CITM

Midterm Exam, December 17, 2015

MATVJII

SOLUTIONS

Exercise 1

Given the next three transformations, find:

- Which of them are linear transformations and argument why.
- Which of them represents rotations and argument why.

3 Point 1.

$$T_1(\boldsymbol{x}) : \mathbb{R}^3 \to \mathbb{R}^2; \quad T_1(\boldsymbol{x}) = \begin{pmatrix} x_1 + x_2 - 3x_3 \\ -x_1 + x_2 \end{pmatrix}$$

4 Point 2.

$$T_2(\boldsymbol{x}) : \mathbb{R}^3 \to \mathbb{R}^3; \quad T_2(\boldsymbol{x}) = \begin{pmatrix} \frac{\sqrt{3}}{2} x_1 + x_2 \\ 3 - x_3 + \frac{1}{\sqrt{3}} \\ x_1 \end{pmatrix}$$

4 Point 3.

$$T_3(\boldsymbol{x}) : \mathbb{R}^3 \to \mathbb{R}^3; \quad T_3(\boldsymbol{x}) = \begin{pmatrix} \frac{1}{2}(x_1 + x_3) - \frac{\sqrt{2}}{2} x_2 \\ \frac{\sqrt{2}}{2}(x_3 - x_2) \\ \frac{1}{2}(x_1 + x_3) + \frac{1}{\sqrt{2}} x_2 \end{pmatrix}$$

Solution:

The $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation (L.T) if and only if:

1. $T(\boldsymbol{a}) + T(\boldsymbol{b}) = T(\boldsymbol{a} + \boldsymbol{b})$ and
2. $T(k\boldsymbol{a}) = kT(\boldsymbol{a})$

If the transformations is linear it can be written as $T(\boldsymbol{x}) = \mathbf{A}\boldsymbol{x}$, with $\mathbf{A}$ constant.

Moreover, a linear transformation is a rotation if and only if, it transforms between spaces of the same dimension $R : \mathbb{R}^m \to \mathbb{R}^m$ and if the columns and rows of the matrix that represents the transformations are unitary and orthogonal.

1. $T_1(\boldsymbol{x})$ can be written as

$$T_1(\boldsymbol{x}) = \underbrace{\begin{pmatrix} 1 & 1 & -3 \\ -1 & 1 & 0 \end{pmatrix}}_{\mathbf{A}} \boldsymbol{x}$$

Therefore it is clear that:

$$T_1(\boldsymbol{a}) = \mathbf{A}\boldsymbol{a}$$

$$T_1(\boldsymbol{b}) = \mathbf{A}\boldsymbol{b}$$

$$T_1(\boldsymbol{a} + \boldsymbol{b}) = \mathbf{A}(\boldsymbol{a} + \boldsymbol{b}) = \mathbf{A}\boldsymbol{a} + \mathbf{A}\boldsymbol{b}$$

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$$T_1(k\boldsymbol{a}) = \mathbf{A}k\boldsymbol{a}$$

$$kT_1(\boldsymbol{a}) = k\mathbf{A}\boldsymbol{a}$$

Since $T_1(\boldsymbol{a}) + T_1(\boldsymbol{b}) = T_1(\boldsymbol{a}+\boldsymbol{b})$ and $T_1(k\boldsymbol{a}) = kT_1(\boldsymbol{a})$, T_1 is a linear transformation. However since it transforms between different spaces, T_1 is not a rotation.

2. $T_2(\boldsymbol{x})$ can be written as

$$T_1(\boldsymbol{x}) = \underbrace{\begin{pmatrix} \frac{\sqrt{3}}{2} & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}}_{\mathbf{A}} \boldsymbol{x} + \underbrace{\begin{pmatrix} 0 \\ 3 - \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}}_{\boldsymbol{f}}$$

