The Algebranomicon

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Contents

We must say that there are as many squares as there are numbers.

Galileo Galilei
Italian physicist and astronomer

Chapter 1

Quadratic Equations

In **??** we learned a set of tools in for solving linear equations. Before, that we saw quadratic patterns for the first time back in **??**. In this chapter, we'll see what makes quadratic equations different from those we've studied until now, and then we'll learn techniques that will help us to solve quadratic equations.

1.1 Challenges to Solving Quadratics

Exploration

Consider the following equation. Do we know what we need to know to solve this equation, without any guesswork? In other words: Do we have POEs, axioms, or other properties that will allow us to isolate x?

$$x^2 + 2x - 15 = 20$$

If you can, solve the equation. Otherwise, identify where you get stuck. What new tools would be helpful for solving this equation?

Our task in this chapter is to be able to solve quadratic equations like the one given in the startup exploration. We won't give the details for how to solve this equation yet – we'll develop those ideas over the next few sections. For now, we will simply point out a few features.

Using what we know so far, it's impossible to isolate x. We can make some progress:

$$\begin{aligned} x^2 + 2x - 15 &= 20 \\ x^2 + 2x &= 35 \qquad \text{APOE, to get all the numbers to the right-hand side} \end{aligned}$$

But now what? We can't combine like terms on the left-hand side, and subtracting anything from that side would give us x's on both sides of the equation (making things worse). We might try undoing the distributive property on the left-hand side. That would give us

$$x(x+2) = 35.$$

But this doesn't seem to be much of an improvement either. Using DPOE on the left-hand side to isolate x would move $(x+2)$ to the right hand side of the equation (and vice versa). It seems that we'll need some other techniques to help us out of this situation.

If we're allowed to just solve the equation my making a clever observation, we might notice that

$$5(7) = 5(5+2) = 35,$$

and so $x = 5$ is a solution to the equation! That's progress. But, it might not be obvious that -7 is *also* a solution to the equation, since

$$-7(-7+2) = -7(-5) = 35.$$

Plus with a different equation, it might not be quite so easy to see a solution just by inspection. Our new techniques should help us overcome these challenges.

1.1.1 Rectangles and Squares

As we saw in **??**, quadratic sequences can be related to rectangles and squares. Our very first quadratic sequence was related to rectangles (fig. 1.1), and so are the familiar perfect squares (fig. 1.2).

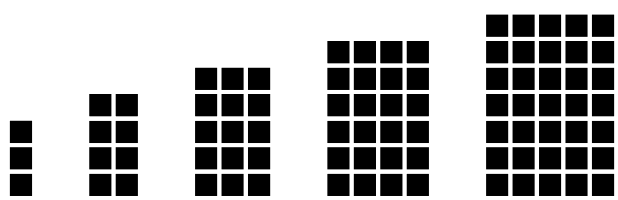


Figure 1.1: Some "rectangular numbers": $3, 8, 15, 24, 35, \ldots$

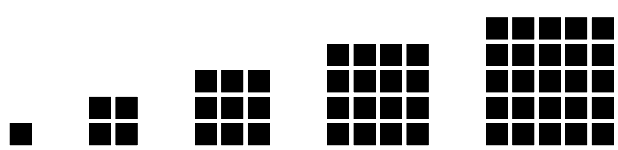


Figure 1.2: The perfect squares: $1, 4, 9, 16, 25, \ldots$

The perfect squares have a straightforward formula. If we let $f(x)$ represent the area of figure x, then the perfect squares are represented by the formula

$$f(x) = x^2.$$

This is the parent function of the quadratic family.

The rectangles in fig. 1.1 also have a formula. If we let $g(x)$ represent the area of figure x, then we might notice that figure x is x units wide and $(x+2)$ units tall. So these rectangles are represented by the formula

$$g(x) = x(x+2).$$

We can simplify this formula using the distributive property:

$$g(x) = x(x+2) = x^2 + 2x$$

We have learned that the highest degree term in a quadratic equation is an x^2 term. The connection between a quadratic rule and rectangles will help us when it comes to solving quadratic equations. In fact, we will solve these equations using the beautiful symmetry of the square.

1.2 Level 1 and Level 2 Quadratics

As we did with linear equations, we'll treat quadratic equations like a game. Level 1 is the easiest kind of quadratic equation to solve, so this is where we'll start. Then, we'll keep things interesting as we increase our skill level by adding challenge and complexity along the way.

1.2.1 Level 1 Quadratics

Startup Exploration: Quadratic Level 1

Determine the value of x given the equation: $x^2 = 64$.

Talk about starting with the easy stuff. We're looking for a number x which, when multiplied by itself, is equal to 64. In other words, we are looking for the "square root of 64". Clearly, $x = 8$ is a solution to this equation. But we can't be too hasty! Notice that $x = -8$ is also a solution, since $(-8)^2 = (-8)(-8) = 64$. So, this equation has two solutions:

$$x = 8 \ \underline{\text{or}} \ -8$$

If we prefer to write our answer in solution set notation, we have

$$\mathcal{S} = \{8, -8\}.$$

So, Level 1 quadratics are pretty easy: we simply take the square root of both sides of the equation.[1] We might have a hard time if the constant value is not a perfect square, as in $x^2 = 12$. But, we'll learn more about handling the square roots of non-perfect-squares soon enough (in chapter 2, to be precise).

This seems like a good time to suggest that it may come in handy to memorize the first 25 or so perfect squares, for quick recognition when they come up in a problem.

$1^2 = 1$	$2^2 = 4$	$3^2 = 9$	$4^2 = 16$	$5^2 = 25$
$6^2 = 36$	$7^2 = 49$	$8^2 = 64$	$9^2 = 81$	$10^2 = 100$
$11^2 = 121$	$12^2 = 144$	$13^2 = 169$	$14^2 = 196$	$15^2 = 225$
$16^2 = 256$	$17^2 = 289$	$18^2 = 324$	$19^2 = 361$	$20^2 = 400$
$21^2 = 441$	$22^2 = 484$	$23^2 = 529$	$24^2 = 576$	$25^2 = 625$

[1] More soon on square roots, including why and under what circumstances "square root of both sides" is, in fact, a property of equality.

Finally, note that zero is also a perfect square, since $0^2 = 0$. Zero, in fact, is the only number that has only one square root. Whereas both 3 and −3 are square roots of 9, the only square root of 0 is 0.

»> Get into negatives under the radical now, or save that for later (it currently appears in the next chapter).

1.2.2 Level 2 Quadratics

Startup Exploration: Quadratic Level 2

Determine the value of w given the equation: $(w + 3)^2 = 16$.

Here, we're told that something-squared is 16. Well, that means that the something in question must either be 4 or negative 4. That is to say, $(w + 3)$ is either 4 or −4. So, this equation is actually two equations at once. We have:

$$w + 3 = 4 \qquad \underline{\text{or}} \qquad w + 3 = -4$$

Solving these equations (using SPOE in both cases), we have that w must be either 1 or −7. So, those are our two solutions: $\mathcal{S} = \{1, -7\}$.

We can record this work in a down-the-page format, like so:

$$\begin{aligned} (w + 3)^2 &= 16 & & \\ w + 3 &= 4 \ \underline{\text{or}}\ -4 & & \text{square root of both sides} \\ w &= 1 \ \underline{\text{or}}\ -7 & & \text{SPOE: subtract 3 throughout} \end{aligned}$$

A few things to note: First, we can't subtract 3 from both sides as the very first step. The parentheses require that we undo the exponent first. Second, beware the use of $\pm$. It may be tempting to use this shorthand notation and write

$$w + 3 = \pm 4,$$

but then it is also tempting to subtract 3 and write

$$w = \pm 1. \quad \text{Nope!}$$

We recommend splitting into two equations, or writing out the two solutions explicitly using the word "or".

Example 1.1

Determine the value of a given the equation: $(a-1)^2 - 2 = 23$.

Solution: This equation requires an extra step, but it's quickly transformed into an equation like the one from the startup exploration.

$$\begin{aligned}
(a-1)^2 - 2 &= 23 && \\
(a-1)^2 &= 25 && \text{APOE} \\
a - 1 &= 5 \ \underline{\text{or}} \ {-5} && \text{square root of both sides} \\
w &= 6 \ \underline{\text{or}} \ {-4} && \text{APOE: add 1 throughout}
\end{aligned}$$

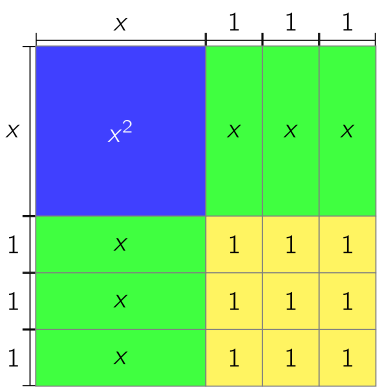
In the end, we have $\mathcal{S} = \{6, -4\}$.

1.3 Level 3 Quadratics

Startup Exploration: Quadratic Level 3

Use the sum to a power property (from **??**) to write the expression $(x+3)^2$ without parentheses. Then, determine the value of x given the equation: $x^2+6x+9=49$.

To expand $(x+3)$, we might think of using algebra tiles to fill in a square with side length $(x+3)$. Or, we could just "sketch" the algebra tiles diagram (shown below, on the right) and calculate the areas of the four regions.



	x	3
x	x^2	$3x$
3	$3x$	9

Note how we have simplified the picture in the "sketch" version. For example, rather than draw three 3-unit-by-x-unit rectangles, we simply write the area of the rectangle $3x$. In the lower right-hand region, we write the area 9 rather than draw nine yellow squares.

Diagrams like this help us to see the relationship between the expressions:

$$(x+3)^2 = x^2+6x+9.$$

Using this fact, we can solve the given equation:

$$\begin{aligned} x^2+6x+9 &= 49 && \\ (x+3)^2 &= 49 && \text{based on the square diagram} \\ x+3 &= 7 \ \underline{\text{or}}\ {}^-7 && \text{square root of both sides} \\ x &= 4 \ \underline{\text{or}}\ {}^-10 && \text{SPOE: subtract 3 throughout} \end{aligned}$$

Can you see the clever trick that we used here? We rewrote an expanded expression as a something-squared expression, and then we solved it like a Level 2 quadratic!

Let's work through another example in detail, for example, solving the Level 3 equation

$$x^2+12x+36=144.$$

Our goal is to write the left-hand side in a something-squared form. To do that, we'll begin by drawing an empty square diagram and filling in the bits that we know. For example, we know that the upper left-hand box must contain x^2, and so the sides of that square must each be x units long.

	x	
x	x^2	

We know that the 36 will appear in the lower right-hand box. But how sould we label the side lenghts here? There are lots of combinations of numbers that multiply together to get 36...

	x	?
x	x^2	
?		36

Remember that the goal is to get an expression of the form something-*squared*, and so we should always strive to create a square. This means that we must choose 6 and 6 as the side lengths. If we chose another pair of factors, like 2 and 18, we would get the proper product in the lower right-hand region, but our overall diagram would no longer be a square.

	x	6
x	x^2	
6		36

To fill in the remaining regions, we multiply the dimensions of each. In both cases we get $6x$. This is good news! Together these make $12x$ (which is what we have in the original, expanded expression). Plus, the two regions contain the same value, and so the beautiful symmetry of the square is preserved.

	x	6
x	x^2	$6x$
6	$6x$	36

So, we have rewritten our expression as a something-squared expression:

$$x^2 + 12x + 36 = (x+6)^2.$$

We can use this to solve the equation that we were given:

$$\begin{aligned} x^2 + 12x + 36 &= 144 & & \\ (x+6)^2 &= 144 & & \text{based on the square diagram} \\ x + 6 &= 12 \ \underline{\text{or}} \ -12 & & \text{square root of both sides} \\ x &= 6 \ \underline{\text{or}} \ -18 & & \text{SPOE} \end{aligned}$$

Example 1.2

Determine the value of n given the equation: $n^2 - 16n + 64 = 1$.

