The Algebranomicon

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Chapter 1

Numbers

> God made the integers, all the rest is the work of man.
>
> — Leopold Kronecker, German mathematician

The number system that we use every day, both in mathematics class and in our regular lives, developed over many generations. Men and women from all over the world, both famous and anonymous, have helped to make mathematics what it is today. Yet, people argue over whether mathematical ideas are "invented" or "discovered".

For example: **imaginary numbers** (which we will discuss briefly in algebra 1, and study in more detail in algebra 2), first appeared on the mathematical scene in the 1500s. Italian mathematician Gerolamo Cardano first wrote about these new numbers in his work with certain types of problems that otherwise would have been impossible to solve. Since he was the first person to describe this new kind of number, we might say that Cardano *invented* imaginary numbers.

But, we now know that imaginary numbers have practical applications in, for example, electrical engineering. The laws of electromagnetism haven't changed since the 1500s. (Well, our understanding of the laws has changed, but the physics has not.) So, maybe imaginary numbers have been there all along, lurking within the fabric of the universe. In this case, Cardano *discovered* imaginary numbers.

It's not clear which is the more accurate description. No matter what side of the debate you find more convincing, there is a certain beautiful interconnectedness to our system of numbers and mathematical laws. To illustrate this, we'd like to tell a story.

1.1 The Story of Numbers

Once upon a time, there was a simple farmer. Knut Krumbli lived in rural Sweden, raising goats and making goat cheese. He and his family led an uncomplicated life and they didn't have much need for mathematics. In fact, they really only needed numbers to count their goats: 1 goat, 2 goats, 3 goats, 4 goats...

But one day, after a terrible storm, Knut went to the field to count the goats and discovered, much to his dismay, that there were no goats to count. He hadn't needed a number to describe this situation before, but now people were asking him hard questions, like "How many of your goats made it through that crazy storm?" (But, you know, in Swedish.)

Knut and his family couldn't very well survive without any goats, so he went to his neighbor for help. The neighbor agreed to loan Knut some goats to restart his herd but, of course, Knut would have to repay his goat-debt later. The village hadn't needed to do much accounting before the storm, but now they needed a system of numbers that could keep track of debts and credits.

Over time, Knut's family got back on their feet and thrived. They paid back their debts and eventually grew to raise more goats (and to make more goat cheese) than they could eat. They began to trade with their neighbors for other foods or services. Of course, everything had relative value: three wheels of goat cheese were worth two bales of hay. So, the village developed a system of numbers for describing exchange rates of this kind.

As the village grew, Knut's family farm led the development of a booming goat cheese industry. They invested their profits into bank accounts that paid interest. In certain situations, everyone was surprised to discover, interest-bearing accounts led to a new system of numbers that no one had seen before.

Eventually, some of Knut's ancestors emigrated to America and, years later, a pair of twins — Knut's great-great-great-grandchildren — would grow up to change the world. But let's not get too far ahead of ourselves. More about the Krumbli twins later...

1.1.1 Dissecting the Story

Mathematically speaking, it's natural to begin our discussion of numbers exactly where Knut began: counting things. The numbers we use to count are called the natural numbers (also known as the counting numbers, for obvious reasons).

Natural Number

A **natural number** is a member of the list of numbers that starts $1, 2, 3, 4, \ldots$ and continues forever. The set of all natural numbers is denoted using the symbol $\mathbb{N}$, so we can write $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$.

We use the {curly braces} here to indicate that we're collecting a group of numbers together as a **set**. A set written in this way is written in **set notation**.

The natural numbers have some interesting properties. If we add two natural numbers, their sum will always be a natural number. The same goes for multiplication: the product of two natural numbers is again a natural number.

Mathematically speaking, we call this **closure**. We say that the natural numbers are closed under the operation of addition. Also, the natural numbers are closed under the operation of multiplication.

Notice that we didn't include 0 among the natural numbers. Zero is a bit tricky because it seems like a counting number. For example, "zero" is (probably) the answer to the counting question, "How many live elephants are there in the room with you right now?" But if there are no elephants to count, can we really count them? That's a philosophical question.[1]

Practically speaking, we usually exclude 0 from the set of natural numbers. We will always be very clear when we come to situation where we want to consider 0 to be a natural number.

The natural numbers are closed under the operations of addition and multiplication, but they are *not* closed under the operation of subtraction. Sometimes, the difference of two natural numbers is a natural number: for example $8 - 6 = 2$, no problem. But some subtraction sentences don't work: for example $10 - 13 = {}^-3$, and $^-3$ is not a natural number.

Integer

An **integer** is a natural number, or the opposite of a natural number, or zero. The set of all integers is denoted $\mathbb{Z}$, so we sometimes write $\mathbb{Z} = \{\ldots, {}^-3, {}^-2, {}^-1, 0, 1, 2, 3, \ldots\}$. We could also write $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$.

The symbol $\mathbb{Z}$ comes from *Zahl*, the German word for number.

Note that every natural number is an integer. We say that the set of natural numbers is a **subset** of the set of integers.[2]

The integers are closed under the operations of addition and multiplication. Plus — bonus! — the integers are closed under the operation of subtraction. Whenever we subtract one integer from another, we always get another integer as the result.

But (as you may have anticipated) we have a problem with division. In certain cases, the quotient of two integers is itself an integer: $15 \div {}^-3 = {}^-5$, no problem. But other times we get a quotient that is not an integer: $^-3 \div 15 = {}^-0.2$ and $^-0.2$ is not an integer. In other words, the integers are not closed under the operation of division.

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[1] Philosophy and mathematics have historically gone hand-in-hand. Many important discoveries (inventions?) in mathematics are attributed to people who are considered both "philosopher" and "mathematician". For example: French mathematician René Descartes, for whom the Cartesian coordinate system is named, is also the philosopher who said "I think therefore I am".

[2] For those who are into mathematical symbols and notation, the sentence "The natural numbers are a subset of the integers" is denoted $\mathbb{N} \subseteq \mathbb{Z}$.

Rational Number

A **rational number** is any number that can be written as the ratio of two integers $\frac{a}{b}$, where b is not zero. This set includes all of your classic fractions, as well as all terminating decimals and all repeating decimals.

The set of all rational numbers is denoted by the symbol $\mathbb{Q}$, which comes from the word *quotient*.

Fractions have a lousy reputation among math students[3], but the rational numbers are great because they are closed under all four of the basic operations. When we add, subtract, multiply, or divide any two rational numbers, the result will always be another rational number. What more could we ask for? In some sense, $\mathbb{Q}$ is a complete number system, and the world was content with the rational numbers for a long time.

But, other numbers exist. Note that the rational numbers include all of the terminating decimals (like 0.5 and 1.678), and all of the repeating decimals (like $0.\overline{3}$ and $-12.34\overline{56}$). Now, consider the number

$$0.10\,110\,1110\,11110\,111110\ldots$$

This number does not terminate, but it does not repeat either.[4] So, this number is *not* a rational number.

Irrational Number

An **irrational number** is a number that cannot be expressed as the ratio of two integers. In decimal form, an irrational number never terminates and never repeats.

You have likely encountered irrational numbers before. A famous example is the number π (pi), which shows up when we study circles. We usually approximate π to be about 3.14, but in fact, the decimal representation of π goes on forever without stopping or repeating:

$$\pi \approx 3.1415926535\ 8979323846\ 2643383279\ 502884197\ 6939937510\ 5820974944\ 5923078164\ldots$$

When we group together all of the rational numbers and all of the irrational numbers, we will have accounted for all possible decimal representations. This combined set of numbers is going to be of key importance to us in algebra 1.

[3] Fractions, the F-word of mathematics?

[4] This number does have a pattern, but that is not the same as “repeating” in the sense of “repeating decimal”.

Real Number

A **real number** is any rational or irrational number. The set of all real numbers is denoted $\mathbb{R}$ which, naturally enough, comes from the word *real*.

Like the set $\mathbb{Q}$, the set $\mathbb{R}$ is closed under the four fundamental operations. $\mathbb{Q}$ and $\mathbb{R}$ is the number systems we will work with in algebra 1. Other types and sets of numbers exist (like $\mathbb{C}$, the set of so-called "complex numbers"), but we won't get into them very much until algebra 2 and beyond.

1.2 Integers

You have probably been working with positive and negative numbers for a while now, so this section will review the most important terms and algorithms for signed numbers. To get the ball rolling, think about this:

Startup Exploration: Integer Comparisons

In each of the expressions below, x and y are natural numbers and $x < y$ (x is less than y). Will the result be greater than 0, less than 0, equal to 0, or is there not enough information to tell? Why?

a. $x + (-y)$ b. $x - (-y)$ c. $x \cdot (-y)$ d. $x \div (-y)$

1.2.1 Language of the signed numbers

As we saw in section 1.1, the integers include the natural numbers, their opposites, and zero. Numbers now include two pieces of information: they have a *size* (or *magnitude*) and a *direction*, either positive or negative.[5] Sometimes, we care only about the magnitude of a number, in which case we refer to a number's *absolute value*.

Absolute Value

The **absolute value** of a number x is its distance away from zero on the number line. To express this in mathematical symbols, we write $|x|$ to mean "the absolute value of x".

For those who like mnemonic devices and memory aids, it may be helpful to think of the absolute value bars as a little numerical shower stall or car wash. A number goes in and all its negativity gets washed away.

[5] Later in mathematics and the physical sciences, we'll encounter mathematical objects with both magnitude and direction again. They're called vectors.

Example 1.1

Compute each of the following.

1. $|4|$ — The absolute value of a positive number is just the original number, so $|4| = 4$.

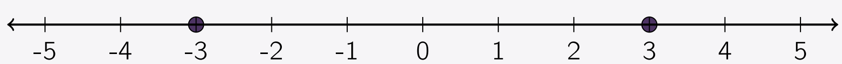
2. $|{-8}|$ — Absolute value ignores the sign and tells us how far a number is from zero. -8 is eight units away from zero, so $|{-8}| = 8$.

3. $-|{-6}|$ — The absolute value bars only apply to what's inside, but then this negative sign *outside* the absolute value bars will make the final answer negative again! So, $-|{-6}| = -6$.

Every nonzero number has a counterpart that is the same distance away from zero, but on the opposite side of the number line. This is a simple idea, but one that is important enough for us to give it a name.

Opposite

The **opposite** of a number x is the number $-x$. In other words, the opposite of a number is the number with the *same absolute value*, but the *opposite sign*. For example, 3 and -3 are opposites. The number 0 is its own opposite.



The sum of opposites is always 0. For this reason, we sometimes use the term **additive inverse** to describe the opposite of a number. (More on inverses in chapter 5.)

1.2.2 Adding Signed Numbers

Like matter and antimatter, combining positive and negative numbers leads to annihilation. (Dramatic, no?)

For example when we bring together $+8$ and -6, we can picture 8 units of "matter" and 6 units of "antimatter". Particles and antiparticles annihilate one another, both disappearing in the process. Since we have more matter than antimatter in this case, all of the antimatter is consumed, leaving behind 2 units of matter.

We might visualize annihilation with a drawing like in fig. 1.1 where black circles are units of matter, red circles are units of antimatter. Or, we could write a number sentence like $8 + -6 = 2$. Of course, no annihilations occur when we scrape together a big pile of matter (or a big pile of antimatter, for that matter). It's only when they mix that anything interesting happens.

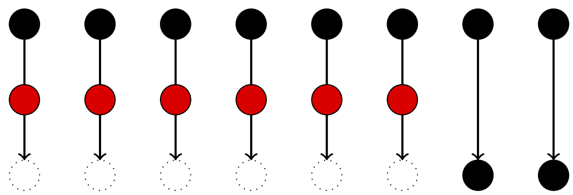


Figure 1.1: Eight units of matter versus six units of antimatter!

Adding Signed Numbers

If the two numbers being added have the same sign, then we add the absolute values of the numbers, and use the sign that they share.

Otherwise, if the two numbers being added have different signs, then we find the difference between their absolute values, and use the sign of the number with larger absolute value.

Example 1.2

Compute each of the following.

1. $-8 + -12$ Both numbers are negative. That's a big pile o' antimatter: $-8 + -12 = -20$

2. $-8 + 12$ The numbers have different signs, so prepare for annihilation! $|12|$ is larger than $|-8|$, so we will have matter left over. Therefore, $-8 + 12 = 4$

1.2.3 Subtracting Signed Numbers

Addition and subtraction are called *opposite* (or *inverse*) *operations* because they "undo" one another. The act of adding 5 of matter can be "undone" by subtracting 5 units of matter. But on the other hand, the act of adding 5 units of matter can also be undone by *adding 5 units of antimatter*.

Subtracting Signed Numbers

Subtracting a number is the same as *adding the opposite of the number*.

When faced with a subtraction problem, change it to an "addition of the opposite" problem and then follow the rules for adding signed numbers.

The benefit of this approach is that we can avoid having to learn a whole new set of rules for subtracting signed numbers. All we need are the rules for addition, plus one new rule about how to change subtraction problems into addition problems!

Example 1.3

Compute each of the following:

1. $8 - 12$ "Subtracting 12" is the same as "adding negative 12", so $8 - 12 = 8 + {}^{-}12$. Then, we can apply the rules for adding signed numbers: $8 + {}^{-}12 \;=\; {}^{-}4$

2. ${}^{-}3 - 14$ This is the same as ${}^{-}3 + {}^{-}14$, and we follow the rules for adding numbers with the same sign: ${}^{-}3 + {}^{-}14 = {}^{-}17$

3. $4 - {}^{-}9$ This is the same as $4 + 9$, which is an easy addition: $4 + 9 = 13$

4. ${}^{-}6 - {}^{-}5$ This is the same as ${}^{-}6 + 5 = {}^{-}1$

When dealing with addition and subtraction of signed numbers in a problem, a good habit is to simplify the signs by rewriting subtraction as addition-of-the-opposite, before doing any computations.

Explaining the Startup Exploration

In the startup exploration for this section, we have two natural numbers x and y where $x < y$. Natural numbers are all positive, so this means that $|\,x\,|$ is less than $|\,y\,|$.

Part (a) asks us to consider $x + (-y)$. Since opposite numbers have the same absolute value, $|\,y\,|$ is the same as $|\,-y\,|$. Therefore when we add, the number with larger absolute value is negative and the sum will take on the negative sign. So, when x and y are natural numbers and $x < y$, we know that $x + (-y)$ is *always negative* so the result is always less than 0.

Part (b) asks us to consider $x - (-y)$. We can change this subtraction expression into addition-of-the-opposite: $x - (-y) = x + y$. Since both x and y are natural numbers, their sum is a natural number.

In other words, when x and y are natural numbers, $x - (^-y)$ is always positive, in other words always greater than 0.

1.2.4 Multiplying and Dividing Signed Numbers

Like addition and subtraction, the operations of multiplication and division are inverse operations. We'll discuss multiplication below, though the rules about the signs of products also apply to the signs of quotients.

Recall from your elementary school days that one way to think about whole number multiplication is as *repeated addition*. We interpret $a \cdot b$ as "a groups with b items in each group". So, $3 \cdot 5$ means "3 groups of 5", which we can write as an addition sentence: $3 \cdot 5 = 5 + 5 + 5$.

Using this interpretation, we can easily explain the product of a positive number and a negative number. The expression $4 \cdot {}^-8$ means "four groups of negative eight": $4 \cdot {}^-8 = {}^-8 + {}^-8 + {}^-8 + {}^-8 = {}^-32$. No problem!

But what about ${}^-5 \cdot 6$? What does it mean to have "negative five groups of six"? Or even worse, what about ${}^-3 \cdot {}^-7$, "negative three groups of negative seven"? Rather than try to twist the metaphor to fit these new situations, let's just admit that multiplication can not *always* be represented by repeated addition.[6]

For the moment, we'll simply review the rules for multiplying (and dividing) signed numbers. We can explain why these rules work using the so-called "field axioms for the real numbers". More on that later.

Multiplying (and Dividing) Signed Numbers

The absolute value of the product of two numbers is the product of their absolute values: $|\, a \cdot b \,| = |\, a \,| \cdot |\, b \,|$. If the two numbers have the same sign, then the product is positive. If the two numbers have opposite signs, then the product is negative.

There are several clever ways to remember this rule. Some people remember that every pair of negatives in a product cancel one another. Other people use a triangle with one positive sign and two negative signs drawn on the vertices. We present another way of looking at it in the next section. Choose whichever mnemonic[7] method is most helpful to you!

[6] The "multiplication as repeated addition" analogy breaks down when we have a negative number of groups, but also for rational and irrational numbers. What's the repeated addition problem for $\frac{2}{3} \cdot \frac{1}{2}$, or for $\sqrt{2} \cdot \sqrt{3}$?

[7] mnemonic ($na \cdot MON \cdot ic$, the first "m" is silent): A learning aid that helps to remember or retain information.

Karmic Multiplication

Karma, an underlying concept of many Eastern religions, is a belief that a person's actions and intentions shape their future. Performing good deeds will contribute to one's "good karma" and will lead to future happiness. Bad deeds contribute to one's "bad karma" and will lead to future suffering.

So, karma suggests that good things happen to good people, and that bad things happen to bad people. Of course we know that the universe does not always operate in accordance with karma.

Karmic Multiplication

When good things happen to good people, that's good!

When bad things happen to good people, that's bad!

When good things happen to bad people, that's bad!

When bad things happen to bad people, that's good!

For example: Mahatma Gandhi used nonviolent means to inspire civil rights movements around the world. Gandhi was a good person. On the other hand, Adolf Hitler was chancellor of Nazi Germany during World War II and orchestrated appalling crimes against humanity. Hitler was a bad person.[8]

In terms of life events, winning the lottery is a good thing. Getting hit by a truck is a bad thing.

If Gandhi had won the lottery, that would have been in accordance with all his good karma. That's good! If Gandhi had been hit by a truck, that would have been in opposition to all of his good karma. That's bad!

If Hitler had won the lottery, that would have been in opposition to his evil karma. That's bad! If Hitler had been hit by a truck, it would have served him right! That's good! Go karma!

Of course, in this metaphor good things and good people represent positive numbers. Bad things and bad people represent negative numbers. When karma is operating as it should, we get a positive result. When the laws of karma are broken, we get a negative result.

[8] understatement ($UN \cdot der \cdot state \cdot ment$): The act of representing something in a weak or restrained way, to a lesser degree than is borne out by the facts.

Example 1.4

Compute each of the following:

1. $8 \cdot {}^-12$ — We multiply the absolute values and, since the two factors have different signs, we know the answer is negative: $8 \cdot {}^-12 = {}^-96$

2. ${}^-72 \div {}^-3$ — We divide absolute values and, since the two factors have the same sign (both negative in this case, so that's "Hitler gets his by a truck"), the answer is positive: ${}^-72 \div {}^-3 = 24$

Note: When multiplying (or dividing) more than two numbers, we can approach things in two different ways. We might simplify the product two factors at a time, and keep track of the sign as we go. Or, we could treat the signs as a separate problem: first multiply all of the absolute values, then go back and count up the negative signs.

Example 1.5

Multiply: $(2)({}^-2)(1)({}^-2)({}^-2)(1)(1)({}^-2)({}^-1)(2)$

Solution: If we count up the negative signs, we find there are five. Pairs of negatives will have a positive product, so we'll have two pairs of negatives plus one left over. Our final answer, then, will be negative. All that remains is to multiply the 2s (of which there are six):

$$(2)({}^-2)(1)({}^-2)({}^-2)(1)(1)({}^-2)({}^-1)(2) = {}^-64$$

Explaining the Startup Exploration

In the startup exploration for this section, x and y are natural numbers and $x < y$.

Since x are y are natural numbers they are both positive, and then $-y$ is negative. Part (c) asks us to consider $x \cdot (-y)$. We have a positive number times a negative number, so the product is *always negative*, always less than 0.

Part (d) asks us to consider $x \div (-y)$. Again, we have a positive number and a negative number, so the quotient is *always* less than 0.

1.3 Rational Numbers

As with integers, you've probably been working with fractions and decimals for a while now. In this section, we review the key terms and algorithms for working with rational numbers. As we get going, think about this:

Startup Exploration: Rational Comparisons

In each of the expressions below, a and b are rational numbers where $0 < a < b < 1$. Will the result be greater than 1, less than 1, equal to 1, or is there not enough information to tell? Why?

a. $a + b$ b. $a - b$ c. $a \cdot b$ d. $a \div b$

1.3.1 The Language of Rational Numbers

Sometimes in life we discover, much to our surprise, that some ridiculous and insignificant thing as been given a name.[9] We find ourselves wondering, "Who decided to give *that* a name? Why bother?" Prepare for one of those moments:

Vinculum

A **vinculum** (plural: vincula) is a horizontal bar used in mathematics to show grouping. For example, the fraction bar in the middle of $\frac{5}{2}$ is a vinculum.

Vincula are used in other contexts as well. For example, we use a vinculum to represent a repeating decimal such as $0.\overline{3}$.

With that definition in mind, we can continue with two of the most daunting words in elementary mathematics. You're definitely not alone if you have ever been confused about these.

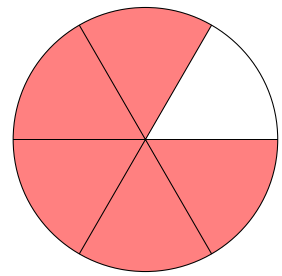
[9] For instance, did you know that the little plastic sheath at the end of your shoelaces is called an "aglet"? Now you know.

Numerator and Denominator

In a fraction, the number above the vinculum is called the **numerator** of the fraction, and the number below the vinculum is called the **denominator** of the fraction.

For example in the number $\frac{5}{6}$, the numerator is 5 and the denominator is 6.

If we ask "*how many* sixths?", the numerator tells us "*five* sixths". The word numerator is related to the word "number", and the numerator counts the pieces.



If we ask "five *whats*?", the denominator tells us "five *sixths*". The word denominator is related to the word "nominate" (as in "to nominate someone for president") which means "to name". The denominator *names* the fraction.

1.3.2 Multiplying Rational Numbers

Don't worry, your version of the 𝔄𝔩𝔤𝔢𝔟𝔯𝔞𝔫𝔬𝔪𝔦𝔠𝔬𝔫 isn't missing any sections. Most textbooks would discuss adding and subtracting rational numbers, but we're going to start by studying the most helpful of the rational number operations: multiplication.

Suppose that at the cheese market, Knut Krumbli is selling chunks from a 10-bound block of cave-aged goat cheese.[10] Half of the original block of cheese is left, and a local weaver asks to buy three fourths of it. The original block had a value of 800 Swedish kronor. How much should Knut charge the weaver?

Let's draw a picture (fig. 1.2). In the images below, the square represents the original block of cheese. We divide the square in half vertically, and the region shaded yellow represents how much of the cheese remains. We can then divide the cheese into fourths horizontally and shade in three fourths (the amount that the weaver wants to buy) in blue.

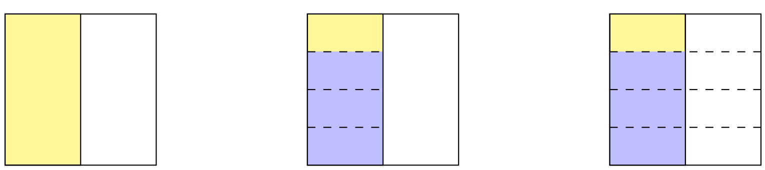


Figure 1.2: Selling three fourths of half of a block of cheese.

What fraction of the whole block does this blue region represent? If we extend the lines, we can see that we've divided up the whole block of cheese into 8 equally-sized pieces. So the weaver is buying $\frac{3}{8}$ of the whole block and Knut should charge 300 kronor.

[10] When cheese is aged in the cool, humid air of underground caves, it can develop a denser texture and a more complex flavor, since small salt crystals form throughout its interior.

In the end, we solved a fraction multiplication problem: "How much is three fourths of one half of a whole block of cheese?"

$$\frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$$

Multiplying Rational Numbers

To multiply rational numbers, we multiply numerators to find the numerator of the product. We multiply denominators to find the denominator of the product.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Explaining the Startup Exploration

In the startup exploration for this section, $0 < a < b < 1$, and part (c) asks us to consider $a \cdot b$. Let's think about this from a block-of-cheese perspective. We start with less than a whole block of cheese (since b is less than 1) and we only want to buy a fraction of what's there (since a is less than 1).

So, we are certainly buying less than a whole block of cheese! Since both a and b are less than 1, we know that their product must be less than one.

Fancy Versions of One

Recall that any number times 1 is itself. This simple fact, along with the fraction multiplication procedure, gives us an extremely powerful tool that we'll use in various different ways throughout algebra. The key idea is that multiplying something by 1 doesn't change its value, even if we use a "fancy version of 1". Consider, for example:

$$\begin{aligned} \frac{5}{8} &= \frac{5}{8} \cdot 1 && \text{Multiplication by 1 doesn't change the number.} \\ &= \frac{5}{8} \cdot \frac{7}{7} && \text{We can rewrite 1 however we like, here } 1 = \frac{7}{7}. \\ &= \frac{35}{56} && \text{Multiply fractions.} \end{aligned}$$

In the end, we have two equivalent fractions $\frac{5}{8} = \frac{35}{56}$. The representation has changed, but the value is the same! The "fancy one" we chose here is $\frac{7}{7}$, but any other version of 1 would work the same way: $\frac{-140}{-140}$, $\frac{2\pi}{2\pi}$, $\frac{\sqrt{3}}{\sqrt{3}}$... The possibilities are endless.

The first application of the "fancy one" has to do with one of the themes we encounter throughout algebra: the idea of finding a "completely simplified" solution to a problem. Fractions introduce us to the first criteria for something being simplified.

Simplified Rational Numbers #1

Fractions should be simplified to **lowest terms**, meaning that the numerator and denominator of the fraction are **relatively prime** integers.

Two integers are said to be relatively prime (or coprime) if they have no common factors other than 1. So, the fraction $\frac{21}{34}$ is in lowest terms, since the factors of 21 are $\{1,3,7,21\}$ and the factors of 34 are $\{1,2,17,34\}$. They have no factors in common, other than 1.

On the other hand, $\frac{18}{84}$ is not in lowest terms. Both 18 and 84 are even, and so both the numerator and denominator are divisible by at least 2. To write this fraction in lowest terms, we "undo" fraction multiplication and search for some fancy ones that we can eliminate:

$$\frac{18}{84}=\frac{2\cdot 3\cdot 3}{2\cdot 2\cdot 3\cdot 7}=\frac{2}{2}\cdot\frac{3}{3}\cdot\frac{3}{2\cdot 7}=1\cdot 1\cdot\frac{3}{14}=\frac{3}{14}$$

Once we know how and why this works, we can take a shortcut and "cancel" common factors from the numerator and denominator:

$$\frac{18}{84}=\frac{2\cdot 3\cdot 3}{2\cdot 2\cdot 3\cdot 7}=\frac{\cancel{2}\cdot\cancel{3}\cdot 3}{\cancel{2}\cdot 2\cdot\cancel{3}\cdot 7}=\frac{3}{2\cdot 7}$$

Simplify Before You Multiply

We can also use the "fancy one" to save ourselves some work! What if we have to multiply:

$$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}$$

Let's write the product out "the long way" before we actually do any computations. The numerator of the product will be the product of the numerators, and likewise for the denominator:

$$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}=\frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{2\cdot 3\cdot 4\cdot 5\cdot 6}$$

Then, a common factor in the numerator and denominator is like multiplication by 1, so we can make some simplifications:

$$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}=\frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{2\cdot 3\cdot 4\cdot 5\cdot 6}=\frac{1\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{5}}{\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{5}\cdot 6}=\frac{1}{6}$$

The alternative would have been to multiply all of those numbers together by hand and then simplify the fraction to lowest terms[11], which is a bunch more work. It pays to be clever!

[11] That would have been $\frac{120}{720}$.

Example 1.6

Multiply: (a) $\frac{3}{4} \cdot \frac{7}{16}$ (b) $\frac{15}{8} \cdot \frac{4}{7} \cdot \frac{14}{3} \cdot \frac{1}{5}$

Problem (a) is nothing special. No common factors are shared between numerators and denominators, so we simply multiply as usual.

$$\frac{3}{4} \cdot \frac{7}{16} = \frac{3 \cdot 7}{4 \cdot 16} = \frac{21}{64}$$

Since there were no common factors to cancel before multiplying, we know the product is already in lowest terms.

To tackle (b), the first helpful step is to factor the individual numerators and denominators to expose all of the factors that are going to be in the product.

$$\frac{3 \cdot 5}{2 \cdot 2 \cdot 2} \cdot \frac{2 \cdot 2}{7} \cdot \frac{2 \cdot 7}{3} \cdot \frac{1}{5}$$

Then, a common factor in the numerator and denominator is like multiplication by 1, so common factors can be crossed out. Look at what happens in this case!

$$\frac{\cancel{3} \cdot \cancel{5}}{\cancel{2} \cdot \cancel{2} \cdot \cancel{2}} \cdot \frac{\cancel{2} \cdot \cancel{2}}{\cancel{7}} \cdot \frac{\cancel{2} \cdot \cancel{7}}{\cancel{3}} \cdot \frac{1}{\cancel{5}} = 1$$

Note that all of the factors cancel out in the denominator. A common mistake at this point is to replace the “empty” denominator with 0 — but remember, a common factor in the numerator and denominator is like a factor of 1, which is the fraction $\frac{1}{1}$, not $\frac{1}{0}$, so there's always a “phantom one” lurking there even if we don't write it.

1.3.3 Adding and Subtracting Rational Numbers

Knut Krumbli would tell you that brining together a herd of 8 goats and a herd of 11 goats results in a herd of 19 goats. It's goat herding, not rocket science.

But, combining a herd of 4 goats and a flock of 13 sheep doesn't really give us 17 of anything until we can find some shared characteristic that is common to both groups. For instance, we could say we have a group of “17 mammals”, or “17 quadrupeds”.

So it goes with fractions. When our fractions have a shared name (a common denominator) we can total up how many things we have with that name: 8 thirds and 11 thirds makes 19 thirds. In math symbols, we write

$$\frac{8}{3} + \frac{11}{3} = \frac{19}{3}$$

When our quantities *don't* share a common unit (a common denominator), we have to find one before we can add in a meaningful way.

Given a fraction, we can use multiplication by a "fancy one" to generate a new fraction that has the same value, but a different, perhaps more helpful, denominator.

Some people like to try and find the *least common denominator*, but that's not strictly necessary. Any common denominator will do. In fact, a guaranteed common denominator of any two fractions is the *product of their denominators*.

Adding and Subtracting Rational Numbers

To add rational numbers, we must have a common denominator, for example the product of the original denominators. Then we add the numerators, and keep the common denominator.

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + b \cdot c}{b \cdot d}$$

To subtract rational numbers, change the subtraction problem to an "addition of the opposite" problem and then follow the algorithm for addition.

Here's the derivation of fraction addition in a bit more detail. Note how we use multiplication by 1 to find a common denominator $b \cdot d$:

$$\frac{a}{b} + \frac{c}{d} = \left(\frac{a}{b} \cdot 1\right) + \left(1 \cdot \frac{c}{d}\right) = \left(\frac{a}{b} \cdot \frac{d}{d}\right) + \left(\frac{b}{b} \cdot \frac{c}{d}\right) = \frac{a \cdot d}{b \cdot d} + \frac{b \cdot c}{b \cdot d} = \frac{a \cdot d + b \cdot c}{b \cdot d}$$

Negative Fractions

Where should we put the negative sign when we have a negative fraction? Does it matter? Consider the following three possibilities. Are they all equivalent?

$$-\frac{3}{4} = \frac{-3}{4} = \frac{3}{-4}$$

A fraction is a way of writing a division problem. If Knut's four children share three bowls of lingonberries equally, then each child will get $3 \div 4 = \frac{3}{4}$ of a bowl of berries.[12] The fraction $\frac{3}{4}$ is just another way of writing $3 \div 4$.

So, all three of the fractions above have the same value. In the first example, the whole fraction has been negated. In the second and third examples, the numbers have opposite signs and so the quotient will be negative. In other words it actually doesn't matter where we put the negative sign. We can put it where it is most convenient for the problem (very often, that's in the numerator of the fraction).

[12] Lingonberries are a popular fruit in Scandinavia and throughout northern, central, and eastern Europe. The berries are quite tart, and so they are usually mixed with sugar and preserved as jam or compote. In Sweden and Norway, reindeer is traditionally served with gravy and lingonberry sauce. Yes, Scandinavians eat reindeer.

Mixed Numbers

Improper fractions have a numerator that is greater than or equal (in absolute value) to their denominator, like $\frac{5}{3}$ or $\frac{-84}{16}$. Improper fractions have been scorned by many elementary school mathematics teachers, who instead prefer mixed numbers: $1\frac{2}{3}$ or $-5\frac{1}{4}$. But, improper fractions are often much easier to work with than mixed numbers.[13]

Simplified Rational Numbers #2

Simplified **improper fractions** are preferred over **mixed numbers** and decimals. Only convert to a mixed number or decimal when the context (or the directions) require it.

We usually prefer exact fraction answers over decimal approximations. Writing the decimal $\frac{10}{7}$, for instance, is much preferred over 1.43, and better even than the exact answer $1.\overline{428571}$ (yep, that's a big chuck o' repeating decimal).

But, be sure to read questions carefully! There are exceptions to these rules. When working in a real-world context, a certain number format may make more sense. For example, when solving a problem about money, the answer \$3.50 makes a lot more sense than $\$\frac{7}{2}$. Likewise, the answer "$1\frac{1}{3}$ pounds of cheese" is better than "$\frac{4}{3}$ pounds of cheese". In ambiguous cases, we will make it clear what number format is preferred.

When faced with mixed numbers in a problem, we have to be careful. When adding, we can convert all mixed numbers to improper fractions, or work with them "as is". Subtracting with mixed numbers is tricky, however, because we we may have to handle regrouping. Multiplication is even trickier.

Since we prefer improper fractions as final answers anyway, we recommend converting all mixed numbers to improper fractions before you start computations. To convert a mixed number to an improper fraction, all we have to do is think about the mixed number as an addition problem:

$$3\frac{5}{8} = 3 + \frac{5}{8} = \frac{3}{1} + \frac{5}{8} = \frac{3 \cdot 8 + 1 \cdot 5}{1 \cdot 8} = \frac{24+5}{8} = \frac{29}{8}$$

Here we used the fact that any integer has a "phantom one" in its denominator: $3 = \frac{3}{1}$. We don't usually write it, but it's there when we need it.

[13] One situation where improper fractions are superior is when describing the slope of a line, as we will see in chapter 7.

Example 1.7

Compute each of the following:

1. $\frac{3}{4}+\frac{5}{6}$

These fractions do not have a common denominator, so we'll have to find one. We could use their least common denominator (which is 12) or use the product of the denominators (which is 24). Let's use 24:

$$\frac{3}{4}+\frac{5}{6}=\frac{3\cdot 6+4\cdot 5}{4\cdot 6}=\frac{18+20}{24}=\frac{38}{24}=\frac{19}{12}$$

2. $1\frac{2}{5}-1\frac{7}{8}$

First, we'll convert to improper fractions and change the subtraction to addition-of-the-opposite, putting the negative sign in the numerator of the fraction. Then we add. In the end, we can adjust the negative sign again:

$$\frac{7}{5}-\frac{15}{8}=\frac{7}{5}+\frac{{}^{-}15}{8}=\frac{7\cdot 8+5\cdot {}^{-}15}{5\cdot 8}=\frac{56+{}^{-}75}{40}=\frac{{}^{-}19}{40}=-\frac{19}{40}$$

Explaining the Startup Exploration

In the startup exploration for this section, $0<a<b<1$.

Part (a) asks about $a+b$. Since both numbers are positive, their sum is positive, but we don't have enough information to tell whether the sum is greater than 1. If a and b are both less than $\frac{1}{2}$, for example, then their sum will be less than 1. On the other hand, if they are both greater than $\frac{1}{2}$, then their sum will be greater than 1.

Part (b) asks us to consider $a-b$. Since a is less than b, we're subtracting a larger number from a smaller number, and so the answer must be negative.

We can reason this out in another way: $a-b$ is the same as $a+(-b)$. The absolute value of b is the same as the absolute value of ${}^{-}b$. And so this sum will be negative because the negative number is the one with the greater absolute value.

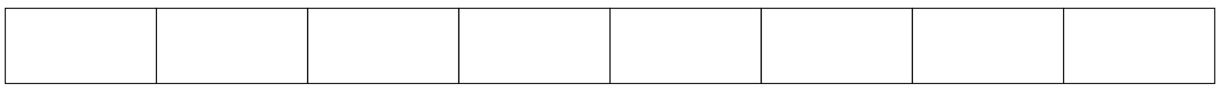
In any case, $a-b$ is negative and so we know for sure that it is less than 1.

1.3.4 Dividing Rational Numbers

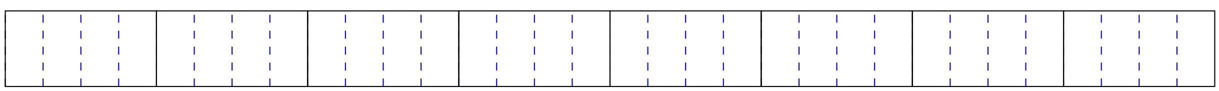
Fraction division may be the most poorly understood operation in all of arithmetic. The algorithm for dividing fractions seems arbitrary, and it's often difficult to judge whether our answers make sense. Let's pause for a moment to think about what fraction division means.

Suppose Jorunn Krumbli, Knut's wife, is making scarves for the goats (Scandinavian winters are chilly). She has 8 meters of burlap, and each scarf requires $\frac{3}{4}$ of a meter. How many scarves can she make?

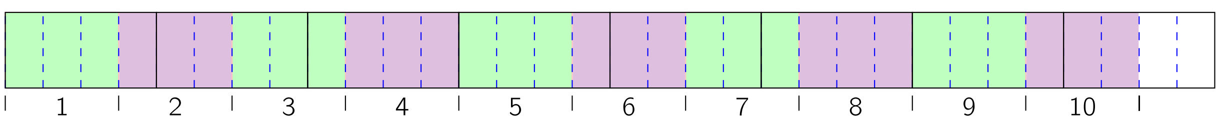
This question is asking us to compute $8 \div \frac{3}{4}$, but let's try to solve the problem by drawing a picture. Suppose this rectangle represents Jorunn's 8 meters of burlap.



To figure out how many pieces of $\frac{3}{4}$ meter are in there, let's first determine the number of pieces of size $\frac{1}{4}$ meter. To do that, we'll break each meter into four pieces. That gives us $8 \cdot 4 = 32$ pieces total.



Now, let's gather these pieces up into groups of three. We'll be able to make 10 whole groups, and we'll have 2 pieces left over. Since we have two pieces, but need three, we have $\frac{2}{3}$ of a group. So, Jorunn can make $10\frac{2}{3}$ scarves for the goats.



Let's retrace our steps. We set out to solve $8 \div \frac{3}{4}$, but in our picture we first multiplied to find the total number of fourths, and then we divided to find how many groups of three we could make. In other words: $8 \div \frac{3}{4}$ must have the same answer as $\frac{8 \cdot 4}{3}$. Does that look familiar?

Dividing Rational Numbers

Dividing by a number is the same as multiplying by the reciprocal of the number. So, we can change a given fraction division problem into an equivalent fraction multiplication problem and then use the rules for fraction multiplication.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$$

Recall that the **reciprocal** of a fraction is the fraction that interchanges the numerator and the denominator of the original fraction. The reciprocal of an integer (which is sitting on a phantom 1) is "one over the original integer".

Example 1.8

Compute each of the following:

1. $\frac{5}{6} \div -\frac{3}{4}$

   We don't need a common denominator or anything, so we can just jump right in with fraction division. We don't even need to move the negative sign, since we know the answer will be negative.

$$\frac{5}{6} \div -\frac{3}{4} = \frac{5}{6} \cdot -\frac{4}{3} = -\frac{20}{18} = -\frac{10}{9}$$

2. $3\frac{3}{4} \div 5$

   First, we'll convert to improper fractions, then we'll implement fraction division At the end, we can simplify before we multiply!

$$3\frac{3}{4} \div 5 = \frac{15}{4} \div \frac{5}{1} = \frac{15}{4} \cdot \frac{1}{5} = \frac{3 \cdot 5}{4} \cdot \frac{1}{5} = \frac{3 \cdot \cancel{5}}{4} \cdot \frac{1}{\cancel{5}} = \frac{3}{4}$$

Let's think about this last answer for a second: does it makes sense? Look again at the problem we were given. If we interpret this as the question"how many groups of 5 are in $3\frac{3}{4}$?", then we can see that we can't even make one whole group: $3\frac{3}{4}$ is less than 5! So, it makes sense for our answer to be less than 1.

