intercept form. We now look at a final linear form, so-called **standard form**.

Startup Exploration: Goats and Chickens

The chickens got into the goat pen on François's farm! When François asked Hermine how many of each animal were mixed up, she coyly replied that there were a total of 44 feet in the pen.

If the pen contains only (anatomically normal) goats and chickens, how many of each animal could there be inside? How many different combinations are possible?

We know that a healthy goat as 4 feet, and so if the pen contains x goats, they account for $4x$ feet. Similarly, a health chicken as 2 feet. If there are y chickens in the pen they account for $2y$ feet. Finally, we know the total number of feet. So, we have the equation

$$4x + 2y = 44.$$

We can tinker with a little guess and check to find combinations that work. For instance there could be 8 goats and 6 chickens, since $4(8)+2(6) = 44$. If you find a few combinations, you might start to notice some patterns in the solutions (of which there are 12).

We'll come back to problems like "goats and chickens" in the future. For now, we'll simply point out that our equation above is an example of a linear equation in standard form.

Standard Form

If A, B, and C are Integers where A and B are not both zero, then $Ax + By = C$ is a linear equation in standard form.

Note that this is not a "$y =$" form. This has its challenges and its benefits. In some situations, standard form equations requires a bit of work before they are very useful.[4] For example, A, B, and C don't tell us much about the line. Especially when compared to point-slope form and slope-intercept form, in which important features of the line are visible right there in the equation.

A benefit of this different format is that any line – whether it is a function or not – has a standard form equation. The definition says that A and B can't *both* be zero (at the same time), but one or the other of them could

[4] Standard form: the fixer-upper form of a linear equation.

be zero. This is how we get equations for vertical and horizontal lines. Vertical lines are of the form $Ax = C$ and horizontal lines are of the form $By = C$.

Example 7.12

Are the following equations written in standard form? If not, why not?

(a) $3y = 4x + 2$

(b) $-2x + 6y = 17$

(c) $3x = 15$

(d) $\frac{1}{2}x + 5y = -12$

(e) $2x - 4(y - 2) = 8$

(f) $10 = 3x - 3y$

Solution: Lines (a), (d), and (e) are *not* in standard form. To be in standard form, the x term and the y term must be on the same side of the equal sign; line (a) violates this rule. Standard form states that the coefficients must be integers; line (d) violates this requirement. Standard form has no parentheses; line (e) violates this.

Lines (b), (c), and (f) are in standard form. Note that (c) has $B = 0$, but that's OK; this is the equation for a vertical line. Line (f) is written with the variables on the right-hand side, but otherwise it meets the requirements.

7.4.1 Converting To and From Standard Form

If we are given a line in some other form, then we can convert to standard form using the properties of equality. This is an application of "transforming formulas" that we saw in section 5.8.

Example 7.13

Convert the lines $y = \frac{3}{2}x - \frac{4}{3}$ and $y = \frac{1}{4}(x - 8) + 2$ to standard form.

Solution: The first line is in slope-intercept form, but there's a fraction in there which is no good for standard form.

$$
\begin{aligned}
y &= \frac{3}{2}x - \frac{4}{3} \\
6y &= 9x - 8 && \text{multiply through by 6 to eliminate fractions} \\
-9x + 6y &= -8 && \text{put } x \text{ and } y \text{ on the same side}
\end{aligned}
$$

The second line is in point-slope form, and we can start out by doing the distributive property.

$$
\begin{aligned}
y &= \frac{1}{4}(x-8)+2 \\
y &= \frac{1}{4}x-2+2 && \text{distributive property} \\
y &= \frac{1}{4}x && \text{combine like terms} \\
4y &= x && \text{multiply through by 4 to eliminate fractions} \\
-x+4y &= 0 && \text{put } x \text{ and } y \text{ on the same side}
\end{aligned}
$$

Of course, we can convert in the other direction as well, if needed – at least to slope-intercept form (to point-slope form, not so much).

Example 7.14

Convert the line $6x - 14y = 21$ to slope-intercept form.

Solution: All we need to do is transform this equation to isolate y.

$$
\begin{aligned}
6x-14y &= 21 \\
-14y &= -6x+21 && \text{SPOE} \\
y &= \frac{-6}{-14}x+\frac{21}{-14} && \text{DPOE} \\
y &= \frac{3}{7}x-\frac{3}{2} && \text{simplify fractions}
\end{aligned}
$$

7.4.2 Simplified Standard Form

It turns out that we can generate infinitely many equivalent equations in standard form just by multiplying both sides of a given form by an integer. For example, if we have the line $3x - y = 5$, we might through by 5 and get $15x - 5y = 25$. Or we could multiply through by -1 and get $-3x + y = -5$. All of these lines are the same because we've applied a property of equality (MPOE) to create them.

Since there are so many possible ways to write an equation in standard form, we have a "standardized" version of it. Standardized standard form sounds like an oxymoron, so we call it "simplified standard form".

There are two additional criteria for simplified standard form.[5] In addition to the rules given in the definition

[5] These rules are just a convention, and they are somewhat arbitrary. In future math courses, many of these restrictions about standard form, for example what kinds of coefficients are allowed, may be relaxed.

above, we require that A is non-negative (it must be greater than or equal to zero). We also require that A, B, and C share no common factors between them other than 1.

The second of these new rules is the trickier one. The line $10x + 6y = 12$ violates this second requirement since each term is divisible by 2. To simplify this, we divide all of the terms by 2 and have an equation that is in simplified standard form: $5x + 3y = 6$. Note that here 3 and 6 share a common factor, but we're OK since the only factor that *all three* numbers share is 1.

The line we wrote to describe the startup exploration $4x + 2y = 44$ does not meet the requirements for simplified standard form. We must divide through by 2 to fix this: $2x + y = 22$ (does this new equation have an interpretation in the context of the problem?).

Example 7.15

Write an equation, in simplified standard form, for the line that goes through the points $(-3, 6)$ and $(7, 12)$.

Solution: We'll start out as usual, finding the slope between these points:

$$m = \frac{\Delta y}{\Delta x} = \frac{12 - 6}{7 - -3} = \frac{6}{10} = \frac{3}{5}$$

Then, we can use either point to write a line in point-slope form:

$$y = \frac{3}{5}(x + 3) + 6.$$

Since we need integer coefficients we can multiply through by 5 at the start. This will avoid having to carry out the distributive property with that fraction.

$$\begin{aligned}
y &= \frac{3}{5}(x+3) + 6 && \\
5y &= 3(x+3) + 30 && \text{multiply through by 5} \\
5y &= 3x + 9 + 30 && \text{distributive property} \\
5y &= 3x + 39 && \text{combine line terms} \\
-3x + 5y &= 39 && \text{SPOE: subtract } 3x \text{ from both sides} \\
3x - 5y &= -39 && \text{multiply through by } -1
\end{aligned}$$

This last step is required because simplified standard form requires that the coefficient of the x-term be non-negative.

7.4.3 Graphing standard form

To graph standard form on most graphing calculators, the only thing we can do is convert to slope-intercept form by solving for y. Many online tools (like Desmos) allow graphing in standard form without any transforming required on our part.

When graphing by hand, there is a clever alternative technique!

Example 7.16

Graph the line $3x - 2y = 12$.

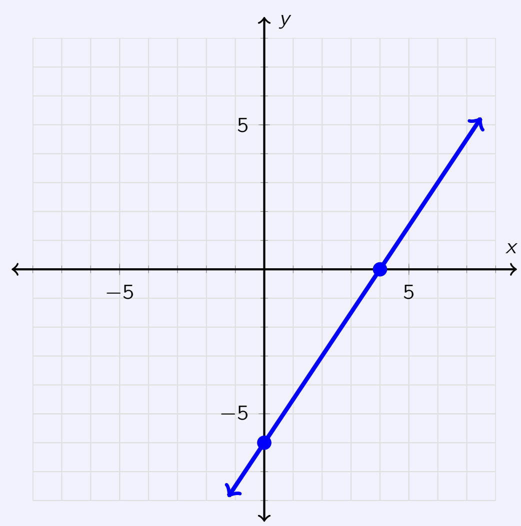
Solution: We can always plot some points by substituting in some values for x or y. We could choose random values, but here's a clever idea: plugging in zero will eliminate a term and simplify our calculations! If we let $x = 0$, then:

$$3(0) - 2y = 12 \qquad \Longrightarrow \qquad {}^{-}2y = 12$$

That's a one-step equation, and $y = {}^{-}6$. If we plug in $y = 0$, we have:

$$3x - 2(0) = 12 \qquad \Longrightarrow \qquad 3x = 12,$$

and so $x = 4$. So, we have found two points on the line: the y-intercept $(0, {}^{-}6)$, and the x-intercept $(4, 0)$. That's enough to draw the graph!



The only time this approach doesn't work is when the y-intercept and the x-intercept are the same. What does that sort of a line look like? How can we spot such a line from its standard form equation?

Speaking of x-intercepts, they haven't gotten too much of our attention so far. Our work with linear equations has highlighted the importance of the y-intercept. We'll learn much more about the significance of x-intercepts in future chapters.

7.5 Parallel and Perpendicular Lines

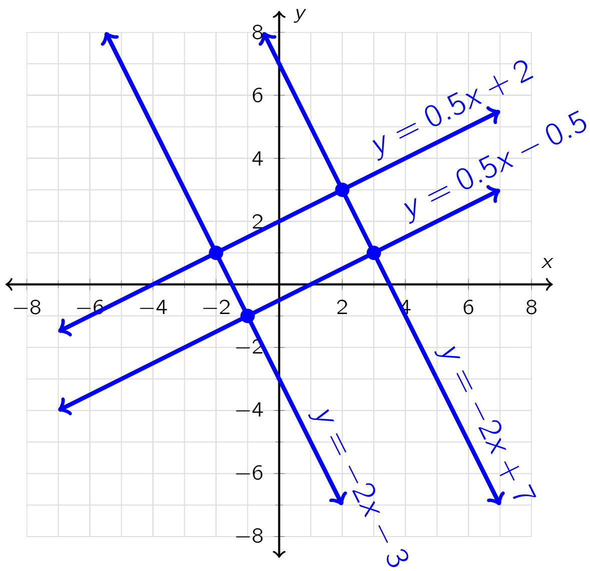
This part of this chapter is dedicated to a geometric connection to linear functions: parallel and perpendicular lines. Our goal for now is to learn how to tell if two lines are parallel or perpendicular, and how to write equations for lines that are parallel or perpendicular to a given line. In the future (like in a geometry class), you might use concepts from this lesson to prove some geometric concepts algebraically.

Name of Startup Exploration

The four points below form the vertices of a rectangle. Graph the rectangle and write four equations in slope-intercept form for the lines containing the four sides of the rectangle.

$$(2, 3) \quad (3, 1) \quad (-1, -1) \quad (-2, 1)$$

We won't go into the details of how to write these equations. Those details are explained in earlier sections. The key thing to study in this example is how the lines are related to each other.



7.5.1 Parallel Lines

Parallel lines never intersect. They exist in the same plane and stay the same distance apart for their entire, infinite length. In the rectangle we drew as part of the startup exploration, opposite sides are parallel. Examine the slopes of each pair of parallel lines. What do you notice?

Algebraically, parallel lines have the *same slope* and *different y-intercepts*. Note that it is not sufficient to just look for having the same slope. If two lines have the same slope and also have the same y-intercept, then they aren't actually *two lines*. Those are the same line... and you can't be parallel to yourself![6]

Parallel will play an important role as we begin to study systems of linear equations in chapter 8.

In our rectangle, opposite sides of the rectangle are parallel. For example, the two lines that have positive slope:

$$y = \frac{1}{2}x + 2 \qquad \text{and} \qquad y = \frac{1}{2}x - \frac{1}{2}.$$

Notice that the slopes of these two lines are the same and that their y-intercepts are different. The same is true for the other pair of lines.

Example 7.17

Which of these lines, if any, are parallel to each other?

(a) $y = \frac{3}{4}x + 5$ (b) $y = \frac{3}{4}(x + 8) - 1$

(c) $y = \frac{3}{4}(x - 12) + 1$ (d) $4x - 3y = 9$

Solution: Parallel lines will have the same slope and different y-intercepts, so we should extract this information. We'll have to do the a bit of work first, converting all of the lines to slope-intercept form:

(a) $y = \frac{3}{4}x + 5$ (b) $y = \frac{3}{4}x + 5$

(c) $y = \frac{3}{4}x - 8$ (d) $y = \frac{4}{3}x - 3$

From this we can see that (a) and (b) are actually the same line, and that this line is parallel to line (c). Line (d) has a different slope, so it is not parallel to any of the others.

Example 7.18

Write an equation for the line that contains $(-2, 1)$ and is parallel to the line $y = -3(x - 4) + 7$.

Solution: Parallel lines have the same slope, so the slope of our line must be -3, just like the slope of

6 Don't start with the interpretive dance about this. You can't be parallel to yourself, no matter how you hold your arms.

the given line. Since it doesn't specify which form to use, we can write our answer in point-slope form:

$$y = -3(x + 2) + 1.$$

The amount of work we will have to do for questions like the ones in this chapter will depend on the format the equations take. We will have to work a little harder if the lines are given in standard form, or when we are asked to write our answer in standard form.

Example 7.19

Write an equation *in simplified standard form* for the line that contains $(2, -1)$ and is parallel to the line $5x + 3y = 2$.

Solution: First, we need to remember that we're after a point and a slope to write the equation of our line. The line we are given is in standard form, so we'll have to convert it to slope-intercept by solving for y.

Using the techniques from section 7.4, we convert $5x + 3y = 2$ into

$$y = -\frac{5}{3}x + \frac{2}{3},$$

and we can see that the slope of the line is $-\frac{5}{3}$.

Our line will have the same slope as the given line (since our line is parallel to it), and we know a point on the line. So, we can write an equation for our line in point-slope form:

$$y = -\frac{5}{3}(x - 2) - 1.$$

But, we are asked for an equation in standard form, so we have to convert.

First we distribute and combine like terms, getting a line in slope-intercept form:

$$y = -\frac{5}{3}x + \frac{7}{3}.$$

Then, we get rid of the fractions and rearrange into standard form:

$$5x + 3y = 7 \qquad \text{Phew!}$$

And now for some good news: There is a clever shortcut that we can use when searching for a standard form equation for a line parallel to a line given in standard form.

You may have noticed that when we convert a standard form equation into a slope-intercept equation, we are doing the same procedure every time:

$$\begin{aligned} Ax + By &= C && \text{standard form} \\ By &= -Ax + C && \text{SPOE: subtract } Ax \text{ from both sides} \\ y &= -\frac{A}{B}x + \frac{C}{B} && \text{DPOE: divide through by } B \text{ to isolate } y \end{aligned}$$

This means that the slope of the line is always $-\frac{A}{B}$. That's a recipe for finding the slope of a line in standard form.

Since A and B are the only parts of the standard form equation that relate to slope, and given that parallel lines have the *same slope*, we can reason that all parallel lines have the same A and B values in standard form.

Look at the previous example: the given line and our answer have the same A and B values, but with different C values! In other words, any line parallel to $5x + 3y = 2$ is going to look like $5x + 3y = C$. In the example, we have to ensure that our line goes through the point $(2, -1)$, so we just need to figure out what C value will accomplish that goal.

How can we figure out what C value will make the line $5x + 3y = C$ go through the given point? We can substitute in these values for x and y and solve the equation to identify C! Observe:

$$\begin{aligned} 5x + 3y &= C && \text{our almost-there equation} \\ 5(2) + 3(-1) &= C && \text{substitute in a known point on the line} \\ 7 &= C && \text{simplify the left-hand side using the order of operations} \end{aligned}$$

So, our standard form equation is $5x + 3y = 7$.

7.5.2 Perpendicular Lines

Perpendicular lines intersect at right angles (that is, $90°$ angles). In the rectangle we drew in the startup exploration, adjacent sides are perpendicular. Examine the slopes of each pair of perpendicular lines. What do you notice?

In our rectangle, the lines the meet in the second quadrant are the lines

$$y = \frac{1}{2}x + 2 \qquad \text{and} \qquad y = -2x - 3.$$

Notice that the slopes of these two lines are both opposites and reciprocals. This is true in general: perpendicular lines have slopes that are *both opposites and reciprocals* of each other. This means that the slopes of perpendicular lines will always have a product of 1.

Example 7.20

Which of these lines are perpendicular to each other?

(a) $y = \frac{3}{4}x + 5$

(b) $y = -\frac{3}{4}(x + 8) - 1$

(c) $y = \frac{4}{3}(x - 12) + 1$

(d) $4x + 3y = 9$

Solution: The key is to look for slopes with opposite signs and then to look for reciprocals. First, we ought to convert line (d) into a form that reveals its slope:

$$\text{(d) } y = -\frac{4}{3}x + 3$$

This means that line (a) is perpendicular to line (d), since $\frac{3}{4} \cdot -\frac{4}{3} = -1$. Also, line (b) is perpendicular to line (c) since $-\frac{3}{4} \cdot \frac{4}{3} = -1$.

Example 7.21

Write an equation for the line that contains $(-2, 1)$ and is perpendicular to the line $y = -3(x - 4) + 7$.

Solution: The given slope is -3, the perpendicular slope must be the opposite, reciprocal slope: $\frac{1}{3}$. The point is $(-2, 1)$, and since the format of the equation was not specified, we can answer in point-slope form:

$$y = \frac{1}{3}(x + 2) + 1.$$

Again, the amount of work we will have to do for questions like the ones in this chapter will depend on the format the equations take.

Example 7.22

Write an equation *in simplified standard form* for the line that contains $(2, -1)$ and is perpendicular to the line $5x + 3y = 2$.

Partial solution: The line we seek is $3x - 5y = 11$. Above, we discussed a clever shortcut for working with parallel lines in standard form. Can you invent a similar shortcut for working with perpendicular lines in standard form? Use the given solution to test out your ideas!

Still need to find a quote that works for this chapter. In the meantime, we have this.

Author
Description of author

Chapter 8

Linear Systems

Every chapter should have a lead paragraph – even just a short one – that appears before the first heading. This is a placeholder paragraph which will at some point be replaces by actual content.

8.1 Mathematical Modeling

Mathematics has many applications to the sciences, medicine, business, and economics. Often, though, we have to "translate" some problem in the real world so that it can be interpreted mathematically. It is usually necessary to break down the situation into a set of variables and equations, then look at them all together and see how they interact with each other. Doing this is called **mathematical modeling**.

We can model the financials of a business, weather patterns, transportation infrastructure, airline schedules, missile guidance systems, factory efficiency — and many other phenomena — in this way.

Startup Exploration: I ♥ YeardleighCorp

Yeardleigh is a good business woman. She realizes that in order to grow her business, she must brand her corporation by selling things like t-shirts with her company logo on it. She spends \$270 on a design. Since she plans on printing a huge number of shirts, she finds a supplier who will sell her blank cotton shirts in a variety of sizes and colors for \$3.00 per shirt, and a printer who will print the logo in full color for \$1.50 per shirt. If Yeardleigh decides to sell the shirts for \$12.00 each, how many shirts must Yeardleigh sell before she can start making a profit on t-shirt sales?

8.1.1 Simultaneous Equations

One approach to mathematical modeling is to model problems using multiple functions, and then exploring how those functions relate to one another.

System of Equations

A set of two or more equations with the same variables is called a **system of equations**. They are also sometimes called *simultaneous equations*.

In the startup exploration, the amount of money Yeardleigh earns will depend on how many t-shirts she buys and sells. So, suppose we let x, the independent variable, represent the number of t-shirts. Then we let y, the dependent variable, represent the amount of money that those t-shirts cost (or that they generate for the company).

Yeadrleigh earns \$12.00 for every t-shirt she sells, so one equation involved with the scenario is

$$y = 12.00x \qquad \text{Yeardleigh's income equation.}$$

She has already spent \$270 on her design, and every t-shirt she prints costs an additional \$4.50 (that's \$3.00 for the shirt and \$1.50 for printing). So, another equation in the scenario is

$$y = 270 + 4.50x \qquad \text{Yeardleigh's cost equation.}$$

We now have to linear equations. Observe what happens when we graph these two equations on the same set of axes (fig. 8.1). What does this graph tell us about the problem?

On the graph in the figure, the red line is Yeardleigh's cost equation and the green line is Yeardleigh's income equation. The x-coordinate shows the independent variable (number of shirts), and the y-coordinate shows the dependent variable (amount of money spent or earned).

Notice that these two lines intersect (in other words: they cross over one another). Does this intersection point have any special significance?

The intersection shows that for a certain number of t-shirts, cost and income are the same. The points to the left of the intersection show where cost is greater than income, points to the right of the intersection show where income is greater than cost. So that intersection point is really important for the problem!

The coordinates of the intersection point are $(36, 432)$. So, if Yeardleigh sells 36 t-shirts, she will make 432 in income... and it will also cost her exactly that much to produce the t-shirts. If she sells more than 36 t-shirts, she will earn more than it costs to make the shirts.

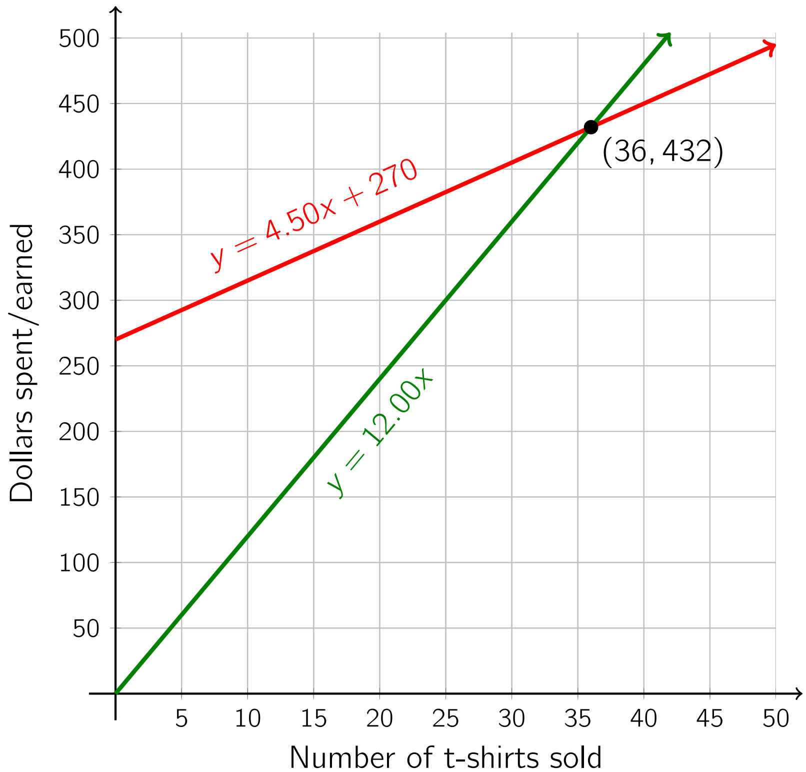


Figure 8.1: Graph showing both Yeardleigh's cost and income equations.

We have found Yeardleigh's *break-even point*, a very important idea in economics and business. It is the point where a company's costs and income are the same. Before this point, the company will be losing money. After this point, the company will start to make a profit. In the context of our problem, Yeardleigh will make a profit on t-shirt sales only if she sells more than 36 shirts.

8.1.2 Writing a System of Equations

When we have two (or more) equations that we want to treat as a system, we usually write them stacked vertically, and we very often add a big curly brace (usually on the left-hand side only) to show that they are meant to be considered a group. In the case of the startup exploration, we write:

$$\begin{cases} y = 12.00x \\ y = 4.50x + 270 \end{cases}$$

8.1.3 Solving Systems of Linear Equations

Solution to a System of Equations

The set of all points that are common to all equations in the system. Graphically, these are the points where all of the graphs of the functions intersect. Note: a system of equations may have no solution.

The solution to a system of linear equations is the intersection point of the graphs of those functions. Can a system of equations have more than one solution?

Imagine that we have a system of two linear equations. Try to visualize the types of solutions we might get. What are the different ways that two straight lines in the plane can intersect?

- The lines might intersect at a single point. Such a system has a unique solution.
- The lines might be parallel and not intersect at all. In this case, the system has no solution.
- The lines might overlap completely (that is, the "two lines" might actually be the same line). In this case, the system has infinitely many solutions: every point one line is also on the other line — they're the same line!

Since we have straight lines, these are the only possibilities. For example, a system of linear equations cannot have "exactly two soltuions". (Can you explain why not?) Later (in algebra 2, for instance), systems will get more complex. Then, we won't have to limit ourselves to linear equations! A system of two quadratic equations, for example, might have no solution, one solution, two solutions, or infinitely many solutions. (Can you picture how each of those might happen? Look back at some of the graphs we drew in chapter 3 and chapter 4.)

8.1.4 Checking a Solution to a System

Suppose we have a point that we think might be a solution to a system of equations. How can we check?

Example 8.1

Determine whether or not the point $(-1, 5)$ is a solution to each of the given systems.

$$\text{System A: } \begin{cases} x + y = 4 \\ x = -1 \end{cases} \qquad \text{System B: } \begin{cases} y = -x + 4 \\ y = -\frac{1}{5}x \end{cases}$$

To check whether the point is a solution to System A, we'll sustitute x and y into both equations. If the point makes both equations true, then the point is a solution to the system.

Checking System A, we have:

$$\begin{aligned} x + y &= 4 & \text{and} \qquad x &= -1 \\ (-1) + 5 &\stackrel{?}{=} 4 & -1 &\stackrel{\checkmark}{=} -1 \\ 4 &\stackrel{\checkmark}{=} 4 \end{aligned}$$

The given point makes both of the equations true, so yes, the point $(-1, 5)$ is a solution to System A.

Checking System B, we have:

$$\begin{aligned} y &= -x + 4 & \text{and} \qquad y &= -\tfrac{1}{5}x \\ 5 &\stackrel{?}{=} -(-1) + 4 & 5 &\stackrel{?}{=} -\tfrac{1}{5}(-1) \\ 5 &\stackrel{\checkmark}{=} 5 & 5 &\neq \tfrac{1}{5} \end{aligned}$$

The given point works for the first equation, but not for the second. To be a solution, the point has to satisfy both equations, so no, the point $(-1, 5)$ is not a solution to System B.

8.1.5 Writing the Solution to a System

NEED SOME VISUALS HERE...?

When we were solving equations — meaning one equation at a time — we use set notation to record our solutions. Since the solution to a system involves two numbers, we have to be mindful about how we write our answers.

Two cases are pretty straightforward. If the system has a unique point as its solution, we can write that point as the solution set. For example, the solution to System A in the previous example is $\mathcal{S} = \{(-1, 5)\}$. Note that we've just enclosed the point $(-1, 5)$ inside the curly braces to show that the set includes that point.

The second easy case is when a system has no solution. Then, we can use the same approach for when a single equation has no solution: $\mathcal{S} = \{\ \}$ or $\mathcal{S} = \emptyset$.

The trickier case is when two lines overlap completely. We can't say that the solutions in the case are "all real numbers" (as we said for a single equation).

8.1.6 Techniques for solving linear systems

Over the next few sections, we will study three different techniques for solving systems of linear equations. A fourth technique, which we only mention here, will be an important topic in algebra 2.

1. **Graphing the system.** To solve a system by graphing, we graph the equations on the same set of coordinate axes and look for the intersection point. If we are lucky, we'll get "nice" points with integers for their coordinates. Graphing with technology works a little better, since trace functionality can give us good estimates for the coordinates if they are not integers.

2. **Collapsing the system by substitution.** To collapse a system is to do something algebraically that fuses two equations into one. If done correctly, we can turn two equations in two variables into one equation in one variable! One method uses the property of substitution to achieve this.

3. **Collapsing the system by elimination.** This is an alternative approach to collapsing a system, that is based on a mathematical structure called a matrix (although we won't go into detail about matrix operations.)

4. **Modeling the system as a matrix.** A *matrix* (plural: matrices) is a rectangular arrangment of numbers. Matrix operations are good for solving systems with more than two variables, which we won't see in algebra 1. We will learn much more about matrices (in general) in algebra 2, along with a number of ways of using them to solve systems.

The different approaches have pros and cons, and you may find that you prefer a particular approach. Generally speaking, you should solve problems using whatever techniques make the most sense to you, and some assignments will be open and allow you to choose your own method. Other assignments may ask you to use a specific technique, though, so its important to read directions carefully.

8.2 Solving Linear Systems By Graphing

We solved the startup exploration at the start of *section* 8.1 by graphing two equations on the same set of axes. That's the idea behind this solution technique.

Graphing is easy with the right technology.

Graphing on the calculator is actually a good way to check your answer, not the only one though.

However, you might not be able to get the exact answer if it is fractional. Don't become dependent on graphing. Some are actually solved easier and more accurately algebraically. Also, if you are prone to type things in wrong, you may have a problem. Scale can also be an issue. I'm not going to solve Yeardleigh's t-shirt problem via graphing.

Things to be careful of: a. Be careful if you have to convert the equation from one form into another, like standard into slope-intercept b. Don't rely on "trace" if it appears that the answer is fractional. Technique: If you are given a word problem, write it as a system of equations. Then, graph the equations to find any intersection points. You can either graph by hand or with a graphing calculator. Your work is the graph. The answer is the intersection point!

a. Graphing By Hand: Use graph paper. You can't use the short-cut way of graphing standard form either. The points will be "nice" if you are asked to graph by hand. Remember that the graph is not the answer. It is the work. This means that it doesn't need to be a high quality graph. Your answer is the intersection point.

b. Graphing By Calculator: They have to be in function form for the graphing calculators that we have in class. The "trace" function is pretty useless as I expect exact answers. You should look at the table to find exact answers. On the table you will notice that the solution will show up as 1 x-value with the same y-value for the two equations.

Example 2 Problem: Solve these systems by graphing. If a system has infinite solutions, give one point that will satisfy the system. y=1x 3+ (a) ïčť 3 (b) y=x y=5x (c) y=1 y=x (d) 2x+y=3 (e) 3xâĹŠy=7 xâĹŠ2y=4 y=3x 7âĹŠ ïčťy=1x 3 âĹŠ ïčť 3

Solution: If you graph with the calculator, the equation has to be in function notation (either slope-intercept or point-slope). IF you do use a calculator, you must sketch a graph to show your work. If you do not use a calculator, you have to make sure to graph very precisely, using graph paper. If I ask you to solve with graphing, I will try to make sure your solutions are integers. If not, approximate them. (a) S = Null Set (lines are parallel) (b) S = {(0,0)} (c) S = {(1,1)} (d) S = {(2, -1)} (e) Any point on the line y = 3x-7, examples (0, -7); (1, -4) in set notation, the solution looks like S=∀(x,y)∈ 3xâĹŠ y= 7 . {}

Please remember, I want you to think about the problem and map out your strategy before you jump right into the work. If you can determine, upon inspection, that lines are parallel, or if you transform some of the equations and notice that they are parallel or the same line, you don't have to do all of the work of graphing the lines (or in the future substitution or elimination). Just write the solution set and a little statement like "upon inspection, the lines are parallel" or "the same line".

8.2.1 Finding an intersection point using a calculator

When finding an intersection point using a calculator, we can use the `trace` function on the calculator. But, `trace` can only give us a decimal approximation, and this could be a problem for some solutions. (For example, a solution that is not whole numbers or a fraction with “convenient” denominator.)

Using `trace` is good for approximating a solution, but then we can use an alternative way: looking at the table. Here, we can search the table for a place where `y1` and `y2` are the same for a particular value of `x`.

For the t-shirt example, go to the `table` screen on your calculator, and scroll to where `x` is 36. You should see that both `y1` and `y2` are 432. This will be how the intersection point will show up in a table.

8.3 Solving Linear Systems By Substitution

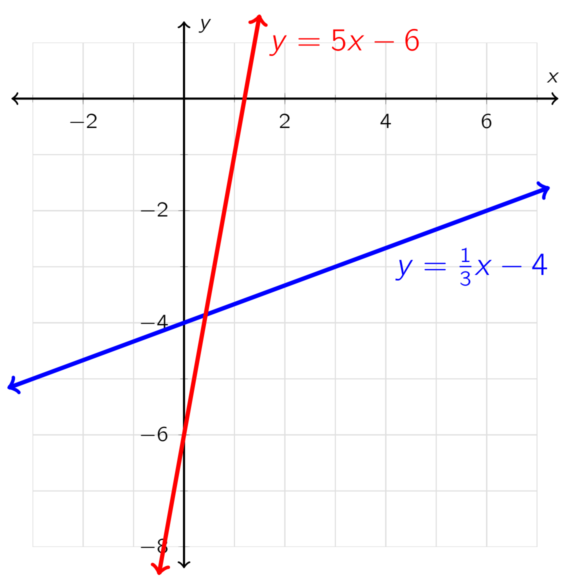
Startup Exploration: Ugly Graphing

Solve the following system by graphing.

$$\begin{cases} y = \frac{1}{3}x - 4 \\ y = 5x - 6 \end{cases}$$

What are the challenges that arise when solving this system by graphing?

The challenge we uncover in the startup exploration is that the intersection point does not have integer coordinates. The x-coordinate is between 0 and 1, and the y-coordinate is between -3 and -4, but beyond that the going is pretty rough even using graphing technology (since the decimal values of the intersection point are not all that pretty).



In this section we will explore the first of two algebraic methods for solving systems of equations. These algebraic approahes will avoid the challenges demonstrated by the startup exploration. The first algebraic technique is the *substitution method*.

We've been substituting numbers for variables for a long time. For example, given $f(x) = 3x + 1$, we know how to find $f(5)$ by substituting 5 for x in the equation:

$$f(x) = 3x + 1 \quad \Longrightarrow \quad f(5) = 3(5) + 1 = 15 + 1 = 16$$

The clever idea behind solving a system using substitution is that rather than just replacing variables with letters, we can substitute one algebraic expression for another.

Substitution Method

A method for solving a system of equations that involves solving one of the equations for one variable and substituting the resulting expression into a different equation.

Example 8.2

Solve the following system using substitution.

$$\begin{cases} y = 3x \\ y = 2x - 4 \end{cases}$$

Solution: We are looking for a point where the two equations have the same x value and the same y value. So, let's assume that the y's in these two equations are equal. If the two y's are equal, then it means "$3x$" is equal to "$2x - 4$" and that is a Level 4 linear equation (one with variables on both sides)!

$$\begin{aligned} 3x &= 2x - 4 && \text{substitution} \\ x &= -4 && \text{SPOE: subtract } 2x \text{ from both sides} \end{aligned}$$

So, if the two y values are equal, as we assumed they were, then it must be that $x = -4$. We just found the x-coordinate of the intersection point! How can we find the y-coordinate?

Notice that the first equation says $y = 3x$. So, if $x = -4$, it must be that $y = 3x = 3(-4) = -12$. So, the solution to the system is the point $(-4, -12)$. Or we can write $\mathcal{S} = \{(-4, -12)\}$.

To check out work, we can either graph the system, or plug the x and y values into the other equation to verify that equality holds.

$$\begin{aligned} y &= 2x - 4 && \text{the second equation we were given} \\ -12 &\stackrel{?}{=} 2(-4) - 4 && \text{substitute our proposed solution} \\ -12 &\stackrel{?}{=} -8 - 4 && \text{simplify right-hand side} \\ -12 &\stackrel{\checkmark}{=} -12 \end{aligned}$$

When we assume that the two y's were equal — as they must be if we have found a solution to the system — then substitution means that we can replace one y with the other.

We can carry this out for the problem in the startup exploration. Since we have two equations in $y =$ format, and since any point that satisfies the system has a y-coordinate that satisfies both equations, we can set the equations equal to one another:

$$\begin{aligned}
\frac{1}{3}x - 4 &= 5x - 6 && \text{substitution} \\
2 &= \frac{14}{3}x && \text{SPOE: variable terms to the right, constants to the left} \\
\frac{6}{14} &= x && \text{MPOE} \\
\frac{3}{7} &= x && \text{simplify the fraction}
\end{aligned}$$

Aha! Even though this x-value is inconvenient to graph by hand (since it's not an integer) or using technology (since it's not a very tidy decimal)... it emerges with no trouble using substitution.

To find the y-coordinate, we use our newly-discovered x-coordinate and one of the two original equations. Why not pick the second equation to minimize the fractions?

$$\begin{aligned}
y &= 5x - 6 && \text{the second equation} \\
y &= 5\left(\frac{3}{7}\right) - 6 && \text{substitute in our new } x\text{-value} \\
y &= \frac{15}{7} - 6 && \text{multiply fractions} \\
y &= \frac{15}{7} - \frac{42}{7} && \text{write with a common denominator} \\
y &= \frac{-27}{7} && \text{simplify the fraction}
\end{aligned}$$

To check our work, we can substitute our candidate values for x and y back into the first equation. Note that since we used the second equation already to figure out the answer, we use the first equation to check it.

$$\begin{aligned}
y &= \frac{1}{3}x - 4 && \text{the first equation} \\
\frac{-27}{7} &\stackrel{?}{=} \frac{1}{3}\left(\frac{3}{7}\right) - 4 && \text{substitute in our candidate solution} \\
\frac{-27}{7} &\stackrel{?}{=} \frac{3}{21} - 4 && \text{multiply fractions} \\
\frac{-27}{7} &\stackrel{?}{=} \frac{3}{21} - \frac{84}{21} && \text{write with a common denominator} \\
\frac{-27}{7} &\stackrel{?}{=} \frac{-81}{21} && \text{subtract fractions} \\
\frac{-27}{7} &\stackrel{\checkmark}{=} \frac{-27}{7} && \text{simplify the right-hand side}
\end{aligned}$$

So, we have found our solution $\mathcal{S} = \left\{\left(\frac{3}{7}, -\frac{27}{7}\right)\right\}$.

In the two examples we have seen so far, both equations have been in $y =$ format. This is not a requirement for using substitution, as the following two examples will demonstrate.