Solution: Let's build the square diagram. The upper left-hand corner contains n^2, as above. To maintain symmetry, we should split the $-16n$ exactly in half. Don't worry about the negative coefficient, we can work with that. We should be on guard though for sign-related issues.

$$\begin{array}{c|c|c|} & n & \\ \cline{2-3} n & n^2 & -8n \\ \cline{2-3} & -8n & \\ \cline{2-3} \end{array}$$

This tells us that the remaining portion of the square's side length must be -8. Let's not worry too much about the fact that negative distances are impossible... the idea still works. This implies that the remaining region contains 64 (note, that positive 64). This agrees with the equation we were given.

$$\begin{array}{c|c|c|} & n & -8 \\ \cline{2-3} n & n^2 & -8n \\ \cline{2-3} -8 & -8n & 64 \\ \cline{2-3} \end{array}$$

Now, we can solve the equation:

$$\begin{aligned} n^2 - 16n + 64 &= 1 & & \\ (n-8)^2 &= 1 & & \text{based on the square diagram} \\ n - 8 &= 1 \ \underline{\text{or}} \ -1 & & \text{square root of both sides} \\ n &= 9 \ \underline{\text{or}} \ 7 & & \text{SPOE} \end{aligned}$$

So, we have $\mathcal{S} = \{7, 9\}$. We can check our work by substituting our proposed solutions back into the original equation. First, we'll test $n = 7$:

$$\begin{aligned} n^2 - 16n + 64 &= 1 \\ (7)^2 - 16(7) + 64 &\stackrel{?}{=} 1 \\ 49 - 112 + 64 &\stackrel{?}{=} 1 \\ 1 &\stackrel{\checkmark}{=} 1 \end{aligned}$$

First, we'll check $n = 9$:

$$\begin{aligned} n^2 - 16n + 64 &= 1 \\ (9)^2 - 16(9) + 64 &\stackrel{?}{=} 1 \\ 81 - 144 + 64 &\stackrel{?}{=} 1 \\ 1 &\stackrel{\checkmark}{=} 1 \end{aligned}$$

1.4 Level 4 Quadratics

Startup Exploration: Quadratic Level 4

Determine the value of x given the equation: $x^2 + 8x + 15 = 99$.

Let's try to build the square diagram. The upper left-hand corner contains x^2, as before. To maintain symmetry, we should split the $8x$ exactly in half. This tells us that the large square must be $(x + 4)$ units on a side.

	x	4
x	x^2	$4x$
4	$4x$	

The problem is that, according to our square diagram, the lower right-hand corner should be 16... but the equation we are given tells us to put 15 in that space. Now what? Our equation does not represent a complete square!

	x	4
x	x^2	$4x$
4	$4x$	15

POEs to the rescue! Why not add 1 to both sides of the given equation to complete the square? This will give us the number we want on the left-hand side and, since we add 1 to both sides, we have an equivalent equation. So, instead of solving the equation

$$x^2 + 8x + 15 = 99,$$

we will add one to both sides and solve the equation

$$x^2 + 8x + 16 = 100.$$

Now, we can draw the square diagram exactly as we wanted.

	x	4
x	x^2	$4x$
4	$4x$	16

And now that we have a proper square diagram, we can solve the equation! Here's the full process:

$$\begin{aligned} x^2+8x+15&=99 && \\ x^2+8x+16&=100 && \text{APOE: add 1 to both sides} \\ (x+4)^2&=100 && \text{based on the square diagram} \\ x+4&=10 \ \underline{\text{or}}\ {}^{-}10 && \text{square root of both sides} \\ x&=6 \ \underline{\text{or}}\ {}^{-}14 && \text{SPOE} \end{aligned}$$

This is a clever application of the POEs. Rather than use the properties to eliminate terms from one side of the equation, we can use the properties to change one side into a particular, more-helpful form.[2]

Example 1.3

Determine the value of x given the equation: $x^2-6x+11=27$.

Solution: When we start the square diagram, we split the $-6x$ as usual. This means that the square has sides of length $(x-3)$. This, in turn, implies that the lower right-hand region should be 9. Our equation has 11 as its constant term: not what we want.

	x	-3
x	x^2	$-3x$
-3	$-3x$	9

To fix, this we can subtract 2 to each side of our equation. This will give us a constant term of 9, which is what we need to make our square diagram work. So, we have:

$$\begin{aligned} x^2-6x+11&=27 && \\ x^2-6x+9&=25 && \text{SPOE: subtract 2 from both sides} \\ (x-3)^2&=25 && \text{based on the square diagram} \\ x-3&=5 \ \underline{\text{or}}\ {}^{-}5 && \text{square root of both sides} \\ x&=8 \ \underline{\text{or}}\ {}^{-}2 && \text{SPOE} \end{aligned}$$

So, our solutions are $\mathcal{S}=\{8,{}^{-}2\}$.

[2] In fact, this is exactly what we've been doing all along: changing one side of an equation into a form that is more helpful. In this chapter we are expanding our notion of what it means for a change to be "helpful".

1.5 Level 5 Quadratics

The last few levels have had "polite" middle terms, which have split evenly into two pieces. What happens if we get a linear term with an odd coefficient?

Startup Exploration: Quadratic Level 5

Determine the value of x given the equation: $x^2 + 3x + 1 = 5$.

If we jump right in and try the square method, we start to get into fraction territory. If we want to split the $3x$ term exactly in half, then each piece would be $\frac{3}{2}$. Then, the square method would predict $\frac{9}{4}$ in the lower-right corner.

	x	$\frac{3}{2}$
x	x^2	$\frac{3}{2}x$
$\frac{3}{2}$	$\frac{3}{2}x$	$\frac{9}{4}$

The lower-right corner isn't what we have in our equation, so we could use APOE and add $\frac{5}{4}$ to adjust both sides... Hmm. Not pretty. To be clear, the square method will not let us down: if we keep going with the fractions, we will arrive at the correct answer! But, perhaps there is an alternative approach that avoids the fractions.

Here's a clever idea: We could use MPOE and multiply through by 2. This would give us a linear term with an even coefficient! In other words:

$$x^2 + 3x + 1 = 5 \xrightarrow{\text{multiply through by 2}} 2x^2 + 6x + 2 = 10$$

This fixes our odd coefficient problem, since now we can break the $6x$ up into two sets of $3x$. But, what do we do with that $2x^2$? We can't use x and $2x$ as the side lengths, for although that gives is the correct product, we would no longer have a square.

	?	3
?	$2x^2$	$3x$
3	$3x$	

Now, here's a *really* clever idea. Let's multiply through by 2 *again*. In other words, we will multiply the original equation by 4:

$$x^2 + 3x + 1 = 5 \xrightarrow{\text{multiply through by 4}} 4x^2 + 12x + 4 = 20$$

We still have an even linear coefficient, and now we can write $4x^2$ as $2x$ times $2x$. Note that we have to take that factor of 2 into account when we're figuring out the other dimensions of the square. Study our new diagram closely and be sure you understand where each of the labels comes from.

	$2x$	3
$2x$	$4x^2$	$6x$
3	$6x$	9

We now find ourselves in a Level 4 situation: our equation has 4 as the constant term, whereas the square model predicts 9 as the constant term. No problem! We can add 5 to both sides, and continute the process as we did with Level 4 quadratics. Here's a summary of the whole process:

$x^2 + 3x + 1 = 5$	original equation
$4x^2 + 12x + 4 = 20$	MPOE: multiply both sides by 4
$4x^2 + 12x + 9 = 25$	APOE: add 5 to both sides
$(2x + 3)^2 = 25$	based on the square diagram
$2x + 3 = 5$ <u>or</u> -5	square root of both sides
$2x = 2$ <u>or</u> -8	SPOE
$x = 1$ <u>or</u> -4	DPOE

Let's pause and review. When faced with an odd coefficient for the x term, our strategy is to multiply through by 4. This will give us an even coefficient for the x term, and at the same time keep the coefficient of the x^2 term in a state where it can be written as something times itself: $4x^2 = 2x \cdot 2x$. After we do this, we'll have a Level 4 quadratic on our hands, and we can apply techniques for handling those.

Example 1.4

Determine the value of x given the equation: $x^2 - 5x + 12 = 62$.

Solution: Faced with an odd linear coefficient, we multiply through by 4. This gives us the revised equation

$$4x^2 - 20x + 48 = 248.$$

We set up the square diagram and see whether our equation represents a complete square.

	$2x$	-5
$2x$	$4x^2$	$-10x$
-5	$-10x$	25

Our (revised) equation does not make a proper square: the constant term in the equation is 48, but the square diagram predicts 25. We can fix this problem using SPOE: subtract 23 from both sides:

$$
\begin{aligned}
x^2 - 5x + 12 &= 62 && \text{original equation} \\
4x^2 - 20x + 48 &= 248 && \text{MPOE: multiply both sides by 4} \\
4x^2 - 20x + 25 &= 225 && \text{SPOE: subtract 23 from to both sides} \\
(2x-5)^2 &= 225 && \text{based on the square diagram} \\
2x - 5 &= 15 \ \underline{\text{or}} \ -15 && \text{square root of both sides} \\
2x &= 20 \ \underline{\text{or}} \ -10 && \text{SPOE} \\
x &= 10 \ \underline{\text{or}} \ -5 && \text{DPOE}
\end{aligned}
$$

So in the end, we have solutions $\mathcal{S} = \{10, -5\}$.

1.6 Level 6 Quadratics

We've arrived at the highest level of the quadratic equation challenge. Until now, all of our equations have started with just x^2. What if the coefficient of the leading term is something other than 1?

Startup Exploration: Quadratic Level 6

Determine the value of x given the equation: $3x^2 + 8x + 1 = 12$.

What shall we do in this scenario? A clever idea is to use DPOE and divide through by 3. That would make the leading coefficient 1, as in the earlier problems. The downside is that most of the other numbers turn into fractions.

$$3x^2 + 8x + 1 = 12 \xrightarrow{\text{divide through by 3}} x^2 + \frac{8}{3}x + \frac{1}{3} = 4$$

The square method will absolutely work on an equation like this, but perhaps we'd prefer an approach that avoided all the fractions.

If scaling the equation down doesn't help, why not try to scale it up? Could we multiply through by some helpful value? Recall that the goal will be to write the x^2 term as something times itself. Since there's already a 3 there, we can solve our problem if we multiply through by 3.

$$3x^2 + 8x + 1 = 12 \xrightarrow{\text{multiply through by 3}} 9x^2 + 24x + 3 = 36$$

Note that now we are in a good position, since $9x^2 = 3x \cdot 3x$. So, let's fill out our square diagram. We have an even coefficient on the linear term, so we can split that evenly. Note that we have to take all the coefficients into account when completing the diagram. For example, when figuring out the dimensions of a box containing $12x$.

	$3x$	4
$3x$	$9x^2$	$12x$
4	$12x$	16

The square model predicts 16 as the constant term, and so our equation is not a complete square. We can use

APOE to fix that, adding 13 to both sides. Here's how it goes:

$$\begin{aligned}
3x^2 + 8x + 1 &= 12 && \text{original equation} \\
9x^2 + 24x + 3 &= 36 && \text{MPOE: multiply through by 3} \\
9x^2 + 24x + 16 &= 49 && \text{APOE: add 13 to both sides} \\
(3x + 4)^2 &= 49 && \text{based on the square diagram} \\
3x + 4 &= 7 \ \underline{\text{or}} \ {}^{-}7 && \text{square root of both sides} \\
3x &= 3 \ \underline{\text{or}} \ {}^{-}11 && \text{SPOE} \\
x &= 1 \ \underline{\text{or}} \ -\tfrac{11}{3} && \text{DPOE}
\end{aligned}$$

In summary, we multiplied through by the coefficient of the x^2 term, which gave us a coefficient that was a perfect square. In the final example for this chapter, we put it all together.

