UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

Centre de la Imatge i la Tecnologia Multimèdia

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Quaternions and Why to use them

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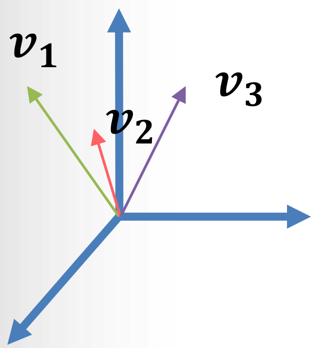
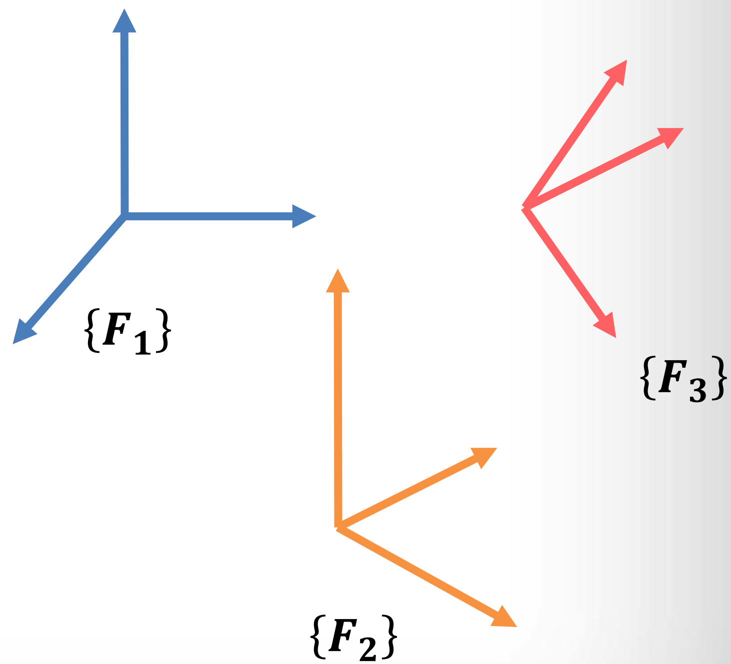
What happen if:

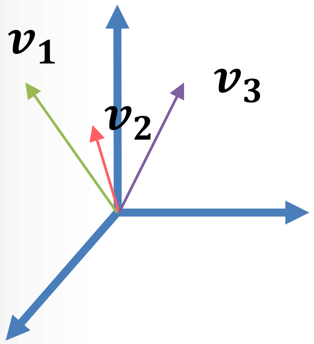




$\boldsymbol{v_1}$ to $\boldsymbol{v_2}$ by $(\boldsymbol{u_1}, \phi_1)$ or $\boldsymbol{r_1}$ or $(\phi_1, \theta_1, \psi_1)$
$\boldsymbol{v_2}$ to $\boldsymbol{v_3}$ by $(\boldsymbol{u_2}, \phi_2)$ or $\boldsymbol{r_2}$ or $(\phi_2, \theta_2, \psi_2)$

Which values have
$(\boldsymbol{u_3}, \phi_3)$ or $\boldsymbol{r_3}$ or $(\phi_3, \theta_3, \psi_3)$ that
transforms $\boldsymbol{v_1}$ to $\boldsymbol{v_3}$?

Example:



$$v_1 = (3 \quad 2 \quad -1)^T$$

v_1 to v_2 by $(u_{1=}(0 \quad 0,7071 \quad 0,7071)^T, \phi_1 = 10\ deg)$

v_2 to v_3 by $(u_{2=}(0,7071 \quad 0 \quad 0,7071)^T, \phi_2 = 20\ deg)$

Which values have (u_3, ϕ_3) that transforms v_1 to v_3?