Example 8.3

Solve the following system using substitution.

$$\begin{cases} y = 3x - 5 \\ x - y = 4 \end{cases}$$

Solution: Here, we have one equation in point-intercept form, and one equation in standard form. No problem! We simply substitute y from the first equation for y in the second equation. It's a good idea to use patentheses when doing the substitution: watch what happens with the negative signs in this example.

$$\begin{aligned} x - y &= 4 && \text{start with the second equation} \\ x - (3x - 5) &= 4 && \text{substitue } y \text{ from first equation, using parentheses} \\ x - 3x + 5 &= 4 && \text{distributive property} \\ -2x + 5 &= 4 && \text{combine like terms} \\ -2x &= -1 && \text{SPOE} \\ x &= \frac{1}{2} && \text{DPOE} \end{aligned}$$

This tells us that if the y's are equal, then it must be that $x = \frac{1}{2}$. To find the y-coordinate, we substitute the x value we just found back into one of the original equations. Let's choose the second equation, which seems a bit easier:

$$\begin{aligned} x - y &= 4 && \text{the second equation} \\ \frac{1}{2} - y &= 4 && \text{substitue } x = \tfrac{1}{2}\text{, which we just computed} \\ -y &= \frac{7}{2} && \text{SPOE} \\ y &= -\frac{7}{2} && \text{MPOE} \end{aligned}$$

So, $\mathcal{S} = \left\{\left(\frac{1}{2}, -\frac{7}{2}\right)\right\}$. To check we can plug these values back into the other equation (the first equation, in this case, since we used the second equation to find y).

$$\begin{aligned} y &= 3x - 5 && \text{the first equation} \\ -\frac{7}{2} &\stackrel{?}{=} 3\left(\frac{1}{2}\right) - 5 && \text{substitue our proposed solution} \\ -\frac{7}{2} &\stackrel{?}{=} \frac{3}{2} - 5 && \text{simplify on the right-hand side} \\ -\frac{7}{2} &\stackrel{?}{=} \frac{3}{2} - \frac{10}{2} \\ -\frac{7}{2} &\stackrel{\checkmark}{=} \frac{-7}{2} \end{aligned}$$

Notice that after our substitution step, we've been getting linear equations like the ones we saw in chapter 5.

In the first example in this section, we got an equation with the variable x on both sides. Recall that when equations have variables on both sides, we sometimes find that the equation had one unique solution, sometimes "no solution", and sometimes "infinitely many solutions". Do you see how these correspond to the different possible solutions to a linear system?

The key to the substitution method is that we need at least one equation to be in $x =$ or $y =$ form. What if neither equation is in this form? We can use our skills at transforming formulas to create what we want!

Example 8.4

Solve the system by substitution.

$$\begin{cases} 3x - 12y = 15 \\ x - 4y = 7 \end{cases}$$

Solution: We could transform either equation, but notice that there's less work to do in the second equation, since the x term has coefficient 1. So, let's transform the second equation to $x = 7 + 4y$ using APOE. Then, we can substitute that expression into the first equation.

$$\begin{aligned} 3x - 12y &= 15 && \text{the first equation} \\ 3(7 + 4y) - 12y &= 15 && \text{substitute our expression for } x \\ 21 + 12y - 12y &= 15 && \text{distributive property on the left-hand side} \\ 21 &= 15 && \text{D'oh.} \end{aligned}$$

We have arrived in one of those impossible situations that tells us the equation we created has no solution. In this case, it means that the system has no solution. In other words, that these lines were parallel all along. $\mathcal{S} = \varnothing$

That was the first "special case". The other special case is when the two lines completely overlap. What do you suppose will happen when we solve the system that lets us know that we're in this special case?

8.4 Solving Linear Systems By Elimination

Startup Exploration: Ugly Substitution

Study the following system of linear equations.

$$\begin{cases} 4x + 3y = 22 \\ 3x + 5y = 11 \end{cases}$$

What challenges would you face if you tried to solve this system by graphing? What challenges would you face if you tried to solve it by substitution?

The equations in the startup exploration would require quite a bit of work to graph them accurately by hand (even using our x- and y-intercept strategy for graphing lines in standard form), and we'd have to do some work so that one could be substituted into the other. The second algebraic approach to solving a system, called the **elimination method**, addresses these concerns.

This technique is particularly helpful when the given equations are both in standard form. The premise behind elimination is that we use the properties of equality to, say, add two equations together. Recall that the addition property of equality says that we can add the same quantity to both sides of an equation. If we know two things are equal, we can add one thing to one side of an equation and the other thing to the other side, and still maintian equality.

Example 8.5

Solve the following system using elimination.

$$\begin{cases} y = 3 \\ 5x - y = 12 \end{cases}$$

Solution: The first equation tells us that y is equal to 3. Of course, we could add y to both sides of an equation... or we could add 3 to both sides of an equation...but since we know y and 3 are equal we can add y to one side of an equation and 3 to the other side of that equation!

In other words, we can add y to the left-hand side of the second equation we are given while simultaneously add 3 to the right-hand side. Since we know $y = 3$, we know equality will be maintained! Watch what happens:

$$\begin{aligned} 5x - y &= 12 && \text{the second equation} \\ 5x - y {\color{red}+y} &= 12 {\color{red}+3} && \text{add on the first equation} \\ 5x &= 15 && \text{combine like terms: } y \text{ has been eliminated!} \\ x &= 3 && \text{DPOE} \end{aligned}$$

Having found x, and having y given in the original system, we have $\mathcal{S} = \{(3, 3)\}$.

Elimination Method

A method for solving a system of equations that involves adding two equations to eliminate a variable.

The method is called "elimination" because it eliminates a variable.[1]

Example 8.6

Solve the following system by elimination.

$$\begin{cases} 2x + 2y = 7 \\ 3x - 2y = 8 \end{cases}$$

Solution: Notice how the coefficients on the y terms are opposites. If we add these two equations together, the y's will be eliminated:

$$\begin{array}{rr} & 2x + 2y = 7 \\ + & 3x - 2y = 8 \\ \hline & 5x + 0y = 15 \\ & 5x = 15 \\ & x = 3 \end{array}$$

We then substitute $x = 3$ back into one of the equations to find $y = \frac{1}{2}$. So, $\mathcal{S} = \left\{\left(3, \frac{1}{2}\right)\right\}$.

The key to the elimination approach is to "add the equations together" so that one variable vanishes, just as the y disappeared on the third in the previous example. In order to faciliatate this elimination, one or both of the equations must sometimes be multiplied by a constant before we add them (an application of MPOE).

[1] This name is also related to a technique from the study of matrices called *Gaussian elimination*.

This approach is required to handle the startup exploration. Recall, that in that problem we were asked to solve the following system by elimination.

$$\begin{cases} 4x + 3y = 22 \\ 3x + 5y = 11 \end{cases}$$

Adding these two equations won't eliminate any variables, as we have encountered in the earlier examples. But, with a few clever applicatons on MPOE, we can put ourselves into exactly the situation that we desire.

Let's make it our goal to eliminate x. What could we do to these equations so that we have equivalent equations but ones in which the x terms have opposite coefficients?

One way (but not the only way) to accomplish this is to multiply the first equation by 3 and multiply the second equation by -4.

$$\begin{aligned} 4x + 3y = 22 \quad &\xrightarrow{\text{multiply through by 3}} \quad 12x + 9y = 66 \\ 3x + 5y = 11 \quad &\xrightarrow{\text{multiply through by -4}} \quad -12x - 20y = -44 \end{aligned}$$

Then, we add equations:

$$\begin{array}{rr} & 12x + 9y = 66 \\ + & -12x - 20y = -44 \\ \hline & -11y = 22 \\ & y = -2 \end{array}$$

Substituting $y = -2$ into the second of the original equations, we get $3x + 5(-2) = 11$. This implies that $x = 7$. So, $\mathcal{S} = \{(7, -2)\}$.

8.5 Applications of Systems

[TODO] Systems: The word problems in this chapter should all connect into one story, IMHO... perhaps advancing the overall story.

"A train leaves New York traveling south at 180 miles per hour. . ." If you were to read that sentence out loud in a room full of adults, you might cause some folks to start sweating and mumbling nervously to themselves. People all over the world, for generations, have been scarred by algebra word problems.

But, word problems have a bad reputation.

Yes, word problems require us to ponder a bit, decipher the information we're given, and stumble around as we figure out how to proceed. But this is the process of problem solving! The algebra skills we have learned so far, along with a little determination and creativity, will serve us well as we tackle these algebra classics.

We discuss several kinds of problems below. Our general pattern is this: We'll think about the unknown quantities and relevant facts given in the problem. We'll choose variables to represent the unknowns and model the situation using equations. (Drawing pictures and making charts might help!) Then, we'll solve the equations and check to ensure your answer makes sense.

As we go along, and as you work on various assignments, remember to think back over these examples. If you have solved a similar problem before, that will give you a place to start when faced with something new.

Two unknowns, two facts

The first examples we did with substitution and elimination are of this type. We are given two unknowns and two different facts about them. The facts usually have different units, which hint at what two equations you need to write for your model.

One way to approach this type of problem is by creating a table to organize the data.

Evil Entertainment. . . Eviltainment?

Yeardleigh receives a shipment of entertainment centers, the basic model weighing 30 kilograms each and the deluxe weighing 50 kilograms each, has a total weight of 880 kilograms. If there are 20 centers altogether, how many weigh 50 kilograms?

The two unknowns are the number of 30kg and 50kg centers. One set of facts has to do with quantity, the other has to do with the weight. We make these into which are rows and columns in the table. I added "total" columns and rows. Let's use T to represent the number of 30kg centers, and F to represent the number of 50kg centers.

	Number of centers	Weight per unit	Total Weight
30kg centers	T	30	$30T$
50kg centers	F	50	$50F$
Combined	20	–	880

Once the table is set up, out two equations are there in the second and fourth columns.

$$\begin{cases} T + F = 20 \\ 30T + 50F = 880 \end{cases}$$

We can now solve this system using our favorite method. It looks like a good candidate for elimination! (Why?)

Mixture problems

The type of mixture problem we have previously solved were very straight forward. Every number we needed was given to us directly in the problem. This is a different type of problem. There is information in the problem that is not directly stated that we will have to infer. This is where the table is really helpful.

Evil Trail Mix

YeardleighCorp opens health food stores, which sell an Evil Trail Mix of raisins and roasted nuts. Raisins sell for \$3.50 per kg, and roasted nuts sell for \$4.75 per kg. How many kg of each should be mixed to make 20 kg of Evil Trail Mix worth \$4.00 per kg?

Our unknowns are number of kg of raisins (let's call that R) and number of kg of nuts (let's call that N). We also have the prices of each ingredient per kg, but nothing about a total cost.

	Amount (kg)	Unit price	Total price
Raisins	R	3.50	$3.50R$
Nuts	N	4.75	$4.75N$
Mixture	20	4.00	??

How can we figure out total cost? Well, look at that row of the table: 20 kg of Evil Trail Mix at \$4.00 per kg = \$80 worth of trail mix! We can calculate this value, even though that information is not given directly in the problem. Sneaky!

$$\begin{cases} R + N = 20 \\ 3.50R + 4.75N = 80 \end{cases}$$

And now, we can solve the system. Once again, this looks like a job for elimination. (Why?)

RTD: Same direction

Rate-time-distance problems (RTD for short) come in a variety of flavors. In "same direction" RTD problems, the travelers are – surprise! – going in the same direction. The often involve a "race" of some kind where someone gets a head start, and our task is to determine when someone catches up or passes the other.

It will be helpful to draw a simple picture at the start to understand how the problem is set up. Many times, there is something not directly stated that is important to the solution, but which you can figure out from the situation. The picture can help you with that.

Same Direction

At 5:00 pm Bob leaves the Chtulhu-Chip factory driving 25 miles per hour. At 5:30, Yeardleigh leaves the Cthulhu-Chip factory and follows the same route, driving 40 miles per hour. At what time does Yeardleigh pass Bob?

The problem is asks when Yeardleigh will pass Bob. In order to know when she will pass, we need to know when they have gone exactly the same amount of distance from the factory (this is the moment she passes him). So we'll write two distance equations, one for Bob and one for Yeardleigh, and compare them to try and figure out at what time their distances from the factories are equal.

[TODO] Systems: Some pictures of the word problems would be helpful.

Our table columns are – naturally enough – rate, time, and distance. The rows deal with the people driving. The problem asks a "when" question, so we must be solving for time. Let's use t to represent the amount of time Bob has been driving, in hours.

Note that we didn't make the variable represent actual time "on the clock". (Why not? What challenges would that introduce into the mathematics?)

Yeardleigh started 30 minutes later, which means she has 30 minutes less travel time than Bob. Note that we have to be careful about units: 30 minutes is 1/2 of an hour. Since the speeds are in miles *per hour*, it makes sense to work in hours: $t - \frac{1}{2}$. If we want to use $t30$ instead, we can. . . but we'll have to also convert the given speeds into miles *per minute*.

	Rate (mph)	Time (hours)	Distance (miles)
Bob	25	t	$25t$
Yeardleigh	40	$t - \frac{1}{2}$	$40\left(t - \frac{1}{2}\right)$

$$\begin{cases} d = 25t \\ d = 40\left(t - \dfrac{1}{2}\right) \end{cases}$$

Solving this system (substitution looks like a good choice!) will give us a value for t. What does that value represent? How can we use it to determine when (on the clock) Yeardleigh will pas Bob?

RTD: Opposite directions

Here we might have travelers starting from the same location and traveling away from one another (in which case you usually are asked to determine when they will be a certain distance apart). Or, we might have travelers who are coming from two distinct locations and moving towards each other (in which we are usually asked to figure out when or where they will meet).

A common mistake with this second type is assuming that the travelers will meet at exactly the half-way point. They don't necessarily. It depends on when they leave and speed at which they traveling!

Opposite Directions

Yeardleigh hamster-naps Feta and holds her for ransom. The twins agree to meet in a neutral location. Bob and Yeardleigh each set out at noon from points 60 km apart and drive toward each other, meeting at 1:30pm. Bob's speed was 4 km/h greater than Yeardleigh's. How fast was each of them driving?

We could choose either Bob's speed or Yeardleigh's speed as the reference point. Let's use Yeardleigh's speed and call it v (for velocity). Then, Bob's speed is 4 km/h greater than that.

	Rate (mph)	Time (hours)	Distance (miles)
Yeardleigh	v	1.5	$1.5v$
Bob	$v+4$	1.5	$1.5(v+4)$

Now, to figure out how to make that table into some helpful equations. Since we have one variable, we only need one equation. The missing key component is that the distance Bob and Yeardleigh traveled together is 60 km, which gives us the extra piece of the puzzle: The two distances add up to 60 km:

$$1.5v + 1.5(v+4) = 60$$

RTD: Roundtrip

"Round trip" means that someone travels from a given location to some other location and back again, as in the following example.

Round Trip

Yeardleigh uses the ransom money from hamster-napping Feta to go on an evil ski trip. An evil ski lift carries Yeardleigh up the slope at the rate of 6 km/h, and then she skies back down the same slope at 34 km/h. The round trip takes 30 minutes. How long is the ski slope, and much time does it take

Yeardlight to ski down it?

Let's let t represent the amount of time that Yeardleigh spends skiing. The whole trip takes 30 minutes (that's $\frac{1}{2}$ hour, remember to watch out for the units!) so her time on the chair lift is $\frac{1}{2} - t$. Here we chop the total time into two pieces using one variable, just like we did with distance in the last problem.

	Rate (km/h)	Time (hours)	Distance (km)
On the ski lift	6	$\frac{1}{2} - t$	$6\left(\frac{1}{2} - t\right)$
Skiing	34	t	$34t$

In a round trip problem, the traveler goes over the same path twice. So, the distance that Yeardleigh travels on the ski lift, is the same as when she is skiing. We don't know what that distance is (yet!) but we know the two distances in our table above are the same. So, our equation is:

$$6\left(\frac{1}{2} - t\right) = 34t$$

Solve this, and we'll get the amount of time she was skiing. Then, we can use this time, together with her skiing speed, to figure out the distance traveled.

Uniform motion in a current

Don't let that complicated-sounding name fool you, problems of this type are not that complicated to solve.

Have you ever tried to walk *up* the *down* escalator at the mall? (Be honest!) When you walk down the down elevator – that is, when you walk in the same direction that the escalator is moving – the motion of the escalator helps you to go faster than if you were just walking down regular stairs. On the other hand, if you try walk in the wrong direction on an escalator, the motion of the escalator will make you go slower than you would walk on regular stairs.

This is the idea of motion in a current. When you go "with the current" (or "downstream", if you're floating on a river) the current adds to your usual rate of motion. Going "against the current" (or "upstream"), the current reduces your usual rate of motion.

Motion in a Current

Bob decides to go on vacation. He takes a day trip on a river boat. The boat travels 60 km upstream (against the current) in 5 hours. The boat travels the same distance downstream in 3 hours. What is the rate of the boat in still water? What is the rate of the river's current?

Notive that Bob makes a round trip journey. So, the distance upstream is the same as the distance downstream. We don't know the speed of the water or the speed of the boat. So, let's let b represent the speed of the boat, and c represent the speed of the current.

The boat goes $b - c$ km/hour upstream, because the current is taking away from how fast the boat can go. The boat will go $b + c$ km/hour downstream, because the current will help the boat go faster!

	Rate (km/h)	Time (hours)	Distance (km)
Upstream	$b-c$	5	60
Downstream	$b+c$	3	60

Our system is:

$$\begin{cases} 5(b-c) = 60 \\ 3(b+c) = 60 \end{cases}$$

To solve this, we'll need the distributive property and the elimination method!

Work problems

In a work problem, a group of workers split up a job. The key is knowing that everyone does a fraction of the job, and that all together they complete one whole job. We will use an equation very similar to DRT:

$$\text{work rate } \times time = \text{ work done.}$$

The "work rate" we will think of "number of jobs per unit of time". If it takes me 3 days to paint an apartment, my work rate for painting is "one third of an apartment per day".

Take a whack at this example:

Work

Ivan can chop a cord of wood* in 4 days, and his father, François, can chop a cord of wood in 2 days. How long will it take them to split a cord of wood if they work together?

(One cord, which is a unit of measure of volume for firewood in the US and Canada, is about 128 cubic feet. . . but that's not really important to this problem.)

If we think of the "job" as "chopping one cord of wood", then Ivan's work rate is "1 job in 4 days", or "$\frac{1}{4}$ job per day". Similarly, François's work rate is "$\frac{1}{2}$ job per day". When the two guys work together, they will finish the job in some amount of time. Let's call that t. The rows in our table are for the individual workers.

	Work rate	Work time	Amount done
Ivan	$\frac{1}{4}$	t	$\frac{t}{4}$
François	$\frac{1}{2}$	t	$\frac{t}{2}$

Together, they complete 1 whole job (they chop one whole cord of wood). This gives us the equation:

$$\frac{t}{4} + \frac{t}{2} = 1$$

[TODO] Systems: Some conclusion might be good?

Still need to find a quote that works for this chapter. In the meantime, we have this.

Author
Description of author

Chapter 9

Inequalities

Every chapter should have a lead paragraph – even just a short one – that appears before the first heading. This is a placeholder paragraph which will at some point be replaces by actual content.

9.1 Equations Versus Inequalities

Startup Exploration: Honest Senators

Imagine that the following statements are released from a government watchdog group: "(1) There exists at least one honest senator. (2) Given any two senators, at least one of them is dishonest."

If both of these statements are true, what can we say about our 100 senators? Determine the number of honest senators or, explain why we don't have enough information to determine exactly how many are honest.

An equation states that two expressions are equal and thus includes an equal sign "=". When we solve an equation (by using one of the properties to simplify it), we create a series of equivalent equations until we can determine the solution set. Recall that the solution set is the collection of all the numbers that make the original equation "true".

Inequalities express an unequal relationship and thus include a mathematical symbol of inequality. Rather than "x is equal to 5" we might have "x is less than 5". To solve an inequality is to find all of the values that make it true. But whereas an equation may have only one solution, there could be many values that make an inequality true. Perhaps infinitely many! All of these are part of the solution set.

9.1.1 Inequality Symbols

Equations all include the equal sign, but we have more options for expressing inequality:

$\neq$	not equal to
$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to

Using these symbols we can write true and false statements. The statement $4 < 5$ is true statement, and the statement $3 > 10$ is false.[1] Note that the statement $12 < 12$ (12 is less than 12) is false. In contrast, the statement $12 \leq 12$ (12 is less than or equal to 12) is true, because the "or equal to".

The math sentence $x < 5$ is an inequality that has infinitely many solutions. For example, we can replace x with 4, π, 3, 2, 1.889, 1, $\frac{1}{2}$, 0, $-6\ldots$ in fact, every number x that is less than the "endpoint" 5 is included in the solution set! But again, 5 itself is not included in the solution set for this inequality.

Because of their behavior around the endpoints, the symbols $<$ and $>$ are said to be "exclusive inequalities", meaning they do not include the endpoint as a solution: $5 < 5$ is a false statement. These are often described as "strictly less than" or "strictly greater than". In contrast, the symbols $\leq$ and $\geq$ are said to be "inclusive" because they do include the end point as a solution: $5 \leq 5$ is a true statement.[2]

[1] Remember your elementary school days: the symbols are like hungry alligators who open their mouths to eat the larger number. Nom, nom, nom.

[2] Another way to describe these "or equal to" inequalities inequalities is using the terms "at most" or describe "less than or equal to", and "at least" to describe "greater than or equal to". For example, if I ask you to give me "at least 5 dollars", then I'm asking you for "greater than or equal to 5 dollars".

9.2 One-variable Inequalities

Startup Exploration: Wire Triangle

A nine-inch piece of wire is bent at two points such that its ends come together to form a triangle. If the bending points must be on the inch marks, how many possible choices of bending points are there?

Consider the equation $3x = 12$. This equation is true when x has the value 4 and we say the solution set $\mathcal{S} = \{4\}$. Now consider the inequality $3x \leq 12$. What values of x make this inequality true? The solution set will be the set of all of those values.

This means that the solution set contains "all real numbers less than or equal to 4" (notice that 4 is included in the solution set). To write this in set notation, we intorduce a few new symbols:

$$\mathcal{S} = \{x \in \mathbb{R} \mid x \leq 4\}$$

The symbols in this set notation can all be translated into regular English. The symbol $\in$, which looks like a little stylized 'e', means "is an element of". So, the phrase $x \in \mathbb{R}$ means "x is an element of $\mathbb{R}$", or in other words: "x is a real number".

The little vertical line has a meaning, too! We can think of the bar as saying "such that" of "for which". So what we have is a math sentence that reads "$\mathcal{S}$ is the set of real numbers x such that x is less than or equal to 4", or a bit more succinctly "all real numbers less than or equal to 4".

9.2.1 Graphing One-variable Inequalities

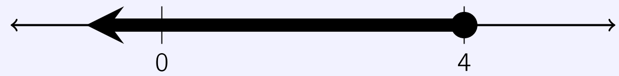
Instead of writing the set notation, another way to show the solution to a one-variable inequality is to draw a graph of the solution. These graphs aren't like the two-dimensional graphs of linear functions. Since we have just one variable, we'll have a one-dimensional graph! Rather than graphing on the coordinate plane, we draw these graphs on a number line.

These graphs may seem pretty basic, but stay tuned! In algebra 2 we will solve "combined" and "polynomial" inequalities and we won't be able to figure out the answers without graphs like these!

Example 9.1

Draw a graph of $x \leq 4$.

Solution: First, we'll show the graph, and then we'll explain what decisions we made in drawing it.



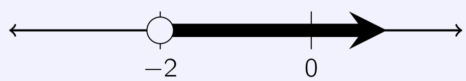
We first found the boundary (which is 4) and plotted it on the number line using a filled in dot. Then we drew an arrow (a ray) pointing to the left because we are interested in values that are less than 4, in addition to 4 itself. We also placed the zero on the number line, for reference.

Contrast the previous example with the next example, and you will probably have a good idea about how to graph any one-variable inequality!

Example 9.2

Draw a graph of $x > -2$.

Solution: Here's the graph. What features are different compared to the previous example?



First, we found the boundary (it's -2) and plotted it on the number line, along with 0. The boundary is excluded from the solution set, so we used an hollow circle to show that -2 itself is not part of the solution. Then we used an arrow (a ray) pointing to the right because we are interested in values that are greater than -2.

The following comments may not be necessary, given the previous to examples, but we'll summarize them here for clarity. There are two main things to consider when graphing a one-variable inequality.

Consideration #1: Boundary. What is the greatest or least possible value of the solution, and is that point included in the solution set?

If we have an inclusive inequality ($\leq$, $\geq$, and $=$), then the boundary point is included in the solution set. On the graph, we use a filled in circle or "closed" endpoint to show that the boundary is part of the solution. We call the boundary point an *inclusive boundary*.

If, on the other hand, we have an exclusive inequality ($<$, $>$, and $\neq$), then the boundary point is not included in the solution set. On the graph we draw a hollow or "open" endpoint to show that the boundary is not part of the solution. We call the boundary an **exclusive boundary**.

Consideration #2: Direction. Are the solutions greater than the boundary (stretching to the right, towards positive infinity) or are they less than the bounday (stretching to the left towards negative infinity)? We draw a heavy arrow in that direction.

We always label the boundary point, and also place the zero on the number line for reference.

9.2.2 (;,;) Interval Notation

In addition to set notation and a number line graph, there is a third way to describe an inequality. In many higher level math classes (like algebra 2 and, later, calculus), we use a technique called *interval notation*. With interval notation, we describe the endpoints of a region on the number line.

Inclusive boundaries are denoted with square brackets: [and]. Exclusive boundaries are denoted using round brackets: (or). Lines that extend forever have no endpoint, so we use positive infinity ∞ or negative infinity $-\infty$ to indicate that the set extends forever in the positive (or negative) direction.

In the first example above, we have $x \leq 4$. To write this in interval notation, we write $x \in (-\infty, 4]$. To write the solution to the second example, $x > -2$, in interval notation, we write $x \in (-2, \infty)$.

Try drawing the graph corresponding to $x \in [8, \infty)$? What about $x \in (-\infty, -10)$? What do you suppose the graph of $x \in [-3, 7)$ might look like?

9.3 Solving Inequalities

Startup Exploration: TODO

TODO

When we solve an equation using the POEs, we create a series of equivalent equations until we can determine the solution set. The big question of this section is: Can we do something similar for inequalities? What does it mean to have *equivalent inequalities*?

Equivalent inequalities is not an oxymoron.[3] It just means that two inequality statements have the same solution set! For example, $3x < 12$ and $x < 4$ are equivalent because the exact same values of x will work in both statements. They have the same graph!

But, how can we transform an inequality into an equivalent inequality? Can we use the same properties as we used for solving equations? In essence: are there Properties of Inequalities like there are Properties of Equality (POEs)?

Strictly seaking, an inequality doesn't only state that two numbers are not-equal.[4] An inequality really shows that two numbers appear in a certain *order on the number line*. If a certain inequality states a "less than" relationship, we want to find properties that maintain that.

Since we are looking for rules which "maintain order", we will call these rules the "Properties of Order", or POOs.[5]

9.3.1 Properties of Order (POOs)

So what are the POOs, and do they work like POEs? Yes... almost.

Suppose we start with an inequality that we know is true, say, $5 < 6$. What operations can we perform to this inequality that maintain its ordered relationship?

---

[3] An oxymoron is a figure of speech that brings together two things that appear to be contradictory. For example, the titles of the movie "Night of the Living Dead" and the song "Along Together" contain oxymorons.

[4] Okay, yes, this is exactly what the $\neq$ symbol does. But, we're not going to do much with the $\neq$ symbol. Solving problems with $\neq$ isn't usually very interesting, as we'll see later in this chapter.

[5] We considered using "POI" to stand for "Property of Inequality", but POO is both more mathematically accurate, and a lot funnier.

Could we add (or subtract) the same things on both sides and maintain order? To take just one example, adding 3 to both sides gives $5+3<6+3$, which maintains order since $8<9$. If we now add -8 to both sides we have $0<1$, and order is again maintained.

In fact, adding the same thing to both sides does nothing to change their relative order on the number line. Adding simply translates the original inequality to the right or left on the number line. (Note that translations to the left happen when adding a negative number, which is the same as subtracting!)

Properties of Order: Addition and Subtraction

The **addition property of order** (APOO) states that for all real numbers a, b, and c

$$\text{if } a<b, \text{ then } a+c<b+c$$

The **subtraction property of order** (SPOO) states that for all real numbers a, b, and c

$$\text{if } a<b, \text{ then } a-c<b-c$$

Note! Rather than write out the same sentence with all the different inequality symbols, we simply note here that APOO and SPOO hold for $a \leq b$, $a>b$, and $a \geq b$.

What about multiplication and division? If we take our sample inequality $5<6$ and multiply both sides by 4, we have $5 \cdot 4<6 \cdot 4$, which maintains order, since $20<24$. If we now divide both sides by 2 we have $10<12$, and order is maintained.

This is encouraging! But... we have to be a bit cautious. Notice that we have not explored multiplication by a negative number. Observe that if we start with $5<6$ and multiply both sides by -4, we get -20 and -24 and $-20>-24$ — not the other way around. The order of the inequality has changed!

There's a geometric interpretation to this: Multiplication (and division) perform a scaling of the points away from (or towards) zero. Multiplying by a positive number performs that scaling on the same side of zero as the original numbers. But, multiplying by a negative number first reflects our inequality about zero. That reflection changes the order of the numbers!

Properties of Order: Multiplication and Division

The **multiplication property of order** (MPOO) states that for all real numbers a, b, and c:

$$\text{if } a<b \text{ and } c>0, \text{ then } ac<bc$$

$$\text{if } a<b \text{ and } c<0, \text{ then } ac>bc$$

The **division property of order** (DPOO) states that for all real numbers a, b, and c:

$$\text{if } a<b \text{ and } c>0, \text{ then } \frac{a}{c}<\frac{b}{c}$$

$$\text{if } a<b \text{ and } c<0, \text{ then } \frac{a}{c}>\frac{b}{c}$$

Note that, as before, MPOO and DPOO behave the same for all of the inequality symbols, not just $<$.

For the most part, we can solve an inequality just like we solve an equation. The addition and subtraction POOs are no different than the POEs. However, we have to be a bit more careful with multiplication and division. When multiplying or dividing both sides of an inequality by a negative number, you must change the direction of the inequality.

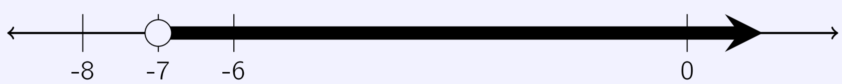
Example 9.3

Solve and graph: $-3x < 21$.

Solution: We can divide both sides by -3 and change the direction of the inequality: $x > -7$. To check, we might try plugging in values that are on both sides of the boundary (like -8 and -6), into the original inequality:

$-3 \cdot -8 = 24$ and $24 > 21$, so -8 *does not* make the original inequality true. The solution set cannot include this value.

$-3 \cdot -6 = 24$ and $18 < 21$, so -6 *does* make the original inequality true. The solution set must include this value, so the solutions must extend from the boundary (-7) toward to the right (hitting -6 and all the points toward positive infinity).



9.3.2 What About the Field Axioms?

Recall that the field axioms were our other tools for simplifying equations, but they were used on one side of an equation only. Since they are not performed on both sides, there's no risk of them messing with order.

So, we can distribute and combine like terms without it having any effect on the truth of the inequality. We can use them to simplify, just like we did when we were solving equations

Example 9.4

Solve and state the solution set: $4x + 3 - 2(3x + 1) > 13$.

Solution: We simplify on the left-hand side first, and then apply the POOs.

$$\begin{aligned} 4x + 3 - 2(3x + 1) &> 13 \\ 4x + 3 - 6x - 2 &> 13 && \text{distributive propoerty (check your signs!)} \\ -2x + 1 &> 13 && \text{combine like terms} \\ -2x &> 12 && \text{APOO} \\ x &< -6 && \text{MPOO with a negative number} \end{aligned}$$

In set notation, we write $\mathcal{S} = \{x \in \mathbb{R} \mid x < -6\}$.

9.3.3 Special Case Solutions

As with equations, inequalities with variables on both sides can have no solution or all real numbers as its solution.[6] The variables may vanish, as they sometimes do, but now we have to look at the inequality that is left over to determine whether it is always true or always false. If the remaining inequality is a true statement, then our solution set is *all real numbers*. If the remaining inequality is false, then our solution set is empty.

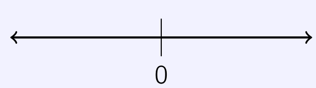
Example 9.5

Solve and state the solution set: $6x - 2 > 2(3x + 1)$.

$$\begin{aligned} 6x - 2 &> 2(3x + 1) \\ 6x - 2 &> 6x + 2 && \text{distributive propoerty} \\ -2 &> 2 && \text{subtract } 6x \text{ from both sides (APOO)} \end{aligned}$$

When we examine the remaining inequality, we see that it's false: -2 is not greater than 2. Therefore, the original inequality has no solution. To write the answer in set notation: $\mathcal{S} = \varnothing$.

The graph of "no solution" is just an empty number line. In that case, we can either write "no graph" or "empty graph" or just draw a blank number line. Boring, perhaps, but mathematically accurate.



The mirror-image situation, as you might expect, is demonstrated by the following example.

[6] We can't simply say "infinitely many solutions", since an inequality like $x < 0$ has infinitely many solutions.

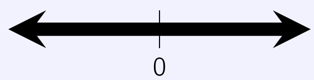
Example 9.6

Solve and state the solution set: $5x + 4 \geq 2x + 3x - 1$.

$$\begin{aligned} 5x + 4 &\geq 2x + 3x - 1 & \\ 5x + 4 &\geq 5x - 1 & \text{combine like terms} \\ 4 &\geq -1 & \text{subtract } 5x \text{ from both sides (APOO)} \end{aligned}$$

This time when we interpret the remaining inequality, we see that it's true: 4 is greater than or equal to -1. Therefore, every real number is a solution to the original inequality. To write the answer in set notation: $\mathcal{S} = \mathbb{R}$.

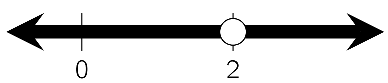
The graph of all real numbers is a graph with the entire number line colored in.



9.3.4 Not-Equal

We haven't talked much about inequalities that use $\neq$, but that's because they're pretty uninteresting. For example: Solve $8x + 3 \neq 19$. We can subtract 3 from both sides, then divide both sides by 8, yielding: $x \neq 2$. So, the only number that *doesn't* work in this inequality is when $x = 2$.

In set notation, we write: $\mathcal{S} = \{x \in \mathbb{R} \mid x \neq 2\}$. The graph is the whole number line, except that single point:



9.4 Two-Variable Inequalities

Startup Exploration: TODO

TODO

Having spent all of that time talking about one-variable inequalities, we can now get on with something more interesting: turning a linear equation like $y = -2x + 6$ into a two-variable inequality like $y \le -2x + 6$.

What does this inequality mean? What counts as solution? To explore this a bit more, suppose we want to check whether the random points $(4, 5)$ and $(-3, 2)$ satisfy the inequality $y \le -2x + 6$.

To accomplish this, we substitute the values $x = 4$ and $y = 5$ into the given inequality, and then see whether the resulting inequality is true or false:

$$\begin{aligned}
y &\le -2x + 6 && \text{original inequality} \\
5 &\overset{?}{\le} -2(4) + 6 && \text{substitute candidate point} \\
5 &\overset{?}{\le} -8 + 6 \\
5 &\not\le -2
\end{aligned}$$

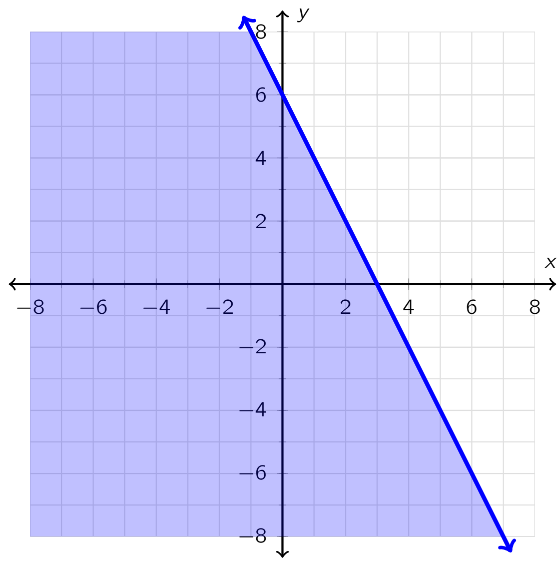
The final inequality is false, so the answer is no, the point $(4, 5)$ is not part of the solution set to the inequality $y \le -2x + 6$. If we test the other point, $(-3, 2)$, we have:

$$\begin{aligned}
y &\le -2x + 6 && \text{original inequality} \\
2 &\overset{?}{\le} -2(-3) + 6 && \text{substitute candidate point} \\
2 &\overset{?}{\le} 6 + 6 \\
2 &\le 12
\end{aligned}$$

This statement is true, and so the points $(-3, 2)$ is part of the solution set. But our work with one-variable inequalities suggests that this probably isn't the *only* point in the solution set. What other points might be included?

Note that the inequality is inclusive, so any point on the line $y = -2x + 6$ must be part of the solution set (that's the line y *equals* $-2x + 6$).

In addition, the solution set must include any point that is "less than" that line. In other words, any point on the same side of the line as $(-3, 2)$, the point that we checked above, and which we know is in the solution set. So, our solution must look like the graph below.



So, graphs of two-variable inequalities share some of the features of one-variable inequalities. We will have a boundary, but rather than a dot or enpoint, it will be a line (or a curve). We will shade a section of the graph to show all the solutions, but this will be a whole region of the plane, rather than a ray on the number line.

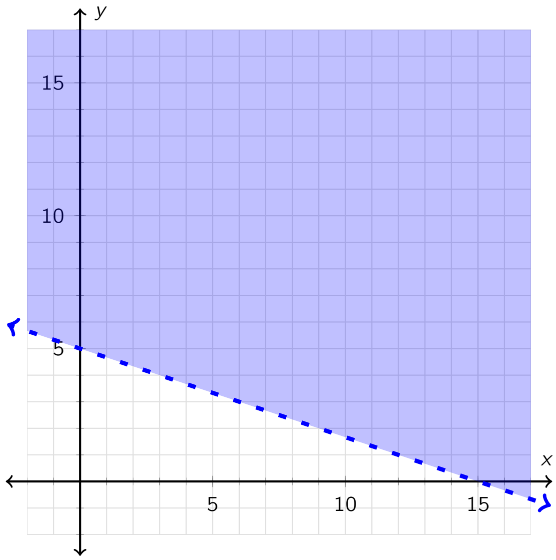
Let's work through another example in detail: graphing the inequality

$$x + 3y > 15.$$

Here we have a line in standard form. An easy way to graph standard form is to find the x- and y-intercepts. When we plug in $x = 0$, we have $y = 5$. When we plug in $y = 0$, we have $x = 15$. So, we can draw our graph by connecting the points $(0, 5)$ and $(15, 0)$.