Example 1.5

Determine the value of x given the equation $-5x^2 - x + 18 = 0$.

Solution: Since we have a leading coefficient that is not a perfect square, we multiply through by that coefficient, -5 in this case. This gives us

$$25x^2 + 5x - 90 = 0$$

(careful with the negative signs). This is an improvement, but we have a linear coefficient that is odd. So, we multiply by 4 to fix that:

$$100x^2 + 20x - 360 = 0$$

Notice that multiplying through by 4 (a perfect square) gives us a leading coefficient that is still a perfect square. This is because the product of two perfect squares is itself a perfect square! (Can you prove that this statement is always true using the properties of exponents?)

We now have a revised equation that we can bring to the square method.

	$10x$	1
$10x$	$100x^2$	$10x$
1	$10x$	1

To get our constant terms to agree, we must add 361 to both sides of our revised equation. This gives us

$$10x^2 + 10x + 1 = 361.$$

And from here, we can complete a familiar process.

$$\begin{aligned}
10x^2 + 10x + 1 &= 316 & & \\
(10x+1)^2 &= 316 & & \text{based on the square diagram} \\
10x + 1 &= 19 \ \underline{\text{or}} \ {}^{-}19 & & \text{square root of both sides} \\
10x &= 18 \ \underline{\text{or}} \ {}^{-}20 & & \text{SPOE} \\
x &= \tfrac{18}{10} \ \underline{\text{or}} \ {}^{-}2 & & \text{DPOE}
\end{aligned}$$

After we simplify our one fraction answer, we have a final solution: $\mathcal{S} = \left\{\frac{9}{5}, {}^{-}2\right\}$.

1.6.1 (;,;) The Quadratic Formula

Now that we have the square method at our disposal, we can tackle any quadratic equation that is thrown at us. We might be tempted to really get generic, and solve the all at once.

Consider a completely generic quadratic equation of the form

$$ax^2 + bx + c = 0.$$

This quadratic has three coefficients: a is the coefficient of the quadratic term, b is the coefficient of the linear term, and c is the constant term. We've set it equal to zero since.... HOW TO EXPLAIN THIS?

What happens if we apply the square method to this totally generic equation? We don't really know anything about a, so to be safe, let's multiply the whole equation by a.

$$a^2x^2 + abx + ac = 0.$$

This will ensure that the leading term is a perfect square, and that's what we need for the box method. Now, the linear coefficient is ab and this might be an odd number for all we know. So, we multiply through by 4, as we have often done above.

$$4a^2x^2 + 4abx + 4c = 0.$$

This is not a pretty sight, but we'll try to make it work with the square method.

	$2ax$	b
$2ax$	$4a^2x^2$	$2abx$
b	$2abx$	b^2

Much of this looks workable, but what about the lower right-hand corner? The square method predicts b^2, but our equation has $4c$. Well, we'll do what we usually do: we'll use the POEs to turn our left-hand side into the form we want. So:

$$\begin{aligned} 4a^2x^2 + 4abx + 4c &= 0 & & \\ 4a^2x^2 + 4abx &= -4c & & \text{SPOE: subtract } 4c \text{ from both sides} \\ 4a^2x^2 + 4abx + b^2 &= b^2 - 4c & & \text{APOE: add } b^2 \text{ to both sides} \end{aligned}$$

It might seem like things are getting worse, but now the left-hand side is what we need to use our square diagram. Let's go!

$$\begin{aligned} (2ax+b)^2 &= b^2 - 4c & & \text{based on the square diagram} \\ 2ax + b &= \pm\sqrt{b^2-4c} & & \text{square root of both sides} \\ 2ax &= \pm\sqrt{b^2-4c} - b & & \text{SPOE: subtract } b \text{ from both sides} \\ x &= \frac{\pm\sqrt{b^2-4c} - b}{2a} & & \text{DPOE: divide both sides by } 2a \\ x &= \frac{-b \pm \sqrt{b^2-4c}}{2a} & & \text{Rearranging the numerator} \end{aligned}$$

COMMENTS

Chapter Summary

The properties of equality which we learned before this chapter aren't enough to solve any random quadratic equation. To overcome the quadratic obstacle, we learned new techniques for solving quadratic equations. These new techniques are based on the symmetry of a square, and we learned various techniques for adjusting a quadratic equation so that we can use the square diagram approach.

We have, however, avoided the necessity of finding the square root of a number that was not a perfec square. This was a helpful simplification – it allowed us to focus on the process of equation solving – but not all quadratics will have answers that come out so "nice". We turn our attention to square roots in the next chapter.

> There is geometry in the humming of the strings, there is music in the spacing of the spheres.
>
> *Pythagoras*
> *Ancient Greek philosopher*

Chapter 2

Radical Expressions

In the last chapter, we found integer solutions to nearly all of the equations that we studied. This was good for understanding the workings of quadratic equations, but not all equations will necessarily be so "polite". The main focus of our work in this chapter is around understanding more about all of those square roots that don't come out evenly. We begin by looking more closely at the sets $\mathbb{Q}$ and $\mathbb{R}$.

2.1 Real numbers

Startup Exploration: Share the Cheese

Middle Market sells mini-wheels of cheese for snacking. Mini-wheels can be sold one at a time, or in boxes of 10. The cheese arrives at the warehouse in crates of 10 boxes (containing 100 wheels of cheese in total).

The warehouse workers need to divide their stock of mini-wheels up among three trucks, each of which will deliver to a local Middle Market branch. The warehouse has a total of 13 crates, 7 boxes, and 9 mini-wheels in stock.

The manager allows the workers to open crates or boxes, if needed, to divide the supply evenly. Describe a process for dividing up the cheese that requires opening the minimum number of boxes.

In **??**, we made some comments which might, at first, appear contradictory. On the one hand, we saw that the set of real numbers, $\mathbb{R}$, includes every possible decimal number. We also saw that the set of rational numbers, $\mathbb{Q}$, includes "terminating decimals and repeating decimals".

On the other hand, we know that $\mathbb{Q}$ is the set of fractions, meaning those numbers that can be expressed in the form

$$\frac{a}{b} \text{ where } a \text{ and } b \text{ are integers, and } b \text{ is not zero.}$$

These statements raise a few questions. What is the relationship between "fraction" and "terminating or repeating decimal"? What can we say about decimal numbers that are neither terminating nor repeating?

2.1.1 Fractions into decimals

Recall that a fraction is simply a divison problem in disguise. If we execute the division problem, we can easily turn a fraction into a decimal. It's especially easy if we have a calculator handy... otherwise, we're in for some long division.

Long Division

Don't worry if your long division is a bit rusty, just take another look at the startup exploration. The warehouse workers must divide 1379 mini-wheels of cheese among the three trucks – that's $1379 \div 3$ – but they must do this with a minimum amount of regrouping.

One solution is to put 4 crates on each of the three trucks. This takes care of 12 crates (1200 mini-wheels in all), but leaves one crate. They have no choice but to open this crate and treat it as 10 boxes. Of course they already had 7 boxes in stock, so now they have 17 boxes in all. They can put 5 boxes on each truck (accounting for 15 boxes, or 150 mini-wheels), but they will have 2 boxes left over. They open these two boxes, revealing 20 mini-wheels. They add these to the 9 mini-wheels they had already, giving 29 mini-wheels in all. Each truck gets 9 of these (using up 27), and they have 2 left over.

So, in the end: each truck gets 4 crates, 5 boxes, and 9 mini-wheels – that's 459 mini-wheels in all – and there are 2 left behind. Now have a look at the long division for this problem: can you spot each of the steps that we took above in the work below?

$$\begin{array}{r|rrrr}
 & & 4 & 5 & 9 \\ \cline{2-5}
3 \,) & 1 & 3 & 7 & 9 \\
 & 1 & 2 & 0 & 0 \\ \cline{2-5}
 & & 1 & 7 & 9 \\
 & & 1 & 5 & 0 \\ \cline{3-5}
 & & & 2 & 9 \\
 & & & 2 & 7 \\ \cline{4-5}
 & & & & 2
\end{array}$$

In the cheese example, it makes sense to stop here with a remainder of 2. In general, though, we could continue the process of division and create a number that extends to the right of the decimal point.

Example 2.1

Convert $\frac{3}{4}$ and $\frac{1}{6}$ into their decimal representations.

Solution: Recall that the fraction three-fourths is equivalent to the division problem $3 \div 4$. To do this by long division, we put the dividend (that's 3) inside the "division house" and leave the divisor (that's 4) outside.

$$\begin{array}{r|lll}
 & 0 & .7 & 5 \\ \hline
4\,) & 3 & .0 & 0 \\
 & 2 & 8 & \\ \cline{2-3}
 & & 2 & 0 \\
 & & 2 & 0 \\ \cline{3-4}
 & & & 0
\end{array}$$

So the decimal representation of $\frac{3}{4} = 0.75$ (you may have known that already). To tackle one-sixth, we note that it is equivalent to $1 \div 6$. The long division starts out like this:

$$\begin{array}{r|llll}
 & 0 & .1 & 6 & 6 \\ \hline
6\,) & 1 & .0 & 0 & 0 \\
 & 0 & 6 & & \\ \cline{2-3}
 & & 4 & 0 & \\
 & & 3 & 6 & \\ \cline{3-4}
 & & & 4 & 0 \\
 & & & 3 & 6 \\ \cline{4-5}
 & & & & 4
\end{array}$$

We might as well stop here, though, because we're stuck in a loop! The 6's in the answer are going to repeat forever. (Can you see why?) So, the decimal representation of $\frac{1}{6} = 0.1\overline{6}$.

We say that 0.75 is a terminating decimal, because the process of long division stops with a remainder of zero. On the other hand, $0.1\overline{6}$ is called a repeating decimal because the long division process gets stuck in a loop. Note that we've used a vinculum over the 6 to indicate which digits repeat.

When it comes to a division problem like $1 \div 6$ on a calculator, the display will likely show `0.166666667`, where the 6's repeat for a while and are followed by a 7. Don't be fooled by this 7: the 6's really do go on forever! That 7 is the calculator rounding up. Always be skeptical about the rightmost digit on your calculator screen.

A Bold Claim

This process of division leads us to make a pretty bold claim: every rational number can be represented *either* as a terminating decimal or a repeating decimal. How can we be sure that every crazy fraction, for instance $\frac{19}{81}$, either terminates or repeats?

Let's think about how long division works: the "subtraction step" in particular. Here's how the process of long division starts out for $\frac{3}{4}$:

$$\begin{array}{r|lll} & & .7 & \\ \cline{2-4} 4 & 3 & .0 & 0 \\ & 2 & 8 & \\ \cline{2-3} & & 2 & \end{array}$$

In the subtraction step, we get 2 as the remainder, and so we know that we have to keep dividing. If we ever get the remainder 0, then we know that we're done with division. This is what happens eventually with $\frac{3}{4}$. On the other hand, if we ever get a remainder that we've gotten before, then we know that we're stuck in a loop. This is what happened with $\frac{1}{6}$.

Now here's a simple yet profound idea: the remainder is always less than the divisor. When dividing by 4, the remainder has to be less than 4. When dividing by 6, the remainder has to be less than 6. (Can you explain why that is? For example, when dividng mini-wheels of cheese among three trucks, could the workers have seven mini-wheels of cheese left over?)

The result is that we have a limited number of choices for the remainder. When dividing by 4, the remainder can only be 0, 1, 2, or 3. When dividing by 6, the remainder can only be : 0, 1, 2, 3, 4, or 5.