Therefore we can calculate:

$$T_2(\boldsymbol{a}) = \mathbf{A}\boldsymbol{a} + \boldsymbol{f}$$

$$T_2(\boldsymbol{b}) = \mathbf{A}\boldsymbol{b} + \boldsymbol{f}$$

$$T_2(\boldsymbol{a}+\boldsymbol{b}) = \mathbf{A}\left(\boldsymbol{a}+\boldsymbol{b}\right) + \boldsymbol{f} = \mathbf{A}\boldsymbol{a} + \mathbf{A}\boldsymbol{b} + \boldsymbol{f}$$

$$T_2(k\boldsymbol{a}) = \mathbf{A}k\boldsymbol{a} + \boldsymbol{f}$$

$$kT_2(\boldsymbol{a}) = k\mathbf{A}\boldsymbol{a} + k\boldsymbol{f}$$

Now we can see that $T_2(\boldsymbol{a}) + T_2(\boldsymbol{b}) \neq T_2(\boldsymbol{a}+\boldsymbol{b})$ and $T_2(k\boldsymbol{a}) \neq kT_2(\boldsymbol{a})$. Therefore, T_2 is not a linear transformation.

Since T_2 is not a linear transformation it can not be a rotation.

3. $T_3(\boldsymbol{x})$ can be written as

$$T_1(\boldsymbol{x}) = \underbrace{\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \end{pmatrix}}_{\mathbf{A}} \boldsymbol{x}$$

Therefore we can calculate:

$$T_3(\boldsymbol{a}) = \mathbf{A}\boldsymbol{a}$$

$$T_3(\boldsymbol{b}) = \mathbf{A}\boldsymbol{b}$$

$$T_3(\boldsymbol{a}+\boldsymbol{b}) = \mathbf{A}\left(\boldsymbol{a}+\boldsymbol{b}\right) = \mathbf{A}\boldsymbol{a} + \mathbf{A}\boldsymbol{b}$$

$$T_3(k\boldsymbol{a}) = \mathbf{A}k\boldsymbol{a}$$

$$kT_3(\boldsymbol{a}) = k\mathbf{A}\boldsymbol{a}$$

Since $T_3(\boldsymbol{a}) + T_3(\boldsymbol{b}) = T_3(\boldsymbol{a}+\boldsymbol{b})$ and $T_3(k\boldsymbol{a}) = kT_3(\boldsymbol{a})$, T_3 is a linear transformation. However it is not a rotation since the rows and columns of $\mathbf{A}$ are not unitary and some of them are not orthogonal.

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Exercise 2

The three vectors $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$, have known components on a basis $\{A\}$

$$^A\boldsymbol{v}_1 = (1,\, 0,\, 0)^\intercal$$
$$^A\boldsymbol{v}_2 = (1,\, 1,\, 0)^\intercal$$
$$^A\boldsymbol{v}_3 = (1,\, 1,\, 1)^\intercal$$

The same vectors observed from a rotated (no displacement) basis $\{B\}$ are given by:

$$^B\boldsymbol{v}_1 = \tfrac{1}{2}\left(\sqrt{3},\, 1,\, 0\right)^\intercal$$
$$^B\boldsymbol{v}_2 = \tfrac{1}{2}\left(\sqrt{3}-1,\, 1+\sqrt{3},\, 0\right)^\intercal$$
$$^B\boldsymbol{v}_3 = \tfrac{1}{2}\left(\sqrt{3}-1,\, 1+\sqrt{3},\, 2\right)^\intercal$$

$\boxed{\text{20 Point}}$ 1. Calculate the components of the vector $\boldsymbol{p}$ in the frame $\{A\}$, if it is known that in the frame $\{B\}$ the vector is given by:

$$^B\boldsymbol{p} = \left(3,\, -2,\, \frac{1}{2}\right)^\intercal$$

Hint: Use the properties of linear transformations.

Solution:

On this exercise, we are working with vectors and their images. To fulfil the main goal it is needed a rotation matrix that allows to go from frame $\{B\}$ to $\{A\}$.

We saw in class that linear transformations can be written as matrix where

$$\mathbf{R} = \begin{pmatrix} \vdots & \vdots & \vdots \\ T(\boldsymbol{e}_1) & T(\boldsymbol{e}_2) & T(\boldsymbol{e}_3) \\ \vdots & \vdots & \vdots \end{pmatrix}$$

being $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ the basis vector of the frame that is being transformed.

As consequence if the images of the basis vectors of a frame are known, the transformation matrix can be easily constructed.