Before we draw the line, though, note that we have an exclusive inequality: "greater than". For one-variable inequalities, we drew a hollow dots which allowed us to show where the boundary was without actually including the boundary itself. In the graph of a two-variable inequality, we will draw a dotted or dashed line to show that the line itself is not included in the solution set.

Since our inequality is "greater than" it seems reasonable to shade the region above the line. The resulting graph is shown below.



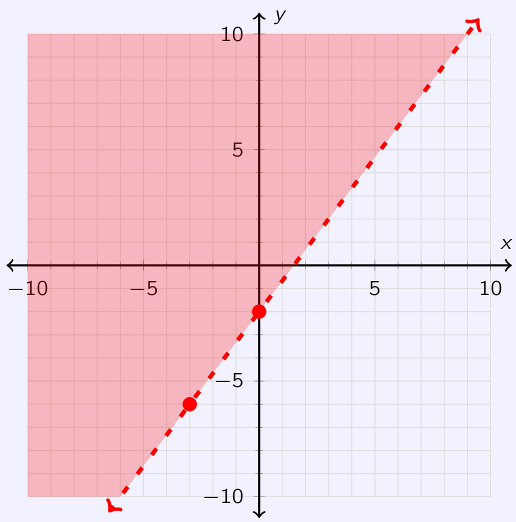
We can check that we've shaded the correct side by testing a point. A convenient point to test is the origin — all of that multiplication by zero makes this easy! If we plug in $x = 0$ and $y = 0$ to our equation, we have

$$\begin{aligned} x + 3y &> 15 && \text{original inequality} \\ 0 + 3(0) &\overset{?}{>} 15 && \text{substitute candidate point} \\ 0 &\not> 15 \end{aligned}$$

We have a false statement in the end, so the origin is not included in the solution set. This means the solution set must lie on the other side of the line. Our graph agrees with this.

Example 9.7

Write the inequality pictured in the graph below.



Solution: We can use the two given points to find that the slope of the line is $\frac{4}{3}$ (perhaps using a slope triangle). We're also given the y-intercept, so we can write the equation for the boundary line: $y = \frac{4}{3}x - 2$.

Since the line is dotted, we know that we're dealing with an exclusive inequality, and the shading lines above the line, so this graph depicts the inequality

$$y > \frac{4}{3}x - 2.$$

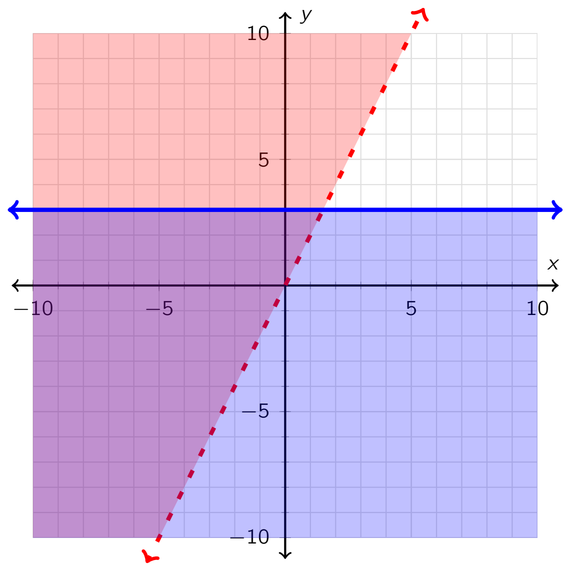
Finally: Remember to keep in mind the criteria for high-quality graphs (graphs should be done on graph paper, we should be mindful about how we place and scale the axes, and so on.)

9.5 Systems of Inequalities

Startup Exploration: TODO

TODO

By now you can perhaps anticipate where this is heading! Here you see a graph of a system of inequalities. The solution to the system is the set of points that the graphs share. In other words, the region where they "overlap".



System of Inequalities

A set of two or more inequalities with the same variables.

Solution to a System of Inequalities

The set of all points common to each inequality in a system. Graphically, the region where the graphs overlap.

The only way to find the solution to a system of inequalities is to graph the system! That is, to graph both inequalities on the same set of axes. There is really no meaningful set notation for the solution; the solution is the graph. In that visual we can see the region of the plane that contains the solution set, boundaries and all.

Graphing Tips

Systems of inequalities can be quite fun to graph. We can use different colors for each inequality, for example yellow and blue colored pencils. Where the graphs overlap, we'll get green!

If we're using just a regular pencil and have only one color choice, we might shade one inequality with horizontal lines and shade the other inequality with vertical lines. The region where we see the checkerboard shading is the solution set.

In any case, it will be important to make sure the solution region is clear. If the colors or shading are muddled and hard to interpret, consider using an arrow or label to identify the solution set.

Example 9.8

Write the system pictured in the graph at the beginning of this section.

Solution: The horizontal line goes through the point $(0, 3)$. It is a solid line and shaded below, so that means we have the inequality "less than or equal to". So, the blue graph depicts $y \leq 3$.

The red graph is a direct variation (a straight line through the origin) through the point $(1, 2)$. So the equation for the line is $y = 2x$. This line is dashed and shaded above, so we have the inequality $y > 2x$.

So, together we have the system of inequalities

$$\begin{cases} y \leq 3 \\ y > 2x \end{cases}.$$

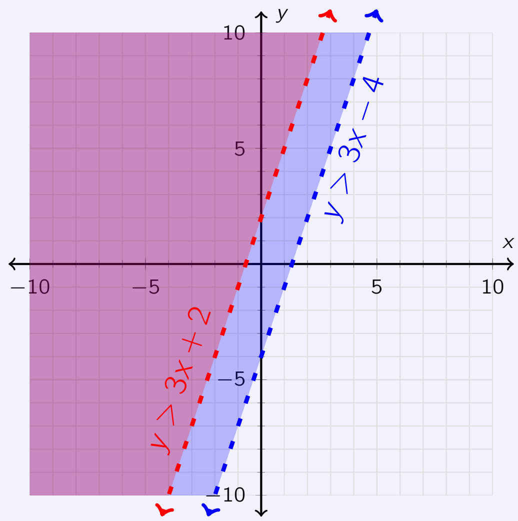
9.5.1 Special Case Systems

We have been graphing lines for a while now, and shading on one side of a line doesn't usually add that much of a challenge. Sometimes, however, we encounter some unexpected images.

Example 9.9

Example. Graph the system $\begin{cases} y > 3x - 4 \\ y > 3x + 2 \end{cases}$

Solution: Before we start graphing, note that the lines are parallel! Our first instinct might be to say that this system has "no solution". That would be correct, if we were dealing with a system of *equations*, but here we have inequalities! Let's take a look at the graph.



Note that the red region is entirely covered over by the blue region! So the overlapping region is, in fact, the red inequality $y > 3x + 2$.

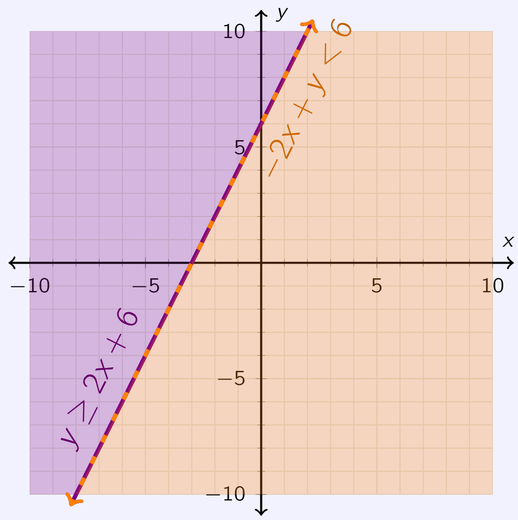
As the previous example shows, a pair of parallel lines can create a pair of inequalities which overlap. But, could the shading have gone differently? There are four different scenarios that can arise when turning two parallel lines into inequalities (one of this is shown in the example). Can you draw pictures of the other three scenarios? How does the solution set look in each case?

Example 9.10

Graph the system:

$$\begin{cases} y \geq 2x + 6 \\ -2x + y < 6 \end{cases}.$$

Solution: Let's fast-forward to the graph and see what's happening here.



It might not have looked like it at first, but we have two different ways of expressing the same line! Both inequalities have the same boundary, but they are shaded in opposite directions, so those regions do not overlap anywhere except the boundary itself.

Since the boundary is exclusive for one of the inequalities, the boundary cannot be part of the solution set. So in this case, there is no part of the graph where the two shaded regions over lap. This system of inequalities has no solution!

What are the other cases that might arise when we have two inequalities that share the same boundary line? Can you draw graphs that express these different possibilities?

9.5.2 Checking a Solution

We jumped right in to graphing a system, but suppose we want simply to check to see if a given point is a solution to a given system of inequalities?

Example 9.11

Determine whether the point $(-2, 1)$ is a solution to the system:

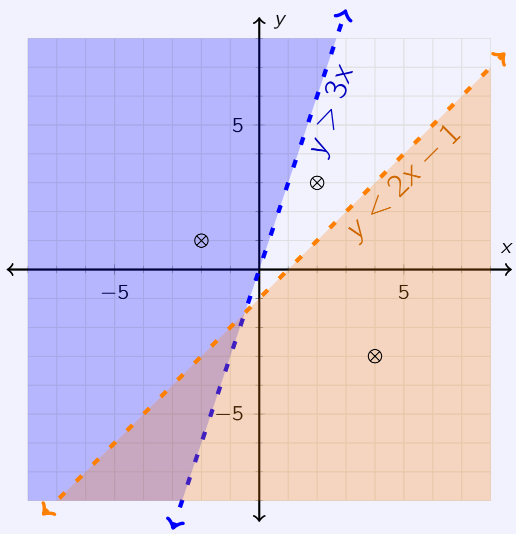
$$\begin{cases} y > 3x \\ y < 2x - 1 \end{cases}.$$

Solution: To answer this question, we don't have to graph. We can just substitute the point into each inequality and check to see whether it makes both true. Let's check $(-2, 1)$:

$$\begin{aligned} &y > 3x && \text{and} && y < 2x - 1 \\ &1 \stackrel{?}{>} 3(-2) && && 1 \stackrel{?}{<} 2(-2) - 1 \\ &1 \stackrel{\checkmark}{>} -6 && && 1 \not< -5 \end{aligned}$$

The point satisfies the first inequality, but not the second. So, $(-2, 1)$ is not a solution to the system.

Graphing is not required to answer this question, but it may be helpful to see a visual. Note where the given point lies on the graph. Predict what would happen if we tested the non-solution $(2, 3)$. Do the same for the non-solution $(4, -3)$.



9.6 Optimization Using Linear Programming

Startup Exploration: Evil Vegan Appliances

A subsidiary of YeardleighCorp, Evil Vegan Appliances, manufactures solar-powered soymilk makers. They manufacture 2 types: a large capacity model for commercial use, and a smaller one for home use. Since these soymilk makers delicate technology, the factory can only hand-make a total of 16 machines per day.

In order to keep demand up — and because she enjoys toying with the emotions of her customers — Yeardleigh decides to restrict production in another way: She decides that they should build no more than 10 commercial models and no more than 12 family models per day, just to keep everyone wanting more.

If Evil Vegan Appliances makes $75 dollar profit on each family size soy milk maker and $100 profit on each commercial model, how many of each type should they build each day to maximize profit? What will their maximum daily profit be in this case?

Businesses want to maximize their profits and minimize their costs. At the same time, businesses have constraints on the availablity and cost of resources. They have to hire workers, buy materials, build production facilities, and so on. A key challenge for any business is to navigate the various constraints so that they can configure their operations in the optimal way.

Optimization

Maximizing or minimizing a quantity, given a set constraints.

There are multiple techniques for optimization. The approach we will learn here is called **linear programming**. Real-world linear programming problems have many variables, sometimes numbering in the millions. Since we're only in algebra 1, we'll stick with just a few variables.

9.6.1 The Process of Linear Programming

To begin the process of modeling the startup exploration, our first goal is to identify what quantity we are trying to optimize, and what variables are at play in the scenario.

In our case, the quantity that we wish to optimize is *profit*, and the variables are the number of commercial machines and number of family machines that can be made per day.

We use the variables and information from the problem to write what is called the *objective function*. This is the equation we use to calculate the quantity we want to optimize.

Let P represent the total profit, let C represent the number of commercial machines that can be made per day, and let F represent the number of family machines that can be made per day. The company makes \$100 per commercial machine and \$75 per family machine, so our objective function (profit function) is

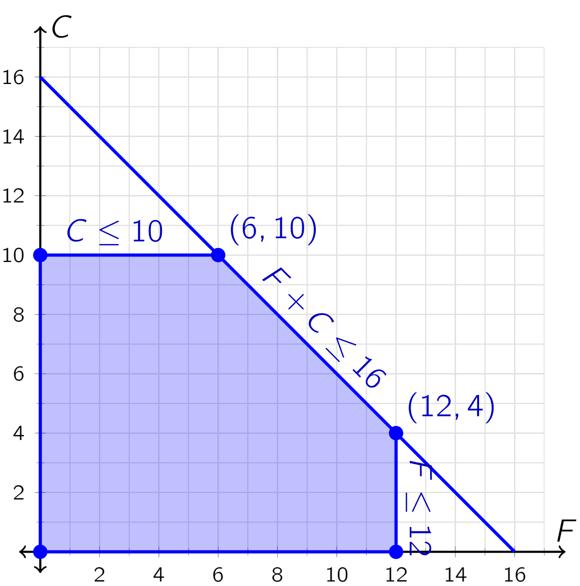
$$P = 100C + 75F.$$

We then model the constraints as a system of inequalities. The constraints on our system include: Per day they can make at most 16 machines (of any kind). In particular, no more than 10 commercial machines, and no more than 12 family machines per day. There are also natural "minimum" constraints at zero, since the factory can't make a negative number of machines.

Now, we combine our variables with our constraints to write a system of inequalities:

$$\begin{cases} F + C \leq 16 & \text{limit on the total number of machines} \\ C \leq 10 & \text{limit on the number of commerical machines} \\ F \leq 12 & \text{limit on the number of family machines} \\ C \geq 0 & \text{floor on the number of commercial machines} \\ F \geq 0 & \text{floor on the number of family machines} \end{cases}$$

We graph all five of these inequalities on the same set of axes. F and C aren't really dependent or independent variables, so it doesn't matter which we place on which axis. We'll put F on the horizontal axis.



The graph is a polygon, and every point inside this region represents a number of family size machines and a number of commercial size machines that the factory could possibly make, given the limits on production. Now, the task is to figure out which point earns the most profit.

The key insight here is that we should find the intersection points on the system. These intersection points show us where we utilize multiple resources to their fullest extent. It turns out that one of these intersection points will be the point where we optimize the objective.[7]

The intersection points are $(0, 0)$; $(0, 10)$; $(12, 0)$; $(6, 10)$; and $(12, 4)$. Those coordinates are of the form (F, C). It's important to keep track of what the numbers represent!

Now, we plug these coordinates into our objective function for total profit, $P = 100C + 75F$. One of these points will give the maximum profit and one will give the minimum profit.

- The point $(0, 0)$ means 0 Family, 0 Commercial: $P = 75(0) + 100(0) = \$0$
- The point $(0, 10)$ means 0 Family, 10 Commercial: $P = 75(0) + 100(10) = \$1000$
- The point $(12, 0)$ means 12 Family, 0 Commercial: $P = 75(12) + 100(0) = \$900$
- The point $(6, 10)$ means 0 Family, 0 Commercial: $P = 75(6) + 100(10) = \$1450$
- The point $(12, 4)$ means 0 Family, 0 Commercial: $P = 75(12) + 100(4) = \$1300$

So, now we have our solution: To maximize daily profit, the factory needs to make 6 family and 10 commercial machines per day, for a maximum daily profit of $1450.

Discussion

The first vertex $(0, 0)$ is kind of a silly point to test. You might not be surprised to see that it's the scenario that creates the minimum profit.

The points $(0, 10)$ and $(12, 0)$ have the company maximizing only one constraint — the number of a certain type of machine they can make — but completely ignoring the fact that they can make a total of 16 machines per day.

The other two vertices show the company making all 16 machines they can make per day, and maximizing one of the other constraints. We can't maximize all three constraints at the same time. (Can you explain why not?)

Finally, our solution makes sense because it seems reasonable that we would want to find the solution that maximizes overall production, but also maximizes production of the machines that make more money for the company!

In a real linear programming problem, there would likely be many more constraints. For example, the maximum number of machines per day might depend on the number of employees we have, the amount of time each

[7] See "History of Linear Programming" below to learn a little about the people who created and proved that this process works.

employee can work, how productive those workers are during that time, the their salaries, how much money there is in the payroll account, how many parts we need, how many parts we have in stock, how much it costs to make new parts, how long it will take to make them,... well, this could go on for a while. At any rate: business work quite hard to figure out challenges like this!

9.6.2 History of Linear Programming

A Soviet mathematician named Leonid Kantorovich was the first to use linear programming in 1939. During World War II he was helping the Soviet army minimize their costs while at the same time maximize losses for their enemies, Nazi Germany. His technique was effective and the Soviets kept it a secret. Because of his work, Kantorovich won a Nobel Prize in Economics, the only Soviet economist to ever win in that field.

In 1947, George Dantzig, and American mathematician, created and published his own algorithm for linear programming. The example he used was a problem that would otherwise have taken vast amounts of computing time to figure out because the number of possibilities to account for was more than the number of particles in the universe! In 1947, they did not have computers that could handle that much data. Rather than test each of the vast number of possibilities, Dantzig's "simplex" method for linear programming took just a few moments and a few calculations.

A funny side-note about Dantzig: one time he was late to a college statistics class at Berkeley. He saw two problems written on the board. He just thought they were really hard homework problems, so he solved them and turned them into his professor.

It turns out that the two problems were actually "open problems" in mathematics. An open problem is a problem in mathematics that no one has been able to solve yet. His professor spent the beginning portion of the class discussing these problems, but since Dantzig was late, he missed all of that.

Also in 1947, another American mathematician, John van Neumann, who contributed quite a bit to mathematics and physics, developed another way to approach linear programming called the "theory of duality". There are other mathematicians who advanced the process over the years. Now, every business and government agency uses linear programming to determine the best way to use their resources.

The greatest shortcoming of the human race is our inability to understand the exponential function.

Albert A. Bartlett
American physicist

Chapter 10

Exponential Functions

Every chapter should have a lead paragraph – even just a short one – that appears before the first heading. This is a placeholder paragraph which will at some point be replaces by actual content.

10.1 Exponential Relationships

Exploration

YeardleighCorp genetic engineers have developed a new kind of fast-growing, nutrient-rich seaweed for the turtle pens. The length of a strand of this seaweed doubles every day.

Scientists measure a particular strand of seaweed to be 15 centimeters long. How long will this strand be in 1 day? 2 days? 3 days? How long was this strand 1 day ago? 2 days ago? 3 days ago?

We are told that the seaweed doubles in length each day, so after one day the strand will be 30 cm long. Continuing the doubling pattern, the strand will be 60 cm after two days, and then 120 cm after three days.

We can analyze this doubling pattern using the skills we learned in chapter 2. The recursive rule rule for the length of the strand is: "start with 15 centimeters, and multiply the previous value by 2". If we think of the starting length as the "stage 0" length, then we can write a rule for the pattern:

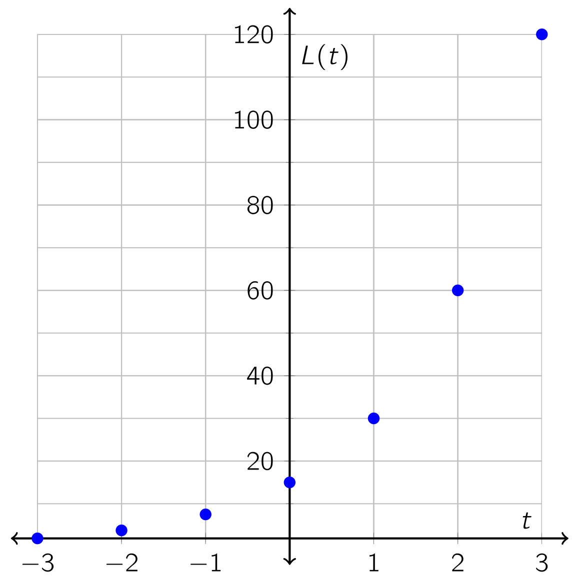
$$L(t) = 15 \cdot 2^t,$$

where $L(t)$ represents the length of the seaweed strand in centimeters after t days.

To make this doubling pattern go "backwards in time", we must divide by 2. One day ago the strand must have been 7.5 cm long, since doubling this value get us to the original measurement of 15 cm. Going further, the strand was 3.75 cm long two days ago, and 1.875 cm long three days ago.

We can organize our data in a table, and make a scatter plot. Note that we use negative numbers to represent days in the past (days before day 0, on which we took the starting measurement).

Time (days) t	Length (cm) $L(t)$
-3	1.875
-2	3.75
-1	7.5
0	15
1	30
2	60
3	120



From these three representations, it is clear that we have an exponential relationship on our hands: The rule has the independent variable used as an exponent, the table shows repeated multiplication, the graph as that smooth J-like shape that we saw in chapter 4.

A new wrinkle that comes up here, though, is the idea of using negative numbers as the exponent. If we're right about our rule, for example, then it must be that

$$15 \cdot 2^{-2} = 3.75,$$

which is not an obvious statement! Let's look more closely at this idea.

10.1.1 Negative Exponents

Recall that when we talk about exponents and exponential expressions, we're talking about expressions of the form: a^b. This is read "a to the power of b" or "a to the b^{th} power". In such an expression, a is called the *base* and b is called the *exponent*. Since a is the base, we call the whole thing a *power of a*.

One interpretation of this expression, as we have seen, is as repeated multiplication:

$$a^b = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{b \text{ times}}$$

This representation is a bit limited — it really only works when the exponent b is a positive integer — but it is still helpful for gaining some intuition about how exponents work.

For clarity, we will name the different forms of the exponential expression. We will use the term "exponent form" to describe a^b, and "expanded form" to describe the form where the multiplication is written out. So, for example, 12^4 is in exponent form, and

$$12 \cdot 12 \cdot 12 \cdot 12$$

is the same number written in expanded form. Of course, if the base is a number, we can compute this value:

$$12^4 = 12 \cdot 12 \cdot 12 \cdot 12 = 20\,736.$$

This last value (the number) is in "standard form" or "decimal form". Expressions like a^b have no decimal form until we know what numbers a and b represent.

10.1.2 Extending Repeated Multiplication

Recall that in section 1.4 we mentioned a rule for using zero as an exponent:

Zero Exponent

For any nonzero real number a,

$$a^0 = 1.$$

Our goal now is to extend this idea to include exponents that are negative integers. We'll do this informally by looking at exponential patterns. We'll start with a table that just shows the non-negative integral powers of 5:

$$\begin{aligned}
5^5 &= 1 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 &&= 3125\\
5^4 &= 1 \cdot 5 \cdot 5 \cdot 5 \cdot 5 &&= 625\\
5^3 &= 1 \cdot 5 \cdot 5 \cdot 5 &&= 125\\
5^2 &= 1 \cdot 5 \cdot 5 &&= 25\\
5^1 &= 1 \cdot 5 &&= 5\\
5^0 &= 1 &&= 1
\end{aligned}$$

Moving "upwards" in this table involves multiplication by 5. Each new row that we pile on top of the table would add 1 to the exponent and put a new factor of 5 in the middle column. If moving upwards in the table means multiplication by 5, then moving "downwards" in the table must mean division by 5, or in other words, multiplication by $\frac{1}{5}$ (the reciprocal of 5).

So, we can extend the table downwards, as shown below. We take away 1 from the exponent in each step, and introduce repeated multiplication by $\frac{1}{5}$.

$$\begin{aligned}
5^5 &= 1\cdot 5\cdot 5\cdot 5\cdot 5\cdot 5 &&= 3125\\
5^4 &= 1\cdot 5\cdot 5\cdot 5\cdot 5 &&= 625\\
5^3 &= 1\cdot 5\cdot 5\cdot 5 &&= 125\\
5^2 &= 1\cdot 5\cdot 5 &&= 25\\
5^1 &= 1\cdot 5 &&= 5\\
5^0 &= 1 &&= 1\\
5^{-1} &= 1\cdot \tfrac{1}{5} &&= \tfrac{1}{5}\\
5^{-2} &= 1\cdot \tfrac{1}{5}\cdot\tfrac{1}{5} &&= \tfrac{1}{25}\\
5^{-3} &= 1\cdot \tfrac{1}{5}\cdot\tfrac{1}{5}\cdot\tfrac{1}{5} &&= \tfrac{1}{125}\\
5^{-4} &= 1\cdot \tfrac{1}{5}\cdot\tfrac{1}{5}\cdot\tfrac{1}{5}\cdot\tfrac{1}{5} &&= \tfrac{1}{625}\\
5^{-5} &= 1\cdot \tfrac{1}{5}\cdot\tfrac{1}{5}\cdot\tfrac{1}{5}\cdot\tfrac{1}{5}\cdot\tfrac{1}{5} &&= \tfrac{1}{3125}
\end{aligned}$$

There are a number of important patterns to see in this table. For example, the pattern "divide by 5 as we move downwards in the table" helps us to explain why $5^0 = 1$. Also note which numbers in the table are reciprocals of one another. This suggests a general rule for working with negative exponents.

Negative Exponents

For any nonzero real number a, and any positive integer n,

$$a^{-n} = \frac{1}{a^n}.$$

In other words, a^{-n} is the reciprocal of a^n (the base raised to the opposite power).

One aspect of negative exponents that can be confusing is that the negative exponent does not change the sign of the evaluated expression. For example, all of the numbers in the table above are positive numbers, even though they have a negative number in the exponent!

Also note that this rule explicitly states that a must be a *nonzero* real number. Can you explain why is this important? Why can't we raise 0 to a negative exponent?

Example 10.1

Write each of the following in standard form (as a fraction in lowest terms).

(a) 36^{-1} (b) 2^{-6} (c) $\left(\dfrac{3}{4}\right)^{-2}$

Solution: In each case, we can take the reciprocal of the number and raise it to the positive version of the power, so

$$36^{-1} = \left(\frac{1}{36}\right)^1 = \frac{1}{36}$$

The next two require a bit more computation, but the idea is the same. Problem (b) becomes:

$$2^{-6} = \left(\frac{1}{2}\right)^6 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{64}$$

And problem (c) becomes:

$$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2 = \left(\frac{4}{3}\right)\left(\frac{4}{3}\right) = \frac{16}{9}$$

It pays to take time when working on problems like this. Consider the next examples carefully!

Example 10.2

Rewrite each of the following expressions in an equivalent form that uses only positive exponents.

(a) $7m^{-2}$ (b) $\frac{3^{-1}}{4}$

Solution: We might make a mistake if we rush into these expressions too quickly. Recall that we have to be clear about exactly what an exponent is attached to. In the first expression, the exponent applies only to the m, not the whole expression (there are really two operations here — what are they?). And so,

$$7m^{-2} = 7\left(\frac{1}{m}\right)^2 = \left(\frac{7}{1}\right)\left(\frac{1}{m}\right)\left(\frac{1}{m}\right) = \frac{7}{m^2}$$

In the second expression, the exponent applies only to the 3 in the numerator, and not the whole fraction (there's a subtle grouping symbol here — what is it?). This expression becomes:

$$\frac{3^{-1}}{4} = \frac{\frac{1}{3}}{4} = \frac{1}{3} \div 4 = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) = \frac{1}{12}$$

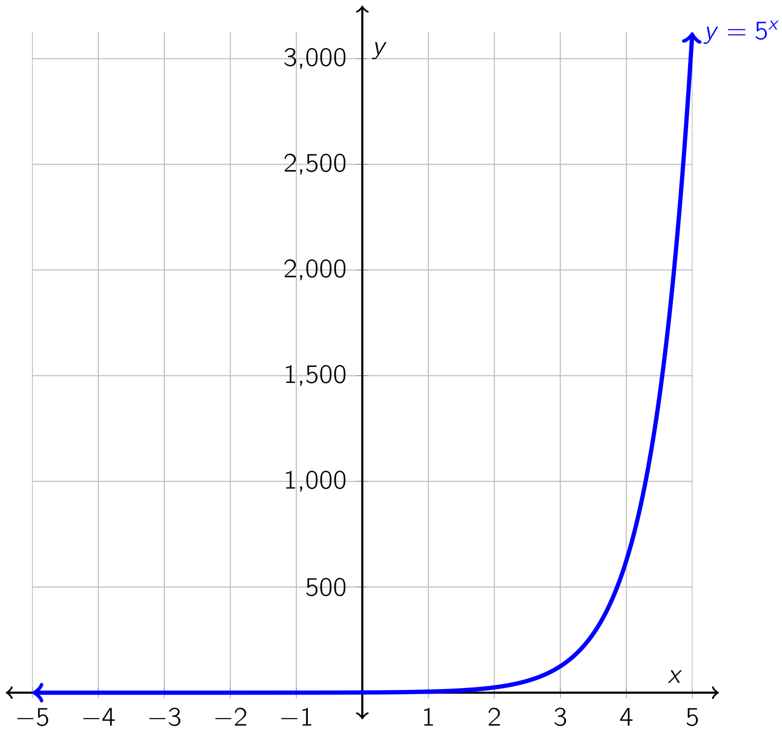
The second expression went through a rather ugly, fraction-in-a-fraction phase, but don't worry: we'll soon learns some simplification rules that will help us make quick work of expressions like these.

10.1.3 Smaller and Smaller

An interesting consequence of this negative exponent business is that we get a progression of numbers that get smaller and smaller, but never disappear. For example, in the sequence of powers of 5 above, as n gets larger, the number 5^n grows huge! By the time $n = 10$, the number 5^{10} is nearly 10 million.

This means that 5^{-10} is 1-over-something-close-to-10-million. That number is incredibly small. But note that it is still a positive number. As n gets larger and larger 5^{-n} gets closer and closer to zero, but never actually reaches zero. There is always a tiny fraction which makes the number 5^{-n} a wee bit greater than zero.

If we look at the graph of $y = 5^x$, we see that the curve flattens out as we look to the left, along the negative x-axis. The line approaches 0 but never reaches it. To capture this idea, we say the graph as a *horizontal asymptote* at $x = 0$.[1]



We can also see how steep the curve gets as we go along the positive x-axis. An important aspect of understanding exponential relationships is understanding this dual nature: incredibly steep on one end, incredibly flat on the other.

[TODO] Idea: include Zeno's paradox here? Or as a journal prompt or activity of some kind?

Startup Exploration: Zeno's Paradox

A paradox is a statement from which we can follow a logical path of reasoning to an impossible of self-contradictory end.

The ancient Greek philosopher Zeno proposed a set of paradoxes about motion. One of these says:

A moving object must arrive at the half-way point on its path before it reaches its goal.

This seems like an obvious statement, right? If you shoot an arrow at a target, the arrow must go half of the distance between you and the target before is goes the whole way to the target!

[1] We'll learn more about asymptotes, and other unusual graph behavior, at the very end of this course and in algebra 2.

Let's call the half-way point "point A" and suppose the arrow is there. Now consider this: before the arrow can fly to the target from point A, it has to go half of the remaining distance. So, the arrow arrives at a point three-fourths of the way along the path. Let's call this point B.

Now, before the arrow can go from point B to the target, it has to go half-way to the target. Now it's seven-eighths of the way there.

Before it can cover the remaining distance it has to cover half of the remaining distance... and this is always true. So, Zeno says, the result is that the arrow never actually reaches the target: there's always a little but more to go!

In fact, it's even worse! Before the arrow can go halfway to the target, the arrow has to go one-fourth of the way. But before it can do that, it has to go one-eighth of the way. In fact, before the arrow can cover any distance, it has to cover half of that distance... and so the arrow never moves in the first place!

Zeno says that movement is impossible, since the above statement about "going half-way" is true. Of course, we know that movement is possible! We do it all the time! This contradiction is what makes the paradox.

Many resolutions to the paradox have been proposed over the year, but it wasn't until the 19th century — around 2000 years after Zeno lived — that mathematics had a clear and rigorous solution to the paradox.

What do you think about these ideas?

10.2 Exponential Growth

Extended Exploration: A Problem With Turtles

[TODO] LINK.

Startup Exploration: Cantor Set

German mathematician Georg Cantor suggested a way to construct a very interesting set of numbers. We start in Stage 0 with a line segment covering the interval from 0 to 1 on the number line. The recursive procedure is to remove the middle third from each of the line segments in the previous step. The first few iterations are shown.

Let $f(x)$ represent the number of segments in stage x of this construction. Write an equation for $f(x)$. How many segments are in stage 10 of the figure? What is the first stage to contain more than 100,000 segments?

Bonus puzzler. The set of points from the original line segment that are not deleted at any step of the (infinite) process is called the *Cantor ternary set*. Can you give an example of a point that is in this set? What can we say about the points that survive this process?

10.2.1 Modeling Exponential Growth

The number of segments in the construction of the Cantor set doubles each time, so the pattern follows the rule "start with 1, multiply the previous value by 2". We can write out the repeated multiplication in expanded form, or use an exponent.

Stage No.	No. of Segments		
0	1	1	$1 \cdot 2^0$
1	2	$1 \cdot 2$	$1 \cdot 2^1$
2	4	$1 \cdot 2 \cdot 2$	$1 \cdot 2^2$
3	8	$1 \cdot 2 \cdot 2 \cdot 2$	$1 \cdot 2^3$
4	16	$1 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	$1 \cdot 2^4$

If we compare the first and last columns in the table, we can see that the explicit rule for the number of segments depends only on the stage number. We have the rule

$$f(x) = 1 \cdot 2^x.$$

Now that we have an equation, it's quite easy to find the number of segments in stage 10 of the figure:

$$f(10) = 1 \cdot 2^{10} = 1024.$$

To determine the stage in which the figure first has more than 100,000 segments, we need to solve the following equation for x:

$$100,000 = 2^x$$

Hmm. This is a bit of a problem, since we don't have a POE that helps us to get x out of the exponent.[2] For now, we'll have to settle for doing a bit of detective work to find this value.

One approach is to use technology. With a graphing calculator or other tool that will display a table of values when given a rule, we can punch in the rule and search through the table for the first row in which the number of segments exceeds 100,000.

If we're working by hand (or working with a regular four-function calculator) we can extend our table manually, or adopt an educated guess-check-revise strategy. In this case, we work forward from $f(10) = 1024$. We don't have to hunt for very long:

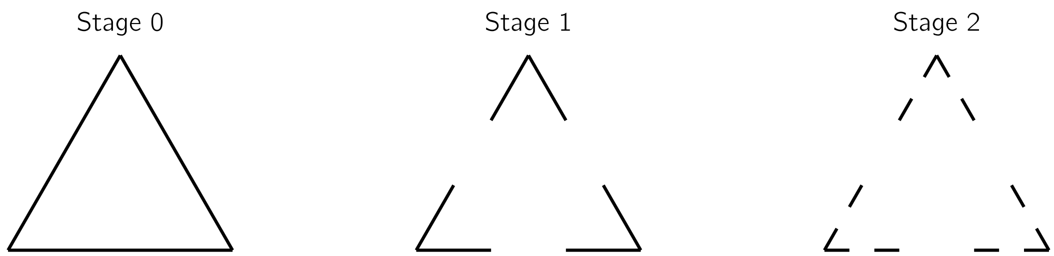
$$f(11) = 2048$$
$$f(12) = 4096$$
$$f(13) = 8192$$
$$f(14) = 16\,384$$
$$f(15) = 32\,768$$
$$f(16) = 65\,536$$
$$f(17) = 131\,072$$

[2] We don't have such a POE *yet*. There are POEs that will help us in situations like this, but they're going to have to wait until algebra 2, when we will learn about the concept of a *logarithm*.

So, stage 17 is the first stage to contain more than 100,000 segments. Are you surprised by this? The figure grows from around 1000 segments in stage 10 to more than 100 times that number of segments less than 10 stages later! Surprising results like this are part of what make exponential relationships so much different from linear relationships.

10.2.2 Extending the Growth Model

Suppose we build an equilateral triangle in which each side is a Cantor segment. Then we let $g(x)$ represent the number of segments in stage x of this new figure. Let's try to answer the same questions as above: how many segments in stage 10, and in what stage will the number of segments first exceed 100,000?



The only difference here is that we've changed the initial number of segments. Now, we have the rule "start with 3, multiply the previous value by 2". The updated table is shown below.

Stage No.	No. of Segments		
0	3	3	$3 \cdot 2^0$
1	6	$3 \cdot 2$	$3 \cdot 2^1$
2	12	$3 \cdot 2 \cdot 2$	$3 \cdot 2^2$
3	24	$3 \cdot 2 \cdot 2 \cdot 2$	$3 \cdot 2^3$
4	48	$3 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	$3 \cdot 2^4$

Notice how we only need to change one part of the first rule that we wrote:

$$g(x) = 3 \cdot 2^x$$

To find the number of segments in stage 10, we compute $g(10) = 3 \cdot 2^{10} = 3072$. Then, we do some detective work to find the stage number in which we first exceed 100,000 segments. Stage 15 is close with $g(15) = 98\,304$, but it is stage 16 that first breaks the barrier: $g(16) = 196\,608$.

10.2.3 Exponential Equations

If we summarize our findings from the explorations above, we have the "general form" of an exponential function.

General Form of the Exponential Function

A equation of the form

$$f(x) = a \cdot b^x$$

is an exponential function if $a \neq 0$, $b \neq 1$, and $b > 0$. The constant value a is the *initial value* of the function (when $x = 0$). The constant value b is called the *constant multiplier*.

Note that there are some constraints placed on the "legal" values of a and b. Why do we need these restrictions?

Consider what happens when $a = 0$. In this scenario, the function $f(x) = ab^x$ degenerates to $f(x) = 0$. That isn't an exponential function, it's a horizontal line along the x-axis! Simiarly, if $b = 1$, then $f(x) = ab^x$ degenerates into $f(x) = a$. That's a horizontal line through the point $(0, a)$!

