UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Centre de la Imatge i la Tecnologia Multimèdia

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Centre de la Imatge i la Tecnologia Multimèdia

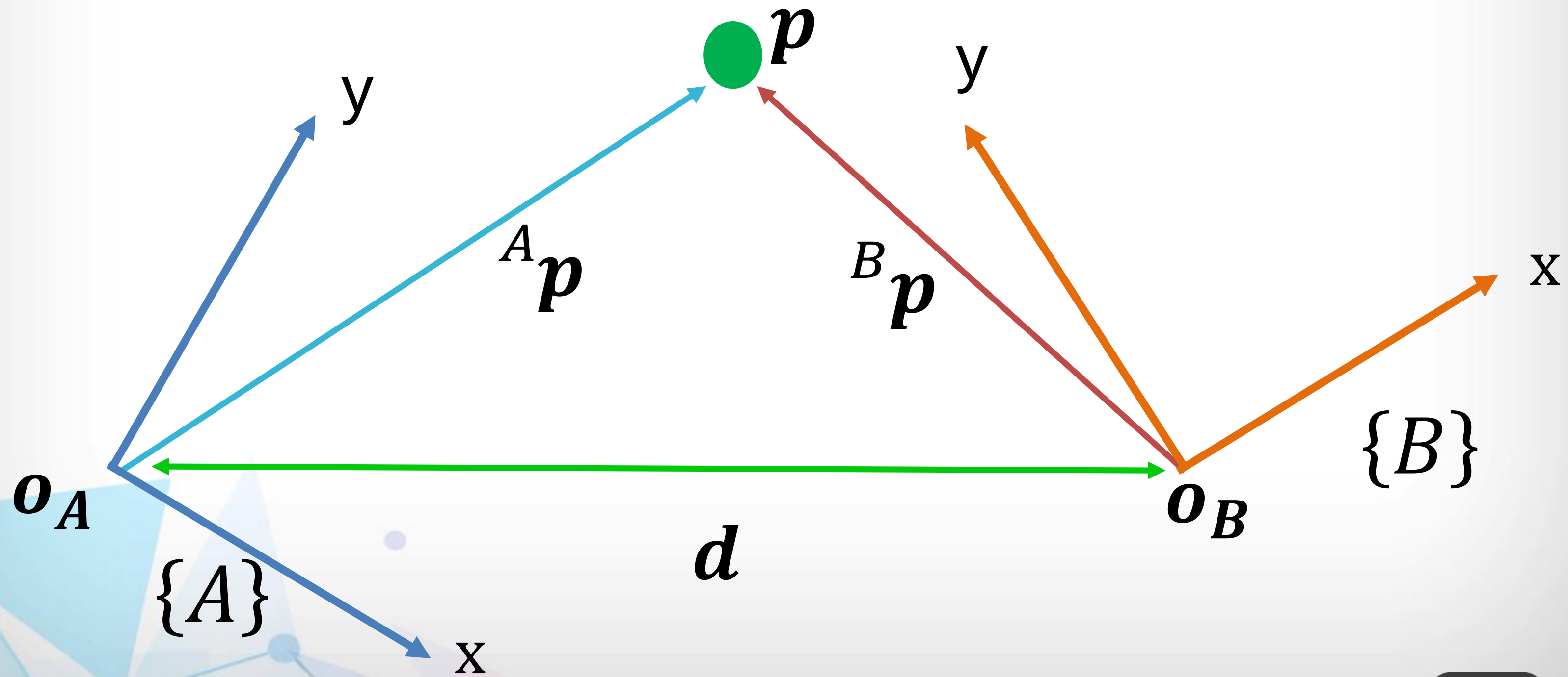
Affine transformations

Julen Cayero

Terrassa, May 18th, 2016

Affine Transformations

Rotation plus displacement (2D view)



Affine Transformations

Rotation plus displacement

$$ {}^{A}\boldsymbol{p} = {}^{A}\mathbf{R}_{\mathrm{B}}\ {}^{B}\boldsymbol{p} + {}^{A}\boldsymbol{d}_{A\to B} $$

Where

- ${}^{A}\boldsymbol{p}$ is the position of $\boldsymbol{p}$ seen from the frame A
- ${}^{B}\boldsymbol{p}$ is the position of $\boldsymbol{p}$ seen from the frame B
- ${}^{A}\mathbf{R}_{\mathrm{B}}$ is the rotation matrix that relates the orientation of both frames
- ${}^{A}\boldsymbol{d}_{A\to B}$ is the displacement between frames seen from A.