Looking at the vectors in $\{A\}$ and claiming the property of linear transformations $T(\boldsymbol{a}) + T(\boldsymbol{b}) = T(\boldsymbol{a}+\boldsymbol{b})$

We can construct the rotation matrix as

$$^B\mathbf{R}_A = \begin{pmatrix} \vdots & \vdots & \vdots \\ T(^A\boldsymbol{v}_1) & T(^A\boldsymbol{v}_2 - {}^A\boldsymbol{v}_1) & T(^A\boldsymbol{v}_3 - {}^A\boldsymbol{v}_2) \\ \vdots & \vdots & \vdots \end{pmatrix}$$

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where

$$T({}^A\boldsymbol{v}_1) = {}^B\boldsymbol{v}_1 = \tfrac{1}{2}\left(\sqrt{3},\, 1,\, 0\right)^\intercal$$
$$T({}^A\boldsymbol{v}_2 - {}^A\boldsymbol{v}_1) = {}^B\boldsymbol{v}_2 - {}^B\boldsymbol{v}_1 = \tfrac{1}{2}\left(-1,\, \sqrt{3},\, 0\right)^\intercal \; .$$
$$T({}^A\boldsymbol{v}_3 - {}^A\boldsymbol{v}_2) = {}^B\boldsymbol{v}_3 - {}^B\boldsymbol{v}_2 = (0,\, 0,\, 1)^\intercal$$

Therefore,

$${}^B\mathbf{R}_A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To obtain ${}^A\boldsymbol{p}$ we need to transform from frame $\{B\}$ to $\{A\}$, so we need the inverse rotation of the matrix above.

$${}^A\mathbf{R}_B = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then, the solution of the exercise can be achieved by simply multiplying the matrix by the vector:

$${}^A\boldsymbol{p} = {}^A\mathbf{R}_B\, {}^B\boldsymbol{p} = \left(3\frac{\sqrt{3}}{2} - 1,\, -\frac{3}{2} - \sqrt{3},\, \frac{1}{2}\right)^\intercal \tag{1}$$

Exercise 3

The rotation matrix

$$\mathbf{R} = \begin{pmatrix} 0.9746 & -.01816 & 0.1309 \\ 0.1309 & 0.9366 & 0.3251 \\ -0.1816 & -0.2998 & 0.9366 \end{pmatrix}$$

represents a rotation of $\phi = 22.5\,\text{deg}$ about an axis $\boldsymbol{u}$ parallel to the vector $(-2, 1, 1)^\intercal$.

Give:

$\boxed{\text{3 Point}}$ 1. $\mathbf{R}_1$ the matrix that represents a rotation of $-\phi = 22.5\,\text{deg}$ about $\boldsymbol{u}$.

$\boxed{\text{3 Point}}$ 2. $\mathbf{R}_2$ the matrix that represents a rotation of $-\phi = 22.5\,\text{deg}$ about $-\boldsymbol{u}$.

$\boxed{\text{3 Point}}$ 3. $\mathbf{R}_3$ the matrix that represents a rotation of $\phi = 22.5\,\text{deg}$ about $-\boldsymbol{u}$.

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Solution:

1. Imagine that we use $\mathbf{R}$ to rotate a generic vector $\boldsymbol{v}$ to obtain $\boldsymbol{v}'$. It is clear that if we apply $\mathbf{R}_1$ over $\boldsymbol{v}'$, the result will be again $\boldsymbol{v}$. As consequence $\mathbf{R}_1 = \mathbf{R}^{-1} = \mathbf{R}^\intercal$.
2. A rotation on the opposite direction and opposite angle represents a rotation in the same direction by the same amount. $\mathbf{R}_2 = \mathbf{R}$.
3. Rotate the same angle about the oposite direction is equivalent to perform the inverse rotation, therefore $\mathbf{R}_3 = \mathbf{R}^{-1} = \mathbf{R}^\intercal$.

