UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

Centre de la Imatge i la Tecnologia Multimèdia

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Rotation matrices
Euler axis/angle representation

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Euler theorem

Theorem A (Euler's theorem on rotations) When a sphere is moved around its center it is always possible to find a diameter whose direction in the displaced position is the same as in the initial position.

or an alternative version:

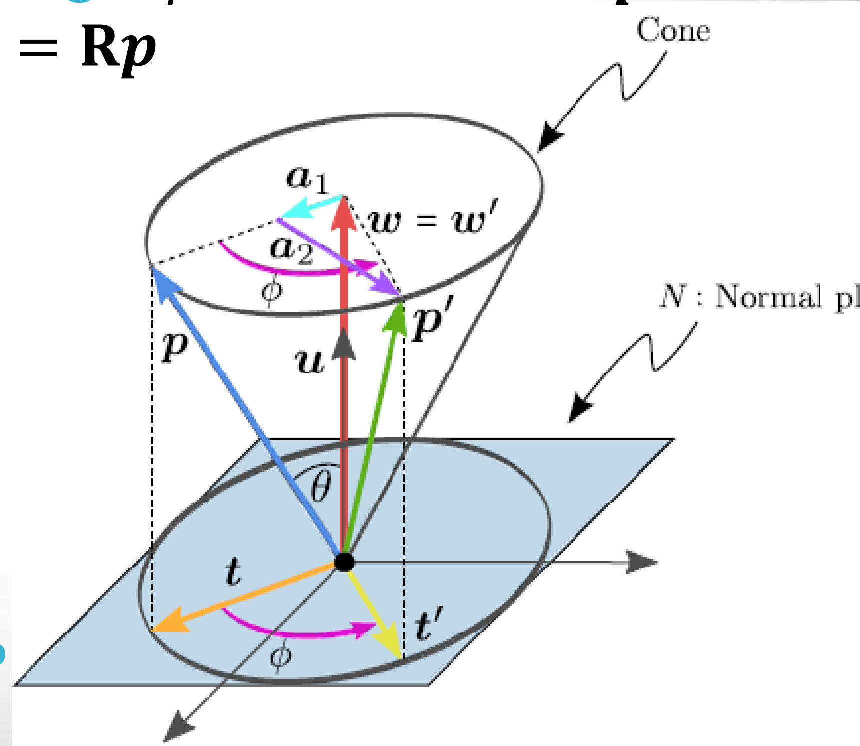
Theorem B (Euler's theorem on rotations) Any motion of a rigid body such that a point, let's say ”O”, on the rigid body remains fixed, is equivalent to a single rotation about some axis that runs through O.

Practical use: If a body is rotating it turns about an axis that pases by points with 0 velocity. Since we are dealing with systems that does not change the origin (rotates maintaining it), the principal axis pass through the origin.

Principal Axis/Angle rotation

How a Rotation about the axis $\boldsymbol{u}$ an angle ϕ affects a vector $\boldsymbol{p}$?

$$\boldsymbol{p}' = \mathrm{T}(\boldsymbol{p}) = \mathbf{R}\boldsymbol{p}$$



Equivalent question: Can we find $\mathbf{R}$?

Principal Axis/Angle rotation

How a Rotation about the axis $\boldsymbol{u}$ an angle ϕ affects a vector $\boldsymbol{p}$?

$$\boldsymbol{p}' = \mathrm{T}(\boldsymbol{p}) = \mathbf{R}\boldsymbol{p}$$

2 simpler cases:

- $\boldsymbol{p}$ is parallel to $\boldsymbol{u}$
- $\boldsymbol{p}$ is orthogonal to $\boldsymbol{u}$

Principal Axis/Angle rotation

Case 1: $\boldsymbol{p}$ is parallel to $\boldsymbol{u}$

If $\boldsymbol{p}$ is parallel to $\boldsymbol{u}$, since they share the origin, $\boldsymbol{p}$ can be described by

$$\boldsymbol{p} = k\boldsymbol{u}$$

Since the rotation is a linear transformation

$$\boldsymbol{p}' = \mathrm{T}(\boldsymbol{p}) = \mathrm{T}(k\boldsymbol{u}) = k\,\mathrm{T}(\boldsymbol{u}) = k\,\boldsymbol{u} = \boldsymbol{p}$$

Therefore $\boldsymbol{p}$ does not change of direction (nor size, it is a rotation) when it is parallel to the rotation axis $\boldsymbol{u}$