Affine Transformations

Equivalently you can use

$$ {}^{B}\boldsymbol{p} = {}^{B}\mathbf{R}_{\mathrm{A}}\ {}^{A}\boldsymbol{p} + {}^{B}\boldsymbol{d}_{B\rightarrow A} $$

Affine Transformations

Let's isolate ${}^{B}\boldsymbol{p}$ from the equation

$$ {}^{A}\boldsymbol{p} = {}^{A}\mathbf{R}_{B}\ {}^{B}\boldsymbol{p} + {}^{A}\boldsymbol{d}_{A\to B} \to $$

$$ {}^{B}\boldsymbol{p} = {}^{A}\mathbf{R}_{B}^{-1}\left({}^{B}\boldsymbol{p} - {}^{A}\boldsymbol{d}_{A\to B}\right) \to $$

$$ {}^{B}\boldsymbol{p} = {}^{A}\mathbf{R}_{B}^{-1}\,{}^{B}\boldsymbol{p} - {}^{A}\mathbf{R}_{\mathrm{B}}^{-1}\,{}^{A}\boldsymbol{d}_{A\to B} $$

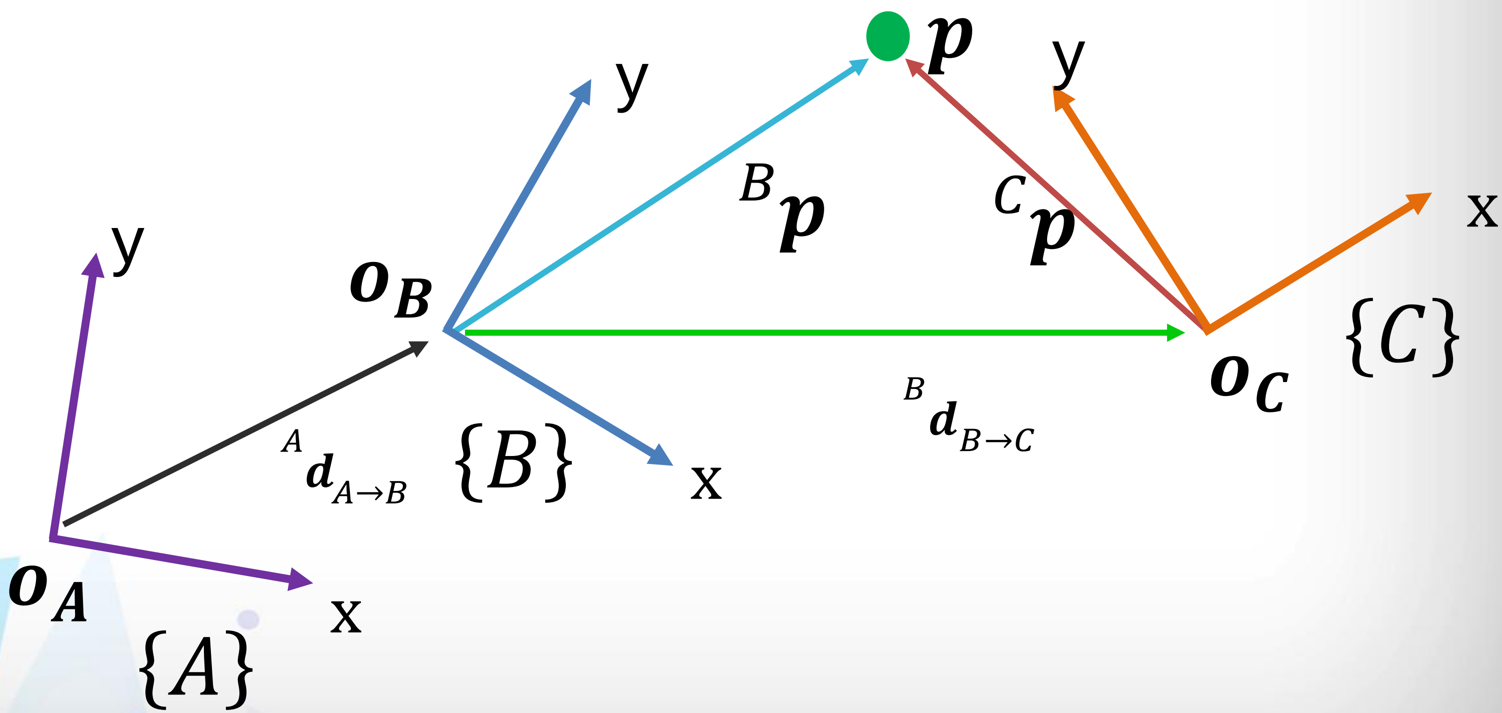
So it must be clear that since

$$ {}^{B}\boldsymbol{p} = {}^{B}\mathbf{R}_{A}\ {}^{A}\boldsymbol{p} + {}^{B}\boldsymbol{d}_{B\to A} $$

