CITM

Midterm Exam, April 22nd 2016

MATVJII

SOLUTIONS

Exercise 1

Given the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 7 & 4 \\ 3 & 1 & -1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -1 & -1 & -1 \\ -4 & -1 & 2 \\ 8 & 4 & 5 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -1 & 1 \\ -3 & 2 & 1 \end{pmatrix}$$

1. Expand the next equation and find the value of the matrix $\mathbf{X}$

$$\mathbf{A} + (\mathbf{CB})^\intercal = \mathbf{B}^\intercal \mathbf{X}$$

Solution:

Expanding the equation one can obtain

$$\mathbf{A} + \mathbf{B}^\intercal\mathbf{C}^\intercal = \mathbf{B}^\intercal\mathbf{X} \to \mathbf{X} = \left(\mathbf{B}^\intercal\right)^{-1}\left(\mathbf{A} + \mathbf{B}^\intercal\mathbf{C}^\intercal\right) \to \mathbf{X} = \left(\mathbf{B}^\intercal\right)^{-1}\mathbf{A} + \mathbf{C}^T$$

It is easy to solve for $\mathbf{X}$ if we have the value of $\left(\mathbf{B}^\intercal\right)^{-1}$.

$$\mathbf{B}^\intercal = \begin{pmatrix} -1 & -4 & 8 \\ -1 & -1 & 4 \\ -1 & 2 & 5 \end{pmatrix}$$

Now we can calculate its inverse by making appear the identity matrix using the Gauss-Jordan procedure on the left hand size of the next joint matrix

$$\begin{pmatrix} -1 & -4 & 8 & | & 1 & 0 & 0 \\ -1 & -1 & 4 & | & 0 & 1 & 0 \\ -1 & 2 & 5 & | & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} {\color{red}-1} & -4 & 8 & | & 1 & 0 & 0 \\ 0 & -3 & 4 & | & 1 & -1 & 0 \\ -1 & 2 & 5 & | & 0 & 0 & 1 \end{pmatrix} \simeq$$
$$\simeq \begin{pmatrix} {\color{red}-1} & -4 & 8 & | & 1 & 0 & 0 \\ 0 & -3 & 4 & | & 1 & -1 & 0 \\ 0 & -6 & 3 & | & 1 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} -1 & -4 & 8 & | & 1 & 0 & 0 \\ 0 & {\color{red}-3} & 4 & | & 1 & -1 & 0 \\ 0 & 0 & 15 & | & 3 & -6 & 3 \end{pmatrix} \simeq$$
$$\simeq \begin{pmatrix} 3 & 0 & -8 & | & 1 & -4 & 0 \\ 0 & -3 & 4 & | & 1 & -1 & 0 \\ 0 & 0 & 15 & | & 3 & -6 & 3 \end{pmatrix}$$

We can divide by 3 the last row and continue pivoting

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$$\left(\begin{array}{ccc|ccc} 3 & 0 & -8 & 1 & -4 & 0 \\ 0 & -3 & 4 & 1 & -1 & 0 \\ 0 & 0 & 5 & 1 & -2 & 1 \end{array}\right) \simeq \left(\begin{array}{ccc|ccc} 3 & 0 & -8 & 1 & -4 & 0 \\ 0 & -15 & 0 & 1 & 3 & -4 \\ 0 & 0 & 5 & 1 & -2 & 1 \end{array}\right) \simeq$$

$$\simeq \left(\begin{array}{ccc|ccc} 15 & 0 & 0 & 13 & -36 & 8 \\ 0 & -15 & 0 & 1 & 3 & -4 \\ 0 & 0 & 5 & 1 & -2 & 1 \end{array}\right)$$

Which implies that

$$(\mathbf{B}^{\intercal})^{-}1 = \begin{pmatrix} 13/15 & -12/5 & 8/15 \\ -1/15 & -1/5 & 4/15 \\ 1/5 & -2/5 & 1/5 \end{pmatrix}$$

