Rotation Matrices:
Rotates a vector or change the basis?

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The answer of the question in the title is, both, rotation matrices can be used for this two purposes. However what is interesting is to understand what are the effect of the operation that we are performing.

Since now we have seen how to obtain rotation matrices from four different parametrizations:

- Euler principal axis and angle,

$$\mathbf{R} = \mathbf{R}(\phi, \boldsymbol{u}) = \mathbf{I}\cos(\phi) + (1-\cos(\phi))\left(\boldsymbol{u}\boldsymbol{u}^\intercal\right) + \sin(\phi)\left[\boldsymbol{u}\right]_\times \tag{1}$$

  or

$$\mathbf{R} = \mathbf{R}(\phi, \boldsymbol{u}) = \mathbf{I} + \sin(\phi)\left[\boldsymbol{u}\right]_\times + (1-\cos(\phi))\left[\boldsymbol{u}\right]_\times^2 \tag{2}$$

- Rotation vector

$$\mathbf{R} = \mathbf{R}(\boldsymbol{r}) = \mathbf{I}\cos(\|\boldsymbol{r}\|) + \frac{(1-\cos(\|\boldsymbol{r}\|))}{\|\boldsymbol{r}\|^2}\left(\boldsymbol{r}\boldsymbol{r}^\intercal\right) + \frac{\sin(\|\boldsymbol{r}\|)}{\|\boldsymbol{r}\|}\left[\boldsymbol{r}\right]_\times \tag{3}$$

  or

$$\mathbf{R} = \mathbf{R}(\boldsymbol{r}) = \mathbf{I} + \frac{\sin(\|\boldsymbol{r}\|)}{\|\boldsymbol{r}\|}\left[\boldsymbol{u}\right]_\times + \frac{(1-\cos(\|\boldsymbol{r}\|))}{\|\boldsymbol{r}\|^2}\left[\boldsymbol{u}\right]_\times^2 \tag{4}$$

- Euler angles

$$\mathbf{R} = \mathbf{R}(\psi, \theta, \phi) = \begin{pmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\psi c_\phi s_\theta + s_\psi s_\phi \\ c_\theta s_\psi & s_\psi s_\theta s_\phi + c_\phi c_\psi & c_\phi s_\psi s_\theta - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{pmatrix} \tag{5}$$

- Quaternions

$$\mathbf{R} = \mathbf{R}(\mathring{q}) = \left(q_0^2 - \boldsymbol{q}^\intercal\boldsymbol{q}\right)\mathbf{I}_3 + 2\boldsymbol{q}\boldsymbol{q}^\intercal + 2q_0\left[\boldsymbol{q}\right]_\times \tag{6}$$

Let the simple rotation matrix

$$\mathbf{R} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \tag{7}$$

given equivalently by:

- A rotation about $\boldsymbol{u} = (0,\ 0,\ 1)^\intercal$ of magnitude $\phi = 45\,\text{deg}$,
- A rotation vector $\boldsymbol{r} = \left(0, 0, \frac{\pi}{4}\right)^\intercal$,
- A set of euler angles $\psi = 45\,\text{deg}$, $\theta = 0\,\text{deg}$, $\phi = 0\,\text{deg}$ or
- A quaternion $\mathring{q} = \frac{1}{\sqrt{2}}\,(1, 0,\ 0,\ 1)^\intercal$.

So... what this transformation does? Let the vector ${}^{A}\boldsymbol{p}$ to be a vector on a frame $\{A\}$. And let $\{A\}$ be the frame from which we have defined $\mathbf{R}$.

If we suppose that $\mathbf{R}$ only affects the vector ${}^{A}\boldsymbol{p}$, we can admit that the image of the vector is another new vector, affected by the rotation. As example let ${}^{A}\boldsymbol{p} = (1,\ 0\,1)^\intercal$ which image after transformation is

$$ {}^{A}\boldsymbol{p}' = \mathbf{R}\,{}^{A}\boldsymbol{p} = \frac{1}{\sqrt{2}}\left(1,\ 1\,\sqrt{2}\right)^\intercal \tag{8} $$

Fig. 1 shows the original vector (blue arrow) and the image after the rotation (yellow arrow).

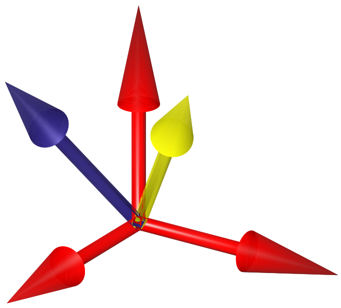

Figure 1: Original vector ${}^{A}\boldsymbol{p}$ in blue and image vector ${}^{A}\boldsymbol{p}' = \mathbf{R}\,{}^{A}\boldsymbol{p}$ (in yellow)

Now, consider that the the vector $\boldsymbol{p}$ is static object in the space, it is that it does not move. From $\boldsymbol{p}$ we know its coordinates on the frame $\{A\}$. Now consider the transformation given by ${}^{B}\boldsymbol{p} = \mathbf{R}\,{}^{A}\boldsymbol{p}$. In this case, the resultant vector ${}^{B}\boldsymbol{p} = \frac{1}{\sqrt{2}}\left(1,\ -1\,\sqrt{2}\right)^\intercal$ coincides with the previous ${}^{A}\boldsymbol{p}'$. However it may interpreted in a different way. Now the vector ${}^{B}\boldsymbol{p}$ can be thought as the coordinates of the vector $\boldsymbol{p}$ expressed in a new reference system. This reference system is depicted in green on Fig. 2, and coincides with a reference frame rotated in this case $-45\,\text{deg}$ about the axis $(0,\ 0\,1)^\intercal$.