- ${}^{A}\mathbf{R}_{\mathrm{B}}^{-1} = {}^{A}\mathbf{R}_{\mathrm{B}}^{\mathrm{T}} = {}^{B}\mathbf{R}_{A}$
- $-\,{}^{B}\mathbf{R}_{A}\ {}^{A}\boldsymbol{d}_{A\to B} = {}^{B}\boldsymbol{d}_{B\to A}$

Composing affine transformations

Example:

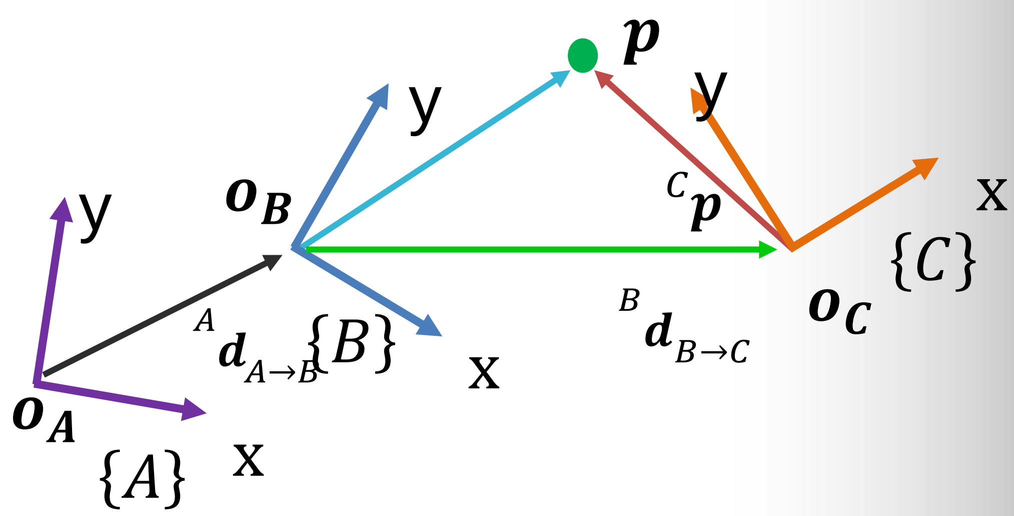


Composing affine transformations

Example:

Known

- ${}^{A}\mathbf{R}_B,\ {}^{B}\mathbf{R}_C$
- ${}^{A}\boldsymbol{d}_{A\to B},\ {}^{B}\boldsymbol{d}_{B\to C}$
- ${}^{C}\boldsymbol{p}$



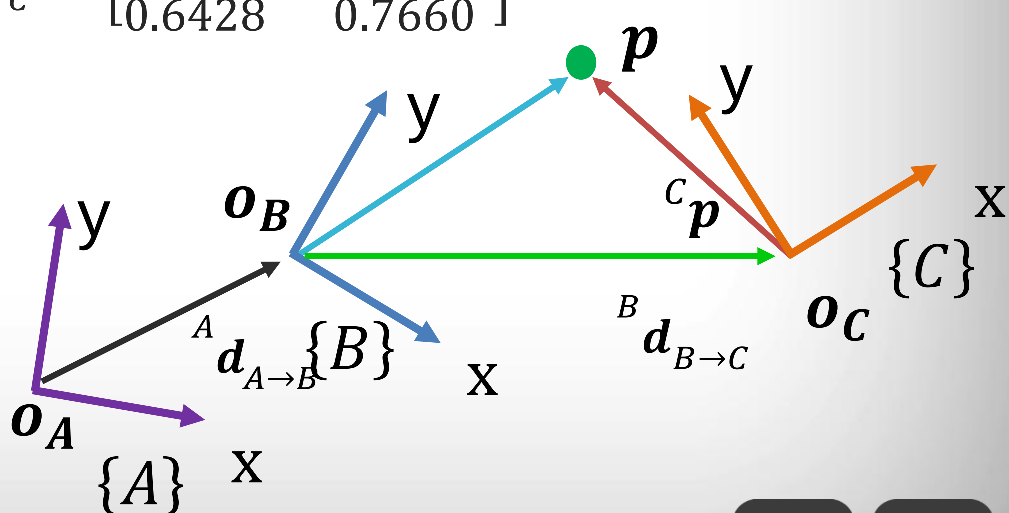
UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Centre de la Imatge i la Tecnologia Multimèdia