Fractions Inside Fractions

As if fractions on their own weren't enough, you may soon face the turducken of arithmetic, the dreaded "fraction in a fraction".[14] Note that the first criteria for simplified rational numbers states that the numerator and denominator must be *integers*.

[14] A "turducken" is a food product where a deboned chicken is stuffed inside a deboned duck, which is then stuffed inside a deboned turkey. In culinary terminology, this is an example of "engastration", a cooking method in which one animal is stuffed inside the gastric cavity of another... which is probably yet another phenomenon that you didn't know had a name.

These creatures can look hard to handle, but don't be intimidated. Recall that a fraction is just a division problem. So, we can rewrite things using a division sign and divide as usual.[15]

Example 1.9

Simplify each of the following:

1. $\dfrac{4}{\left(\frac{1}{3}\right)}$

   All we have to do is rewrite the fraction-in-a-fraction as "numerator ÷ denominator", and then divide as usual.

$$\frac{4}{\left(\frac{1}{3}\right)} = 4 \div \frac{1}{3} = 4 \cdot \frac{3}{1} = 4 \cdot 3 = 12$$

2. $\dfrac{\left(\frac{2}{5}\right)}{\left(1\frac{3}{8}\right)}$

   Here we must rewrite using the obelus, then convert to improper fractions, then divide!

$$\frac{\left(\frac{2}{5}\right)}{\left(1\frac{3}{8}\right)} = \frac{2}{5} \div 1\frac{3}{8} = \frac{2}{5} \div \frac{11}{8} = \frac{2}{5} \cdot \frac{8}{11} = \frac{16}{55}$$

Explaining the Startup Exploration

In the startup exploration for this section, we have rational numbers a and b where $0 < a < b < 1$. Part (d) asks us to consider $a \div b$. This division problem is another way of writing $\frac{a}{b}$. Since a is less than b, though, we know that this fraction is a proper fraction (as opposed to an improper fraction), which means it is less than 1. So, $a \div b$ is less than 1.

1.3.5 Evil and Wrong

Everyone makes mistakes, but not all mistakes are created equal. Some mistakes are not just wrong, the are **Evil and Wrong**. "Wrong" because they are mistakes, "evil" because they are subtle, and sneaky, and tempting.

What we mean here is that sometimes we feel drawn to perform certain arithmetic or algebraic maneuvers — some things seem so logical, so easy, so natural, so tempting — when in reality, they are a total trap. For example:

[15] The fancy mathematical name for the ÷ symbol is *obelus* (plural: obeli)... another thing that you probably didn't know had a name).

WARNING!

Armed with the idea of "simplify before you multiply", we might want to try and pull this stunt in other places:

$$\frac{2+3}{2+7} = \frac{\cancel{2}+3}{\cancel{2}+7} = \frac{3}{7}$$

Seems logical, right? But so wrong! There is no "simplify before you add" maneuver, tempting though it may be. Such a thing is **Evil and Wrong**.

You may be thinking, "Nah, I'd never to that," and with numerical expressions like these, you may be right. But later, when faced with variable expressions like

$$\frac{x+3}{x+7}$$

this temptation may come back, in disguise. We'll draw attention to these **Evil and Wrong** mistakes as we go along because they are so tempting. Beware!

1.4 Order of Operations

We turn now to something else that it probably familiar, the **order of operations**. Consider the following problem as we get started:

Warm-up Problem (TODO: Better name for these?)

The four Krumbli kids each performed the following computation:

$$14 - 6 \cdot 7 + 10$$

Sini got $^{-}38$, Siri got $^{-}18$, Sten got 66, and Stig got 136. Which of them performed the computation correctly? What mistakes did the others make?

The order of operations gives us a standard procedure for simplifying numeric expressions. A **numeric expression** is the algebra way of describing something you may have called just a "math problem" in elementary school. For example $12 - 5$ is a numeric expression. It is not in its simplest form because we can evaluate $12 - 5$ and write 7 instead.

Simplification Rule #1

A numeric expression is completely simplified if all operations have been evaluated and all grouping symbols have been eliminated. The resulting quantity is called the *value* of the expression.

As we go along in algebra, we will learn many rules that maintain "mathematical equivalence." Expressions are mathematically equivalent if they represent the same quantity. For example, $\frac{4}{8}$ is equivalent to $\frac{1}{2}$, and $12 - 5$ is equivalent to 7. Our job when we simplify is to maintain the equivalence from one step to the next. The order of operations is a set of rules for how to do that.

Order of Operations

The **order of operations** is an agreed-upon order for simplifying numeric expressions. The big idea is that "more powerful" operations take priority over "less powerful" operations. When we want to alter the usual rules of precedence, we introduce grouping symbols to make our intentions clear.

When simplifying an expression, we evaluate things in the following order:[a]

First: Grouping symbols like (parentheses), [square brackets], {curly braces}, as well as other subtle grouping symbols like absolute value and the vinculum. Here we work from the innermost set of groupers[b] to the outermost.

Then: Exponents (and, later, roots and logarithms). In the case of a "stack" of exponents, work from the top down.

Then: Multiplication and division. Recall that dividing is the same as "multiplying by the reciprocal". So, these two operations have the same priority and we work from left to right.

Finally: Addition and subtraction. Recall that "subtracting" is the same as "adding the opposite". So, these two operations have the same priority and we work from left to right.

[a] PEMDAS is an mnemonic acronym for remembering the order of operations: Parentheses, Exponents, Multiplication, Division, Addition, Subtraction. In Canada it's BEDMAS (B for Brackets). In the UK and Australia it's BIDMAS or BODMAS (Indices or Orders, which are other words for exponent). A better acronym might be GEMS or PEMA, which group together the pairs of operations that have the same priority (the G is for Grouping symbols).

[b] As in grouping symbols, not the fish.

Explaining the Startup Exploration

Simplify: $14 - 6 \cdot 7 + 10$

Solution: The correct answer follows the order of operations, like so

$$\begin{aligned}
& 14 - 6 \cdot 7 + 10 & \\
&= 14 - 42 + 10 && \text{work multiplication before addition/subtraction} \\
&= {}^{-}28 + 10 && \text{work addition and subtraction from left to right} \\
&= {}^{-}18 &
\end{aligned}$$

So, it is Siri who had the right answer.

Students who just memorize a clever mnemonic device might be tempted to do the <u>A</u>ddition before the <u>S</u>ubtraction, but don't be fooled! Those two operations have the same priority. (Sini made this mistake and got $^{-}38$.)

Sten worked the operations straight through from left to right, all at the same priority and got 66. Stig did something very creative, computing $(14 - 6) \cdot (7 + 10) = 8 \cdot 17 = 136$.

Notice how we showed the work going down the page, simplifying the problem one step at a time. Some people prefer to work across, and that's OK too. The point is that it's important to show work in an organized way when we solve a problem so that our reasoning and thought process are clear.

The work you show is the road map to your solution. Whoever reads over your work must be able to follow and understand your steps, without having to make any assumptions about what you actually did to reach your answer. It's a bad habit to skip steps, and it's no help to people reading your work to say you did a step in your head. Every step you take needs to be written down clearly and neatly.

Abuse of the Equal Sign

Consider the following work, written out by a student to simplify $8 \cdot 4 + 10$

$$8 \cdot 4 = 32 + 10 = 42 \qquad \text{OK or not OK?}$$

The student has reached the correct value, and it may even be clear what the student is thinking: "8 times 4 is 32, plus 10 more makes 42". This, however, is a heinous abuse of the equal sign! Look at what the first part of the work says:

$$8 \cdot 4 = 32 + 10 \qquad \text{These are not equal!}$$

One way to avoid this misuse of the equal sign is to write your work going down the page (as shown above). If you prefer to write across the page, be sure to write out the whole problem as you perform each simplification:

$$8 \cdot 4 + 10 = 32 + 10 = 42$$

In the next few sections, we'll look more closely at some trickier aspects of the order of operations.

1.4.1 About Grouping Symbols

As expressions get complicated, we may have grouping symbols inside of other grouping symbols. If there are different symbols, we can more easily see where different groups begin and end. But, we may just find a bunch of parentheses, like in the example below. In that case, we have to be a bit careful about what's being grouped together.

Example 1.10

Simplify: $12 + (3 - (4 - 2) + 5)$

Solution: When faced with multiple grouping symbols, we must start with the innermost set of grouping symbols and evaluate our way to the outermost. Once we have simplified the expression inside a set of grouping symbols down to a single quantity, we can write that quantity without the groupers.

$$\begin{aligned}
&12 + (3 - (4 - 2) + 5) \\
&= 12 + (3 - 2 + 5) \\
&= 12 + (1 + 5) \\
&= 12 + 6 \\
&= 18
\end{aligned}$$

The Vinculum Is a Grouping Symbol

The **vinculum** (as in the fraction bar) is a grouping symbol. For example, if the task is to simplify a fraction such as:

$$\frac{20 + 2^2 \cdot (14 - 9)}{(2 - 4)^3}$$

then we must think of the expression in the numerator as a group, and likewise for the denominator. In other words, like so:

$$\Big(20 + 2^2 \cdot (14 - 9)\Big) \div \Big((2 - 4)^3\Big)$$

We have two options for simplifying this. We might keep it in a fraction the whole time, or we could simplify the numerator and denominator separately and then squish them back into a fraction at the end.

Example 1.11

Simplify: $\dfrac{20 + 2^2 \cdot (14 - 9)}{(2 - 4)^3}$

Solution: Let's dismantle this thing and handle it in two pieces. The numerator works like this:

$$\begin{aligned}
&20 + 2^2 \cdot (14 - 9) \\
&= 20 + 2^2 \cdot 5 \\
&= 20 + 4 \cdot 5 \\
&= 20 + 20 \\
&= 40
\end{aligned}$$

The denominator works like this: $(2 - 4)^3 = (^-2)^3 = {}^-8$. Then, we can put the pieces back into their original fraction configuration:

$$\frac{20 + 2^2 \cdot (14 - 9)}{(2 - 4)^3} = \frac{40}{-8} = -5$$

1.4.2 About Exponents

An expression of the form a^b is read "a to the power of b" or "a to the b^{th} power". In such an expression, a is called the **base** and b is called the **exponent**. We call the whole thing a **power** of a, since a is the base.

We will get into more detail about exponents later on, but we'll pause here to mention two key ideas. First, recall that we can think about an exponent as shorthand for a repeated multiplication.[16]

$$a^b = \underbrace{a \cdot a \cdot a \cdots a}_{b \text{ times}}$$

That part you probably knew already. This next fact may be new.

Raising to the Power Zero

For any nonzero number a, the expression $a^0 = 1$. In other words, any nonzero number raised to the power 0 equals 1.

When we say that a is a "nonzero number" this means, naturally enough, that a cannot be 0. The expression $0^0 \neq 1$. What *does* it equal? That's a tricky question that will have to wait for later. 0^0 is an unusual mathematical creature![a]

[a] It's not the only one, either. In 1872, Karl Weierstrass (or Weierstraß, if you prefer the German double-S) shook the foundations of calculus with his mathematical monster, now called the "Weierstraß function". It's a bit complicated to get into the details, but he described a mathematical rule which behaves like the geometric "fractals" that we'll study in chapter 2... and it made some other mathematicians very upset.

We'll get into the "hows and whys" of exponents in **??**, and we'll return to the idea of the zero exponent. In the meantime, here's an example of how this fact might come in handy.

[16] As with "multiplication is repeated addition", this interpretation breaks down eventually. Expressions like 5^{-3} and $16^{\frac{1}{2}}$ don't really translate well into "repeated multiplication". Don't panic about the idea of a negative number or a fraction up there in the exponent! All will be revealed as the course goes on.

Example 1.12

Simplify: $\left(\dfrac{120-(24-5^2)}{7^2 \cdot 400 \div 6^3}\right)^0$

Solution: If we go on "auto-pilot" we might follow all of the simplification rules, work from the inside to the outside, simplify the numerator, simplify the denominator. . .

But, the expression is raised to the power 0. So the answer is probably 1! We must check that we don't have 0^0, but we can use a little number sense to do a quick check of the numerator in the fraction, and see that it will not equal zero. (Can you see why, without having to work it all out?)

We must also check that the denominator is not zero. A quick check there shows that it is not zero either. (Can you see why?)

So, this one's easy:

$$\left(\frac{120-(24-5^2)}{7^2 \cdot 400 \div 6^3}\right)^0 = 1$$

The lesson here is to look at the entire problem and plan an efficient solution strategy *before* just jumping in and crunching numbers. In this case, we can save ourselves a lot of work with a bit of careful observation.

Fractions Versus Exponents

If we have a fraction raised to a power, we must be very mindful of the notation and to what, exactly, the exponent applies.

Example 1.13

Simplify each of the following:

1. $\left(\dfrac{2}{3}\right)^4$ The parentheses indicate that we are multiplying together 4 copies of the fraction:

$$\left(\frac{2}{3}\right)^4 = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{16}{81}$$

2. $\frac{2^4}{3}$ Remember that the numerator is a group, and so the exponent applies only to the 2, not the whole fraction:

$$\frac{2^4}{3} = \frac{2 \cdot 2 \cdot 2 \cdot 2}{3} = \frac{16}{3}$$

One of the Trickiest Concepts in Algebra 1

What is the difference between the following three expressions?

$$(-3)^2 \qquad -(3^2) \qquad -3^2$$

Tricky, right? This is an important difference that will come back over and over again, and will look a little different every time.

In the first case, the parentheses make it clear: $(-3)^2$ means "raise negative three to the second power".

$$(-3)^2 = -3 \cdot -3 = 9$$

As usual, the product of two negatives is positive (Hitler gets hit by a truck). The second case is clear as well. The parentheses indicate that we should simplify the exponent first and then take the opposite of the result:

$$-(3^2) = -(3 \cdot 3) = -(9) = -9$$

The third case, -3^2, is the tricky one. It looks kind of ambiguous, since there are no parentheses. But, suppose we wrote the problem like this:

$$0 - 3^2$$

Now, it's clear what to do even without the parentheses, and that is the key.

Opposites of Numbers to a Power

The expression $-a^n$, written without parentheses, is equivalent to the parenthesized expression $-(a^n)$. For example: $-3^2 = -(3^2) = -9$.

You might be saying to yourself, "No sweat. I get it." But (if history and human nature are any indication) you may find yourself tripping over this concept at some point. Confusion around the notation can pop up, for instance, when we're typing expressions into a graphing calculator.[17]

1.4.3 About Multiplication

In algebra we frequently use x (the letter) to stand for a unknown or variable quantity. So, we never use $\times$ to show multiplication. It would be too confusing to have a mix of letter-x's and multiplication-$\times$'s in the same expression. Can you imagine trying to decode something like:

$$x \times 3 + 4 \times x = x + 4x \times x \times x \qquad \text{Yikes!}$$

No more x for multiplication!

To show the operation of multiplication, use an asterisk, a dot, or parentheses. Instead of writing 3×4 to represent "3 times 4", we write: $3 * 4$, or $3 \cdot 4$, or $3(4)$.

Implied Operations

A key aspect of algebra will be learning to read the notation and use it correctly when writing expressions and equations. Algebra is very much like a language, and all languages have special rules. For example, in English we sometimes smash up two words as a contraction: instead of writing "do not" we can write "don't".

We used contractions (of a kind) in mathematics as well. We don't usually write a positive sign in front of positive numbers. We don't usually write the phantom 1 that is in the denominator of an integer.

Another kind of mathematical contraction is the use of an **implied operation**. This arises most often when dealing with multiplication. Here's an example of how implied operations can sneak into a problem.

[17] A note to old-school parents who are used to working with calculators that use reverse Polish notation (RPN) or a stack, and might want to argue that $-3^2 = 9$. When using an RPN calculator we punch in -3 and press enter to push that quantity onto the stack. The number and the negative are both in the stack together, so squaring the entry on the top of the stack means squaring negative three, which is equivalent to $(-3)^2$ and not the same as -3^2.

Example 1.14

Simplify: $12 - 5(2 + 8)$

Solution:

$$\begin{aligned} & 12 - 5(2+8) \\ &= 12 - 5(10) && \text{Aha! That 5(10) means multiplication!} \\ &= 12 - 50 \\ &= -38 \end{aligned}$$

A very common mistake is to do the $12 - 5$ first, instead of the $5(10)$, but that would totally violate the order of operations!

In this chapter we have thought deeply about fundamental ideas regarding numbers and operations, and perhaps you have seen some familiar ideas in a new light. Armed with this knowledge, we now venture deeper into the algebraic wilderness.

Chapter 2

Sequences

A mathematician, like a painter or a poet, is a maker of patterns.

— G. H. Hardy, British mathematician

2.1 Sequences and Recursion

Startup Exploration: Communicating a Pattern

Predict the next few numbers in number pattern:

$$2, 5, 8, 11, 14, 17, \ldots$$

How would you describe this pattern to a partner who could not see it? Could you communicate the pattern *without actually listing all the numbers*? What's the minimum amount of information your partner would need to recreate the pattern?

Informally, we call this a number pattern. Mathematically speaking, an ordered list of numbers like this is called a **sequence**. Each of the numbers in the list is called a **term** of the sequence.

Sequences often have patterns within them. Perhaps, when thinking about how you'd describe this sequence to a partner, you thought about a rule like "add 3" or "+3". (Do you see how this applies to the given pattern?)

But "add 3" is not enough to recreate the sequence. Consider the sequence: $1, 4, 7, 10, 13, \ldots$ And what about $-10, -7, -4, -1, 2, \ldots$ The phrase "add 3" also applies to these sequences, even though they are different from the sequence in the startup exploration.

To distinguish these different sequences, we must include the starting value in our description. We can describe the original sequence clearly and unambiguously by saying something like "Start with 2, then add 3 to the previous value".

So, when describing the pattern of a sequence we are really describing how to generate the sequence from scratch. To do that, we have to answer these two questions: First, how does the sequence begin? Second, what must we do to the *current* term to find the *next* term of the sequence? This is called the **recursive** description of the sequence.

Recursive

Describes a procedure that is applied over and over again, starting with a number or a geometric figure, to produce a sequence of numbers or figures.

As the definition says, we can start a recursive procedure with a number or a geometric figure. We'll start our exploration of sequences by studying geometric figures called **fractals**.

Fractals

Fractal

A geometric figure that has undergone infinitely many applications of a recursive procedure and exhibits the property of self-similarity.

Fractal geometry is often called "the geometry of nature". If we look around the natural world, it is not like we see a lot of perfectly straight lines, rigid rectangles, and regular pentagons. But, the growth of a tree can be described using a recursive procedure: grow towards the sun for a bit, branch off at an angle, repeat. Trees exhibit self-similarity. If we break off a branch of a tree and stick it in the ground, looks just like a little tree!

Clouds, coastlines, mountains, trees, romanesco broccoli, the folds of your brain, your vascular system, your bronchial tubes, the lining of your small intestines... all of these are a kind of fractal.[1] Technically speaking, natural fractals only have their recursive procedure applied a handful of times (we say the procedure has a handful of *iterations*) so they aren't true mathematical fractals. A mathematical fractal undergoes an infinite number of iterations.[2]

[1] In 1968, Hungarian biologist Aristid Lindenmayer developed a method for writing recursive rules that can be used to model the growth of algae. Called "Lindenmayer systems" or "L-systems" today, his methods have been used to model more complex organisms, as well as purely mathematical structures.

[2] Fractals may play an interesting role later on in your study of mathematics, for example the Mandelbrot set is a fractal that involves the complex numbers. Do an internet search for "Mandelbrot set" and check out the pictures!

We will begin our study of sequences by looking at some famous mathematical fractals that were first studied by Polish mathematician Wacław Sierpiński.

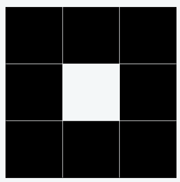
Extended Exploration: Sierpiński's Triangle

[TODO] Click here to visit the extended exploration: Sierpiński's Triangle

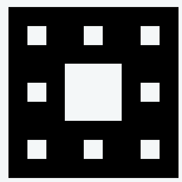
Startup Exploration: Sierpiński's Carpet

To draw Sierpiński's carpet, we begin with a square called "stage 0". We subdivide this square into nine congruent sub-squares and remove the one in the center. We repeat the process with the remaining eight sub-squares. Stages 0 through 3 are shown below.

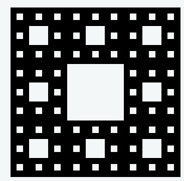
Stage 0



Stage 1



Stage 2



Stage 3

There is one solid square in stage 0, and there are eight (smaller) solid squares in stage 1. How many of the smallest solid squares are there in each stage 2? What about stage 3?

2.1.1 Algebra of Sierpiński's Carpet

Sierpiński's carpet generates some interesting sequences of numbers. For example, if we consider the number of (smallest) squares at each stage of the fractal. We have one square in stage 0, and eight squares in stage 1.

To create stage 2, we divide each of the eight stage-one squares into 9 pieces, and then remove the center square. So each of the eight squares from stage 1 turns into eight new, tiny squares in stage 2. So there are $8 \cdot 8 = 64$ tiny squares in stage 2. To create stage 3, each of these sixty-four tiny squares becomes 8 super-tiny squares, so there are $64 \cdot 8 = 512$ super-tiny squares in stage 3. Together, we have a sequence that begins:

$$1, \ 8, \ 64, \ 512, \ \ldots$$

What is a recursive rule for this sequence? The sequence starts with 1, and then to go from one number to the next, we multiply by 8. So, that's our rule: "start with 1, multiply the previous value by 8". In other words, to find the next term in the sequence, take the previous term (the last term in the sequence that we know) and multiply by 8.

We can use this rule to generate the next few terms of our sequence, but watch out! We quickly end up with a lot of squares and you don't want to get any on your shoes.

$$1,\ 8,\ 64,\ 512,\ 4096,\ 32\,768,\ 262\,144,\ 2\,097\,152,\ 16\,777\,216,\ \ldots$$

Now-Next Rules

We can always write a recursive rule as a sentence, as we did above. Another way to capture a recursive procedure is using a "now-next" rule, sometimes called a "start-now-next" rule. If we have the sentence "start with 1, multiply the previous value by 8", we could write the now-next rule as follows:

START = 1
NEXT = NOW · 8

It's pretty obvious that the first part of the rule says where to start. The second part of the rule says, "To find the NEXT number in the pattern, we take the number we have NOW and multiply by 8."

2.1.2 Recursive Rules and Recursive Formulas

Recursive rules are easy to write in sentence form, and now-next equations are nice and succinct, but there is a more mathematical way. We are going to write what we call a recursive formula.

We use the variable a with a subscript to represent a specific term of the sequence. So, a_1 represents the first term of the sequence, a_2 represents the second term of the sequence, and a_{98} would represent the 98th term of the sequence.

For example: given the sequence $4, 12, 36, 108, \ldots$, we have:

$$a_1 = 4$$
$$a_2 = 12$$
$$a_3 = 36$$
$$a_4 = 108$$

We use a_n to represent any old term of the sequence. Then, a_{n+1} represents the *next* term in the sequence. (Can you explain why?)

We can write the recursive rule either as "start with 4, multiply the previous term by 3", or "START = 4, NEXT = NOW · 3". Here's how we can translate this into a recursive formula.

"Start with 4" means that the first term of the sequence is 4. We write $a_1 = 4$, since a_1 represents the first term of the sequence. This is just like "START = 4" in the now-next rule. To translate "NEXT = NOW · 3", we write $a_{n+1} = a_n \cdot 3$.

A note about notation: When multiplying a number and a letter, we usually write the number first. Also, we *don't* usually write a multiplication symbol in between.[3] So, we have created the recursive formula:

$$a_1 = 4$$
$$a_{n+1} = 3\,a_n$$

Example 2.1

Write the recursive formula for the sequence $1, 5, 25, 125, 625, \ldots$

Solution: With a little exploration, we see that the sentence version of this rule is "Start with 1, multiply previous by 5", and the now-next version is "START = 1, NEXT = NOW · 5". So, we have the recursive formula: $a_1 = 1,\ a_{n+1} = 5\,a_n$.

Example 2.2

Write out the first five terms of the sequence generated by each rule.

1. "Start with 128, multiply previous by $\frac{1}{2}$"

   Solution: The rule states clearly that the first term is 128, no trouble. Then, to find the second term, we multiply the first term by $\frac{1}{2}$, that means $128 \cdot \frac{1}{2} = 64$. To find the third term, we multiply the second term by one half: $64 \cdot \frac{1}{2} = 32$. We repeat for the next few terms, which gives:

   $$128, 64, 32, 16, 8, \ldots$$

2. $a_1 = 12,\ a_{n+1} = -2 \cdot a_n$

   Solution: The first term is a_1, and the formula says that's 12. Then, to find a_2, the second term, we have

   $$a_2 = -2 \cdot a_1 = -2 \cdot 12 = -24.$$

   We continue to multiply by -2 each step of the way and get:

   $$12, -24, 48, -96, 192, \ldots$$

[3] More on working with letters, or variables, in chapter 3.

Recursive rules and formulas are handy for describing a sequence, but suppose we want to skip around and find random terms of the sequence. In this situation, the recursive rule is the worst possible rule to have!

For example, how could we use the rule $a_1 = 4$, $a_{n+1} = 3 \cdot a_n$ to find the value of the 1000th term in the sequence, a_{1000}? The rule tells us that $a_{1000} = 3 \cdot a_{999}$. But, what's a_{999}?

Well, $a_{999} = 3 \cdot a_{998}$ But, what's a_{998}?

Hmm. $a_{998} = 3 \cdot a_{997}$ But... oh boy. Can you see the problem here?

If we want to skip around and find random terms in a sequence, it's much easier to use a different kind of formula, called an "apparent" or "explicit formula". More on those in the next section!

2.2 Geometric Sequences

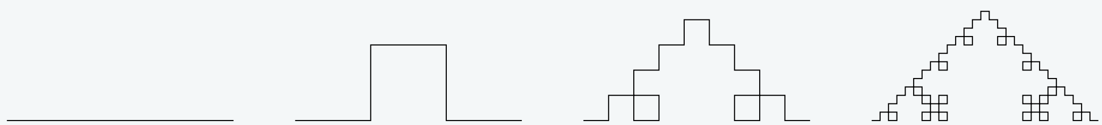
We are going to continue our study of sequences by looking at another fractal. In 1904, Swedish mathematician Helge von Koch first described a kind of fractal that has since come to be known as a Koch curve. There are several variants of the Koch curve. We'll look at two different forms in the following explorations.

Extended Exploration: Koch Curve, Triangular Version

[TODO] Click here to visit the extended exploration: Koch Curve

Startup Exploration: Koch Curve, Square Version

We begin in stage 0 with a line segment of length 1. To create stage 1, we alter the segment as follows: cut it into three pieces, and replace the center piece with three sides of a square. We repeat the process for each line segment in the previous figure to create stages 2 and 3.



Stage 0 Stage 1 Stage 2 Stage 3

Write a recursive formula describing the number of segments in each stage of the fractal.

2.2.1 The Algebra of the Koch Curve

The Koch curves are beautiful things, at once incredibly simple and incredibly complex. As the square-based version above grows, each line segment is replaced by five shorter segments. The recursive rule is "start with 1, multiply the previous value by 5".

To compute the number of segments in each stage, we might organize our work in a list like this:

$$\begin{array}{rl}
1 & \text{segment in stage 0} \\
(1)\cdot 5 & \text{segments in stage 1} \\
(1\cdot 5)\cdot 5 & \text{segments in stage 2} \\
(1\cdot 5\cdot 5)\cdot 5 & \text{segments in stage 3} \\
(1\cdot 5\cdot 5\cdot 5)\cdot 5 & \text{segments in stage 4} \\
(1\cdot 5\cdot 5\cdot 5\cdot 5)\cdot 5 & \text{segments in stage 5}
\end{array}$$

We can use a bit of shorthand, and write this repeated multiplication using an exponent.

$$\begin{aligned} 1 &= 5^0 && \text{segment in stage 0} \\ 1 \cdot 5 &= 5^1 && \text{segments in stage 1} \\ 1 \cdot 5 \cdot 5 &= 5^2 && \text{segments in stage 2} \\ 1 \cdot 5 \cdot 5 \cdot 5 &= 5^3 && \text{segments in stage 3} \\ 1 \cdot 5 \cdot 5 \cdot 5 \cdot 5 &= 5^4 && \text{segments in stage 4} \\ 1 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 &= 5^5 && \text{segments in stage 5} \end{aligned}$$

Notice that the exponent is equal to the stage number. This "5 to a power" notation works even for stage 0, since since $5^0 = 1$.

So, if we want to know how many segments are in stage 8 of the fractal, we can use this pattern to predict that there will be $1 \cdot 5^8$ segments. If we let x represent the stage number, then stage x of the fractal will have 5^x segments.

We have discovered a way of calculating the number of segments that is *not* recursive, because it doesn't rely on our knowing any of the previous terms. Instead, to produce the value of a certain term, all we need is the *number of the term*. We can compute the number of line segments in stage x without having to know anything about the stages that came before it.

2.2.2 Explicit Formulas for Sequences

In our discussion of fractals, we have always described the first image as "stage 0" of the fractal. But, when we write out a sequence, the first term is, well, the *first* term (not the *zero*th term).[4]

In other words, the same pattern of values has a slightly different numbering, depending on whether we're describing stages of a fractal or terms in a sequence.

Value	1	5	25	125
Fractal Stage Number	0	1	2	3
Sequence Term Number	1	2	3	4

So, if we want to write a recursive formula for the terms of a sequence, we have to make a little adjustment:

$$\begin{aligned} a_1 &= 1 && = 5^0 \\ a_2 &= 5 && = 5^1 \\ a_3 &= 25 && = 5^2 \\ a_4 &= 125 && = 5^3 \end{aligned}$$

[4] In some scientific disciplines, it is customary to start counting with zero: for example, in computer science. Jason, one of the authors of the 𝔄𝔩𝔤𝔢𝔟𝔯𝔞𝔫𝔬𝔪𝔦𝔠𝔬𝔫, is a computer scientist by training and thinks this way. Jason also prefers to include 0 as one of the natural numbers. Patty, the other author of the 𝔄𝔩𝔤𝔢𝔟𝔯𝔞𝔫𝔬𝔪𝔦𝔠𝔬𝔫, is a mathematician by training and prefers to start counting at 1... most of the time.

Can you see the relationship between the subscript and the exponent? If we let a_n represents any term of the sequence, then our rule is:

$$a_n = 5^{n-1}$$

Rules of this kind are called apparent formulas or explicit formulas. One benefit of rules like this is that if we want to know, say, the number of segments in the curve at stage 1904, we can compute simply:

$$a_{1904} = 5^{1903}$$

By the way, this number is enormous. It's more than 1300 digits long!

Example 2.3

Write explicit formulas for the other features of the Koch curve.

Solution: *Length of one segment.* Each segment in a certain stage is one third the length of the segment in the stage before. So, the sequence generated by the length of one segment in each stage is

$$\left(\frac{1}{3}\right)^0, \left(\frac{1}{3}\right)^1, \left(\frac{1}{3}\right)^2, \ldots \quad \text{and so} \quad a_n = \left(\frac{1}{3}\right)^{n-1}$$

.

Total length of the curve. Since we know the number of segments and the length of each segment, we can multiply to find the total length of the curve. We have

$$\left(\frac{5}{3}\right)^0, \left(\frac{5}{3}\right)^1, \left(\frac{5}{3}\right)^2, \ldots \quad \text{and so} \quad a_n = \left(\frac{5}{3}\right)^{n-1}$$

.

Example 2.4

What if the stage 0 figure had been a segment of length 7, rather than length 1? How would that change our formula?

Solution: The number of segments would not change, but the length of each segment (and the total length of the curve) would! The new sequence for the length of one segment would be generated as follows:

$$\begin{aligned} 7 &= 7 * \left(\frac{1}{3}\right)^0 && \text{length of one segment in stage 0} \\ 7 * \left(\frac{1}{3}\right) &= 7 * \left(\frac{1}{3}\right)^1 && \text{length of one segment in stage 1} \\ 7 * \left(\frac{1}{3}\right) * \left(\frac{1}{3}\right) &= 7 * \left(\frac{1}{3}\right)^2 && \text{length of one segment in stage 2} \\ 7 * \left(\frac{1}{3}\right) * \left(\frac{1}{3}\right) * \left(\frac{1}{3}\right) &= 7 * \left(\frac{1}{3}\right)^3 && \text{length of one segment in stage 3} \end{aligned}$$

Again we can use an exponent to simplify the repeated multiplication of $\frac{1}{3}$. This generates the sequence

$$7 * \left(\frac{1}{3}\right)^0, \ 7 * \left(\frac{1}{3}\right)^1, \ 7 * \left(\frac{1}{3}\right)^2, \ \ldots$$

If we let n represent the term number, then the recursive formula for this sequence is

$$a_n = 7 * \left(\frac{1}{3}\right)^{n-1}$$

2.2.3 Geometric Sequences

So far, all of our sequences have had recursive rules like "start with A, *multiply* the previous term by B". Sequences with recursive rules of this type are called **geometric sequences**. Geometric sequences belong to the family of *exponential relationships*, because the $(n-1)$ expression appears as an exponent.

To generate the next term of a geometric sequence, we multiply the previous term by a fixed value. This fixed value is sometimes called, naturally enough, the *constant multiplier*. More often, it is called the *common ratio*.

Geometric Sequence

A sequence where the ratio between each pair of successive terms is constant. The constant ratio is called the *common ratio*, usually denoted r. Geometric sequences are exponential relationships.

Example 2.5

Determine whether or not the sequence $4, 12, 36, 108, \ldots$ is a geometric sequence.

Solution: If this is a geometric sequence, then it must have a rule of the form "start with A, multiply the previous term by B"? Let's check.

To go from 4 to 12, we multiply by $\frac{12}{4} = 3$.

To go from 12 to 36, we multiply by $\frac{36}{12} = 3$. Looking good so far!

To go from 36 to 108, we multiply by $\frac{108}{36} = 3$. Nice! Based on the four terms given, the sequence is geometric.

Now, look at what we did to determine this: we created ratios of successive terms, and found that they were all the same.

$$\frac{12}{4} = \frac{36}{12} = \frac{108}{36} = 3$$

So, the *common ratio* for this sequence is 3.

Example 2.6

Write recursive and explicit formulas for the geometric sequence

$$32,\ 24,\ 18,\ 13\tfrac{1}{2}, \ldots$$

Solution: To get from 32 to 24, our first instinct might be to subtract: $32 - 8 = 24$. But, we're told in the problem that this is a *geometric* sequence, and that means that the recursive rule involves multiplication, not subtraction.

How can get from 32 to 24 using multiplication? The constant multiplier must be less than one (can you explain why?), and we can divide to find what it is:

$$\frac{24}{32} = \frac{3}{4}$$

So, $\frac{3}{4}$ is a good candidate for the constant ratio of the sequence. Let's check the other terms to see if we're right. We multiply 24 by $\frac{3}{4}$ to see if that gives us the next term in the sequence:

$$24 \cdot \frac{3}{4} = \frac{24}{1} \cdot \frac{3}{4} = \frac{\cancel{4} \cdot 6}{1} \cdot \frac{3}{\cancel{4}} = \frac{6}{1} \cdot \frac{3}{1} = 18 \qquad \text{Check!}$$

Now see if 18 times $\frac{3}{4}$ gives the next term:

$$18 \cdot \frac{3}{4} = \frac{18}{1} \cdot \frac{3}{4} = \frac{\cancel{2} \cdot 9}{1} \cdot \frac{3}{\cancel{2} \cdot 2} = \frac{9}{1} \cdot \frac{3}{2} = \frac{27}{2} = 13\tfrac{1}{2} \qquad \text{Check!}$$

So, we have found the correct constant multiplier based on the information we were given. The recursive formula is

$$a_1 = 32,\ a_{n+1} = \frac{3}{4} \cdot a_n,$$

and the explicit formula is

$$a_n = 32 \cdot \left(\frac{3}{4}\right)^{n-1}.$$

If we look back over the explicit rules for the sequences in this section, we might notice that the formulas have a formula! In other words, the apparent rule for a geometric sequence always has a certain structure, which we summarize here.

Apparent Formula for a Geometric Sequence

Given a geometric sequence with first term a_1 and common ratio r, in other words, a sequence of the form

$$a_1\ ,\ a_1 * r\ ,\ a_1 * r^2\ ,\ a_1 * r^3\ ,\ \ldots$$

The apparent or explicit formula for the sequence is

$$a_n = a_1 * r^{n-1}$$

2.3 Arithmetic Sequences

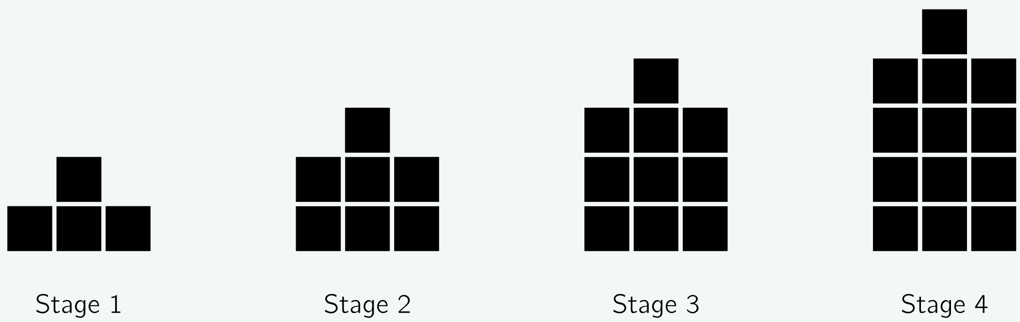
Not all sequences are geometric sequences, of course. Let's explore some other types of sequences.

Extended Exploration: Squares, Triangles, Segments

[TODO] Click here to visit the extended exploration: Squares, Triangles, Segments

Startup Exploration: Tile Pattern #1

The pictures below represent stages 1, 2, 3, and 4 for a pattern of square tiles.



Draw pictures representing stages 5 and 6 in the pattern. Write a sentence or two to describe the pattern in the pictures. What would the stage 0 figure look like?

Write out the sequence for the number of tiles at each stage (starting with stage 1). Write a recursive rule to describe your sequence. How is this rule different from the rules in section 2.2?

The number of tiles in each stage of the pattern creates the sequence

$$4,\ 7,\ 10,\ 13,\ 16,\ 19,\ \ldots$$

We might write recursive rules that go something like "start with 4, add 3 to the previous term", or "START = 4, NEXT = NOW + 3". The fact that we're adding in the rule is a clear difference from the rules we saw when studying geometric sequences.

Sequences like these are called **arithmetic sequences**.[5] Instead of having a common ratio, these sequences have a *common difference*. Arithmetic sequences belong to the family of *linear relationships*.

[5] A word about pronunciation. The branch of mathematics that deals with calculations and operations on numbers is called "arithmetic". When used as a noun in this way, the word is pronounced with the emphasis on the second syllable: *a·RITH·me·tic*. The sequences we're talking about in this section are "arithmetic sequences". When the word is used as an adjective, the emphasis is on the third syllable: *a · rith · ME · tic*.

Arithmetic Sequence

A sequence where the difference between each pair of successive terms is constant. The constant difference is called the *common difference*, usually denoted d. Arithmetic sequences are linear relationships.

Example 2.7

Verify that the given sequence is arithmetic and write a recursive formula for it: $12,\ 17,\ 22,\ 27,\ \ldots$

Solution: In order for a sequence to be arithmetic, we must add the same quantity as we go from term to term. We can check this by subtracting (which is why the thing we add is called a "common difference"). So, let's check:

$$17 - 12 = 5$$
$$22 - 17 = 5$$
$$27 - 22 = 5$$

Check! This is an arithmetic sequence with a common difference of 5.

To write the recursive formula we know that the common difference is added to the current term in order to find the next term. We also know the first term. So:

$$a_1 = 12,\ a_{n+1} = a_n + 5$$

is the recursive formula for the sequence.

