**color**

- we have been using r, g, b..
- why
- what is a color?
- can we get all colors this way?
- how does wavelength fit in here, what part is physics, what part is physiology
- can i use r, g, b for simulation of reflection
- how should colors be stored

**different meanings of color**

- neural response of cone cells: retinal color
- this is processed giving us: *perceived color* that we actually experience and base judgments upon.
  - note we do not have direct experiential access to retinal color
  - see constancy slides
  - ted's image and video
- The perceived color is often associated with the object we are observing, which we might call the *object color*.
- we organize colors and name them as well

**what can we say about color**

- deep: what is the experience of color
- simple: are two experiences identical

**our plan**

- mostly retinal color
- well understood, and is the starting point
- we will describe bio-physically
- we will re-describe using perceptual experiments and formal definitions

**light beams**

- Visible light is electromagnetic radiation that falls roughly in the *wavelengths* $380 < \lambda < 770$, measured in nanometers.
- A *pure beam* $l_\lambda$ has one "unit" of light of a specific wavelength $\lambda$.
- A *mixed beam* $l(\lambda)$ has different amounts of various wavelengths.
  - These amounts are determined by the function $l(\cdot) : R \to R_+$,
  - The value is always non-negative since there is no "negative light".

**cones**

- retinal has 3 kinds of light sensitive cone cells
  - color blind people have only 2 kinds
- called long, medium and short (after the wavelengths of light they are most sensitive to).
- three sensitivity functions $k_l(\lambda)$, $k_m(\lambda)$ and $k_s(\lambda)$.
  - describes how strongly one type of cone "responds" to pure beams of light of different wavelengths.

  - see figure

**visualize**

- each pure beam of light results in three cone response values on the retina,
- visualize this response as a single point in a 3D space.
- define a 3D linear space, with coordinates labeled $[L, M, S]^t$
- for a fixed $\lambda$, we can draw the retinal response as a single vector with coordinates $[k_l(\lambda), k_m(\lambda), k_s(\lambda)]^t$.

**lasso curve**

- As we let $\lambda$ vary, such vectors will trace out a *lasso* curve in space (see demo)
- The lasso curve lies completely in the positive octant since all responses are positive.
- The curve both starts and ends at the origin since these extreme wavelengths are at the boundaries of the visible region, beyond which the responses are zero.
- The curve spends a short time on the $S$ axis (shown with blue tinted points)
- finally comes close to the $L$ axis (shown in red).
- curve never comes close to the $M$ axis, as there is no light that stimulates these cones alone.

**color space**

- $[L, M, S]^t$ coordinates of the pure light beam describe the (retinal) color sensation
- We use the symbol $\vec{c}$ to represent this sensation
- in this lasso visualization we can think of each 3D vector as *potentially* representing some color.
  - we are not sure which ones are really achievable (negatives for starters, are not).
- Vectors on the lasso curve are the *actual* colors of pure beams.

**mixed beams**

- for summed light $\sum_i l(i) l_{\lambda_i}$, the three responses $[L, M, S]^t$ are

$$
\begin{aligned}
L &= \sum_i l(i)\ k_l(\lambda_i) \\
M &= \sum_i l(i)\ k_m(\lambda_i) \\
S &= \sum_i l(i)\ k_s(\lambda_i)
\end{aligned}
$$

- for a mixed beam of light $l(\lambda)$, the three responses $[L, M, S]^t$ are

$$
\begin{aligned}
L &= \int_\Omega d\lambda\ \ l(\lambda)\ k_l(\lambda) \\
M &= \int_\Omega d\lambda\ \ l(\lambda)\ k_m(\lambda) \\
S &= \int_\Omega d\lambda\ \ l(\lambda)\ k_s(\lambda)
\end{aligned}
$$

  where $\Omega = [380..770]$.

**mixed in vis**

- As we look at all possible mixed beams $l(\lambda)$, the resulting $[L, M, S]^t$ coordinates sweep out some set of vectors in 3D space.

- Since $l(\lambda)$ can be any positive function, the swept set is comprised of all positive linear combinations of vectors on the lasso curve.
- Thus, the swept set is the *convex cone* over the lasso curve, which we call the *color cone*.
- Vectors inside the cone represent actual achievable color sensations.
- Vectors outside the cone, such as the vertical axis do not arise as the sensation from any actual light beam, whether pure or composite.