What about the restriction $b > 0$? Consider what happens when b is negative, for example $f(x) = (-2)^x$. When x is an even integer, $f(x)$ is positive. When x is an odd integer, $f(x)$ is negative. This back-and-forth means that the graph doesn't have the same smooth curving shape that we associate with the exponential family. Plus, we haven't even talked about values of x that are not integers! Those have all sorts of other problems!

When b is greater than 1, we have a situation like the ones we have seen in this chapter, called *exponential growth*. Note that b could also be a positive fraction: numbers between 0 and 1 are allowed. In the case that b is a fraction, the exponential behaves a bit differently. We'll study this situation, called *exponential decay*, in section 10.5.

10.3 Percent Change

Earlier explorations have involved an amount doubling as time went by. But what if we have a quantity that grows at a different rate? For example, the United Nations estimates that the population of the Earth will grow by approximately 0.77% per year between now and 2050. How do our rules have to change to accomodate an arbitrary rate of growth?

Startup Exploration: Hermine's Choice

Hermine is shopping for a mother's day gift. The vase she wants to buy is on sale for 25% off. She also has a 10% off coupon that can be applied to any item in the store. The store manager agrees that she can use both discounts, and gives her two options: *Option 1*: Take 25% off the full price of the vase, then apply the coupon to the discounted price. *Option 2*: Apply the 10% coupon to the full price of the vase, and then take 25% off the discounted price.

Which option should Hermine choose? Why?

Later, Hermine told Bob this story. He said, "Wow! 35% off! You got a great deal!" How would you respond to Bob?

In chapter 6, we discussed the percent proportion:

$$\frac{\text{part}}{\text{whole}} = \frac{\text{percent}}{100}.$$

A key use of percents, as we saw, is to make clear and accurate mathematical comparisons. One comparison that is often illuminating is to compare a certain quantity with itself. If the quantity has increased or decreased over time, we may want to know *by what percent* has it changed?

To calculate a percent change, we compare the amout that a quantity has changed to its starting value. In other words,

$$\frac{\text{amount of change}}{\text{original amount}} = \frac{\text{percent change}}{100}.$$

If we wish to isolate percent change, we can multiply both sides by 100.

Of course, a value may grow larger or smaller, and so we must remember to be clear and describe the percent change as either a "percent increase" or "percent decrease".

Example 10.3

Yeardleigh gives Hermine \$300 worth of YeardleighCorp stock as a birthday gift. After one month, the value of YeardleighCorp stock had increased by 15%. How much were Hermine's shares worth at that point?

After two months, Hermine's shares are worth \$240. What is the percent change from her original \$300 gift? What is the percent change from the value of her shares after one month?

Solution: To compute the value after one month, we find %15 of the value: $300(0.15) = 45$, and then we add this on to find the new total value. So, after one month Hermine's shares are worth $300 + 45 = 345$ dollars.

Since the stock value goes down in the second month, we're looking for percent decrease. The drop from the original value is $300 - 240 = 60$ dollars. We compare this to the original price to find the percent change from that price:

$$\frac{300}{60} = 0.2 = 20\%$$

We can make the same comparison with the price after one month. The amount of the change is $345 - 240 = 105$ dollars, so the percent change from that price is:

$$\frac{105}{345} \approx 0.304 \approx 30\%$$

So Hermine's shares have undergone a 20% decrease from their starting value, and a 30% decrease from their price one month ago.

Note that to calculate the percent increase we found 15% of 300 and then added that on to the original 300. So the new amount is

$$300 + 300(0.15).$$

We can apply the distributive property to this and get

$$300(1 + 0.15) = 300(1.15).$$

If we interpret this as a percent question in its own right, we are saying that the new value is 115% of the original value. This might seem like an unusual statement, but it makes sense: we have 100% of the amount (all of what we started with) and then we add on an additional 15%.

Now consider the case of a percent decrease, such as Hermine's discount coupon in the startup exploration. If Hermine wants to buy a vase that costs \$75 and has a 10% off coupon, we can find 10% of the value of the vase and subtract that from the original value:

$$75 - 75(0.10)$$

Applying the distributive property as we did above, we have:

$$75 - 75(0.10) = 75(1 - 0.10) = 75(0.90)$$

This makes sense, too: if Hermine gets 10% off the price of the vase, then she is only paying 90% of the original price. Armed with these ideas, we can determine which option Hermine should choose in the startup exploration.

Explaining the Startup Exploration

Solution: It may have been helpful to make up a price for the vase (a convenient number, like \$100), but we'll use a variable here. Let V represent the price of the vase.

If Hermine chooses option 1, she will first get a 25% discount. In other words, the price of the vase will be reduced to 75% of its original value and Hermine would pay $(0.75)V$. The next step is to take this value and apply the 10% coupon. If we take 10% off, that's like paying for 90% of the vase. So in the end the new price is:

$$(0.90)\big((0.75)V\big) = \big((0.90)(0.75)\big)V = (0.675)V$$

Hermine pays for 67.5% of the vase, which is a 32.5% discount.

If Hermine chooses option 2, she gets the 10% discount first. This makes her cost 90% of the original price. Then we take 75% of this value. So option 2 means the new price would be:

$$(0.75)\big((0.90)V\big) = \big((0.75)(0.90)\big)V = (0.675)V,$$

which is exactly the same as option 1.

So it doesn't matter which option Hermine chooses. She will pay the same amount for the vase either way. In response to Bob's comment, this is not a 35% reduction in the price, but a 32.5% reduction.

Note how the associative and commutative properties of multiplication play a role in this problem! The field axioms come to the rescue again!

10.3.1 Using Growth Rate

In section 10.1, we studied a kind of fast-growing seaweed under development at YeardleighCorp labs. Suppose that YeardleighCorp genetic engineers have a different kind seaweed which grows at a rate of 25% per day.

Scientists measure a particular strand to be 6 centimeters long. How long will the strand be in 1 week? How long was the strand 1 week ago? When will this strand be 100 meters long?

To find the length of the strand after one day, we compute $6(1.25) = 7.5$ cm. The "1.25" here means that we've got 100% of the original length of the strand, plus an extra 25%. On day 2, the strand will grow an additional 25% and be $7.5(1.25) = 9.375$ cm long.

To find the length day-by-day, we could use the recursive rule "start with 6 centimeters, multiply the previous value by 125%". We can model this as an exponential equation. If $L(t)$ represents the length of the seaweed in centimeters after t days, then we have

$$L(t) = 6(1.25)^t$$

To find the length of the strand after one week (7 days), we compute

$$L(7) = 6(1.25)^7 \approx 28.610 \text{ cm}$$

To find the length one week ago, we compute

$$L(-7) = 6(1.25)^{-7} \approx 1.258 \text{ cm}$$

How long before the strand is 100 m long? Our rule gives the length in centimeters, so we want to know when the strand is 10,000 cm long.[3] If we do a bit of detective work, perhaps using some graphing technology to examine a table of values, we can discover:

$$f(33) \approx 9466.331 \text{ cm}$$
$$f(34) \approx 11\,832.914 \text{ cm}$$

So this strand of seaweed reaches 10,000 cm in length sometime between day 33 and day 34. For the record: this kind of approximation is a perfectly acceptable way to answer for this type of question, in algebra 1.[4]

You may be suspicious about the fact that we predict a 6 cm piece of seaweed will grow to be longer than an American football field in a little more than one month. A note on realism: Exponential functions assume unlimited growth. But, in the real world, this type of growth cannot be sustained: a piece of seaweed cannot grow to be 10 miles long, a flock of geese cannot grow to include millions of members. Constraints (like the availability of food or space) restrict unlimited growth. The kind of function that models the way things *really* grow is a bit too complicated for algebra 1. For our purposes, we will assume that the creatures and businesses we study have unlimited resources and can grow without bound.

We summarize our approach here, and present a second way to express the equation for exponential growth.

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3 Recall that there are 100 centimeters in one meter.

4 If you're feeling detail-oriented, you can use your calculator to narrow down the answer by adjusting the table settings to that x-values increase by 0.1 or 0.01, rather than by 1 whole unit. If you want to go there, you can actually narrow down the interval considerably (if we assume the seaweed grows uniformly, then the actual value is around 33.246 days). This is not required: an approximation between two integers is fine for now.

Exponential Growth, Given Growth Rate

A equation of the form

$$f(x) = a \cdot (1 + r)^x$$

is an exponential function where a is the initial value, and r is the growth rate (the percent increase each "generation").

If we compare this form of the equation to the previous form, $y = ab^x$, we can see that $b = (1 + r)$. This value is called the *growth factor*. Be mindful that "growth rate" and "growth factor" are different the "growth factor" is "1 + growth rate".

10.4 Personal Finance

This material in this section is quite possibly the most important thing we will discuss in all of algebra 1! The topics here apply not just to students who plan on studying science, technology, engineering, mathematics (or other "mathematically intense" disciplines). The concept of personal finance is important for anyone who plans to have a job or a bank account, anyone who wants to own a car or a home.

10.4.1 Interest

Cars are expensive. Most people don't have enough money lying around to just go out and buy a car. Instead, most people enter in to an agreement with a bank. The bank agrees to lend you the money you need to buy the car, and you agree to pay back that money over time.

Of course, a bank is a business and banks want to make money. So, they charge you for the "convenience" of being able to get all the money you need, all at once. This means that the bank requires you to pay back *more* than you borrowed. This extra amount you will have to pay is called **interest**.

Interest is usually calculated as a percentage of the amount you borrowed. The percentage is called the **interest rate**. The interest rate is usually given as an annual percentage rate, or APR (you might have heard this term in a car commercial).

There are two kinds of interest in the world: good interest and bad interest.[5] The "extra money" that you have to pay to the bank is "bad interest". On the other hand, some savings accounts earn interest, meaning that every so often the bank will add on a percentage of however much money is in the account. This is "good interest": free money!

As a future consumer, it is important to learn about how interest works.[6] A good rule of thumb is to try and minimize "bad interest" which works *against* you, and to try and get as much "good interest" as possible working *for* you.

10.4.2 Simple Interest

The simplest way to calculate interest is called, naturally enough, **simple interest**. It is calculated based only on the original amount borrowed or invested, called the **principal amount**, or just *principal*.

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[5] There are 10 kinds of people in the world: People who understand binary numbers and people who don't.

[6] When you go to the bank to get a car loan, you won't have to do all the math yourself. The bank will do the interest calculations up front and tell you how much you should expect to pay. But: credit cards are different, and — in the opinion of the authors — much more sinister.

Simple Interest

We calculate simple interest using the formula

$$I = Prt,$$

where I represents the interest earned, P represents the principal (the initial amount invested or borrowed), r represents the annual percentage rate, and t represents the time (in years).

The formula $I = Prt$ only calculates the interest, meaning the extra amount that we add on. The formula doesn't tell us the total amount that we own (or owe). To calculate the total amount, we can create an alternative formula for A the total amount:

$$A = P + Prt.$$

Let's take an example: Suppose Bob wants to buy a car and needs to borrow \$20,000. The bank offers Bob 6.5% APR on a five-year loan. In other words: at the end of each year, 6.5% of \$20,000 will be added to the amount Bob owes, and Bob agrees to pay back the total amount in 5 years. Table 10.1 shows a schedule of how Bob's balance changes over the five years.

No. Years (x)	Calculations	Total Amount (y)
0	—	20000
1	20000 + 20000 (0.065) (1)	21300
2	20000 + 20000 (0.065) (2)	22600
3	20000 + 20000 (0.065) (3)	23600
4	20000 + 20000 (0.065) (4)	25200
5	20000 + 20000 (0.065) (5)	26500

Table 10.1: Calculating simple interest of 6.5% on \$20000.

In the end, Bob will sign a contract agreeing to pay back a total of \$26,500, which is \$6500 more than the amount he wants to borrow. This is the "extra money" that goes to the bank.

Notice that the total goes up by the same amount each year. This makes sense, because the calculation for each year is done based on the original 20,000. The growth in the total amount is linear.

Let's compare this to the evil-genius that is *compound interest*.

10.4.3 Compound Interest

The key idea of compound interest is that it isn't calculated based just on the principal amount, but based on the principal amount *plus all of the interest that has been earned up to that point*.

Let's see how this impacts Bob. Suppose his $20,000 car loan has an APR of 6.5% "compounded annually". Each time we calculate his interest, we will need to add on 6.5% of the previous year's total amount, not just the principal amount.

No. Years (x)	Calculations	Total Amount (y)
0	—	20000.00
1	20000.00 (1.065)	21300.00
2	21300.00 (1.065)	22684.50
3	22684.50 (1.065)	24158.99
4	24158.99 (1.065)	25729.33
5	25729.33 (1.065)	27401.74

Table 10.2: Calculating (annual) compound interest of 6.5% on $20000.

In the end, Bob will sign a contract agreeing to pay a total of $27,401.74. That's $7401 more than he borrowed, and $901.74 more than he would pay in the simple interest scenario! That's the incentive banks have to use compound interest!

Notice that to calculate the amount we owe in the next year, we take the current year and multiply by 1.065. In other words, we have the rule NEXT = NOW $\cdot 1.065$, and that's an exponential relationship!

Over the 5 years of his car loan, Bob pays an extra $900 or so compared to simple interest. That's bad news, but it might not hurt Bob's bottom line that much: it's just $15 extra per month. This is part of the evil genius of compound interest: the difference isn't huge in the short run. If there were a huge difference, then no one would agree to pay compound interest! We would all protest and boycott organizations that used it!

Those that know how to use compound interest, however, know that they can change what is known as the "period of compounding." This changes how often interest is calculated. Rather than calculating interest once a year, we break up the annual percentage into smaller pieces and calculate the interest more often.

Let's return to the story od Bob and his car loan, and suppose that interest is compounded semi-annually. That means that we take the 6.5% APR, cut it into two pieces, and calculate interest twice a year.

No. Years (x)	Calculations	Total Amount (y)
0	—	20000.00
1	$20000(1.065/2)^2$	21321.13
2	$20000(1.065/2)^4$	22729.52
3	$20000(1.065/2)^6$	24230.95
4	$20000(1.065/2)^8$	25831.55
5	$20000(1.065/2)^{10}$	27537.89

Table 10.3: Calculating (semi-annual) compound interest of 6.5% on $20000.

The exponents here show that in one year we have gone through 2 compounding periods, in two years we have gone through 4 compounding periods, and so on. Generally, if we compound interest semi-annually, we will have $2t$ compounding periods in t years.

Notice that the difference is just a little bit higher at the end of 5 years: around $136. Now, let's look at what happens when we change to monthly compounding, meaning we divide the interest rate into 12 pieces, but that we recompute interest 12 times per year.

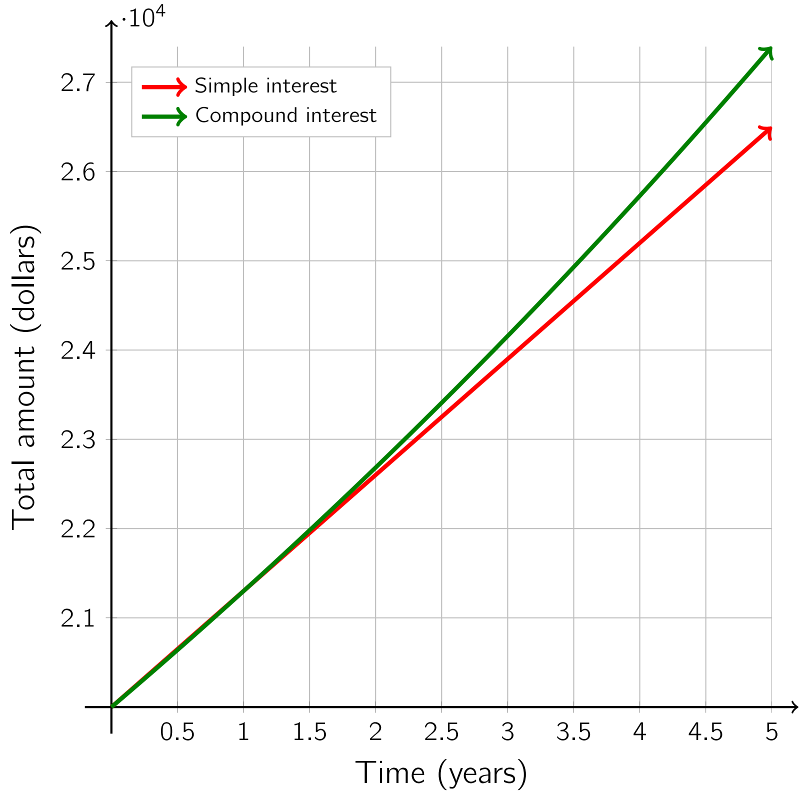


Figure 10.1: Comparing 6.5% simple and compound interest over 5 years.

After 5 years, Bob will owe

$$20000(1.065/12)^{5\cdot 12} = 27656.65$$

which is slightly more than anything we've seen so far. What happens if we change to daily compounding? In this case the interest rate will be divided into 365 pieces (let's ignore leap years) and we will calculate interest 365 times per year.[7] After 5 years, Bob will owe

$$20000(1.065/365)^{5\cdot 365} = 27679.81$$

Once again, notice that the difference isn't that great compared to the other calculations we've made. But, its higher than anything we've computed so far. Bob would end up oweing the bank $7679.81 more than he borrowed!

We can distill these calculatons into a formula for compound interest.

[7] Daily compound interest is how credit card companies charge interest — seriously!

Compound Interest

We calculate compound interest using the formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where A represents the total amount, P represents the principal, r represents the annual interest rate, t represents the time in years, and n represents the number of compounding periods per year.

For example in the scenario of "monthly compounding", $n = 12$ and so we use the equation

$$A = P\left(1 + \frac{r}{12}\right)^{12t}$$

to compute interest after t years. Note that we have $\frac{r}{12}$, meaning we've divided the rate into 12 pieces, and $12t$ in the exponent meaning that in each year t we have 12 compounding periods.

Credit Cards and Compound Interest

Credit card companies charge compound interest on your balance, and interest is compounded daily. Plus, credit cards often charge a higher interest rate than other types of financial products (car loans and student loans generally charge a much lower rate of interest).

These two things — high interest rates and daily compounding — combine to make it very hard to pay of large amounts of credit card debt. It's troubling that the average American consumer owes several thousand dollars in credit card debt, with American consumers together owing billions of dollars to credit card companies.

10.4.4 (;,;) Continuous Compounding

At this point, someone usually wonders whether a bank could make an unlimited amount of money using this technique. What happens if a bank were so evil that they compounded interest every minute of every day? What about every second of every day? What about every millisecond?

Interestingly, there is a limit to the amount of money that a bank can rake in using compound interest. Notice that the amounts we have seen — under annual, semi-annual, monthly, and daily compounding — have all been increasing, but that they have been increasing by less and less. The increases are sort of "trailing off" and, over time, the change between compounding schemes gets closer and closer to zero.

Exploration

[TODO] Could we link to an exploration about this?

Imagine that we invest a principal amount P at an evil bank which charges 100% interest. Experiment with the amount we would owe after one year given different compounding periods. What is the maximum that we would be expected to pay back after one year?

This scenario — continuous compounding for one year at 100% interest — gives rise to a very interesting number which will be of great importance in future mathematics courses, including algebra 2.[8]

[8] Actually, this number has already been mentioned in the 𝔄𝔩𝔤𝔢𝔟𝔯𝔞𝔫𝔬𝔪𝔦𝔠𝔬𝔫! Go on a little scaveneger hunt through the footnotes in chapter 4 and see what you can find!

10.5 Exponential Decay

Extended Exploration: A Problem With Pintonium

LINK

Startup Exploration: No Pressure

Patrons to the Cheeseville Zoo can take a hot air balloon ride over the park. They have to fly high enough to be at a safe distance above the most dangerous areas of the zoo: Lake Mascarpone (the reptile exhibit) and NAME THE ISLAND (the primate enclosure).

Of course humans can't fly too high: atmospheric pressure (the pressure of air around you) decreases as altitude increases, by about 6% for every 500 meters. Above a certain altitute, people need breathing aids if they aren't in a pressurized environment (like an airplane).

Mountain climbers who scale Mount Everest say that "altitude sickness" sets in around 2500 m, and they say that climbing about 8000 m is entering the "death zone" (a little morbid, but an effective deterrent).

What percentage of normal air pressure is safe for humans? Specifically: at what percentage does a person risk altitude sickness? At what percentage does a person risk death?

[TODO] What's the name of the monkey enclosue where the gorilla mauling happens?

At an altitude of 0 meters (sea level), we experience 100% of normal atmospheric pressure. When we rise to a height of 500 meters, the atmospheric pressure decreases by 12%. In other words, we are experiencing approximately 88% of normal air pressure at that altitude.

Rising to 1000 feet, we reduce by an additional 12%. To calculate the new atmospheric pressure value, we could find 12% of 88%, and subtract this from the original 88%:

$$88 - 88(0.12) = 88 - 10.56 = 77.44$$

Or, we could think about the percent remaining and find 88% of the previous value 88%:

$$1 \cdot 0.88 \cdot 0.88 = 1 \cdot (0.88)^2 = .7744 = 77.44\%$$

This second approach tells us that we've got an exponential relationship. This time things are a little different because the numbers are decreasing, but the general idea is the same.

If we let $P(a)$ represent the atmospheric pressure at altitude a, then our relationship is

$$P(a) = 1 \cdot (0.88)^a$$

Note that the altitude value goes in steps of 500 meters, so when $a = 2$ that represents an altitude of $2 \cdot 500 = 1000$ meters.

To calculate the pressure at 2500 meters, we should notice that this is 5 "steps" of 500 meters each. So, $a = 5$ and the pressure at that altitude is:

$$(P(5) = 1 \cdot (0.88)^5 \approx 0.528$$

According to our model, altitude sickness sets in at around 52.8% of normal atmospheric pressure. This number seems overly precise; it's probably better to say "around 50%".

The death zone is at 8000 meters, or when $a = 16$ in our model. The pressure at that altitude is:

$$P(16) = 1 \cdot (0.88)^{16} \approx 0.129$$

Our model predicts that atmospheric pressure of around 13% is life-threatening.[9]

10.5.1 Exponentially Decreasing

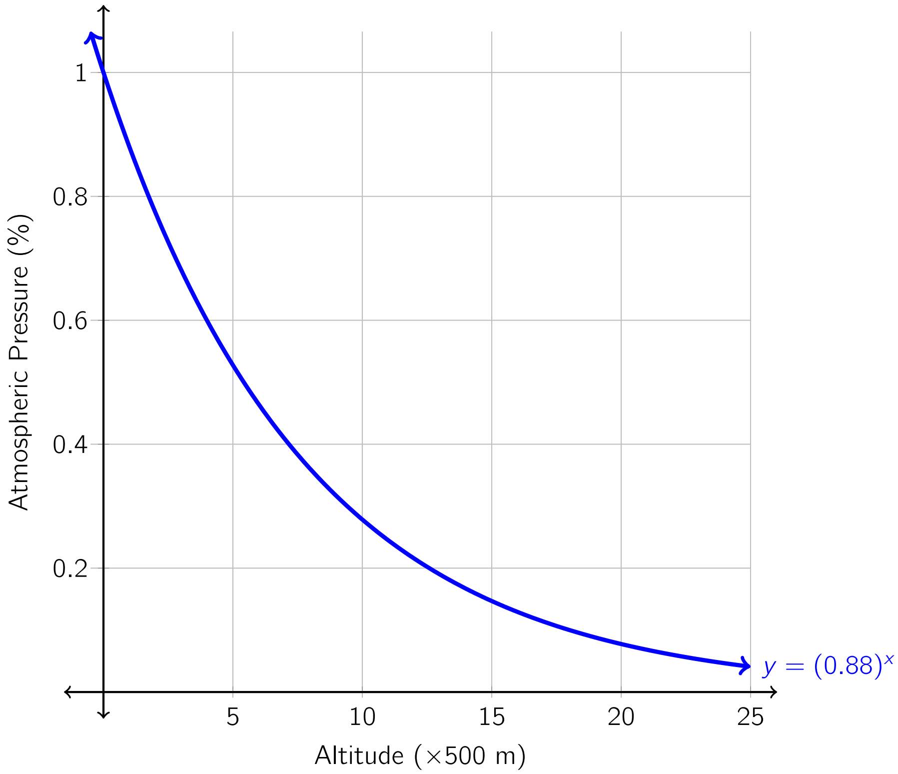
Exponential decay describes a relationship which decreases exponentially. These have the same general formula for exponential relationships — we have an initial value a and a constant multiplier b — but now the constant multiplier is a fraction between 0 and 1.

Exponential Decay

An equation of the form $f(x) = a \cdot b^x$ represents exponential decay when $a \neq 0$ and $0 < b < 1$.

The graph of our atmospheric pressure example shows the familiar J-like shape, only backwards: a decreasing trend.

9 For reference, most commercial airlines cruise at an altitude of around 12,000 m and pressurize their cabin to approximate an altitude of around 2100 m.



As with exponential growth, we distinguish between *decay factor* and *decay rate*. The decay rate is the percent by which some value decreases. It tells us the amount that has "disappeared". The decay factor is another word for the constant multiplier. It will tell us what to multiply by to get what is "left over".

If air pressure decreases by 12% per step change in elevation, then the decay rate is 12%. This means that 88% of the pressure "remains", and so this is the decay factor. The relationship between these two values is straight forward: *decay factor* = 1 − *decay rate*. One way to write an equation for exponential decay uses the decay rate.

Definition

Exponential decay is described by an equation of the form

$$f(x) = a(1 - r)^x$$

where a represents the initial value, and r represents the decay rate (a percent, written as a decimal).

Example 10.5

In the late 1990's, the Cheeseville Zoo saw some hard times. After a record high attendance of 1,000,000 visitors in 1995, attendance declined at a rate of 26% per year. Approximatley how many people attended the zoo in 2000? In what year did attendance first drop below 100,000?

Solution: A decay rate of 26%, means we can write a rule using the decay factor $1 - 0.26 = 0.74$ or 74%. So, let's create a rule

$$A(t) = 1\,000\,000(0.74)^t$$

to represent the attendance after t years (where 1995 is "year 0").

In this model, the year 2000 is when $t = 5$. In that year, the attendance was:

$$A(5) = 1\,000\,000(0.74)^5 \approx 221\,900$$

To find when attendance first dropped below 100,000, we have to do some hunting. We find the two values:

$$A(7) \approx 121\,513$$
$$A(8) \approx 89\,919$$

This tells us that "year 8", meaning calendar year 2003, was the first time attendance dropped below 100,000 visitors.

10.5.2 Linear Versus Exponential Change

Before this chapter, we spent most of our time discussing linear equations. We have seen situations that can be represented quite effectively using linear functions. For example, exchanging money between two different currencies, the amount a person might earn from working a certain number of hours at a particular pay rate, or how much distance a person can cover when moving at a constant speed.

As we have seen, though, many important natural phenomena don't follow straight lines. Fundamental aspects of biology and physics are build around relationships that follow curved lines: population growth over time, the decrease of atmospheric pressure with altitude.

It's sometimes hard to appreciate the difference between a curved line and a straight line, and it might not seem like all that big of a deal. After all, a straight line could keep pace with an exponential curve, if the line were steep enough... right?

To compare linear and exponential growth, we close this chapter with a famous fable first told by the Persian poet Ferdowsi.

The Chessboard

Many centuries ago, so the story goes, the inventor of the game of chess showed his invention to the king. The king was so delighted by the new game that he told the inventor that he could have whatever he liked as compensation.

The inventor proposed that the king give him one grain of wheat for the first square of the chessboard, two grains of wheat for the second square, four grains of wheat for the third square, and so on, doubling the number of grains for each of the 64 squares on the board.

The king was shocked that the inventor should ask such a low price and offered instead to pay *one million grains of wheat* for every single square of the chessboard! The inventor refused, saying he much preferred his original proposal.

The king agreed to pay the inventor's price, chuckling to himself that the foolish man had asked for so little.

But when the time came for the king to pay the debt, he was no longer chuckling. What happened?

Suppose we use some algebra to compare the different proposals. The king is offering to pay 1 million grains of rice every for every square on the chessboard. We can represent his proposal with the equation

$$f(x) = 1\,000\,000\,x,$$

where $f(x)$ is number of grains of wheat that the inventor earns from x squares on the chessboard. This is a straight line with slope of one million. The enormous slope means that the line is so steep it would be indistinguishable from a vertical line on most graphs.

The inventor's proposal is a relatively tame exponential function: start with 1, double the previous value. That is the function

$$g(x) = 2^x,$$

where $g(x)$ represents the number of grains on square x of the chessboard. Note that $f(x)$ and $g(x)$ are measuring different things: f gives us a running total after x squares, whereas g gives us the amount sitting on the single square x (to find the total number of grains, we'd have to add up these values).

At first glance, there hardly seems to be a competition between these two: the super-steep linear function should dominate the mild exponential function... But have a close look at the graphs in section 10.5.2, which show the behavior of these graphs across 8, 16, 32, and 64 squares of the chessboard. What are these graphs communicating?

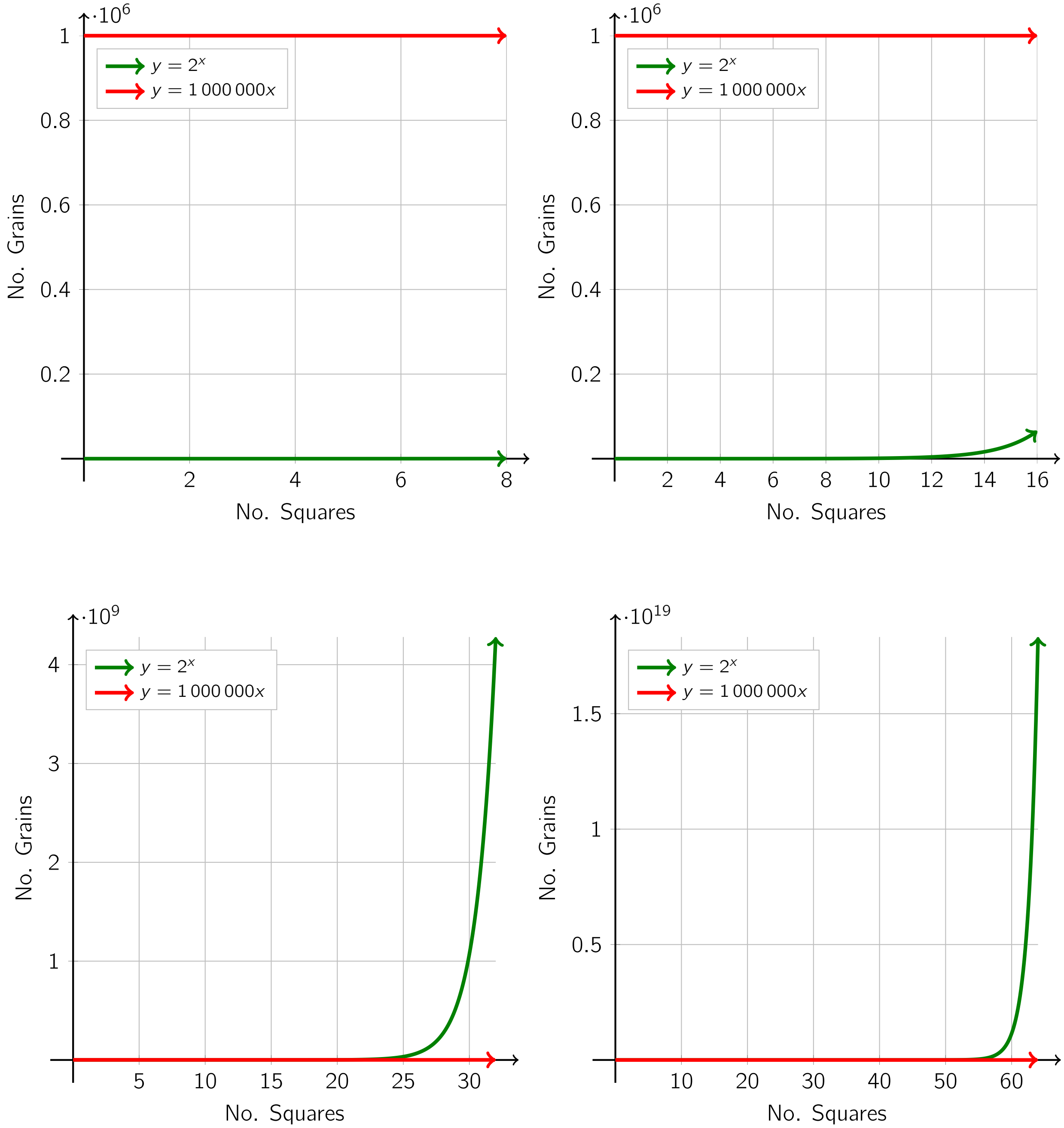


Figure 10.2: Growth in the chessboard fable: 8, 16, 32, and 64 squares.

Still need to find a quote that works for this chapter. In the meantime, we have this.

Author
Description of author

Chapter 11

Exponential Expressions

Every chapter should have a lead paragraph – even just a short one – that appears before the first heading. This is a placeholder paragraph which will at some point be replaces by actual content.

11.1 Fundamentals of Exponents

11.2 Simplified Algebraic Expressions: Exponents Edition

In earlier chapters we learned properties that could be used to simplify linear expressions. The criteria we learned in chapter 5 for "simplified algebraic expressions" still apply, but now that we are dealing with exponents, there are a few new criteria that we need to add. We will summarize the criteria here and then learn the details over the next few sections.

Simplified Exponential Expressions

An exponential expression is considered completely simplified if...

1. It contains only positive exponents
2. Bases appear at most once per term
3. Powers with a numeric base are written in standard form (if that's reasonable)
4. It contains no explicit grouping symbols

Criteria #1 asks for only positive exponents. In section 10.1.1, we saw how to use the reciprocal to rewrite a negative exponent using a positive exponents. We'll see more about negative exponents in section 11.4.

Criteria #2 prohibits a term to be written in the form $x^2 \cdot x^3$, since this is a term that has the same base (x) appearing twice. In a way, we're looking for a kind of "combining like terms" that will allow us to write this as x raised to a single exponent. Properties of this kind will be the main focus in this chapter.

Criteria #3 asks us to evaluate expressions like 2^3 and write 8 instead... if that's reasonable. We have already seen that exponential expressions can yield some enormous numbers, so it's better to write 6^{12} than 2 176 782 336. Use your best judgement here: a number that's larger than, say, five digits should probably be left in exponential form. It's also best to continue our convention of using simplified improper fractions instead of decimals.

Criteria #4 is already in effect: explicit grouping symbols have been banned from simplified algebraic expressions since chapter 5. We mention it again here because some special care is required when it comes to eliminating groupers that are tangled up with exponents.

Remember: Order of Operations

The expression $2 \cdot 3^x$ is completely simplified. We might be tempted to write 6^x, but this would be Evil and Wrong. Since exponents must be evaluated before multiplication, it is impossible to fully evaluate this expression, We don't know what x is, and so it is impossible to resolve the expression 3^x, and so we cannot do the multiplication. So, we stop here.

Exponent Properties

In the next few sections we will derive the properies for simplifying exponential expressions and achieveing simplified exponential form.[1] With each new property, work out the initial examples and think about how to state a general simplification rule.

[1] These aren't all of the definitions and properties for exponents, but they are the key ones we need for now. We will learn a few more tools in later chapters when we look at quadratics, polynomials, and radicals. There are also a few saved for algebra 2, when we will see non-integer exponents and logarithms.

11.3 Product Properties

11.3.1 The Product Rule

Startup Exploration: Derivation #1

Use expanded form to write each of the expressions below as a single base raised to a singe exponent.

$$x^4 \cdot x^3 \qquad \text{and} \qquad z^5 \cdot z^8$$

Make a conjecture about a general rule for simplifications of this type.

This shortcut is called the product rule, since it applies to the situation in which we are finding the product of two exponential expressions with the same base.

We can write the first example out in expanded form, and then rewrite the long string of x's under a single exponent:

$$x^4 \cdot x^3 = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x = x^7$$

The same goes for the z's in the second expression:

$$z^5 \cdot z^8 = z \cdot z \cdot z \cdot z \cdot z \cdot z \cdot z \cdot z \cdot z \cdot z \cdot z \cdot z \cdot z = z^{13}$$

The new exponent will be the sum of the original exponents!

Product Rule of Exponents

For any nonzero real number a and any integers m and n,

$$a^m \cdot a^n = a^{m+n}.$$

The intuition behind this rule is that $a^m \cdot a^n$ means we have m factors of a multiplied by n factors of a, and so we have $m + n$ factors of a in all. That means a^{m+n}. This is pretty straightforward! There's really no need to "memorize" this rule. We can always re-derive it if we forget exactly how it works.

Example 11.1

Simplify $3b^2 \cdot 4b^5$.

Solution: In this example, we will need to use the commutative property of multiplication to move the coefficients together and the variable parts together. In other words:

$$3b^2 \cdot 4b^5 = 3 \cdot 4 \cdot b^2 \cdot b^5 = 12b^{2+5} = 12b^7.$$

11.3.2 The Power Rule

Startup Exploration: Derivation #2

Use expanded form to write each of the expressions below as a single base raised to a singe exponent.

$$(x^3)^4 \qquad \text{and} \qquad (z^5)^8$$

Make a conjecture about a general rule for simplifications of this type.

This simplification is called the power rule, since it applies to situations in which we have a power which is itself being raised to a power.

Writing the first expression in expanded form gives us a product. From there we can apply the product rule that we learned in the last section:

$$(x^3)^4 = x^3 \cdot x^3 \cdot x^3 \cdot x^3 = x^{3+3+3+3} = x^{4 \cdot 3} = x^{12}$$

Notice that we get a repeated addition in the exponent, which we can rewrite as a multiplication. In the second expression, we have:

$$(z^5)^8 = z^5 \cdot z^5 \cdot z^5 \cdot z^5 \cdot z^5 \cdot z^5 \cdot z^5 \cdot z^5 = z^{8 \cdot 5} = z^{40}$$

The new exponent is the product of the original exponents!

Power Rule of Exponents

For any nonzero real number a and any integers m and n,

$$(a^m)^n = a^{n \cdot m}$$

In this case, the intuition is that the outermost exponent means we will multiply together n factors of a^m. Since each of those factors has m factors of a, we have "n groups of m factors of a" in all, which is another way of saying $n \cdot m$ factors of a.