And the solution is

$$\mathbf{X} = \begin{pmatrix} -1/3 & -14 & -184/15 \\ 4/3 & -2 & 13/15 \\ 1 & -2 & -3/5 \end{pmatrix}$$

Exercise 2

Describe the solution of the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ in terms of the parameter λ, when

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 & 2 \\ 1 & 1 & 6-\lambda & 4 \\ 0 & \frac{\lambda}{2} & -2 & -1 \\ -1 & 3 & -\lambda-2 & 0 \end{bmatrix}$$

1. Discuss the dimension of the sub/space spanned for the columns vectors of $\mathbf{A}$
2. If there exist any value of λ for which the dimension of the column space of $\mathbf{A}$ is less than 4, it is still possible to solve the system? Give at least 1 example for every case of λ.

Solution:

The Dimension of the column space spanned by the vector columns is $\mathbf{A}$ is equal to the number of pivots available if we perform a gauss triangulation on $\mathbf{A}|0$.

Let's then to perform the triangulation:

$$\begin{bmatrix} 1 & -1 & 4 & 2 & | & 0 \\ 1 & 1 & 6-\lambda & 4 & | & 0 \\ 0 & \frac{\lambda}{2} & -2 & -1 & | & 0 \\ -1 & 3 & -\lambda-2 & 0 & | & 0 \end{bmatrix} \simeq \begin{bmatrix} 1 & -1 & 4 & 2 & | & 0 \\ 0 & 2 & 2-\lambda & 2 & | & 0 \\ 0 & \frac{\lambda}{2} & -2 & -1 & | & 0 \\ 0 & 2 & 2-\lambda & 2 & | & 0 \end{bmatrix} \simeq$$

$$\simeq \begin{bmatrix} 1 & -1 & 4 & 2 & | & 0 \\ 0 & 2 & 2-\lambda & 2 & | & 0 \\ 0 & 0 & \lambda^2-2\lambda-8 & -2\lambda-4 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

After pivoting we can observe that the system's dimension is 3 at best. However depending of the value of λ the system dimension can be even lower.

We can see that by solving:

$$\lambda^2 - 2\lambda - 8 = 0 \rightarrow \lambda_{1,2} = 4, -2 \tag{1}$$

If we substitute both solutions in the last non-null row we can see that for $\lambda = 4$, the system is still dimension 3, however for $\lambda = -2$, the row vanishes and then the system's dimension is 2.

In this cases we can still find a solution of the system, however $\boldsymbol{b}$ has to accomplish some constraints. Particularly, if $\boldsymbol{b}$ is a lineal combination of the vectors that span the subspace we always can find a solution for the system of equations, but obviously infinite solutions exists.

As example, for $\lambda = -2$ the vector $\boldsymbol{b} = (0, 2, 1, 2)^\intercal$ is a possible solution and in fact any vector of the form

$$\boldsymbol{x} = \begin{pmatrix} -6Z-3T \\ -2Z-T \\ Z \\ T \end{pmatrix} \tag{2}$$

is a solution.

In the case that $\lambda \neq -2$ the column space of $\mathbf{A}$ has dimension 3, and taking for example $\lambda = 4$ it can be demonstrated that vectors in the form

$$\boldsymbol{x} = \begin{pmatrix} -3Z \\ Z \\ Z \\ 0 \end{pmatrix} \tag{3}$$

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	t	w	r	H	h
exp 1	38	0	20	40	10
exp 2	20	2	30	20	0
exp 3	11	2	20	10	0
exp 4	0	4	10	0	0

are possible solutions and as consequence the vector $\boldsymbol{x} = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ is also a solution. For every other value of λ we always can find a vector depending only of one free parameter.