**Map of Color Space**

- Scales of vectors in the cone correspond to brightness changes in our perceived color sensation, so lets normalize by scale
  - see figure
- we only draw colors in the gamut of the RGB monitor
- Colors along the boundary of the cone are vivid and are perceived as "saturated".
- As as we circle around the boundary, we move through the different "hues" of color.
- Starting from the $L$ axis, we move along the rainbow colors from red to green to violet.
  - achievable by pure beams
- color cone's boundary has a planar wedge (a line segment in the 2D figure).
  - The colors on this wedge are the pinks and purples.
  - They do not appear in the rainbow and can only be achieved by appropriately combining beams of red and violet.
- As we move in from the boundary towards the central region of the cone, the colors, while maintaining their hue, de-saturate, becoming pastel and eventually grayish or whitish.

**metamers**

- There are an infinite number of vectors making up the lasso curve,
  - certainly more than three!
- Thus, for vectors strictly inside the color cone, there are many ways to generate some fixed $[L, M, S]^t$ coordinates using positive linear combinations of vectors on the lasso curve.
- Each of these is equivalent to some light beam that produces this fixed response.
- Thus, there must be many physically distinct beams of light, with different amounts of each wavelengths, that generate the same color sensation. We call any two such beams *metamers*.
- webdemo (wiki metamerism)

**Mathematical Model**

- this basic model was established using some perceptual experiments and math.
- no microscopes
- we will re-derive this model, and get a clearer idea about color as a vector space.
- to do experiments we use a basic setup (see fig)
- ask user if two patches match
- simple scene ==> perceived color is the same as retinal.

**transitivity**

- In our very first experiment, we test that the metameric relation is transitive

- In particular we find that, if $l_1(\lambda)$ is indistinguishable to $l'_1(\lambda)$, and $l'_1(\lambda)$ is indistinguishable to $l''_1(\lambda)$, then $l'_1(\lambda)$ will always be indistinguishable to $l''_1(\lambda)$.
- Due to this transitivity, we actually *define* $\vec{c}(l_1(\lambda))$, "the color of the beam $l_1(\lambda)$", as the collection of light beams that are indistinguishable to a human observer from $l_1(\lambda)$.
- in this language $\vec{c}(l_1(\lambda)) = \vec{c}(l'_1(\lambda)) = \vec{c}(l''_1(\lambda))$.
- a (retinal) color is an equivalence class of light beams

**linear structure**

- we want to use linear algebra to work on colors
- this will give us a theory of primary light mixing
  - different from paint mixing.
- but this will take a little bit of abstraction.
- We know from physics that when two light beams, $l_1(\lambda)$ and $l_2(\lambda)$, are added together, they simply form a combined beam with light distribution $l_1(\lambda) + l_2(\lambda)$.
- Thus, we attempt to define the *addition* of two colors, as the color of the addition of two beams.

$$\vec{c}(l_1(\lambda)) + \vec{c}(l_2(\lambda)) := \vec{c}(l_1(\lambda) + l_2(\lambda))$$

- For this to be well defined, we must experimentally verify that it does not make a difference which beam we choose as representative for each color.
  - if $\vec{c}(l_1(\lambda)) = \vec{c}(l'_1(\lambda))$, then we must verify that, for all $l_2(\lambda)$, we have $\vec{c}(l_1(\lambda) + l_2(\lambda)) = \vec{c}(l'_1(\lambda) + l_2(\lambda))$
- experiment confirms!

**nonneg scalar mult**

- since we can multiply a light beam by a nonneg scalar, we try the definition

$$\alpha\vec{c}(l_1(\lambda)) := \vec{c}(\alpha l_1(\lambda))$$

- Again, we need to verify that the behavior of this operation does not depend on our choice of beam.
- experiment confirms

**annoying technicality**

- we do not have a definition for scalar multiply with a negative number.
  - since there is no such thing as negative light
  - later we will see that we will really need negative combinations to create a desired color.
- lets think about subtraction: when we say $\vec{c}_1 - \vec{c}_2 = \vec{c}_3$, we could define this to mean $\vec{c}_1 = \vec{c}_3 + \vec{c}_2$.
  - negatives jump over the equal sign and give us something well defined.
- now we can give meaning to negative colors
- actual and negative colors will be part of a larger space of things called extend colors, which will be a full linear space.