Exercise 4

It is desired to rotate the vector

$$\boldsymbol{p} = (-1,\ -3,\ 2)^\intercal$$

about an axis $\boldsymbol{u}$ by an amount of π rad. It is known that the projection of $\boldsymbol{p}$ over $\boldsymbol{u}$ is given by

$$\boldsymbol{w} = (0,\ 0,\ 2)^\intercal$$

and the projection of $\boldsymbol{p}$ over the plane perpendicular to $\boldsymbol{u}$ is

$$\boldsymbol{t} = (-1,\ -3,\ 0)^\intercal\ .$$

15 Point 1. What is the image of $\boldsymbol{p}$ after rotating?

Solution:

Since the projection of $\boldsymbol{p}$ over $\boldsymbol{w}$ is

$$\boldsymbol{w} = (0,\ 0,\ 2)^\intercal$$

The rotation axis can be identified as:

$$\boldsymbol{u} = (0,\ 0,\ 1)^\intercal\ .$$

Therefore the rotation is performed about the z axis by 180 deg. The projection of $\boldsymbol{p}$ over $\boldsymbol{u}$, it is $\boldsymbol{w}$, does not change with the rotation. And is the projection over the perpendicular plane, $\boldsymbol{t}$ that rotates 180 deg.

As consequence, the image of $\boldsymbol{t}$, $\boldsymbol{t}^\intercal = (1,\ 3,\ 0)^\intercal$ (see Figure 1).

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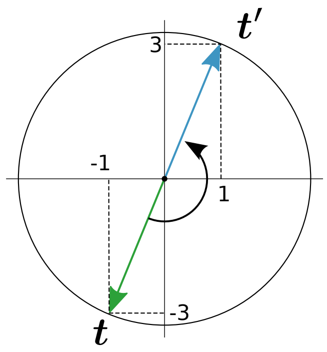

Figure 1: rotation of $\boldsymbol{t}$.

As consequence the image of $\boldsymbol{p}$ can be recovered as

$$\boldsymbol{p}' = (1,\, 3,\, 2)$$

There was a typo on the statement the day of the exam. $\boldsymbol{p}$ appeared as $\boldsymbol{p} = (-1,\, 3,\, 2)$. Clearly this wasn't coherent with the projection over the plane. The information given was redundant and you could finish the exercise by using any of both, the information of $\boldsymbol{p}$ or using its projections. The solution $\boldsymbol{p}' = (1,\, -3,\, 2)$ is also a valid solution.

Exercise 5

Let three different frames be defined as $\{F_1\}$, $\{F_2\}$ and $\{F_3\}$.

It is known that the origin of the frame $\{F_2\}$ relative to the frame $\{F_1\}$ is given by the vector $\boldsymbol{t}_1 = {}^{F_1}\boldsymbol{t}_{F_2}$. In the same way, the origin of frame $\{F_3\}$ is known in the frame $\{F_2\}$ and represented by the vector $\boldsymbol{t}_2 = {}^{F_2}\boldsymbol{t}_{F_3}$.

Moreover it is known that a vector in the basis $\{F_1\}$ can be expressed in the directions of the frame $\{F_2\}$ by the rotation matrix $\mathbf{R}_1 = {}^{F_2}\mathbf{R}_{F_1}$ and in a similar way a vector in the basis $\{F_3\}$ can be expressed in the directions of the frame $\{F_2\}$ by the rotation matrix $\mathbf{R}_2 = {}^{F_2}\mathbf{R}_{F_3}$.

30 Point 1. Give the homogeneous affine transformation that allows to express the position of a point $\boldsymbol{p}$, known in $\{F_3\}$ into the frame $\{F_1\}$ in function of $\boldsymbol{t}_1$, $\boldsymbol{t}_2$, $\mathbf{R}_1$ and $\mathbf{R}_2$.

Solution:

This problem can be subdivided into two smaller parts. One that stablish how to relate what happens in the frame three with what happens in the frame two and a second part doing the same with frames two and frame one. Once this transformations are obtained the composition is easy.

