UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

Centre de la Imatge i la Tecnologia Multimèdia

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Linear Transformations and Rotation matrices

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Linear Transformation

¿WHAT IS?

Multivariate function that transforms the space maintaining the origin

$$\mathrm{T}: \mathbb{R}^{\boldsymbol{n}} \rightarrow \mathbb{R}^{\boldsymbol{m}}$$

If $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^{\boldsymbol{n}}$ T **is a linear transformation iff**

$$\mathrm{T}(\boldsymbol{a}+\boldsymbol{b}) = \mathrm{T}(\boldsymbol{a}) + \mathrm{T}(\boldsymbol{b})$$
$$\mathrm{T}(\lambda\boldsymbol{a}) = \lambda\mathrm{T}(\boldsymbol{a})$$

Note:

- T maintain the origin fix.
- Points on a line remain alineated after the transformation

Linear Transformation

Some exercises:

Find A, s.t. $\mathrm{T}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$

1) $\mathrm{T}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathrm{T}(\boldsymbol{x}) = \mathrm{T}\begin{pmatrix} x_1 + 3x_3 \\ 3x_1 + 5x_2 \\ -2\,x_3 \end{pmatrix}$

2) $\mathrm{T}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathrm{T}(\boldsymbol{x}) = \mathrm{T}\begin{pmatrix} \frac{3x_1}{x_2} + x_3 \\ 2x_1x_3 \end{pmatrix}$

3) $\mathrm{T}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathrm{T}(\boldsymbol{x}) = \mathrm{T}\begin{pmatrix} x_2 \\ x_2 + 1 \\ -x_1 \end{pmatrix}$

If it is known that

$$\mathrm{T}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \quad \mathrm{T}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$$

$$\mathrm{T}\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix}$$

find $\boldsymbol{x}$ s.t. $\mathrm{T}(\boldsymbol{x}) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Rotation

¿WHAT IS?

Especial linear transformation that preserves:

- **Distances (norm of vector)**
- **Angles (dot product)**
- **Volumes (triple product)**

$$\mathrm{R}: \mathbb{R}^{\boldsymbol{n}} \rightarrow \mathbb{R}^{\boldsymbol{n}}$$

An aside

- **Vector norm:** length of a vector

$$\|\boldsymbol{x}\| = \sqrt{x_1 + x_2 + \cdots + x_n} = \sqrt{\boldsymbol{x}^T\boldsymbol{x}}$$

- **Dot product:** angle between vectors

$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}^T\boldsymbol{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \|\boldsymbol{x}\|\|\boldsymbol{y}\| \cos(\alpha)$$

- **Triple product:** volume

$$\boldsymbol{x}^T(\boldsymbol{y} \times \boldsymbol{z})$$

Rotation

Fixing ℝ^3

Which are the parameters that defines a rotation?

$$R = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots \\ \mathrm{T}(\boldsymbol{e}_1) & \mathrm{T}(\boldsymbol{e}_2) & \mathrm{T}(\boldsymbol{e}_3) \\ \vdots & \vdots & \vdots \end{bmatrix}$$

9 elements. However we have **6** restrictions (orthonormal basis)… Since $\boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{e}_3$ form a orthonormal basis, $\mathrm{T}(\boldsymbol{e}_1)$, $\mathrm{T}(\boldsymbol{e}_2)$ and $\mathrm{T}(\boldsymbol{e}_3)$ have to form a orthonormal basis

$$\mathrm{T}(\boldsymbol{e}_j)^T \mathrm{T}(\boldsymbol{e}_i) = 1 \; if \; j = i$$
$$\mathrm{T}(\boldsymbol{e}_j)^T \mathrm{T}(\boldsymbol{e}_i) = 0 \; if \; j \neq i$$

This leads to **3** DoF. Rotations can be defined by using only **3 parameters**

An aside II

- **Determinant of a 3x3 matrix:** **Grassman rule**

$$\det(\mathbf{M}) = |\mathbf{M}| =$$

$$\begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} = (m_{11}m_{22}m_{33} + m_{21}m_{32}m_{13} + m_{12}m_{23}m_{31}) -$$

$$- (m_{31}m_{22}m_{13} + m_{23}m_{32}m_{11} + m_{21}m_{12}m_{33})$$

Rotation

Fixing $\mathbb{R}$^3

The **6 restrictions over $\mathbf{R}$ are equivalent to state that**

$$\det(\mathbf{R}) = 1$$
$$\mathbf{R}^{\mathbf{T}} = \mathbf{R}^{-\mathbf{1}}$$

Rotation

Geometric meaning

Let $\mathrm{T}(\boldsymbol{v})$ represent a rotaion of 90 degs about the z axis

What is the value of the matrix $\mathbf{R}$ such that $\mathbf{R}\,\boldsymbol{v}$ rotates the vector $\boldsymbol{v}$ 90 degs about the z axis?

$$\mathbf{R} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Which are the images of the next vectors?

$$\boldsymbol{v}_1 = (\mathbf{1} \quad \mathbf{0} \quad \mathbf{0})^T$$
$$\boldsymbol{v}_2 = (\mathbf{0} \quad \mathbf{1} \quad \mathbf{0})^T$$
$$\boldsymbol{v}_3 = (\mathbf{0} \quad \mathbf{0} \quad \mathbf{1})^T$$
$$\boldsymbol{v}_4 = (-\mathbf{1} \quad \mathbf{2} \quad \mathbf{2})^T$$

Which $\mathbf{R}$ performs the inverse rotation?

Rotations & change of basis

Change of Basis

$B_2 \xrightarrow{\mathbf{C}} B_1$ **:** Where $\mathbf{C}$ is a matrix whose columns where the vectors of basis B_2 seen form B_1

$\mathbf{C}\boldsymbol{x}$ had the meaning: I take a vector of B_2 and it goes to B_1

Now with the same result $\mathbf{R}\boldsymbol{x}$ rotates the vector $\boldsymbol{x}$.

Concatenation of LinearTrans

Can you calculate

$$\mathrm{T}_2\left(\mathrm{T}_1(\boldsymbol{x})\right) = ?$$

If we know that $\mathrm{T}_1(\boldsymbol{x}) = \mathbf{A}\,\boldsymbol{x}$, and that $\mathrm{T}_2(\boldsymbol{x}) = \mathbf{B}\,\boldsymbol{x}$

$$\mathrm{T}_2\left(\mathrm{T}_1(\boldsymbol{x})\right) = \mathbf{BA}\boldsymbol{x}$$

Note the order!!! Is not the same $\mathbf{BA}$ than $\mathbf{AB}$

$$\mathbf{BA} \neq \mathbf{AB}$$

The same applies to rotations. Take $\mathbf{R_1}$, which rotates a given vector. After the first rotation, the resultant vector (The image vector) is again rotated by a rotation matrix $\mathbf{R_2}$. The final vector is represented by:

$$\boldsymbol{v}' = \mathbf{R_2 R_1}\boldsymbol{v}$$