Having a limited number of choices means that eventually we have to recycle one of those remainders! We can't go on forever without either using the remainder 0 (in which case the decimal terminates) or reusing one of the nonzero remainders (in which case the decimal repeats).

Even when dividing something ugly like $1903 \div 8167$, the remainders in the subtraction steps will always be less than 8167. We might have to divide for a long time, but we know it can't carry on forever. Eventually we'll either use the remainder 0, or reuse a remainder we've used already. So the fraction $\frac{1903}{8167}$ has a decimal representation that either terminates or repeats.

Our argument applies to any denominator, and so to any rational number. Therefore, it's true that every rational number has a decimal representation that either terminates or repeats! Have we blown your mind yet? If not, stay tuned.

2.1.2 Decimals into fractions

What about the other way around? Does every terminating-or-repeating decimal have a corresponding fraction representation?

Terminating decimals into fractions

Consider a terminating decimal like 0.375. Can we turn this decimal into a fraction, meaning a ratio of two integers? If so, how?

Recall the notion of *place value*, and how the individual digits in a number are each standing in some "place" that is named after a power of ten.[1] The key to turning a terminating decimal into a fraction is recalling how to read a decimal using its place value.

thousands	hundreds	tens	units		tenths	hundredths	thousandths	ten-thousandths	hundred-thousandths	millionths
4	1	2	6	•	3	7	5	0	9	8

To read the decimal 0.375, we can say "zero point three seven five", but this isn't very helpful. Instead, we read the number using place value and say "three hundred seventy-five *thousandths*". Now, if someone were to say that number aloud, it sounds just like the fraction

$$\frac{375}{1000}.$$

In fact, this decimal number and this fraction represent exactly the same value. Of course, the fraction isn't in simplest form yet, but that's easy to fix:

$$0.375 = \frac{375}{1000} = \frac{3}{8}$$

.

We have accomplished the goal of turning a terminating decimal into a fraction. The technique is simply to read the decimal aloud using its place value, and then write down the fraction we hear.

Repeating decimals into fractions

The "read the number with its place value" technique won't work for repeating decimals. (Why not?) Instead, we'll use some clever applications of the techniques we learned when solving equations.

Suppose we try to write the repeating decimal $0.\overline{4}$ as a fraction. Let's give this number a name so that we can do some algebraic manipulations.

$$x = 0.\overline{4}$$

[1] We really are blowing the cobwebs off of some old mathematics in this chapter: Long division! Place value! It goes to show that even simple mathematical ideas can have deep and meaningful consequences.

Our goal will be to find an alternative way of writing x. To do that, we're going to make two clever moves.

The first clever move is to use MPOE: we will multiply both sides of this equation by 10. Multiplying $0.\overline{4}$ by 10 moves the decimal point one place to the right. But remember, the 4's repeat *forever*, so there are *still infinitely many* 4's to the right of the decimal point! We have:

$$10x = 4.\overline{4}$$

The second clever move is to use an idea from when we were solving systems of equations: the elimination method. Watch what happens when we subtact the first equation we wrote from the second equation:

$$\begin{array}{rr} & 10x = 4.\overline{4} \\ - & x = 0.\overline{4} \\ \hline & 9x = 4.0 \end{array}$$

Notice that the two numbers on the righthand side of our equations are exactly four units apart. In other words: the infinitely long tail of 4's disappears when we subtract! Now all we have to do is use DPOE to isolate x:

$$9x = 4 \quad \implies \quad x = \frac{4}{9}$$

If you have a calculator handy, you can perform this division and see that we have accomplished the goal of turning our repeating decimal into a fraction:

$$0.\overline{4} = \frac{4}{9}$$

This process is sometimes called *killing the tail*, since our goal is to subtract two different decimal forms that have the same repeating part, thereby eliminating the infintely long tail of digits.

Example 2.2

Convert the repeating decimal $0.\overline{63}$ to its decimal representation.

Solution. We'll kill the tail again, but note that we have two digits after the decimal which repeat. This will require a slight adjustment. We'll start as we did before, by assigning an algebraic name to our number:

$$x = 0.6363\ldots$$

If we multiply both sides by 10, we'll have

$$10x = 6.3636\ldots$$

which is also a repeating decimal, but with a *different* repeating tail. We could work with this, but it's a bit easier to multiply by 10 again (in other words, to multiply the original equation by 100):

$$100x = 63.6363\ldots$$

Now we have an equation in which the number on the righthand side has exactly the same tail as in the original equation. So, we subtract:

$$\begin{array}{rrr} & 100x = & 63.\overline{63} \\ - & x = & 0.\overline{63} \\ \hline & 99x = & 63.00 \end{array}$$

We divide both sides by 99, and then simplify our fraction to lowest terms. In the end, we have:

$$0.\overline{63} = \frac{63}{99} = \frac{7}{11}$$

The moral of the story is that we may have to adjust our method and choose the "just right" powers of 10. Consider how we might use kill the tail to turn $0.1\overline{6}$ back into $\frac{1}{6}$? (Note that the 6 repeats in the decimal form, but the 1 does not.)

Let's pause to reflect. In the first part of this section, we explained why every fraction can be written as either a terminating or repeating decimal. We can make this conversion using long division. Then we went on to show the reverse: that every terminating decimal can be written as a fraction (by reading it with its place value) and every repeating decimal can be written as a fraction (by killing the tail).

Armed with these tools, we might get the idea that *every decimal* number can be turned into a fraction. Unfortunately (or fortunately, depending on how you look at it), this is not the case.

2.1.3 Existence of irrational numbers

The **irrational numbers** are all of the real numbers that are not rational numbers. In other words, those decimal numbers that cannot be expressed as either a terminating or repeating decimal.

Back in **??** we gave an example of such a number:

$$0.10\,110\,1110\,11110\,111110\ldots$$

This number clearly has a pattern. We might explain it by saying: "After the decimal point write one, then zero, the 2 ones, then zero, then 3 ones, then zero, and so on, always writing 1 more one than you did the last time."

The problem is that is does not terminate (our pattern will continue forever), but it doesn't repeat either. The strings of 1's get longer and longer. There is never a set of always-repeating digits to group under a vinculum.

This single number is enough to prove that irrational numbers exist. Of course, there are lots of them. The famous number π is irrational, and in some sense is even more diabolical.

$$\pi \approx 3.1415926535\ 8979323846\ 2643383279\ 502884197\ 6939937510\ 5820974944\ 5923078164\ldots$$

This number doesn't even have a pattern that we can use to describe it (as far as we know). The digits go on infinitely, and come in a random sequence.

You may be wondering, "How do we know π is irrational?" After all, it may be clear why the first number with the ones and zeros is irrational, but how to we know for sure that π never terminates and never repeats?

Unfortunately, explaining the irrationality of π requires a bit more mathematics that we can get into here. However, we have learned enough to prove that certain other numbers are irrational. More on that at the end of section 2.2.

2.2 Square Roots

We have already worked quite a bit with exponents, but every exponent so far has been an integer. Could we have a rational number as an exponent? If so, what would it mean?

Startup Exploration: Half Power

Consider the expression

$$9^{1/2},$$

that is, "nine to the power one-half". What are some possible interpretations of this?

Use a calculator to explore what happens when we raise certain numbers to the one-half power (start with the natural numbers between 1 and 20). What patterns do you notice? What conjectures do you have about what's happening?

Suppose we let $x = 9^{1/2}$. We'd like to find an alternative way of expressing x that uses only integer exponents. One approach is to multiply each side of this equation by itself. Then we'd have:

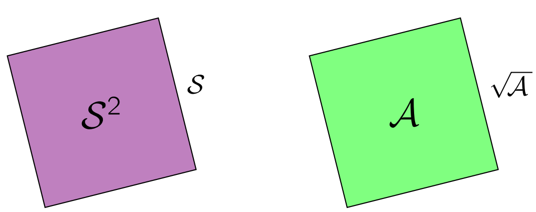
$$\begin{aligned} x &= 9^{1/2} & \\ x \cdot x &= 9^{1/2} \cdot 9^{1/2} & \text{multiply each side by itself} \\ x \cdot x &= 9^{(1/2)+(1/2)} & \text{product rule for exponents} \\ x \cdot x &= 9^{1} & \\ x \cdot x &= 9 & \end{aligned}$$

So, x is the number that when multiplied by itself gives 9 as the result. That could be either 3 or -3, since $3 \cdot 3 = -3 \cdot -3 = 9$. Using some vocabulary that we already know: we say that $9^{1/2}$ is a *square root* of 9.

Square Root

A **square root** of a is a number b such that $b \cdot b = a$.

The reason this is called a "square root" has to do with the geometric interpretation of this operation. If we have a square of side length $\mathcal{S}$, then the area of the square is $\mathcal{S}^2$. Conversely, if we have a square with area $\mathcal{A}$, then the *square root of* $\mathcal{A}$ gives us the side length of the square.



Positive numbers have two square roots: one positive, one negative. Both 3 and −3 are square roots of 9, since $(3)(3) = 9$ and $(-3)(-3) = 9$. Zero is the only number that has exactly one square root: the square root of 0 is 0. Negative numbers cause some problems when it comes to square roots (stay tuned for more on that).

Most of the time, we'll be working with the positive square root of a number, called the **principal square root** For example, the principal square root of 9 is 3 and the principal square root of 25 is 5.

The checkmark-ish symbol we use to denote the square root of a is called the **radical**: $\sqrt{a}$. This symbol means "the principal square root of a". So, $\sqrt{9} = 3$ and $\sqrt{25} = 5$. So, although −3 is a square root of 9, it is incorrect to write $\sqrt{9} = -3$ or $\sqrt{9} = \pm 3$. This isn't quite rise to the level of Evil and Wrong, but it's not right.[2]

When simplifying an expression or solving an equation, we won't always give both square roots. A good guideline as we go along is to pay close attention and use the notation given in the problem. For example:

$$\pm\sqrt{100} = \pm 10 \qquad \text{and} \qquad -\sqrt{36} = -6$$

In these cases, the notation specifically indicates that we want both the positive and negative square root, or only the negative square root.

2.2.1 Imaginary Numbers

Consider the expression $\sqrt{-16}$. This is a bit of a problem. We're meant to find the principal square root of −16, but the closest we can get is $4 \cdot -4 = -16$. It's true that 4 and −4 have the same absolute value, but they're not the same number, which means we're not *squaring* anything. So, there is no real number equal to $\sqrt{-16}$. Of course, −16 isn't special, the same argument applies to any negative number.

Square Root of a Negative Number

If $a < 0$, then there is no real number equal to $\sqrt{a}$.

Note that this *doesn't* mean that −16 doesn't have any square roots. It simply means that the square roots are not real numbers. The square roots of −16 are members of a set called the *complex numbers* $\mathbb{C}$, but not members of the real numbers $\mathbb{R}$.

To build up the complex numbers, we introduce the so-called imaginary unit i which has the property that $i^2 = 1$. Remember, the domain of algebra 1 is the set real numbers. But, the domain of algebra 2 and beyond is the set of complex numbers. So, if you're intrigued about imaginary numbers, just hang in there.

[2] An algebra note from the future! When we simplify the expression $\sqrt{m^2}$, the result is $|\,m\,|$, the absolute value of m. Do you see why? The value that comes out of the radical must be positive, because that notation gives the principal square root.

Our work in algebra 1 will bring us mostly in contact with *square roots*, but other roots are possible. For example a "cube root" of a number a is a number b such that $b \cdot b \cdot b = a$. We write $\sqrt[3]{a}$ to denote the cube root of a. Negative numbers under the cube root symbol are no problem:

$$\sqrt[3]{-8} = -2 \qquad \text{since} \qquad -2 \cdot -2 \cdot -2 = -8$$

2.2.2 (;,;) Irrationality of the Square Root of Two

We have shown that irrational numbers exist. Consider the mathematical argument below, which explains why the square root of 2 is irrational.