2.3.1 Explicit Formulas for Arithmetic Sequences

Of course, we can write an apparent or explicit formula (that is, a non-recursive formula) for an arithmetic sequence. Consider the sequence from the startup exploration: $4, 7, 10, 13, \ldots$ We know where each of the terms come from:

$$a_1 = 4$$
$$a_2 = (4) + 3$$
$$a_3 = (4 + 3) + 3$$
$$a_4 = (4 + 3 + 3) + 3$$
$$a_5 = (4 + 3 + 3 + 3) + 3$$

Notice the repeated addition of 3. This is a case where we can reinterpret repeated addition as multiplication:

$$\begin{aligned}
a_1 &= 4 & &= 4 + 3 \cdot 0 \\
a_2 &= 4 + 3 & &= 4 + 3 \cdot 1 \\
a_3 &= 4 + 3 + 3 & &= 4 + 3 \cdot 2 \\
a_4 &= 4 + 3 + 3 + 3 & &= 4 + 3 \cdot 3 \\
a_5 &= 4 + 3 + 3 + 3 + 3 & &= 4 + 3 \cdot 4
\end{aligned}$$

Notice now that these multiplications are 3 times "one less than the stage number"! Therefore, we can write

$$a_n = 4 + 3(n - 1)$$

As with geometric sequences, there is a formula for these formulas, too:

Apparent Formula for an Arithmetic Sequence

Given an arithmetic sequence with first term a_1 and common difference d, in other words, a sequence of the form

$$a_1\ ,\ a_1 + d\ ,\ a_1 + 2d\ ,\ a_1 + 3d\ ,\ \dots$$

The apparent or explicit formula for the sequence is

$$a_n = a_1 + (n - 1) * d$$

2.3.2 Using Stage Zero

There is another way to write the apparent rule for an arithmetic sequence. We can use this approach when we know (or can find) the "zeroth" term. Then, we interpret the stage 1 figure not as the *start*, but rather as though we are joining a sequence that is "already in progress".

For example, in the tile sequence from the startup exploration, to find the stage 0 figure we have to "back up a step". Since the pattern goes forward by adding 3, to back up one step we must subtract 3. So, the stage 0 figure is just 1 square tile.

Stage	Value	Start with stage 1?		Start with stage 0?	
1	4	4	$= 4 + 3(0)$	$1 + 3$	$= 1 + 3(1)$
2	7	$4 + 3$	$= 4 + 3(1)$	$1 + 3 + 3$	$= 1 + 3(2)$
3	10	$4 + 3 + 3$	$= 4 + 3(2)$	$1 + 3 + 3 + 3$	$= 1 + 3(3)$
4	13	$4 + 3 + 3 + 3$	$= 4 + 3(3)$	$1 + 3 + 3 + 3 + 3$	$= 1 + 3(4)$

One benefit of this new rule is that we find ourselves multiplying the constant difference by the term number itself (before we multiplied by one less than the term number). In other words, we can write the apparent rule as follows:

Apparent Formula for an Arithmetic Sequence (Zero Version)

Given an arithmetic sequence with first term a_1 and common difference d, we can write the apparent or explicit formula for the sequence is

$$a_n = a_0 + n * d$$

Where a_0 represents the "zeroth" term of the sequence (the term that comes before the first term).

In later chapters, we will explore in more detail the connections between the "stage 1 version" and the "stage 0 version" of the rule for arithmetic sequences, and we will learn techniques for writing "stage 0 versions" of the rules for geometric sequences.

Example 2.8

Write a stage zero version of the explicit rule for the arithmetic sequence: $43,\ 35,\ 27,\ 19,\ \ldots$

Solution: This sequence is decreasing, so we must be adding a negative number in the rule. In other words, the common difference must be negative. Subtracting neighboring terms, we can find that the common difference is $^-8$.

To write a zero-based rule, we have to know the zeroth term, and to find that we have to back up from the first term. So, we have $a_0 = 43 - {}^-8 = 43 + 8 = 51$. This value makes sense: Since the sequence is decreasing, the zero term should be larger than the first term.

Knowing the common difference and the zero term, we can write a zero-based explicit rule:

$$a_n = a_0 + n * d = 51 + n * {}^-8 = 51 + {}^-8n$$

2.4 Other Types of Sequences

Startup Exploration: Tile Pattern #2

The pictures below represent stages 1, 2, 3, and 4 for a new pattern of square tiles.

Stage 1 Stage 2 Stage 3 Stage 4

Draw pictures representing stages 5 and 6 in the pattern. Write a sentence or two to describe the pattern in the pictures. What would the Stage 0 figure look like?

Write out the sequence for the number of tiles at each stage (starting with stage 1). Write a recursive rule to describe your sequence. How is this rule different from the rules in the last few sections?

You may have noticed that these sequences are a bit harder to work with. They belong to the family of *quadratic relationships*, and we'll study lots more about quadratic relationships later.

The figures in the startup exploration generate the sequence:

$$3,\ 8,\ 15,\ 24,\ 35,\ 48,\ \ldots$$

Is this sequence geometric? Let's check for a common ratio: the ratio between the first two terms is $\frac{8}{3}$, and the ratio between the next two terms is $\frac{15}{8}$. Those are different ratios, since if we write them with a common denominator, we have $\frac{8}{3} = \frac{64}{24}$ and $\frac{15}{8} = \frac{45}{24}$. So, the sequence is *not geometric*.

Is the sequence arithmetic? Let's check for a common difference:

$$8 - 3 = 5$$
$$15 - 8 = 7$$
$$24 - 15 = 9$$
$$35 - 24 = 11$$
$$48 - 35 = 13$$

The sequence does not have a common difference, so it is *not arithmetic*. But take a look at those differences! The differences have a pattern of their own: They go up by 2 every time. In other words, the differences form an arithmetic sequence! It's a sequence in a sequence! The turducken of sequences![6]

To describe this sequence with a recursive rule, we'll need to give the starting value, as usual: "start with 3". Then, we must describe the pattern in the differences: in this case, we're adding consecutive odd numbers (starting with 5). So, one way to express this recursive rule is "start with 3, add consecutive odd numbers (starting with 5) to the previous term". Note that we kind of sneak in two starting places: one for the start of the sequence (3, in this case) and one for the start of the sequence of numbers that are being added on (5, in this case). Tricky!

Example 2.9

Verify that the given sequence is quadratic, and write a recursive rule: $1, 4, 10, 19, 31, 46 \ldots$

Solution: In order for a sequence to be quadratic the differences between successive terms must form an arithmetic sequence. Let's check:

$$4 - 1 = 3$$
$$10 - 4 = 6$$
$$19 - 10 = 9$$
$$31 - 19 = 12$$
$$46 - 31 = 15$$

The differences are: $3, 6, 9, 12, 15, \ldots$, and that's an arithmetic sequence with common difference 3. So, yes, the original sequence is quadratic.

Now let's try to write a recursive rule (in sentences). Clearly, we start with 1. Then, we add consecutive multiples of three, starting with 3. So, our rule is "start with 1, add consecutive multiples of three (starting with 3) to the previous term".

At this point, our goal is just to recognize that these sequences are neither arithmetic nor geometric, but follow a different kind of pattern. Writing the formulas for them can be quite challenging — but our brains grow when we stretch them around new ideas! Let's give it a shot.

2.4.1 (;,;) Formulas for Quadratic Sequences

---

6 This sequence-in-a-sequence stuff can get pretty involved. Here, we found an arithmetic sequence in the *differences* between the terms in our quadratic sequence. But why not build the sequence $1, 4, 12, 27, 51, \ldots$, in which the sequence of differences is our quadratic sequence! Of course we could keep building sequences like this for as long as we wanted. This isn't just a turducken, it's a *rôti sans pareil*, or "roast without equal", which is a recipe that calls for 17 different birds, each one stuffed into the body cavity of the next.

Extension Sections

Sections marked with the Cthulhu (;,;) emoticon, like this one, are extension sections that might be a bit more intense than the norm. We encourage you to explore the concepts, but don't feel discouraged if you find the material challenging.

Your math brain grows when you think deeply about mathematics, so hard work is valuable, even if things aren't completely clear right away. Many of the ideas in the optional sections will appear again in later chapters, so you'll probably find that your confidence grows as time goes on.

In the last section, we looked at the sequence, which came from a rectangular pattern of tiles:

$$3,\ 8,\ 15,\ 24,\ 35,\ \ldots$$

We wrote the recursive rule in sentences: "start with 3, add consecutive odd numbers (starting with 5) to the previous term". Can we translate this into a recursive formula?

The first step is easy: $a_1 = 2$. Hooray for small victories!

In order to describe the recursive step, we need to describe the sequence of differences: $5, 7, 9, 11, \ldots$. Since this is an arithmetic sequence, we know how to write its explicit rule. Let's use the symbol b, so we don't get our sequences confused. Then this sequence is $b_n = 5+(n-1)\cdot 2$ or, if we use a zero-based rule, $b_n = 3+n\cdot 2$.

Let's try and put these together:

$$\begin{aligned}
a_1 &= 3 & & \\
a_2 &= 8 & &= a_1 + 5 & &= a_1 + b_1 \\
a_3 &= 15 & &= a_2 + 7 & &= a_2 + b_2 \\
a_4 &= 24 & &= a_3 + 9 & &= a_3 + b_3 \\
a_5 &= 35 & &= a_4 + 11 & &= a_4 + b_4
\end{aligned}$$

So, our recursive step is that $a_{n+1} = a_n + b_n$. Since we have an explicit formula for the b_n's, we can replace that part with their explicit rule! Altogether we have:

$$a_1 = 3,\ \ a_{n+1} = a_n + 3 + n \cdot 2$$

How can we check to see if we're right? One way is to use the rule to try and re-generate the sequence. Our rule states that $a_1 = 3$. To find a_2, we can use the rule with $n = 1$ and $n+1 = 2$:

$$a_2 = a_1 + 3 + 1 \cdot 2 = 3 + 3 + 1 \cdot 2 = 3 + 3 + 2 = 8.$$

Then, we can take one step forward and apply the rule again. Now, $n = 2$ and $n+1 = 3$:

$$a_3 = a_2 + 3 + 2 \cdot 2 = 8 + 3 + 2 \cdot 2 = 8 + 3 + 4 = 15.$$

Let's go one more step and try $n = 3$ and $n + 1 = 4$:

$$a_4 = a + 3 + 3 + 3 \cdot 2 = 15 + 3 + 3 \cdot 2 = 15 + 3 + 6 = 24.$$

Phew! It pays to be patient when working out a convoluted rule like this, but in the end, we can see that our rule is working as intended!

Example 2.10

Write a recursive rule for the quadratic sequence: $1, 4, 9, 16, 25, \ldots$

Solution: A bit of tinkering leads us to the rule "start with 1, add consecutive odd numbers (starting with 3) to the previous term". So $a_1 = 1$.

How do we write the apparent formula for the odd number pattern? The common difference is 2, and the pattern starts at 3, so $b_n = 3 + (n-1) \cdot 2$ is the apparent formula for the differences.

Putting the pieces together:

$$\begin{aligned}
a_1 &= 1 & & \\
a_2 &= 4 &= a_1 + 3 &= a_1 + b_1 \\
a_3 &= 9 &= a_2 + 5 &= a_2 + b_2 \\
a_4 &= 16 &= a_3 + 7 &= a_3 + b_3 \\
a_5 &= 25 &= a_4 + 9 &= a_4 + b_4
\end{aligned}$$

So, again, we have $a_{n+1} = a_n + b_n$. Then, we can replace the b_n with the apparent formula we created for the sequence of differences.

$$a_1 = 1, \;\; a_{n+1} = a_n + 3 + (n-1) \cdot 2$$

If we would rather use a zero-based rule for the pattern in the differences, we could write

$$a_1 = 1, \;\; a_{n+1} = a_n + 1 + n \cdot 2$$

Note that even though these two rules look quite different, they are equivalent ways of describing the sequence. In later chapters, we will learn techniques that will help us to explain why these two different-looking rules give us the same result.

2.4.2 (;,;) Writing Explicit Formulas For Quadratic Sequences

It seems only proper to discuss a method for writing a non-recursive formula for a quadratic sequence.

There are, in fact, multiple methods for writing rules like this. There is a way that requires knowledge of calculus, there is a method that uses *systems of equations* (more on those in **??**), there is the not-so-efficient method of guess and check, and so on. Most of these require knowledge of the structure of a quadratic relationship which (seeing as how we're only here in chapter 2) we haven't discussed yet.

But, there is a clever approach that requires a bit of pattern-hunting and detective work. That's the approach we'll explore here.

Let us once again consider the sequence $1, 4, 9, 16, 25, \ldots$. You might have recognized these numbers are the **perfect squares**. That name comes from the idea that we can view these numbers as the areas of squares, as shown in fig. 2.1. The first number is the area of a 1-by-1 square, the second is the area of a 2-by-2 square, then a 3-by-3 square, and so on. Knowing this, we can write any term of the sequence:

$$a_n = n \cdot n$$

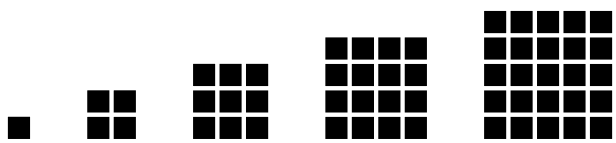


Figure 2.1: The perfect squares: $1, 4, 9, 16, 25, \ldots$

With other quadratic sequences, we can sometimes crack the code if we think about areas of rectangles and triangles. We'll look for pairs of integers that could give us the areas we're after, and then look for arithmetic sequences among those integers. The next thing you know, we'll have a non-recursive formula for the quadratic!

Let's go back to the other example we've been studying, the sequence $3, 8, 15, 24, 35 \ldots$. This sequence came from the tile pattern at the start of section 2.4. Go back and take another look at those pictures. What do you notice?

Stage	Number of Squares	Dimensions of Rectangle
1	3	1×3
2	8	2×4
3	15	3×5
4	24	4×6

In stage n, the size of the rectangle is n units tall and $(n + 2)$ units wide! So, we can use those two values to write an explicit formula:

$$a_n = n \cdot (n + 2)$$

How cool is that?

Example 2.11

Write a non-recursive formula to generate the sequence $6, 12, 20, 30, 42, \ldots$

Solution: It's not obvious how rectangles are related to these numbers, but if we assume we're looking for rectangles with integer side lengths, then there are a limited number of options.

For example, if the first number represents the area of a rectangle with integer side lengths, then it could be either a 1×6 rectangle or a 2×3 rectangle. Let's organize the different options in a table:

Stage	Value	Possible Rectangles							
1	6	1×6	or	2×3					
2	12	1×12	or	2×6	or	3×4			
3	20	1×20	or	2×10	or	4×5			
4	30	1×30	or	2×15	or	3×10	or	5×6
4	42	1×42	or	2×21	or	3×14	or	6×7

Now comes the detective work. We are looking for patterns in the factors as they progress through the terms. We've highlighted the key patterns below.

Stage	Value	Possible Rectangles							
1	6	1×6	or	2×3					
2	12	1×12	or	2×6	or	3×4			
3	20	1×20	or	2×10	or	4×5			
4	30	1×30	or	2×15	or	3×10	or	5×6
4	42	1×42	or	2×21	or	3×14	or	6×7

Notice that the first set of factors (in yellow) form the arithmetic sequence $2, 3, 4, 5, \ldots$ and the second set (in green) form the arithmetic sequence $3, 4, 5, 6, \ldots$

The yellow sequence is always one more than the term number. The green sequence is always two more than the term number. So, we have our non-recursive formula!

$$a_n = (n+1) \cdot (n+2)$$

If these last two sections felt a bit overwhelming, don't worry. After we have some more algebraic tools in our toolbox, we'll return to quadratic relationships and describe them in more detail.

Chapter 3

Graphs and Data

> There is a magic in graphs. The profile of a curve reveals in a flash a whole situation — the life history of an epidemic, a panic, or an era of prosperity. The curve informs the mind, awakens the imagination, convinces.
>
> — Henry D. Hubbard, US National Bureau of Standards

3.1 Visualizing a Sequence

In chapter 2 we investigated pictures patterns (fractals by Koch and Sierpiński, patterns of made of square tiles) and used those pictures to generate sequences of numbers. We begin this chapter with an discussion of another way to represent our sequences of numbers visually: by making a coordinate graph.

3.1.1 Coordinate Graphing

Figure 3.1 summarizes the familiar landmarks of the **coordinate plane**. We see the horizontal **x-axis** and the vertical **y-axis**. Using the two axes as number lines, we can locate **ordered pairs** of numbers using the convention (x, y). The points $(7, -2)$ and $(-6, 3)$ have been plotted as examples. The point $(0, 0)$ where the the two axes meet, is a special point called the **origin**.

The axes chop the plane into four regions called **quadrants**, which are numbered starting in the upper right and moving counter-clockwise (as shown in the figure). The signs in parentheses indicate the signs of x- and y-coordinates in each quadrant. The x-coordinates of the points in Quadrants I and IV are positive, while points in Quadrants II and III have x-coordinates that are negative. Points with positive y-coordinates lie in Quadrants I and II, while points with negative y-coordinates fall in Quadrants III and IV.

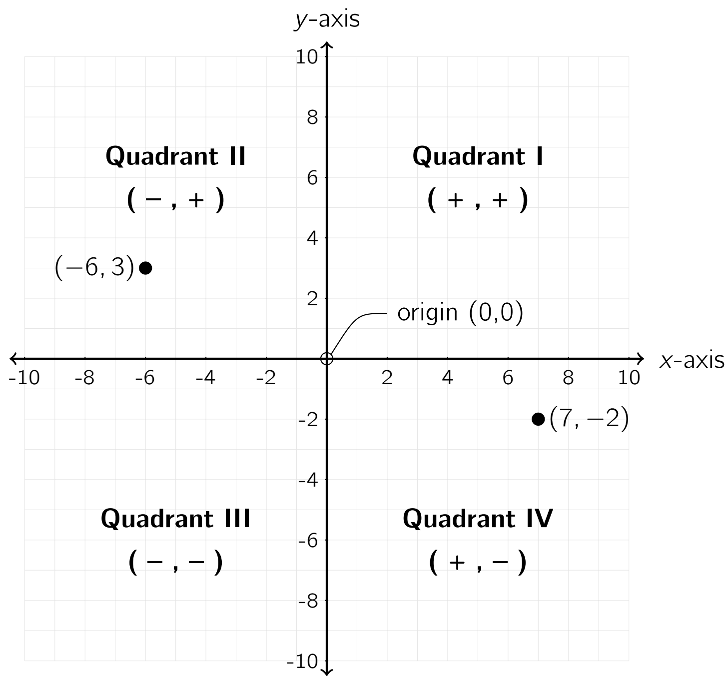


Figure 3.1: The coordinate plane and its important landmarks.

Startup Exploration: Pizza Intersections

Bob's friend Melvin "Hambone" Jones once worked delivering pizzas in his hometown of Euclid, Ohio. The town has streets running north-south and east-west.[a]

Hambone is currently parked at the intersection we will call $(0, 0)$. If he drives one block east, he will arrive at the intersection $(1, 0)$. If he then turns right and drives one block south, he will arrive at the intersection $(1, -1)$.

Starting from $(0, 0)$, describe are all the intersections that Hambone can reach by driving a total distance of exactly 10 blocks.

[a] Euclid, Ohio is also the home of the Polka Hall of Fame, though that has nothing to do with this problem. Also, Euclid isn't laid out in a grid as this problem implies, though it should be, given its name.

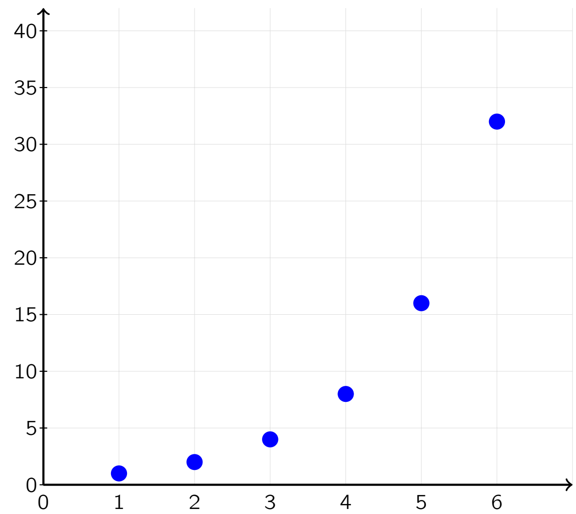
3.1.2 Graphing a Sequence

Our goal is to make a visual representation of a sequence on this coordinate plane.

"But," you may be asking yourself, "a sequence is just a list of numbers. How do we make a coordinate graph, which needs coordinate pairs of numbers? It takes *two numbers* to make a *pair*!"

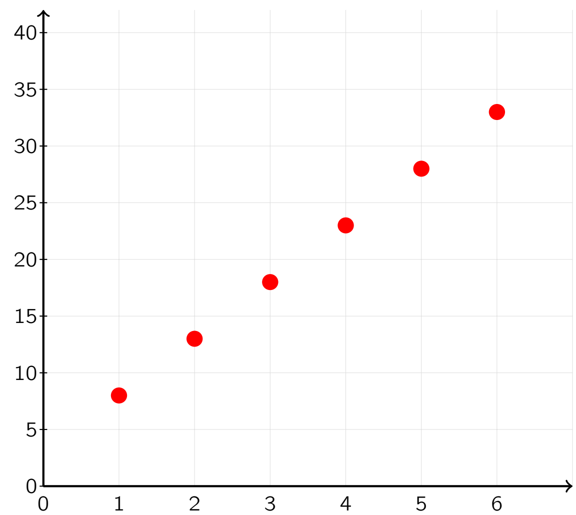
To graph a sequence, we use the term number as the x-coordinate and the term's value as the y-coordinate. Consider the sequence $1, 2, 4, 8, 16, 32, \ldots$. The first term is 1, the second term is 2, and so on. We can organize this in a table, and write out the ordered pairs. Then, we can plot those ordered pairs and see a visual representation of our sequence!

Term No. x	Term Value y	Coord. Pair (x, y)
1	1	$(1, 1)$
2	2	$(2, 2)$
3	4	$(3, 4)$
4	8	$(4, 8)$
5	16	$(5, 16)$
6	32	$(6, 32)$



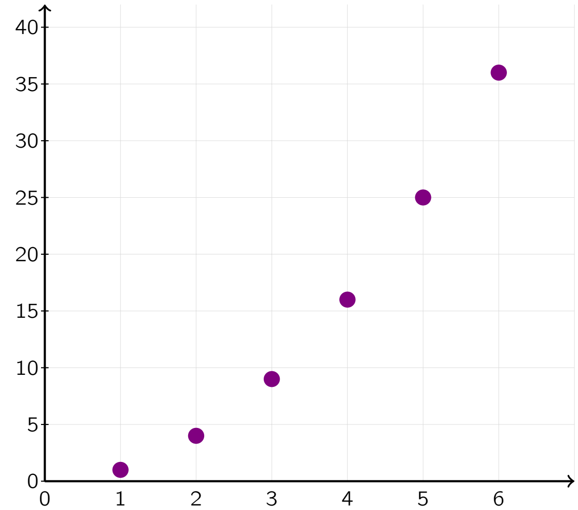
Recall that that sequence above, where the terms have a constant ratio, is called a geometric sequence. Let's look at an example of an arithmetic sequence, in which the terms have a constant difference. For example, consider the sequence $8, 13, 18, 23, 28, 33, \ldots$. This sequence produces the following table and graph:

Term No. x	Term Value y	Coord. Pair (x, y)
1	8	$(1, 8)$
2	13	$(2, 13)$
3	18	$(3, 18)$
4	23	$(4, 23)$
5	28	$(5, 28)$
6	33	$(6, 33)$



Finally, let's see what a quadratic pattern looks like. For example, the perfect squares form a quadratic sequence $1, 4, 9, 16, 25, 36, \ldots$. Their table and graph go like this:

Term No. x	Term Value y	Coord. Pair (x, y)
1	1	$(1, 1)$
2	4	$(2, 4)$
3	9	$(3, 9)$
4	16	$(4, 16)$
5	25	$(5, 25)$
6	36	$(6, 36)$



Take a moment to compare the graphs of these three sequences. How are they alike? How are they different?

3.1.3 Features of the Graph of a Sequence

Note that since the term number is always greater than 0, our graphs only show the positive part of the x-axis. We'll soon see that this is an artificial limitation: the negative part of the x-axis is just as important as the positive part.

Note also that we haven't connected the dots: a sequence has a first term and a second term, but no one-and-a-halfth term. We'll soon see that this is an artificial limitation, too. The fractional values in between the points we've plotted are also important.

In fact, when we generate a sequence we use only the natural numbers as the input values to the formula. However, if we change our point of view and allow the input to be any real number, we will have turned our sequence into a mathematical relationship called a **function**.[1] We'll discuss the details of what functions are (and what they are not) in chapter 4.

[1] Technically, a sequence is *already* a function. The distinction we make here has to do with what kinds of input values are allowed. Later, once we have some additional concepts under our belts, we'll talk about a sequence as "a function whose domain is the natural numbers".

3.2 Algebraic Expressions

Startup Exploration: Don't Take All Year

Find three natural numbers x, y, and z which satisfy the equation $28x + 30y + 31z = 365$. Can you find more than one set of numbers x, y, z that satisfy the equation?

In chapter 1 we used the order of operations to simplify numeric expressions, which are made up of numbers and arithmetic operators. For example,

$$3 \cdot 4 - 8(4^2 - 1)$$

is a numeric expression. It contains only numbers and operators and, in the end, it simplifies down to a single number.[2] An **algebraic expression** on the other hand can contain letters in addition to numbers and operators, for example

$$3x - 5y + 18.$$

These letters, called *variables*, stand in for numbers that we don't know or which may change.

Variable

A **variable** whose value can change. In algebra, variables are often represented by letters. We usually use letters from the Latin alphabet $(a, b, c, d, \ldots)$, but sometimes also use other symbols, such as letters from the Greek alphabet $(\alpha, \beta, \gamma, \delta, \ldots)$.

Algebraic Expression

A symbolic representation of mathematical operations that can involve both numbers and variables.

3.2.1 Numbers and Variables

In your mathematical career so far, you have probably worked with letters that stand in for numbers. Recall the formula for the circumference of a circle:

$$C = \pi d.$$

[2] Spoiler alert: It's -108.

This formula explains the relationship between d, which stands in for the diameter of some circle, and C, which stands for the circumference of that circle. The letters d and C are variables. They stand in for numbers that can change, depending on which circle we're talking about.[3]

Once we start to introduce letters into our expressions, we have to discuss some standard notation and terminology. As we have seen, we don't usually write any multiplication symbol when multiplying a number times a variable. Rather than writing $3 \cdot x$ or $3(x)$, we can write $3x$ without anything in between.

An algebraic expression that is built using only multiplication (or division) is called a **term**. For example, $3x$ and $\frac{1}{2}m$ are terms. On the other hand, the expression $3x + 2y$ is not a term because it includes addition. In fact, this expression is the sum of two terms.

Term

An algebraic expression that represents only multiplication and division between variables and constants.

When we have the product of a number and a variable — like $3x$ or $^{-}11g$ — the number part is called the **coefficient** of the term. So, the coefficient of $3x$ is 3, and $^{-}11$ is the coefficient of $^{-}11x$. If we have a variable all alone without a number attached — like y or w — then we picture a "phantom 1" lurking there as the coefficient: y is the same as $1y$ and $1w$ is the same as w.

Coefficient

The numerical factor in a term with a variable. If no number is explicitly written, the coefficient is understood to be 1.

3.2.2 Evaluating Algebraic Expressions

A variable is a "placeholder" that stands in for a number. We can only determine the value of an algebraic expression if we know what numbers the different variables represent.

Consider the expression $3x$. If we know that x represents 15, then we can **evaluate** the expression $3x$ in the case that $x = 15$. In that case, it must be that $3x$ represents $3(15) = 45$.

When we evaluate an algebraic expression, we substitute in values for its variables, and then simplify the resulting numeric expression using the order of operations. It is a really good habit always to use parentheses when substituting numeric values for variables. This can avoid confusion about negative numbers!

[3] Note that π is *not* a variable. We use a (Greek) letter in this case not because the value of π might change, but because it's an irrational number that is impossible to write out in full. We often use letters to stand in for mathematical objects that are inconvenient, sometimes impossible, to write down in another way: e, i, ϕ, and $\aleph_0$ each has special mathematical meaning.

Example 3.1

Evaluate the expressions $6x+4$ and x^2-5 for the x values 3, $^-1$, and $\frac{1}{2}$.

Solution:

(a) To evaluate the expression $6x+4$ for the given values of x, we simply substitute and follow the order of operations.

When $x = 3$:
$$\begin{aligned} 6x+4 &= 6(3)+4 \\ &= 18+4 \\ &= 22 \end{aligned}$$

When $x = {}^-1$:
$$\begin{aligned} 6x+4 &= 6({}^-1)+4 \\ &= {}^-6+4 \\ &= {}^-2 \end{aligned}$$

When $x = \frac{1}{2}$:
$$\begin{aligned} 6x+4 &= 6\left(\tfrac{1}{2}\right)+4 \\ &= 3+4 \\ &= 7 \end{aligned}$$

(b) We do the same in order to evaluate x^2-5 for the given x values.

When $x = 3$:
$$\begin{aligned} x^2-5 &= (3)^2-5 \\ &= 9-5 \\ &= 4 \end{aligned}$$

When $x = {}^-1$:
$$\begin{aligned} x^2-5 &= ({}^-1)^2-5 \\ &= 1-5 \\ &= {}^-4 \end{aligned}$$

When $x = \frac{1}{2}$:
$$\begin{aligned} x^2-5 &= \left(\tfrac{1}{2}\right)^2-5 \\ &= \tfrac{1}{4}-5 \\ &= -\tfrac{19}{4} \end{aligned}$$

Note how the parentheses help out when $x = {}^-1$! Without those parentheses, we would have run the risk of making the most common mistake in algebra 1: remember the difference between $({}^-1)^2$ and ${}^-1^2$.

3.3 Graphing a Function

The graphs of sequences that we created earlier were quite limited. Since sequences use only the natural numbers as input values, the only points we had available to plot were the points where $x = 1, 2, 3, 4, \ldots$. But now, knowing how to evaluate algebraic expressions, we can create more complete graphs by choosing a wider range of x values.

Startup Exploration: Extending Our Sequences

Write a zero-based explicit rule for the arithmetic sequence shown below (this is the second example from section 3.1). First write the rule in terms of n and a_n, then translate your rules into a graphable format in terms of x and y.

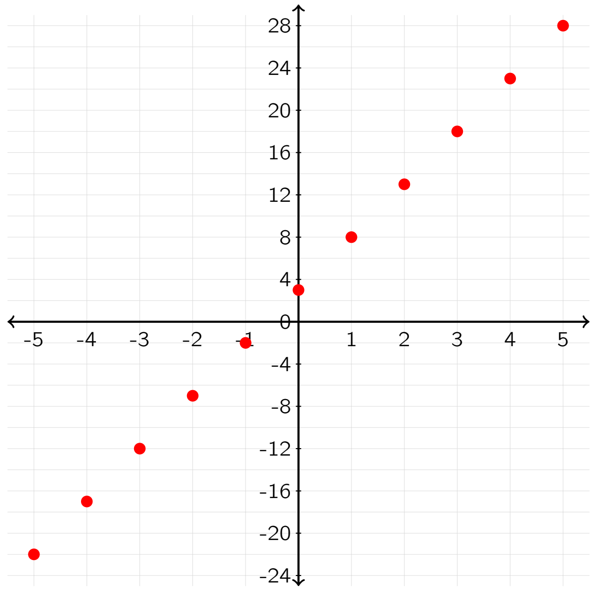
$$8, 13, 18, 23, 28, \ldots$$

The given sequence represents the y-values of the rule for the x-values 1, 2, 3, 4, and 5. Evaluate your rule for the x-values 0, -1, -2, -3, -4, and -5. Then, plots these 11 points on a coordinate grid.

The rule for the sequence in the startup exploration is $a_n = 5n + 3$, or in terms of x and y, we have the rule $y = 5x + 3$. To create the coordinate graph, we can substitute the different x-values into the rule and compute the y-values. The middle column in the table below is our "process column" in which we substitute an x-value and compute the corresponding y-value.

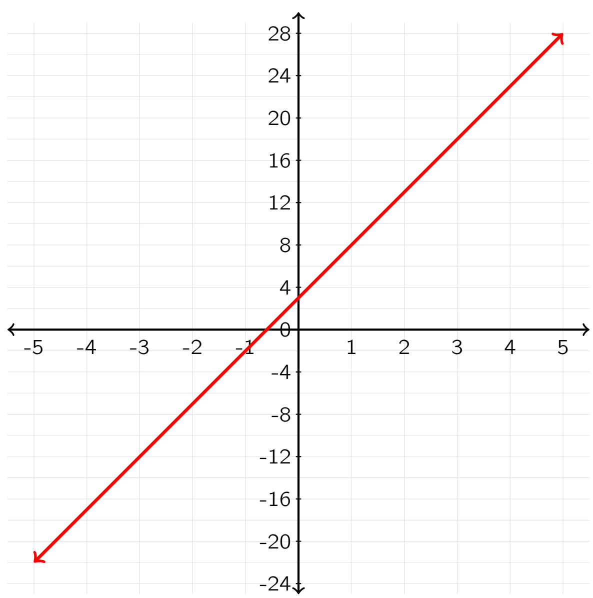
x	$y = 5x + 3$	(x, y)
0	$y = 5(0) + 3 = 0 + 3 = 3$	$(0, 3)$
-1	$y = 5(-1) + 3 = -5 + 3 = -2$	$(-1, -2)$
-2	$y = 5(-2) + 3 = -10 + 3 = -7$	$(-2, -7)$
-3	$y = 5(-3) + 3 = -15 + 3 = -12$	$(-3, -12)$
-4	$y = 5(-4) + 3 = -20 + 3 = -17$	$(-4, -17)$
-5	$y = 5(-5) + 3 = -25 + 3 = -22$	$(-5, -22)$

In the end, we generate 6 new coordinate pairs, which we can graph alongside the five points that we were given.

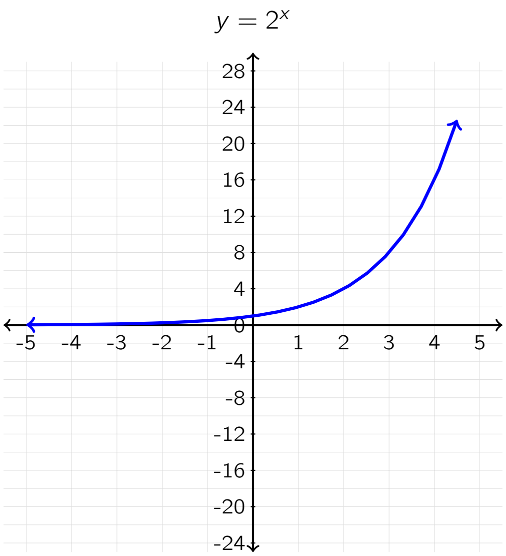


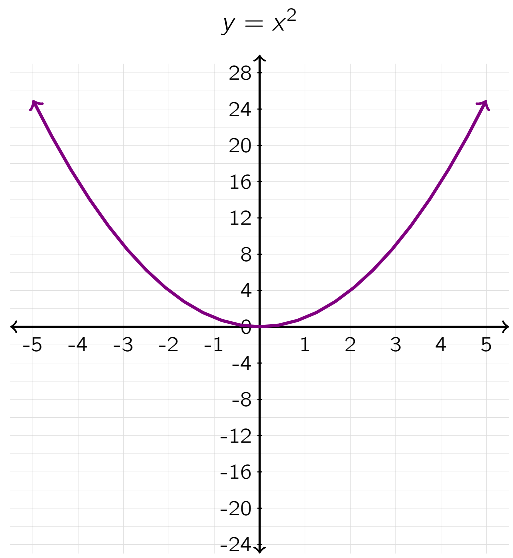
Can you anticipate the location of the point that we plot when $x = \frac{1}{3}$? What about when $x = -\frac{5}{2}$? What would our graph look like if we used all of the points in $\mathbb{R}$ as the x-values?

When we use all of the real numbers as input to the rule $y = 5x + 3$, the resulting graph is a straight line. Of course, it would be impossible to actually plot *all* of the points (there are infinitely many of them), but the pattern holds true, and so we can replace the dots with a continuous line.



The rules below correspond to the other two sequences we studied in section 3.1. Under each rule is the graph that is created when we plot the rule over $\mathbb{R}$.





Are you surprised by these two graphs? Back in section 3.1, the graphs of these two sequence looked similar. But, we were only looking at the first quadrant! Their graphs are very different for negative values of x.

The moral of the story is that when we are asked to graph an equation by hand, we need to use a variety of different x-values, including negative numbers and fractions. Often, a problem will clearly indicate exactly what values to use.

Extended Exploration: Big Graphs

[TODO] Click here to visit the extended exploration: Big Graphs

[TODO] Criteria for high quality graphs.

3.4 Patterns in Data

The graphs that we have been working with so far have been very orderly. Technical and scientific data, however, are not always so tidy. Data can be noisy, messy, and incomplete. We will need some tools that can help us to see and describe patterns that may (or may not) exist in experimental data.

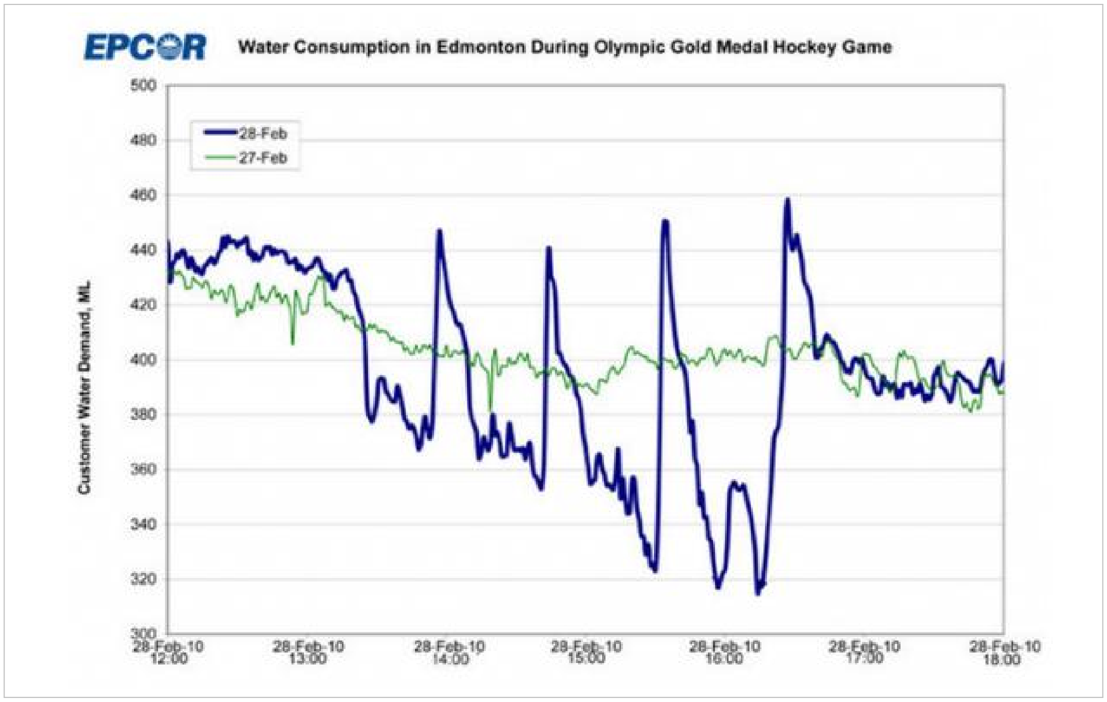


Figure 3.2: Water consumption in Edmonton during the 2010 Olympic men's ice hockey final

Startup Exploration: Water Consumption

The graph shown in fig. 3.2 depicts water consumption in Edmonton, capital of the Canadian province of Alberta, during the gold medal men's ice hockey game at the 2010 Winter Olympics in Vancouver. The game was played between Canada and United States.

Water consumption during the game is shown in blue, while data from the same time period on the previous day is shown in green.

Write down anything you notice or wonder about the data presented in this graph.

A few notes: Ice hockey games are played in three 20-minute periods with breaks in between. In this particular game, the score was tied at the end of regular play. Canada scored the game winning goal in overtime and was awarded the gold medal.

Extended Exploration: Who Is the Best Age Guesser?

[TODO] Click here to visit the extended exploration: Who Is the Best Age Guesser?

In the graph of Edmonton water consumption, the amount of water being used varies depending on the time of day (not the other way around). We say that "time of day" is the *independent variable* and "water demand" is the *dependent variable*.

Independent Variable

A variable whose values affect the values of another variable. In a graph of the relationship between two variables, the quantity represented on the horizontal axis (the x-axis) usually represents the independent variable.

Dependent Variable

A variable whose values depend on the values of another variable. In a graph of the relationship between two variables, the quantity represented on the vertical axis (the y-axis) usually represents the dependent variable.

3.4.1 Correlation

We can compare just about any two quantities. One way to do this is with a graph called a **scatter plot**.