Exercise 3

We want to model the temperature t of a house in function of the next measurable parameters: h, the heat sources power; H the humidity; r the sun radiation; and w the quantity of re-circulation air coming form outside. The linear function that we are expecting to obtain is

$$t = ah + bH + cr + dw$$

We have measure those quantities on some experiments and the results are provided in the array below

1. Calculate the value of the model parameters
2. Which temperature do you expect for $a = 5$, $b = 5$, $c = 5$ and $d = 1$?

Solution:

The measures taken, give us a system of 4 equations and 4 unknowns. It is

$$\begin{aligned} 10h + 40H + 20r &= 38 \\ 20H + 30r + 2w &= 20 \\ 10H + 20r + 2w &= 11 \\ 10r + 4w &= 0 \end{aligned} \tag{4}$$

We can now apply the Gauss procedure to triangulate the system and find the values of the parameters.

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$$\left(\begin{array}{cccc|c} 10 & 40 & 20 & 0 & 38 \\ 0 & 20 & 30 & 2 & 20 \\ 0 & 10 & 20 & 2 & 11 \\ 0 & 0 & 10 & 4 & 0 \end{array}\right) \simeq \left(\begin{array}{cccc|c} 10 & 40 & 20 & 0 & 38 \\ 0 & 20 & 30 & 2 & 20 \\ 0 & 0 & 100 & 20 & 20 \\ 0 & 0 & 10 & 4 & 0 \end{array}\right) \simeq$$
$$\simeq \left(\begin{array}{cccc|c} 10 & 40 & 20 & 0 & 38 \\ 0 & 20 & 30 & 2 & 20 \\ 0 & 0 & 100 & 20 & 20 \\ 0 & 0 & 0 & 200 & -200 \end{array}\right) \tag{5}$$

By backtraking the triangular system we can fin that the solution is:

$$\begin{aligned} h &= 1 \\ H &= 0.5 \\ r &= 0.4 \\ w &= -1 \end{aligned} \tag{6}$$

With these values we can now estimate the temperature in the room

$$t = 1a + 0.5b + 0.4c - 1d = 8.5 \tag{7}$$

Exercise 4

Two different Basis are

$$\mathfrak{B}_0 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \right\}$$
$$\mathfrak{B}_1 = \left\{ \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\}$$

1. Demonstrate that the given basis are basis of $\mathbf{R}^3$.

Solution:

The vectors that forms both basis are defined in a third common base e.g. the base $\mathfrak{B}_c$. The matrices that defines the transformation that goes from $\mathfrak{B}_0$ and $\mathfrak{B}_1$ to $\mathfrak{B}_c$ are

$$\mathbf{C}_0 = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & -2 \\ 3 & 1 & -1 \end{pmatrix} \quad \mathbf{C}_1 = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & -1 \end{pmatrix} \tag{8}$$

respectively, which columns are the basis of $\mathfrak{B}_0$ and $\mathfrak{B}_1$ seen from $\mathfrak{B}_c$.

Both $\mathfrak{B}_0$ and $\mathfrak{B}_1$ are valid basis if and only if the dimension of the space that they span is equal to 3 or what is equivalent if the three vectors of every base are linearly independent or what is equivalent if we are able to find 3 pivots in the Gauss triangulation process.

By triangulating the matrix $\mathbf{C}_0$ we found that

$$\mathbf{C}_0 = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & -2 \\ 3 & 1 & -1 \end{pmatrix} \simeq \begin{pmatrix} 2 & 2 & 1 \\ 0 & 0 & -5 \\ 0 & -4 & -5 \end{pmatrix} \tag{9}$$

Which, by shifting the second and third row forms a triangular system.

In the same fashion,

$$\begin{aligned} \mathbf{C}_1 = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & -1 \end{pmatrix} \simeq \begin{pmatrix} 2 & -1 & 2 \\ 0 & 7 & -4 \\ 0 & -4 & -5 \end{pmatrix} \simeq \\ \simeq \begin{pmatrix} 2 & -1 & 2 \\ 0 & 7 & -4 \\ 0 & 0 & -8 \end{pmatrix} \end{aligned} \tag{10}$$

Therefore, we can stablish that $\mathfrak{B}_0$ and $\mathfrak{B}_1$ are valid basis of $\mathbf{R}^3$.