Step 1. Assume (for the moment) that $\sqrt{2}$ is, in fact, rational. In other words, that it can be written as the ratio of two integers. Then, we can write

$$\sqrt{2} = \frac{a}{b}$$

where a and b are relatively prime. That is to say, our fraction is in lowest terms.

Step 2. Square both sides of the equation above.

$$2 = \frac{a^2}{b^2}$$

Step 3. Multiply both sides of this equation by b^2.

$$2b^2 = a^2$$

This means that a^2 is an even number. If a^2 is an even number, then a must be an even number.

Step 4. If a is an even number, then it is divisible by 2. In other words, there is an integer m such that $a = 2m$.

Step 5. Substitute $2m$ for a in the equation from Step 3, and simplify.

$$\begin{aligned} 2b^2 &= (2m)^2 \\ 2b^2 &= 4m^2 \\ b^2 &= 2m^2 \end{aligned}$$

This means that b^2 is an even number. If b^2 is an even number, then b is an even number.

Step 6. If both a and b are even numbers, then the original fraction $\frac{a}{b}$ was not in lowest terms, which was our assumption! This contradiction shows that our original assumption cannot be true.

Thus, $\sqrt{2}$ cannot be written as a fraction. In other words, $\sqrt{2}$ is an irrational number.

Thoughts to Chew On

Steps 3 and 5 both include assertions about numbers being even. How do we know when a number is even? Specifically, how do we know that a^2 and b^2 are even?

Steps 3 and 5 both go on to say something like "if a^2 is even, then a is even". How do we know this is true? Under what circumstances will the square of a number be even or odd?

Step 6 argues that $\frac{a}{b}$ is not in lowest terms. How do we know this is true?

This is an example of *proof by contradiction*. We assume that some statement is true, then show that this assumption leads to some kind of impossible situation. The impossibility means we have to reject the original assumption. What was our original assumption in this proof? What is the contradiction that results from that assumption?

2.3 Simplified Radical Form

When we take the square root of a perfect square we get an integer as the answer. But, things are not so easy when taking the square root of a number that is not a perfect square. In fact, the square root of a non-square natural number will be an irrational number, like $\sqrt{2}$.

The *exact value* of an irrational number can only be represented using some kind of symbol, like π or $\sqrt{2}$. Writing out a decimal value – no matter how many decimals you write down – will always be an approximation. So, it's a good habit of mind to think "should I be giving an exact answer to this problem, or is a decimal approximation good enough". Very often, the context (or the directions) will make this choice clear.

To help us standardize the way we write radical expressions, we all agree to comply with *simplified radical form*.

Simplified Radical Form

A radical expression is considered completely simplified if...

1. Like radical terms have been combined.
2. The expression under the radical has no perfect square factors other than 1.
3. There are no fractions under the radical.
4. There are no radicals in the denominator of a fraction.

Over the next few sections, we will discuss each of these criteria and the algebraic manipulations that we can use to make sure our expressions comply. The first criteria is quite straightforward, so let's get right to it.

2.3.1 Like Radical Terms

Criteria #1 states that like radical terms must be combined. We combine radical terms as we do variable terms. For example, we are very familiar with the simplification

$$x + x = 2x.$$

We combine radical terms in exactly the same way:

$$\sqrt{5} + \sqrt{5} = 2 \cdot \sqrt{5} = 2\sqrt{5}.$$

Note, in particular, that the sum here is not $\sqrt{10}$. Similarly, $3x + 4x = 7x$ and so with radicals, we have $3\sqrt{21} + 4\sqrt{21} = 7\sqrt{21}$. When we have a multiplication of a number times a radical, we can omit the multiplication symbol.

WARNING!

When we say like radical terms "can be combined", don't go thinking you can add the numbers under the radical. To add the values like this is **Evil and Wrong**.

$$\sqrt{3} + \sqrt{3} \neq \sqrt{6}$$

While we're at it, don't get any ideas about splitting the radical-of-a-sum into the sum-of-radicals. This, too, is **Evil and Wrong**.

$$\sqrt{2 + 14} \neq \sqrt{2} + \sqrt{14}$$

2.3.2 Product Properties of Radicals

Startup Exploration: Building Blocks

The number 1 is the *additive building block* of the natural numbers. In other words: If we want to "build" any natural number using only addition, the only number we need is the number 1. Every natural number can be written as the sum of a bunch of 1's.

What are the *multiplicative building blocks* of the natural numbers? Note that we have to say *blocks* (plural) since the number 1 is not enough: multiplying together a bunch of 1's always gives us 1 as the product. What is the smallest collection of natural numbers that we need in order to build the rest using only multiplication?

The second criteria for simplified radical form states that the expression under the radical may have no perfect square factors other than 1. This may seem strangely worded. It clearly handles the idea that there should be no perfect squares under the radical, and that makes sense. Expressions like $\sqrt{4}$ and $\sqrt{25}$ can pretty obviously be simplified.

But, this criteria also catches expressions like $\sqrt{24}$ and $\sqrt{50}$ because those numbers, neither of which is a perfect square, each have a perfect square as a factor: 24 has 4 as a factor, and 50 has 25 as a factor.

How can we simplify an expression like $\sqrt{50}$ so that it has no perfect square factors under the radical? For help, we turn to:

Product Rule of Radicals

For any $a \geq 0$ and $b \geq 0$,

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}.$$

Note: In algebra 1 we only use the square root version of this property, though in fact it applies to radicals of any degree: cube roots, fourth roots, and so on.

This property looks an awful lot like the product rule for exponents, which makes sense since here we are undoing the power of a product rule, where the power is the exponent one-half!

Example 2.3

Express $\sqrt{50}$ in simplified radical form.

Solution: We know $\sqrt{50}$ is not yet in simplified radical form because 50 is divisible by a perfect square, $50 = 25 \cdot 2$. We apply the multiplication property of radicals like so:

$$\begin{aligned} \sqrt{50} &= \sqrt{25 \cdot 2} && \text{rewrite 50 to show its perfect square factor} \\ &= \sqrt{25} \cdot \sqrt{2} && \text{product rule of radicals} \\ &= 5 \cdot \sqrt{2} && \text{simplify the square root of a perfect square} \end{aligned}$$

So, $\sqrt{50} = 5\sqrt{2}$. These two expressions are equal, but only the second expression satisfies the criteria of simplified radical form.

2.3.3 Different Approaches to Simplifying

There are a number of ways to go about applying this property to simplify expressions. Use whatever approach makes the most sense to you! Here are some alternatives, though you might find a different approach that fits you better. In any case, it will probably be helpful to learn a variety of methods. Depending on the problem, some methods may be easier to use than others.

For example, let's examine different ways to get $\sqrt{108}$ into simplified radical form.

Strategy 1: Largest Square Factor

In this strategy, we find the largest perfect square factor and simplify it using the product rule for radicals. We might notice that $108 = 3 \times 36$:

$$\sqrt{108} = \sqrt{36 \cdot 3} = \sqrt{36} \cdot \sqrt{3} = 6\sqrt{3}$$

Strategy 2: One Square at a Time

It might not be obvious what the largest perfect square is, so in this strategy we look for *any* prefect square factor and work one square at a time. For instance, we might notice that 108 is divisible by 9 (how can we quickly spot divisibility by 9?). Then:

$$\sqrt{108} = \sqrt{9 \cdot 12} = \sqrt{9} \cdot \sqrt{12} = 3 \cdot \sqrt{12} = 3 \cdot \sqrt{4 \cdot 3} = 3 \cdot \sqrt{4} \cdot \sqrt{3} = 3 \cdot 2 \cdot \sqrt{3} = 6\sqrt{3}$$

In this approach, we have to keep checking to see whether the number under the radical is "square-free" or not. After our first simplification, we have 12 under the radical. But 12 has 4 as a factor, so we have to do another simplification step.

This process might take a little longer, but it is sometimes easier to identify smaller perfect square factors and chip away at the problem, than it is to identify the largest perfect square factor and finish the problem in a single step.

Strategy 3: The Sniper Method

The idea here is to write the *prime factorization* of the number under the radical, and then look for pairs of factors.[3] The factorization of $108 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$, so:

$$\sqrt{108} = \sqrt{\underline{2 \cdot 2} \cdot \underline{3 \cdot 3} \cdot 3} = \underline{2} \cdot \underline{3} \cdot \sqrt{3} = 6\sqrt{3}$$

We've given this strategy the memorable (though perhaps gruesome) name *the sniper method*. Think of the radical as a prison. There are snipers outside and any number that tries to escape needs to have a decoy. A single factor of 2 is stuck inside for life, but if the 2 has a partner (that is, if there's a $2 \cdot 2$ under the radical), then 2 can make a break for it!

But, only one of the partners survives the jailbreak. The snipers take out the decoy. In the example above, one 2 makes it out, and so does one 3. The final factor of 3 is partnerless, and left trapped inside its radical prison.

[3] For a handy, if unusually-formatted, list of prime numbers, see appendix A.

Example 2.4

TO DO.

2.3.4 Quotient Properties of Radicals

Criteria #3 and #4 for simplified radical form are both pretty antiquated. They came about in the pre-calculator days when folks had to do a lot more calculation by hand and use large data tables to approximate radical values. Yet, these last two properties are still considered "standard" for simplified radical form.

Rules were made to be broken, though, and there will be times when it's OK to break away from these criteria (#4 especially). But we'll burn that bridge when we come to it. For now, all four criteria are in effect.

Both of these have to do with interactions between radicals and fractions. Criteria #3 disallows fractions under the radical, and criteria #4 forbids radicals in the denominator of a fraction.

To tackle Criteria #3, for example when faced with expressions like

$$\sqrt{\frac{4}{49}} \quad \text{or} \quad \sqrt{\frac{24}{25}},$$

we turn to:

Quotient Rule of Radicals

For any $a \geq 0$ and $b \geq 0$,

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

Again, this property applies to radicals of any degree (though for now we'll focus on square roots). And again, this property is just like the quotient rule for exponents, but with a rational exponent.

Example 2.5

Express $\sqrt{\frac{4}{49}}$ and $\sqrt{\frac{24}{25}}$ in simplified radical form.

Solution: Here we have a fairly clear application of the rule:

$$\begin{aligned} \sqrt{\frac{4}{49}} &= \frac{\sqrt{4}}{\sqrt{49}} && \text{quotient rule of radicals} \\ &= \frac{2}{7} && \text{simplify sqaure roots} \end{aligned}$$

In the second example, the numerator doesn't contain in a perfect square, so we must apply the product rule.

$$\begin{aligned} \sqrt{\frac{24}{25}} &= \frac{\sqrt{24}}{\sqrt{25}} && \text{quotient rule of radicals} \\ &= \frac{\sqrt{24}}{5} && \text{simplify denominator} \\ &= \frac{\sqrt{4 \cdot 6}}{5} && \text{product rule in the numerator} \\ &= \frac{\sqrt{4} \cdot \sqrt{6}}{5} \\ &= \frac{2\sqrt{6}}{5} \end{aligned}$$

2.3.5 Rationalizing the Denominator

When simplifying an expression using the division property, we may encounter something like the following:

$$\sqrt{\frac{9}{2}} = \frac{\sqrt{9}}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

Back in the pre-calculator days, this led to criteria #4, no radicals in the denominator of a fraction.

When we have a radical in the denominator of a fraction we have an *irrational denominator*. Our goal is to fix this by creating an equivalent fraction with a *rational denominator*. The process of making this translation is called **rationalizing the denominator**.

We will employ the trusty *identity property of multiplication*. Remember, multiplying a number by a fancy 1 does not change the value of the number. The trick will be to choose the way our version of 1 looks. We are going to choose a fancy version of 1 that when multiplied by our irrational denominator gives us a rational number (in fact, an integer).