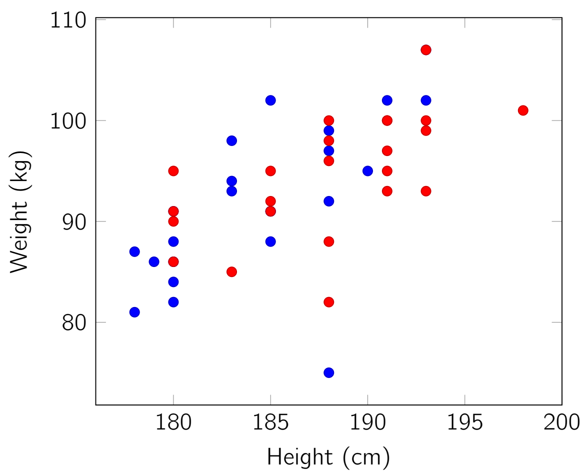
Scatter Plot

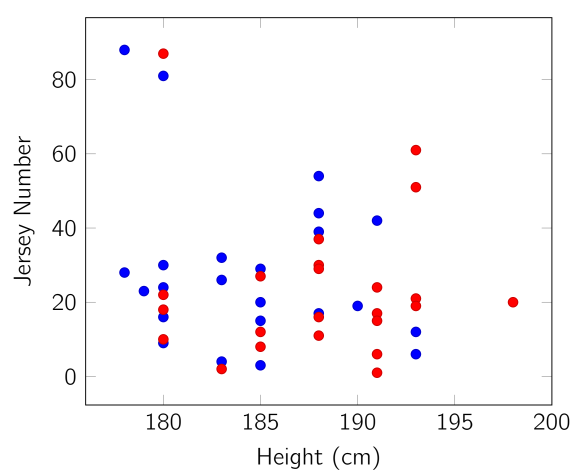
A graph that relates data of two different sets. The two sets of data are displayed as ordered pairs.

Suppose we wish to compare, say, the height and weight for all of the players on the top two 2010 Olympic men's ice hockey teams. Let's make a graph comparing every player's height and weight as ordered pairs: (height, weight). This graph is shown on the left below. Blue dots represent players on the United States team, red dots represent players on team Canada.[4]

While we're at it, let's make another comparison. The graph on the right plots every player's height and jersey number as ordered pairs (height, jersey number). What do you notice about these two graphs? How are they the same? How are they different?

[4] Data from the International Olympic Committee, as reported in Wikipedia.





Notice that the graph of weight and height has a clear upward slant. This seems reasonable: the taller someone is, the more we might expect that person to weigh. There are some data points that don't fit the trend, but generally speaking the data points are increasing as we look toward the right on the graph. We say that this data shows a positive **correlation**.

Correlation

A trend between two sets of data, as seen in a scatter plot. A trend can show positive, negative, or no correlation. Positive correlation shows an **increasing** trend in data. Negative correlation shows a **decreasing** trend in data.

The graph of jersey number versus height is more of a blob. There's no clear trend in this data, and so we say that is shows *no correlation*. This makes sense, too: there's no logical connection between a player's height and the number they wear on their shirt.

A third possibility would be data that shows a negative correlation, meaning that the data are decreasing as we look towards the right on the graph. Can you imagine two variables that might show a negative correlation when compared on a scatter plot?

Graphing experimental data on a scatter plot helps us to see if there is a relationship between variables. If there is, a pattern will emerge in the graph. The points will fall (approximately) in a line or a curve and will have a correlation.

If the scatter plot shows a positive correction, it means that as the independent variable increases, the dependent variable increases. If the scatter plot shows a negative correlation, it means that as the independent variable increases, the dependent variable decreases. If a scatter plot shows no correlation, it indicates that there is no relationship between the two variables.

A key thing to remember when it comes to looking at data is that "correlation does not imply causation". In other words: If we see that two variables are correlated, we might be tempted to assume that the change in one variable *causes* the change in the other. This is sometimes true, but not always.

For example, it seems reasonable to believe that a change in height will cause a change in weight. But, there is data that shows a positive correlation between "consumption of mozzarella cheese per person" and "number of civil engineering doctorates awarded". This has to be a coincidence! There's no (good) reason to think that changing one of these variables would cause a change in the other one.[5]

3.4.2 Continuous and Discrete Data

When we drew the graph of a sequence, we didn't connect the dots. A sequence has a first term and a second term, but no one-and-a-halfth term. The x-values have no "in-betweens".

Similarly, imagine a graph showing "time" as the independent variable and "number of hockey players on the ice" as the dependent variable. In this case, the y-values would have no "in-betweens". There could be 11 players or 12 players on the ice, but never 11.5 players.

Discrete Data

Data for which it doesn't make sense for measurements to exist between given data points. Discrete data often involves *counting items*, such at the number of cars in a parking lot over time.

On the other hand, when we started to picture the graph of a rule that could accept any real number as input, we drew a continuous line on the graph. The graph of water consumption in Edmonton, is jagged, spiky, and irregular — but it's a continuous line. We can measure how much water has been used at any point in time, and we can measure the amount of water in fractions of a unit.

Continuous Data

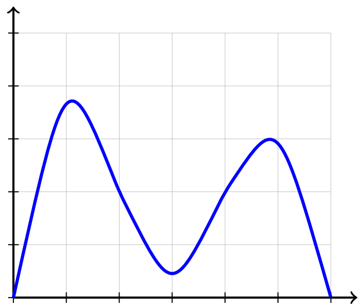
Data that has no holes, gaps, or breaks. Continuous data often involves *measuring some value* where measurements exist (and may change) between data points. For example, a person's height over time.

The key to knowing the difference between continuous and discrete data is to ask whether the data involves measuring or counting. Consider a data collection scenario in which we want to graph "Bob's distance from home at any given time (in kilometers)". Bob is always a certain distance away from home (perhaps 0 km, if he is at home), and he could be any distance (even fractions of a kilometer). So, this is continuous data and our graph should be an unbroken line.

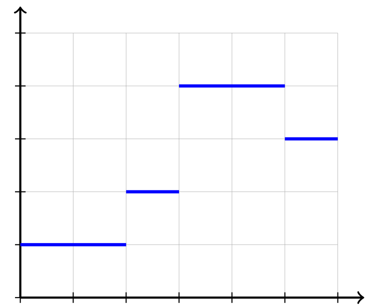
Now consider the scenario in which we graph "number of customers in line at the cheese counter at Middle Market." Although there is always a certain number of people in line (maybe zero), there can never be 5.7

[5] This fact is courtesy of the website Spurious Correlations, which has many graphs of interesting and ridiculous data that show correlation but not causation.

people in line. We must count people, and so this data is discrete. Our graph would have to “jump” from 5 people to 6 people without going through the in-between values.



Distance from home
(continuous data)



Number of customers
(discrete data)

3.5 Interpreting graphs

In this section, our goal is to hone our skills at understanding what a graph is *communicating*.

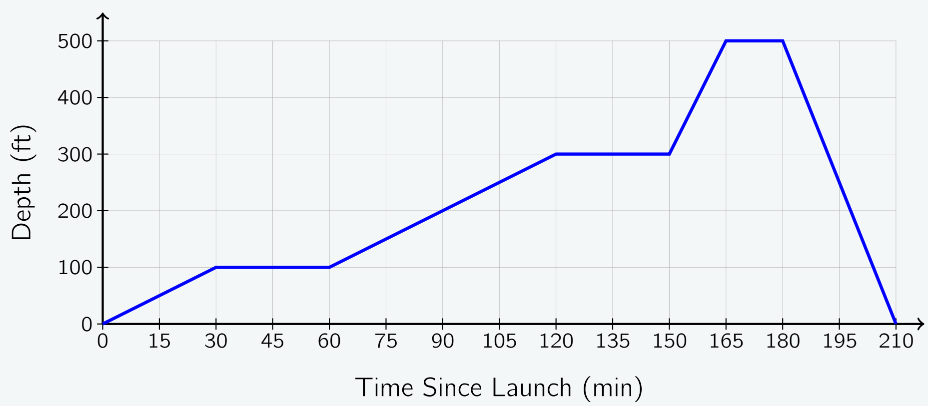
Extended Exploration: Interpreting Graphs with Distance Match

[TODO] Click here to visit the extended exploration: Interpreting Graphs with Distance Match

Startup Exploration: Yeardleigh's Submersible

Always questing after the most delicious ingredients, Yeardleigh buys an underwater submersible vehicle so that she can hunt the ocean floor for interesting sea plants. The graph below shows the depth of Yeardleigh's submersible over time.

Study the graph. What can you tell about what's happening? Write a short paragraph telling, in words, the same story that the graph is telling visually.



At the beginning of the trip, Yeardleigh's submersible dives a total of 100 feet in the first 30 minutes. At the end of the trip, it returns to the surface, rising 500 feet in the last 30 minutes. This tells us that the depth of the submersible was *changing much faster* at the end of the journey compared to the beginning.

Note that the graph reflects this: the line is quite steep at the end of the trip and not so steep at the beginning. The steeper the line, the faster the dependent variable is changing with respect to the independent variable.[6]

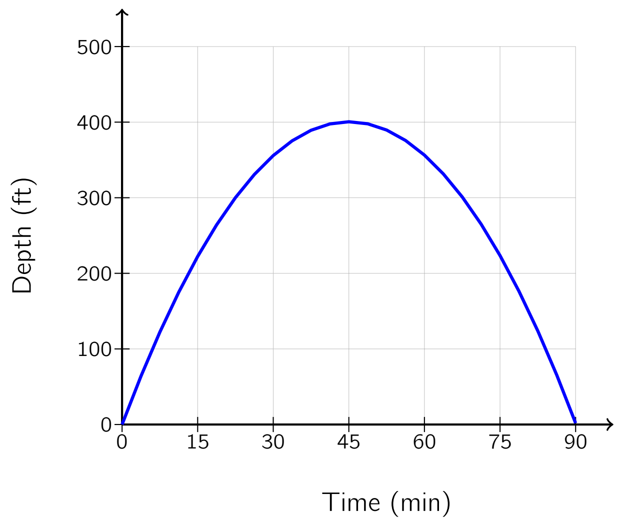
Plus, we can tell from the graph when the submersible is getting deeper (the depth is increasing; the line shows a positive trend) and when it is getting shallower (the depth is decreasing over time; the line shows a negative trend).

[6] We call this the "rate of change", and it will become an important focus of our work in chapter 7.

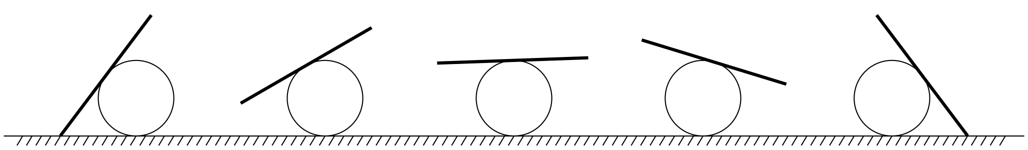
Notice that other parts of the graph are horizontal, for example between 30 and 60 minutes. This tells us that the depth of the submersible stayed the same during that time. Of course Yeardleigh could still be moving around below the surface, but she remains at a constant depth, neither diving nor surfacing.

3.5.1 Interpreting Curves

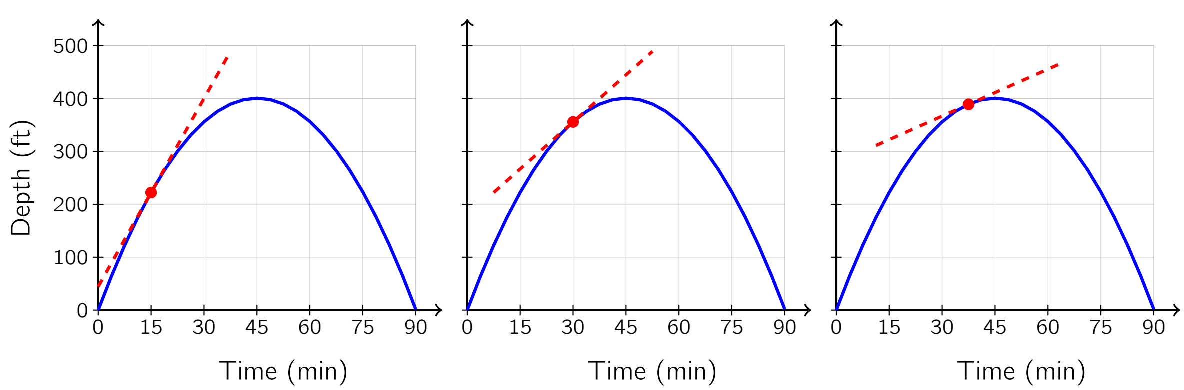
Straight lines are fine, but what if the graph shows curved lines? Consider this graph showing the submersible's depth over time. What story would we tell about this graph?



One way to get a feel for what's happening is to imagine leaning a ruler or pencil against the rounded edge of a soda can. Then picture the ruler rolling along the side of the can, and how the angle of the ruler will change as it rolls.



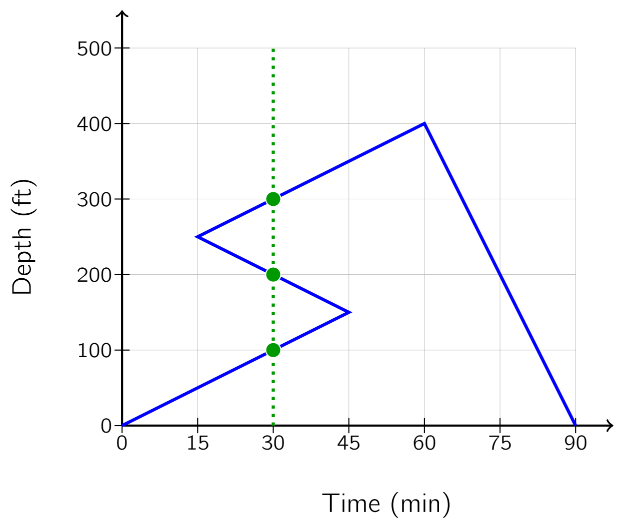
Now picture a straight line rolling along the surface of the curved line in the graph above. The straight line approximates the curved line at the point where they touch.



Using the straight line, we can see that the submersible starts out diving at a fairly high rate. It gradually slows its rate of descent until it eventually stops diving. Then it gradually accelerates as it returns to the surface.[7]

3.5.2 Impossible Interpretations

Consider this graph showing the submersible's depth over time. What's going on here?



This graph is a problem, given the context of "depth of the submersible over time". Consider this question: How deep it the submersible 30 minutes after launch?

According to the graph, the submersible is 100 feet deep... and 200 feet deep... *and* 300 feet deep... all at the same time! That's impossible!

This kind of problem could pop up in other places. For example, in a distance-time graph, points that line up vertically mean that something is in more than one place at one instant in time. Though we might wish reality were different, nothing can be in two (or more) places at the same time.

Graphs like this — where several y-values stack up vertically over the same x-value — violate a certain requirement that we will learn about in the next chapter, as we delve into the important mathematical idea of a *function*.

[7] Believe it or not, this idea — approximating a curved line with a straight line — is one of the fundamental motivating ideas in calculus. As you work to interpret these curved graphs, you're growing your calculus brain, right here in algebra 1. How cool is that?

Chapter 4

Fundamentals of Functions

4.1 A Modest Proposal

I'm having a hard getting this chapter to flow. I've got over it and over it, and I just haven't been able to take the version 1 material and get it to work... whereas the other parts of the book flow so well.

I've been thinking about our talk the other week about the course coming in waves, and how this chapter is kind of the crest of the wave that began in chapter 1, and that after this point we start the new linear wave.

I've thought long and hard and I have a modest proposal.

I say we cut this chapter, redistribute the concepts to other parts of the book, and replace it with a "capstone section", consisting only of the "exploring the three families" activity.

In the capstone section, we use the activity to preview transformations, and talk about things informally. But, we don't get into any kind of $f(x) + c$ detail until we get into each of the individual families. For example, the transformations talk would first appear after we've done direct variation, and gotten into the linear forms. We focus at that time on transformations only of linear functions. Transformations of exponentials happens in the exponential unit.

Note that "big graphs" is still there in the previous chapter, and so we'd still have informal language available about shifting up/down, steeper/shallower, wider/narrower (between big graphs and exploring the families).

We save domain and range discussions until we need them. Linear functions have nothing interesting to say about domain/range, so save it. When have some more experience about how exponential equations work, the infinite progression of fractions leading to an asymptote, and how that means that there's a distinction between output values of linear and exponential functions — then we discuss domain and range.

Finally, all the "is it a function" stuff feels like a kind of distraction. Why is the concept so important at this very moment? I like the idea of calling things by their proper names, and if we're going to study "functions" we should define that term. But, I don't think we need all the detail.

So, here's what I picture: A short introductory bit that defines function, gives some examples and nonexamples, but basically says: We're going to study functions in algebra 1. There are things that aren't functions, but we'll learn about those things later.

I say we introduce function notation, since for the most part it's just a renaming. Plus, it gives a handy notation for evaluation.

Then, I say we do the activity "exploring the families" (with the sorting of equations, graphs, and informal discussions of transformations) and explain the key features of each family (table, equation graph).

And then we start the linear unit.

Thoughts?

Chapter 5

Solving Equations

Mathematics is the art of giving the same name to different things.

– Henri Poincaré, French mathematician

5.1 Equivalence

Equivalence is an important concept in algebra, and indeed throughout mathematics. The kernel of the idea is that we can manipulate mathematical objects in ways that change their appearance without changing their value.

Startup Exploration: Never the Tween Shall Meet?

Bob thinks: "Since $\frac{3}{7}$ and $\frac{4}{7}$ are 'right next to each other' on the number line, there must be no fractions in between them." Use what you know about equivalent fractions to find a fraction in between $\frac{3}{7}$ and $\frac{4}{7}$.

Given any two fractions $\frac{a}{b}$ and $\frac{c}{d}$, describe a procedure for finding a fraction that lies between them on the number line. Does your procedure work for the fractions $\frac{3}{5}$ and $\frac{5}{8}$.

Name of Extended Exploration

[TODO] Click here to visit the extended exploration: NAME

We've studied equivalence before and know, for instance, that multiplication by 1 is an operation which maintains equivalence. When we multiply some quantity by 1 — even a very fancy version of 1 — the quantity remains unchanged. This is the idea behind equivalent fractions:

$$\frac{3}{7} = \frac{3}{7} \cdot 1 = \frac{3}{7} \cdot \frac{18}{18} = \frac{54}{126}$$

We also know the order of operations, which gives us a process for simplifying numerical expressions. The order of operations guarantees we always have exactly the same quantity we started with, even though it may look different.

As we progress through this chapter (and beyond), we will learn more tools that we can use to simplify algebraic expressions and solve equations. We will have to be careful to use the tools correctly, though, so that the maneuvers we perform are sure to maintain equivalence. The devil is in the details, so it is a good habit to pay attention and proceed carefully.

The first key piece of the equivalence puzzle is something that probably seems obvious:

Substitution

We may replace one quantity with another quantity that we know has the same value.

This is a property about numbers that we use all the time, for example when we replace 4+3 with 7. We do a series of substitutions when we solve a problem using the order of operations. Given $2 + 3 \cdot 6$ we first substitute 18 for $3 \cdot 6$, giving $2 + 18$. Then we substitute 20 for $2 + 18$.

5.1.1 The Field Axioms

An **axiom** is a statement that is accepted as true without proof. The real numbers $\mathbb{R}$ are built upon several axioms, called the **field axioms**, which are the properties and laws that make arithmetic and algebra work.[1]

[1] A *field* is a mathematical object that pairs up a set of numbers with some operations on those numbers. The number system we know so well — the real numbers $\mathbb{R}$, along with the operations of addition and multiplication — are a field. But many other fields exist! If you continue to study algebra in college, the focus gradually shifts from studying familiar fields like the real numbers, to studying the properties of fields in general.

Axiom: Closure Properties

When we add two real numbers, their sum is a real number. Mathematically speaking, we say that *the real numbers are closed under the operation of addition*.

Similarly, when we multiply two real numbers, their product is a real number. We say that *the real numbers are closed under the operation of multiplication*.

This might seem obvious, but we saw in Chapter 1 that closure is not guaranteed for all sets and all operations. For example, the integers are not closed under the operation of division.

Axiom: Identity Properties

The **identity property of addition** states that for any real number a,

$$0 + a = a + 0 = a$$

The number 0 is called the **additive identity**.

The **identity property of multiplication** states that for any real number a,

$$1 \cdot a = a \cdot 1 = a$$

The number 1 is called the **multiplicative identity**.

As we saw above, the identity property of multiplication is the axiom that allows us to create equivalent fractions.

Axiom: Inverse Properties

The **inverse property of addition** states that for any real number a, there exists a real number ^-a such that

$$a + {}^-a = {}^-a + a = 0$$

The number ^-a is called the **additive inverse** of a. Very often we will call it the **opposite** of a.

The **inverse property of multiplication** states that for any nonzero real number a, there exists a real number $\frac{1}{a}$ such that

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$$

The number $\frac{1}{a}$ is called the **multiplicative inverse** of a. Very often we will call it the **reciprocal** of a.

Note that when discussing the multiplicative inverse of a, this axiom states that a has to be nonzero. In ordinary arithmetic, division by zero is undefined, and expressions such as $5 \div 0$ and $\frac{5}{0}$ have no meaning.

Tangent: Explaining Division By Zero

Consider the following two equations. How are they related?

$$72 \div 8 = \boxed{} \qquad \text{and} \qquad \boxed{} \cdot 8 = 72$$

Now consider these two equations. How are they related?

$$5 \div 0 = \boxed{} \qquad \text{and} \qquad \boxed{} \cdot 0 = 5$$

The first pair of equations demonstrates the relationship between multiplication and division, which tells us that the same number goes in both boxes. Whatever number makes the first equation true, must also make the second equation true.[a]

This second pair of equations is related in the same way. But, there's a problem with the multiplication sentence $\boxed{} \cdot 0 = 5$. *Anything times zero is zero.* So, no number exists that could make that multiplication sentence true! Therefore, there can be no answer in the related division sentence.

At this point, troublemakers always like to ask about $0 \div 0 = \boxed{}$ because that sentence, they say, is related to the multiplication sentence $\boxed{} \cdot 0 = 0$, and *every number in the universe* makes that sentence true. They're right, of course. We say that $a \div 0$ is "undefined" when $a \neq 0$, whereas $0 \div 0$ is called an "indeterminate form". It doesn't really matter as far as we're concerned, though. Dividing *anything* by zero means that we don't get a clear answer. It's mathematically off-limits.

[a] Spoiler alert: it's 9.

Axiom: Commutative Properties

The **commutative property of addition** states that for real numbers a and b:

$$a + b = b + a$$

The **commutative property of multiplication** states that for real numbers a and b:

$$a \cdot b = b \cdot a$$

The commutative properties allow us to rearrange the order of things when we add or multiply. For instance, when adding up a string of numbers, it's sometimes helpful to "make a ten":

$$\begin{aligned} & 12 + 6 + 8 && \text{grouping up the 12 and the 8 would be easier than adding left-to-right} \\ = {} & 12 + 8 + 6 && \text{commutative property of addition says } 6 + 8 = 8 + 6\text{, so we can swap them} \\ = {} & 20 + 6 \\ = {} & 26 \end{aligned}$$

Note that this property is for *addition and multiplication*, but not subtraction or division! To move things around in, say, a subtraction problem, we must use the definition of subtraction as "addition of the opposite" to create an equivalent addition problem.

$$6 - 5 \neq 5 - 6 \qquad \text{but} \qquad 6 + (^-5) = (^-5) + 6$$

Axiom: Associative Properties

The **associative property of addition** states that for real numbers a, b, and c:

$$(a + b) + c = a + (b + c)$$

The **associative property of multiplication** states that for real numbers a, b, and c:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

The associative properties allow us to move around certain parentheses and regroup addition and multiplication. Here, the numbers don't move; the parentheses do.

$$\begin{aligned} & 5 \cdot (8 \cdot 7) && \text{order of operations would force us to do what's inside the parentheses first} \\ &= (5 \cdot 8) \cdot 7 && \text{associative property of multiplication says we can move the parentheses} \\ &= 40 \cdot 7 \\ &= 280 \end{aligned}$$

A few words of warning: We can only use this property around addition or multiplication. Subtraction, for example, would have to be transformed into addition of the opposite.

$$(8 - 4) + 3 \neq 8 - (4 + 3) \qquad \text{but} \qquad (8 + {^-4}) + 3 = 8 + (^-4 + 3)$$

Also, we can't apply the associative property when there are different operations involved:

$$6 \cdot (5 + 4) \neq (6 \cdot 5) + 4$$

Here's a sneaky one, can you explain why these two equations are not equivalent?

$$5 + 6(3 + 4) \neq (5 + 6)3 + 4$$

To handle situations like these last two, we need the final field axiom:

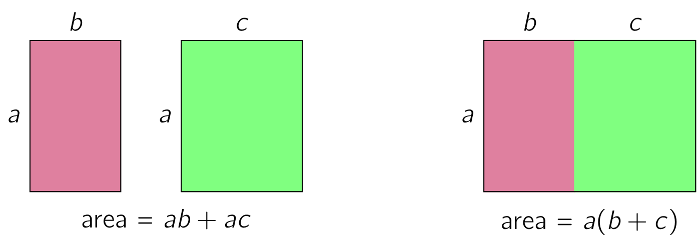
Axiom: Distributive Property

The **distributive property** states that for real numbers a, b, and c:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

This property explains how multiplication and addition are related to each other. To get a feel for this, suppose we have two rectangles: one that is a units tall and b units wide, and a second rectangle that is a units tall and c units wide. Together, their combined area is $a \cdot b + a \cdot c$.

But, since the two rectangles have the same height, we could glue them together perfectly and create one conjoined rectangle with height a and width $(b + c)$. The area of this rectangle is $a \cdot (b + c)$. But of course, the area hasn't changed, so these two different ways of expressing the area are equal.



Another way to think about the distributive property is to think of taking the a (in this case) and sprinkling it over the parentheses so that it becomes a part of each of the terms inside.

$$a \cdot c$$
$$a \cdot b$$
$$a\,(\,b + c\,) = ab + ac$$

Both the rectangle-area metaphor and the sprinkling metaphor can be extended to include expressions like

$$a(b + c + d) = ab + ac + ad,$$

and to expressions with any number of terms inside the parentheses.

5.2 Simplified Algebraic Expressions

The axioms are a set of tools that we can use to manipulate algebraic objects such that the result is equivalent to the original. That's great... but why would we ever need to do that?

Clarity and simplicity are important when it comes to communicating mathematical ideas. The fraction $\frac{16}{24}$ is fine, but the simplified fraction $\frac{2}{3}$ communicates the same information more clearly. For example, it's much easier to visualize a circle with $\frac{2}{3}$ shaded than it is to visualize a circle with $\frac{16}{24}$ shaded.

If a movie starts at 8 o'clock, we would all prefer to see that simple number rather than have to compute $(4+5)\cdot 2-10$ o'clock (yes, even algebra teachers feel this way). A simpler form will almost always be easier to understand.

So, as we did with rational numbers and arithmetic expressions, we will define what it means for an algebraic expression to be "simplified". Recall that an *algebraic expression* could be a single term, or the sum of difference of terms. A *term* could be a number, or a variable, or the product/quotient of numbers and variables.

Criteria for Simplified Algebraic Expressions

An algebraic expression is considered completely simplified if...

1. Numerical expressions have been evaluated
2. Redundant negative signs have been rewritten
3. Terms have been arranged in order of decreasing degree, with coefficients written first
4. It contains no explicit grouping symbols
5. Different variable terms appear at most once

Several of these criteria are old news. Criteria #1 just means that we have to simplify numerical expressions using the order of operations: the expression $2\cdot 3\cdot x$ is not simplified until we evaluate "$2\cdot 3$" and write "6" instead. The equivalent expression $6\cdot x$ or $6x$ is simplified.

Criteria #2 is also something we know how to handle. In an expression like $x+{}^{-}6$, where there are two signs in a row, we can simplify using the definition of subtraction: $x+{}^{-}6=x-6$.

Criteria #3 is just a bit of cosmetics. We say that the **degree of a term** is the power to which the variable is raised in a variable term. So, if we have an expression that includes a variable raised to different powers, it's often helpful to see the terms arranged in order from greatest degree to smallest degree. For example, instead of:

$$2x+3x^4+7-6x^3-18x^2$$

it's often nicer to write:

$$3x^4-6x^3-18x^2+2x+7$$

Note that when there's a variable with no exponent written, as in the $2x$ here, we think of it has having a "phantom 1" in the exponent: $2x = 2x^1$. Also note that the term 7 has no variable part at all. In this case, we picture a different kind of "phantom 1": $7 = 7 \cdot 1 = 7x^0$. If we make these substitutions, we can really see how the terms are arranged by degree:

$$3x^4 - 6x^3 - 18x^2 + 2x^1 + 7x^0$$

Also note that this criteria asks us to write the coefficients first. So, $2x$ is preferred over $x2$. This is, primarily, because it might be tricky to tell the difference between $x2$ and x^2. It would be especially confusing in a term has both a coefficient and an exponent: x^43 looks a bit too much like x^{43}, whereas $3x^4$ is much clearer.

The only criteria left are criteria #4 and criteria #5. These are more interesting, and so each gets its own section.

5.2.1 The Distributive Property

To eliminate grouping symbols, as expressed by criteria #4, we can rely on the field axioms for help.[2] The expression $(x + 2) + 2$ is not simplified, but this can be fixed easily enough using the associative property of addition.

$$(x + 2) + 2 = x + (2 + 2) = x + 4$$

The expression $3(x + 4)$ is not simplified either, and in this case the distributive property comes to the rescue. Get sprinkling!

$$3(x + 4) = 3 \cdot x + 3 \cdot 4 = 3x + 12$$

There are subtleties to using the distributive property. Distribution and negative numbers can lead to some easy-to-miss mistakes. We have to be on the lookout, so study the following examples carefully.

Example 5.1

Simplify: (a) $^{-}3(2x - 4)$ and (b) $8 - 5(6x - 9)$

Solution: In problem (a) we distribute a negative number. Notice what happens with the signs of the

[2] Some special grouping symbols, like the vinculum, can remain in the final expression. Generally, however, we must get rid of grouping symbols that do not double as another operation.

terms in the result.

$$\begin{aligned}
& {}^-3(2x - 4) && \\
={}& {}^-3(2x + {}^-4) && \text{change subtraction to addition of the opposite} \\
={}& {}^-3 \cdot 2x + {}^-3 \cdot {}^-4 && \text{distributive property} \\
={}& ({}^-3 \cdot 2)x + {}^-3 \cdot {}^-4 && \text{commutative property of multiplication} \\
={}& {}^-6x + 12 && \text{substitution (in both terms)}
\end{aligned}$$

The first thing we did here was to change the subtraction to addition of the opposite. This was helpful because — look! — we end up with *positive 12* in the final expression. That might otherwise have been an easy thing to miss.

In problem (b) notice that we have an implied operation that will require us to apply the distributive property. In other words, the first step is *not* to do $8 - 5$! As with the last problem, we'll start by changing to all-addition.

$$\begin{aligned}
& 8 - 5(6x - 9) && \\
={}& 8 + {}^-5(6x + {}^-9) && \text{change subtraction to addition of the opposite} \\
={}& 8 + {}^-5 \cdot 6x + {}^-5 \cdot {}^-9 && \text{distributive property} \\
={}& 8 + {}^-30x + 45 && \text{substitution} \\
={}& {}^-30x + 8 + 45 && \text{commutative property of addition} \\
={}& {}^-30x + 53 && \text{substitution}
\end{aligned}$$

Again here, we get a term ${}^-5 \cdot {}^-9 = 45$. Changing to all-addition before distributing is a helpful technique for getting these signs right.

Example 5.2

Simplify: $2x - (x - 4)$

Solution: This one is subtle and sneaky! The negative sign is stuck on the parentheses, which means we'll have to "distribute the negative sign" or — more accurately — we again have an implied operation, hiding in there as a "phantom one". This expression is the same as $2x - 1(x - 4)$.

Observe:

$$
\begin{aligned}
& 2x-(x-4) & \\
&= 2x-1(x-4) && \text{rewrite to expose the "phantom one"}\\
&= 2x+{}^-1(x+{}^-4) && \text{change subtraction to addition of the opposite}\\
&= 2x+{}^-1x+{}^-1\cdot{}^-4 && \text{distributive property}\\
&= 1x+4 && \text{substitution}\\
&= x+4 && \text{rewrite to hide the "phantom one"}
\end{aligned}
$$

Did you try this problem on your own before reading the solution? Did you get $x-4$? If so, don't be too upset: you're in good company. This is one of the most common mistakes in algebra 1. Always be on the looking when you see the distributive property mixed up with subtraction and negative signs.

5.2.2 Combining Like Terms

Expression Sort

Take a moment to sort the following mathematical objects into different groups, creating whatever categories make the most sense to you. After sorting, describe how you made your decisions. What features define each of the groups you have created?

$17x$	$2xy$	$-4x^2$	8	$0.5xy^2$	$-y$	$-15z$
-3	$8x^2y$	$4y$	xy^2	xyz	$11z$	x
$4xy^2$	$-2x^2y$	$3xyz$	x^2	$-6y$	$4x$	$3x^2$

One way to sort the terms above is to group **like terms** together. Like terms have the same variable factors raised to the same powers. For example, $17x$ and $4x$ are like terms. Also, x^2 and $-4x^2$ are like terms. But, $17x$ and $11z$ are *not* like terms since they have different variable parts. Though it may seem confusing at first, x and x^2 are *not* like terms. They both have x's, but those x's are raised to different powers.

Explaining the Startup Exploration

The terms in the startup exploration can be grouped using many different schemes. The categorization below is based on concept of grouping "like terms" together.

- Terms with no variable: -3 and 8
- Terms with x as the variable: x, $4x$, and $17x$
- Terms with x^2 as the variable: x^2, $3x^2$, and $-4x^2$
- Terms with y as the variable: $-y$, $4y$, and $-6y$
- Terms with z as the variable: $-15z$, and $11z$
- Terms with xy as the variable: $2xy$
- Terms with x^2y as the variable: $-2x^2y$, and $8x^2y$
- Terms with xy^2 as the variable: $0.5xy^2$, xy^2, and $4xy^2$
- Terms with xyz as the variable: xyz and $3xyz$

Just as 3 goats and 2 goats combine to make 5 goats, so too do $3x$ and $2x$ combine to make $5x$. Similarly, $8x^2 + x^2 = 9x^2$ (in this case, x^2 really means $1x^2$: there's a phantom 1 lurking there as the coefficient).

The expression $3xy^2 + 3x^2y + 3x^2y^2$ is simplified, since the variables and their corresponding powers do not match (look closely!).

The process of uniting like terms under a single coefficient is called **combining like terms**.[3] This is not technically a property in itself, but a "shortcut" for a somewhat longer chain of events:

$$\begin{aligned} & 3x + 2x & \\ &= x(3+2) & \text{the distributive property, in reverse} \\ &= x(5) & \text{substitution} \\ &= 5x & \text{commutative property of multiplcation} \end{aligned}$$

At first, it's best to commute the terms and group them up. Then we can add up the coefficients of the like terms. We'll see this approach, and a shortcut, in the examples below.

[3] Also sometimes called "collecting" like terms, or "gathering" like terms, and sometimes abbreviated "CLT".

Example 5.3

Simplify: $3x^2 + 2x - 2 + x^3 + 4x^2 - 3x - 2x^3 + 8$

Solution: To make sure we keep track of the signs, we'll convert to all-addition and then move the terms around to put like terms next to each other.

$$\begin{aligned} & 3x^2 + 2x - 2 + x^3 + 4x^2 - 3x - 2x^3 + 8 & \\ &= 3x^2 + 2x + {}^-2 + x^3 + 4x^2 + {}^-3x + {}^-2x^3 + 8 & \text{change to all-addition} \\ &= x^3 + {}^-2x^3 + 3x^2 + 4x^2 + 2x + {}^-3x + {}^-2 + 8 & \text{commutative property of addition} \\ &= {}^-1x^3 + 7x^2 + {}^-x + 6 & \text{combine like terms} \\ &= x^3 + 7x^2 - x6 & \text{simplify signs} \end{aligned}$$

Example 5.4

Simplify $3x - 4y + 2 - 5x + 7y - 6 + 8y$

Solution: There is no commutative property of subtraction, remember, so we can rewrite all subtraction into adding the opposite, as in the previous example. A shortcut is to remember that *the sign goes with the term* when you rearrange. Really, this is why we rewrite subtraction in the first place! If we're careful, though, we can avoid the rewriting step.

If we look at the given expression, we see that there are three "types" of terms that need combining: terms with x, terms with y and constant terms (numbers).

Let's pick a type of term (let's choose ones with x) and group them up — signs and all! In the given expression, we have $3x - 5x$, to we have ${}^-2x$ in all.

Pick another type of term and repeat as necessary! If we look at the y terms, we have $-4y + 7y + 8y$ so that's $11y$ in all. Looking at the constant terms, we have $2 - 6 = {}^-4$.

Putting it together, we have ${}^-2x + 11y + {}^-4$ or (after simplifying signs) ${}^-2x + 11y - 4$.

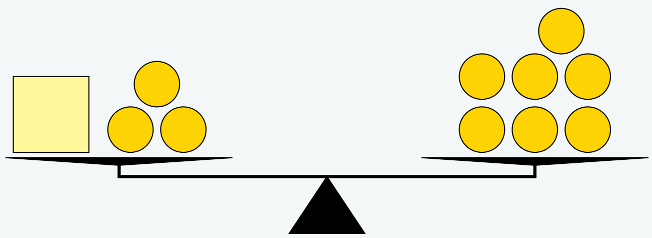
5.3 The Properties of Equality (POEs)

Extended Exploration: Mystery Numbers

[TODO] Click here to visit the extended exploration: Mystery Numbers

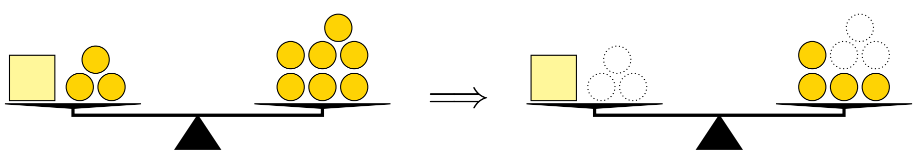
Startup Exploration: Cheese Scale

Bob put a block of edam cheese and three wheels of gouda on one side of a two-pan balance. On the other pan he put seven wheels of gouda. He discovered that the scale was in perfect balance.



If all of the wheels of gouda are identical in weight, how many wheels will balance the block of edam?

If Bob removes a wheel of gouda from the right-hand pan, it will tip the scales... unless he also removes an equivalent amount of gouda from the left-hand pan. In particular, the scale will still be in balance after Bob removes three wheels from each pan of the balance.[4]



Taking three wheels from each side of the balance is helpful because it means that we will have gotten the block of edam alone on one side. From there it's easy to see that the block weighs the same as four wheels.

The idea of maintaining balance is the heart of the algebra of solving equations.

[4] Edam and gouda are traditional Dutch cheeses. Every summer the city of Alkmaar, in the Netherlands, hosts an open-air cheese market. Cheese merchants demonstrate how cheese was bought and sold in medieval times. It operates on Waagplein, which means "weighing square" in Dutch, because once the buyer and seller agreed on a price, the huge wheels of cheese were taken to the scales by official "cheese carriers", and then weighed very carefully to determine their value. The cheese carriers' guild was established in 1593.

Equation

An **equation** is a number sentence which states that two algebraic expressions are equal.

To create an algebraic version of the balance problem, we need a way to represent the weight of the block of edam. In other words, we need a way to write down the weight of the block *before we know what the weight actually is*.

Unknown

An **unknown** is a quantity whose value is not known. We usually represent an unknown using a letter.

Both variables and unknowns are represented by letters though, technically speaking, an unknown is different from a variable. A variable's value can vary or change, whereas an unknown has a fixed value (or set of values) that must be determined.

Let's use B to stand for the unknown weight of the block, and let's assume that each wheel weighs 1 unit. Then, we can translate our balance problem into algebraic symbols:

$$B + 3 = 7.$$

If we take three (wheels) from each side of the balance, we have

$$B = 4$$

which tells us that B, the weight of the block, is four units. Our key task here was transforming the first equation into the second equation.

In section 5.1 we used the field axioms to simplify individual algebraic expressions. An equation is a math sentence with an expression on each side of the equal sign. So, we need some rules that will allow us to manipulate two expressions at the same time, but in such a way that their equality remains intact.

5.3.1 The Properties of Equality

An equation states that the two pans of a balance (the expressions on either side of the equal sign) are in alignment. If we were to, say, add five to the expression on one side of the equation, then we would tip the scales out of balance... unless we also add five to the expression on other side of the equation.

