UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

Centre de la Imatge i la Tecnologia Multimèdia

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Centre de la Imatge i la Tecnologia Multimèdia

# Euler pa/a, Rotation vector and Euler angles

Julen Cayero

Terrassa, April 28th, 2016

# Rotation Mat to axis and angle

Given the rotation matrix we can find the euler's principal axis

$$\mathbf{R} = \left(\mathbf{I}_3 \cos\phi + (1-\cos\phi)\left(\boldsymbol{u}\boldsymbol{u}^T\right) + \sin\phi\,[\boldsymbol{u}]_x\right)$$

Since $[\boldsymbol{u}]_x$ is skew symmetric its trace is null, which implies that

$$\mathrm{trace}(\mathbf{R}) = 3\cos\phi + (1-\cos\phi)(u_1^2 + u_2^2 + u_3^2) \rightarrow$$

$$\mathrm{trace}(\mathbf{R}) = 1 + 2\cos\phi \rightarrow$$

$$\phi = \cos^{-1}\left(\frac{\mathrm{trace}(\mathbf{R}) - 1}{2}\right)$$

# Rotation Mat to axis and angle

## Given the rotation matrix we can find the Euler's principal angle

$$\mathbf{R} = (\mathbf{I_3}\cos\phi + (1-\cos\phi)(\boldsymbol{u}\boldsymbol{u}^{\boldsymbol{T}}) + \sin\phi\,[\boldsymbol{u}]_x)$$

$$\mathbf{R}^{\mathbf{T}} = (\mathbf{I_3}\cos\phi + (1-\cos\phi)\left(\boldsymbol{u}\boldsymbol{u}^{\boldsymbol{T}}\right) - \sin\phi\,[\boldsymbol{u}]_x)$$

As consequence

$$\mathbf{R} - \mathbf{R}^{\mathbf{T}} = 2\sin\phi\,[\boldsymbol{u}]_{\boldsymbol{x}}$$

Which implies that

$$[\boldsymbol{u}]_{\boldsymbol{x}} = \frac{\mathbf{R} - \mathbf{R}^{\mathbf{T}}}{2\sin\phi}$$

# Rotation vector

**Euler principal angle and axis:** We have used 4 parameters. 3 for the unitary principal axis + one additional to encode the rotation magnitude.

However rotations are defined by 3 parameters. Therefore, how about making a vector

$$\boldsymbol{r} = \phi \boldsymbol{u}$$

$\boldsymbol{r}$ is a 3 dimensional vector, which norm is equal to the rotation magnitude, ant its direvtion is the principal axis. Then:

$$\boldsymbol{u} = \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|}; \quad \phi = \|\boldsymbol{r}\|$$

# Rotation vector and rotation matrices

We can use the past relations over the Rodrigues formula to achieve:

$$\boldsymbol{p}' = \left[\mathbf{I}_3 \cos(\|\boldsymbol{r}\|) + \frac{(1-\cos(\|\boldsymbol{r}\|))}{\|\boldsymbol{r}\|^2}\left(\boldsymbol{r}\boldsymbol{r}^{\boldsymbol{T}}\right) + \frac{\sin(\|\boldsymbol{r}\|)}{\|\boldsymbol{r}\|}[\boldsymbol{r}]_x\right]\boldsymbol{p}$$

# Euler angles

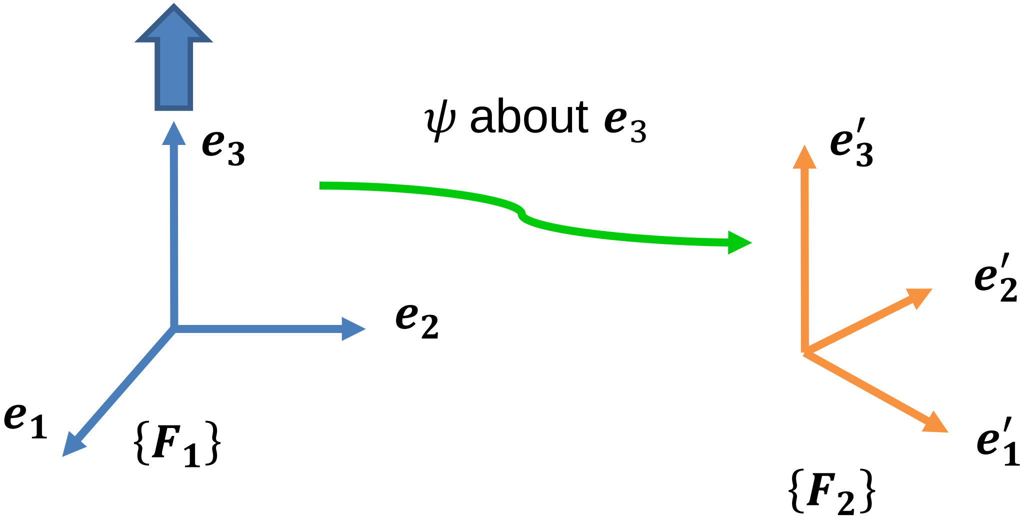
**If we are working with orthonormal bases, can we take benefit of knowing the direction of the three base axis directions?**

We can use the composition of 3 succesive (simple) rotations about different axis to generate any possible rotation.

