Relations between Rotation Matrix and Euler axis-angle parametrization

Julen Cayero

April 28, 2016

Since now we have seen that rotations are linear transformations. And that every linear transformation can be expressed as

$$\boldsymbol{p}' = T(\boldsymbol{p}) = \mathbf{R}\boldsymbol{p}. \tag{1}$$

In addition we have stated the Euler's theorem about rotations:

Theorem 0.1 (Euler's theorem on rotations) *When a sphere is moved around its center it is always possible to find a diameter whose direction in the displaced position is the same as in the initial position.*

or an alternative version:

Theorem 0.2 (Euler's theorem on rotations) *Any motion of a rigid body such that a point, let's say "$\boldsymbol{O}$", on the rigid body remains fixed, is equivalent to a single rotation about some axis that runs through $\boldsymbol{O}$.*

In this document we are going to give the geometric interpretation that allows to formulate rotation matrices from euler axis-angle information.

1 Euler axis-angle → Rotation matrix

It is desired to find the rotation matrix $\mathbf{R}$ that transform a vector $\boldsymbol{p}$ into its image $\boldsymbol{p}' = \mathbf{R}\boldsymbol{p}$. The image is the result of rotating $\boldsymbol{p}$ about an axis in the direction of the unitary vector $\boldsymbol{u}$ an angle ϕ. For this section take ϕ, $\boldsymbol{u}$ and $\boldsymbol{p}$ as known quantities.

Let's examine two simple examples for which we can conceptually interpret the result of the rotation:

1. Let the vector $\boldsymbol{w}$ to be parallel to $\boldsymbol{u}$, i.e. $\boldsymbol{w} = k\boldsymbol{u}$, being k an scalar. Which is the effect of the rotation over $\boldsymbol{w}$?. Since $T(\boldsymbol{w})$ represents a linear transformation, and u rotated by itself does not modified its direction,

$$T(\boldsymbol{w}) = T(k\boldsymbol{u}) = kT(\boldsymbol{u}) = k\boldsymbol{u} = \boldsymbol{w} \tag{2}$$

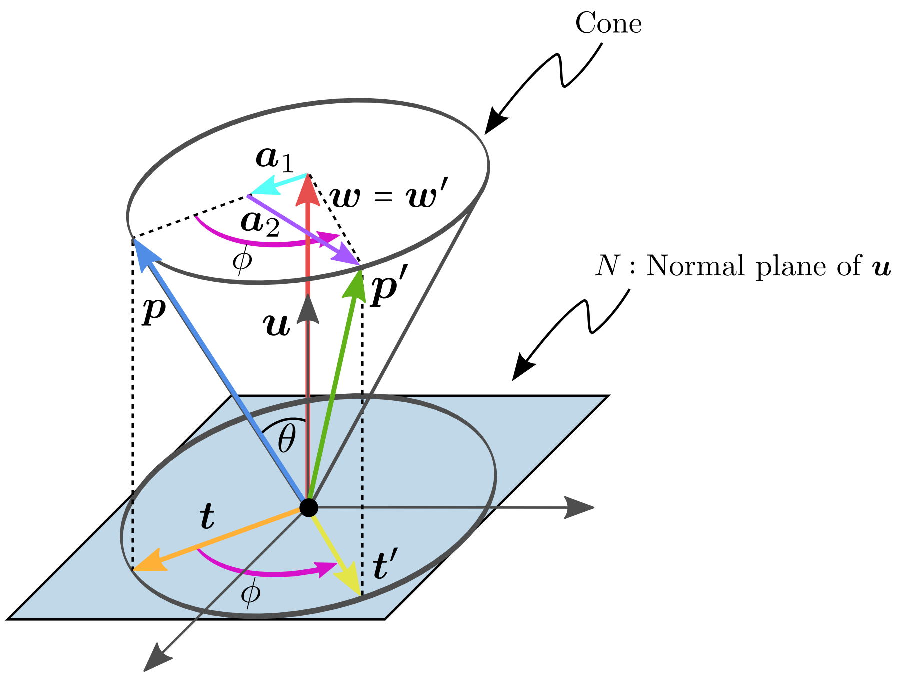

Figure 1: Geometrical interpretation of the rotation.

2. Let the vector $\boldsymbol{t}$ to be orthogonal to $\boldsymbol{u}$, i.e. $\boldsymbol{t}^\intercal \boldsymbol{u} = 0$. Therefore, a rotation of $\boldsymbol{t}$ an angle ϕ about u, produces a new vector that is orthogonal to u (it lies on the normal plane to $\boldsymbol{u}$) and $\boldsymbol{t}^\intercal \boldsymbol{t}' = \|\boldsymbol{t}\|^2 \, cos\phi$.

We can combine the two basic results above to understand what is the effect of applying a rotation over a generic vector that is neither parallel nor orthogonal to $\boldsymbol{u}$.

Let a generic 3-dimensional vector given by $\boldsymbol{p} = \boldsymbol{t} + \boldsymbol{w}$, where $\boldsymbol{w}$ is the projection of $\boldsymbol{p}$ over $\boldsymbol{u}$

$$\boldsymbol{w} = (\boldsymbol{p}^\intercal \boldsymbol{u})\, \boldsymbol{u}, \tag{3}$$

and $\boldsymbol{t}$ is the projection of $\boldsymbol{p}$ over the normal plane to $\boldsymbol{u}$,

$$\boldsymbol{t} = \boldsymbol{p} - \boldsymbol{w} = \boldsymbol{p} - (\boldsymbol{p}^\intercal \boldsymbol{u})\, \boldsymbol{u}. \tag{4}$$