Composing affine transformations

Example with numbers:

Known

- $^{A}\mathbf{R}_B = \begin{bmatrix} 0.9397 & 0.3420 \\ -0.3420 & 0.9397 \end{bmatrix}$, $^{B}\mathbf{R}_C = \begin{bmatrix} 0.7660 & -0.6428 \\ 0.6428 & 0.7660 \end{bmatrix}$
- $^{A}\boldsymbol{d}_{A\to B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $^{B}\boldsymbol{d}_{B\to C} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- $^{C}\boldsymbol{p} = \begin{bmatrix} -0,5 \\ 1 \end{bmatrix}$

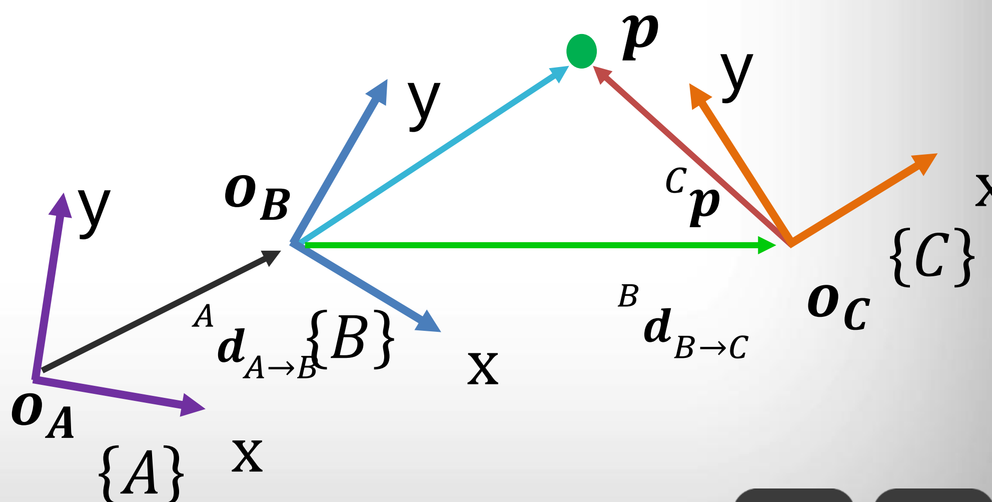


Composing affine transformations

Example with numbers:

Could you find:

- ${}^{A}\mathbf{R}_{C}$ and ${}^{A}\boldsymbol{d}_{A\to C}$?



Affine transformations, homogeneous coordinates

- 2D vector

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- 3D vector

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- 2D homogeneous vector

$$\hat{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}$$

- 3D vector

$$\hat{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix}$$

Affine transformations, homogeneous coordinates

- Translate a vector

$$\hat{\boldsymbol{x}}' = \begin{pmatrix} \mathbf{I}_{3\text{x}3} & \boldsymbol{t} \\ \mathbf{0}_{1\text{x}3} & 1 \end{pmatrix} \hat{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{x} + \boldsymbol{t} \\ 1 \end{pmatrix}$$

- Rotate a vector

$$\hat{\boldsymbol{x}}' = \begin{pmatrix} \mathbf{R} & \mathbf{0}_{3\text{x}1} \\ \mathbf{0}_{1\text{x}3} & 1 \end{pmatrix} \hat{\boldsymbol{x}} = \begin{pmatrix} \mathbf{R}\boldsymbol{x} \\ 1 \end{pmatrix}$$

Affine transformations, Affine matrix

$\mathbf{A}$ has inverse

$$\mathbf{A}^{-1} = \begin{pmatrix} \mathbf{R}^{\mathrm{T}} & -\mathbf{R}^{\mathrm{T}}\, t \\ \mathbf{0}_{1\mathrm{x}3} & 1 \end{pmatrix}$$

Prove that :