We will use the 3,2,1 global to local notation.

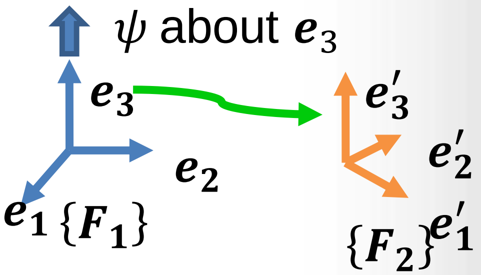
# Euler angles

## Simple rotation about the z axis.



# Euler angles

## Simple rotation about the z axis.

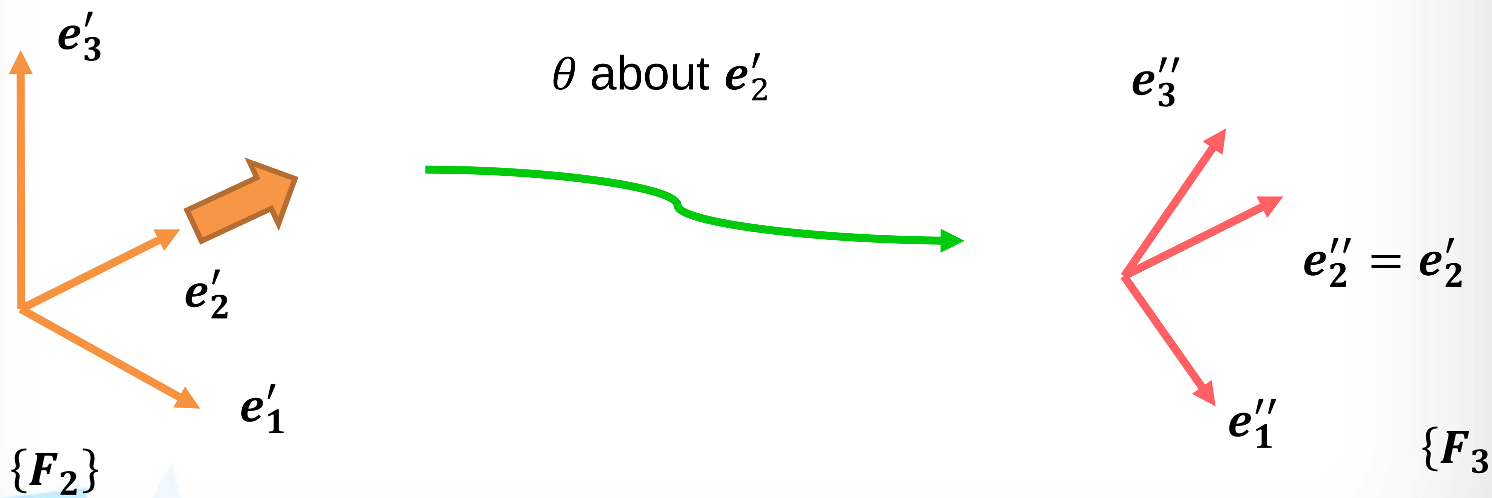



$$\boldsymbol{v}_{\{\boldsymbol{F}_1\}} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{v}_{\{\boldsymbol{F}_2\}}$$

$$\boxed{\boldsymbol{v}_{\{\boldsymbol{F}_2\}} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{v}_{\{\boldsymbol{F}_1\}}}$$

# Euler angles

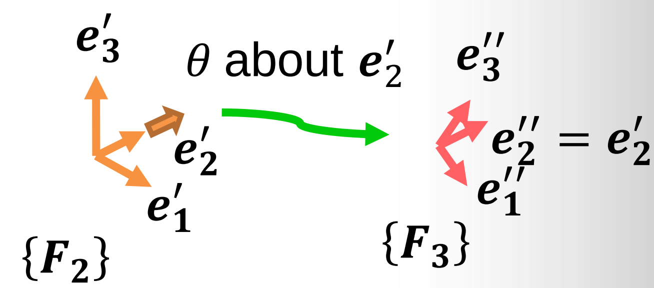
## Simple rotation about the z axis.