This idea, and other like it, are captured mathematically is a set of rules called the Properties of Equality, or POEs for short.

Properties of Equality: Addition and Subtraction

The **addition property of equality** (APOE) states that for all real numbers a, b, and c

$$\text{if } a = b, \text{ then } a + c = b + c$$

The **subtraction property of equality** (SPOE) states that for all real numbers a, b, and c

$$\text{if } a = b, \text{ then } a - c = b - c$$

Note that since "subtraction" is the same as "addition of the opposite", SPOE is really just APOE in disguise.

Properties of Equality: Multiplication and Division

The **multiplication property of equality** (MPOE) states that for all real numbers a, b, and c

$$\text{if } a = b, \text{ then } a \cdot c = b \cdot c$$

The **division property of equality** (DPOE) states that for all real numbers a, b, and c, where $c \neq 0$

$$\text{if } a = b, \text{ then } \frac{a}{c} = \frac{b}{c}$$

Here again, note that since "division" is the same as "multiplication by the reciprocal", DPOE is really just MPOE in disguise. Note too that c must not equal 0 when using DPOE, lest we be flummoxed upon division by zero.

Let's use the properties of equality to solve the problem of Bob's cheese balance.

Example 5.5

Bob put a block of edam cheese and three wheels of gouda on one side of a two-pan balance. On the other pan he put seven wheels of gouda. He discovered that the scale was in perfect balance. If all of the wheels of gouda are identical in weight, how many wheels will balance the block of edam?

Solution: Let B represent the weight of the block of edam. Then, the given information suggests the equation $B + 3 = 7$. We want to isolate B.

$$\begin{aligned} B + 3 &= 7 \\ B + 3 - 3 &= 7 - 3 && \text{SPOE: subtract 3 from both sides of the equation} \\ B &= 4 \end{aligned}$$

Note that in our work above, we started out by defining what variable we would use to represent our unknown. Then, we wrote an equation based on the information given in the problem. Our goal was to get B by itself on one side of the equation, or to "isolate B". Since 3 had been added to B, we had to "undo" this. So, we subtracted 3.

Inverse Operations (a.k.a. Opposite Operations)

Operations that will "undo" one another. For example, addition and subtraction are inverse operations. Multiplication and division are inverse operations.

When we solve an equation, we use inverse operations to undo the order of operations. Once we get a hang of the rules, solving equations can become a game. Plus, like the best video games, equation-solving has different levels of challenge so that the game stays interesting even as you get better and better at it.

5.4 Solving Level 1 and Level 2 Linear Equations

5.4.1 Level 1 Equations

Level 1 linear equations are the simplest kind to solve because it takes only one step to isolate the variable. The example we saw in the last section was a Level 1 linear equation.

Example 5.6

Consider each of the equations below. What step should we perform to isolate the variable?

$$x \div 4 = 21 \qquad w - 9 = 50 \qquad y + 13 = {}^{-}24 \qquad 8x = 72$$

Partial solution: In the first equation we must multiply both sides of the equation by 4 (MPOE).

$$\begin{aligned} x \div 4 &= 21 \\ \frac{x}{4} &= 21 && \text{definition of division} \\ \frac{x}{4} \cdot 4 &= 21 \cdot 4 && \text{MPOE: multiply both sides of the equation by 4} \\ x &= 84 \end{aligned}$$

In the second equation we should add 9 to both sides (APOE). In the third equation, we should subtract 13 (SPOE). Alternatively, we can think of this as APOE in which we add $^{-}13$ to both sides.

In the last example, we should divide both sides of the equation by 8 (DPOE). Or, we can think of this as MPOE in which we multiply both sides by $\frac{1}{8}$.

5.4.2 Level 2 Equations

Here's an example of a level 2 equation. How is it different from Level 1?

$$3x + 16 = 43$$

Notice that in this equation, two things were done to the unknown x. First it was multiplied by 3, and then 16 was added to the result. To isolate the variable, the rule of thumb is to undo whatever was done to the unknown. Usually this means that we will have to undo the order of operations in reverse. Most of the time, we can rely on the "underpants analogy".

The Underpants Analogy

When you get dressed for school in the morning, you put your underpants on *before* you put on your jeans. When you get undressed to go to bed, you do the *opposite* of each step, and you do the steps in the *opposite order*. In other words: you have to take your jeans off first, before you can take your underpants off.

To undo the order of operations, we have to work upside-down and backwards:

$$3x + 16 = 43$$

$$3x + 16 - 16 = 43 - 16 \qquad \text{SPOE: subtract 16 from both sides of the equation}$$

$$3x = 27 \qquad \text{Simplify and substitute}$$

$$\frac{3x}{3} = \frac{27}{3} \qquad \text{DPOE: divide both sides of the equation by 3}$$

$$x = 9 \qquad \text{Simplify and substitute}$$

In the end we have isolated x, and so this final equation tells us that 9 is a solution to the original equation.

Solution & Solution Set

A **solution** to an equation is any value that will make the equation true.

Sometimes an equation will have more than one solution. The set of all solutions to an equation are called a **solution set**. To write a solution set, we use set notation and write $\mathcal{S} = \{list\ of\ solutions\}$.

As we go through the process of solving an equation, each application of a Property of Equality produces an equation that is equivalent to the original.

Equivalent Equations

Equations that have the same solution set.

5.4.3 Showing and Checking Our Work

As with problems that involved the order of operations, it is often best to show your work going down the page as we have done in the earlier sections. It is also very helpful to anyone reading your work if you write beside each step the property that you used to justify turning one equation into an equivalent equation.

A note on vocabulary: we *simplify and expression* and we *solve an equation*. Even though the work we show might be very similar, the goal of our steps is quite different.

Having found a solution, it is a great habit to check the solution by substituting it back into the original equation. In the example above, we had the equation $3x + 16 = 43$ and found that this was equivalent to the equation $x = 9$. To check our solution, we plug 9 back in for x in the original equation:

$$\begin{aligned} 3x + 16 &= 43 && \text{original equation} \\ 3(9) + 16 &\stackrel{?}{=} 43 && \text{substitute in 9 for } x \\ 27 + 16 &\stackrel{?}{=} 43 && \text{carry out the order of operations} \\ 43 &\stackrel{\checkmark}{=} 43 && \text{Check!} \end{aligned}$$

When we substitute in 9, we put little question marks over out equal sign, since we are checking the solution — we're not sure whether the two expressions are equal yet! In the end, when we discover that the left-hand side is equal to the right-hand side, we replace our question with a check mark of confirmation!

The key here is to realize that, ultimately, the process we use to solve an equation yields *candidates* for the value of the unknown. We are not guaranteed that all of these candidates actually satisfy the original equation, until we check them to know for sure.[5]

Once we have verified that our solution satisfies the original equation, we should write our answer in set notation. The equation $x = 5$ isn't our solution, but an equivalent equation — in fact, it's the simplest equation that has the same solution as the original!

In other words, the equation solving process creates simpler and simpler equations that all have the same solution. Eventually, we get to the an equation that shows the solution clearly. We use set notation to record our solution because it is a notation that suggests the solution works for all of the intermediate equations.

Plus, solution set notation works nicely with all types of equations. Soon we will have equations with multiple solutions (even infinitely many!) and the $x =$ notation starts to be become quite awkward.

So, in our ongoing example, we write $\mathcal{S} = \{5\}$. Yes, a mathematical "set" may contain just a single element![6] The alternative to using solution set notation is to write the sentence "The solution to the equation is 5."

---

[5] There will come a time when we will do everything correctly, and still some of our solution-candidates will have to be rejected! Get in the habit now of checking solutions!

[6] This is another situation in which we find a conflict between everyday language and the language of mathematics. Lots of words have a special meaning when used in a mathematical context: similar, odd, mean, product... What other examples can you think of?

5.5 Solving Level 3 Linear Equations

Level 1 equations are called, naturally enough, *one-step equations*, and Level 2 equations are called — you guessed it! — two-step equations. In Level 3, we increase complexity by combining the tasks of expression-simplifying and equation-solving.

Extended Exploration: Evil Mystery Numbers

[TODO] Click here to visit the extended exploration: Evil Mystery Numbers

Startup Exploration: Linear Level 3

Determine the value x, given the equation

$$2x + 7 - 5x + 12 = 15.$$

Here we find an equation that must be simplified using the field axioms and related tools. Let's start out by trying to simplify the left-hand side.

$$\begin{aligned}
2x + 7 - 5x + 12 &= 15 \\
2x + 7 + {}^{-}5x + 12 &= 15 && \text{rewrite subtraction as adding the opposite} \\
2x + {}^{-}5x + 7 + 12 &= 15 && \text{commutative property of addition} \\
{}^{-}3x + 19 &= 15 && \text{combine like terms — this is now a Level 2 equation!} \\
{}^{-}3x + 19 - 19 &= 15 - 19 && \text{SPOE} \\
{}^{-}3x &= {}^{-}4 \\
\frac{{}^{-}3x}{{}^{-}3} &= \frac{{}^{-}4}{{}^{-}3} && \text{DPOE} \\
x &= \frac{4}{3} && \text{Simplify}
\end{aligned}$$

Notice that after a few lines, we had turned the Level 3 equation into a Level 2 equation. We know how to solve those! So, our goal will be to try to turn Level 3 equations into a Level 1 or Level 2 equations. But, since there are more steps in the process, there's more chance we can make a mistake. So, it's a good habit to remember to check our answer at the end.

We can check our answer for the last problem by substituting our solution back into the original equation and simplifying using the order of operations.

$$
\begin{aligned}
2x + 7 - 5x + 12 &= 15 && \text{original equation} \\
2\left(\frac{4}{3}\right) + 7 - 5\left(\frac{4}{3}\right) + 12 &\stackrel{?}{=} 15 && \text{substitute in our solution } x = \tfrac{4}{3} \\
2\left(\frac{4}{3}\right) + 7 + {}^{-}5\left(\frac{4}{3}\right) + 12 &\stackrel{?}{=} 15 && \text{rewrite subtraction as adding the opposite} \\
\frac{8}{3} + 7 + \frac{{}^{-}20}{3} + 12 &\stackrel{?}{=} 15 && \text{multiply fractions} \\
\frac{8}{3} + \frac{21}{3} + \frac{{}^{-}20}{3} + \frac{36}{3} &\stackrel{?}{=} 15 && \text{rewrite left-hand side with common denominator} \\
\frac{45}{3} &\stackrel{?}{=} 15 && \text{add fractions} \\
15 &\stackrel{\checkmark}{=} 15 && \text{boom!}
\end{aligned}
$$

Writing our solution as a solution set, we have $\mathcal{S} = \{15\}$.

Example 5.7

Determine the value of w, given the equation

$$7w + 2({}^{-}3w + 1) = 12.$$

Solution: Our goal is to try and simplify the left-hand side so that it looks like a Level 1 or Level 2 equation.

$$
\begin{aligned}
7w + 2({}^{-}3w + 1) &= 12 \\
7w + {}^{-}6w + 2 &= 12 && \text{distributive property} \\
1w + 2 &= 12 && \text{combine like terms — that's a Level 1 equation!} \\
w + 2 - 2 &= 12 - 2 && \text{SPOE} \\
w &= 10
\end{aligned}
$$

Let's check our solution:

$$
\begin{aligned}
7w + 2(-3w + 1) &= 12 && \text{original equation} \\
7(10) + 2(-3(10) + 1) &\stackrel{?}{=} 12 && \text{substitute in our solution } w = 10 \\
70 + 2(-30 + 1) &\stackrel{?}{=} 12 && \text{carry out the order of operations on the left-hand side} \\
70 + 2(-29) &\stackrel{?}{=} 12 \\
70 + -58 &\stackrel{?}{=} 12 \\
12 &\stackrel{\checkmark}{=} 12
\end{aligned}
$$

We're going to look at one more example, and discuss two alternative ways to approach it.

Example 5.8

Determine the value of g, given the equation

$$-4(2g - 7) = 36.$$

Solution: Approach #1. Those parentheses are just asking to be simplified using the distributive property. Be mindful of the signs, though!

$$
\begin{aligned}
-4(2g - 7) &= 36 \\
-8g + 28 &= 36 && \text{distributive property — that's a Level 2 equation!} \\
-8g + 28 - 28 &= 36 - 28 && \text{SPOE} \\
-8g &= 8 \\
\frac{-8g}{-8} &= \frac{8}{-8} && \text{DPOE} \\
g &= -1
\end{aligned}
$$

Approach #2. Let's solve the same equation again. This time, notice that we can divide both sides by

-4 as the first step, which eliminates the need for the distributive property.

$$\begin{aligned}
-4(2g-7) &= 36 \\
\frac{-4(2g-7)}{-4} &= \frac{36}{-4} && \text{DPOE} \\
2g-7 &= -9 && \text{simplify — that's a Level 2 equation!} \\
2g-7+7 &= -9+7 && \text{APOE} \\
2g &= -2 \\
\frac{2g}{2} &= \frac{-2}{2} && \text{DPOE} \\
g &= -1
\end{aligned}$$

The two different approaches taken in the previous example are not always available. Compare the solutions to examples 5.3 and 5.4 in this section. Can we avoid the distributive property in example 5.3 Why or why not?

5.5.1 A Note on Checking Work

Every step we take in solving an equation generates an equivalent equation that has the same solution set. If we make a mistake, we accidentally create an equation with a different solution set. Knowing this can help us check our answers on the more complex equations (ones with more steps).

Example 5.9

Solve for x, given the equation $2x - 3(4x - 1) = -53$.

Let's say we mess up the signs when carrying out the distributive property — a very common mistake.

$$\begin{aligned}
2x-3(4x-1) &= -53 && \text{Line 1.} \\
2x-12x-3 &= -53 && \text{Line 2. That's a mistake!} \\
-10x-3 &= -53 && \text{Line 3.} \\
-10x &= -50 && \text{Line 4.} \\
x &= 5 && \text{Line 5.}
\end{aligned}$$

Now, we go back to check our work by substituting the solution into the original equation. The answer doesn't work in line 1, so we know it is not the correct solution.

$$2(5) - 3\big(4(5) - 4\big) = {}^{-}53 \implies {}^{-}38 \neq {}^{-}53 \quad \text{(something went wrong!)}$$

But, our answer *does* work in line 2 (and also lines 3,4, and 5).

$$2(5) - 12(5) - 3 = {}^{-}53 \implies {}^{-}53 = {}^{-}53 \quad \text{(that works!)}$$

This is the clue helps us pinpoint the location of the mistake. It tells us the mistake must have happened when transforming line 1 into line 2.

If, on a different problem, we find that our solution doesn't work for steps 1, 2, or 3, but *does work* in step 4, then it means that our mistake must have happened when transforming step 3 into step 4!

5.6 Solving Level 4 Linear Equations

In Level 4, we add a wrinkle, which can lead to some very unusual results.

Extended Exploration: Super Evil Mystery Numbers

[TODO] Click here to visit the extended exploration: Super Evil Mystery Numbers

Startup Exploration: Linear Level 4

Determine the value of x given the equation

$$13x - 5x - 5 = x + 7 + x.$$

The goal is the same: to isolate the variable on one side of the equal sign. We begin by simplifying each side.

$$\begin{aligned} 13x - 5x - 5 &= x + 7 + x && \\ 8x - 5 &= 2x + 7 && \text{combine like terms on both sides} \\ 8x - 5 - 2x &= 2x + 7 - 2x && \text{SPOE: subtract } 2x \text{ from both sides — clever!} \\ 6x - 5 &= 7 && \text{combine like terms again — that's a Level 2 equation!} \\ 6x &= 12 && \text{APOE: add 5 to both sides} \\ x &= 2 && \text{DPOE: divide both sides by 6} \end{aligned}$$

In this example, we subtracted $2x$ from both sides of the equation. This might seem like a tricky move, but it's a completely legal application of SPOE. We are allowed to do the same thing to both sides of the balanced equation, even if that means subtracting an unknown amount from both sides.[7]

Let's see what happens if we isolate the unknown on *the other side of the equal sign*. (It had better give us the

[7] Of course, you have to subtract the same unknown amount from both sides. It's no fair to subtract x from one side of an equation and y from the other side. This will put the equation out of balance unless we know that these two unknown amounts are equal... that is, unless we know that $x = y$. Caution: can you think of a situation in which it might be dangerous to apply a POE using an unknown? Think about what might happen if we use DPOE to divide both sides by an unknown value x.

same answer!)

$$
\begin{aligned}
13x - 5x - 5 &= x + 7 + x && \text{same equation as before} \\
8x - 5 &= 2x + 7 && \text{combine like terms on both sides, as before} \\
8x - 5 - 8x &= 2x + 7 - 8x && \text{SPOE: subtract } 8x \text{ from both sides} \\
{}^-5 &= {}^-6x + 7 && \text{combine like terms} \\
{}^-12 &= {}^-6x && \text{SPOE: subtract 7 from both sides} \\
2 &= x && \text{DPOE: divide both sides by } {}^-6
\end{aligned}
$$

So, it doesn't matter which side we choose to eliminate the unknown. As long as we apply APOE or SPOE correctly, the solution will be the same. One strategy is to choose the side that will result in a *positive coefficient* for the variable. This isn't required, but it helps avoid the chances of losing a negative sign along the way (maybe that's happened to you).[8]

If you don't like the side of the equation that the unknown is on, you can always rewrite the equation. For example if you have the equation $3 = 4 + x$ and you really want the x on the left-hand side, you can just rewrite the equation as $4 + x = 3$. Those are equivalent equations!

Example 5.10

Determine the value of x given the equation

$$3(x - 2) + 3x = 4x - 6.$$

Solution: Let's jump right in and start simplifying the left-hand side.

$$
\begin{aligned}
3(x - 2) + 3x &= 4x - 6 && \\
3x - 6 + 3x &= 4x - 6 && \text{distributive property} \\
6x - 6 &= 4x - 6 && \text{combine like terms} \\
2x - 6 &= -6 && \text{SPOE: subtract } 4x \text{ from both sides} \\
2x &= 0 && \text{APOE: add 6 to both sides} \\
x &= 0 && \text{DPOE: divide both sides} by 2
\end{aligned}
$$

[8] Seems like a good place for a footnote... but about what?

Yes, we can get 0 as the solution to an equation. We can always check to be sure:

$$\begin{aligned} 3(x-2)+3x &= 4x-6 && \text{original equation} \\ 3(0-2)+3(0) &\stackrel{?}{=} 4(0)-6 && \text{substitute in our solution } x=0 \\ 3(^-2)+0 &\stackrel{?}{=} {}^-6 && \text{simplify} \\ {}^-6 &\stackrel{\checkmark}{=} {}^-6 \end{aligned}$$

5.6.1 Special Cases

There are some things to be careful about with when it comes to solving Level 4 linear equations. Here is an example of something strange that can happen when we have unknowns on both sides of the equal sign.

Example 5.11

Determine the value of x given the equation

$$12x+36=2(6x-10).$$

Solution: We proceed as we have done in earlier problems:

$$\begin{aligned} 12x+36 &= 2(6x-10) \\ 12x+36 &= 12x-20 && \text{distributive property} \\ 12x+36-12x &= 12x-20-12x && \text{SPOE: subtract } 12x \text{ from both sides} \\ 36 &= {}^-20 && \text{Huh?} \end{aligned}$$

In this case the variable term disappears completely from both sides of the equation. And then, what is left is obviously *not equal*. In a sense, the original "equation" is not an equation at all. The two given expressions are not equal, and never will be.

This means that there is no value of x what will ever work to make the equation true. We say that this equation *has no solution*.

As unsatisfying as it might sound, it is possible to have an equation that does not have a solution.[9]

[9] Equations without solutions are very important in the history of mathematics. The French mathematician Pierre de Fermat made a conjecture in 1637 stating that a certain equation had no integer solutions. His claim, which came to be known as Fermat's Last Theorem, wasn't officially proven to be correct for more than 350 years. British mathematician Andrew Wiles published the first complete proof of Fermat's Last Theorem in 1995. The theorem states that the equation $a^n+b^n=c^n$ has no integer solutions for a, b, and c when the exponent n is a natural number greater than 2.

How do you write down the solution to a problem that has no solution? This is not quite as philosophical as it sounds. We can, of course, write the words "no solution". Or, we could write a solution set that contains no numbers, $\mathcal{S} = \{\ \}$.

Or, there is a third option. You may or may not be surprised to learn that mathematicians have invented a symbol for just such an occasion. We use the mathematical symbol $\varnothing$, called the "empty set" or "null set", as a way to show that that our solution set is empty.[10] We write $\mathcal{S} = \varnothing$ to mean "the solution set is empty".

Notice that we don't write the curly braces around the empty set. In other words, we write $\mathcal{S} = \varnothing$ and not $\mathcal{S} = \{\varnothing\}$. The former says "$S$ is the empty set", which is what we mean when we have an equation with no solutions. The latter says "S is the set which contains the empty set" which — believe it or not — is not the same thing.[11]

For the record, the empty set is not the same as zero. So, an equation can have the solution set $\mathcal{S} = \{0\}$, like example 5.6. This is different from an equation having the solution set $\mathcal{S} = \{\ \}$, like example 5.7.

If you look back at the reason why the last example has no solution, you may be wondering about whether there is another possibility. Study the following example.

Example 5.12

Determine the value of x given the equation

$$5x - 3(x + 2) = 2x - 6.$$

Solution: Let's go for it:

$$\begin{aligned}
5x - 3(x + 2) &= 2x - 6 && \\
5x - 3x - 6 &= 2x - 6 && \text{distributive property, watch the signs} \\
2x - 6 &= 2x - 6 && \text{combine like terms — something is fishy already.} \\
2x - 6 - 2x &= 2x - 6 - 2x && \text{SPOE: subtract } 2x \text{ from both sides} \\
-6 &= -6 && \text{But, of course.}
\end{aligned}$$

In this case the variable term disappears again, and the leftovers are *obviously equal*. This is the opposite

[10] The symbol $\varnothing$, which was introduced by French mathematician André Weil in 1939, is a letter in the Norwegian alphabet.

[11] The empty set is not the same as zero, and it is not the same as nothing. It is a set with nothing inside it, like an empty bag. So, writing $\mathcal{S} = \varnothing$ is stating that the solution set is like an empty bag. Extending this metaphor, $\{\varnothing\}$ is a bag with an empty bag in it and, therefore, a set which contains the empty set is not empty. Om.

of what happened earlier. If that example had no solutions, this one has all of them. We can replace x with literally any real number and the equation will be true. That means that we have a solution set that contains every real number there is.

We say that this equation's solutions are *all real numbers*.

In this example, our solution set is the set of real numbers. We can write this out in words, "all real numbers", or use the shorthand symbol that stands for the real numbers and write, $\mathcal{S} = \mathbb{R}$. Notice no curly braces are used here either.

Checking Our Answers in Special Cases

In the case that we find "all real numbers" as as the solution to an equation, we know that every number is going to work. So to check, we might pick a couple of different numbers (easy ones, like 0 and 1) and substitute those into the original equation. Any number that you choose should satisfy the equation.

If we find that an equation has "no solutions", then any number we use to check will fail to satisfy the equation. Not very informative. In this case, it may be best just to check back over our work to make sure that we did each of the steps correctly.

5.7 Solving Level 5 Linear Equations

Over the last few sections we have added complexity to the equation-solving picture. We have learned properties that can be used to "undo" whatever has been "done" to the variable. Sometimes this has led to situations where the equation has no solutions, or infinitely many solutions. We turn now to equations that involve the absolute value of an unknown.[12]

Extended Exploration: Dreadful Mystery Numbers

[TODO] Click here to visit the extended exploration: Dreadful Mystery Numbers

Startup Exploration: Linear Level 5

Recall the definition of absolute value. Determine the value of x given the equation

$$|\,x\,| = 6.$$

Recall that the absolute value of a number is that number's distance away from zero on the number line. Distances are always positive: $^-6$ is 6 units away from 0, and so $|\,^-6\,| = 6$. Of course, $|\,6\,| = 6$ as well. So, the equation above has two solutions $x = 6$ <u>or</u> $^-6$. In solution set notation, we write $\mathcal{S} = \{6, {}^-6\}$.

The presence of the absolute value gives us the possibility of having two solutions, since whatever is inside the absolute value bars could be either the positive or the negative version of the value.

Here's an extension to the idea. Determine the value of x given the equation

$$|\,2x\,| = 16$$

This is telling us that the number "$2x$" is 16 units away from zero on the number line. In other words

$$2x = 16 \ \underline{\text{or}} \ {}^-16.$$

This is really two equations written at once: $2x = 16$ and $2x = {}^-16$. We can apply DPOE and divide everything by 2, which will isolate x. That is, we have

$$x = 8 \ \underline{\text{or}} \ {}^-8.$$

The following two examples carry this idea a bit further.

[12] We're discussing absolute value equations here in the chapter on linear equations, though we put "linear" in quotes. Absolute value equations of the kind that we're discussing here are linear-like enough that they fit here. Other types of absolute value equations are possible, though we won't get into the full range of variations in this course.

Example 5.13

Determine the value of x given the equation

$$|{-3x} + 9| = 12.$$

Solution: The stuff inside the absolute value bars can equal 12 or -12. So, this means, we have two equations:

$$\begin{aligned} -3x + 9 &= 12 \quad \underline{\text{or}} \quad & -3x + 9 &= -12 & \\ -3x &= 3 & -3x &= -21 & \text{SPOE: subtract 9 throughout} \\ x &= -1 & x &= 7 & \text{DPOE: divide by } -3 \text{ throughout} \end{aligned}$$

This equation has solutions $\mathcal{S} = \{-1, 7\}$.

Note that the steps we took for solving were the same for both equations (in this case, first we used SPOE, and then DPOE). So, we can streamline our work a bit, as shown in the next example.

The next example also has things going on outside of the absolute value symbols. For these types of problems, we want to isolate the absolute value expression first (using the properties of equality), then separate into two equations (if necessary).

Example 5.14

Determine the value of x given the equation: $3|{-4x} + 4| = 48$.

Solution:

$$\begin{aligned} 3|{-4x} + 4| &= 48 & \\ |{-4x} + 4| &= 16 & \text{DPOE, to isolate the absolute value expression} \\ -4x + 4 &= 16 \ \underline{\text{or}} \ -16 & \text{definition of absolute value} \\ -4x &= 12 \ \underline{\text{or}} \ -20 & \text{SPOE: subtract 4 throughout} \\ x &= -3 \ \underline{\text{or}} \ 5 & \text{DPOE: divide by } -4 \text{ throughout} \end{aligned}$$

So, this equation has solutions $\mathcal{S} = \{-3, 5\}$.

5.7.1 Special Considerations

A few sneaky things can pop up when it comes to absolute value equations. Recall that the only number that has absolute value zero is zero itself: $|0| = 0$. So, not all absolute value equations have *two* solutions. Consider

$$|3x + 51| = 0.$$

We can't split this into two different equations. All we have is

$$3x + 51 = 0,$$

and that's a Level 2 linear equation with just a single solution, $\mathcal{S} = \{-17\}$. To demonstrate another tricky aspect, consider the equation

$$|4x| + 16 = 5.$$

No problem! We begin by subtracting 16 from both sides:

$$|4x| = -11$$

But, the absolute value of a number can never be negative. So, this equation has no solution. In other words, we have $\mathcal{S} = \emptyset$.

Don't be too quick with the "no solutions" talk, though. If we encounter the equation

$$-3|x + 2| = -12,$$

we may see the -12 and assume there is no solution. But recall that we must isolate the absolute value expression first. To do that we will divide both sides by -3 and then we will have

$$|x + 2| = 4,$$

an equivalent equation that surely has a solution (two solutions, in fact). So, don't jump to conclusions!

WARNING!

Some folks like to use the notation "plus/minus" in situations that involve absolute value. For example, writing "± 6" with the little stacked-up plus and minus signs to stand for "6 or -6". We recommend avoiding $\pm$ notation when solving equations.

Here's why. See if you can spot the error in the following work.

$$\begin{aligned} |\,x-5\,| &= 12 \\ x-5 &= \pm 12 \\ x &= \pm 17 \qquad \text{APOE: add 5 to both sides} \end{aligned}$$

And so, $\mathcal{S} = \{17, {-17}\}$. This work looks reasonable, but let's check the two solutions.

$$|\,17-5\,| \stackrel{?}{=} 12 \implies |\,12\,| \stackrel{?}{=} 12 \implies 12 \stackrel{\checkmark}{=} 12$$

$$|\,{-17}-5\,| \stackrel{?}{=} 12 \implies |\,{-22}\,| \stackrel{?}{=} 12 \implies 22 \neq 12$$

So, using the $\pm$ notation has led us to get one correct solution and one incorrect solution for the equation! This is why we recommend writing out the word "or" explicitly, or breaking into two separate equations.

(By the way, the correct solution is $\mathcal{S} = \{17, {-7}\}$. Try working out the problem again using a more reliable technique. Can you see exactly where the work using $\pm$ went astray?)

5.8 Applications of Equation Solving

Now that we know how to handle different types of equations, let's explore a few applications. First, we'll discuss writing (and then solving) equations based on problem situations. Then, we'll see how to apply the properties of equality to manipulate formulas from science and other disciplines. We'll close this chapter with an extension section about explaining some fundamental ideas using the field axioms.

5.8.1 Writing Equations to Solve Word Problems

There are lots of ways to approach problems like this: educated guessing-and-checking, wishful thinking, making an organized list, and so on. One way is to model the situation with (and then solve) an equation. Converting a word problem into an equation is an important skill for your mathematical toolbox.

Extended Exploration: Mixed Bag 'O Problems

[TODO] Click here to visit the extended exploration: Mixed Bag 'O Problems

Startup Exploration: Trick or Treat

On Halloween night, Hildegaard and Ingvar (Bob and Yeardleigh's parents) had 200 candies in their trick-or-treat bowl. They gave 5 candies to each kid who came to their door, and at the end of the night they had 10 candies left. How many kids came trick-or-treating?

The key in word problems like this is first to identify the unknown in the problem. We choose a variable to stand in for the unknown, and write an equation based on the other information given in the problem.

In our example, the unknown is "how many kids came trick-or-treating", so let's use k to represent the number of kids. Hildegaard and Ingvar gave 5 candies to every kid, so they gave away $5k$ candies. They started with 200 and gave away $5k$. Giving away suggests subtraction, so they were left with $200 - 5k$ candies at the end of the night. The problem tells us that they had 10 candies let over, so:

$$200 - 5k = 10.$$

That's a Level 2 equation! We'll leave it for you to solve and put the solution in the footnote.[13]

Here's a second example. "Consecutive integer problems" are a clever type of mystery number problem that might at first appear impossible to solve. There doesn't seem to be enough information!

[13] Solving this equation using the POEs leads to the result $k = 38$. So, the answer to the question is that 38 kids came trick-or-treating at the Krumbli house on Halloween night.

Name of Startup Exploration

The sum of three consecutive integers is 39. What are the numbers?

What does it mean to have three consecutive integers? Consecutive means "in an unbroken sequence", so three consecutive integers are three integers in a row, such as $4, 5, 6$.

Let N represent the smallest of the three consecutive integers. Then, the next integer is $N+1$, and the integer after that is $N+2$. The problem says their sum is 39. So we have

$$\begin{aligned} N+(N+1)+(N+2) &= 39 & \\ 3N+3 &= 39 & \text{combine like terms} \\ 3N &= 36 & \text{SPOE} \\ N &= 12 & \text{DPOE} \end{aligned}$$

At this point, many students would be tempted to draw a box around their answer and call it a day. After all, we solved the equation, right? But, look back at the question: it asks us to find the *numbers*, plural. Our answer to the question should list all three of the numbers!

What did we actually compute when we solved the equation? We chose N to represent the smallest of the three consecutive integers, and that's 12. So, the full answer is that the three integers are 12, 13, and 14. (A quick check shows that these three do, in fact, add up to 39.)

The moral of the story here is always to check that we are answering the question that has been asked. This step is a helpful way to avoid giving an irrelevant or incomplete solution.

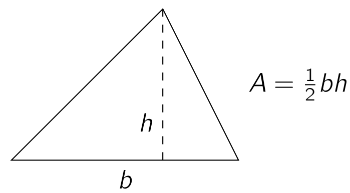
Finally, when it comes to stating the solution to a word problem, we don't usually use set notation. A handy rule of thumb is to think about how we'd say the answer out loud to someone. When we solve an equation with no context, we'd probably say "The solution is 4." Set notation is appropriate, since the answer is just a number, $\mathcal{S} = \{4\}$.

On the other hand, if we were answering a word problem that asked how many pounds of cream cheese Bob ate, we'd say "Bob ate 4 pounds of cream cheese." When the context of the problem implies a unit on the answer (pounds, dollars, goats, miles per hour), we write the answer and the unit rather than use solution set notation: 4 pounds of cream cheese.

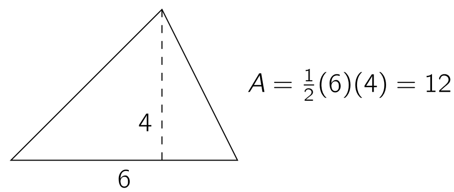
5.8.2 Transforming Formulas

In all of the examples in this chapter so far, we have solved equations and found the value of an unknown. In this section, we'll discuss a different use the field axioms and the properties of equality.

In the past, you may have learned a formula for the area of a triangle. If we have a triangle with base of length b and height h, then its area A has a tidy little formula.



If we know the base and height of a specific triangle, we can find its area.



But what if we are given the area and the base length: Could we work backwards to find the height? Could we "transform" the area formula so that it computes h instead of A?

Transforming a formula means to isolate a specific variable, even though the other side of the equal sign might not simplify down to a single number. The only difference between this and what we have been doing before now is that the equations may have multiple variables.

Example 5.15

Transform the triangle area formula, $A = \frac{1}{2}bh$, to isolate h.

Solution: We'll start with the given formula, and then apply the properties of equality to get h by itself on one side of the equal sign.

$$\begin{aligned}
A &= \frac{1}{2}bh \\
2 \cdot A &= 2 \cdot \frac{1}{2}bh && \text{MPOE, to eliminate the fraction} \\
2A &= bh && \text{simplify on the righthand side} \\
\frac{2A}{b} &= \frac{bh}{b} && \text{DPOE, to isolate } h \\
\frac{2A}{b} &= h && \text{simplify on the righthand side: mission accomplished!}
\end{aligned}$$

So, our transformed formula expresses the height of a triangle in terms of its area and the length of a base:

$$h = \frac{2A}{b}$$

This is a skill used in science (especially physics) and higher levels of mathematics. Generally, we have to study the equation we're given and find the variable we are trying to isolate. Then, think about what has been done to that variable which must be undone. We undo these using the POEs.

[TODO] Remember: Since we moved transforming formulas before the linear equations formulae, we should add a bit about transforming standard from later.

Even though our answers won't be just a number, they should be simplified as much as possible, for instance we must avoid fractions-in-fractions. Parentheses, too, can lead to tricky situations. It often pays to formulate a plan before rushing to work.

Example 5.16

$P = 2(w + d)$ for w

Solution: Sometimes we have to distribute, other times it is easier to avoid it all together. Compare the two options below.

Option 1: Avoiding the distributive property

$$\begin{aligned} P &= 2(w+d) && \\ \tfrac{P}{2} &= w - d && \text{DPOE} \\ \tfrac{P}{2} - d &= w && \text{SPOE} \end{aligned}$$

Option 2: Distributive property first

$$\begin{aligned} P &= 2(w+d) && \\ P &= 2w + 2d && \text{distributive property} \\ P - 2d &= 2w && \text{SPOE} \\ \tfrac{P-2d}{2} &= w && \text{DPOE} \\ \tfrac{P}{2} - \tfrac{2d}{2} &= w && \text{undo fraction subtraction} \\ \tfrac{P}{2} - d &= w && \text{write second fraction in lowest terms} \end{aligned}$$

In the previous example, the second approach is a bit more work. The distributive property can't always be avoided though, as we saw back in example 5.3. So, it pays to be observant and do a bit of thinking before we start crunching the numbers.

5.8.3 (;,;) Proving Foundational Results Using the Field Axioms

As we mentioned in section 5.1, an axiom is a statement that is accepted without proof. In some sense, we "take it for granted" that adding 0 to any number doesn't change the number: $a + 0 = a$ for any real number a.

Similarly, there's a field axiom about multiplication by 1. But, that there is no axiom about multiplication by 0. Why not? When thinking about the field axioms, an interesting thing to consider is why some rules are omitted.[14]

Surprisingly, we can use the axioms given in section 5.1 to explain other basic results of arithmetic. In other words: our axioms are the fundamental building blocks. We can use them to build complex structures (like Level 5 equations) or simple structures. Sometimes, we discover that we can build really simple — even obvious-looking — mathematical statements from *even simpler statements*!

Multiplication by Zero

We know that $a \cdot 0 = 0$, but there's no axiom stating this. That's because we can explain multiplication by zero using the axioms that we already have. Consider this argument, where we state the property that we used to transform each line:

$$\begin{aligned} 0 + 0 &= 0 && \text{identity property of addition} \\ a \cdot (0 + 0) &= a \cdot 0 && \text{MPOE: multiply both sides by } a \\ a \cdot 0 + a \cdot 0 &= a \cdot 0 && \text{distribuive property on left-hand side} \\ a \cdot 0 &= 0 && \text{SPOE: subtract } a \cdot 0 \text{ from both sides} \end{aligned}$$

In the end, we used only the given axioms and the properties of equality to demonstrate that any number times 0 is zero. We don't need an axiom for this rule, because we can *build it* from the given set of axioms!

Multiplication and Negative Numbers: Property #1

We are given an axiom stating that $1 \cdot a = a$ (that's the identity property of multiplication). But, we have no axiom stating that $^-1 \cdot a = {^-a}$. It turns out that we don't need one:

$$\begin{aligned} 1 + {^-1} &= 0 && \text{inverse property of addition} \\ a \cdot (1 + {^-1}) &= a \cdot 0 && \text{MPOE: multiply both sides by } a \\ a \cdot 1 + a \cdot {^-1} &= a \cdot 0 && \text{distribuive property on left-hand side} \\ a + a \cdot {^-1} &= 0 && \text{simplifications: } a \cdot 1 = a \text{ and } a \cdot 0 = 0 \\ a \cdot {^-1} &= {^-a} && \text{SPOE: subtract } a \text{ from both sides} \end{aligned}$$

We could apply the commutative property of multiplication on the left-hand side, if we want, to have the new rule $^-1 \cdot a = {^-a}$.

[14] Remember that the Cthulhu icon (;,;) indicates that this is an extension section. Don't feel bad if some of the ideas feel like swimming in the deep end! Stick with it!

Multiplication and Negative Numbers: Property #2

Another fundamental result that seems obvious is that if we take the "double negative" of a number, we get back to the original number. In other words, the opposite of the opposite of a is a itself: $^-(^-a) = a$.

Consider this: we know that $a + {}^-a = 0$ (that's the inverse property of addition). We know that ^-a also has an additive inverse: $^-a + {}^-(^-a) = 0$. Since these are both equal to 0, they are equal to each other. So:

$$\begin{aligned} a + {}^-a &= {}^-a + {}^-(^-a) && \text{both equal 0, so they equal each other} \\ a &= {}^-(^-a) && \text{SPOE: subtract } {}^-a \text{ from both sides!} \end{aligned}$$

Multiplication and Negative Numbers: Property #3

We can use the axioms to explain why the product of a negative number and a positive number is negative. Let a and b be positive numbers:

$$\begin{aligned} a \cdot 0 &= 0 && \text{multiplication by zero} \\ a \cdot (b + {}^-b) &= 0 && \text{rewrite 0 using the additive inverse property} \\ a \cdot b + a \cdot {}^-b &= 0 && \text{distributive property} \\ a \cdot {}^-b &= {}^-(a \cdot b) && \text{SPOE: subtract } a \cdot b \text{ from both sides} \end{aligned}$$

So, if we have "the opposite of a" times b, the result is the opposite of the product of a and b.

Multiplication and Negative Numbers: Property #4

We can use property #3 to prove that the product of two negative numbers is positive. Let a and b be positive numbers:

$$\begin{aligned} {}^-a \cdot 0 &= 0 && \text{multiplication by zero} \\ {}^-a \cdot (b + {}^-b) &= 0 && \text{rewrite 0 using the additive inverse property} \\ {}^-a \cdot b + {}^-a \cdot {}^-b &= 0 && \text{distributive property} \\ {}^-(a \cdot b) + {}^-a \cdot {}^-b &= 0 && \text{property \#3} \\ {}^-a \cdot {}^-b &= a \cdot b && \text{APOE: add } a \cdot b \text{ to both sides!} \end{aligned}$$

So, the two products are the same! The product of two negative numbers is the same as the product of their opposites (that is, their positive partners).