ANSWER: $(u_{3=}(0.5017 \quad 0.2323 \quad 0.8333)^T, \phi_3 = 26.44\ deg$

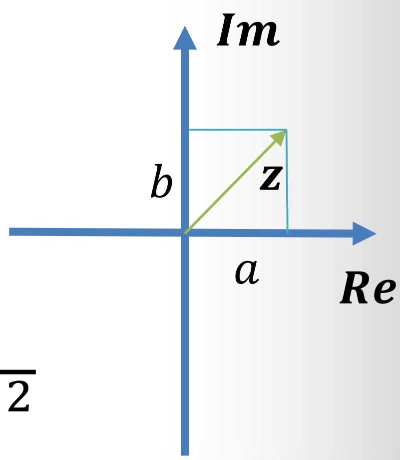
What we know since now

- Rotation matrix -> 9 components. Easy to compose rotations
- Euler principal axis and angle -> 4 components. Compose rotations by transforming to rotation matrices
- Rotation vector -> 3 components. Compose rotatiopns by transforming to rotation matrices
- Euler angles -> 3 components. Compose rotations by transforming to rotation matrices

Good for memory storage but… not so good if we have to operate with them

Complex numbers in 2D

Unitary complex numbers in 2D retains information about direction.



$z = a + bi$

Norm:

$$\sqrt{(z\bar{z})} = \sqrt{a^2 + abi - abi \ - b^2 i^2} = \ \sqrt{a^2 + b^2}$$

Where $i^2 = -1$

Complex numbers in 2D

What happens if we multiply two complex numbers?

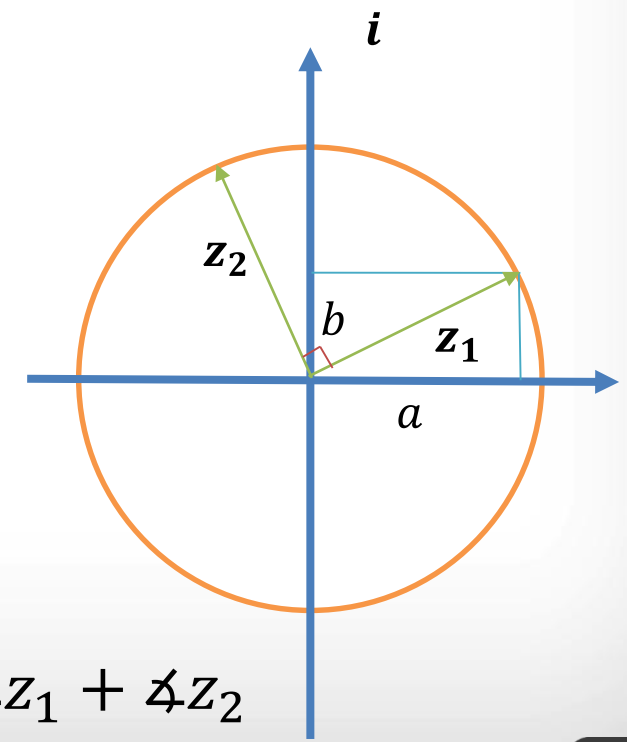
$$z_1 = a + bi$$

$$z_2 = i$$

$$z_1 z_2 = ai - b = -b + ai$$

Rotates the vector 90 degs.

In fact the multiplication makes



$$\measuredangle z_1 z_2 = \measuredangle z_1 + \measuredangle z_2$$

Complex numbers in 2D

What happens if we multiply two complex numbers?

$$z_1 = a + bi$$

$$z_2 = \cos\alpha + i\ \sin\alpha = e^{i\alpha}$$

$$z_1 z_2 = a\cos\alpha + i\ b\cos\alpha +$$

$$+ ia\sin\alpha + i^2 b\sin\alpha\ \rightarrow$$

$$z_1 z_2 = \ a\cos\alpha - b\sin\alpha$$

$$+i\ (b\cos\alpha + \ a\sin\alpha\ )$$

$$z_1 z_2 = \begin{bmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{pmatrix}a \\ b\end{pmatrix}$$

$$z_1 = 2 + 3i$$

$$z_2 = \cos 30 + i\ \sin 30 = \frac{\sqrt{3}}{2} + i\frac{1}{2}$$

$$z_1 z_2 = \sqrt{3} - \frac{3}{2} + i\left(\frac{3\sqrt{3}}{2} + 1\right) = 0.2321 + 3.598i$$

$$z_1 z_2 = \begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{pmatrix}2 \\ 3\end{pmatrix} = \begin{pmatrix}0.2321 \\ 3.598\end{pmatrix}$$