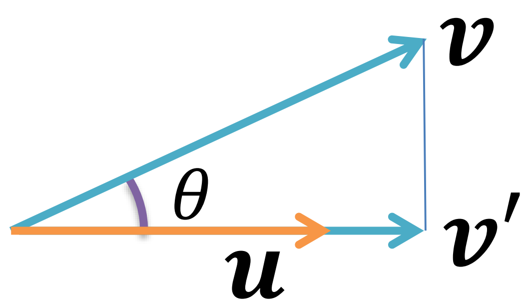
Principal Axis/Angle rotation

Case 2: $\boldsymbol{p}$ is perpendicular to $\boldsymbol{u}$

If $\boldsymbol{p}$ is perpendicular to $\boldsymbol{u}$, since they share the origin, $\boldsymbol{p}$ will describe a circumference arc about $\boldsymbol{u}$ of an angle ϕ.

An aside

What is the projection of a vector over a direction?



$$\boldsymbol{v}' = \frac{\boldsymbol{v}^T \boldsymbol{u}}{\|\boldsymbol{u}\|^2}\,\boldsymbol{u} = \frac{\|\boldsymbol{v}\|\|\boldsymbol{u}\|}{\|\boldsymbol{u}\|^2}\cos\theta\,\boldsymbol{u} = \|\boldsymbol{v}\|\,\cos\theta\,\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}$$

If $\boldsymbol{v}$ **and** $\boldsymbol{u}$ are unit length vectors… $\boldsymbol{v}' = (\boldsymbol{v}^T\boldsymbol{u})\boldsymbol{u} = \cos\theta\,\boldsymbol{u}$

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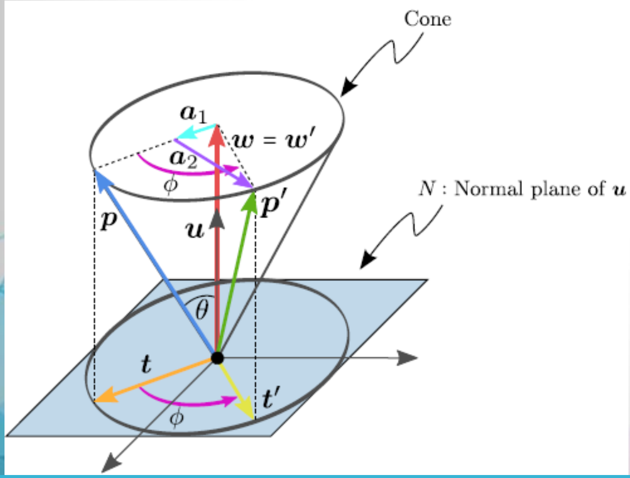
Principal Axis/Angle rotation

General Case

$$\boldsymbol{p} = \boldsymbol{t} + \boldsymbol{w}$$

- $\boldsymbol{w}$ parallel
- $\boldsymbol{t}$ perpendicular



$$\boldsymbol{w} = (\boldsymbol{p}^T\boldsymbol{u})\,\boldsymbol{u}$$
$$\boldsymbol{t} = \boldsymbol{p} - (\boldsymbol{p}^T\boldsymbol{u})\,\boldsymbol{u}$$
$$\|\boldsymbol{t}\| = \boldsymbol{p}\sin\boldsymbol{\theta}$$

Principal Axis/Angle rotation

General Case

$$\boldsymbol{p}' = \mathrm{T}(\boldsymbol{p}) = \mathrm{T}(\boldsymbol{t}+\boldsymbol{w}) = \mathrm{T}(\boldsymbol{t}) + \mathrm{T}(\boldsymbol{w}) = \boldsymbol{w} + \mathrm{T}(\boldsymbol{t})$$

- $\boldsymbol{w}$ parallel
- $\boldsymbol{t}$ perpendicular

$$\boldsymbol{w} = (\boldsymbol{p}^T\boldsymbol{u})\,\boldsymbol{u}$$
$$\boldsymbol{t} = \boldsymbol{p} - (\boldsymbol{p}^T\boldsymbol{u})\,\boldsymbol{u}$$

Principal Axis/Angle rotation

So… what is $\mathrm{T}(\boldsymbol{t})$?