# Euler angles

## Simple rotation about the y axis.

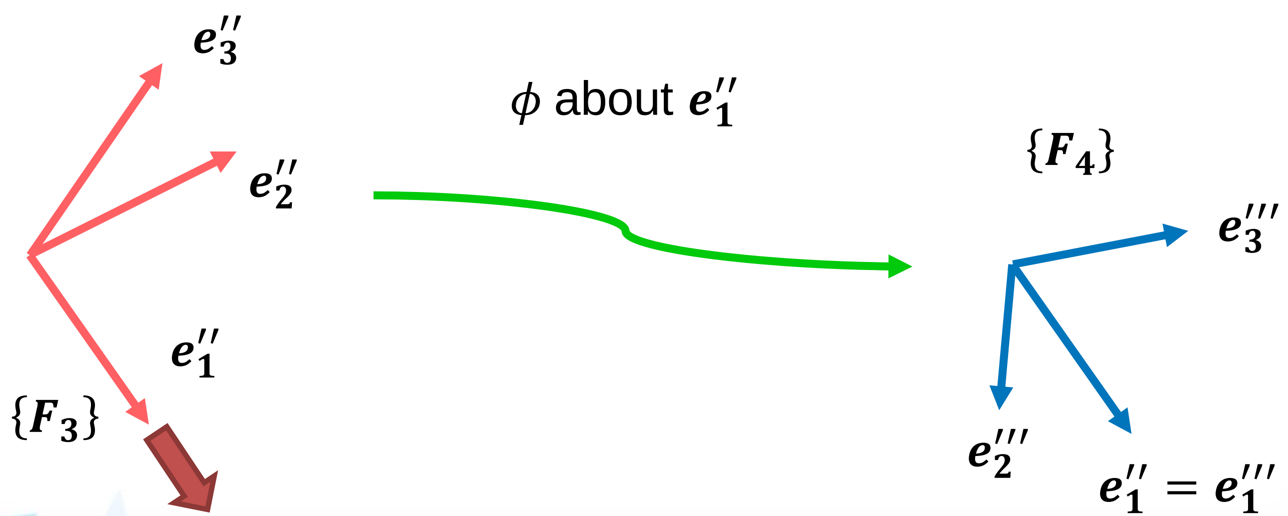
$$\boldsymbol{v}_{\{F_2\}} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \boldsymbol{v}_{\{F_3\}}$$




$$\boldsymbol{v}_{\{F_3\}} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \boldsymbol{v}_{\{F_2\}}$$

# Euler angles

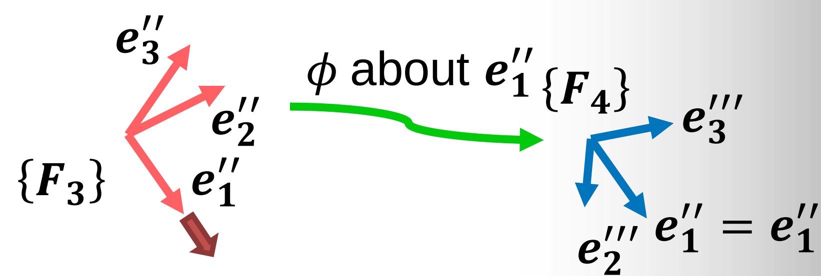
## Simple rotation about the x axis



# Euler angles

## Simple rotation about the x axis.

$$\boldsymbol{v}_{\{\boldsymbol{F}_3\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \boldsymbol{v}_{\{\boldsymbol{F}_4\}}$$




$$\boxed{\boldsymbol{v}_{\{\boldsymbol{F}_4\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \boldsymbol{v}_{\{\boldsymbol{F}_3\}}}$$

# Euler angles

## Composition of rotations

$$v_{\{F_2\}} = \mathrm{R}_\psi v_{\{F_1\}}; \quad v_{\{F_3\}} = \mathrm{R}_\theta v_{\{F_2\}}; \quad v_{\{F_4\}} = \mathrm{R}_\phi v_{\{F_3\}};$$

$$v_{\{F_4\}} = \mathrm{R}_\phi \mathrm{R}_\theta \mathrm{R}_\psi v_{\{F_1\}}$$

# Euler angles

Rotation matrix are equivaltent to the change of basis matrix from target frame to initial frame

$$\boldsymbol{v}_{\{F_1\}} = \left(\mathbf{R}_\phi \mathbf{R}_\theta \mathbf{R}_\psi\right)^T \boldsymbol{v}_{\{F_4\}} \rightarrow \mathbf{R} = \mathbf{R}_\psi^{\mathbf{T}} \mathbf{R}_\theta^{\mathbf{T}} \mathbf{R}_\phi^{T}$$

$$\mathbf{R} = \begin{pmatrix} \cos\theta\cos\psi & \cos\psi\sin\theta\sin\phi - \cos\phi\ \sin\psi & \cos\psi\cos\phi\sin\theta + \sin\psi\sin\phi \\ \cos\theta\sin\psi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix}$$