Quaternions

Definition

$$\bar{\bar{q}} = q_0 + iq_1 + jq_2 + k\,q_3$$

$$i^2 = j^2 = k^2 = -1$$

$$ij = k; \quad jk = i; \quad ki = j;$$

$$ji = -k; \quad kj = -i; \quad ik = -j$$

Quaternions

Quaternion multiplication

$\bar{\bar{q}}\bar{\bar{p}} = (q_0 + iq_1 + jq_2 + kq_3)\,(p_0 + ip_1 + jp_2 + kp_3)$=…

Quaternions

It is not easy to maintain this notation on the computer so

$$\bar{\bar{q}} = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} q_0 \\ \boldsymbol{q} \end{pmatrix}$$

$$\bar{\bar{q}}\bar{\bar{p}} = \begin{pmatrix} q_0 p_0 - \boldsymbol{q}^T \boldsymbol{p} \\ q_0 \boldsymbol{p} + p_0 \boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{r} \end{pmatrix} = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \bar{\bar{p}} = \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & p_3 & -p_2 \\ p_2 & -p_3 & p_0 & p_1 \\ p_3 & p_2 & -p_1 & p_0 \end{bmatrix} \bar{\bar{q}}$$

$$\bar{\bar{q}}\bar{\bar{p}} = \overbrace{\begin{bmatrix} q_0 & -\boldsymbol{q}^T \\ \boldsymbol{q} & q_0 \mathbf{I}_3 + [\boldsymbol{q}]_x \end{bmatrix}}^{\mathbf{Q}(\bar{\bar{q}})} \bar{\bar{p}} = \overbrace{\begin{bmatrix} p_0 & -\boldsymbol{p}^T \\ \boldsymbol{p} & p_0 \mathbf{I}_3 - [\boldsymbol{p}]_x \end{bmatrix}}^{\widetilde{\mathbf{Q}}(\bar{\bar{p}})} \bar{\bar{q}}$$

Quaternions

Given:

$$\bar{\bar{q}} = \begin{pmatrix} 1 & 2 & 2 & 1 \end{pmatrix}^T$$

$$\bar{\bar{p}} = \begin{pmatrix} -1 & 1 & 2 & -2 \end{pmatrix}^T$$

Calculate:

$$\bar{\bar{q}}\bar{\bar{p}} =$$

$$\bar{\bar{p}}\bar{\bar{q}} =$$

Quaternions

Since quaternions imaginary numbers, they have **conjugate**:

$$\tilde{\bar{\bar{q}}} = q_0 - iq_1 - j\,q_2 - kq_3 = \begin{pmatrix} q_0 \\ -\boldsymbol{q} \end{pmatrix}$$

Quaternions have norm:

$$\|\bar{\bar{q}}\|^2 = \bar{\bar{q}}\tilde{\bar{\bar{q}}} = q_0^2 + \boldsymbol{q}^{\boldsymbol{T}}\boldsymbol{q} = \begin{pmatrix} q_0 & \boldsymbol{q}^{\boldsymbol{T}} \end{pmatrix}\begin{pmatrix} q_0 \\ \boldsymbol{q} \end{pmatrix}$$

Exist the identity quaternion:

$$\bar{\bar{q}}\,\bar{\bar{q}}_{\mathrm{I}} = \bar{\bar{q}} = \bar{\bar{q}}_{\mathrm{I}}\bar{\bar{q}} \rightarrow \bar{\bar{q}}_{\mathrm{I}} = \begin{pmatrix} 1 \\ \boldsymbol{0} \end{pmatrix}$$

And also the inverse:

$$\bar{\bar{q}}\,\bar{\bar{q}}^{-1} = \bar{\bar{q}}_{\mathrm{I}} \rightarrow \bar{\bar{q}}^{-1} = \frac{\tilde{\bar{\bar{q}}}}{\|\bar{\bar{q}}\|^2}$$