**extended color space**

- let us call any of our original equivalence classes of light beams using the term: *actual color*.
- Let us define an *extended color* as a formal expression of the form

$$\vec{c}_1 - \vec{c}_2$$

  where the $\vec{c}$ are actual colors.

- We define two extended colors $\vec{c}_1 - \vec{c}_2$ and $\vec{c}_3 - \vec{c}_4$, to be equal if $\vec{c}_1 + \vec{c}_4 = \vec{c}_3 + \vec{c}_2$,
- Any extended color that is not an actual color will be called an *imaginary color*.

**ecs ops**

- Multiplication by $-1$ is $-(\vec{c}_1 - \vec{c}_2) := (\vec{c}_2 - \vec{c}_1)$
- addition is $(\vec{c}_1 - \vec{c}_2) + (\vec{c}_3 - \vec{c}_4) := (\vec{c}_1 + \vec{c}_3) - (\vec{c}_2 + \vec{c}_4)$.
- With these operations, we indeed have a linear space of extended colors!
- some extended colors are actual colors.
- we will typically drop the word "extended'
- going back to figure, vectors inside the cone are actual colors, while vectors outside the cone are imaginary colors.

**Color Matching**

- goal1: establish the dimensionality of color space
- goal2: give us a form for mapping light beams to color coordinates
- user watches two screens
- on left they are shown a a pure *test beam* $l_\lambda$
- on the right, they observe a light that is made up of positive combinations of three pure *matching beams*, with wavelengths 435, 545 and 625 nanometers.
- observer must adjust three intensity knobs on the right side, $k_{435}(\lambda)$, $k_{545}(\lambda)$ and $k_{625}(\lambda)$ to get a match
- If the user cannot succeed, then they are allowed to move one or more of the matching beams over to the left side
  - like letting the intensity become negative.
- This process is repeated for all $\lambda$
- webdemo

**results**

- user can indeed succeed in obtaining a match for *all* visible wavelengths.
- so color space is 3D
- we get 3 so-called matching functions $k_{435}(\lambda)$, $k_{545}(\lambda)$ and $k_{625}(\lambda)$, (see Figure)
- Notice that, at each of the wavelengths 435, 545, and 625, one of the matching functions is set to 1, while the other two are set to 0.
- in summary

$$\vec{c}(l_\lambda) = [\vec{c}(l_{435})\ \vec{c}(l_{545})\ \vec{c}(l_{645})] \begin{bmatrix} k_{435}(\lambda) \\ k_{545}(\lambda) \\ k_{625}(\lambda) \end{bmatrix}$$

- and for mixed beams we get

$$\vec{c}(l(\lambda)) = [\vec{c}(l_{435})\ \vec{c}(l_{545})\ \vec{c}(l_{645})] \begin{bmatrix} \int_\Omega d\lambda\ \ l(\lambda)\ k_{435}(\lambda) \\ \int_\Omega d\lambda\ \ l(\lambda)\ k_{545}(\lambda) \\ \int_\Omega d\lambda\ \ l(\lambda)\ k_{625}(\lambda) \end{bmatrix}$$

- and we can compute the mapping from light to color

**visualize**

- we can visualized like LMS space
- lasso passes through axes
- lasso does leave the first octant

**summary**

- light comes in beams $l(\lambda)$.
- actual colors are equivalence classes of metameric beams
- these live inside of the mathematical "color space"
  - color space includes imaginary colors as well.
- this is a 3d linear space
- one basis is $[\vec{c}(l_{435})\ \vec{c}(l_{545})\ \vec{c}(l_{645})]$
- we have an equation to compute color coordinates in this basis given a beam.
- This linear space is consistant with beam addition
- note: this linear space is not consistant with paint mixing!