Study the following examples:

Example 2.6

Write $\dfrac{3}{\sqrt{2}}$ in simplified radical form.

Solution: Note the clever use of multiplication by a fancy version of 1.

$$\begin{aligned}
\frac{3}{\sqrt{2}} &= \frac{3}{\sqrt{2}} \cdot 1 && \text{identity property of multiplication} \\
&= \frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} && \text{substitute a fancy version of 1} \\
&= \frac{3\sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} && \text{multiply fractions} \\
&= \frac{3\sqrt{2}}{2} && \text{definition of square root (in the denominator)}
\end{aligned}$$

Note that we chose as our fancy 1 *exactly what we needed* to make the denominator of our fraction turn into an integer. This might seem like cheating, but it's a completely legal move, algebraically speaking.

Be sure to pay close attention. Sometimes we can use the division property in reverse to get rid of radicals in the denominator. We'll work the next example in two different ways to show the comparison.

Example 2.7

Write $\dfrac{\sqrt{84}}{\sqrt{6}}$ in simplified radical form.

Solution: First, we'll rationalize the denominator using a fancy version of 1.

$$
\begin{aligned}
\frac{\sqrt{84}}{\sqrt{6}} &= \frac{\sqrt{84}}{\sqrt{6}} \cdot 1 && \text{identity property of multiplication} \\
&= \frac{\sqrt{84}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} && \text{substitute a fancy version of 1} \\
&= \frac{\sqrt{84} \cdot \sqrt{6}}{\sqrt{6} \cdot \sqrt{6}} && \text{multiply fractions} \\
&= \frac{\sqrt{84 \cdot 6}}{6} && \text{product rule for radicals} \\
&= \frac{\sqrt{2 \cdot 2 \cdot 3 \cdot 7 \cdot 3 \cdot 2}}{6} && \text{simplify numerator using the sniper method} \\
&= \frac{2 \cdot 3\sqrt{2 \cdot 7}}{6} \\
&= \frac{6\sqrt{14}}{6} \\
&= \sqrt{14}
\end{aligned}
$$

Now, an alternative approach: We'll use the division property of radicals in reverse first, and then simplify the fraction under the radical.

$$
\begin{aligned}
\frac{\sqrt{84}}{\sqrt{6}} &= \sqrt{\frac{84}{6}} && \text{division property of radicals} \\
&= \sqrt{\frac{14}{1}} && \text{simplify the fraction} \\
&= \sqrt{14} && \text{Voilà.}
\end{aligned}
$$

The second approach is much easier in this case, though it may not always be this easy. (Under what circumstances will we be able to use the kind of shortcut?)

WARNING!

Answers with fractions must be simplified, but folks sometimes get overly aggressive with the simplification. Consider the following:

$$\frac{2}{\sqrt{6}} = \frac{2}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{2\sqrt{6}}{\sqrt{6} \cdot \sqrt{6}} = \frac{2\sqrt{6}}{6}$$

At this point, we can do one more simplification:

$$\frac{2\sqrt{6}}{6} = \frac{\sqrt{6}}{3} \qquad \text{Yes!}$$

But we might be tempted to try and simplify even more:

$$\frac{\sqrt{6}}{3} = \frac{\sqrt{2}}{1} \qquad \text{No!}$$

It's tempting, but we can't simplify using things *under* the radical and things *outside* the radical. That 6 under the radical cannot cancel with the 3 outside! To attempt such a simplification is **Evil and Wrong**.

Example 2.8

Determine the value of x given the equation: $2x^2 + 4x - 3 = 40$.

Solution: We'll use everything that we learned in chapter 1! Our first step is to ensure that the first term is a perfect square, so we multiply through by 2.

$$4x^2 + 8x - 6 = 80$$

This also gives us an even linear coefficient, so it looks like we're ready for the square method.

	$2x$	2
$2x$	$4x^2$	$4x$
2	$4x$	4

The square method predicts 4 as the constant term, but our equation has -6. APOE to the rescue:

$$\begin{aligned} 4x^2 + 8x - 6 &= 80 && \\ 4x^2 + 8x + 4 &= 90 && \text{APOE: add 10 to both sides} \\ (2x+2)^2 &= 90 && \text{based on the square diagram} \\ 2x + 2 &= \pm\sqrt{90} && \text{square root of both sides} \\ 2x &= -2 \pm \sqrt{90} && \text{SPOE} \\ x &= \frac{-2 \pm \sqrt{90}}{2} && \text{DPOE} \end{aligned}$$

Almost there! Remember, we need to simplify our radicals! We can use the sniper method in this case:

$$\sqrt{90} = \sqrt{3 \cdot 3 \cdot 2 \cdot 5} = 3\sqrt{2 \cdot 5} = 3\sqrt{10}.$$

And so in the end, we have

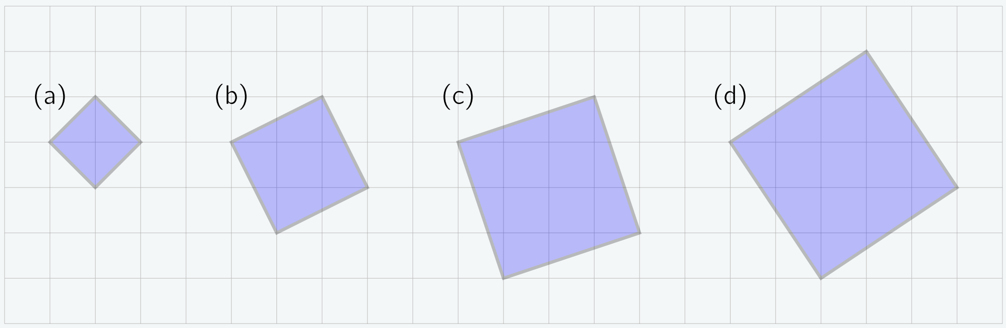
$$\mathcal{S} = \left\{ \frac{-2 \pm 3\sqrt{10}}{2} \right\}$$

We'll admit that these answers ain't pretty (note that there are two answers there!), but they are the values that satisfy our original quadratic equation.

2.4 Coordinate Geometry

Startup Exploration: Squarea

As we saw in section 2.2, a square with area $\mathcal{A}$ has side length $\sqrt{\mathcal{A}}$. Consider the figures below.



What is the area of each square? What is the side length of each square? Can you draw a square with side length $\sqrt{8}$? What about $\sqrt{13}$?

By the way, how do we know that each of these figures is, in fact, a square? (Hint: Think back to the slopes of parallel lines.)

As an application of radicals and radical expressions, which are closely connected to the side lengths of squares, it's natural to discuss concepts from geometry. We'll begin with one of the most famous and important statements in mathematics.

2.4.1 The Pythagorean Theorem

It's a good bet that have seen the Pythagorean Theorem before, and that you will see it in every high school mathematics class you take, and many of the mathematics classed you take in college. In fact, the Pythagorean theorem is a foundational piece of an entire branch of mathematics based on the properties of triangles called *trigonometry*.[4]

[4] Trigonometry begins with the study of triangles. *Trigon* is another way of saying *triangle* – in fact it might be a better way of naming that shape! Most of the other polygons we know (pentagons, hexagons, octagons) have that *-gon* suffix, and the prefix *tri-* means "three" (as in tricycle).

The Pythagorean Theorem

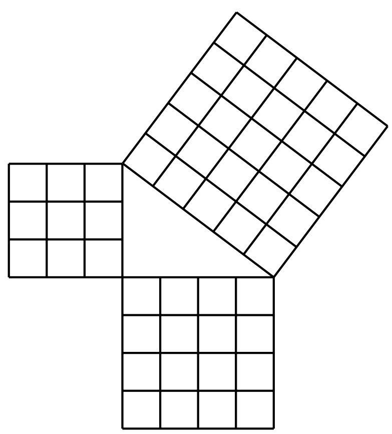
The sum of the squares of the lengths of the **legs** of a right triangle is equal to the square of the length of the **hypotenuse**.

In other words, if a and b represent the lengths of the legs (the perpendicular sides) of a right triangle, and c represents the length of the hypotenuse (the longest side, opposite the right angle), then

$$a^2 + b^2 = c^2.$$

The theorem is named after Greek philosopher and mathematician Pythagoras of Samos, who lived around 570–495 BCE. However, there is substantial evidence that the theorem was known to many different cultures from many different time periods. There is evidence, for instance, that the ancient Babylonians knew about Pythagorean triples (see the next section) more than 1000 years BCE.

Let's jump in with a famous right triangle: one with legs of length 3 and 4, and with hypotenuse of length 5. If we draw squares on the sides of the triangle, we can see that the sums of the areas of the two smaller squares (9+16) is exactly equal to the area of the largest square (25).



This relationship holds true for all right triangles as does its *converse*. The converse of the theorem states that if we have three numbers that satisfy the Pythagorean theorem, then we know that they must form the sides of a right triangle.

You might be asking yourself, "How do we know that the theorem is true for *every right triangle ever*?" Those who are curious might enjoy exploring this question at the end of this section.

2.4.2 Pythagorean Triples

Pythagorean Triple

A *Pythagorean triple* is a set of three positive integers satisfying the Pythagorean theorem.

There are infinitely many Pythagorean triples, and it is quite handy to know a few by heart. (This is because they are used quite a bit in problems, and those delightful standardized tests we all know and love.) Here are a few common Pythagorean triples:

$$(3,4,5) \qquad (5,12,13) \qquad (7,24,25) \qquad (9,40,41) \qquad (8,15,17)$$

The benefit of memorizing a few of these is that, if you see that a right triangle that has leg lengths of 7 and 24, you know that the hypotenuse has length 25 without having to do any calculations.

The triples above are called **primitive Pythagorean triples** because the three values are relatively prime. You can generate new Pythagorean triples by scaling up a triple that you know. For example $(6,8,10)$ is a triple formed by scaling up the $(3,4,5)$ triple by a factor of 2.

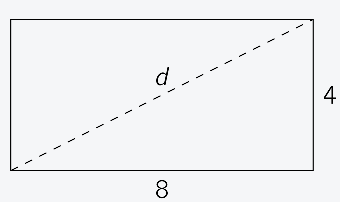
Find-the-Missing-Side Problems

The classic application of the Pythagorean theorem is to find a missing side length, either a leg or a hypotenuse, and you may have solved problems like this before. Now that we have some algebra skills, however, our answers should be given in simplified radical form! No more decimal approximations (unless the directions state otherwise)!

Example 2.9

Find the length of the digaonal of a 4-by-8 rectangle.

Solution: This problem might not, at first, seem to have anything to do with the Pythagorean theorem. The theorem, after all, is about right triangles, and this question is about a rectangle! Drawing a picture helps to reveal the connection:



If we let d represent the length of the diagonal, then we can see that it is the hypotenuse of a right triangle with legs of length 4 and 8. So:

$$\begin{aligned}
d^2 &= 4^2 + 8^2 && \text{Pythagorean theorem} \\
d^2 &= 16 + 64 \\
d^2 &= 80 \\
d &= \sqrt{80} && \text{square root of both sides} \\
d &= \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5} && \text{simplify using the sniper method} \\
d &= 2 \cdot 2 \cdot \sqrt{5} \\
d &= 4\sqrt{5}
\end{aligned}$$

So, a 4-by-8 rectangle has a diagonal which is $4\sqrt{5}$ units long.

To get a feel for whether this answer is reasonable, we could find a decimal approximation using a calculator (it's about 9 units long, which seems OK), or we could reason as follows. We know that $\sqrt{4} = 2$, and so $\sqrt{5}$ must be a bit more than 2. So, $4\sqrt{5}$ must be a bit more than 8. This seems reasonable.