Rotation using Quaternions

Insert the vector $\boldsymbol{v}$ in a quaternion as follow $\bar{\bar{v}} = \begin{pmatrix} 0 \\ \boldsymbol{v} \end{pmatrix}$

Now calculate

$$\bar{\bar{w}} = \bar{\bar{q}}\,\bar{\bar{v}}\,\tilde{\bar{\bar{q}}} = \mathbf{Q}(\bar{\bar{q}})\left(\bar{\bar{v}}\,\tilde{\bar{\bar{q}}}\right) = \mathbf{Q}(\bar{\bar{q}})\left(\tilde{\mathbf{Q}}(\tilde{\bar{\bar{q}}})\bar{\bar{v}}\right) = \mathbf{Q}(\bar{\bar{q}})\tilde{\mathbf{Q}}(\tilde{\bar{\bar{q}}})\bar{\bar{v}}$$

$$\mathbf{Q}(\bar{\bar{q}})\tilde{\mathbf{Q}}(\tilde{\bar{\bar{q}}}) = \begin{bmatrix} q_0^2 + \boldsymbol{q}^T\boldsymbol{q} & 0 & 0 & 0 \\ 0 & q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 0 & 2q_1q_2 + 2q_0q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 0 & 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

$$\mathbf{R}(\bar{\bar{q}})$$

If $\bar{\bar{q}}$ is a unit quaternion, then $\mathbf{R}(\bar{\bar{q}})$ is an orthonormal matrix.

Unit Norm Quaternions

$$\|\bar{\bar{q}}\|^2 = \bar{\bar{q}}\tilde{\bar{\bar{q}}} = q_0^2 + \boldsymbol{q}^T\boldsymbol{q} = 1$$

How can I select the components of $\bar{\bar{q}}$ to be unitary?

$$\bar{\bar{q}} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \boldsymbol{u}\sin\frac{\theta}{2} \end{pmatrix}$$

$$\|\bar{\bar{q}}\| = \cdots$$

Rotation using Quaternions

$\mathbf{R}(\bar{\bar{q}})$ is a rotation matrix

$$v' = \mathbf{R}(\bar{\bar{q}})v$$

$$\mathbf{R}(\bar{\bar{q}}) = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}$$

Note that $\mathbf{R}(\bar{\bar{q}})$ can also be written as

$$\mathbf{R}(\bar{\bar{q}}) = \left(q_0^2 - \boldsymbol{q}^T\boldsymbol{q}\right)\mathbf{I}_3 + \mathbf{2}\boldsymbol{q}\boldsymbol{q}^T + \mathbf{2}q_0[\boldsymbol{q}]_x$$

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Rotation using Quaternions

What is the rotation encoded in $\mathbf{R}(\bar{\bar{q}})$?

$$\boldsymbol{v}' = \mathbf{R}(\bar{\bar{q}})\boldsymbol{v}$$

$$\mathbf{R}(\bar{\bar{q}}) = (q_0^2 - \boldsymbol{q}^T\boldsymbol{q})\mathbf{I}_3 + \mathbf{2}\boldsymbol{q}\boldsymbol{q}^T + \mathbf{2}q_0[\boldsymbol{q}]_x$$

Let's take $\boldsymbol{q} \parallel \boldsymbol{v} \rightarrow \boldsymbol{v} = \lambda\boldsymbol{q}$

$$\boldsymbol{v}' = (q_0^2 - \boldsymbol{q}^T\boldsymbol{q})\boldsymbol{v} + \mathbf{2}(\boldsymbol{v}^T\boldsymbol{q})\boldsymbol{q} + \mathbf{2}q_0(\boldsymbol{q} \times \boldsymbol{v}) \rightarrow$$

$$\boldsymbol{v}' = (q_0^2 - \boldsymbol{q}^T\boldsymbol{q})\lambda\boldsymbol{q} + \mathbf{2}(\lambda\boldsymbol{q}^T\boldsymbol{q})\boldsymbol{q} = \lambda(q_0^2 - \boldsymbol{q}^T\boldsymbol{q})\boldsymbol{q} = \boldsymbol{v}$$