Mind Blown!

We've been taking the properties of equality for granted, but *even the properties of equality* can be derived from the field axioms. The POEs are the properties that allow us to make a cancellation, for example taking

$$x + 4 = 10$$

and thinking about it as

$$x + 4 = 6 + 4.$$

We can cancel the 4 from both sides and get $x = 6$ (we call this SPOE, the subtraction property of equality). Suppose we have real numbers a, b, and c, and we know that $a + c = b + c$. Then, consider this chain of reasoning:

$$\begin{aligned}
a &= a + 0 && \text{additive identity property} \\
&= a + (c + {}^{-}c) && \text{rewrite 0 using the additive inverse property} \\
&= (a + c) + {}^{-}c && \text{associative property of addition} \\
&= (b + c) + {}^{-}c && \text{substitution: we assumed that } a + c = b + c \\
&= b + (c + {}^{-}c) && \text{associative property of addition} \\
&= b + 0 && \text{additive inverse property} \\
&= b && \text{additive identity property}
\end{aligned}$$

What does this tell us? Notice that we started our equation with a and ended up with b. This means $a = b$. Altogether, we have shown that if we know $a + c = b + c$, then it must be that $a = b$. That's one of our POEs!

This section has gotten into some pretty esoteric stuff. Don't worry if it feels overwhelming at first. Let these ideas sink in and read through this section again in a few weeks. It takes time for our brains to incorporate abstract ideas!

Chapter 6

Proportional Reasoning

> Give me the place to stand, and I shall move the earth.
>
> — Archimedes, Greek mathematician and philosopher

6.1 Proportions as Equations

Mathematicians, like Yeardleigh's team of gourmet chefs, won't use tools, techniques, or ingredients unless they know exactly where they come from. This is the attitude we adopt, for the most part, in algebra. This point of view, however, may upset some students' mathematical status quo.

Case in point: solving proportions. Many of us have been taught a mysterious method called "cross multiplication" as a means of solving a proportion. Unless we can explain it's inner workings – Where does it come from? Why does it give us the correct answer? – then the technique must be considered off-limits.

We'll explain so-called cross-multiplication in this chapter, and explore alternative (easier) methods of handling proportions.

Extended Exploration: Multiply and Conquer

[TODO] Click here to visit the extended exploration: Multiply and Conquer

Startup Exploration: Hamster and Superhamster

Genetic engineers at YeardleighCorp have genetically engineered superhamsters that weigh 15 ounces. Tests indicate that one superhamster can carry a 40-ounce packet of food on its back. If a human could carry the same amount as a superhamster, relative to body mass, how many pounds could a 120-pound teenager carry?

We have mentioned that the study of mathematical relationships is a central idea of algebra. One helpful skill for studying relationships is the ability to make sound mathematical comparisons.

Ratio

A comparison of two quantities, often expressed as a fraction.

There are two main types of ratios: part-to-part ratios and part-to-whole ratios. For example, "number of girls in class to number of boys in class" is a part-to-part ratio, whereas "number of girls in class to number of students in class" is a part-to-whole ratio.

Proportion

An equation stating that two ratios are equal.

Proportions are just equations, and so we don't need any special techniques like "cross-multiplication" to solve them. We can use the trusty properties of equality, just as we do with any other equation.

Example 6.1

Solve for x: $\dfrac{x}{15} = \dfrac{44}{60}$

Solution: Our suggestion is not to think of the left side as a fraction at all. Think of it as x divided by 15. That's just a Level 1 linear equation! We need to get rid of the "divide by 15" and so we multiply both sides by 15.

$$15 \cdot \frac{x}{15} = 15 \cdot \frac{44}{60}$$

$$\cancel{15} \cdot \frac{x}{\cancel{15}} = \cancel{15} \cdot \frac{44}{\cancel{15} \cdot 4}$$

$$x = 11$$

After a bit of simplifying, we have isolated x with just one application of MPOE.

We call this approach "clear the denominator". Of course, it's not the only way to solve a proportion.

6.1.1 Adding a Bit More Complexity

In their most basic form, proportions have the unknown in the numerator of one of the ratios. These are just Level 1 (one-step) equations. All we need to do to solve them is multiply both sides of the equation by the denominator of the unknown (that's MPOE in action).

But what if the unknown is in the denominator of a fraction?

Example 6.2

The startup exploration problem says that a 15-ounce superhamster can carry a 40-ounce packet of food on its back. At that rate of strength, how many pounds could a 120-pound teenager carry?

Solution: Let's write a proportion comparing the ratio of weight to carrying capacity, and let us use P to represent the amount that the teenage can carry, in pounds. Then we have:

$$\frac{15 \text{ ounces}}{40 \text{ ounces}} = \frac{120 \text{ pounds}}{P \text{ pounds}}$$

One approach at this point is to take the reciprocal of both sides. That will put the unknown conveniently in the numerator!

$$\frac{40 \text{ ounces}}{15 \text{ ounces}} = \frac{P \text{ pounds}}{120 \text{ pounds}}$$

$$120 \cdot \frac{40}{15} = \frac{P}{120} \cdot 120$$

$$320 = P$$

So, a 120-pound teenage of superhamster strength could carry 320 pounds of food. For comparison, according to the US census, the average American ate approximately 257 pounds of fruit in 2009 (including fresh fruit, dried fruit, and fruit juice). So this teenager could carry 25% more fruit than they would typically eat in a year.

Very often in mathematical comparisons, it is helpful to create a comparison "per unit" of some quantity: miles *per hour*, dollars *per pound*, and so on. These "per unit" comparisons mean "per *one* unit". For example, if Bob buys 3 pounds of Swiss cheese for $19.50, then we can find an equivalent ratio

$$\frac{19.50}{3} = \frac{6.50}{1}$$

and see that the cheese costs "6.50 dollars per (one) pound".

Unit Rate

A **unit rate** is a ratio in which the value of one of the quantities is 1. For example "miles per hour", meaning "miles traveled in *one* hour".

Before we move on, the maneuver we made in the last example — taking the reciprocal of both sides — bears another moment of reflection.

Tangent: Explaining Reciprocal of Both Sides

Is "take the reciprocal of both sides" a property of equality? In other words, if we know that $a = b$, do we know for sure that $\frac{1}{a} = \frac{1}{b}$ is always true?

The fact that a and b end up in the denominator of a fraction means that we must require that a and

b are non-zero at the start. If a and b are both non-zero, then we can use DPOE with both numbers.

$$\begin{aligned}
a &= b && \text{We assume that this is true, and that } a \text{ and } b \text{ are non-zero}\\
\frac{a}{a} &= \frac{b}{a} && \text{DPOE, using } a\\
1 &= \frac{b}{a} && \text{Simplify}\\
\frac{1}{b} &= \frac{b}{a \cdot b} && \text{DPOE, using } b\\
\frac{1}{b} &= \frac{\cancel{b}}{a \cdot \cancel{b}} && \text{Cancel the common factor}\\
\frac{1}{b} &= \frac{1}{a} && \text{Voilà}
\end{aligned}$$

So, in the end, if two numbers (both nonzero) are equal, then we know that their reciprocals are equal as well. "Take the reciprocal of both sides" is a property of equality.

Of course, we might be faced with numerators and denominators that are more complicated. Here's an example with variables on both sides of the equation. Our approach is to use MPOE to clear *both of the denominators*.

Example 6.3

Solve for x: $\dfrac{x+1}{2} = \dfrac{x+2}{5}$

Solution:

$$\begin{aligned}
\frac{x+1}{2} &= \frac{x+2}{5}\\
\cancel{2} \cdot 5 \cdot \left(\frac{x+1}{\cancel{2}}\right) &= 2 \cdot \cancel{5} \cdot \left(\frac{x+2}{\cancel{5}}\right) && \text{MPOE two times!}\\
5(x+1) &= 2(x+2)\\
5x+5 &= 2x+4 && \text{distributive property}\\
3x &= 1 && \text{SPOE two times!}\\
x &= \frac{1}{3} && \text{DPOE}
\end{aligned}$$

To write our answer in set notation, we have $\mathcal{S} = \left\{\frac{1}{3}\right\}$.

The previous example may give you some insight into how cross-multiplication works, and might come in handy when working on the problems and exercises.

WARNING!

Solve: $\dfrac{x+2}{5} = \dfrac{7}{6}$

There may be a temptation to subtract 2 from both sides in the first step.

$$\frac{x+2-2}{5} = \frac{7-2}{6} \qquad \Longrightarrow \qquad \frac{x}{5} = \frac{5}{6}$$

But, to do so would be **Evil and Wrong**! The x+2 is grouped together by the fraction bar (vinculum) and that little group is divided by 5. Before we can manipulate the group, we have to undo the division by 5. Then, the 2 is free to be subtracted.

Note that this is the same kind of structure as

$$4(x+2) = 16$$

Here we have to divide both sides by 4, before we can use SPOE on the $x+2$. For some reason the temptation to subtract first is stronger when in fraction form (maybe because the grouping isn't obvious without the parentheses). In any case: beware!

6.2 Applications of Proportional Reasoning

Proportional reasoning encompasses a key set of skills that are important to have in your problem solving toolbox. In this section we'll discuss a few of the classic ways that ratio and proportion show up in problem solving contexts.

6.2.1 Percent

Startup Exploration: Spam "Less Sodium"

Bob is counting cans of Spam in his pantry. For every 3 cans of "low-sodium" Spam, he has 5 cans of Spam with the usual (higher) amount of sodium. What percent of the Spam in his pantry is low-sodium?

Break the word "percent" into it component parts, "per cent", and think about what each part means. The word "cent" indicates 100 (there are 100 cents in a dollar, and 100 years in a *cent*ury), while the word "per" indicates that we are making a comparison (earning \$10 *per* hour at a job means you earn \$10 for every hour that you work).

Putting the words back together, percent means "per 100" or "for every 100". When we write a percent as 75%, or say out loud "seventy-five percent", we're really mean "75 per 100" or "75 out of 100". We could write the decimal 0.75, or the fraction $\frac{75}{100}$.

The fraction $\frac{75}{100}$ isn't in lowest terms, so we could write instead:

$$\frac{3}{4} = \frac{75}{100}$$

Proportion is the heart of percent, and percent problems are all really about the percent proportion:

$$\frac{\text{part}}{\text{whole}} = \frac{\text{percent}}{100}$$

So given a percent question, we just have to ask ourselves which pieces of the percent proportion we know, and which we are trying to find.

Example 6.4

Write proportions that could be used to solve each of the following:

1. What number is 10% of 20?

2. What percent of 75 is 50?

3. 15 is 25% of what number?

Partial solution:

1. $\dfrac{x}{20} = \dfrac{10}{100}$ We're given the percent (10%) and the whole (20). The word "of" is one hint that 20 is the whole. We're looking for the percent, so that's where we'll write the unknown (x, or whatever variable you like). Solve the proportion for the unknown, and we'll have our answer.

2. $\dfrac{50}{75} = \dfrac{x}{100}$ Here we are asked for the percent. The other information has to be untangled a bit, but we can see that 75 is the whole and 50 is the part.

3. $\dfrac{15}{x} = \dfrac{15}{100}$ We're given the percent again, only this time the question asks "of what number", indicating that we're seeking the whole. The unknown ends up in the denominator, but that's no problem if we remember to reciprocalize!

In Bob's case, he has 3 cans of "low-sodium" Spam for every 5 cans of "regular sodium". That's a part-to-part comparison. To write that as a part-to-whole ratio, note that 3 cans are "low-sodium" for every 8 cans that he counts. We setup the percent proportion to find the percent of cans that were "low-sodium":

$$\frac{3 \text{ less sodium cans}}{8 \text{ cans}} = \frac{p \text{ ``low-sodium'' cans}}{100 \text{ cans}}$$

Which means 37.5% of the cans in Bob's pantry are "low-sodium".

6.2.2 Probability

Questions about chance and probability are very often just proportional reasoning questions in fancy clothes. We won't get into much detail about probability in this course, except to hit some of the highlights.

Startup Exploration: 1d12

Suppose we were to roll a standard 12-sided die (with faces showing the numbers 1–12) 1000 times. How many times would we expect to roll a prime number?

There are 12 numbers on the die, and these numbers comprise the **sample space** of the experiment:

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

Five of the numbers in the sample space are prime numbers $\{2, 3, 5, 7, 11\}$. So, the probability of rolling a prime number on a single toss of the die is $\frac{5}{12}$.

Sample Space

The sample space of a probability experiment is the set of all possible outcomes of the experiment.

Generally speaking, the probability of a certain desired outcome occurring in an experiment is the ratio

$$\frac{\text{number of ways the desired outcome can occur}}{\text{number of outcomes in the sample space}}$$

This probability $\frac{5}{12}$ tells us that if we threw the die 12 times and wrote down the numbers, we'd expect to see approximately 5 prime numbers among our data. To predict how many prime numbers we might see when throwing the die 1000 times, we set up a proportion. Let p be the number of primes we expect in 1000 trials.

$$\frac{5 \text{ primes}}{12 \text{ trials}} = \frac{p \text{ primes}}{1000 \text{ trials}}$$

Solving for p, we see that $p \approx 417$. So, we predict 417-ish primes in 1000 trials.[1]

Startup Exploration: 2d6

Suppose we were to roll *two* standard 6-sided dice and add the two numbers. If we perform this experiment 1000 times, how many times would we expect to roll a prime number?

Determining the sample space for this experiment requires a bit of thinking. It might be tempting to simply list all of the possible sums — the lowest is 2, the highest is 24 — but these sums are not all equally likely. There's only one way to roll a 2, but there are multiple ways to roll a 9. A clever strategy here is to organize the outcomes in a table like the one shown in table 6.1.

[1] Of course, nothing is guaranteed when it comes to probability experiments. It could be that we get *1000* prime numbers! It's not very likely that this could happen, in fact it's practically impossible. In smaller experiments — throwing the die 50 times, say — it's quite likely that we'd get many more (or many less) than the 20 or some primes we would expect in that case.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Table 6.1: Sample space for rolling two 6-sided dice.

From the table we can see that there are, for instance, four ways to roll a 9. With a bit of counting in the table, we can see that there are 18 cells that contain a prime number (out of 36 cells total). So, the probability of rolling a prime number in this experiment is $\frac{15}{36}$. We set up and solve our proportion:

$$\frac{15}{36} = \frac{p}{1000} \qquad \Longrightarrow \qquad p \approx 417.$$

In 1000 trials, then, we would expect to see approximately 417 prime numbers when rolling two 6-sided dice and finding their sum.[2]

The examples above discuss *theoretical probability*. We were working in a "perfect world" and basing our predictions and conclusions on properties of the dice. The alternative would have been to conduct an experiment and then use the data from that experiment to predict the outcome of future experiments. We call that *experimental probability*, and an example from that department will close out this section.

6.2.3 Experimental Data

In an ongoing project about goat safety, cryptozoologists at YeardleighCorp Labs have proposed a study of the hunting behavior of the chupacabra.[3] The "scientists" plan to release a chupacabra into the hills around YeardleighCorp Labs, and then study the impact on the local wild goat population.

Before the study can begin, the scientists need to estimate the existing population of goats. It's clearly impossible to actually count all of the wild goats, so to make an estimate the biologists propose to use the *capture-recapture* (or *capture-mark-recapture*) process.

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[2] Isn't it interesting that the probability of rolling a prime number on a single 12-sided die is the same as rolling a prime number on a pair of 6-sided dice? Is this a coincidence, or is there a mathematical connection between these two probabilities that suggests why they are the same?

[3] Cryptozoology (which comes from the Greek "crypto", meaning "hidden", and "zoology", the study of animals) is a pseudoscience (the prefix "pseudo" means "false") based on the study of animals that have not been proven to exist. The "animals" in question are called "cryptids". Famous cryptids include the Loch Ness monster, bigfoot, the yeti, and the chupacabra. The chupacabra, Spanish for "goat sucker" is, allegedly, a creature that attacks and drinks the blood of livestock, especially goats.

First, they will capture a sample of goats and mark them with ear tags. Then these tagged goats will be released and allowed to mingle with all the untagged goats. The biologists will wait a few weeks for the goats to mix thoroughly, then capture second sample of goats.

The assumption is that the proportion of "tagged goats to total goats *in the second sample*" is the same as the proportion of "tagged goats to total goats *in the study area*". (A second assumption is that the population remains constant during this process.) Solving the resulting proportion will give an estimate for the number of goats in the study area.

Example 6.5

Suppose this team of cryptozoologists initially captures and tags 25 goats, then releases them. Several weeks later, they capture a second sample of 100 goats, 6 of which have tags. What is a reasonable estimate for the size of the goat population?

Solution: Our assumption is that the ratio of *tagged goats* to *all goats* represented by the second sample is equivalent to the ratio for the whole population. Originally, 25 goats were tagged. Then the second sample of 100 contained 6 tagged goats. So, we can estimate the total population as follows:

$$\frac{\text{6 tagged in second sample}}{\text{100 total in second sample}} = \frac{\text{25 tagged in population}}{g\text{ total in population}}$$

Solving the proportion:

$$\begin{aligned}
\frac{6}{100} &= \frac{25}{g} \\
\frac{100}{6} &= \frac{g}{25} && \text{take reciprocal of both sides} \\
\frac{100}{6}\cdot 25 &= \frac{g}{25}\cdot 25 && \text{MPOE} \\
417 &\approx g
\end{aligned}$$

Based on the data, a reasonable estimate is that there are approximately 417 goats in the study area.

There is a great deal more that could be said about probability, some of which you have probably seen before. There are other ways to represent a sample space (like a tree diagram), there are other kinds of experiments in which certain events influence other events (you may recall that it sometimes matters whether or not you "put the marble back into the bag")... but for now, we just wish to mention the aspects of probability include a taste of proportional reasoning.

We now turn to another application of proportional reasoning, and our first in-depth look at a certain kind of linear function.

6.3 Direct Variation

Name of Startup Exploration

In her search for delicious new flavors, Yeardleigh travels to Indian to learn about traditional cooking techniques and flavor combinations. In preparing for her visit, she looks up looks up the currency exchange rate and learns that 1 US dollar (USD) is equal to approximately 60 Indian rupees (INR).

Make a table and a graph showing how many rupees Yeardleigh will get in exchange for 100, 200, 300, 400, 500 USD.

How many rupees will Yeardleigh get for exchanging 250 USD? How much is 22 500 INR worth in USD?

6.3.1 Variation and Direct Variation

When scientists begin to understand a phenomenon, like a law of physics, it's often quite helpful to understand it in terms of a relationship: how one quantity changes with respect to another.

For example: The greater the mass of an object, the greater the gravitational force acting on that object.[4] We say that the force varies directly with mass. There is an equation that describes this relationship, but the key idea is that the force acting on an object due to gravity is directly proportional to the object's mass.

Direct Variation

The variables x and y are said to be **directly proportional** if their values have a constant ratio. In other words, $\frac{y}{x} = k$, where k is a constant called the **constant of variation**.

An equation of the form $y = kx$ is called a **direct variation**. The quantities x and y are directly proportional (can you see how this equation is related to the equation in the previous paragraph?).

Since we have a proportional relationship, we can solve direct variation problems using our knowledge of proportions. We will also see that the graphs, tables, and equations for direct variations share special characteristics.

This also means that all of the proportion problems we solved earlier can be solved by looking at them as linear functions. If we want to estimate how many goats live in the chupacabra study area, we can model the situation using a linear equation and find the answer as a point on the line.

[4] When the masses of the two objects are very different (for example, if we are comparing your mass to the mass of the Earth), then the larger object's mass dominates the interaction. The equation describing this relationship is part of Isaac Newton's "Law of Universal Gravitation".

Let's look back at Yeardleigh's currency exchange. The rate "60 rupees per dollar" is constant. The number of Indian rupees Yeardleigh receives is directly proportional to the number of US dollars she exchanges. The two quantities are in direct variation.

If we let y represent the number of rupees, and x represent the number of dollars, then we have the equation

$$y = 60x,$$

or we can write this equation to highlight the ratio

$$\frac{y}{x} = 60$$

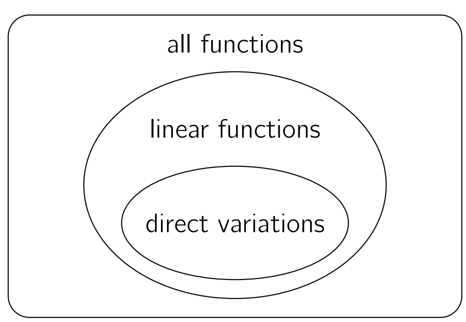
and see that the constant of variation in this scenario is the exchange rate of 60 rupees per dollar.

More about k (constant of variation)

k = value of dependent = y value of independent x In a word problem context, you divide a value of your dependent variable by the corresponding value of the independent variable. If you are given a table, data point, or graph (out of context) you just take a y- value and divide it by an x-value. This constant is the most important part of a direct variation. With it you can write the equation, you can solve problems, you can test to see if data is a direct variation, you can describe the graph (which quadrants it appear in, correlation, steepness, etc.)

6.3.2 Direct Variation: a Type of Linear Function

The family of direct variations is a subset of the family of linear functions, in the same way that the integers $\mathbb{Z}$ are a subset of the rational numbers $\mathbb{Q}$.



Every direct variation is a linear function, but not every linear function is a direct variation. What makes direct variation "special"?

Graph of a Direct Variation

This is the easiest way to spot a direct variation. It must be a straight line that goes through the origin. That's it, really! Since every direct variation is of the form $y = kx$, it's quite easy to see that when $x = 0$, we have

$y = k \cdot 0 = 0$. So the point $(0, 0)$ is always on the graph of a direct variation.

Equation of a Direct Variation

The equation of a direct variation will be of the form $y = kx$. It seems simple enough, but trickiest thing that might come up here is if we encounter an equation which, at first glance, doesn't appear to be a direct variation but which, with a little manipulation (using the POEs), can be transformed into a direct variation.

Example 6.6

Which of the following rules show that x and y are directly proportional? If a rule is a direct variation, state the constant of variation.

(a) $\dfrac{y}{x} = \dfrac{1}{2}$ (b) $2y = 8x$

(c) $y + 7 = 2x + 4$ (d) $3y + 1 = x + 1$

Solution: Equations (a), (b), and (d) represent direct variation; equation (c) does not. Equation (a) is clearly a direct variation with $k = \frac{1}{2}$. With equation (b) we can use DPOE to divide both sides by 2. The result is an equation of the form $y = 4x$, and that's another direct variation with $k = 4$.

With equation (c) we can use SPOE to get y by itself, but the result is $y = 2x - 3$, and this is not a direct variation. When we isolate y with equation (d) — first using SPOE and then DPOE — we have $y = \frac{1}{3}x$. That's a direct variation with $k = \frac{1}{3}$.

The lesson here is: if we can transform and equation using the POEs into and equation of the form $y = kx$, then we have a direct variation.

Data Table of a Direct Variation

Up until this point, we have studied techniques for identifying a table of data (or a sequence) as linear, exponential, or quadratic. We've always been given tables that count sequentially through x-values $(0, 1, 2, 3, \dots)$. As we study the different types of functions in more depth, we will be able to identify functions from tables which aren't sequential, or which jump around.

For directly proportional relationships, we should be able to recognize not just that they are "linear", but that they are specifically "direct variation". The test for a direct variation goes back to the defining characteristic of a direct variation: the fact that $\frac{y}{x}$ is constant.

Example 6.7

Do the following data tables show direct variation? If so, write the equation for the direct variation.

(a)

x	y
-7	21
9	-27
13	-39

(b)

x	y
40	10
16	64
-20	-5

Solution: Table (a) shows a direct variation, since for each line of the table, we have the same ratio:

$$\frac{y}{x} = \frac{21}{-7} = \frac{-27}{9} = \frac{-39}{13} = -3$$

The equation for this direct variation is $\frac{y}{x} = -3$ or $y = -3x$.

Table (b) does not represent a direct variation. The first and last rows have the same ratio, but the ratio of the middle row is different:

$$\frac{10}{40} = \frac{-5}{-20} = \frac{1}{4} \quad \text{but this is different from} \quad \frac{64}{16} = 4$$

The ratio $\frac{y}{x}$ has to be the same for all points, otherwise we do not have a direct variation. This is one of the things that makes a direct variation special. A table can still be linear and not be a direct variation: there are lots of straight lines that *do not* go through the origin! In chapter 7 we will learn a different way to determine whether a table of data represents a (generic) linear function.

Here's one more example of the kind of question we might encounter about direct variation.

Example 6.8

Find the missing value in each case: (a) A direct variation includes the points $(16, -2)$ and $(x, 4)$. Determine the value of x. (b) A different direct variation includes the points $(6, 30)$ and $(-10, y)$. Determine the value of y.

Solution: We are told that we have a direct variation, and so we know that the two points must share the same ratio. We can therefore set up a proportion to solve part (a):

$$\frac{-2}{16} = \frac{4}{x}$$

and solve to find the value of x. We find that $x = -32$.

Let's use a different approach to solve part (b). Knowing one point is enough to write the equation for the direct variation. We know that the point $(6, 30)$ is on the direct variation, and so

$$k = \frac{y}{x} = \frac{30}{6} = 5.$$

Therefore, we have the equation $y = 5x$. We can then substitute -10 for x and solve to discover that $y = -50$.

A final note: Technically, we can write all of these ratio with x over y and find the answers we are looking for. The reason we stick so closely to "y over x" is the relationship that this quantity has to the unit rate in the problem. Plus, we want the notation that we use with direct variations to be consistent with the notation we use with linear functions generally.

6.4 Inverse Variation

Extended Exploration: Teeter Totter Nickels

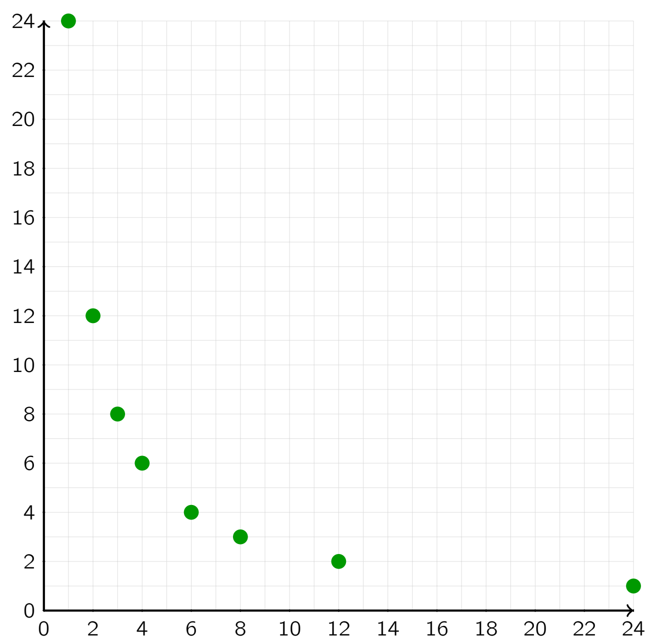
[TODO] Click here to visit the extended exploration: Teeter Totter Nickels

Startup Exploration: Herb Garden

François has enough mulch to plant an herb garden covering 24 square meters. Plus, he has whole barn full of snap-together fencing units that are each 1-meter long. How many different herb gardens can François create, if he wants to use all the mulch and surround the garden with a rectangular fence?

To solve François's problem, we need to find a rectangle that has natural number side lengths and whose area is 24 square meters. One candidate is the long, skinny rectangle that is 1 meter by 24 meters. Of course, that's not the only one! We list the possible rectangles below and plot a graph of the data.

Length (x)	Width (y)
1	24
2	12
3	8
4	6
6	4
8	3
12	2
24	1



The first thing that should jump out about this graph is that it is definitely not linear!

Full Disclosure

Even though we're beginning our discussion of linear functions, inverse variation is *not* a linear relationship. We discuss it here just as a way to compare and contrast inverse and direct variation.

Inverse Variation

The variables x and y are said to be *inversely proportional* if their values have a constant product. In other words, $xy = k$, where k is a constant called the *constant of variation*.

An equation of the form $y = \frac{k}{x}$ is called an *inverse variation*. The quantities x and y are inversely proportional.

In an inverse variation (where the variables are inversely proportional), the most important thing to remember is the fact that $x \cdot y$ is constant. For every point (x, y) the product of x and y is always the same number.

When we force the product of two numbers to be the same, certain patterns arise. Most notably, if the value of one of the quantities goes up, the other one has to go down.

Data Table of an Inverse Variation

If we step away from context and think purely about data, the test for whether some data shows an inverse variation is whether $xy = k$, some constant value.

Example 6.9

Do the tables below show inverse variation? If so, write the equation for the direct variation.

(a)

x	y
2.5	20
-10	-5
8	5.5

(b)

x	y
3.5	4
1	14
-7	-2

Solution: Table (a) does not represent an inverse variation. The first and second rows of the table have the same product (50), but the last row has a different product (44) Table (b) does represent an inverse variation: all three rows have the same product (14). So, table (b) has equation $xy = 14$.

Example 6.10

Find the missing values in the table, given that the data represent an inverse variation.

(a)

x	y
3	a
9	4
2	b
c	8

Solution: The second row of the table gives us the clue we need: $9 \cdot 4 = 36$, so that must be the constant of variation for this inverse variation. Then, it is simply a matter of finding the partner of each value that multiplies to get 36.

In the first row, $3a = 36$ implies that $a = 12$. In the third row, $2b = 36$ implies that $b = 18$. In the fourth row, $8c = 36$ implies that $c = 4.5$.

Equation of an Inverse Variation

We have seen that the equation for an inverse variation is either $yx = k$ or $y = \frac{k}{x}$. Writing the equation for an inverse variation brings out two important features that we must be aware of. Consider the equation

$$y = \frac{k}{x}$$

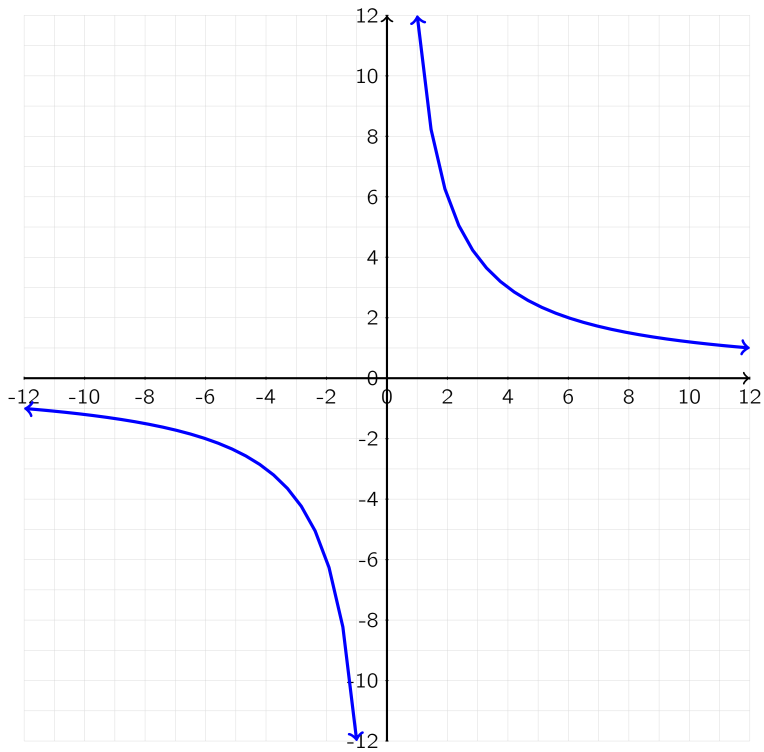
First, what happens when $x = 0$? Yikes! Remember, division by zero is undefined, and so 0 is an illegal value for x. We say that 0 must be *restricted from the domain* of the function.

Second, what happens when $y = 0\ldots$ or is that even possible? Note that as x gets larger and larger, the value of the fraction $1/x$ gets smaller and smaller (closer and closer to zero). But, no matter how big x gets, $1/x$ will never be *equal to* zero. It will get close to zero. Super close! Impossibly close! But it will never equal zero.

We can see this unusual behavior reflected in the graph.

Graph of an Inverse Variation

Not being able to have an x or y value of zero does very interesting things to the graphs. Consider the inverse variation $xy = 12$ or $y = \frac{12}{x}$:

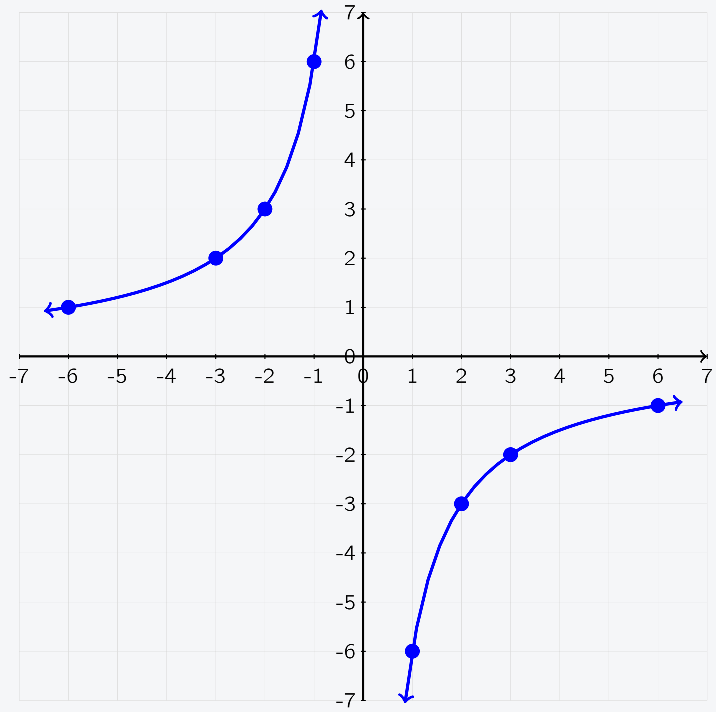


Notice a few things: First, the graph is in two pieces, and since it never crosses the x- or y-axis, it appears entirely in the first and third quadrants. It appears in those particular quadrants because of the defining feature of inverse variation: x and y have a constant product.

This graph is the picture of all points whose coordinates have a product of (positive) 12. The only way to get a positive product is to either multiply two positive numbers, or two negative numbers — exactly the description of the coordinates in the first and third quadrants! What do you suppose the graph of $y = \frac{-12}{x}$ might look like?

Example 6.11

Write the equation for the inverse variation pictured in the graph below.



Solution: Since we know that this is an inverse variation, all we need is to find the coordinates of one point. Some are plotted for us, so that's good news. If we use the point $(-2, 3)$, we can see that $k = xy = -2 \cdot 3 = -6$. Once we have found the constant of variation, we can easily write the equation: $xy = -6$ or $y = \frac{-6}{x}$.

About the "Family" of Inverse Variation

You may have noticed that inverse variation doesn't seem to fit into any of the function families we have looked at so far. This is true. Inverse variation doesn't belong to linear, quadratic, exponential families of functions, and so we won't do much more with inverse variation for quite a while.

Inverse variation is actually a member of the super-family of functions called *rational functions*, which is more of a focus in algebra 2. This rational super-family has a number of interesting and surprising features, and they can take on a whole variety of bizarre shapes.

Inverse variation graphs do have something in common with exponential functions: asymptotes. As we saw, inverse variations get impossibly close to both the x- and y-axis, but never reach them. Generally, these graphs have two asymptotes, one horizontal, one vertical.

Chapter 7

Linear Functions

A curve does not exist in its full power until contrasted with a straight line.

— Robert Henri, American painter

So far we have learned quite a bit about the family of linear functions. From our work with sequences we learned how to tell if a sequence was arithmetic. We can write recursive rules for arithmetic sequences, and we can change those recursive rules into into explicit formulas.

We turned arithmetic sequences into tables of data, and when we graphed some linear functions, we saw that all were straight lines. We saw a preview of how different transformations might change the way its graph looks.

Most recently we learned about a specific type of linear function, the direct variation. In this chapter we will bring together all of the pieces we have learned previously into a single coherent, and very important, idea.

7.1 Rate of Change and Slope

Extended Exploration: Calculating Speed

[TODO] Click here to visit the extended exploration: Calculating Speed

Startup Exploration: CCC

Bob enrolls at Cheeseville Community College, and the admissions counselor there gives him the following table estimating the cost to attend. The total cost of a semester at CCC is made up of two expenses: (a) the fixed fees, which are the same for every student every semester (technology fee, printing allowance, and so on), and (b) the tuition rate, which varies per credit hour.

Most courses at CCC are either 3 or 4 credit hours per semester. The table summarizes several combinations of 3- and 4-hour courses.

No. Credit Hours	3	4	6	7	10
Cost (dollars)	501	654	960	1113	1572

What can you learn from the table? How much is tuition per credit hour? How much does CCC estimate Bob will pay in fees?

7.1.1 Unit Rates in Data

Since we're fresh from a discussion of proportional reasoning, perhaps our first guess might be that this is a direct variation. Unfortunately, a quick check of the ratios $\frac{y}{x}$ shoots that idea down:

$$\frac{501}{3} = 167 \qquad \text{but} \qquad \frac{654}{4} = 163.5$$

Maybe we have an arithmetic sequence? If so, then the terms of the "cost" sequence will have a constant difference. But, if we take the differences between neighboring terms, we have different values:

$$654 - 501 = 153 \qquad \text{but} \qquad 906 - 654 = 306$$

But, hang on. Look at the corresponding "number of credit hours". These numbers don't increase by the same amount at every step, like they did for our arithmetic sequences. There's a jump from 4 to 6. We're going to have to be a bit more clever to account for this!

Notice that when "credit hours" increases by 1, "cost" increases by \$153. And then, when "credit hours" increases by 2, "cost" increases by \$306 — the increase in both values has doubled! This even works at the end of the table: When the number of hours jumps by 3 and the cost jumps by \$459, the rate "\$dollars per 1 credit hour" says the same:

$$\frac{153}{1} = \frac{306}{2} = \frac{459}{3} = \frac{\text{change in the total cost}}{\text{change in the number of hours}}$$

So, even though the table skips over some numbers, the *change* in the cost still corresponds to the *change* in the number of hours. So, the tuition rate at CCC must be \$153 per credit hour.

Example 7.1

Does the data in the table below represent someone moving at a constant speed? If so, what is the speed?

Time (sec)	Distance (m)
5	17.5
12	42.0
31	108.5

Solution: Comparing the first two data points, we see that the change in distance is $42.0 - 17.5 = 24.5$ meters, and that this occurs over $12 - 5 = 7$ seconds. This ratio "24.5 meters per 7 seconds" corresponds to the unit rate "3.5 meters per second".

Comparing the second pair of data points, we see that a change in distance of $108.5 - 42.0 = 66.5$ meters happens over $31 - 12 = 19$ seconds. The ratio "66.5 meters per 19 seconds" corresponds to the unit rate "3.5 meters per second".

So, yes, this table does represent movement at a constant speed of 3.5 meters per second.

7.1.2 Rate of Change

We want to be able to tell if *any* data set is linear. The examples of speed and hourly tuition begin to suggest a general approach. We are seeking a constant **rate of change**.

Rate of Change

Rate of change is a measurement of how quickly the dependent variable changes relative to a one-unit change in the independent variable.

To compute the rate of change of y with respect to x, we want to "unitize" a change in y (the dependent variable) with respect to the corresponding change in x (the independent variable).

To write this out using mathematical notation, we use an abbreviation for the idea of "change": the Greek letter delta, Δ. This makes the definition of rate of change look like the following:

$$R.O.C. = \frac{\Delta \text{ dependent variable}}{\Delta \text{ independent variable}} = \frac{\Delta y}{\Delta x}$$

The symbol "Δy" is pronounced "delta y" or "change in y". We sometimes say that rate of change is "delta y over delta x".

Speed is an example of a rate of change. It is a measurement of how distance changes relative to a one-unit change in time: meters per (one) second. Bob's tuition at Cheeseville Community College is another example that measures how his total expense changes relative to a unit-change in the number of credit hours he takes: dollars per (one) hour.

7.1.3 Slope

Rate of change is a very important concept in mathematics and it is a general term that doesn't just apply to straight lines. Different types of functions have different patterns emerge in their rates of change. The study of rate of change led to the development of calculus!

But, the rate of change of a straight line is special: it's always the same. When it comes to linear relationships, the key is knowing that the *rate of change for linear data is constant*. We have a special term for the rate of change in a linear relationship.

Slope

Slope is a measurement of the steepness of a line.

For a line, slope and rate of change are describing the same thing. We don't really talk about the "slope" of a curve (like the graph of an exponential and quadratic functions), since the steepness changes, as we saw in section 3.5.