Note in addition that

$$\|\boldsymbol{t}\| = \|\boldsymbol{p}\| \sin(\theta) \tag{5}$$

and

$$\|\boldsymbol{w}\| = \|\boldsymbol{p}\| \cos(\theta)\,. \tag{6}$$

Again by using the properties of linear transformations:

$$\boldsymbol{p}' = T(\boldsymbol{p}) = T(\boldsymbol{t}) + T(\boldsymbol{w}) \tag{7}$$

This result means that given a vector $\boldsymbol{p}$, we can project it over the direction of $\boldsymbol{u}$ and its perpendicular, interpreted here as $\boldsymbol{w}$ and $\boldsymbol{t}$ respectively, and calculate its transformations to reconstruct later $T(\boldsymbol{p})$. Since the result of applying the rotation only modifies $\boldsymbol{t}$, the result of the transformation is given by

$$\boldsymbol{p}' = T(\boldsymbol{p}) = \boldsymbol{w} + T(\boldsymbol{t}), \tag{8}$$

which make the vector $\boldsymbol{p}'$ to describe a cone when varying ϕ as can be seen on Fig. 1.

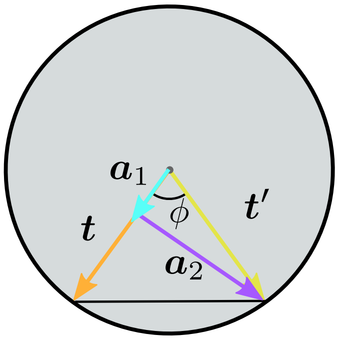

Figure 2: Base of the cone.

In order to formulate the rotation, at this point, we need to find how $T(\boldsymbol{t})$ looks like. To this end, take a look to Fig. 2. On Fig. 2, it can be appreciated that $\boldsymbol{t}$ and $\boldsymbol{t}'$ are part of an isosceles triangle. Of this triangle we know the direction and norm of $\boldsymbol{t}$. Moreover we can fin an orthogonal direction to $\boldsymbol{t}$, given by

$$\boldsymbol{n} = \frac{\boldsymbol{u} \times \boldsymbol{p}}{\|\boldsymbol{u} \times \boldsymbol{p}\|}. \tag{9}$$

With this two identified directions (directions of $\boldsymbol{t}$ and $\boldsymbol{n}$) we can describe the vector $\boldsymbol{t}'$ as

$$T(\boldsymbol{t}) = \boldsymbol{t}' = \underbrace{k_1 \frac{\boldsymbol{t}}{\|\boldsymbol{t}\|}}_{\boldsymbol{a}_1} + \underbrace{k_2 \boldsymbol{n}}_{\boldsymbol{a}_1} \tag{10}$$

By simply analysing the triangle given by $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{t}'$ we can see that

$$\boldsymbol{a}_1 = \|\boldsymbol{t}\| \cos(\phi) \frac{\boldsymbol{t}}{\|\boldsymbol{t}\|} = \cos(\phi)\, \boldsymbol{t} \tag{11}$$

and

$$\boldsymbol{a}_2 = \|\boldsymbol{t}\| \sin(\phi)\, \boldsymbol{n} \tag{12}$$

As result we can state that:

$$\boldsymbol{p}' = T(\boldsymbol{p}) = \boldsymbol{w} + \cos(\phi)\, \boldsymbol{t} + \|\boldsymbol{t}\| \sin(\phi) \frac{\boldsymbol{u} \times \boldsymbol{p}}{\|\boldsymbol{u} \times \boldsymbol{p}\|} \tag{13}$$

Using Eq. (3) and Eq. (4) and noting that $\|\boldsymbol{t}\| = \|\boldsymbol{u} \times \boldsymbol{p}\|$ the transformation can be formulated as

$$\begin{gathered} \boldsymbol{p}' = T(\boldsymbol{p}) = (\boldsymbol{p}^\intercal \boldsymbol{u})\, \boldsymbol{u} + \cos(\phi)\, (\boldsymbol{p} - (\boldsymbol{p}^\intercal \boldsymbol{u})\, \boldsymbol{u}) + \sin(\phi)\, (\boldsymbol{u} \times \boldsymbol{p}) = \\ \boldsymbol{p} \cos(\phi) + (1 - \cos(\phi))\, (\boldsymbol{p}^\intercal \boldsymbol{u})\, \boldsymbol{u} + \sin(\phi)\, (\boldsymbol{u} \times \boldsymbol{p})\,. \end{gathered} \tag{14}$$

But now... where is $\mathbf{R}$?

We can factorize the past expression in function of $\boldsymbol{p}$ to arrive to

$$\boldsymbol{p}' = T(\boldsymbol{p}) = \underbrace{\left(\mathbf{I}\cos(\phi) + (1-\cos(\phi))\left(\boldsymbol{u}\boldsymbol{u}^T\right) + \sin(\phi)\left[\boldsymbol{u}\right]_\times\right)}_{\mathbf{R}}\boldsymbol{p}. \tag{15}$$

To achieve the result we have use the next equality (test it in MatLab !!)

$$\left(\boldsymbol{p}^\intercal\boldsymbol{u}\right)\boldsymbol{u} = \left(\boldsymbol{u}^\intercal\boldsymbol{u}\right)\boldsymbol{p} \tag{16}$$

And the definition of the skew symmetric matrix

$$[\boldsymbol{x}]_\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \tag{17}$$

The formulation of the rotation matrix as function of the euler axis and angle of rotations is known as Rodrigues' formula

$$\mathbf{R}\left(\boldsymbol{u},\phi\right) = \left(\mathbf{I}\cos(\phi) + (1-\cos(\phi))\left(\boldsymbol{u}\boldsymbol{u}^T\right) + \sin(\phi)\left[\boldsymbol{u}\right]_\times\right) \tag{18}$$