Example 11.2

—

11.3.3 Product to a Power

Startup Exploration: Derivation #3

Use expanded form to write each of the expressions below without parentheses.

$$(xy)^4 \qquad \text{and} \qquad (abc)^8$$

Make a conjecture about a general rule for simplifications of this type.

In this case, we are taking the power of a product so we call this shortcut "power of a product" or "product to a power". We can write out the first expression in expanded form, and then use the commutative property to rearrange the factors:

$$(xy)^4 = (xy)(xy)(xy)(xy) = xy\,xy\,xy\,xy = xxxx\,yyyy = x^4y^4$$

The second example works just the same way:

$$\begin{aligned}(abc)^8 &= (abc)(abc)(abc)(abc)(abc)(abc)(abc)(abc)\\ &= abc\;abc\;abc\;abc\;abc\;abc\;abc\;abc\\ &= aaaaaaaa\;bbbbbbbb\;cccccccc\\ &= a^8b^8c^8\end{aligned}$$

We can remove the parentheses by attaching the exponent to each of the factors inside!

Product to a Power Rule

For any nonzero real numbers a and b and any integer m,

$$(ab)^m = a^m b^m$$

The idea is that the exponent gives us m copies of the product, which means m copies of each of the factors in that product. The commutative property of multiplication allows us to move these factors around and regroup them.

Note that the thing in the parentheses is a product, *not a sum*. We'll learn how to handle sums raised to powers, for example $(x + y)^2$, in section 11.5.

Example 11.3

—

11.4 Quotient Properties

The rules in the last section were all based on multiplication. We'll look at some division-related laws, which behave in a very similar way (since multiplication and division are inverse operations). We pull these rules out into their own section since they bring us into closer contact with negative exponents, which take some getting used to.

11.4.1 Simplifying Negative Exponents

Startup Exploration: Derivation #4

Use the definition of a negative exponent to write each of the expressions using only positive exponents.

$$\frac{a^{-2}}{b} \qquad \text{and} \qquad \frac{2x^3y^{-2}}{m^{-5}q^2}$$

Make a conjecture about a general rule for simplifications of this type.

We'll use the definition of negative exponents to work these out, but sometimes that can involve numerous steps. For example in the first expression, we have:

$$\frac{a^{-2}}{b} = \frac{\frac{1}{a^2}}{b} = \frac{1}{a^2} \div b = \frac{1}{a^2} \cdot \frac{1}{b} = \frac{1}{a^2 b}$$

After going through all of that work, we simply end up with a^2 in the denominator. In the second example, we can see that the 2, the x^3, and the q^2 have positive exponents already and are going to stay right where they are. Let's extract the expressions with negative exponents and treat those separately:

$$\frac{2x^3y^{-2}}{m^{-5}q^2} = \frac{2x^3}{q^2} \cdot \frac{y^{-2}}{1} \cdot \frac{1}{m^{-5}} = \frac{2x^3}{q^2} \cdot \frac{1}{y^2} \cdot \frac{m^5}{1} = \frac{2x^3m^5}{q^2y^2}$$

When we see a negative exponent, we can move that base to the other part of the fraction and change the sign of the exponent.

Beware of the Headless Fraction

We may write a number without a denominator. The number 41, for example, has no denominator — or rather, it has a phantom 1 in the denominator which we don't write unless we need to.

We cannot write a number without a numerator. For example, if we are asked to simplify the expression g^-6, we can use the shortcut and put g^6 in the other part of the fraction, but now we have an empty numerator...

$$g^{-6} = \frac{\quad}{g^6} \quad \longleftarrow \text{ what goes here?}$$

We can't leave the numerator empty because then we don't have a number that makes sense. "Empty numerator" might imply "numerator = 0", but that would make the whole fraction equal to 0, which would be wrong: if $g \neq 0$, then $\frac{1}{g} \neq 0$.

It must be that the numerator is 1. Indeed this makes sense, for we can introduce a phantom 1 before we apply the shortcut:

$$g^{-6} = 1 \cdot g^{-6} = \frac{1}{g^6}$$

Notice that in the first example of derivation #4, we filled the empty numerator with 1, just as we did here.

11.4.2 Quotient Rule

Startup Exploration: Derivation #5

Use expanded form to write each of the expressions below as a single base raised to a singe exponent.

$$\frac{m^6}{m^2} \qquad \text{and} \qquad \frac{z^5}{z^8}$$

Make a conjecture about a general rule for simplifications of this type.

This simplification is called the quotient rule, since it applies when we are finding the quotient of two exponential expressions with the same base.

When we write out the first expression in expanded form, we have some common factors of x which "cancel" from the numerator and denominator.

$$\frac{m^6}{m^2} = \frac{m\,m\,m\,m\,m\,m}{m\,m} = \frac{m\,m\,m\,m\,\cancel{m}\,\cancel{m}}{\cancel{m}\,\cancel{m}} = \frac{m\,m\,m\,m}{1} = m^4$$

Here, we replace the empty denominator with a soon-to-be phantom 1, which promptly vanishes. In the second example, the leftovers are in the denominator.

$$\frac{z^5}{z^8} = \frac{z\,z\,z\,z\,z}{z\,z\,z\,z\,z\,z\,z\,z} = \frac{\cancel{z}\,\cancel{z}\,\cancel{z}\,\cancel{z}\,\cancel{z}}{\cancel{z}\,\cancel{z}\,\cancel{z}\,\cancel{z}\,\cancel{z}\,z\,z\,z} = \frac{1}{z^3} = z^{-3}$$

The new exponent is the difference of the original exponents: numerator minus denominator!

Quotient Rule

For any nonzero real number a and any integers m and n,

$$\frac{a^m}{a^n} = a^{m-n}$$

The intuition here is that when we have a quotient involving two expressions with the same base, there are going to be common factors in the numerator and denominator. Some of these will "cancel", and in fact either the whole numerator or the whole denominator (or both!) will be reduced simply to the number 1.

Since multiplication and division are inverse operations, this rule can be seen as a version of the product rule if we allow negative exponents:

$$\frac{a^m}{a^n} = a^m \cdot a^{-n} = a^{m+^-n} = a^{m-n}$$

This rule also helps us to understand the zero exponent. We know that anything divided by itself is 1, and so for instance:

$$\frac{a^m}{a^m} = 1 \qquad \text{since a number divided by itself is 1}$$

If we interpret this using the quotient rule, we have:

$$\frac{a^m}{a^m} = a^{m-m} = a^0 \qquad \text{application of the quotient rule}$$

This means we have two different interpretations of the same quantity, and so these two interpretations must be equal. In other words, it must be that:

$$a^0 = 1.$$

11.4.3 Quotient to a Power

Startup Exploration: Derivation #6

Use expanded form to write each of the expressions below without parentheses.

$$\left(\frac{x}{y}\right)^2 \qquad \text{and} \qquad \left(\frac{ab}{c}\right)^7$$

Make a conjecture about a general rule for simplifications of this type.

By now, you may be getting the hang of these simplification properties! In this case, we have

$$\left(\frac{x}{y}\right)^2 = \left(\frac{x}{y}\right)\left(\frac{x}{y}\right) = \frac{x \cdot x}{y \cdot y} = \frac{x^2}{y^2}$$

The second example is longer, but works the same:

$$\begin{aligned}
\left(\frac{ab}{c}\right)^7 &= \left(\frac{ab}{c}\right)\left(\frac{ab}{c}\right)\left(\frac{ab}{c}\right)\left(\frac{ab}{c}\right)\left(\frac{ab}{c}\right)\left(\frac{ab}{c}\right)\left(\frac{ab}{c}\right) \\
&= \frac{ab\,ab\,ab\,ab\,ab\,ab\,ab}{c\,c\,c\,c\,c\,c\,c} \\
&= \frac{(ab)^7}{c^7} \\
&= \frac{a^7b^7}{c^7}
\end{aligned}$$

In general, when we have a quotient raised to a power, we can apply the exponent to both the numerator and denominator. To eliminate the parentheses entirely in the second example, we had to use the power of a product property to simplify the numerator.

Quotient to a Power Rule

For any nonzero real numbers a and b and any integer m,

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

The exponent means that we have m copies of the fraction $\frac{a}{b}$ which, when we multiply, become a single fraction with m copies of a in the numerator and m copies of b in the denominator.

11.4.4 Combo Problems

The second example in derivation #6 required us first to simplify the power of a quotient, and then the power of a product. Smashup problems like this require some careful dissection. Consider the following nasty-looking expression

$$\frac{(5x^3)(3x^{-2})}{30x^{12}}.$$

With a monster like this, it is sometimes hard to know how to begin. A good strategy is to proceed one step at a time, carefully applying the properties from this section to try and make this slightly simpler with each step.

For instance, we might start by multiplying in the numerator. We'll have a coefficient $(5 \cdot 3 = 15)$, and we'll use the product property to multiply the variables $(x^3 \cdot x^{-2} = x^{3+-2} = x^1)$. This gives us:

$$\frac{(5x^3)(3x^{-2})}{30x^{12}} = \frac{15x}{30x^{12}}.$$

Now, we'll need the quotient property to simplify the variables. The coefficients also share a common factor of 15. This will eliminate the numerator entirely!

$$\frac{(5x^3)(3x^{-2})}{30x^{12}} = \frac{15x}{30x^{12}} = \frac{1}{2x^{11}}.$$

That final answer might not look all that "simple", but it meets the requirements of a simplified exponential expression, and so we're done. Try the following examples, and then study the solutions carefully.

Example 11.4

Simplify each of the following.

$$\left(\frac{x^3y^{-2}}{x^{-1}y^5}\right)^3 \qquad \text{and} \qquad \frac{8x^2y}{y^2} \div \frac{16xy^2}{(4y^2)^2}$$

Solution: We recommend simplifying what's inside those parentheses before dealing with the outermost exponent. To simplify the insides, we'll need to move some negative exponents, and then apply the product property.

$$\left(\frac{x^3y^{-2}}{x^{-1}y^5}\right)^3 = \left(\frac{x^3x^1}{y^5y^2}\right)^3 = \left(\frac{x^{3+1}}{y^{5+2}}\right)^3 = \left(\frac{x^4}{y^7}\right)^3 = \frac{x^{4\cdot 3}}{y^{7\cdot 3}} = \frac{x^{12}}{y^{21}}$$

The last thing we did was apply the power of a quotient to get our final answer. To tackle the second problem, recall that division is like multiplication by the reciprocal.

$$\frac{8x^2y}{y^2} \div \frac{16xy^2}{(4y^2)^2} = \frac{8x^2y}{y^2} \cdot \frac{(4y^2)^2}{16xy^2}$$

Then, before we can apply any of the product properties, we need to fix the product of a power, which will in turn require the power property: $(4y^2)^2 = 4^2 \cdot (y^2)^2 = 16y^{2\cdot 2} = 16y^4$. From here, we can simplify the individual fractions, then apply the product property to multiply them, and then apply the quotient property (again) to simplify the result.

$$\begin{aligned}
\frac{8x^2y}{y^2} \div \frac{16xy^2}{(4y^2)^2} &= \frac{8x^2y}{y^2} \cdot \frac{(4y^2)^2}{16xy^2} \\
&= \frac{8x^2y}{y^2} \cdot \frac{16y^4}{16xy^2} \\
&= \frac{8x^2}{y} \cdot \frac{y^2}{x} \\
&= \frac{8x^2y^2}{yx} \\
&= 8xy
\end{aligned}$$

Some of the simplification steps we took above could have been completed in a different order. There are multiple paths to the most simplified form!

11.5 Raising Sums to Powers

Startup Exploration: Check This Work!

The following is a very common error:

$$(a+b)^2 = a^2 + b^2 \qquad \text{(nice try, but wrong).}$$

This equation looks reasonable, but it is not true most of the time. Find three different pairs of values a and b which demonstrate that this equation does not work in general. Can you find any "lucky" values of a and b that do satisfy this equation?

As we see in the startup exploration, $(x+3)^2 \neq x^2 + 3^2$. The temptation to "sprinkle" an exponent over a sum — using a kind of exponent distributive property — is Evil and Wrong. We *will* use the distributive property to work out what it means to raise a sum to a power, but there's more to it than a simple sprinkle.

We learned the distributive property in section 5.1, and it has come in quite handy since then, for example when converting the point-slope form of a line to slope-intercept form. Recall that the distributive property tells us that for real numbers a, b, and c:

$$a \cdot (b+c) = a \cdot b + a \cdot c$$

But suppose "a" isn't just a number, but an expression. For example, given the expression

$$(x+3)^2 = (x+3)(x+3)$$

we can use the distributive property (twice, in fact) to simplify the right-hand side. Observe:

$$\begin{aligned}
(x+3)^2 &= (x+3)(x+3) \\
&= (x+3) \cdot x + (x+3) \cdot 3 && \text{distribute the expression (x+3)} \\
&= x \cdot x + 3 \cdot x + x \cdot 3 + 3 \cdot 3 && \text{distribute in each of the resulting expressions} \\
&= x^2 + 3x + 3x + 9 && \text{simplify} \\
&= x^2 + 6x + 9 && \text{combine like terms}
\end{aligned}$$

We do a kind of "double distribution": first with one set of parentheses over the other, and then distributing over the results.

Example 11.5

Simplify $(3x-5)^2$.

Solution: The exponent means that we're multiplying this expression by itself, so we will carry out "double distribution":

$$\begin{aligned}(3x-5)^2 &= \underline{(3x-5)}(3x-5) \\ &= 3x\underline{(3x-5)} - 5\underline{(3x-5)} \qquad \text{distribute in each of the resulting expressions}\end{aligned}$$

So, in the end, $(3x-5)^2 = 9x^2 - 30x + 25$.

11.5.1 Old School Multiplication

There can be a lot of stuff to keep track of in these multiplications. A technique that can be helpful is to set up the multiplication like an "old school" (literally) multiplication problem from back in your elementary school days.

Recall how to multiply 43×43 via long "multiplication". We interpret "43" in a more expanded form as "40+3", then we break this multiplication into two steps. We first multiply 3×43 to get the first *partial product* of 129. Then, we multiply 40×43 to get the second partial product 1720.

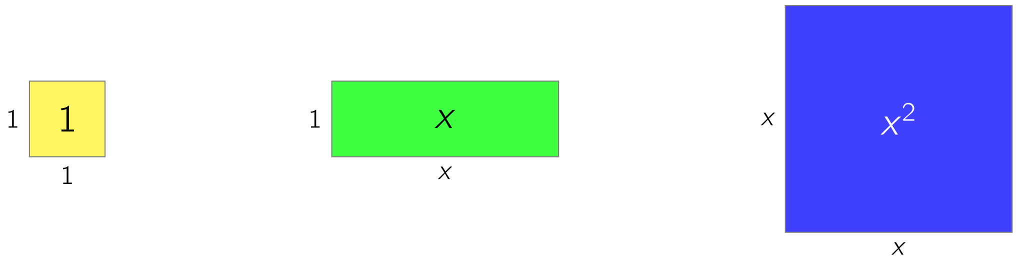
$$\begin{array}{ccccc l} & & & 4 & 3 & \\ & \times & & 4 & 3 & \\ \hline & & 1 & 2 & 9 & \quad \color{red}{(3 \times 43)} \\ + & 1 & 7 & 2 & 0 & \quad \color{red}{(40 \times 43)} \\ \hline & 1 & 8 & 4 & 9 & \end{array}$$

We can use this helpful organization scheme to multiply $(4x+3)(4x+3)$.

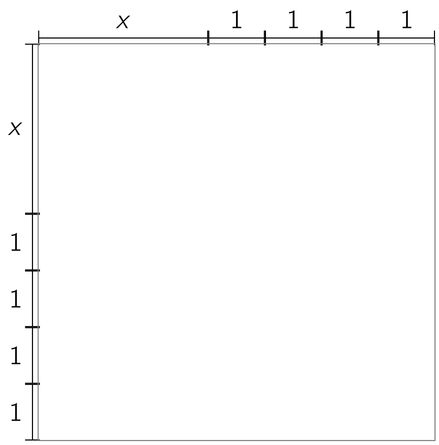
$$\begin{array}{cccccccl} & & & 4x & + & 3 & \\ & \times & & 4x & + & 3 & \\ \hline & & & 12x & + & 9 & \quad \color{red}{3(4x+3)} \\ + & 16x^2 & + & 12x & & & \quad \color{red}{4x(4x+3)} \\ \hline + & 16x^2 & + & 24x & + & 9 & \end{array}$$

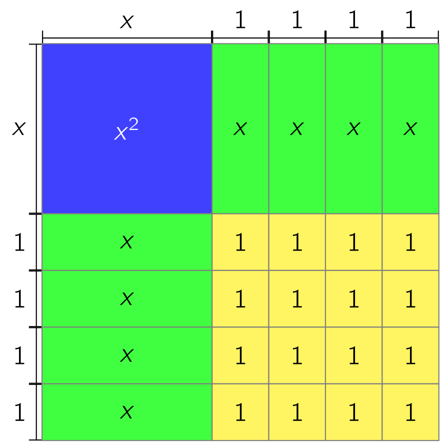
11.5.2 Algebra Tiles

Another helpful visual may be to draw "algebra tiles".[2] In a set of algebra tiles, we pick a short length to represent 1 and a long length to represent x. Then, we create a set rectangular tiles:



To multiply, say, $(x+4)^2$, we create a rectangle (a square, actually) that is "x+4" units on each side, and we fill in the area with our tiles.





The area is filled in with one x^2-tile, eight x-tiles, and sixteen 1-tiles. So, we have a total area of $x^2+8x+16$.

11.5.3 Beyond Squares

Of course, the methods above all work for multiplying other expressions as well — not just something times itself. Can you extend the models above to find the product of, say,

$$(3x+4)(2x+3)$$

When we multiply, we get the product $6x^2+17x+12$. How can we get to this product using the "double distribution" approach? What about the old-school method? What would the algebra tiles area model look like?

[2] Sometimes drawing a picture is enough. It can also be helpful, though, to actually manipulate physical or virtual tiles. For example you could cut a set of algebra tiles from cardstock, or do an internet search for an online algebra tiles simulator.

We can also raise an expression to a power other than 2. For instance, we could raise an expression to the third power, as in the example below. There's a lot going on here, read through the work slowly and see if you can understand each step.

$$
\begin{aligned}
(2x+5)^3 &= (2x+5)(2x+5)(2x+5) && \\
&= (4x^2+20x+25)(2x+5) && \text{multiply the first two terms} \\
&= 2x(4x^2+20x+25)+5(4x^2+20x+25) && \text{distribute the three-term expression!} \\
&= \underline{8x^3+40x^2+50x}+\underline{20x^2+100x+125} && \text{distribute twice more} \\
&= 8x^3+60x^2+150x+125 && \text{combine like terms}
\end{aligned}
$$

We must say that there are as many squares as there are numbers.

Galileo Galilei
Italian physicist and astronomer

Chapter 12

Quadratic Equations

In chapter 5 we learned a set of tools in for solving linear equations. Before, that we saw quadratic patterns for the first time back in chapter 2. In this chapter, we'll see what makes quadratic equations different from those we've studied until now, and then we'll learn techniques that will help us to solve quadratic equations.

12.1 Challenges to Solving Quadratics

Exploration

Consider the following equation. Do we know what we need to know to solve this equation, without any guesswork? In other words: Do we have POEs, axioms, or other properties that will allow us to isolate x?

$$x^2 + 2x - 15 = 20$$

If you can, solve the equation. Otherwise, identify where you get stuck. What new tools would be helpful for solving this equation?

Our task in this chapter is to be able to solve quadratic equations like the one given in the startup exploration. We won't give the details for how to solve this equation yet – we'll develop those ideas over the next few sections. For now, we will simply point out a few features.

Using what we know so far, it's impossible to isolate x. We can make some progress:

$$\begin{aligned} x^2 + 2x - 15 &= 20 \\ x^2 + 2x &= 35 \qquad \text{APOE, to get all the numbers to the right-hand side} \end{aligned}$$

But now what? We can't combine like terms on the left-hand side, and subtracting anything from that side would give us x's on both sides of the equation (making things worse). We might try undoing the distributive property on the left-hand side. That would give us

$$x(x+2) = 35.$$

But this doesn't seem to be much of an improvement either. Using DPOE on the left-hand side to isolate x would move $(x+2)$ to the right hand side of the equation (and vice versa). It seems that we'll need some other techniques to help us out of this situation.

If we're allowed to just solve the equation my making a clever observation, we might notice that

$$5(7) = 5(5+2) = 35,$$

and so $x = 5$ is a solution to the equation! That's progress. But, it might not be obvious that -7 is *also* a solution to the equation, since

$$-7(-7+2) = -7(-5) = 35.$$

Plus with a different equation, it might not be quite so easy to see a solution just by inspection. Our new techniques should help us overcome these challenges.

12.1.1 Rectangles and Squares

As we saw in chapter 2, quadratic sequences can be related to rectangles and squares. Our very first quadratic sequence was related to rectangles (fig. 12.1), and so are the familiar perfect squares (fig. 12.2).

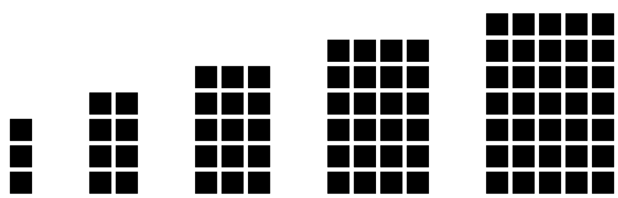


Figure 12.1: Some "rectangular numbers": $3, 8, 15, 24, 35, \ldots$

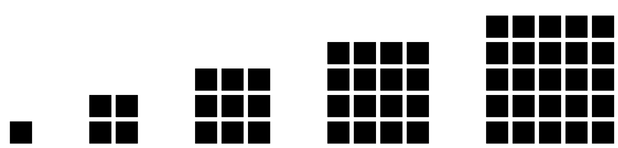


Figure 12.2: The perfect squares: $1, 4, 9, 16, 25, \ldots$

The perfect squares have a straightforward formula. If we let $f(x)$ represent the area of figure x, then the perfect squares are represented by the formula

$$f(x) = x^2.$$

This is the parent function of the quadratic family.

The rectangles in fig. 12.1 also have a formula. If we let $g(x)$ represent the area of figure x, then we might notice that figure x is x units wide and $(x+2)$ units tall. So these rectangles are represented by the formula

$$g(x) = x(x+2).$$

We can simplify this formula using the distributive property:

$$g(x) = x(x+2) = x^2 + 2x$$

We have learned that the highest degree term in a quadratic equation is an x^2 term. The connection between a quadratic rule and rectangles will help us when it comes to solving quadratic equations. In fact, we will solve these equations using the beautiful symmetry of the square.

12.2 Level 1 and Level 2 Quadratics

As we did with linear equations, we'll treat quadratic equations like a game. Level 1 is the easiest kind of quadratic equation to solve, so this is where we'll start. Then, we'll keep things interesting as we increase our skill level by adding challenge and complexity along the way.

12.2.1 Level 1 Quadratics

Startup Exploration: Quadratic Level 1

Determine the value of x given the equation: $x^2 = 64$.

Talk about starting with the easy stuff. We're looking for a number x which, when multiplied by itself, is equal to 64. In other words, we are looking for the "square root of 64". Clearly, $x = 8$ is a solution to this equation. But we can't be too hasty! Notice that $x = -8$ is also a solution, since $(-8)^2 = (-8)(-8) = 64$. So, this equation has two solutions:

$$x = 8 \ \underline{\text{or}} \ -8$$

If we prefer to write our answer in solution set notation, we have

$$\mathcal{S} = \{8, -8\}.$$

So, Level 1 quadratics are pretty easy: we simply take the square root of both sides of the equation.[1] We might have a hard time if the constant value is not a perfect square, as in $x^2 = 12$. But, we'll learn more about handling the square roots of non-perfect-squares soon enough (in chapter 13, to be precise).

This seems like a good time to suggest that it may come in handy to memorize the first 25 or so perfect squares, for quick recognition when they come up in a problem.

$1^2 = 1$	$2^2 = 4$	$3^2 = 9$	$4^2 = 16$	$5^2 = 25$
$6^2 = 36$	$7^2 = 49$	$8^2 = 64$	$9^2 = 81$	$10^2 = 100$
$11^2 = 121$	$12^2 = 144$	$13^2 = 169$	$14^2 = 196$	$15^2 = 225$
$16^2 = 256$	$17^2 = 289$	$18^2 = 324$	$19^2 = 361$	$20^2 = 400$
$21^2 = 441$	$22^2 = 484$	$23^2 = 529$	$24^2 = 576$	$25^2 = 625$

[1] More soon on square roots, including why and under what circumstances "square root of both sides" is, in fact, a property of equality.

Finally, note that zero is also a perfect square, since $0^2 = 0$. Zero, in fact, is the only number that has only one square root. Whereas both 3 and −3 are square roots of 9, the only square root of 0 is 0.

»> Get into negatives under the radical now, or save that for later (it currently appears in the next chapter).

12.2.2 Level 2 Quadratics

Startup Exploration: Quadratic Level 2

Determine the value of w given the equation: $(w + 3)^2 = 16$.

Here, we're told that something-squared is 16. Well, that means that the something in question must either be 4 or negative 4. That is to say, $(w + 3)$ is either 4 or −4. So, this equation is actually two equations at once. We have:

$$w + 3 = 4 \qquad \underline{\text{or}} \qquad w + 3 = -4$$

Solving these equations (using SPOE in both cases), we have that w must be either 1 or −7. So, those are our two solutions: $\mathcal{S} = \{1, -7\}$.

We can record this work in a down-the-page format, like so:

$$\begin{aligned} (w+3)^2 &= 16 && \\ w + 3 &= 4 \ \underline{\text{or}}\ -4 && \text{square root of both sides} \\ w &= 1 \ \underline{\text{or}}\ -7 && \text{SPOE: subtract 3 throughout} \end{aligned}$$

A few things to note: First, we can't subtract 3 from both sides as the very first step. The parentheses require that we undo the exponent first. Second, beware the use of $\pm$. It may be tempting to use this shorthand notation and write

$$w + 3 = \pm 4,$$

but then it is also tempting to subtract 3 and write

$$w = \pm 1. \quad \text{Nope!}$$

We recommend splitting into two equations, or writing out the two solutions explicitly using the word "or".

Example 12.1

Determine the value of a given the equation: $(a-1)^2 - 2 = 23$.

Solution: This equation requires an extra step, but it's quickly transformed into an equation like the one from the startup exploration.

$$\begin{aligned} (a-1)^2 - 2 &= 23 && \\ (a-1)^2 &= 25 && \text{APOE} \\ a - 1 &= 5 \ \underline{\text{or}} \ {}^-5 && \text{square root of both sides} \\ w &= 6 \ \underline{\text{or}} \ {}^-4 && \text{APOE: add 1 throughout} \end{aligned}$$

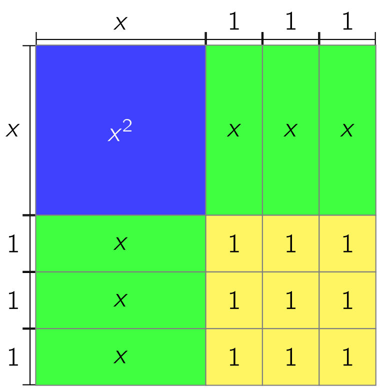
In the end, we have $\mathcal{S} = \{6, {}^-4\}$.

12.3 Level 3 Quadratics

Startup Exploration: Quadratic Level 3

Use the sum to a power property (from section 11.5) to write the expression $(x+3)^2$ without parentheses. Then, determine the value of x given the equation: $x^2 + 6x + 9 = 49$.

To expand $(x+3)$, we might think of using algebra tiles to fill in a square with side length $(x+3)$. Or, we could just "sketch" the algebra tiles diagram (shown below, on the right) and calculate the areas of the four regions.



	x	3
x	x^2	$3x$
3	$3x$	9

Note how we have simplified the picture in the "sketch" version. For example, rather than draw three 3-unit-by-x-unit rectangles, we simply write the area of the rectangle $3x$. In the lower right-hand region, we write the area 9 rather than draw nine yellow squares.

Diagrams like this help us to see the relationship between the expressions:

$$(x+3)^2 = x^2 + 6x + 9.$$

Using this fact, we can solve the given equation:

$x^2 + 6x + 9 = 49$	
$(x+3)^2 = 49$	based on the square diagram
$x + 3 = 7$ <u>or</u> -7	square root of both sides
$x = 4$ <u>or</u> -10	SPOE: subtract 3 throughout

Can you see the clever trick that we used here? We rewrote an expanded expression as a something-squared expression, and then we solved it like a Level 2 quadratic!

Let's work through another example in detail, for example, solving the Level 3 equation

$$x^2 + 12x + 36 = 144.$$

Our goal is to write the left-hand side in a something-squared form. To do that, we'll begin by drawing an empty square diagram and filling in the bits that we know. For example, we know that the upper left-hand box must contain x^2, and so the sides of that square must each be x units long.

	x	
x	x^2	

We know that the 36 will appear in the lower right-hand box. But how sould we label the side lenghts here? There are lots of combinations of numbers that multiply together to get 36...

	x	?
x	x^2	
?		36

Remember that the goal is to get an expression of the form something-*squared*, and so we should always strive to create a square. This means that we must choose 6 and 6 as the side lengths. If we chose another pair of factors, like 2 and 18, we would get the proper product in the lower right-hand region, but our overall diagram would no longer be a square.

	x	6
x	x^2	
6		36

To fill in the remaining regions, we multiply the dimensions of each. In both cases we get $6x$. This is good news! Together these make $12x$ (which is what we have in the original, expanded expression). Plus, the two regions contain the same value, and so the beautiful symmetry of the square is preserved.

	x	6
x	x^2	$6x$
6	$6x$	36

So, we have rewritten our expression as a something-squared expression:

$$x^2 + 12x + 36 = (x + 6)^2.$$

We can use this to solve the equation that we were given:

$$\begin{aligned} x^2 + 12x + 36 &= 144 & & \\ (x+6)^2 &= 144 & & \text{based on the square diagram} \\ x + 6 &= 12 \ \underline{\text{or}} \ -12 & & \text{square root of both sides} \\ x &= 6 \ \underline{\text{or}} \ -18 & & \text{SPOE} \end{aligned}$$

Example 12.2

Determine the value of n given the equation: $n^2 - 16n + 64 = 1$.

Solution: Let's build the square diagram. The upper left-hand corner contains n^2, as above. To maintain symmetry, we should split the $-16n$ exactly in half. Don't worry about the negative coefficient, we can work with that. We should be on guard though for sign-related issues.

	n	
n	n^2	$-8n$
	$-8n$	

This tells us that the remaining portion of the square's side length must be -8. Let's not worry too much about the fact that negative distances are impossible... the idea still works. This implies that the remaining region contains 64 (note, that positive 64). This agrees with the equation we were given.

	n	-8
n	n^2	$-8n$
-8	$-8n$	64

Now, we can solve the equation:

$$\begin{aligned} n^2 - 16n + 64 &= 1 & & \\ (n-8)^2 &= 1 & & \text{based on the square diagram} \\ n - 8 &= 1 \ \underline{\text{or}} \ -1 & & \text{square root of both sides} \\ n &= 9 \ \underline{\text{or}} \ 7 & & \text{SPOE} \end{aligned}$$

So, we have $\mathcal{S} = \{7, 9\}$. We can check our work by substituting our proposed solutions back into the original equation. First, we'll test $n = 7$:

$$\begin{aligned} n^2 - 16n + 64 &= 1 \\ (7)^2 - 16(7) + 64 &\stackrel{?}{=} 1 \\ 49 - 112 + 64 &\stackrel{?}{=} 1 \\ 1 &\stackrel{\checkmark}{=} 1 \end{aligned}$$

First, we'll check $n = 9$:

$$\begin{aligned} n^2 - 16n + 64 &= 1 \\ (9)^2 - 16(9) + 64 &\stackrel{?}{=} 1 \\ 81 - 144 + 64 &\stackrel{?}{=} 1 \\ 1 &\stackrel{\checkmark}{=} 1 \end{aligned}$$

12.4 Level 4 Quadratics

Startup Exploration: Quadratic Level 4

Determine the value of x given the equation: $x^2 + 8x + 15 = 99$.

Let's try to build the square diagram. The upper left-hand corner contains x^2, as before. To maintain symmetry, we should split the $8x$ exactly in half. This tells us that the large square must be $(x + 4)$ units on a side.

	x	4
x	x^2	$4x$
4	$4x$	

The problem is that, according to our square diagram, the lower right-hand corner should be 16... but the equation we are given tells us to put 15 in that space. Now what? Our equation does not represent a complete square!

	x	4
x	x^2	$4x$
4	$4x$	15

POEs to the rescue! Why not add 1 to both sides of the given equation to complete the square? This will give us the number we want on the left-hand side and, since we add 1 to both sides, we have an equivalent equation. So, instead of solving the equation

$$x^2 + 8x + 15 = 99,$$

we will add one to both sides and solve the equation

$$x^2 + 8x + 16 = 100.$$

Now, we can draw the square diagram exactly as we wanted.

	x	4
x	x^2	$4x$
4	$4x$	16

And now that we have a proper square diagram, we can solve the equation! Here's the full process:

$$\begin{aligned}
x^2 + 8x + 15 &= 99 && \\
x^2 + 8x + 16 &= 100 && \text{APOE: add 1 to both sides} \\
(x+4)^2 &= 100 && \text{based on the square diagram} \\
x + 4 &= 10 \ \underline{\text{or}} \ {}^-10 && \text{square root of both sides} \\
x &= 6 \ \underline{\text{or}} \ {}^-14 && \text{SPOE}
\end{aligned}$$

This is a clever application of the POEs. Rather than use the properties to eliminate terms from one side of the equation, we can use the properties to change one side into a particular, more-helpful form.[2]

Example 12.3

Determine the value of x given the equation: $x^2 - 6x + 11 = 27$.

Solution: When we start the square diagram, we split the $-6x$ as usual. This means that the square has sides of length $(x - 3)$. This, in turn, implies that the lower right-hand region should be 9. Our equation has 11 as its constant term: not what we want.

	x	-3
x	x^2	$-3x$
-3	$-3x$	9

To fix, this we can subtract 2 to each side of our equation. This will give us a constant term of 9, which is what we need to make our square diagram work. So, we have:

$$\begin{aligned}
x^2 - 6x + 11 &= 27 && \\
x^2 - 6x + 9 &= 25 && \text{SPOE: subtract 2 from both sides} \\
(x-3)^2 &= 25 && \text{based on the square diagram} \\
x - 3 &= 5 \ \underline{\text{or}} \ {}^-5 && \text{square root of both sides} \\
x &= 8 \ \underline{\text{or}} \ {}^-2 && \text{SPOE}
\end{aligned}$$

So, our solutions are $\mathcal{S} = \{8, {}^-2\}$.

[2] In fact, this is exactly what we've been doing all along: changing one side of an equation into a form that is more helpful. In this chapter we are expanding our notion of what it means for a change to be "helpful".

12.5 Level 5 Quadratics

The last few levels have had "polite" middle terms, which have split evenly into two pieces. What happens if we get a linear term with an odd coefficient?

Startup Exploration: Quadratic Level 5

Determine the value of x given the equation: $x^2 + 3x + 1 = 5$.

If we jump right in and try the square method, we start to get into fraction territory. If we want to split the $3x$ term exactly in half, then each piece would be $\frac{3}{2}$. Then, the square method would predict $\frac{9}{4}$ in the lower-right corner.

	x	$\frac{3}{2}$
x	x^2	$\frac{3}{2}x$
$\frac{3}{2}$	$\frac{3}{2}x$	$\frac{9}{4}$

The lower-right corner isn't what we have in our equation, so we could use APOE and add $\frac{5}{4}$ to adjust both sides... Hmm. Not pretty. To be clear, the square method will not let us down: if we keep going with the fractions, we will arrive at the correct answer! But, perhaps there is an alternative approach that avoids the fractions.

Here's a clever idea: We could use MPOE and multiply through by 2. This would give us a linear term with an even coefficient! In other words:

$$x^2 + 3x + 1 = 5 \quad \xrightarrow{\text{multiply through by 2}} \quad 2x^2 + 6x + 2 = 10$$

This fixes our odd coefficient problem, since now we can break the $6x$ up into two sets of $3x$. But, what do we do with that $2x^2$? We can't use x and $2x$ as the side lengths, for although that gives is the correct product, we would no longer have a square.

	?	3
?	$2x^2$	$3x$
3	$3x$	

Now, here's a *really* clever idea. Let's multiply through by 2 *again*. In other words, we will multiply the original equation by 4:

$$x^2 + 3x + 1 = 5 \xrightarrow{\text{multiply through by 4}} 4x^2 + 12x + 4 = 20$$

We still have an even linear coefficient, and now we can write $4x^2$ as $2x$ times $2x$. Note that we have to take that factor of 2 into account when we're figuring out the other dimensions of the square. Study our new diagram closely and be sure you understand where each of the labels comes from.

	$2x$	3
$2x$	$4x^2$	$6x$
3	$6x$	9

We now find ourselves in a Level 4 situation: our equation has 4 as the constant term, whereas the square model predicts 9 as the constant term. No problem! We can add 5 to both sides, and continute the process as we did with Level 4 quadratics. Here's a summary of the whole process:

$$\begin{aligned}
x^2 + 3x + 1 &= 5 && \text{original equation} \\
4x^2 + 12x + 4 &= 20 && \text{MPOE: multiply both sides by 4} \\
4x^2 + 12x + 9 &= 25 && \text{APOE: add 5 to both sides} \\
(2x + 3)^2 &= 25 && \text{based on the square diagram} \\
2x + 3 &= 5 \ \underline{\text{or}} \ {}^-5 && \text{square root of both sides} \\
2x &= 2 \ \underline{\text{or}} \ {}^-8 && \text{SPOE} \\
x &= 1 \ \underline{\text{or}} \ {}^-4 && \text{DPOE}
\end{aligned}$$

Let's pause and review. When faced with an odd coefficient for the x term, our strategy is to multiply through by 4. This will give us an even coefficient for the x term, and at the same time keep the coefficient of the x^2 term in a state where it can be written as something times itself: $4x^2 = 2x \cdot 2x$. After we do this, we'll have a Level 4 quadratic on our hands, and we can apply techniques for handling those.