In mathematical notation, the letter most often used to abbreviate slope is m.[1] So, we can update our notation from earlier:

$$m = \frac{\Delta y}{\Delta x}$$

Slope from Data

As we have seen in the earlier examples, finding slope from a data table is rather easy. We find the change between two y-values and find the change between the corresponding x-values. When we unitize this ratio — in other words, when we divide Δy by Δx — we've got slope. Woot!

This works even if the data is not given in table form. You may have heard the old saying, "The shortest distance between two points is a straight line." Well... what's the slope of that line?

[1] "Why m?" we bet you're wondering. Many people have looked into this question, digging back through mathematical writing to try and learn where this abbreviation comes from. It seems that the earliest recorded use of m for slope was in 1757 by Italian mathematician Vincenzo Riccati. But his work includes no explanation of *why* the letter m was used.

Example 7.2

Find the slope of the line connecting the points $(11, 4)$ and $(16, 29)$.

Solution: The change in the y-values is $\Delta y = 29 - 4 = 25$, and the change in the x-values is $\Delta x = 16 - 11 = 5$. So, the slope of the line between these two points is

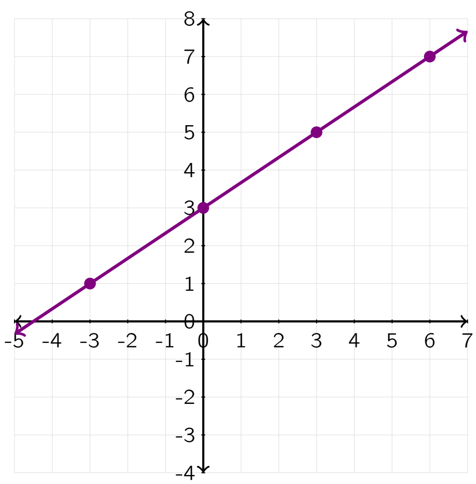
$$m = \frac{\Delta y}{\Delta x} = \frac{25}{5} = 5.$$

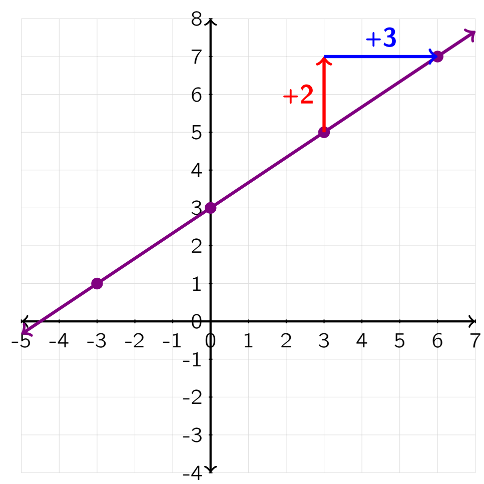
So, the line connecting the points $(11, 4)$ and $(16, 29)$ has a slope of 5.

Slope From a Graph: The Slope Triangle

Suppose we're given the graph of a line and asked to find the slope. We could create a data table using points on the line and find the slope from that data. Or, we could inspect the graph directly using a *slope triangle*. A slope triangle is a right triangle that connects two points with a single vertical move and a single horizontal move.

Here is the graph of a line with some points marked. To find the slope of the line, we draw a slope triangle.

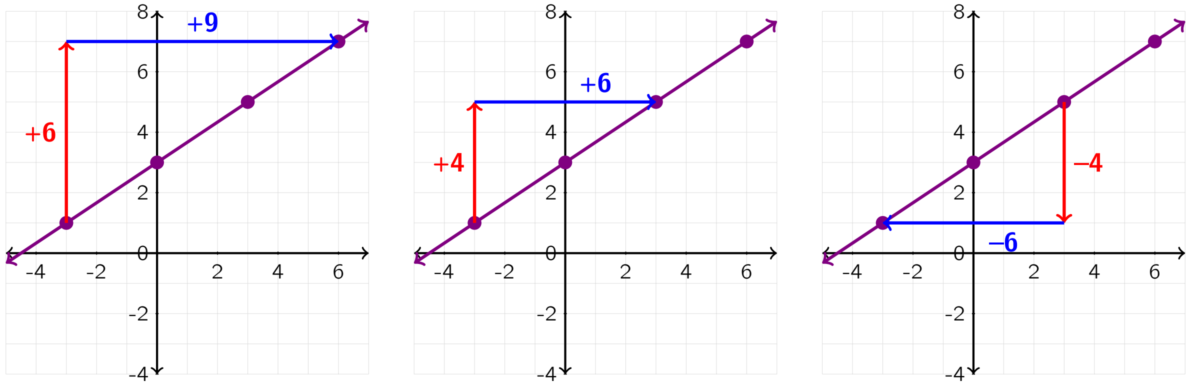




the red arrow shows the vertical change. It starts at 5 and goes to 7, a vertical distance of $7 - 5 = 2$ units. The blue arrow shows the horizontal change. It starts at 3 and goes to 6, a horizontal change of $6 - 3 = 3$ units. Now we can calculate slope

$$m = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{2}{3}.$$

Since the rate of change is constant for a straight line, it doesn't matter which two points we choose, or which point we choose as the starting place. Each of the slope triangles below can be used to compute slope of this line!



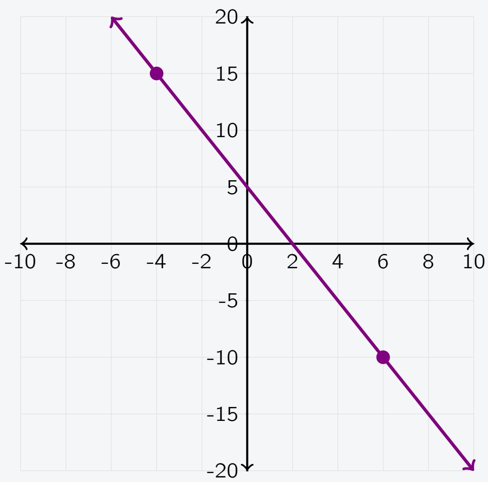
Again, we don't have to use a slope triangle. We can just convert at least two of the points on the graph into a table and then use the techniques for finding slope from a data set.

There are two commonly mixed up things about slope. First: reciprocalizing the ratio. Remember it's change in y over change in x, not the other way around.

Second, folks sometimes have problems with sign. A rule of thumb: if the data has a positive correlation, the slope is positive. If the data has a negative correlation, the slope is negative. It's a good habit always to double-check the sign of a slope to verify that it makes sense in context.

Example 7.3

Find the slope of the line depicted in the following graph.



Solution: This problem exhibits two key features: the slope of the line will be negative (since the line decreases as we move to the right), and we're going to have to be careful about the scale on the axes!

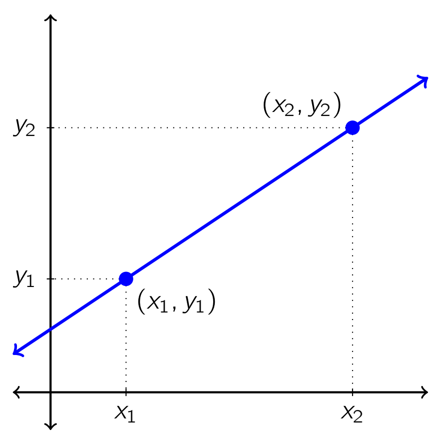
We might see a vertical change of one "tick mark" on the axis, but note that one tick mark is a change of 5 units. Similarly, one tick on the x-axis is a change of two units.

If we start at the point on the left and move to the point on the right, we have a horizontal change of $+10$ units in the horizontal direction and -25 units in the vertical direction (a negative vertical change means moving downwards). So, the slope is

$$m = \frac{-25}{10} = \frac{-5}{2} = -\frac{5}{2}$$

7.1.4 The Slope Formula

The process of computing slope is always basically the same, so we can derive a formula for slope. Study the picture below, showing a line through two "generic points" called (x_1, y_2) and (x_2, y_2).



The vertical change between the two points is $y_2 - y_1$, and the horizontal change is $x_2 - x_1$. So we can computer the slope between these two points using our familiar formula

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

The trickiest part about using this formula is being consistent about which point we call (x_1, y_1) and which point we call (x_2, y_2). It doesn't matter which point is which — but we have to be consistent.

Tangent: The Order of the Points

That last sentence says "it doesn't matter which point is which." Why is that? Explain why it doesn't matter which point we call "point 1" and which we call "point 2". In other words, if we have two points called (x_1, y_1) and (x_2, y_2), explain why

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

(Hint: Experiment with some numbers first, then try to extend your observations to the two "generic" points given.)

7.1.5 Horizontal and Vertical Lines

If we think about how slope is calculated — vertical change over horizontal change — a horizontal line has *no vertical change*, or a vertical change of zero. If we pick two points on a horizontal line and calculate slope, we would get 0 divided by some number, which always equals zero! So the slope of a horizontal line is always 0. This makes sense if we remember the slope is a measure of the steepness: a horizontal line is flat, and that means no "steepness" at all!

If we pick two points on a vertical line and try to calculate slope, we will get a horizontal change of 0. So we'll have some number divided by 0... and that's off limits. We say that the slope of a vertical line is *undefined*, since division by 0 is undefined.

7.1.6 Using Slope

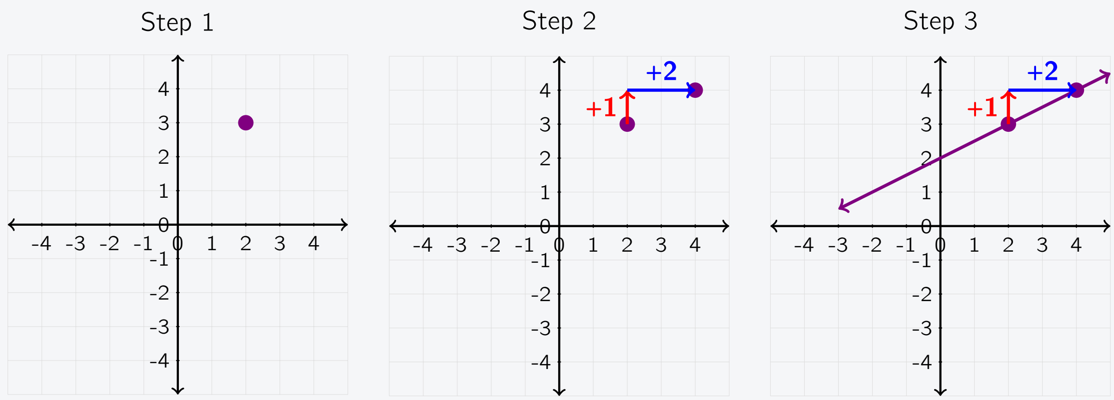
Before, when we were graphing a line we had to take some domain values, like the integers between 3 and 3, plug them in to the equation, and plot every single point, one at a time. Now we have an alternative!

All we really need to graph a line is two points. If we have one point on the line and the *slope*, we can find another point on the line.

Example 7.4

Graph the line that goes through $(2, 3)$ and has a slope of $\frac{1}{2}$.

Solution: Step 1: Plot the given point. Step 2: The slope $\frac{1}{2}$ tells us how to find another point on the line: the numerator is the change in y, so we move up 1 unit. The denominator is the change in x, so we move 2 units to the right. Now we have a second point: $(4, 4)$. Step 3: Connect the dots and you have your graph!



By the way, perhaps now you see why we've been such fans of improper fractions. Seriously, which form is easier to use when creating a graph: a line with slope $\frac{4}{3}$ or a line with slope 1.33333...?

Colinearity

Points that lie on the same line are said to be *colinear*. If we are asked to determine whether two points are colinear... well, that's easy! Any two points will define a line, so any pair of points will be colinear.

But what about three points? How could we determine whether three points are colinear — in other words, whether they all lie on the same line?

Example 7.5

Are the points $(4, 1)$; $(-1, 5)$; and $(1, 2)$ colinear? Why or why not?

Solution: The slope between the first pair of points is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 5}{4 - -1} = \frac{-4}{5} = -\frac{4}{5}.$$

The slope between the second pair of points

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 2}{-1 - 1} = \frac{3}{-2} = -\frac{3}{2}.$$

Those two slopes are not equal, so: nope! These points won't line up in a single line! They are not colinear.

7.2 Point-Slope Form

We are going to learn three different ways to write equations for lines: the point-slope form, the slope-intercept form, and the standard form. Each form tells you something a little different about the line, and each one has its own set of pros and cons. Over the next few sections we will learn about these different forms, what they tell us about lines, what they are good for, and how to convert from one form to another.

Extended Exploration

[TODO] Click here to visit the extended exploration: Deriving the Point Slope Form

Startup Exploration: One Point, One Slope

A line with slope 2 passes through the point $(4, 3)$. Name three other points that also lie on this line. Can you find a point on the line that lies in the second quadrant? The third quadrant? The fourth quadrant?

The first form that we will study is called *point-slope form*. As the name suggests, all we need to write an equation this form is the slope of the line and a point on the line.

In the startup exploration, we are asked to find different points on the line that has slope 2 and passes through the point $(4, 3)$. Finding other points isn't that hard. We saw how to do this in the last example in section 7.1.[2] It is more interesting to consider a generic point (x, y) on this line.

We now have two points on the line $(4, 3)$ and (x, y), and we know the slope is 2. We can arrange all of this information using the slope formula:

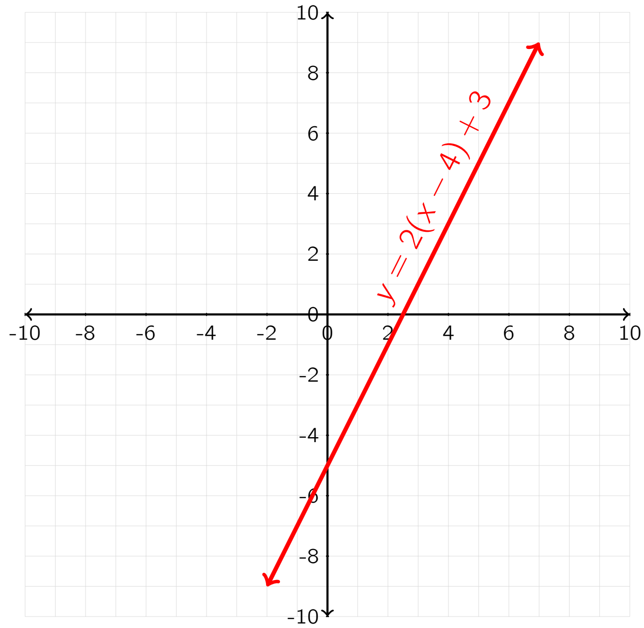
$$2 = \frac{y-3}{x-4}.$$

There should be something familiar about this: we've got an equation with a y and an x in it... so this is starting to look like a graphable equation. All we have to do it transform this equation to isolate y. Let's do that!

$$\begin{aligned} 2 &= \frac{y-3}{x-4} && \\ 2(x-4) &= y-3 && \text{MPOE: multiply both sides by } (x-4) \\ 2(x-4)+3 &= y && \text{APOE: add 3 to both sides} \end{aligned}$$

[2] Well, the line doesn't pass through the second quadrant, so finding a point that meets that requirement is impossible. Something to ponder: are the following statements true or false? (a) A linear function can pass through at most 3 of the four quadrants. (b) A linear function will always pass through exactly 3 of the 4 quadrants.

If we flip this around to write it in a more familiar format, we have $y = 2(x-4)+3$. That's an equation we can graph! If we plot some points (or pull out our favorite piece of graphing technology) we can graph this equation:



What do you know! Graphing this equation gives us exactly the line we were looking for: a line with slope 2 that goes through the point $(4,3)$. So the equation $y = 2(x-4)+3$ describes the line. Can you see the slope and the given point hiding there in the equation?

7.2.1 Deriving the Point-Slope Form

Let's make things even more generic. Suppose we have a line with slope m that goes through the specific point (x_1, y_1). Then let (x, y) be any other random point on the line. We can summarize this using the slope formula, and then transform our equation into a graphable, $y = something$ format:

$$\begin{aligned} m &= \frac{y-y_1}{x-x_1} & \\ m(x-x_1) &= y-y_1 & \text{MPOE: multiply both sides by } (x-x_1) \\ m(x-x_1)+y_1 &= y & \text{APOE: add } y_1 \text{ to both sides} \end{aligned}$$

This last line is what we call the **point-slope form** of a line.

Point-Slope Form

The form of a linear equation that uses the slope and any point on the line. It is written either $y - y_1 = m(x - x_1)$ or $y = m(x - x_1) + y_1$, where m is the slope of the line and (x_1, y_1) is a point on the line.

7.2.2 Using Point-Slope Form

The trickiest part about the point-slope form — as you might anticipate — is handling the plusses and minuses. It can be tricky to remember the $-x_1$ and the $+y_1$, so be on the lookout and always check your signs.

Example 7.6

State the slope of each of the following lines, and name a point on each line.

(a) $y = -3(x - 1) - 4$ (b) $y = 5 - 8(x + 4)$ (c) $y - 8 = 4(x - 2)$

Solution: Let us compare the given equation (a) to the generic point-slope formula

$$y = -3(x - 1) - 4 \qquad \Longleftrightarrow \qquad y = m(x - x_1) + y_1,$$

We can see that the slope is -3. The equation has "-4" whereas the formula wants "$+y_1$", so we change the rule to reflect addition outside the parentheses: $y = -3(x - 1) - 4 = -3(x - 1) + {}^-4$. Now we can see that a point on the line is $(3, {}^-4)$.

With (b), note that things are a bit out of order, but we can commute the terms (taking the sign along, remember!) and have $y = {}^-8(x + 4) + 5$. Now, we have addition in the parentheses, whereas the rule wants subtraction in there. We can make that adjustment easily enough: $y = {}^-8(x - {}^-4) + 5$. From this version, we can see that the slope is -8 and a point is $({}^-4, 5)$.

The definition of point-slope forms gives us this form as an alternative (check the box above)! So, the slope of this line is 4 and a point on the line is $(2, 8)$.

Back in chapter 3, we had to pick some values for the domain, substitute them into the equation, find the values of the range, and plot each point. In section 7.1, we learned that we really just need a point and a slope. This information is ready to be extracted from the point-slope form!

Example 7.7

Graph (by hand) the line $y = 3(x-4) - 5$.

Partial solution: We can see that the slope is 3 and that a point on the line is $(4, {}^-5)$. With this information we find a second point, and then graph the line!

How would we go about writing an equation in point-slope form for data given to us in a table, or as a collection of points?

Example 7.8

Write the equation of the line that goes through the points $(6, 10)$ and $(-1, -4)$.

Solution: First, we find the slope of this line:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10 - {}^-4}{6 - {}^-1} = \frac{14}{7} = 2.$$

Then, we can use either of the two points as the "point" in point-slope form:

$$y = 2(x-6) + 10 \quad \underline{\text{or}} \quad y = 2(x+1) - 4$$

7.2.3 Point-Slope as a Transformation of a Direct Variation

There is another way to look at the point-slope form that relates to transformations in a plane. In section 6.3, we studied direct variations of the form $y = kx$. This is a straight line through the origin, and k is the slope. So let's rewrite this as $y = mx$.

If we want to move the graph of the direct variation up or down, we add something to the equation. For example, if we want to move the graph 12 units up, the equation would become $y = mx + 12$. If we want to move down 7 units, the equation would become $y = mx + {}^-7$. In general, if I wanted to move the graph vertically y_1 units the equation would become $y = mx + y_1$.

If we want to move the graph of the direct variation to the right 5 units, we would replace x with $(x-5)$ and our new equation would be $y = m(x-5)$. If we want to move the graph to the left 8 units, we have the equation $y = m(x+8)$. In general, if we wanted to move the graph horizontally x_1 units, the equation would become $y = m(x - x_1)$.

If we want to cause *both* a horizontal and vertical shift, we would do both transformations to get $y = m(x - x_1) + y_1$. We can think of this transformation as "moving the origin" of the direct variation from $(0, 0)$ to (x_1, y_1).

7.3 Slope-Intercept Form

Point-slope form is handy and quite easy to write in many cases. But, it has those parentheses in it which means that we could simplify it even further. We explore that possibility in this section.

Name of Startup Exploration

Graph the line that goes through the points $(-1, 7)$ and $(2, -2)$. Find the slope through these two points and write two equations in point-slope form for the line.

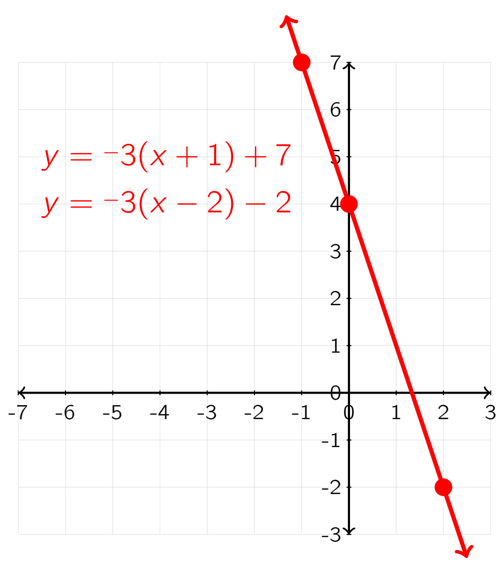
Use your toolbox of equivalence rules to eliminate the parentheses and rewrite each of your point-slope equations. What happens? How is your new equation reflected in the graph?

Point-slope form is the easiest form for algebra students to use when writing the equation of a line. Once we know the slope of the line we only need one point on the line — any point at all — write the equation.

However, point-slope form is a pain for algebra *teachers* to grade. Why? Well, how many points are there on a line? Each one of those points can be used to write an equation for that line. So, there are infinitely many different point-slope equations for a single line! If a student gets creative about what point to use, the teacher may have to do some work to determine whether the student's answer is correct or not.[3]

One way to simplify things would be to designate a kind of "standard point" that we always use at *the point* when writing a point-slope form equation. Luckily, there's a natural point to set as the standard.

The startup exploration asks us to graph, and write two different equations for, the given line.



[3] Don't get any ideas.

We can eliminate the parentheses using the distributive property, and then we can combine like terms as needed. If we use the point $(-1, 7)$, then we have:

$$\begin{aligned} y &= {}^-3(x+1)+7 \\ &= {}^-3x + {}^-3(1) + 7 && \text{distributive property} \\ &= {}^-3x + {}^-3 + 7 && \text{substitution} \\ &= {}^-3x + 4 && \text{substitution/combining like terms} \end{aligned}$$

If we use the point $(2, -2)$, then we have:

$$\begin{aligned} y &= {}^-3(x-2)-2 \\ &= {}^-3(x+{}^-2)+{}^-2 && \text{adjust negative signs, for safety} \\ &= {}^-3x + {}^-3({}^-2) + {}^-2 && \text{distributive property} \\ &= {}^-3x + 6 + {}^-2 && \text{substitution} \\ &= {}^-3x + 4 && \text{substitution/combining like terms} \end{aligned}$$

After this simplification process, the two different point-slope form equations have turned into the *same equation*! It turns out that this always happens: if we simplify two different point-slope equations for the same line (by getting rid of the parentheses and combining like terms), we will always end up with the same equation. This simplified version is called the **slope-intercept form**.

Slope-Intercept Form

The **slope-intercept form** of a linear equation is an equation of the form

$$y = mx + b.$$

The value m is the slope of the line, and b is the value at which the line crosses the y-axis. The point $(0, b)$ is called the y-intercept of the line.

Note from the definition that the equation $y = {}^-3x + 4$ contains two pieces of information. It tells us the slope of the line is $^-3$, and it tells us that the line passes through the point $(0, 4)$. We can verify this in the graph above.

Since a linear function has a constant slope and and can only have one y-intercept (two straight lines can't cross more than once!), a linear function has only one slope-intercept equation.

Graphing a line in slope-intercept form is exactly the same as graph a line in point-slope form. In fact, it's easier! Figuring out the "point" in point-slope form can lead to some challenges with positive and negative numbers. With slope-intercept form, the "point" is much easier to identify. Then, we can use the slope to find a second point and graph the line.

7.3.1 Writing Equations in Slope-Intercept Form

Suppose we are given the slope of a line and a point on the line that is *not* the y-intercept. There are two primary ways to get to an equation for a line in slope-intercept form.

Method 1: Simplify point-slope form. This method works just as we saw earlier. We use the slope and the point to write an equation in point-slope form, which we then simplify using the distributive property and combining like terms. The result is slope-intercept form.

Method 2: The "old school" method for folks who don't like (or never learned) point-slope form. In this approach, we use the slope and the x and y coordinates of the point to write an equation that we can use to calculate b.

Example 7.9

A line as a slope of 2 and passes through the point $(3, 7)$. Write the slope-intercept equation of the line.

Solution: We're given $m = 2$, so we can start writing an equation in slope-intercept form: $y = 2x + b$. We don't know b yet, but we do know a pair (x, y) that is on the line! So, we plug in the x and y values from the point: $x = 3$ and $y = 7$. In other words:

$$7 = 2(3) + b$$

We can now solve for b by subtracting 6 from both sides (SPOE). We find that $b = 1$. So, the slope-intercept equation is $y = 2x + 1$.

These two methods are equally valid mathematically, so pick the approach that you understand the best and feel most confident about.

Example 7.10

Write an equation in slope-intercept form for the line containing the following data points.

x	y
5	12
6	8
7	4
8	0

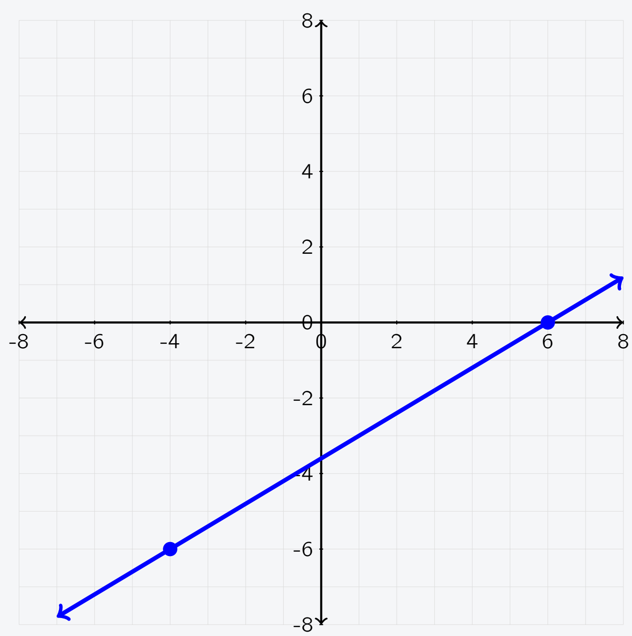
Solution: If a table of data ever includes the y-intercept — the point with x-coordinate 0 — then our job is quite easy. In this case, though, we don't have that. To write the equation, we could use one of the two methods above, or we could do a bit of detective work and find the y-intercept using the pattern in the table.

Notice that the y-values decrease by 4 as x increases by 1. That means the slope of the line is $^{-}4$. It also means we can work our way backwards in the table to the point where $x = 0$ and $y = 32$. So, our equation is $y = {}^{-}4x + 32$.

These equations should look familiar. We saw equations just like these when we were studying arithmetic sequences! The value of the sequence at stage 0 was just the y-intercept. The common difference (the value we added at each step) was the slope!

Example 7.11

Write an equation in slope-intercept form for the line pictured in the graph below.



Solution: If a graph has a clear y-intercept then our job is quite easy. In this case, though, we don't have that. We might try to guess about the point — maybe it's $-3.5\ldots$ish? Would you bet on that? — but we need another approach to be sure.

So, we'll use a slope triangle to find the slope between the two given points $\frac{6}{10} = \frac{3}{5}$. Then, we'll write an equation in point-slope form using the point $(6, 0)$: $y = \frac{3}{5}(x - 6) + 0$. Simplifying using the distributive property and combining like terms gives:

$$y = \frac{3}{5}x - \frac{18}{5},$$

and $-\frac{18}{5} = -3.6$, which is close to -3.5, but not the same thing!

7.3.2 (;,;) Equivalence of the Point-Slope and Slope-Intercept Forms

In this extension section, we'll look in more detail at a claim we made earlier in this chapter. If the content here feels like too much, take a break and come back to it later. Your brain keeps working on things even when you're not thinking about them consciously!

In section 7.3 we made the claim that *always get the same answer* when we start with an equation in point-slope form and we convert to an equation in slope-intercept form (using the distributive property and combining like terms).

Have you started to become suspicious of claims like this? Have you started to wonder, "How do we know that we *always* get the same answer?" This is a good question to ask. After all there are infinitely many different point-slope equations we could write. Surely we're not going to try and convert them all and check that we get the same answer every time.

Here's one way to think about this comparison. We have a line. Since we are going to assume we have a linear function, we know that this line is not vertical. So, that means it must cross over the y-axis at some point $(0, b)$. Suppose (x_1, y_1) is another point on that line.

This line has a slope that we can represent using the slope formula. We'll use the y-intercept as "point 2":

$$m = \frac{b - y_1}{0 - x_1} = \frac{b - y_1}{^{-}x_1}$$

Trust us on this next bit: this will be helpful. We're going to use the POEs to isolate b so that we can make a comparison in a moment.

$$\begin{aligned} m &= \frac{b - y_1}{^{-}x_1} & \\ ^{-}mx_1 &= b - y_1 & \text{MPOE: multiply both sides by } ^{-}x_1 \\ {\color{blue}^{-}mx_1 + y_1} &= b & \text{APOE: add } y_1 \text{ to both sides} \end{aligned}$$

We're going to come back to this last line in a moment.

Our goal is to show that point-slope form and slope-intercept form are equivalent. So, let's write the point-slope form of the equation using the generic point (x_1, y_1) and try to convert it to slope-intercept form.

$y = m(x - x_1) + y_1$	point-slope form
$y = m(x + {}^{-}x_1) + y_1$	rewrite subtraction
$y = mx + {}^{-}mx_1 + y_1$	distributive property
$y = mx + {}^{-}mx_1 + y_1$	the expression in blue is equal to b
$y = mx + b$	slope-intercept form

So, in the end, we have shown that if we have a line with y-intercept $(0, b)$, then the point-slope equation for the line — using any point on the line! — can be converted into the slope-intercept equation of the line.

7.4 Standard Form

So far, we know a lot about writing equations for lines. We can find the slope of a line, we can write equations in point-slope form, and we can convert point-slope form into slope-intercept form. We now look at a final linear form, so-called **standard form**.

Startup Exploration: Goats and Chickens

The chickens got into the goat pen on François's farm! When François asked Hermine how many of each animal were mixed up, she coyly replied that there were a total of 44 feet in the pen.

If the pen contains only (healthy) goats and chickens, how many of each animal could there be inside? How many different combinations are possible?

We know that a healthy goat as 4 feet, and so if the pen contains x goats, they account for $4x$ feet. Similarly, a health chicken as 2 feet. If there are y chickens in the pen they account for $2y$ feet. Finally, we know the total number of feet. So, we have the equation

$$4x + 2y = 44.$$

We can tinker with a little guess and check to find combinations that work. For instance there could be 8 goats and 6 chickens, since $4(8) + 2(6) = 44$. If you find a few combinations, you might start to notice some patterns in the solutions (of which there are 12).

We'll come back to problems like "goats and chickens" in the future. For now, we'll simply point out that our equation above is an example of a linear equation in standard form.

Standard Form

If A, B, and C are Integers where A and B are not both zero, then $Ax + By = C$ is a linear equation in standard form.

Note that this is not a "$y =$" form. This has its challenges and its benefits. In some situations, standard form equations requires a bit of work before they are very useful.[4] For example, A, B, and C don't tell us much about the line. Especially when compared to point-slope form and slope-intercept form, in which important features of the line are visible right there in the equation.

A benefit of this different format is that any line — whether it is a function or not — has a standard form equation. The definition says that A and B can't *both* be zero (at the same time), but one or the other of

[4] Standard form: the fixer-upper form of a linear equation.

them could be zero. This is how we get equations for vertical and horizontal lines. Vertical lines are of the form $Ax = C$ and horizontal lines are of the form $By = C$.

Example 7.12

Are the following equations written in standard form? If not, why not?

(a) $3y = 4x + 2$

(b) $-2x + 6y = 17$

(c) $3x = 15$

(d) $\frac{1}{2}x + 5y = -12$

(e) $2x - 4(y - 2) = 8$

(f) $10 = 3x - 3y$

Solution: Lines (a), (d), and (e) are *not* in standard form. To be in standard form, the x term and the y term must be on the same side of the equal sign; line (a) violates this rule. Standard form states that the coefficients must be integers; line (d) violates this requirement. Standard form has no parentheses; line (e) violates this.

Lines (b), (c), and (f) are in standard form. Note that (c) has $B = 0$, but that's OK; this is the equation for a vertical line. Line (f) is written with the variables on the right-hand side, but otherwise it meets the requirements.

7.4.1 Converting To and From Standard Form

If we are given a line in some other form, then we can convert to standard form using the properties of equality. This is an application of "transforming formulas" that we saw in section 5.8.

Example 7.13

Convert the lines $y = \frac{3}{2}x - \frac{4}{3}$ and $y = \frac{1}{4}(x - 8) + 2$ to standard form.

Solution: The first line is in slope-intercept form, but there's a fraction in there which is no good for standard form.

$$\begin{aligned} y &= \frac{3}{2}x - \frac{4}{3} \\ 6y &= 9x - 8 && \text{multiply through by 6 to eliminate fractions} \\ -9x + 6y &= -8 && \text{put } x \text{ and } y \text{ on the same side} \end{aligned}$$

The second line is in point-slope form, and we can start out by doing the distributive property.

$$\begin{aligned} y &= \frac{1}{4}(x-8) + 2 \\ y &= \frac{1}{4}x - 2 + 2 && \text{distributive property} \\ y &= \frac{1}{4}x && \text{combine like terms} \\ 4y &= x && \text{multiply through by 4 to eliminate fractions} \\ -x + 4y &= 0 && \text{put } x \text{ and } y \text{ on the same side} \end{aligned}$$

Of course, we can convert in the other direction as well, if needed.

Example 7.14

Convert the line $6x - 14y = 21$ to slope-intercept form.

Solution: All we need to do is transform this equation to isolate y.

$$\begin{aligned} 6x - 14y &= 21 \\ -14y &= -6x + 21 && \text{SPOE} \\ y &= \frac{-6}{-14}x + \frac{21}{-14} && \text{DPOE} \\ y &= \frac{3}{7}x - \frac{3}{2} && \text{simplify fractions} \end{aligned}$$

7.4.2 Simplified Standard Form

It turns out that we can generate infinitely many equivalent equations in standard form just by multiplying both sides of a given form by an integer. For example, if we have the line $3x - y = 5$, we might through by 5 and get $15x - 5y = 25$. Or we could multiply through by -1 and get $-3x + y = -5$. All of these lines are the same because we've applied a property of equality (MPOE) to create them.

Since there are so many possible ways to write an equation in standard form, we have a “standardized” version of it. Standardized standard form sounds like an oxymoron, so we call it “simplified standard form”.

There are two additional criteria for simplified standard form.[5] In addition to the rules given in the definition above, we require that A is non-negative (it must be greater than or equal to zero). We also require that A, B, and C share no common factors between them other than 1.

The second of these new rules is the trickier one. The line $10x + 6y = 12$ violates this second requirement since each term is divisible by 2. To simplify this, we divide all of the terms by 2 and have an equation that is in simplified standard form: $5x + 3y = 6$. Note that here 3 and 6 share a common factor, but we're OK since the only factor that *all three* numbers share is 1.

The line we wrote to describe the startup exploration $4x + 2y = 44$ does not meet the requirements for simplified standard form. We must divide through by 2 to fix this: $2x + y = 22$ (does this new equation have an interpretation in the context of the problem?).

Example 7.15

Write an equation, in simplified standard form, for the line that goes through the points $(-3, 6)$ and $(7, 12)$.

Solution: We'll start out as usual, finding the slope between these points:

$$m = \frac{\Delta y}{\Delta x} = \frac{12 - 6}{7 - -3} = \frac{6}{10} = \frac{3}{5}$$

Then, we can use either point to write a line in point-slope form:

$$y = \frac{3}{5}(x + 3) + 6.$$

Since we need integer coefficients we can multiply through by 5 at the start. This will avoid having to

[5] These rules are just a convention, and they are somewhat arbitrary. In future math courses, many of these restrictions about standard form, for example what kinds of coefficients are allowed, may be relaxed.

carry out the distributive property with that fraction.

$$\begin{aligned}
y &= \frac{3}{5}(x+3)+6 \\
5y &= 3(x+3)+30 && \text{multiply through by 5} \\
5y &= 3x+9+30 && \text{distributive property} \\
5y &= 3x+39 && \text{combine line terms} \\
-3x+5y &= 39 && \text{SPOE: subtract } 3x \text{ from both sides} \\
3x-5y &= -39 && \text{multiply through by } -1
\end{aligned}$$

This last step is required because simplified standard form requires that the coefficient of the x-term be non-negative.

7.4.3 Graphing standard form

To graph standard form on most graphing calculators, the only thing we can do is convert to slope-intercept form by solving for y. Many online tools (like Desmos) allow graphing in standard form without any transforming required on out part.

When graphing by hand, there is a clever alternative technique!

Example 7.16

Graph the line $3x - 2y = 12$.

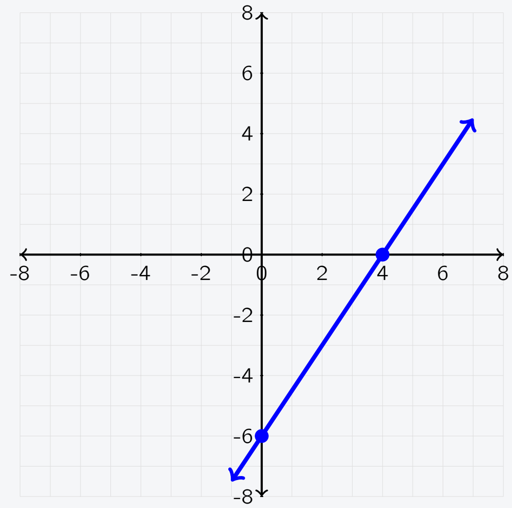
Solution: We can always plot some points by substituting in some values for x or y. We could choose random values, but here's a clever idea: plugging in zero will eliminate a term and simplify our calculations! If we let $x = 0$, then:

$$3(0) - 2y = 12 \qquad \Longrightarrow \qquad {}^{-}2y = 12$$

That's a one-step equation, and $y = {}^{-}6$. If we plug in $y = 0$, we have:

$$3x - 2(0) = 12 \qquad \Longrightarrow \qquad 3x = 12,$$

and so $x = 4$. So, we have found two points on the line: the y-intercept $(0, {}^{-}6)$, and the x-intercept $(4, 0)$. That's enough to draw the graph!



The only time this approach doesn't work is when the y-intercept and the x-intercept are the same. What does that sort of a line look like? How can we spot such a line from its standard form equation?

Speaking of x-intercepts, they haven't gotten too much of our attention so far. Our work with linear equations has highlighted the importance of the y-intercept. We'll learn much more about the significance of x-intercepts in future chapters.

7.5 Parallel and Perpendicular Lines

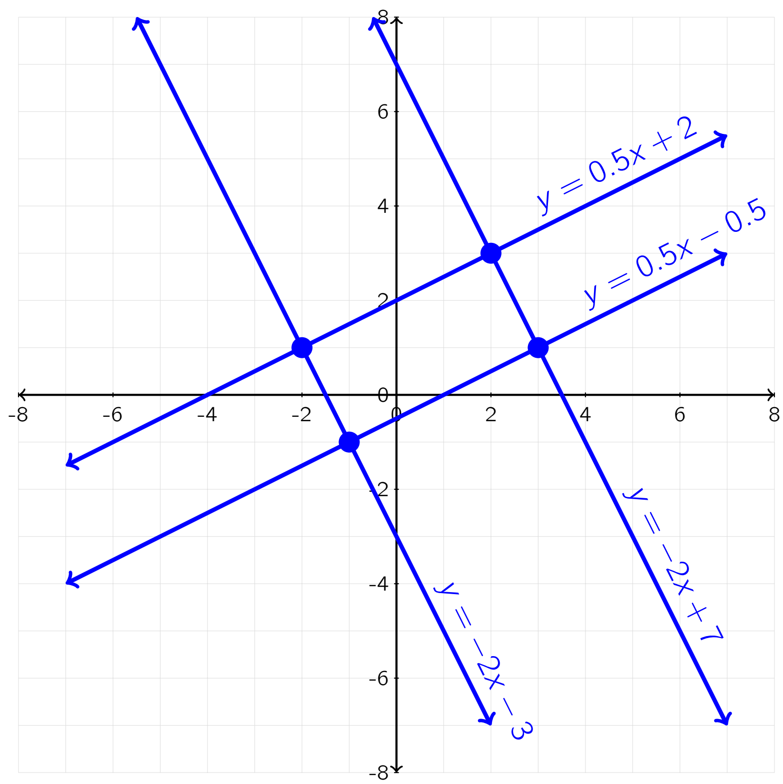
This part of this chapter is dedicated to a geometric connection to linear functions: parallel and perpendicular lines. Our goal for now is to learn how to tell if two lines are parallel or perpendicular, and how to write equations for lines that are parallel or perpendicular to a given line. In the future (like in a geometry class), you might use concepts from this lesson to prove some geometric concepts algebraically.