**Bases**

- we can insert any (non singular) 3-by-3 matrix $M$ and its inverse to obtain

$$
\begin{aligned}
\vec{c}(l(\lambda)) &= \left([\vec{c}(l_{435})\ \vec{c}(l_{545})\ \vec{c}(l_{645})]M^{-1}\right)\left(M\begin{bmatrix}\int_\Omega d\lambda\ \ l(\lambda)\ k_{435}(\lambda)\\ \int_\Omega d\lambda\ \ l(\lambda)\ k_{545}(\lambda)\\ \int_\Omega d\lambda\ \ l(\lambda)\ k_{625}(\lambda)\end{bmatrix}\right)\\
&= [\vec{c}_1\ \vec{c}_2\ \vec{c}_3]\begin{bmatrix}\int_\Omega d\lambda\ \ l(\lambda)\ k_1(\lambda)\\ \int_\Omega d\lambda\ \ l(\lambda)\ k_2(\lambda)\\ \int_\Omega d\lambda\ \ l(\lambda)\ k_3(\lambda)\end{bmatrix}
\end{aligned}
$$

- where the $\vec{c}_i$ describe a new color basis defined as

$$[\vec{c}_1\ \vec{c}_2\ \vec{c}_3] = [\vec{c}(l_{435})\ \vec{c}(l_{545})\ \vec{c}(l_{645})]M^{-1}$$

- the $k(\lambda)$ functions form the new associated matching functions, defined by

$$\begin{bmatrix}k_1(\lambda)\\ k_2(\lambda)\\ k_3(\lambda)\end{bmatrix} = M\begin{bmatrix}k_{435}(\lambda)\\ k_{545}(\lambda)\\ k_{625}(\lambda)\end{bmatrix}$$

**basis specification**

- Starting from any fixed basis for color space, such as $[\vec{c}(l_{435})\ \vec{c}(l_{545})\ \vec{c}(l_{645})]$,
- 1: specify an invertible 3-by-3 matrix $M$.
- 2: specify three actual colors $\vec{c}_i$.
  - Each such $\vec{c}_i$ can be specified by some light beam $l_i(\lambda)$ that generates it.
  - plug each such light beam into above calculation to obtain its 456 color coordinates, determining the matrix.
  - plugging in each new light gives you one column of $M^{-1}$.
- directly specify three new matching functions.
  - To be valid matching functions, they must arise from a basis change like the above Equation, and so each matching function must be some linear combination of $k_{435}(\lambda)$, $k_{545}(\lambda)$ and $k_{625}(\lambda)$
  - each new matching fuction corresponds to a row of $M$.
  - else we will not respect metamerism

- some cameras can mess this up
- see fig

**LMS revisited**

- the LMS matching functions we saw originally describe a basis for color space
- the coordinates of a color are called $[L, M, S]^t$.
- The actual basis is made up of three colors we can call $[\vec{c}_l, \vec{c}_m, \vec{c}_s]$.
- The color $\vec{c}_m$ is a very imaginary color
  - there is no real light beam with LMS color coordinates $[0, 1, 0]^t$.

**Gamut**

- observe: we cannot find three vectors that both hit the lasso curve and contain the entire curve in their positive span.
- so if we want a basis where all actual colors have non-negative coordinates, at least one of the basis vectors defining this octant must lie outside of the cone of actual colors.
  - Such a basis vector must be an imaginary color.
- Conversely, if all of our basis vectors are actual colors, and thus within the color cone, then there must be some actual colors that cannot be written with non-negative coordinates
- in this basis. We say that such colors lie outside the *gamut* of this color space.

**XYZ space**

- central standardized space
- specified by the three matching functions called $k_x(\lambda)$, $k_y(\lambda)$ and $k_z(\lambda)$, (see figure).
- The coordinates for some color with respect to this basis is given by a coordinate vector that we call $[X, Y, Z]^t$.
- These particular matching functions were chosen such that they are always positive, and so that the Y-coordinate of a color represents its overall perceived "luminance". Thus, $Y$ is often used as a black and white representation of the color.
- The associated basis $[\vec{c}_x, \vec{c}_y, \vec{c}_z]$ is made up of three imaginary colors; the axes in are outside of the color cone.

**RGB**

- there are a variety of RGB standards
- current one is called *Rec. 709 RGB space.*
- basis $[\vec{c}_r, \vec{c}_g, \vec{c}_b]$ is made up of three actual colors intended to match the colors of the three phosphors of an ideal monitor/tv display.
- Colors with non-negative RGB coordinates can be produced on a monitor and are said to lie inside the *gamut* of the color space. These colors are in the first octant of the Figure.
- some actual colors lie outside the gamut
- Additionally, on a monitor, each phosphor maxes out at "1", which also limits the achievable outputs.
- images with colors outside the gamut need some kind of mapping/clipping to keep in the gamut. (advanced topic)
- later we see another type of animal called sRGB