Example 12.4

Determine the value of x given the equation: $x^2 - 5x + 12 = 62$.

Solution: Faced with an odd linear coefficient, we multiply through by 4. This gives us the revised equation

$$4x^2 - 20x + 48 = 248.$$

We set up the square diagram and see whether our equation represents a complete square.

$$\begin{array}{c|c|c|}
 & 2x & -5 \\ \hline
2x & 4x^2 & -10x \\ \hline
-5 & -10x & 25 \\ \hline
\end{array}$$

Our (revised) equation does not make a proper square: the constant term in the equation is 48, but the square diagram predicts 25. We can fix this problem using SPOE: subtract 23 from both sides:

$$\begin{aligned}
x^2 - 5x + 12 &= 62 && \text{original equation} \\
4x^2 - 20x + 48 &= 248 && \text{MPOE: multiply both sides by 4} \\
4x^2 - 20x + 25 &= 225 && \text{SPOE: subtract 23 from to both sides} \\
(2x-5)^2 &= 225 && \text{based on the square diagram} \\
2x - 5 &= 15 \ \underline{\text{or}} \ {-15} && \text{square root of both sides} \\
2x &= 20 \ \underline{\text{or}} \ {-10} && \text{SPOE} \\
x &= 10 \ \underline{\text{or}} \ {-5} && \text{DPOE}
\end{aligned}$$

So in the end, we have solutions $\mathcal{S} = \{10, -5\}$.

12.6 Level 6 Quadratics

We've arrived at the highest level of the quadratic equation challenge. Until now, all of our equations have started with just x^2. What if the coefficient of the leading term is something other than 1?

Startup Exploration: Quadratic Level 6

Determine the value of x given the equation: $3x^2 + 8x + 1 = 12$.

What shall we do in this scenario? A clever idea is to use DPOE and divide through by 3. That would make the leading coefficient 1, as in the earlier problems. The downside is that most of the other numbers turn into fractions.

$$3x^2 + 8x + 1 = 12 \xrightarrow{\text{divide through by 3}} x^2 + \frac{8}{3}x + \frac{1}{3} = 4$$

The square method will absolutely work on an equation like this, but perhaps we'd prefer an approach that avoided all the fractions.

If scaling the equation down doesn't help, why not try to scale it up? Could we multiply through by some helpful value? Recall that the goal will be to write the x^2 term as something times itself. Since there's already a 3 there, we can solve our problem if we multiply through by 3.

$$3x^2 + 8x + 1 = 12 \xrightarrow{\text{multiply through by 3}} 9x^2 + 24x + 3 = 36$$

Note that now we are in a good position, since $9x^2 = 3x \cdot 3x$. So, let's fill out our square diagram. We have an even coefficient on the linear term, so we can split that evenly. Note that we have to take all the coefficients into account when completing the diagram. For example, when figuring out the dimensions of a box containing $12x$.

	$3x$	4
$3x$	$9x^2$	$12x$
4	$12x$	16

The square model predicts 16 as the constant term, and so our equation is not a complete square. We can use

APOE to fix that, adding 13 to both sides. Here's how it goes:

$3x^2 + 8x + 1 = 12$	original equation
$9x^2 + 24x + 3 = 36$	MPOE: multiply through by 3
$9x^2 + 24x + 16 = 49$	APOE: add 13 to both sides
$(3x + 4)^2 = 49$	based on the square diagram
$3x + 4 = 7 \ \underline{\text{or}} \ {}^-7$	square root of both sides
$3x = 3 \ \underline{\text{or}} \ {}^-11$	SPOE
$x = 1 \ \underline{\text{or}} \ {}^-\frac{11}{3}$	DPOE

In summary, we multiplied through by the coefficient of the x^2 term, which gave us a coefficient that was a perfect square. In the final example for this chapter, we put it all together.

Example 12.5

Determine the value of x given the equation $-5x^2 - x + 18 = 0$.

Solution: Since we have a leading coefficient that is not a perfect square, we multiply through by that coefficient, -5 in this case. This gives us

$$25x^2 + 5x - 90 = 0$$

(careful with the negative signs). This is an improvement, but we have a linear coefficient that is odd. So, we multiply by 4 to fix that:

$$100x^2 + 20x - 360 = 0$$

Notice that multiplying through by 4 (a perfect square) gives us a leading coefficient that is still a perfect square. This is because the product of two perfect squares is itself a perfect square! (Can you prove that this statement is always true using the properties of exponents?)

We now have a revised equation that we can bring to the square method.

	$10x$	1
$10x$	$100x^2$	$10x$
1	$10x$	1

To get our constant terms to agree, we must add 361 to both sides of our revised equation. This gives us

$$10x^2 + 10x + 1 = 361.$$

And from here, we can complete a familiar process.

$$\begin{aligned}
10x^2 + 10x + 1 &= 316 && \\
(10x+1)^2 &= 316 && \text{based on the square diagram} \\
10x + 1 &= 19 \ \underline{\text{or}}\ {}^-19 && \text{square root of both sides} \\
10x &= 18 \ \underline{\text{or}}\ {}^-20 && \text{SPOE} \\
x &= \tfrac{18}{10} \ \underline{\text{or}}\ {}^-2 && \text{DPOE}
\end{aligned}$$

After we simplify our one fraction answer, we have a final solution: $\mathcal{S} = \left\{\frac{9}{5}, {}^-2\right\}$.

12.6.1 (;,;) The Quadratic Formula

Now that we have the square method at our disposal, we can tackle any quadratic equation that is thrown at us. We might be tempted to really get generic, and solve the all at once.

Consider a completely generic quadratic equation of the form

$$ax^2 + bx + c = 0.$$

This quadratic has three coefficients: a is the coefficient of the quadratic term, b is the coefficient of the linear term, and c is the constant term. We've set it equal to zero since.... HOW TO EXPLAIN THIS?

What happens if we apply the square method to this totally generic equation? We don't really know anything about a, so to be safe, let's multiply the whole equation by a.

$$a^2x^2 + abx + ac = 0.$$

This will ensure that the leading term is a perfect square, and that's what we need for the box method. Now, the linear coefficient is ab and this might be an odd number for all we know. So, we multiply through by 4, as we have often done above.

$$4a^2x^2 + 4abx + 4c = 0.$$

This is not a pretty sight, but we'll try to make it work with the square method.

	$2ax$	b
$2ax$	$4a^2x^2$	$2abx$
b	$2abx$	b^2

Much of this looks workable, but what about the lower right-hand corner? The square method predicts b^2, but our equation has $4c$. Well, we'll do what we usually do: we'll use the POEs to turn our left-hand side into the form we want. So:

$$
\begin{aligned}
4a^2x^2 + 4abx + 4c &= 0 & \\
4a^2x^2 + 4abx &= -4c & \text{SPOE: subtract } 4c \text{ from both sides} \\
4a^2x^2 + 4abx + b^2 &= b^2 - 4c & \text{APOE: add } b^2 \text{ to both sides}
\end{aligned}
$$

It might seem like things are getting worse, but now the left-hand side is what we need to use our square diagram. Let's go!

$$
\begin{aligned}
(2ax + b)^2 &= b^2 - 4c & \text{based on the square diagram} \\
2ax + b &= \pm\sqrt{b^2 - 4c} & \text{square root of both sides} \\
2ax &= \pm\sqrt{b^2 - 4c} - b & \text{SPOE: subtract } b \text{ from both sides} \\
x &= \frac{\pm\sqrt{b^2 - 4c} - b}{2a} & \text{DPOE: divide both sides by } 2a \\
x &= \frac{-b \pm \sqrt{b^2 - 4c}}{2a} & \text{Rearranging the numerator}
\end{aligned}
$$

COMMENTS

Chapter Summary

The properties of equality which we learned before this chapter aren't enough to solve any random quadratic equation. To overcome the quadratic obstacle, we learned new techniques for solving quadratic equations. These new techniques are based on the symmetry of a square, and we learned various techniques for adjusting a quadratic equation so that we can use the square diagram approach.

We have, however, avoided the necessity of finding the square root of a number that was not a perfec square. This was a helpful simplification – it allowed us to focus on the process of equation solving – but not all quadratics will have answers that come out so "nice". We turn our attention to square roots in the next chapter.

There is geometry in the humming of the strings, there is music in the spacing of the spheres.

Pythagoras
Ancient Greek philosopher

Chapter 13

Radical Expressions

In the last chapter, we found integer solutions to nearly all of the equations that we studied. This was good for understanding the workings of quadratic equations, but not all equations will necessarily be so "polite". The main focus of our work in this chapter is around understanding more about all of those square roots that don't come out evenly. We begin by looking more closely at the sets $\mathbb{Q}$ and $\mathbb{R}$.

13.1 Real numbers

Startup Exploration: Share the Cheese

Middle Market sells mini-wheels of cheese for snacking. Mini-wheels can be sold one at a time, or in boxes of 10. The cheese arrives at the warehouse in crates of 10 boxes (containing 100 wheels of cheese in total).

The warehouse workers need to divide their stock of mini-wheels up among three trucks, each of which will deliver to a local Middle Market branch. The warehouse has a total of 13 crates, 7 boxes, and 9 mini-wheels in stock.

The manager allows the workers to open crates or boxes, if needed, to divide the supply evenly. Describe a process for dividing up the cheese that requires opening the minimum number of boxes.

In chapter 1, we made some comments which might, at first, appear contradictory. On the one hand, we saw that the set of real numbers, $\mathbb{R}$, includes every possible decimal number. We also saw that the set of rational numbers, $\mathbb{Q}$, includes "terminating decimals and repeating decimals".

On the other hand, we know that $\mathbb{Q}$ is the set of fractions, meaning those numbers that can be expressed in the form

$$\frac{a}{b} \text{ where } a \text{ and } b \text{ are integers, and } b \text{ is not zero.}$$

These statements raise a few questions. What is the relationship between "fraction" and "terminating or repeating decimal"? What can we say about decimal numbers that are neither terminating nor repeating?

13.1.1 Fractions into decimals

Recall that a fraction is simply a divison problem in disguise. If we execute the division problem, we can easily turn a fraction into a decimal. It's especially easy if we have a calculator handy... otherwise, we're in for some long division.

Long Division

Don't worry if your long division is a bit rusty, just take another look at the startup exploration. The warehouse workers must divide 1379 mini-wheels of cheese among the three trucks – that's $1379 \div 3$ – but they must do this with a minimum amount of regrouping.

One solution is to put 4 crates on each of the three trucks. This takes care of 12 crates (1200 mini-wheels in all), but leaves one crate. They have no choice but to open this crate and treat it as 10 boxes. Of course they already had 7 boxes in stock, so now they have 17 boxes in all. They can put 5 boxes on each truck (accounting for 15 boxes, or 150 mini-wheels), but they will have 2 boxes left over. They open these two boxes, revealing 20 mini-wheels. They add these to the 9 mini-wheels they had already, giving 29 mini-wheels in all. Each truck gets 9 of these (using up 27), and they have 2 left over.

So, in the end: each truck gets 4 crates, 5 boxes, and 9 mini-wheels – that's 459 mini-wheels in all – and there are 2 left behind. Now have a look at the long division for this problem: can you spot each of the steps that we took above in the work below?

$$
\begin{array}{rrrrr}
 & & 4 & 5 & 9 \\ \cline{2-5}
3 \big) & 1 & 3 & 7 & 9 \\
 & 1 & 2 & 0 & 0 \\ \cline{2-5}
 & & 1 & 7 & 9 \\
 & & 1 & 5 & 0 \\ \cline{3-5}
 & & & 2 & 9 \\
 & & & 2 & 7 \\ \cline{4-5}
 & & & & 2
\end{array}
$$

In the cheese example, it makes sense to stop here with a remainder of 2. In general, though, we could continue the process of division and create a number that extends to the right of the decimal point.

Example 13.1

Convert $\dfrac{3}{4}$ and $\dfrac{1}{6}$ into their decimal representations.

Solution: Recall that the fraction three-fourths is equivalent to the division problem $3 \div 4$. To do this by long division, we put the dividend (that's 3) inside the "division house" and leave the divisor (that's 4) outside.

$$
\begin{array}{rccc}
 & 0 & .7 & 5 \\ \cline{2-4}
4\,) & 3 & .0 & 0 \\
 & 2 & 8 & \\ \cline{2-3}
 & & 2 & 0 \\
 & & 2 & 0 \\ \cline{3-4}
 & & & 0
\end{array}
$$

So the decimal representation of $\frac{3}{4} = 0.75$ (you may have known that already). To tackle one-sixth, we note that it is equivalent to $1 \div 6$. The long division starts out like this:

$$
\begin{array}{rcccc}
 & 0 & .1 & 6 & 6 \\ \cline{2-5}
6\,) & 1 & .0 & 0 & 0 \\
 & 0 & 6 & & \\ \cline{2-3}
 & & 4 & 0 & \\
 & & 3 & 6 & \\ \cline{3-4}
 & & & 4 & 0 \\
 & & & 3 & 6 \\ \cline{4-5}
 & & & & 4
\end{array}
$$

We might as well stop here, though, because we're stuck in a loop! The 6's in the answer are going to repeat forever. (Can you see why?) So, the decimal representation of $\frac{1}{6} = 0.1\overline{6}$.

We say that 0.75 is a terminating decimal, because the process of long division stops with a remainder of zero. On the other hand, $0.1\overline{6}$ is called a repeating decimal because the long division process gets stuck in a loop. Note that we've used a vinculum over the 6 to indicate which digits repeat.

When it comes to a division problem like $1 \div 6$ on a calculator, the display will likely show `0.166666667`, where the 6's repeat for a while and are followed by a 7. Don't be fooled by this 7: the 6's really do go on forever! That 7 is the calculator rounding up. Always be skeptical about the rightmost digit on your calculator screen.

A Bold Claim

This process of division leads us to make a pretty bold claim: every rational number can be represented *either* as a terminating decimal or a repeating decimal. How can we be sure that every crazy fraction, for instance $\frac{19}{81}$, either terminates or repeats?

Let's think about how long division works: the "subtraction step" in particular. Here's how the process of long division starts out for $\frac{3}{4}$:

$$
\begin{array}{rlll}
 & & .7 & \\ \cline{2-4}
4 \big) & 3 & .0 & 0 \\
 & 2 & 8 & \\ \cline{2-3}
 & & 2 &
\end{array}
$$

In the subtraction step, we get 2 as the remainder, and so we know that we have to keep dividing. If we ever get the remainder 0, then we know that we're done with division. This is what happens eventually with $\frac{3}{4}$. On the other hand, if we ever get a remainder that we've gotten before, then we know that we're stuck in a loop. This is what happened with $\frac{1}{6}$.

Now here's a simple yet profound idea: the remainder is always less than the divisor. When dividing by 4, the remainder has to be less than 4. When dividing by 6, the remainder has to be less than 6. (Can you explain why that is? For example, when dividng mini-wheels of cheese among three trucks, could the workers have seven mini-wheels of cheese left over?)

The result is that we have a limited number of choices for the remainder. When dividing by 4, the remainder can only be 0, 1, 2, or 3. When dividing by 6, the remainder can only be : 0, 1, 2, 3, 4, or 5.

Having a limited number of choices means that eventually we have to recycle one of those remainders! We can't go on forever without either using the remainder 0 (in which case the decimal terminates) or reusing one of the nonzero remainders (in which case the decimal repeats).

Even when dividing something ugly like $1903 \div 8167$, the remainders in the subtraction steps will always be less than 8167. We might have to divide for a long time, but we know it can't carry on forever. Eventually we'll either use the remainder 0, or reuse a remainder we've used already. So the fraction $\frac{1903}{8167}$ has a decimal representation that either terminates or repeats.

Our argument applies to any denominator, and so to any rational number. Therefore, it's true that every rational number has a decimal representation that either terminates or repeats! Have we blown your mind yet? If not, stay tuned.

13.1.2 Decimals into fractions

What about the other way around? Does every terminating-or-repeating decimal have a corresponding fraction representation?

Terminating decimals into fractions

Consider a terminating decimal like 0.375. Can we turn this decimal into a fraction, meaning a ratio of two integers? If so, how?

Recall the notion of *place value*, and how the individual digits in a number are each standing in some "place" that is named after a power of ten.[1] The key to turning a terminating decimal into a fraction is recalling how to read a decimal using its place value.

thousands	hundreds	tens	units		tenths	hundredths	thousandths	ten-thousandths	hundred-thousandths	millionths
4	1	2	6	•	3	7	5	0	9	8

To read the decimal 0.375, we can say "zero point three seven five", but this isn't very helpful. Instead, we read the number using place value and say "three hundred seventy-five *thousandths*". Now, if someone were to say that number aloud, it sounds just like the fraction

$$\frac{375}{1000}.$$

In fact, this decimal number and this fraction represent exactly the same value. Of course, the fraction isn't in simplest form yet, but that's easy to fix:

$$0.375 = \frac{375}{1000} = \frac{3}{8}$$

.

We have accomplished the goal of turning a terminating decimal into a fraction. The technique is simply to read the decimal aloud using its place value, and then write down the fraction we hear.

Repeating decimals into fractions

The "read the number with its place value" technique won't work for repeating decimals. (Why not?) Instead, we'll use some clever applications of the techniques we learned when solving equations.

Suppose we try to write the repeating decimal $0.\overline{4}$ as a fraction. Let's give this number a name so that we can do some algebraic manipulations.

$$x = 0.\overline{4}$$

[1] We really are blowing the cobwebs off of some old mathematics in this chapter: Long division! Place value! It goes to show that even simple mathematical ideas can have deep and meaningful consequences.

Our goal will be to find an alternative way of writing x. To do that, we're going to make two clever moves.

The first clever move is to use MPOE: we will multiply both sides of this equation by 10. Multiplying $0.\overline{4}$ by 10 moves the decimal point one place to the right. But remember, the 4's repeat *forever*, so there are *still infinitely many* 4's to the right of the decimal point! We have:

$$10x = 4.\overline{4}$$

The second clever move is to use an idea from when we were solving systems of equations: the elimination method. Watch what happens when we subtact the first equation we wrote from the second equation:

$$\begin{array}{rr} & 10x = 4.\overline{4} \\ - & x = 0.\overline{4} \\ \hline & 9x = 4.0 \end{array}$$

Notice that the two numbers on the righthand side of our equations are exactly four units apart. In other words: the infinitely long tail of 4's disappears when we subtract! Now all we have to do is use DPOE to isolate x:

$$9x = 4 \quad \Longrightarrow \quad x = \frac{4}{9}$$

If you have a calculator handy, you can perform this division and see that we have accomplished the goal of turning our repeating decimal into a fraction:

$$0.\overline{4} = \frac{4}{9}$$

This process is sometimes called *killing the tail*, since our goal is to subtract two different decimal forms that have the same repeating part, thereby eliminating the infintely long tail of digits.

Example 13.2

Convert the repeating decimal $0.\overline{63}$ to its decimal representation.

Solution. We'll kill the tail again, but note that we have two digits after the decimal which repeat. This will require a slight adjustment. We'll start as we did before, by assigning an algebraic name to our number:

$$x = 0.6363\ldots$$

If we multiply both sides by 10, we'll have

$$10x = 6.3636\ldots$$

which is also a repeating decimal, but with a *different* repeating tail. We could work with this, but it's a bit easier to multiply by 10 again (in other words, to multiply the original equation by 100):

$$100x = 63.6363\ldots$$

Now we have an equation in which the number on the righthand side has exactly the same tail as in the original equation. So, we subtract:

$$\begin{array}{lrr} & 100x = & 63.\overline{63} \\ - & x = & 0.\overline{63} \\ \hline & 99x = & 63.00 \end{array}$$

We divide both sides by 99, and then simplify our fraction to lowest terms. In the end, we have:

$$0.\overline{63} = \frac{63}{99} = \frac{7}{11}$$

The moral of the story is that we may have to adjust our method and choose the "just right" powers of 10. Consider how we might use kill the tail to turn $0.1\overline{6}$ back into $\frac{1}{6}$? (Note that the 6 repeats in the decimal form, but the 1 does not.)

Let's pause to reflect. In the first part of this section, we explained why every fraction can be written as either a terminating or repeating decimal. We can make this conversion using long division. Then we went on to show the reverse: that every terminating decimal can be written as a fraction (by reading it with its place value) and every repeating decimal can be written as a fraction (by killing the tail).

Armed with these tools, we might get the idea that *every decimal* number can be turned into a fraction. Unfortunately (or fortunately, depending on how you look at it), this is not the case.

13.1.3 Existence of irrational numbers

The **irrational numbers** are all of the real numbers that are not rational numbers. In other words, those decimal numbers that cannot be expressed as either a terminating or repeating decimal.

Back in chapter 1 we gave an example of such a number:

$$0.10\,110\,1110\,11110\,111110\ldots$$

This number clearly has a pattern. We might explain it by saying: "After the decimal point write one, then zero, the 2 ones, then zero, then 3 ones, then zero, and so on, always writing 1 more one than you did the last time." The problem is that is does not terminate (our pattern will continue forever), but it doesn't repeat either. The strings of 1's get longer and longer. There is never a set of always-repeating digits to group under a vinculum.

This single number is enough to prove that irrational numbers exist. Of course, there are lots of them. The famous number π is irrational, and in some sense is even more diabolical.

$$\pi \approx 3.1415926535\ 8979323846\ 2643383279\ 502884197\ 6939937510\ 5820974944\ 5923078164\ldots$$

This number doesn't even have a pattern that we can use to describe it (as far as we know). The digits go on infinitely, and come in a random sequence.

You may be wondering, "How do we know π is irrational?" After all, it may be clear why the first number with the ones and zeros is irrational, but how to we know for sure that π never terminates and never repeats?

Unfortunately, explaining the irrationality of π requires a bit more mathematics that we can get into here. However, we have learned enough to prove that certain other numbers are irrational. More on that at the end of section 13.2.

13.2 Square Roots

We have already worked quite a bit with exponents, but every exponent so far has been an integer. Could we have a rational number as an exponent? If so, what would it mean?

Startup Exploration: Half Power

Consider the expression

$$9^{1/2},$$

that is, "nine to the power one-half". What are some possible interpretations of this?

Use a calculator to explore what happens when we raise certain numbers to the one-half power (start with the natural numbers between 1 and 20). What patterns do you notice? What conjectures do you have about what's happening?

Suppose we let $x = 9^{1/2}$. We'd like to find an alternative way of expressing x that uses only integer exponents. One approach is to multiply each side of this equation by itself. Then we'd have:

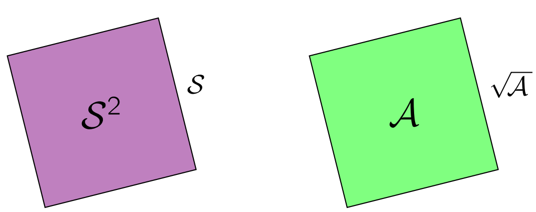
$$\begin{aligned} x &= 9^{1/2} \\ x \cdot x &= 9^{1/2} \cdot 9^{1/2} && \text{multiply each side by itself} \\ x \cdot x &= 9^{(1/2)+(1/2)} && \text{product rule for exponents} \\ x \cdot x &= 9^{1} \\ x \cdot x &= 9 \end{aligned}$$

So, x is the number that when multiplied by itself gives 9 as the result. That could be either 3 or -3, since $3 \cdot 3 = -3 \cdot -3 = 9$. Using some vocabulary that we already know: we say that $9^{1/2}$ is a *square root* of 9.

Square Root

A **square root** of a is a number b such that $b \cdot b = a$.

The reason this is called a "square root" has to do with the geometric interpretation of this operation. If we have a square of side length $\mathcal{S}$, then the area of the square is $\mathcal{S}^2$. Conversely, if we have a square with area $\mathcal{A}$, then the *square root of* $\mathcal{A}$ gives us the side length of the square.



Positive numbers have two square roots: one positive, one negative. Both 3 and -3 are square roots of 9, since $(3)(3) = 9$ and $(-3)(-3) = 9$. Zero is the only number that has exactly one square root: the square root of 0 is 0. Negative numbers cause some problems when it comes to square roots (stay tuned for more on that).

Most of the time, we'll be working with the positive square root of a number, called the **principal square root** For example, the principal square root of 9 is 3 and the principal square root of 25 is 5.

The checkmark-ish symbol we use to denote the square root of a is called the **radical**: $\sqrt{a}$. This symbol means "the principal square root of a". So, $\sqrt{9} = 3$ and $\sqrt{25} = 5$. So, although -3 is a square root of 9, it is incorrect to write $\sqrt{9} = -3$ or $\sqrt{9} = \pm 3$. This isn't quite rise to the level of Evil and Wrong, but it's not right.[2]

When simplifying an expression or solving an equation, we won't always give both square roots. A good guideline as we go along is to pay close attention and use the notation given in the problem. For example:

$$\pm\sqrt{100} = \pm 10 \qquad \text{and} \qquad -\sqrt{36} = -6$$

In these cases, the notation specifically indicates that we want both the positive and negative square root, or only the negative square root.

13.2.1 Imaginary Numbers

Consider the expression $\sqrt{-16}$. This is a bit of a problem. We're meant to find the principal square root of -16, but the closest we can get is $4 \cdot -4 = -16$. It's true that 4 and -4 have the same absolute value, but they're not the same number, which means we're not *squaring* anything. So, there is no real number equal to $\sqrt{-16}$. Of course, -16 isn't special, the same argument applies to any negative number.

Square Root of a Negative Number

If $a < 0$, then there is no real number equal to $\sqrt{a}$.

Note that this *doesn't* mean that -16 doesn't have any square roots. It simply means that the square roots are not real numbers. The square roots of -16 are members of a set called the *complex numbers* $\mathbb{C}$, but not members of the real numbers $\mathbb{R}$.

To build up the complex numbers, we introduce the so-called imaginary unit i which has the property that $i^2 = 1$. Remember, the domain of algebra 1 is the set real numbers. But, the domain of algebra 2 and beyond is the set of complex numbers. So, if you're intrigued about imaginary numbers, just hang in there.

[2] An algebra note from the future! When we simplify the expression $\sqrt{m^2}$, the result is $|\,m\,|$, the absolute value of m. Do you see why? The value that comes out of the radical must be positive, because that notation gives the principal square root.

Our work in algebra 1 will bring us mostly in contact with *square roots*, but other roots are possible. For example a "cube root" of a number a is a number b such that $b \cdot b \cdot b = a$. We write $\sqrt[3]{a}$ to denote the cube root of a. Negative numbers under the cube root symbol are no problem:

$$\sqrt[3]{-8} = -2 \qquad \text{since} \qquad -2 \cdot -2 \cdot -2 = -8$$

13.2.2 (;,;) Irrationality of the Square Root of Two

We have shown that irrational numbers exist. Consider the mathematical argument below, which explains why the square root of 2 is irrational.

Step 1. Assume (for the moment) that $\sqrt{2}$ is, in fact, rational. In other words, that it can be written as the ratio of two integers. Then, we can write

$$\sqrt{2} = \frac{a}{b}$$

where a and b are relatively prime. That is to say, our fraction is in lowest terms.

Step 2. Square both sides of the equation above.

$$2 = \frac{a^2}{b^2}$$

Step 3. Multiply both sides of this equation by b^2.

$$2b^2 = a^2$$

This means that a^2 is an even number. If a^2 is an even number, then a must be an even number.

Step 4. If a is an even number, then it is divisible by 2. In other words, there is an integer m such that $a = 2m$.

Step 5. Substitute $2m$ for a in the equation from Step 3, and simplify.

$$\begin{aligned} 2b^2 &= (2m)^2 \\ 2b^2 &= 4m^2 \\ b^2 &= 2m^2 \end{aligned}$$

This means that b^2 is an even number. If b^2 is an even number, then b is an even number.

Step 6. If both a and b are even numbers, then the original fraction $\frac{a}{b}$ was not in lowest terms, which was our assumption! This contradiction shows that our original assumption cannot be true.

Thus, $\sqrt{2}$ cannot be written as a fraction. In other words, $\sqrt{2}$ is an irrational number.

Thoughts to Chew On

Steps 3 and 5 both include assertions about numbers being even. How do we know when a number is even? Specifically, how do we know that a^2 and b^2 are even?

Steps 3 and 5 both go on to say something like "if a^2 is even, then a is even". How do we know this is true? Under what circumstances will the square of a number be even or odd?

Step 6 argues that $\frac{a}{b}$ is not in lowest terms. How do we know this is true?

This is an example of *proof by contradiction*. We assume that some statement is true, then show that this assumption leads to some kind of impossible situation. The impossibility means we have to reject the original assumption. What was our original assumption in this proof? What is the contradiction that results from that assumption?

13.3 Simplified Radical Form

When we take the square root of a perfect square we get an integer as the answer. But, things are not so easy when taking the square root of a number that is not a perfect square. In fact, the square root of a non-square natural number will be an irrational number, like $\sqrt{2}$.

The *exact value* of an irrational number can only be represented using some kind of symbol, like π or $\sqrt{2}$. Writing out a decimal value – no matter how many decimals you write down – will always be an approximation. So, it's a good habit of mind to think "should I be giving an exact answer to this problem, or is a decimal approximation good enough". Very often, the context (or the directions) will make this choice clear.

To help us standardize the way we write radical expressions, we all agree to comply with *simplified radical form*.

Simplified Radical Form

A radical expression is considered completely simplified if...

1. Like radical terms have been combined.
2. The expression under the radical has no perfect square factors other than 1.
3. There are no fractions under the radical.
4. There are no radicals in the denominator of a fraction.

Over the next few sections, we will discuss each of these criteria and the algebraic manipulations that we can use to make sure our expressions comply. The first criteria is quite straightforward, so let's get right to it.

13.3.1 Like Radical Terms

Criteria #1 states that like radical terms must be combined. We combine radical terms as we do variable terms. For example, we are very familiar with the simplification

$$x + x = 2x.$$

We combine radical terms in exactly the same way:

$$\sqrt{5} + \sqrt{5} = 2 \cdot \sqrt{5} = 2\sqrt{5}.$$

Note, in particular, that the sum here is not $\sqrt{10}$. Similarly, $3x + 4x = 7x$ and so with radicals, we have $3\sqrt{21} + 4\sqrt{21} = 7\sqrt{21}$. When we have a multiplication of a number times a radical, we can omit the multiplication symbol.

Warning!

When we say like radical terms "can be combined", don't go thinking you can add the numbers under the radical. To add the values like this is **Evil and Wrong**.

$$\sqrt{3}+\sqrt{3} \neq \sqrt{6}$$

While we're at it, don't get any ideas about splitting the radical-of-a-sum into the sum-of-radicals. This, too, is **Evil and Wrong**.

$$\sqrt{2+14} \neq \sqrt{2}+\sqrt{14}$$

13.3.2 Product Properties of Radicals

Startup Exploration: Building Blocks

The number 1 is the *additive building block* of the natural numbers. In other words: If we want to "build" any natural number using only addition, the only number we need is the number 1. Every natural number can be written as the sum of a bunch of 1's.

What are the *multiplicative building blocks* of the natural numbers? Note that we have to say *blocks* (plural) since the number 1 is not enough: multiplying together a bunch of 1's always gives us 1 as the product. What is the smallest collection of natural numbers that we need in order to build the rest using only multiplication?

The second criteria for simplified radical form states that the expression under the radical may have no perfect square factors other than 1. This may seem strangely worded. It clearly handles the idea that there should be no perfect squares under the radical, and that makes sense. Expressions like $\sqrt{4}$ and $\sqrt{25}$ can pretty obviously be simplified.

But, this criteria also catches expressions like $\sqrt{24}$ and $\sqrt{50}$ because those numbers, neither of which is a perfect square, each have a perfect square as a factor: 24 has 4 as a factor, and 50 has 25 as a factor.

How can we simplify an expression like $\sqrt{50}$ so that it has no perfect square factors under the radical? For help, we turn to:

Product Rule of Radicals

For any $a \geq 0$ and $b \geq 0$,

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}.$$

Note: In algebra 1 we only use the square root version of this property, though in fact it applies to radicals of any degree: cube roots, fourth roots, and so on.

This property looks an awful lot like the product rule for exponents, which makes sense since here we are undoing the power of a product rule, where the power is the exponent one-half!

Example 13.3

Express $\sqrt{50}$ in simplified radical form.

Solution: We know $\sqrt{50}$ is not yet in simplified radical form because 50 is divisible by a perfect square, $50 = 25 \cdot 2$. We apply the multiplication property of radicals like so:

$$\begin{aligned} \sqrt{50} &= \sqrt{25 \cdot 2} && \text{rewrite 50 to show its perfect square factor} \\ &= \sqrt{25} \cdot \sqrt{2} && \text{product rule of radicals} \\ &= 5 \cdot \sqrt{2} && \text{simplify the square root of a perfect square} \end{aligned}$$

So, $\sqrt{50} = 5\sqrt{2}$. These two expressions are equal, but only the second expression satisfies the criteria of simplified radical form.

13.3.3 Different Approaches to Simplifying

There are a number of ways to go about applying this property to simplify expressions. Use whatever approach makes the most sense to you! Here are some alternatives, though you might find a different approach that fits you better. In any case, it will probably be helpful to learn a variety of methods. Depending on the problem, some methods may be easier to use than others.

For example, let's examine different ways to get $\sqrt{108}$ into simplified radical form.

Strategy 1: Largest Square Factor

In this strategy, we find the largest perfect square factor and simplify it using the product rule for radicals. We might notice that $108 = 3 \times 36$:

$$\sqrt{108} = \sqrt{36 \cdot 3} = \sqrt{36} \cdot \sqrt{3} = 6\sqrt{3}$$

Strategy 2: One Square at a Time

It might not be obvious what the largest perfect square is, so in this strategy we look for *any* prefect square factor and work one square at a time. For instance, we might notice that 108 is divisible by 9 (how can we quickly spot divisibility by 9?). Then:

$$\sqrt{108} = \sqrt{9 \cdot 12} = \sqrt{9} \cdot \sqrt{12} = 3 \cdot \sqrt{12} = 3 \cdot \sqrt{4 \cdot 3} = 3 \cdot \sqrt{4} \cdot \sqrt{3} = 3 \cdot 2 \cdot \sqrt{3} = 6\sqrt{3}$$

In this approach, we have to keep checking to see whether the number under the radical is "square-free" or not. After our first simplification, we have 12 under the radical. But 12 has 4 as a factor, so we have to do another simplification step.

This process might take a little longer, but it is sometimes easier to identify smaller perfect square factors and chip away at the problem, than it is to identify the largest perfect square factor and finish the problem in a single step.

Strategy 3: The Sniper Method

The idea here is to write the *prime factorization* of the number under the radical, and then look for pairs of factors.[3] The factorization of $108 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$, so:

$$\sqrt{108} = \sqrt{\underline{2 \cdot 2} \cdot \underline{3 \cdot 3} \cdot 3} = \underline{2} \cdot \underline{3} \cdot \sqrt{3} = 6\sqrt{3}$$

We've given this strategy the memorable (though perhaps gruesome) name *the sniper method*. Think of the radical as a prison. There are snipers outside and any number that tries to escape needs to have a decoy. A single factor of 2 is stuck inside for life, but if the 2 has a partner (that is, if there's a $2 \cdot 2$ under the radical), then 2 can make a break for it!

But, only one of the partners survives the jailbreak. The snipers take out the decoy. In the example above, one 2 makes it out, and so does one 3. The final factor of 3 is partnerless, and left trapped inside its radical prison.

Example 13.4

TO DO.

[3] For a handy, if unusually-formatted, list of prime numbers, see appendix A.

13.3.4 Quotient Properties of Radicals

Criteria #3 and #4 for simplified radical form are both pretty antiquated. They came about in the pre-calculator days when folks had to do a lot more calculation by hand and use large data tables to approximate radical values. Yet, these last two properties are still considered "standard" for simplified radical form.

Rules were made to be broken, though, and there will be times when it's OK to break away from these criteria (#4 especially). But we'll burn that bridge when we come to it. For now, all four criteria are in effect.

Both of these have to do with interactions between radicals and fractions. Criteria #3 disallows fractions under the radical, and criteria #4 forbids radicals in the denominator of a fraction.

To tackle Criteria #3, for example when faced with expressions like

$$\sqrt{\frac{4}{49}} \quad \text{or} \quad \sqrt{\frac{24}{25}},$$

we turn to:

Quotient Rule of Radicals

For any $a \geq 0$ and $b \geq 0$,

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

Again, this property applies to radicals of any degree (though for now we'll focus on square roots). And again, this property is just like the quotient rule for exponents, but with a rational exponent.

Example 13.5

Express $\sqrt{\frac{4}{49}}$ and $\sqrt{\frac{24}{25}}$ in simplified radical form.

Solution: Here we have a fairly clear application of the rule:

$$\begin{aligned} \sqrt{\frac{4}{49}} &= \frac{\sqrt{4}}{\sqrt{49}} && \text{quotient rule of radicals} \\ &= \frac{2}{7} && \text{simplify sqaure roots} \end{aligned}$$

In the second example, the numerator doesn't contain in a perfect square, so we must apply the product rule.

$$\begin{aligned}\sqrt{\frac{24}{25}} &= \frac{\sqrt{24}}{\sqrt{25}} && \text{quotient rule of radicals}\\ &= \frac{\sqrt{24}}{5} && \text{simplify denominator}\\ &= \frac{\sqrt{4\cdot 6}}{5} && \text{product rule in the numerator}\\ &= \frac{\sqrt{4}\cdot\sqrt{6}}{5}\\ &= \frac{2\sqrt{6}}{5}\end{aligned}$$

13.3.5 Rationalizing the Denominator

When simplifying an expression using the division property, we may encounter something like the following:

$$\sqrt{\frac{9}{2}} = \frac{\sqrt{9}}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

Back in the pre-calculator days, this led to criteria #4, no radicals in the denominator of a fraction.

When we have a radical in the denominator of a fraction we have an *irrational denominator*. Our goal is to fix this by creating an equivalent fraction with a *rational denominator*. The process of making this translation is called **rationalizing the denominator**.

We will employ the trusty *identity property of multiplication*. Remember, multiplying a number by a fancy 1 does not change the value of the number. The trick will be to choose the way our version of 1 looks. We are going to choose a fancy version of 1 that when multiplied by our irrational denominator gives us a rational number (in fact, an integer).