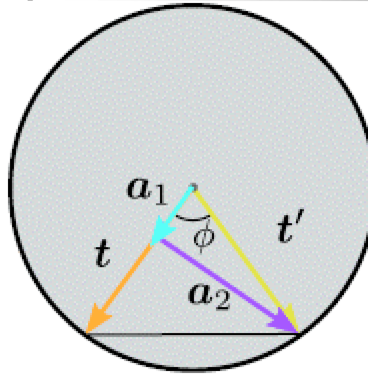
$$\mathrm{T}(\boldsymbol{t}) = \boldsymbol{t}' = \boldsymbol{a_1} + \boldsymbol{a_2}$$

$$\boldsymbol{a}_1 = \frac{\boldsymbol{t}}{\|\boldsymbol{t}\|}\|\boldsymbol{t}'\|\cos\phi = \frac{\boldsymbol{t}}{\|\boldsymbol{t}\|}\|\boldsymbol{t}\|\cos\phi = \boldsymbol{t}\cos\phi$$

$$\boldsymbol{a}_2 = \frac{\boldsymbol{u}\times\boldsymbol{p}}{\|\boldsymbol{u}\times\boldsymbol{p}\,\|}\|\boldsymbol{t}'\|\sin\phi = \frac{\boldsymbol{u}\times\boldsymbol{p}}{\|\boldsymbol{u}\times\boldsymbol{p}\,\|}\|\boldsymbol{t}\|\sin\phi \rightarrow$$

$$\boldsymbol{a}_2 = \frac{\boldsymbol{u}\times\boldsymbol{p}}{\|\boldsymbol{p}\|\sin\theta}\|\boldsymbol{p}\|\sin\theta\sin\phi \rightarrow$$

$$\boldsymbol{a_2} = (\boldsymbol{u}\times\boldsymbol{p})\sin\phi$$



Principal Axis/Angle rotation

General Case

$$\boldsymbol{p}' = \mathrm{T}(\boldsymbol{p}) = \mathrm{T}(\boldsymbol{t} + \boldsymbol{w}) = \mathrm{T}(\boldsymbol{t}) + \mathrm{T}(\boldsymbol{w}) = \boldsymbol{w} + \mathrm{T}(\boldsymbol{t})$$
$$\boldsymbol{p}' = (\boldsymbol{p}^T\boldsymbol{u})\,\boldsymbol{u} + (\boldsymbol{p} - (\boldsymbol{p}^T\boldsymbol{u})\,\boldsymbol{u})\cos\phi + (\boldsymbol{u} \times \boldsymbol{p})\sin\phi \rightarrow$$

$$\boldsymbol{p}' = \boldsymbol{p}\cos\phi + (1 - \cos\phi)(\boldsymbol{p}^T\boldsymbol{u})\,\boldsymbol{u} + (\boldsymbol{u} \times \boldsymbol{p})\sin\phi$$

Remember:

$$\boldsymbol{w} = (\boldsymbol{p}^T\boldsymbol{u})\,\boldsymbol{u}$$
$$\boldsymbol{t} = \boldsymbol{p} - (\boldsymbol{p}^T\boldsymbol{u})\,\boldsymbol{u}$$

Principal Axis/Angle rotation

Matrix representation

$$\boldsymbol{p}' = \mathrm{T}(\boldsymbol{p}) = \boldsymbol{p}\cos\phi + (1-\cos\phi)(\boldsymbol{p}^{T}\boldsymbol{u})\,\boldsymbol{u} + (\boldsymbol{u}\times\boldsymbol{p})\sin\phi$$

Can we write that as $\boldsymbol{p}' = \mathbf{R}\boldsymbol{p}$?

$$\boldsymbol{p}' = \underbrace{(\mathbf{I}_3\cos\phi + (1-\cos\phi)(\boldsymbol{u}\boldsymbol{u}^{T}) + [\boldsymbol{u}]_x \sin\phi)}_{\mathbf{R}}\boldsymbol{p}$$

Prove:

$$(\boldsymbol{p}^{T}\boldsymbol{u})\,\boldsymbol{u} = (\boldsymbol{u}\boldsymbol{u}^{T})\boldsymbol{p}$$

Principal Axis/Angle rotation

Matrix representation

$$\boldsymbol{p}' = (\mathbf{I}_3 \cos\phi + (1-\cos\phi)(\boldsymbol{u}\boldsymbol{u}^T) + [\boldsymbol{u}]_x \sin\phi)\boldsymbol{p}$$

Rodrigues' Formula

Equivalently:

$$\boldsymbol{p}' = (\mathbf{I}_3 + (1-\cos\phi)[\boldsymbol{u}]_x^2 + [\boldsymbol{u}]_x \sin\phi)\boldsymbol{p}$$

Note that $(\boldsymbol{u}\boldsymbol{u}^T) = [\boldsymbol{u}]_x^2 + (\boldsymbol{u}^T\boldsymbol{u})\mathbf{I}_3$

Exercises:

Midterm Exam

- 1 to 4

Mock Exam

- 1 & 2

Final Exam

- 1

Re-eval

- 1 to 4