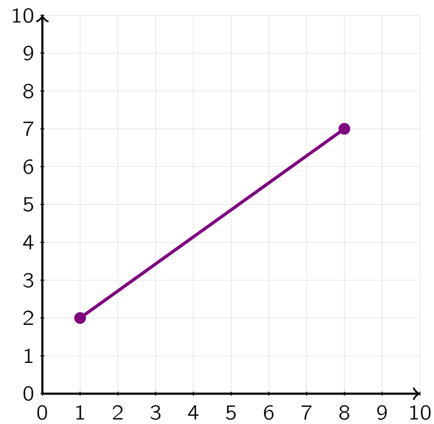
On the other hand, if we had gotten $4\sqrt{8}$ we might reason that $\sqrt{9} = 3$ and so $4\sqrt{8}$ must be a bit less than $4\sqrt{9} = 4 \cdot 3 = 12$. This is too long for the diagonal of a 4-by-8 rectangle, since the two sides together are only 12 units long in total!

2.4.3 The Distance Formula

Startup Exploration: Grid Distance

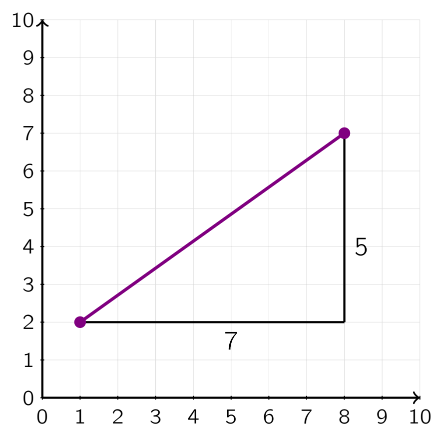
Find the length of the line segment connecting the points $(1, 2)$ and $(6, 7)$.

One important application of the Pythagorean theorem is called the **distance formula**. It is a formula that we can use to calculate the distance between two points on a coordinate grid. In the startup exploration, we have the segment pictured below:



If we think of the line segment into the hypotenuse of a right triangle, then we can use the Pythagorean theorem! The legs are the vertical and horizontal distances between the points, as in a slope triangle.

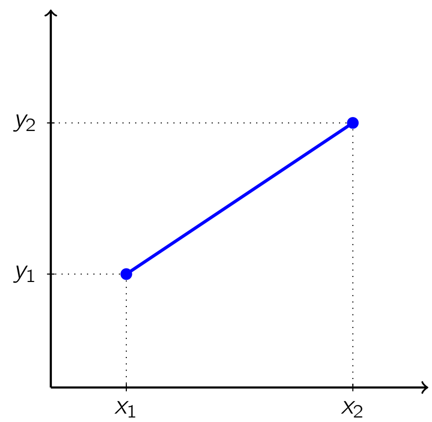
Draw the triangle, find the horizontal and vertical distances, then apply the Pythagorean theorem.



The horizontal distance is 7 units, and the vertical distance is 5 units. Those are the legs of the right triangle. Then, we use the theorem to find the length of the hypotenuse:

$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
5^2 + 7^2 &= c^2 \\
25 + 49 &= c^2 \\
74 &= c^2 \\
\sqrt{74} &= c
\end{aligned}
$$

Since the process is the same every time, we can generalize to find the distance between any two points (x_1, y_1) and (x_2, y_2) in the plane.



Distance Formula

Given two points in the plane (x_1, y_1) and (x_2, y_2), the length d of the line segment connecting the points is given by the formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This might look like a bunch of alphabet soup, much harder to remember than the Pythagorean theorem. But remember: this *is* the Pythagorem theorem! If you forget the formula, don't panic! Just remember that the line segment between the two points is the hypotenuse of a right triangle.

Example 2.10

Find the distance between the points $(5, 12)$ and $(-4, -2)$.

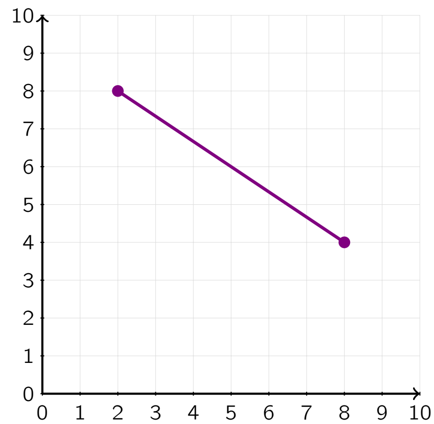
Solution: We will use the distance formula, but note that we have both subtraction and negative numbers. Watch those minus signs!

$$\begin{aligned} d &= \sqrt{(5 - -4)^2 + (12 - -2)^2} \\ &= \sqrt{9^2 + 14^2} \\ &= \sqrt{81 + 196} \\ &= \sqrt{277} \end{aligned}$$

Since 277 is prime, we know our answer complies with simplified radical form, so we're all done. The distance betnwee the two points is $\sqrt{277}$ units.

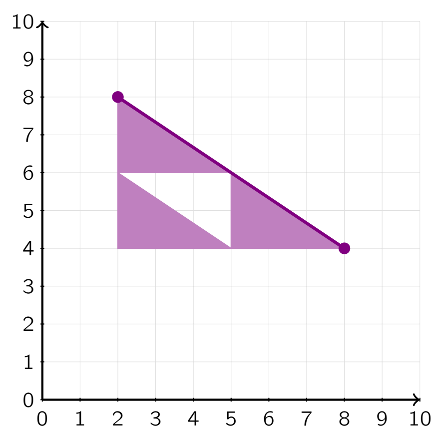
2.4.4 Midpoint Formula

When working with the distance formula, there is a related formula for finding the coordinates of *midpoint* of a given line segment. Suppose we wanted to find the midpoint of the line segment connecting $(2, 8)$ and $(8, 4)$.



Well, it sure looks like the midpoint of this segment is the point $(5, 6)$... but can we be sure?

One way to explain this is by drawing in the right triangle, and then chopping that triangle into four congruent sub-triangles. (Remember our work with the Sierpiński triangle ages ago?)



This diagram suggests that the x-coordinate of the midpoint of the hypotenuse is exactly halfway between the x-coordinates of the legs. The same goes for the y-coordinate. So, to find the coordinates of the midpoint, all we have to do is average the coordinates of the endpoints! Once again, you don't have to memorize this formula if you remember where it comes from!

The Midpoint Formula

Given a line segment with endpoints (x_1, y_1) and (x_2, y_2), the coordinates of the midpoint of the segment are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

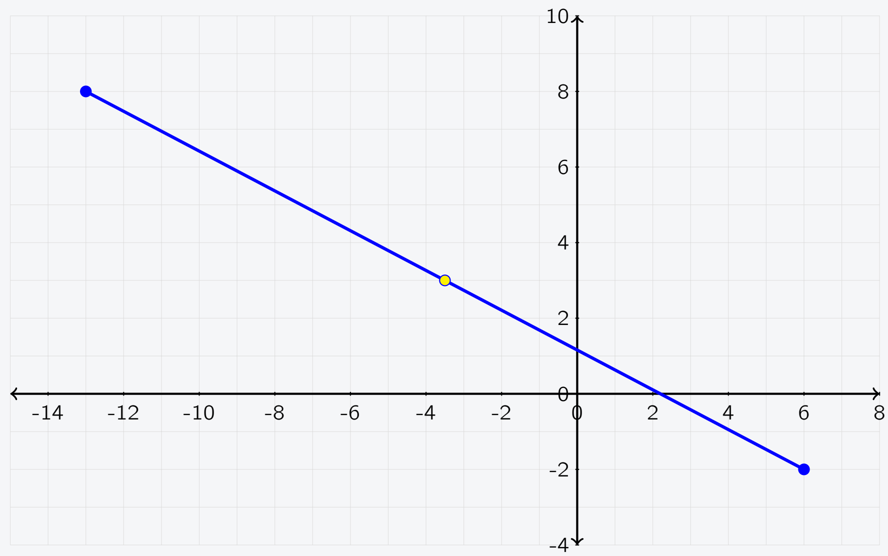
Example 2.11

Find the midpoint of the segment connecting the points $(-13, 8)$ and $(6, -2)$.

Solution: Let's calculate the midpont first, and then check our answer by drawing a picture. The formula is pretty straightforward. The midpoint should be located at:

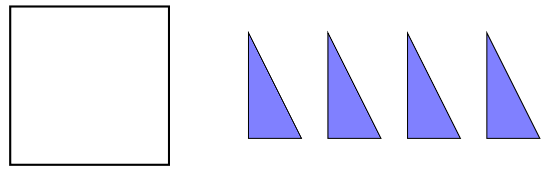
$$\left(\frac{-13+6}{2}, \frac{8+-2}{2}\right) = \left(\frac{-7}{2}, \frac{6}{2}\right) = (-3.5, 3)$$

Here's a graph to check that our answer is reasonable:

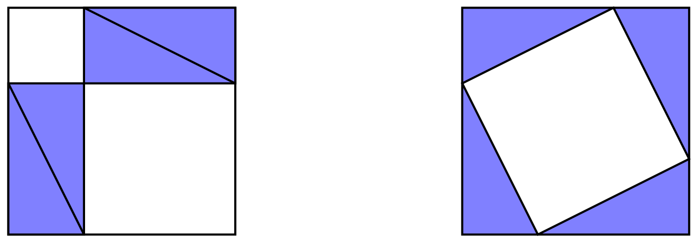


2.4.5 (;,;) Proof of the Pythagorean Theorem

Imagine a pizza box and four identical slices of pizza... where the pizza slices are right triangles. (Not typical for pizza slices, we know, but go with it.)



Consider two different arrangements of the same four pizza slices inside the same pizza box. The slices are arranged to that they fit inside without overlapping, and they lay flat on the bottom of the box.



Step 1: Consider the image on the left: Write an expression for the area of the box that is left *uncovered*. You may find it helpful to label the sides of the triangles, or the sides of the box (or both).

You may be tempted to say that the uncovered regions are squares. How do we know – for sure – that these two regions are actually *squares*, as opposed to some other quadrilateral?

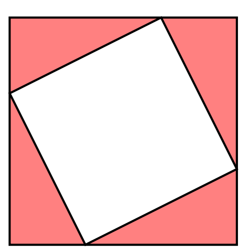
Step 2: Consider the image on the right: Write an expression for the area of the box that is left *uncovered*. Use the same labels you assigned when studying the other picture.

Again, you may be tempted to say that the uncovered region is a square. How do we know that it is a square (and not, say, a rhombus)?

Step 3: The area that is uncovered by the pizza slices is the same. (How do we know this?) What does this tell us about the two expressions for the uncovered area?

An Alternative Approach

In fact, we can explain the theorem using only the right-hand diagram. Let a and b be the lengths of the legs of one of the triangles, and let c be the length of the hypotenuse.



Step 1: Express the sides of the largest square (the pizza box itself) in terms of a and b.

Step 2: Express the area of the largest square in terms of a and b. (Hint: You'll need the sum to a power rule.)

Step 3: The area of the largest square can also be expressed at the sum of the areas of the four triangles plus the area of the tiled square. Express the area of the largest square in this way (in terms of a, b, and c).

Step 4: We now have two ways of expressing the area of the largest square. What happens when set them equal to each other?

Generalizing

All the figures in this discussion have been drawn using a particular square and a particular triangle. Explain why these arguments are proof that the Pythagorean theorem is true for any right triangle, not just the specific triangle that is pictured in these diagrams.

Chapter Summary

Each chapter should have some kind of closing remarks that summarize the chapter briefly and give some hints about the future direction.

Nature creates curved lines while humans create straight lines.

Hideki Yukawa
Japanese theoretical physicist

Chapter 3

Quadratic Functions

In the last few chapters, we have studied quadratic equations, the methods for solving them, and the various algebraic rules for manipulating the kinds of answers we get (square roots). We turn now to quadratic relationships and situations that can be represented using quadratic functions. We'll look more closely at the equations and graphs of quadratic functions, and look closely at one of the classic applications from physics.

