Euler Angles from Rotation Matrix

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We are interested into convert from rotation matrices to Euler angles. Since now we have seen how taking the euler angles we can formulate the rotation matrix,

$$\mathbf{R} = \mathbf{R}(\psi, \theta, \phi) = \begin{pmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\psi c_\phi s_\theta + s_\psi s_\phi \\ c_\theta s_\psi & s_\psi s_\theta s_\phi + c_\phi c_\psi & c_\phi s_\psi s_\theta - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{pmatrix} \tag{1}$$

This matrix can be interpreted (in addition of the explained in class) as a consecutive composition of the next simpler rotations:

1. A rotation of ϕ about the x axis,
2. A rotation of θ about the y axis and
3. A rotation of ψ about the z axis.

From a given matrix $\mathbf{R}$, we could expect to extract the euler angles by

$$\begin{aligned} \theta &= -\arcsin(R_{31}) \\ \phi &= \arctan(R_{32}/R_{33}) \\ \psi &= \arctan(R_{21}/R_{11}) \end{aligned} \tag{2}$$

However since the $\sin(\bullet)$ and $\tan(\bullet)$ functions are double valued on the interval of $[-\pi, \pi]$ (two different angles produce the same image), **eight** different solutions arise from combining both possible angles of every line in Eq. (2), and not all them are valid.

In fact, in a general case only two combinations produce the same rotation matrix. To isolate the valid cases consider the definition of the next function

$$\alpha = atan2(y, x) = \begin{cases} atan(|y|/|x|)\mathrm{sign}(y) & \text{if } x > 0 \\ (\pi - atan(|y|/|x|))\,\mathrm{sign}(y) & \text{if } x < 0 \\ \frac{\pi}{2}\mathrm{sign}(y) & \text{if } x = 0 \end{cases} \tag{3}$$

$atan2(y, x)$ is defined in many programming languages and of course it is implemented in MatLab (see documentation). This function allows to find the angle between in the interval $[-\pi, \pi]$ that satisfies the values of x and y.

The process to extract the euler angles can be simplified by using this function. Let $\cos(\theta) \neq 0$

1. Calculate the angle *theta* as

$$\theta = -\arcsin(R_{31}) \tag{4}$$

   or equivalently use

$$\theta = \pi + \arcsin(R_{31}) \tag{5}$$

2. Calculate ϕ as

$$\phi = atan2\left(\frac{R_{32}}{\cos(\theta)}, \frac{R_{33}}{\cos(\theta)}\right) \tag{6}$$

3. Calculate ψ as

$$\psi = atan2\left(\frac{R_{21}}{\cos(\theta)}, \frac{R_{11}}{\cos(\theta)}\right) \tag{7}$$

In the case that $\theta = -\pi/2$, $\cos(\theta) = 0$ and $R_{31} = 1$, and the rotation matrix is simplified to

$$\mathbf{R} = \mathbf{R}(\psi, \theta, \phi) = \begin{pmatrix} 0 & -c_\psi s_\phi - c_\phi s_\psi & -c_\psi c_\phi + s_\psi s_\phi \\ 0 & -s_\psi s_\phi + c_\phi c_\psi & -c_\phi s_\psi - c_\psi s_\phi \\ 1 & 0 & 0 \end{pmatrix} \tag{8}$$

In the case that $\theta = \pi/2$, $\cos(\theta) = 0$ and $R_{31} = -1$, and the rotation matrix is simplified to

$$\mathbf{R} = \mathbf{R}(\psi, \theta, \phi) = \begin{pmatrix} 0 & c_\psi s_\phi - c_\phi s_\psi & c_\psi c_\phi + s_\psi s_\phi \\ 0 & s_\psi s_\phi + c_\phi c_\psi & c_\phi s_\psi - c_\psi s_\phi \\ -1 & 0 & 0 \end{pmatrix} \tag{9}$$

For both cases, the procedure to extract the euler angles will not hold since the matrix entries where we seek information are now 0. We talked in class about the gimbal lock. An effect that appears when $theta = \pi/2 + k\pi$ with $k = 1, 2, 3....$ The gimbal lock problem make the ϕ and ψ angles dependent. This effect can be understood looking at non 0 elements on Eq. (8) and Eq. (9), which can be related with the sum angle $(\psi + phi)$ by using trigonometric identities.

Under this circumstances, ϕ and ψ can not be identified separately any more. The usual way to tackle this problem is to take one of them equal to a known value, and then use $atan2$ to find the value for the unknown angle.

e.g. taking $\phi = 0$,

$$\psi = \begin{cases} atan2(-R_{23}, -R_{13}) & \text{if } R_{31} = 1 \\ atan2(R_{23}, R_{13}) & \text{if } R_{31} = -1 \end{cases} \tag{10}$$