Study the following examples:

Example 13.6

Write $\frac{3}{\sqrt{2}}$ in simplified radical form.

Solution: Note the clever use of multiplication by a fancy version of 1.

$$\begin{aligned} \frac{3}{\sqrt{2}} &= \frac{3}{\sqrt{2}} \cdot 1 && \text{identity property of multiplication} \\ &= \frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} && \text{substitute a fancy version of 1} \\ &= \frac{3\sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} && \text{multiply fractions} \\ &= \frac{3\sqrt{2}}{2} && \text{definition of square root (in the denominator)} \end{aligned}$$

Note that we chose as our fancy 1 *exactly what we needed* to make the denominator of our fraction turn into an integer. This might seem like cheating, but it's a completely legal move, algebraically speaking.

Be sure to pay close attention. Sometimes we can use the division property in reverse to get rid of radicals in the denominator. We'll work the next example in two different ways to show the comparison.

Example 13.7

Write $\frac{\sqrt{84}}{\sqrt{6}}$ in simplified radical form.

Solution: First, we'll rationalize the denominator using a fancy version of 1.

$$\begin{aligned} \frac{\sqrt{84}}{\sqrt{6}} &= \frac{\sqrt{84}}{\sqrt{6}} \cdot 1 && \text{identity property of multiplication} \\ &= \frac{\sqrt{84}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} && \text{substitute a fancy version of 1} \\ &= \frac{\sqrt{84} \cdot \sqrt{6}}{\sqrt{6} \cdot \sqrt{6}} && \text{multiply fractions} \\ &= \frac{\sqrt{84 \cdot 6}}{6} && \text{product rule for radicals} \\ &= \frac{\sqrt{2 \cdot 2 \cdot 3 \cdot 7 \cdot 3 \cdot 2}}{6} && \text{simplify numerator using the sniper method} \\ &= \frac{2 \cdot 3\sqrt{2 \cdot 7}}{6} \\ &= \frac{6\sqrt{14}}{6} \\ &= \sqrt{14} \end{aligned}$$

Now, an alternative approach: We'll use the division property of radicals in reverse first, and then simplify the fraction under the radical.

$$\begin{aligned} \frac{\sqrt{84}}{\sqrt{6}} &= \sqrt{\frac{84}{6}} && \text{division property of radicals} \\ &= \sqrt{\frac{14}{1}} && \text{simplify the fraction} \\ &= \sqrt{14} && \text{Voilà.} \end{aligned}$$

The second approach is much easier in this case, though it may not always be this easy. (Under what circumstances will we be able to use the kind of shortcut?)

Warning!

Answers with fractions must be simplified, but folks sometimes get overly aggressive with the simplification. Consider the following:

$$\frac{2}{\sqrt{6}} = \frac{2}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{2\sqrt{6}}{\sqrt{6} \cdot \sqrt{6}} = \frac{2\sqrt{6}}{6}$$

At this point, we can do one more simplification:

$$\frac{2\sqrt{6}}{6} = \frac{\sqrt{6}}{3} \qquad \text{Yes!}$$

But we might be tempted to try and simplify even more:

$$\frac{\sqrt{6}}{3} = \frac{\sqrt{2}}{1} \qquad \text{No!}$$

It's tempting, but we can't simplify using things *under* the radical and things *outside* the radical. That 6 under the radical cannot cancel with the 3 outside! To attempt such a simplification is **Evil and Wrong**.

Example 13.8

Determine the value of x given the equation: $2x^2 + 4x - 3 = 40$.

Solution: We'll use everything that we learned in chapter 12! Our first step is to ensure that the first term is a perfect square, so we multiply through by 2.

$$4x^2 + 8x - 6 = 80$$

This also gives us an even linear coefficient, so it looks like we're ready for the square method.

	$2x$	2
$2x$	$4x^2$	$4x$
2	$4x$	4

The square method predicts 4 as the constant term, but our equation has -6. APOE to the rescue:

$$\begin{aligned}
4x^2 + 8x - 6 &= 80 \\
4x^2 + 8x + 4 &= 90 && \text{APOE: add 10 to both sides} \\
(2x+2)^2 &= 90 && \text{based on the square diagram} \\
2x + 2 &= \pm\sqrt{90} && \text{square root of both sides} \\
2x &= -2 \pm \sqrt{90} && \text{SPOE} \\
x &= \frac{-2 \pm \sqrt{90}}{2} && \text{DPOE}
\end{aligned}$$

Almost there! Remember, we need to simplify our radicals! We can use the sniper method in this case:

$$\sqrt{90} = \sqrt{3 \cdot 3 \cdot 2 \cdot 5} = 3\sqrt{2 \cdot 5} = 3\sqrt{10}.$$

And so in the end, we have

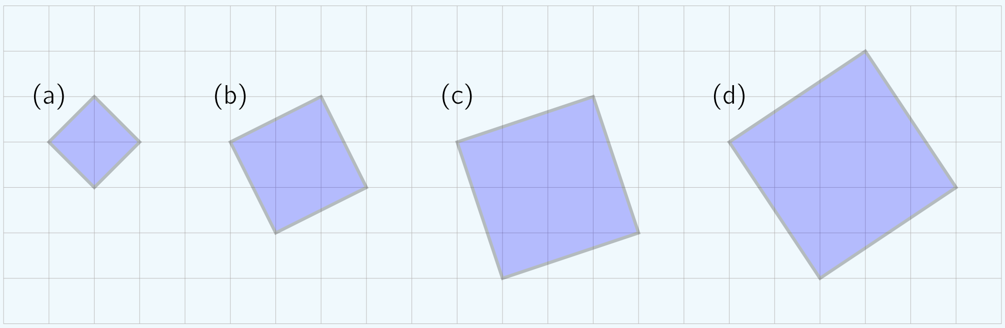
$$\mathcal{S} = \left\{ \frac{-2 \pm 3\sqrt{10}}{2} \right\}$$

We'll admit that these answers ain't pretty (note that there are two answers there!), but they are the values that satisfy our original quadratic equation.

13.4 Coordinate Geometry

Startup Exploration: Squarea

As we saw in section 13.2, a square with area $\mathcal{A}$ has side length $\sqrt{\mathcal{A}}$. Consider the figures below.



What is the area of each square? What is the side length of each square? Can you draw a square with side length $\sqrt{8}$? What about $\sqrt{13}$?

By the way, how do we know that each of these figures is, in fact, a square? (Hint: Think back to the slopes of parallel lines.)

As an application of radicals and radical expressions, which are closely connected to the side lengths of squares, it's natural to discuss concepts from geometry. We'll begin with one of the most famous and important statements in mathematics.

13.4.1 The Pythagorean Theorem

It's a good bet that have seen the Pythagorean Theorem before, and that you will see it in every high school mathematics class you take, and many of the mathematics classed you take in college. In fact, the Pythagorean theorem is a foundational piece of an entire branch of mathematics based on the properties of triangles called *trigonometry*.[4]

[4] Trigonometry begins with the study of triangles. *Trigon* is another way of saying *triangle* – in fact it might be a better way of naming that shape! Most of the other polygons we know (pentagons, hexagons, octagons) have that *-gon* suffix, and the prefix *tri-* means "three" (as in tricycle).

The Pythagorean Theorem

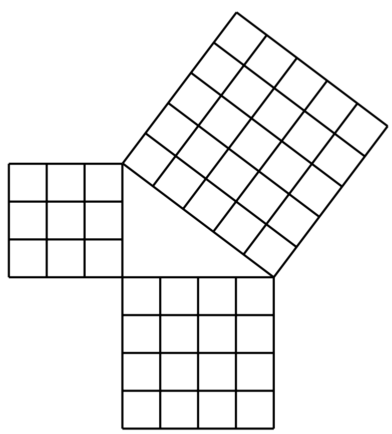
The sum of the squares of the lengths of the **legs** of a right triangle is equal to the square of the length of the **hypotenuse**.

In other words, if a and b represent the lengths of the legs (the perpendicular sides) of a right triangle, and c represents the length of the hypotenuse (the longest side, opposite the right angle), then

$$a^2 + b^2 = c^2.$$

The theorem is named after Greek philosopher and mathematician Pythagoras of Samos, who lived around 570–495 BCE. However, there is substantial evidence that the theorem was known to many different cultures from many different time periods. There is evidence, for instance, that the ancient Babylonians knew about Pythagorean triples (see the next section) more than 1000 years BCE.

Let's jump in with a famous right triangle: one with legs of length 3 and 4, and with hypotenuse of length 5. If we draw squares on the sides of the triangle, we can see that the sums of the areas of the two smaller squares (9+16) is exactly equal to the area of the largest square (25).



This relationship holds true for all right triangles as does its *converse*. The converse of the theorem states that if we have three numbers that satisfy the Pythagorean theorem, then we know that they must form the sides of a right triangle.

You might be asking yourself, "How do we know that the theorem is true for *every right triangle ever*?" Those who are curious might enjoy exploring this question at the end of this section.

13.4.2 Pythagorean Triples

Pythagorean Triple

A *Pythagorean triple* is a set of three positive integers satisfying the Pythagorean theorem.

There are infinitely many Pythagorean triples, and it is quite handy to know a few by heart. (This is because they are used quite a bit in problems, and those delightful standardized tests we all know and love.) Here are a few common Pythagorean triples:

$$(3,4,5) \qquad (5,12,13) \qquad (7,24,25) \qquad (9,40,41) \qquad (8,15,17)$$

The benefit of memorizing a few of these is that, if you see that a right triangle that has leg lengths of 7 and 24, you know that the hypotenuse has length 25 without having to do any calculations.

The triples above are called **primitive Pythagorean triples** because the three values are relatively prime. You can generate new Pythagorean triples by scaling up a triple that you know. For example $(6,8,10)$ is a triple formed by scaling up the $(3,4,5)$ triple by a factor of 2.

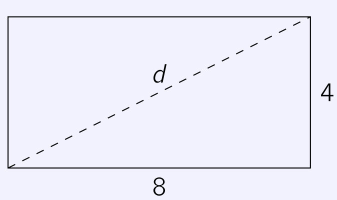
Find-the-Missing-Side Problems

The classic application of the Pythagorean theorem is to find a missing side length, either a leg or a hypotenuse, and you may have solved problems like this before. Now that we have some algebra skills, however, our answers should be given in simplified radical form! No more decimal approximations (unless the directions state otherwise)!

Example 13.9

Find the length of the digaonal of a 4-by-8 rectangle.

Solution: This problem might not, at first, seem to have anything to do with the Pythagorean theorem. The theorem, after all, is about right triangles, and this question is about a rectangle! Drawing a picture helps to reveal the connection:



If we let d represent the length of the diagonal, then we can see that it is the hypotenuse of a right triangle with legs of length 4 and 8. So:

$$\begin{aligned}
d^2 &= 4^2 + 8^2 && \text{Pythagorean theorem} \\
d^2 &= 16 + 64 \\
d^2 &= 80 \\
d &= \sqrt{80} && \text{square root of both sides} \\
d &= \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5} && \text{simplify using the sniper method} \\
d &= 2 \cdot 2 \cdot \sqrt{5} \\
d &= 4\sqrt{5}
\end{aligned}$$

So, a 4-by-8 rectangle has a diagonal which is $4\sqrt{5}$ units long.

To get a feel for whether this answer is reasonable, we could find a decimal approximation using a calculator (it's about 9 units long, which seems OK), or we could reason as follows. We know that $\sqrt{4} = 2$, and so $\sqrt{5}$ must be a bit more than 2. So, $4\sqrt{5}$ must be a bit more than 8. This seems reasonable.

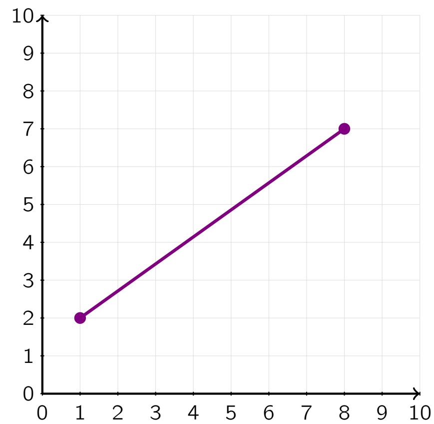
On the other hand, if we had gotten $4\sqrt{8}$ we might reason that $\sqrt{9} = 3$ and so $4\sqrt{8}$ must be a bit less than $4\sqrt{9} = 4 \cdot 3 = 12$. This is too long for the diagonal of a 4-by-8 rectangle, since the two sides together are only 12 units long in total!

13.4.3 The Distance Formula

Startup Exploration: Grid Distance

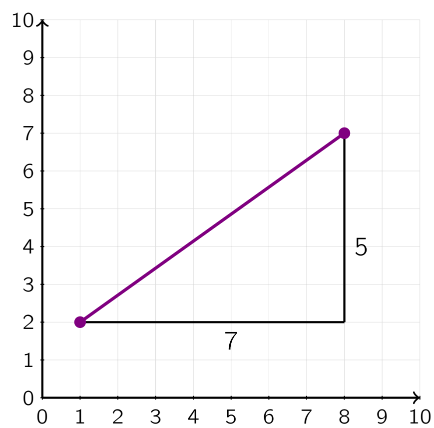
Find the length of the line segment connecting the points $(1, 2)$ and $(6, 7)$.

One important application of the Pythagorean theorem is called the **distance formula**. It is a formula that we can use to calculate the distance between two points on a coordinate grid. In the startup exploration, we have the segment pictured below:



If we think of the line segment into the hypotenuse of a right triangle, then we can use the Pythagorean theorem! The legs are the vertical and horizontal distances between the points, as in a slope triangle.

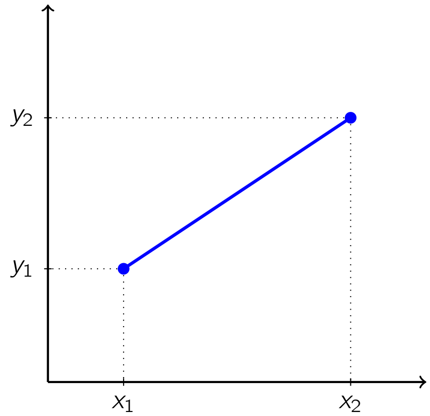
Draw the triangle, find the horizontal and vertical distances, then apply the Pythagorean theorem.



The horizontal distance is 7 units, and the vertical distance is 5 units. Those are the legs of the right triangle. Then, we use the theorem to find the length of the hypotenuse:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 5^2 + 7^2 &= c^2 \\ 25 + 49 &= c^2 \\ 74 &= c^2 \\ \sqrt{74} &= c \end{aligned}$$

Since the process is the same every time, we can generalize to find the distance between any two points (x_1, y_1) and (x_2, y_2) in the plane.



Distance Formula

Given two points in the plane (x_1, y_1) and (x_2, y_2), the length d of the line segment connecting the points is given by the formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This might look like a bunch of alphabet soup, much harder to remember than the Pythagorean theorem. But remember: this *is* the Pythagorem theorem! If you forget the formula, don't panic! Just remember that the line segment between the two points is the hypotenuse of a right triangle.

Example 13.10

Find the distance between the points $(5, 12)$ and $(-4, -2)$.

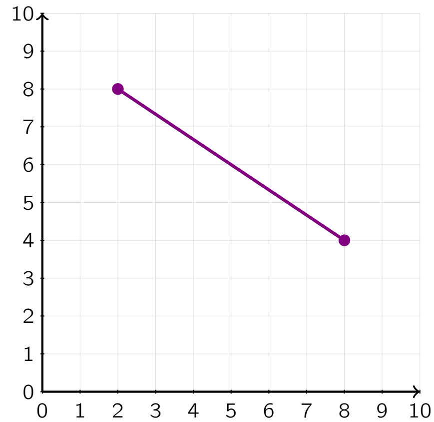
Solution: We will use the distance formula, but note that we have both subtraction and negative numbers. Watch those minus signs!

$$\begin{aligned} d &= \sqrt{(5 - {}^-4)^2 + (12 - {}^-2)^2} \\ &= \sqrt{9^2 + 14^2} \\ &= \sqrt{81 + 196} \\ &= \sqrt{277} \end{aligned}$$

Since 277 is prime, we know our answer complies with simplified radical form, so we're all done. The distance betnwee the two points is $\sqrt{277}$ units.

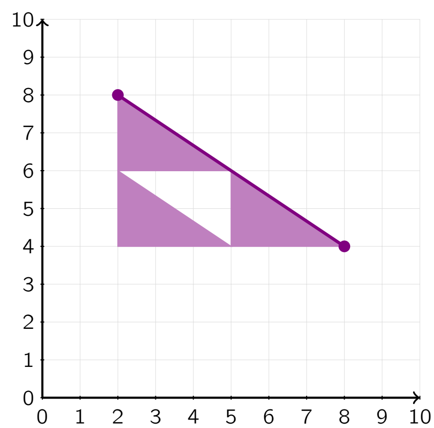
13.4.4 Midpoint Formula

When working with the distance formula, there is a related formula for finding the coordinates of *midpoint* of a given line segment. Suppose we wanted to find the midpoint of the line segment connecting $(2, 8)$ and $(8, 4)$.



Well, it sure looks like the midpoint of this segment is the point $(5, 6)$... but can we be sure?

One way to explain this is by drawing in the right triangle, and then chopping that triangle into four congruent sub-triangles. (Remember our work with the Sierpiński triangle ages ago?)



This diagram suggests that the x-coordinate of the midpoint of the hypotenuse is exactly halfway between the x-coordinates of the legs. The same goes for the y-coordinate. So, to find the coordinates of the midpoint, all we have to do is average the coordinates of the endpoints! Once again, you don't have to memorize this formula if you remember where it comes from!

The Midpoint Formula

Given a line segment with endpoints (x_1, y_1) and (x_2, y_2), the coordinates of the midpoint of the segment are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

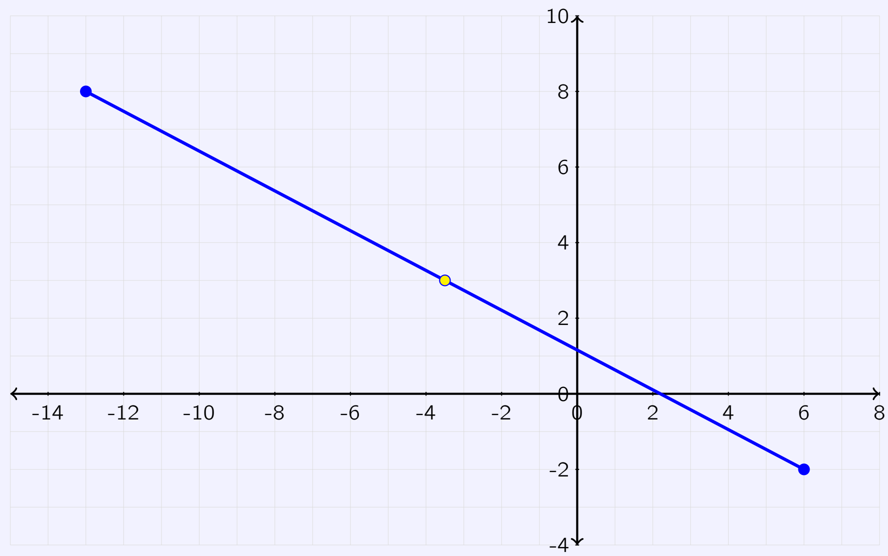
Example 13.11

Find the midpoint of the segment connecting the points $(-13, 8)$ and $(6, -2)$.

Solution: Let's calculate the midpont first, and then check our answer by drawing a picture. The formula is pretty straightforward. The midpoint should be located at:

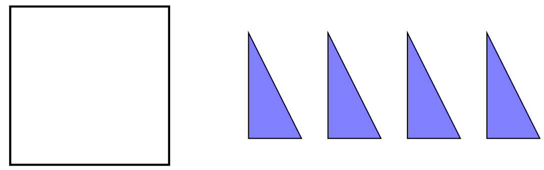
$$\left(\frac{-13 + 6}{2}, \frac{8 + -2}{2}\right) = \left(\frac{-7}{2}, \frac{6}{2}\right) = (-3.5, 3)$$

Here's a graph to check that our answer is reasonable:

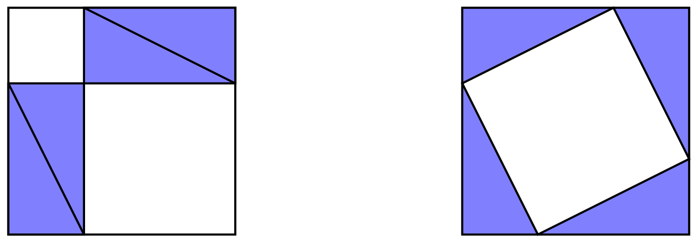


13.4.5 (;,;) Proof of the Pythagorean Theorem

Imagine a pizza box and four identical slices of pizza... where the pizza slices are right triangles. (Not typical for pizza slices, we know, but go with it.)



Consider two different arrangements of the same four pizza slices inside the same pizza box. The slices are arranged to that they fit inside without overlapping, and they lay flat on the bottom of the box.



Step 1: Consider the image on the left: Write an expression for the area of the box that is left *uncovered*. You may find it helpful to label the sides of the triangles, or the sides of the box (or both).

You may be tempted to say that the uncovered regions are squares. How do we know – for sure – that these two regions are actually *squares*, as opposed to some other quadrilateral?

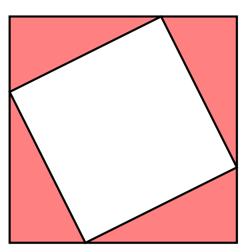
Step 2: Consider the image on the right: Write an expression for the area of the box that is left *uncovered*. Use the same labels you assigned when studying the other picture.

Again, you may be tempted to say that the uncovered region is a square. How do we know that it is a square (and not, say, a rhombus)?

Step 3: The area that is uncovered by the pizza slices is the same. (How do we know this?) What does this tell us about the two expressions for the uncovered area?

An Alternative Approach

In fact, we can explain the theorem using only the right-hand diagram. Let a and b be the lengths of the legs of one of the triangles, and let c be the length of the hypotenuse.



Step 1: Express the sides of the largest square (the pizza box itself) in terms of a and b.

Step 2: Express the area of the largest square in terms of a and b. (Hint: You'll need the sum to a power rule.)

Step 3: The area of the largest square can also be expressed at the sum of the areas of the four triangles plus the area of the tiled square. Express the area of the largest square in this way (in terms of a, b, and c).

Step 4: We now have two ways of expressing the area of the largest square. What happens when set them equal to each other?

Generalizing

All the figures in this discussion have been drawn using a particular square and a particular triangle. Explain why these arguments are proof that the Pythagorean theorem is true for any right triangle, not just the specific triangle that is pictured in these diagrams.

Chapter Summary

Each chapter should have some kind of closing remarks that summarize the chapter briefly and give some hints about the future direction.

Nature creates curved lines while humans create straight lines.

Hideki Yukawa
Japanese theoretical physicist

Chapter 14

Quadratic Functions

In the last few chapters, we have studied quadratic equations, the methods for solving them, and the various algebraic rules for manipulating the kinds of answers we get (square roots). We turn now to quadratic relationships and situations that can be represented using quadratic functions. We'll look more closely at the equations and graphs of quadratic functions, and look closely at one of the classic applications from physics.

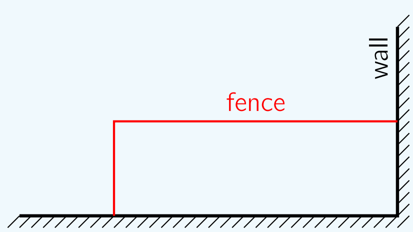
14.1 Quadratic Relationships

Extended Exploration: Staking a Claim

LINK

Startup Exploration: Compost Heap

Uncle François wants to make a compost heap in the corner of his vegetable garden. He plans to enclose a rectangular area using the two existing walls of the garden, plus 8 yards of leftover fencing.



If he wants to enclose the largest possible area, how should he arrange the fence? What is the maximum area he can enclose?

Thoughts:

This startup is similar to staking a claim, but I hope different enough. I couldn't come up with a new quadratic scenario that so clearly demonstrated all the features of quadratics (all in the first quadrant) AND had such a natural way to write the equation.

The plan will be to do the staking a claim treatment with this problem and hit the highlights about symmetry, the vertex, and all that.

14.2 Vertex Form

Build on transformations to discuss vertex form, and graphing in vertex form.

14.3 Graphing Quadratics

Graphing quadratics in standard form, equation for line of symmetry in standard form, finding vertex in standard form (using LOS).

14.4 Projectile Motion

Projectile motion is a classic application of quadratic functions, and we'll study both the mathematics and the science in this section.

Startup Exploration: Rocket Launch

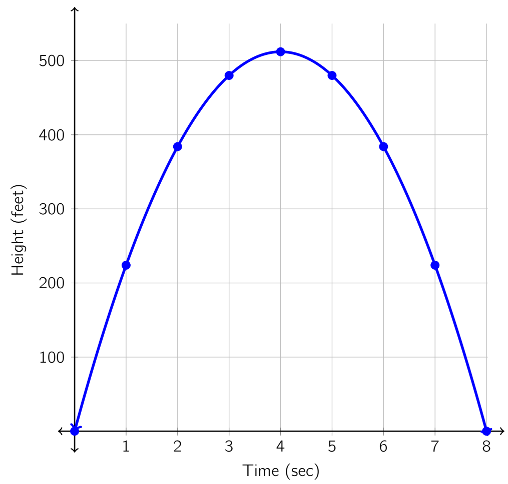
When an object is launched vertically, it flies straight up and then falls straight down. Height data for the flight of one of Ivan's model rockets is given below.

Time (sec)	Height (feet)
0	0
1	224
2	384
3	480
4	512
5	480
6	384
7	224
8	0

Study the table and make a graph of the data to see what you can learn. Then, write a few sentences describing the flight of Ivan's rocket in as much detail as you can.

From the data itself we can see the up-and-down pattern. Plus, we can see a nice symmetry: The object starts at a height of 0 meters (so it starts on the ground) and lands on the ground 8 seconds later. One second after launch it is at the same height as 1 second before it lands. This mirror image pattern continues until the object reaches its maximum height 4 seconds after launch.

A graph of the data shows the same up-and-down pattern, and the same symmetry: we could fold this graph along the vertical line at $x = 4$ and the two halves would match up exactly.



These features all suggest that we may have a quadratic pattern.[1] Scientifically speaking, Ivan's rocket is a *projectile*, and quadratic functions are the underpinning of the physics that describes the trajectory of a projectile.

Projectile

Any object that moves through the air or through space acted on only by the force of gravity.

We use projectiles in games: thowing a baseball, basketball, or dart. We use projectiles as weapons: shooting an arrow or a cannonball, throwing a spear, or launching something from a catapult.

Not all flying things are projectiles. Airplanes, helicopters, and birds use energy to keep themselves in the air, and they can use that energy to steer and change direction in midair. Projectiles are not "powered flight", they are launched and the allowed to follow their path, acted on only by gravity.

Trajectory

An imaginary tracing of a projectile's position as it moves through space.

Ivan's rocket was launched, but we can use the term "launch" rather loosely. An item that has been dropped is, according to our definition, a projectile. This is the first kind of motion we'll investigate closely, beginning with a discussion about gravity.

[1] The title of this chapter might also have been a hint.

14.4.1 Falling Objects

An object that has been dropped and falls through space is one example of a projectile. Gravity causes the object to *accelerate* towards the ground. In other words, falling objects *increase in velocity* (speed) as time goes by.

The acceleration of gravity is −9.8 meters per second per second, or −32 feet per second per second. No, that's not a typo: we're talking about meters per second *per second*. Acceleration is a change in speed. Speed might be measured in "meters per second", and so if an object is accelerating its speed might be changing at a rate of 9.8 meters per second, *per second*. We some times abbreviate "meters per second per second" as m/s^2.

The acceleration of gravity is a negative quantity because acceleration has both a magnitude and a direction, and that direction is downwards (toward the center of the Earth). Gravity will slow down an object that is thrown upwards, and pulls dropped objects towards the ground.

This graph shows the height of an object that has been dropped over time. Notice how the curve of the graph gets steeper the longer it is falling. What can you discern from the graph? How high up was the object when it was dropped? When does the object hit the ground?

»> GRAPH

»> The motion of a projectile can be explained using a quadratic equation.

»> Misconceptions about the graph. Looks like the object is moving sideways, but the graph shows height over time, not x- and y- displacement. Footnote about how this comes later, parametric functions.

Example 14.1

Mr. Campbell, science teacher and inventor of the Spam cannon, drops a can of Spam off a bridge that is 80 feet high. When will the Spam hit the ground?

Solution: To do.

Technically, when we solve a quadratic equation like this, we often get both a positive and a negative value. But the negative solution doesn't make sense in the context of the problem: that would be a point in time before the Spam was dropped.

14.4.2 Horizontal Launch

Dropping objects is boring! Let's throw something! But, we're going to add a constraint: we will throw things horizontally, parallel to the ground.

Objects that are thrown or launched horizontally travel in two dimensions: both horizontally and vertically. However, the vertical and horizontal motion are independent. Gravity pulls the object downwards, but has no influence on the projectile's horizontal movement.

To describe the vertical part of the projectile's motion, we use the equation for a falling object. To describe the horizontal motion, use the usual equation $d = r \cdot t$, where r is the object's initial horizontal velocity. Notice how we have separated the two aspects of the projectile's motion: the vertical component is just like dropping, the horizontal component is just like movement along the ground.

Example 14.2

Mr. Campbell fires a can of SPAM horizontally off the edge of an 80-meter high cliff with an initial horizontal velocity of 120 meters per second. When will the SPAM hit the ground? How far from the base of the cliff will it land?

Solution: To do.

14.4.3 Vertical Launch

Objects that are thrown or launched vertically travel in only one dimension. The rising and falling components of their motion are symmetrical. We saw an example of this vertical motion at the start of this chapter.

This equation is just like the falling object equation, but it has the "vt" added to it. This represents the distance traveled because of the upward motion from the launch.

Example 14.3

Mr. Campbell fires a can of Spam vertically from the ground with an initial velocity of 96 ft/s. When will the Spam reach its maximum height? What is the maximum height of the can of Spam?

Example 14.4

Mr. Campbell fires a can of Spam vertically from the top of the 80-meter (240-foot) cliff with initial velocity of 128 ft/s. When will the SPAM reach its maximum height? What is the maximum height of the can of SPAM?

As it falls, the Spam drifts slightly, falls past the launch site, and falls all the way to the base of the cliff. What is its total flight time?

Solution: To do.

»> Something about how we're not launching at an angle yet.

»> Summary section and other closing remarks in transition to polynomials...

Still need to find a quote that works for this chapter. In the meantime, we have this.

Author, description of author

Chapter 15

Polynomials

Much of the beginning of this section is review and renaming as we have actually been working on polynomials all year long!

All real numbers can be represented as polynomials. So, like numbers, we will be learning how to add, subtract, multiply, divide, factor, raise to powers, etc. with polynomials.

15.1 Building Polynomials

Exploration

Before reading he official definition of a polynomial, consider the two lists below.

Polynomials	Not Polynomials
$2x+4$	2^x+4
$3x$	$\sqrt{3x}$
$1.2y^2$	$1.2y^{-2}$
$5x^2+3x-7$	$\lvert 5x+3 \rvert - 7$
$-2w^3-8w^2+w$	$-2w^3-8x^2+y$
$\frac{1}{4}x+15$	$\frac{4}{x}+15$

Based on this information, conjecture about what it means to be a polynomial. Which of the following (if any) are polynomials?

$$6x-8+7x^{-1} \qquad 150 \qquad x^2+x^4-3x^3+8-x$$

Recall that a **term** is a number, a variable, or the product (or quotient) of numbers and variables. The following expressions are all terms:

$$4 \qquad 3x \qquad 14x^6$$

We have been dealing with terms throughout this course. We know how to combine *like terms*. When we perform the distributive propoerty, we are multiplying one term with each of several terms.

Remember that terms are separated by addition and subtraction, and that an **algebraic expression** is the sum or difference of terms. An expression can be a single term all by itself, since x and 0 are both terms, $x + 0$ (which is the same as x) is an expression.

A **polynomial** is a special type of algebraic expression.

Polynomial

A algebraic expression which contains only a single variable, in which the coefficients are all rational numbers, and the degree of each term is a non-negative integer.

Recall that the **degree of a term** is the exponent on the variable of a term. It is usually quite easy to identify the degree of a term: the degree of $5x^3$ is 3. Remember, though, that some special situations arise.

If we have a variable with no exponent, we imagine a phantom 1 in the exponent. So for example, the term $18x$ is of degree 1, since $18x = 18x^1$. If we have a number with no variable, we can imagine a phanom 1 of a different kind: a variable raised to the zero power. So, the term 10 is of degree 0, since $10 = 10x^0$.

This means that a polynomial can always be written in the form

$$a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_nx^n,$$

where this sum stops at some integer n, and in which the a-values (the coefficients) are rational numbers, some of which might be zero. Usually, we prefer to write a polynomial with the terms in order of decreasing degree, that is:

$$a_nx^n + \cdots + a_3x^3 + a_2x^2 + a_1x^1 + a_0x^0$$

In other words: a polynomial may not have a variable in the exponent, and may not include a variable raised to a non-integer exponent (so, no variables under radicals), nor a negative exponent (so, variables in a denominator). Also, the variable may not be enclosed in an absolute value expression.

Look back over the list of polynomials and non-polynomials in the startup exploration. The expressions in the list of polynomials all meet these requirements. The non-polynomials all fail on at least one point (can you identify the point of failure for each one?). Of the three expressions we are asked to classify, the first one is not a polynomial, since it includes the variable x raised to a negative exponent.

The other two expressions are polynomials. The number 150 is a polynomial since it is of the form $150x^0$. The third expression given meets the definition of a polynomial, and can be rewritten as follows:

$$x^4 - 3x^3 + x^2 - x + 8$$

Degree of a Polynomial

The degree of the highest-degree term in the polynomial.

The degree of a polynomial is the highest exponent in a polynomial. This is very important because that highest degree has the biggest impact on the value of the expression. If we were to put a "y=" in front of the polynomial to turn it into a function, that highest degreed term would tell you which family it belongs in, what the graph is going to look like, etc.

Vocabulary

Standard Form of a Polynomial A polynomial written so that the degree of the terms decreases from left to right and no terms have the same degree.

Very much common sense at this point in the year. Basically Standard form is simplified (like terms are combined and no parenthesis), you just have to make sure to write the terms in the correct order. The highest degree comes first, the rest follow in order of decreasing degree, which you've been required to do for a while now.

15.1.1 Naming Polynomials

In the life sciences, biological classification is the practice of grouping organisims into species, and then grouping various species togeher into groups, and then grouping those groups together, and so on, to produce a systematic classification of living things. Such a classification is important so that biologists can then be clear when communicating to one another.

For example, there are many different types of fruit fly, and so the term "fruit fly" is ambiguous. In English, we make a distinction between the so-called "common fruit fly" and the "Asian fruit fly". Biologists in Korea, however, might not use these terms in the same way (is the Asian fruit fly more common in Korea than the common fruit fly?). Biologists have therefore agreed to use a different, more universal, system for describing organisms so that they can distinguish between *Drosophila melanogaster* (what we call the common fruit fly) and *Drosophila suzukii* (what we call the Asian fruit fly).

The Latinized names for organisms identify the genus and species. In mathematics, we classify polynomials using Latin and Greek names that identify the degree and the number of terms.

Degree	Name
0	Constant
1	Linear
2	Quadratic
3	Cubic
4	Quartic
5	Quintic

Table 15.1: List of polynomial names by degree.

The first name is by degree, and this is related to the name you would give to that family of functions. We worked closely with two examples in this course so far. A polynomial of degree 1 is called a *linear* polynomial, and a polyonmial of degree 2 is called a *quadratic* polynomial.

For polynomials with degree higher than 5, we usuall say "polynomial of degree n", or "nth degree polynomial". For example $3x^8 + 4$ is an eighth degree polynomial, or a polynomial of degree 8.

When naming a polynomial, the second name tells us how many terms it has.

No. Terms	Name
1	Monomial
2	Binomial
3	Trinomial

Table 15.2: List of polynomial names by number of terms.

For polynomials with more than three terms, we say "polynomial with n terms" or "an n-term polynomial". For example,

$$11x^8 + x^5 + x^4 - 3x^3 + 5x^2 - 3$$

is a polynomial with 6 terms, or a 6-term polynomial.

Note that some of these prefixes are familiar. *Tri-* means "three", as in triangle and tricycle. *Bi-* means "two", as in bicycle. There are less familiar word parts, too: *mono-* means one, and *-nomial* means "names" (indicating the number of terms)

When giving the full name of a polynomial, we include both the name by degree and the name by the number of terms. Here are some examples:

Polynomial	Name
x	Monomial
y	Binomial
z	Trinomial

Table 15.3: Examples of polynomial names.

Polynomial Name -14x3 -1.2x2 -1 7x - 2 3x3 + 2x - 8 2x2 - 4x + 8 x4 + 3 cubic monomial quadratic monomial constant monomial linear binomial cubic trinomial quadratic trinomial Quartic binomial

Above are examples of the full names of polynomials. Remember we use the family (degree) name first. It is the most important name.

15.2 Adding, Subtracting, and Multiplying Polynomials

Adding and Subtracting Polynomials

To add or subtract polynomials, simply combine like terms. So, nothing really new here. We've been doing this for a long while. I'm just going to add more terms than you are used to and terms of higher degree.

Example 1

Simplify: 1. (5x2 - 3x + 7) + (2x2 + 5x - 7) 2. (3x3 + 6x - 8) + (4x2 + 2x - 5)

Solution: 1. 7x2 + 2x 2. 3x3 + 4x2 + 8x - 13

At this point, you should be able to mentally group the terms together by degree. You can also write the rearrangement down by commuting the terms if you need to. Just be sure to combine the terms with the correct terms. For example, the second example, it is common for student to accidentally combine the cubic term and the quadratic term.

Example 2

Simplify: 1. (2x3+ 4x2- 6) - (3x3+ 2x - 2) 2. (7x3- 3x + 1) - (x3- 4x2- 2)

Solution: 1. (2x3+ 4x2- 6) + (-3x3+ -2x - -2) = -x3+ 4x2- 2x - 4

2. (7x3 - 3x + 1) + (-x3- -4x2- -2) = 6x3+ 4x2- 3x + 3

More examples, this time with subtraction. You have to remember that every term of that second polynomial is being subtracted. So it is useful to change the problem to an addition one before moving on remembering that subtraction is defined by adding the opposite. You basically have to change every sign in the polynomial that you are subtracting. In essence, you are distributing a negative one.