Name of Startup Exploration

The four points below form the vertices of a rectangle. Graph the rectangle and write four equations in slope-intercept form for the lines containing the four sides of the rectangle.

$$(2,3) \quad (3,1) \quad (-1,-1) \quad (-2,1)$$

We won't go into the details of how to write these equations. Those details are explained in earlier sections. The key thing to study in this example is how the lines are related to each other.



7.5.1 Parallel Lines

Parallel lines never intersect. They exist in the same plane and stay the same distance apart for their entire, infinite length. In the rectangle we drew as part of the startup exploration, opposite sides are parallel. Examine the slopes of each pair of parallel lines. What do you notice?

Algebraically, parallel lines have the *same slope* and *different y-intercepts*. Note that it is not sufficient to just look for having the same slope. If two lines have the same slope and also have the same y-intercept, then they aren't actually *two lines*. Those are the same line... and you can't be parallel to yourself!

Parallel will play an important role as we begin to study systems of linear equations in **??**.

In our rectangle, the lines the meet in the second quadrant are the lines

$$y = \frac{1}{2}x + 2 \qquad \text{and} \qquad y = -2x - 3.$$

Notice that the slopes of these two lines are both opposites and reciprocals. This is true in general: perpendicular lines have slopes that are *both opposites and reciprocals* of each other. This means that the slopes of perpendicular lines will always have a product of 1.

Example 7.17

Which of these lines, if any, are perpendicular to each other?

(a) $y = \frac{3}{4}x + 5$ (b) $y = \frac{3}{4}(x+8) - 1$

(c) $y = \frac{3}{4}(x-12) + 1$ (d) $4x - 3y = 9$

Solution: Parallel lines will have the same slope and different y-intercepts, so we should extract this information. We'll have to do the a bit of work first, converting all of the lines to slope-intercept form:

(a) $y = \frac{3}{4}x + 5$ (b) $y = \frac{3}{4}x + 5$

(c) $y = \frac{3}{4}x - 8$ (d) $y = \frac{4}{3}x - 3$

From this we can see that (a) and (b) are actually the same line, and that this line is parallel to line (c). Line (d) has a different slope, so it is not parallel to any of the others.

Example 7.18

Write an equation for the line that contains $(-2, 1)$ and is parallel to the line $y = -3(x-4) + 7$.

Solution: Parallel lines have the same slope, so the slope of our line must be -3, just like the slope of the given line. Since it doesn't specify which form to use, we can write our answer in point-slope form:

$$y = -3(x + 2) + 1.$$

The amount of work we will have to do for questions like the ones in this chapter will depend on the format the equations take. We will have to work a little harder if the lines are given in standard form, or when we are asked to write our answer in standard form.

Example 7.19

Write an equation *in simplified standard form* for the line that contains $(2, -1)$ and is parallel to the line $5x + 3y = 2$.

Solution: First, we need to remember that we're after a point and a slope to write the equation of our line. The line we are given is in standard form, so we'll have to convert it to slope-intercept by solving for y.

Using the techniques from section 7.4, we convert $5x + 3y = 2$ into

$$y = -\frac{5}{3}x + \frac{2}{3},$$

and we can see that the slope of the line is $-\frac{5}{3}$.

Our line will have the same slope as the given line (since our line is parallel to it), and we know a point on the line. So, we can write an equation for our line in point-slope form:

$$y = -\frac{5}{3}(x - 2) - 1.$$

But, we are asked for an equation in standard form, so we have to convert.

First we distribute and combine like terms, getting a line in slope-intercept form:

$$y = -\frac{5}{3}x + \frac{7}{3}.$$

Then, we get rid of the fractions and rearrange into standard form:

$$5x + 3y = 7 \qquad \text{Phew!}$$

And now for some good news: There is a clever shortcut that we can use when searching for a standard form equation for a line parallel to a line given in standard form.

You may have noticed that when we convert a standard form equation into a slope-intercept equation, we are doing the same procedure every time:

$$\begin{aligned} Ax + By &= C && \text{standard form} \\ By &= -Ax + C && \text{SPOE: subtract } Ax \text{ from both sides} \\ y &= -\frac{A}{B}x + \frac{C}{B} && \text{DPOE: divide through by } B \text{ to isolate } y \end{aligned}$$

This means that the slope of the line is always $-\frac{A}{B}$. That's a recipe for finding the slope of a line in standard form.

Since A and B are the only parts of the standard form equation that relate to slope, and given that parallel lines have the *same slope*, we can reason that all parallel lines have the same A and B values in standard form.

Look at the previous example: the given line and our answer have the same A and B values, but with different C values! In other words, any line parallel to $5x + 3y = 2$ is going to look like $5x + 3y = C$. In the example, we have to ensure that our line goes through the point $(2, -1)$, so we just need to figure out what C value will accomplish that goal.

How can we figure out what C value will make the line $5x + 3y = C$ go through the given point? We can substitute in these values for x and y and solve the equation to identify C! Observe:

$$\begin{aligned} 5x + 3y &= C && \text{our almost-there equation} \\ 5(2) + 3(-1) &= C && \text{substitute in a known point on the line} \\ 7 &= C && \text{simplify the left-hand side using the order of operations} \end{aligned}$$

So, our standard form equation is $5x + 3y = 7$.

7.5.2 Perpendicular Lines

Perpendicular lines intersect at right angles (that is, $90°$ angles). In the rectangle we drew in the startup exploration, adjacent sides are perpendicular. Examine the slopes of each pair of perpendicular lines. What do you notice?

In our rectangle, the lines the meet in the second quadrant are the lines

$$y = \frac{1}{2}x + 2 \qquad \text{and} \qquad y = -2x - 3.$$

Notice that the slopes of these two lines are both opposites and reciprocals. This is true in general: perpendicular lines have slopes that are *both opposites and reciprocals* of each other. This means that the slopes of perpendicular lines will always have a product of 1.

Example 7.20

Which of these lines are perpendicular to each other?

(a) $y = \frac{3}{4}x + 5$

(b) $y = -\frac{3}{4}(x + 8) - 1$

(c) $y = \frac{4}{3}(x - 12) + 1$

(d) $4x + 3y = 9$

Solution: The key is to look for slopes with opposite signs and then to look for reciprocals. First, we ought to convert line (d) into a form that reveals its slope:

$$\text{(d) } y = -\frac{4}{3}x + 3$$

This means that line (a) is perpendicular to line (d), since $\frac{3}{4} \cdot -\frac{4}{3} = -1$. Also, line (b) is perpendicular to line (c) since $-\frac{3}{4} \cdot \frac{4}{3} = -1$.

Example 7.21

Write an equation for the line that contains $(-2, 1)$ and is perpendicular to the line $y = -3(x - 4) + 7$.

Solution: The given slope is -3, the perpendicular slope must be the opposite, reciprocal slope: $\frac{1}{3}$. The point is $(-2, 1)$, and since the format of the equation was not specified, we can answer in point-slope form:

$$y = \frac{1}{3}(x + 2) + 1.$$

Again, the amount of work we will have to do for questions like the ones in this chapter will depend on the format the equations take.

Example 7.22

Write an equation *in simplified standard form* for the line that contains $(2, -1)$ and is perpendicular to the line $5x + 3y = 2$.

Solution: The line we seek is $3x - 5y = 11$. Above, we discussed a clever shortcut for working with parallel lines in standard form. Can you invent a similar shortcut for working with perpendicular lines in standard form? Use the given solution to test out your ideas!

Hint: What must we do to A and B in the standard form equation to ensure that the slopes will be both opposites and reciprocals?

Glossary

Symbols · **A** · **B** · **C** · **D** · **E** · **F** · **G** · **H** · **I** · **L** · **M** · **N** · **O** · **P** · **Q** · **R** · **S** · **T** · **U** · **V** · **X** · **Y** · **Z**

Symbols

Δ Delta, the fourth letter of the Greek alphabet. Used to represent change. They symbol Δx is read "delta x" or "the change in x".

A

abscissa The x-coordinate of a point in the coordinate plane.

absolute value For real numbers, it is the distance a number is away from zero on a number line. It is a scalar quantity, meaning it just has a magnitude and no direction (sign). The absolute value of a number is always non-negative. In the order of operations, it works like a grouping symbol. The "absolute value of x" is denoted $|x|$.

addition property of equality For all real numbers a, b, and c: If $a = b$, then $a + c = b + c$. This axiom is used when solving equations.

addition property of order For all real numbers a, b, and c: If $a > b$, then $a + c > b + c$. This axiom is used when solving inequalities and also applies to inclusive symbols of order.

additive identity The number which, when added to a given number x, leaves x unchanged. In the real number system, 0 is the additive identity. The existence of the additive identity is a **field axiom**.

additive inverse The number which, when added to a given number x, gives a sum of 0, the additive identity. The opposite of a number is its additive inverse. The existence of the additive inverse is a **field axiom**.

algebraic expression A symbolic representation of mathematical operations that can involve both numbers and variables. There is no equal sign in an expression.

algebraic number A number that is the root on a nonzero polynomial equation in one variable with rational coefficients. The set of algebraic numbers is a subset of the real numbers.

arithmetic sequence A sequence where the difference between each pair of successive terms is constant. The constant difference is called the "common difference", usually denoted d.

associative property of addition For all real numbers a, b, and c: $a+(b+c)=(a+b)+c$. This field axiom allows for the regrouping of longer strings of addition.

associative property of multiplication For all real numbers a, b, and c: $a(bc)=(ab)c$. This field axiom allows for the regrouping of longer strings of multiplication.

asymptote A line that a curve approaches as they both tend towards infinity. There are three types of asymptotes: vertical, horizontal, and oblique (slant). Exponential functions have a horizontal asymptote.

axiom A property or statement that is accepted without proof.

axis One of two perpendicular number lines used to locate points in the coordinate plane. The plural form is "axes".

axis of symmetry The line about which one can reflect an image onto itself. For example, a parabola has an axis of symmetry. Given the graph of a quadratic function, the axis of symmetry is a vertical line through the vertex. When written in standard form, the equation for the line of symmetry is given by $x=-\frac{b}{2a}$.

B

base (1) For triangles: A side of the triangle. (2) For expressions: A term or expression that is raised to a power.

binomial A polynomial with exactly 2 terms.

boundary For a one-variable **inequality**, the boundary is a point on the number line. Inclusive boundaries are drawn as closed or filled in points, and exclusive boundaries are draw as open circles. For a two-variable inequality, the boundary is a line or curve. Inclusive boundaries are drawn as solid lines/curves, and exclusive boundaries are lines/curves drawn with a dashed or dotted line. For a linear inequalities, the boundary line separates the plane into two **half-planes**, one of which will contain the solutions to the inequality.

C

Cartesian plane See **coordinate plane**.

closure A set is said to be to "have closure" (or to "be closed") under an operation performing the operation on members from the set always yields a result that is also a member of the set. The **natural numbers**, for example, are closed under the operation of addition, since the sum of any two natural numbers is itself a natural number. The natural numbers are not closed under the operation of subtraction.

coefficient The numerical factor in a term with a variable. If the number is not explicitly written, the coefficient is understood to be 1.

colinear To be on the same line.

combining like terms A short-cut used to add terms that have exactly the same variables raised to the same exponents.

common difference In an arithmetic sequence, it is the constant difference between successive terms.

common monomial factor A monomial that is a factor of every term in a polynomial expression.

common ratio In an geometric sequence, it is the constant ratio between successive terms.

commutative property of addition For all real numbers a and b: $a + b = b + a$. This field axiom allows for the reordering longer strings of addition.

commutative property of multiplication For all real numbers a and b: $ab = ba$. This field axiom allows for the reordering longer strings of multiplication.

completing the square Using the properties of equality on a quadratic equation to convert one side into a perfect square trinomial. Completing the square can be used as a technique to solve quadratic equations.

complex number A member of the set of numbers that consists of real and imaginary numbers. The set is denoted $\mathbb{C}$.

compound interest A way to calculate interest based on both the principal amount and any interest already accrued. This type of interest is an exponential relationship. The formula is $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where P is the principal amount, r is the rate of interest, t is the amount of time over which interest is to be computed, and n is the number of compounding periods per unit of time.

constant A value that does not change.

constant function A function whose graph is a horizontal line. It is of the form $f(x) = c$, where c is a constant. Constant functions are polynomial functions of degree zero.

constant multiplier In a sequence that grows or decays exponentially, the number each term is multiplied by to get the next term. Also known as the "common multiplier", or "common ratio".

constant of variation The constant ratio in a direct variation or the constant product in an inverse variation. It is designated with the variable k.

constant term A term that includes no variable.

constraint The limitations on the values of the variables in a problem. Equations, inequalities and systems are used model the constraints in real-world situations.

continuous data Data that has no breaks and has measurements that can change between data points. Graphically, the measured data points are connected with lines or curves.

continuous function A function that has no breaks in the domain or range. The graph of a continuous function is a line or curve with no holes, gaps, or vertical asymptotes.

converse of the Pythagorean theorem If a triangle has sides a, b, and c, such that $a^2 + b^2 = c^2$, then the triangle is a right triangle with a hypotenuse of length c.

conversion factor A ratio used to convert measurement from one unit to another.

coordinate plane A plane with a pair of scaled, perpendicular axes allowing one to locate points with ordered pairs and to represent lines and curves by equations. Also known as the Cartesian plane, named for its creator, French philosopher René Descartes.

coprime See **relatively prime**.

correlation Used in describing data graphed in a scatter plot. It is a trend between two variables. A trend can show positive, negative, or no correlation. Positive correlation shows an **increasing** trend in data. Negative correlation shows a **decreasing** trend in data.

cubic A function, number, or expression raised to the third power. Called cubic as it relates to the volume of a cube.

D

decay factor In exponential decay, the constant multiplier used to calculate the amount of decay after each unit of time. In the formula $y = (1 - r)^x$, it is the quantity $(1 - r)$. It represents the quantity remaining and is the common multiplier in the exponential relationship.

decay rate In exponential decay, the fraction or percentage by which a population decreases for each unit of time. In the formula $y = (1 - r)^x$, it is the quantity r.

decreasing A function is said to be decreasing if as x increases, y decreases. Lines with negative slopes are decreasing.

degree of a polynomial The degree of the term in a polynomial with the highest degree.

degree of a term The power (exponent) to which the variable is raised in a variable term. If there is no exponent explicitly written on a variable in a term, the term is understood to be of degree 1. The degree of a constant term is zero.

denominator The number or expression below the **vinculum** in a **rational number** or **rational expression**. For example, in the number $\frac{5}{2}$, the denominator is 2.

dependent variable A variable whose values depend on the values of another variable. In a graph of the relationship between the two variables, the values on the vertical axis represent the values of the dependent variable. The generic variable used is y.

difference of squares A binomial of the form $a^2 - b^2$.

dimensional analysis A strategy for converting a measurement from one unit to another using multiplication by a string of conversion factors. The key is to include the units with the numbers. It is used often in science.

direct variation In algebra 1, a relationship in which the ratio of two variables is constant. A direct variation has an equation of the form $y = kx$. The quantities represented by x and y are said to be **directly proportional**. The value k is called the **constant of variation**.

directly proportional Used to describe two variables whose values have a constant ratio.

discrete data Data that can only take on certain values. Discrete data usually involves a count of items.

discrete function A function whose domain and range have breaks or are made up of distinct values rather than intervals of real numbers. The graph of a discrete function will have breaks or will be made up of distinct points.

discriminant The expression under the square root in the quadratic formula, used to determine the number and nature of the roots of a quadratic. If a quadratic equation is written in standard form, then the discriminant is $b^2 - 4ac$. If the value of the discriminant is greater than 0, there are two real solutions to the quadratic equation. If it is equal to zero, there is one real solution. If it is less than zero, there are no real solutions to the quadratic equation.

distance formula A formula based on the Pythagorean theorem that uses the coordinates of two points to calculate the distance between the two points. The formula for the distance d between any two points (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

distributive property For all real numbers a, b, and c: $a(b + c) = ab + ac$. This field axiom allows one to simplify an expression without having the evaluate the sum inside the grouping symbol first.

division property of equality For all real numbers a, b, and c, where $c \neq 0$: if $a = b$, then $\frac{a}{c} = \frac{b}{c}$. This property is a version of the multiplication property of equality. It is used when solving equations.

division property of order For all real numbers a, b, and c where $c > 0$: if $a < b$, then $\frac{a}{c} < \frac{b}{c}$. If, on the other hand, $c < 0$, then $a < b$ implies $\frac{a}{c} > \frac{b}{c}$. This property is a version of the multiplication property of order. It is used when solving inequalities.

domain The set of all input values of a function, or the x-values. In a problem context it is represented by the independent variable.

domain restriction Values that cannot be used in the domain of a function. Radical and rational functions have domain restrictions.

doubling time In exponential growth, the amount of time it takes for a population, or amount, to double in size. It is constant for an exponential relationship.

E

elimination method A method for solving a system of equations that involves adding or subtracting multiples of the equations in order to eliminate a variable. It is based on Gaussian Elimination, a method to solve systems of equations that have been converted into matrices.

equation A statement that says the value of one expression is the same as the value of another expression.

equivalent equations Equations that have the same solution set.

equivalent inequalities Inequalities that have the same solution set.

evaluate To find the value of an expression. If an expression contains variables, values must be substituted for the variable before the expression can be evaluated.

exclusive boundary See **boundary**.

exclusive inequality See **inequality**.

exponent A number or variable written as a small superscript to a number or a variable, called the **base**, that indicates how many times the base is being used as a factor.

exponential decay A decreasing pattern in which amounts decrease by a constant percent.

exponential equation An equation in which a variable appears in the exponent.

exponential form The form of an expression in which repeated multiplication is written using exponents.

exponential function A function that repeatedly multiplies an initial amount by the same positive number. They can all be modeled using $y = ab^x$ where a is the initial amount and b is the constant multiplier.

exponential growth An increasing pattern in which amounts increase by a constant percent.

extraneous solution An apparent solution of an equation that does not satisfy the original equation. They occur when the transformation of an equation changes the solution set of the original equation, for example squaring both sides of an equation or multiplying by a quantity that can be zero.

F

factor One of the numbers, variables, or expressions multiplied to obtain a product.

factored form The form of an expression when it is written as the product of factors. The factors can be numbers, variables, or expressions. Factored form is not simplified.

factoring The process of rewriting an expression as a product of factors.

family of functions Similar functions that are all transformations of the same parent function.

field axiom One of a set of axioms including closure, identity, inverse, associative, commutative, and distributive properties. Along with a few definitions and properties of equality, they create the foundation upon which algebra is built.

FOIL A mnemonic for remembering the procedure to multiply two binomials. F stands for multiplying the first term in each binomial. O stands for multiplying the outer terms of the binomials. I stands for multiplying the inner terms of the binomials. L stands for multiplying that last term in each binomial.

fractal A geometric figure that has undergone infinite applications of a recursive procedure and which exhibits the property of self-similarity.

function A relation in which there is exactly one output value for each input value. The graph of a function must pass the vertical line test.

function notation A notation in which a function is named by a letter and the input is shown in parenthesis after the function name, generically, $f(x)$, read "f of x". The variables used may be changed to better represent quantities in a problem, for example $d(t)$ may represent distance d as a function of time t. When graphing in the **coordinate plane**, $f(x)$ is another way to write y. When x is replaced by a number, it indicates that one should evaluate the function at that value. The notation was first used by Swiss mathematician Leonhard Euler.

function rule An expression that represents the relationship between the variables of a function.

G

GCF Greatest Common Factor

geometric sequence A sequence where the ratio between each pair of successive terms is constant. The constant ratio is called the "common ratio", usually denoted r. Geometric sequences are exponential.

growth factor In exponential growth, the constant multiplier used to calculate the amount of growth after each unit of time. In the formula $y = (1 + r)^x$, it is the quantity $(1 + r)$. It is the common multiplier in the exponential relationship.

growth rate In exponential growth, the fraction or percentage by which a population increases for each unit of time. In the formula $y = (1 + r)^x$, it is the quantity r.

H

half-life The time needed for an amount of a substance to exponentially decay to half the original amount. Half-life is constant for an exponential relationship.

half-plane The set of points on a plane that fall on one side of a boundary line. Part of the solution of a linear inequality in two variables is a half-plane.

hypotenuse The side of a right triangle opposite the right angle. It is the longest side of the triangle.

I

identity When solving equations with variables on both sides, identities occur when the equation is true for every value of the variable. The solution set S is written as $S = \mathbb{R}$.

identity property of addition The sum of any number and 0 is that number. For every real number a, $a+0 = a$ and $0 + a = a$. The existence of the **additive identity** is a **field axiom**.

identity property of multiplication The product of any number and 1 is that number. For every real number a, $a \cdot 1 = a$ and $1 \cdot a = a$. The existence of the multiplicative identity is a **field axiom**.

imaginary number A member of the set of numbers that is created by taking the square root of a negative number. In the set of imaginary numbers, the square root of -1 is represented by the letter i. The set of imaginary numbers is a subset of the complex number system. The sets of real and imaginary numbers are disjoint, meaning they have no common members.

implied operation An operation that is not explicitly written. For example, in $3(x + 4)$ the multiplication between 3 and $(x + 4)$ is an implied operation, since no multiplication symbol is explicitly written in between.

improper fraction A fraction whose **numerator** is greater than its **denominator**. For example, $\frac{5}{2}$ is an improper fraction. A fraction that is not an improper fraction is called a **proper fraction**. See also **mixed number**.

inclusive boundary See **boundary**.

inclusive inequality See **inequality**.

increasing A function is said to be decreasing if as x increases, y increases. Lines with positive slopes are decreasing.

independent variable A variable whose values affect the values of another variable. In a graph of the relationship between the two variables, the values on the horizontal axis represent the values of the dependent variable. The generic variable used is x.

inequality A statement that one quantity is less than or greater than another. An inequality may exclusive or inclusive. The exclusive inequalities are $<$ and $>$, read “less than” and “greater than”. The inclusive inequalities are $\leq$ and $\geq$, read “less than or equal to” and “greater than or equal to”.

initial value The starting value of a sequence or exponential function.

integer A member of the set of natural numbers, their opposites, and zero. The set is denoted $\mathbb{Z}$, and we may write $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$. The integers are a subset of the rational numbers.

intercept The point which a graph intersects one of the axes.

interest A percentage of the balance added to an account at regular time intervals.

interest rate The percentage used to calculate interest.

inverse property of addition For any real number a, there exists a real number ^{-}a such that $a + {}^{-}a = 0$. The number ^{-}a is called the **additive inverse** of a. Very often we will call it the **opposite** of a.

inverse property of multiplication For any nonzero real number a, there exists a real number $\frac{1}{a}$ such that $a \cdot \frac{1}{a} = 1$. The number $\frac{1}{a}$ is called the **multiplicative inverse** of a. Very often we will call it the **reciprocal** of a.

inverse variation In algebra 1, a relationship in which the product of two variables is constant. An inverse variation has an equation in the form $xy = k$, or $y = \frac{k}{x}$. The quantities represented by x and y are said to be **inversely proportional**. The value k is called the **constant of variation**.

inversely proportional Used to describe two variables whose values have a constant product.

irrational number A number that cannot be expressed as the ratio of two integers. In decimal form, an irrational number has an infinite number of digits and does not repeat. The set of irrational numbers consist of algebraic and transcendental numbers. The set of irrational numbers is a subset of the real numbers.

irreversible operation An operation performed when solving an equation that changes the solution set of the equation. Multiplying or dividing both sides of an equation by an expression that might equal zero are considered irreversible operations.

L

leg One of the perpendicular sides of a right triangle.

like terms Terms with exactly the same variable factors in a variable expression. The variables and the powers to which the variables are raised must be identical for the terms to be considered like terms.

limited domain The restricted domain of a function. Domains are usually limited in real world contexts. For example, we rarely allow negative values for a variable that represents "time". For this reason it is often referred to as a reasonable domain.

line of best fit A line used to model a set of data. A line of best fit shows general direction of the data. When hand-drawn, one should have about the same number of data points above and below the line. When using the linear regression tool on the calculator, the correlation coefficient will show how well the line fits the data.

line of symmetry See **axis of symmetry**.

linear In the shape of a line or represented by a line. In mathematics, a linear equation or expression has variables raised only to the power of 1.

linear function A function characterized by a constant rate of change. The graph of a linear function is a non-vertical line. It is a polynomial of degree one.

linear inequality An inequality of two variables whose boundary is formed by a linear function. It describes a region of the coordinate plane that consists of a boundary line and a half-plane.

linear programming A method to optimize a quantity that uses an objective function to represent the quantity and a system of linear inequalities to represent the constraints on the variables involved. The system of inequalities are graphed to represent a set of feasible solutions and the vertices of the region will describe the optimal amount of the quantity.

linear relationship A relationship that can be represented by a linear function. A linear relationship is characterized by a constant rate of change.

linear term A term of degree 1.

lowest terms The form of a fraction in which the numerator and denominator are **relatively prime**. A fraction in lowest terms is also called a reduced fraction.

M

mapping diagram A diagram used to determine if a relation is a function. The values of the domain and range are written in circles. Arrows are drawn from the elements of the domain to the corresponding elements of the range. It is a visual that shows how the members of the domain map to the members of the range.

mathematical equivalence The idea that numbers, expressions, equations, functions, or other mathematical objects can be algebraically manipulated, using specific rules, such that their representations and appearance are changed while other fundamental properties remain unchanged.

mathematical modeling Translating a real-world scenario with a given set of constraints into an abstract representation that can be manipulated and studied mathematically. For example, creating a set of variables and equations to solve a **linear programming** problem.

maximum The greatest value. In a quadratic function, the vertex will be a maximum if the coefficient of the quadratic term is negative.

midpoint The point on a line segment halfway between the endpoints. The coordinates of the midpoint are found by averaging the abscissas and ordinates of the endpoints.

midpoint formula The formula that can be used to compute the midpoint of a line segment. Given a line segment with endpoints (x_1, y_1) and (x_2, y_2), the midpoint of the segment has coordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

minimum The smallest value. In a quadratic function, the vertex will be a minimum if the coefficient of the quadratic term is positive.

mixed number The sum of a nonzero **integer** and a **proper fraction**. For example $2\frac{3}{5}$ is a mixed number. See also **improper fraction**.

monomial A polynomial with only one term.

multiplication property of equality For all real numbers a, b, and c: if $a = b$ then $ac = bc$. This property is used to solve equations.

multiplication property of order For all real numbers a, b, and c and $c > 0$: if $a < b$ then $ac < bc$. If, on the other hand, $c < 0$, then $a < b$ implies $ac > bc$. This property is used to solve equations.

multiplicative identity The number which, when multiplied by a given number x, leaves x unchanged. In the real number system, 1 is the multiplicative identity. The existence of the multiplicative identity is a **field axiom**.

multiplicative inverse The number which, when multiplied by a given nonzero number x, gives a product of 1, the multiplicative identity. The reciprocal of a number is its multiplicative inverse. The existence of the multiplicative inverse is a **field axiom**.

N

natural number A member of the set $\{1, 2, 3, 4, \ldots\}$, denoted $\mathbb{N}$. Also called the counting numbers. The number 0 is sometimes included as a natural number.

negative correlation See **correlation**.

null set A set that contains no elements. Also called the empty set. Used to show that there is no solution to an equation. Denoted $\varnothing$ or $\{\}$.

numerator The number or expression above the **vinculum** in a **rational number** or **rational expression**. For example, in the number $\frac{5}{2}$, the numerator is 5.

numeric expression An expression containing only numbers and mathematical operations.

O

obelus The division symbol $\div$.

one-variable data Data that measures only one trait or quantity. A one-variable data set consists of single values (as opposed to ordered pairs) and is graphed on a number line. Compare with: **two-variable data**.

opposite See **additive inverse**.

optimization To maximize or minimize a quantity given constraints. For example a company will want to optimize (maximize) their profits while faced with constraints such as the cost and availability of labor and materials.

order of magnitude A way of expressing the size of an very large or very small number by giving the power of 10 associated with the number.

order of operations The agreed-upon order in which operations are carried out when evaluating an expression.

ordered pair A pair of numbers named in an order that matters. The coordinates of a point are given as an ordered pair in which the first number is the x-coordinate (abscissa) and the second number is the y-coordinate (ordinate).

ordinate The y-coordinate of a point in the coordinate plane.

origin The point where the coordinate axes intersect. In a coordinate plane it has the coordinates $(0, 0)$.

P

parabola The set of all points whose distance from a fixed point (called the focus) is equal to the distance from a fixed line (called the directrix). Also known as the smooth "U" shaped curve of a quadratic function.

parallel lines Lines in the same plane that never intersect. They are always the same distance apart in Euclidean geometry. The slopes of parallel lines are the same.

parent function The most basic form of a function. A parent function can be transformed to create a family of functions.

percent change The percent by which an amount differs from its original amount. It is calculated by taking the amount of the change and dividing it by the original amount.

perfect cube A number that is equal to the cube of an integer, or a polynomial that is equal to the cube of another polynomial.

perfect square A number that is equal to the square of an integer, or a polynomial that is equal to the square of another polynomial.

perfect square trinomial A trinomial generated by squaring a binomial. For example, squaring the binomial $(a + b)$ yields $(a + b)^2 = a^2 + 2ab + b^2$. Thus, $a^2 + 2ab + b^2$ is a perfect square trinomial.

period of compounding The number of times interest is calculated during a year for compound interest. It is represented by n in the **compound interest** formula.

perpendicular lines Lines that intersect at a right angle. The slopes of perpendicular lines are opposites and reciprocals. The slopes of perpendicular lines multiply to -1.

point-slope form The form of a linear equation that uses the slope and any point on the line. It is written either $y - y_1 = m(x - x_1)$ or $y = m(x - x_1) + y_1$, where m is the slope of the line and (x_1, y_1) is a point on the line. It can be derived from the slope formula and represents the transformation of the line $y = mx$ where a vertical shift of y_1 and a horizontal shift of x_1 has occurred.

polynomial A sum of terms that have positive integer exponents. In algebra 1, all polynomials are in one variable.

positive correlation See **correlation**.

power An expression of the form a^n is called a power of a.

principal amount The original amount invested in a situation that involves accumulating interest. It is represented by P in the **compound interest** and simple interest formulas.

principal square root The positive square root of a number.

proper fraction A fraction whose **numerator** is less than its **denominator**. For example, the fraction $\frac{7}{9}$ is a proper fraction. A fraction that is not a proper fraction is called an **improper fraction**.

proportion An equation stating that two ratios are equal.

Pythagorean theorem A formula that expresses the relationship between the sides of a right triangle. It states that the sum of the squares of the legs of a right triangle is equal to the square of the **hypotenuse**.

Q

quadrant One of the four regions that a coordinate plane is divided into by the two axes. The quadrants are numbered I, II, III, and IV, starting in the upper right and moving counterclockwise.

quadratic formula The formula used to find the exact solution to any quadratic equation. Given that $ax^2 + bx + c = 0$, the formula states

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is derived by completing the square on the standard form quadratic equation.

quadratic function A function with an equation of the form $y = ax^2 + bx + c$ where $a \neq 0$. The graph of a quadratic function is a **parabola**.

quadratic term A term of degree 2.

R

radical The root symbol $\sqrt{}$, used to denote square roots, cube roots, and so on. The symbol $\sqrt[n]{x}$ is read "nth root of x." If n is not stated, as in $\sqrt{x}$, it is understood to be 2 and the radical indicates the square root.

radical expression An expression containing a radical (square root, cube root, or any nth root).

radical function A function where the independent variable is under a radical (square root, cube root, or any nth root).

radioactive decay The process by which an unstable element loses mass with a release of energy, transforming it into a different element or isotope.

range (1) In statistics it is the difference between the greatest value in a data set and the smallest value in a data set. (2) In the study of functions it is the set of all output values of a function. It is represented by the dependent variable.

rate A **ratio** that measures two quantities with different units.

rate of change A measurement of how quickly one quantity changes relative to another quantity. Given values (x_1, y_1) and (x_2, y_2), the rate of change of y with respect to x is $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$. The patterns of the rate of change of a set of data can be used to determine what type of data is represented by the pattern. For example, the rate of change of linear data is constant.

ratio A comparison between two quantities, often written in fraction form.

rational expression An expression that can be written as a ratio of two polynomials. The value of the variable cannot make the denominator 0.

rational function A function that is expressed as the ratio of two polynomial expressions. The values of the independent variables that make the denominator zero are restricted from the domain.

rational number A number that can be written as a ratio of two integers $\frac{a}{b}$ where $b \neq 0$. Their decimal forms are either terminating or repeating. The set of rational numbers is denoted $\mathbb{Q}$. The rational numbers are a subset of the real numbers.

rationalizing the denominator The process of making the denominator of a fraction a rational number without changing the value of the expression. It is used to eliminate a radical from the denominator of a fraction.

real number Denoted $\mathbb{R}$, the set of real numbers include the integers, rational numbers, and irrational numbers, but not imaginary numbers. This is the number set used in algebra 1. The set is closed under the operations of addition and multiplication. Members can be graphed on the standard number line. The real numbers is a subset of the complex numbers.

reciprocal The multiplicative inverse. The reciprocal of a given number is the number it must be multiplied by to get 1 (the multiplicative identity). To find the reciprocal of a number, we can write the number as a fraction and then invert the fraction. The reciprocal of n is $\frac{1}{n}$.

recursive Describes a procedure that is applied over and over again, starting with a number or a geometric figure, to produce a sequence of numbers or figures. The procedure requires previous entries in the pattern to find subsequent entries.

recursive rule Instructions for producing each stage of a sequence from the previous stage. It must contain a description of "stage 0", or the starting value.

recursive sequence An ordered list of numbers defined by a starting value and a recursive rule. We generate a recursive sequence by applying the rule to the starting value, then applying the rule to the resulting value, and so on.

relation Any set of ordered pairs.

relatively prime Two numbers are said to be relatively prime (or coprime) if they have no common factors other than 1. For example, 16 and 21 are relatively prime. In contrast, 21 and 24 are not relatively prime, since both numbers are divisible by 3.

repeating decimal A decimal representation of a rational number with a digit or group of digits after the decimal point that repeat infinitely.

root A zero or an x-intercept of a function.

S

sample space The set of all possible outcomes of a probability experiment.

scatter plot A two-variable data display in which values on a horizontal axis represent values of one variable and values on the vertical axis represent values of the other variable. The coordinates of each point represent a pair of data values.

scientific notation A notation in which a number is written as the product of a number greater than or equal to 1 but less than 10, multiplied by an integer power of 10.

sequence A function whose domain is the set of positive integers. A sequence is an ordered list of objects, like numbers. The individual objects are called terms. Unlike a set, order matters, and terms may be repeated.

set An unordered collection of items. Often denoted by listing the elements inside a set of braces.

set notation Using curly braces { and } to designate quantities that belong to a set. Certain sets do not require the use of braces, as they have symbols used to denote them, like the **null set**, the set of **integers**, and the set of **real numbers**.

simple interest Interest calculated using the formula $I = Prt$. The interest is only ever calculated using the initial investment (called the **principal amount**) and show linear growth.

simplified radical form A radical written so that (1) no perfect square factors exist under the radical (2) no fractions are under the radical and (3) there are no radicals in the denominator of the fraction.

simplify Using algebraic laws and properties which maintain equivalence in order to write an answer so that it fits a set of criteria. The criteria depend on what is being simplified.

slope The measurement of the steepness of a line, or the rate of change of a linear relationship. Often denoted m, and referred to as "rise over run." Given points (x_1, y_1) and (x_2, y_2), the slope of the line between the points is calculated as $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.

slope-intercept form The form $y = mx + b$ of a linear equation. The value of m is the slope and the value of b is the y-intercept. It is the simplified version of **point-slope form**.

solution A solution to an equation (or inequality) is any value of the variable (or variables) in the equation (or inequality) that make the equation (or inequality) true. The solution to a system of equations (or inequalities) is the set of all of the points common to all equations in the system. If there is no solution, the system is said to be inconsistent. If there are infinitely many solutions to a system, the system is said to be dependent. If there is a single solution, the system is said to be independent. In a system of two equations in two variables, the solution is the intersection point of the two lines.

solution set The set of values that make an equation, inequality, or system true.

solution set notation One way to denote the solution set to an equation, written as $S = \{\ solutions\ \}$.

solve To find the solution set of an equation.

square root The square root of a number a, denoted $\sqrt{a}$, is the number b such that that $b \cdot b = a$. Every positive number has two square roots, a **principal square root** and a negative square root. The set of real numbers is not closed under the operation of square root.

standard form (1) For linear equations, it is an equation of the form $Ax + By = C$, in which A and B are not both 0. (2) For a polynomial, it is an expression written such that it is simplified and the terms are written in decreasing order of degree (highest degree term appears first). (3) For quadratic equations, it is an equation of the form $ax^2 + bx + c$, where $a \neq 0$.

subset A subset is a set that consists entirely of members from another set. If a set A is a subset of a set B, then every item in A is in B.

substitution To replace a quantity with another one that is equivalent.

substitution method A method for solving a system of equations that involves solving one of the equations for one variable and substituting the resulting expression into the other equation. See also: **elimination method**.

subtraction property of equality For all real numbers a, b, and c: if $a = b$ then $a - c = b - c$. This property is a restatement of the **addition property of equality** and is used to solve equations.

subtraction property of order For all real numbers a, b, and c: If $a > b$, then $a - c > b - c$. This property is a restatement of the **addition property of order** and is used to solve inequalities.

system of equations A set of two or more equations with the same variables. The equations act as constraints on the variables.

system of inequalities A set of two or more inequalities with the same variables. The inequalities act as constraints on the variables.

T

term An algebraic expression that represents only multiplication and division between variables and constants.

terminating decimal A decimal number with a finite number of nonzero digits after the decimal point.

transcendental number An irrational number that is not algebraic. The number π is transcendental because it is not the root of a polynomial equation in one variable with rational coefficients.

trinomial A polynomial with exactly three terms.

two-variable data A collection of data that measure two traits or quantities. A two-variable data set consists of pairs of values. Compare with: **one-variable data**.

U

unit rate A ratio in which one of the quantities has the value of 1.

unknown A quantity in an equation whose value is not known. In algebra, letters are often used to represent unknowns.

V

variable A trait or quantity whose value can change, or vary. In algebra, letters are often used to represent variables.

vector A quantity that has both a size (or magnitude) and a direction Vectors play an important role in physics and engineering, since many physical quantities (such as velocity, acceleration, and force) are best represented using vectors.

vertex Of a parabola, the point where the graph changes direction from increasing to decreasing or from decreasing to increasing.

vertex form A form of a quadratic equation. Given that (h, k) is the vertex, this form is written either as $y - k = a(x - h)^2$ or $y = a(x - h)^2 + k$. It can be derived by completing the square on standard form and represents the transformation of $y = ax^2$ by translation h units horizontally and k units vertically.

vertical line test A method for determining whether a graph on the coordinate plane represents a function. If all possible vertical lines cross the graph only once or not at all, the graph represents a function. If even one vertical line crosses the graph in more than one point, the graph does not represent a function.

vertical motion formula When an object is dropped or launched vertically, its height can be expressed using $h(t) = at^2 + vt + h_0$, where $h(t)$ is the object's height at time t, v is its initial vertical velocity, h_0 is its starting height, an a is the acceleration of gravity. For dropped objects, v is zero. This formula is used in the study of projectile motion.

vinculum A bar used in mathematics to show grouping. Examples of vincula include: the fraction bar (as in $\frac{1}{x+2}$), the bar used to show repeating digits (as in $0.\overline{3}$), and the horizontal bar of a radical (as in $\sqrt{2+5}$).

X

x-axis The horizontal number line on a coordinate graph. The independent variable is drawn on the x-axis.

x-intercept Any point at which a graph intersects the x-axis.

Y

y-axis The vertical number line on a coordinate graph. The dependent variable is drawn on the y-axis.

y-intercept Any point at which a graph intersects the y-axis.

Z

zero Of a function, the values of the independent variable that make the corresponding values of the function equal to zero, also known as the **roots** or x-intercepts of the function.

zero product property Property of real numbers stating that if the product of two or more factors equals zero, then at least one of the factors must equal zero. This property is used along with factoring as a method for solving a polynomial equation.

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