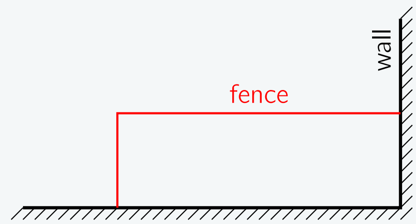
3.1 Quadratic Relationships

Extended Exploration: Staking a Claim

LINK

Startup Exploration: Compost Heap

Uncle François wants to make a compost heap in the corner of his vegetable garden. He plans to enclose a rectangular area using the two existing walls of the garden, plus 8 yards of leftover fencing.



If he wants to enclose the largest possible area, how should he arrange the fence? What is the maximum area he can enclose?

Thoughts:

This startup is similar to staking a claim, but I hope different enough. I couldn't come up with a new quadratic scenario that so clearly demonstrated all the features of quadratics (all in the first quadrant) AND had such a natural way to write the equation.

The plan will be to do the staking a claim treatment with this problem and hit the highlights about symmetry, the vertex, and all that.

3.2 Vertex Form

Build on transformations to discuss vertex form, and graphing in vertex form.

3.3 Graphing Quadratics

Graphing quadratics in standard form, equation for line of symmetry in standard form, finding vertex in standard form (using LOS).

3.4 Projectile Motion

Projectile motion is a classic application of quadratic functions, and we'll study both the mathematics and the science in this section.

Startup Exploration: Rocket Launch

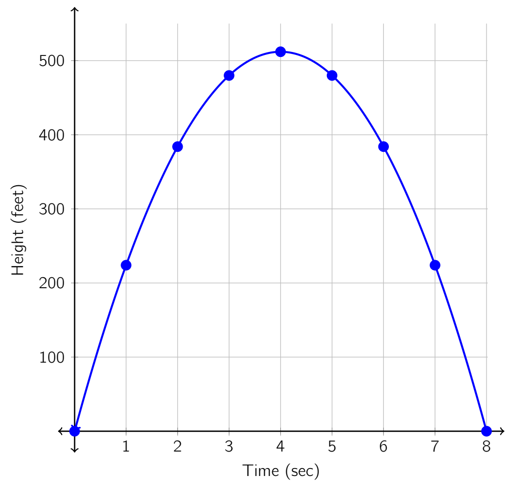
When an object is launched vertically, it flies straight up and then falls straight down. Height data for the flight of one of Ivan's model rockets is given below.

Time (sec)	Height (feet)
0	0
1	224
2	384
3	480
4	512
5	480
6	384
7	224
8	0

Study the table and make a graph of the data to see what you can learn. Then, write a few sentences describing the flight of Ivan's rocket in as much detail as you can.

From the data itself we can see the up-and-down pattern. Plus, we can see a nice symmetry: The object starts at a height of 0 meters (so it starts on the ground) and lands on the ground 8 seconds later. One second after launch it is at the same height as 1 second before it lands. This mirror image pattern continues until the object reaches its maximum height 4 seconds after launch.

A graph of the data shows the same up-and-down pattern, and the same symmetry: we could fold this graph along the vertical line at $x = 4$ and the two halves would match up exactly.



These features all suggest that we may have a quadratic pattern.[1] Scientifically speaking, Ivan's rocket is a *projectile*, and quadratic functions are the underpinning of the physics that describes the trajectory of a projectile.

Projectile

Any object that moves through the air or through space acted on only by the force of gravity.

We use projectiles in games: thowing a baseball, basketball, or dart. We use projectiles as weapons: shooting an arrow or a cannonball, throwing a spear, or launching something from a catapult.

Not all flying things are projectiles. Airplanes, helicopters, and birds use energy to keep themselves in the air, and they can use that energy to steer and change direction in midair. Projectiles are not "powered flight", they are launched and the allowed to follow their path, acted on only by gravity.

Trajectory

An imaginary tracing of a projectile's position as it moves through space.

Ivan's rocket was launched, but we can use the term "launch" rather loosely. An item that has been dropped is, according to our definition, a projectile. This is the first kind of motion we'll investigate closely, beginning with a discussion about gravity.

[1] The title of this chapter might also have been a hint.

3.4.1 Falling Objects

An object that has been dropped and falls through space is one example of a projectile. Gravity causes the object to *accelerate* towards the ground. In other words, falling objects *increase in velocity* (speed) as time goes by.

The acceleration of gravity is -9.8 meters per second per second, or -32 feet per second per second. No, that's not a typo: we're talking about meters per second *per second*. Acceleration is a change in speed. Speed might be measured in "meters per second", and so if an object is accelerating its speed might be changing at a rate of 9.8 meters per second, *per second*. We some times abbreviate "meters per second per second" as m/s^2.

The acceleration of gravity is a negative quantity because acceleration has both a magnitude and a direction, and that direction is downwards (toward the center of the Earth). Gravity will slow down an object that is thrown upwards, and pulls dropped objects towards the ground.

This graph shows the height of an object that has been dropped over time. Notice how the curve of the graph gets steeper the longer it is falling. What can you discern from the graph? How high up was the object when it was dropped? When does the object hit the ground?

»> GRAPH

»> The motion of a projectile can be explained using a quadratic equation.

»> Misconceptions about the graph. Looks like the object is moving sideways, but the graph shows height over time, not x- and y- displacement. Footnote about how this comes later, parametric functions.

Example 3.1

Mr. Campbell, science teacher and inventor of the Spam cannon, drops a can of Spam off a bridge that is 80 feet high. When will the Spam hit the ground?

Solution: To do.

Technically, when we solve a quadratic equation like this, we often get both a positive and a negative value. But the negative solution doesn't make sense in the context of the problem: that would be a point in time before the Spam was dropped.

3.4.2 Horizontal Launch

Dropping objects is boring! Let's throw something! But, we're going to add a constraint: we will throw things horizontally, parallel to the ground.

Objects that are thrown or launched horizontally travel in two dimensions: both horizontally and vertically. However, the vertical and horizontal motion are independent. Gravity pulls the object downwards, but has no influence on the projectile's horizontal movement.

To describe the vertical part of the projectile's motion, we use the equation for a falling object. To describe the horizontal motion, use the usual equation $d = r \cdot t$, where r is the object's initial horizontal velocity. Notice how we have separated the two aspects of the projectile's motion: the vertical component is just like dropping, the horizontal component is just like movement along the ground.

Example 3.2

Mr. Campbell fires a can of SPAM horizontally off the edge of an 80-meter high cliff with an initial horizontal velocity of 120 meters per second. When will the SPAM hit the ground? How far from the base of the cliff will it land?

Solution: To do.

3.4.3 Vertical Launch

Objects that are thrown or launched vertically travel in only one dimension. The rising and falling components of their motion are symmetrical. We saw an example of this vertical motion at the start of this chapter.

This equation is just like the falling object equation, but it has the "vt" added to it. This represents the distance traveled because of the upward motion from the launch.

Example 3.3

Mr. Campbell fires a can of Spam vertically from the ground with an initial velocity of 96 ft/s. When will the Spam reach its maximum height? What is the maximum height of the can of Spam?

Example 3.4

Mr. Campbell fires a can of Spam vertically from the top of the 80-meter (240-foot) cliff with initial velocity of 128 ft/s. When will the SPAM reach its maximum height? What is the maximum height of the can of SPAM?

As it falls, the Spam drifts slightly, falls past the launch site, and falls all the way to the base of the cliff. What is its total flight time?

Solution: To do.

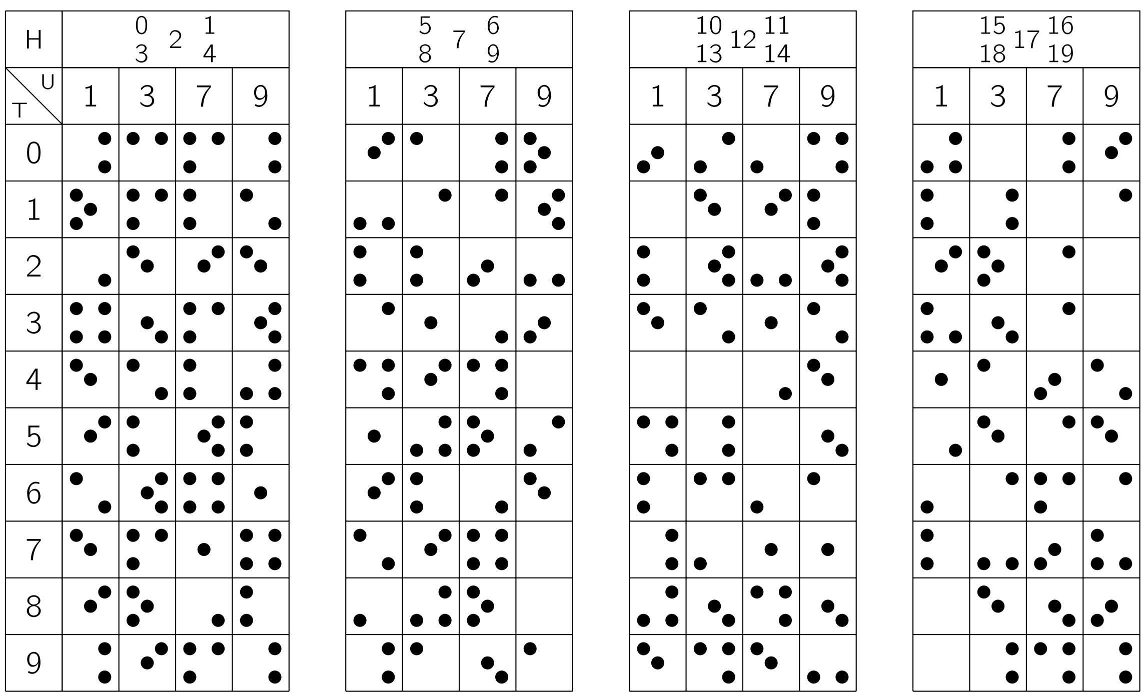
»> Something about how we're not launching at an angle yet.

»> Summary section and other closing remarks in transition to polynomials...

Appendix A

Table of Prime Numbers to 2000

In additon to the numbers 2 and 5, the tables below show all of the prime numbers up to 2000.



How to Use the Table

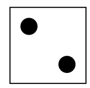
Suppose we want to know whether the number 1439 is prime. Here's how we use the table to determine this:

1. The box on top of each table shows a group of hundreds (that's what the H stands for at the left side). We should think of the number 1439 as having "14" in the hundreds place. The table heading of the third table includes the number 14, so 1439 will appear in this section of the table.

2. The rows of the table indicate the digit in the tens place (that's what the T stands for at the left side). Since 1439 has a 3 in the tens place, we'll use that row of the table.

3. The columns in each table indicate the units place (that's what the U stands for at the left side). We need the column with heading 9, since that's the number in the units place of 1439.

4. We have narrowed our selection down to a single cell in the table. That cell looks like this:

   The dots indicate which of the numbers associated with this cell are prime, using the same layout as in the topmost box. This pattern indicates that 1439 is prime. The box also tells us that 1039 is prime, and that 1139, 1239, and 1339 are all composite.

Notice and Wonder

Notice that the columns include only the headings 1, 3, 7, and 9. Why do you suppose that is?

Notice that none of the boxes have all five dots. Why do you suppose that is? If we continue the table, do you suppose we will ever find a box with all five dots? Why or why not?

Notice that if a box contains four dots, it's always the four corners. Why do you suppose that is? If we continue the table, do you suppose we will ever find a box with four dots in a different arrangement? Why or why not?

Back Matter

Glossary, bibliography, and index removed during editing process.