How you show your work is up to you. It might be easier to keep track of everything if you set the problem up vertically like an "old school" addition or subtraction problem. To do so, just think of the degree of the term like the place value of a digit. You can leave spaces blank if there is a degree missing. I put the subtraction in parenthesis to remind myself that every term needs to be subtracted. Sometimes if you don't , you just think that the coefficient of the first term is negative.

Example 3

Simplify: 1. (7y2- 3y + 4) + (8y2+ 3y - 4)

2. (2x3- 5x2+ 3x - 1) - (8x3- 8x2 + 4x + 3)

Solution:

1. 7y2- 3y + 4 + 8y2+ 3y - 4 15y2 + 0y + 0 = 15y2

2. 2x3- 5x2+ 3x - 1 - (8x3- 8x2 + 4x + 3) -6x3+ 3x2- x - 4

Example 4

Simplify: 1. (7y3 + 2y2+ 5y - 1) + (5y3+ 7y)

2. (b4 - 6 + 5b + 1) + (8b4 + 2b - 3b2

Solution:

1. 7y3 + 2y2 + 5y - 1 + 5y3+ 0y2 + 7y + 0 12y3+ 2y2 + 12y - 1

2. b4 + 0b3 + 0b2 + 5b - 5 + 8b4 + 0b3- 3b2 + 2b + 0 9b4 + 0b3 - 3b2 + 7b - 5 = 9b4 - 3b2+ 7b - 5

Multiplying Polynomials

You already know how to multiply polynomials. You just use the distributive property, either single or multiple. You can also set it up Leo B. style. In this section, we will do a little review and add something that will help with factoring, the area model (algebra tiles.)

Algebra tiles are tools that help one represent polynomials. They can, at most, be used to represent quadratics. There are six tiles. The red tiles represent negative quantities. The other colors represent the positive quantities. The small square is the unit square and represents 1. The thin rectangle represents x. The large square represents x2.

You can use the tiles to show a picture of a polynomial, to show how addition/subtractions work, but what they are really good for is showing how multiplication and factoring work. Let's start with the basics. Example 5: What polynomial is represented below?

Solution: So, what is the polynomial represented above? 2x2 + 2x - 4x + 8 - 3 Simplified (because red/green and red/yellow cancel) 2x2 - 2x + 5

When you multiply, you are really finding the areas of rectangles. The length and width are the factors and the product is the area.

Example 6: Multiply 2x (3x + 1)

They are really good at having you understand how to multiply polynomials. Each factor is a dimension of a rectangle. So 2x is the width and 3x + 1 is the length. Then you add the pieces to make a rectangle... x * x is x2 so you put little x2 pieces wherever you see an x *x. When you see an x*1 you get x, so you put a little green tile there. You end up with six x2 and two x, which is the product we knew it would be to begin with, because you already know how to multiply via the distributive property. This seems annoying now, but will help you out when I ask you to undo multiply two binomials.

Example 7: Use an area model to show the following multiplication. (x + 2) (x + 1)

1. What are the factors? 2. What type of factors are they? 3. What will their product look like? 4. Use the tiles to multiply

Solution: Let's try to use the area model to multiply two binomials together. This sort of multiplication works a little differently than the regular distributive property, as you saw when we covered raising binomials to powers and the Leo B. method, so we will see what happens when we use the tiles.

1. x+2 and x + 1 2. They are both linear binomials 3. Should be a quadratic trinomial. 4.

Use the tiles just like we did before. Each factor is a dimension of the rectangle. The width is x + 2 and the length is x + 1. Multiply the x's and the units. What is the area of the rectangle? x2 + x + 2x + 2. Simplify and get x2 + 3x + 2. The multiplication yielded 4 terms, two from multiplying x times (x+1) and two from multiplying 2 times (x+1). When we combined like terms, we got our quadratic trinomial.

If you "learned" algebra from a book, Kumon, or some outside math class, you probably learned how to multiply using FOIL. It only works with two binomials, which is why I don't like it. If I only can remember one way to multiply, it should be something that works for everything. FOIL does not.

It is called F.O.I.L. which stands for "First, Outer, Inner, Last" F = first terms in each binomial O = the two outer terms I = the two inner terms L = last terms in each binomial

If you want practice, go back and simplify the ones you did with the tiles using FOIL. This is really just a double distribution. F and O come from distributing the first x over the second binomial. The I and L come from distributing the second term of the first binomial over the second binomial.

15.3 The GCF

The Basics of Factoring

Factoring is not dividing. It is re-writing a number as a multiplication problem. You can factor 6 and rewrite it as $2 \cdot 3$. You can do the same to polynomials. Your goal is to rewrite the polynomial as a multiplication problem. It has the same value, but looks different. There are different types of factoring. We will start with the type that undoes the multiplication of the most basic form of the distributive property you learned with the Field Axioms.

You have factored before. In 6th grade you learned how to "prime factorize" numbers. We are going to be doing something similar with polynomials. For polynomials, factoring is working backwards from multiplication. You have the product as an answer. I want you to undo the polynomial multiplication to find the factors that made that product.

Undoing the Distributive Property by Factoring Out the GCF

This is what we are familiar with. This is how we simplify by distributing.

Simplify: 3x(2x + 5) 3x*2x + 3x * 5 6x2 + 15x

Now factoring takes the 6x2+ 12x (the product) and rewrites it so that you have the original, un-simplified multiplication problem. There is a lot of logic involved. You have to find the Greatest Common Factor of the terms of a polynomial. To find the GCF of a polynomial 6x2+ 15x, you have to look at each term. What do you notice about each of the terms? You should notice that both terms have x and both terms are multiples of 3. This gives you a clue to what some common factors are... 3 and x. Now, to find the greatest common factor. You can do this systematically and prime factorize each term.

6x2: $2 \cdot 3 \cdot x \cdot x$ 15x: $3 \cdot 5 \cdot x$

Circle the common factors. All of the common terms multiply together to form the GCF. What do you notice between the GCF and the previous problem? The GCF is that number that got sprinkled during the distribution. This means that if you can find the GCF, you can undo the distributive property and factor a polynomial.

Example 1

Find the GCF of 6x4 + 4x3 + 8x2 Solution: What do you notice about each of the terms? You may be able to find the GCF that way, or, if you can't, prime factorize each term. 6x4: $2 \cdot 3 \cdot x \cdot x \cdot x \cdot x$ 4X3 : $2 \cdot 2 \cdot x \cdot x \cdot x$ 8x2 : $2 \cdot 2 \cdot 2 \cdot x \cdot x$

All of the terms have an x2 and all of them are even, so 2x is the GCF. Example 2

Find the GCF of each of the following polynomials. 1. 12x4+ 18x3 2. 32y4- 16y2 3. -4y2- 8y - 12 Solution: 1. 6x3 2. 16y2 3. 4 or -4 (-4 technically isn't the greatest but it is sometimes convenient to use the negative factor to change all of the signs)

Factoring out the GCF

Technically, when you undo the distributive property you are "factoring out a monomial term." Now, if you can find the GCF, this is really easy. The GCF is the thing being sprinkled. The stuff left over after the GCF is taken out goes in parenthesis. It is the thing that gets the sprinkling. Now, the cool thing, is that this is so easy to check. If you want to see that you factored correctly, all you have to do is redistribute.

We will use 6x3 + 4x2 + 8x. Start by finding the GCF, which we know is 2x from example 1. The GCF is the monomial that goes outside of the ( ) in the distribution problem., whatever factors are left over from the prime factorization makes up the polynomial that goes inside of the ( ). 2x (3x2+ 2x + 4). Now, check to see if they are equivalent. Distribute the 2x or just graph both on your graphing calculator.

If you factor out the GCFs of the polynomials in example 2, you get the following:

1. 6x3(2x + 3) 2. 16y2 (2y2 - 1) 3. 4(-y2 - 2y -3) or -4(y2 + 2y + 3)

Often, when you are asked to factor a polynomial, the first thing you should look for is a GCF that can be factored out of the problem. Actually, one of the most missed things on the assessment over factoring is forgetting to factor out a GCF.

15.4 Factoring the Special Case x2 + bx + c

This is how to factor using algebra tiles. We are basically taking the multiplication and doing it in reverse. Instead of starting with the dimensions, we are going to start with the tiles and arrange them into a rectangle. The length and width of the rectangle will be the factors we are looking for. How can these tiles be arranged into a rectangle?

First, you must know the polynomial being represented by the tiles. Then, experiment and turn them into a single rectangle. After that, find the length and width of the rectangle. Remember that the "x2" tiles have to be in the top left corner and the unit tiles have to form a rectangle in the lower right corner. The "x" tiles fill in the spaces.

Now all you have to do is determine what the factors are that create this rectangle. Look along the left side. It is made up of 1x and 1 unit, so the binomial is (x+1). Finally look along the top. It is made up of 1x and 2 units, so the binomial is (x+2). Therefore the factored form of x2 + 3x+2 = (x+1)(x+2). This is easy to check. Just multiply the binomials back out or use a graphing calculator.

The tiles are manipulatives that are used to show the area model of a polynomial. It is just as easy to realize that a quadratic trinomial results from the product of two binomials from experience. The multiplication of 2 binomials results in 4 terms, two of which are combined in the final step. Instead of drawing the individual tiles, you can just draw a rectangle and break it up into 4 sections. The top left section represents the quadratic term, the bottom right is the section containing the constant term. The other two sections are the two that combined to form the linear term. If you use this instead, you have to figure out how to break up the linear term. There is only one combinations that is going to work. In addition, once you figure out how to break it up and have your rectangle drawn, you find the GCF of each column to find one factor. Find the GCF of each row to find the other factor.

Quadratic Term Part of linear term Part of linear term Constant Term

Of course, we want to be able to do this without the tiles and without drawing rectangles. In essence we are anti-foiling, or anti-double distributing. Think about where all of the terms in the quadratic come from in terms where/when during the process of multiplication.

1. Where does quadratic term come from? 2. Where does constant term come from? 3. Where does linear term come from?

Knowing the answers to the 3 questions above, you can factor using logic.

1. The quadratic term comes from multiplying x by x. So we know our factors have to start (x. . . .)(x. . .) 2. Since all of the terms are positive, we know our factors have to be (x+. . .)(x+. . .) 3. We also know that the factors have to look like (x+ some number)(x + some other number) 4. So what could those numbers be? Well, we know they multiply together to give us the "c" term. Since the c in this case is 2, we know the two numbers have to be 1 and 2. 5. It looks like our factors are going to be (x+1) and (x+2), but we have to be sure. The

way to check is to look at the "bx" term. We know that comes from multiplying the "outer" and "inner" terms together and adding those products together. Let's check to see if this is going to work. The "outer" product is 2x, the inner is 1x, the sum is 3x. That is what we were given originally, so x2+ 3x + 2 = (x+1)(x+2)

If you want to factor without the tiles, you have to "guess and check". Here's an organized way to do just that. It is, of course, based on the above sequence of logical steps.

1. Find the factors of the "c" term. 2. The factors that add up to the "b" term are the correct ones. 3. Check the signs to make sure they will multiply correctly! 4. Check your answer!

That is the actual procedure we are going to use to factor quadratic trinomials of the form x2 + bx + c. Remember to incorporate the sign of the factors of c. The final step is to always check to make sure everything works right. It is really easy to pick the wrong factors or the wrong sign on the factors. I'll go through the reasoning only on some of these, but give the solution to all of them. Remember, for this special case, I am looking for factors of c that add to b.

Example 1

Factor:

1. x2 + 7x + 12 2. x2 + 8x + 12 3. x2 + 2x - 3 4. x2 - 6x + 8 5. x2 + x - 12 6. x2 - 3x - 10 7. x2- 8x + 15 8. x2- 3x - 18 9. x2- 3x + 2 10. x2- 10x + 21

Solution: 1. We need to look for factors of +12 that add to 7. This means they both have to be positive. The possible factor pairs are (1, 12), (2, 6), and (3,4). Which of these pairs add up to 7? That would be 3 and 4. It seems that (x + 3)(x + 4) might be the factors. They add up to 7 and multiply out to 12. Time to check (x+3)(x+4) =x2 + 4x + 3x + 12 = x2+ 7x + 12. 2. (x + 6)(x+2) 3. We need to look for factors of -3 that add to 2. This means that one is negative and one is positive. Also that the positive one is "bigger." The possible factor pairs are (1, -3) and (-1, 3). Which of these add up to +2? (-1, 3). (x - 1) (x+3) looks to be the answer. Time to check (x-1)(x+3) = x2+ 3x - x - 3 = x2+ 2x - 3 4. We need to look for factors of +8 that add to -6. This means that they both are negative. The possible factor pairs are (-1, -8), (-2, -4). Which of these add up to -6? (-2, -4). (x - 2)(x -4) might be the solution. Check: (x-2)(x-4) = x2- 4x - 2x + 8 = x2-6x + 8 5. (x + 4)(x - 3) 6. (x-5)(x+2) 7. (x - 3)(x-5) 8. (x - 6)(x + 3) 9. (x - 2)(x-1) 10. (x - 7)(x-3)

15.5 Perfect Square Trinomials and Difference of Squares

Perfect Square Trinomials

A trinomial formed by squaring a binomial is called a Perfect Square Trinomial. In the exponential unit, we called this raising a sum to a power. Over the course of the past few weeks, you should have noticed that there is a pattern to squaring a sum.

Example 1

Simplify and find the patterns in the resulting trinomial:

1. (x + 5)2 2. (2x - 3)2 3. (x - 4)2 4. (5x + 2)2

Solution:

1. x2 + 10x + 25 2. 4x2 - 12x + 9 3. x2 -8x + 16 4. 25x2 + 20x + 4

The perfect square trinomial has some special features. These features make it easier to multiply these out and easier to factor them too. You should notice that the first term and last term of the trinomial is a perfect square. You might have also noticed that the "bx" term is twice the product of the terms in the parenthesis.

Perfect Square Trinomial

(a +b)2 = a2 + 2ab + b2 (a - b)2 = a2 - 2ab + b2

The key is knowing when you have a perfect square trinomial. You can't just look at the quadratic and constant terms. *Remember, when you square root, you can get a positive or a negative.* The way to check to see if you have a PST A. Look at the ax2 term. Is it a perfect square? If so, move on to #2. B. Look at the c terms. Is it a perfect square? If so, move on to #3. C. Take the square root of ax2 and the square root of c. Multiply the roots together. Then double them. Is this the bx term? If so, you have a perfect square trinomial.

Example 1

Determine whether the following polynomials are perfect square trinomials. If so, factor it.

1. x2 - 12x + 36 2. 9x2 + 34x + 25 3. x2 + 18x + 81 4. 64x2- 20x + 1

Solution:

1. x2 is a perfect square. 36 is a perfect square. The square root of x2 is x. The square root of 36 is 6. 6 times x is 6x. Double it and get 12x. Hmm, I forgot that I can get -6 as a square root of 36, so go through step 3 again, and I get -12x. . . and that is what bx is, so, yes, x2 - 12x + 36 is a perfect square. So this will factor to (x - 6)2 2. No because 3x times 5 times 2 is not 34. We can't factor this yet. 3. Yes because x times 9 times 2 = 18. So this will factor to (x + 9)2 4. No because 8x times -1 times 2 is not -20. We can't factor this yet.

Difference of Squares

The difference of squares is a binomial that is formed by subtracting two perfect squares, literally the difference of 2 squares. Below are some examples:

1. x2 - 4 2. x2- 625 3. 4x2 - 25 4. 16x2- 81

Notice, it is not the sum of squares!!!!!!!! THE SUM OF SQUARES DO NOT FACTOR OVER REAL NUMBERS. This means that you will see "trick" questions that ask you to factor the sum of squares. Those are not factorable in algebra 1.

Now, let's look at these. They are always a "perfect square - another perfect square". Another reason why it is good to know the first 25 perfect squares. We can definitely factor the first two to see what the factors are like. They fit the special case of x2 + bx + c, it is just that b = 0.

1. We are looking for factors of -4 that add to 0. It has to be 2, and -2. So this will factor to (x-2)(x+2) 2. We are looking for factors of -625 that add to zero. It has to be +25, -25, so (x-25)(x+25).

Using the factors we have just found, what do you notice? The factors are almost identical, except for the sign separating the terms. They have to be opposite for that "bx" term to drop out of the final product. So, let's think about what #3 and #4 will be. Almost identical, except for the sign between the terms of the factor. The terms of the factors are square roots of the terms of the original.

3. (2x - 5) (2x + 5) 4. (4x - 9)(4x + 9)

Difference of Squares

a2 - b2= (a + b)(a - b)

This is the general form of a difference of squares. "a" and "b" can be any terms. When you see a difference of squares, to factor, just square root the two terms. Make one binomial have a "+" and the other a "-". Check, and you are done.

Examples! Be careful. #2 - #4 are tricky. You should know by now that your math teacher loves the tricky problems!

Example 2

Factor each of the following completely.

1. x2- 100 2. x4 - 16 3. 100x2 - 400 4. 3x2 - 75 5. 225x2 - 121y2

Solution:

1. Normal... (x + 10)(x -10) 2. Hmm, x4 and 16 are both perfect squares... so (x2 + 4)(x2 - 4)... but wait a minute! One of those factors is a perfect square. I can factor it even more. (x2 + 4)(x -2)(x+2)... but

beware! Don't go "factor crazy" and try and factor the x2 + 4. It is one of those sums of squares that don't factor. I like to call this the "Goldilocks" problem. 3. Hmmm, again... 100 is a perfect square and so is 400, but I notice that the polynomials has a GCF of 100. I can factor that out first. 100(x2 - 4). Now I can factor x2- 4. So 100(x-2)(x+2) 4. Not a difference of squares, but maybe I can do what I did in #3 to factor this one. The GCF is 3. I can factor this to 3(x2 - 25) = 3(x-5)(x+5) 5. Not a polynomial, but It is a difference of squares. So (15x - 11y)(15x + 11y)

15.6 Factoring by Grouping

This is a general form of factoring that will allow you to factor just about anything that can be factored. We will be starting with things that are technically not like the polynomials we study as functions, but only because it is easier to understand the mechanics of factoring by grouping. Factoring by grouping is literally the multiple distribution/Leo B. method in reverse. Let's refresh our memories.

Use the "multiple distribution method" to simplify (a + b) (c + d).

When you multiply these expressions you need to multiply a (c + d) and then b (c + d). You write out ac +ad + bc + bd. If you examine these 4 terms, you should notice that you have groups of terms that have a common gcf, ac + ad has a in common and bc+bd has b in common. Derp... Because that is what we distributed. This means that both a and b were both distributed over (c +d).

So, let's try to factor this example: gh + gt + vh + vt

1. Examine the expression you are given. You need to find equal "groups" of terms that have common factors. gh + gt + vh + vt... there are two "groups" of terms here. The terms with "g" as a factor and terms with "v" as a factor. 2. Separate them out into their two groups gh +gt + vh + vt. I usually do this by just underlining them. 3. Then you need to factor out the gcf of each group you formed. g(h + t) + v(h + t) 4. You should be left with things in ( ) that are identical. That (expression) is one of the factors, so (h + t) is one of the factors. The gcf's form the other factor... so g + v. 5. Finally you check your answer. (g + v)(h + t) = gh + gt + vh + vt.

Note: Sometimes you will have to rearrange the factors to find the correct groups. You may also have to factor out a negative to get the leftovers to be identical.

Example 1

Factor completely.

1. ad + ac - d - c 2. 3x2 - 4x - 6x + 8 3. xy + 3y + 4x + 12 4. 3x (x - 4) + 2(x - 4) 5. y2 (2x + 5) - (2x + 5)

Solution:

1. The two groups are ad + ac and -d - c . Factor out "a" from the first group. For the second group you will need to factor out a -1 to change the signs. Whenever you need to change the signs of an expression, you can factor out a negative 1. –> (a - 1)(d + c) 2. The two groups are 3x2 - 4x and -6x + 8. Factor out an "x" from the first group. For the second group, you will need to factor out a -2, not just a two, once again to make the signs correct. –>(x -2)(3x -4) 3. The two groups are xy + 3y and 4x + 12. Factor out "y" from the first group. For the second group you will need to factor out a 4. –> (y + 4)(x + 3) 4. The first few steps were already done for us. –> (x - 4)(3x +2) 5. The first few steps were already done for us. –> (2x + 5) (y2 - 1)... but this last one is tricky! First, when one of the groups has "nothing" in front , or only a + or -, that means that

there is a “1” there that becomes part of the factor. Second. . . my answer is “completely factored”. You might have noticed that y2 - 1 is a difference of squares so the answer is –> (2x + 5)(y+1)(y-1)

Now, let’s look at polynomials. Up until this point, we have only been able to factor “special case” polynomials. We really haven’t looked at factoring polynomials of the form “ax2 + bx + c”. Factoring by grouping is going to allow us to factor any polynomial, but I might have to do something first.

We are going to start with an example that doesn’t need to be factored by grouping. I can factor x2 + 5x + 6 by grouping, but, right now, I don’t have enough terms. So I have to break that middle term up into 2 different terms. Basically, I need to “un-combine like terms” to figure out what was distributed over each group. My options are x + 4x, 2x + 3x and only one of them will work. Which one? Well let’s try both to find out.

Option 1: x2+ x + 4x + 6 –> x(x +1) + 2(2x + 3) –> what is in the ( ) is not identical. So Not this one. Option 2: x2+ 2x + 3x + 6 –> x(x + 2) + 3 (x + 2) –> woo hoo! –> (x + 3)(x +2)

Let’s try: 2x2+ 13x + 15

If you hate grouping, you don’t have to use it. You may have to spend a lot of time guessing and checking though. If I were to use pure guess and check, I have the following options:

(x + 15)(2x + 1) (x +3)(2x +5) (x +5)(2x +3) (x+1)(2x + 15)

I need to find two linear binomials so that the first term is 2x2 (formed by 2x * 1x) and the last term is 15 (formed by either 1*15 or 3*5). I then need to find the combination that will give me a 13x. This one isn’t too bad because there aren’t very many options. It turns out that (x+5)(2x+3) is the combination that will work. You have to be aware of the sign and check your answer!

If you want something more streamline and systematic than just guess and check, you have to factor by grouping. I just have to figure out what to break 13x into to get two groups. By inspection it is obvious that I want to break it into 3x + 10x. By grouping 2x2 + 3x + 10x + 15 = x(2x + 3) + 5(2x +3) = (x + 5)(2x +3).

Now, of course, there is a structured way to find what to break the terms up into. It’s not always. . . “just look at it and figure it out”

The Procedure to factor a quadratic trinomial with grouping. (ax2 + bx + c)

1. Multiply a and c 2. Find the factors of the product “ac” that will add to b. 3. Split “bx” into two terms using the numbers found in step 2. 4. Group 5. Factor out the GCF from each group 6. Factor into binomials

Now, let’s see this with an example that would be horrible to use guess and check with: 20x2 + 7x - 6

1. 20 (-6) = -120 2. Option: 1, 120; 2, 60; 3, 40; 5, 24; 6, 20; 8, 15; 10, 12 –> since it is negative, that means one is positive and one is negative, but they have a difference of +7 so it has to be -8 and 15. 3. 20x2 - 8x + 15x - 6 –> (it doesn’t matter which way you split the terms) 4. (20x2 - 8x) + (15x - 6) 5. 4x (5x -2) + 3 (5x - 2) 6. (4x + 3)(5x -2)

Example 2

Factor each of the following by grouping

1. 6x2 -7x - 5 2. 10x2 + 17x + 3 3. 8x2 + 6x - 5 4. 6x2- 19X + 10

Solution:

1. (2x +1)(3x -5) 2. (5x+1)(2x+3) 3. (2x-1)(4x+5) 4. (3x-2)(2x-5)

15.7 Factoring Completely

Sometimes it seems that a polynomial can't be factored. Sometimes they can't and sometimes they can. One way to check to see if a polynomial is factorable is to check the Discriminant, if it is quadratic. If the Discriminant is a perfect square, then it is likely factorable.

Another thing to check for is a GCF. Sometimes polynomials, especially special cases, are disguised by a GCF. For example, 3x2 - 75. If you notice it has 2 terms which makes it a candidate for a difference of squares, but it isn't formed by squares. The terms have a common factor, factor it out and see what you are left with. Each term has a factor of 3, so 3 (x2- 25)... oh look, the leftovers are a difference of squares. So 3(x-5)(x+5).

When you are instructed to factor completely, you have to factor until you can't factor anymore. If you leave a common factor, you will get the problem wrong. Here are the basic steps you need to remember when asked to factor.

1. Factor out any GCFs 2. Factor the resulting polynomial. -Check for special cases! 3. Check to make sure each factor is factored.

Example 1

Factor Completely: 1. 7x3- 343x 2. 4x4 - 64 3. 2x2y - 8xy + 8y Solution:

1. 7x(x-7)(x+7) 2. 4(x2 + 4)(x-2)(x+2) 3. 2y (x - 2)(x-2)

15.8 The Big Solving Day

Making Factoring Useful

The whole point of factoring is to make solving higher order equations easier. So how do you use factoring to solve an equation?

The Zero Product Property

If aâĂćb = 0, then a = 0 or b = 0 Example

3x = 0, therefore x = 0

The Zero product property seems pretty lame and obvious. It says that if you have a product that is equal to zero, that means at least one of the factors had to be zero. So, if I have a standard form polynomial equation of the form polynomial = 0 and can factor it, one of those factors have to be zero. This makes factoring a useful way to solve equations. Let's see how this works.

Solve x2+ 5x + 6 = 0. It is factorable (x +2)(x+3) = 0. So either x + 2 = 0 or x + 3 = 0. Solve each of those linear equations formed by setting each factor to zero via the zero product property, and you get x = -2 or -3! We could have figured that out by graphing or the quadratic formula too.

How about x2- 25 = 0. It is factorable (x + 5)(x -5) = 0. So either x + 5 = 0 or x - 5 = 0. Solve each of those linear equations, and you get x = +5 or -5! We could have used the PoEs and opposite operations, but remember how people forget to add the +/-. Well, if you solve by factoring, you don't need to add that.

Finally, let's try to solve 6x2 -7x = 5. First, it is called the zero product property and not the five product property, so put it in standard from. 6x2- 7x - 5 = 0. It is factorable (2x +1)(3x -5). So either 2x + 1 = 0 or 3x - 5 = 0. You get x = -1/2 or 5/3. Alternatively, you could have used the quadratic formula.

Note: If something isn't factorable, that does not mean it has no solution set. There are quadratics that have solutions (irrational) that aren't factorable over integers. Now, if a quadratic has no solution, then it is definitely not factorable. You can actually use the discriminant to determine if a quadratic is factorable as stated in the previous section.

Another Note: If you have a higher order equation, like a cubit, the only way you have to solve it, that isn't graphing, is factoring.

Example 1 Problem: 1. (3x + 4)(x - 5) = 0 2. 2x(6x - 9) = 0 3. x2 - 4x = -4 4. x2 + 7x + 12 = 0 5. 8x2 + 6x = 5 6. 7x3 = 343x Solution: 1. S = -4/3 or 5 2. S = 0 or 3/2 3. S = 2 4. S=-3, -4 5. S = 1/2 , -5/4 6. S = 0,7,-7 Revisiting Radical Equations

If you remember before spring break we did a little lesson on solving equations where the unknown was under a radical. I also said that there was one type we would have to wait to solve. Well, now is the time to solve that final type of radical equation. These equations have the unknown both under a radical and outside the radical. These types generate a quadratic equation when you square both sides, which is why we had to wait until now.

Type 4: (the reason we are doing this now instead of with the radical unit)

Example 2

Solve:

Solution: Just like the first example, I just squared both sides of the equation, creating a quadratic after the first step.

Work to find candidates for solution Check of candidates to find solution set

Check 6

Check -1

Since the original equation wanted the positive root, it appears that only 6 works and -1 is an extraneous solution. S = 6. -1 ext

One simple change in the equation creates the opposite solution set. has the solution set x= -1, 6 ext.

So be careful and check the candidates to make sure they belong in the solution set. When you solve one of the 4th type and get 2 candidates, both might work, 1 might work, or none of them will work, depending on the signs and how things play out in the solution process.

Note for EOC... Sometimes you will see that the EOC will ask you to find"roots" or "zeroes" of a function. That means that you are trying to find x-intercepts. By now, you should have noticed that finding x-intercepts means to set the y to zero and solve the corresponding equation.

15.9 Completing the Square

Solving Quadratic Equations

During this lesson we are going to learn a final method for solving a quadratic equation, a method that gives us the Quadratic Formula and is the method we use to convert standard form to vertex form.

Reminders about ways to Solve Quadratics:

1. Graphing (finding the x-intercepts, not always exact) –> doesn't work for irrational solutions 2. Factoring/Zero-Product Property (not everything is factorable) –> only works if you can factor 3. Quadratic Formula –> always works, but simplifying can be annoying 4. Completing the Square –> always works

Completing the square

First, we are going to do a couple of example problems to solve equations that look a specific way, kind of like vertex form. Why? Because completing the square turns all standard form quadratics equations that look like these.

Example 1 (1) (2) Solve each of the following with a radical. (1) (2)

Solution: Remember, when you chose to perform a root in the process of solving, you have to add Âś.

Now, if you can solve all of those, then, you are ready to learn how to complete the square. The only thing you have to do is convert the quadratic to the appropriate form, a perfect square trinomial on one side a number on the other side. The first step is to convert into . It seems like voodoo, but it is based on really knowing the pattern for a PST. First, you get the coefficient of the quadratic term to be 1 by division. Then you remove the term that is preventing the quadratic from being a PST, the constant term. You add or subtract it off, then figure out what you have to add to make a PST, based on the linear term. You know that in order for the quadratic to be a PST, you have to halve the linear term and square it for the constant term. There is a very set procedure to do this.

Steps to follow to complete the square.

1. Move the constant using SPoE to the side opposite the variables. This should leave the linear and quadratic terms. 2. Divide to make sure the coefficient of the quadratic term is 1. You should have something that looks like x2 + #x = # 3. Add the square of ? the coefficient of the linear term to both sides to make sure the side with the variables is a perfect square. 4. Factor the left side.

Let's work an example to show this process.

Example 2

Solve:

1. Move the constant term 2. no need to divide, quadratic coefficient =1 3 & 4. Complete the Square, divide linear coefficient by 2, square it, add it to both sides. Then factor the polynomial side. 5. Solve

Solution:

The beauty of completing the square is that it combines factoring and opposite operations to solve and the solution is pretty much simplified by the time you find it. You just have to be really comfortable with figuring out what to add to a quadratic to turn it into a PST.

Here are some more examples.

Example 3

Solve:

(1) (2) (3)

Solution:

1. 2. 3.

15.10 Deriving the Quadratic Formula

Since completing the square is such a rigid procedure, it is possible to do it for the general form quadratic and derive a formula for solving quadratics. See if you can follow the steps below, which is the derivation of the quadratic formula. As you follow the steps, you should see parts of the formula starting to emerge.

I'm going to follow the same steps for completing the square.

First, subtract the constant term.

Then divide by a to make the coefficient of the quadratic term 1.

Add the square of half of the coefficient of the linear term.

There is some work simplifying the right side, common denominators and rearrangement into a familiar form.

Then factor the polynomial side. This gives you a perfect square, so square root both sides. Don't forget the +/-.

Simplify the radical remaining, remembering that you have only multiplication and division properties for radicals, and none for addition/subtraction.

Finally, get that x by itself... and there you have the quadratic formula.

Converting from Standard to Vertex

Once you've figured out how to complete the square, you can try to convert a standard from quadratic function into a vertex form. There is a reason why it is sometimes called "completed square" form. You have to complete the square. The y on the other side does complicate things slightly. You can't divide both sides by the coefficient of the quadratic term. You have to factor it out and be very careful about what you add to both sides to complete the square.

Example 1

Convert into vertex form.

Solution: Since the coefficient of the quadratic is 1, you don't have to worry about the division.

Example 2

Convert into vertex form.

Solution: Since the coefficient of the quadratic is 3, you need to factor the 3 out. When you add to complete the PST, you don't just add 1, you add 3 to both sides.

15.11 You can divide polynomials?

Polynomial Division

Polynomials represent real numbers, so if you can divide real numbers, you can divide polynomials. Polynomial division is just long division. You are going to have to remember the algorithm for long division and apply it to polynomials. If you think about what we've done with polynomials, we've avoided division. We've factored, but the actual act of division and getting remainders, that is something we haven't dealt with. You'll see what this is used for later, when you have to graph rational functions in algebra 2.

As I said, it is just like long division. , which means if I ask you to divide I want to see as the answer. Remember, this works just like old fashioned long division. So if I ask for the quotient, I just want the quotient. If I just want the remainder, I don't need to write it as a fraction over the divisor. If there is no remainder, then that means that the Divisor is a factor of the Dividend.

Example 1

Divide by

Solution: First, check and make sure the polynomials are correctly arranged in decreasing degree and completely simplified. If a specific term is missing, then add it on with a zero for a coefficient. This is crucial, but unnecessary for our example. The degrees are like place values. So Set it up like –> . To actually divide you look at the first term in the divisor and think "What do I have to multiply that term by to get the first term in the dividend?" In this case, you have to multiply x by x2 to get x3. . Then, just follow the algorithm of long division. Multiply the entire divisor by x2, line up these terms with their counterparts under the dividend and subtract. Bring down terms as necessary and continue until what you are left with is of lower degree than the divisor. If it is not zero, then that is your remainder.

Therefore the Quotient is x2 - 3x - 2 and the remainder is -6. Written correctly, .

To check your solution, multiply the quotient and the divisor, then add the remainder. That should give you the dividend.

A good way to practice is to take a couple of quadratic expressions that you know are factorable. Factor them. Then divide the polynomial by on of the factors and make sure the other factor is your answer. In algebra 1, you will have to factor, at most, a quadratic by a linear.

Example 2

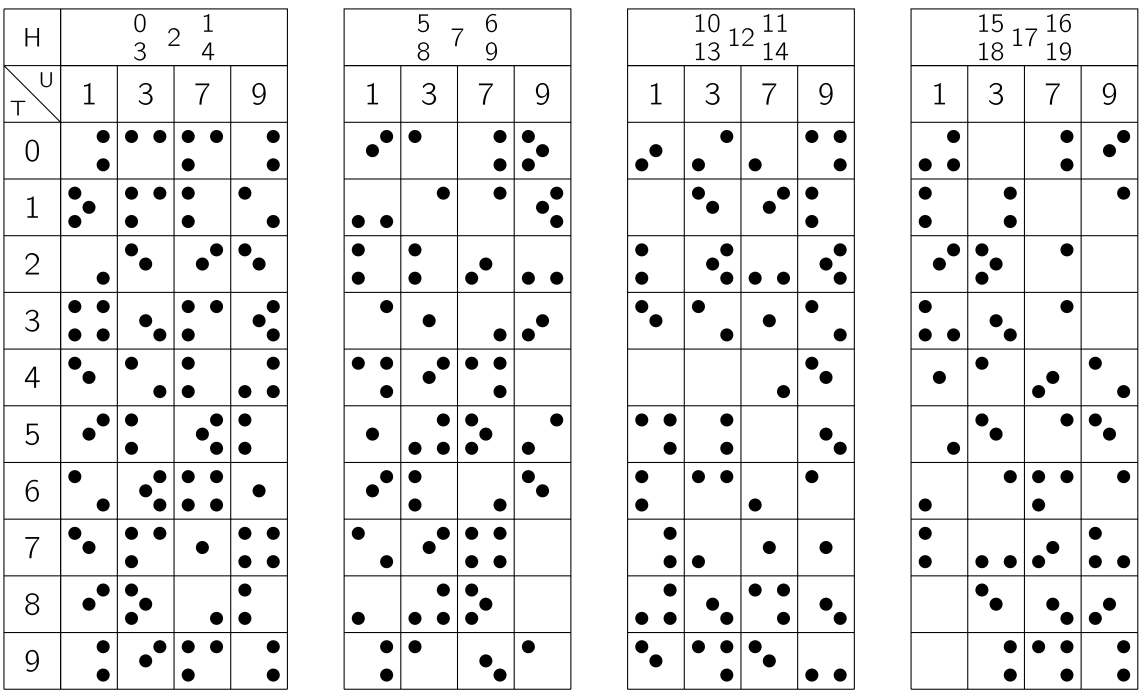
Divide by

Solution: I know that x2 - 4 is factorable and the answer should be x + 2. If I can divide, I should get x + 2. Don't forget, you must have all of the degrees. Since the linear term is missing, I have to add it in as a place-value holder.

Appendix A

Table of Prime Numbers to 2000

In additon to the numbers 2 and 5, the tables below show all of the prime numbers up to 2000.



How to Use the Table

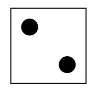
Suppose we want to know whether the number 1439 is prime. Here's how we use the table to determine this:

1. The box on top of each table shows a group of hundreds (that's what the H stands for at the left side). We should think of the number 1439 as having "14" in the hundreds place. The table heading of the third table includes the number 14, so 1439 will appear in this section of the table.

2. The rows of the table indicate the digit in the tens place (that's what the T stands for at the left side). Since 1439 has a 3 in the tens place, we'll use that row of the table.

3. The columns in each table indicate the units place (that's what the U stands for at the left side). We need the column with heading 9, since that's the number in the units place of 1439.

4. We have narrowed our selection down to a single cell in the table. That cell looks like this:

   The dots indicate which of the numbers associated with this cell are prime, using the same layout as in the topmost box. This pattern indicates that 1439 is prime. The box also tells us that 1039 is prime, and that 1139, 1239, and 1339 are all composite.

Notice and Wonder

Notice that the columns include only the headings 1, 3, 7, and 9. Why do you suppose that is?

Notice that none of the boxes have all five dots. Why do you suppose that is? If we continue the table, do you suppose we will ever find a box with all five dots? Why or why not?

Notice that if a box contains four dots, it's always the four corners. Why do you suppose that is? If we continue the table, do you suppose we will ever find a box with four dots in a different arrangement? Why or why not?

Back Matter

Glossary, bibliography, and index removed during editing process.