If the position of a point is known on the frame $\{F_3\}$, we can relate it with the position of the point seen on $\{F_2\}$ by:

$${}^{F_2}\boldsymbol{p} = {}^{F_2}\mathbf{R}_{F_3}\, {}^{F_3}\boldsymbol{p} + {}^{F_2}\boldsymbol{t}_{F_3}.$$

And we can do the same to pass from $\{F_2\}$ to $\{F_1\}$

$$^{F_1}\boldsymbol{p} = {}^{F_1}\mathbf{R}_{F_2}\,{}^{F_2}\boldsymbol{p} + {}^{F_1}\boldsymbol{t}_{F_2}$$

On these equations we can substitute the given data to obtain

$$^{F_1}\boldsymbol{p} = \mathbf{R}_1^{\mathsf{T}}\,{}^{F_2}\boldsymbol{p} + \boldsymbol{t}_1$$
$$^{F_2}\boldsymbol{p} = \mathbf{R}_2\,{}^{F_3}\boldsymbol{p} + \boldsymbol{t}_2$$

Composing both results we can extract that the target transformation is given by:

$$T(\boldsymbol{x}) = \mathbf{R}_1^{\mathsf{T}}\mathbf{R}_2\boldsymbol{x} + \mathbf{R}_1^{\mathsf{T}}\boldsymbol{t}_2 + \boldsymbol{t}_1$$

From this result the homogeneous affine matrix can be extracted as:

$$\mathbf{A} = \begin{pmatrix} \mathbf{R}_1^{\mathsf{T}}\mathbf{R}_2 & \mathbf{R}_1^{\mathsf{T}}\boldsymbol{t}_2 + \boldsymbol{t}_1 \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} \tag{2}$$

An alternative solution is to provide the affine transformation matrices to go from $\{F_3\}$ to $\{F_2\}$

$$^{F_2}\mathbf{A}_{F_3} = \begin{pmatrix} \mathbf{R}_2 & \boldsymbol{t}_2 \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} \tag{3}$$

and to go from $\{F_2\}$ to $\{F_1\}$

$$^{F_1}\mathbf{A}_{F_2} = \begin{pmatrix} \mathbf{R}_1^{\mathsf{T}} & \boldsymbol{t}_1 \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} \tag{4}$$

And then multiply it to obtain the same result that has been presented on Eq. (2),

$$\mathbf{A} = {}^{F_1}\mathbf{A}_{F_2}\,{}^{F_2}\mathbf{A}_{F_3} \tag{5}$$

Exercise 6

Given the next two quaternions:

$$\mathring{p} = \frac{\sqrt{3}}{2}\left(1,\, \frac{1}{3},\, \frac{1}{3},\, \frac{1}{3}\right)^{\mathsf{T}}$$

$$\mathring{q} = \frac{1}{2}\left(-1,\, 1,\, 1,\, 1\right)^{\mathsf{T}}$$

15 Point 1. Calculate the quaternion that corresponds to the intermediate orientation between $\mathring{p}$ and $\mathring{q}$.

Help:

α	0	30	45	60	90	120	135	150	180	210	225	240	270	300	315	330	360
$\cos(\alpha)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\sin(\alpha)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0

Solution:

Every of both quaternions represent a different attitude. In order to interpret it, we can try to obtain the angle and the axis of rotation. Every unit quaternion can be represented by

$$\mathring{r} = (\cos(\theta_r/2)\,,\, \sin(\theta_r/2)\, \boldsymbol{u}_r)^\intercal$$

We can use the scalar and the norm of the vector part to identify the angle.

In this case:

$$\cos(\theta_p/2) = \tfrac{\sqrt{3}}{2} \qquad \sin(\theta_p) = \tfrac{1}{2}$$
$$\cos(\theta_q/2) = -\tfrac{1}{2} \quad \sin(\theta_q) = \tfrac{\sqrt{3}}{2}$$

By using the tables provided it can be observed that

$$\theta_p = 60\,\text{deg}$$
$$\theta_q = 240\,\text{deg}$$

The axis of rotation is easily obtained by normalizing the vectorial part. And for both quaternions the axis of rotation is given by $\boldsymbol{u} = \frac{1}{\sqrt{3}}\,(1,\,1,\,1)^\intercal$.

Since we are seeking the quaternion that represents the intermediate orientation it will be given by the quaternion with the same axis of rotation and the angle $\theta_l = \frac{1}{2}\,(\theta_p + \theta_p)$:

$$\mathring{l} = \left(\cos(150)\,,\, \frac{\sin(150)}{\sqrt{3}} 1,\, 1,\, 1\right)^\intercal = \frac{1}{2}\left(-\sqrt{3},\, \frac{1}{\sqrt{3}},\, \frac{1}{\sqrt{3}},\, \frac{1}{\sqrt{3}}\right)^\intercal$$