Rotation using Quaternions

What is the rotation encoded in $\mathbf{R}(\bar{\bar{q}})$?

$$\boldsymbol{v}' = \mathbf{R}(\bar{\bar{q}})\boldsymbol{v}$$

$$\mathbf{R}(\bar{\bar{q}}) = (q_0^2 - \boldsymbol{q}^T\boldsymbol{q})\mathbf{I}_3 + 2\boldsymbol{q}\boldsymbol{q}^T + 2q_0[\boldsymbol{q}]_x$$

Let's take $\boldsymbol{q} \perp \boldsymbol{v}$

$$\boldsymbol{v}' = (q_0^2 - \boldsymbol{q}^T\boldsymbol{q})\boldsymbol{v} + 2(\boldsymbol{v}^T\boldsymbol{q})\boldsymbol{q} + 2q_0(\boldsymbol{q} \times \boldsymbol{v}) \rightarrow$$

$$\boldsymbol{v}' = (q_0^2 - \boldsymbol{q}^T\boldsymbol{q})\boldsymbol{v} + 2q_0(\boldsymbol{q} \times \boldsymbol{v}) \rightarrow$$

$$\|\boldsymbol{q}\|\boldsymbol{v}_\perp$$

$$\boldsymbol{v}' = \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)\boldsymbol{v} + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\boldsymbol{v}_\perp = \cos\theta\,\boldsymbol{v} + \sin\theta\,\boldsymbol{v}_\perp$$

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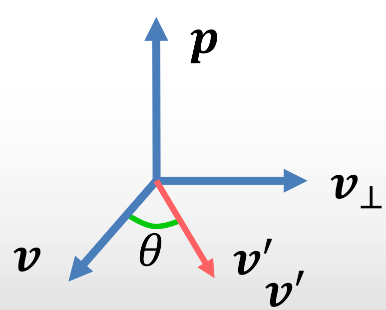

Rotation using Quaternions

The quaternion

$$\bar{\bar{q}} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \boldsymbol{u}\sin\frac{\theta}{2} \end{pmatrix}$$

Performs a rotation of θdegs obout the unitary axis $\boldsymbol{u}$

By substituting the components on

$$\mathbf{R}(\bar{\bar{q}}) = \left(q_0^2 - \boldsymbol{q}^T\boldsymbol{q}\right)\mathbf{I_3} + 2\boldsymbol{q}\boldsymbol{q}^T + 2q_0[\boldsymbol{q}]_x \rightarrow$$

$$\mathbf{R}(\bar{\bar{q}}) = \left(\cos^2\frac{\theta}{2} \quad - \sin^2\frac{\theta}{2}\right)\mathbf{I}_3 + 2\sin^2\frac{\theta}{2}\,\boldsymbol{u}\boldsymbol{u}^T + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}[\boldsymbol{u}]_x \rightarrow$$

$$\mathbf{R}(\bar{\bar{q}}) = \cos\theta\,\mathbf{I}_3 + (1-\cos\theta)\boldsymbol{u}\boldsymbol{u}^T + \sin\theta\,[\boldsymbol{u}]_x$$

Composing rotations

If we can rotate a vector by using a unit quaternion as

$$\bar{\bar{w}} = \bar{\bar{q}}\,\bar{\bar{v}}\,\tilde{\bar{\bar{q}}}$$

What happen if we rotate the image by using another unit quaternion $\bar{\bar{p}}$?

$$\bar{\bar{t}} = \bar{\bar{p}}\,\bar{\bar{w}}\,\tilde{\bar{\bar{p}}} = \bar{\bar{p}}\,\bar{\bar{q}}\,\bar{\bar{v}}\,\tilde{\bar{\bar{q}}}\,\tilde{\bar{\bar{p}}}$$

Which implies that the quaternion that rotate $\boldsymbol{v}$ first by using the quaternion $\bar{\bar{q}}$ and second $\bar{\bar{p}}$ is $\bar{\bar{p}}\bar{\bar{q}}$

Exercises

Midterm Exam

- 1 to 4

Mock Exam

- 1 & 3

Final Exam

- Exercise 1

Re-eval

- All except 5