2. Find the components of the vector $\boldsymbol{u}$ defined in the basis $\mathfrak{B}_0$ as

$$\boldsymbol{u}_{\mathfrak{B}_0} = (1, -1, 2)^\intercal$$

in the basis $\mathfrak{B}_1$

Solution:

$$\begin{tikzcd} \mathfrak{B}_0 \arrow[dr, "\mathbf{C}_0"] & \\ & \mathfrak{B}_c \\ \mathfrak{B}_1 \arrow[ur, "\mathbf{C}_1"'] & \end{tikzcd}$$

Figure 1: Relation between basis.

By following the directions of Fig. 2, we can relate the same vector expressed in both basis as

$$\boldsymbol{u}_{\mathcal{B}_c} = \mathbf{C}_0 \boldsymbol{u}_{\mathcal{B}_0} = \mathbf{C}_1 \boldsymbol{u}_{\mathcal{B}_1} \rightarrow \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & -2 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & -1 \end{pmatrix} \boldsymbol{u}_{\mathfrak{B}_1}$$

Which leads to the linear system

$$\begin{pmatrix} 2 & -1 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & -1 \end{pmatrix} \boldsymbol{u}_{\mathfrak{B}_1} = \begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix}$$

So we can solve the system by Gauss triangulation as follows

$$\begin{pmatrix} 2 & -1 & 2 & | & 2 \\ 3 & 2 & 1 & | & -4 \\ 1 & 2 & -1 & | & 0 \end{pmatrix} \simeq \begin{pmatrix} 2 & -1 & 2 & | & 2 \\ 0 & 7 & -4 & | & -14 \\ 0 & 5 & -4 & | & -2 \end{pmatrix}$$
$$\simeq \begin{pmatrix} 2 & -1 & 2 & | & 2 \\ 0 & 7 & -4 & | & -14 \\ 0 & 0 & -8 & | & 56 \end{pmatrix}$$

Which implies that

$$\boldsymbol{u}_{\mathfrak{B}_1} = \begin{pmatrix} 5 \\ -6 \\ -7 \end{pmatrix}$$

3. Find the components of the vector $\boldsymbol{u}$ defined in the basis $\mathfrak{B}_1$ as

$$\boldsymbol{u}_{\mathfrak{B}_1} = (2, -2, 1)^{\intercal}$$

in the basis $\mathfrak{B}_0$

Solution:

Applying the same equation that we have presented before but knowing now $\boldsymbol{u}_{\mathfrak{B}_1}$

$$\mathbf{C}_0 \boldsymbol{u}_{\mathcal{B}_0} = \mathbf{C}_1 \boldsymbol{u}_{\mathcal{B}_1} \rightarrow \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & -2 \\ 3 & 1 & -1 \end{pmatrix} \boldsymbol{u}_{\mathfrak{B}_0} = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

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Which leads to the linear system

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & -2 \\ 3 & 1 & -1 \end{pmatrix} \boldsymbol{u}_{\mathfrak{B}_0} = \begin{pmatrix} 8 \\ 3 \\ -3 \end{pmatrix}$$

So we can solve the system by Gauss triangulation as

$$\begin{pmatrix} 2 & 2 & 1 & | & 8 \\ 1 & 1 & -2 & | & 3 \\ 3 & 1 & -1 & | & -3 \end{pmatrix} \simeq \begin{pmatrix} 2 & 2 & 1 & | & 8 \\ 0 & 0 & -5 & | & -2 \\ 0 & -4 & -5 & | & -30 \end{pmatrix}$$

Which implies that

$$\boldsymbol{u}_{\mathfrak{B}_0} = \begin{pmatrix} -3.2 \\ 7 \\ 0.4 \end{pmatrix}$$