RAGNAMUS

PHYSICS II - CLASSICAL MECHANICS

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Introduction

This is based off notes from various classical mechanics books.

Variational Calculus

Define the integral

$$I = \int_a^b F(y, y', x)dx \tag{1}$$

as a functional that acts on a curve $y(x)$. Its use is as a cost function, be it time or distance or some other measurable quantity. The curve is chosen to make the integral stationary (minimum or maximum). Hence consider

$$y(x) \to y(x) + \alpha \nu(x) \tag{2}$$

where the parameter α is small and $\nu(x)$ is an arbitrary function. We can redefine the stationary requirement as

$$\left.\frac{dI}{d\alpha}\right|_{\alpha=0} = 0 \tag{3}$$

for any function. Substitute equation 2 into 1,

$$I(y, \alpha) = \int_a^b F(y + \alpha\nu, y' + \alpha\nu', x)dx \tag{4}$$

Now expand as a Taylor series in α, only writing out up to first order terms

$$I(y, \alpha) = \int_a^b \left(\frac{\partial F}{\partial y}\alpha\nu + \frac{\partial F}{\partial y'}\alpha\nu'\right) dx + O(\alpha^2) \tag{5}$$

Set the first-order variations to zero

$$\delta I = \int_a^b \left(\frac{\partial F}{\partial y}\alpha\nu + \frac{\partial F}{\partial y'}\alpha\nu'\right) dx = 0 \tag{6}$$

The second term can be dealt with by integration by parts

$$\left[\nu\frac{\partial F}{\partial y'}\right]_a^b + \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] \nu(x)dx = 0 \tag{7}$$

We can now apply the restriction that the end points are fixed and the function must pass through then so that $\nu(a) = \nu(b) = 0$. We also recall that it works for arbitrary functions. Therefore the Euler-Lagrange equation comes straight out

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \tag{8}$$

There are certain special cases, if F is constant in one or more of the variables. If F does not contain y explicitly, the Euler-Lagrange equation trivially reduces to

$$\frac{\partial F}{\partial y'} = C \tag{9}$$

If F does not contain x explicitly, multiply equation 8 by y'

$$y'\frac{\partial F}{\partial y} = y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \tag{10}$$

Now write

$$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + y''\frac{\partial f}{\partial y'} \tag{11}$$

$$y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'} = \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) \tag{12}$$

The left hand side of the equation is a total derivative of F with respect to x, therefore we can integrate both sides to obtain

$$F = y'\frac{\partial F}{\partial y'} = C \tag{13}$$

If we have several dependent variables $F = F(y_1, y_1', y_2, y_2', ..., y_n, y_n')$ where $y_i = y_i(x)$, the analysis changes to n separate but simultaneous equations for each $y_i(x)$.

$$\frac{\partial F}{\partial y_i} = \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) \tag{14}$$

If we have independent variables $y = y(x_1, x_2, ..., x_n)$, the function becomes

$$\frac{\partial F}{\partial y} = \sum_{i=1}^{n}\left(\frac{\partial F}{\partial y_{x_i}}\right) \tag{15}$$

where $y_{x_i} = \frac{\partial y}{\partial x_i}$. If we have higher order derivatives, $F = F(y, y', ..., y^{(n)}, x)$ then the Euler-Lagrange equation can be written

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - ... + (-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y(n)}\right) = 0 \tag{16}$$

One can use the method of Lagrange undetermined multipliers to solve for constrained variation. If the constraint takes the form

$$J = \int_a^b G(y, y', x)dx \tag{17}$$

Problems

1. Shortest curve joining two points

2. The brachistochrome

3. Fermat's principle

4. Total derivatives

Bibliography