In the same way we can see what happens with the inverse transformation, i.e. what is the effect of $\mathbf{R}^\intercal$. Since it is known that $\mathbf{R}^\intercal = \mathbf{R}^{-1}$, we can expect that $\mathbf{R}^\intercal$ performs the inverse operation in fact

$$ {}^{A}\boldsymbol{p} = \mathbf{R}^\intercal\,{}^{A}\boldsymbol{p}' \tag{9} $$

represents a derotation of vector ${}^{A}\boldsymbol{p}'$, and

$$ {}^{A}\boldsymbol{p} = \mathbf{R}^\intercal\,{}^{B}\boldsymbol{p} \tag{10} $$

gives the coordinates of vector $\boldsymbol{p}$ in $\{A\}$, knowing the coordinates of $\boldsymbol{p}$ in $\{B\}$, ${}^{B}\boldsymbol{p}$.

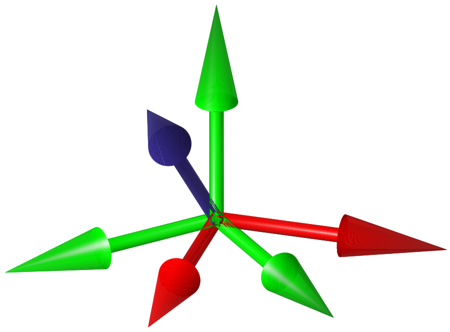

Figure 2: The vector $\boldsymbol{p}$ in blue is seen by the frame $\{A\}$ (in red) as ${}^{A}\boldsymbol{p} = (1,\, 0\, 1)^{\intercal}$ and from the transformed reference frame $\{B\}$ (in green) as ${}^{B}\boldsymbol{p} = \frac{1}{\sqrt{2}}\,(1,\, 1\,\sqrt{2})^{\intercal}$

But, how can be interpreted the transformation given by $\mathbf{R}^{\intercal}\,{}^{A}\boldsymbol{p}$?

We can think of it from the two perspectives presented before. First, we can take ${}^{A}\boldsymbol{p} = {}^{C}\boldsymbol{p}'$ as known image of a rotated vector in an unspecified basis $\{C\}$. As consequence

$$ {}^{C}\boldsymbol{p} = \mathbf{R}^{\intercal}\,{}^{C}\boldsymbol{p}' = \frac{1}{\sqrt{2}}\left(1,\, -1\,\sqrt{2}\right)^{\intercal} \tag{11} $$

results in a derotation, it is a rotation of $-45\,\mathrm{deg}$ the $z$axis of the vector ${}^{A}\boldsymbol{p}$. This can be appreciated on Fig. 3.

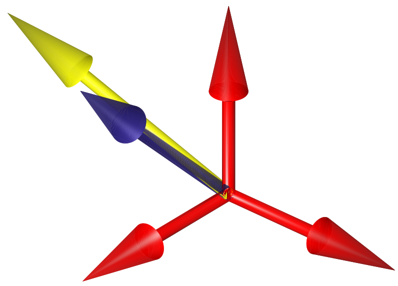

Figure 3: Original vector ${}^{C}\boldsymbol{p}$ in blue and image vector ${}^{C}\boldsymbol{p}' = {}^{A}\boldsymbol{p} = \mathbf{R}^{T}\,{}^{A}\boldsymbol{p}$

However if $\boldsymbol{p}$ is seen as a fixed vector

$$ {}^{C}\boldsymbol{p} = \mathbf{R}^{\intercal}\,{}^{A}\boldsymbol{p} \tag{12} $$

gives the coordinates of vector $\boldsymbol{p}$ in $\{C\}$, knowing the coordinates of $\boldsymbol{p}$ in $\{A\}$, ${}^{A}\boldsymbol{p}$, as can be seen on Fig. 4. It is equivalent to say that ${}^{C}\boldsymbol{p}$ are the coordinates of $\boldsymbol{p}$ as seen from a reference that results from rotating the system $\{A\}$, $45\,\mathrm{deg}$ about the axis $\frac{1}{\sqrt{2}}\,(0,\, 0\, 1)^{\intercal}$
.

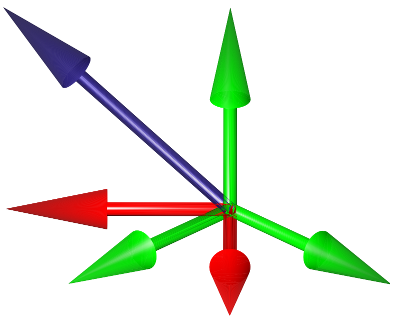

Figure 4: The vector $\boldsymbol{p}$ in blue is seen by the frame $\{A\}$ (in red) as ${}^{A}\boldsymbol{p} = (1,\ 0\,1)^{\intercal}$ and from the transformed reference frame $\{B\}$ (in green) as ${}^{B}\boldsymbol{p} = \frac{1}{\sqrt{2}}\left(1,\ -1\,\sqrt{2}\right)^{\intercal}$