# Different Euler's angles

**Is posible to select other axis permutations rather than the 3-2-1 order presented.**

# An aside

If we know the values of the $\sin\alpha$ and $\cos\alpha$ of the angle $\alpha$ it is posible to identify $\alpha$

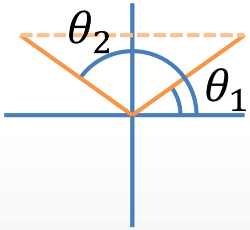
| | |
|---|---|
| $\cos\alpha < 0$<br>$\sin\alpha > 0$ | $\cos\alpha > 0$<br>$\sin\alpha > 0$ |
| $\cos\alpha < 0$<br>$\sin\alpha < 0$ | $\cos\alpha > 0$<br>$\sin\alpha < 0$ |

# Rotation Matrix to Euler angles

$$\mathbf{R} = \begin{pmatrix} \cos\theta\cos\psi & \cos\psi\sin\theta\sin\phi - \cos\phi\ \sin\psi & \cos\psi\cos\phi\sin\theta + \sin\psi\sin\phi \\ \cos\theta\sin\psi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix}$$

**With :**

$$\boldsymbol{r_{31}} = -\sin\theta \rightarrow \theta = \sin^{-1}(-\boldsymbol{r_{31}})$$




**2 possibilities: Select one**

# Rotation Matrix to Euler angles

$$\mathbf{R} = \begin{pmatrix} \cos\theta\cos\psi & \cos\psi\sin\theta\sin\phi - \cos\phi\ \sin\psi & \cos\psi\cos\phi\sin\theta + \sin\psi\sin\phi \\ \cos\theta\sin\psi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix}$$

With $\theta$ known;

$$\phi = atan2\left(\frac{r_{32}}{\cos\theta}, \frac{r_{33}}{\cos\theta}\right)$$

$$\psi = atan2\left(\frac{r_{21}}{\cos\theta}, \frac{r_{11}}{\cos\theta}\right)$$

# Rotation Matrix to Euler angles

An example:

$$\mathbf{R} = \begin{pmatrix} 0.9254 & 0.018 & 0.3785 \\ 0.1632 & 0.8826 & -0.4410 \\ -0.3420 & 0.4698 & 0.8138 \end{pmatrix}$$

Determine $\phi$, $\theta$ and $\psi$.

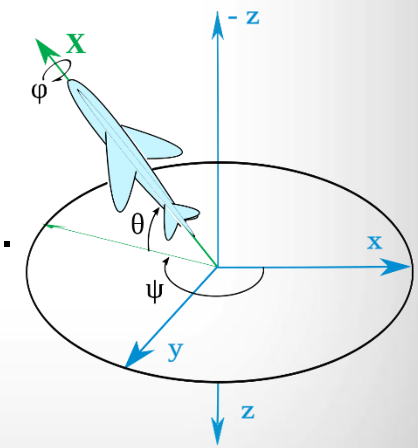
# Rotation Matrix to Euler angles

**When** $\theta = \pm k\frac{\pi}{2}$ for $k$ odd

$$\mathbf{R} = \begin{pmatrix} 0 & \cos\psi\sin\theta\sin\phi - \cos\phi\,\sin\psi & \cos\psi\cos\phi\sin\theta + \sin\psi\sin\phi \\ 0 & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi \\ \pm 1 & 0 & 0 \end{pmatrix}$$

And the past equations are not usable anymore.

If $\theta = \pm k\frac{\pi}{2}$ for $k$ odd, the x and z axis will be aligned and the last rotation will be conteracting the first one. In this case, we can not distinguish between $\psi$ and $\phi$.



# Rotation Matrix to Euler angles

Therefore we can assign, $\phi$ or $\psi$ a value and determine the other by using the remaining entries of the matrix.

As example:

$$\mathbf{R} = \begin{pmatrix} 0 & 0.3420 & 0.9397 \\ 0 & 0.9397 & -0.3420 \\ -1 & 0 & 0 \end{pmatrix}$$

Determine $\phi$, $\theta$ and $\psi$.

# Composing rotations

To concatenate rotations with the parametrizations given, we need to pass for the rotation matrices.

$$\mathbf{R_3} = \mathbf{R_2 R_1}$$

This brings inefficiency to the code because we need constantly to relate different 3-4 dimensional parametrizations with the rotation matrices to make operations.