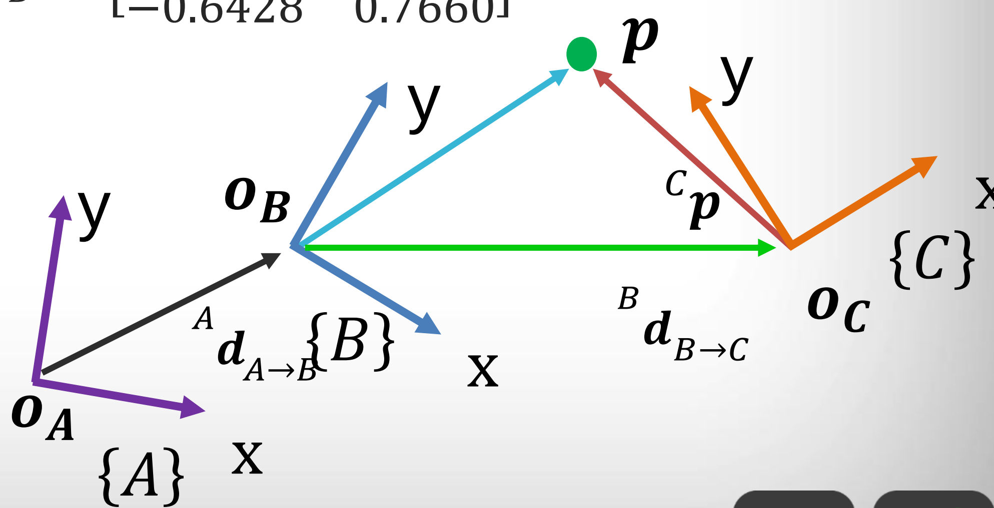
$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Composing affine transformations

Example with numbers:

Known

- $^{A}\mathbf{R}_{B} = \begin{bmatrix} 0.9397 & 0.3420 \\ -0.3420 & 0.9397 \end{bmatrix}$, $^{C}\mathbf{R}_{B} = \begin{bmatrix} 0.7660 & 0.6428 \\ -0.6428 & 0.7660 \end{bmatrix}$
- $^{A}\boldsymbol{d}_{A\to B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $^{B}\boldsymbol{d}_{B\to C} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- $^{A}\boldsymbol{p} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

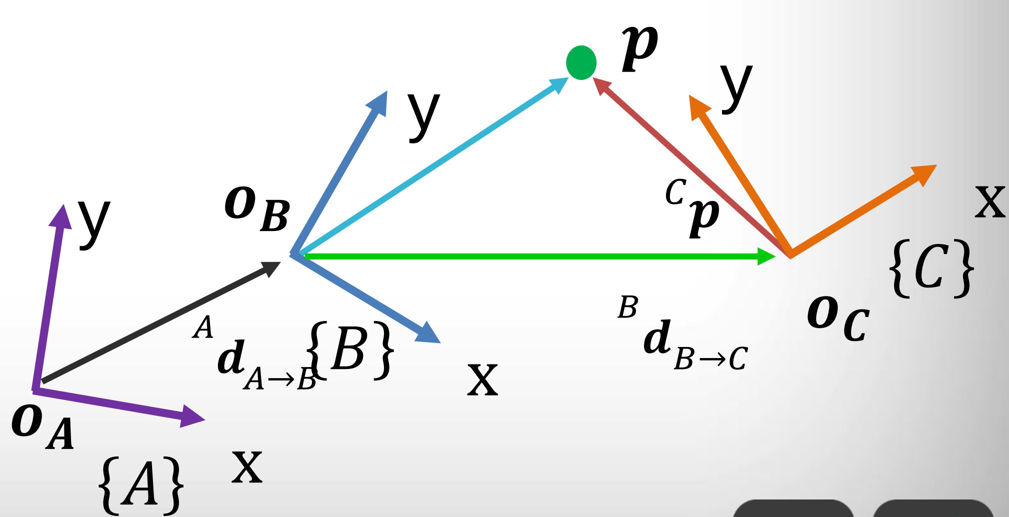


Composing affine transformations

Example with numbers:

Could you find:

- ${}^{A}\mathbf{R}_{C}$ and ${}^{C}\boldsymbol{d}_{C \to A}$?



Exercises

Midterm Exam

- Exercise 5

Final Exam

- Exercise 2

Additional one:

El origen del marco de referencia $\{B\}$ es conocido desde el marco de referencia $\{A\}$ a través del vector $\boldsymbol{t}$. La orientación del frame $\{A\}$ con respecto a $\{B\}$ se encuentra codificada en la matriz de rotación $\mathbf{R}$, tal que ${}^{B}\boldsymbol{p} = \mathbf{R}\,{}^{A}\boldsymbol{p}$. Representando ${}^{A}\boldsymbol{p}$ las componentes del vector $\boldsymbol{p}$ expresadas en el marco de referencia $\{A\}$ y ${}^{B}\boldsymbol{p}$ las componentes del vector $\boldsymbol{p}$ expresadas en el marco de referencia $\{B\}$ si $\{A\}$ y $\{B\}$ tuvieran un origen común. ¿Cu ál es la matriz de transformación afín que permite expresar un vector conocido en $\{B\}$, en el marco de referencia $\{A\}$?